LINEAR SYSTEMS REDUCTION AND ITS RELATIONSHIP TO MULTIVARIABLE CONTROL SYSTEMS SYNTHESIS

BY

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A thesis submitted to the
University of Sheffield
in partial fulfilment for the degree of
Doctor of Philosophy in Control Systems

Faculty of Engineering (Control)
University of Sheffield

Ph.D.
August 1979

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Summary

It is recognised for sometime that reduced models play an important role in control systems synthesis; as a result, much effort has been devoted to model reduction. This thesis is concerned with the study of reducing the order of the mathematical model, representing a linear system, and, the consequence of using the reduced model in the synthesis of linear and nonlinear multivariable control systems.

The effects of using the reduced model are studied in terms of stability, performance degradation, departure from optimality, sensitivity reduction, etc., of the final system. The research has a dual nature; for the results pertinent to reduced model applications are also valid for the original model of the plant in the design of the real plant. This is due to inaccuracies in the model, resulting from modelling errors, as the order of the plant cannot be accurately determined.

The mathematical methods used for obtaining analytical results, or model reduction and its applications, are linear algebra and functional analysis.

The contributions to this thesis are: the formulation of some feasible time and frequency domain reduction techniques; the establishment of multivariable theorems for reduced model applications; the derivation of some bounds for the original linear and nonlinear multivariable systems, and, the adaptation of these bounds for use in conjunction with reduced models; and the integration of the above results with certain recent mathematical developments in control systems theory.

A case study of an industrial boiler is also included, where different reduction techniques are used for obtaining low order models. These are then used for control systems design, and, their effectiveness in application is assessed by observing the final system response.

Declaration

This thesis is the original work of the author. Where results of other authors are mentioned, this is clearly stated.

To my parents

(Woon Foong How and Tan (nee) Teck Heng)

and grandparents

for their courage and inspiration.

ACKNOWLEDGEMENTS

I should like to thank Professor H. Nicholson, for his encouragement and interest in the work, his advice on preparation, and his patience and kindness. My indebtedness to him is obvious.

I should like to express thanks to, Professor S.A. Marshall (now with the University of Wollongong, New South Wales), who kindly and enthusiastically supervised this work for two years, and to Dr. D.H. Owens, for stimulating and useful discussions, on both specific and general issues.

I should also like to thank the following people for interesting discussions,and for drawing my attention to certain references:

Professor L. Mirsky, Dr. A.S.I. Zinober (Sheffield University), Dr. P.A. Cook (UMIST), Dr. S. Barnett (Bradford University) and Dr. M.J. Grimble (Sheffield City Polytechnic).

The financial assistance provided by Wison Associate Foundation in the form of a scholarship is gratefully acknowledged.

Thanks are also due to, my colleagues, Messrs. Tony Field and Vince Fernando, for their interest and discussions, to Messrs. Patrick T.K. Fung and T.P. Chiam for their assistance, and, to Mrs. J. Czerny and Mrs. A.L. Barrett for their careful typing of the thesis.

List of Common Symbols

S	Original system(model)
S_r	Reduced system(model)
$q_{ij}(s)$	ij^{th} element of matrix $Q(s)$ (sometimes written $Q_{ij}(s)$)
$Q^t(s)$	transpose of $Q(s)$
$Q^*(s)$	complex conjugate of $Q(s)$, i.e. $Q(-s)$
$Q^H(s)$	Hermitian part of $Q(s)$, i.e. $Q^t(-s)$
$\hat{Q}(s)$	inverse of $Q(s)$, i.e. $Q^{-1}(s)$ (applies in general to all matrices and scalars, unless otherwise stated)
$A > B$	reads $A-B>0$, i.e. $A-B$ is positive definite (unless otherwise stated, for nonnegative matrices A and B, $A-B>0$ means $a_{ij}-b_{ij}>0$, i.e. positive, for all i,j.)
$\in$	belongs to a certain region or contains in a certain set
$\cup$	union of sets
$\cap$	intersection of sets
$\exists$	there exists
$\forall$	for all
$\langle x,y \rangle$	inner product defined in Euclidian space, E_n
$\langle x,y \rangle_{H_n}$	inner product defined in Hilbert space, $L_n^2(0,T]$
$x\rangle\langle y$	dyadic product of vectors x and y
E	exponentiation to base 10, (e.g. 0.1E-02 means 0.1×10^{-2})
D	unless otherwise stated, the contour consisting the imaginary axis and the semi circle of infinite radius in the right half s - plane
O	unless otherwise stated, the contour consisting of the unit circle on the origin in the z-plane
iff	if and only if
Rge	range of
Nsp	null space of
i.s.L.	in the sense of Lyapunov

CONTENTS

Many queries needed to be answered

Ten thousand years are too long

The mountain is not obstructed with iron

This day we shall take the first step
to scale its summit.

I

INTRODUCTION TO THE THESIS

CHAPTER I.

INTRODUCTION TO THE THESIS

Introduction

This thesis attempts to study the various methods of reducing the order of a linear dynamical system and the consequences resulting from the application of the reduced system (or appropriately called, reduced model). The reduced system can be used in simulation and design work, and, the effects of its application can be studied in terms of stability, performance degradation, departure from optimality and sensitivity, etc. of the overall system.

The nature and applications of linear systems reduction are briefly presented in the next section, from which a new problem to the 'model reduction theme' is formulated, the latter giving justification to the motivation for research.

To assist the reader, the general organisation and outline of the thesis are then explained. (It is recommended that this chapter be read before reading the other chapters.)

The main contributions to the thesis are stated at the end of the chapter.

1.1 Linear systems reduction.

1.1.1 Its nature.

The necessity to reduce the order of a mathematical model has been conscious to engineers for a long time. It is felt that a lower order model, that approximates the transient response of the original (higher order) model, could be used as a substitute for the latter, in simulation studies and synthesis of control systems. This is advantageous, as it results in computation and time savings, and, economising computer storage.

In classical control studies, 'reduced (lower order) models' were used sub-consciously in design and approximate analysis, the simplest example, being that first and second order models were used freely in frequency response and root locus plots. The earliest reduced models used, were obtained by unsystematic 'common sense' rules. It was not until the early 1950's, encouraged by the frequent use of electronic calculating machines, that detailed and systematic 'ways of approach' were paid to model reduction, so much so, that the latter has become a 'state of the art'. Up to date, more and more methods evolved, some are different from others but most are variations of the same theme. As most methods are technique oriented, and emphasis placed on sophistication in reduction, rather than on practicability and effectiveness, the topic of model reduction, where it is supposed to be a valuable tool than to be a mere toy, has come under some criticisms.

Closely related to model reduction is the topic of approximate models, due to inexact modelling. Very often, in modelling a plant from its physical equations, or from experimental data, obtained from its input-output characteristics, inexact models are only obtained. This is because all detailed plant information cannot be precisely known, and, its behaviour can change due to internal parameter variations or external influences. Thus the order of the plant is different, usually lower, from that of the actual plant. However, past studies seemed to have treated model reduction and inexact modelling as two different types, and, very little effort had been made to correlate the two.

1.1.2 Its applications.

Reduced models can be used in the following areas:

(1) Simulation studies.

(2) Optimal control systems synthesis.

(3) Multivariable systems design using linear state feedback or frequency response.

(4) Model reference adaptive systems.

(5) Sensitivity analysis study.

There are many cases in practice where complex models can be replaced by their lower order equivalents for simulation purposes. Examples can be drawn from the field of power systems and chemical engineering. The order of the derived model may be over a hundred, where a large number of grid points may be taken that produce a large set of differential equations, or, in chemical process plants, too many state variable points may produce a complicated model.

In optimal control studies, computational methods are often employed to obtain non-analytic solutions for complicated cost functions. The chief examples are, the use of mathematical programming techniques to obtain the optimal state trajectories, and, the use of iterative algorithms to compute nonlinear performance indices. Essentially they are step by step techniques that require a great deal of computing time and storage. Model reduction is useful here in the sense that, when the high order model is replaced by its reduced counterpart, the iterative process is not only speeded up, but, the problem also becomes more simplified.

For multivariable systems, a controller is usually required for use in a closed loop system to perform a specific task. If the model of the system is high, designing the controller would be tedious. Hence if a reduced model is used, the design would become simpler and faster. The design approach can be taken from the time domain or frequency domain. In the time domain design can be done using linear state feedback theory. A reduced model would thus produce a sub-optimal controller, (if design is done using optimal regulator theory) for the original system, whose dimension is lesser than that that could be designed without using the reduced model. The difference in dimension is proportional to the difference in order between the two models. In the frequency domain, where design is done using the extended Bode or Nyquist plots, considerable simplification in design is possible by using first or second order models, to approximate the frequency and phase characteristics of the high order model. This is very desirable as frequency response methods are more tolerable to model inaccuracies and less sensitive to errors.

In model reference adaptive systems (MRAS), the original plant is desired to be controlled by using a parallel model of the same order as the plant. The objective is to require the controlled plant to respond exactly as the model. Here the model is used as the reference and the plant loop is adaptive in such a way as to accommodate changes in the plant to changes in the model or to external disturbance. An interesting aspect of MRAS is Linear Model following systems (LMFS) where control is applied in such a way that the response of the plant follows that of its linear model. Model reduction finds application here, in that a reduced model of the plant, can be used to effect control of the system as a whole. This strategy is justified in the sense that the plant model is only a close approximation to the plant, for the latter's dynamics are not precisely known. Under these conditions the synthesis of the control system will be much simplified. The situation can also be viewed from another way by using the plant model to represent the actual plant, and the reduced model to represent the plant model. This arrangement is ideal for fast,'off-line' simulation studies and economical controller synthesis.

Feedback systems tend to reduce parameter sensitivity and attenuate external disturbances. In a controlled system, parameter sensitivity plays a great part in determining system performance and stability.

Using reduced models in the feedback scheme will reduce the sensitivity of parameters as there are fewer parameters in the reduced model. In some aspects of optimal control, where parameter sensitivity is an acute problem, a 'parameter sensitivity' model is modelled separately from the dynamical model to enable better analysis of the whole system. Thus, a reduced 'parameter sensitivity' model can be used instead.

1.2 Problem formulation and motivation for research.

It is observed, from the applications of model reduction, that the stability of the original system is uncertain under the influence of the reduced model controller. It is also expected that degradation of performance would result when reduced models are used to simulate, control or design the system.

Considering the nature of reduction, reduced models and 'inexact modelling' cannot be viewed as two different topics, but should be amalgamated. Throughout the thesis, where S and S_r represent the original and reduced models respectively, they could also be taken to represent the plant (assumed linear, though this restriction can be removed in certain cases) and plant model (equivalent to the inexact model) respectively. Thus all analytical results obtained from the study of reduced and original models are also applicable to the plant model and plant. The results are thus less restricted, hence they have wider interpretations.

From the above formulations, it is motivating to pursue research along the following lines.

Any new reduction methods that are introduced, must be as feasible as possible, and, their theoretical framework must be justified from a practical viewpoint, and related to close concepts in dynamical systems.

An attempt should be made to justify the reduction techniques by applying them to practical and very high order models. The reduced models, so obtained, should be used to design a controller for the original system, and, the effects that follow, such as performance, sensitivity and stability, should be observed. This would make the problem more meaningful and realistic.

An attempt should be made to build a general theoretical framework, chiefly in the areas of error bounds and stability, of a system under the influence of a controller synthesized by a reduced model. General stability theorems, which indirectly infer the stability of the original system from the stability of the reduced system, irrespective of the method of reduction used, are needed. This, so far, has been a neglected topic. Adaptation of certain existing design methods, to incorporate design using reduced models, is beneficial.

The sensitivity and optimality of a system under the influence of a reduced model controller must be studied. It is interesting to explain the departure from sensitivity or optimality of a system in terms of the characteristics of its reduced counterpart.

The idea of using reduced models in nonlinear systems design, especially in Model Reference Adaptive Systems, and the subsequent effect on stability, deserves some investigation. This is a novel approach, and it seems has never been introduced before, nevertheless, it is worthwhile pursuing as nonlinear systems play an integral part in practice.

Any work that has been done in the above areas will be reviewed in Chapter II.

1.3 General organisation and outline of the thesis.

1.3.1 Layout and nomenclature.

Each chapter is more or less independent and self contained, emphasis on strict continuity throughout the chapters is slightly relaxed. For ease of reference, the bibliography pertinent to a particular chapter is given at the end of the chapter concerned. Diagrams, where appropriate, are given adjacent to the page of description, otherwise, they are included at the end of the chapter if references to them are being made constantly throughout the pages. In general, the subscript r, of a quantity, indicates that the quantity describes the reduced model, and, the same quantity without the subscript, means that it describes the original model. The words, 'system' and 'model', are used interchangeably throughout the thesis, distinction only being made if the meaning becomes impaired. Thus S and S_r are taken to represent the original and reduced models respectively, or, the original and reduced systems respectively. Standard nomenclature in control systems theory are used throughout the text, explanation is given only when there is ambiguity. Examples are:

$\hat{Q}(s)$ means $Q^{-1}(s)$ (the inverse of the matrix $Q(s)$)

$\hat{G}_r(s)$ means $G_r^{-1}(s)$ (the inverse of $G_r(s)$)

$\hat{k}$ does not necessarily mean $1/k$ (unless stated otherwise), it means $\hat{k}$ is associated with $\hat{Q}(s)$ as k is associated with $Q(s)$.

The main results are given in the form of theorems, and, their proofs are given as simple as possible. Where convenient, illustrations are given to aid the proofs.

1.3.2 Outline of the thesis.

Chapter II gives a brief but thorough survey of model reduction techniques and their applications. The survey is by no means detailed or exhaustive.

A new frequency domain reduction method using harmonic synthesis method for single input-single output system is given in Chapter III. The idea is extended to multivariable systems by using the Characteristic Loci concept approach.

Chapter IV details some new time domain reduction methods, from the complex plane and state space matrices point of view. The methods are 'design oriented' and use approximating techniques for curve fitting in the time domain. The criterion for reduction is not specified, and, it is open to the subjective judgement of the designer. One method relies on the estimation of residues and modes and the other on the pseudo-inverse of a matrix. However, both methods work on the principle of sequential approximation, where the model is continuously reduced and updated.

All reduction methods in Chapters III and IV are tested using a seventh order and a fourth order model.

The stability of linear multivariable systems under the influence of a reduced model controller is studied in Chapter V. In all cases stability of S is expressed in that of S_r. Some stability theorems for reduced model applications are derived using frequency response methods and Characteristic Loci criterion. Stability theorems are also derived by using the theory of M-matrices to study linear multivariable systems. This, together with the use of the contraction mapping principle, from functional analysis, widens some results, and, confirms some, derived earlier. The study on the characteristic polynomial of a system leads to some interesting results, and, the newly formed concept of Reimann surfaces and multivariable root loci is used in linking the stability of S_r and S. Lyapunov theory is also used to express certain stability bounds for the reduced system. The above methods have one factor in common; in that they all study the distribution of the characteristic roots in the large. The chapter ends with the adaptation of the methods for discrete systems.

Chapter VI studies bounds on multivariable systems designed using reduced models. The analytical bounds are mainly expressed in transfer function matrices form, but could also be adapted to time domain design methods. Some modified bounds for the general multivariable system and a stability theorem for reduced models are also derived. The error estimates for using reduced models are expressed in the form of linear inequalities, using matrix theory and functional analysis. Some existing multivariable bounds are also modified, to cater for the inclusion of reduced models. The rest of the chapter adapts these error bounds, together with the stability results in Chapter V, to some existing multivariable design techniques; example Inverse Nyquist Array method, Characteristic Loci method, pole shifting method and multivariable root loci method, etc., for use with reduced models. The results in Chapters V and VI form a general design philosophy of systems using reduced models.

The effect on the sensitivity and optimality characteristics of a system, when reduced models are used, is investigated in Chapter VII. The emphasis is stressed on Comparison Sensitivity, in Bode's sense, in the frequency domain. Results are derived for using reduced models in sensitivity reduction design where the closed loop system behaves better than the corresponding open loop case. Conditions for the stability of a sub-optimal system are also investigated, by imposing bounds on the matrix Riccati equations. The relationship between sensitivity and optimality in the application of reduced models is derived. The chapter rounds off by adapting some sub-optimal design methods, for use with the general reduced model, such that stability is guaranteed.

Bounds for nonlinear multivariable systems, especially in the presence of a reduced model, are studied in Chapter VIII; the main emphasis is again on stability. The nonlinear system considered here is composed of a linear block, followed by a nonlinear block where the nonlinearities of the latter are either sector restricted, functionally approximated or bounded by integral constraints. Studies are also made on the stability of the original nonlinear system, in the absence of the reduced model, using the describing function method and the extended graphical method of Popov. It is then shown that these results can be more easily adapted for use with reduced models, thus increasing their power, than existing methods. Using reduced models in Model Reference Adaptive Control Systems design is also investigated, using the concept of positive realness of a matrix and the hyperstability criterion. Some existing MRACS design methods are adapted along this line to suit the use of reduced models.

A case study of an industrial boiler is done in Chapter IX. Prior to reduction, the transfer function of the boiler model is first obtained and manipulated into various amenable forms. The order of the boiler model is very high (thirty three) and various reduction methods, both of time and frequency domains, are used to obtain lower order models. From this, a critical assessment is made on reduction methods; on their accuracies, difficulties and advantages. The reduced models so obtained are next used to design controllers for the original model. Two different design philosophies are considered; one an 'algorithmic computer oriented' method from the time domain, the other 'a state of the art' method from the frequency domain. Comparisons are then made on the boiler response under different controller actions. This enables the validity of reduced models to be assessed, and also gives a further assessment of their reduction methods.

Chapter X concludes the thesis.

1.4 Contributions to the thesis.

All theorems, listed numerally and accompanying an alphabet, example, theorem 5A, 5B, 6A, 7B, etc. are known multivariable system theorems. Only a limited number are stated, when useful, for reference. All other theorems derived, that are pertinent to linear systems reduction, or related to the general system, are obtained by the author, as far as multivariable control systems theory is concerned, and, to the best of the author's knowledge, have not been used before in the control literature. All other results that have been obtained by other authors are also clearly stated.

The frequency domain reduction technique, for single input-single output systems, presented from a different viewpoint, based on harmonic synthesis, in Chapter III is original. A novel approach to multivariable systems reduction is introduced, by considering the behaviour of the system's characteristic loci, as the latter is known to govern the dynamics of the system. This concept is believed to be new, and, to the best of the author's knowledge, has never appeared before in the literature. A part of this chapter has appeared in a letter in Electronics Letters.

The two reduction methods, sequential approximation in the time domain, presented in Chapter IV are original. One method works on the principle of 'mode and residue' aggregation, the other uses 'least square error minimisation' from the state space point of view. Both methods are combinations of existing methods, removing some of their undesirable features and inherent difficulties, and they work best when adapted to use with interactive graphics. Stability of the reduced model with the methods is always guaranteed.

In Chapter V, the stability theorems given in the frequency domain for a multivariable system under the influence of a reduced model *controller, irrespective of the reduction method employed, are general, original and believed to be new. Some of the ideas have appeared as a letter, in Electronics Letters. (An existing theorem, due to Aoki, Vittal Rao, et al is only applicable to systems that used a restricted class of reduced models, namely those obtained by projection methods, only.) These theorems are advantageous since they are flexible and can be used directly with any design methods. The idea of using the methods of M-matrices and the contraction mapping principle to study stability in terms of the reduced system is original, although the methods have been used independently before, to investigate stability for the original linear system directly. The contribution here lies in the interpretation of these results in graphical form and the linking of them to the stability theorems established earlier, thus widening their interpretations. The adaptation of these theorems to the recently developed multivariable root loci concept, for stability investigation, is original, and, believed to be new. Studying reduced model stability, from the inertia of the matrix of the characteristic equation and obtaining new graphical interpretations for the stability margins, are also original.*

The modified bounds derived for the original linear multivariable system in Chapter VI appear to be new, though they are known to exist in various related forms. The advantage of these bounds is that they are expressed in simple forms, in various degress of sharpness, and require modest computational effort for their evaluation. A stability theorem for reduced model application is also given here as an alternative to that similarly given in Chapter V. The advantage of this theorem is that its bounds are easier to evaluate, although this may be compensated by loss in sharpness. However, modifications are also made where the *sharpness can be adjusted in relation to computational complexity. The error bounds, due to using reduced models, expressed in the frequency domain are original. The method of preserving interaction structure of the system during reduction is introduced in this chapter. It is proved that this has desirable stability and performance characteristics. The idea has appeared as part of a letter in Electronics Letters.*

In Chapter VII, some theorems are given regarding the role of reduced models in sensitivity reduction design. A theorem is given here with gives priori conditions, for the stability of sub-optimal control systems. This is believed to be new and, to the author's best knowledge, no results regarding sub-optimal stability, for the multivariable system using a general class of reduced model, have appeared before in the literature.

A criterion given in Chapter VIII for the absence of limit cycles in nonlinear systems is new. It is believed that the proposed criterion is more flexible and less conservative than existing criteria, and, its main power is the ease to which it can be adapted to work with reduced models. Part of this idea has appeared as a letter in Electronics Letters. The graphical interpretations of the circle criterion and the Popov criterion for multivariable systems, different from existing interpretations, are original and believed to be new. The bounds are more flexible and adjustable, hence less conservative. Formulated in this way they are also very convenient for use with reduced models. An original approach to estimating performance bounds between S and S_r in nonlinear systems design, using integral inequality estimates, is believed to be novel. The advantages of the estimates is that they can be evaluated easily, requiring modest computational effort.

All simulation results and evaluations given in Chapter IX are original.

Conclusions.

Most of the mathematical results, where possible, are explained in simple graphical forms, in terms of frequency response and characteristic root loci plots. This would prove to be of advantage to the designer, who is familiar with graphical methods.

II

A GENERAL SURVEY OF MODEL REDUCTION TECHNIQUES AND THEIR APPLICATIONS.

CHAPTER II.

A GENERAL SURVEY OF MODEL REDUCTION TECHNIQUES AND THEIR APPLICATIONS.

Introduction:

Approximate models were used by Evans[45], Biernson[44] and others[46] for control system design in the early 1950's. They were obtained by 'rules of thumb', and, it is believed that a systematic mathematical procedure for model reduction was started by Dudnikov[4], Kalyaev[6], Kardashov[7] and Golant[5] et al, after the mid 1950's, in the U.S.S.R. There was a great interest in model reduction in this country, after Nicholson[84] used a reduced model to obtain control strategies for a boiler model. Up to date there is a vast number of reduction techniques in the literature and they can be classified into the following 'schools of approach'.

(i) modal synthesis

(ii) geometrical methods

(iii) time domain curve fitting

(iv) frequency domain curve fitting

(v) Pade approximation techniques

(vi) general methods.

The above headings are only taken as a guide, and are not necessarily in chronological order. No hard and fast rules are laid on them, as in most cases they overlap, and one method can fall in either category. The methods in the last heading are either general combinations of those of the above headings, or, do not fall into any headings at all.

2.1 Survey of reduction techniques[1-81].

(i) Modal synthesis[8-13].

The basic philosophy here is the retention of dominant modes and rejection ofnondominant ones, based on the fact that poles near the origin dominate the response and transients of faraway poles decay rapidly. Davison[8] gave a first mathematical formulation of the problem by transforming S(A,B,C) into its canonical structure and rearranging states in order of dominance.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ \dot{z} &= \Lambda z + Fu \\ \Lambda &= TAT^{-1}, \quad x = Uz \end{aligned} \tag{2.1}$$

By retaining the first ℓ state variables a reduced model is obtained as $A_r = A_o + A_1 \Lambda_1 \Lambda_o^{-1}$. It was pointed out by Chidambara[9], that Davison's method produces a steady state error, and, this later caused the authors and Fossard[10], to modify the method, to eliminate steady state error. Marshall[11] also produced a similar method that yields zero steady state (s.s.) error by partitioning the matrices in eqn.(2.1) as

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}, \quad F = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$$

$$U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \quad \text{and z partitioned as} \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

By equating $\dot{z}_2 = 0$ a s.s. reduced model is obtained as

$$A_r = U_1 \Lambda_1 U_1^{-1}, \quad B_r = B_1 - A_2 V_4^{-1} \Lambda_2^{-1} (V_3 B_1 + V_4 B_2)$$

Kappulajuru and Elangoven[12] noticed that the modal method of reduction has certain inaccuracies in certain regions of the transient response. They divided the response time into three regions, and used different methods to approximate the response, by considering dominant, sub-dominant and non-dominant modes in order.

Wilson[13], Fisher et al. modified the modal methods of Davison and Marshall for discrete systems, by considering the equation

$$x_{k+1} = A\,x_k + B\,u_k$$

(ii) Geometrical methods[14-19]

Geometrical methods of reduction rely on the geometry of vector spaces where a reduced model is obtained by manipulating the model structure in vector spaces. Essentially geometrical methods are related to time domain curve fitting methods. Examples of geometrical methods are due to Anderson, Mitra, Nordall, De Sarkar[16] et al. and Shaked et al.

Anderson[14] formulated the problem of reduction from the time domain response equations,

$$x^{(r)}\{(k+1)T\} = \Phi(T)\,x^{(r)}(kT) + \Delta(T)\,u(kT) \tag{2.2}$$

i.e.
$$b_q = Mc_q \tag{2.3}$$

$$\bar{x}\{(k+1)T\} = \bar{\Phi}\,\bar{x}(kT) + \bar{\Delta}(T)\,u(kT) \tag{2.4}$$

i.e.
$$\bar{b}_q = Mc_q \tag{2.5}$$

where eqns (2.3) and (2.5) represent eqns (2.2) and (2.4) in matrix form, with $b_q^t = (x_q^{(r)}(T),\ldots x_q^{(r)}((k+1)T))$, $c_q^t = (\emptyset_{q1},\ldots\emptyset_{qr},\Delta_{q1},\ldots\Delta_{qm})$ and M is a rectangular matrix with an $(i+1)^{th}$ row given by $M_{i+1} = x_1^{(r)}(iT)$, $x_2^{(r)}(iT),\ldots x_r^{(r)}(iT)$, $u_1(iT)$, $u_2(iT),\ldots u_m(iT)$. The aim of the reduction is to choose a vector c_q such that equation (2.5) is satisfied and also minimizes the inner product $\langle(b_q-\bar{b}_q), (b_q-\bar{b}_q)\rangle$. Eqn (2.5) can be generalized to the form B = MC, and from the theory of linear spaces, the' solution is given by the taking the pseudo-inverse of M in the equation

$$B = MC = M\,[\bar{\Phi}(T)\ \bar{\Lambda}(T)]^t \tag{2.6}$$

$$C^t = [\bar{\Phi}(T)\ \bar{\Lambda}(T)] = B^t M (M^t M)^{-1} = B^t M^+ \tag{2.7}$$

where $C^t = (c_1, c_2, \ldots c_r)$, $B^t = (b_1, b_2, \ldots b_r)$ as in eqn (2.3). The pseudo-inverse M^+ is equivalent to least square curve fitting in the time domain and the time interval T is chosen large enough to ensure that M^+ exists.

Mitra[15,77,82] considered reduction by projecting a vector onto a best linear sub-space, and, minimizing the projection error. The reduction is done in two stages as follows.

$$\begin{matrix}\text{Controllable} \\ \text{system } S\end{matrix} \xrightarrow[\text{reduction}]{\text{approximate}} \begin{matrix}\text{Uncontrollable} \\ \text{system } \hat{S}\end{matrix} \xrightarrow[\text{reduction}]{\text{strict}} \begin{matrix}\text{Reduced} \\ \text{system } S_r\end{matrix} \tag{2.8}$$

$\hat{S}$ has a controllable subspace of dimension n-m and is governed by $\hat{S}$: $\dot{\hat{x}}(t) = \hat{A}\,\hat{x} + \hat{B}\,u$, $\hat{y} = \hat{C}\,\hat{x}$. The optimal projection matrix P is such that $\hat{x} = Px$ where the projection is on and along the sub-spaces ε_1 and ε_2 of dimensions n-m and m. $\hat{S}$ is related to S by $\hat{A} = PA$, $\hat{B} = PB$. Here P is chosen as $P = I_n - T_1\,(T_1^t\,W^{-1}(\infty)\,T_1)^{-1} T_1^t\,W^{-1}(\infty)$, where $dW(t)/dt = \exp(A(t_1-t))BC^{-1}B^t\exp(A^t(t_1-t))$ and T_1 is an orthogonal matrix obtained from the modal matrix of A by the Gram Schmidt orthogonalization procedure. The controllable subspace ε_1 is given by solutions of the equation $(T_1^t\,W^{-1}(\infty)T_1)^{-1}T_1^t\,W^{-1}(\infty)x = 0$. If any n-m independent solutions of x form the columns of the matrix, F, the final stage of strict reduction yields S_r as

$$A_r = (CF)\,(F_{11}^{-1}\,\hat{A}_{11}F_{11} + F_{11}^{-1}\,\hat{A}_{12}F_{21})\,(CF)^{-1}\ ,\ B_r = (CF)\,F_{11}^{-1}\,\hat{B}_{11} \tag{2.9}$$

where

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \hat{B} = \begin{bmatrix} \hat{B}_{11} \\ \hat{B}_{21} \end{bmatrix}, F = \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix}, \hat{A}_{11} \triangleq (n-m)x(n-m), \ \hat{B}_{11} \triangleq (n-m)x\ell. \quad F_{11} \triangleq (n-m)x(n-m)$$

The chief disadvantage of Mitra's method is that it is computationally expensive, and is difficult to implement.

Nordahl[17] and Melsa constructed a reduced model by matching the shape of the hypersurfaces of the Lyapunov functions, V and V_r, as the latter determine the response of the corresponding system. The procedure involves minimising angles, between hypersurfaces in r-space, but, due to the geometrical complexity, the method is not reliable for high order models.

De Sarker[16] and Dharma Rao used algebraic rather than geometrical approach in the above procedure. For the autonomous system $\dot{x} = Ax$, the associated Lyapunov matrix equation can be written as $PA + \frac{1}{2}Q = -A^tP - \frac{1}{2}Q = S$, where $A = P^{-1}(S - \frac{1}{2}Q)$. A reduced model A_r is constructed such that its Lyapunov function

$$\dot{V}_r/V_r \triangleq \dot{V}/V \qquad (2.10)$$

in r-dimensional space. The matrices P_r and S_r are determined by deleting n-r smallest rows and columns of P and S respectively and the reduced model is given by

$$A_r = P_r^{-1} (S_r - \tfrac{1}{2}Q_r) \qquad (2.11)$$

For the forced system $\dot{x} = Ax + Bu$, x is partitioned as $x^t = (x_1, x_2)$ and B_r is obtained by fitting original and reduced models, at steady state point, i.e. setting $\dot{x}_2 = 0$, giving

$$B_r = A_r (C_{11} B_{11} + C_{12} B_{12}) \qquad (2.12)$$

where $B = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}$, $C = A^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$

The main disadvantage of the above method, lies in the difficulty of choosing the P matrix, and, the choice is related to the nature of the system, just as the response depends on the latter.

Shaked[18] et al, looked at reduction from the transmission zeroes and zero directions, point of view. The method exploits the structural properties of the state space matrices, and, a reduced model is obtained by making zero direction vectors, of the reduced model, to be obtained, to coincide with that of the original model.

With geometrical methods, they are largely intuitive in approach, and, do not give an indication of the 'goodness' of the reduced model so obtained.

(iii) time domain curve fitting[20-36].

Time domain curve fitting, involves fitting the time response of S_r to that of S, usually via minimization of a performance index. Wilson[20] considered reduction, from the state space matrices where $S_r(A_r,B_r,C_r)$ is to be obtained from S(A,B,C) via

$$J = \min \int_0^\infty \langle e(t), Qe(t)\rangle \, dt \triangleq \text{trace } (PS) \tag{2.13}$$

or $$J = \min \lim_{t\to\infty} E\{\langle e(t), Qe(t)\rangle\} \triangleq \text{trace } (RM) \tag{2.14}$$

where $e(t) = y(t) - y_r(t)$. The input can either be a vector of impulse functions, or, a white noise vector, with zero mean, and a covariance matrix $E\{u(t)u^t(s)\} = N\delta(t-s)$, where $N > 0$ and symmetric. Associated with eqns (2.13) and (2.14) are the Lyapunov equations

$$FR + RF^t + S = 0 \tag{2.15}$$

$$F^tP + PF + M = 0 \tag{2.16}$$

where

$$F = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}, \quad S = \begin{bmatrix} BNB^t & BNB_r^t \\ B_rNB^t & B_rNB_r^t \end{bmatrix}, \quad M = \begin{bmatrix} C^tC & -C^tC_r \\ -C_r^tC & C_r^tC_r \end{bmatrix} \tag{2.17}$$

and P and R are symmetric matrices of the Lyapanov equations and can be suitably partitioned in conformity with eqn.(2.17). Taking the derivations $\partial J/\partial a_r$, $\partial J/\partial b_r$, $\partial J/\partial c_r$, where a_r, b_r, c_r are the elements of A_r, B_r and C_r, one obtains

$$B_r = -P_{22}^{-1} P_{12}^t B \; ; \; C_r = CR_{12} R_{22}^{-1} \; ; \; A_r = -P_{22}^{-1} P_{12}^t AR_{12} R_{22}^{-1} \tag{2.18}$$

If the eigenvalues for A_r are prespecified, thus fixing a canonical form for A_r, then the solution for S_r is a linear one. If the eigenvalues are not specified, then a minimization algorithm must be used iteratively to obtain S_r, for single input-single output systems. For multivariable systems[76], the principle of superposition and eqns (2.15) to (2.18) can be used to obtain (A_r,B_r,C_r) iteratively.

Aplevich[21] formulated the same problem, by considering a discrete time case, along the conceptual lines of controllability and observability. The cost function to be minimized is,

$$J = \sum_{k=1}^{\infty} \| \hat{Y}_k - \hat{Y}_k \|^2 \tag{2.19}$$

where $\hat{Y}_k = CA^{k-1} B$, k=1,2..., and, Y_k is a weighting matrix of S(A,B,C), at time k. It is assumed that S is completely controllable and observable, and, that the controllable and observable part of $S_r(A_r,B_r,C_r)$, is at least of order r. By differentiating J w.r.t. the elements of A_r,B_r,C_r, one obtains the Lyapunov equations,

$$E^tEA_rDD^tA_r^t - E^tEDD^t + E^tEB_rB_r^t = 0 \tag{2.20}$$

$$E^t\hat{E}A\hat{D}D^tA_r^t - E^t\hat{E}\hat{D}D^t + E^t\hat{E}BB_r^t = 0 \qquad (2.21)$$

where $D = (B_r, A_rB_r, A_r^2B_r, \ldots)$, $\hat{D} = (B, AB, A^2B \ldots)$, $E^t = (C_r, C_rA_r, C_rA_r^2, \ldots)$ and $\hat{E}^t = (C, CA, CA^2, \ldots)$. Like Wilson's method, a minimisation routine, example a gradient search method, can be used to obtain S_r (A_r, B_r, C_r). Alternatively, Aplevich[22] offered an approximate solution for S_r, using simple matrix manipulations, via the modified form of B.L. Ho's algorithm and optimality. The matrices $\hat{Y}_k$ are arranged as

$$\begin{bmatrix} \hat{Y}_1 & \hat{Y}_2 \cdots\cdots \hat{Y}_n \\ \vdots & \\ \hat{Y}_n & \hat{Y}_{n+1} \cdots \hat{Y}_{2n-1} \\ \hat{Y}_{n-1} & \cdots \quad \hat{Y}_{2n} \end{bmatrix} \xrightarrow[\text{column operations yield}]{\text{where row and}} \begin{bmatrix} I_r \; A_r \; B_r \\ \cdots\cdots\cdots\cdots \\ \\ \end{bmatrix}$$

The chief disadvantage of state space formulations by minimising the error criterion, is that, it involves too much labour in manipulating matrices, and, solving large order matrix equations. Another formidable problem is core storage, example, for a system whose order > 15, solution by Aplevich's approximate method is formidable and uneconomical; and, to reduce the core storage, would result in a poorer approximation. Also, with a large matrix, the great amount of computation can yield a final solution whose accuracy is questionable.

Galiana's[23] method is also very similar to that of Wilson and Aplevich, in that, he generated the reduced state space matrices, from input/output data. Bandler[24], et al, used gradient methods that are developed recently for reduced model solution. These algorithms provide faster rate of convergence than the steepest descent method, used by Aplevich. Chidambara[25] also provided a method of reduction, based on error minimisation, in the time domain. A set of non-linear equations result in the end as usual.

Instead of using a state space formulation, Renganathan[26] represented the n^{th} order model by a single differential equation

$$y^{(n)} + a_{n-1}\, y^{(n-1)} + \dots\ a_o y = 0 \tag{2.22}$$

and a reduced model

$$y_r^{(m)} + b_{n-1}\, y_r^{(m-1)} + \dots\ b_o y_r = 0 \tag{2.23}$$

is constructed by equating term by term the integrals $\int_o^\infty t\, [y^{(i)}]^2\, dt$ and $\int_o^\infty t[y_r^{(i)}]^2\, dt$ of eqns (2.22) and (2.23).

Sinha and Pille[27] considered reduction via using least square curve fitting of the responses of the discrete models. The reduced discrete model is expressed in terms of the pulse transfer function $H(z) = C(z)/R(z) = (a_o + a_1 z^{-1} + \dots\ a_m z^{-m})/(b_o + b_1 z^{-1} + \dots\ b_n z^{-n})$, or eqivalently,

$$c_i = \sum_{j=o}^{m} z_d\, r_{i-j} + \sum_{j=1}^{n} b_j\, c_{i-j} \tag{2.24}$$

Eqn (2.24), in matrix form is $\quad A_k\, \emptyset = C_k \qquad (2.25)$

where $A_k = \begin{bmatrix} r_o & r_{-1} \cdots & r_{-m} & c_{-1} & c_{-2} \cdots & c_{-n} \\ r_1 & r_{1-m} & r_{1-m} & c_o & & \\ \vdots & & & \vdots & & \\ r_k & r_{k-1} & r_{k-m} & c_{k-1} & & c_{k-n} \end{bmatrix}$, $\quad \emptyset^t = (a_o a_1 \dots a_m b_1 b_2 \dots b_n)$

$C_k^t = (c_1 c_2 \dots c_k)$

The least squares solution of eqn (2.25) is

$$\hat{\emptyset}_k = [A_k^t A_k]^{-1} A_k^t C_k = A_k^+ C_k \tag{2.26}$$

The continuous transfer function is obtained by taking the inverse z-transform of H(z) prece'ded by a zero order hold. By superposition, the above can easily be adapted to multivariable systems.

Meier and Luenberger[28] attempted reduction from the continuous transfer function of the system. They took into account random and deterministic inputs, and, for the former, they showed that model reduction is also related to the Wiener filtering problem. In fact the state space method of Wilson is a replica of their method. The error criterion to be minimised is

$$J = E\ \{|Y(t) - \hat{Y}(t)|^2\} = (1/2\pi j) \int_{-j\infty}^{j\infty} |T(s)-\hat{T}(s)|^2 \Phi_x(s)\ ds \qquad (2.27)$$

where $Y(t)$, $\hat{Y}(t)$ are outputs, $\Phi_x(s)$ is the rational power spectral density (bilateral Laplace transform of $E\{X(\tau)X(t+\tau)\}$ of a stationary random process $X(t)$, with $E[X(t)] = 0$.) The transfer function $\hat{T}(s) = \sum_{i=1}^{m} \hat{r}_i/s-\hat{p}_i$, and the optimum parameters $\alpha = \hat{r}_i,\hat{p}_i$ $i=1,..\ m$ is obtained by setting $\partial J/\partial\alpha = 0$. If the poles are specified, the set of equations $\partial J/\partial\alpha = 0$ can be reduced to a linear set; otherwise, a minimisation algorithm is needed to evaluate the parameters.

Riggs and Edgar[29] used a simplified version, by also expressing the transfer functions, inpartial fraction forms, $H(s) = \Sigma\ r_i/s-p_i$, $H_r(s) = \Sigma\ r_{ri}/s-p_{ri}$, with the modes written in time domain as,

$$y(t) = \sum_{j=1}^{n} r_j \exp\ \{p_j t\}\ ,\ y_r(t) = \sum_{j=1}^{m} r_{rj} \exp\ \{p_{rj} t\} \qquad (2.28)$$

and minimising the functional $J = \int_a^b \langle(y-y_r),Q(y-y_r)\rangle\ dt$. A set of non-linear equations, which becomes linear when the poles are specified, is obtained by setting $\partial J/\partial r_{ri} = \partial J/\partial p_{ri} = 0$.

Instead of obtaining the optimal parameters from the non-linear equations, Sinha[73] and Bereznai obtained them by using a hill climbing technique to minimize the quadratic performance index J.

Another time domain reduction approach is by moments approximation which does not minimise a performance index. Moments approximation or moments matching, which is also equivalent to the Pade approximant and continued fraction methods[32] of reduction, that will be discussed in a later section, is a very old analytical tool, used by Paynther[30] and Ba Hli[31] in network synthesis. Horowitz[34], later also Bosley[32] and Lees[33] employed moments approximation to model reduction. The transfer function is expanded as

$$G(s) = \int_o^\infty f(t) \exp\{-st\}\, dt = \alpha_o - \alpha_1 s + \alpha_2^2 s^2/2! + \ldots \qquad (2.29)$$

$$G_r(s) = \int_o^\infty f_r(t) \exp\{-st\}\, dt = \alpha_{ro} - \alpha_{r1} s + \alpha_{r2}^2 s^2/2! + \ldots \qquad (2.30)$$

where $\alpha_i = \int_o^\infty t^i f(t)dt = (-1)^i [d^i G(s)/ds^i]_{s=o}$ is the i^{th} unnormalised moment about the origin of the impulse response f(t). The transfer functions can also be written as $G(s) = (b_o + b_1 s + .. b_{n-1} s^{n-1})/(a_o + a_1 s + \ldots a_n s^n)$, $G_r(s) = (d_o + d_1 s + \ldots d_{m-1} s^{m-1})/(c_o + c_1 s + \ldots c_m s^m)$. By matching the first q moments, i.e. $\alpha_i = \alpha_{ri}$, $i = 0,1..q$, of the original and reduced models; gives the linear equation

$$\sum_{i=0}^{p} a_{p-i}\, d_i - \sum_{j=0}^{p} b_{p-j}\, c_j = 0 \qquad (2.31)$$

$$p = 0,1,\ldots 2m-1$$

which could be solved to obtain the reduced model. It can also be shown that if $\alpha_i = \alpha_{ri}$ $i = 0, 1, 2, \ldots m+n$ and if $G_r(s)$ is asymptotically stable, then $G_r(s)$ is the m/n Pade approximant of G(s). Using this approach, Zakian[35] extended the moments matching method to multivariable systems.

Kitamori[36] obtained a reduced model by fitting to the impulse response, h(t), of G(s), a linear combination of orthogonal functions $h_r(\theta)_\alpha = \alpha_1 f_1(\theta) + \alpha_2 f_2(\theta) + .. \alpha_r f_r(\theta)$, where f_i is Laplace transformable and α_i is chosen to minimize,

$$E(\alpha) = \int_0^\infty (h(\theta) - h_r(\theta)_\alpha)^2 \rho(\theta)\, d\theta \tag{2.32}$$

where $\rho(\theta)$ is a positive scalar. By setting $\partial E(\alpha)/\partial\alpha_i = 0$, with $\int f_i(\theta) f_j(\theta)\rho(\theta)\, d\theta = \delta_{ij}$ from the orthogonality condition, gives $\alpha_i = \int h(\theta) f_i(\theta)\rho(\theta) d\theta$. Hence $G_r(s)$ is given by the Laplace transform of $h_r(\theta)_\alpha = \sum_{i=1}^{r} \alpha_i f_i(\theta)$.

The reduced models, S_r, obtained in state space forms are related to S via the aggregation matrix, Z, (following Aoki)[83] by

$$A_r Z = ZA\ , \quad B_r = ZB\ , \quad C_r Z = C \tag{2.33}$$

where $x_r = Zx$. In general, there are more than one solution for Z, except for special classes of reduced models, example projection methods[19], where a unique Z exists.

(iv) frequency domain curve fitting.[37-46, 2-7]

A number of research workers have attempted to obtain a reduced model, by approximating its frequency response, to that of the original model. In 1952, Levy fitted a transfer function to a set of frequency data, at about the same time the Russian workers, Kardashov, Simoyu, Dudnikov etc., considered a similar approach to model reduction.

In Levy's[37] approach the transfer functions $G(s) = K\{1+a_1 s+\ldots a_m s^m\}/\{1+b_1 s+\ldots b_n s^n\}$ is rewritten as

$$G(s) = K\{R+j\omega I\}/\{Q+j\omega L\} = K\ P(\omega)/T(\omega) \tag{2.34}$$

where $R = (1 - a_2\omega^2+a_4\omega^4-\ldots)$, $I = (a_1 - a_3\omega^2-\ldots)$, $Q = (1-b_2\omega^2+b_4\omega^4-\ldots)$ and $L = (b_1-b_3\omega^2+b_5\omega^4-\ldots)$. The generated data also assumes the form of eqn (2.34) and the error in fit at ω_k is $e_k = G(j\omega_k)-N(\omega_k)/D(\omega_k)$. The error functional to be minimized is

$$E = \sum_{k=o}^{m} |e_k\ D(\omega_k)|^2 = |G(j\omega_k)D(\omega_k) - N(\omega_k)|^2 \qquad (2.35)$$

By differentiating eqn (2.35) w.r.t. the unknown coefficients, $\partial E/\partial a_i = \partial E/\partial b_i = 0$ and using a simplification procedure, results in a set of linear equations, which when solved yields the unknown coefficients.

Sanathanan[38] and Koerner modified eqn (2.35), to improve fitting, over a region that spans over several decades, and, also in the low frequency region; by using an iterative procedure on eqn (2.35). Sumner[39] and Payne[40] used a hill climbing algorithm, to minimize eqn (2.35), and found that the method sometimes yields right hand open loop poles, and singularities can also occur in the non-linear least square equations.

Vittal Rao[41] and Lamba adapted Levy's method to model reduction. Following eqn (2.34), $G_r(s)$ can be written as,

$$G_r(s) = K\ (\alpha+j\omega\beta)/(\sigma+j\omega\tau) = KN(\omega)/D(\omega) \qquad (2.36)$$

The error at a particular frequency fitting is $e(\omega) = G(s) - G_r(s)$ $= K(P(\omega)D(\omega)-T(\omega)N(\omega))/T(\omega)D(\omega)$ and the error functional to be minimized is chosen as

$$E = \int_{\Omega} |T(\omega)D(\omega)e(\omega)|^2\ d\omega \qquad (2.37)$$

The derivatives of E, w.r.t. the unknown coefficients, of $G_r(s)$ give the complicated set of linear equations.

The Levy method needs a considerable amount of computation and its equations are rather tedious. Reddy[42] offered a simplified version by minimizing the error in the phase of two transfer functions together with magnitude. Referring back to eqns (2.34) and (2.35) the four error functionals are

$$E_1 = \int_\Omega (R-\alpha)^2\, d\omega, \quad E_2 = \int_\Omega (I-\beta)^2\, d\omega, \quad E_3 = \int_\Omega (Q-\sigma)^2 d\omega, \quad E_4 = \int_\Omega (T-\tau)^2\, d\omega \tag{2.38}$$

The derivatives equated to zero are $\partial E_1/\partial c_i = \int (R-\alpha)(d\alpha/dc_i)d\omega = 0$, $dE_2/dc_i = \int (R-\beta)\omega^2(d\beta/dc_i)\, d\omega = 0$, $\partial E_3/\partial d_i = \int (Q-\sigma)(d\sigma/dd_i)\, d\omega = 0$, $\partial E_4/\partial d_i = \int (T-\tau)\omega^2(d\tau/dd_i)\, d\omega = 0$. This simplifies the complexity of the work in solving for the p+q unknown coefficients, in the p+q linear equations.

Hsia[43] looked at reduction by taking as criterion the magnitude ratio of the frequency response of the models that deviate the least at various frequencies. The magnitude ratio can be expanded by Taylor series as

$$\lambda(\omega) = |G(j\omega)/G_r(j\omega)|^2 = (M_o+M_2+M_4+\ldots)/(\Delta_o+\Delta_2+\Delta_4 +\ldots) \tag{2.39}$$

By requiring the ratio to be unity, i.e. $\lambda(\omega)=1$, $G_r(s)$ can be constructed by comparing the coefficients M_{2j} with Δ_{2j}.

Kardashov[7] considered minimising the time domain criterion $J = \int_o^\infty (y-y_r)^2\, dt$, where Parseval's theorem gives

$$J = (1/2\pi j) \int_{-j\infty}^{j\infty} |\alpha(s)E(s)|^2\, ds \tag{2.40}$$

where $E(s) = G(s) - G_r(s)$, $G(s) = D(s)/K(s)$, $G_r(s) = D_r(s)/K_r(s)$. Linearisation is made possible by the factor $\alpha(s) = G(s)K(s)/G_r(s)K_r(s)$ in eqn (2.43) Setting $\partial J/\partial c_k = \partial J/\partial d_j = 0$, and, using complex integration in the frequency domain, after simplification, yields a set of linear equations for the unknown parameters of $G_r(s)$.

Simoiu[3] employed the method of moments in a geometrical sense in frequency response fitting. The transfer functions are expanded into Taylor series form, $G(s) = 1 + \sum_{i=1}^{\infty} a_i\, s^i$, $G_r(s) = 1 + \sum_{i=1}^{r} c_i\, s^i$; and $G_r(s)$ is approximated by discarding certain terms. The unknown coefficient c_i can be evaluated by successive integration using the recursive formula ,

$$S_i = M_{i-1} + \sum_{j=o}^{i-2} S_{i-1-j} M_j \qquad (2.41)$$

where $M_i = \int_o^\infty (k-\emptyset(t))(-t)^i/i!\, dt$, $S_1 = \int_o^\infty (k-\emptyset(t))dt = \lim_{s\to o}$ $\int_o^\infty (k-\emptyset(t)) \exp(-st)\, dt = \lim_{s\to o} (k(s)-\emptyset(s))$ and $\emptyset(s) = k/sG(s)$.

Dudnikov[4] gave a method of approximating a transfer function by expansion in continued fraction form. A reduced transfer function, is obtained by truncating certain quotients, and the coefficients are found from graphical plots and tables of experimentally obtained amplitude phase characteristics. Emphasis is placed on the initial portion of the characteristic.

$G_o(s) = (b_o+b_1s+..b_ks^k)/(1+a_1s+..a_ns^n)$ can be expanded into continued fraction form as

$$G_o(s) = A_o + \cfrac{1}{B_o^{-1}(j\omega) + A_1 + \cfrac{1}{B_1^{-1}(j\omega) + A_2 + \ddots + \cfrac{1}{B_{n-1}^{-1}(j\omega) + A_n}}} \qquad (2.42)$$

where $G_o(j\omega) = A_o + \emptyset_1(j\omega)$, $G_1(j\omega) = 1/\emptyset_1(j\omega)$, $G_1(j\omega) = A_1 + B_o^{-1}(j\omega) + \emptyset_2(j\omega)$ etc. By plotting $G_o(j\omega),\ldots G_n(j\omega)$ as Nyquist diagrams, constants $A_o, B_o, A_1, B_1, ..B_{n-1}..A_n$ can be determined from their real and imaginary parts. In fact many forms of continued fraction can be used, example, Golant[5] and Dudnikov later used the Cauer's J-type continued fraction in place of the ordinary continued fraction.

Chen[46] and Biernson[44] considered reduction, from the Bode plot by using rules of thumb, example, the corner frequencies can be ignored if the gains are below -15 dB or above +15 dB. Evans[45] similarly approximated high order models from the root locus plot in the complex domain.

A disadvantage of reduction by frequency methods is that the time domain response of the reduced model is not known directly, and, large steady state errors can result. Furthermore, the linearised equations do not guarantee stable reduced models, and, implementation on the computer is normally uneconomical.

(v) Pade approximation techniques[47-67]

Model reduction by Pade approximation is a very popular technique, as it is reported to have a high degree of success, and, to yield accurate reduced models. Reduction by time moments[32,33,35] (discussed earlier), Markov parameters and continued fraction expansions etc., are all equivalent methods, and, they form a sub-set of the general Pade approximation methods.

Chen and Shieh[47] introduced continued fraction in the s-domain for model reduction, by considering that

$$G(s) = (A_{21}+A_{22}s+A_{23}s^2+\ldots A_{2n}s^{n-1})/(A_{11}+A_{12}s+A_{13}s^2+\ldots A_{1,n+1}s^{n}) \qquad (2.43)$$

can be expanded into Cauer type continued fraction (equivalent to Taylor series expansion about $s = 0$),

$$G(s) = 1/(h_1+(s)\ /(h_2+(s)/(h_3+(s)/(h_4+(s)/\ldots)))) \qquad (2.44)$$

and $G_r(s)$ is constructed by retaining the first $2r$ h_i parameters and discarding the remaining inner nested parameters. The h_i parameters are determined from the first column of the Routh table,[48]

$$\begin{matrix} A_{11} & A_{12} & A_{13} & A_{14} & \cdots \\ A_{21} & A_{22} & A_{23} & A_{24} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots & \\ A_{41} & A_{42} & \cdots & & \\ \vdots & & & & \end{matrix} \qquad h_i = A_{i,1}/A_{i+1,1} \quad i=1,2,\ldots \qquad (2.45)$$

where $A_{ij} = A_{i-2,j+1} - (A_{i-2,1}\, A_{i-1,j+1})/A_{i-1,1}$, $i=3,4,..2n+1$, $j=1,2,...$ An algorithmic method for converting the truncated continued fraction transfer function back to the original form is also given. Despite the simplicity of the method, the reduced model so obtained can be unstable. Also as pointed by Wright[49], the Routh table can sometimes fail when the coefficients $A_{i1} = A_{i+1,1}$ and $A_{ij} = A_{i+1,j}$. This is equivalent to the fact that no Pade approximant exists for the particular transfer function.

Chuang[50,51,52] modified the method of Chen and Shieh by rewriting G(s) in descending powers of s, and, employed the expansion scheme as before, but, this time it is equivalent to Taylor series expansion about $s = \infty$. The modified procedure yields better initial transient response. By using the modified Pade approximation (MPA) method, Chuang[51] also showed that reduction can be achieved by expanding

$$\begin{aligned} G(s) &= p_o + p_1 s + p_2 s^2 + \ldots && \text{about } s = 0 \\ &= q_o + q_1 s^{-1} + q_2 s^{-2} + \ldots. && \text{about } s = \infty \end{aligned} \tag{2.46}$$

$G_r(s) = (r_o + r_1 s + \ldots r_{r-1} s^{r-1})/(d_o + d_1 s + \ldots d_r\, s^r)$ is constructed by expanding likewise, and matching the coefficients up to the s^{r-1} (resp.s^{-r}) terms, gives a set of linear equations in terms of r_i and d_i. A biased reduced model is obtained in this way, with equal emphasis on initial transient and final transient response. However, the reduced model obtained, is again not guaranteed stable.

Instead of using the Cauer form, Shamash[53,55] employed a J-type continued fraction expansion, where G(s) is expanded as

$$G(s) = 1/(h_1 + (s)/(h_2 s + (1)/(h_3 + (s)/(h_4 s + (1)/\ldots)))) \tag{2.47}$$

The above scheme was proved to be computationally cheaper and more efficient than ordinary expansion. He also showed that by retaining certain dominant[54] modes in the expansion, and obtaining others by continued fraction, a stable reduced model can be obtained. For discrete systems[56] $G(z) = (a_0+a_1z+..a_mz^m)/(b_o+b_1z+..b_nz^n)$, the linear transformation $z = p + 1$ can be used to transform $G(z)$ into continuous form and expanded as

$$G(z) = 1/(h_1+(z-1)/(h_2+(z-1)/(\ldots(z-1)/h_{2k}))) \qquad (2.48)$$

A reduced model is obtained by truncating as before. Other than expanding $G(s)$ about $s = 0$, or $s = \infty$, Davison[58] and Lucas expanded it about any general point. From $G(s) = \int_0^\infty g(t) \exp(-st)\, dt$, putting $s=a+z$, then $F(z) = G(a+z) = \int_0^\infty g(t) \exp\{-(a+z)t\}\, dt = \Sigma (-1)^i M_i z_i/i!$ where $M_i = \int t^i \exp(-at)\, g(t)\, dt$ is the i^{th} time moment. A reduced model $F_r(z)$ is constructed by matching the moments of $F(z)$ and $F_r(z)$, i.e.

$$\int_0^\infty t^i g(t) \exp(-at)\, dt = \int_0^\infty t^i g_r(t) \exp(-at)\, dt \qquad (2.49)$$

$$i=0,1, \ldots 2m-1$$

where $a=2/T$ and $T = \int_0^\infty t^2 g^2(t)\, dt / \int_0^\infty g^2(t)\, dt$

Lal[59] and Mitra also used moments evaluation algorithm for model reduction, and, made comparisons on Pade approximation simplification techniques. Brown[60] and Brown[61] et al. used moments matching reduction techniques to pulse transfer function, and, Hutton[62] and Friedland used Routh table approximation methods, which is the same as continued fraction and moments matching, for model simplification.

Shamash[57] and Hickin[63] and Sinha etc., also showed that minimal realization algorithms, example those of B.L. Ho or Silverman's can be used to obtain reduced models. $G(s) = C(sI-A)^{-1} B$ can be expanded about $s=0$ and $s=\infty$ as $G(s) = \sum_{i=1}^{\infty} C_i s^{i-1}$ and $G(s) = \sum_{i=1}^{\infty} D_i s^{-i}$ respectively, where

$C_i = C\,A^{-i}\,B \triangleq i^{th}$ time moment of system/i! and $D_i = C\,A^{i-1}B \triangleq i^{th}$ Markov parameters of the system. From the Hankel matrices,

$$F = \begin{bmatrix} C_1 & C_2 \cdots & C_r \cdots & C_n \\ \vdots & & & \\ C_r & C_{r+1} \cdots & C_{2r-1} & \cdots \\ \vdots & & & \\ C_n & C_{n+1} \cdots & C_{n+r-1} & C_{2n-1} \end{bmatrix} \qquad F^* = \begin{bmatrix} C_2 & C_3 \cdots & C_{r+1} \cdots & C_{n+1} \\ \vdots & & & \\ C_{r+1} & \cdots & C_{2r} & \\ \vdots & & & \\ C_{n+1} & C_{n+2} \cdot & C_{n+r} \cdots & C_{2n} \end{bmatrix}$$

$$H = \begin{bmatrix} D_n & \cdots D_r & \cdots D_2 & D_1 \\ \vdots & & & \\ D_{n-r+1} & \cdots D_1 & \cdots C_{r-2} & C_{r-1} \\ \vdots & & & \vdots \\ D_1 & C_1 \cdots & C_{n-2} & C_{n-1} \end{bmatrix} \qquad H^* = \begin{bmatrix} D_{n-1} & \cdots D_{r-1} & \cdots D_1 & C_1 \\ D_{n-r} & \cdots C_1 & \cdots C_{r-1} & C_r \\ & & & \vdots \\ C_1 & C_2 \cdots & C_{n-1} & C_n \end{bmatrix}$$

$$F_1 = F_2{}^t = (C_1\ C_2 \cdots C_r \vdots \cdots C_n) \qquad H_1 = H_2{}^t = (D_n\ D_{n-1} \cdots \vdots\ D_r \cdots D_2\ D_1)$$

Silverman's algorithm gives $S(A,B,C)$ as $A = F\,F^{*-1}$, $B = F_2$, $C = F_1 F^{*-1}$. A reduced model of order r is given by $S_r\ (A_r, B_r, C_r)$ where $A_r = F_r(F_r{}^*)^{-1}$, $B_r = F_{2r}$, $C_r = F_{1r}\ (F_r{}^*)^{-1}$ where the matrices F_r, $F_r{}^*$ are obtained by appropriate partitioning. S_r is equivalent to the (r-1, r) Pade approximant of G(s), which is equivalent to the Cauer type continued fraction expansion of Chen and Shieh. Similarly, considering the Markov parameters, S can be represented by $A = H(H^*)^{-1}$, $B = H_2$, $C = (H_1\ H^{-1})\ A^n$ and S_r is given by $A_r = H_r(H_r{}^*)^{-1}$, $B_r = F_{2r}$, $C_r = (F_{1r}\ F_r{}^{-1})A_r{}^r$.

Most of the continued fraction methods described above are originally meant for single input single output system. Chen[64] first extended the continued fraction method to matrix continued fractions for multivariable systems. The expansion scheme is as follows.

$$G(s) = (H_1 + s(H_2 + s\ (H_3 + \ldots s\ (H_{2n})^{-1} \ldots)^{-1})^{-1})^{-1} \qquad (2.50)$$

A reduced order transfer function matrix is obtained by truncating the inner nested brackets. For example, the first simplest model is H_1^{-1}, the second simplest model is $(H_1 + s(H_2)^{-1})^{-1}$ etc.[65] Following Chen, other authors, example Chuang[51], Shamash[55] and Hutton[62] et al. extended the continued fraction methods to matrix continued fractions, for multivariable systems; and Shieh[66] and Galiano modified the Routh table for evaluating the coefficients of the continued fractions.

Unfortunately, model reduction by Pade approximation of multivariable systems is not as successful as for single input/single output systems. The reduced transfer matrix obtained can be physically unrealizable in that the order of its numerator elements can be higher than that of its denominator elements, and the order of the reduced transfer matrix is normally higher than the order specified[67]. Sometimes the reduced model can also be unstable.

vi) General methods[68-75]

The methods stated below do not strictly come into any one of the above categories, but, however, can be loosely fitted into some of them.

Arumugam[68] and Ramamoorthy used a Schwarz canonical form for systems reduction. The single input/single output system $S(A,b,c)$, assumed controllable, is transformed into Schwarz form $S(B,f,c)$. The reduced model is constructed by making a ratio test of the elements of the Schwarz matrix B and the order r is determined by the last successful ratio test, j, i.e. $r = n - j$. An inverse transformation is used to convert $S_r(B_r,f_r,c_r)$ back to $S_r(A_r,b_r,c_r)$. However, the method cannot be extended to multivariable systems, and, does not give any indication of goodness of the reduced model so obtained.

Towill[70] et al. considered reduction by replacing certain aggregated far away poles by a single pole. This fact was also considered by Isermann[71], who gave some rules of thumb methods for model simplification, example, by neglecting small time constants, replacement of time constants by time delays and replacement of different time constants by equal time constants, etc.

Another method, due to Nagarajan[72], is by dividing the characteristic equation repeatedly by the largest remaining eigenvalue until the range ratio of the smallest and largest eigenvalues falls to some recommended figure. In this way, the smaller time constants are successively eliminated. The method only works when there are no dynamics in the numerator transfer function.

Brown[74] used a time varying low order model to approximate a high order time invariant model. The procedure is based on an approximate minimization of the difference between the time rate of change of the variable of S and S_r, subjected to random signal inputs, and minimising the conditional ensemble expectation, $R = E \{<(\dot{x}_{r1}-\dot{x}_1), (\dot{x}_{r1}-\dot{x}_1)>|u\}$, where $\dot{x}_{r1}$ and $\dot{x}_1$ are the measurable states of S_r and S, respectively.

Other reduction methods include that of Brierly[75] et al., where reduction is done with the aid of computer graphics. This is a trial and error method, where the root locus of S_r is shaped, by shifting poles manually, to approximate that of S.

2.2 Survey of applications[82-110]

Reduced models can be used in the following areas,

(i) Simulation studies

(ii) Optimal control synthesis

(iii) Multivariable feedback systems

(iv) Model reference adaptive control systems

(v) Sensitivity studies

Only a cursory review of the listed references will be given.

In the area of simulation studies, reduced models find wide applications, example in power systems engineering[86,87,89] and chemical plants[88]. Here the time or frequency response of the system is required, and, a lower order differential equation[85] is sought to approximate the higher order one, the latter being obtained by considering too many grid points or state variables.

In optimal control synthesis, numerical techniques, such as dynamic programming[84] and iterative algorithms, are employed to obtain solutions for non-analytic cost functions[96]. Using reduced models[92,93] here will result in speeding up the iterative processes and economising storage and computation. Allwright[91] has used low order models to determine stopping criterion for optimisation algorithms and Chapman[94] has used them to speed up iterative algorithms. Mitra[82] and Aoki are probably the first to perform an analytical study of using reduced models in optimal control problems. Specific examples, like the linear optimal regulator[82] and linear tracking problems, are studied in terms of performance deterioration[83,96] when the sub-optimal controller, designed using reduced models, is used[97,99,105]. Aoki[83] also first gave a stability theorem for sub-optimal controllers, designed using a class of reduced models, the latter being obtained by projective reduction methods only. Following Mitra and Aoki, some papers began to appear, investigating the performance and stability of sub-optimal controllers involving certain classes of projective reduction methods only[95,98].

Perhaps the most popular use of reduced models is in designing a controller for the general multivariable systems. The effectiveness of using reduced models in controller design, in terms of stability and performance, using frequency response methods have not yet been thoroughly investigated[1]. Associated with multivariable frequency design, is model reference adaptive control design[108] (MRAC). The design philosophy of Linear Model Following systems is to control the plant in such a manner such that its response follows that of a model[110]. Due to practical limitations'perfect model following'is impossible, hence adaptive characteristics are implemented in the control loop to accommodate changes in the plant, due to parameter changes or external disturbance, so that the response of the plant still follows that of the model as close as possible[109]. Shaked[107] has studied the stability of linear model following systems, using reduced and original models, but, the application of reduced models in MRAC systems is still novel, and, few investigations have been made[108].

Parameter sensitivity[100-104,106] is a study of how closed loop parameters vary as a whole when an external disturbance is injected. In a control system, it plays a great part in determining system performance and stability, and, the application of feedback tends to reduce sensitivity[100,101]. Using reduced models[101] in the feedback scheme, will reduce the sensitivity of parameters as there are few parameters in the reduced model. Towill[104] and Mehdi have used low order models to predict the sensitivity response of an aircraft system. In some aspects of optimal control[106,103], a'parameter sensitivity' model is modelled separately from the dynamical model to enable better analysis of the whole system. Kokotovic[100] and Sannuti have used reduced models in optimal control studies, where a reduced analytical model would result in a reduced 'parameter sensitivity' model.

Conclusions

The survey covered above is thorough, but not exhaustive or detailed. Neither is it intended to be. References 1 and 2 list a wide reference of reduction techniques and reference 1 gives some account of their applications. The ratio of published materials[1] on 'reduced model applications' to that on reduction techniques is comparatively small. Regarding the former, it seems that more emphasis is devoted to studying sub-optimal controllers, involving reduced models of a restrictive nature only.

The next few chapters investigate some methods of reduction, and the use of reduced models, obtained by any reduction techniques, in the general multivariable control systems problem, in terms of performance bounds, sensitivity and stability.

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III

SOME FREQUENCY RESPONSE METHODS FOR LINEAR SYSTEMS REDUCTION.

CHAPTER III.

SOME FREQUENCY RESPONSE METHODS FOR LINEAR SYSTEMS REDUCTION

Introduction

In this chapter some frequency response methods are presented for linear systems reduction. The current trend in designing multivariable systems in the frequency domain yields promising results, thus some insight may be gained relating design and reduction when the latter is considered in the frequency sense.

Frequency response methods for reduction of single input-single output systems are not new, but, the approach given here is different from those reviewed in the last chapter. All frequency methods used in the past consider the amplitude and phase characteristics of a single sinusoid passing through a linear system and finding some ways to construct a reduced model whose response would approximate the characteristics[9]. Guillemin's approach in approximating feedback system's design, considers input-output time waveforms, and, decomposes them into their odd and even Fourier components, with individual amplitudes and phases[1]. An approximate system is synthesized by considering the effect due only to certain harmonics. However, the method is rather crude as a pure time delay function has to be approximated, and, tedious, if the system is not a low pass filter. Also, at the end of the synthesis it can yield an unstable system with right hand side (r.h.s.) s-plane poles.

The method given here is based on harmonic decomposition of time waveforms for single input-single output systems, and, the approach to the problem is entirely different from that of Guillemin's. After reduction a low-order stable reduced model is obtained. It is shown that the method can be extended to multivariable systems in two ways. The first way is by using the principle of superposition and cause-effect relationship[5]. The second way is via the characteristic loci[6] of a square transfer function, the latter being a natural extension of the Nyquist locus for a single input-single output system.

3.1. Harmonic synthesis of single input-single output systems.

The problem considered is shown in fig. 3.1 where $H_o(j\omega)$ and $H_r(j\omega)$ represent the original and reduced model transfer functions of order n and r, respectively. The time domain input signal is extended to be periodic, of period 2T, from $-\infty$ to $+\infty$, and, is assumed to satisfy Dirichlet's condition, i.e., having a finite number of discontinuities in any interval. By superposition, the steady state response of the kth output Fourier harmonic is related to the same input harmonic by the cause effect relationship,

$$\begin{aligned} A_k \operatorname{Cos}(\omega_k t) &\rightarrow A_k |H_r(j\omega_k)| \operatorname{Cos}(\omega_k t + \arg H_r(j\omega_k)) \\ B_k \operatorname{Sin}(\omega_k t) &\rightarrow B_k |H_r(j\omega_k)| \operatorname{Sin}(\omega_k t + \arg H_r(j\omega_k)) \end{aligned} \tag{3.1}$$

where $\omega_k = \omega_o k$. If $H_r(s)$ is represented in the form

$$H_r(s) = \frac{K_r \prod_{i=1}^{p} (s+z_i) \prod_{i=1}^{\ell-p} (s^2+2a_i s+a_i^2+b_i^2)}{\prod_{j=1}^{q} (s+p_j) \prod_{j=1}^{r-q} (s^2+2c_j s+c_j^2+d_j^2)} \tag{3.2}$$

its amplitude-phase characteristics can be conveniently written as

$$|H_r(j\omega_k)|^2 = \frac{K_r \prod_{i=1}^{p} (\omega_k^2+z_i^2) \prod_{i=1}^{\ell-p} [(a_i^2+b_i^2-\omega_i^2)^2+(2a_i\omega_k)^2]}{\prod_{j=1}^{g} (\omega_k^2+p_j^2) \prod_{j=1}^{r-q} [(c_j^2+d_j^2-\omega_k^2)^2+(2c_j\omega_k)^2]} \tag{3.3}$$

$$\arg H_r(j\omega_k) = \sum_{i=1}^{p} \arg(j\omega_k+z_i) - \sum_{j=1}^{q} \arg(j\omega_k+p_j) + \sum_{i=1}^{\ell-p} \arg(j2\omega_k a_i+a_i^2+b_i^2-\omega_k^2)$$

$$- \sum_{j=1}^{r-q} \arg(j\omega_k c_j+c_j^2+d_j^2-\omega_k^2) \tag{3.4}$$

$H_o(s)$ can similarly be represented as in eqn.(3.2) and both $H_o(s)$ and $H_r(s)$ are assumed to be strictly proper, i.e. $|H_r(s)| \to 0$, $|H_o(s)| \to 0$ as $s \to \infty$. Thus $H_o(s)$ and $H_r(s)$ can be considered as low pass filters and heavily attenuate high frequency signal amplitudes; the degree of attenuation depending on the pole/zero distribution and the value of $r-\ell$. A reduced model is constructed by considering the effect due to the first few harmonics, as the response due to the higher harmonics are relatively less significant, and, minimizing the error criterion,

$$E = \int_o^T (y_o(t) - Y_r(t))^2 \, dt \tag{3.5}$$

The upper bound in the integral is finite due to the periodic nature of the input. The output due to the first h harmonics can be written as, eqn.(3.1),

$$y_r(t) = \sum_{k=1}^{h} f_{rk} \operatorname{Sin}(\omega_k t + \phi_{rk}) \tag{3.6}$$

$$y_o(t) = \sum_{k=1}^{h} f_{ok} \operatorname{Sin}(\omega_k t + \phi_{ok}) \tag{3.7}$$

where
$$f_{rk} = (A_k^2 + B_k^2)^{\frac{1}{2}} |H_r(j\omega_k)| \tag{3.8}$$

$$\phi_{rk} = \arg H_r(j\omega_k) + \tan(A_k/B_k) \tag{3.9}$$

and similarly for f_{ok} and ϕ_{ok}, where appropriate substituting the subscript r by o in eqn.(3.8). The optimum value of an unknown parameter θ of $H_r(s)$ is given by

$$\partial E/\partial\theta = \int_0^T (y_o(t) - y_r(t))\partial y_r(t)/\partial\theta = 0 \qquad (3.10)$$

As there are no discontinuities in the output waveform, its derivatives converge, hence from eqn. (3.6),

$$\partial y_r(t)/\partial\theta = \sum_{k=1}^{h} F_{rk} \operatorname{Sin}(\omega_k t + \phi_{rk} + \psi_{rk}) \qquad (3.11)$$

where $$F_{rk}^2 = (\partial f_{rk}/\partial\theta)^2 + (f_{rk}\partial\phi_{rk}/\partial\theta)^2 \qquad (3.12)$$

$$\tan\psi_{rk} = \{f_{rk}\ \partial\phi_{rk}/\partial\theta\} / \{\partial f_{rk}/\partial\theta\} \qquad (3.13)$$

A maximum number of $2(\ell+r)-(p+q)+1$ equations in the form of eqn. (3.11) can be obtained with $\theta \triangleq z_i(i=1,..p)$, a_i, $b_i(i=1,...\ell-p)$, $p_j(j=1,...q)$, c_j, $d_j(j=1,...r-q)$ and K_r. The derivatives $\partial f_{rk}/\partial\theta$ and $\partial\phi_{rk}/\partial\theta$ are obtained from eqns.(3.8) and (3.9) via (3.3) and (3.4),

$$\partial f_{rk}/\partial K_r = f_{rk}/2K_r \quad , \quad \partial\phi_{rk}/\partial K_r = 0$$

$$\partial f_{rk}/\partial z_i = z_i f_{rk}/(\omega_k^2+z_i^2) \quad , \quad \partial\phi_{rk}/\partial z_i = -\omega_k/(z_i^2+\omega_k^2)$$

$$\partial f_{rk}/\partial a_i = 2a_i N_{fai} f_{rk}/T_{ni} \quad , \quad \partial\phi_{rk}/\partial a_i = 2\omega_k N_{\phi ai}/T_{ni}$$

$$\partial f_{rk}/\partial b_i = 2b_i N_{fbi} f_{rk}/T_{ni} \quad , \quad \partial\phi_{rk}/\partial b_i = -4a_i b_i \omega_k/T_{ni}$$

$$\partial f_{rk}/\partial P_j = -p_j f_{rk}/(p_j^2+\omega_k^2) \quad , \quad \partial\phi_{rk}/\partial p_j = \omega_k/(p_j^2 + \omega_k^2)$$

$$\partial f_{rk}/\partial c_j = -2c_j N_{fcj} f_{rk}/T_{dj} \quad , \quad \partial\phi_{rk}/\partial c_j = -2\omega_k N_{\phi cj}/T_{dj}$$

$$\partial f_{rk}/\partial d_j = -2d_j N_{fdj} f_{rk}/T_{dj} \quad , \quad \partial\phi_{rk}/\partial d_j = 4c_j d_j \omega_k/T_{dj}$$

where

$$
\begin{aligned}
N_{fai} &= a_i^2 + b_i^2 + \omega_k^2 \\
N_{fbi} &= a_i^2 + b_i^2 - \omega_k^2 \\
N_{fcj} &= c_j^2 + d_j^2 + \omega_k^2 \\
N_{fdj} &= c_j^2 + d_j^2 - \omega_k^2 \\
N_{\phi ai} &= b_i^2 - a_i^2 - \omega_k^2 \\
N_{\phi cj} &= d_j^2 - c_j^2 - \omega_k^2 \\
T_{ni} &= N_{fbi}^2 + (2\alpha_i\omega_k)^2 \\
T_{dj} &= N_{fdj}^2 + (2c_j\omega_k)^2
\end{aligned}
\tag{3.14}
$$

The following trigonometric relations are used in the simplification below

$$A \sin(x+\alpha)+B \operatorname{Sin}(x+\beta) = C \operatorname{Sin}(x+\gamma) \tag{3.15a}$$

$$2 \operatorname{Sin}(x) \operatorname{Sin}(y) = \operatorname{Cos}(x-y) - \operatorname{Cos}(x+y) \tag{3.15b}$$

$$A \operatorname{Sin}(x+\alpha) + B \operatorname{Cos}(x+\beta) = R \operatorname{Cos}(x+\lambda) \tag{3.15c}$$

Substituting eqns. (3.6), (3.7) and (3.11) into eqn.(3.10) gives,

$$\int_o^T \{\sum_{k=1}^{h} f_{ok}\operatorname{Sin}(\omega_k t+\phi_{ok}) - \sum_{k=1}^{h} f_{rk}\operatorname{Sin}(\omega_k t+\phi_{rk})\}\{\sum_{k=1}^{h} F_{rj}\operatorname{Sin}(\omega_j t+\phi_{rj}+\psi_{ri})\}\, dt \tag{3.16}$$

Simplifying the first bracketed expression in eqn.(3.16) using eqn. (3.15a) yields,

$$\int_{o}^{T} \sum_{k=1}^{h} \sum_{j=1}^{h} \Delta_k F_{rj} \mathrm{Sin}(\omega_k t+\delta_k)\mathrm{Sin}(\omega_j t+\phi_{rj}+\psi_{rj})dt = 0 \tag{3.17}$$

where $$\Delta_k^{\,2} = f_{ok}^{\,2} + f_{rk}^{\,2} - 2f_{ok}f_{rk}\mathrm{Cos}(\phi_{ok}-\phi_{rk}) \tag{3.18}$$

$$\tan(\delta_k) = \{f_{ok}\mathrm{Sin}(\phi_{ok})-f_{rk}\mathrm{Sin}(\phi_{rk})\}/\{f_{ok}\mathrm{Cos}(\phi_{ok})-f_{rk}\mathrm{Cos}(\phi_{rk})\} \tag{3.19}$$

Using eqn.(3.15b) in (3.17) and integrating over the half period T,

$$\frac{1}{2}\sum_{k=1}^{h}\sum_{\substack{j=1\\ j\neq k}}^{h} \{\mathrm{Sin}[(\omega_k-\omega_j)T-\phi_{rj}-\psi_{rj}+\delta_k] -\mathrm{Sin}(-\phi_{rj}-\psi_{rj}+\delta_k)\}F_{rj}\Delta_k/(\omega_k-\omega_j)$$

$$-\frac{1}{2}\sum_{k=1}^{h}\sum_{\substack{j=1\\ j\neq k}}^{h} \{\mathrm{Sin}[(\omega_k+\omega_j)T+\phi_{rj}+\psi_{rj}+\delta_k] -\mathrm{Sin}(\phi_{rj}+\psi_{rj}+\delta_k)\}\ F_{rj}\Delta_k/(\omega_k+\omega_j)$$

$$-\frac{1}{2}\sum_{k=1}^{h} \{\mathrm{Sin}(2\omega_k T+\phi_{rk}+\psi_{rk}+\delta_k)-\mathrm{Sin}(\phi_{rk}+\psi_{rk}+\delta_k)\}\ F_{rk}\Delta_k/2\omega_k$$

$$+\frac{1}{2}T\sum_{k=1}^{h} F_{rk}\,\Delta_k\,\mathrm{Cos}(\delta_k-\phi_{rk}-\psi_{rk}) = 0 \tag{3.20}$$

Simplification with eqn.(3.15a) yields,

$$\sum_{k=1}^{h}\sum_{\substack{j=1\\ j\neq k}}^{h} M_{kj}F_{rj}\,\Delta_k\,\mathrm{Sin}(S_{kj}-\phi_{rj}-\psi_{rj}+\delta_k)-\sum_{k=1}^{h}\sum_{\substack{j=1\\ j\neq k}}^{h} P_{kj}F_{rj}\,\Delta_k\mathrm{Sin}(C_{kj}+\phi_{rj}+\psi_{rj}+\delta_k)$$

$$-\sum_{k=1}^{h} P_{kk}F_{rk}\Delta_k\mathrm{Sin}(C_{kk}+\phi_{rk}+\psi_{rk}+\delta_k)+T\sum_{k=1}^{h} F_{rk}\Delta_k\mathrm{Cos}(\delta_k-\phi_{rk}-\psi_{rk}) = 0 \tag{3.21}$$

where

$M_{kj}=\{2-2\mathrm{Cos}[(\omega_k-\omega_j)T]\}/(\omega_k-\omega_j)$, $\tan(S_{kj})=\mathrm{Sin}[(\omega_k-\omega_j)T]/\{\mathrm{Cos}[(\omega_k-\omega_j)T] -1\}$

$$P_{kj} = \{2-2\mathrm{Cos}[(\omega_k+\omega_j)T]\}/(\omega_k+\omega_j)\ ,\ \tan(C_{kj})=\mathrm{Sin}[(\omega_k+\omega_j)T]/\{\mathrm{Cos}[(\omega_k+\omega_j)T]-1\} \tag{3.22}$$

Using eqn.(3.15a) on the first two expressions and eqn.(3.15c) on the last two expressions in eqn.(3.21), further simplification yields,

$$\sum_{k=1}^{h}\sum_{\substack{j=1\\j\neq k}}^{h} \nabla_{kj}\, F_{rj}\, \Delta_k \mathrm{Sin}(\eta_{kj}+\delta_k) + \sum_{k=1}^{h} F_{rk}\, \Delta_k\, \varepsilon_k\, \mathrm{Cos}\,(\delta_k+\gamma_k) = 0 \tag{3.23}$$

$$\text{where } \nabla_{kj}^2 = M_{kj}^2 + P_{kj}^2 - 2\, M_{kj}\, P_{kj}\, \mathrm{Cos}\,[2(\phi_{rj}+\psi_{rj}) + C_{kj}-S_{kj}] \tag{3.24}$$

$$\tan\,(\gamma_k)=\{T\mathrm{Sin}(\phi_{rk}+\psi_{rk})+P_{kk}\mathrm{Cos}(\phi_{rk}+\psi_{rk}+C_{kk})\}/\{T\mathrm{Cos}(\phi_{rk}+\psi_{rk}) - P_{kk}\mathrm{Sin}(\phi_{rk}+\psi_{rk}+C_{kk})\} \tag{3.25}$$

$$\varepsilon_k^2 = T^2 + P_{kk}^2 - 2TP_{kk}\ \mathrm{Sin}\ [2(\phi_{rk}+\psi_{rk})+C_{kk}] \tag{3.26}$$

$$\tan(\eta_{kj})=\{M_{kj}\mathrm{Sin}(S_{kj}-\phi_{rj}-\psi_{rj})-P_{kj}\mathrm{Sin}(C_{kj}+\phi_{rj}+\psi_{rj})\}/\{M_{kj}\mathrm{Cos}(S_{kj}-\phi_{rj}-\psi_{rj}) - P_{kj}\mathrm{Cos}(C_{kj}+\phi_{rj}+\psi_{rj})\} \tag{3.27}$$

Eqn.(3.23) represents a set of non-linear equations with a maximum number of $2(\ell+r)-(p+q)+1$ equations and the same number of unknowns, whose solution gives the optimal parameters of eqn.(3.2) in the sense that eqn.(3.5) is satisfied. Although eqn.(3.23) is complete and compact by itself, a simpler approximate solution to the model reduction problem can be obtained by discretizing and minimizing eqn.(3.5), i.e.

$$E = \min < \hat{y}_o(t) - \hat{y}_r(t),\ Q(\hat{y}_o(t)-\hat{y}_r(t))> \tag{3.28}$$

where $\hat{y}_o(t)=(y_o(t_1),y_o(t_2),\ldots(y_o(t_N)),\ \hat{y}_r(t)=(y_r(t_1),y_r(t_2),\ldots y_r(t_N))$ and $Q = \mathrm{diag}\ \{q_1,q_2,\ldots q_N\}$ is a weighting matrix of N samples. From eqns. (3.16) and (3.17) $y_o(t_i)$ and $y_r(t_i)$ can be expressed as

$$y_o(t_i)-y_r(t_i) = \sum_{k=1}^{k} \Delta_k \operatorname{Sin}(\omega_k t_i+\delta_k), \quad 1 < i < N \qquad (3.29)$$

3.2 Choice of operating frequency, inputs, and numerical feasibilities.

The operational(fundamental) frequency is chosen as $\omega_o=\pi/T$, $T \geqslant \tau_\ell$, τ_ℓ being the largest time constant of the process, such as to ensure that all modes are excited sufficiently to give good dynamics. For a system with dominant poles, it is necessary only to consider the first few important harmonics. The corresponding 'frequency range of interest' or the 'critical range', which lies above ω_o, where curve fitting takes place,is as shown shaded in fig.3.2. If the system has non-dominant modes, the critical range can be widened or shifted over a particular region so as to give better accuracy to the time domain response of the system. This of course depends on other factors, chiefly, on the spread of the eigenvalues. The critical range is proportional to the latter, as ω_o is proportional to the smallest eigenvalue.

The Levy[1] and associated methods[9] attempt to fit frequency data of similar amplitude and phase over the entire spectrum. This is undesirable, in the sense that difficulties are encountered when the band spreads over several decades, and, poor fitting is achieved in the low frequency region (corresponding to the sub-harmonic region, in fig.3.2). Both difficulties, however, can be alleviated by studying the eigenvalue distribution, then choosing a sufficiently large ω_o, and, focusing attention over a small, but relatively high bandwidth of interest. It can be argued with good justification that ignoring the high frequency region will result in poor initial time transient fitting, while ignoring the low frequency region will result in large steady state errors.

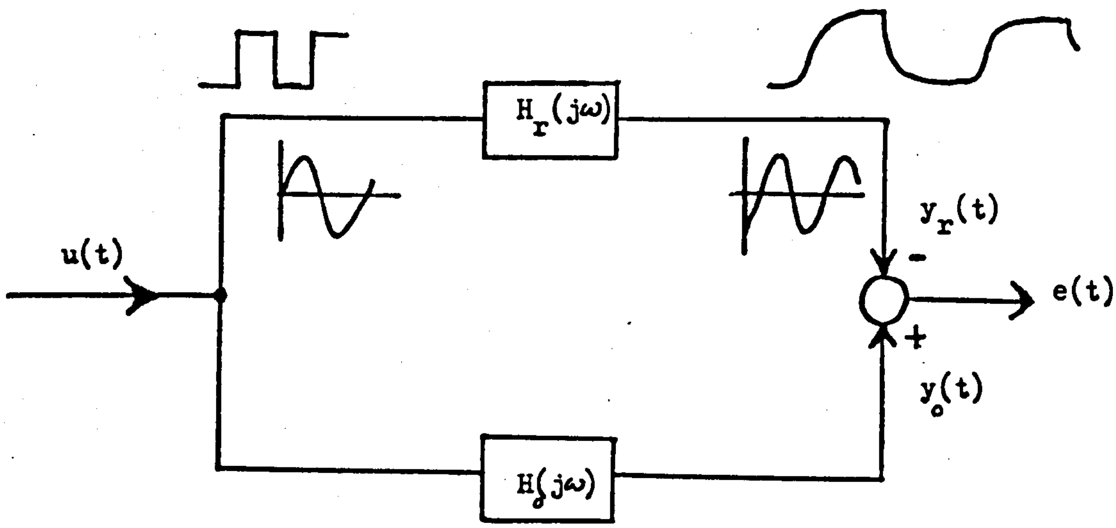
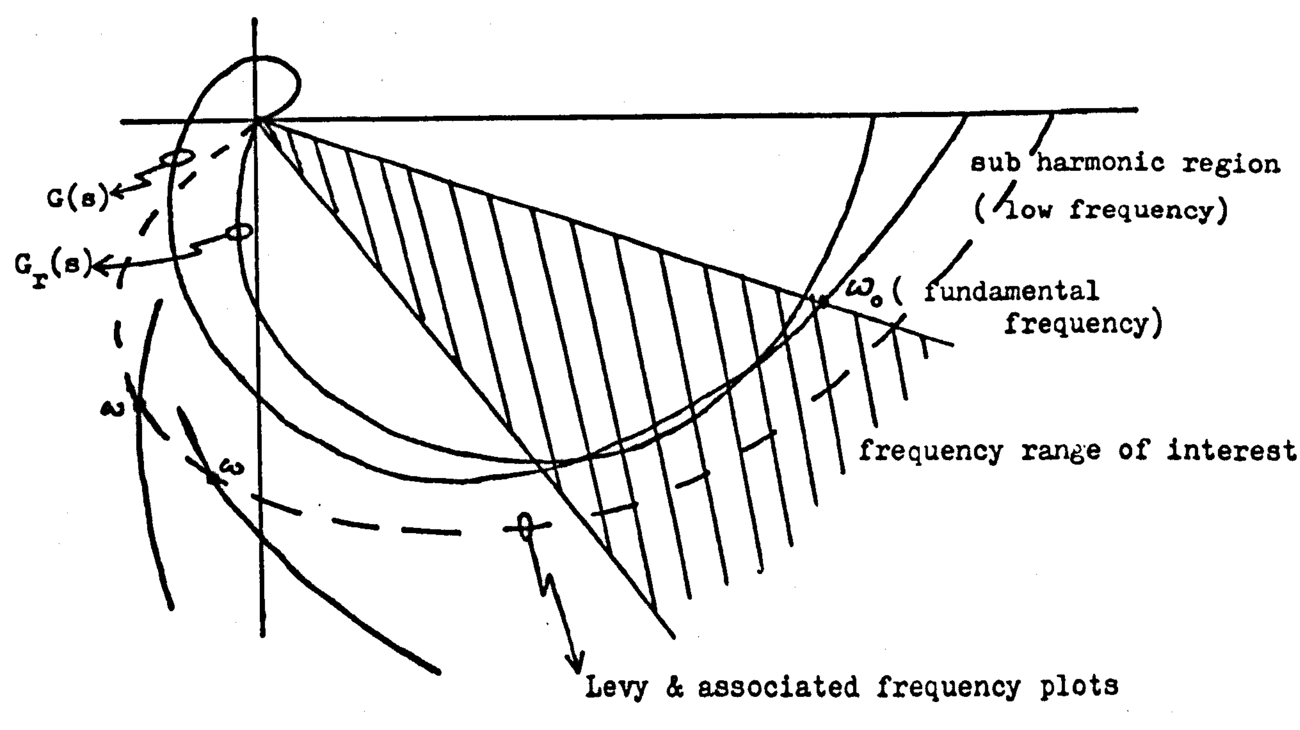


Fig. 3.1 Arrangement for Reduction by Harmonic Synthesis

Fig. 3.2 Diagram to illustrate Harmonic Synthesis method and bandwith of interest



The input function can be any convenient test signal but if it has a finite number of singularities, example for a square wave, the Gibbs phenomenon can occur at points of discontinuities in the Fourier representation of the signal. To suppress the oscillations due to the Gibbs phenomenon, the Lanczos damping factor $(\mathrm{Sin}\ \alpha)/\alpha$, can be weighted to the signal,

$$f(t) = \tfrac{1}{2}A_o + \sum_{k=1}^{h} \{\mathrm{Sin}(k\pi/2h)/(k\pi/2h)\}\{A_k \mathrm{Cos}(\omega_k t) + B_k \mathrm{Sin}(\omega_k t)\} \tag{3.30}$$

This is necessary as the output accuracy depends on the faithful representation of the input. The modification to eqn.(3.30) will reduce the oscillations by a factor of nine. Alternatively, the singularity problem can be overcome, by joining two discontinuous points by a straight line of very high but finite gradient. Impulse inputs can be approximated by triangular functions of very high amplitude and low base. The distance between two 'impulses' would depend on the time constants of the system. Stochastic signals can be difficult to simulate, but, as was indicated (Meier 1967)[9], white noise and impulse inputs are equivalent.

The reduction method presented, differs from the Levy and associated techniques, in the sense that integration is done in the time domain, on the Fourier representation of the signal of different amplitudes and phases. The Levy and associated methods[9] consider system response, due to a single sinusoid of constant amplitude and phase, and integration is done in the frequency domain over a spectrum of interest. A correlation of the method and that of Levy is shown by the dotted line joining various frequency points on the solid line, in fig.3.2.

To alleviate numerical difficulties associated with the large number of parameters in the nonlinear equations, ℓ dominant poles can be retained, and the remaining r-ℓ poles and other zeroes found one at a time to give the best transient response. Thus the order of the model is updated sequentially, one by one, from unity to r. Since the success of most minimization routines and the solution of nonlinear equations depend very much on initial parameter estimates and proper problem presentation, the initial parameters can be estimated with the aid of the phase-amplitude criterion in the s-plane by noting the response due to the fundamental harmonic. The local optimum is then expected to lie in the vicinity.

Most singularity problems associated with the nonlinear least squares equations are merely computational, in that part of the intergrand of eqn.(3.17) can take an infinite value at some point, when the integral itself at that particular point is finite. Associated with eqns. (3.23) and (3.28) are eqns.(3.1) and (3.14), and, it is observed that singular points can occur in the form arc tan $[N(\theta)/D(\theta)]$ (where $N(\theta) \triangleq \text{Real}(\theta)$, $D(\theta) \triangleq p_j$, or $\triangleq a_i^2+b_i^2-\omega_k^2$, or $\triangleq c_j^2+d_j^2-\omega_k^2$), when a pole or a zero is at the origin, when a pole vector equals to a harmonic value, or when a pair of complex poles lie on the imaginary axis causing resonance. The singularity problem can be overcome in the first few cases by assigning arc tan $[N(\theta)/D(\theta)] = \pi/2$ and the phenomenon of resonance can be overcome by constraining complex poles off the imaginary axis.

To ensure that a stable reduced model is obtained, poles are constrained in the l.h.s. s-plane, and where possible, also off the imaginary axis, as an oscillatory model is often undesirable, when a regulator system is needed. The reduction method is such that initial real pole/zero estimates will remain as final real pole/zeros and, initial complex pole/zeros will remain complex in the end, or take the form of double poles. This is advantageous to the designer working in the frequency domain in the sense that system performance can be specified by pole allocations.

3.3 Multivariable systems and characteristic loci.

The single input-single output reduction method above can be extended to multivariable systems, as follows[5]. For a(mxm) multivariable transfer function matrix $G_r(s)$, the i^{th} output can be written as

$$y_{ri}(s) = \sum_{j=1}^{m} g_{rij}(s)u_j(s), \forall i \qquad (3.31)$$

where $g_{rij}(s)$ is the ij^{th} element of $G_r(s)$. By superposition, each term in the r.h.s. of eqn.(3.31) can be considered independently, but, also preserving the interaction structure of $G_o(s)$, i.e. $f_{rij}(\omega) = f_{oij}(\omega)$ and $m_{rij}(\omega) = m_{rij}(\omega)$, where $f_{rij}(\omega) = |g_{rij}(s)|_{i\neq j}/|g_{rii}(s)|$, $f_{oij}(\omega) = |g_{oij}(s)|_{i\neq j}/|g_{oii}(s)|$, $m_{rij}(\omega) = |g_{rij}(s)|_{i\neq j}/|g_{rjj}(s)|$ and $m_{oij}(\omega) = |g_{oij}(s)|_{i\neq j}/|g_{ojj}(s)|$. This is important in terms of stability for the original model when design is made on the reduced model (the proof and derivation will be given in Chapter VI). The diagonal elements $g_{rii}(s)$ are found as outlined above in section 3.1, and, $g_{rij}(s)$ is generated by minimizing

$$J_\omega = \int_\Omega \{[f_{rij}(\omega)-f_{oij}(\omega)]^2+[m_{rij}(\omega)-m_{oij}(\omega)]^2\} \, d\omega \qquad (3.32)$$

over a selected bandwidth of interest, Ω.

The second method of extending the frequency response reduction method to multivariable systems is by considering the characteristic loci, $\rho_{oj}(s)$, of $G_o(s)$ (assuming $G_o(s)$ is square), as $\rho_{oj}(s)$ is a natural extension of the Nyquist plot for single input-single output systems[6]. Since $\rho_{oj}(s)$ determines the dynamical behaviour of $G_o(s)$, $G_r(s)$ can be constructed such that ρ_{rj} approximates $\rho_{oj}(s)$ over all possible frequency ranges. A feasible method of achieving this is to expand $\rho_{oj}(s)$ about a certain point by perturbation, and, then construct $\rho_{rj}(s)$ by neglecting certain terms of the expansion.[3,4]

Let $G_r(s) \triangleq G_r(s,\varepsilon_o) = G_o(s,\varepsilon)$ be analytic in the region $|\varepsilon-\varepsilon_o|<R$, where ε is a variable in terms of s such that $G_o(s,\varepsilon)\rightarrow G_o(s,\varepsilon_o) \triangleq G_o(s)$ as $\varepsilon\rightarrow\varepsilon_o$ and ε_o can be taken as zero where abritrary. Since $\rho_{oj}(s,\varepsilon)$ of $G_o(s,\varepsilon)$ depends continuously on $G_o(s,\varepsilon)$, such that $\rho_{rj}(s) \triangleq \rho_{oj}(s,\varepsilon)\rightarrow\rho_{oj}(s,\varepsilon_o)$ as $\varepsilon\rightarrow\varepsilon_o$, then $\rho_{oj}(s,\varepsilon)$ is also analytic in $|\varepsilon-\varepsilon_o|< R$. If $\rho_{oj}(s,\varepsilon)$ is unrepeated for all s, it can be expanded as

$$\rho_{oj}(s,\varepsilon) = \sum_{i=o}^{\infty} (\varepsilon-\varepsilon_o)^i \rho_{oj}^{(i)}(s,\varepsilon_o) \tag{3.33}$$

where $\rho_{oj}^{(i)}(s,\varepsilon_o)$ is the i^{th} Taylor derivative, with $\rho_{oj}^{(o)}(s,\varepsilon_o) = \rho_{oj}(s,\varepsilon_o)$, $G_o(s,\varepsilon)$ and the characteristic vectors, $\alpha_{oj}(s,\varepsilon) \in \mathrm{Nsp}[\rho_{oj}(s,\varepsilon)I-G_o(s,\varepsilon)]$, $\beta_{oj}(s,\varepsilon) \in \mathrm{Nsp}[\rho_{oj}(s,\varepsilon)I-G_o^t(s,\varepsilon)]$, where Nsp means 'null space of' can similarly be written as,

$$G_o(s,\varepsilon) = \sum_{i=o}^{\infty} (\varepsilon-\varepsilon_o)^i G_o^{(i)}(s,\varepsilon_o) \tag{3.34}$$

$$\alpha_{oj}(s,\varepsilon) = \sum_{i=o}^{\infty} (\varepsilon-\varepsilon_o)^i \alpha_{oj}^{(i)}(s,\varepsilon_o) \tag{3.35}$$

$$\beta_{oj}(s,\varepsilon) = \sum_{i=o}^{\infty} (\varepsilon-\varepsilon_o)^i \beta_{oj}^{(i)}(s,\varepsilon_o) \tag{3.36}$$

Furthermore, $\alpha_{oj}(s,\varepsilon)$ can be defined by normalizing such that

$$\sum_{i=o}^{\infty} (\varepsilon-\varepsilon_o)^i \beta_{oj}^t(s,\varepsilon_o)\alpha_{oj}^{(i)}(s,\varepsilon_o) = 1 \tag{3.37}$$

and since $\beta_{oj}^t(s,\varepsilon_o)\alpha_{oj}(s,\varepsilon_o) = 1$, it follows from eqn.(3.37) that

$$\beta_{oj}^t(s,\varepsilon_o)\,\alpha_{oj}^{(i)}(s,\varepsilon_o) = 0, \quad i \neq 0, \ \forall i \tag{3.38}$$

The index ε_o in the coefficients above represents the unperturbed components and for notational convenience it will be dropped, as henceforth it is understood that $G_o(s)$, $\rho_{oj}(s)$ mean $G_o(s,\varepsilon_o)$, $\rho_{oj}(s,\varepsilon_o)$ etc. Since

$$G_o(s,\varepsilon)\ \alpha_{oj}(s,\varepsilon) = \rho_{oj}(s,\varepsilon)\ \alpha_{oj}(s,\varepsilon) \tag{3.39}$$

substituting the perturbed quantities from eqns. (3.33) to (3.36) in eqn. (3.39) gives

$$\sum_{\ell=o}^{\infty}\sum_{k=o}^{\infty}(\varepsilon-\varepsilon_o)^{\ell+k}G_o^{(\ell)}(s)\alpha_{oj}^{(k)}(s) = \sum_{\ell=o}^{\infty}\sum_{k=o}^{\infty}(\varepsilon-\varepsilon_o)^{\ell+k}\rho_{oj}^{(\ell)}(s)\ \alpha_{oj}^{(k)}(s) \tag{3.40}$$

Equating coefficients of $(\varepsilon-\varepsilon_o)^m$ gives

$$(\rho_{oj}(s)I-G_o(s))\alpha_{oj}^{(m)}(s) = h(s) \tag{3.41}$$

where $\quad h(s) = \sum_{i=1}^{m}(G_o^{(i)}(s)-\rho_{oj}^{(i)}(s)I)\alpha_{oj}^{(m-i)}(s)$.

and $m \neq o$, $h(s) = 0$. Premultiplying both sides of eqn.(3.41) by $\beta_{oj}^{t}(s)$ yields

$$\sum_{i=1}^{m}\beta_{oj}^{t}(s)(G_o^{(i)}(s)-\rho_{oj}^{(i)}(s)I)\alpha_{oj}^{(m-i)}(s) = 0 \tag{3.42}$$

as $\beta_{oj}(s) \in \text{Nsp}[\rho_{oj}(s)I-G_o^{t}(s)]$. The orthogonal condition of eqn.(3.38) and the quasi-biorthogonal condition $\beta_{oj}^{t}(s)\alpha_{oj}(s) = 1$ give

$$\rho_{oj}^{(m)}(s) = \sum_{i=1}^{m}\beta_{oj}^{t}(s)G_o^{(i)}(s)\ \alpha_{oj}^{(m-i)}(s) \tag{3.43}$$

Since a unique solution vector for $\alpha_{oj}^{(m)}(s)$ in eqn.(3.41) exists if $h(s) \in \text{Rge}[\rho_{oj}(s)I-G_o(s)]$, where Rge means 'range of', it can be formulated in terms of the spectral properties of $G_o(s)$ in $\text{Nsp}[\rho_{oj}(s)I-G_o^{t}(s)]$ with $\beta_{oj}^{(t)}(s)$ quasi-orthogonal with $h(s)$. Hence from eqn. (3.41), $\alpha_{oj}^{(m)}(s)$ is given by

$$\alpha_{oj}^{(m)}(s) = \sum_{\substack{k=1\\k\neq j}}^{\rho}\sum_{q=1}^{d_k}(q-1)!\ Z_{kq}(s)h(s)/(\rho_{oj}(s)-\rho_{ok}(s))^{q} \tag{3.44}$$

where $\rho_{oj}(s)$ has linear elementary divisor, p is the number of distinct loci and d_k is the index of $\rho_{ok}(s)$

Proof: Define

$$E(s) = \sum_{\substack{k=1\\k\neq j}}^{p} \sum_{q=1}^{d_k} (q-1)!\, Z_{kq}(s)/(\rho_{oj}(s)-\rho_{ok}(s))^q \tag{3.45}$$

where $Z_{kq}(s)$ are linearly independent component matrices of $G_o(s)$. By the spectral resolution theorem,

$$f(G_o(s)) = \sum_{k=1}^{p} \sum_{q=1}^{d_k} f_k^{(q-1)}(s)\, Z_{kq}(s) \tag{3.46}$$

where $f_k^{(q)}(s) = \partial^q f_k(s)/\partial \rho_o(s)$ (evaluated at $\rho_o(s) = \rho_{ok}(s)$), $G_o(s)$ can be expressed as

$$G_o(s) = \sum_{k=1}^{p} (\rho_{ok}(s)\, Z_{k1}(s) + Z_{k2}(s)) \tag{3.47}$$

$$\text{Now}\quad E(s)(\rho_{oj}(s)I-G_o(s)) = \{\sum_{\substack{k=1\\k\neq j}}^{p} \sum_{q=1}^{d_k} (q-1)!\rho_{oj}(s)Z_{kq}(s)/(\rho_{oj}(s)-\rho_{ok}(s))^q\}$$

$$-\{\sum_{\substack{k=1\\k\neq j}}^{p} \sum_{q=1}^{d_k} (q-1)!Z_{kq}(s)/(\rho_{oj}(s)-\rho_{ok}(s))^q\}\sum_{\ell=1}^{p} (\rho_{o\ell}(s)Z_{\ell 1}(s) + Z_{\ell 2}(s)) \tag{3.48}$$

Using the properties that $Z_{kp}(s)Z_{\ell r}(s) = 0$ if $k \neq \ell$, the second expression on the r.h.s. of eqn. (3.48) simplifies into

$$\{\sum_{\substack{k=1\\k\neq j}}^{p} \sum_{q=1}^{d_k} (q-1)!\rho_{ok}(s)Z_{kq}(s)Z_{k1}(s)/(\rho_{oj}(s)-\rho_{ok}(s))^q\}-\{\sum_{\substack{k=1\\k\neq j}}^{p} \sum_{q=1}^{d_k} (q-1)!Z_{kq}(s)Z_{k2}(s)/$$

$$(\rho_{oj}(s)-\rho_{ok}(s))^q\} \tag{3.49}$$

Now $$Z_{kq}(s) = (G_o(s) - \rho_{ok}(s)I)^{q-1}\, Z_{k1}(s)/(q-1)! \tag{3.50}$$

Combining the first term on the r.h.s. of eqn.(3.48) with that of exp.(3.49), and using eqn.(3.50), and remembering that $Z_{k1}(s)$ is idempotent and commutes with $G_o(s)$, the r.h.s. of eqn.(3.48) becomes

$$\sum_{\substack{k=1\\k\neq j}}^{p}\sum_{q=1}^{d_k} (G_o(s) - \rho_{ok}(s)I)^{q-1}\, Z_{k1}(s)/(\rho_{oj}(s)-\rho_{ok}(s))^{q-1} - \sum_{\substack{k=1\\k\neq j}}^{p}\sum_{q=1}^{d_k} (G_o(s)-\rho_{ok}(s)I)^{q} Z_{k1}(s)/(\rho_{oj}(s)-\rho_{ok}(s))^{q} \tag{3.51}$$

which can be written as

$$\sum_{k=1}^{p} Z_{k1}(s) - Z_{j1}(s) + \sum_{\substack{k=1\\k\neq j}}^{p}\sum_{q=2}^{d_k} (G_o(s)-\rho_{ok}(s)I)^{q-1}\, Z_{k1}(s)/(\rho_{oj}(s)-\rho_{ok}(s))^{q-1}$$

$$- \sum_{\substack{k=1\\k\neq j}}^{q}\sum_{q=2}^{d_k} (G_o(s)-\rho_{ok}(s)I)^{q-1} Z_{k1}(s)/(\rho_{oj}(s)-\rho_{ok}(s))^{q-1} - \sum_{\substack{k=1\\k\neq j}}^{p} (G_o(s)-\rho_{ok}(s)I)^{d_k}\, Z_{k1}(s)\Big/(\rho_{oj}(s) - \rho_{ok}(s))^{q-1} \tag{3.52}$$

Since $\mathrm{Rge}\,[Z_{k1}(s)] = \mathrm{Nsp}\,[(G_o(s)-\rho_{ok}(s)I)^{d_k}]$, it follows that $(G_o(s)-\rho_{ok}(s)I)^{d_k}\, Z_{k1}(s)y(s) = 0$, for some $y(s) \neq 0$, hence $(G_o(s)-\rho_o(s)I)^{d_k}$ $Z_{k1}(s) = 0$. Therefore the last term of eqn. (3.52) vanishes, and, since $\sum_{k=1}^{p} Z_{k1}(s) = I$, eqn.(3.48) is simplified into

$$E(s)[\rho_{oj}(s) - G_o(s)] = I - Z_{j1}(s) \tag{3.53}$$

Premultiplying eqn. (3.41) by E(s) and substituting eqn. (3.53)

$$(I-Z_{j1}(s))\alpha_{oj}^{(m)}(s) = E(s)h(s) \tag{3.54}$$

If the index d_j of $\rho_{oj}(s) = 1$, from the spectral resolution theorem and Lagrange polynomials $Z_{j1}(s)$ is the sum of constituent matrices $\alpha_{oj}(s)\beta_{oj}{}^t(s)$ associated with $\rho_{oj}(s)$, thus $Z_{j1}(s)\alpha_{oj}{}^{(m)}(s) = \sum_{i=1} \alpha_{oi}(s)$ $\beta_{oi}{}^t(s)\alpha_{oj}{}^{(m)}(s)$. A solution vector $\alpha_{oj}{}^{(m)}(s)$ is sought such that $\beta_{oi}{}^t(s)\alpha_{oj}{}^{(m)}(s) = 0$, (eqn.(3.38)), for every $\beta_{oi}(s) \in Nsp[\rho_{oi}(s)I-G_o{}^t(s)]$. Hence $Z_{j1}(s)\alpha_{oj}{}^{(m)}(s) = 0$, thus eqn.(3.54) reduces to

$$\alpha_{oj}{}^{(m)}(s) = E(s)h(s)$$

which is eqn.(3.44).

The idempotent matrix $Z_{k1}(s)$ in the r.h.s. of eqn.(3.50) can be conveniently computed by the Lagrange Sylvester interpolatory polynomial as

$$Z_{k1}(s) = \prod_{\substack{k=1\\k\neq j}}^{p} (G_o(s)-\rho_{ok}(s)I) / \prod_{\substack{k=1\\k\neq j}}^{p} (\rho_{oj}(s)-\rho_{ok}(s)) \qquad (3.55)$$

Eqns. (3.43) and (3.44) can be used alternatively with m=1,2,... to find the coefficients $\rho_{oj}{}^{(i)}(s)$ and $\alpha_{oj}{}^{(i)}(s)$ in eqns. (3.33) and (3.35), using also eqns. (3.50) and (3.55)

For simple $G_o(s)$, $d_k = 1$ for all $\rho_{ok}(s)$, then eqn.(3.44) simplifies into

$$\alpha_{oj}{}^{(m)}(s) = \prod_{\substack{k=1\\k\neq j}}^{p} Z_{kq}(s)h(s)/(\rho_{oj}(s)-\rho_{ok}(s))^q \qquad (3.56)$$

The first coefficients $\rho_{oj}{}^{(1)}(s)$ and $\alpha_{oj}{}^{(1)}(s)$ in eqns. (3.43) and (3.44) can be simplified as

$$\rho_{oj}{}^{(1)}(s) = \langle \beta_{oj}(s),\ G_o{}^{(1)}(s)\ \alpha_{oj}(s) \rangle$$

$$\begin{aligned}\alpha_{oj}{}^{(1)}(s) &= E(s)h(s) = E(s)(G_o{}^{(1)}(s)-\rho_{oj}{}^{(1)}(s)I)\alpha_{oj}(s)\\ &= E(s)G_o{}^{(1)}(s)\alpha_{oj}(s)\end{aligned} \qquad (3.57)$$

as $E(s)\alpha_{oj}(s) = E(s)\sum_{i=1}(\alpha_{oi}(s)\beta_{oi}^{t}(s))\alpha_{oj}(s) = E(s)Z_{j1}(s)=0$, after substituting for $E(s)$ in eqn. (3.45) as $Z_{kq}(s)Z_{j1}(s) = 0$. Higher coefficients $\alpha_{oj}^{(m)}(s)$ can thus be built on the simplified forms of $\alpha_{oj}^{(1)}(s)$ in eqn. (3.57)

3.4 Relation to the model reduction problem.

A reduced model $G_r(s)$ can be constructed from $G_o(s)$ by approximating $\rho_{oj}(s)$ by $\rho_{rj}(s)$ and minimizing the error criterion

$$\min \int_{\Omega} < \rho_e(\omega), Q\rho_e(\omega) > \quad d\omega \tag{3.58}$$

over a spectrum Ω, where the dimension of the error vector $\rho_e(\omega)$ is equal to the dimension, m, of $G_o(s)$; and $\rho_e(\omega) = (\rho_{e1}(\omega), \rho_{e2}(\omega), \ldots \rho_{em}(\omega))^t$. where $\rho_{ej}(\omega) = |\rho_{rj}(s) - \rho_{oj}(s)|$

Further from eqn. (3.33),

$$\rho_{ej}(\omega) = \left| \sum_{i=1}^{\infty} (\varepsilon-\varepsilon_o)^i \rho_{oj}^{(i)}(s) \right| \tag{3.59}$$

and $Q = \mathrm{diag}\{q_1, q_2, \ldots q_n\}$ is a weighting matrix.

However, $G_r(s)$ is non-unique for a given set of $\rho_{rj}(s)$, but, is unique for a given set of $\rho_{rj}(s)$ and the corresponding characteristic vectors $\alpha_{rj}(s), \beta_{rj}(s)$. Hence to obtain a unique $G_r(s)$, from a geometrical point of view, it is intuitively satisfactory to have $\alpha_{oj}(s), \alpha_{rj}(s)$ and $\beta_{oj}(s)$, $\beta_{rj}(s)$ in close alignment.

Since $\mathrm{Cos}(\theta) = <\alpha_{oj}(s), \alpha_{rj}(s)> / \|\alpha_{oj}(s)\|_E \|\alpha_{rj}(s)\|_E$, it follows that a suitable error criterion is to minimize the angle between the vectors

$$\min \int_{\Omega}\{e_\alpha^2(\omega) + e_\beta^2(\omega)\} \quad d\omega \tag{3.60}$$

where

$$e_\alpha(\omega) = <\alpha_{oj}(s), \alpha_{rj}(s)> - \|\alpha_{oj}(s)\|_E \|\alpha_{rj}(s)\|_E \tag{3.61}$$

$$e_\beta(\omega) = <\beta_{oj}(s), \beta_{rj}(s)> - \|\beta_{oj}(s)\|_E \|\beta_{rj}(s)\|_E$$

The perturbation coefficients $\rho_{oj}^{(i)}(s)$ in eqn. (3.59) can be calculated in terms of $\rho_{oj}(s)$ and $\alpha_{oj}(s)$, however $G_o^{(i)}(s)$ in eqn. (3.34) can be found recursively by numerical extrapolation taking small step lengths

$$G_o^{(i+1)}(s) \approx [G_o^{(i)}(s,\varepsilon+\Delta\varepsilon) - G_o^{(i)}(s,\varepsilon)]/\Delta\varepsilon|_{\varepsilon=\varepsilon_o} \tag{3.62}$$

and $\Delta\varepsilon$ can be chosen as

$$\Delta\varepsilon = \|G_o(s)\| - \|G_{rk}(s)\| \tag{3.63}$$

such that $G_{rk}(s) \to G_o(s)$ as $\Delta\varepsilon \to 0$ where $G_{rk}(s)$ represents the reduced model, $G_r(s)$, in the k^{th} iteration of the routine.

If the perturbations of $G_o(s)$ are assumed linear in ε, i.e. if $G_r(s) = G_o(s,\varepsilon) = G_o(s) + \varepsilon G_{ek}(s)$ where $G_{ek}(s)$ is independent of ε, then $G_o^{(1)}(s) = G_{ek}(s)$, where $G_{ek}(s) = G_o(s)-G_{rk}(s)$, and, higher derivatives $G_o^{(i)}(s) = 0$, for $i = 2,3,\ldots.(i\neq 1)$. Eqns. (3.43) and (3.44) yield

$$\begin{aligned} \rho_{oj}^{(m)}(s) &= \langle \beta_{oj}(s), G_o^{(1)}(s)\alpha_{oj}^{(m-1)}(s) \rangle \\ \alpha_{oj}^{(m)}(s) &= E(s)\ \{G_o^{(1)}(s)-\rho_{oj}^{(1)}(s)I\}\alpha_{oj}^{(m-1)}(s) \end{aligned} \tag{3.64}$$

Using eqn. (3.64) avoids solving eqn. (3.62). Further savings in computation are achieved if the perturbations in $\rho_{oj}(s)$ and $\alpha_{oj}(s)$ are also assumed linear i.e. $\rho_{oj}^{(i)}(s) = \alpha_{oj}^{(i)}(s) = 0$, $\forall i$, $i \neq 1$.

A minimization routine with numerical integration can be used to solve eqns. (3.58) and (3.60) simultaneously to obtain $G_r(s)$. The solution can further be simplified by assigning constant values to $e_\alpha(s)$ and $e_\beta(s)$, thus treating eqn. (3.61) as an equality constraint and neglecting eqn. (3.60). Also, to ensure that $\rho_{rj}(s)$ stays within the Gershgorin discs of $G_r(s)$, the inequality constraints,

$$|\rho_{rk}(s) - g_{rii}(s)| \leq \sum_{i \neq j} |g_{rij}(s)|$$

$$|\rho_{rk}(s) - g_{rjj}(s)| \leq \sum_{j \neq i} |g_{rji}(s)| \qquad (3.65)$$

$$\forall i, \forall j, \forall k$$

are introduced.

The accuracy in reduction can be determined by comparing the difference in open-loop response. Now

$$e(s) = G_e(s)u(s) \qquad (3.66)$$

i.e.

$$e^H(s) = u^H(s)G_e^{\;H}(s) \qquad (3.67)$$

where $e(s) = y_o(s) - y_r(s)$, $G_e(s) = G_o(s) - G_r(s)$, $u(s)$ is the input vector, and, the superscript H denotes the Hermitian matrix, example $G_e^{\;H}(s) = G_e^{\;t}(-s)$. Premultiplying eqn.(3.66) by $e^H(s)$ and postmultiplying eqn.(3.67) by $e(s)$ and combining gives,

$$e^H(s)e(s)e^H(s)e(s) = e^H(s)H(s)e(s) \qquad (3.68)$$

i.e.

$$\|e(s)\|_E^2 = \langle e(s), H(s)e(s) \rangle / \langle e(s), e(s) \rangle \qquad (3.69)$$

where $H(s) = G_e(s)U(s)G_e^H(s)$ and $U(s) = u(s)u^H(s)$. The r.h.s. of eqn. (3.69) is a Rayleigh quotient, and, since $H(s)$ is positive definite Hermitian, its eigenvalues $\eta_j(s)$ are real and positive for each $s=j\omega$. The Courant Fischer extremal bounds give

$$\{\min \eta_j(s)\}^{\frac{1}{2}} \leq \|e(s)\|_E \leq \{\max \eta_j(s)\}^{\frac{1}{2}} \qquad (3.70)$$

Equality is observed in eqn.(3.70) in the ideal case when $G_r(s) = G_o(s)$, which implies $y_r(s) = y_o(s)$, from which it is deduced that all $\eta_j(s) = 0$, and, the error in output response, $e(s) = 0$.

For a multivariable system, it is difficult to assess the order of the model from the common denominator of its transfer function matrix.(unless the system is single input-single output) Working on the elements of the transfer function matrix leaves the order of $G_r(s)$ uncertain, and, in general, the order of the final $G_r(s)$ will be different from the order prespecified. If importance is stressed on a definite order of the final $G_r(s)$, this can be achieved by working on the state space matrices $S_r(A_r, B_r, C_r)$ of $G_r(s)$.

For convenience, if S_r is chosen in canonical form,

$$\begin{aligned} A_r &= \operatorname{diag}(a_{r11}, a_{r22}, \ldots a_{rrr}) \\ B_r^t &= (b_{r1}, b_{r2}, \ldots b_{rr}) \\ C_r &= (c_{r1}, c_{r2}, \ldots c_{rr}) \end{aligned} \qquad (3.70a)$$

then $G_r(s)$ can be written as

$$G_r(s) = \sum_{i=1}^{n} (c_{ri} \rangle\langle b_{ri}) / (s - a_{rii}) \qquad (3.70b)$$

assuming a_{rii} is distinct. It is also desirable to impose the constraint

$$\begin{aligned} b_{ri} &\neq 0 \quad , \forall i \\ c_{ri} &\neq 0 \quad , \forall i \end{aligned} \qquad (3.70c)$$

such that S_r is controllable and observable, hence a minimal realisation of $G_r(s)$. The order of $G_r(s)$ can thus be prespecified by the dimension of A_r.

If the steady state values were to be matched in $G(s)$ and $G_r(s)$, then application of the final value theorem to eqn.(3.70b) gives

$$-\left(\sum_{i=1}^{n} (c_{ri} \rangle\langle b_{ri}) / a_{rii}\right) u = y_{s.s.} \qquad (3.70d)$$

Some parameters a_{rii}, b_{ri}, or c_{ri} can be chosen to satisfy the equality constraint, eqn(3.70d), and the rest found by a minimization routine used in matching the Characteristic Loci.

For A_r not chosen in diagonal form, then $G_r(s)$ can be written as

$$G_{rij}(s) = \langle c_{ri},\ \mathrm{Adj}\,\Phi(s)\,b_{rj}\rangle / \det\Phi(s) \qquad (3.70e)$$

where 'Adj' is the adjoint of $\Phi(s)$ $(= (sI - A)^{-1})$.

Non square systems

If $G(s)$ is non square, i.e. the number of inputs is not equal to the number of outputs, reduction by the Loci method is still possible by 'squaring' $G(s)$, and proceed as before. This is done by assigning appropriate fictitious rows to B or columns to C such that $G(s)$ is square. The corresponding fictitous rows of B_r or columns of C_r can be neglected afterwards to yield a non square $G_r(s)$. The fictitious input or output vectors must be appropriatly chosen, such that their contributions are very small compared to the real input and output vectors.

Computational algorithm.

The reduction can be done by the following algorithm

(i) Retain some dominant modes of $G_o(s)$. Fix model order, ensure that it is physically realizable and choose initial parameters for $G_r(s)$, via $S_r(A_r,B_r,C_r)$. To simplify the problem some parameters may be fixed, example, to satisfy s.s. error constraint.

(ii) Evaluate $G_o^{(i)}(s)$ by eqn. (3.62) and approximate $G_r(s)(\triangleq G_o(s,\varepsilon))$ by eqn.(3.34), till error term in the series expansion $\|E_n(s)\| \approx 0$.

(iii) Use eqns. (3.43) and (3.44) to obtain $\rho_{oj}^{(i)}(s)$ and $\alpha_{oj}^{(i)}(s)$ $i=1,2,\ldots$ For $G_o^{(i)}(s) = 0$, $i=2,3,4,\ldots$, use the simplified form of eqn.(3.64).

(iv) Evaluate eqns. (3.58) and (3.60) by a minimization routine in view of the inequality constraint of eqn.(3.65).

(v) If optimal $G_{rk}(s)$ found, go to (vi), otherwise update $G_{rk}(s)$ and go to (ii), with $k = k+1$.

(vi) Test for accuracy of $y_r(s)$ by eqn.(3.70). If satisfactory, exit; otherwise,go to (i).

The advantage of using perturbation analysis, in the above, is that the characteristic loci can be approximated to any degree of accuracy required, and, the eigenvalues of $G_r(s)$ need not be calculated directly, by an eigenvalue subroutine, in every iteration of the algorithm; as the latter is tedious. However, the characteristic loci of $G_o(s)$ must be known for the reduction process.

Repeated characteristic loci.

If $G_o(s)$ is nonsimple, i.e. if $\rho_{oj}(s)$ has multiplicity m such that $\rho_{ok}(s,\varepsilon)\to\rho_{oj}(s)$ as $\varepsilon\to 0$ for $k = \gamma_1,\gamma_2,\ldots\gamma_m$, then $\rho_{ok}(s,\varepsilon)$ is an analytic function of $\varepsilon^{1/m}$, $l\leq m$, in the region $|\varepsilon-\varepsilon_o|<r$, and can be represented by the Puiseux series in fractional powers of ε, $O(|\varepsilon|^{1/\ell})$.

It can be shown that only in special cases when $\rho_{oj}(s)$ has index 1, then $\rho_{ok}(s,\varepsilon)$ can be reduced, by Kato's reduction method, to the form[3,4],

$$\rho_{ok}(s,\varepsilon) = \rho_{oj}(s) + h_{1k}(s)\varepsilon + O\{|\varepsilon|^{1+(1/\ell)}\} \tag{3.71}$$

where $h_{1k}(s)$ is an eigenvalue of $Z_{11}(s)G_o^{(1)}(s)Z_{11}(s)$. Further, if $h_{1k}(s)$ is also of index 1, a further reduction can yield

$$\rho_{ok}(s,\varepsilon) = \rho_{oj}(s) + h_{1k}(s)\varepsilon + h_{2k}(s)\varepsilon^2 + O\{|\varepsilon|^{2+(1/\ell)} \tag{3.72}$$

where $h_{2k}(s)$ is a characteristic value of $Z_{11}(s)\{G_o^{(1)}(s)E(s)G_o^{(1)}(s)+G_o^{(2)}(s)\}$ $Z_{11}(s)$. In the special case, when $G_o(s,\varepsilon)$ is normal throughput in the region $|\varepsilon-\varepsilon_o|<r$, then $h_{1k}(s)$ and $h_{2k}(s)$ have indices 1 for all k. By repeated use of the reduction process it is possible to make $O\{|\varepsilon|^{n+(1/n)}\}\to 0$, then $\rho_{ok}(s,\varepsilon)$ in eqn.(3.72) becomes a Taylor series in integer powers of ε.

In general the Puiseux series is numerically awkward to evaluate, at best the eigenvalues must be computed directly in every iteration of the minimization routine if $G_r(s)$ becomes nonsimple at some value of frequency.

3.5 Examples.

The reduction methods discussed above are illustrated by two examples below.

(a) The first example is a seventh order single input-single output system considered by Dorf,

$$\begin{aligned} H(s) = (375E{+}03s + 312E{+}02)/(s^7&+0.8364E07s^6+0.4097E04s^5 \\ &+ 0.7034E05s^4 + 0.8537E05s^3 + 0.28143E07s^2 \\ &+ 0.3311E07s + 0.28125E06) \end{aligned}$$

whose poles are

$$-0.09191,\ -2.0244 \pm j\ 0.9646,\ -7.6709 \pm j\ 13.442,\ -32.08 \pm j38.87$$

and zero is -0.08333.

The harmonic synthesis reduction method of section 3.1 was used, choosing the half period T = 20 seconds (> 1/0.09191). A second order model was required, and choosing initial pole estimates as $-3.0\pm j2.0$, the reduction procedure gave the final poles as $-1.7101\pm j1.509$, yielding the reduced model,

$$H_r(s) = 0.612/(s^2 + 3.42s + 5.2)$$

with the sum of squares error equal to 0.05183.

The response of $H(s)$ and $H_r(s)$ to a unit step input is shown in fig.3.3. The curve fitting is good with small steady state error. For comparison purposes, the continued fraction method of Chen and Shieh was also used, which yielded a model $H_{cr}(s) = (0.13s + 0.011)/(s^2 + 146s + 0.0994)$ whose response is shown in the same figure.

(b) To illustrate reduction by characteristic loci method, the multivariable system, $G(s)$, considered by Macfarlane in commutative controller design[7], was used.

$$g_{11}(s)=(2s^2+3s-1)/(s(s+1)(s+2)), \qquad g_{12}(s)= -2(s^2+s-1)/s(s+1)^2(s+3)$$

$$g_{21}(s)=(s^2+s-1)/s(s+2) \qquad , \qquad g_{32}(s)= -(s^2-2)/s(s+1)(s+2)$$

The characteristic loci and characteristic vectors of $G(s)$ are

$$\rho_1(s) = 1/(s+1), \quad \rho_2(s) = 1/s(s+2)$$

$$\alpha_1^t(s) = (2/(s+1),\ 1) \quad , \quad \alpha_2^t(s) = (1,\ s+1)$$

and its characteristic polynomial is c.p. = $s(s+1)(s+2)$, hence $G(s)$ is a 3rd order model.

A reduced second order model, $G_r(s)$, was required; and observing $G(s)$, there is an infinite time constant present, which means that the response to a step input is unbounded. Since the infinite time constant dominates the response of $G(s)$, it was retained in $G_r(s)$. The canonical structure of S_r was chosen as, $A_r = \mathrm{diag}(\ 0,\ a_{r22}\ )$,

$$B = \begin{bmatrix} b_{r11} & b_{r12} \\ 1.0 & 0.4 \end{bmatrix} , \qquad C = \begin{bmatrix} 1.0 & 1.0 \\ c_{r21} & c_{r22} \end{bmatrix}$$

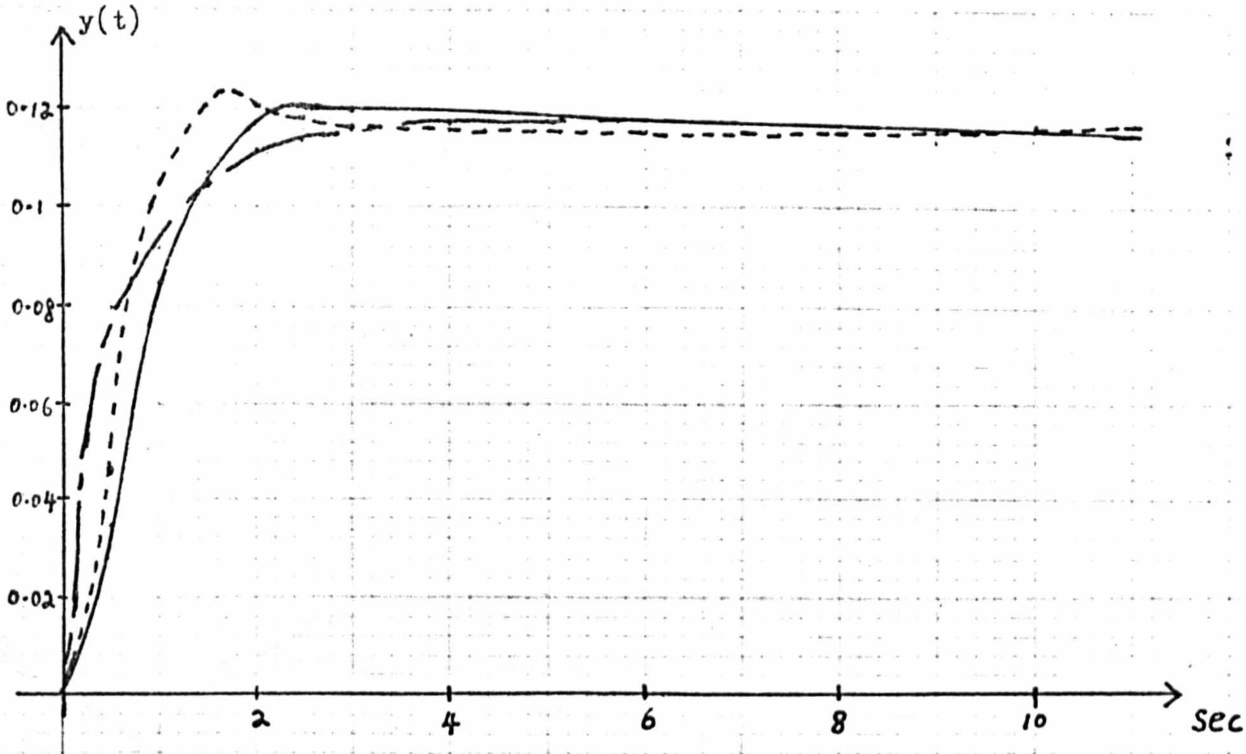


Fig. 3.3

——————— original model 7th order

- - - - - - - reduced model obtained by Harmonic Synthesis method, 2nd order.

——— - ——— - reduced model by Chen & Shieh's method, 2nd order.

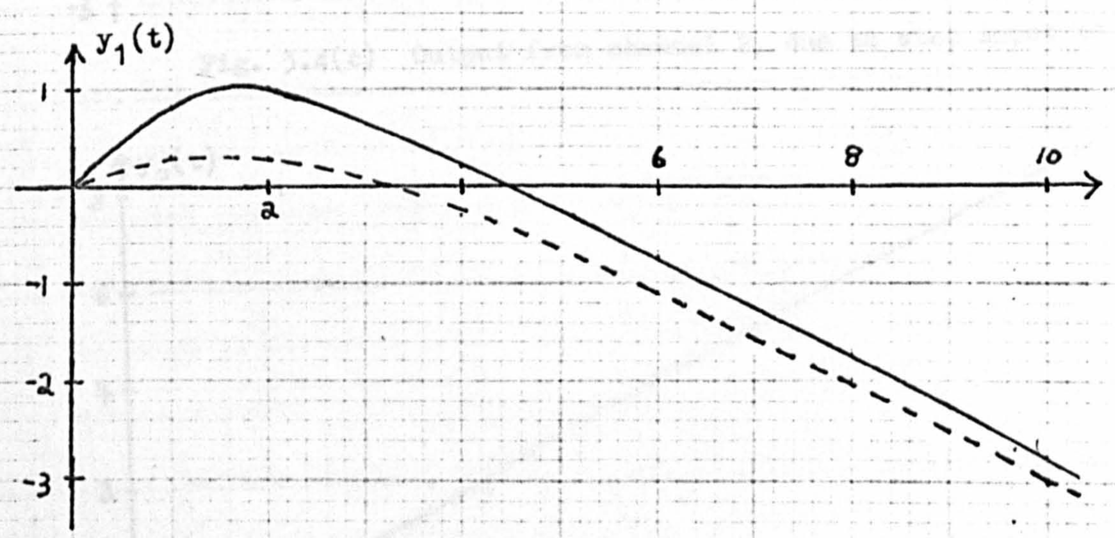


Fig. 3.4(a) Output from channel 1, due to step input to channel 1

——————— original model

- - - - - - - simplified model obtained by Characteristic Loci method, 2nd order

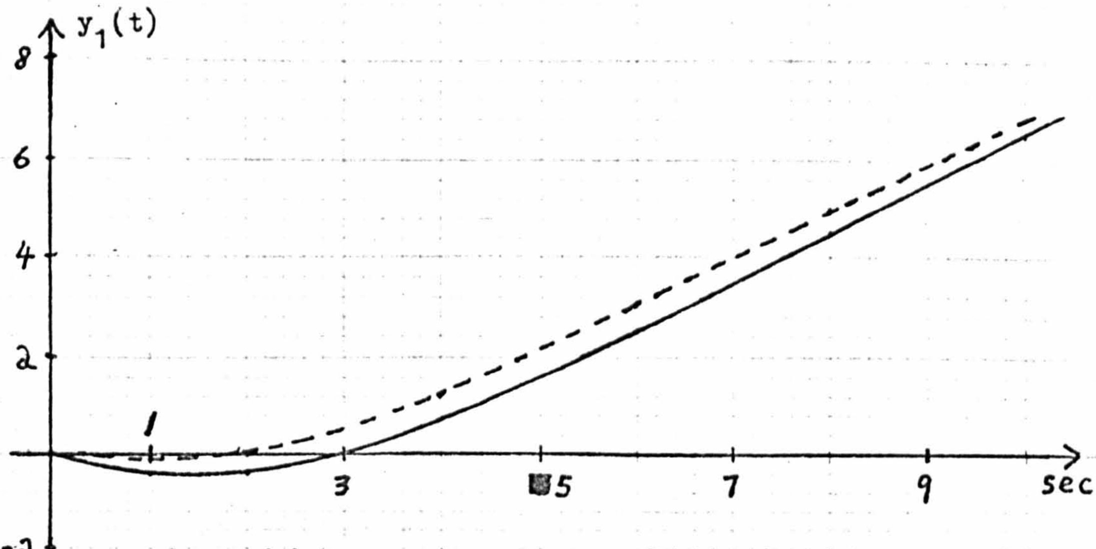


Fig. 3.4(b) Output from channel 1, due to step input to channel 2

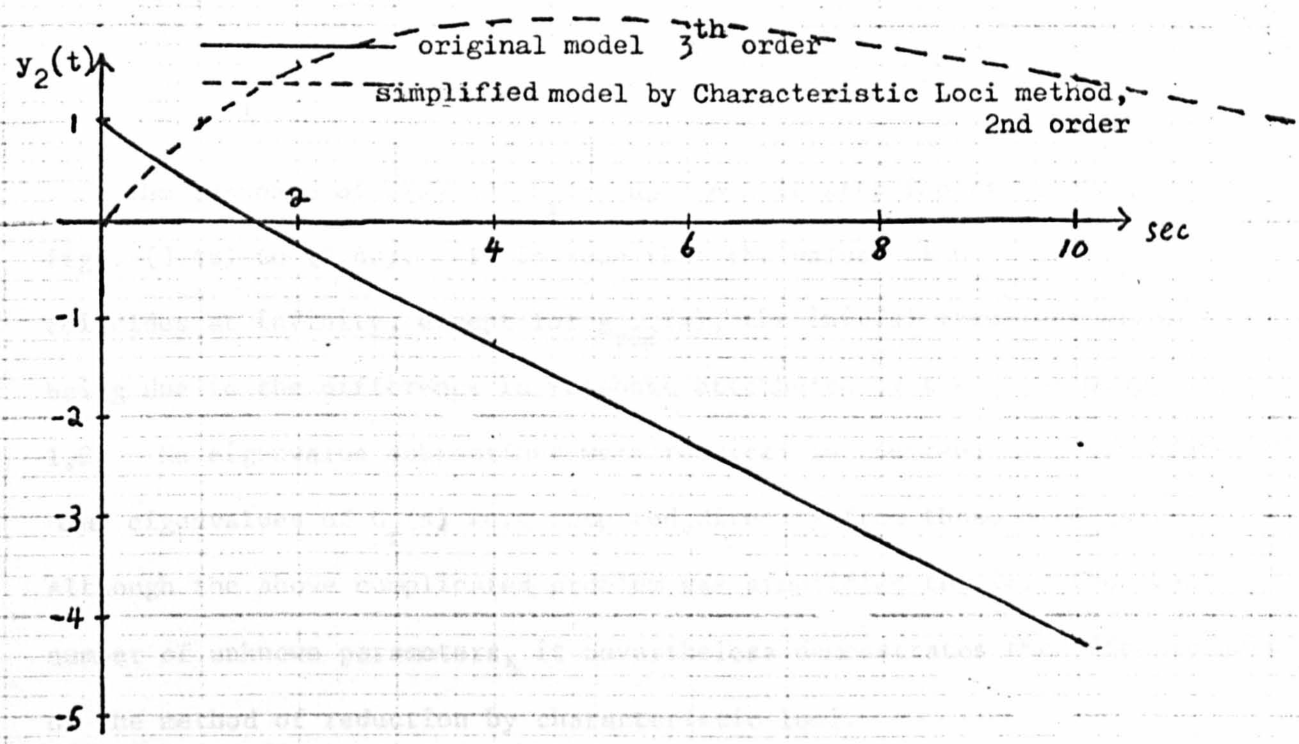


Fig. 3.4(c) Output from channel 2, due to step input to channel 1

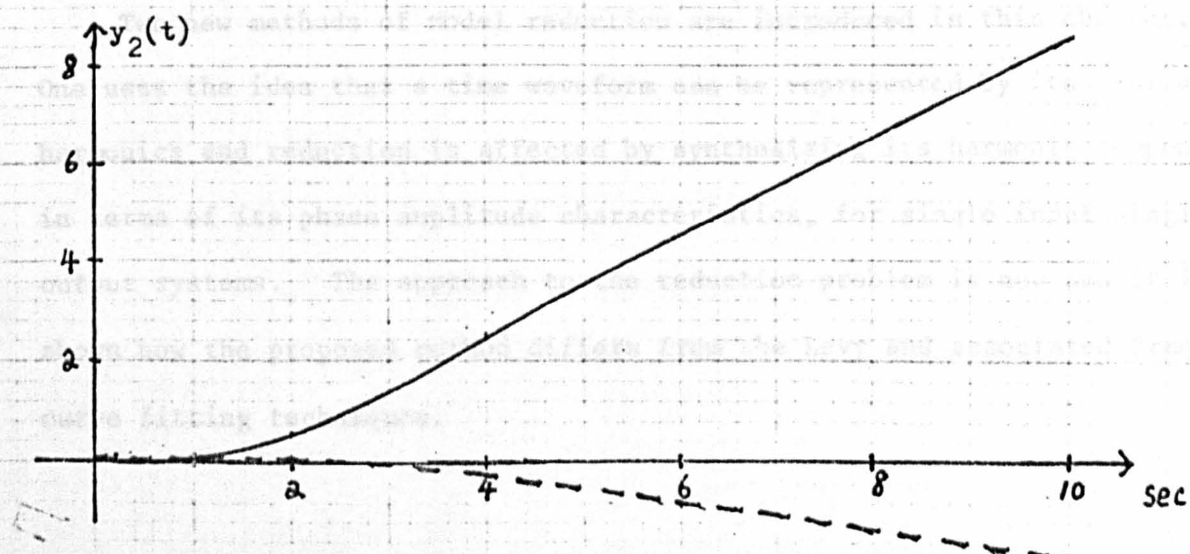


Fig. 3.4(d) Output from channel 2, due to step input to channel 2

with some parameters fixed, to reduce the number of unknown variables.

The initial unknown values were chosen as $a_{r22} = -1.0$, $b_{r11} = 1.0$, $b_{r12} = 1.0$, $c_{r21} = 1.0$, $c_{r22} = 1.0$. Using a minimization routine, with the reduction procedure of section 3.4, gave the final S_r as;

$$A_r = \text{diag}(0, -0.508), \quad B = \begin{bmatrix} -0.4923 & -1.006 \\ 1.0 & 0.4 \end{bmatrix}, \quad C_r = \begin{bmatrix} 1.0 & 1.0 \\ 0.3149 & 1.566 \end{bmatrix}$$

which is controllable and observable. The final $G_r(s)$ is,

$$G_r(s) = \begin{bmatrix} (0.5077\,s-0.2501)/(s+0.508)s & (-0.606s-0.5109/(s+0.508)s \\ (1.411s-0.07875)/(s+0.508)s & (0.3096s-0.1609)/(s+0.508)s) \end{bmatrix}$$

The response of G(s) and $G_r(s)$ due to unit step inputs is shown in figs. (3.4a) to (3.4d). It is seen that the values of G(s) and $G_r(s)$ coincides at infinity, except for $g_{r22}(s)$, the initial transient error being due to the difference in response attributed by the modes -0.508 and -1,-2. No eigenvalue subroutines were required in the reduction procedure, the eigenvalues of $G_r(s)$ were computed directly from those of $G_o(s)$. Although the above complicated problem was simplified to yield the least number of unknown parameters, it nevertheless demonstrates the effectiveness of the method of reduction by characteristic loci.

3.6 Conclusions.

Two new methods of model reduction are introduced in this chapter. One uses the idea that a time waveform can be represented by its Fourier harmonics and reduction is affected by synthesizing its harmonic components, in terms of its phase amplitude characteristics, for single input single output systems. The approach to the reduction problem is new and it is shown how the proposed method differs from the Levy and associated frequency curve fitting techniques.

The other method, also novel, uses the concept of the characteristic function (eigenvalue) of a transfer function matrix, for a multivariable system, over the field of rational fractions in s. Since the dynamics of a system are dependent on its transfer matrix characteristic loci, it is intuitively felt that reduction in this direction provides a sound theoretical framework. This is justified by the fact that the characteristic loci, is also related to multivariable root loci, and, is a natural extension of the frequency response and root locus of single input single output systems.

The methods are each illustrated by an example.

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IV

SOME TIME DOMAIN METHODS FOR LINEAR SYSTEMS REDUCTION.

CHAPTER IV.

SOME TIME DOMAIN METHODS FOR LINEAR SYSTEMS REDUCTION

Introduction

Time domain methods of reduction like Mitra's and Wilson's (Chapter II) suffer from computational difficulties when large order systems are involved. This is due to the fact that the methods produce large matrices in matrix equations, and, problems arise in computer storage and numerical aspects of the solutions. The method of Marshall[2] is also computationally unattractive for high order systems as it requires the computation of eigenvalues and eigen-vector matrices, and, partitioning them for further manipulation. Besides, the method only yields accurate results if the neglected fast modes have small residues compared to those of the dominant modes. Otherwise, significant transient error can occur if the residues of the fast modes are numerically larger than those of the dominant ones. The same thing can be said of the method due to Davison[1]. In practice, a system can have dominant residues associated with non-dominant modes.

On the other hand, the Pade[7] approximation and continued fraction methods are probably the most accurate and simplest methods, in terms of modest core requirements, ease in software implementation and number of computations. However, the methods can also yield unstable models and can fail in cases where the Pade approximation, for the required model order, does not exist.

Towill[4] and others have approximated a large system by replacing certain poles by an equivalent pole or time delay. Marshall[3] et al used pole selection, time response and root-locus technique for systems reduction via interactive graphics.

In this chapter, combinations of some of the above techniques are used to give some new methods of reduction where the model is approximated sequentially to yield the best type of response. Computational difficulties and storage problems inherent in the above methods are overcome here. Further, one of the methods is design oriented, in that the poles of the reduced model can be made to lie in certain constrained regions in the l.h.s. s-plane.

4.1 Moments and equivalent time constants.

A single stable transfer function expression in terms of the ratio of two polynomials, can be isolated into its real poles and complex poles and their associated residues, respectively. The reduction analysis given below covers five parts, viz (a) reduction of real modes to real modes, (b) complex modes to real modes, (c) complex modes to complex modes, (d) real modes to complex modes, (e) existence of repeated modes.

(a) Reduction of real mode to real mode.

Consider a first order transfer function of the form

$$\{R_a/(s+a)\} + \{R_b/(s+b)\} \tag{4.1}$$

where a and b are real poles with real residues R_a, R_b (assume $R_a > 0$, $R_b > 0$). The impulse response of eqn. (4.1) is

$$y(t) = R_a \exp(-at) + R_b \exp(-bt) \tag{4.2}$$

Consider representing y(t) by a single mode whose pole and residue are time dependent, i.e.

$$R_a \exp(-at) + R_b \exp(-bt) = R_c(t)\exp(-c(t)t) \tag{4.3}$$

The r.h.s. of eqn. (4.3) represents a time varying model. Replacing $R_c(t)$ and c(t) by constant values R_c and c respectively, would result in an error $e(t) = y(t) - R_c \exp(-ct)$. One possible method of determining optimal values for R_c and c is by minimizing the criterion

$$\int_0 e^2(t)\ dt \tag{4.4}$$

for an impulse input. This problem was studied by Meier and Luenberger and others in the complex domain and by Wilson in the time domain. (see Chapter II). A set of non-linear equations result, and, if the poles are prespecified, the equations will reduce to a linear set with unknown residues. However, the error criterion in eqn. (4.4) is subjective and only restricted to impulse inputs. Also, parameters that are optimal in the mathematical sense may not be so in the engineering sense.

Below, some approximation techniques are used to choose prospective values for R_c and c in eqn. (4.3) such that the response, y(t), is satisfactory to the designer's judgement.

For convenience, $R_c(t)$ is kept constant at R_c, such that the initial error or steady state error, due to an impulse or step input, can be set to zero by the initial or final value theorem, (with later correction), respectively, i.e.

$$\begin{aligned} R_c &= R_a + R_b \\ \text{or} \qquad R_c &= R_a/a + R_b/b \end{aligned} \tag{4.5}$$

From eqn. (4.3)

$$c(t) = -\{[\ln\ (R_a/R_c)\exp(-at)+(R_b/R_c)\exp(-bt)]\}/t \tag{4.6}$$

The variation of c(t) with t is shown in fig.4.1. To select a constant value of c(t), a first approximation would be to take the average value of c(t)

$$c_1 = \{1/(T-t_o)\} \int_{t_o}^{T} c(t)\ dt \tag{4.7}$$

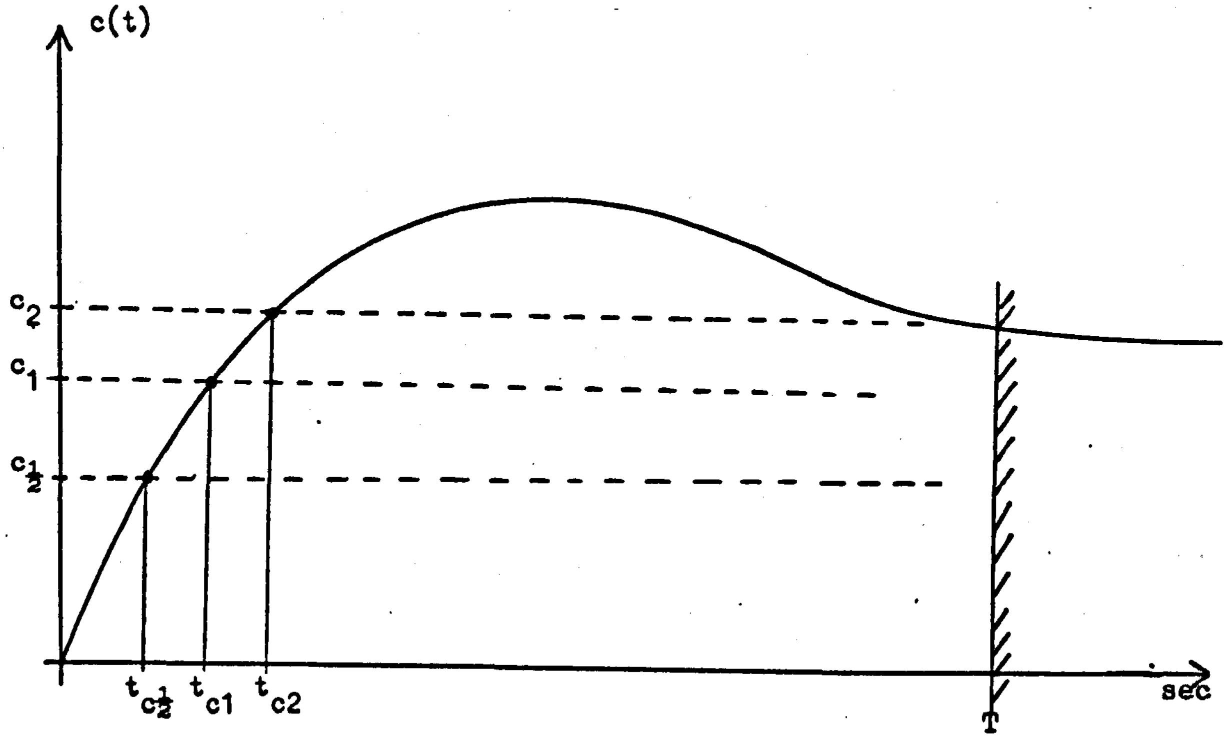

Fig. 4.1 Variation of c(t) with time

c_2, c_1, $c_{\frac{1}{2}}$ correspond to root mean square, mean and square mean root values of c(t) respectively

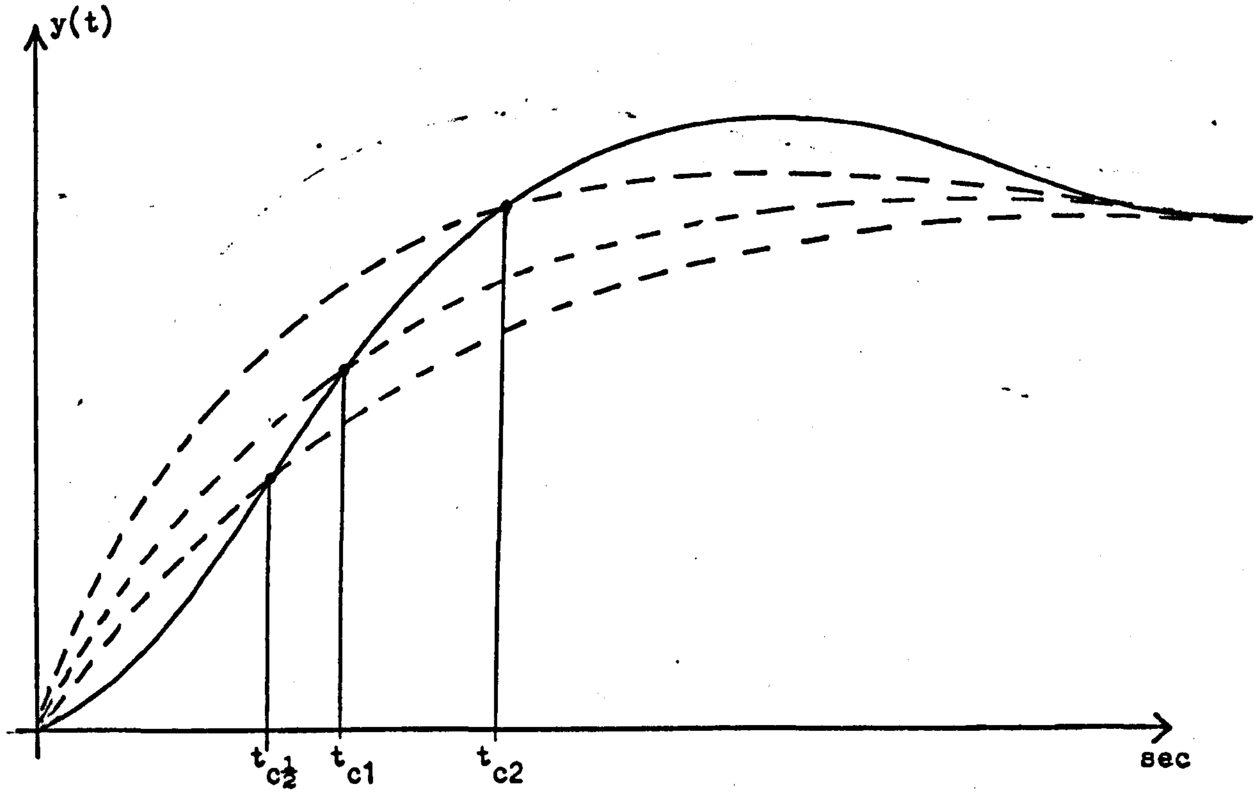

Fig. 4.2 Curve fitting of a single exponential mode with different time constants to a sum of various exponential modes

------ single mode

——— sum of different modes

where T is the settling time of two exponentials, $T \gg 1/a + 1/b$, and t_o is the initial time, preferably $t_o = 0$. Expanding eqn. (4.6),

$$c(t) = a - (1/t)\,\ell n(R_a/R_c) - (1/t)\ell n\,\{1+(R_b/R_a)\exp(-(b-a)t)\} \quad (4.8)$$

where

$$\ell n\{1+(R_b/R_a)\exp(-(b-a)t)\} = \sum_{i=1}^{\infty} ((-1)^{i+1}/i)(R_b/R_a)^i \exp(-i(b-a)t) \quad (4.9)$$

assuming $b > a$ and $R_a > R_b$. Now,

$$\int_{t_o}^{T} (1/t)\exp(-k(b-a)t)dt = \sum_{\substack{\ell=1\\ n,\text{even}}}^{n} (-1)^{\ell}(\ell-1)!\,[\exp(-k(b-a)t)/(k(b-a)t)^{\ell}]_{t_o}^{T}$$

$$+ \int_{t_o}^{T} n!\,\exp(-k(b-a)t)/(k(b-a))^n t^{n+1}\,dt \quad (4.10)$$

Using the recursive relationship of eqn.(4.10) and from eqns (4.8) and (4.9), evaluating eqn. (4.7) gives,

$$c_1 = a - \ell n(T/t_o)\ell n(R_a/R_c)/(T-t_o) - (1/(T-t_o)) \sum_{k=1}^{m} \sum_{\substack{\ell=1\\ n,\text{even}}}^{n} (-1/k)^{k+\ell}(K_b/K_a)^k$$

$$\times\, [(\ell-1)!/(k(b-a))^{\ell}]\{(1/T^{\ell})\exp(-k(b-a)T) - (1/t_o{}^{\ell})\exp(-k(b-a)t_o)\} + O(t) \quad (4.11)$$

where the error term,

$$O(t) = (-n!/(b-a)^n) \sum_{k=1}^{m} (R_b/R_a)^k((-1)^{k+1}/k^{n+1}) \int_{t_o}^{T} (1/t^{n+1})\exp(-k(b-a)t)\,dt \quad (4.12)$$

For $n \to \infty$ i.e. more terms are taken, the integral in eqn. (4.12) $\to$ zero, hence it is possible for $O(t) \to 0$ as $n \to \infty$.

Suppose, now there are p exponential modes to be replaced by a single mode. The equivalent form of eqn. (4.3) is

$$R_c(t)\exp(-c(t)) = \sum_{i=1}^{p} R_i \exp(-a_i t) \tag{4.13}$$

where

$$R_c(t) = R_c \triangleq \sum_{i=1}^{p} R_i$$

$$\text{or} \quad R_c \triangleq \sum_{i=1}^{p} R_i/a_i \tag{4.14}$$

from the initial and final value theorems. Thus from eqn. (4.13)

$$c(t) = (-1/t)\,\ln\{(1/R_c)\sum_{i=1}^{p} R_i \exp(-a_i t)\} = a_j - (1/t)\,\ln(R_j/R_c)$$

$$-\,(1/t)\,\ln\{1+(1/R_j)\sum_{\substack{i=1\\ i\neq j}}^{p} R_i \exp(-(a_i-a_j)t)\} \tag{4.15}$$

where

$$\ln\{1+(1/R_j)\sum_{\substack{i=1\\ i\neq j}}^{p} R_i\exp(-(a_i-a_j t)\} = \sum_{k=1}^{\infty}\sum_{\substack{i=1\\ i\neq j}}^{p} ((1-1)^{k+1}R_i/kR_j^{\,k})\exp(-k(a_i-a_j)t) \tag{4.16}$$

assuming $a_i > a_j$ and $R_i > R_j$. Hence, as before, substituting eqns.(4.15) and (4.16) into eqn.(4.7) and using the recursive integral relation similar to eqn.(4.10), eqn.(4.7) is evaluated as

$$c_1 = a_j - (1/(T-t_o))\ln(T/t_o)\ln(R_j/R_c) - (1/(T-t_o))\sum_{k=1}^{m}\sum_{\substack{\ell=1\\ n,\text{even}}}^{n}\sum_{i=1}^{p}((-1)^{k+1}/k^{\ell+1})$$

$$((\ell-1)!/(a_i-a_j)^{\ell})\{(1/T^{\ell})\exp(-k(a_i-a_j)T) - (1/t_o^{\,\ell})\exp(-k(a_i-a_j)t_o)\} + O(t) \tag{4.17}$$

where

$$O(t) = -n!\sum_{k=1}^{m}\sum_{i\neq j}^{p}((-1)^{k+1}R_i/k^{n+1}(a_i-a_j)^n R_j^{\,k})\int_{t_o}^{T}(1/t^{n+1})\exp(-k(a_i-a_j)t)\,dt$$

In eqn.(4.17), by taking suitable number of terms O(t) can be made to converge. It is also noted that c_1 is the mean value or the first moment of c(t). Subsequent approximations to c(t) can be made by considering higher or lower order moments, m_k, (k>1 or k<1).

$$m_k = E[c^k(t)] = (1/(T-t_o)) \int_{t_o}^{T} |c^k(t)| \, dt$$

and c_k can be taken as (4.18)

$$c_k = (m_k)^{1/k} \quad , \text{ k an integer}$$

By taking different values of m_k, the single exponential curve of the lower order model can be made to fit as close as possible, to the curves of the exponentials of the higher order model; see fig.4.2. This freedom also allows the designer to obtain a biased reduced model, i.e., one giving a better dynamic response in any stage of the transient, or giving a better steady state response. From eqns. (4.13) and (4.18),

$$m_k = (1/(T-t_o)) \int_{t_o}^{T} |(-1/t)^k \ell n^k \{(1/R_c) \sum_{i=1}^{p} R_{ai} \exp(-a_i t)\}| \, dt \tag{4.19}$$

Unlike the case, k=1, eqn.(4.19) is best evaluated by numerical quadrature, example, Simpson's rule. If $t_o = 0$, then the integral, c(t), is singular at that point if $\Sigma R_{ai} = R_c$, i.e. matching initial values of impulse response. However, from L'Hopital's rule,

$$\lim_{t \to 0} c(t) = (\sum_{i=1}^{p} a_i R_{ai}/R_c)^k \tag{4.20}$$

T, in this case is chosen as $T \gg (1/a_i)$, where a_i is the smallest exponential mode. Regarding eqn.(4.19), it is necessary to introduce the constraint

$$\Sigma R_{ai}(\exp(-a_i t))/R_c > 0 \tag{4.21}$$

so that the logarithm of the r.h.s. of exp.(4.21) is defined only for positive values. Also the modulus of c(t) in eqn.(4.19) is taken so as to ensure that $c_k > 0$, i.e. $-c_k$ lies in the l.h.s. s-plane thus producing a stable reduced model. The inequality of exp.(4.21) is automatically satisfied for all $R_{ai} > 0$ or all $R_{ai} < 0$, given all $a_i > 0$. The constraint can be violated if the R_{ai} are of mixed signs, thus in grouping modes with positive and negative residues, care must be taken to ensure that eqn. (4.21) is satisfied.

(b) Reduction of complex modes to real modes.

Complex modes can occur in either of two forms

(i) $$R_a(s+\alpha)/\{(s+a)^2+\beta^2\} \leftrightarrow (R_a u/\beta)\exp(-at)\mathrm{Sin}(\beta t+\phi) \tag{4.22}$$

where $\tan(\phi) = \beta/(\alpha - a)$, $u^2 = (\alpha-a)^2+\beta^2$

(ii) $$R_a\beta^2/(s^2+2\varepsilon\beta+\beta^2) \leftrightarrow (R_a\beta/\eta)\exp(-\varepsilon\beta t)\mathrm{Sin}(\beta\eta t) \tag{4.23}$$

where $\eta^2 = (1-\varepsilon^2)$

If eqn.(4.22) were replaced by a single time constant,

$$R_c(t)\exp(-c(t)t) = (R_a u/\beta)\exp(-at)\mathrm{Sin}(\beta t+\phi)$$

from which

$$c(t) = (-1/t)\ln\{(R_a u/\beta)\exp(-at)\mathrm{Sin}(\beta t+\phi)/R_c\} \tag{4.24}$$

Similarly, for eqn.(4.23),

$$c(t) = (-1/t)\ln\{(R_a\beta/R_c\eta)\exp(-\varepsilon\beta t)\mathrm{Sin}(\beta\eta t)\} \tag{4.25}$$

The constant value c_k of $c(t)$ can be evaluated by considering appropriate moments, m_k. The constant residue R_c can be fixed as before, from the initial or final value theorem of the impulse or step response, respectively. For the case where there are more than one pair of complex modes, eqns. (4.24) and (4.25) generalize to,

$$c(t) = (-1/t)\ell n \{ \sum_{i=1}^{n} (R_{ai}u_i/R_c\beta_i)\exp(-a_it)\mathrm{Sin}(\beta_it+\phi_i) \qquad (4.26)$$

$$c(t) = (-1/t)\ell n \{ \sum_{i}(R_{ai}\beta_i/R_c\eta_i)\exp(-\varepsilon_i\beta_it)\mathrm{Sin}(\beta_i\eta_it)\} \qquad (4.27)$$

(c) Reduction of complex mode to complex mode.

For the case of complex modes where the single time constant is also required to be complex, the real and imaginary parts can be equated separately. Specifically, if

$$y(s) = \sum_{i=1}^{n} (R_{ai}+jb_i)/(s+a_i+j\beta_i) + \sum_{i=1}^{n} (R_{ai}-jb_i)/(s+a_i-j\beta_i) \qquad (4.28)$$

where the r.h.s. of eqn.(4.28) represents the response in conjugate pairs, and it is desirable to approximate y(s) by a conjugate pair of complex modes,

$$y(t) = (R_c(t)+jb_c(t))\exp\{-(c(t)+j\beta_c(t))t\} + (R_c(t)-jb_c(t))\exp\{-(c(t) - j\beta_c(t))t\} \qquad (4.29)$$

From eqns. (4.28) and (4.29), by symmetry, equating only the modes with positive imaginary parts gives,

$$(R_c(t)+jb_c(t))\exp\{-(c(t)+j\beta_c(t))t\} = \sum_{i=1}^{n} (R_{ai}+jb_i)\exp(-(a_i+j\beta_i)t) \qquad (4.30)$$

The constant value of residue R_c and jb_c can as before be determined from the initial or final value theorem. The initial value theorem gives

$$R_c + jb_c = \sum_{i=1}^{n} (R_{ai} + jb_i) \tag{4.31}$$

from which $\quad R_c = \sum_{i=1} R_{ai}, \; b_c = \sum_{i=1} b_i$

The final value theorem gives

$$R_c + jb_c = \sum_{i=1} (R_{ai}+jb_i)/(a_i+j\beta_i) \quad \text{from which}$$

$$R_c = \Sigma(R_{ai}a_i+b_i\beta_i)/(a_i^2+\beta_i^2), \quad b_c = \Sigma(b_ia_i-R_{ai}\beta_i)/(a_i^2+\beta_i^2)$$

To determine $\beta_c(t)$ and $c(t)$, equating real and imaginary parts of eqn.(4.30)

$$\exp(-c(t)t)\{R_c\mathrm{Cos}(\beta_c(t)t)+b_c\mathrm{Sin}(\beta_c(t)t\} = \sum_{i=1}^{n} \exp(-a_it)\{R_{ai}\mathrm{Cos}(\beta_ct)+b_i\mathrm{Sin}(\beta_it)\} \tag{4.32}$$

$$\exp(-c(t)t)\{b_c\mathrm{Cos}(\beta_c(t)t)-R_c\mathrm{Sin}(\beta_c(t)t)\} = \sum_{i=1}^{n} \exp(-a_it)\{b_i\mathrm{Cos}(\beta_it)-R_{ai}\mathrm{Sin}(\beta_it)\} \tag{4.33}$$

Eqn.(4.32) divided by eqn.(4.33), gives,

$$\beta_c(t) = (1/t) \arg\tan\{(N(t)b_c-D(t)R_c)/(N(t)R_c + D(t)b_c)\} \tag{4.34}$$

where N(t) and D(t) represent the r.h.s. of expressions of eqns. (4.32) and (4.33), respectively. Hence, from eqn.(4.32),

$$c(t) = (-1/t)\ell n\, \{N(t)/[R_c\mathrm{Cos}(\beta_c(t)t)+b_c\mathrm{Sin}(\beta_c(t))]\} \tag{4.35}$$

As before, the best approximation of $\beta_c(t)$ can be found by considering different moments of $\beta_c(t)$, similar to eqn.(4.18), i.e.

$$\beta_{ck} = \{(1/(T-t_o)) \int_{t_o}^{T} |\beta_c^{\,k}(t)| \, dt \}^{1/k} \tag{4.36}$$

(d) Reduction of real modes to complex modes.

Consider a group of real modes $\sum_{i=1}^{n} R_{ai}/(s+a_i)$ to be replaced by a complex pair. The impulse response matching yields,

$$\sum_{i=1}^{n} R_{ai} \exp(-a_i t) = (R_c(t) + jB_c(t))\exp\{-(c(t)-j\beta(t))t\}$$

$$+ (R_c(t)-jB_c(t))\exp\{-(c(t)+j\beta(t))\} \tag{4.37}$$

A constant value R_c can be assigned to $R_c(t)$ by levelling the initial value of the impulse responses, i.e.

$$R_c = \frac{1}{2} \sum_{i=1}^{n} R_{ai} \tag{4.38}$$

or choose $\qquad R_c = \frac{1}{2} \Sigma R_{ai}/a_i$

from the final value theorem of a step input. It is also desirable to choose constant values for $B_c(t)$ and $\beta(t)$ in eqns. (4.37). B_c can be chosen around the same magnitude as R_c, and, β should be chosen according to the severity of oscillations present in the original response. Tentative values of B_c and β are best made by trial and error.

From, eqn.(4.37),

$$c(t) = (-1/t)\ell n\{\Sigma R_{ai} \exp(-a_i t)/2(R_c \mathrm{Cos}(\beta t) - B_c \mathrm{Sin}(\beta t))\} \tag{4.39}$$

and the approximations c_k can be found from eqn.(4.18), taking into account, the constraint

$$\Sigma R_{ai} \exp(-a_i t)/(R_c \mathrm{Cos}(\beta t) - B_c \mathrm{Sin}(\beta t)) > 0 \tag{4.40}$$

be satisfied.

(e) Existence of repeated modes.

For a mode a_i to repeat, say, m times, its partial fraction transfer function can be represented as

$$\sum_{i=1}^{m} R_{aj}/(s+a_i)^j \tag{4.41}$$

where R_{aj} and a_i are real, or, if complex, have their conjugate counterparts similarly written. The time domain representation of the impulse response due to a single mode is

$$R_c(t) \exp(-c(t)t) - \sum_{j=1}^{m} R_{aj}(t^{j-1}/(j-1)!)\exp(-a_i t) \tag{4.42}$$

The initial value theorem (i.v.t.) yields $R_c = R_{a1}$ and the final value theorem (f.v.t.) gives $R_c = \sum_{j=1}^{m} R_{aj}/a_i^j$. Further, from eqn. (4.42)

$$c(t) = (-1/t)\ell n \{ \sum_{j=1}^{m} (R_{aj}/R_c)(t^{j-1}/(j-1)!)\exp(-a_i t)\} \tag{4.43}$$

In general, if there are n modes, each repeated p_j times, existing with other unrepeated modes, and, if all the n modes were to be replaced by a single mode, the general form of eqn.(4.43) is

$$c(t) = (-1/t)\ell n \{ \sum_{i=1}^{n} \sum_{j=1}^{p_j} (R_{aji}/R_c)(t^{j-1}/(j-1)!)\exp(-a_i t)\} \tag{4.44}$$

with

$$\left.\begin{aligned} R_c &= \sum_{j=1}^{n} R_{aj1} && \text{(i.v.t.)} \\ \text{or} \quad R_c &= \sum_{i=1}^{n} \sum_{j=1}^{p_j} (R_{aji}/a_i^j) && \text{(f.v.t.)} \end{aligned}\right. \tag{4.45}$$

As before, the constraints $\ell n \{ \} > 0$ must be imposed on eqns. (4.43) and (4.44). If the repeated modes a_i and the residue R_{aji} were complex, then the synthesis must be done along the similar lines given in sections (b) and (c) above. A pair of complex modes can also be used to approximate repeated real or complex modes by following the synthesis of section (d).

The value of R_c determined from the initial value theorem always gives zero initial error, after reduction, of an impulse response. However, by applying the final value theorem of the step response to both reduced and original models, and equating the two, we obtain

$$R_c'/c_k = \sum_{i=1}^{n} R_i/a_i = R_c \tag{4.46}$$

where R_c' is the residue that will give zero steady state error, and, R_c is the exaggerated residue, that gives small s.s. errors, determined by applying the final value theorem to the original model. (see eqn.(4.14)). Thus, after reduction,

$$R_c' = c_k R_c \tag{4.47}$$

4.2 Extension to multivariable systems.

The synthesis given above can be extended to multivariable transfer functions in two ways. The procedure can be applied to every individual element of the matrix as if it is a single input-single output system. However, after reduction, it is not guaranteed that the order of the transfer matrix is the same as that of the required model; as the individual elements may not have a common denominator. The order of the model must be determined from the degree of the characteristic polynomial of the transfer matrix. (assumed proper).

An alternative method is to represent the transfer matrix in partial fraction form with matrix residues, i.e.

$$G(s) = \sum_{i=1}^{n} \sum_{j=1}^{p_j} R_{aji}/(s+a_i)^j \tag{4.48}$$

where R_{aji} is a matrix residue (in general, can be complex). The same synthesis given in section 4.1 applies to eqn.(4.48), except that R_{aji} is treated as a matrix. The matrix equivalent form of say eqn.(4.15) becomes

$$C(t)I = (-1/t)\ \ell n\ \{ R_c^{-1} \sum_{i=1}^{p} R_i \exp(-a_i t) \} \tag{4.49}$$

assuming R_c is non-singular, and, is similarly computed from the form of eqn.(4.14) and if necessary corrected by the form of eqn.(4.47). The $\ell n\ \{\ \}$ expression in eqn.(4.49) can be evaluated as $\ell n\ \{\ \} = \ell n(Y_1 Y_2 \ldots Y_k)$ $= \ell n\ (I+X_1) + \ell n(I+X_2) + \ldots\ \ell n(I+X_k)$ such that each matrix X_j $j=1, \ldots k$ has eigenvalues with modulus less than unity. This assures convergence of the expression of $\ell n(I+X_j)$ by power series.

The matrix equivalent equations of eqn.(4.32), (4.33) and (4.37) etc. can similarly be written, care being taken to replace the reciprocal of a scalar quantity by the matrix inverse of the same quantity, and, multiply a scalar quantity by the unit matrix, I. When the unknown variable is a complex mode, the oscillatory part, $\beta_c(t)$, must be assigned a tentative value to simplify the solutions of the matrix equations. (Example, the matrix equivalent form of eqn.(4.32)). The real part of the mode c(t) need only be found, and can be represented in the form of eqn.(4.49) and evaluated easily, using eqn.(4.18). (The matrix equivalent form of eqn.(4.34) does not exist, hence it is avoided.)

Using the above method will require some slight modification in the approximation synthesis for the single input/single output case. This is to simplify matters for the solution of the matrix equations. Although the method will yield the required reduced model, the final interaction structure of the transfer function may not be preserved. Also, the overall computational demands are high, compared to the single input/single output case.

4.3 Reduction by Sequential Approximations.

The procedures outlined above can be applied to the reduction of a high order single input/single output system in a systematic manner. Since the choice of parameters is opened to the designer's engineering judgement, the reduction is best done with interactive graphics where root locus transient response and other performance specifications can be computed and displayed. Suppose, r, is the reduced model order. A systematic procedure of reduction is suggested below.

Algorithm:

1st reduction:

1. Separate out repeating and non-repeating modes and treat them separately. Divide modes into dominant, moderately dominant and non-dominant groups.

2. From the dominant tuple $\{R_{11}\ \exp(-a_1 t), \ldots\ R_{i1}\ \exp(-a_i t)\}$, replace it by a single mode $R_c\ \exp(-ct)$ such that the transient response and s.s. error are satisfactory. Repeated modes are replaced by a single time constant as far as possible. Preferably, group modes together whose residues have the same sign. The dominant tuple can thus be divided into sub-tuples. Check that inequality and numerical constraints are not violated.

3. Repeat procedure for the moderately dominant and non-dominant tuples.

4. Check transient response and root locus plots etc. If unsatisfactory, GO TO 3. Otherwise, proceed as below.

2nd reduction:

Suppose ℓ modes are chosen from the 1st stage of reduction ($r<\ell$). Group the ℓ modes into sub-tuples and proceed synthesis as above.

3rd and subsequent reduction:

If necessary, continue reduction stage till r modes are left in the final tuple, giving a model whose dynamics are satisfactory to the designer's requirements.

Iteratively, from the original model,

$$I_1 \triangleq \{R_{a1}\exp(-a_1t)\ R_{a2}\exp(-a_2t),\ \ldots\ R_{an}\exp(-a_nt)\}$$

select p modes, say $p = \ell$ from above grouping

$$I_2 \triangleq \{R_{b1}\exp(-b_1t),\ R_{b2}\exp(-b_2t),\ldots R_{bl}\exp(-b\ t)\}$$

divide into sub-tuples again and take p modes or less.

$$\vdots$$

$$I_j \triangleq \{R_{c1}\exp(-c_1t),\ \ldots\ R_{cr}\exp(-c_rt)\}$$

(final tuple)

The squared error between original and reduced model due to a step response, assuming zero s.s. error is,

$$e^2(t) = \int_0^\infty \{\sum_{i=1}^{n}(R_{ai}/a_i)(1-\exp(-a_it)) - \sum_{i=1}^{r}(R_{ci}/c_i)(1-\exp(-c_it))\}^2\,dt$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}(R_{ai}R_{aj}/a_ia_j)\{(1/(a_i+a_j))-(1/a_i)-1/a_j\} + \sum_{i=1}^{r}\sum_{j=1}^{r}(R_{ci}R_{cj}/c_ic_j)\{$$

$$(1/(c_i+c_j))-(1/c_i)-1/c_j\} - 2\sum_{i=1}^{n}\sum_{j=1}^{r}(R_{ai}R_{cj}/a_ic_j)\{(1/(a_i+c_j))-(1/a_i)-1/c_j\} \tag{4.50}$$

and that for an impulse response is,

$$e^2(t) = \int_0^\infty \{\sum_{i=1}^{n} R_{ai}\exp(-a_it) - \sum_{i=1}^{r} R_{ci}\exp(-c_it)\}^2\,dt \tag{4.51}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} R_{ai}R_{aj}/(a_i+a_j) + \sum_{i=1}^{r} \sum_{j=1}^{r} R_{ci}R_{cj}/(c_i+c_j) - 2 \sum_{i=1}^{n} \sum_{j=1}^{r} R_{ai}R_{cj}/(a_i+c_j)$$

Figures 4.3 to 4.5 illustrate the proposed computational algorithm in flowchart form. The supervisor programme has three modules; Selection and Bank, Reduction, and, Store and Simulation. All three modules interact freely with each other and the supervisor programme. The Selection and Bank module accumulates all modes that are constructed and found to be good. Further reduction by selecting tuples from the bank is possible. The Reduction module reduces the modes by computing effective time constants and residues. The Store and Simulation module regroups and temporarily stores all modes after a subsequent iteration and simulates the transient response.

For multivariable systems[6], if the above procedure is to be applied to every element of G(s) it is desirable to have the interaction structure of G(s) preserved as much as possible in $G_r(s)$, i.e. $f_{rij}(\omega) = f_{oij}(\omega)$ and $\eta_{rij}(\omega) = \eta_{oij}(\omega)$, where $f_{rij}(\omega) = |g_{rij}(s)|_{i\neq j}/|g_{rii}(s)|$ and $\eta_{rij}(s) = |g_{rij}(s)|_{i\neq j}/|g_{rij}(s)|$, and similarly for f_{oij} and η_{oij}. It will be proved in Chapter VI that this will ensure that S_o will be stable within the same range of gain, as S_r in the presence of the controller, K_r.

Computational algorithm:

(i) Obtain $g_{rii}(s)$, $\forall i$, by procedures suggested earlier.

(2) Obtain $g_{rij}(s)$, $i\neq j$, $\forall i,j$, as in (1)

(3) Compute $M(\omega) = \int_\Omega (f_{oij}(\omega)-f_{rij}(\omega))^2\, d\omega$ and $N(\omega) = \int_e (\eta_{oij}(\omega)-\eta_{rij}(\omega))^2\, d\omega$

(4) If $M(\omega)$ and $N(\omega)$ are small, EXIT, otherwise GO TO step 2 and correct any large deviations between $g_{rij}(s)$ and $g_{oij}(s)$.

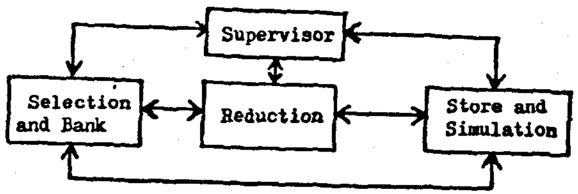

Fig. 4.3 Basic structure of the Algorithm

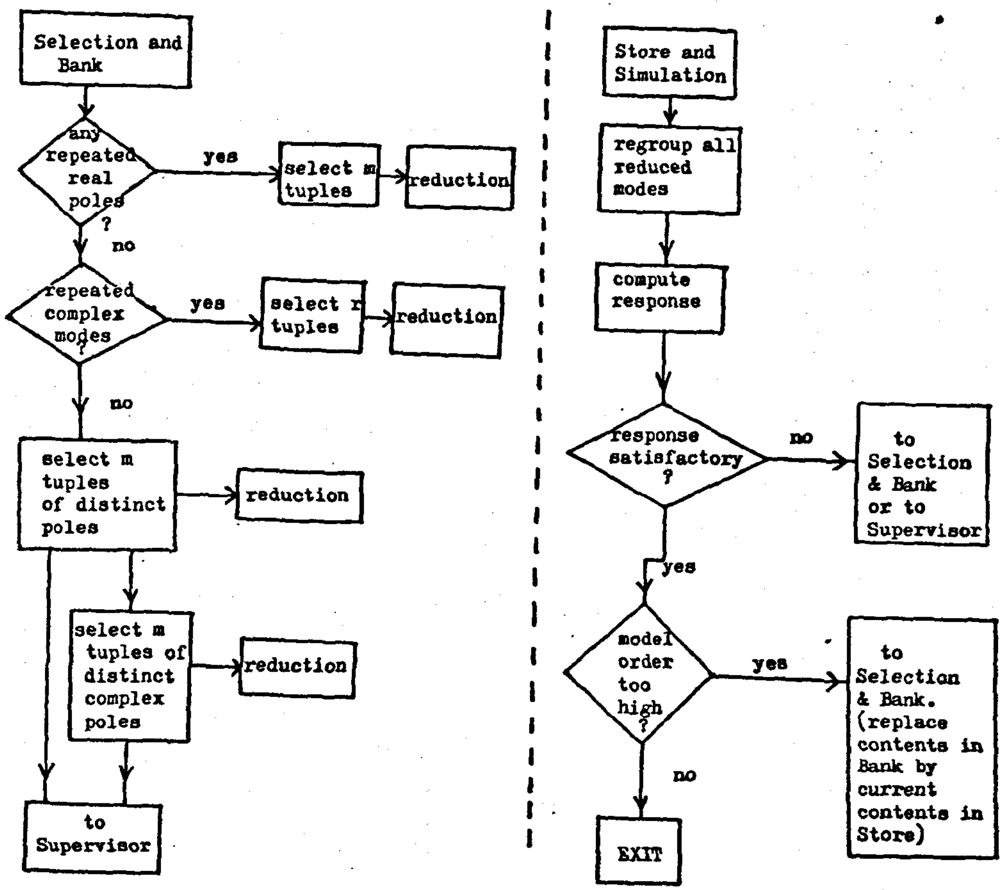

Fig. 4.4 Flow chart of Selection & Bank and Store & Simulation modules

The Reduction program consists of 4 sub-programs:

1) from real mode to real mode reduction
2) from complex mode to complex mode reduction
3) from complex mode to real mode reduction
4) from real mode to complex mode reduction

Different routines are written for the four depending on the formula involved, only the first will be illustrated.

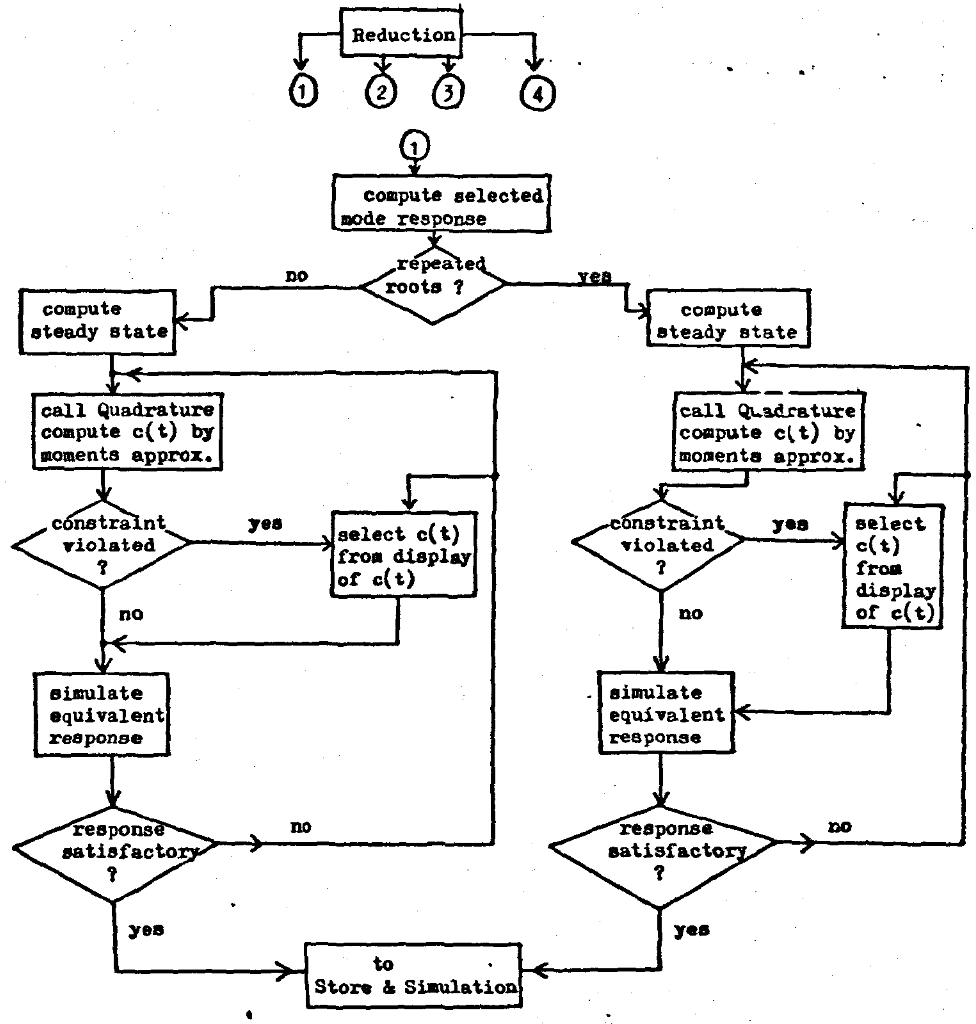

Fig.4.5 Flow chart of a Reduction sub-program

4.4 Reduction by state space methods.

Consider an n^{th} order single input-single output system $S(A,b,c)$, with $x(0) = 0$, represented by

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= c^t x \end{aligned} \qquad (4.52)$$

i.e.

$$y\{(k+1)T\} = c^t\ \nabla\{(k+1)T\}\ bu\ \{(k+1)T\} \qquad (4.53)$$

where

$$\nabla\{(k+1)T\} = \exp(AT)x(0) + \Delta\{(k+1)T\}$$

$$\Delta(T) = \int_o^T \exp(A(t-\lambda))\ u(\lambda)\ d\lambda$$

Thus for an impulse input, $\Delta(T) = \exp(AT)$, and, for a step input, $\Delta(T) = A^{-1}(\exp(AT)-I)$. To obtain a reduced model, $S_r(A_r,b_r,c_r)$, from eqn. (4.52), it is desirable to equate $y_r(t) = y(t)$, and from there construct S_r. Hence from eqn. (4.53), for k = 0, 1, 2, ...

$$\begin{aligned} &< c_r, \nabla_r(T)\ b_r > = y(T) \\ &< c_r, \nabla_r(2T) b_r > = y(2T) \\ &\vdots \\ &< c_r, \nabla_r(kT)\ b_r > = y(kT) \qquad k >> r \end{aligned} \qquad (4.54)$$

where $y\{(k+1)T\}$ is given by eqn. (4.53). In matrix form eqn. (4.54) is

$$M\ b_r = q \qquad (4.55)$$

where M is a (kxr) matrix whose i^{th} row is $c_t^{\ t}\nabla_r(iT)$, and, $q^t = \{y(T),\ldots y(kT)\}$. In general eqn. (4.55) is incompatible, and a b_r cannot be found which satisfies q. An approximate solution is given by

$$b_r = M^+ q \qquad (4.56)$$

where M^+ is the pseudo-inverse of M. In general, $M^+ = R(R^HR)^{-1}(F^HF)^{-1}F^H$ where M = FR. F and R are $(k \times \ell)$ and $(\ell \times r)$ matrices, and, the columns of F may be any ℓ linearly independent columns of M, where rank (M) = ℓ. Thus, when ℓ=r,(the order of the reduced model), $M^+ = (M^tM)^{-1}M^t$. The solution given in eqn.(4.56) is such that $< (Mb_r-q),(Mb_r-q) >$ is a minimum, i.e. $< (q_r-q),(q_r-q) >$ is a minimum.

An alternative convenient form can be written for eqn. (4.55). Transposing eqn.(4.54)

$$\begin{aligned} &< V_r(T)b_r, c_r > &= y(T) \\ &< V_r(2T)b_r, c_r > &= y(2T) \\ &\vdots \\ &< V_r(kT)b_r, c_r > &= y(kT) \end{aligned} \tag{4.57}$$

i.e. $$Nc_r = q \tag{4.58}$$

where N is a (kxr) matrix whose i^{th} row is given by $b_r^t V_r^t(iT)$. Thus from eqn.(4.58),

$$c_r = N^+q \tag{4.59}$$

Eqns. (4.56) and (4.59) can be used to find b_r and c_r respectively. Both equations demand that the pair(A_r,c_r) or (A_r,b_r) need be known for the remaining vector b_r or c_r to be found. In any case, a canonical structure for A_r must be fixed for either eqns. (4.56) or (4.59). Thus the time constants of the modes are specified. In general, choosing a c_r in eqn. (4.56) and solving for b_r, does not guarantee the same c_r when b_r, from eqn.(4.56), is substituted into eqn.(4.59). Similarly choosing a b_r in eqn.(4.59) may not yield the same b_r in the l.h.s. of eqn.(4.56). To find the 'best' choice of the pair (b_r,c_r) for a given A_r, the following scheme is suggested.

Initially, choose a c_r, called $c_r^{(1)}$ and find b_r from eqn.(4.56). Call this value $b_r^{(1)}$. Substitute $b_r^{(1)}$ in eqn. (4.59) and calculate c_r. Call this value $c_r^{(2)}$. Use $c_r^{(2)}$ in eqn. (4.56) to find $b_r^{(2)}$ and then use $b_r^{(2)}$ to find $c_r^{(3)}$ in eqn. (4.59). Repeat the process till $b_r^{(i)}$ is very close to $b_r^{(i+1)}$ and $c_r^{(i)}$ is very close to $c_r^{(i+1)}$, i.e. $\| b_r^{(i)} - b_r^{(i+1)} \| \approx 0$ and $\| c_r^{(i)} - c_r^{(i+1)} \| \approx 0$. The iterative scheme will converge if

$$\lim_{i\to\infty} \| b_r^{(i+1)} - c_r^{(i+1)} \| < \| M_i^+ - N_i^+ \| \qquad (4.60)$$

If exp.(4.60) does not exist, then $S_r(A_r,b_r,c_r)$ can only be found for a given pair (A_r,b_r) or (A_r,c_r).

The reduction can be done sequentially by the following computational algorithm:

(1) Choose a set of dominant eigenvalues $(\lambda_{r1},..\lambda_{rr})$ for A_r from the set $(\lambda_1,...\lambda_n)$ of A. Find a canonical structure for A_r.

(2) Choose an initial value of c_r or b_r and find the final pair (b_r,c_r) using the iterative scheme of eqns. (4.56) and (4.59).

(3) Compute and display the time response $y_r(t)$ of $S_r(A_r,b_r,c_r)$.

(4) If satisfactory, exit. Otherwise modify $(\lambda_{r1},...\lambda_{rr})$ or modify the canonical structure of A_r or choose a different starting point for (b_r,c_r) and GOTO (2).

The canonical structure of A_r can be formulated in many ways, example the convenient companion form, after choosing a set of eigenvalues; or choosing also a set of basic eigenvectors $V_r=(v_1,v_2,...v_r)$, A_r can be constructed as $A_r = V_r \operatorname{diag}(\lambda_1,...\lambda_r) V_r^{-1}$. V_r can be constructed from the original eigenvectors, V, of A, corresponding to $\lambda_1,...\lambda_r$ by suitable partitioning such that V_r is of full rank. Alternatively, by employing

Gershgorin's theorem, A_r can be formed without actually knowing in detail the eigenvalues of A. Row and column Gershgorin circles for A can be drawn, and, by inspection of their intersection, suitable new circles of different radii pertinent to A_r can be constructed such that they contain the dominant eigenvalues of A. To guarantee stability, the final intersection of circles must lie in the l.h.s. plane.

From the 2r circles, centre $a_{11}, a_{22}, \ldots a_{rr}$ and respective row and column radii $R_{11}, R_{22}, \ldots R_{rr}, R_{11}', \ldots R_{rr}'$, the set of 2r equations

$$\begin{aligned} \sum_{i \neq j} |a_{ij}| &= R_{ij} \qquad \forall j \\ \sum_{j \neq i} |a_{ji}| &= R_{ji}' \qquad \forall i \end{aligned} \tag{4.61}$$

gives r(r-1) unknowns. Hence r(r-3) of them can be assigned arbitrary values and the rest found accordingly.

If it is desirable to have the state vector, x_r, similar to the first r elements of the original state vector, x, then c_r can be chosen as $c_r = c^{(r)}$ where $c^{(r)}$ represents the first r elements of c. Hence b_r can be computed from the pair (A_r, c_r). In general if $S_r(A_r, b_r, c_r)$ is controllable and observable, $S_r(F_r, g_r, h_r)$ can be constructed from a similarity transformation such that $h_r = c^{(r)}$.

4.5 Extension to multivariable systems.

For multivariable systems, the analogous form of eqn.(4.53) is

$$y(kT) = C\nabla(kT)Bu(kT) \quad k=1,2,\ldots \tag{4.62}$$

and in component form can be written as

$$y_j(kT) = \sum_{i=1}^{\ell} < c_j, \nabla(kT)b_i > u_i(kT) \qquad \forall j \tag{4.63}$$

where $y = (y_1, y_2, \ldots y_r)$, $u = (u_1, u_2, \ldots u_r)$, c_j^t and b_i are the row and column vectors of matrices B and C. By superposition, if all inputs are 'shut down' except the j^{th} input, eqn. (4.63) gives $y_j(kT) = \langle c_j, \nabla(kT) b_j \rangle u_j(kT)$ which is identical to eqn. (4.53). Thus the same reduction technique can be applied to eqn. (4.63) by considering one input and one output at a time, i.e. for $j = 1, 2, \ldots r$, $u_i(kT) = 0$, $(i \neq j)$. In this way, matrices B_r and C_r are generated column wise and row wise, sequentially $B_r = (b_{r1}, \cdot b_{rr})$, $C_r^t = (c_{r1}, c_{r2}, \ldots c_{rr})$. The degree of interaction in $S_r(A_r, B_r, C_r)$ can be defined from the time domain matrix

$$\begin{bmatrix} y_{r1} \\ \cdot \\ \cdot \\ \cdot \\ y_{rr} \end{bmatrix} = \begin{bmatrix} \langle c_{r1}, \nabla_r b_{r1} \rangle & \cdots & \langle c_{r1}, \nabla_r b_r \rangle \\ \cdot & & \\ \cdot & & \\ , & & \\ \langle c_{rr}, \nabla_r b_1 \rangle & \cdots & \langle c_{ri}, \nabla_r b_{ri} \rangle \end{bmatrix} \begin{bmatrix} u_{r1} \\ \cdot \\ \cdot \\ \cdot \\ u_{rr} \end{bmatrix} \qquad (4.64)$$

$$\begin{aligned} &\text{as rowwise} && f_{rij}(t) = \langle c_{ri}, \nabla_r b_{rj} \rangle_{i \neq j} / \langle c_{rj}, \nabla_r b_{rj} \rangle \\ &\text{columnwise} && m_{rij}(t) = \langle c_{ri}, \nabla_r b_{rj} \rangle_{i \neq j} / \langle c_{rj}, \nabla_r b_{rj} \rangle \end{aligned} \qquad (4.65)$$

and $m_{ij}(t)$, $f_{ij}(t)$ for the original model is similarly defined. Whenever necessary, it is desirable to preserve the interaction structure in S(A,B,C) and $S_r(A_r, B_r, C_r)$ by checking that the squared error

$$\int_o^t \{(m_{ij}(t) - m_{rij}(t))^2 + (f_{ij}(t) - f_{rij}(t))^2\} \, dt \qquad (4.66)$$

is small.

An alternative approach to multivariable reduction is to consider eqn.(4.62) as a single input-multiple output or its dual, multi input-single output, problem. From eqn. (4.62), consider observation being made on the output vector, $y^{(j)}(kT)$, when all inputs are set to zero, except the j^{th} input;

i.e., $u_i(kT) = 0$, $\forall i$, except $i = j$. For $j = 1, 2, ..$, eqn. (4.62) can sequentially be written as,

$$\begin{aligned} y^{(1)}(kT) &= C_r\nabla_r(kT)b_{r1} \\ y^{(2)}(kT) &= C_r\nabla_r(kT)b_{r2} \\ &\vdots \\ y^{(\ell)}(kT) &= C_r\nabla_r(kT)b_{r\ell} \end{aligned} \tag{4.67}$$

which can be arranged in matrix form as

$$Y(kT) = C_r\nabla_r(kT)B_r \tag{4.68}$$

where $Y(kT) = [y^{(1)}(kT)y^{(2)}(kT)...y^{(\ell)}(kT)]$ is an $(\ell x\ell)$ matrix, evaluated at the k^{th} sample, of $S(A,B,C)$. For $k = 1,2,...$ the sequence of matrices generated by eqn. (4.68) can be arranged as

$$(Y^t(T) \vdots Y^t(2T) \vdots .. Y^t(kT)) = B_r^t\ (\nabla_r^t(T)C_r^t \vdots \nabla_r^t(2T)C_r^t \vdots ... \nabla_r^t(kT)C_r^t) \tag{4.68}$$

i.e. $$Q^t = B_r^tK^t$$

giving

$$B_r = K^+Q = (K^tK)^{-1}K^tQ \tag{4.70}$$

where Q^t and K^t are $(\ell x\ell k)$ and $(rx\ell k)$ matrices as shown.

Analogous to eqn.(4.69), the sequence of matrices generated by eqn. (4.68) for k=1,2... can also be written as

$$(Y(T) \vdots Y(2T) \vdots ... Y(kT)) = C_r(\nabla_r(T)B_r \vdots \nabla_r(2T)B_r \vdots ... \nabla_r(kT)B_r)$$

i.e. $$P^t = C_rL^t \tag{4.71}$$

where P^t and L^t are $(\ell x\ell k)$ and $(rx\ell k)$ matrices as shown,

Eqn.(4.71) gives

$$C_r = L^+P = (L^tL)^{-1}L^tP \tag{4.72}$$

The solutions given by eqns.(4.70) and (4.72) are such that the matrices B_r and C_r^t have least norms which minimize $\|KB-Q\|_E^2$ and $\|LC^t-P\|_E^2$ respectively. Sufficient samples (kT) are taken such that $k >> r$ so as to ensure that K^tK and L^tL are non-singular. However, if they still become singular, then the generalized form of the pseudo-inverse must be used. Like the single input-single output case the pair (A_r,C_r) or (A_r,B_r) must be known before eqns.(4.70) or (4.72) gives the remaining triplet. The pair (B_r,C_r) can also be obtained iteratively, for a given A_r, from eqns. (4.70) and (4.72) if

$$\lim_{i\to\infty} \| B_r^{(i+1)} - C_r^{t(i+1)} \| < \| K_i^+ - L_i^+ \| \qquad (4.73)$$

Reduction by using the method of pseudo-inverse was also employed by Anderson[7] and Sinha and Pille[7]. Anderson chose the reduced state vector, $x_r(t)$, from the first r elements of the original vector x(t) and attempted a least square fit between $x_r(t)$ and x(t). The matrices A_r and B_r obtained in the end are expressed in the forms of state transition and driving matrices, $\Phi(\tau)$ and $\Delta(\tau)$, respectively; and reconstructing (A_r,B_r) sometimes can be difficult.

The pseudo-inverse method suggested in this chapter assumes an arbitrary state vector $x_r(t)$ and a chosen canonical form, A_r. The method is iterative and works with interactive graphics. The triple $S_r(A_r,B_r,C_r)$ can be updated as necessary, subjected to designer's requirements.

Sinha and Pille's method is very similar to Anderson's, except that they avoid the problem of matrix inversion in the pseudo-inverse by using an iterative algorithm. However, convergence properties and large amount of matrix computations do not justify the method proposed. Also, in the end like Anderson's, the reduced model is expressed in its discretized form and an inverse z-transform is necessary to convert it to continuous form. Thus extra time and effort are necessary in the conversion of high order transfer function matrices.

Unlike the method proposed in this chapter, the methods of Anderson, Sinha et al do not guarantee a stable reduced model. For multivariable systems, their methods do not provide a means for preserving the interaction structure of the model in the reduction.

4.6 Example.

(a) The sequential moments approximation reduction method is illustrated by the following 7th order single input-single output example,

$$H(s)=0.375\ E06(s+1)/(s^7+59.5s^6+1838s^5+35{,}120s^4 + 0.4485\ E06s^3+0.3772\ E07s^2+0.1963\ E08s+0.4876\ E08)$$

and in partial fraction form is

$$H(s) = \{1.234\ E{-}06/(s+8.5)\} + \{(-1.124E{-}05-j8.241E{-}05)/(s+8.5-j11.0)\}^* + \{(1.037E{-}04+j2.54E{-}05)/(s+9.0-j10.0)\}^* + \{(-9.303E{-}05+j5.543E{-}05)/(s+8.0-j10.0)\}^*$$

where the expression $\{\ \}^*$ denotes that the mode occurs in complex conjugate pairs. Inspecting the expression for H(s), it is found that pairs of complex modes dominate the response, hence it is useful to retain a pair of complex modes, giving a second order model. The 3 pairs of complex modes were reduced sequentially to a single pair. Finally, the real mode and the single pair of complex modes were replaced by an equivalent pair, giving the reduced model as

$$H_r(s) = \{-j3.252/(s+4.472-j0.41)\}^*$$
$$= 0.612/(s^2 + 3.42 + 5.2)$$

In the above reduction procedure, it was found that taking the first moment i.e. $m_k = m_1$ of the effective pole, c(t) was sufficient to produce a good response. The responses of H(s) and $H_r(s)$ are shown in fig.4.6, following a unit step input.

(b) Next, a fourth order boiler model was considered, whose state space matrices, S(A,B,C), are

$$A = 10^{-3} \times \begin{bmatrix} 2.47 & 8.16 & 5.42 & 3.36 \\ -3.37 & -12.6 & -6.77 & -3.11 \\ 2.34 & 6.65 & 2.53 & 2.61 \\ -4.83 & -7.00 & -5.77 & -7.07 \end{bmatrix}, B = 10^{-3} \times \begin{bmatrix} -0.605 & -0.00361 \\ 1.35 & 0.0309 \\ -2.54 & 0.0309 \\ 1.64 & -0.151 \end{bmatrix}$$

$$C = \begin{bmatrix} 3.0 & 3.0 & 5.0 & 4.0 \\ 4.0 & 5.0 & 3.0 & 2.0 \end{bmatrix}$$

It was desirable to obtain a second order model $S_r(A_r,B_r,C_r)$ using the method of least square minimization by matrix pseudo-inverse. The eigenvalues of A are

$$\lambda_1 = -0.213E-02$$
$$\lambda_2 = -0.414E-02$$
$$\lambda_3 = -0.783E-02$$
$$\lambda_4 = -0.625E-03$$

and the dominant modes λ_1 and λ_4 were retained in the reduced model. For convenience, C_r was set as $C_r = I$, the unit matrix. Choosing A_r = diag $\{\lambda_1, \lambda_4\}$ in its simplest form, the pair (A_r,C_r) yielded a B_r where

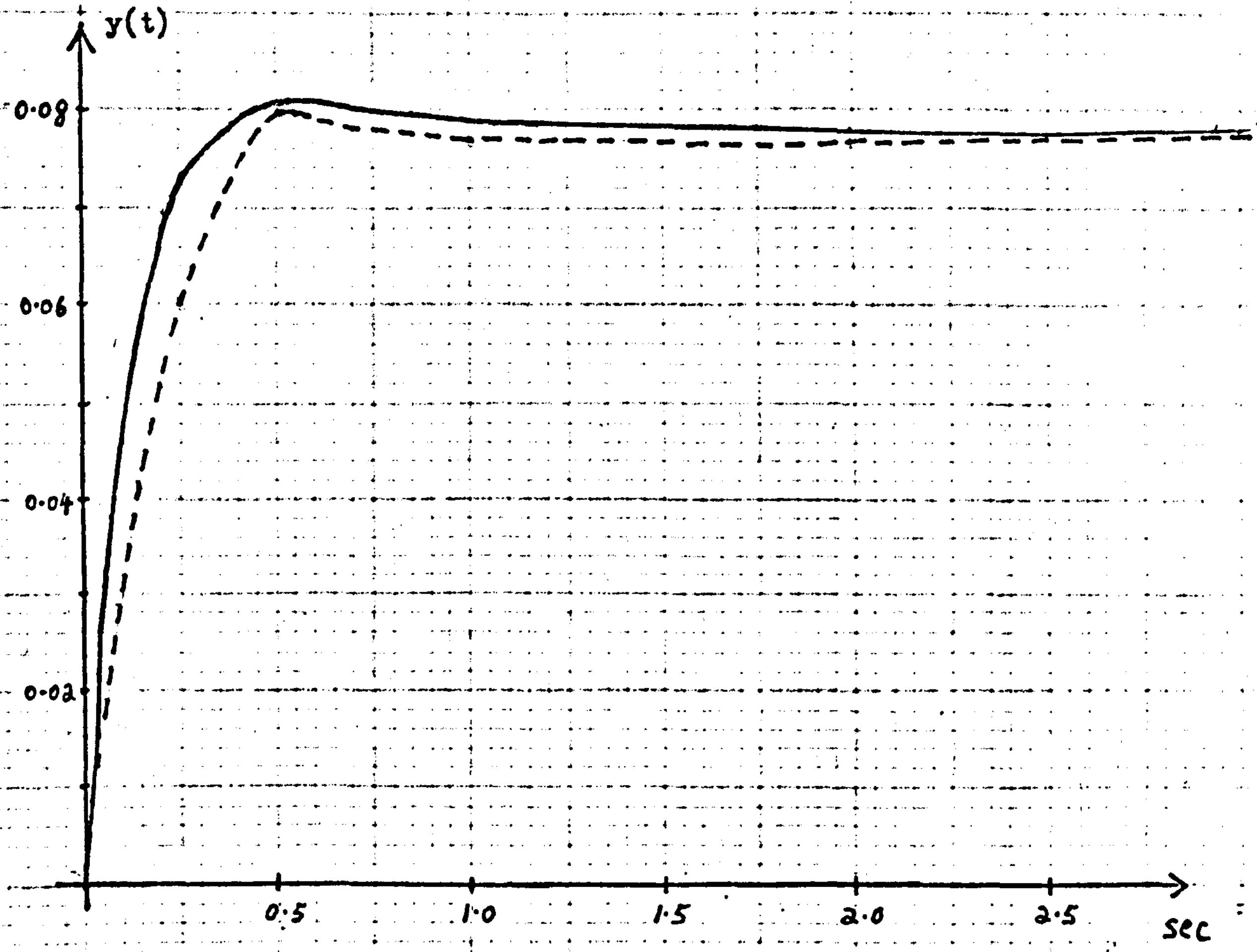

Fig. 4.6

——————— original model 7th order

- - - - - - - reduced model obtained by sequential mode aggregation method, 2nd order

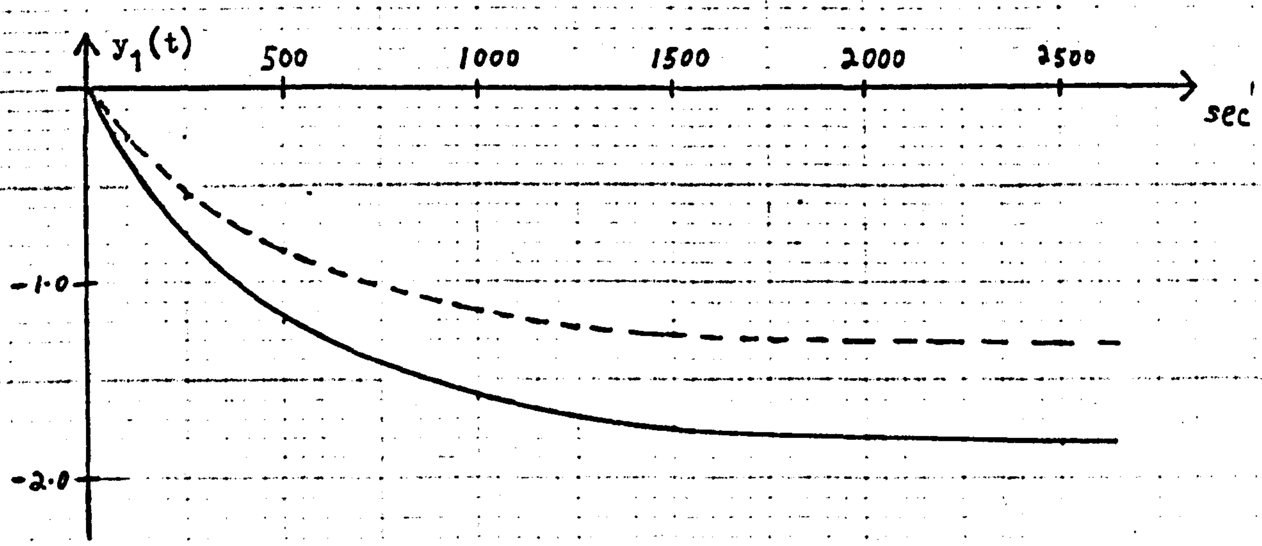

Fig. 4.7(a) Output from channel 1, due to step input to channel 1

——————— original model, 4th order

- - - - - - - reduced model obtained by sequential 'state space' approximation method, 2nd order

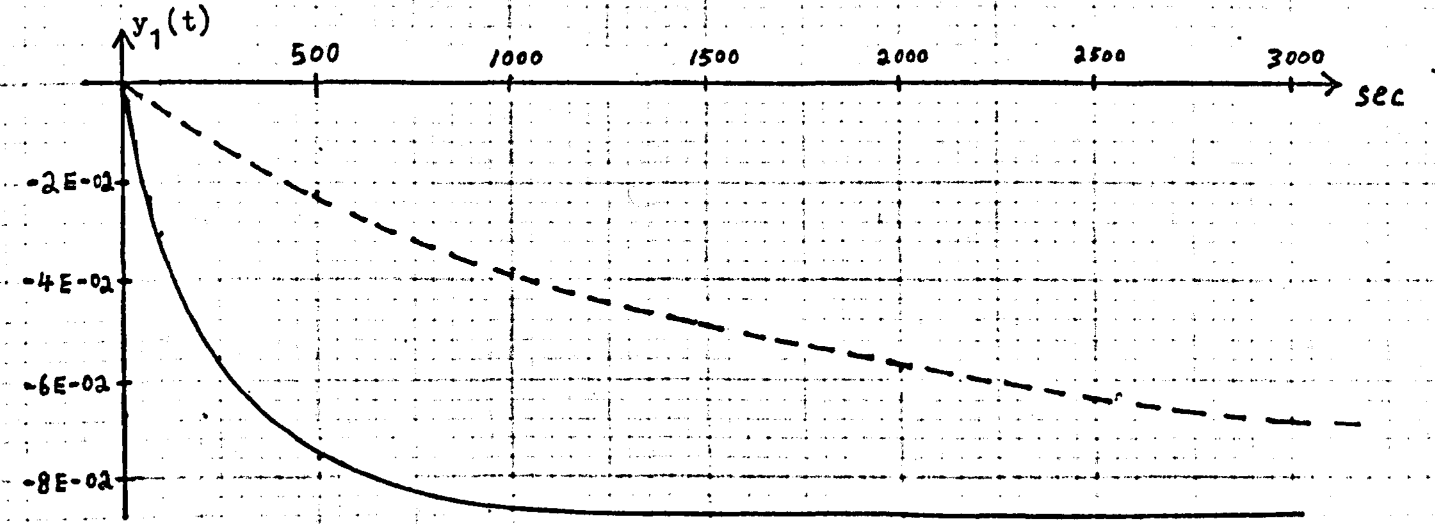

Fig. 4.7(b) Output from channel 1, due to step input to channel 2

original model 4th order

reduced model obtained by sequential 'state space' approximation method, 2nd order

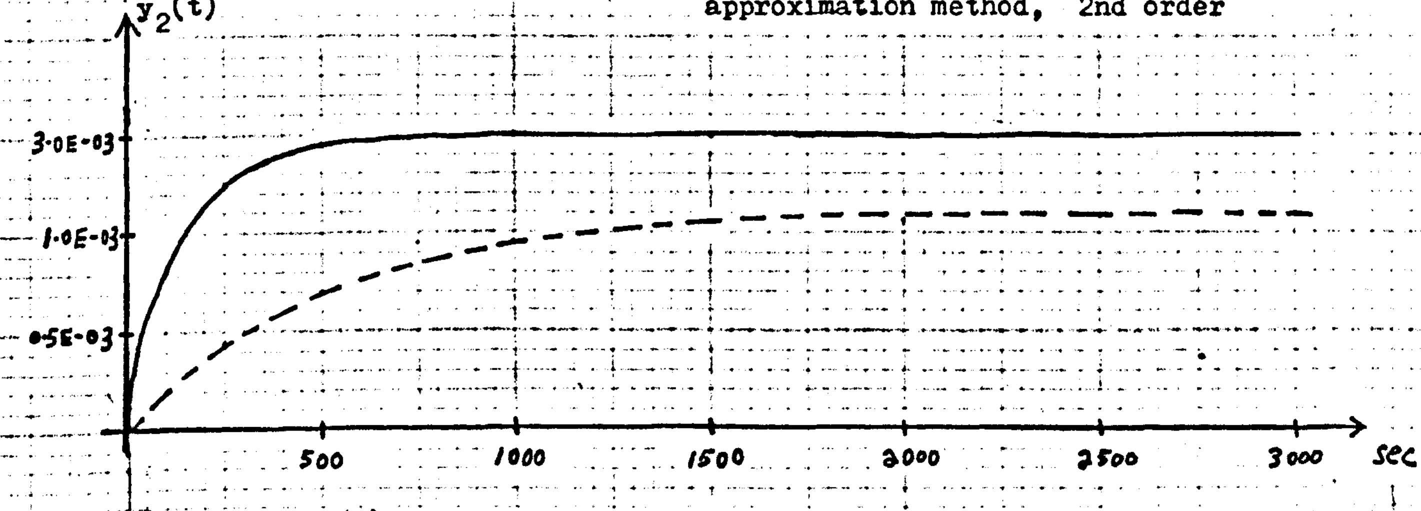

Fig. 4.7(c) Output from channel 2, due to step input to channel 1

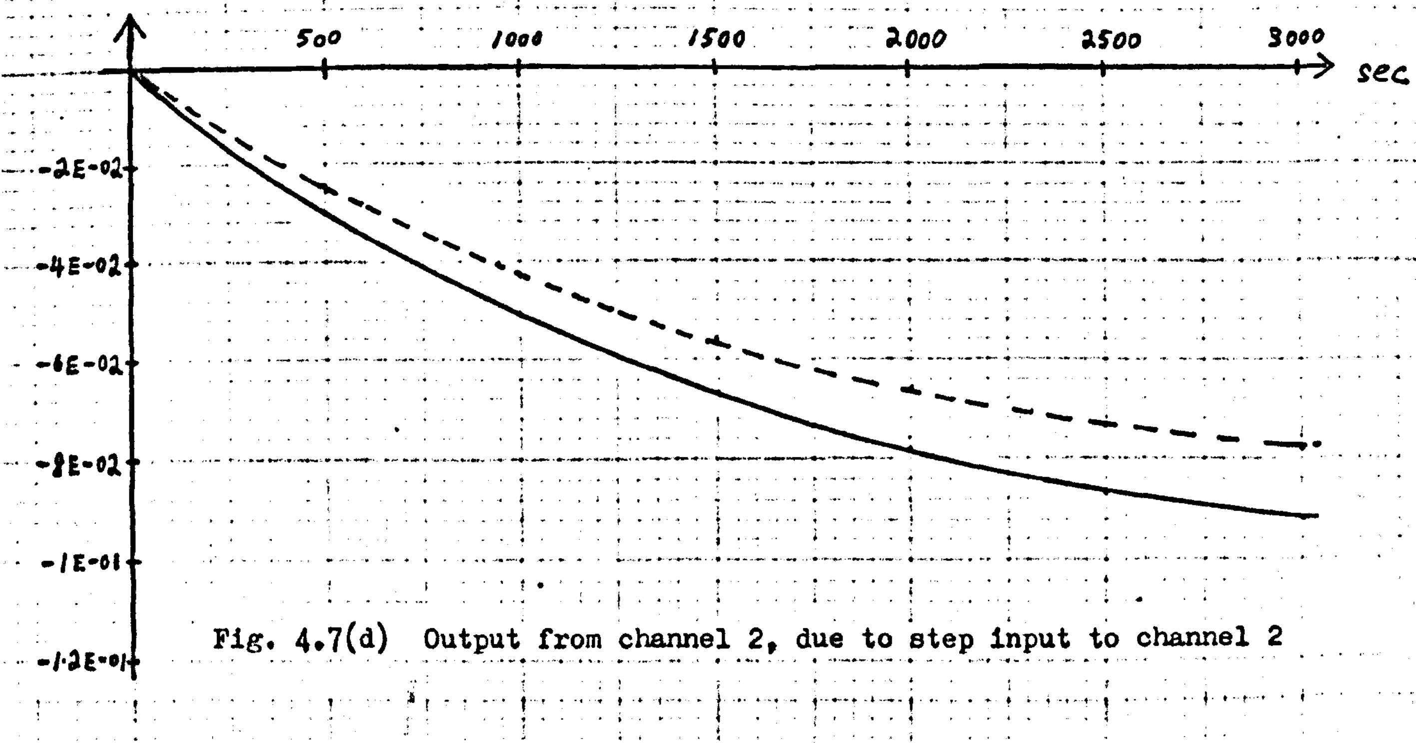

Fig. 4.7(d) Output from channel 2, due to step input to channel 2

the response of S_r was fairly close to that of S, but the interacting terms did not match closely. Modifying the structure of A_r, but keeping the eigenvalues roughly the same, gave the triple (A_r, B_r, C_r) whose response was close to that of the original system, following separate inputs to both channels. The final values of S_r are

$$A_r = \begin{bmatrix} -2.13E-03 & 1.31E-03 \\ 2.51E-06 & -6.30E-04 \end{bmatrix}, \quad B_r = \begin{bmatrix} -0.28E-02 & -5.06E-05 \\ 4.69E-06 & -0.58E-04 \end{bmatrix}, \quad C_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The responses of S and S_r to unit step inputs are shown in figs. 4.7(a) to 4.7(d). They can be further improved by updating A_r, but this does not justify the effort involved. The sampling time was taken as T=100 seconds and k = 20 samples were taken.

4.7 Conclusions.

Both methods of reduction discussed in this chapter are iterative in nature, and, to obtain the best reduced model finally, is wholly based on a trial and error scheme. No single reduction method works well for the general practical system, hence it is best to use a method, systematic in its approach but at the same time flexible in its orientation, that meets the particular characteristics of the system. This is justified by the fact that end results are important, with the view that the model is needed in a specific task in control studies[5].

Like all iterative methods, the 'initial guess' plays a crucial role in deciding how far is the final solution.

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CHAPTER V

STABILITY OF LINEAR MULTIVARIABLE SYSTEMS DESIGNED USING REDUCED MODELS

Introduction

One of the chief uses of reduced models is in the design of controllers for multivariable systems. The main problem associated with this is the stability of the original system, S, after design. More often than not, after the controller, K_r, is implemented, S becomes unstable, although the 'reduced' system, S_r, with same K_r implemented is stable. The designer is thus faced with a serious problem, and, the validity of using reduced models is somewhat limited. Few work has been done in investigating the stability of systems using reduced models*. Although Mitra[32] and Aoki[23] had studied the role of reduced models in some classes of optimal control problems, the stability of systems associated with reduced models is still not satisfactorily solved; hence the stability question is still left open. Following Aoki's[23] formulation, Vittal Rao and Lamba[24] had partially solved the problem by showing that S can be stable, provided that the reduced state space model used is obtained in a certain way. However, this is restricted to a very limited class of reduced models only, and, S considered, must be in the time domain with full state feedback. For the general class of reduced models, not obtained by 'projective techniques', the stability of S is not guaranteed[23]. Chen and Shieh[32] had used a reduced model for single input-single output system's design, and, although S is stable, it is due to the fact that intuitive engineering judgement is used in the design, rather than analysis based on formal mathematical treatment.

This chapter investigates the stability of S based on that of S_r, under the action of the same K_r, figs. 5.1(a) and 5.1(b). New results are obtained in the frequency and time domains, which to the author's best knowledge, have not appeared before in the literature. The stability results given are general, for they apply to all reduced models obtained by any valid reduction techniques.

* before the publication of this thesis

Before going into the main results, two well known multivariable stability theorems are reviewed for reference. It is well known from the Hsu-Chen theorem[2], that provided $\det(I + Q(\infty)\ H(\infty)) \neq 0$, then

$$\det F(s) = \det Q(s)/\det R(s) = \det \hat{R}(s)/\det \hat{Q}(s) = \Delta_c(s)/\Delta_o(s) \qquad \ldots (5.1)$$

where $R(s) = \hat{F}(s)\ Q(s)$, $F(s) = I + Q(s)\ H(s)$, is the return difference matrix and $Q(s) = G(s)\ K(s)$. The superscript 'hat' denotes inverses, example $\hat{F}(s) = F^{-1}(s)$, $\hat{R}(s) = R^{-1}(s)$ etc. $\Delta_c(s) = \det(sI - A + BK)$, $\Delta_o(s) = \det(sI - A)$ are the closed loop characteristic polynomial (c.l.c.p.) and open loop characteristic polynomial (o.l.c.p.) of the controllable and observable triple S(A,B,C), respectively. Let D be the contour in the s-plane consisting of the imaginary axis from $-j\omega$ to $+j\omega$ and the semi-circle of radius ω in the right half s-plane. Here the radius is chosen large enough to enclose every zero of $\det Q(s)$ and $\det R(s)$ and indentations are made around every pole on the imaginary axis to include them in the right half plane. Let p_c and p_o be the number of right-half plane zeroes of $\Delta_c(s)$ and $\Delta_o(s)$ respectively. Also let $\det Q(s)$ (resp. $\det \hat{Q}(s)$) map D into Γ_o (resp. $\hat{\Gamma}_o$) encircling the origin n_o times clockwise (resp. counter clockwise). Similarly, let $\det R(s)$ (resp. $\det \hat{R}(s)$) map D into Γ_c (resp. $\hat{\Gamma}_c$) encircling the origin n_c times clockwise (resp. counter clockwise). Let $\det F(s)$ map D into Γ_f encircling the origin n_f times clockwise. Finally, let t_j the characteristic loci of F(s), where $\prod_{j=1}^{m} t(s) = \det F(s)$ map D into Γ_{fj}, encircling the origin n_{fj} times clockwise.

Then by application of the principle of the argument to eqn. (5.1) $\Sigma n_{fj} = n_f = n_o - n_c = p_c - p_o$.

Theorem 5A[6,5,3]

For closed loop stability, $p_c = 0$, it is required that

$$\sum_{j=1}^{m} n_{fj} = n_f = n_o - n_c = -p_o \qquad \ldots (5.2)$$

Eqn. (5.2) is a well known result and expresses stability in terms of characteristic loci encirclements in the Nyquist sense. Since $F(s) = I + Q(s)\,H(s)$ and if $\rho_j(s)$ is the characteristic loci of the return ratio matrix $Q(s)\,H(s)$, then $t_j(s) = 1 + \rho_j(s)$. Thus the same stability theorem applies when working with $\rho_j(s)$ with the critical point being shifted from the origin to '-1'. Since $\hat{R}(s) = I + \hat{Q}(s)$ and if $\hat{\rho}_j(s)$, $\hat{\lambda}_j(s)$ are the characteristic loci of $\hat{Q}(s)$ and $\hat{R}(s)$ respectively, then $\hat{\lambda}_j(s) = 1 + \hat{\rho}_j(s)$. Returning to eqn.(5.1) it is seen that $\det F(s) = \Pi(1 + \hat{\rho}_j(s))/\Pi\hat{\rho}_j(s)$. Thus for stability $\Sigma\hat{n}_{ci} - \Sigma\hat{n}_{oi} = -p_o$, where $\hat{n}_{ci}$ and $\hat{n}_{oi}$ are the number of counter clockwise encirclements of the origin by the loci $\hat{\lambda}_j(s)$ and $\hat{\rho}_j(s)$ respectively. Following Rosenbrock, the eigenvalues can be expressed in terms of the individual elements $\hat{q}_{ij}(s)$, $\hat{r}_{ij}(s)$ etc. of the matrices $\hat{Q}(s)$, $\hat{R}(s)$ by diagonal dominance. Let $\hat{q}_{ii}(s)$ and $\hat{r}_{ii}(s)$ $(i = 1, \ldots m)$ map D into $\hat{\Gamma}_{oi}$ and $\hat{\Gamma}_{ci}$ encircling the origin n_{oi} and n_{ci} times respectively. Then it has been shown[5],

Theorem[5] 5B

Assume system is open loop stable, i.e. $p_o = 0$ in eqn.(5.2). A sufficient condition for closed loop stability is that the following be satisfied :

(i) $\quad \sum_{i=1}^{m} n_{oi} = \sum_{i=1}^{m} n_{ci}$

(ii) $\quad |\hat{q}_{ii}(s)| > d_i \;$ or $\; |\hat{q}_{ii}(s)| > \delta_i \quad \forall s \in D,\ \forall i$

(iii) $\quad |\hat{r}_{ii}(s)| > d_i \;$ or $\; |\hat{r}_{ii}(s)| > \delta_i \quad \forall s \in D,\ \forall i$ ...(5.3)

where $d_i = \sum_{i \neq j} |\hat{q}_{ij}(s)|$, $\delta_i = \sum_{j \neq i} |\hat{q}_{ji}(s)|$, $\forall s \in D$ means "for all s on contour D".

5.1 Some frequency domain multivariable stability theorems for reduced models

Without loss of generality, it is assumed that the feedback matrix $H_r(s) = I$. Some stability theorems are derived below.

Theorem 5.1

Let $\rho_{rj}(s)$ (resp. $\hat{\rho}_{rj}(s)$) be the characteristic loci of $Q_r(s) \equiv G_r(s)\,K_r(s)$ (resp. $\hat{Q}_r(s) \equiv \hat{K}_r(s)\,\hat{G}_r(s)$) where $Q_r(s)$ (resp. $\hat{Q}_r(s)$) is simple. Then all loci $\rho_j(s)$ (resp. $\hat{\rho}_j(s)$) of $Q(s) \equiv G(s)\ K_r(s)$ (resp. $\hat{Q}(s) \equiv \hat{K}_r(s)\ \hat{G}(s)$) lie in the union of discs with

centre, $\rho_{rj}(s)$ (resp. $\hat{\rho}_{rj}(s)$) $\quad \forall j$

radii, $\| G_e(s)\ K_r(s) \| \inf \kappa(P(s))$ $\{$resp. $\| \hat{K}_r(s)\ \hat{G}_e(s) \| \inf \kappa(\hat{P}(s))\}$

where $G_e(s) = G(s) - G_r(s)$ (resp. $\hat{G}_e(s) = \hat{G}(s) - \hat{G}_r(s)$, in general $\hat{G}_e(s) \neq G^{-1}(s)$). $\kappa(P(s)) = \| P(s) \|\ \| \hat{P}(s) \| = \kappa(\hat{P}(s))$ and $\inf \kappa(P(s)$ is taken w.r.t. all matrices $P(s)$ (resp. $\hat{P}(s)$) for which $\hat{P}(s)\ Q_r(s)\ P(s)$ (resp. $\hat{P}(s)\ \hat{Q}_r(s)\ P(s)$) is diagonal $[P(s)\ \hat{D}(s)$ (resp. $\hat{D}(s)\ \hat{P}(s)$) where $D(s)$ (resp. $\hat{D}(s)$) is diagonal, also diagonalises $Q_r(s)$ (resp. $\hat{Q}_r(s)$)$]$. Here $\| \ \|$ is a matrix norm induced by an absolute vector norm $h(\ )$. $[\ \| W(s) \| = \sup_{x(s) \neq 0} h(W(s)x(s))/h(x(s))$ is an induced norm$]$.

Proof : The above is a result similar to Bauer and Fike's[7]. For convenience only $Q_r(s)$, $\rho_{rj}(s)$ etc. are used in the proof, as similar results follow by working with their inverses. The following lemma is also used.

Lemma 5.1

If $W(s)\ \alpha(s) = V(s)\ \alpha(s)$ $\quad$ ...(5.4)

$$\text{then} \quad \inf_{\alpha(s) \neq 0} \frac{h(W(s)\ \alpha(s))}{h(\alpha(s))} \leq \frac{h(W(s)\alpha(s))}{h(\alpha(s))} = \frac{h(V(s)\ \alpha(s))}{h(\alpha(s))} \leq \| V(s) \| \quad \ldots$$

from which

$$\| \hat{W}(s) \|^{-1} \leq \| V(s) \| \quad \ldots(5.5)$$

Returning to the proof, since $\rho_k(s)$ is a characteristic value of $Q(s)$, then

$$(Q_r(s) + Q(s) - Q_r(s))\ \gamma(s) = \rho_k(s)\ \gamma(s) \quad \ldots(5.6)$$

$\gamma(s) \neq 0$, from which

$$P(s)\{\text{diag}(\rho_{r1}(s), \ldots \rho_{rm}(s)) + \hat{P}(s)\ G_e(s)\ K_r(s)\ P(s)\}\ \hat{P}(s)\gamma(s) = \rho_k(s)\gamma(s) \quad \ldots(5.7)$$

Eqn. (5.7) gives

$$[\rho_k(s)\ I - \text{diag}\{\rho_{r1}(s), \ldots \rho_{rm}(s)\}]\ \alpha(s) = \hat{P}(s)\ G_e(s)\ K_r(s)\ P(s)\ \alpha(s) \quad \ldots(5.8)$$

where $\alpha(s) = \hat{P}(s)\,\gamma(s) \neq 0$. Applying lemma 5.1 to eqn.(5.8) yields

$$\min_j |\rho_k(s) - \rho_{rj}(s)| \leq \| G_e(s)\, K_r(s)\| \inf \kappa(P(s)) \qquad \ldots(5.9)$$

The l.h.s. of eqn.(5.9) is obtained by $\min_j |d_j| = 1/\max_j |1/d_j|$ and the r.h.s. is obtained since $\| \hat{P}(s)\, G_e(s)\, K_r(s)\, P(s)\| \leq \| G_e(s)\, K_r(s)\| \; \| P(s)\| \; \| \hat{P}(s)\|$ and $\inf \kappa(P(s))$ is taken as eqn. (5.9) is true for every $P(s)$ which diagonalises $Q_r(s)$. Hence all $\rho_j(s)$ lie in the union of discs defined in theorem 5.1. Q.E.D.

Theorem 5.1 is used to derive some multivariable stability theorems for reduced models. For the moment, S_r and S are assumed to be open loop stable i.e. $G_r(s)\, K_r(s)$ and $G(s)\, K_r(s)$ have no right-half plane poles.

Theorem 5.2

Let $\rho_{rj}(s)$ (resp. $\hat{\rho}_{rj}(s)$) map the D contour in the s-plane into the frequency response contour Γ_{rj} (resp. $\hat{\Gamma}_{rj}$). Then sufficient conditions for S to be closed-loop stable are:

(1) every Γ_{rj} (resp. $\hat{\Gamma}_{rj}$) locus individually satisfies the Nyquist criterion (NC)

(2) $|1 + \rho_{rj}(s)| > \| G_e(s)\, K_r(s)\| \inf \kappa(P(s)) \quad \forall s \in D, \ \forall j \qquad \ldots(5.10)$
$[\text{resp.}\ |1 + \hat{\rho}_{rj}(s)| > \| \hat{K}_r(s)\, \hat{G}_e(s)\| \inf \kappa(\hat{P}(s))] \ \forall s \in D, \ \forall j$

Proof : The geometrical interpretation of theorem 5.2 is shown in fig. 5.2. If the above conditions are satisfied it is seen that the band of circles cannot overlap the critical point and by theorem 5.1 no locus Γ_j, due to $\rho_j(s)$ can enclose it, hence S is closed loop stable. The dual theorem for the inverse locus is illustrated in fig. 5.3.

A less sharper bound is obtained from eqn.(5.10) if

$$|1 + \rho_{rj}(s)| > \| G_e(s)\| \; \| K_r(s)\| \; \kappa(P(s)) \quad \forall j, \ \forall s \in D$$
$$[\text{resp.}\ |1 + \hat{\rho}_{rj}(s)| > \| \hat{G}_e(s)\| \; \| \hat{K}_r(s)\| \; \kappa(\hat{P}(s))] \quad \forall j, \ \forall s \in D \qquad \ldots(5.11)$$

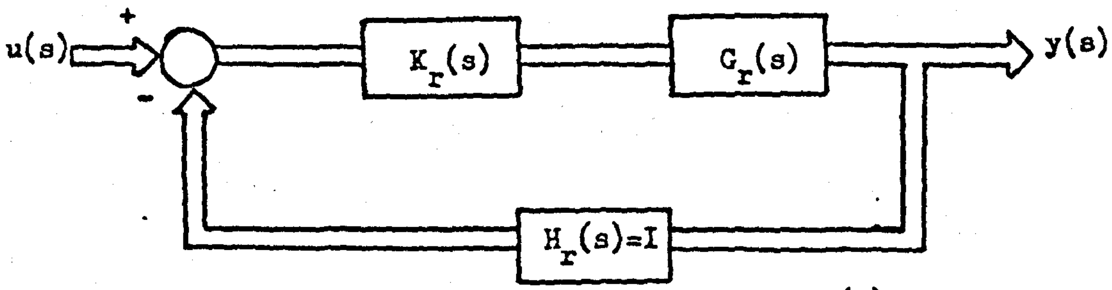


Fig. 5.1(a) Reduced system S_r

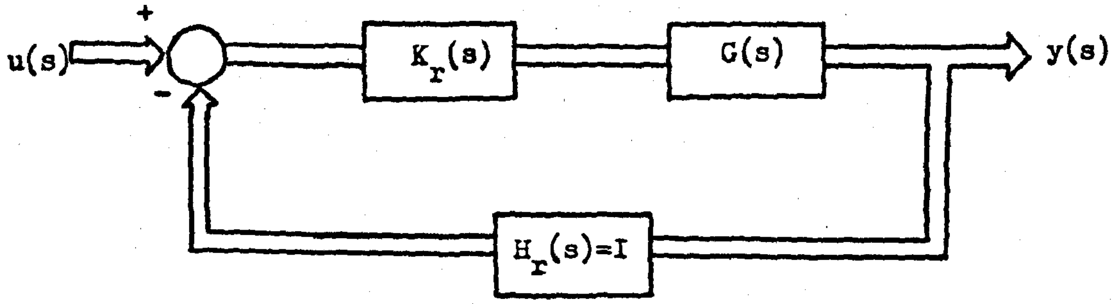
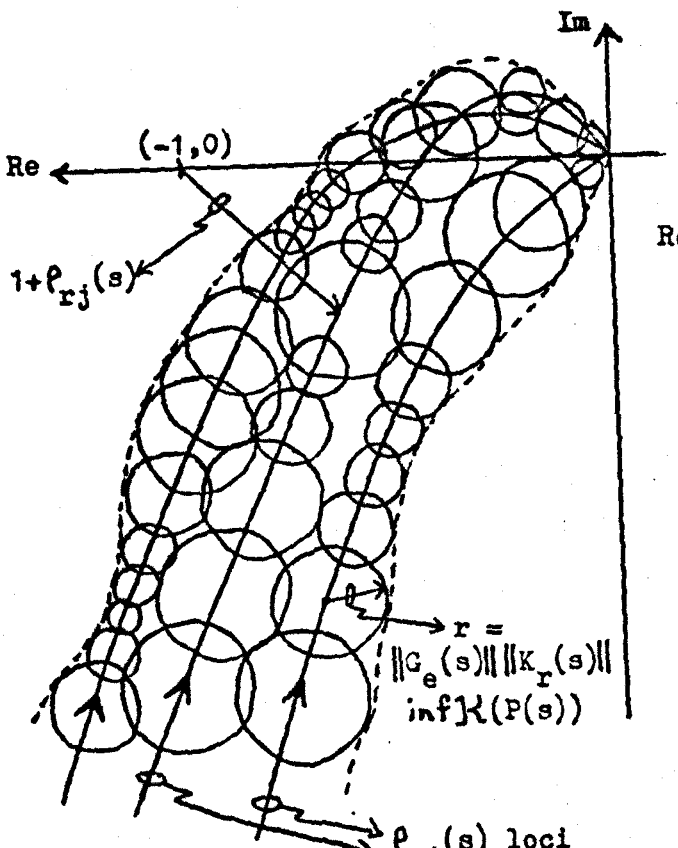


Fig. 5.1(b) Original system S



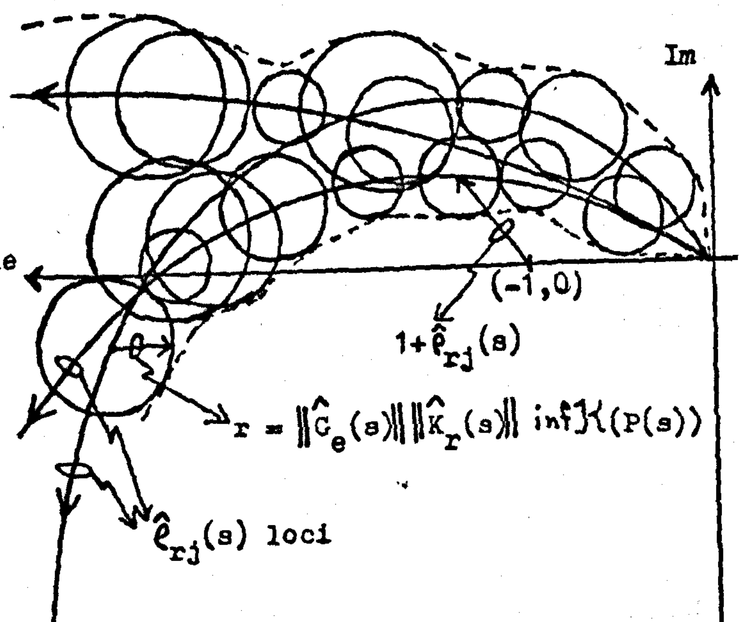


Fig. 5.3 Illustrating stability for original system using inverse characteristic loci

(system is stable if band avoids -1 point

Fig. 5.2 Illustrating stability for original system using direct loci (theorem 5.2)

(system is stable if band avoids -1 point)

All loci are expressed in terms of return ratio matrices as $\| G_e(s) K_r(s)\| \leq \| G_e(s)\| \| K_r(s)\|$, but the form given in eqn.(5.11) is sometimes useful as a guideline in design, since $\| K_r(s)\|$ is isolated and limiting values can be estimated for $K_r(s)$. Inf $\kappa(P(s))$ (resp. inf $\kappa(\hat{P}(s))$) is estimated by post (resp. pre) multiplying a diagonal $D(s)$ (resp. $\hat{D}(s)$) to $P(s)$ (resp. $\hat{P}(s)$) such that $\kappa(P(s)\, D(s))$ (resp. $\kappa(\hat{D}(s)\, \hat{P}(s))$) is a minimum.

If $G(s) = G_r(s)$, then $G_e(s) = 0$, and as expected from theorem 5.1 $\rho_j(s) = \rho_{rj}(s)$, $\forall j$. As the area of the disc depends on $\| \;\|$ inf $\kappa(P(s))$, their optimum size can be found by the following modification to allow greater freedom in choosing $K_r(s)$.

Theorem 5.3

Let $\rho_{rj}(s) + \beta$ (resp. $\hat{\rho}_{rj}(s) + \alpha$) map D into $\Gamma_{rj} + \beta$ (resp. $\hat{\Gamma}_{rj} + \alpha$), where β and α are arbitrary complex numbers. Then sufficient conditions for S to be closed loop stable are:

(1) $\Gamma_{rj} + \beta$ (resp. $\hat{\Gamma}_{rj} + \alpha$) individually satisfies NC $\quad \forall j$

(2) $$|1 + \beta + \rho_{rj}(s)| > \| G_e(s)\, K_r(s) - \beta I\| \inf \kappa(P(s)) \quad \forall j,\ \forall s \in D \qquad \ldots(5.12)$$

$$\left[\text{resp.}\ |1 + \alpha + \hat{\rho}_{rj}(s)| > \| \hat{K}_r(s)\, \hat{G}_e(s) - \alpha I\| \inf \kappa(\hat{P}(s))\right] \quad \forall j,\ \forall s \in D$$

or

(3) $$|1 + \beta + \rho_{rj}(s)| > (\| G_e(s)\| \| K_r(s)\| + |\beta|\, \|I\|) \inf \kappa(P(s)) \quad \forall j,\ \forall s \in D \qquad \ldots(5.13)$$

$$\left[\text{resp.}\ |1 + \alpha + \hat{\rho}_{rj}(s)| > (\| \hat{G}_e(s)\| \| \hat{K}_r(s)\| + |\alpha|\, \|I\|) \inf \kappa(\hat{P}(s))\right] \quad \forall j,\ \forall s \in D$$

Proof : Since $P(s)$ diagonalises $Q_r(s)$, $P(s)$ will also diagonalise $\beta I + Q_r(s)$, and since

$$[\beta I + Q_r(s)] + [-\beta I + G_e(s)\, K_r(s)] = Q_r(s) + G_e(s)\, K_r(s) \qquad \ldots(5.14)$$

substituting the l.h.s. of eqn. (5.14) into eqn. (5.6) and following the arguments of eqns. (5.7) and (5.8)

$$P(s)\, [\text{diag}\{\rho_{r1}(s), \ldots \rho_{rm}(s)\} + \beta I + \hat{P}(s)\, G_e(s)\, K_r(s)\, P(s) - \beta I]\, \hat{P}(s)\, \gamma(s) = \rho_k(s)\, \gamma(s)$$

from which

$$(\rho_k(s)I - \mathrm{diag}\{\rho_{r1}(s),\ldots\rho_{rm}(s)\} + \beta I)\alpha(s) = (\hat{P}(s)\ G_e(s)\ K_r(s)\ P(s) - \beta I)\alpha(s) \quad \ldots(5.15)$$

Applying lemma 5.1 to eqn. (5.15) yields

$$\min_j\ |\rho_k(s) - (\beta + \rho_{rj}(s))| \leqslant \|\, G_e(s)\ K_r(s) - \beta I \,\| \inf \kappa(P(s)) \quad \ldots(5.16)$$

Comparing eqns. (5.16) and (5.9) it is seen that the geometrical interpretation of theorem 5.3 is similar to that of theorem 5.2 except that the discs centres are shifted by β (resp. α for the inverse loci) with appropriate modifications in radii.

Here β can be chosen to minimize $\|\quad\|$ of eqn. (5.16), hence minimizing the disc areas, by requiring

$$\partial \,\|\quad\|\, /\partial \mathrm{Re}(\beta) = \partial \|\quad\| /\partial \mathrm{Im}(\beta) = 0 \quad \ldots(5.17)$$

or arbitrary chosen, by designer, subject to engineering constraints. Eqn. (5.12), deduced from eqn. (5.16), gives a sharper bound than eqn. (5.13) as

$$\|\, G_e(s)\ K_r(s) - \beta I \,\| \leqslant \|\, G_e(s) \,\|\ \|\, K_r(s) \,\| + |\beta|\ \|\, I \,\|$$

but the form given in eqn. (5.13) is useful in estimating $\|\, K_r(s) \,\|$ in design and it offers a wider stability margin.

If $\eta_1(s), \eta_2(s),\ldots\eta_m(s)$ are the characteristic loci of $G_e(s)\ K_r(s)$ and if $V(s)$ diagonalises $G_e(s)\ K_r(s)$ another useful stability theorem can be obtained as follows

Theorem 5.4

Let all conditions be the same as in theorem 5.3 together with the above modifications. Then sufficient conditions for closed loop stability of S are:

(1) $\Gamma_{rj} + \beta$ (resp. $\hat{\Gamma}_{rj} + \alpha$) individually satisfies NC $\forall j$

(2) $|1 + \beta + \rho_{rj}(s)| > \max_j |\, \eta_j(s) - \beta \,| \inf_{P,V} \kappa(P(s)\ \kappa(V(s))\quad \forall j,\ \forall s \in D$

$[\text{resp. } |1 + \alpha + \hat{\rho}_{rj}(s)| > \max_j |\, \hat{\eta}_j(s) - \alpha \,| \inf_{P,V} \kappa(\hat{P}(s))\kappa(\hat{V}(s))]\ \forall j, \forall s \in D$

...(5.18)

Proof : The r.h.s. of eqn. (5.16) becomes

$$\| V(s)\ U(s)\ \hat{V}(s) - \beta I\| \inf\kappa(P(s)) \leq \| U(s) - \beta I\|\ \ \| V(s)\|\ \ \|\hat{V}(s)\| \inf \kappa(P(s))$$
$$\leq \max_j |\eta_j(s) - \beta| \inf_{P,V} \kappa(P(s))\ \kappa(V(s))$$

where $U(s) = \text{diag}\{\eta_1(s), \eta_2(s), \ldots \eta_m(s)\}$, hence eqn . (5.16) becomes

$$\min_j | \rho_k(s) - (\beta + \rho_{rj}(s))| \leq \max_j |\eta_j(s) - \beta| \inf_{P,V} \kappa(P(s))\ \kappa(V(s)) \qquad \ldots(5.19)$$

The proof follows.

As in theorem 5.3 β(resp. α) can be chosen to minimize $\max_j |\eta_j(s) - \beta I|$ (resp. $\max_j |\hat{\eta}_j(s) - \alpha I|$) or as in theorem 5.2 taken to be zero. Although theorem 5.4 is useful in testing for stability if $\eta_j(s)$ (resp. $\hat{\eta}_j(s)$) can be easily calculated, it has poor connections with design, as $K_r(s)$ (resp. $\hat{K}_r(s)$) is not directly assessable.

The stability conditions stated above are expressed in terms of the return ratio matrix $T_r(s) = Q_r(s)\ H_r(s) = Q_r(s)$ (resp. $\hat{T}_r(s) = \hat{Q}_r(s)$. Similar results to the above can be expressed in terms of the return difference matrix, $F_r(s)$ (resp. $\hat{R}_r(s)$).

Theorem 5.5

Let $\nu_{rj}(s) + \beta$(resp. $\hat{\lambda}_{rj}(s) + \alpha$) map D into $\Delta_{rj} + \beta$ (resp. $\hat{\Delta}_{rj} + \alpha$) where $\nu_{rj}(s)$ (resp. $\hat{\lambda}_{rj}(s)$) is the characteristic loci of $F_r(s)$ (resp. $\hat{R}_r(s)$) and β(resp. α) is a complex number. Then sufficient conditions for S to be closed loop stable are :

(1) $\Delta_{rj} + \beta$(resp. $\hat{\Delta}_{rj} + \alpha$) individually satisfies NC $\forall j$

(2) $|\nu_{rj}(s) + \beta| > \| G_e(s)\ K_r(s) - \beta I\|\ \inf \kappa(P(s)) \qquad \forall j,\ \forall s \in D \ \ldots(5.20)$

$[\text{resp. } |\hat{\nu}_{rj}(s) + \alpha| > \| \hat{K}_r(s)\ \hat{G}_e(s) - \alpha I\| \inf \kappa\ (\hat{P}(s))] \quad \forall j,\ \forall s \in D$

or

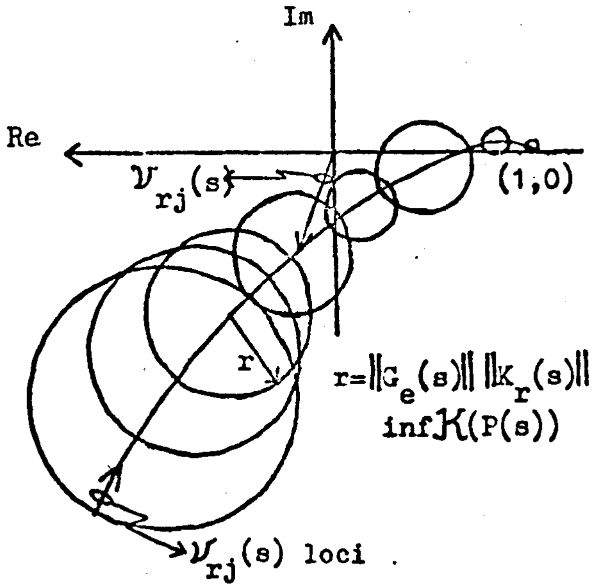
(3) $|\nu_{rj}(s) + \beta| > \max_j |\eta_j(s) - \beta| \inf_{P,V} \kappa(P(s))\ \kappa(V(s)) \quad \forall j,\ \forall s \in D \ \ldots(5.21)$

$[\text{resp. } |\hat{\lambda}_{rj}(s) + \alpha| > \max_j |\hat{\eta}_j(s) - \alpha| \inf_{P,V} \kappa(\hat{P}(s))\ \kappa(\hat{V}(s))] \quad \forall j,\ \forall s \in D$

Proof : Since $F(s) = I + Q(s)$ (resp. $\hat{R}(s) = I + \hat{Q}(s)$), it follows that $\| F(s) - F_r(s) \| = \| Q(s) - Q_r(s) \| = \| G_e(s)\, K_r(s) \|$ (resp. $\| \hat{R}(s) \| = \| \hat{Q}(s) - \hat{Q}_r(s) \| = \| \hat{K}_r(s)\, \hat{G}_e(s) \|$) and $P(s)$ (resp. $\hat{P}(s)$) which diagonalises $Q_r(s)$ (resp. $\hat{Q}_r(s)$) will also diagonalise $F_r(s)$ (resp. $\hat{R}_r(s)$). Thus substituting $\| F(s) - F_r(s) \|$ (resp. $\| \hat{R}(s) - \hat{R}_r(s) \|$) into eqn. (5.6) will produce the same circular disc as defined in theorem 5.1. Further, by the eigenvalue shift theorem, $\nu_{rj}(s) = 1 + \rho_{rj}(s)$ (resp. $\hat{\lambda}_{rj}(s) = 1 + \hat{\rho}_{rj}(s)$). Thus, the conditions for stability are exactly the same as in theorems 5.3 and 5.4, except that the critical point is shifted from '-1' to the origin, with the disc areas remaining unaltered. The proof for stability is exactly the same as given in theorems 5.2 to 5.4. The result follows, see figs. 5.4 and 5.5.

Special cases : In the special case when $Q_r(s)$ (resp. $\hat{Q}_r(s)$) or $G_e(s)\, K_r(s)$ (resp. $\hat{K}_r(s)\, \hat{G}_e(s)$ is a normal matrix i.e. $Q_r^H(s)\, Q_r(s) = Q_r(s)\, Q_r^H(s)$, then $P(s)$ (resp. $\hat{P}(s)$) or $V(s)$ (resp. $\hat{V}(s)$) can be chosen to be unitary, and using the spectral norm $\| \;\; \|_2$, $\kappa(P(s)) = 1$ or $\kappa(V(s)) = 1$ (since $P(s)\, P^H(s) = I$, therefore $\| P(s) \|_2 = \| \hat{P}(s) \|_2 = \{\lambda_{max} (P(s)\, P^H(s))\}^{\frac{1}{2}} = 1$). Thus the stability boundaries in theorems 5.2 to 5.5 are much simplified for computation.

Open loop unstable systems : The stability conditions given in theorems 5.2 to 5.5 are sufficient but not necessary. The first condition requires that the reduced system, S_r, be closed loop stable in the sense of the Nyquist criterion. However, this is not necessary for closed loop stability of original system, S. S is only closed loop stable if its characteristic loci, say, $\rho_j(s)$ do not enclose the critical point. Also as far as closed loop stability of S and S_r is concerned, either or both S_r and S can be open-loop unstable. Assume now that S_r is open-loop unstable and has z right-half plane zeroes in its open-loop characteristic polynomial (o.l.c.p.). Then,



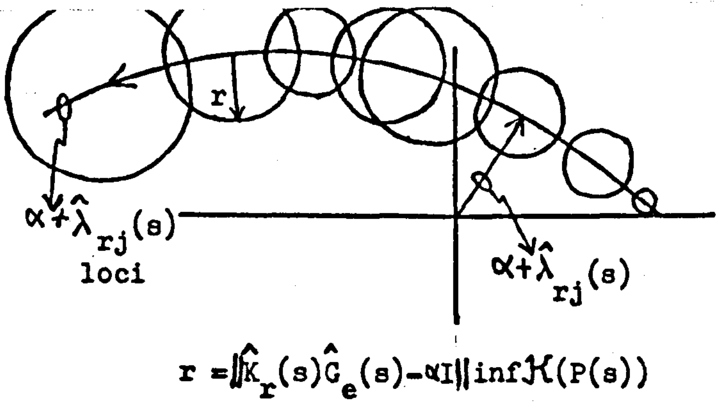


Fig. 5.4 Illustrating stability for S using return difference direct loci(theorem 5.5)

Fig.5.5 Illustrating stability for S using return difference inverse loci

(In both cases, for stability band must not touch or overlap origin)

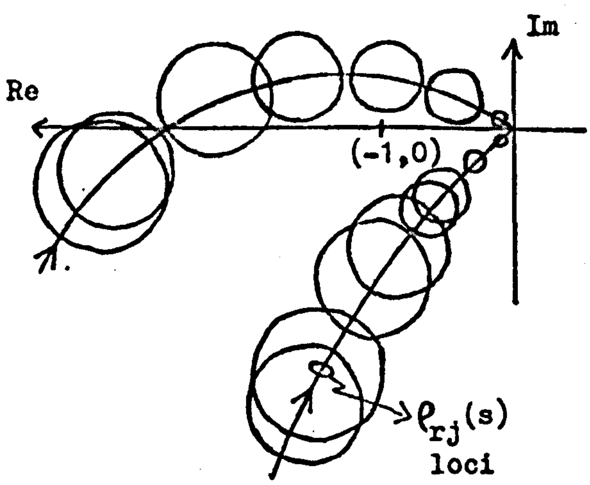


Fig. 5.6 Illustrating closed loop stability for S when p_o = 1 (theorem 5.6)

(i.e. right hand side pole = 1)

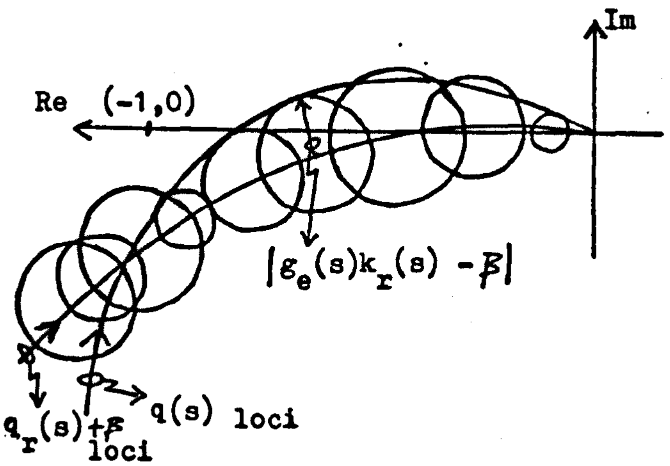


Fig. 5.7 Illustrating stability for S when it is single input-single output (theorem 5.7)

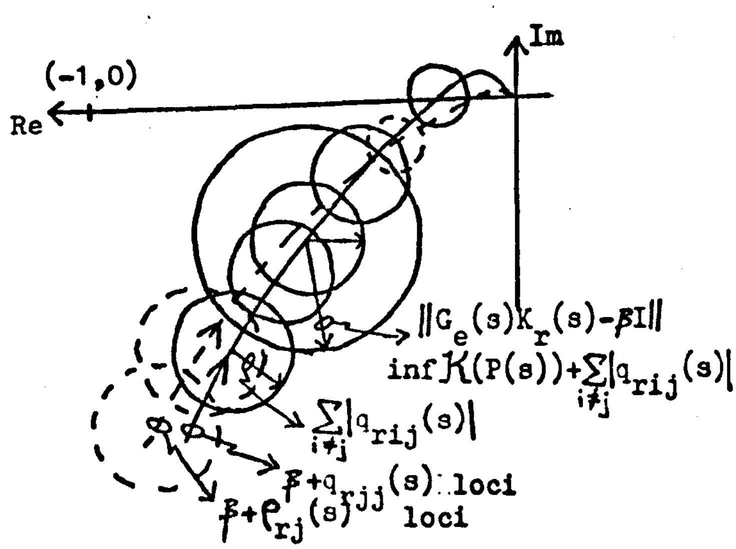


Fig. 5.8 Illustrating stability for S by diagonal dominance and Characteristic Loci (theorem 5.8)

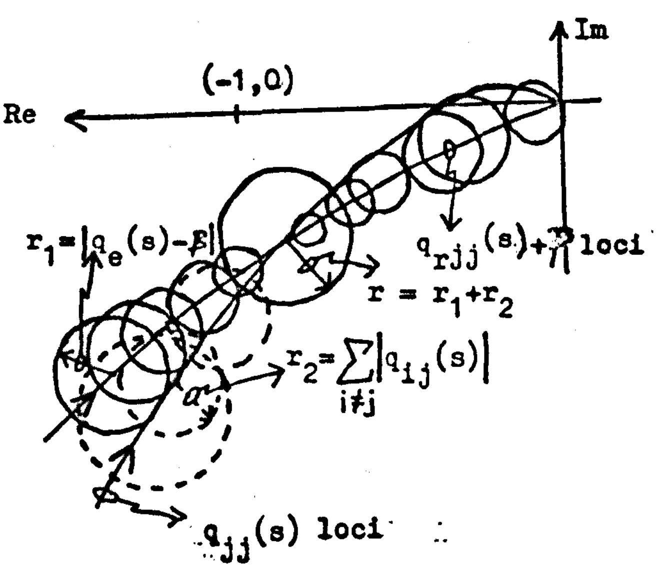


Fig. 5.9 Illustrating stability for S using diagonal dominance and Nyquist loci of S_r (theorem 5.9)

Theorem 5.6

Sufficient conditions for S to be closed-loop stable are:

(1) S_r satisfies NC in the sense $\sum_{i=1}^{m} n_{fj} = -z$ (resp. $\sum_{i=1}^{m} \hat{n}_{ci} - \sum_{i=1}^{m} \hat{n}_{oi} = -z$)

(2) Condition (2) of one of the theorems 5.2 to 5.5 be satisfied

(3) S has z right-half plane zeroes in its o.l.c.p.

Proof : Since S_r is closed-loop stable, the total number of encirclements of the critical point is $\Sigma n_{fj} = -p_o = -z$. It follows that if condition (2) is satisfied, i.e. the bands do not overlap the critical point, it is seen that the loci of S encircle the critical point the same number of times as that of S_r. Hence by theorem 5A, S is closed-loop stable if $p_o = z$ in its o.l.c.p. This is illustrated graphically in fig. 5.6.

Theorem 5.6 is the general case of theorems 5.2 to 5.5 where both S and S_r are open-loop stable. However, if the number z is different in the o.l.c.p's of S_r and S, then closed-loop stability of S is uncertain, and, to determine the latter in terms of that of S_r, will, in general, be difficult.

In practice, normally a stable G(s) is given, and a stable $G_r(s)$ is derived from it. If $G_r(s)$ derived is unstable, then the reduction method is ineffective and G(s) is not closely approximated, hence, its dynamics, together with the overall dynamics when controller $K_r(s)$ is implemented, cannot be properly studied. Thus in terms of dynamical interest in the design, $G_r(s)$ must be stable. However, if G(s) given is unstable, then during the design process, extraneous right-half plane zeroes must be added to $G_r(s)$ or $K_r(s)$ such that the number of right-half plane zeroes in the o.l.c.p. of S_r is equal to that in the o.l.c.p. of S. This is necessary before theorem 5.6 can be applied where closed-loop stability of S is to be deduced from that of S_r during design.

Non-simple matrices : The stability conditions of theorems 5.1 to 5.6 are only valid when $Q_r(s)$ is simple, i.e. similar to a diagonal matrix. If $Q_r(s)$

has repeated characteristic loci, but is not normal, then theorem 5.1 breaks down. In computing the loci, usually rounding errors make them distinct. However, at a specific ω_c, $\rho_{rj}(\omega_c)$ can be non distinct or the structure of $Q_r(s)$ can be such that $\rho_{rj}(s)$ is repeating for all $s = j\omega$. The modification to theorem 5.1 of Bauer and Fike is added below for completeness.

Corollary 5.1

Let $Q_r(s)$ (resp. $\hat{Q}_r(s)$) be a general, not necessary simple, matrix with $\rho_{rj}(s)$ having index $m_j > 1$ (resp. $\hat{m}_j > 1$, where in general $\hat{m}_j \neq 1/m_j$). Then $\rho_j(s)$ (resp. $\hat{\rho}_j(s)$), $\forall j$, lie in the union of discs with

centre $\rho_{rj}(s)$ (resp. $\hat{\rho}_{rj}(s)$)

radii $\| G_e(s)\, K_r(s) + P(s)\, H_p\, \hat{P}(s) \|\ \inf \kappa(P(s))$

$[$resp. $\| \hat{K}_r(s)\, \hat{G}_e(s) + \hat{P}(s)\, H_{\hat{p}} P(s) \|\ \inf \kappa(\hat{P}(s))]$

where H_p (resp. $H_{\hat{p}}$) is a matrix with entry unity or zero depending on m_j (resp. $\hat{m}_j$) along its superdiagonal and null elsewhere. Inf $\kappa(P(s))$ (resp. inf $\kappa(\hat{P}(s))$) is taken as the minimum of $\kappa(P(s)\ D(s))$ (resp. $\kappa(\hat{D}(s)\ \hat{P}(s))$) for which $P(s)$ (resp. $\hat{P}(s)$) quasi-diagonalises $Q_r(s)$ (resp. $\hat{Q}_r(s)$) to diag$\{C_{r1}(s), \ldots C_{rp}(s)\}$ (resp. diag $\{J_{r1}(s), \ldots J_{rq}(s)\}$). [$P(s)\ D(s)$ (resp. $\hat{D}(s)\ \hat{P}(s)$) where $D(s)$ (resp. $\hat{D}(s)$) is diagonal also quasi-diagonalises $Q_r(s)$ (resp. $\hat{Q}_r(s)$)]. $C_{rj}(s)$ (resp. $J_{rj}(s)$) is a Jordan Block, p (resp. q) being the number of elementary divisors of $\rho_{rj}(s)\ I - Q_r(s)$ (resp. $\hat{\rho}_{rj}(s)\ I - \hat{Q}_r(s)$).

Proof : Following eqn. (5.7),

$$P(s)\left[\mathrm{diag}\{C_{r1}(s), \ldots C_{rp}(s)\} + \hat{P}(s) G_e(s) K_r(s) P(s)\right] \hat{P}(s) \gamma(s) = \rho_k(s) \gamma(s) \quad \ldots (5.22)$$

Now write

$$\mathrm{diag}\ \{C_{r1}(s), \ldots C_{rp}(s)\} = \mathrm{diag}\{\rho_{r1}(s), \ldots \rho_{rm}(s)\} + H_p \quad \ldots (5.23)$$

Substituting eqn. (5.23) into (5.22) and manipulating gives,

$$\left[\rho_k(s) I - \mathrm{diag}\{\rho_{r1}(s), \ldots \rho_{rm}(s)\}\right] \alpha(s) = \left[\hat{P}(s) G_e(s) K_r(s) P(s) + H_p\right] \alpha(s) \quad \ldots (5.24)$$

where $\alpha(s) = \hat{P}(s)\, \gamma(s) \neq 0$. Applying lemma 5.1 gives

$$\min_j |\rho_k(s) - \rho_{rj}(s)| \leq \| G_e(s)\, K_r(s) + P(s)\, H_p\, \hat{P}(s)\| \ \inf \kappa(P(s)) \qquad \ldots(5.25)$$

The corollary follows.

Since P(s) will also quasi-diagonalise $\beta I + Q_r(s)$ to diag $\{C'_{r1}(s), \ldots C'_{rp}(s)\}$ where $C'_{rj}(s) = \beta I + C_{rj}(s)$ and β is a complex number, eqns. (5.15) is still valid if diag$\{\rho_{r1}(s),\ldots\rho_{rm}(s)\}$ is replaced by diag$\{\rho_{r1}(s),\ldots\rho_{rm}(s)\}$ $+H_p$. Rearranging the modified form of eqn. (5.15) and applying lemma 5.1 yields the modified form of eqn. (5.16),

$$\min_j |\rho_k(s) - (\beta+\rho_{rj}(s))| \leq \| G_e(s)\, K_r(s) - \beta I + P(s)\, H_p\, \hat{P}(s)\| \inf\kappa(P(s)) \qquad \ldots(5.26)$$

Further, if V(s) quasi-diagonalises $G_e(s)\, K_r(s)$ to $U(s) + H_v$ where H_v is similarly defined as H_p and $U(s) = \text{diag}\{\mu_1(s)\ldots\mu_m(s)\}$, the r.h.s. of eqn. (5.26) becomes

$$\| V(s)\, U(s)\, \hat{V}(s) - \beta I + V(s)\, H_v\, \hat{V}(s) + P(s)\, H_p\, \hat{P}(s)\| \inf\kappa(P(s)) \leq$$
$$\{\max_j |\mu_j(s) - \beta| + H_v + \hat{W}(s)\, H_p\, W(s)\} \inf_{p,v}\kappa(P(s))\, \kappa(V(s)) \qquad \ldots(5.27)$$

where $W(s) = \hat{P}(s)\, V(s)$. Thus eqn. (5.26) becomes

$$\min_j |\rho_k(s) - (\beta+\rho_{rj}(s))| \leq \{ \max_j |\mu_j(s) - \beta| + H_v + \hat{W}(s)\, H_p\, W(s)\} \inf_{p,v}\kappa(P(s))\kappa(V(s)) \qquad \ldots(5.28)$$

Since $\|F_r(s)\| = \| I + Q_r(s)\|$, it follows that P(s) will also reduce $F_r(s)$ to its Jordan form, and by the eigenvalue shift theorem, $\nu_{rj}(s) = 1+\rho_{rj}(s)$, the critical point is shifted from '-1' to the origin. Hence change in critical point is invariant to the nature of $Q_r(s)$. Similar results apply to the inverse case, $\| \hat{R}_r(s)\| = \| I + \hat{Q}_r(s)\|$.

Corollories 5.2, 5.3, 5.4, 5.5 and 5.6

Sufficient conditions for closed-loop stability of S are the same as those given in theorems 5.2 to 5.6 except that the terms $P(s)\, H_p\, \hat{P}(s)$ (resp. $P(s)\, H_{\hat{p}}\, \hat{P}(s)$), H_v, $V(s)\, H_v\, \hat{V}(s)$ (resp. $V(s)\, H_{\hat{v}}\, \hat{V}(s)$) are included in the $\|\ \ \|$

expressions as appropriate, thus increasing the radii of the discs.

Proof : The proof and geometrical interpretation follows exactly as those given in theorems 5.2 to 5.6.

Hence when $Q_r(s)$ and $G_e(s)\ K_r(s)$ are non-simple, the stability bounds are less sharp than those when the matrices are simple, ($H_p = H_v = 0$, for simple $Q_r(s)$ and $G_e(s)\ K_r(s)$) assuming that the bounds are expressed in the same form for both cases. The bounds can also be adjusted to isolate $\| K_r(s) \|$ (resp. $\| \hat{K}_r(s) \|$) though at the expense of diminishing sharpness, during design and then 'tuned' down by the scaling factor β (resp. α) in the final stage to offer the maximum stability margins. For example, the sharpest band, eqn. (5.25), to satisfy the Nyquist criterion with Γ_{rj} locus doing likewise, is

$$| 1 + \beta + \rho_{rj}(s) | > \inf \| \hat{P}\ G_e(s)\ K_r(s)\ P(s) - \beta I + H_p \| \qquad \ldots(5.29)$$

although computationally the r.h.s. of exp. (5.29) may be unfeasible to evaluate. Also, it is noticed that $\| H_p \|_1 = \| H_p \|_\infty = \| H_p \|_2 = 1$ (the same applies for H_v), thus when isolating $\| K_r(s) \|$, it is convenient not to isolate $\| H_p \|$ as an added term near the critical point, as overlap of the critical point by the disc is likely to occur. The norms that are used above can be computationally obtained as

$$\| A(s) \|_1 = \| A^H(s) \|_\infty = \max_j \sum_{i=1}^{m} | a_{ij}(s) |$$

$$\| A(s) \|_2 = [\lambda_{max} (A^H(s) A(s))]^{\frac{1}{2}}$$

Single input-single output systems : The stability conditions given in theorems 5.2 to 5.6 are stated for multivariable systems having m inputs and m outputs. In the special case, when m = 1, the single input-single output system is obtained. The stability conditions are as follows,

Theorem 5.7

Let $q_r(s) + \beta$ (resp. $\hat{q}_r(s) + \alpha$) map D into $\Gamma_r + \beta$ (resp. $\hat{\Gamma}_r + \alpha$). The sufficient conditions for closed-loop stability of S are :

(1) $\Gamma_r + \beta$ (resp. $\hat{\Gamma}_r + \alpha$) satisfies NC

(2) $|1 + \beta + q_r(s)| > |g_e(s)\ k_r(s) - \beta| \quad \forall s \in D$ ..(5.30)

$\left[\text{resp. } |1 + \alpha + q_r^{-1}(s)| > |\hat{g}_e(s)\ k_r^{-1}(s) - \alpha|\right] \forall s \in D$

where $q_r(s) = g_r(s)\ k_r(s)$, $g_e(s) = g(s) - g_r(s)$, $\hat{g}_e(s) = g^{-1}(s) - g_r^{-1}(s)$

Proof : The geometrical interpretation of theorem 5.7 is shown in fig. 5.7.

Eqn. (5.30) is the special case of eqn. (5.12) for m = 1. The transfer function matrices $Q_r(s)$, $G_e(s)\ K_r(s)$ reduce to single expressions $q_r(s)$, $g_e(s)\ k_r(s)$ etc. Thus the characteristic locus becomes the single loop classical Nyquist locus. Stability is obvious if the band does not overlap the critical point. Theorem 5.7 can also be derived by considering $Q_r(s)$ and $G_e(s)\ K_r(s)$ diagonal. Then from eqn. (5.12), $\rho_{rj}(s)$ becomes $q_{rjj}(s)$, $\| G_e(s)\ K_r(s) - \beta I\|$ becomes $\max_j |g_{ej}(s)\ k_{rj}(s) - \beta I|$ and $\inf\kappa(P(s)) = 1$ as P = I. Hence the condition

$$|1 + \beta + q_{rjj}(s)| > \max_j |g_{ej}(s)\ k_{rj}(s) - \beta I|, \quad \forall j, \quad \forall s \in D$$

is required for stability, which reduces to eqn. (5.30) for j = 1.

As before, the 'tuning' factor β (resp. α) can be chosen to minimize $|g_e(s)\ k_r(s) - \beta|$ (resp. $|\hat{g}_e(s)\ k_r^{-1}(s) - \alpha|$) to obtain the narrowest band, or chosen to be zero as appropriate.

Sometimes it is convenient to express stability of S in terms of the diagonal elements of $Q_r(s)$ and the structure of $Q_r(s)$, example via Gershgorin's theorem, theorem 5B. Thus,

Theorem 5.8

Let $q_{rjj}(s) + \beta$ (resp. $\hat{q}_{rjj}(s) + \alpha$) map D into $\Gamma_{rj} + \beta$ (resp. $\hat{\Gamma}_{rj} + \alpha$). Then sufficient conditions for S to be closed loop stable are :

(1) $\Gamma_r + \beta$ (resp. $\hat{\Gamma}_r + \alpha$) satisfies NC, $\quad \forall j$

(2) $|1+\beta+q_{rjj}(s)| > \sum_{\substack{i=1 \\ i \neq j}}^{m} |q_{rij}(s)|$ (resp. $|1+ +\hat{q}_{rjj}(s)| > \sum_{i \neq j} |\hat{q}_{rij}(s)|$), $\forall j, \forall s \in D$ ...(5.31)

(3) $|1+\beta+q_{rjj}(s)| > \| G_e(s)K_r(s)-\beta I\| \inf\kappa(P(s)) + \sum_{i\neq j} |q_{rij}(s)|$, $\forall j$, $\forall s \in D$

...(5.32)

$[$resp. $|1+\alpha+\hat{q}_{rjj}(s)| > \| \hat{K}_r(s)\hat{G}_e(s)-\alpha I\| \inf\kappa(P(s)) + \sum_{i\neq j}|\hat{q}_{rij}(s)|\,]$ $\forall j$, $\forall s \in D$

Proof : The graphical interpretation of theorem 5.8 is shown in fig. 5.8. Condition 2 requires that the S_r be diagonally dominant, hence by Gershgorin's theorem, the characteristic loci, $\rho_{rj}(s)$ of S_r are trapped in the Gershgorin band. Condition 3 is an application of Gershgorin's theorem and theorem 5.1, where the loci $\rho_j(s)$ of S are trapped in the band centred on $\rho_{rj}(s)$. Hence it is easily seen that if all three conditions are satisfied, the band defined by condition (3) cannot overlap the critical point, meaning no loci of S can enclose the critical point, thus S is closed loop stable.

Theorem 5.8 expresses stability of S in terms of that of S_r via Gershgorin's theorem and theorem 5.1. The next theorem gives stability of S in terms of theorem 5.7 and diagonal dominance of S.

Theorem 5.9

Let Γ_{rj} be as defined in theorem 5.8. Sufficient conditions for closed loop stability of S are :

(1) Γ_{rj} (resp. $\hat{\Gamma}_{rj}$) satisfies NC, $\forall j$

(2) $|1+\beta+q_{rjj}(s)| > |q_{ejj}(s)-\beta|$ $[$resp. $|1+\alpha+\hat{q}_{rjj}(s)| > |\hat{q}_{ejj}(s)-\beta|\,]$ $\forall j$, $\forall s \in D$

(3) $|1+\beta+q_{rjj}(s)| > |q_{ejj}(s)-\beta| + \sum_{\substack{i=1\\ i\neq j}}^{m} |q_{ij}(s)|$ $[$resp. $|1+\alpha+\hat{q}_{rjj}(s)| > |\hat{q}_{ejj}(s)-\alpha|$

$+ \sum_{i\neq j} |\hat{q}_{ij}(s)|\,]$, $\forall j$, $\forall s \in D$ ...(5.32)

where $Q_e(s) = Q(s) - Q_r(s) = G_e(s)\,K_r(s)$, $\hat{Q}_e(s) = \hat{Q}(s) - \hat{Q}_r(s)$

Proof : The graphical interpretation is shown in fig. 5.9. Conditions (1) and (2) ensure that the Nyquist loci Γ_j of S satisfies the Nyquist criterion.

Condition (3) ensures that S be diagonally dominant, hence by theorem 5B, S is closed loop stable.

Theorem 5.9 does not require the stability of S_r to determine the stability of S. It works on the diagonal elements of $Q_r(s)$ and computes the off-diagonal elements of Q(s). This is particularly attractive since $q_{rjj}(s)$ is used in the design. To determine the stability of S_r, would be an independent piece of work, example by theorem 5B, S_r need be independently diagonal dominant.

The theorems above constrain the characteristic loci in some circular discs. Another useful result can be obtained by employing a theorem of Hirsch[31] where, instead of circular disc, the eigenvalues are constrained in a rectangular region.

Theorem 5.10

Let every loci defined by Cartesian co-ordinates, in the complex plane, $\beta + (\mu_{ri}(\omega), \nu_{rj}(\omega))$ $[\text{resp.}\ \alpha + (\hat{\lambda}_{ri}(\omega), \hat{\sigma}_{rj}(\omega))]$ map D into $\beta + \Gamma_{rij}$ (resp. $\alpha + \hat{\Gamma}_{rij}$) where $\mu_{ri}(\omega)$, $\nu_{rj}(\omega)$ (resp. $\hat{\lambda}_{ri}(\omega)$, $\hat{\sigma}_{rj}(\omega)$) are the real eigenvalues of the Hermitian matrices $B_r(s) = \frac{1}{2}(Q_r(s) + Q_r^H(s))$, $C_r(s) = -j\frac{1}{2}(Q_r(s) - Q_r^H(s))$ $[\text{resp.}\ \hat{B}_r(s) = \frac{1}{2}(\hat{Q}_r(s) + \hat{Q}_r^H(s)),\ \hat{C}_r(s) = -j\frac{1}{2}(Q_r(s) - Q_r^H(s))]$ and β (resp. α) is a frequency dependent complex number. Let $B_e(s) = B(s) - B_r(s) = \frac{1}{2}(G_e(s)$ $K_r(s) + K_r^H(s)\ G_e(s))$ (resp. $\hat{B}_e(s) = \hat{B}(s) - \hat{B}_r(s)$), $C_e(s) = C(s) - C_r(s)$ (resp. $\hat{C}_e(s) = \hat{C}(s) - \hat{C}_r(s)$) and $t_{rij}^2(\omega) = \mu_{ri}^2(\omega) + \nu_{rj}^2(\omega)$ (resp. $\hat{\ell}_{rij}^2(\omega) = \hat{\lambda}_{ri}^2(\omega)$ $+ \hat{\sigma}_{rj}^2(\omega)$). Then sufficient conditions for closed loop stability of S are :

(1) $\Gamma_{rij} + \beta$ (resp. $\hat{\Gamma}_{rij} + \alpha$) individually satisfies NC, $\forall i$, $\forall j$

(2) $|1+\beta+t_{rij}(\omega)| > \|B_e(s) - \beta I\| \inf\kappa(P(s))$, $\forall i,j$, $\forall s \in D$ ... (5.33)

$[\text{resp.}\ |1+\alpha+\hat{\ell}_{rij}(\omega)| > \|\hat{B}_e(s) - \alpha I\| \inf\kappa(P(s))]\ \forall i,j\ \ \forall s \in D$

(3) $|1+\beta+t_{rij}(\omega)| > \|C_e(s) - \beta I\| \inf\kappa(W(s))$, $\forall ij$, $\forall s \in D$ ... (5.34)

$[\text{resp.}\ |1+\alpha+\hat{\ell}_{rij}(\omega)| > \|\hat{C}_e(s) - \alpha I\| \inf\kappa(W(s))]$. $\forall i,j$, $\forall s \in D$

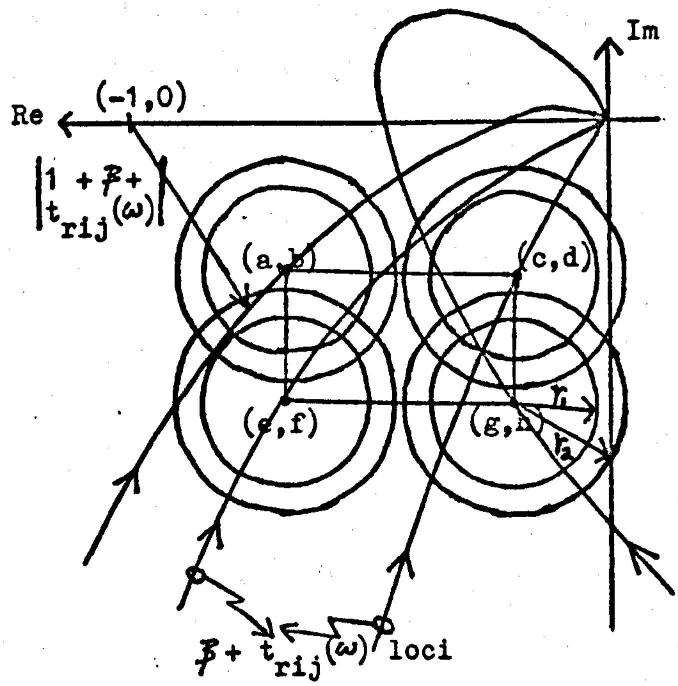
where P(s) and W(s) diagonalise $B_e(s)$ and $C_e(s)$ respectively. (If they quasi-diagonalise $B_e(s)$ and $C_e(s)$, then $P(s)\ H_p\ \hat{P}(s)$, $W(s)\ H_w\ \hat{W}(s)$ must be added in the respective $\|\ \ \|$ expressions).

Proof : The graphical interpretation is shown in fig. 5.10 for m = 2. It is well known from Hirsch theorem that $\max_i \mu_{ri}(\omega) \leqslant Re(\rho_r(s)) \leqslant \min_i \mu_{ri}(\omega)$ and $\max_j \nu_{rj}(\omega) \leqslant I_m(\rho_r(s) \leqslant \min_j \nu_{rj}(\omega)$. In other words all characteristic loci $\rho_r(s)$ of $Q_r(s)$ lie in the rectangular region defined by Cartesian co-ordinates $(\min \mu_{ri}(\omega), \min \nu_{rj}(\omega))$, $(\max \mu_{ri}(\omega), \min \nu_{rj}(\omega))$, $(\max \mu_{ri}(\omega), \max \nu_{rj}(\omega))$ and $(\min \mu_{ri}(\omega), \max \nu_{rj}(\omega))$. Now at every co-ordinate point $(\mu_{ri}(\omega), \nu_{rj}(\omega))$, for i, j = 1, ...m, two concentric circles of radii given by the r.h.s. of eqns. (5.33) and (5.34) are drawn. From theorem 5.1, the union of circles contains all $\mu_i(\omega)$ of B(s) and all $\nu_j(\omega)$ of C(s). Hence the rectangular region defined by the grid points $(\min \mu_i(\omega), \min \nu_j(\omega))$, $(\max \mu_i(\omega), \min \nu_j(\omega))$, $(\max \mu_i(\omega), \max \nu_j(\omega))$ and $(\min \mu_i(\omega), \max \nu_j(\omega))$ lie within the circles. Thus S is closed loop stable if expres. (5.33) and (5.34) are satisfied, i.e. no circles, hence no rectangles will overlap the critical point. Q.E.D.

For stability investigation, only the circle of greater radius in exprs. (5.33) and (5.34) need be drawn and a total of m^2 circular bands are required. The above theorem assumes open loop stability of S and S_r, for simplicity. If they are open-loop unstable, the similar form of theorem 5.6 can be used.

5.2 Stability of reduced systems using M-matrices

M-matrices are quite useful in stability studies of dynamical systems (Siljak, Cook, Araki etc.). Recently Araki et al[10] derived a stability theorem, which generalises Rosenbrock's diagonal dominance theorem, for multivariable systems using the theory of M-matrices. [A is called an M-matrix (semi-M-matrix) if its off-diagonal elements a_{ij}, $i \neq j$, are non-positive and its principal minors are positive (or non-negative)] .



$(a,b) = (\max \mu_{ri}(\omega), \max \nu_{rj}(\omega))$

$(c,d) = (\min \mu_{ri}(\omega), \max_{rj} \nu(\omega))$

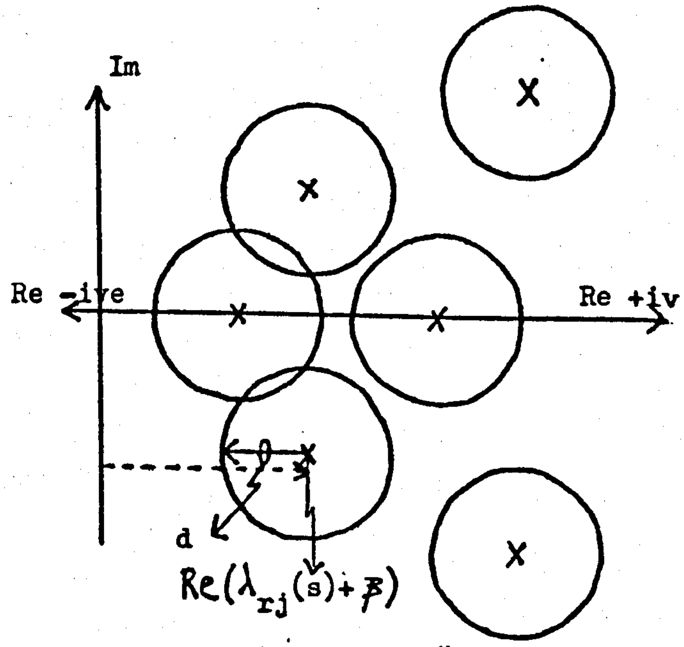
$(e,f) = (\max \mu_{ri}(\omega), \min \nu_{rj}(\omega))$

$(g,h) = (\min \mu_{ri}(\omega), \min \nu_{rj}(\omega))$

$r_1 = \|C_e(s) - \beta I\| \inf \mathcal{H}(W(s))$

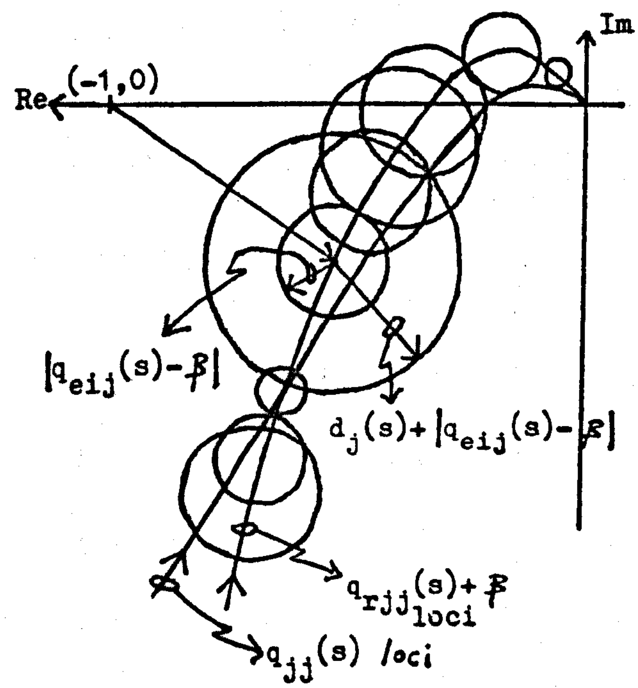
$r_2 = \|B_e(s) - \beta I\| \inf \mathcal{K}(P(s))$

Fig. 5.10 Illustrating stability for S for theorem 5.10



$d = \|C_e(s) - \beta I\| \inf \mathcal{H}(W(s))$

Fig. 5.11 Illustrating stability for S (circles must avoid the Im axis), condition (iv), theorem 5.12, also theorem 5.19



$d_j(s) \geqslant \sum_{i \neq j}^{m} |h_{rij}(s)|\, x_j/x_i$

Fig. 5.12 Illustrating stability for S; theorem 5.13

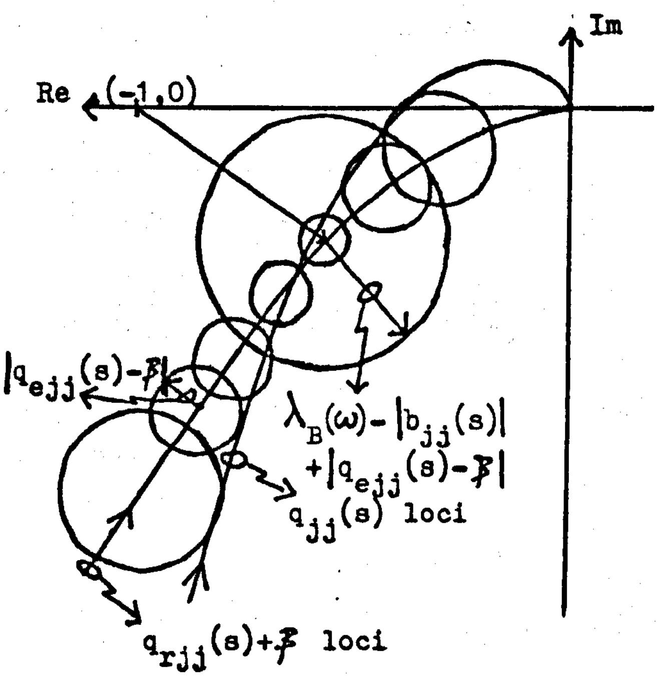


Fig. 5.13 Illustrating stability for S; theorem 5.14

(for all cases, circular bands must avoid critical point for stability)

Just as Gershgorin's theorem links theorem 5B (diagonal dominance) to theorem 5A (characteristic loci), the M-matrix stability theorem is connected to theorem 5A by a theorem of Ky Fan.

Thus following Araki[10], a unity feedback open loop stable system is closed loop stable if,

Theorem 5C[10]

(1) The Nyquist (resp. inverse Nyquist) diagram of $q_{jj}(s)$ (resp. $\hat{q}_{jj}(s)$) does not encircle the critical point.

(2) $C(s) = A(s) - B(s)$ (resp. $\hat{C}(s) = \hat{A}(s) - \hat{B}(s)$) is an M-matrix, $s \in \Omega : \{ \infty < \omega < \infty\}$, where $b_{ij}(s) = |q_{ij}(s)|_{i\neq j}$, $b_{ii}(s) = 0$, $a_{ii}(s) = |1+q_{ii}(s)|$, $a_{ij}(s) = 0$, $(i\neq j)$ (resp. for the elements of $\hat{A}(s)$ and $\hat{B}(s)$.

To determine that $C(s)$, ($c_{jj}(s) = |1+q_{jj}(s)|$, $c_{ij}(s) = -|q_{ij}(s)|_{i\neq j}$) is an M-matrix it is necessary and sufficient that all $\det(L_j(s)) > 0$, $j = 1,\ldots m$ where $L_j(s)$ is a leading principal sub-matrix of $C(s)$, or, that all eigenvalues of $C(s)$ have positive real parts.

Hence when designing S based on S_r, the stability of S must be assured based on the stability of S_r. Using M-matrices, it follows that

Theorem 5.11

Let $q_{rjj}(s) + \beta$ (resp. $\hat{q}_{rjj}(s) + \alpha$) map D into $\Gamma_{rj} + \beta$ (resp. $\hat{\Gamma}_{rj} + \alpha$). Sufficient conditions for closed loop stability of S are :

(1) $\Gamma_{rj} + \beta$ (resp. $\hat{\Gamma}_{rj} + \alpha$) individually satisfies NC, $\forall j$

(2) $C_r(s)$ (resp. $\hat{C}_r(s)$) is an M-matrix.

(3) $|1 + \beta + q_{rjj}(s)| > |q_{ejj}(s) - \beta| \quad \forall j, \ \forall s \in D$ ...(5.35)

$\left[\text{resp.} \left| 1 + \alpha + \hat{q}_{rjj}(s)\right| > |\hat{q}_{ejj}(s) - \beta| \right]$

(4) $|1 + q_{jj}(s)| \geqslant |1 + q_{rjj}(s)|$, $|q_{ij}(s)|_{i\neq j} \leqslant |q_{rij}(s)| \quad \forall ij, \ \forall s \in D$ ...(5.36)

$\left[\text{resp.} \left| 1 + \hat{q}_{jj}(s)\right| \geqslant |1 + \hat{q}_{rjj}(s)|, \ |\hat{q}_{ij}(s)|_{i\neq j} \leqslant |\hat{q}_{rij}(s)| \right]$

Proof : Conditions (1) and (2) require that S_r be stable in the sense of M-matrix requirements. Condition (3) ensures that the loci Γ_j of $q_{jj}(s)$ of S satisfies the Nyquist criterion by theorem 5.7. Condition (4) ensures that the matrix C(s) of S is an M-matrix. The latter makes use of a theorem by Ostrowskii[11]. It states that if A is an M-matrix and if some elements are perturbed such that $a_{ii} \leqslant |b_{ii}|$ and $|a_{ij}| \geqslant |b_{ij}|$, $\forall i, i \neq j$, then the value of any principal minor does not decrease. Thus it follows that if some elements of an M-matrix, A, are increased without changing their signs, the new matrix, B, is also an M-matrix, i.e. $b_{ii} \geqslant a_{ii}$, $\forall i$, and $0 \geqslant b_{ij} \geqslant a_{ij}$, $i \neq j$. Hence C(s) is an M-matrix in condition (4), thus S is closed loop stable.

Q.E.D.

A criterion in estimating the error between S and S_r in the design can be chosen by comparing the $C_r(s)$, $C(s)$ (resp. $\hat{C}_r(s)$, $\hat{C}(s)$) matrices. From a result of Fan[18], for two M-matrices C(s) and $C_r(s)$ (resp. $\hat{C}(s)$ and $\hat{C}_r(s)$) with $c_{rij}(s) \leqslant c_{ij}(s)$ (resp. $\hat{c}_{rij}(s) \leqslant \hat{c}_{ij}(s)$),

$$\det L_r(\theta \cup \gamma)(\det L(\theta)/\det L(\theta \cup \gamma))\det L_r(\theta) \leqslant 1 \qquad \ldots(5.37)$$

$$\left[\text{resp. } \det \hat{L}_r(\theta \cup \gamma)(\det \hat{L}(\theta)/\det \hat{L}(\theta \cup \gamma))\det \hat{L}_r(\theta) \leqslant 1\right], \text{ and,}$$

where $\theta, \gamma \in \{1, 2, \ldots m\}$. For the null set, ϕ, $\det L(\phi) = \det L_r(\phi) = 1$, and, $\det L(\theta \cup \gamma)$ is the principal minor formed by the union of sets $\theta \cup \gamma$. Thus for $\theta = \phi$ and $\gamma = (1,2,\ldots,)$, exp. (5.37) reduces to $\det C_r(s)/\det C(s) \leqslant 1$. (Equality is observed if there is no error involved in the reduction).

As a special case, Rosenbrock's diagonal dominance criterion for stability can be deduced. Thus if S_r is stable in the sense that $Q_r(s)$ is diagonal dominant, then a sufficient condition for S to be stable is that Q(s) is also diagonal dominant. This is seen as follows. If $|1 + q_{rjj}(s)| > |q_{rjj}(s)|$ and $|q_{rjj}(s)| > \sum_{i \neq j} |q_{rij}(s)|$, it is easily seen that $\det L_{rj}(s) > 0$, $\forall j$, by Gershgorin's theorem, hence $C_r(s)$ is an M-matrix. But condition (4) of theorem 5.11 gives $|1+q_{jj}(s)| > \sum_{i \neq j} |q_{ij}(s)|$, which gives C(s) as an

M-matrix. Now if $|1+q_{jj}(s)| > |q_{jj}(s)| > \sum_{i \neq j} |q_{ij}(s)|$, it is easily seen that S is stable in the sense of diagonal dominance requirements.

Theorem 5.11 expresses the results in 'analytical form'. A graphical interpretation, suitable to the designer, can be obtained as follows,

Theorem 5.12

Let Γ_{rj}, β be as defined as in theorem 5.11 and let $\lambda_{rj}(s)$ (resp. $\hat{\lambda}_{rj}(s)$) be an eigenvalue of $C_r(s)$ (resp. $\hat{C}_r(s)$). Then sufficient conditions for S to be closed loop stable are :

(1) $\Gamma_{rj} + \beta$ (resp. $\hat{\Gamma}_{rj} + \alpha$) individually satisfies NC, $\forall j$

(2) $C_r(s)$ (resp. $\hat{C}_r(s)$) is an M-matrix

(3) $| 1 + \beta + q_{rjj}(s)| > |q_{ejj}(s) - \beta| \quad \forall j, \ \forall s \in D$ ...(5.38)

$\left[\text{resp.} | 1 + \alpha + \hat{q}_{rjj}(s)| > |\hat{q}_{ejj}(s) - \alpha| \right]$

(4) $\mathrm{Re}(\lambda_{rj}(s)+\beta) > \|C_e(s) - \beta I\| \inf \kappa(W(s)) \quad \forall j, \quad \forall s \in D$ ...(5.39)

$\left[\text{resp. } \mathrm{Re}(\hat{\lambda}_{rj}(s) + \alpha) > \| \hat{C}_e(s) - \alpha I\| \inf \kappa(W(s)) \right]$

where $C_e(s) = C(s) - C_r(s)$ (resp. $\hat{C}_e(s) = \hat{C}(s) - \hat{C}_r(s)$)

Proof : Conditions (1), (2) and (3) are as explained in theorem 5.11. Condition (4) derived from theorem 5.1, is based on the fact that all $\mathrm{Re}(\lambda_j(s))$ must be positive. Since all eigenvalues of an M-matrix have positive real parts, it follows that a matrix with positive real part eigenvalues, and nonpositive off-diagonal elements is an M-matrix. Thus if $\lambda_j(s)$ lie in the right half plane, and, since by definition, all off-diagonal elements of C(s) are non-positive, it follows that C(s) is an M-matrix.

Stability of S follows from theorem 5C.

Theorem 5.12 is illustrated in fig.5.11. As a special case; when $Q_r(s)$ and Q(s) are diagonal dominant, from Gershgorin's theorem all $\lambda_{rj}(s)$ and $\lambda_j(s)$ lie in the r.h.s. plane. Thus it is possible to find a β (scaling factor) such that condition (4) of theorem 5.12 is satisfied. The advantage of the above theorem is that circular discs can be displayed graphically, and be tuned by β, so as to enable the designer to have more freedom and ease of determining stability margins for S when design is made on S_r. The M-matrix stability requirement can also be interpreted by a theorem of Ky Fan[13]. In this case, like the Inverse Nyquist Array method, circular bands are drawn on the diagonal elements of $Q_r(s)$.

Theorem 5.13

Let Γ_{rj}, β be as defined as in theorem 5.11. Sufficient conditions for S to be closed loop stable are :

(1) $\Gamma_{rj} + \beta$ (resp. $\hat{\Gamma}_{rj} + \alpha$) individually satisfies NC, $\forall j$

(2) $|1 + \beta + q_{rjj}(s)| > |q_{eij}(s) - \beta| \quad \forall j, \ \forall s \in D$ ...(5.40)

$[\text{resp. } |1 + \alpha + \hat{q}_{rjj}(s)| > |\hat{q}_{ejj}(s) - \alpha|]$

(3) $|1 + \beta + q_{rjj}(s)| > d_j(s) + |q_{ejj}(s) - \beta| \quad \forall j, \ \forall s \in D$ ...(5.41)

$[\text{resp.} |1 + \alpha + \hat{q}_{rjj}(s)| > \hat{d}_j(s) + |\hat{q}_{ejj}(s) - \alpha|]$

(4) $d_j(s) \geq \sum_{\substack{i=1 \\ i \neq j}}^{m} |q_{rij}(s)| \, x_j/x_i \ ; \ d_j(s) \geq \sum_{\substack{i=1 \\ i \neq j}}^{m} |q_{ij}(s)| \, x_j/x_i \quad \forall j, \ \forall s \in D$ ...(5.42)

$[\text{resp. } \hat{d}_j(s) \geq \sum_{\substack{i=1 \\ i \neq j}}^{m} |\hat{q}_{rij}(s)| \, y_j/y_i ; \ \hat{d}_j(s) \geq \sum_{\substack{i=1 \\ i \neq j}}^{m} |\hat{q}_{ij}(s)| \, y_j/y_i]$

Proof : Conditions (1) and (2) ensure that $q_{rjj}(s)$ and $q_{jj}(s)$ satisfy the Nyquist criterion. Conditions (3) and (4) satisfy the remaining conditions of a theorem of Ky Fan which ensures that the characteristic loci of S satisfy NC provided those of S_r also satisfy NC. The theorem states that if $A \triangleq a_{ij}$ is indecomposable and non-negative, $a_{ij} > 0$ and $a_{ii} = 0$, and if the matrix $B = b_{ij}$ with $|b_{ij}| = a_{ij}$, $i \neq j$, has every eigenvalue inside the circle set $|\lambda - b_{ii}| \leq d_i$, $\forall i$, then there exists n positive numbers $x_1, \ldots x_n$ such that $d_i \geqslant \sum_{i\neq j} a_{ij}\, x_j/x_i$, $\forall i$. The graphical interpretation of theorem 5.13 is shown in fig. 5.12. If conditions (3) and (4) are satisfied, no band, hence no loci, can overlap the critical point, thus S is closed loop stable.

Fan's theorem can also be used to prove the stability result due to Araki et al, i.e. theorem 5C. From $d_i = \sum_{j\neq i} a_{ij}\, x_j/x_i$, $\forall i$, it follows that $d_i x_i - \sum_{j\neq i} a_{ij}\, x_j \geqslant 0$, $\forall i$, which implies $(d_i\, I - A)x \geqslant 0$ where x is a positive vector and $A \triangleq a_{ij}$, $i \neq j$. Hence $d_i\, I - A$ is a semi M-matrix (d_i I-A has nonpositive off-diagonal elements and $(d_i I-A)x \geqslant 0$ for $x > 0$). Recall the matrix C(s) of theorem 5C. Comparing C(s) with $d_i I-A$ it is seen that $d_i = |1+q_{ii}(s)|$, $a_{ii} = 0$, $a_{ij} = |q_{ij}(s)|$, $i \neq j$. Choose $b_{ii} = q_{ii}(s)$. Thus it follows that C(s) is a semi M-matrix if $|\rho_k(s) - q_{ii}(s)| \leqslant |1+q_{ii}(s)|$ where $\rho_k(s)$ is an eigenvalue of Q(s). Hence C(s) is a M-matrix if no $\rho_k(s)$ loci enclose the critical point, thus system is closed loop stable.

A further useful stability theorem for S and S_r can be derived from a result of Kotelyanskii[26,31].

Theorem 5.14

Let Γ_{rj}, β be as defined in theorem 5.11. Sufficient conditions for closed loop stability of S are;

(1) $\Gamma_{rj} + \beta$ (resp. $\hat{\Gamma}_{rj} + \alpha$) individually satisfy NC, $\forall j$

(2) $|1 + \beta + q_{rjj}(s)| > |q_{ejj}(s) - \beta| \quad \forall j, \quad \forall s \in D$ ...(5.43)

$[\text{resp.} |1 + \alpha + \hat{q}_{rjj}(s)| > |\hat{q}_{ejj}(s) - \alpha|]$

(3) $|1 + \beta + q_{rjj}(s)| > \lambda_B(\omega) - |b_{jj}(s)| + |q_{ejj}(s)-\beta| \ \forall j, \ \forall s \in D$ ...(5.44)

$[\text{resp.} |1 + \alpha + \hat{q}_{rjj}(s)| > \hat{\lambda}_B(\omega) - |\hat{b}_{jj}(s)| + |\hat{q}_{ejj}(s)-\alpha|]$

(4) $q_{rij}(s) \leqslant |b_{ij}(s)| \ ; \quad q_{ij}(s) \leqslant |b_{ij}(s)| \ \forall i \neq j, \ \forall s \in D$ ...(5.45)

$[\text{resp.}\ \hat{q}_{rij}(s) \leqslant |\hat{b}_{ij}(s)| \ ; \quad \hat{q}_{ij}(s) \leqslant |\hat{b}_{ij}(s)|]$

where $B \triangleq |b_{ij}(s)|$ is a non-negative matrix and $\lambda_B(\omega)$ is the spectral radius or Perron-Frobenius characteristic root of B, $\lambda_B(\omega) = |\lambda_{max}(B)|$.

Proof : Kotelyanskii theorem states that if $|q_{rij}(s)| < |b_{ij}(s)|$, $i \neq j$, then all characteristic loci of $Q_r(s)$ lie in the union of discs $|\rho_{rj}(s) - q_{rjj}(s)| \leq \lambda_B(\omega) - |b_{jj}(s)|$. Theorem 5.14 is shown graphically in fig. 5.13 which is similar to fig.5.12 of theorem 5.13. Conditions (1) and (2) ensure that $q_{rjj}(s)$ and $q_{jj}(s)$ satisfy the Nyquist criterion. The remaining conditions (3) and (4) result from the application of Kotelyanskii's theorem to both S_r and S. If all conditions are satisfied, it is seen that no loci of S_r, hence that of S, can enclose the critical point, thus ensuring stability.

Theorems 5.13 and 5.14 expressed stability of S in terms of that of S_r. This is convenient, as S_r is used to design S, but at the expense of lesser sharpness in the stability conditions for S. As the conditions are only sufficient, in practice, separate tests can be made on S to invoke stronger stability conditions. However, the latter may be too costly and involved, and do not provide a correlation between the stabilities of S and S_r. Whereas in the former case, scaling factors can be suitably chosen to reduce the width of the bands to provide an economical and simple controller that guarantees the stabilities of S_r and S simultaneously.

The stability theorems from theorem 5.11 to 5.14 are based on encirclements by diagonal elements $q_{rjj}(s)$ (resp. $\hat{q}_{rjj}(s)$) of $Q_r(s)$ (resp. $\hat{Q}_r(s)$). This is advantageous if design is made on the diagonal elements instead of on the characteristic loci. For open-loop unstable systems, theorems 5.11 to 5.14 can be modified in the following way.

Theorem 5.15

Suppose S_r has z right half plane zeroes in its o.l.c.p. Let $\Delta_{rj} + \beta$ (resp. $\hat{\Delta}_{rj} + \alpha$), $\Gamma_{rj} + \beta$(resp. $\hat{\Gamma}_{rj} + \alpha$) be the images of D, under the mapping $r_{rjj}(s) + \beta$(resp. $\hat{r}_{rjj}(s) + \alpha$), $q_{rjj}(s) + \beta$(resp. $\hat{q}_{rjj}(s) + \alpha$), and encircle the origin n_{ci} (resp. $\hat{n}_{ci}$), n_{oi} (resp. $\hat{n}_{oi}$) times clockwise (resp. counter clockwise). Then sufficient conditions for closed loop stability of S are :

(1) S_r satisfies NC in the sense $\sum_{i=1}^{m} n_{ci} - \sum_{i=1}^{m} n_{oi} = -z$ (resp. $\sum_{i=1}^{m} \hat{n}_{ci} - \sum_{i=1}^{m} \hat{n}_{oi} = -z$).

(2) S has z right half plane zeroes in its o.l.c.p.

(3) The remaining conditions of one of the theorems 5.11 to 5.14 be satisfied.

Proof : The proof follows parallel to that of theorem 5.6 (see fig. 5.6). Condition (1) is a direct consequence of the Hsu-Chen theorem and Gershgorin's theorem, theorems 5A and 5B. The M-matrix theorems of 5.11 and 5.12 are related to the characteristic loci encirclement theorem by Ky Fan's theorem, hence conditions (2) and (3) are as similarly proved as in theorem 5.6.

Since $\hat{r}_{rjj}(s) = 1 + \hat{q}_{rjj}(s)$, from condition (1), $\sum_{i=1}^{m} \hat{n}_{ci}$ can be interpreted as the number of encirclements of the '-1' point by all loci $\sum_{j=1}^{m} \hat{\Gamma}_{rj_m} + \alpha$. If the characteristic loci are considered, $\hat{\lambda}_{rj}(s) = 1 + \hat{\rho}_{rj}(s)$, then $\sum_{i=1} \hat{n}_{ci}$ in theorem 5.6 is interpreted as the total number of encirclements of the '-1' point by all the loci $\hat{\rho}_{rj}(s)$.

5.3 Stability of reduced systems using contraction mapping principle.

Freeman[19] has used the idea of contraction mapping in functional analysis to study the stability of linear multivariable systems. It is based on the fact that the system of fig. 5.1 is stable when inputs $u_i(t)$ which are integrable on finite intervals and bounded as $|u_i(t)| \leq M_i \exp(-\sigma_i t)$ for $\sigma_i > 0$ and $\forall i$ produce output $y_i(t)$, for all initial conditions with finite L_∞ norms, which are integrable on finite intervals and are bounded by $|y_i(t)| \leq N_i \exp(-\alpha_i t)$ for some $\alpha_i > \sigma > 0$ and $\forall i$. These assumptions assume that the Laplace transforms of the inputs exist in a domain $\Omega = \{s : \mathrm{Re}\ s > \sigma\}$, which is an open connected set, and the transforms are infinitely differentiable in Ω. Thus the input and output transforms of a stable system are holomorphic (regular, single-valued) functions in the domain Ω, hence the problem of finding a condition for stability reduces to deriving conditions under which the operation on input u(s), maps u(s) transforms which are on the space of holomorphic functions on Ω into y(s) transforms which are also in the space of holomorphic functions on Ω.

Hence in a linear metric space, Y, if

$$d\{T(y_1(s)),\ T(y_2(s))\} \leqslant M\ d\{y_1(s),\ y_2(s)\} \qquad \ldots(5.46)$$

where d{ } represents metrics in Y, $T : Y \to Y$ (T is an operator which transforms the elements of Y into the elements of the same space Y) and M is a constant such that $M < 1$, then a contraction is obtained and by the fixed point theorem there exists exactly one point $y(s) \in Y$ such that $y(s) = Ty(s)$, where in Banach space, T is a matrix norm and y(s) can be solved iteratively in the space of stable transforms.

The outputs of figs. 5.1(a) and 5.1(b) can be written as

$$y_r(s) = Q_r(s)\ u(s) - Q_r(s)\ y_r(s) + z_r(s) \qquad \ldots(5.47)$$

$$y(s) = Q(s)\ u(s) - Q(s)\ y(s) + z(s) \qquad \ldots(5.48)$$

where $z_r(s)$, z(s) are effects due to initial conditions. Adding A(s) y(s) to both sides of eqn. (5.48) and assuming (I + A(s)) has an inverse,

$$y(s) = (I+A(s))^{-1}\ Q(s)\ u(s) - W(s)\ y(s) + (I+A(s))^{-1}\ z(s) \qquad \ldots(5.49)$$

where $W(s) = (I+A(s))^{-1}\ (Q(s)-A(s))$ and $Q(s) = G(s)\ K_r(s)$. From eqn. (5.49), considering two points $y_1(s)$, $y_2(s)$ in Y,

$$\| y_1(s)-y_2(s) \| = \| -W(s)(y_1(s)-y_2(s)) \| < \| W(s) \|\ \| y_1(s)-y_2(s) \| \quad \ldots(5.50)$$

and identifying eqn. (5.50) with (5.46), a contraction is obtained if $\| W(s) \| < 1$. Thus

Theorem 5D[19]

A sufficient condition for S to be closed loop stable is

(1) $\| W(s) \| < 1$ $\qquad \ldots(5.51)$

If A(s) is chosen diagonal, $A(s) = \text{diag}\ \{q_{11}(s), \ldots q_{mm}(s)\}$ and

$\| \;\; \|_1$ or $\| \;\; \|_\infty$ are taken, then

$$\| W(s) \| = \max_i \sup_{s\in\partial\Omega} \sum_{\substack{j=1 \\ j\neq i}}^{m} |q_{ij}(s)/(1+q_{ii}(s))| < 1 \qquad \ldots(5.52)$$

which is a form of theorem 5B, the diagonal dominance theorem; shown in fig. 5.14.

Following theorem 5D, stability conditions for both S_r and S can be derived.

Theorem 5.16

Let S_r be closed loop stable in the sense of theorem 5D, i.e. its bounded input transforms produce bounded output transforms and are holomorphic in the domain Ω. A sufficient condition for S to be similarly stable is any one of the following :

(1) $\| W_r(s) \| < 1 - \| W(s) - W_r(s) \| < 1 \qquad \ldots(5.53)$

If $A_r(s)$ is chosen such that $A(s) = A_r(s)$, then

(2) $\| W_r(s) \| < 1 - \inf_A \| (I+A(s))^{-1} G_e(s) K_r(s) \| < 1 \qquad \ldots(5.54)$

If $\| A_r(s) \| = \| A(s) \| < 1$ and $\| I \| = 1$, then

(3) $\| W_r(s) \| < 1 - (\| G_e(s) K_r(s) \| / (1 - \| A_r(s) \|)) < 1 \qquad \ldots(5.55)$

If $A_r(s) = A(s)$, then

(4) $\| Q_r(s) - A_r(s) \| < \mathrm{glb}(I+A(s)) - \| G_e(s) K_r(s) \| < \mathrm{glb}(I+A(s)) \qquad \ldots(5.56)$

(5) $\| Q_r(s) \| < \| A_r(s) \| + \mathrm{glb}\, A(s) - \| G_e(s) K_r(s) \| < \| A_r(s) \| + \mathrm{glb}\, A(s) \qquad \ldots(5.57)$

If $A_r(s) = A(s) = 0$, $K_r(s) \neq 0$, then,

(6) $\| Q_r(s) \| < 1 - \| G_e(s) K_r(s) \| < 1 \qquad \ldots(5.58)$

(7) $\| G_r(s) \| < \| K_r(s) \|^{-1} - \| G_e(s) \| < \| K_r(s) \|^{-1} \qquad \ldots(5.59)$

where $W_r(s) = (I+A_r(s))^{-1} (Q_r(s)-A_r(s))$, $Q_r(s) = G_r(s) K_r(s)$, $G_e(s) = G(s) - G_r(s)$ and the greatest lower bound $\mathrm{glb}\, A = \| A^{-1} \|^{-1}$.

Proof : Since $W(s)$ can be written as $W_r(s) + W(s) - W_r(s)$, for stability, theorem 5D gives $\| W_r(s) + W(s) - W_r(s) \| < 1$. Since $\| W_r(s) + W(s) - W_r(s) \| \leqslant \| W_r(s) \| + \| W(s) - W_r(s) \| = E(s)$, hence provided $E(s) < 1$, it is sufficient to deduce that $\| W(s) \| < 1$. Since $\| \quad \| \geqslant 0$, then $1 - \| \quad \| \leqslant 1$, condition (1) follows, thus S is closed loop stable. The extreme inequality $\| W_r(s) \| < 1$ in condition (1) indicates that S_r is likewise stable. However, if $\| E(s) \| > 1$, the status of $\| W(s) \| < 1$ or $\| W(s) \| > 1$ is uncertain.

Condition (2) is a consequence of condition (1) when $A_r(s) = A(s)$, and $\inf_A \| \quad \|$ is taken w.r.t. all matrices $A(s)$ which make $\| \quad \|$ a minimum. The proof for stability of S, i.e. $\| W(s) \| < 1$ is exactly the same as in condition (1) as condition (2) is a special case of condition (1).

The remaining conditions (3) to (7) are special cases of condition (1). Hence stability of S i.e. $\| W(s) \| < 1$ is guaranteed if condition (1) is true, as the proof for stability of S in the remaining conditions make use of that for condition (1). The remaining conditions are now derived.

Since $\| W(s) - W_r(s) \| = \| (I + A(s))^{-1} G_e(s) K_r(s) \| \leqslant \| (I + A(s)^{-1} \| \| G_e(s) K_r(s) \|$ and that $\| (I + A(s)^{-1} \| \leqslant 1/(1 - \| A(s) \|)$ if $\| A(s) \| < 1$ and $\| I \| = 1$, condition (3) follows.

Condition (4) is obtained from condition (2) by isolating $\| (I+A(s))^{-1} \|$ as $\| W_r(s) \| \leqslant \| (I + A(s))^{-1} \| \; \| Q_r(s) \|$ and multiplying the inequality throughout by $1/\| (I + A(s))^{-1} \|$.

Condition (5) follows by isolating $\| Q_r(s) \|$ from $\| Q_r(s) - A(s) \|$. Condition (6) follows from condition (2) by putting $A_r(s) = 0$, thus $W_r(s) = Q_r(s)$, and condition (7) follows from condition (6) by isolating $\| G_r(s) \|$ from $\| Q_r(s) \|$.

Q.E.D.

Some of the bounds are stronger than others, particular strong ones being conditions (1), (2) and (6). However their main advantage is that they provide a range of flexible margins to the designer and are expressed in different forms suitable for computation. The stability bounds above are analytical in nature but can also be interpreted in graphical forms. For example, for closed loop stability of S_r and S it is seen that $\| W_r(s)\|$ in conditions (1), (2), (3) and (6) must lie in the inner circle, concentric with the unit circle, centred on the origin of the Nyquist plane. The design and graphical interpretation can be made more flexible, for example by choosing $W_r(s)$ such that S_r is diagonal dominant.

If $A_r(s) = A(s) = \text{diag}\{q_{r11}(s),\ldots q_{rmm}(s)\}$, then from the definition of $W_r(s)$, $W(s)$ and eqn. (5.52)

$$\| W(s) - W_r(s)\| = \max_i \sup_{s\in\partial\Omega} \sum_{j=1}^{m} |q_{eij}(s)|/|1 + q_{rij}(s)| \qquad \ldots(5.60)$$

Substituting eqn. (5.60) in exp. (5.53) (condition (1) of theorem 5.16) yields

$$\max_i \sup_{s\in\delta\Omega} \sum_{\substack{j=1\\ j\neq i}}^{m} |q_{rij}(s)| < \max_i \sup_{s\in\partial\Omega}\{|1+q_{rjj}(s)| - \sum_{j=1}^{m} |q_{eij}(s)|\}$$
$$< |1+q_{rij}(s)| \qquad \ldots(5.61)$$

The graphical interpretation of eqn. (5.61) can be evaluated as follows. Define $R_i(s) = |1+q_{rjj}(s)| - \sum_{j=1}^{m} |q_{eij}(s)|$. Consider the Nyquist loci of the diagonal elements $q_{rjj}(s)$ of $Q_r(s)$ be drawn, with appropriate Gershgorin circles drawn around them. Then the systems S_r and S are closed loop stable if the second band of circles of radius $R_i(s)$ centred on $q_{rjj}(s)$ contains the first band (the Gershgorin band) and does not overlap the (-1, 0) point. This is shown in fig. 5.14. Interpreted in this way, it is seen that condition (1) of theorem 5.16 is similar to the graphical interpretation of theorems 8, 9, 13 and 14. (figs. 8, 9, 12 and 13). For stability of S, the second band must not contain the critical point.

The other conditions of theorem 5.16, expr. (5.54) to (5.59), can be interpreted in a similar way. Other than obtaining a diagonal dominance condition in S_r, it has also been suggested that by choosing various canonical structures for A(s), example diagonal or upper or lower triangular, different

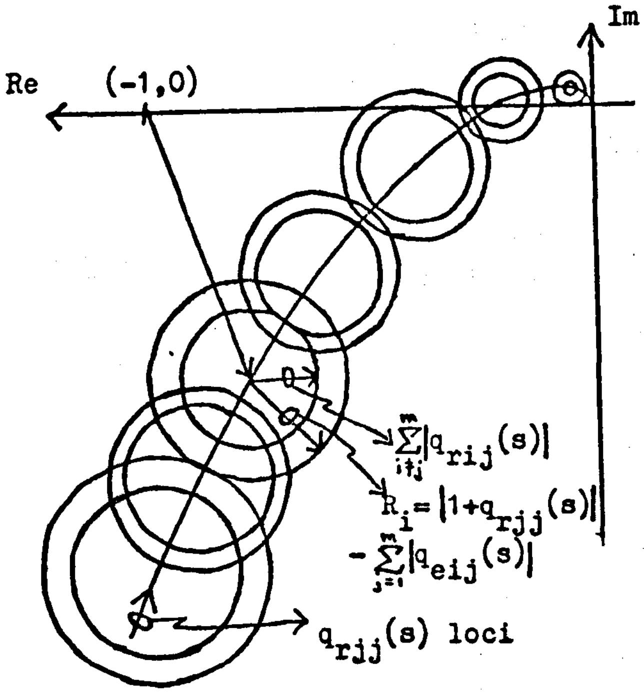


Fig. 5.14 Illustrating stability for S and S_r; special case of theorem 5.16, eqn. (5.61)

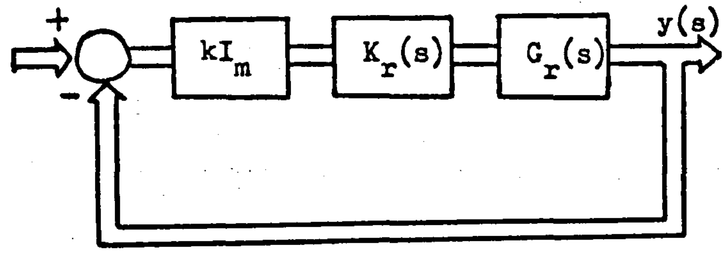


Fig. 5.15 Set-up for investigating behaviour of characteristic frequency loci(multivariable root loci.)

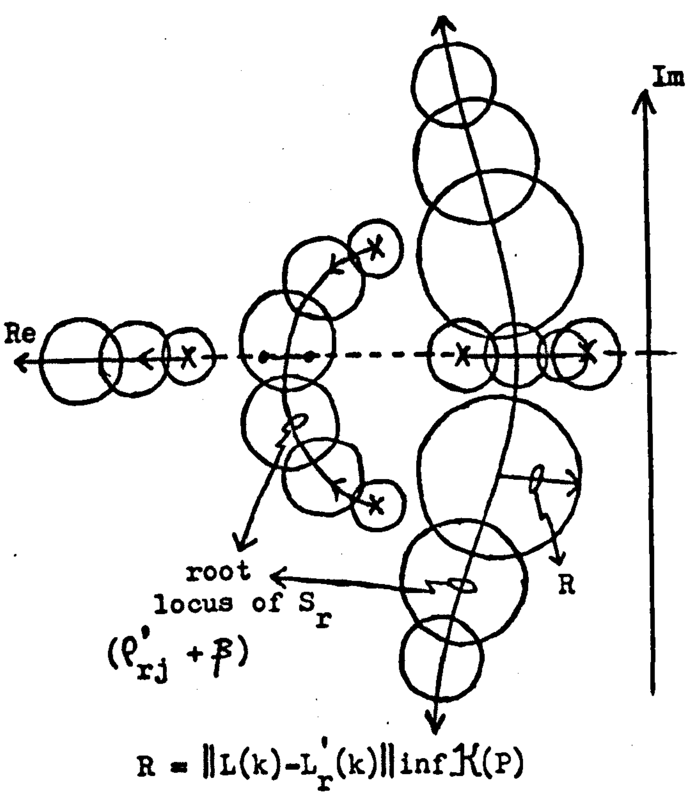


Fig. 5.16 Illustrating stability for S using the root locus method for single input-single output system (special case of multivariable root loci) (for stability, circular band must avoid Imaginary axis), theorem 5.21

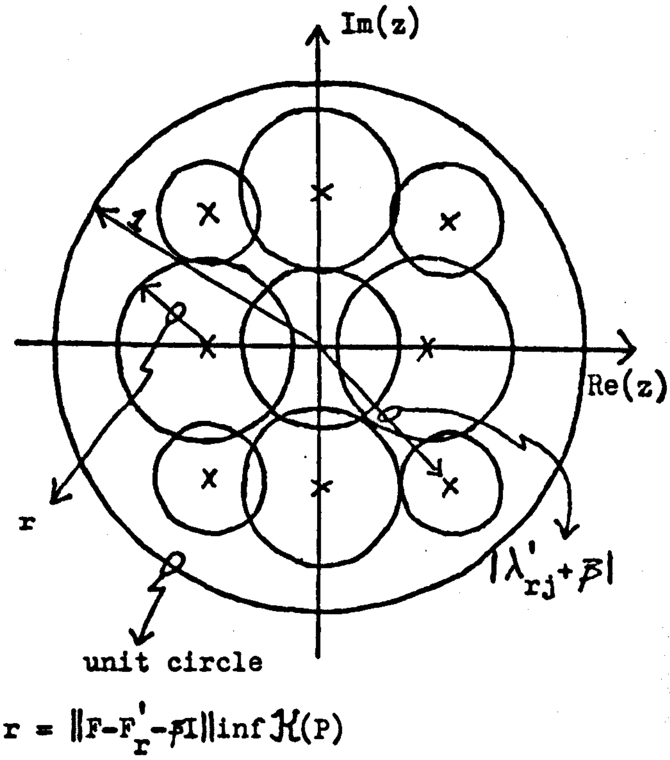


Fig. 5.17 Illustrating stability for S for discrete system; (the smaller circles must stay inside the unit circle for stability), theorem 5.29

versions of Nyquist loci with bands centred on them can be obtained. This will offer greater flexibility and choice in designs, but the stability conditions of theorem 5.16 will still hold for S_r and S.

It is noted that the contraction mapping principle does not derive the Nyquist criterion. The latter is automatically satisfied if the system is stable. Even if S_r and S or both are open-loop unstable, but using the contraction mapping principle, are shown to be both closed loop stable, then the Nyquist criterion is automatically satisfied by Chen-Hsu or Rosenbrock's stability theorem, theorems 5A and 5B.

Stability of sub-optimal controllers by contraction mapping principle.

Consider the optimal linear regulator problem for systems S and S_r, with performance indices

$$J = \langle x, Qx \rangle_{H_n} + \langle u, Ru \rangle_{H_m} \qquad \ldots(5.62)$$

$$J_M = \langle x_r, Q_M x_r \rangle_{H_r} + \langle u, Ru \rangle_{H_m} \qquad \ldots(5.63)$$

where $x \in H_n$, $x_r \in H_r$, $u \in H_m$ and H_n, H_r, H_m are (nxn), (rxr) and (mxm) Hilbert spaces with $Q \geqslant 0$, $Q_M \geqslant 0$ and $R > 0$. It is well known that for a controllable pair S_r (A_r, B_r), the control law $u = -K_r X_r$ where

$$K_r = R^{-1} B_r^t P \qquad \ldots(5.64)$$

and P is the solution of

$$A_r^t P + PA_r - PB_r R^{-1} B_r^t P + Q_M = 0 \qquad \ldots(5.65)$$

minimizes eqn. (5.63), $J_M^* = \langle x_r, Px_r \rangle$ and when implemented, the closed loop system S_r $(A_r - B_r K_r, 0)$ is stable i.s.L. (in the sense of Lyapunov). Now consider controlling S(A,B) with $u_s = -K_r x_r$ i.e. $u_s = -K_r Z\dot{x}$ where $X_r = Zx$ and Z is the aggregation matrix of S_r and S. It can be shown that the use of the sub-optimal controller, u_s, in S results in the value of the criterion function $J = \langle x, Tx \rangle$, where T satisfies[23],

$$(A-BK_rZ)^tT + T(A-BK_rZ) + Z^tK_r^t RK_rZ + Q = 0 \qquad \ldots(5.66)$$

provided $A-BK_rZ$ is a stability matrix. This is not always the case[23]. Sufficient conditions are now derived to show when $A-BK_rZ$ is stable.

For the systems described by the state space equations $S(A,B)$, $S_r(A_r,B_r)$, with $u_s = -K_rZX$, $U = -K_rx_r$, the equivalent form of eqns. (5.47) and (5.48) are

$$A_r(s)\ x_r(s) + x_r(s) = A_r(s)\ x_r(s) - \Phi_r(s)\ B_r\ K_r\ x_r + \Phi(s)\ x_r(0) \qquad \ldots(5.67)$$

$$A(s)\ x(s) + x(s) = A(s)\ x(s) - \Phi(s)\ B\ K_r\ Z\ x(s) + \Phi(s)\ x(0) \qquad \ldots(5.68)$$

where $\Phi_r(s) = (sI-A_r)^{-1}$, $\Phi(s) = (sI-A)^{-1}$, $x_r(0)$, $x(0)$ are initial conditions and the terms $A_r(s)\ x_r(s)$, $A(s)\ x(s)$ are as before added to both sides of the equations to invoke greater flexibility in the stability margins. Rewriting eqns. (5.67) and (5.68) in the form $y(s) = Wy(s)$, where W is a mapping from the Banach space Y of bounded holomorphic functions defined in the interior Ω of a semicircle of radius R and boundary $\partial\Omega$ in the right half complex plane into itself, one obtains,

$$W_r(s) = (I+A_r(s))^{-1}\ (A_r(s) - \Phi_r(s)B_rK_r) \qquad \ldots(5.69)$$

$$W(s) = (I+A(s))^{-1}\ (A(s) - \Phi(s)BK_rZ) \qquad \ldots(5.70)$$

where $W_r(s)$ and $W(s)$ are $(r\mathrm{x}r)$ and $(n\mathrm{x}n)$ matrices respectively. Consider augmenting $W_r(s)$ by block matrices such that

$$W_r'(s) = \left[\begin{array}{c:c} W_r(s) & X_{12}(s) \\ \hdashline X_{21}(s) & X_{22}(s) \end{array}\right] \qquad \ldots(5.71)$$

such that $W_r'(s)$ is $(n\mathrm{x}n)$ and $X_{12}(s)$, $X_{21}(s)$, $X_{22}(s)$ are general matrices of appropriate dimensions.

Thus from exp. (5.53) of theorem 5.16, a sufficient condition for closed loop stability of S i.e. for $A-BK_rZ$ to be a stability matrix is

$$\| W_r'(s) \| < 1 - \| W(s) - W_r'(s) \|$$

i.e.

$$\max_i \sup_{s\in\partial\Omega} \sum_{j=1}^{n} W'_{rij}(s) < 1 - \max_i \sup_{s\in\partial\Omega} \sum_{j=1}^{n} \{W_{ij}(s) - W'_{rij}(s)\} \quad \ldots(5.72)$$

The $X_{ij}(s)$ matrices in eqn. (5.71) can be chosen to test if exp. (5.72) is satisfied. Although exp. (5.72) is conservative, nevertheless it illustrates that the contraction mapping method is a rigorous method of invoking sufficient stability conditions. Because optimal control is restricted in design, once a performance index is chosen, there is no room for alteration if the sub-optimal controller happens to be unstable. At best the values of Q and R must be adjusted afterwards in eqn. (5.62) such that exp (5.72) is satisfied. But then the sub-optimal control is only stable and near optimal to a performance index not specified earlier.

5.4 Stability investigation by system eigenvalues and closed loop characteristic polynomial.

The stability of closed loop systems can be studied from investigating the behaviour of its c.l.c.p. (closed loop characteristic polynomial).

For

$$R_r(s) = (I + Q_r(s)\, H_r(s))^{-1}\, Q_r(s) \quad \ldots(5.73)$$

the c.l.c.p. is the least common denominator of all minors of $R_r(s)$ and the o.l.c.p. (open loop characteristic polynomial) is that of $Q_r(s)$.

For stability it is required that the c.l.c.p. of $R_r(s)$ have inertia In(0, r, 0) i.e. it is a Hurwitz polynomial. The Routh-Hurwitz (R-H) stability test involves finding conditions where all roots are confined in a certain region in the complex plane, namely, in the left half complex plane (the problem in the large). However, establishing conditions in the R-H test for stability between S and S_r is difficult, and if such conditions exist, the latter may be impractical. Below, two known polynomial theorems are modified and adapted to investigate the stability between S and S_r.

Let the c.l.c.p. of S_r be

$$d_r(s) = s^r + c_r s^{r-1} \ldots + c_1 \qquad \ldots(5.74)$$

and assume that it is a Hurwitz polynomial of inertia In(0,r,0).

Now consider n-r factors $\prod_{i=1}^{n-r} (s+\lambda_i)$, where $Re(\lambda_i) > 0$, $\forall i$, and if $Im(\lambda_i)$ exists, it occurs in conjugate pairs. Consider the modified c.l.c.p of S_r as

$$d_r'(s) = d_r(s) \prod_{i=1}^{n-r} (s+\lambda_i) = s^n + a_{rn} s^{n-1} + \ldots a_{r1} \qquad \ldots(5.75)$$

and let the c.l.c.p. of S be

$$d(s) = s^n + a_n s^{n-1} + \ldots a_2 s + a_1 \qquad \ldots(5.76)$$

and let

$$qb_i = a_i - a_{ri} \qquad \ldots(5.77)$$

Theorem 5.17

Sufficient conditions for closed loop stability of S if S_r is likewise stable are:

(1) $$q_{kk} = 2 \sum_{i=\beta}^{\theta} (-1)^{i+k} a_i b_j > 0, \quad k = 1,\ldots n \qquad \ldots(5.78)$$

with $j = 2k-i$,

$\beta = 1$ for $2k-1 \leqslant n$,

$= 2k-n$ for $2k-1 > n$,

$\theta = 2k-1$ for $2k-1 \leqslant n$,

$= n+1$ for $2k-1 > n$

(2) $$q > - \min_k q_{kk} / \left(2 \sum_{i=1}^{n} b_i^2\right) \qquad \ldots(5.79)$$

Proof : The above is based on a theorem given by Shane and Barnett[20]. If each coefficient a_{ri} of eqn. (5.75) is changed by the addition of $q\,b_i$ of eqn. (5.77) and if exps. (5.78) and (5.79) are satisfied, then d(s) of eqn. (5.76) is a Hurwitz polynomial of inertia In(0,n,0) given $d_r'(s)$ is a Hurwitz polynomial of inertia In(0,n,0). Thus S is closed loop stable.

Q.E.D.

Let A be the companion form matrix of $d_r'(s)$, i.e. $A = \begin{bmatrix} 0 & \vdots & I_{n-1} \\ \cdots & \cdots & \cdots \\ -a_{r1}, & \cdots & -a_{rn} \end{bmatrix}$

and let $P = (p_{ij})$ and Q be positive definite matrices. Then,

Theorem 5.18

Sufficient conditions for closed loop stability of S if S_r is likewise stable are :

(1) $A^tP + PA = -Q$ ...(5.80)

(2) $a_i - a_{ri} = qp_{ni}$, $i = 1,2,\ldots n$ ...(5.81)

(3) $-\lambda_{min}(Q)/(2\sum_{i=1}^{n} p_{ni}^2) < q$ ...(5.82)

where $\lambda_{min}(Q)$ is the smallest eigenvalue of Q.

Proof : The above is also based on a known polynomial result.[20] If the conditions of theorem 5.18 are satisfied, then the inertia of d(s) of S is the same as that of $d_r'(s)$ of S_r. Hence d(s) is a Hurwitz polynomial given $d_r'(s)$ is a Hurwitz polynomial.

The disadvantage of theorems 5.17 and 5.18 is that the theorems are expressed in algebraic forms which makes the stability regions complicated and unattractive to design. It is also noticed that the augmentation of n-r spurious roots $\prod_{i=1}^{n-r}(s+\lambda_i)$ to $d_r(s)$ to obtain $d_r'(s)$ in eqn. (5.57) involves computational labour. A theorem is next given where stability margins are easily expressed in graphical form and more amenable in design.

For convenience, define the companion canonical structure matrices for d(s) and $d_r(s)$, although other forms may be assumed,

$$M = \begin{bmatrix} 0 & \vdots & I_{n-1} \\ \cdots & \cdots & \cdots \\ -a_1 & \vdots & -a_2, \ldots -a_n \end{bmatrix} , \quad M_r = \begin{bmatrix} 0 & \vdots & I_{r-1} \\ \cdots & \cdots & \cdots \\ -a_{r1} & \vdots & -a_{r2}, \ldots -a_{rn} \end{bmatrix} \quad \ldots(5.83)$$

where M and M_r are of dimensions (nxn) and (rxr) respectively. Define M_r' as

$$M_r' = \begin{bmatrix} M_r & X = 0 \\ Y & W \end{bmatrix} \quad \ldots (5.84)$$

where X, Y and W are arbitrary augmented matrices such that dimension of M_r' is (nxn). If X = 0, from Schur's formula[29]

$$\det M_r' = \det(M_r) \det(W - YM_r^{-1}X) \qquad \cdots (5.85)$$
$$= \det(M_r) \det(W)$$

which means the n eigenvalues of M_r' are those r of M and the spurious n-r of W.

Theorem 5.19

Let λ'_{rj} be the eigenvalues of M_r', j = 1,...n and let β be an arbitrary complex number. Then S is closed loop stable, i.e. M is a stability matrix if

(1) $\quad \text{Re}(\lambda'_{rj}+\beta) < 0, \quad \forall j$ $\qquad \cdots (5.86)$

(2) $\quad \| M - M_r' - \beta I + PH_p\hat{P} \| \inf\kappa(P) < |\text{Re}(\lambda'_{rj}+\beta)|, \quad \forall j$ $\qquad \cdots (5.87)$

where P in general quasi-diagonalises M_r'.

Proof : The above is a direct consequence of theorem 5.1. If the circles centred on $\text{Re}(\lambda'_{rj}+\beta)$ do not overlap the imaginary axis all eigenvalues λ_j of M lie in the left half plane, thus c.l.c.p. of S is a Hurwitz polynomial, hence S is stable.

Q.E.D.

The graphical interpretation of the above theorem is similar to fig. 5.11. The theorem is more flexible than the polynomial theorems of 5.17 and 5.18 in that M_r' need not be a stability matrix. Provided β is so chosen such that $\text{Re}(\lambda'_{rj}+\beta) < 0$, it is possible for exp. (5.87) to be satisfied. Suppose $d_r(s)$ is Hurwitzian, i.e. M_r is a stability matrix. To use theorem 5.19, n-r spurious eigenvalues $\lambda_{r+1},\ldots\lambda_n$ of W must be chosen in conjunction with r eigenvalues $\lambda_1,\ldots\lambda_r$ of M_r.

A convenient canonical structure for W is usually chosen where its eigenvalues can be easily determined and Y can be chosen such that $\| \quad \|$ in exp. (5.87) is minimized. Other canonical forms for M_r' can be chosen by using the similarity transformation $TM_r'T^{-1}$, but the simplest form is $M_r' = \mathrm{diag}\{\lambda_{r1}', \ldots \lambda_{rn}'\}$ assuming the eigenvalues of M_r' are found beforehand. Assuming M_r' is simple, condition (2) of theorem 5.19 becomes

$$\| M - \mathrm{diag}\{\lambda_{r1}'+\beta, \ldots \lambda_{rn}'+\beta\} \| < |\mathrm{Re}(\lambda_{rj}'+\beta)|, \quad \forall j \qquad \ldots (5.88)$$

and β can be chosen to obtain the sharpest bound.

If the dimension $R_r(s)$ of eqn. (5.73) is large, then determining the c.l.c.p. is not computationally attractive. An alternative procedure is to make a minimal realization for $Q_r(s)$ and $Q(s)$. For simplicity assume the feedback matrix H_r is a constant. If $Q_r(S)$ and $Q(s)$ are minimally realized by

$$\begin{aligned} S_r :\ \dot{x}_r &= A_r x_r + B_r u & \qquad S :\ \dot{x} &= Ax + Bu \\ y &= C_r x_r & y &= Cx \\ u &= -H_r y & u &= -H_r y \end{aligned} \qquad \ldots (5.89)$$

then eqn. (5.89) is controllable and observable, its system eigenvalues equal to the zeroes of the c.l.c.p. Thus $d_r(s) = \det(sI - A_r + B_r H_r C_r)$ and $d(s) = \det(sI - A + BH_r C)$. M and M_r' in eqn. (5.83) are identified as

$$M = A - BH_r C, \qquad M_r' = \left[\begin{array}{c:c} A_r - B_r H_r C_r & O \\ \hdashline Y & W \end{array}\right] \qquad \ldots (5.90)$$

Theorem 5.19 can then be applied.

The state space matrices in eqn. (5.89) can be obtained from, example, Ho and Kalman's algorithm[28]. In general, if

$$Q(s) = Q(\infty) + H_o s^{-1} + H_1 s^{-2} + H_2 s^{-3} + \ldots.$$
$$h(s) = s^r + f_1 s^{r-1} + \ldots f_r$$

where Q(s) is dimension (qxp) and h(s) is the least common denominator of Q(s), define

$$T = \begin{bmatrix} H_o & H_1 & \cdots & H_{r-1} \\ H_1 & H_2 & \cdots & H_r \\ H_{r-1} & H_r & \cdots & H_{2r-2} \end{bmatrix} \qquad M = \left[\begin{array}{c:c} 0 & \text{block diag } (I_q) \\ \hdashline -f_r I_q, & -f_{r-1} I_q, \ldots -f_1 I_q \end{array}\right]$$

Suppose K and L are (qr x qr), (pr x pr) nonsingular constant matrices such that $KTL = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = I^t_{n,qr} \; I_{n,pr}$, where $I_{n,qr} = (I_n 0)$, n being the rank of T. Then S(A,B,C) is given by : $A = I_{n,qr} \; KMTLI^t_{n,pr}$; $B = I_{n,qr} \; KTI^t_{p,pr}$; $C = I_{q,qr} \; TLI^t_{n,pr}$.

Theorem 5.19 can also be adapted to investigate stability of S by the multivariable root loci (characteristic frequency loci of $Q_r(s)$) of S_r. Suppose $Q_r(s)$ is the return ratio matrix for a linear multivariable feedback system, S_r, then following MacFarlane[22], S_r can be associated with a set of characteristic algebraic functions by means of the characteristic equation

$$\Delta_r(\ell,s) = \det\left[\ell_r I_m - Q_r(s)\right] = 0 \qquad \ldots(5.91)$$

where m is the dimension of $Q_r(s)$ and $\Delta(\ell,s)$ is irreducible over the field of rational functions in the complex variable s. If $\Delta_r(\ell,s)$ is regarded as a polynomial in ℓ_r with coefficients which are rational functions of s, i.e. $\ell_r(s)$, then $\ell_r(s)$ is called the characteristic gain function of $Q_r(s)$ (in the above sections, $\ell_r(s)$ is termed $\rho_r(s)$, the characteristic loci of $Q_r(s)$). However, if $\Delta_r(\ell,s)$ is regarded as a polynomial in s with coefficients which are rational functions of ℓ, i.e. $s_r(\ell)$, then $s_r(\ell)$ is called the characteristic frequency function (multivariable root loci) of $Q_r(s)$. Here the algebraic functions $\ell_r(s)$, $s_r(\ell)$ can be defined on their appropriate Riemann surfaces.

It is also known that the multivariable root loci are given by the solution of the equation

$$p_{xr}(s) \; p_{dr}(s) \; e_r(s) \; b_{mr}(s,k) = 0 \qquad \ldots(5.92)$$

for s in terms of the scalar gain parameter k. Here $b_{mr}(s,k)$ is such that

$$\det\left[f_r(s,k)I_m - F_r(s,k)\right] = b_{or}(s)f_r^m(s,k) + b_{1r}(s)f_r^{m-1}(s,k)$$
$$+ \ldots .\ b_{mr}(s,k) = 0$$

where $F_r(s,k) = I_m + k\,Q_r(s)$, $e_r(s)$ is the least common denominator of all non-zero non-principal minors of $Q_r(s)$ with all factors common to $b_{or}(s)$ removed, $p_{dr}(s)$ has as its zeroes the decoupling zeroes of S_r, and $p_{xr}(s)$ has as its zeroes those poles of $G_r(s)$, $K_r(s)$ (assumed feedback matrix $H_r(s)$ $= I$) which are lost when $Q_r(s)$ is formed. It can also be shown that provided $p_{xr}(s) \neq p_{dr}(s) \neq e_r(s) \neq 1$, the multivariable root loci for the system of fig. 5.15 are given by

$$p_{xr}(s) = 0 \qquad \ldots(5.93a)$$

$$p_{dr}(s) = 0 \qquad \ldots(5.93b)$$

$$e_r(s) = 0 \qquad \ldots(5.93c)$$

$$\ell_{rj}(s) = -1/k \qquad \ldots(5.93d)$$

The last equation, (5.93d) is a direct generalization of the classical single log root locus. The first three equations give single point solutions that are invariant with gain k. A similar set of equations can be written for S. Assuming the non-existence of single points, i.e. $p_x(s) = p_d(s) = e(s) = 1$, then $\ell(s) = -1/k$ gives the multivariable root loci for S. A theorem relating the multivariable root loci of S_r to that of S can be now constructed as follows.

Theorem 5.20

Let $L_{rj}(k)$ and $L_j(k)$ be (nxn) and (rxr) companion form matrices of the polynomials $\ell_{rj}(s) = -1/k$ and $\ell_j(s) = -1/k$, respectively, as k varies.

Define

$$L'_{rj}(k) = \begin{bmatrix} L_{rj}(k) & 0 \\ X & Z \end{bmatrix} \quad \text{where } L'_{rj}(k) \text{ is (nxn) and}$$

X and Z are arbitrary matrices of compatible dimensions. Let ρ'_{rji} and ρ_{ji} be eigenvalues of $L'_{rj}(k)$ and $L_j(k)$ respectively. Then S is closed loop stable if:

(1) $\mathrm{Re}\,(\rho'_{rji} + \beta) < 0 \quad \forall j, \quad \forall i$ (5.94)

(2) $\| L_j(k) - L'_{rj}(k) \| \inf \kappa(P_j) < |\mathrm{Re}(\rho'_{rji} + \beta)| \quad \forall j, \quad \forall i$ (5.95)

(3) Existence of singular point loci (if any) occurs in the left hand plane.

Proof: The eigenvalues of $L_{rj}(k)$ and $L_j(k)$ are the solutions of the polynomials $\ell_{rj}(s) = -1/k$ and $\ell_j(s) = -1/k$, respectively, as k varies.

By theorem 5.1, conditions (1) and (2) require that the multivariable root loci of S be confined in the left half ℓ-surface.

Since there are m sets of root loci $\ell_j(s) = -1/k$, j=1,....m, each set can be shown on an appropriate Riemann surface. However, for the case m=1, then the single loop classical root locus defined on a single sheet of the complex plane is obtained.

Single input-single output systems

In this case $p_x(s) = p_d(s) = e(s) = 1$. Hence

Theorem 5.21

S is closed loop stable if

(1) $\mathrm{Re}(\rho'_{rj} + \beta) < 0, \quad \forall j$ (5.96)

(2) $\| L(k) - L'_r(k) \| \inf \kappa(P) < |\mathrm{Re}(\rho'_{rj} + \beta)| \quad \forall j$ (5.97)

Proof: Follows from theorem 5.21 with j = 1. (see fig. 5.16)

For multivariable systems, theorem 5.20 requires computing the eigenvalues of $L'_{rj}(k)$ m times. A different approach to computing the root loci is via the state space matrices.

If the triple (A_r, B_r, C_r) is a minimal realization of $Q_r(s)$ in fig. 5.15, then the closed loop poles are given by

$$\det(sI_r - A_r + k\, B_r\, C_r) = 0 \qquad \ldots(5.98)$$

At a closed loop pole, eqn. (5.93d), $\ell_{rj}(s) = -1/k$ holds, hence

$$\det(sI_r - A_r - B_r\, C_r/\ell_{rj}) = 0 \qquad \ldots(5.99)$$

The multivariable root loci can then be calculated as the eigenvalues of $(A_r + (B_rC_r/\ell_{rj}))$ as ℓ_{rj}, (hence as k) varies. The solution of eqn. (5.99) also satisfies eqns. (5.93a) to (5.93c), hence obtaining the multivariable root loci by this method also includes single point loci. Similarly, the eigenvalues of $(A + (BC/\ell_j))$ are the multivariable root loci for S where (A, B, C) is a minimal realization for the transfer function of S. Here A is of dimension (nxn) while A_r is (rxr). To establish a theorem for the root loci of S_r and S define,

$$T_j = A + (BC/\ell_j) \quad , \quad T'_{rj} = \left[\begin{array}{c:c} A_r + (B_rC_r/\ell_{rj}) & 0 \\ \hdashline Y & W \end{array}\right] \qquad \ldots(5.100)$$

where Y and W are arbitrary matrices with compatible dimensions such that T'_{rj} is (nxn). Then,

Theorem 5.22

Let λ'_{rji} be the eigenvalues of T'_{rj}, $j = 1,\ldots m$, $i = 1,\ldots r$.

Then S is closed loop stable if :

(1) $\quad \mathrm{Re}(\lambda'_{rji} + \beta) < 0 \quad \forall j, \quad \forall i \qquad \ldots(5.101)$

(2) $\quad \| T_j - T'_{rj} - \beta I \| \ \inf \kappa(P) < |\mathrm{Re}(\lambda'_{rji} + \beta)| \quad \forall j, \quad \forall i \qquad \ldots(5.102)$

Proof : The proof follows exactly as given in theorem 5.19. If the above conditions are satisfied, then T_j is a stability matrix, i.e. the eigenvalues λ_{ji} of the multivariable root loci of S lie in the left half plane, thus S is closed loop stable.

It is noticed that theorem 5.22 is more flexible than theorem 5.20, in the sense that numerically it is less complicated, and, the bands can be made sharper by tuning the extra parameter matrices, Y and W.

State feedback systems and eigenvalue assignment

Consider the system S_r with state or output feedback.

$$\dot{x}_r = A_r x_r + B_r u \quad ; \quad y = C_r x_r \qquad \ldots(5.103)$$

$$u = -K_r x_r \quad ; \quad u = -K_o y$$

It is desired to implement u on S such that

$$\dot{x} = Ax + Bu \quad ; \quad y = Cx \qquad \ldots(5.104)$$

$$u = -K_r Z x \;,\; u = -K_o y$$

where Z is the aggregation matrix between S_r and S, with $x_r = Zx$. The relationship[23] between eqns. (5.103) and (5.104) is given by

$$\begin{aligned} A_r Z &= ZA \\ B_r &= ZB \\ C_r Z &= C \end{aligned} \qquad \ldots(5.105)$$

Aoki[23] first showed that the feedback system $A + BK_r Z$ is stable if Z is chosen from the modal matrix of A. Generalizing a result of Vittal Rao and Lamba, Hickin and Sinha[24] showed that this is true for all classes of projective reduction techniques, i.e. for $Z = (E_{11}^{-1} \vdots\ 0\ )P$ where $K^+ = E^t$

$= (E_{11}^t E_{21}^t)$, the r columns of E are chosen to span the controllable subspace of the pair (PA, PB) and P is a projection matrix.

Theorem 5E[24]

For state feedback, let $A_f = A - BK_r Z$, $A_{rf} = A_r - B_r K_r$, then

(a) $A_f v_i = \lambda_i v_i \qquad i = r+1, r+2, \dots n$

(b) If $A_f v_i' = \lambda_i' v_i'$, then $A_{rf} v_i'' = \lambda_i' v_i''$ for $i = 1, 2, \dots r$

where λ_i is an eigenvalue of A and λ_i' an eigenvalue of A_{rf}

For output feedback, suppose λ''_i is an eigenvalue of $A-BK_oC$ and μ_i is an eigenvalue of $A_r - B_rK_oC_r$, then

(c) $\lambda_i'' \approx \lambda_i \qquad i = r+1, r+2, \dots n$

(d) $\lambda_i'' \approx \mu_i \qquad i = 1, 2, \dots r$

Theorem 5E is not true for nonprojective reduction techniques.

To determine stability between S_r and S for the latter class of reduction techniques, a similar form of theorem 5.19 can be employed.

For state feedback, let

$$F = A-BK_rZ , \quad F_r = A_r-B_rK_r , \quad F_r' = \begin{bmatrix} F_r & 0 \\ Y & W \end{bmatrix}$$

for output feedback, let

$$F = A-BK_oC, \quad F_r = A_r-B_rK_oC_r \text{ and } F_r' = \begin{bmatrix} F_r & 0 \\ Y & W \end{bmatrix}$$

then

Theorem 5.23

Let λ_{rj}' be the eigenvalues of F_r' (i.e. λ_{rj}' consists of all the r eigenvalues of F_r and all the n-r eigenvalues of W). Then S is closed loop stable if:

(a) $\mathrm{Re}(\lambda_{rj}' + \beta) < 0 \qquad \forall j$ ...(5.106)

(b) $\|F_r' - F_r - \beta I\| \inf\kappa(P) < |Re(\lambda'_{rj} + \beta)| \quad \forall j$ ...(5.107)

Proof : The proof follows exactly that of theorem 5.19. The above conditions establish the fact that F_r' is a stability matrix, when the inertia of F_r is known (possibly F_r is a stability matrix).

Thus for the class of nonprojective reduction techniques, theorem 5.24 offers a computationally feasible method for assuring stability of S when design is made on K_r. It is also noted that the theorem is applicable for both state or output feedback whereas theorem 5E of Sinha and Hickin offers only approximate (hence uncertain) solution of stability for systems with output feedback. The only disadvantage of theorem 5.24 is that it cannot be used for eigenvalue assignment, but then so is theorem 5E for nonprojective reduction techniques.

5.5 Stability investigation using Lyapunov theory

Stability determination using the Lyapunov approach is very useful because of its generality[30]. In order to adapt Lyapunov's methods to stability investigation of S_r and S, a domain of stability, V_c, that is common to both S_r and S is sought.

Theorem 5.24

Consider S^{nxn} : $\dot{x} = f(x)$ and S_r^{rxr} : $\dot{x}_r = f(x_r)$. Augment S_r by n-r states i.e. $\dot{x}_{ar} = f(x_{ar})$ where $x_{ar} = (x_1, \ldots x_r, x_{r+1}, \ldots x_n)^t$. Both S_r and S are asymptotically stable if there is a $V_c(x) \triangleq V(x) \cap V_{ar}(x_{ar}) > 0$ such that $\dot{V}_c(x) < 0$ for both S_r and S.

Proof : S_{ar} is asymptotically stable if there is a $V_{ar}(x_{ar}) > 0$ such that $\dot{V}_{ar}(x_{ar}) < 0$, and, S is similarly stable if there is a $V(x) > 0$ such that $\dot{V}(x) < 0$. Hence S_r and S have a common region of stability for $V_c(x) \triangleq V(x) \cap V_{ar}(x_{ar})$. The fictitious n-r augmented states can be chosen such that the augmented equations are stable.

If a $V_c(x)$ exists, theorem 5.24 only provides a sufficient condition for stability of S_r and S, however the theorem is quite general in its presentation. For linear systems described by state space equations (A, B, C), Barnett and Storey have obtained sufficient conditions for stability when (A, B, C) is perturbed to (A', B', C'), which is a special case of theorem 5.24.

For the systems of eqns. (5.103) and (5.104), in the sequel, let

$$F_{r11} = F_r = A_r - B_r K_r$$

$$F = A-BK_rC = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} , \quad F'_r = \begin{bmatrix} F_{r11} & F_{r12} \\ F_{r21} & F_{r22} \end{bmatrix} \quad \ldots (5.108)$$

where F_{r12}, F_{r21} and F_{r22} are arbitrary augmented matrices of compatible dimensions. Then,

Theorem 5.25

Sufficient conditions for F to be a stability matrix provided that F'_r is also a stability matrix, i.s.L., are either of the following :

(a)
$$\begin{aligned} S_{11} &= R_{11} + P_{11} F_{e11} + P_{12} F_{e21} \\ 2S_{12} &= P_{11} F_{e12} + P_{12} F_{e22} - F^t_{e11} P_{12} - F^t_{e21} P_{22} \\ -2R_{12} &= P_{11} F_{e12} + P_{12} F_{e22} + F^t_{e11} P_{12} + F^t_{e21} P_{22} \\ S_{22} &= R_{22} + P_{22} F_{e22} + P^t_{12} F_{e12} \end{aligned} \quad \ldots (5.109)$$

(b)
$$\begin{aligned} S_{11} &= R_{11} + F_{e11} Q^{-1}_{11} - (S_{12}-R_{12}) Q^t_{12} Q^{-1}_{11} \\ S_{12} &= R_{12} + F_{e12} Q^{-1}_{22} - (S_{11}-R_{11}) Q_{12} Q^{-1}_{22} \\ -R^t_{12} &= S^t_{12} + F_{e21} Q^{-1}_{11} - (S_{22}-R_{22}) Q^t_{12} Q^{-1}_{11} \\ S_{22} &= R_{22} + F_{e22} Q^{-1}_{22} + (S^t_{12}+R^t_{12}) Q_{12} Q^{-1}_{22} \end{aligned} \quad \ldots (5.110)$$

where P, Q are arbitrary positive definite matrices that satisfy the Lyapunov equations $F'_r P + P F'^t_r = -N$, $F'_r Q + QF'^t_r = -M$, where $N > 0$, $M > 0$ and S and R are arbitrary skew and positive semidefinite.

Proof : From a result of Barnett and Storey[27], A+B is a stability matrix given A is a stability matrix if $B = P^{-1}(S-R)$ or if $B = (S-R)Q$ where S and R are arbitrary skew and positive semidefinite, and P and Q satisfy $AP + PA^t = -N$, $AQ + QA^t = -M$, the usual Lyapunov equations. Thus F_r' of eqn. (5.108) is a stability matrix if

$$F_r' P + PF_r'^t = -Q \qquad \ldots(5.111)$$

Identifying B with $F_e = F - F_r'$ in eqn. (5.108) it is seen that F is a stability matrix given F_r' is a stability matrix if

$$F_e = P^{-1}(S-R) \qquad \ldots(5.112)$$

$$\text{i.e.} \quad PF_e = (S-R) \qquad \ldots(5.113)$$

$$\text{or} \quad F_e = (S-R)Q \qquad \ldots(5.114)$$

$$\text{i.e.} \quad F_e Q^{-1} = (S-R) \qquad \ldots(5.115)$$

Partitioning the matrices as

$$F_e = \begin{bmatrix} F_{e11} & F_{e12} \\ F_{e21} & F_{e22} \end{bmatrix} = \begin{bmatrix} F_{11} - F_{r11} & F_{12} - F_{r12} \\ F_{21} - F_{r21} & F_{22} - F_{r22} \end{bmatrix}, P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^t & P_{22} \end{bmatrix}$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^t & Q_{22} \end{bmatrix}, S = \begin{bmatrix} S_{11} & S_{12} \\ -S_{12}^t & S_{22} \end{bmatrix}, R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^t & R_{22} \end{bmatrix}$$

and substituting in eqns. (5.113) and (5.114) with direct expansion yields eqns. (5.109) and (5.110). A similar set of equations can be written for eqns. (5.112) and (5.115).

These sets are only sufficient conditions for F to be a stability matrix based on F_r' being a stability matrix. It is noticed that the unaugmented matrix $F_r = F_{r11}$ need not necessarily be a stability matrix. This is made possible by the interconnecting elements, given by the 'off diagonal terms' of the sub-matrices P_{12}, P_{21}, Q_{12}, Q_{21} etc. that coupled F_r to F_r'. The 'interconnecting elements' correspond to the tuning factors α and β in the frequency domain stability theorems. To simplify matters F_{r12} or F_{r21} in F_r' can be set to zero. Thus F_r' is a stability matrix if F_{r11} and F_{r22} are stability matrices. A canonical structure for F_{r22} can easily be set up with known eigenvalues.

Much simplification is still obtained if F_r' is decoupled, i.e. the 'off diagonal elements' sub-matrices of F_r', P and Q are set to zero. In this case, eqn. (5.111) becomes

$$\begin{bmatrix} F_{r11} & 0 \\ 0 & F_{r22} \end{bmatrix} \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} F_{r11} & 0 \\ 0 & F_{r22} \end{bmatrix}^t = - \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}$$

where $F_{r11} P_{11} + P_{11} F_{r11}^t = -Q_{11}$ indicates closed loop stability for reduced model and F_{r22} and P_{22} are so chosen to satisfy $F_{r22} P_{22} + P_{22} + F_{r22}^t = -Q_{22}$ and eqns. (5.109) and (5.110).

Although decoupling simplifies the stability conditions and lessens the choice of parameters, it is compensated by weakening the stability conditions of eqns. (5.109) and (5.110). These conditions do not offer an easy graphical interpretation, hence in certain cases, their applications to design are restricted. Being of an algebraic nature, their main value lies in theoretical utility, like providing a link in proving theorems.

5.6 Extensions to discrete systems

In dealing with discrete systems the unit modulus circle in the z-plane plays the fundamental role in contrast to the D semi-circle in the s-plane for continuous systems. It is well known that det F(z) = c.l.c.p./o.l.c.p. where c.l.c.p. and o.l.c.p. are polynomials in terms of the variable z. Thus if det (F(z)) mapped the O contour, the unit circle centred at the origin and the outer circle of infinite radius, in the z-plane into the closed curve Γ

in the frequency response plane, the well-known stability theorems in terms of return difference and return ratio matrices and characteristic loci apply exactly for the discrete as for the continuous system[3]. The stability theorems in the earlier sections can be modified below.

Theorem 5.26

Let $\rho_{rj}(z)$ (resp. $\hat{\rho}_{rj}(z)$), $t_{rj}(z)$ (resp. $\hat{\lambda}_{rj}(z)$) be the characteristic loci of $Q_r(z)$ (resp. $\hat{Q}_r(z)$), $F_r(z)$ (resp. $\hat{R}_r(z)$) where the symbols have their usual meaning as in theorems 5.1 to 5.15. Further let $\beta+\rho_{rj}(z)$ (resp. $\alpha+\hat{\rho}_{rj}(z)$) and $\beta+t_{rj}(z)$ (resp. $\alpha+\hat{\lambda}_{rj}(z)$) map 0 into $\beta+\Gamma_{rj}$ (resp. $\alpha+\hat{\Gamma}_{rj}$) and $\beta+\Delta_{rj}$ (resp. $\alpha+\hat{\Delta}_{rj}$) respectively.

Then the conditions for closed loop stability of the discrete multivariable system, S, are exactly the same as those given in theorems 5.1 to 5.15, except that quantities in s-variables are replaced by quantities in z-variables.

Proof : The proofs follow exactly the same lines of argument as those given in their continuous counterpart, since det $F(z)$ = c.l.c.p.(z)/o.l.c.p.(z) and det $F(s)$ = c.l.c.p.(s)/o.l.c.p.(s).

Alternatively, instead of direct mapping from z-plane into Γ, the contour 0 can be mapped into a fictitious 's' plane via the bilinear transformation $z = (1+W)/(1-W)$. This W plane has the same stability boundary as the ordinary s-plane, and a second mapping by $\rho_{rj}(W)$, det $F_r(W)$ etc., will transform the fictitious 'D contour' into Γ, the usual Nyquist frequency domain. This extra effort has the advantage that the classical Nyquist contours, obtained in this way, offer familiar design techniques to the designer.

The state space matrices S(A,B,C,) describing the discrete multivariable system as

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k \qquad\qquad k = 1,2\ldots \qquad\qquad \ldots(5.116) \\
u_k &= -K_r\, x_k
\end{aligned}
$$

and the associated Lyapunov equation is

$$F_d^t \; R \; F_d - R = -M \qquad \ldots(5.117)$$

where F > 0 and M > 0 and $F_d = (A + BK_r Z)$ is the discrete closed loop system matrix (assuming state feedback). For F_d to be a stability matrix, it is required that all eigenvalues, $|\lambda_j(F_d)| < 1$.

The continuous version of theorem 5.25 can be easily adapted to the discrete system of eqn. (5.116) by using the transformation

$$F = (F_d - I)\;(F_d + I)^{-1} \qquad \ldots(5.118)$$

Here the interior of the unit circle is mapped into the left half region of the complex plane, hence the stability of eqn. (5.116) can be determined by the direct application of theorem 5.26 via eqn. (5.118).

Stability of the discrete system can also be determined from the Lyapunov equation of (5.117) and the c.l.c.p. of the controllable and observable state space system. Suppose

$$a_r(\lambda) = \lambda^r + c_r\lambda^{r-1} + \ldots . c_1 \qquad \ldots(5.119)$$

in the c.l.c.p. of S_r. Let F_r represent the companion matrix representation of eqn. (5.119). Augment $F_r(r x r)$ such that

$$F'_r = \begin{bmatrix} F_r & 0 \\ X & W \end{bmatrix} \qquad \ldots(5.120)$$

is (nxn) and is a stability matrix, i.e. F'_r satisfies

$$F_r^{'t} \; PF'_r - P = -Q \qquad \ldots(5.121)$$

Similarly let $\quad F_e = F - F'_r \qquad \ldots(5.122)$

where F is the discrete system matrix of $S(A+BK_rZ)$. Then,

Theorem 5.27

F is a stability matrix given F_r' is a stability matrix if :

(1) $F_e = (R - \frac{1}{2}P)^{-1} (RY + \frac{1}{2}PY + PF_r')$ ...(5.123)

where Y is symmetric and satisfies the Riccati-type algebraic equation

$$YPY + F_r'^t PY + YPF_r' + Q_o = 0 \qquad \text{...(5.124)}$$

where Q_o is any symmetric matrix for which $(Q+Q_o) > 0$, R is any skew-symmetric matrix and P any solution of eqn. (5.121).

Proof : Theorem 5.27 is an adaptation of a matrix result of Shane and Barnett[20]. If the roots of F_r' are inside the unit circle, then the root distribution of $F_r' + B$ is unchanged if B satisfied eqn. (5.123).

Another result can also be stated in terms of the c.l.c.p.'s of S_r and S. Suppose $a_r(\lambda)$ in eqn. (5.119) is augmented by n-r spurious roots all with modulus less than unity, i.e.

$$a_r'(\lambda) = a_r(\lambda) \prod_{i=1}^{n-r} (\lambda+d_i) = \lambda^n + b_{rn}\lambda^{n-1} + \ldots\ldots b_{r1} \qquad \text{...(5.125)}$$

and let $a(\lambda) = \lambda^n + b_n\lambda^{n-1} + \ldots b_1$ ...(5.126)

where $a(\lambda)$ is the c.l.c.p. of F. Then

Theorem 5.28

$S(A+BK_rZ)$ is stable if $S_r(A_r+B_rK_r)$ is stable and if :

(1) $b_i - b_{ri} = q\alpha_{in}/(2+qp_{nn}) \qquad i = 1,\ldots n$ ...(5.127)

where either

(2) $q > 0$ or $-2q/(2+qp_{nn})^2 < \lambda_{min}(Q) \sum_{i=1}^{n} \alpha_{in}^2$ ...(5.128)

where $P = (p_{ij})$ is a solution of eqn. (5.120) (F_r' in this case is the companion matrix for $a_r'(\lambda)$) and α_{in} is the in[th] element of $F_r'^t P$.

Proof : The above is also an adaptation of a known polynomial result[20]. The root distribution (inside the unit circle) of the polynomial $a_r'(\lambda)$ is unaltered, if $a_r'(\lambda)$ is perturbed to $a(\lambda)$, where $a_r'(\lambda)$ and $a(\lambda)$ are of the same degree, and conditions (1) and (2) are satisifed. Hence $|\lambda_j| < 1, \quad \forall j$, if $|\lambda_{rj}'| < 1, \quad \forall j$. Thus the discrete system S is stable given S_r is stable.

The results of theorems 5.27 and 5.28 are in algebraic form, and can be very conservative when it comes to design. A graphical result is thus needed that is more amenable to computation and design. To this end, an equivalent form of theorem 5.19 for discrete systems is sought.

Theorem 5.29

The discrete system $S(A-BK_rZ)$ is stable if :

(1) $|\lambda_{rj}' + \beta| < 1, \quad \forall j$ ...(5.129)

(2) $\|F - F_r' - \beta I\| \ \inf\kappa(P) < 1 - |\lambda_{rj}' + \beta|, \quad \forall j$ ...(5.130)

If $F_r' = \mathrm{diag}(\lambda_{r1}', \dots \lambda_{rn}')$, eqn. (5.130) can be replaced by

(3) $\|F - \mathrm{diag}(\lambda_{r1}' + \beta, \dots \lambda_{rn}' + \beta)\| < 1 - |\lambda_{rj}' + \beta| \quad , \forall j$ ...(5.131)

Proof : Theorem 5.29 is the discrete version of theorem 5.19. Here the circular bands are all contained within the unit circle instead of lying in the left half plane. Thus if the conditions of the theorem are satisifed, no band can overlap the boundary of the unit circle, hence all roots of F lie in the unit circle, thus S is closed loop stable.

Q.E.D.

The graphical interpretation of theorem 5.29 is shown in fig. 5.17.

Again, as with the continuous case, theorem 5.29 is more flexible and easy to implement than theorems 5.27 and 5.28. Also, theorem 5.29 has the advantage that the reduced model S_r need not be closed loop stable to determine stability of S.

Conclusions

The results in this chapter are mainly of theoretical interest, however, they are of great practical value if used in conjunction with interactive graphics. The main result was given in terms of original model stability based on reduced model stability. The bounds for original model stability could be adjusted by 'tuning factors' α and β, and engineering constraints can be accommodated. The optimal choice of α and β can be found mathematically, but the best choice in the engineering sense is 'an art'. Here trial and error adjustment is the best procedure in relation to system stability and integrity. As graphical interpretations of these bounds are necessary and evaluation of them involves computation, computer aided design with graphics is indespensable.

The stability conditions given are very general and they are applicable to a wide class of design techniques, both vector methods and frequency response methods. Adaptation of these results to design will be studied in the next chapter.

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VI

BOUNDS FOR MULTIVARIABLE SYSTEMS DESIGNED USING REDUCED MODELS.

CHAPTER VI

BOUNDS FOR MULTIVARIABLE SYSTEMS DESIGNED USING REDUCED ORDER MODELS.

Introduction

The last chapter investigates the stability of multivariable systems designed using reduced order models. This chapter is concerned with the performance and integrity of multivariable control systems, designed using such models, and, the adaptation of reduced models to various design techniques.

Multivariable design can be classified broadly into two categories; namely, the state vector method and frequency response method[4]. The first method involves such techniques as 'optimal control design', where the linear regulator problem with quadratic cost function is a popular example, pole shifting and unity rank feedback, etc. where the central point of focus here is the manipulation of the closed loop characteristic polynomial. Recently, attempts have been made to develop the multivariable root loci design in the s-plane, a natural extension of the classical single loop root locus[40,41]. The second method, using frequency response, are extensions of the Bode and Nyquist plots for single loop systems in terms of return ratio and return difference matrices[4]. Examples of such methods are Inverse Nyquist Array, Characteristic Loci and Sequential Return Difference. These techniques are not algorithmic in nature, in contrast to pole shifting and optimal control, for they rely very much on design experience, but, they give more room for flexibility. Also, in contrast to time domain methods, they often yield simpler controllers.

The chief interest in using reduced models in multivariable systems design is to determine the degree of accuracy in the final phase of the design. To this end, it is desirable to find conditions to impose error bounds on the transient response of the system. Aoki[44] and Mitra[45] had studied the role of reduced models in multivariable systems in terms of the sub-optimal control problem. Ganderson and George had made error estimates for the performance of approximate dynamic systems using the theory of differential inequalities and Lyapunov functions in the time domain[19].

In this chapter, error bounds are given in the frequency domain for multivariable systems, designed using reduced models. The bounds are general, and, they are applied to all systems designed using reduced models, the latter being obtained by any valid reduction technique . Some new bounds for multivariable frequency response designs are also given, where they are not related to the topic of model reduction.

6.1 Some modified bounds for multivariable systems.

It is well known that stability of multivariable systems can be determined from characteristic loci encirclements[46,4]. Alternatively, conditions can be imposed on the structure of the transfer function matrix to constrain the characteristic loci in certain regions in the complex plane to yield sufficient conditions for stability, of which Rosenbrock's diagonal dominance criteria is an example[1]. Cook's[2] modification of Rosenbrock's result using Gershgorin mean bands offers significant improvement, on stability bounds, though it is more suitable for non-linear multivariable systems, which will be discussed in Chapter Eight. Cook's result is related to various theorems of Ostrowski [5], but in fact, there are various theorems scattered here and there that determine stability bounds, with some giving sharper results than others. All these methods have one point in common, in that it concerns solving the zeroes of the c.ℓ.c.p. in the large. The recent results of Araki et al.[3] using M-matrix theory for investigating stability is related to the loci encirclement criteria through a theorem of Ky Fan[12,18], hence they are also concerned with solving the zeroes of the c.ℓ.c.p.in the large. One disadvantage of using M-matrix approach is that a graphical interpretation of the results is difficult to obtain.

Some results for determining stability bounds are derived below, of which some are new and some known in other forms. They are related to results due to Ostrowski[5], Brauer[11], Lederman, Kotelyanski and Ky Fan[10,12].

Theorem 6.1

Let every $q_{ii}(s)$ (resp. $\hat{q}_{ii}(s)$) of $Q(s)$ (resp. $\hat{Q}(s)$ map D into Γ_i (resp. $\hat{\Gamma}_i$). Then sufficient conditions for closed loop stability are:

(1) Γ_i (resp. $\hat{\Gamma}_i$) satisfies NC (Nyquist criterion), $\forall i$

either (2) $|1 + q_{ii}(s)| > r(A(s)) - |a_{ii}(s)|$, $\forall i, \forall s \epsilon D$

$$[\text{resp. } |1 + \hat{\hat{q}}_{ii}(s)| > r(\hat{A}(s)) - |\hat{a}_{ii}(s)|] \tag{6.1}$$

or (3) $|1+q_{ii}(s)| > \max_i \sum_{i \neq j} |q_{ij}(s)| - \min_{ij} |q_{ij}(s)| (1-\sqrt{\beta(s)})$, $\forall ij$, $\forall s \epsilon D$

$$[\text{resp. } |1 + \hat{q}_{ii}(s)| > \max_i \sum_{i \neq j} |\hat{q}_{ij}(s)| - \min_{ij} |\hat{q}_{ij}(s)| (1-\sqrt{\hat{\beta}}(s))] \tag{6.2}$$

or (4) $|1+q_{ii}(s)| > \max_i \sum_{i \neq j} |q_{ij}(s)| - \min_{ij} |q_{ij}(s)| (1-\sigma(s))$, $\forall ij, \forall s \epsilon D$

$$[\text{resp. } |1+\hat{q}_{ii}(s)| > \max_i \sum_{i \neq j} |\hat{q}_{ij}(s)| - \min_{ij} |\hat{q}_{ij}(s)| (1-\hat{\sigma}(s))] \tag{6.3}$$

or (5) $|1+q_{ii}(s)| > \max_i \sum_{i \neq j} | q_{ij}(s)| - \min_{ij} |q_{ij}(s)| (1-\Delta(s))$, $\forall ij$, $\forall s \epsilon D$

$$[\text{resp. } |1+\hat{q}_{ii}(s)| > \max_i \sum_{i \neq j} |\hat{q}_{ij}(s)| - \min_{ij} |\hat{q}_{ij}(s)| (1-\hat{\Delta}(s))] \tag{6.4}$$

or (6) $|1+q_{ii}(s)| > \max_i \sum_{i \neq j} |q_{ij}(s)| - \varepsilon(s) (t(s)-\gamma(s))$, $\forall ij$, $\forall s \epsilon D$

$$[\text{resp. } |1+\hat{q}_{ii}(s)| > \max_i \sum_{i \neq j} |\hat{q}_{ij}(s)| - \hat{\varepsilon}(s) (\hat{t}(s)-\hat{\gamma}(s))] \tag{6.5}$$

where $r(A(s)) = \inf_{\|\ \| \in N} [\sup_{x \neq 0} \| A(s)x(s) \| / \| x(s) \|]$ is the spectral radius of the non-negative matrix $A(s)$, $t_i(s) = \sum_{j=1}^{n} |q_{ij}(s)|$, $\beta = \max (t_i(s)/t_j(s))$ with $t_i(s) < t_j(s)$, $t(s) = \max_i t_i(s)$, $\tau(s) = \min_i t_i(s)$, $\sigma^2(s) = (\tau(s) - \min_{ij} |q_{ij}(s)|)/(t(s) - \min_{ij} |q_{ij}(s)|)$

$$\Delta(s) = 2(\tau(s) - \min |q_{ij}(s)|)/(t(s) - 2\min |q_{ij}(s)| + [t^2(s) - 4\min |q_{ij}(s)| (t(s) - \tau(s))]^{\frac{1}{2}})$$

$$\gamma(s) = \Sigma t_i(s)/m, \quad \varepsilon(s) = [(\min_{i \neq j} |q_{ij}(s)| / t(s)) - \min |q_{ii}(s)|]^{m-1}$$

and the superscript 'hat' ^ symbol denotes quantities relevant to the inverse loci theorem.

Proof:

A theorem due to Ky Fan[12,5] and Kotelyanskii[10,5] states that the characteristic loci of a matrix $Q(s)$ lie in the union of discs

$$|\rho_i(s) - q_{ii}(s)| \leq \eta - b_{ii} \qquad (6.6)$$

where η is the spectral radius of a real non-negative matrix B (i.e. $b_{ij} > 0$, $\forall i,j$) where $|q_{ij}(s)| \leq b_{ij}$ $(i,j = 1,\ldots m,\ i \neq j)$.

Condition (2) of theorem 6.1 is the direct consequence of the Fan-Kotelyanskii theorem, and, if condition (1) is satisfied, no loci can encircle the critical point, thus ensuring stability. Other conditions are obtained as follows.

By the Perron-Frobenius theorem[8] for non-negative B (i.e. $B \geq 0$) and assumed irreducible,

$$\min_i \sum_{j=1}^{n} b_{ij} \leq \eta \leq \max_i \sum_{j=1}^{n} b_{ij} \qquad (6.7)$$

Hence

$$\eta - b_{ii} \leq \max_i \sum_{j \neq i} b_{ij} \qquad (6.8)$$

To obtain the sharpest bound, b_{ij} is chosen as $b_{ij} = |q_{ij}(s)|$. Hence from eqn. (6.6)

$$|\rho_i(s) - q_{ii}(s)| \leqslant \max_i \sum_{j\neq i} |q_{ij}(s)| \tag{6.9}$$

which is a modified form of Rosenbrock's diagonal dominance criterion (Gershgorin's theorem). Sharper bounds to eqn. (6.8) can be obtained by using the results due to Ledermann, Ostrowskii[5] and Brauer[5] as follows

$$\eta \leqslant \max_i \ \Sigma|q_{ij}(s)| - \min_{ij} |q_{ij}(s)|(1-\sqrt{\beta(s)}) \tag{6.10}$$

$$\eta \lesseqgtr \max_i \ \Sigma|q_{ij}(s)| - \min_{ij} |q_{ij}(s)|(1 - \sigma(s)) \tag{6.11}$$

$$\eta \leqslant \max_i \ \Sigma|q_{ij}(s)| - \min_{ij} |q_{ij}(s)|(1 - \underline{\Delta}(s)) \tag{6.12}$$

$$\eta \leqslant \max_i \ \Sigma|q_{ij}(s)| - \varepsilon(s) \ (t(s) - \gamma(s)) \tag{6.13}$$

Substitution of eqns. (6.10) to (6.13) individually into eqn.(6.6)yields a sharper form of (6.9). By similar argument as in condition (1), sufficient conditions for stability can be ensured. The results follow, eqns. (6.2) to (6.5). Theorem 6.1 is shown graphically in fig.6.1.

Q.E.D.

Consider the return difference matrix $F(s) = I + Q(s)$ $(H(s) = I)$. By the Fan-Kotelyenskii theorem,[5,12]

$$|\lambda_i(s) - f_{ii}(s)| \leqslant \eta' - b'_{ii} \tag{6.14a}$$

where $\lambda_i(s)$ is the characteristic loci of $F(s)$ and $|f_{ij}(s)| \leqslant b'_{ij}$

Theorem 6.2

Let $f_{ii}(s)$ (resp. $\hat{f}_{ii}(s)$) of $F(s)$ (resp. $\hat{R}(s)$) map D into Γ_i (resp. $\hat{\Gamma}_i$), $\forall i$. Then sufficient conditions for closed loop stability are exactly the same as those given in theorem 6.1, except that the critical point is shifted from '-1' to the origin.

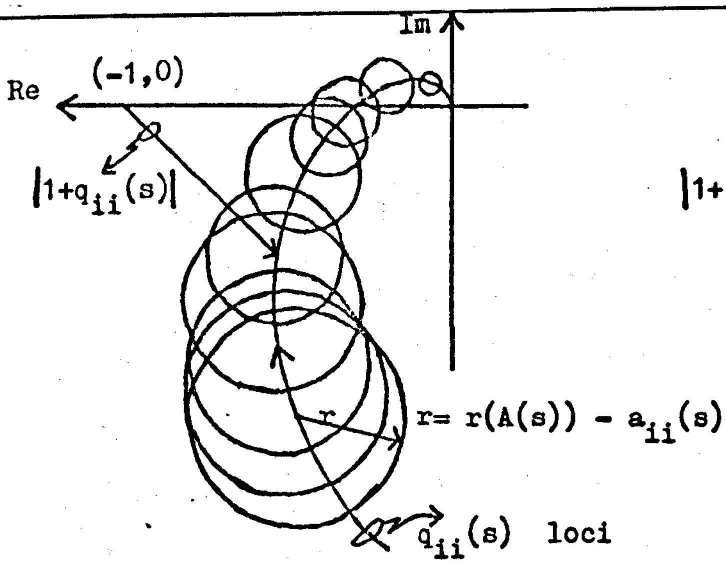

Fig. 6.1 Extended bounds(stability) for linear multivariable systems (Illustrating theorem 6.1)

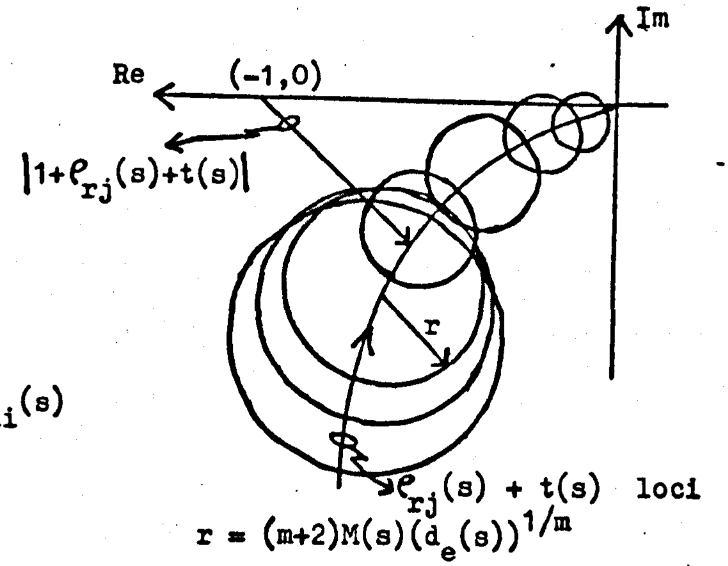

Fig. 6.2 Illustrating theorem 6.4 for stability of S_r and S

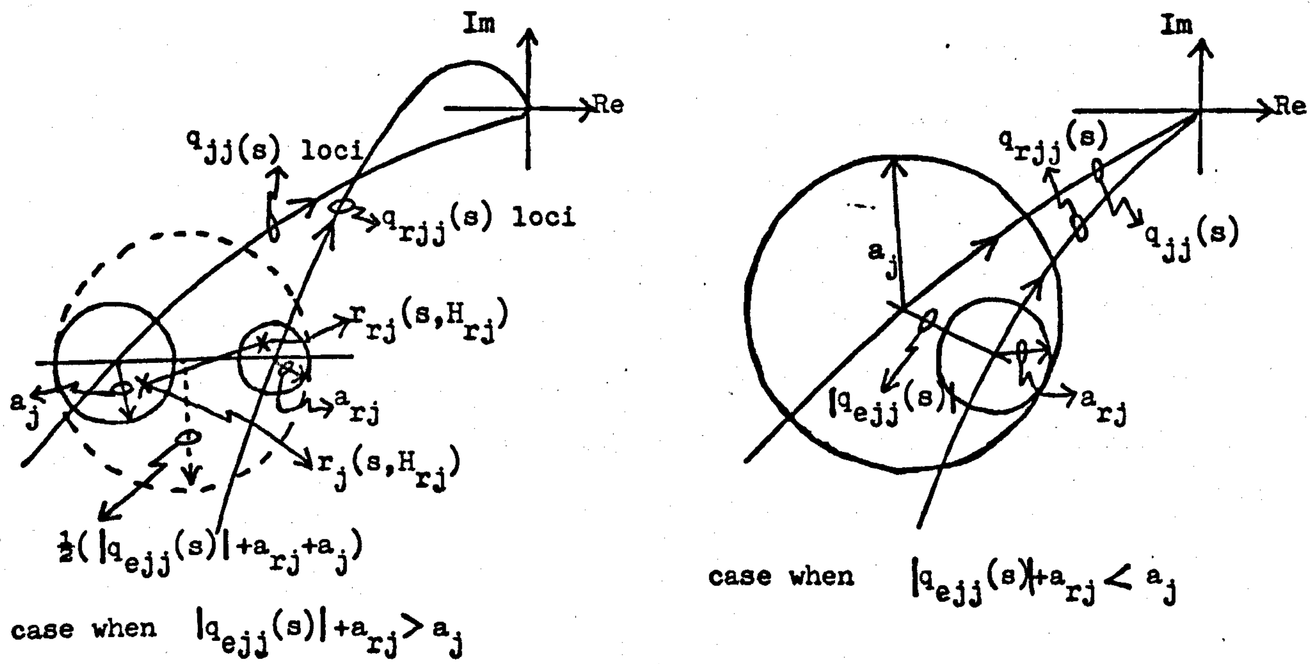

Fig. 6.3 Bounds for S_r and S; illustrating theorem 6.9

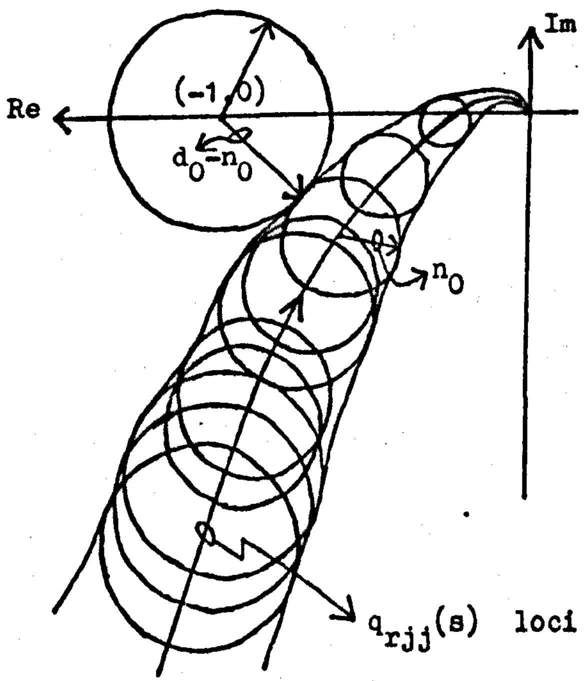

Fig. 6.4 Contraction mapping error estimates for diagonal system's design (see eqn. (6.45))

Proof: Choose $|f_{ij}(s)| = |q_{ij}(s)| = b'_{ij}(i \neq j)$. Since $|1+q_{ii}(s)| \leq 1+|q_{ii}(s)|$, choose $|f_{ii}(s)| = b'_{ii} = 1 + |q_{ii}(s)|$. Hence η' is the spectral radius of $B' = I + B$. By the eigenvalue shift theorem $\eta' = 1 + \eta$, the r.h.s. of exp. (6.14) becomes $\eta' - b'_{ii} = 1 + \eta - (1+b_{ii}) = \eta - b_{ii}$. Comparing this to exp. (6.6) it is deduced that the radii of the bands remain unchanged, but the critical point is shifted from '-1' to the origin. A similar argument applies to the inverse case, since $\hat{R}(s) = I + \hat{Q}(s)$ and by the eigenvalue shift theorem, $\hat{\eta}' = 1 + \hat{\eta}$ where $\hat{\eta}'$ and $\hat{\eta}$ are the spectral radii of $\hat{R}(s)$ and $\hat{Q}(s)$ respectively.

Since the eigenvalues of a matrix are the same as its transpose, columns instead of rows can also be considered in the above, with the eigenvalues lying in the intersection of the bands, the sharper band being taken. The results are sharper than Rosenbrock's diagonal dominance criterion, in that only the maximum or minimum row or column need be dominant, even so, very coarse 'diagonal dominance' is required, rendered possible by the correction factors $\min |q_{ij}(s)|(1 - \Delta(s))$ etc. The results are also related to that of Araki[3] et al. who used M-matrices[6] but the latter has a computationally unattractive graphical evaluation.

6.2 A multivariable theorem for reduced models.

An important multivariable theorem for reduced models will be derived below, using a result due to Ostrowski[10]. This theorem is very similar to theorem 5.1 of Chapter V except that the radii of the discs are evaluated by using the elements of $Q_r(s)$ and $Q(s)$ in a more straightforward and simple manner.

Theorem 6.3

Let $\rho_{rj}(s)$ (resp. $\hat{\rho}_{rj}(s)$) be the characteristic loci of $Q_r(s)$ (resp. $\hat{Q}_r(s)$) and, further, let $M(s) = \max\{|q_{ij}(s)|, |q_{rij}(s)|\}$ (resp. $\hat{M}(s) =$

$\max\{|\hat{q}_{ij}(s)|, |\hat{q}_{rij}(s)|\}$) and $d_e(s) = (\sum_{i=1}^{m}\sum_{j=1}^{m} |q_{ij}(s) - q_{rij}(s)|)/M(s)$ (resp. $\hat{d}_e(s) = (\sum_{i=1}^{m}\sum_{j=1}^{m} |\hat{q}_{ij}(s) - \hat{q}_{rij}(s)|)/\hat{M}(s)$). Then $\rho_j(s)$ of $Q(s)$ (resp. $\hat{\rho}_j(s)$ of $\hat{Q}(s)$) lies in the union of disc.

$$\text{centre:} \quad \rho_{rj}(s) \quad (\text{resp. } \hat{\rho}_{rj}(s)) \tag{6.14b}$$

$$\text{radii:} \quad (m+2)M(s)\,\{d_e(s)\}^{1/m} \quad (\text{resp. } (m+2)\hat{M}(s)\,\{\hat{d}_e(s)\}^{1/m})$$

Proof: The above is a direct consequence of Ostrowskii's theorem[10]; if α_j, β_j are eigenvalues of A and $B \in C^{m\times m}$, then $|\beta_j - \alpha_j| < (m+2)M\,d^{1/m}$ where $M = \max\{|a_{ij}|, |b_{ij}|\}, \forall ij$, and $d = \Sigma\,|b_{ij} - a_{ij}|$, $\forall ij$. The result follows.

Ostrowskii's theorem can be modified to adjust the sharpness of the bounds. If $Q_r(s)$ is replaced by $Q_r(s) + t(s)I$, then $\rho_{rj}(s)$ is replaced by $\rho_{rj}(s) + t(s)$. Then $\rho_j(s)$ is contained in the disc $|\rho_j(s) - (\rho_{rj}(s) + t(s))| < (m+2)M(s)\,\{d_e(s)\}^{1/m}$, where $d_e(s) = (\sum_{i,j} |q_{ij}(s) - q_{rij}(s) - t(s)\delta_{ij}|)/M(s)$, $\forall ij$, where the Kronecker $\delta_{ij}=1, \forall i=j$, $\delta_{ij}=0, \forall i\neq j$, and, $M(s) = \max\{|q_{rij}(s) + t(s)\delta_{ij}|, \; |q_{ij}(s)|\}, \forall ij$.

The disc centres are replaced by t(s) with appropriate changes in discs radii. The tuning factor t(s) can thus be selected to minimize the area of the discs i.e. $\min\{M(s)(d_e(s))^{1/m}\}$.

A stability theorem based on the encirclement criterion of $\rho_{rj}(s)$ can also be derived from theorem 6.3.

Theorem 6.4

Let $t(s) + \rho_{rj}(s)$ (resp. $\hat{t}(s) + \hat{\rho}_{rj}(s)$) map D into $\Gamma_{rj} + t(s)$ (resp. $\hat{\Gamma}_{rj} + \hat{t}(s)$) where $t(s)$, $\hat{t}(s)$ are convenient 'tuning' factors (in general $t(s) \neq 1/\hat{t}(s)$). Then sufficient conditions for closed loop stability of S are:

(1) $\Gamma_{rj} + t(s)$ (resp. $\hat{\Gamma}_{rj} + \hat{t}(s)$) satisfies NC, $\forall j$

(2) $|1 + \rho_{rj}(s) + t(s)| > (m+2)M(s)\ \{d_e(s)\}^{1/m}$, $\forall j$, $\forall s \epsilon D$ (6.15)

(resp. $|1 + \hat{\rho}_{rj}(s) + \hat{t}(s)| > (m+2)\hat{M}(s)\ \{\hat{d}_e(s)\}^{1/m}$)

Proof: The proof is very similar to that given in theorems 5.2 and 6.1.

By theorem 6.3, condition (2) ensures that all loci of Q(s) cannot overlap the critical point, and by condition (1) stability is assured. The graphical interpretation is shown in figure 6.2.

A similar theorem can be stated in terms of the return difference matrix. Since $Q_r(s)$ is replaced by $I + Q_r(s)$, then $\rho_{rj}(s)$ is replaced by $1 + \rho_{rj}(s)$, which means shifting the critical point from '-1' to the origin. The radii of the discs are governed by $M'(s) = \max\ \{|\delta_{ij} + q_{rij}(s)|,$ $|\delta_{ij} + q_{ij}(s)|\} = \max\ \{|q_{rij}(s)|,\ |q_{ij}(s)|\} = M(s)$, assuming $\max\ \{|1 + q_{rii}(s)|,$ $|1 + q_{ii}(s)|\} \leq \max\{|q_{rij}(s)|,\ |q_{ij}(s)|\}$, $\forall ij$

Theorem 6.4 is very general in its nature and can be used as an alternative to theorem 5.2 and associated theorems where reduced models are concerned.

6.3 Error estimates for multivariable systems designed by reduced models.

It is desirable to relate bounds for multivariable systems of reduced models to that of the original model. This not only aids design considerations, in terms of stability and performance etc., but also gives a measure of validity using reduced models in design. Gunderson and George[19] have made error estimates, on using reduced models in the time domain, via Lyapunov functions, and differential inequalities. This section examines error bounds from the frequency domain point of view.

For a multivariable model with feedback described by

$$x(s) = u(s) - H(s)\ y(s)$$
$$y(s) = Q(s)\ x(s) \tag{6.16}$$

$y(s)$ is related to $u(s)$ by $y(s) = R(s)u(s)$, where $R(s)=Q(s)[I+Q(s)H(s)]^{-1}$ $= Q(s)\ [I+H(s)Q(s)]^{-1}$ where for simplicity assume $H(s) = \text{diag}\ (h_1,h_2,\ldots h_m)$. Consider $r_j(s,H_j)$, that relates the j^{th} output y_j to the j^{th} input u_j, where the j^{th} loop is open but all the other loops closed. Here $H_j = \text{diag}\ \{h_1(s)\ldots h_{j-1}(s),\ 0,\ h_{j+1}(s),\ldots\ h_m(s)\}$. Araki et al.[3] have given a bound for $|r_j(s,F_j) - q_{jj}(s)|$ using M-matrix theory. The well known theorem is reproduced below.

Theorem 6A[3].

Let B (resp $\hat{B}$) be an mxm matrix that satisfies $b_{jj} = 0$ (resp. $\hat{b}_{jj} = 0$) and $b_{ij} \geq |q_{ij}(s)|$ (resp. $\hat{b}_{ij} \geq |q_{ij}(s)|$), $\forall i,\ i\neq j$. Choose $A = \text{diag}\ (a_1,\ldots a_m)$ (resp. $\hat{A} = \text{diag}\ (\hat{a}_1,\ldots\hat{a}_m)$ such that $A - B$ (resp. $\hat{A} - \hat{B}$) is a semi-M-matrix.

For

$$|h_i^{-1}(s) + q_{ii}(s)| > a_i \qquad (\text{resp.}\ \ |h_i(s) + \hat{q}_{ii}(s)| > \hat{a}_i),\ \forall i,\ i\neq j$$

then

$$\begin{aligned} |r_j(s,H_j) - q_{jj}(s)| &< a_j \qquad && \forall a_j > 0 \\ &= 0 && \forall a_j = 0 \\ \text{resp.}\ |r_j(s,H_j)^{-1} - \hat{q}_{jj}(s)| &< \hat{a}_j && \forall \hat{a}_j > 0 \\ &= 0 && \forall \hat{a}_j = 0 \end{aligned} \tag{6.17}$$

It was shown that A can be chosen as small as possible such that A-B is a semi-M-matrix, i.e. A-B is chosen locally minimum in the sense of $A-B-\Delta$ is not a semi-M-matrix for any diagonal matrix $\Delta = \text{diag}\ \{\delta_1,\ \ldots\delta_m\}$ with $\delta_j \geq 0$ and $\Sigma\ \delta_j > 0,\ \forall j$. For a given B, A can be chosen as follows:

(1) By a simultaneous permutation of rows and columns, B can be brought to the normal form (composed of lower triangular sub-matrices)

$$B = \begin{bmatrix} B_1 & & \\ B_{21} & B_2 & 0 \\ B_{v1} & \cdots & B_v \end{bmatrix}$$

where each B_j ($j = 1, .. v$) is an $M_v x M_v$ irreducible matrix[8].

(2) Choose a weight matrix W = block diag $\{W_1, .. W_v\}$ where W_j is a diagonal matrix with positive diagonal elements ($j = 1, ... v$).

(3) Compute the maximum eigenvalue max (λ_j) of $B_j W_j^{-1}$ for $j = 1,.. v$. Then A = block diag $\{\max(\lambda_1)W_1, ... \max(\lambda_v)W_v\}$ makes A-B a locally minimum semi-M-matrix.

It was also shown that trade-off among loops is possible in that $a_i' a_j' = a_i a_j$, $i, j = 1,.. m$, $i \neq j$, where $A' = \text{diag}\{a_1', ..a_m'\}$. Then A'-B is a semi-M-matrix if A-B is a semi-M-matrix.

It is also clear that similar bounds can be written for the reduced model, S_r, using theorem 6A. However, the bounds for S_r will exist independently from those of S, in general, therefore, it is meaningful only to make a comparison between the bounds, by relating S_r to S. In the sequel, $Q_r(s) = G_r(s)K_r(s)$, $q_{rij}(s) = \sum_{k=1}^{m} g_{rik}(s)k_{rkj}(s)$, $H_r(s)=H(s)=\text{diag}\{h_{r1}(s),..h_{rm}(s)\}$ etc.

Theorem 6.5.

Let S_r be bounded as in theorem 6A. Suppose $b_{rij} = |q_{rij}(s)|$, $b_{rii} = 0$ and $A_r = \text{diag}(a_{r1},..a_{rm})$ such that $A_r - B_r$ is a semi-M-matrix and locally minimum. If,

(1a) $a_i \geq a_{ri}$ (resp. $\hat{a}_i \geq \hat{a}_{ri}$) , $\forall i$, $\forall s \in D$ (6.18)

(1b) $|q_{rij}(s)| \geq |q_{ij}(s)|$ (resp. $|\hat{q}_{rij}(s)| \geq |\hat{q}_{ij}(s)|$), $\forall i \neq j$

(2) $|h_{ri}^{-1}(s) + q_{ii}(s)| > a_i$ (resp. $|h_{ri}(s) + \hat{q}_{ii}(s)| > \hat{a}_i$), $\forall i \neq j$, (6.19)

then S is bounded by,

$$|r_j(s,H_{rj}) - q_{jj}(s)| < a_j \ , \quad \forall a_j > 0 \qquad (6.20)$$
$$= 0 \ , \quad \forall a_j = 0$$

$$(\text{resp. } |r_j(s, H_{rj})^{-1} - \hat{q}_{jj}(s)| < \hat{a}_j, \ \forall \hat{a}_j > 0$$
$$= 0 \ , \ \forall \hat{a}_j = 0 \ )$$

Proof: The proof makes use of a theorem of Ostrowski[7,6]. If some elements of a semi-M-matrix are increased without changing their signs, the new matrix is also a semi-M-matrix, i.e. B is a semi-M-matrix if $b_{ii} \geq a_{ii}$, $\forall i$, $0 \geq b_{ij} \geq a_{ij}$, $\forall i \neq j$, given that A is a semi-M-matrix. Condition (1) ensures that A-B is a semi-M-matrix given $A_r - B_r$ is a semi-M-matrix. The result follows from theorem 6A for the original model.

Remarks: In general, $a_j \neq a_{rj}$, hence the models are bounded by circles of different radii on the locus of $q_{jj}(s)$, $q_{rjj}(s)$ respectively. The matrix A is determined in the same way in theorem 6A such that the semi-M-matrix A-B is locally minimum. If the bounds are chosen to be the same radius for both models, i.e. $a_i = a_{ri}$, then eqn. (6.18) is obsolete and the semi-M-matrix $A\text{-}B \equiv A_r\text{-}B$ is not locally minimum. An interesting point to make is that $q_{jj}(s)$ is also bounded by $q_{rjj}(s)$ by bands as given in theorem 5.11 of chapter V.

A special case: Theorem 6A offers bounds on multivariable systems by imposing constraints on the structure of the transfer function matrices in that certain matrices formed must be M-matrices. The diagonal dominance theorem of Rosenbrock for multivariable bounds[1], using a theorem of Ostrowoski[16] is a special case of theorem 6A. Thus,

Theorem 6B[1].

Let $d_i(s) = \sum_{i \neq j} |q_{ij}(s)|$ (resp. $\hat{d}_i(s) = \sum_{i \neq j} |\hat{q}_{ij}(s)|$) and

$\phi_{ri}(s) = \max_{i \neq j} \ d_j(s)/|h_j^{-1}(s) + q_{jj}(s)|$ (resp. $\hat{\phi}_i(s) = \max_{i \neq j} \ \hat{d}_j(s)/|h_j(s)+\hat{q}_{ij}(s)|$)

If $|h_i^{-1}(s) + q_{ii}(s)| > d_i(s)$ (resp. $|h_i(s) + \hat{q}_{ii}(s)| > \hat{d}_i(s)$), $\forall i$ then the multivariable system is bounded by

$$|q_{jj}(s) - r_j(s,H_j)| < \phi_j\ d_j(s) < d_j(s), \quad \forall j \tag{6.21}$$

$$\text{(resp. } |\hat{q}_{jj}(s) - r_j(s,H_j^{-1})| < \hat{\phi}_j\ \hat{d}_j(s) < \hat{d}_j(s) \text{)}$$

Exp. (6.21) represents row dominance of Q (s), and, if the latter is satisfied, the closed loop dynamics are contained in the inner band of circles $\phi_j(s)\ d_j(s)$, given by Ostrowski's theorem for diagonal dominant matrices. The results are also applicable for the case of column dominance by interchanging the subscripts i and j. Theorem 6B can also be proved as a special case of theorem 6A.

Proof: Identifying $d_i(s)$ in exp. (6.21) with a_i in exp. (6.17) and choosing $b_{ij} = |q_{ij}(s)|$, $i \neq j$, where A and B are as defined in theorem 6A, it is seen that A - B is diagonally dominant. Hence its leading principal minors are non-negative, thus A - B is a semi-M-matrix (though it may not necessarily be locally minimum).

A parallel theorem to theorem 6.5 can be obtained for reduced model applications.

Theorem 6.6

Let S_r be bounded as in theorem 6B i.e. w.r.t. 'diagonal dominance' conditions. Then if

(1a) $|q_{ii}(s)| > |q_{rii}(s)|$ (resp. $|\hat{q}_{ii}(s)| > |\hat{q}_{rii}(s)|$), $\forall i$ (6.22)

(1b) $|q_{rij}(s)| > |q_{ij}(s)|$ (resp. $|\hat{q}_{rij}(s)| > |\hat{q}_{ij}(s)|$), $\forall i \neq j$ (6.23)

(2) $|h_{ri}^{-1}(s) + q_{ii}(s)| > |q_{ii}(s)|$ (resp. $|h_{ri}(s) + \hat{q}_{ii}(s)| > |\hat{q}_{ii}(s)|$), $\forall i$ (6.24)

the original model is bounded by

$$|q_{jj}(s) - r_j(s,H_j)| < \phi_j(s) d_j(s) < d_j(s), \quad \forall j \tag{6.25}$$

$$\text{(resp. } |\hat{q}_{jj}(s) - r_j(s,H_j^{-1})| < \hat{\phi}_j(s)\ \hat{d}_j(s) < \hat{d}_j(s) \text{)}$$

Proof: Theorem 6.6 is a special case of theorem 6.5. Choosing $a_i = |q_{ii}(s)|$ and $a_{ri} = |q_{rii}(s)|$, exp (6.18) becomes exp. (6.22), condition (1a) of both theorems. Conditions (1b) and (2) follow exactly as in theorem 6.5, making appropriate symbol changes. Conditions (1) and (2) of theorem 6.6 indicate that S is diagonally dominant, when S_r is likewise so. This is proved in a mathematical way, and it corresponds with intuitive reasoning, that when both models are diagonally dominant, they are bounded in similar forms by exps. (6.21) and (6.25), though the bounds may not necessarily be locally minimum.

The response error, $e(s) = y(s) - y_r(s)$, between original and reduced model can be estimated in a number of ways, using functional analysis and contraction mapping principle, etc. A general theorem can be stated as

Theorem 6.7

The fractional response error between S and S_r is bounded by

$$0 \leq \| y(s) - y_r(s) \| / \| y_r(s) \| \leq \| R(s) - R_r(s) \| \, \| \hat{R}_r(s) \| \tag{6.26}$$

Proof: The lower bound error is obvious, the upper bound will be given as follows:

$$y_r(s) = R_r(s)\, u(s) \tag{6.27}$$

$$y(s)-y_r(s) = (R(s)-R_r(s))u(s)$$

$$\text{i.e.} \quad y(s)-y_r(s) = (R(s)-R_r(s))\hat{R}_r(s)y_r(s) \tag{6.28}$$

$$\therefore \quad \| y(s)-y_r(s) \| \leq \| R(s)-R_r(s) \| \, \| R_r(s) \| \, \| y_r(s) \| \tag{6.29}$$

Exp. (6.26) follows.

Alternative convenient expression to theorem 6.7 can also be obtained, in terms of return difference and return ratio matrices and their inverses. Since $R(s)-R_r(s) = \{\hat{F}(s)G(s) - \hat{F}_r(s)G_r(s)\}\, K_r(s)$ and $\| R_r(s) \| \leq \| \hat{F}_r(s)G_r(s) \| \, \| K_r(s) \|$ and $u(s) = \hat{K}_r(s)\, \hat{G}_r(s)\, F_r(s)$, following the arguments from eqns. (6.27) to (6.29), gives

$$0 \leq \| e(s) \| / \| y_r(s) \| \leq \| \hat{F}(s)G(s) - \hat{F}_r(s)G_r(s) \| \; \| \hat{G}_r(s)F_r(s) \| \, \kappa(K_r(s)\;) \tag{6.30}$$

where $\kappa(K_r(s)) = \| K_r(s) \| \; \| \hat{K}_r(s) \|$. The fractional error in open loop response is

$$0 \leq \| e(s) \| / \| y_r(s) \| \leq \| Q(s) - Q_r(s) \| \quad \| \hat{Q}_r(s) \| \leq \| G_e(s) \quad \| \; \| \hat{G}_r(s) \| \kappa(K_r(s)) \tag{6.31}$$

from exps. (6.29) and (6.30).

Another way of expressing $\|e(s)\|$ is by using the Courant-Fisher min-max relationship[5].

Theorem 6.8

The response error between S and S_r is bounded by

$$\{\min_j \; \eta_j(s)\}^{\frac{1}{2}} \leq \| y(s) - y_r(s) \|_E \leq \{\max_j \; \eta_j(s)\}^{\frac{1}{2}} \tag{6.32}$$

where $\eta_j(s)$ is an eigenvalue of $W(s) = \{R_e(s)u(s)\}\{R_e(s)u(s)\}^H$ where $R_e(s) = R(s) - R_r(s)$.

Proof: From eqn. (6.27), $y(s) - y_r(s) = e(s) = R_e(s)u(s)$

$$\text{i.e.} \qquad e(s)e^H(s) = \{R_e(s)u(s)\} \, \{R_e(s)u(s)\}^H \tag{6.33}$$

Premultiplying both sides of eqn. (6.33) by $e^H(s)$ and postmultiplying by e(s)

$$e^H(s)e(s)e^H(s)e(s) = e^H(s)W(s)e(s)$$

from which

$$\| e(s) \|_E = \langle e(s), W(s)e(s) \rangle / \langle e(s), e(s) \rangle \tag{6.34}$$

Application of the Courant Fisher min-max relationship to eqn (6.34) yields eqn. (6.32).

It is noticed that W(s) is positive definite Hermitian, hence $\eta_j(s) > 0$ and is real, $\forall j$. Also rank (W(s)) = 1, hence the lower error bound $\{\min_j \; \eta_j(s)\} = 0$. When the model orders are equal, $R_e = 0$ i.e. $W(s) = 0$, hence $\max_j \eta_j(s) = 0$, as expected. For the open loop case $R_e(s)$ is replaced by $Q_e(s) = Q(s) - Q_r(s)$ in $W(s)$.

Theorems 6.7 and 6.8 give peak magnitude errors due to sinusoidal inputs. In the time domain, the bounds can be expressed as

$$\| e(t) \| / \| y_r(t) \| \leqslant \| R_e(t) \| \; \| \hat{R}_r(t) \| \tag{6.35}$$

$$\| e(t) \|_E \leqslant \{\max_j \eta_j(t)\}^{\frac{1}{2}}$$

where the Laplace transform operator relates $L\,\{y_r(t)\} = y_r(s)$, $L\{R_r(t)\} = R_r(s)$ etc.

If given the state space matrices $S(A,B,C)$ and $S_r(A_r,B_r,C_r)$, the error $e(t) = y(t) - y_r(t)$ can be computed from the spectral components of these matrices.

Corollary 6.1

$$e(t) = \sum_{i=1}^{n} \exp(-\lambda_i t)[<\beta_i,x(0)>+ \sum_{j=1}^{m} \int_0^t (\exp(-\lambda_i \tau)u_j(\tau)d\tau)$$

$$<\beta_i, b_j>] \; C\alpha_i$$

$$- \sum_{i=1}^{r} \exp(-\lambda_{ri} t)[<\beta_{ri},x_r(0)> + \sum_{j=1}^{m} (\int_0^t \exp(-\lambda_{ri}\tau)u_j(\tau)d\tau)<\alpha_{ri},b_{rj}>$$

$$] \; C_r \; \alpha_{ri}$$

where λ_i, λ_{ri} are the eigenvalues of A, A_r; α_i, β_i^t, α_{ri}, β_{ri}^t are the column and row vectors of the modal matrices of A and A_r respectively, and, b_j, b_{rj} the column vectors of B and B_r. The initial conditions are $x(0)$, $x_r(0)$ and u_j ($j=1, \ldots m$) is the input function. It is seen that for $e(t)$ to be small, λ_{ri} should be close to some dominant λ_i, assuming that the square bracket terms are approximately equal.

Error bounds between S and S_r can also be established in terms of the matrix structural properties of the transfer function.

From $y_r(s) = R_r(s)u(s)$,

$$y_{ri}(s) = r_{rii}(s)u_i(s) + \sum_{j\neq i} r_{rij}(s)u_j(s), \quad \forall i$$

If the j^{th} loop is open but all loops closed, i.e. $H_r(s) = \text{diag}\{h_1(s),.. h_{j-1}(s), 0, h_{j+1}(s),..h_m(s)\}$, then $r_{rii}(s) = r_{rj}(s, H_{rj})$ and the error

$$e_i(s) = r_{ei}(s,H_{ri})u_i(s) + \sum_{j\neq i} r_{eij}(s)u_j(s), \quad \forall i \tag{6.36}$$

where $r_{ei}(s,H_{ri}) = r_i(s,H_{ri}) - r_{ri}(s,H_{ri})$, $e_i(s) = y_i(s) - y_{ri}(s)$, etc. can be expressed in the following bounds.

Theorem 6.9.

If S and S_r are bounded in the sense of M-matrix requirements of theorem 6A, then $r_{ei}(s,H_{ri})$ can be bounded as

(1) $$|r_j(s,H_{rj}) - c(s)| \leq d, \quad \forall j \tag{6.37}$$

(2) $$|r_{rj}(s,H_{rj}) - c(s)| \leq d, \quad \forall j \tag{6.38}$$

(3) $$|r_{ej}(s,H_{rj})| \leq 2d, \quad \forall j \tag{6.39}$$

subjected to $|q_{ejj}(s)|+a_{rj}>a_j$ or $|q_{ejj}(s)|+a_j>a_{rj}$, $\forall j$

If: $|q_{ejj}(s)|+a_{rj}<a_j$, then $d \triangleq a_j$; if: $|q_{ejj}(s)|+ a_j<a_{rj}$, then $d \triangleq a_{rj}$

In the above, $r_{ej}(s,H_{rj}) = r_j(s,H_{rj}) - r_{rj}(s,H_{rj})$, $q_{ejj}(s) = q_{jj}(s)-q_{rjj}(s)$, $c(s)$ is the centre of the circle given by $|c(s)-q_{jj}(s)|+ a_j = |c(s)-q_{rjj}(s)|+a_{rj}$, d is its radius given by $2d = |q_{ejj}(s)| + a_j + a_{rj}$, and, the quantities a_j, a_{rj} are also radii of circles for S and S_r, respectively, as defined in theorem 6A.

Proof: The graphical interpretation of theorem 6.9 is shown in fig.6.3.

Both $r_j(s, H_{rj})$ and $r_{rj}(s,H_{rj})$ are contained in the new circle, centre $c(s)$, and, radii d, where the circumference of the new circle touches the original circles that bound $r_j(s,H_{rj})$ and $r_{rj}(s,H_{rj})$. In the case when one of the circles moves into the other, then the sharper radius $d = a_j$ or or $d = a_{rj}$ is taken. Eqn. (6.39) is explained by the fact that the distance between two points in a circle is always less than or equal to its diameter. Also the same equation, $|r_j(s,H_{rj}) - r_{rj}(s,H_{rj})| \leq 2d$, can also be viewed as $r_j(s,H_{rj})$ being contained in the band of circles, centre $r_{rj}(s,H_{rj})$ and radius, 2d. When $r_{rj}(s,H_{rj}) \to r_j(s,H_{rj})$ meaning $q_{rij}(s) \to q_{ij}(s)$, $\forall ij$, then $2d \to a_j + a_{rj}$.

Corollary 6.2.

Returning to eqn. (6.36), the error in output response can be expressed by theorem 6.7 as

$$|e_i(s)| \leq |r_{ei}(s,H_{ri})||u_i(s)| + \sum_{j\neq i} |r_{eij}(s)||u_j(s)|$$

where from theorem 6.9 and assuming a unit impulse,

$$|e_i(s)| \leq |q_{eii}(s)| + a_i + a_{ri} + \sum_{j\neq i} |r_{eij}(s)| \tag{6.40}$$

In matrix form, eqn. (6.40) becomes

$$\| e(s) \| \leq \| Q_{eii}(s) + (a_i+a_{ri})I \| + \| R_{eij}(s) \| \tag{6.41}$$

where $Q_{eii}(s) = \text{diag}\,\{q_{e11}(s), \ldots q_{emm}(s)\}$ and $R_{eij}(s) \triangleq r_{eij}(s)$, $\forall i\neq j$, $R_{eii}(s) = 0$, $\forall i=j$.

Eqn. (6.40) expresses a bound for a particular output. The radii a_j, a_{rj} in one loop can be made as small as possible by trade-off among loops $a_i^! \; a_j^! = a_i \; a_j$, $a_{ri}^! \; a_{rj}^! = a_{ri} \; a_{rj}$.

Parallel to theorem 6.9 the error bound can also be expressed for diagonal dominant systems.

Theorem 6.10.

Let S and S_r be bounded in the sense of diagonal dominance conditions, theorem 6B. Then $r_{ei}(s,H_{ri})$ is bounded exactly as in theorem 6.9, except that the radius d of the new circle is given by

$$2d = |q_{ejj}(s)| + \phi_j(s)d_j(s) + \phi_{rj}(s)d_{rj}(s) \quad (6.42)$$

(It is noticed here that $d \neq d_{rj}(s)$, where $d_{rj}(s) = \sum_{j\neq i} |q_{rij}(s)|$ and the symbols $\phi_j(s)$, $\phi_{rj}(s)$ have their usual meanings as given in theorem 6B.)

Proof: The proof and geometrical interpretation follow exactly as that given in theorem 6.9.

Corollary 6.3.

When the models are diagonal dominant, then $\sum_{i\neq j} r_{ij}(s) \approx 0$, $\sum_{i\neq j} r_{rij}(s) \approx 0$, hence following eqns. (6.40) and (6.41)

$$|e_i(s)| \leqslant |q_{eii}(s)| + \phi_i(s)d_i(s) + \phi_{ri}(s)d_{ri}(s)$$

$$\text{i.e.} \quad \| e(s)\| \leqslant \| Q_{eii}(s) + (\phi_i(s)d_i(s)+ \phi_{ri}(s)d_{ri}(s))I\|$$

Error estimates by contraction mapping principle.

The response error can also be estimated by using the contraction mapping principle in functional analysis[21]. (see Chapter V) In a Banach space of holomorphic transforms, it is well known that the error involved in the n^{th} iterate, $y_{rn}(s)$ is

$$d\{y_{r*}(s) - y_{rn}(s)\} \leqslant [\| W_{Ar}\|^n /1 -\| W_{Ar}\|] \; d\{y_{r1}(s) - y_{ro}(s)\} \quad (6.43)$$

where $y_{ro}(s)$, is the initial point, $y_{r1}(s) = W_{Ar}y_{ro}(s)$, the first iterate and $y_{r*}(s)$, the final solution. Choosing $y_{ro}(s) = 0$ and $n = 1$, eqn. (5.49) of Chapter V becomes

$$y_{r1}(s) = (I+A_r(s))^{-1} Q_r(s)u(s)+(I+A_r(s))^{-1}z_r(s) \quad (6.44)$$

which means eqn. (6.44) is a first approximation for $y_{r*}(s)$, the approximation being involved, because A only approximately represents $Q_r(s)$. Hence from eqn. (6.43), the fractional error, $e_{rf}(s)$, resulting from designing the system based on $A_r(s)$ is

$$e_{rf}(s) = \| y_{r*}(s) - y_{r1}(s) \| / \| y_{r1}(s) \| \leq \| W_{Ar} \| / 1 - \| W_{Ar} \| \tag{6.45}$$

This has a simple graphical interpretation shown in fig.6.4. Since $W_{Ar} = (I+A_r)^{-1} (Q_r(s)-A_r)$, and, if $A_r(s) = \text{diag}\{q_{rjj}(s)\}$, $\forall j$, then $e_{rf} \leq n_o/(d_o-n_o)$ where $\| W_{Ar} \| = n_o/d_o$. After $K_r(s)$ is implemented on the original model, S, a similar equation to eqn. (6.45) can be written.

$$e_f(s) = \| y_*(s) - y_1(s) \| / \| y_1(s) \| \leq \| W_A \| / 1 - \| W_A \| \tag{6.46}$$

where $e_f(s)$ is the error involved in designing the system Q(s) by the matrix A(s). It is useful to express $e_f(s)$ in terms of $\| W_A - W_{Ar} \|$, $\| W_{Ar} \|$ and $e_{rf}(s)$.

Since $\| W_A \| \leq \| W_{Ar} \| + \| W_A - W_{Ar} \|$, eqn. (6.46) gives

$$e_f(s)/(1+e_f(s)) \leq \| W_{Ar} \| + \| W_A - W_{Ar} \|$$

and rearrangement gives

$$e_f(s) \leq (\| W_{Ar} \| + \| W_{Ae} \|)/(1 - \| W_{Ar} \| - \| W_{Ae} \|) \tag{6.47}$$

where $W_{Ae} = W_A - W_{Ar}$. Letting $\alpha_r = \| W_{Ar} \| / 1 - \| W_{Ar} \|$ and $E = \| W_{Ae} \| / 1 - \| W_{Ar} \|$, eqn. (6.47) gives

$$e_f(s) \leq (\alpha_r + E)/(1 - E) \tag{6.48}$$

It is noticed that for stability of S, $\| W_A \| < 1$ and it was proved that a tighter condition for stability of both S and S_r is $1 - \| W_{Ae} \| < 1$, given that $\| W_{Ar} \| < 1$. Hence $0 < 1 - \| W_{Ar} \| - \| W_{Ae} \| < 1 - \| W_{Ar} \| < 1$. Thus $e_f(s) \geq 0$ in eqn. (6.47). Further $0 < E < 1$, and this with eqns. (6.45) and (6.48) gives

$$e_{rf}(s) \leq (\alpha_r + E)/(1 - E) \tag{6.49}$$

Thus eqns. (6.48) and (6.49) becomes

$$0 \leqslant \max \{e_f(s), e_{rf}(s)\} \leqslant (\alpha_r + E)/(1 - E) \tag{6.50}$$

Eqn. (6.50) states that the response errors for both models are bounded by a common bound, when S is stable, given S_r is stable.

The graphical interpretation of eqn. (6.45) is shown in fig.6.4 for the special case when S_r is diagonal dominant. The graphical interpretation for eqn. (6.50) is similar to that shown in fig.5.14.

6.4 Adaptation of reduced models to multivariable systems design.

This section studies the application of reduced models to existing multivariable design techniques by frequency response and time domain methods. The approach is by considering single input-single output systems, then generalizing to multivariable systems, where convenient, otherwise, the approach is to consider the problem in its multivariable form.

Single input-single output systems (s.i.s.o.).

It was shown in Chapter V that the classical Nyquist and root locus of s.i.s.o. models are bounded by bands of circles centred on those diagrams of the reduced model, theorems 5.7 and 5.21. Stability is guaranteed if the bands satisfy the encirclement theorem or avoid the imaginary axis. These single loop ideas can be used in later multivariable designs, after the reduced models are obtained, say, from the reduction techniques given in Chapters III and IV.

6.4.1 Inverse Nyquist Array (INA) method[24,1]

The reduction and design are achieved by two suggested propositions[26]. It is assumed that S and S_r are open-loop stable.

Proposition 6.1.

It interaction effects are negligible and accurate reduced models are obtained for the diagonal terms of the transfer function matrix, particularly retaining certain dominant modes and maintaining the shape of the root locus diagram in the vicinity of the origin, it is possible to design the diagonal compensator $K_c(s)$ with a predictable amount of stability margin, such that S is stable with good dynamics within the range of design gains in which S_r is likewise stable.

Proof:

It was shown in theorem 5.21 that the root locus of S is contained in a band of circles, centred on the root locus diagram of S_r. The art of the procedure is to choose tuning factors α and β, such that the circles are made as small as possible, such that stability margin is high and performance error between the two models is small. The root locus is basically a 'reduction method' in that a dominant second order system is heavily used in the approximation, the latter being very successful from past design experience. Similarly, by theorem 5.7 the single loop inverse Nyquist design can be used.

Proposition 6.2.

As far as interactions are concerned, it is sufficient to ensure that S will be closed loop stable within the close range of gain values for which S_r is likewise stable, if and only if S_r is diagonal dominant, and the interaction structures of S and S_r are similar, in the sense that the ratio of proportionality of the interacting terms to the diagonal terms of the inverse transfer function is the same.

Proof:

The proof makes use of the extension of the diagonal dominance theorem of theorem 6B. By theorem 5.9 it was shown that a sufficient condition for stability of S is that S should be diagonal dominant. If diagonal dominance is achieved by matrix operations such that $|\hat{q}_{rii}(s)| > \hat{d}_{ri}(s)$, $|\hat{r}_{rii}(s)| > \hat{d}_{ri}(s)$ where $\hat{d}_{ri}(s) = \sum_{i \neq j} |\hat{q}_{rij}(s)|$, then the matrix

$K_a\ K_b(s)$ required to achieve diagonal dominance would be the same for both models as their interaction structures are similar. Hence S is closed loop stable, as far as interactions are concerned, in the presence of $K_a\ K_b(s)$.

Proposition 6.2 is only a sufficient condition[1]. Failure to satisfy it does not imply closed-loop instability, however, if proposition 6.2 is satisfied, proposition 6.1 is both necessary and sufficient. As was shown in theorem, 6B, the stability margin is determined by the Gershgorin circle bands of radii $d_{ri}(s)$, and dynamic information of a loop is concentrated in the narrower band of radii $\phi_{ri}(s)\ d_{ri}(s)$. Since the interaction structure is closely maintained, the radii for both models, by theorem 6.2, would be approximately equal for the range of gain between zero and k_{rj}.

Preserving interaction structure.

The i^{th} output can be written as

$$y_{ri}(s) = \sum_{j=1}^{m} g_{rij}(s) u_j(s)\ ,\ \forall i \tag{6.51}$$

where each term can be considered independently. However, in general, this will not preserve the interaction structure. Neither will it if $G_r(s)$ is obtained in whole. This fact, which is important in design in terms of stability, has been neglected or ignored in many reduction methods.

Davison et al.[25] have investigated the severity of interaction in multivariable closed loop systems by computing the interaction index as a ratio of two performance indices from the state space equations,

$$I^j(\theta_1, \ldots \theta_m) = (J_j^* - J_j)/J_j \tag{6.52}$$

where $J_j = \max \int_0^\infty y_j^2(t)\,dt$, $J_j^* = \int_0^\infty y_j^{*2}(t)\,dt$, $1 \leq j \leq \ell$ and $y_j(t)$, $y_j^*(t)$ satisfy the equations

$$\dot{x} = \left[A + \theta_j \sum_{i=1}^{n} (-k_i^j\ B_i) d_j^t\right] x, \quad y = Dx$$

$$\dot{x} = \left[A + \theta_1 \sum_{i=1} (-k_i^1\ B_i) d_1^t + \theta_2 \sum_{i=1} (-k_i^2\ B_i) d_2^t + \ldots \right. \tag{6.53}$$

$$\left. \theta_m \sum_{i=1} (-k_i^m B_i)\ d_m^t \right] x\ , \quad y^* = Dx$$

In the above, output feedback $u(t) = u_o(t) + \theta_j(-k_1^i,..-k_n^j)^t y_j$, is used and it is assumed that the m controllers are stable. The interaction index, eqn. (6.52) gives a measure of the relative change in the control of $y_j(t)$ when all the controllers are simultaneously applied to the system compared with the j^{th} controller only being applied to the system. Davison's method of computing the interaction index is numerically demanding and it involves the solution of Lyapunov's equation.

A different definition of interaction will be defined here. In the frequency domain, the degree of interaction between two consecutive loops can be given as the ratio of the magnitude response between the off-diagonal and diagonal terms. As the inverse transfer function matrix gives an easy transition from open loop to closed loop systems, $\hat{G}_r(s)$ will be considered here. Define

$$\hat{f}_{rij} = |\hat{g}_{rij}(s)|_{i\neq j}/|\hat{g}_{rii}(s)| \tag{6.54}$$

$$\hat{m}_{rij} = |\hat{g}_{rij}(s)|_{i\neq j}/|\hat{g}_{rjj}(s)| \tag{6.55}$$

To preserve the same interaction pattern, it is desirable that

$$\hat{f}_{rij} = \hat{f}_{ij} \tag{6.56}$$

$$\hat{m}_{rij} = \hat{m}_{ij} \tag{6.57}$$

for $0 < \omega < \infty$. Infinite sets of frequency dependent parameters exist that satisfy eqns. (6.56) and (6.57) separately and if the diagonal ratios are such that $|\hat{g}_{rii}(s)|/|\hat{g}_{ii}(s)| = |\hat{g}_{rij}(s)|/|\hat{g}_{jj}(s)|$, satisfaction of one equation also satisfies the other with the same parameters. In general, a fixed set of constant parameters can be obtained by minimizing the performance index over a spectrum of interest.

$$J_\omega = \int_\Omega \{(\hat{f}_{rij}(\omega)-\hat{f}_{ij}(\omega))^2+(\hat{m}_{rij}(\omega)-\hat{m}_{ij}(\omega))^2\}d\omega \tag{6.58}$$

In the ideal case, the bracketed terms vanish and the error is zero. For severe interactions, $\hat{f}_{ij} > 1$, the following index

$$J'_\omega = \int_\Omega \{(\hat{f}_{rij}/\hat{f}_{ij}) - 1\}^2 + \{(\hat{m}_{rij}/\hat{m}_{ij}) - 1\}^2 \quad d\omega \tag{6.59}$$

is suggested for faster convergence near the optimum. If the diagonal ratios are equal, then only one term in eqn. (6.58) need be considered.

To alleviate computational burden, where possible, certain dominant modes of G(s) should be retained. As usual, non-minimum phase can give rise to design difficulties and any r.h.p. or ℓ.h.p. zeroes must be separately constrained in $G_r(s)$. This uniformity makes it easier to assess stability margins. By virtue of its flexibility, the i.n.a. is more tolerant to model inaccuracies; hence it is not necessary that the interaction pattern be rigidly maintained, but should be such that the same $K_aK_b(s)$ controller matrix is applicable to both models and the remaining stability margins be determined by single-loop approaches alone. Also, any valid s.i.s.o. reduction method can be used on the diagonal terms of G(s), and the off-diagonal terms can then be obtained in conformity with the interaction pattern suggested above.

Algorithm for i.n.a. design using reduced models.

Step 1. Design $\hat{K}_{rb}(s)\hat{K}_{ra}(s)$ to obtain stability by diagonal dominance of S_r and design compensator $\hat{K}_{rc}$ for each individual loop.

Step 2. Test for diagonal dominance of S by theorem 6B and stability of S by theorem 5.9. (Note: Diagonal dominance is automatically achieved if design is done according to Proposition 6.2.) If diagonal dominance achieved, proceed, otherwise go to Step 1 and redesign $\hat{K}_{ra}(s)$.

Step 3. Determine output error between S and S_r by theorems 6.7 or 6.8. If tolerable exit, otherwise go to Step 1.

6.4.2 Characteristic Loci design[28,29]

It was shown in Chapter V that the characteristic loci of S are bounded by bands of circles determined by the characteristic loci of S_r. In design, compensators can be employed to shape the loci of S_r to obtain the required dynamic response. The bands are thus shaped according to how the loci of S_r are shaped. Before going into the algorithm for Characteristic Loci design, an integrity theorem is stated for design using S_r.

Macfarlane et al.[29] have given an integrity theorem in which closed loop stability can be studied resulting from error monitoring, actuator and transducer failures. For a reduced model, let

$$T_{re}(s)\ U = H_r(s)G_r(s)K_r(s)\ U$$

$$T_{ru}(s)\ U = K_r(s)H_r(s)G_r(s)\ U$$

$$T_{ry}(s)\ U = G_r(s)K_r(s)H_r(s)\ U$$

where $T_r(s)$ are return ratio matrices and U is a switch matrix having all diagonal elements 1 (for a normally operating channel) or a small quantity ε (for a failed channel), and, let $\rho_{rij}(s)$, $i=1,\ldots j$ be the characteristic loci of a principal sub-matrix T_{sj}, $j=1,\ldots m$ of $T_{re}(s)$, or $T_{ru}(s)$ or $T_{ry}(s)$. Then Theorem 6C[29].

The reduced model, S_r, has high integrity against a failure condition if, and only if,

(1) The locus $\rho_{rij}(s)$ satisfies NC , $\forall i$, $\forall j$.

A similar theorem can be stated for S, in terms of S_r.

Theorem 6.11.

Let $\rho_{rij}(s) + \beta$ map D into $\Gamma_{rij} + \beta$. Then S has high integrity against all failure conditions if:

(1) $\Gamma_{rij} + \beta$ individually satisfies NC, $\forall i$, $\forall j$

either

(2) $|1+\beta+\rho_{rij}(s)| > \| G_e(s)K_r(s)H_r(s)-\beta I\| \inf\kappa(P(s)), \ \forall i, \forall j, \forall s\epsilon D$ (6.60)

or

(3) $|1+\beta+\rho_{rij}(s)| > (m+2)M(s)\{d_e(s)\}^{1/m}, \forall i, \forall j, \forall s\epsilon D$ (6.61)

where $M(s)=\max\{|t_{rij}(s)+\beta\delta_{ij}|, |t_{ij}(s)|\}$, $d_e(s)=(\sum_{i,j}|t_{rij}(s)-t_{ij}(s)-\beta\delta_{ij}|)/M(s)$

Proof: Conditions (1) and (2) ensure that all characteristic loci of the principal sub-matrices of T(s) satisfy Nyquist criterion (see Theorem 5.1). Similarly, conditions (1) and (3) ensure that all loci of the sub-matrices of T(s) do likewise (see Theorem 6.3). Hence by theorem 6C, S has high integrity against all failure conditions.

Another convenient way of interpreting theorem 6C in terms of reduced models is via the structural properties of $T_r(s)$ and its diagonal elements.

Thus Fan's theorem, Gershgorin's theorem and the theorems of M-matrices (see Chapter V) can be used as an alternative to evaluate theorem 6C.

Theorem 6.12.

Let $t_{rjj}(s) + \beta$ map D into $\Gamma_{rj} + \beta$, where $t_{rjj}(s)$ is a diagonal element of the return ratio matrix $T_r(s)$. Define the matrix $C_r(s)$ where $c_{rjj}(s) = |1 + t_{rjj}(s)|$, $c_{rij}(s) = -|t_{rij}(s)|$, $i\neq j$. Then sufficient conditions for S to have high integrity against a failure condition (depending on how T(s) is specified) are:

(1) $\Gamma_{rj} + \beta$ individually satisfies NC, $\forall j$

(2) $C_r(s)$ is an M-matrix.

(3) $|1+\beta+t_{rjj}(s)| > |t_{ejj}(s)-\beta|, \forall j, \forall s\epsilon D$

(4) $|1+t_{jj}(s)| \geq |1+t_{rjj}(s)|$; $|t_{ij}(s)| \leq |t_{rij}(s)|, \forall j, \forall s\epsilon D$

where $t_{ejj}(s) = t_{jj}(s)-t_{rjj}(s)$ and $t_{jj}(s)$ is a diagonal element of T(s).

Proof:

Conditions (1) and (2) ensure that S_r is stable in the sense of M-matrix requirements (see theorem 5C). Condition (3) ensures that the Nyquist locus of $t_{jj}(s)$ satisfies the encirclement criterion by theorem 5.7. Condition (4) (Ostrowskii's theorem, see the last condition of theorem 5.11) ensures that the matrix C(s) of S is an M-matrix, given that $C_r(s)$ is an M-matrix. Thus S is stable, by theorem 5C, and, by theorem 5.13 (KY Fan's theorem) it is seen that all the characteristic loci of T(s) satisfy the encirclement criterion. Now since every principal sub-matrix of C(s) (an M-matrix) is an M-matrix, it follows from theorems 5C and 5.13 that all characteristic loci of every principal sub-matrix of T(s) satisfy the encirclement criterion. Hence, by theorem 6C, S has high integrity against a failure condition.

As a special case, if the matrix T(s) is diagonal dominant and its diagonal elements satisfy the Nyquist criterion, it can easily be deduced from theorem 5.12 that S has high integrity. (If C(s) is diagonal dominant, then it is an M-matrix). Alternatively, from Gershgorin's theorem it can also be seen that all characteristic loci of the principal sub matrices of T(s) satisfy the Nyquist criterion.

It is well known that the Characteristic Loci design has four phases: (1) stability phase, (2) integrity phase, (3) interaction phase, (4) performance phase.

Stability and integrity are effected by modifying phases of appropriate sets of loci

$$\sum_{i=1}^{m} \text{phase } (\rho_{ri}(s)) = \text{phase}\{\det(K_r(s))\} + \sum_{i=1}^{m} \text{phase } (\lambda_{ri}(s))$$

where $\rho_{ri}(s)$ and $\lambda_{ri}(s)$ are the loci of $T_r(s)$ and $G_r(s)$ respectively.

Interaction is effected by aligning the characteristic directions at high frequencies and balancing the gains at low frequencies to get acceptable interaction. From

$$R_r(s) = \sum_{i=1}^{m} (q_{ri}(s)/1+q_{ri}(s))\alpha_{ri}(s)><\beta_{ri}(s)$$

where $\alpha_{ri}(s)$ and $\beta_{ri}(s)$ are a pair of dyadic vectors, for $s\to\infty$, $|q_{ri}(s)|<<1$, hence $R_r(s)\to\Sigma\ q_{ri}(s)\alpha_{ri}(s)><\beta_{ri}(s)=Q_r(s)$. It follows that to achieve low interaction, it is necessary that $Q_r(s)\to I$ as $s\to\infty$. The angle of alignment is $\text{Cos}\theta_j(s)=|<\alpha_{rj}(s),e_j>|/\|\alpha_{rj}(s)\|$ where e_j is the j^{th} column of a unit matrix. Thus if $\theta_j(s)$ is small at high frequencies, interaction arising from the j^{th} input will be correspondingly small. For low frequencies, usually $|q_{rj}(s)|>>1$, $\forall j$, hence $R_r(s)\to\Sigma\ <\alpha_{ri}(s),\beta_{ri}(s)> = I_m$.

Satisfactory overall performance is achieved by injecting gain into the phase compensated and aligned system.

$$\Sigma\ \ln|\rho_{rj}(s)| = \ln\det|K_r(s)| + \sum_{i=1}^{m} \ln|\lambda_{ri}(s)|$$

The design using S_r, including check on stability and integrity of S, can be done using the following algorithm.

Step 1. Design $K_r(s)=\pi K_{rj}(s), j=1,\ldots$ accumulatively, such that $G_r(s)K_r(s)$ is stable by theorem 5A.

Step 2. Test for stability of S by theorem 5.3. If stable proceed, otherwise go to Step 1.

Step 3. Test for integrity of S (assuming S_r has high integrity) by theorem 6.11. If satisfactory, proceed, otherwise go to Step 1 and redesign $K_{ri}(s)$.

Step 4. Determine output error e(t) between S and S_r by theorems 6.7 or 6.8 (assuming performance phase of S_r is satisfactory). If tolerable, exit, otherwise go to Step 1.

In the above it is assumed that open loop error between S and S_r is fairly small.

6.4.3 Multivariable systems design by sequential method[30].

Mayne introduced a design method where the compensator matrix is designed by a succession of single loop approaches using Nyquist loci. Each loop is designed separately and then closed to see the effect on the overall stability of the system. As the loops can be renumbered, they can be closed in any sequence and design accordingly. The method also allows attenuation of external disturbance and provides security against component failures.

For a full explanation of nomenclature, refer to Mayne's design algorithm[30].

(i) Record inputs and outputs.

(ii) Choose $K^1(s)$. Set $G^{0,\alpha}(s) = G^{\alpha}(s)K^1(s)$, $\alpha=1,...m$.

(iii) Set $i=1$. Set $t_i^{\alpha}(s) = 1 + k_i(s)g_{ii}^{i-1,\alpha}(s)$, $\alpha=1,..m$. Choose $k_i(s)$.

(iv) If $i=m$ stop. Otherwise for $\alpha=1,..m$, set $\bar{k}_i^{\alpha}(s)=k_i(s)/t_i^{\alpha}(s)$.

Set $G^{i,\alpha}(s)=G^{i-1,\alpha}(s)-\bar{k}_i^{\alpha}(s)\ g_{.i}^{i-1,\alpha}(s)\,g_{i.}^{i-1,\alpha}(s)$

(v) Choose $K^{i+1}(s)$ $(K^m(s)=I_m)$. Set $G^{i,\alpha}(s)=G^{i,\alpha}(s)\ K_c^{i+1}(s)$, $\alpha=1,..m$

Set $i=i+1$

(vi) Go to (ii).

The final compensation matrix $K_c(s) = \pi K^i(s)G^i(s)$ and $t^i(s)$ is generated by $G^i(s)=G^{i-1}(s)-\bar{k}\ (s)g_{.i}^{i-1}(s)g_{i.}^{i-1}(s)$ and $t^i(s)=1+k_i(s)g_{ii}^{i-1}(s)$.

The closed loop input-output equation is

$$y(s) = \hat{T}(s)\ G(s)\ K(s)\ y_d(s) + \hat{T}(s)\ n(s)$$

where $\hat{T}(s)$ is the inverse return ratio matrix and n(s) the external disturbance. For the system to reduce the effect of external disturbance in

$\omega \in \Omega$, it is required that $\|\hat{T}(s)\| < 1$ for $s\in\Omega$. Since $\det T_i(s) = \prod_{j=1}^{m} t_j(s)$, by applying theorem 5B, the stability of the system at the stage of closing the i^{th} loop can be investigated by the encirclement criterion of $t_j(s)$, $j=1, \ldots i$.

Algorithm for sequential design using reduced model, S_r

Step 1. Record inputs and outputs.

Step 2. Design i^{th} loop by sequential method, using S_r, such that stability criterion is satisfied and performance satisfactory.

Step 3. Predict stability of original model, S, if $K_r(s)$ is implemented on G(s), when i^{th} loop is closed, by theorems 5.3, 5.9 or 5.11. If stable proceed, otherwise to go Step 2.

Step 4. Determine output error between S and S_r by theorems 6.7 or 6.8. If tolerable, proceed. Otherwise go to Step 2.

Step 5. Repeat for $i=i+1$ from Step 2.

Step 6. Test if $\|\hat{T}(s)\| < 1$. If yes , exit, otherwise go to Step 2.

Design algorithms, using S_r, for other frequency design methods such as dyadic expansion[32] and commutative controller technique[31] can be similarly generated. The stability criterion used is loci encirclement, thus theorems 5.3 and 6.4 are general in their applications.

6.4.4 Design by pole shifting techniques[34-39] using S_r.

Wonham[34] showed that controllability of an open loop system is equivalent to assigning an arbitrary set of poles to the transfer matrix of the closed-loop system by linear state feedback. For the system

$$S : \quad \dot{x} = Ax + Bu \quad , \quad y = Cx \tag{6.62}$$

state feedback $u = Kx$ is introduced such that $A_c = A + BK$ has a specified set of eigenvalues. If some states are inaccessible, Davison[35] has shown that using output feedback, $u = K^*y$, a constant matrix K^* can always be found such that ℓ eigenvalues of the closed-loop system A+BKC are arbitrary close (but not necessary equal) to ℓ preassigned values. Here $\ell \leq n$ where ℓ = rank (c).

MacFarlane et al.[37] gave a simple pole shifting algorithm, derived from the Hsu-Chen theorem, for design. However, K_r is unity rank and sacrifices much design freedom. In general, the problem of determining K or K* is non-unique and non-linear for multivariable systems[39]. Several methods due to Fallside et al.[36], Munro et al.[38] and Sridhar et al.[38] are proposed that circumvent the non linear problem, but these methods restrict the degree of freedom in choosing K at the expense of linearity. Paraskevopoulos[39] and Tzafestas have introduced a general method that yields a family of K and several linear methods of obtaining K are also suggested. Also, the problem of pole assignment with respect to minimum eigenvalue sensitivity to parameter variations was studied by Gourishankar et al.[38].

The main interest here is to study the following problem.

Given

$$S_r \; : \; \dot{x}_r = A_r x_r + B_r u \quad , \quad y = C_r \, x_r \tag{6.63}$$

$$u = K_r \, x_r \quad \text{or} \quad u = K_r^* \; y$$

$$x_r = Zx$$

determine $u = K_r \, x_r$ or $K_r^* \, y$ such that when implemented on the system S of eqn. (6.62), the closed loop system $A_{rc} = B + BK_r \, Z$ (state feedback) or $A_{rc} = A + BK_r^* \, C$ (output feedback) is stable.

It was shown that for reduced models obtained by projection methods, the eigenvalues of $A + BK_r \, Z$ are those of the reduced model $A_r + B_r \, K_r$ and the remaining n-r eigenvalues of the open loop matrix A[43,46]. However, this is not the case for non projective model reduction techniques[46]. In this case, theorem 5.23 can be used in the design algorithm below.

Algorithm for reduced model design by pole shifting.

Step 1. Determine K_r or K_r^* based on a set of assigned eigenvalues such that S_r is stable.

Step 2. Test for stability of S by checking eigenvalues of $A+BK_rZ$ or $A+BK_rC$ by theorem 5.23. If stable go to Step 4. Otherwise proceed.

Step 3. Modify spurious eigenvalues or try to reduce circle size by theorem 5.23. Test for stability of S. If stable, proceed. Otherwise, modify the set of pre-assigned eigenvalues and go to Step 2.

Step 4. Determine output error between S and S_r by theorems 6.7 or 6.8. If satisfactory, exit. Otherwise go to Step 3.

One of the chief inaccuracies of design by pole shifting is that the control of the transmission zeroes of the system is not possible, for the latter plays an important role in dynamic response.[4]

6.4.5 Design by multivariable root loci using S_r[40-42].

Single loop design using the classical root locus method can be done with relative ease on S_r. Theorem 5.21 is of main importance for checking the bounds of the c.ℓ.c.p. of S. The root locus of S is bounded by bands of circles centred on the root locus of S_r. Stability and transient response of S can be assessed at each stage of the design on S_r.

MacFarlane et al.[40,41] extended the root locus technique to multivariable systems in terms of characteristic frequency loci. The algebraic function

$$\Delta(\ell,s) = \det\left[\ell I_m - Q(s)\right] = 0 \qquad (6.64)$$

that is irreducible over the field of rational functions in s, is regarded as a polynomial in s with coefficients that are rational functions of ℓ, i.e.

$s(\ell)$, is considered. The function $s(\ell)$ is called the root loci for multivariable systems and can be defined on their appropriate Riemann surfaces (see also Chapter V, section 5.4).

An alternative expression for obtaining the multivariable root loci is given by

$$\begin{aligned} &\ell_j(s) = -1/k \\ &\det\,(sI-A-BC/\ell_j) = 0 \end{aligned} \qquad (6.65)$$

where $\ell_j(s)$ is the function, $\Delta(\ell,s)$, in eqn. (6.64), this time being considered as a polynomial in ℓ with coefficients that are rational functions in s, and the triple S(A,B,C) is a minimal realization of the transfer function matrix Q(s). The root loci are then calculated as the eigenvalues of $\{A+(BC)/\ell_j\}$ as $\ell_j(s)$ (hence as gain k) varies. This method includes all singular point loci, if any. Theorem 5.22 can be used in this case for design on S using S_r.

Kouvaritakis and Shaked[42] computed the multivariable root loci from the state space matrices and the c.ℓ.c.p. of S, as the gain parameter k varies. The c.ℓ.c.p. is given by

$$\begin{aligned} \Delta(s,k) &= \det(s\,I_n-A+kBKC) \\ &= \det(sI-A)\det(I_m+kC(sI-A)^{-1}BK) \\ &= \det(sI-A)\det(I_m+kG(s)K) \end{aligned} \qquad (6.66)$$

For a given K, the solution of $\Delta(s,k) = 0$, for s in terms of k gives the root loci of the system S. It has also been shown that all the n root loci begin at the open loop poles (k=0), each n_z locus terminates at a system finite zero (invariant zero) and the remaining $n-n_z$ loci tend to infinity. The n_z finite zeroes are the roots of the equation

$$\det\,(sNM - NAM) = 0$$

where the rows and columns of the full rank matrices N and M span the left null-space and kernel of B and C, respectively. The invariant zeroes are also given by[23]

$$z(s) = \det(P(s)) = 0$$

$$\text{where} \quad P(s) = \begin{bmatrix} sI - A & -B \\ C & 0_{m,m} \end{bmatrix}$$

is the Rosenbrock system matrix. Alternatively, z(s) can also be interpreted as the product of the numerator zeroes of M(s), the Smith MacMillan form of G(s) while p(s) is the product of the denominator poles of G(s)[23,41].

In the case of Kouvaratakis et al.[42] a similar form of theorem 5.19 or theorem 6.3 is applicable for design on S using S_r.

Algorithm for multivariable root loci design using S_r.

Step 1. Plot the c.ℓ.c.p. (i.e. multivariable root loci) of S_r, as a function of gain parameter, k by solving the polynomial $\ell_r(s) = -1/k$ (excludes singular point loci), or by computing the eigenvalues of $\{A_r+(B_rC_r)/\ell_{rj}\}$ as $\ell_{rj}(s)$ (hence as k) varies.

Step 2. Draw circle bands on the root loci of S_r. Use theorem 5.20 if $\ell_r(s) = -1/k$ is considered or use theorem 5.22 if loci of S_r are computed from eigenvalues of $A_r+(B_rC_r)/\ell_{rj}$ or from eigenvalues of NA_rM. These circles contain the root loci of S.

Step 3. Reduce circle size, by adjusting 'tuning' factors or modifying spurious eigenvalues, to obtain high stability and good transient characteristics for S, with respect to S_r.

Step 4. Determine output error, e(t) between S and S_r by theorems 6.7 or 6.8. If tolerable, exit. Otherwise go to Step 3.

6.5 Conclusions.

This chapter investigates some error bounds in multivariable systems designed using reduced models. The bounds are expressed mainly in terms of transfer function matrices, however, they can also be modified to cater for time domain design methods. Together, with the stability conditions developed in Chapter V, they form a general design philosophy of systems by reduced models, using any multivariable design techniques. The later half of the chapter adapts these bounds into design algorithms for existing design methods. A consequence of these analytical error bounds is that it offers a good estimation of accuracy when design is made on reduced models, or when the order of the plant is different from that of the model.

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VII

SENSITIVITY AND OPTIMALITY SYNTHESIS USING REDUCED MODELS.

CHAPTER VII

SENSITIVITY AND OPTIMALITY SYNTHESIS USING REDUCED MODELS

Introduction

This chapter investigates the sensitivity and optimality characteristics of a system designed using reduced models. In some aspects the two characteristics are intimately related, hence they are studied together.

It is well known that feedback can provide a reduction of sensitivity, to variations of plant parameters, from their nominal values, and, to added disturbance signals. There are many definitions of sensitivity of which two have been termed classical[7]. The first is the dependence of solutions to differential equations or parameters, and, the second is the sensitivity function, discussed by Bode using return ratio and return difference in feedback systems[7,22]. Cruz and Perkins[1] defined a new form of sensitivity function (comparison sensitivity) in Bode's tradition where the sensitivity of the output of the closed loop multivariable system is related to that of the nominally equivalent open-loop system. Horowitz[7] also used a similar definition of sensitivity in feedback system design and later McMorran[6] extended Bode's sensitivity definition to multivariable systems with particular emphasis on the Inverse-Nyquist-Array design method. Other definitions of sensitivity include trajectory sensitivity reduction[22], performance sensitivity and incorporating a sensitivity function in the performance index in optimal control systems. However, such treatment of sensitivity in state space design is unnecessarily complicated, and, Horowitz had pointed out that comparison sensitivity in the tradition of Bode, used in frequency response design is much superior than those used in state space methods[7].

Kalman[8] had given a frequency domain interpretation of single input optimal systems and the multivariable case was generalized by Anderson[5].

The latter also proved the converse frequency domain optimal theorem for multivariable systems. From Kalman's result, it is known that the optimality of a system is related to its sensitivity (in the sense of Bode). Thus an optimal system is also one that is less sensitive to parameter variations. For the multivariable system, Cruz et al[1-3] and Kwakernaak[4] etc showed that the optimality result of Anderson[5] is consistent with the sensitivity criterion of Bode.

However, when using reduced models in design, the desirable characteristics of optimality and sensitivity may not be maintained. It is shown later in the chapter, that a compromise between optimality and sensitivity is possible in sub-optimal design, in that, although optimality is lost, the design can still satisfy the sensitivity requirement.

For reasons of computational tractability, Meditch[12] proposed using a reduced model to design a sub-optimal controller. Later, Aoki[13] used aggregation matrices to study stability and performance of sub-optimal control systems, and, also studied bounds for the solution of matrix Riccati equations. Mitra[14] also studied the role of reduced models in such systems, namely the minimum energy, the tracking and the linear regulator problem. By extending the concepts of Aoki, on aggregation, Vittal Rao[15] et al made similar studies on certain classes of optimal control systems.

Using a general class of reduced models, the stability of sub-optimal control systems cannot be guaranteed, although, it is possible to obtain stable designs with a restricted class of reduced models, namely, those obtained by projection methods. It will be shown later on in the chapter, that priori condition can be established to ensure stability of sub-optimal multivariable systems design (different from Mee's[9] single input result).

The aims of this chapter are: (i) to study comparison sensitivity in the frequency domain when reduced models are used for design, (ii) find sufficient conditions for stability of sub-optimal control systems, and (iii) study the departure from sensitivity and optimality when reduced models are used.

7.1 Comparison Sensitivity Properties[17]

Let $Q'(s)$, $R'(s)$ be perturbations, caused by parameter variations, from the nominal $Q(s)$ and $R(s)$, respectively, i.e.

$$Q'(s) = Q(s) + \delta Q(s) \tag{7.1}$$

$$R'(s) = R(s) + \delta R(s) \tag{7.2}$$

$$\hat{Q}'(s) = \hat{Q}(s) + \delta\hat{Q}(s) \tag{7.3}$$

$$\hat{R}'(s) = \hat{R}(s) + \delta\hat{R}(s) \tag{7.4}$$

where $Q'\hat{Q}' = I$, $R'\hat{R}' = I$, and in general $\delta\hat{Q} \neq (\delta Q)^{-1}$.

Defining the sensitivity matrix

$$M(s) = (I + Q'(s)H(s))^{-1} \tag{7.5}$$

McMorran[6] has shown that for equations (7.1) to (7.4)

$$\delta\hat{R}(s) = \delta\hat{Q}(s) \tag{7.6}$$

$$\delta\hat{Q}(s) = -\hat{Q}'(s)\delta Q(s)\hat{Q}(s) \tag{7.7}$$

$$\delta\hat{R}(s) = -\hat{R}'(s)\delta R(s)\hat{R}(s) \tag{7.8}$$

$$\delta R(s)\hat{R}(s) = M(s)\delta Q(s)\hat{Q}(s) \tag{7.9}$$

Further, if variation in parameters is in $G(s)$ alone, i.e.

$\delta Q(s) = \delta G(s).K(s)$, equations (7.6) to (7.9) yield

$$\delta R(s)\hat{R}(s) = M(s)(G'(s)\hat{G}(s) - I) \tag{7.10}$$

$$R'(s)\delta\hat{R}(s) = -M(s)(G'(s)\hat{G}(s) - I) \tag{7.11}$$

$$\delta\hat{R}(s) = -\hat{Q}'(s)(G'(s)\hat{G}(s) - I) \tag{7.12}$$

where the last equation is used in the Inverse Nyquist Array design.

Cruz and Perkins[1] have shown that $M(s)$ in equation (7.5) is related to the open loop output error and closed loop output error by

$$e_c(s) = M(s)e_o(s) \tag{7.13}$$

where $e_c(s) = y_c(s) - y_c'(s)$, $e_o(s) = y_o(s) - y_o'(s)$. Here the prime represents the perturbed quantitities and the subscripts o and c represent open-loop and closed loop quantities. For a feedback system to be better than a corresponding open-loop system, in the presence of parameter perturbations, Cruz and Perkins formulated the following criterion to be satisfied, (see figure 7.1)

$$<e_c(t), e_c(t)>_{H_m} \; < \; <e_o(t), e_o(t)>_{H_m} \tag{7.14}$$

where $< >_{H_m}$ is the $L_2^m[0, t_1)$ Hilbert space inner product, and $t_1 \simeq 5\tau$, where τ is the largest time constant of the system. The class of signals is restricted to those such that equation (7.14) exists. Simplifying equation (7.14),

$$\sum_{k=-\infty}^{\infty} E_{ok}\left[M^H(s)M(s) - I\right] E_{ok} \; < \; 0 \tag{7.15}$$

Theorem[1] 7A

A sufficient condition for equation (7.15) to be satisfied is:

$$M^H(s)M(s) - I \;\leq\; 0 \qquad \forall\, s \in \Omega \tag{7.15a}$$

or $$\hat{M}^H(s)\hat{M}(s) - I \;\geq\; 0 \qquad \forall\, s \in \Omega \tag{7.15b}$$

over a frequency band of interest, Ω.

Provided $M(s)$ or $\hat{M}(s)$ is not unitary at all frequencies, equations (7.15a) and (7.15b) will be satisfied. If $M(s)$ or $\hat{M}(s)$ is unitary, then the feedback performance will be equal to the corresponding open-loop performance. For single input-single output systems, theorem 7A requires $|\, M(s) \,| \;<\; 1$ or $|\, \hat{M}(s) \,| \;>\; 1$

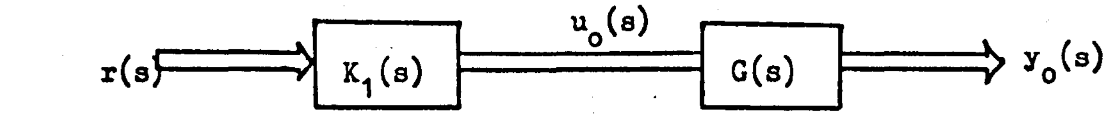
7.2 Reduced Models in Sensitivity Reduction Design

When $K_r(s)$ and $H_r(s)$ are designed using $G_r(s)$, it is interesting to note the effect of parameter variations on S_r and S. Similar to equation (7.5), defined for S_r,

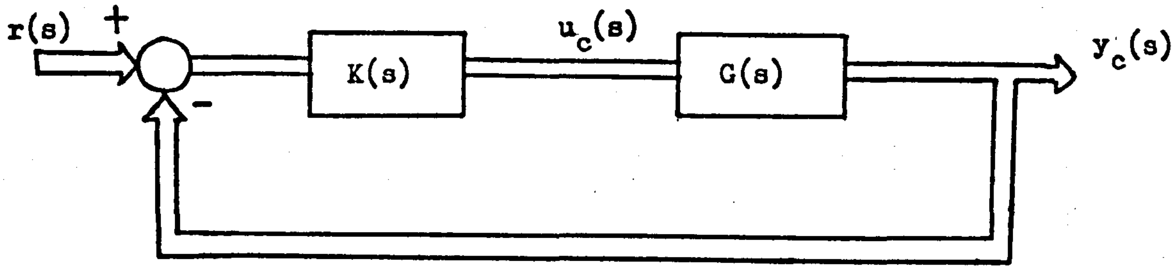
$$M_r(s) \;=\; (I + T_r'(s))^{-1} \tag{7.16}$$

where $T_r'(s) \;=\; Q_r'(s)H_r(s)$. To give a meaningful role of using reduced models in sensitivity reduction design, the following question can be posed.

Suppose a reduced model $G_r(s)$ is obtained from a plant model $G(s)$ and a controller $K_r(s)$ is designed for it. Now assume some plant



the open loop system

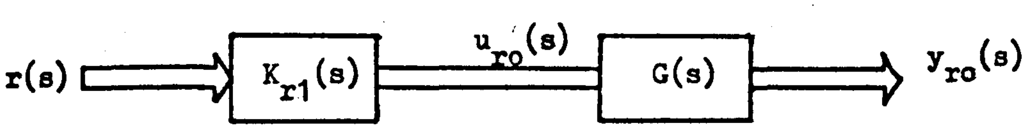


the corresponding closed loop system

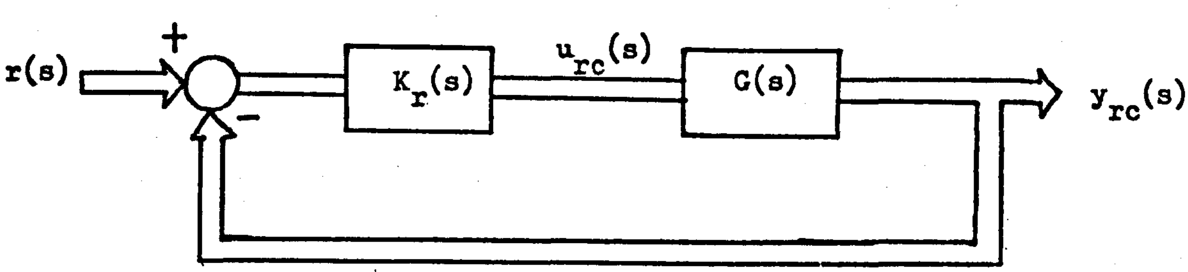
$e_o(s) = y_o(s) - y_o'(s)$ $\qquad$ $e_c(s) = y_c(s) - y_c'(s)$

the prime represents perturbed quantities due to parameter variations

Fig. 7.1(a) Diagram for Bode's 'Comparis on Sensitivity' set-up



the open loop system (reduced model equivalent)



the corresponding closed loop system(reduced model equivalent)

$e_{ro}(s) = y_{ro}(s) - y_{ro}'(s)$ $\qquad$ $e_{rc}(s) = y_{rc}(s) - y_{rc}'(s)$

the prime represents perturbed quantities due to parameter variations

Fig. 7.1(b) Bode's 'Comparis on Sensitivity' set-up using $K_r(s)$, designed using $G_r(s)$, on G (s) parameter changes, due to drift or inaccurate modelling such that $G(s)$ becomes $G'(s)$. The appropriate modification in $G_r(s)$ is then $G_r'(s)$. Now, under what conditions is $K_r(s)$ insensitive when $G(s) \rightarrow G'(s)$ when the same $K_r(s)$ is insensitive when $G_r(s) \rightarrow G_r'(s)$, in terms of $M_r(s)$? In other words, under what conditions is theorem 7A satisfied, for original model S, in terms of that for reduced model, S_r? (see also figure 7.1)

Many sufficient conditions can be found for the above. Below are stated some results that can be easily interpreted graphically. The sensitivity criteria to be satisfied are:

$$M_r^H(s)M_r(s) - I \leqslant 0 \qquad \forall s \in \Omega \tag{7.17}$$

resp. $$\hat{M}_r^H(s)\hat{M}_r(s) - I \geqslant 0 \qquad \forall s \in \Omega \tag{7.18}$$

$$M^H(s)M(s) - I \leqslant 0 \qquad \forall s \in \Omega \tag{7.19}$$

resp. $$\hat{M}^H(s)\hat{M}(s) - I \geqslant 0 \qquad \forall s \in \Omega \tag{7.20}$$

Theorem 7.1

Assuming S_r satisfies the sensitivity requirements of equations (7.17) and (7.18), a sufficient condition for S to satisfy the sensitivity requirements of equations (7.19) and (7.20) is that the matrix

$$E_e(s) = M^H(s)M(s) - M_r^H(s)M_r(s) \leqslant 0$$

(resp $\hat{E}_e(s) = \hat{M}^H(s)\hat{M}(s) - \hat{M}_r^H(s)\hat{M}_r(s) \geqslant 0$)

Proof:

$$M^H(s)M(s) - I = M_r^H(s)M_r(s) - I + E_e(s) \leqslant 0, \text{ if } M_r^H(s)M_r(s) \leqslant 0$$

and $E_e(s) \leqslant 0$ (follows from the fact that if A and B are both negative semidefinite, then A + B is negative semidefinite)

It is noticed that $E_e(s)=E_e^H(s)= M_s^H(s)M_e(s) + M^H(s)M_r(s) - M_r^H(s)M(s)$ where $M_s(s) = M(s) + M_r(s)$, $M_e(s) = M(s) - M_r(s)$, and if $M^H(s)M_r(s)$ is also hermitian, then $E_e(s) = M_s^H(s)M_e(s)$.

The theorems stated below regarding equations (7.17) to (7.20) are given in terms of the eigenvalues of the matrices.

Let λ_r (resp $\hat{\lambda}_r$) be the eigenvalue of $M_r^H(s)M_r(s)$ (resp $\hat{M}_r^H(s)\hat{M}_r(s)$) and similarly λ be eigenvalue of $M^H(s)M(s)$ etc.

Then,

Theorem 7.2

A sufficient condition for equations (7.19) (resp (7.20)) to be satisfied is:

(1) $\mathrm{Re}\{\lambda_{rj} + \beta\} < 1 \quad , \quad \forall j, \forall s\epsilon\Omega$ (7.21)

$(\text{resp } \mathrm{Re}\{\hat{\lambda}_{rj} + \alpha\} > 1)$

(2) $|\mathrm{Re}(\lambda_{rj} + \beta)| > -1 + ||E_e(s) - \beta I||_2 \quad , \quad \forall j, \forall s\epsilon\Omega$ (7.22)

$(\text{resp } |\mathrm{Re}(\hat{\lambda}_{rj} + \alpha)| > 1 + ||\hat{E}_e(s) - \alpha I||_2$

Proof:

The eigenvalues of $M_r^H(s)M(s)$ are real and positive, hence for $M_r^H(s)M_r(s) - I \leqslant 0$, it is necessary that $\lambda_{rj} \leqslant 1, \forall j$. Condition (1) ensures that all eigenvalues lie to the left of the '+1' line and condition (2) (application of theorem 5.1) ensures that $\lambda_j \leqslant 1$, $\forall j$, hence ensuring that $M^H(s)M(s) - I \leqslant 0$. Here α and β are arbitrary complex numbers, and, since $M_r^H(s)M_r(s)$ is normal, the spectral norm $|| \; ||_2$ is used, with $\inf \kappa(p(s)) = 1$

Theorem 7.3

Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \lambda_m$ be as defined above and let $c_1, \ldots c_{m-1}$, $d_1, \ldots d_m$ be $2m - 1$ frequency dependent real numbers. Let $a_{rij}(s)$, $a_{ij}(s)$ be an element of $M_r^H(s)M_r(s)$, $M^H(s)M(s)$ respectively, and let the superflex ^ represent the respective dual quantities. Then a sufficient condition for equations (7.17) and (7.19) to be satisfied is

(1) $c_i(\omega) > 0, \; d_i(\omega) - d_{i+1}(\omega) \geqslant 1/c_i(\omega), \quad \forall i, \forall \omega\epsilon\Omega$ (7.23)

$\text{resp}\left[\, \hat{c}_i(\omega) > 0, \; \hat{d}_i(\omega) - \hat{d}_{i+1}(\omega) \geqslant 1/\hat{c}_i(\omega) \,\right]$

(2) $1 \geq d_i(\omega) \geq |a_{rij}(s)| + c_i(\omega) \sum_{j>i} |a_{rij}(s)|^2, \quad \forall i, \forall \omega \in \Omega$ (7.24)

resp $\hat{d}_i(\omega) \geq |\hat{a}_{rij}(s)| + \hat{c}_i(\omega) \sum_{j>i} |\hat{a}_{rij}(s)|^2, \quad \hat{d}_i(\omega) \geq 1$

(3) $1 \geq d_i(\omega) \geq |a_{ij}(s)| + c_i(\omega) \sum_{j>i} |a_{ij}(s)|^2, \quad \forall i, \forall \omega \in \Omega$ (7.25)

resp $\hat{d}_i(\omega) \geq |\hat{a}_{ij}(s)| + \hat{c}_i(\omega) \sum_{j>i} |\hat{a}_{ij}(s)|^2, \quad \hat{d}_i(\omega) \geq 1$

Proof:

If A is hermitian with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_m$ and if $c_i > 0$, $d_i - d_{i+1} \geq 1/c_i$, $d_i \geq a_{ii} + c_i \sum_{j>i} |a_{ij}|^2$, then $\lambda_i \leq d_i$, $\forall i$.

Using the well known theorem for the localization of eigenvalues in the large[23], conditions (1) to (3) ensure that $\lambda_{ri} \leq 1$, $\lambda_i \leq 1$, $\forall i$ (resp $\hat{\lambda}_{ri} \geq 1$, $\hat{\lambda}_i \geq 1$), hence ensuring that $M_r^H(s)M_r(s) \leq 0$, $M^H(s)M(s) \leq 0$ (resp $\hat{M}_r^H(s)\hat{M}_r(s) \geq 0$, $\hat{M}^H(s)\hat{M}(s) \geq 0$)

Theorem 7.1 or equations (7.17) to (7.20) can also be evaluated in terms of the characteristic loci of $M_r(s)$ and $M(s)$, resp in the special case when $M_r(s)$ and $M(s)$ are both normal.

Consider

$$M_r^H(s)M_r(s) - I < 0 \tag{7.17}$$

$$\hat{M}_r^H(s)\hat{M}_r(s) - I > 0 \tag{7.18}$$

When $M_r(s)$ and $M(s)$ are normal, then $\lambda_{rj}(s) = |\rho_{rj}(s)|^2$, $\lambda_j(s) = |\rho_j(s)|^2$, where $\lambda_{rj}(s)$, $\lambda_j(s)$ and $\rho_{rj}(s)$, $\rho_j(s)$ are the eigenvalues of $M_r(s)M_r^H(s)$, $M(s)M^H(s)$ and $M_r(s)$, $M(s)$ respectively[23]. Thus for equation (7.17) to be satisfied, it is required that $|\rho_{rj}(s)| < 1$. The converse case is also true.

Hence when $M_r(s)$ and $M(s)$ are both normal, and letting $M_e(s) = M(s) - M_r(s)$, $\hat{M}_e(s) = \hat{M}(s) - \hat{M}_r(s)$, Thus

Theorem 7.4

Let $\rho_{rj}(s) + \beta$ (resp $\hat{\rho}_{rj}(s) + \alpha$) map D into $\Gamma_{rj} + \beta$ (resp $\hat{\Gamma}_{rj} + \alpha$), and let C be the unit circle centred on the origin in the Nyquist plane. The sufficient conditions for the sensitivity criteria of equations (7.19) and (7.20) to be satisfied are:

(1) The locus $\Gamma_{rj} + \beta$ (resp $\hat{\Gamma}_{rj} + \alpha$) lies inside (resp outside) C, $\forall j$

either

(2) $|\rho_{rj}(s) + \beta| \leq 1 - ||M_e(s) - \beta I||_2 \quad \forall j, \forall s \in \Omega$ (7.26)

$[\text{resp } |\hat{\rho}_{rj}(s) + \alpha| \geq 1 + ||\hat{M}_e(s) - \alpha I||_2] \quad , \forall j, \forall s \in \Omega$

or

(3) $|\rho_{rj}(s) + \beta| \leq 1 - (m + 2)M_{max}(s)(\Delta(s))^{1/m}, \quad \forall j, \forall s \in \Omega$ (7.27)

(resp $|\hat{\rho}_{rj}(s) + \alpha| \geq 1 + (m + 2)\hat{M}_{max}(s)(\hat{\Delta}(s))^{1/m}$), $\forall j, \forall s \in \Omega$

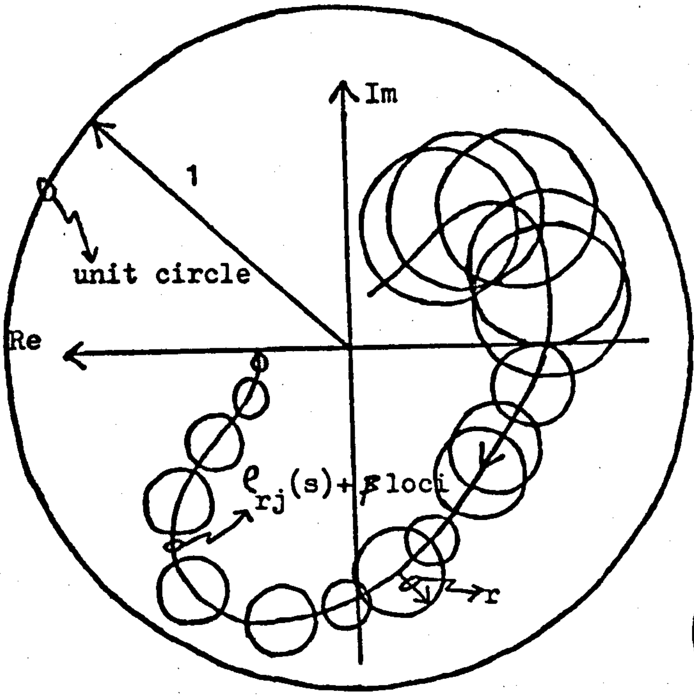
where $M_{max}(s) = \max\{M_{ij}(s), M_{rij}(s) + \delta_{ij}\beta\}$

$\Delta(s) = \{1/M_{max}(s)\} \sum_{i,j} |M_{ij}(s) - M_{rij}(s) + \delta_{ij}\beta|$, where $\delta_{ij} = 1, \forall i = j$, $\delta_{ij} = 0$, otherwise. $\hat{M}_{max}(s)$ and $\hat{\Delta}(s)$ are similarly given, with the respective quantities replaced by their inverses, and, β by α.

Proof:

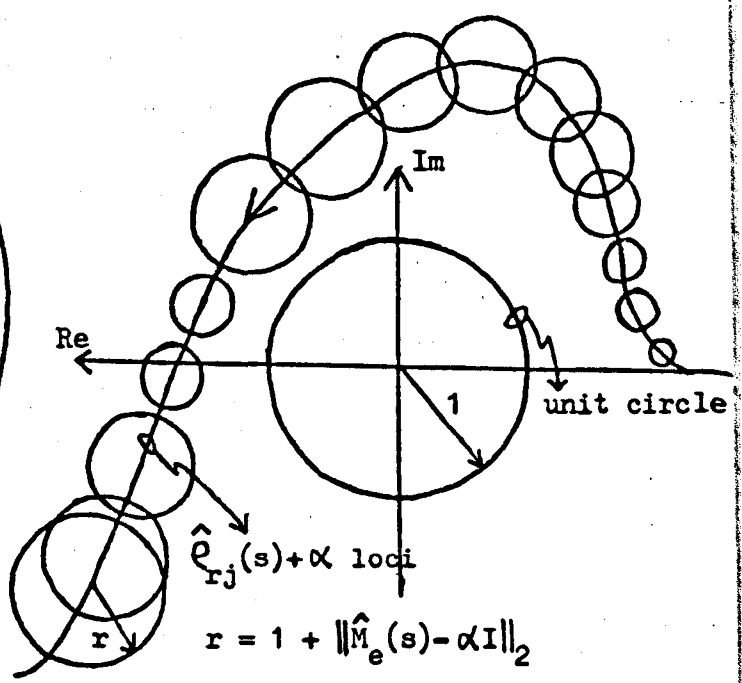
Condition (2) is a consequence of theorem 5.1 and condition (3) is a consequence of theorem 6.4 (Ostrowski's theorem)[20,24]. The graphical interpretation of theorem 7.4 is shown in figures 7.2(a) and 7.2(b). It is required that the band on the loci $\rho_{rj}(s) + \beta$ (resp $\hat{\rho}_{rj}(s) + \alpha$) lies inside (resp outside) C. This means that $|\rho_j(s)| < 1$ (resp $|\hat{\rho}_j(s)| > 1$), $\forall j$, which satisfies $M^H(s)M(s) - I < 0$ (resp $\hat{M}^H(s)\hat{M}(s) - I > 0$).

Since, from equations (7.16), $\hat{M}_r(s) = I + T_r'(s)$, it follows from the eigenvalue shift theorem,



$r = 1 - \|M_e(s) - \beta I\|_2$

Fig. 7.2(a) Illustrating theorem 7.4 (satisfying criteria for Comparision Sensitivity for S_r and S)



$r = 1 + \|\hat{M}_e(s) - \alpha I\|_2$

Fig. 7.2(b) Illustrating theorem 7.4 using inverse loci (satisfying criteria for Comparision Sensitivity for S_r and S)

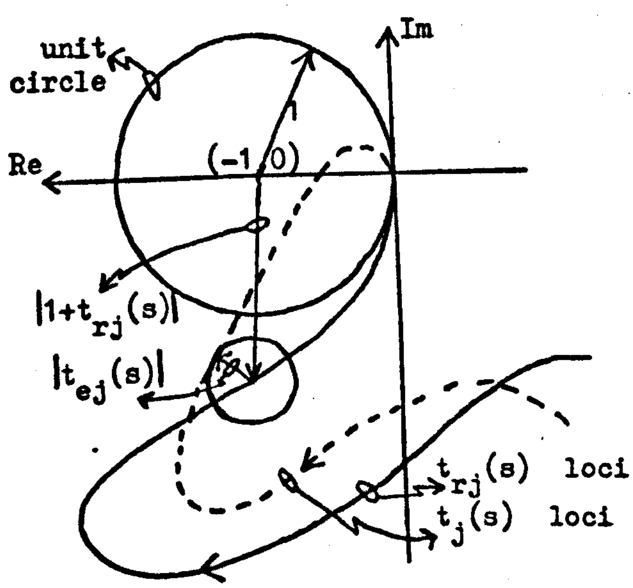
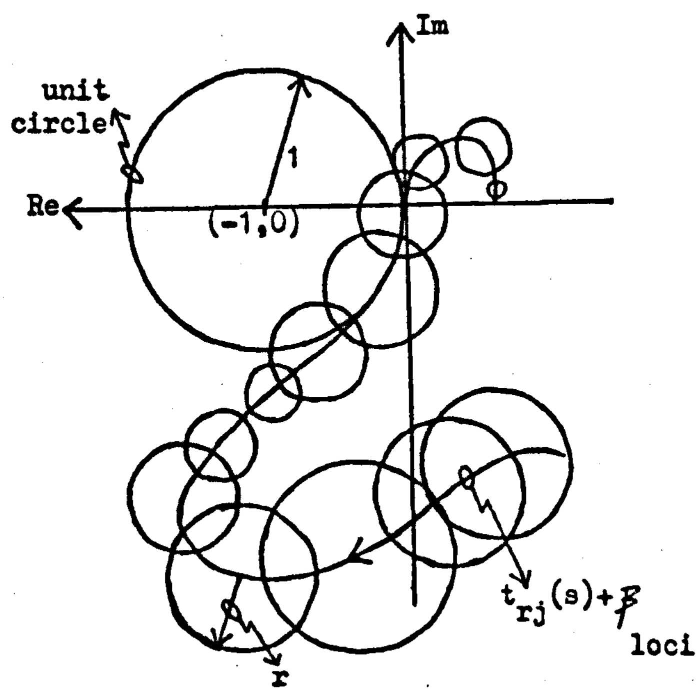


Fig. 7.3(a) Condition for sub optimal stability of S (see eqn (7.47))



$r = \|Q_e(s) - \beta I\| \inf \mathcal{K}(V(s))$

Fig. 7.3(b) Illustrating theorem 7.6 (conditions for stability of optimal S_r and sub optimal S)

$$\hat{\rho}_{rj}(s) = 1 + t_{rj}(s) \tag{7.28}$$

where $t_{rj}(s)$ is the characteristic loci of $T_r'(s)$. Thus when considering $t_{rj}(s)$ instead of $\hat{\rho}_{rj}(s)$, the same result of theorem 7.4 applies, with the centre of the unit circle shifted to the '-1' point and the locus of $\hat{\rho}_{rj}(s) + \alpha$ replaced by the locus of $t_{rj}(s) + \alpha$.

7.2.1 State space descriptive systems

For state space descriptive systems, S(A,B,C) Cruz et al[2] showed that the equivalent form of equation (7.14) is

$$\langle e_c(t), Qe_c(t)\rangle_{H_n} < \langle e_o(t), Qe_o(t)\rangle_{H_n} \tag{7.34}$$

where $\langle\ \rangle_{H_n}$ is the $L_2^n[0,t_1)$ Hilbert space inner product, $Q > 0$ is a state weighting matrix, $e_c(t) = x_c - x_c'$, $e_o(t) = x_o - x_o'$. A sufficient condition for equations (7.34) to be satisfied is

$$M^H(s)QM(s) - Q < 0, \quad \forall s\epsilon\Omega \tag{7.35}$$

or $\quad \hat{M}^H(s)Q\hat{M}(s) - Q > 0$

where in this case,

$$M(s) = (sI - A' - B'K)^{-1}(sI - A') = (I - \Phi'(s)B'K)^{-1} \tag{7.36}$$

is the state sensitivity matrix of order n x n and K is a constant state feedback matrix.

The parallel equations for $S_r(A_r,B_r,C_r)$ are

$$\langle e_{rc}(t),Q_r e_{rc}(t)\rangle_{H_r} < \langle e_{ro}(t),Q_r e_{ro}(t)\rangle_{H_r} \tag{7.37}$$

$$M_r^H(s)Q_rM_r(s) - Q_r < 0, \quad \forall s\epsilon\Omega \tag{7.38}$$

or $\quad \hat{M}_r^H(s)Q_r\hat{M}_r(s) - Q_r > 0$

with

$$M_r(s) = (sI - A_r' - B_r'K_r)^{-1}(sI - A_r') = (I - \Phi_r'(s)B_r'K_r) \tag{7.39}$$

of order r x r. Taking into account design considerations, K is related to K_r by

$$K' = K_r Z \tag{7.40}$$

where $x_r = Zx$. As the dimensions of $M(s)$ and $M_r(s)$ are incompatible in state space systems, the dimensions of $M_r(s)$ must be extended before equation (7.38) can be used in conjunction with equation (7.35).

Define

$$N_r(s) = \begin{bmatrix} M_r(s) & 0 \\ 0 & Y(s) \end{bmatrix} \quad \text{or} \quad N_r(s) = \begin{bmatrix} Y(s) & 0 \\ 0 & M_r(s) \end{bmatrix}$$

$$\hat{N}_r(s) = \begin{bmatrix} \hat{M}_r(s) & 0 \\ 0 & \hat{Y}(s) \end{bmatrix} \qquad Q_r' = \begin{bmatrix} Q_r & 0 \\ 0 & \mathrm{diag}(q_{r+1},\ldots\ldots q_n) \end{bmatrix}$$

where $Y(s)$, $\hat{Y}(s)$, $\mathrm{diag}(q_{r+1},\ldots q_n)$ are such that the $N_r(s)$, $\hat{N}_r(s)$, Q_r' are of dimension $(n \times n)$ and $Q_r' > 0$.

Suppose the characteristic loci of $M_r(s)$ (resp $\hat{M}_r(s)$) are $\lambda_{rj}(s)$ (resp $\hat{\lambda}_{rj}(s)$), $j = 1, \ldots r$ and the arbitrary assigned characteristic loci of $Y(s)$ (resp $\hat{Y}(s)$) are $\lambda_{ri}(s)$ (resp $\hat{\lambda}_{ri}(s)$), $i = r+1,\ldots n$. Thus the characteristic loci of $N_r(s)$ (resp $\hat{N}_r(s)$) are $\lambda_{rj}(s)$ (resp $\hat{\lambda}_{rj}(s)$), $j = 1,\ldots n$.

The modified form of expression (7.38) becomes

$$\begin{aligned} &N_r^H(s)Q_r'N_r(s) - Q_r' < 0 \\ &\hat{N}_r^H(s)Q_r'\hat{N}_r(s) - Q_r' > 0 \end{aligned} \tag{7.41}$$

Expressions (7.35) and (7.38) can be interpreted using the Ostrowski-Schneider-Taussky[18] (OST) theorem of the Lyapunov equation.

The theorem states that for $B \in M_n(C)$, and the inertia of B, $\mathrm{In}(B) = (\pi, v, \delta)$, there exists a Hermitian H such that $BHB^H - H > 0$, if and only if B has no eigenvalues z such that $|z| = 1$. If $BHB^H - H > 0$, then B has $\pi(H)$ [resp. $v(H)$] eigenvalues z such that $|z| > 1$ [resp. $|z| < 1$]. The converse case of the theorem is also true. Suppose B has ℓ (resp. $n-\ell$) eigenvalues outside (resp. inside) the unit circle. Then there exists a Hermitian P such that $BPB^H - P > 0$ where $In(P) = (\ell, n-\ell, 0)$.

The OST theorem can now be applied to expressions (7.35) and (7.38).

Theorem 7.5

Let $\lambda_{rj}(s) + \beta$ (resp $\hat{\lambda}_{rj}(s) + \alpha$) map D into $\Gamma_{rj} + \beta$ (resp $\Gamma_{rj} + \alpha$), and let C be the unit circle as in theorem 7.4. Then expression (7.35) is satisfied, i.e. S satisfy the sensitivity criterion if

(1) $\Gamma_{rj} + \beta$ (resp $\hat{\Gamma}_{rj} + \alpha$) lies inside (resp outside) C, $\forall j$

either

(2) $|\lambda_{rj}(s) + \beta| < 1 - ||N_e(s) - \beta I||\inf \kappa(P(s)), \quad \forall j, \forall s \in \Omega$ (7.42)

$[\text{resp } |\hat{\lambda}_{rj}(s) + \alpha| > 1 + ||\hat{N}_e(s) - \alpha I||\inf \kappa(P(s))], \quad \forall j, \forall s \in \Omega$

or

(3) $|\lambda_{rj}(s) + \beta| < 1 - (m+2)M_{max}(s)(\Delta(s))^{1/m}, \quad \forall j, \forall s \in \Omega$ (7.43)

$[\text{resp } |\hat{\lambda}_{rj}(s) + \alpha| > 1 + (m+2)\hat{M}_{max}(s)(\hat{\Delta}(s))^{1/m}], \quad \forall j, \forall s \in \Omega$

where $N_e(s) = M(s) - N_r(s)$, $\hat{N}_e(s) = \hat{M}(s) - \hat{N}_r(s)$, $M_{max}(s) = \max\{M_{ij}(s), N_{rij}(s) + \delta_{ij}\beta\}$, $\Delta(s) = \{1/M_{max}(s)\}\Sigma_{ij} |M_{ij}(s) - N_{rij}(s) - \delta_{ij}\beta|$, and similarly for the inverse quantities, $\hat{M}_{max}(s)$, $\hat{\Delta}(s)$ etc.

Proof:

The proof is very similar to that given in theorem 7.4.

Regarding expressions (7.41) and (7.35), with $Q_r' > 0$, $Q > 0$, it is seen that $In(Q_r') = In(Q) = (n,0,0)$. Thus when applying the OST theorem to expression (7.41), all $|\hat{\lambda}_{rj}(s)|$ must lie outside the unit disc. The converse case of the OST theorem is used to explain that $\hat{M}^H(s)Q\hat{M}(s) - Q > 0$ for a particular Q. Since the circular bands, given by expression (7.42) or (7.43) also lie outside the unit disc, all eigenvalues $|\lambda_j(s)|$ of $\hat{M}(s)$ also lie outside the unit disc. Thus a Hermitian Q exist such that $\hat{M}^H(s)Q\hat{M}(s) - Q > 0$, or $M^H(s)QM(s) - Q < 0$.

(When considering $M(s)$ instead of $\hat{M}(s)$, all loci $|\lambda_j(s)|$ must lie inside the unit disc.)

The graphical interpretation of theorem 7.5 is similar to that of theorem 7.4. In theorem 7.5 it is noted that the weighting matrices Q_r and Q cannot be chosen arbitrarily. All is known is that a class of matrices Q_r and Q exists such that $\hat{N}_r^H(s)Q_r\hat{N}_r(s) - Q_r > 0$ and $\hat{M}^H(s)Q\hat{M}(s) - Q > 0$. Q_r and Q can be solved directly from these equations. Prespecification of Q_r and Q cannot guarantee the applicability of theorem 7.5. If they are prespecified, then a similar form of theorem 7.2 or 7.3 can be used. However, this is disadvantageous in design, as theorem 7.2 does not work on the characteristic loci of S_r.

7.3 Reduced models and frequency domain optimality

For the observable system $S_r(A_r,B_r,C_r)$ minimizing the functional

$$J = \langle y,Q_r y\rangle_{H_m} + \langle u,Ru\rangle_{H_m} \tag{7.44}$$

where H_m is the $L_2^m[0,\infty)$ Hilbert space $Q_r \geq 0$, $R > 0$, it is well known that the control law $u = -K_r x_r = -R^{-1}B_r^t P_r x_r$, where P_r satisfies the Riccati equation

$$P_r A_r + A_r^t P_r + P_r B_r R^{-1} B_r^t P_r = C_r^t Q_r C_r \tag{7.45}$$

results in the frequency domain equation for optimality,[8,17]

$$F_r^H(s)RF_r(s) = R + G_r^H(s)Q_r G_r(s) \tag{7.46}$$

Equation (7.46) gives a necessary and sufficient condition for optimality. $F_r(s) = I + K_r(sI - A_r)^{-1}B_r$, $G_r(s) = C_r(sI - A_r)^{-1}B_r$ are the return difference matrix and forward path transfer function of the optimal system respectively. It was also shown that a necessary condition for optimality[17] from equation (7.46) is $|\det F_r(s)| \geq 1$ or $|\rho_{rj}(s)| \geq 1$, $\forall_j$, $\forall s = j\omega$, where $\rho_{rj}(s)$ is the eigenvalue of $F_r(s)$. Viewed graphically the characteristic loci $\rho_{rj}(s)$ cannot penetrate the unit circle at the origin, and S_r is stable.

Let $t_{rj}(s)$ be an eigenvalue of $K_r(sI - A_r)^{-1}B_r$, then $\rho_{rj}(s) = 1 + t_{rj}(s)$, thus the unit circle is shifted to the '-1' point if the locus of $t_{rj}(s)$ is considered. Now, suppose sub-optimal control

$$u = -K_r Zx$$

is applied to the original model S, where $x_r = Zx$. The locus $t_j(s)$, resulting from sub-optimal control, would then penetrate the unit circle and, can encircle the critical point, thus causing an unstable system S, (see figure 7.3a). Conditions are sought such that the stability of S is ensured.

From figure 7.3a, S is closed loop stable if

$$|t_{ej}(s)| < |1 + t_{rj}(s)|, \quad \forall j \tag{7.47}$$

where $t_{ej}(s) = t_j(s) - t_{rj}(s)$. Frequency domain theorem established in Chapter V can be adapted to study the stability of S. Let

$$Q_r(s) = K_r(sI - A_r)^{-1}B_r = K_r\Phi_r(s)B_r$$

$$Q(s) = K_rZ(sI - A)^{-1}B = K_rZ\Phi(s)B$$

$$Q_e(s) = Q(s) - Q_r(s)$$

and $t_{rj}(s)$, $t_j(s)$ be eigenvalues of $Q_r(s)$, $Q(s)$ respectively

Theorem 7.6

The sub optimal control system, S, will be stable given that the optimal control system, S_r, is stable, if

(1) The locus $t_{rj}(s) + \beta$ satisfies NC, $\forall j$, $\forall s \in D$

either

(2) $|1 + \beta + t_{rj}(s)| > ||Q_e(s) - \beta I||\inf \kappa(V(s))$, $\forall j$, $\forall s \in D$ (7.48)

or

(3) $|1 + \beta + t_{rj}(s)| > (m + 2)M_{max}(s)\{\Delta(s)\}^{1/m}$, $\forall j$, $\forall s \in D$ (7.49)

where V(s) diagonalizes $Q_r(s)$ and $M_{max}(s)$, $\Delta(s)$ are as defined as in theorem 6.4.

Proof:

Given that condition (1) is satisfied, condition (2) or (3) ensure that the locus $t_j(s)$ satisfies the Nyquist criterion; follows from theorems 5.1 and 6.4 (see figure 7.3b)

If the arbitrary scaling factor β is chosen $\beta = 0$, condition (1) is automatically satisfied, as the optimal system, S_r, is always stable. Writing $Q_e(s) = K_r(Z\Phi(s)B - \Phi_r(s)B_r)$ and since $||AB|| \leq ||A||\ ||B||$ and letting $\beta = 0$ and assuming the special case when $Q_r(s)$ is normal, condition (2) reduces to

$$|1 + t_{rj}(s)| > ||K_r||_2\ ||Z\Phi(s)B - \Phi_r(s)B_r||_2 \quad , \forall j\ , \ \forall s \epsilon D$$

i.e. $$|1 + t_{rj}(s)|/||K_r||_2 > ||Z\Phi(s)B - \Phi_r(s)B_r||_2, \ \forall j\ , \ \forall\ s \epsilon\ D \qquad (7.50)$$

Although the righthand-side of expression (7.50) is independent of P, Q and R, the bound is inconvenient, as it is expressed in terms of $t_{rj}(s)$. A more useful bound can be derived from condition (3). For an optimal S_r, with $\beta = 0$, it is necessary that $|1 + t_{rj}(s)| > 1$. Equation (7.49) is satisfied if

$$1 > (m + 2)M_{max}(s)\{\Delta(s)\}^{1/m} \qquad (7.51)$$

where $M_{max}(s) = \max_{i,j}(|q_{rij}(s)|,\ |q_{ij}(s)|)$,

$\Delta(s) = \{\sum_{i=1}^{m}\sum_{j=1}^{m} |q_{ij}(s) - q_{rij}(s)|\}^{1/m} / M_{max}(s)$. Rearranging expression (7.51)

$$(m + 2)^{m/m-1}M_{max}(s)\{\sum_i \sum_j |q_{ij}(s) - q_{rij}(s)|\}^{1/(m-1)} < 1 \qquad (7.52)$$

Now,

$$\sum_i \sum_j |q_{ij}(s) - q_{rij}(s)| \leq ||Q(s) - Q_r(s)||_c \leq ||K_r(s)||_c||\Phi_r(s)B_r - Z\Phi(s)B||_c \qquad (7.53)$$

where the cubic norm $||A||_c = m \max|a_{ij}|$. For the time being it is assumed that $r = m$ such that K_r and B_r are square matrices. Also,

$$\begin{aligned} M_{max}(s) &= \max_{i,j}(|q_{rij}(s)|,|q_{ij}(s)|) \\ &= (1/m) \max\{||Q_r(s)||_c,||Q(s)||_c\} \leq \\ &\quad (1/m)||K_r||_c \max\{||\Phi_r(s)B_r||,||Z\Phi(s)B||\} \qquad (7.54) \end{aligned}$$

Substituting expressions (7.53) and (7.54), into (7.52), and letting

$$W(s) = (1/m)(m + 2)^{m/(m-1)} \max \{||\Phi_r(s)B_r(s)||_c, ||Z\Phi(s)B||_c\}||\Phi_r(s)B_r - Z\Phi(s)B||^{1/(m-1)} \tag{7.55}$$

one obtains

$$||K_r(s)||_c^{m/(m-1)}||W(s)||_c < 1 \qquad \forall s = j\omega \tag{7.56}$$

Since $||K_r(s)|| = ||R^{-1}P_rB_r^t|| < ||R^{-1}|| \; ||P_r|| \; ||B_r^t||$, evaluating expression (7.56), one obtains

Corollary 7.1

The sub-optimal system S is stable if

$$||P_r||_c < ||W(s)||_c^{m/(1-m)} \; / \; ||R^{-1}||_c||B_r^t||_c \quad \forall s = j\omega \tag{7.57}$$

Expression (7.57) requires that the solution of the matrix Riccati equation P_r be bounded. An effective method to ensure that P_r satisfies expression (7.57) is as follows.

Assume $||P_r||_c < \alpha$ (7.58)

where α is a constant, then from expression (7.57)

$$||R^{-1}|| < ||W(s)||_c^{m/(m-1)} \; /\alpha \; ||B_r^t||_c \quad , \quad \forall s = j\omega \tag{7.59}$$

Algorithm to find P_r

Step 1. Choose a value of α and compute R such that expression (7.59) is satisfied

Step 2. Choose the matrix Q and find P_r by solving equation (7.45)

Step 3. If $||P_r||_c < \alpha$, exit, otherwise go to Step 1 or Step 2.

It is far easier to find a R such that expression (7.59) is satisfied for all frequencies, and then checking that $||P_r|| < \alpha$, then by solving P_r from equations (7.45) and checking that it satisfies expression (7.57). The constant, α, can be chosen from 'open loop' quantities. For the optimal system,

$$<x_r, P_r x_r>_E \;=\; \min_K <x_r, P_K x_r>_E$$

where for the closed loop system

$$P_K = \int_0^\infty \exp\{(A_r - B_rK_r)^t t\}(C_r^tQ_rC_r + K_r^tRK_r)\exp\{(A_r - B_rK_r)t\}\,dt$$

For the open loop system

$$P_{ro} = \int_0^\infty \exp(A_r^t t)(C_r^tQ_rC_r)\exp(A_rt)\,dt \tag{7.60}$$

If the open loop system is stable, then

$$\langle x_r, P_rx_r\rangle_E < \langle x_r, P_{ro}x_r\rangle_E \tag{7.61}$$

From expression (7.61), $P_{ro} - P_r > 0$, i.e. positive definite symmetric. Using a general inequality of Minkowski[33],

$$\lambda_j(A + B) > \lambda_j(A) \qquad j = 1, \ldots m \tag{7.62}$$

for symmetric positive definite matrices A and B, and identifying A with P_r and B with $P_{ro} - P$, it is seen that $\lambda_j(P_{ro}) > \lambda_j(P_r)$, and P_r and P_{ro} are symmetric positive definite. From expression (7.62) it is easily deduced that $\det P_{ro} > \det P_r$.

Hence, if α is chosen as $\alpha = \det P_{ro}$, P_r in expression (7.58) is required to satisfy $||P_r||_c < \det P_r$ or $||P_r||_c < \det P_{ro}$. Other convenient norms or different values of α can be used. Since P_r is symmetric $||P_r||_2 = \lambda_{max}(P_r)$ where the spectral norm of P_r is equal to its spectral radius. From expression (7.62) $\lambda_{max}(P_r) < \lambda_{max}(P_{ro})$, the P_r can be computed, by using the spectral norm instead of the cubic norm, provided that $(1/m)||Q_r(s)||_c < ||Q_r(s)||_2$, $(1/m)||Q(s)||_c < ||Q(s)||_2$ and $\sum_i\sum_j |q_{ij}(s) - q_{rij}(s)| < ||Q_e(s)||_2$, $\forall s = j\omega$.

In the above analysis, all matrices are assumed square, i.e. r = m. In the case of r > m, r - m fictitious inputs can be introduced into the modified system $S_r'(A_r, B_r'C_r)$, where $B_r' = (B_r \vdots 0)$. $R' = \text{diag}\{r_1, r_2, \ldots r_m, 0, \ldots 0\}$, assuming $R = \text{diag}\{r_1, r_2, \ldots r_m\}$. S_r' in this case is still observable. However, if r < m, then m - r fictitious states must be augmented to $S_r(A_r, B_r, C_r)$ such that the modified system $S_r'(A_r', B_r', C_r')$ is observable, with

$$A_r' = \begin{pmatrix} A_r & 0 \\ 0 & \varepsilon I \end{pmatrix} , \quad B_r' = \begin{pmatrix} B_r \\ \cdots \\ \alpha \end{pmatrix} , \quad C_r' = (C_r \,\vdots\, \beta)$$

and $|\varepsilon|$, $||\alpha||$, $||\beta||$ must be chosen very small such that the dynamical response of S_r' is very close to that of S_r. In practice, m is usually low, hence S_r must be modelled such that $r \geqslant m$. The plant order n is usually very high with $n >> m$.

The stability of S can also be expressed in terms of the determinants of $F_r(s)$. For an optimal, S_r, the locus of det $F_r(s)$ cannot penetrate the unit circle centred on the origin. Thus, parallel to equation (7.47) the locus det F(s) of S, that penetrates the unit circle, will satisfy the Nyquist criterion if

$$|\det F_r(s)| > |\det F_r(s) - \det F(s)| \qquad (7.63)$$

where from equation (7.46), pre and post multiplying throughout by $R^{-\frac{1}{2}}$ and taking determinants

$$|\det F_r(s)| = \det^{\frac{1}{2}}(I + R^{-\frac{1}{2}} G_r^H(s) Q_r G_r(s) R^{-\frac{1}{2}})$$

The determinantal bound, however, is complicated and inconvenient.

7.3.2 Reduced Model and Sub-optimal Filtering

It is well known that the solution for the optimal filter[16] is very similar to that for the optimal regulator and the two problems can be treated separately. The optimal filter also has the same frequency domain[10] characterisation as the optimal regulator. Thus the same problems arise in using reduced models for sub-optimal filter design.

For a given system S(A, B, C) suppose the observed signal z(t) is corrupted by an additive noise disturbance v(t), i.e.

$$z(t) = y(t) + v(t)$$

and suppose that the statistical properties of the noise and input are

$$\begin{aligned} \text{cov}\,[u(t),\, u(\tau)] &= Q\delta(t - \tau) \\ \text{cov}\,[v(t),\, v(\tau)] &= R\delta(t - \tau) \\ \text{cov}\,[u(t),\, v(\tau)] &= 0 \end{aligned} \tag{7.64}$$

Then it is well known that the Kalman Bucy[16] filter has a feedback structure with gain

$$K = PC^tR^{-1}$$

where P is the solution of the steady state matrix Riccati equation

$$PA^t + AP - PC^tR^{-1}CP + BQB^t = 0 \tag{7.65}$$

and P is also the covariance matrix for the error in the state estimate $(\hat{x} - x)$. The frequency domain interpretation of the optimal filter[10] is

$$F(s)RF^H(s) = R + G(s)QG^H(s) = \Delta(s)\Delta^H(s) \tag{7.66}$$

where $F(s) = I + C(sI - A)^{-1}K$, $G(s) = C(sI - A)^{-1}B$ and $\Delta(s)$ is the Hurwitz factor of the spectral factorisation of the spectral density matrix of the observation vector z. In equation (7.66) it is known that $|\rho(s)| \geq 1$ and $|\det F(s)| \geq 1$ where $\rho(s)$ is the characteristic loci of F(s). Thus to determine stability in sub-optimal filter design, a similar form of theorem 7.6 and corollary 7.1 can be used. Instead of solving the Riccati equation (7.65), the filter gain K, can also be computed iteratively from the discrete state space system $S(\Phi, \Gamma, C)$

$$\begin{aligned} P_k &= (I - K_kC)\Phi P_{k-1}\Phi^t \\ K_k &= \Phi P_{k-1}\Phi^tC^t(C\Phi P_{k-1}\Phi^tC^t + R_k)^{-1} \end{aligned}$$

for $k = 1, 2, \ldots, N$ with initial condition P_o specified.

It is also essential to know the overall stability of the system, i.e. the optimal system with the optimal filter incorporated. To study the stability of the sub-optimal system incorporating a sub-optimal filter, in terms of the stability of an optimal system incorporating an optimal filter, is complicated, as it involves too many parameters, and the systems are interconnected. The stability of the system and filter arrangement can be best studied from the frequency domain point of view in terms of sub-systems and their interconnections, using a result of Cook[11].

Suppose that the optimal system $y = Su$ is decomposed into an interconnection of sub-systems S_i represented by

$$\begin{aligned} y &= \sum_i L_i y_i \\ y &= S_i u_i \\ u_i &= K_i u + \sum_j R_{ij} y_j \end{aligned} \tag{7.67}$$

Then if S_i, R_{ij}, K_i, L_i are L^p-stable, with $||S_i||_p \neq 0$, $\forall i, \forall j$ and the matrix W, with element, $W_{ij} = ||S_i||_p^{-1} \delta_{ij} - ||R_{ij}||_p$ is an M-matrix, then the total system S is L^p stable. (Here L^p stable means $||S||_p < \infty$ and $||S||_p = \inf [K : ||(Su)_T||_p \leq K||u_T||_p, \forall u, \forall T])$

The stability of the total sub-optimal system S' can be interpreted in terms of that of the optimal system S as follows

Theorem 7.7

Given the total optimal system S is stable, the total sub-optimal system S' is stable if

(1) S_i', R_{ij}', K_i', L_i' are L^p-stable, $\forall i, \forall j$

(2) $W_{ii}' > W_{ii}$, $0 \geq W_{ij}' \geq W_{ij}$, $\forall i, \forall j, \forall i \neq j$ (7.62)

Proof:

Condition (2) determines that W' is an M-matrix and with condition (1) S' is stable.

Theorem 7.7 is general. The M-matrix requirement of W' can be interpreted graphically as diagonal dominance requirement of Rosenbrock, or characteristic loci requirement of Ky Fan (see Chapter V).

7.3.3 Reduced Models and Relationship between Sensitivity and Optimality

The optimality of a system is also intimately related to the sensitivity problem of the system. For a single input system, Kalman showed that an optimal system is also one which is less sensitive to parameter variations. Anderson[5] generalized the result to multivariable systems. S(A, B, C) will only minimize

$$J = \tfrac{1}{2}\langle Q^{\frac{1}{2}}x, Q^{\frac{1}{2}}x\rangle_{H_n} + \langle R^{\frac{1}{2}}u, R^{\frac{1}{2}}u\rangle_{H_m} \qquad (7.63)$$

where $(A, Q^{\frac{1}{2}})$ is observable and $R > 0$, with the control law $u = -Kx$, if and only if, the closed loop system $A - BKC$ is asymptotically stable and

$$F^H(s)F(s) - I \geq 0, \qquad \forall s = j\omega \qquad (7.64)$$

where $F(s) = I + R^{-\frac{1}{2}}K\Phi(s)BR^{-\frac{1}{2}}$, is the return difference matrix of the optimal system. Conversely, if expression (7.64) is satisfied with strict inequality sign, then the closed loop system S_c is optimal, with respect to some performance index, J, under the condition that S_c is stable and that (A, K) is completely observable.

Expression (7.64) is also the criterion for sensitivity reduction in closed loop systems, see expression (7.20) where $\hat{M}(s) = F(s)$. As $s \to \infty$, $F(s) \to I$, thus strict inequality will never be satisfied in expression (7.64) for a strictly proper transfer function matrix $R^{-\frac{1}{2}}K\Phi(s)BR^{-\frac{1}{2}}$.

Hence to satisfy the sensitivity reduction criterion it is necessary $s\ (=\infty) \notin \Omega$, as $|\rho_j(s)| = 1$ at $s = \infty$, where $\rho_j(s)$ is the characteristic loci of F(s), assuming F(s) is normal and Ω is the spectrum of interest where the sensitivity condition is satisfied.

Thus in sub optimal design, the original system, S, assuming F(s) is normal, will weakly satisfy the sensitivity reduction criterion since $|\rho_{rj}(s)| < 1$ for some $s = j\omega$.

7.4 Reduced Models in Sub-optimal Control Systems Design

The linear optimal regulator problem and tracking problem, designed using S_r, have been studied in some detail by Aoki, Mitra, Meditch and Vittal Rao et al[12-15]. Aoki had also used reduced models to obtain upper bounds for the solution of matrix Riccatti equations.

For completeness, this section reviews some of the earlier work of the above authors. The sub-optimal design procedures are then incorporated into a general design algorithm, based on the stability theorems developed in section 7.3 and those in Chapter V.

For controllable S(A, B, C) and $S_r(A_r, B_r, C_r)$, the associated performance indices to be minimized, are

$$J = \langle x, Qx\rangle_{H_n} + \langle u, Ru\rangle_{H_m} \tag{7.65}$$

$$J_r = \langle x_r, Q_r x_r\rangle_{H_r} + \langle u, Ru\rangle_{H_m} \tag{7.66}$$

where $x \in H_n$, $x_r \in H_r$, $u \in H_m$ and H_n, H_r and H_m are (n x n), (r x r) and (m x m) Hilbert spaces, with $Q \geq 0$, $Q_r \geq 0$ and $R > 0$. For equation (7.66), the control law, $u = -K_r x_r(t)$, yields $J_{r,min} = \langle x_r(o), P_r x_r(o)\rangle$ where

$$K_r = R^{-1}B_r^t P_r$$

and P_r is the solution of

$$A_r^t P_r + P_r A_r - P_r B_r R^{-1} B_r^t P_r + Q_r = 0 \tag{7.67}$$

The resulting closed loop system matrix $A_r - B_r K_r$ is also stable and satisfies the closed loop equation

$$(A_r - B_r K_r)^t P_r + P_r(A_r - B_r K_r) = -(K_r^t R K_r + Q_r) \tag{7.68}$$

Similar equations to equations (7.67) and (7.68) can be written for the index J in equation (7.65), i.e.

$$K = R^{-1}B^t P$$

$$A^t P + PA - PBR^{-1}B^t P + Q = 0 \tag{7.69}$$

$$(A - BK)^t P + P(A - BK) = -(K^t RK + Q)$$

for $J_{min} = \langle x(o), Px(o)\rangle$. The chief interest is to use control $u = -K_r x_r$, from S_r, on S, where S and S_r are related by the matrices

$$\begin{aligned} A_r Z &= ZA \\ B_r &= ZB \\ C_r Z &= C \end{aligned} \tag{7.70}$$

with $x_r = Zx$, i.e. it is desirable to use $u_s = -K_r Zx$ on S to obtain a sub-optimal control. If $ZZ^{\dagger} = ZZ^t(ZZ^t)^{-1} = I$, exists, then pre and post multiplying equation (7.67) by Z^t and Z respectively and using equation (7.70)

$$A^t Z^t P_r Z + ZP_r ZA - Z^t P_r ZBR^{-1}B^t Z^t P_r Z + Z^t Q_r Z = 0 \tag{7.71}$$

Comparing equation (7.71) to (7.69) it is seen that $Z^t P_r Z$ corresponds to P if $Z^t Q_r Z$ is made to correspond to Q (they cannot be equated as P and Q are of rank n while $Z^t P_r Z$ and $Z^t Q_r Z$ are at most of rank r). Hence if Q_r is chosen as

$$Q_r = (Z^{\dagger})^t Q Z^{\dagger} \tag{7.72}$$

then the sub optimal control u_s will result in a $J = \langle x(o), Tx(o)\rangle$ where T satisfies

$$(A - BK_r Z)^t T + T(A - BK_r Z) + Z^t K_r R K_r Z + Q = 0 \tag{7.73}$$

provided if and only if $(A - BK_r Z)$ is a stability matrix, and in this situation Aoki[13] showed that

$$T \geq P \geq Z^t P_r Z \tag{7.74}$$

($T \geq P$ means $T - P \geq 0$, i.e. $T - P$ is positive semi-definite). Equation (7.74) can also be used to provide upper and lower bounds on solutions of matrix Riccati equations.

Equations (7.65) and (7.66) deal with the state regulator problem. The output regulator problem[15] is considered by considering the performance index functional,

$$J = \langle y, Qy\rangle_{H_m} + \langle u, Ru\rangle_{H_m} \tag{7.75}$$

for an m input-m output system. Analysis following the same lines from equations (7.65) to (7.74) can be applied to equation (7.75).

For the output tracking problem the functional to be minimized is

$$J = \tfrac{1}{2}\langle (y - f),\ Q(y - f)\rangle_{H_m} + \langle u,\ Ru\rangle_{H_m} \qquad (7.76)$$

assuming A(A, B, C) to be observable. Substituting $y - f = Cx - f$ in equation (7.76) and introducing the Hamiltonian

$$H = \tfrac{1}{2}\langle e,\ Qe\rangle + \tfrac{1}{2}\langle u,\ Ru\rangle + \langle p,\ Ax + Bu\rangle$$

where $e = Cx - y$ and p is the co-state vector it can be shown that by evaluating $\partial H/\partial u = 0$, the following equations pertinent to equation (7.76) can be obtained

$$\begin{aligned} u &= -R^{-1}B^{t}p(t) \\ p(t) &= Kx(t) + W \\ C^{t}QC + KA + A^{t}K - KBR^{-1}B^{t}K &= 0 \\ A^{t}W - KBR^{-1}B^{t}W - C^{t}Qf &= 0 \end{aligned} \qquad (7.77)$$

A similar set of equations can be written for the reduced model S_r. It is desirable to use control $u_r = -R^{-1}B_r^{t}P_r$, derived from S_r, on S. The equations of the two controls are

Sub optimal S: $u_r = -R^{-1}B^{t}Z^{t}K_r Zx - R^{-1}B^{t}Z^{t}W_r$

optimal S_r: $u = -R^{-1}B^{t}Kx - R^{-1}B^{t}W$

It is seen that $Z^{t}K_rZ$ and $Z^{t}W_r$ correspond to K and W respec tively. Pre and post multiplying the Lyapunov equation for S_r

$$C_r^{t}Q_rC_r + K_rA_r + A_r^{t}K_r - K_rB_rR^{-1}B_r^{t}K_r = 0$$

by Z^{t} and Z respectively and using equation (7.70)

$$C^{t}Q_rC + Z^{t}K_rZA + A^{t}Z^{t}K_rZ - Z^{t}K_rZBR^{-1}B^{t}Z^{t}K_rZ = 0$$

Comparing the above to the third equation of (7.77) it is seen that $Z^{t}K_rZ$ corresponds to K if Q corresponds to Q_r. (Here they can be equated as they are of the same dimension and can be of same rank.) Similarly multiplying the last equation of (7.77) by Z^{t} and simplifying

$$A^tZ^tW_r - Z^tK_rZBR^{-1}B^tZ^tW_r - C^tQ_rf_r = 0 \tag{7.78}$$

Comparing equation (7.78) to the last equation of (7.77), with the correspondence of K, W and Q to Z^tK_rZ, Z^tW_r and Q_r as defined above, it is seen that f can be made to correspond to f_r, i.e. $f = f_r$ and $Q = Q_r$ etc. The closed loop equation for the tracking problem is

$$\dot{x} = (A - R^{-1}B^tZ^tK_rZ)x - BR^{-1}B^tZ^tW_ru \tag{7.79}$$

Equation (7.78) is true if and only if equation (7.79) is stable. It was shown in Chapter V (theorem 5E) that if Z is obtained from projection methods of model reduction (Aoki, Hickin et al), then, for state feedback, the eigenvalues of the original closed loop system, $A_f = A - BK_rZ$ or $A_f = A - R^{-1}B^tZ^tK_rZ$, are the r eigenvalues of the reduced closed loop system, $A_{fr} = A_r - B_rK_r$ or $\bar{A}_{fr} = A_r - R^{-1}B_r^tK_r$, and the n-r unretained eigenvalues of the open-loop matrix A. For output feedback, an approximate relationship holds. Theorem 5E is not true if Z is obtained non-specifically.

Here for a general S, in order to assess stability of the sub-optimal control, corollary 7.1 of theorem 7.6 or theorem 5.21 can be used.

Algorithm for designing sub-optimal controllers with stability assessment.

Step 1. Choose $Q \geq 0$, $R > 0$ for performance index J

Step 2. Calculate corresponding Q_r and R_r in J_r

Step 3. Determine sub-optimal control $u = -K_rZx$ or $u = -K_ry$ on S, using Corollary 7.1.

Step 4. Compute transient error $e(t) = y(t) - y_r(t)$. If tolerable, exit, otherwise go to Step 1.

Using Corollary 7.1 in the above algorithm, to determine sub-optimal control action, will guarantee the stability of the sub-optimal system.

7.5 Conclusions

Using reduced models in design, it is seen that the sensitivity properties of the closed loop system (or plant) is important. The controller is thus designed such that the closed loop plant satisfies the sensitivity criterion in terms of the sensitivity function of the reduced model.

Designing sub-optimal control systems via the frequency domain approach, it is seen that a prior condition can be determined, such that the sub optimal feedback system is guaranteed stable. However, gain in stability is compensated by loss in sensitivity. As the characteristic loci of the sub-optimal system penetrates the unit disc, a sub-optimal system is less immune to parameter variations that an optimal system.

The bounds established in this chapter are best evaluated using interactive graphics, when immunity to parameter variation has a high priority in design.

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VIII

BOUNDS FOR NONLINEAR MULTIVARIABLE SYSTEMS AND REDUCED MODEL DESIGN.

CHAPTER VIII

BOUNDS FOR NONLINEAR MULTIVARIABLE SYSTEMS AND REDUCED MODEL DESIGN

Introduction

This chapter looks at nonlinear multivariable systems, and, the adaptation of reduced models in systems design, the main emphasis being on stability. The classical Lur'e problem[2,7,10] is investigated in the multivariable setting, consisting of a linear transfer function, in the forward path, and a slope restricted, nonlinear, time dependent, memoryless feedback structure. The results of Rosenbrock, Cook and Falb et al[4] are adapted to the use of reduced models. Araki[6], following the lines of Zames[5], have extended the idea of conicity and positivity to investigate stability of nonlinear systems, in L_2 space. Following the same lines, stability bounds are derived for reduced models in nonlinear systems.

The describing function method has proved to be an attractive design tool for nonlinear multivariable systems. Sufficient conditions are derived for the absence of limit cycles. The derived criteria are less conservative and more flexible than existing criteria, than those of Mees' and Ramani and Atherton[12]. It is also shown that the new criteria can be easily modified, to accommodate the use of reduced models in design, by widening the bounds.

Rosenbrock[2] and Cook[3] studied the Lur'e problem by giving a graphical interpretation of the 'circle criterion'[10,30] in terms of 'diagonal dominance', starting from Anderson's generalization[19] of the Kalman-Yacobovitch lemma. The other solution for the Lur'e problem is the Popov[7] criterion, and it has been extended to multivariable systems by Jury and Lee[9]. Using Geshgorin's theorem, Shankar and Atherton[11] gave a graphical evaluation of the Jury-Lee criterion, in special cases, their results coincided with those of Cook[3]. Viewing stability from the Characteristic Loci (eigenvalue) concept, the stability results of Falb, Freedman and Zames[4], for normal matrices, are rederived by a shorter method. New graphical interpretations are given for the multivariable Lur'ë problem in terms of the 'circle criterion' and the Jury-Lee criterion. The new graphical results are more flexible, and in certain cases, less conservative than existing ones; however, their main advantage is the ease with which they can be used, to obtain a stability relationship between reduced and original models.

Model Reference Adaptive Control Systems (MRACS)[16,17] is an interesting aspect in system design. The adaptive mechanism is usually nonlinear in nature; to this end, Popov's hyperstability[8] criterion and Lyapunov synthesis method find attractive design applications. Hsia[31] showed that reduced models can be used beneficially in designing MRACS. In the chapter the role of reduced models in MRACS design is studied with emphasis on stability.

8.1 Stability analysis of nonlinear systems designed using reduced models

8.1.1 The circle criterion

This section studies nonlinear control systems shown in figure 8.1. The forward path consists of the linear block $Q_r(s) = G_r(s)K_r(s)$, where $Q_r(s)$ is assumed to be strictly proper and real rational. The nonlinear transfer function in the feedback path $N = \text{diag}\{n_1(t,y),\ldots, n_m(t,y)\}$ consists of time dependent, memoryless nonlinearities, lying in the sector (a_i,b_i), i.e. slope restricted by

$$0 \leq a_i \leq n_i(t,y) \leq b_i \quad , \quad a_i < b_i \quad , \quad \forall i \tag{8.1}$$

The aim is to find the closed loop stability of the systems

$$S_r \;:\; \dot{x}_r = (F_r - G_r N H_r)X_r \tag{8.2}$$

$$S \;:\; \dot{x} = (F - GNH)X \tag{8.3}$$

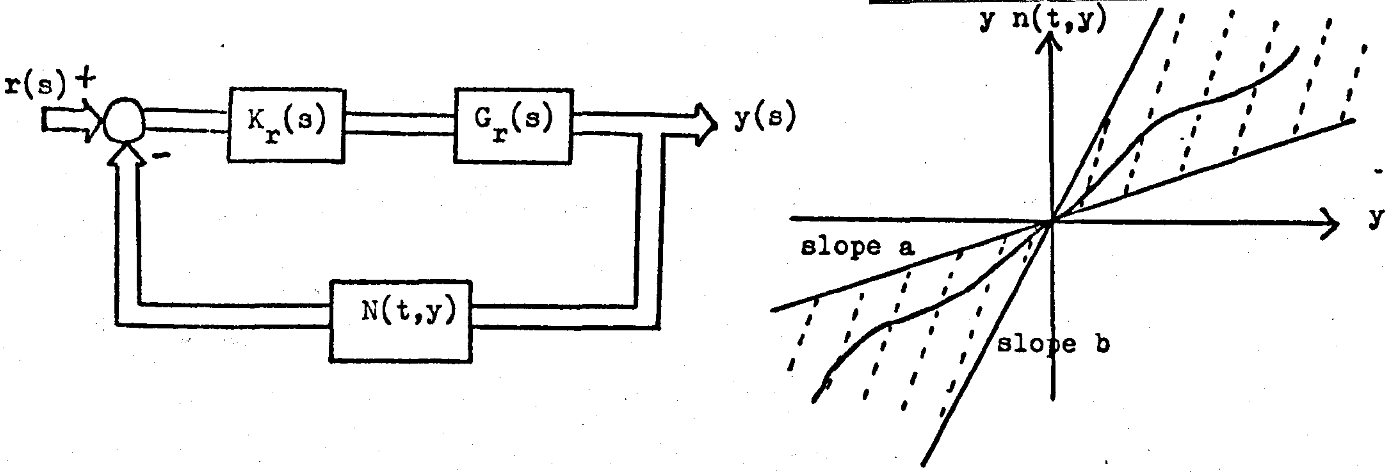
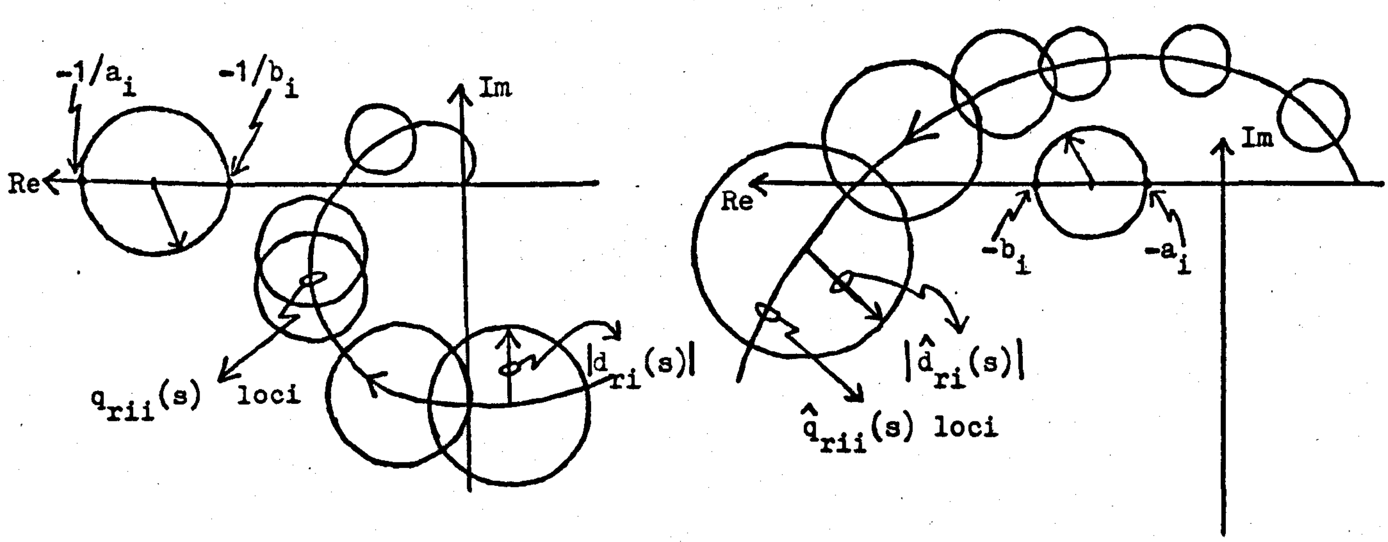


Fig. 8.1 Nonlinear system configuration



direct loci | inverse loci

Fig. 8.2 Illustrating theorem 8B

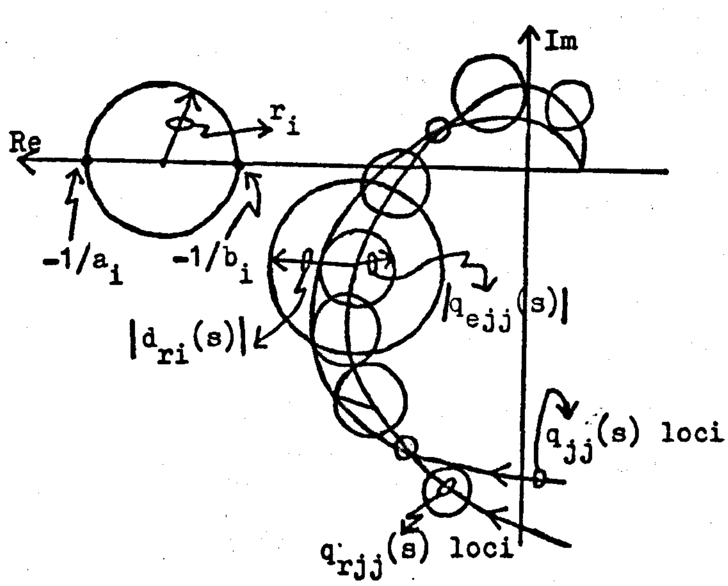


Fig. 8.3 Illustrating theorem 8.2
(bands must avoid critical disc for stability)

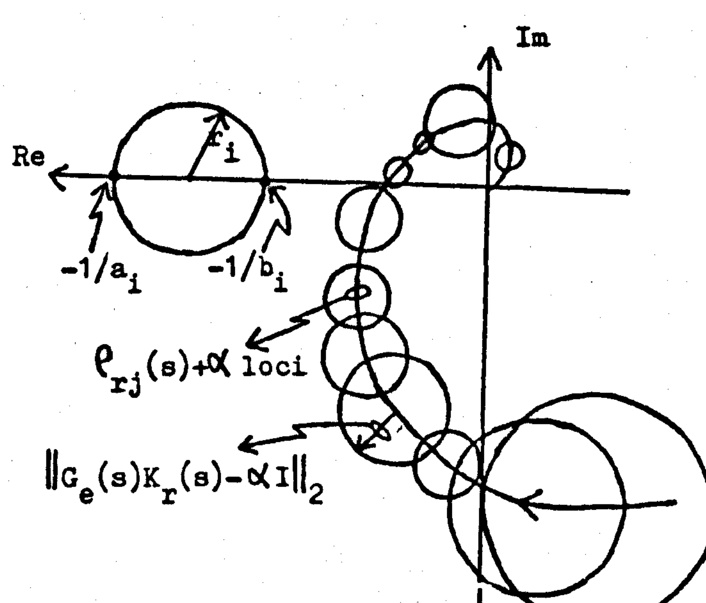


Fig. 8.4 Illustrating theorem 8.3
(case when $G_r(s)K_r(s)$ is normal
bands must avoid critical disc
for nonlinear system stability)

where $S_r(F_r, G_r, H_r)$, $S(F, G, H)$ are minimal realizations of $Q_r(s) = G_r(s)K_r(s)$, $Q(s) = G(s)K_r(s)$ respectively. Graphical solution in the frequency domain is best suited to design and stability assessment of S and S_r in equation (8.2). Many graphical solutions are available and a few will be adapted to study S and S_r. In the sequel, it is assumed that the linear systems defined by $S_r(F_r, G_r, H_r)$, $S(F, G, H)$ with $u = -Ay$ and $u = -By$ where $A = \text{diag}\{a_1, \ldots, a_m\}$, $B = \text{diag}\{b_1, \ldots, b_m\}$, are closed loop stable

Theorem 8A[2]

It is well known that if

$$W_r(s) = \left[B^{-1} + Q_r(s)\right]^{-1}\left[A^{-1} + Q_r(s)\right] \tag{8.4}$$

$$\text{resp} \quad \left[A^{-1} + Q_r(s)\right]\left[B^{-1} + Q_r(s)\right]^{-1} \tag{8.5}$$

$$\text{resp} \quad \left[I_m + Q_r(s)A\right]^{-1}\left[I_m + Q_r(s)B\right] \tag{8.6}$$

$$\text{resp} \quad \left[I_m + BQ_r(s)\right]\left[I_m + AQ_r(s)\right]^{-1} \tag{8.7}$$

exist and be positive real, then the system $S_r(A_r - B_rNC_r)$ is stable in the sense of Lyapunov (i.s.L.). Theorem 8A gives sufficient conditions for stability and can be proved using Anderson's generalization of the Kalman-Yacubovitch lemma.

The stability of S will now be based on that of S_r, via theorem 8A.

Theorem 8.1

Given that S_r is stable i.s.L., in terms of theorem 8A then S is likewise stable if $W_e(s) = W(s) - W_r(s)$ is positive real, where W(s) has the same definition as $W_r(s)$, in equation (8.4), except that the subscript r is dropped.

Proof:

From theorem 8A, it can be deduced that S is stable if W(s) is positive real. Hence if $W_r(s)$ is positive real (thus ensuring stability of S_r) and $W_e(s)$ is also positive real, then $W_r(s) + W_e(s)$ is also positive real (follows from a theorem of positive real matrices)[23]. Thus $W(s)$ is stable i.s.L.

The algebraic content of theorem 8.1 is cumbersome and to test for positive realness of $W_e(s)$ would be tedious. Some graphical methods will be developed in section 8.3 to give graphical interpretations of theorem 8.1. Below are stated some known multivariable stability theorems for S_r, and, they are adapted to study the stability of S via perturbation techniques.

Theorem 8B[3]

Let the linear system $S_r(F_r - G_rBH_r)$ (resp $S_r(F_r - G_rAH_r)$) be asymptotically stable. Then the nonlinear system $S_r(F_r - G_rNH_r)$ is stable i.s.L. if

(1) $|\delta_i + q_{rii}(s)| - |d_{ri}(s)| > r_i \quad , \quad \forall i, \quad \forall s \in D$ (8.8)

(resp $|\hat{\delta}_i + \hat{q}_{rii}(s)| - |\hat{d}_{ri}(s)| > \hat{r}_i$) $\forall i, \quad \forall s \in D$

where $|d_{ri}(s)| = \frac{1}{2}\sum_{i\neq j}\{|q_{rij}(s)| + |q_{rji}(s)|\}$,

$|\hat{d}_{ri}(s)| = \frac{1}{2}\sum_{i\neq j}\{|\hat{q}_{rij}(s)| + |\hat{q}_{rji}(s)|\}$, $\delta_i = (a_i^{-1} + b_i^{-1})/2$,

$\hat{\delta}_i = (a_i + b_i)/2$, $r_i = (a_i^{-1} - b_i^{-1})/2$, $\hat{r}_i = (b_i - a_i)/2$

Theorem 8B is shown graphically in figure 8.2. The locus $q_{rii}(s)$ (resp $\hat{q}_{rii}(s)$) automatically satisfies the Nyquist criterion if $S_r(F_r - G_rBH_r)$ (resp $S_r(F_r - G_rAH_r)$) is stable. Theorem 8B is Cook's modification of Rosenbrock's theorem. Here stability is determined by mean diagonal dominance. The next theorem gives the stability of S in terms of theorem 8B.

Theorem 8.2

Let $\alpha + q_{rjj}(s)$ (resp $\beta + \hat{q}_{rjj}(s)$) map D into $\alpha + \Gamma_{rj}$ (resp $\beta + \hat{\Gamma}_{rj}$). Let $|\eta_{rii}(s)| = |q_{rii}(s)| - |d_{ri}(s)| > 0$ (i.e. diagonal dominant).

Define $q_{ejj}(s) = q_{jj}(s) - q_{rjj}(s)$, $\Delta_e = |q_{jj}(s)| - |q_{rjj}(s)|$, $|d_{ei}(s)| = |d_i(s)| - |d_{ri}(s)|$, where $|d_i(s)|$ has the same definition as $|d_{ri}(s)|$, minus subscript r. (Similar definitions for inverse quantities are defined by adding superflex ^ on top.) Then sufficient conditions for S to be stable i.s.L. are:

(1a) $\Gamma_{rj} + \alpha$ (resp $\hat{\Gamma}_{rj} + \beta$) satisfies NC, with critical point $-b_j^{-1}$ (resp $-a_j$) , $\forall j$

(1b) $|1 + \alpha + q_{rjj}(s)| > |q_{ejj}(s) - \alpha|$, $\forall i$, $\forall s \epsilon D$ (8.9)

(resp $1 + \beta + \hat{q}_{rjj}(s)| > |\hat{q}_{ejj}(s) - \beta|$)

(2) $\Delta_e \geq |d_{ei}(s)|$, or $|\eta_{rii}(s)| > |d_{ei}(s)| - \Delta_e$, $\forall i$, $\forall s \epsilon D$ (8.10)

(resp for inverse quantities).

(3) $|\delta_i + q_{rii}(s) + q_{eii}(s)| > |d_{ri}(s)| + r_i$, $\forall i$, $\forall s \epsilon D$ (8.11)

(resp for inverse quantities)

Proof:

Conditions (1a) and (1b) ensure $q_{jj}(s)$ satisfies NC when $q_{rjj}(s)$ likewise satisfies NC (see theorem 5.7). Condition (2) requires mean diagonal dominance of Q(s) based on that of $Q_r(s)$ and increment $Q_e(s)$. Condition (3) requires system S to satisfy expression (8.8). Hence S is stable i.s.L. by theorem 8B. Theorem 8.2 is shown graphically in figure 8.3.

It is also desirable to express stability of S in terms of its characteristic loci. To this end, when designing multivariable systems with nonlinear feedback, the results of theorems 5.2 or 6.4 can be adapted to a stability condition due to Falb , et al[4].

Theorem 8.3

Let $Q_r(s)$, Q(s) both be normal, and $\rho_{rj}(s)$, $\rho_j(s)$ their characteristic loci. Let $\rho_{rj}(s) + \alpha$ map D into $\Gamma_{rj} + \alpha$. Also let r_i and δ_i be as defined in theorem 8B and other pertinent quantities as defined in theorems 5.2 or 6.4. Then the nonlinear system S is stable if:

(1) $\Gamma_{rj} + \alpha$ satisfies NC (critical point $-b_j^{-1}$) , $\forall j$

either (2) $|\delta_j + \rho_{rj}(s) + \alpha| > \|G_e(s)K_r(s) - \alpha I\| + r_j$,

$\forall j, \quad \forall s \epsilon D$ (8.12)

or (3) $|\delta_j + \rho_{rj}(s) + \alpha| > (m+2)\{M(s)\}^{(m-1)/m}\{\sum_{i,j} |q_{eij}(s) - \alpha I|\}^{1/m}$,

$\forall j, \quad \forall s \epsilon D$ (8.13)

Proof:

Theorem 8.3 shown graphically in figure 8.4, is similar to figure 8.2. Here the critical disc is the same. If $Q_r(s)$ is normal and N(t,y) is slope restricted in the sense of equation (8.1), then S_r is stable i.s.L. if $\rho_{rj}(s)$ does not encircle or penetrate the critical disc. Conditions (1) and (2) or (3) ensure that $\rho_j(s)$ of Q(s) cannot encircle or penetrate the critical disc, and, since Q(s) is normal, S is stable i.s.L.

In the special case when $Q_r(s) \to Q(s)$, $q_{eij}(s) \to 0$, $G_e(s) \to 0$ and the bands on $\rho_{rj}(s)$ vanish, and, letting $\alpha = 0$, theorem 8.3 reduces to the result of Falb et al.

8.1.2 Input-Output Stability in L_2-space[5,6]

The circle criterion deals with global stability i.s.L. Araki[6] proposed a stability criterion, in Hilbert space, following the formulation of Zames[5], on conicity and positivity. However, in many practical systems, stability in the latter case implies stability in the former case. If x(t), y(t) $\in L_2$ space, then the inner product is defined on a Hilbert space, $L_2^m(\infty,0]$, i.e. $\langle x,y\rangle = \langle x,Py\rangle_{H_m}$ with

$P > 0$, and, the system is L_2 stable if bounded inputs produce bounded outputs.

Suppose that the nonlinear part of figure 8.1 is bounded by expression (8.1). Define $u_{rjj} = 0$, $u_{rij} = \max_{\omega\in R}|q_{rij}(s)|$, $i \neq j$, $v_{rj} = \min_{\omega\in R}|q_{rjj}(s) - \delta_j|$, where as before $\delta_j = -\frac{1}{2}(a_j^{-1} + b_j^{-1})$, $r_j = \frac{1}{2}(a_j^{-1} - b_j^{-1})$. Further let matrices be defined such that $V_r = \operatorname{diag}(v_{rj})$, $R = \operatorname{diag}(r_j)$ and $U_r = (u_{rij})$. Then following Araki[6],

<u>Theorem[6] 8C</u>

Case 1: $0 \leq a_j \leq b_j \quad , \quad \forall j$

The system, S_r, figure 8.1 is L_2-stable if the locus of $q_{rjj}(s)$ satisfies the Nyquist criterion (NC), with critical point $(\delta_j, 0)$ and if $V_r - R - U_r$ is an M-matrix.

Case 2: $a_j = 0$, $b_j > 0$, $\forall j$

S_r is L_2-stable if $D_r - B^{-1} - U_r$ is an M-matrix. Here $B = \operatorname{diag}(b_j)$, $D_r = \operatorname{diag}(d_{rj})$, $d_{rj} = \min_{\omega\in R}\operatorname{Re}\{q_{rjj}(s)\}$

Case 3: $a_j < 0$, $b_j < 0$, $\forall j$

S_r is L_2-stable if $R - V'_r - U_r$ is an M-matrix. Here $V'_r = \operatorname{diag}(v'_{rj})$ where $v'_{rj} = \max_{\omega\in R}|q_{rjj}(s) - \delta_j|$

Theorem 8C can be adapted to investigate the L_2 stability of S after K_r is implemented. The results of theorem 5.11and a main result due to Ostrowski will be used. Let $Q(s) = q_{ij}(s)$, $Q_e(s) = q_{eij}(s)$ where $Q(s) = Q_r(s) + Q_e(s)$. Further, assume that the same slope restrictions of equation (8.1) apply to S. (If they are changed, appropriate changes on the bounds a_j, b_j can be made accordingly, and the analysis follows exactly.)

Theorem 8.4

Case 1: $0 < a_j \leq b_j \quad , \quad \forall j$

S is L_2-stable if S_r is likewise stable and if

(1) The locus of $\beta + q_{rjj}(s)$ satisfies NC, critical point $(\delta_j, 0)$, $\forall j$

(2) $|\beta + q_{rjj}(s)| > |q_{ejj}(s) - \beta|$, $\forall j$, $\forall s \epsilon D$ (8.14)

(3) $v_j - r_j - u_{ij} \geq v_{rj} - r_j - u_{rij}$, $\forall j$, $u_{ij} \leq u_{rij}$, $\forall i \neq j$, $\forall s \epsilon D$ (8.15)

where $v_j = \min_{\omega \epsilon R}|q_{jj}(s) - \delta_j|$, $u_j = \max_{\omega \epsilon R}|q_{ij}(s)|$, $\forall i \neq j$

Case 2: $a_j = 0$, $b_j > 0$, $\forall j$

S is L_2-stable, given S_r is likewise stable, if

(1) $d_j - b_j^{-1} - u_{jj} \geq d_{rj} - b_j^{-1} - u_{rij}$, $\forall j$, $u_{ij} \leq u_{rij}$, $\forall i \neq j$, $\forall s \epsilon D$ (8.16)

where $d_j = \min_{\omega \epsilon R} \mathrm{Re}\{q_{jj}(s)\}$, $u_{ij} = \max_{\omega \epsilon R}|q_{ij}(s)|$

Case 3: $a_j < 0$, $b_j > 0$

S is L_2-stable, given S_r is likewise stable if

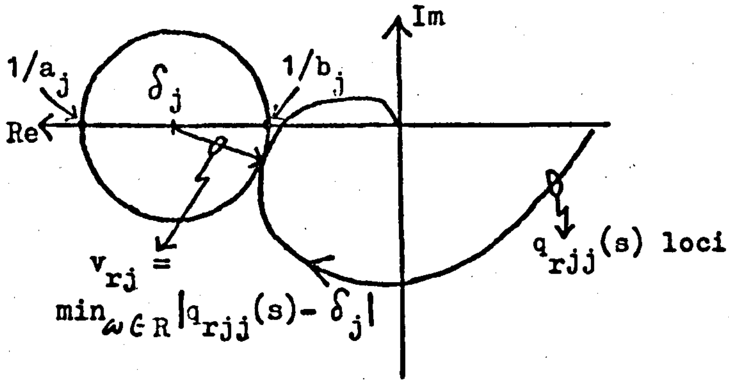
(1) $r_j - v_j' - u_{jj} \geq r_j - v_{rj}' - u_{rjj}$, $\forall j$, $u_{ij} \leq u_{rij}$, $\forall i \neq j$ (8.17)

where $v_j' = \min_{\omega \epsilon R}|q_{jj}(s) - \delta_j|$

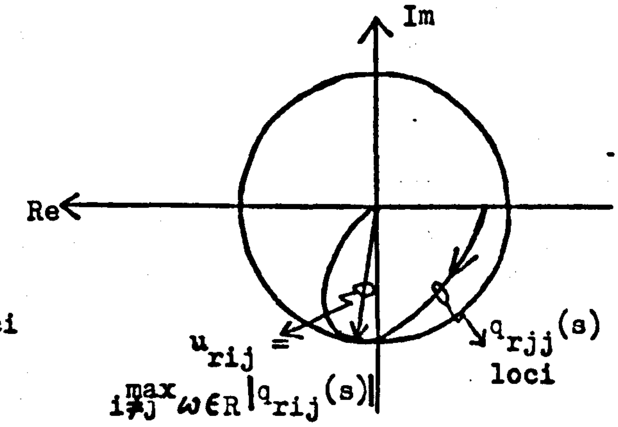
Proof:

In case 1, conditions (1) and (2) state that the loci of $q_{jj}(s)$ must satisfy NC (see theorem 5.8 for single input - single output systems). Conditions (3) of case 1, (1) of case 2 and (1) of case 3, equations (8.15), (8.16) and (8.17) respectively, are obtained from a theorem of Ostrowski. This states that if $F \triangleq (f_{ij})$ is an M-matrix, then $H \triangleq (h_{ij})$ is also an M-matrix if $f_{ii} \leq h_{ii}$, $\forall i$, $f_{ij} \leq h_{ij} \leq 0$, $\forall i \neq j$ (i.e. $|f_{ij}| \geq |h_{ij}|$, $\forall i \neq j$, as the off diagonal elements of F and H are nonpositive). Hence the latter condition states that the respective matrices of S must also be M-matrices. The stability conditions of theorem 8.4 thus follow from theorem 8C.

The graphical method for evaluating theorem 8C is shown in figure 8.5a. Theorems 5.2 and 6.4 can also be adopted to interpret theorem 8.4 in graphical terms.



calculation of $V_r = \mathrm{diag}(v_{rj})$



calculation of $U_r = (u_{rij})$

Fig. 8.5(a) Illustrating theorem 8C case when $0 < a_j \leq b_j$

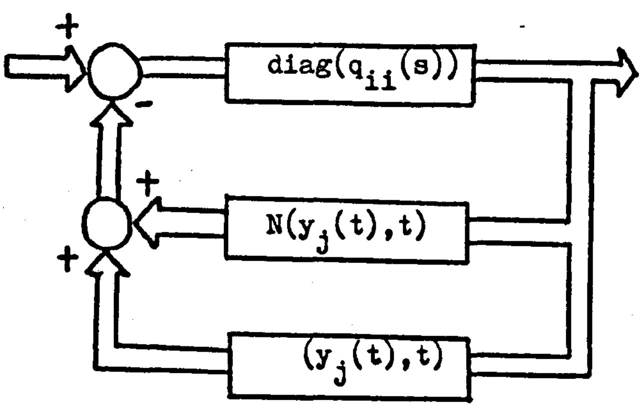
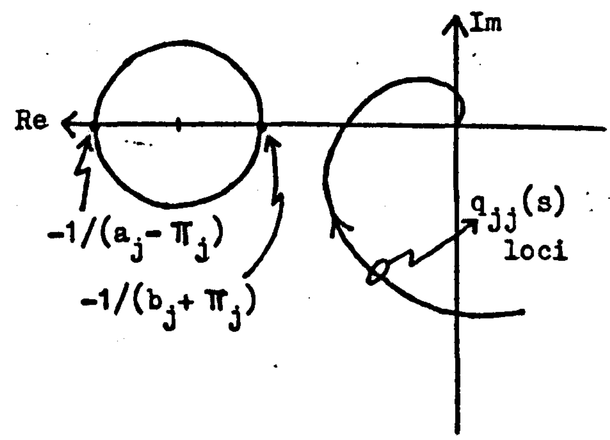
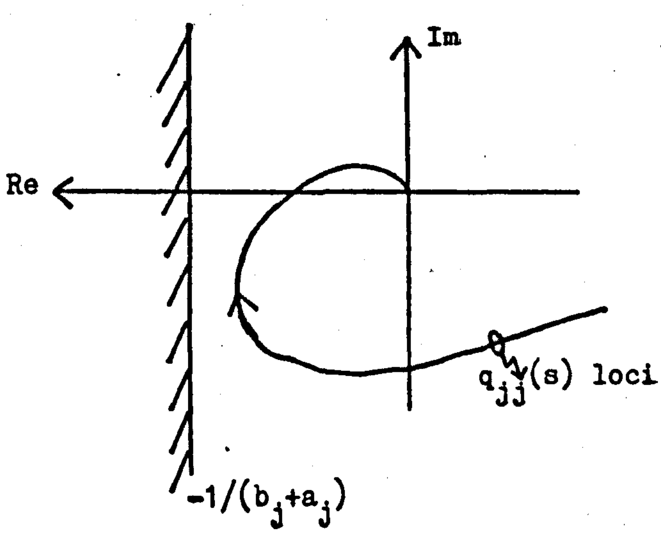


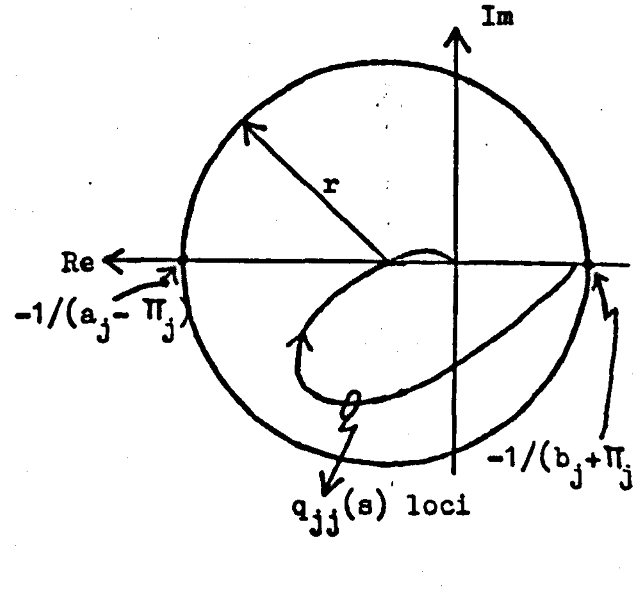
Fig. 8.5(b) Multiinput-multioutput nonlinear system considered in theorem 8D



case(i) $a_j > \pi_j$



case(ii) $a_j = \pi_j$



case(iii) $a_j < \pi_j$

Theorem 8.5

Let λ_{rj}, ρ_{rj} and θ_{rj} be eigenvalues of $V_r - R - U_r$, $D_r - B^{-1} - U_r$ and $R - V'_r - U_r$ respectively. Let α, β and γ be arbitrary scaling factors. Then S is L_2 stable if:

Case 1:

(1) The locus of $\alpha + q_{rjj}(s)$ satisfies NC, critical point $(\delta_j, 0)$, $\forall j$.

(2) $|\alpha + q_{rjj}(s)| > |q_{ejj}(s) - \alpha|$, $\forall j$, $\forall s \in D$ (8.18)

either

(3) $\mathrm{Re}\ (\lambda_{rj} + \alpha) > ||V_{re} - U_{re} - \alpha I||\inf \kappa(P_1(s))$, $\forall j$, $\forall s \in D$ (8.19)

or

(4) $\mathrm{Re}\ (\lambda_{rj} + \alpha) > (m+2)M_1(s)\{d_{e1}(s)\}^{1/m}$, $\forall j$, $\forall s \in D$ (8.20)

Case 2:

(1) $\mathrm{Re}\ (\rho_{rj} + \beta) > ||D_{re} - U_{re} - \beta I||\inf \kappa(P_2(s))$, $\forall j$, $\forall s \in D$ (8.21)

or

(2) $\mathrm{Re}\ (\rho_{rj} + \beta) > (m+2)M_2(s)\{d_{e2}(s)\}^{1/m}$, $\forall j$, $\forall s \in D$ (8.22)

Case 3:

(1) $\mathrm{Re}\ (\theta_{rj} + \gamma) > ||-V'_{re} - U_{re} - \gamma I||\inf \kappa(P_3(s))$, $\forall j$, $\forall s \in D$ (8.23)

or

(2) $\mathrm{Re}\ (\theta_{rj} + \gamma) > (m+2)M_3(s)\{d_{e3}(s)\}^{1/m}$, $\forall j$, $\forall s \in D$ (8.24)

where $V_{re} = V - V_r$, $U_{re} = U - U_r$, $D_{re} = D - D_r$, $V'_{re} = V' - V'_r$

Proof:

Equations (8.19) to (8.24) result from the application of theorems 5.2 and 6.4 to condition (3) of theorem 8.4. The real parts of the eigenvalues of an M-matrix are positive, and since $V - R - U$, $D - B^{-1} - U$, $R - V' - U$ and $R - V' - U$ have nonpositive off-diagonal elements and their eigenvalues are confined to the r.h.s. plane, they are M-matrices. Stability of S thus follows from theorem 8C.

Another situation where M-matrices can be used to determine the stability of composite systems is shown in figure 8.5b. Here the linear system is diagonal $Q(s) = \text{diag}\,\{q_{ii}(s)\}$ and the nonlinear blocks satisfy

$$a_j < n_j(y_j,t) < b_j \tag{8.25}$$

$$|\phi_i(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m,t)| \leq \sum_{j=1}^{m} \theta'_{ij}\,|\varepsilon_j| \tag{8.26}$$

Theorem 8D[6]

The system of figure 8.5 is L_2 stable if, for each j, $\Pi - \Phi'$ is an M-matrix where $\Pi = \text{diag}\,\{\pi_j\}$, $\pi_j > 0$, $\Phi' = (\theta'_{ij})$ and $q_{jj}(s)$ satisfies one of the following conditions:

(1) $a_j > \pi_j$

The Nyquist diagram of $q_{jj}(s)$ does not encircle or intersect the disc with centre $(-\frac{1}{2}(1/(a_j - \pi_j)) + (1/(b_j + \pi_j)),0)$ and radius $\frac{1}{2}|(1/a_j - \pi_j)) - (1/(b_j + \pi_j))|$

(2) $a_j = \pi_j$

The Nyquist diagram lies to the right of the vertical line passing through $(-1/(a_j + b_j),0)$

(3) $a_j < \pi_j$

The Nyquist diagram lies inside the disc with centre and radius as defined in condition (1).

Now assume original and reduced models, S and S_r satisfy expression (8.25) and the following bounds.

$$\text{for } S_r \;:\; |\phi_i(\varepsilon_1,\ldots,\varepsilon_m,t)| \leq \sum_{j=1}^{m} \theta'_{rij}|\varepsilon_j| \tag{8.26a}$$

$$\text{for } S \;:\; |\phi_i(\varepsilon_1,\ldots,\varepsilon_m,t)| \leq \sum_{j=1}^{m} \theta'_{ij}|\varepsilon_j| \tag{8.26}$$

Then:

Theorem 8.6

Let $\Delta + \beta$ be the Nyquist locus of $q_{rjj}(s) + \beta$ with bands of radius $|q_{ejj}(s) - \beta|$ centred on $q_{rjj}(s) + \beta$, $\forall s = j\omega$, $\forall s \in D$. Then S of the configuration of figure 8.5 is L_2 stable given that S_r is likewise stable if:

(1) $\theta'_{ij} \leqslant \theta'_{rij}$, $\forall i$, $\forall j$ (8.26b)

(2) $\Delta + \beta$ satisfies conditions (1), (2) or (3) as appropriate of theorem 8D.

Proof:

Since S_r is L_2 stable $(\pi_j - \theta'_{rij})$, $\forall ij$, is an M-matrix. From condition (1) with $\theta'_{ij} \leq \theta'_{rij}$, it follows that $(\pi_j - \theta'_{ij})$ is also an M-matrix. (Ostrowski's result) Condition (2) requires S to satisfy the remaining conditions of theorem 8D, given that S_r satisfies the same remaining conditions. Hence S is L_2 stable by theorem 8D.

8.2 Describing functions and reduced models in nonlinear multivariable systems synthesis

8.2.1 Limit cycles and describing function (d.f.)[1,13]

The use of describing function techniques have been extended to multivariable nonlinear systems by MacFarlane[14], where it was shown that the d.f. method is a valuable analytical tool in analysis, expressed in the form of return difference and return ratio matrices. Mees[1] extended the first order frequency independent d.f. expression, for single input - single output systems, into the describing function matrix for multivariable systems. The feedback configuration has the form shown in figure 8.1. Using Rosenbrock's idea of diagonal dominance, Mees formulated a criterion for the absence of limit cycles. Ramani[12] and Atherton also gave two criteria for non-limit cycles. The first is algebraic in nature, using a theorem of

Hirsch that constrains the eigenvalues in a rectangle; the second, for a (2 x 2) $Q(s)$, requires all the d.f. and $q_{jj}(s)$ loci and an eigenvalue locus to be plotted.

The use of d.f. techniques in stability analysis is heuristical, and, it does not guarantee stability as does the circle criterion (c.c.). However, compared to the c.c., it has more information to the non-linear part, and, it can also have a non-diagonal nonlinear $N(t,y)$ matrix, and, the nonlinearities are not slope restricted. Thus it can handle hard nonlinearities like hysteresis and backlash. It has been argued that the d.f. and c.c. are complementary to each other and that neither contradicts one another in stability determination[1]. In fact, it has been shown that the inverse d.f. locus lies in the critical disc[1].

For the nonlinear feedback system of figure 8.1, where the nonlinear block $N(a)$ is the first order frequency independent d.f. matrix, and, a, is the vector amplitude of the first harmonic of the vector x, the d.f. equations are

$$(Q(s)N(a) + I)x = 0 \tag{8.27}$$

$$(N(a)Q(s) + I)y = 0 \tag{8.28}$$

or equivalently,
$$(\hat{Q}(s) + N(a))x = 0 \tag{8.29}$$

$$(Q(s) + \hat{N}(a))y = 0 \tag{8.30}$$

For the limit cycles to exist[14], equations (8.27) to (8.30) must have non-trivial solutions, which require det $(\hat{Q}(s) + N(a)) = 0$ or det $(Q(s) + \hat{N}(a)) = 0$, i.e. there must be at least one zero eigenvalue in $\hat{Q}(s) + N(a)$ or $Q(s) + \hat{N}(a)$.

Using Geshgorin's theorem,

<u>Theorem 8E</u>[1,13] (Mees' criterion)

For the absence of limit cycles, in figure 8.1, it is sufficient that

(1) $|(n_{jj}(a_j) + \hat{q}_{jj}(s)| > \sum_{i \neq j} |n_{ij}(a_j)| + \sum_{i \neq j} |\hat{q}_{ij}(s)| \quad , \forall j \quad , \forall a_j$ (8.31)

(resp $|q_{jj}(s) + \hat{n}_{jj}(a_j)| > \sum_{i \neq j} |\hat{n}_{ij}(a_j) + \sum_{i \neq j} |q_{ij}(s)|$),

$\forall j \quad , \forall a_j$ (8.32)

<u>Theorem 8F</u>[12] (Ramani and Atherton's criterion 1, RA1)

When $N(a) = \text{diag}\ \{n_{jj}(a_j)\}$, for the absence of limit cycles, it is sufficient that:

$$W(s) + \hat{U} > 0 \text{ or } W(s) + \hat{L} < 0 \text{ or } V(s) > 0 \text{ or } V(s) < 0$$

where $W(s) = \frac{1}{2}(Q(s) + Q^H(s))$, $V(s) = -j\frac{1}{2}(Q(s) - Q^H(s))$. When N(a) is non-diagonal, modifications can be made such that $(m^2 \times m^2)$ matrices are used.

<u>Theorem 8G</u>[13] (Ramani et al criterion 2, RA2)

When $N(a) = \text{diag}\ \ n_{jj}(a_j)$ and Q(s) is (2 x 2), a limit cycle is possible if, for a consistent pair of (a, $s = j\omega$), K satisfies

$$-\hat{n}_{11}(a_1) - q_{11}(s) = \lambda_j(s) - q_{11}(s)K \qquad (8.33)$$

$$-\hat{n}_{22}(a_2) - q_{22}(s) = \lambda_j(s) - q_{22}(s)K \qquad (8.34)$$

where $\lambda_j(s)$ is an eigenvalue of Q(s)

Using equations (8.27) to (8.30), some new criteria can be derived as follows. The matrices in equations (8.29) and (8.30) can be written as

$$\hat{Q}(s) + N(a) = \{(\hat{\alpha} + \hat{k})I + \hat{Q}(s)\} + N'(s) - \hat{\alpha}I \tag{8.35}$$

$$Q(s) + \hat{N}(a) = \{(\alpha + k)I + Q(s)\} + \hat{N}'(a) - \alpha I \tag{8.36}$$

where

$$\hat{k}I + N'(a) = N(a) \tag{8.37}$$

$$kI + \hat{N}'(a) = \hat{N}(a) \tag{8.38}$$

and α, $\hat{\alpha}$ are tuning factors, k, $\hat{k}$ are arbitrary 'critical points', and, in general, $\hat{\alpha} \neq 1/\alpha$, $\hat{k} \neq 1/k$. Thus,

Theorem 8.7 (New criterion)[14]

The system, S, of figure 8.1 will have no limit cycles if

(1) $|\hat{k} + \hat{\alpha} + \hat{\lambda}_j(s)| > ||N'(a) - \hat{\alpha}I|| \inf \kappa(P(s))$, $\forall j$, $\forall s = j\omega$ (8.39)

resp $|k + \alpha + \lambda_j(s)| > ||\hat{N}'(a) - \alpha I|| \inf \kappa(P(s))$, $\forall j$, $\forall s = j\omega$ (8.40)

where $\lambda_j(s)$ (resp $\hat{\lambda}_j(s)$) is the characteristic loci of $Q(s)$ (resp $\hat{Q}(s)$) and $P(s)$ diagonalizes $Q(s)$.

Proof:

S will have no limit cycles if the matrices in the r.h.s. of equation (8.35) or (8.36) have non-zero eigenvalues. A direct application of a theorem of Bauer and Fike, BF, (used in theorem 5.1) to the r.h.s. of equations (8.35) and (8.36) yields expressions (8.39) and (8.40).

The graphical interpretation of expressions (8.39) and (8.40) means that the circular bands, of halfwidth given by the r.h.s. of expression (8.39) (resp expression (8.40)) centred on $\hat{\alpha} + \hat{\lambda}_j(s)$ (resp $\alpha + \lambda_j(s)$), must not touch or overlap the $-k$ (resp $-\hat{k}$) 'critical point'. The constants k and α are advisable not to be chosen as zero if $Q(s)$ is strictly proper, as in this case $\rho_{rj}(s) \to 0$ as $s \to \infty$.

This means that the circular bands must avoid the origin, $\forall s = j\omega$.

Specifically, when the d.f. locus is required, the l.h.s. of equations (8.29) and (8.30) can be written as

$$\hat{Q}(s) + N(a) = \{(\hat{\alpha} + n_{ii}(a_i))I + \hat{Q}(s)\} + N''(a) - \hat{\alpha}I \tag{8.41}$$

$$Q(s) + \hat{N}(a) = \{(\alpha + \hat{n}_{ii}(a_i))I + Q(s)\} + \hat{N}''(a) - \alpha I \tag{8.42}$$

where

$$n_{ii}(a_i)I + N''(a) = N(a) \tag{8.43}$$

$$\hat{n}_{ii}(a_i)I + \hat{N}''(a) = \hat{N}(a) \tag{8.44}$$

Thus,

Theorem 8.8

S will exhibit no limit cycles if:

(1) $|n_{ii}(a_i) + \hat{\alpha} + \hat{\lambda}_j(s)| > ||N''(a) - \hat{\alpha}I||\inf \kappa(P(s)), \forall j, \forall s = j\omega$ (8.45)

$\{\text{resp } |\hat{n}_{ii}(a_i) + \alpha + \lambda_j(s)| > ||\hat{N}''(a) - \alpha I|| \inf \kappa(P(s))\},$

$\forall j, \forall s = j\omega$ (8.46)

Proof:

Application of BF theorem to the r.h.s. of equations (8.41) and (8.42) yields expressions (8.45) and (8.46).

The graphical interpretation of theorem 8.8 requires that the $-n_{ii}(a_i)$ (resp $-\hat{n}_{ii}(a_i)$) locus must not penetrate the $\{\hat{\alpha} + \hat{\lambda}_j(s)\} + ||N''(a) - \hat{\alpha}I|| \inf \kappa(P(s)) \exp(j\phi)$, $0 < \phi < 2\pi$ (resp $\{\alpha + \lambda_j(s)\} + ||\hat{N}''(a) - \alpha I|| \inf \kappa(P(s)) \exp(j\phi)$, $0 < \phi < 2\pi$) bands. This is shown in figure 8.6, where $\hat{n}_{ii}(a_i)$ is an element of $\hat{N}(a)$.

When $N(a) = \text{diag}\{n_i(a_i)\}$, theorem 8.8 becomes

Corollary 8.1

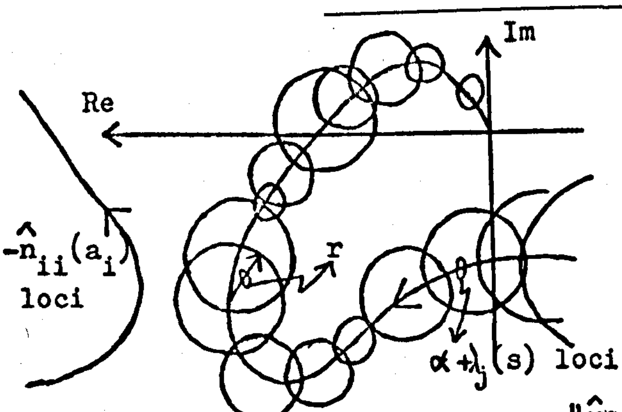
S will exhibit no limit cycle if:

(1) $|n_i(a_i) + \hat{\alpha} + \hat{\lambda}_j(s)| > \max_j |n_j(a_j) - n_i(a_i) - \hat{\alpha}| \inf \kappa(P(s))$

$\forall j \; , \; \forall s = j\omega$ (8.47)

(resp. $|1/n_i(a_i) + \alpha + \lambda_j(s)| > \max_j |1/n_j(a_j) - 1/n_i(a_i) - \alpha| \inf \kappa(P(s))$),

$\forall j \; , \; \forall s = j\omega$ (8.48)



$r = \|\hat{N}''(a) - \alpha I\| \inf \mathcal{K}(P(s))$

Fig. 8.6(a) Illustrating theorem 8.8 (for absense of limit cycles, d.f. locus must avoid circular bands) Theorem 8.12 for reduced model application is similarly explained.

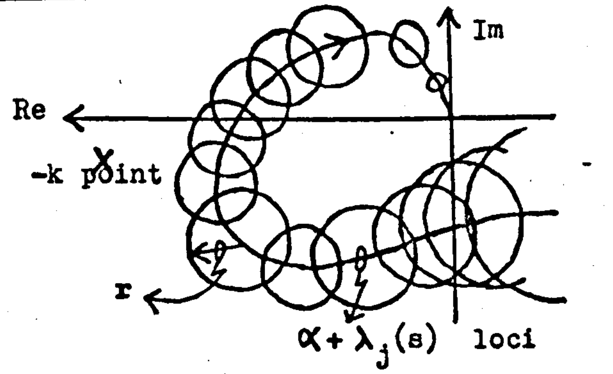
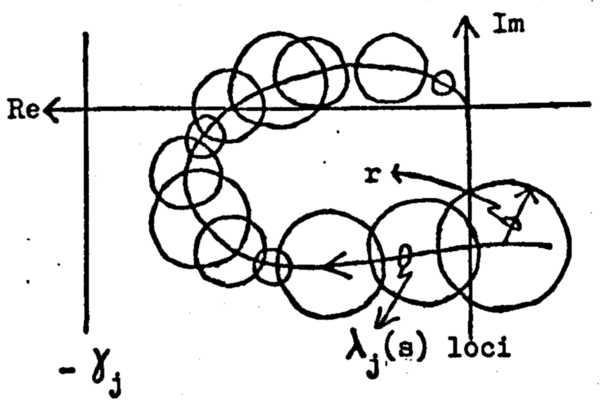


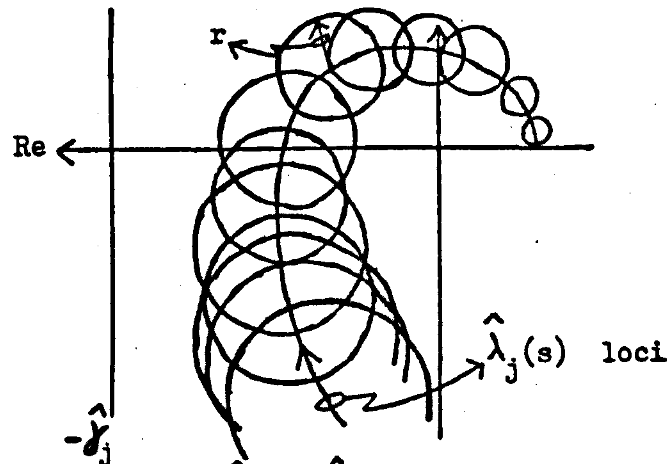
Fig. 8.6(b) Illustrating theorem 8.7 (for nonlimit cycles, the -k point must avoid the band) Similar explanation for theorem 8.11



$r = \|W(s) - \gamma_j I\| \inf \mathcal{K}(V(s))$

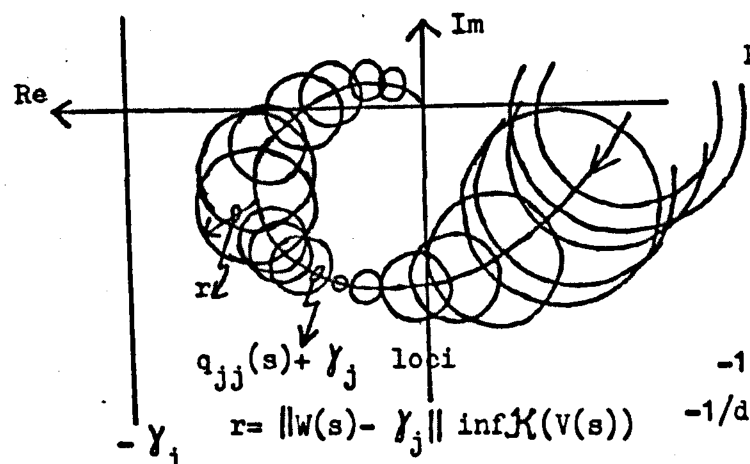
Fig. 8.7(a) Illustrating theorem 8.13 using direct Characteristic Loci (for stability of nonlinear system, the circular band must lie to the right of the $-\gamma_j$ vertical line)

Similar explanation for theorems 8.17 (reduced model application) & 8.18



$r = \|\hat{W}(s) - \hat{\gamma}_j\| \inf \mathcal{K}(V(s))$

Fig. 8.7(b) Illustrating theorem 8.13 using inverse Characteristic Loci (for stability, circular band must lie to the right of the $-\hat{\gamma}_j$ vertical line)



$r = \|W(s) - \gamma_j\| \inf \mathcal{K}(V(s)) + \sum_{i \neq j} |q_{ij}(s)|$

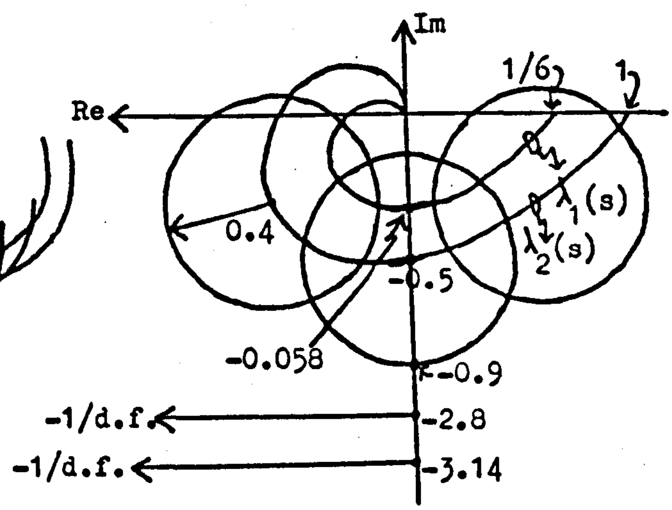


Fig. 8.9 Illustration of example. (See section 8.2.1)

Fig. 8.7(c) Illustrating theorems 8.14 and 8.16('circle criterion') (for stability, band must lie to right side of vertical line) Similar explanation for theorems 8.17 & 8.18 (reduced model applications)

Proof: Follows from theorem 8.8 and equations (8.43) and (8.44)

In the special case, when the transfer function matrix, Q(s), is normal, P(s) can be chosen unitary, and using the spectral norm, $|| \ ||_2$, *inf* $\kappa(P(s)) = 1$, *thus the radii of the bands are frequency independent* when Q(s) is normal. It is also interesting to note that when $n_j(a_j) = n_c \quad , \forall j$, and setting $\alpha = \hat{\alpha} = 0$,

Corollary 8.2

When $N(a) = \text{diag}\,\{n_c\}$, $\hat{N}(a) = \text{diag}\,\{1/n_c\}$, S will exhibit no limit cycles if

(1) $|n_c + \hat{\lambda}_j(s)| > 0 \quad , \quad \forall s = j\omega$ (8.49)

$\left(\text{resp. } |1/n_c + \lambda_j(s)| > 0\right) \quad , \quad \forall s = j\omega$ (8.50)

Proof: Follows from corollary 8.1.

Thus, for nonlimit cycles, the $-1/n_c$ (resp $-n_c$) locus must not intercept with any $\lambda_j(s)$ (resp $\hat{\lambda}_j(s)$) locus. This can also be derived by applying the eigenvalue shift theorem to $\hat{Q}(s) + N(a)$ or $Q(s) + \hat{N}(a)$. The same result was reported in the RA2 criterion, but, using Gershgorin's theorem, the bands will not vanish.

Theorem 6.3 (Ostrowski's theorem) can be adapted to study the eigenvalues of $Q(s) + \hat{N}(a)$ in terms of the eigenvalues of Q(s). Thus parallel theorems and corollaries to theorems 8.7, 8.8, corollaries 8.1 and 8.2 can be stated.

Theorem 8.9

S will exhibit no limit cycles if:

(1) $|\hat{k} + \hat{\alpha} + \hat{\lambda}_j(s)| > (m+2)\{\hat{M}(s)\}^{(m-1)/m}\{\sum_{i,j} |n'_{ij}(a) - \hat{\alpha}\delta_{ij}|\}^{1/m}$

$\forall j \quad , \quad \forall s = j\omega$ (8.51)

$\left(\text{resp. } |k + \alpha + \lambda_j(s)| > (m+2)\{M(s)\}^{(m-1)/m}\{\sum_{i,j} |\hat{n}'_{ij}(a) - \alpha\delta_{ij}|\}^{1/m}\right)$

$\forall j \quad , \quad \forall s = j\omega$ (8.52)

where $\hat{M}(s) = \max \{|(\hat{\alpha} + \hat{k})\delta_{ij} + \hat{q}_{ij}(s)|, |n_{ij}(a) + \hat{q}_{ij}(s)|\}$,
$M(s) = \{|(\alpha + k)\delta_{ij} + q_{ij}(s)|, |\hat{n}_{ij}(a) + q_{ij}(s)|\}$, δ_{ij} is the Kronecker delta ($\delta_{ij} = 1, \forall i = j$, $\delta_{ij} = 0$, otherwise) and m is the dimension of Q(s).

Proof: Direct application of theorem 6.3 to equations (8.35) to (8.38)

Theorem 8.10

S will exhibit no limit cycles if:

$$\text{(1)} \quad |n_{ii}(a_i) + \hat{\alpha} + \hat{\lambda}_j(s)| > (m+2)\{\hat{M}(s)\}^{(m-1)/m}\{\sum_{i,j} |n''_{ij}(a) - \hat{\alpha}\delta_{ij}|\}^{1/m} \qquad \forall j \quad , \quad \forall s = j\omega \qquad (8.53)$$

$$(\text{resp.}\ |\hat{n}_{ii}(a_i) + \alpha + \lambda_j(s)| > (m+2)\{M(s)\}^{(m-1)/m}\{\sum_{i,j} |\hat{n}''_{ij}(s) - \alpha\delta_{ij}|\}^{1/m}) \qquad \forall j \quad , \quad \forall s = j\omega \qquad (8.54)$$

where $\hat{M}(s) = \max \{|(\hat{\alpha} + n_{ii}(a_i))\delta_{ij} + \hat{q}_{ij}(s)|, |n_{ij}(a) + \hat{q}_{ij}(s)|\}$,
$M(s) = \max \{|(\alpha + \hat{n}_{ii}(a_i))\delta_{ij} + q_{ij}(s)|, |\hat{n}_{ij}(a) + q_{ij}(s)|\}$

Proof: Application of theorem 6.3 to equations (8.41) to (8.44)

Corollary 8.3

When $N(a) = \text{diag}\ \{n_i(a_i)\}$, $\hat{N}(a) = \text{diag}\ \{1/n_i(a_i)\}$, S will exhibit no limit cycles if:

$$\text{(1)} \quad |n_i(a_i) + \hat{\alpha} + \hat{\lambda}_j(s)| > (m+2)\{\hat{M}(s)\}^{(m-1)/m}\{\sum_{j=1}^{m} |n_j(a_j) - n_i(a_i) - \hat{\alpha}|\}^{1/m} \qquad \forall j \quad , \quad \forall s = j\omega \qquad (8.55)$$

$$(\text{resp.}\ |1/n_i(a_i) + \alpha + \lambda_j(s)| > (m+2)\{M(s)\}^{(m-1)/m}\{\sum_{j=1}^{m} |1/n_j(a_j) - 1/n_i(a_i) - \alpha|\}^{1/m}) \quad , \quad \forall j \quad , \quad \forall s = j\omega \qquad (8.56)$$

where $\hat{M}(s)$, $M(s)$ have the same interpretation as theorem 8.12.

Proof: Follows from theorem 8.10 and equations (8.43) and (8.44).

When $N(a) = \text{diag}\ \{n_c\}$, $\hat{N}(a) = \text{diag}\ \{1/n_c\}$, and setting $\alpha = \hat{\alpha} = 0$, corollary 8.3 reduces to corollary 8.2. Thus Ostrowski's theorem, in this special case, gives the same result as Bauer and Fike's theorem.

In general, the Ostrowski's bands differ in sharpness from those obtained by the BF theorem, but in many cases, both require different computational demands in their evaluations.

8.2.2. Use of reduced models in d.f. synthesis[14]

The accuracy of d.f. methods increases with the order of the linear system, where the higher harmonics are more heavily attenuated. In design, where reduced model, S_r, in place of the high model, S, is required, this may not be the case, so the accuracy may be doubtful. However, provided S_r is not too low, and, if the transient response error of the models is small, it can be assumed that the higher harmonics of S_r are attenuated in roughly the same proportion as those of S. Hence it is assumed that the d.f. method is reasonably accurate in S_r, and, if no limit cycles exist in the latter, it is reasonable to expect no limit cycles in S, provided conditions, based on S_r characteristics, can be found for it. S can be represented as

$$Q(s) = Q_r(s) + Q_e(s) \tag{8.57}$$

$$\hat{Q}(s) = \hat{Q}_r(s) + \hat{Q}_e(s) \tag{8.58}$$

where $Q_e(s) = G_e(s)K_r(s) = \{G(s) - G_r(s)\}K_r(s)$, $\hat{Q}_e(s) = \hat{K}_r(s)\{\hat{G}(s) - \hat{G}_r(s)\}$. Thus replacing $\hat{Q}(s)$ by $\hat{Q}_r(s) + \hat{Q}_e(s)$, $Q(s)$ by $Q(s) + Q_e(s)$ in equations (8.35), (8.36), (8.41) and (8.42), and, letting $\lambda_{rj}(s)$ (resp $\hat{\lambda}_{rj}(s)$) be the characteristic loci of $Q_r(s)$ (resp $\hat{Q}_r(s)$), similar theorems and corollaries in section 8.2.1. can be constructed for S, in terms of the characteristics of S_r. All other quantities remain unchanged in equations (8.37), (8.38), (8.43) and (8.44).

Theorem 8.11

S will have no limit cycles if one of the following conditions is satisfied:

(1) $|\hat{k} + \hat{\alpha} + \hat{\lambda}_{rj}(s)| > \| N'(a) + \hat{Q}_e(s) - \hat{\alpha}I \| \inf \kappa(P_r(s))$

$\forall j \;,\; \forall s = j\omega$ (8.59)

(resp. $|k + \alpha + \lambda_{rj}(s)| > \| \hat{N}'(a) + Q_e(s) - \alpha I \| \inf \kappa(P_r(s))$)

$\forall j \;,\; \forall s = j\omega$ (8.60)

(2) $|\hat{k} + \hat{\alpha} + \hat{\lambda}_{rj}(s)| > \hat{W}_r(s)\{\sum_{i,j} |n'_{ij}(a) + \hat{q}_{eij}(s) - \hat{\alpha}\delta_{ij}|\}^{1/m}$

$\forall j \;,\; \forall s = j\omega$ (8.61)

(resp. $|k + \alpha + \lambda_{rj}(s)| > W_r(s)\{\sum_{i,j} |\hat{n}_{ij}(a) + q_{eij}(s) - \alpha\delta_{ij}|\}^{1/m}$)

$\forall j \;,\; \forall s = j\omega$ (8.62)

where $Q_e(s) = (q_{eij}(s))$, $\hat{Q}_e(s) = (\hat{q}_{eij}(s))$, $W_r(s) = (m+2)\{M_r(s)\}^{(m-1)/m}$, $\hat{W}_r(s) = (m+2)\{\hat{M}_r(s)\}^{(m-1)/m}$, $M_r(s) = \max\{|(\alpha + k)\delta_{ij} + q_{rij}(s)|,$ $|\hat{n}_{ij}(a) + q_{ij}(s)|\}$, $\hat{M}_r(s) = \max\{|(\hat{\alpha} + \hat{k})\delta_{ij} + \hat{q}_{rij}(s)|, |n_{ij}(a) + \hat{q}_{ij}(s)|\}$

Proof:

Application of theorems 5.1 and 6.3 to the modified form of equations (8.35) and (8.36).

Theorem 8.12

S will have no limit cycles if one of the following conditions is satisfied:

(1) $|n_{ii}(a_i) + \hat{\alpha} + \hat{\lambda}_{rj}(s)| > \| N''(a) + \hat{Q}_e(s) - \hat{\alpha}I \| \inf \kappa(P_r(s))$

$\forall j \;,\; \forall s = j\omega$ (8.63)

resp $|\hat{n}_{ii}(a_i) + \alpha + \lambda_{rj}(s)| > \| \hat{N}''(a) + Q_e(s) - \alpha I \| \inf \kappa(P_r(s))$

$\forall j \;,\; \forall s = j\omega$ (8.64)

(2) $|n_{ii}(a_i) + \hat{\alpha} + \hat{\lambda}_{rj}(s)| > \hat{W}_r(s)\{\sum_{i,j} |n''_{ij}(a) + \hat{q}_{eij}(s) - \hat{\alpha}\delta_{ij}|\}^{1/m}$

$\forall j \;,\; \forall s = j\omega$ (8.65)

resp $|\hat{n}_{ii}(a_i) + \alpha + \lambda_{rj}(s)| > W_r(s)\{\sum_{i,j} |\hat{n}''_{ij}(a) + q_{eij}(s) - \alpha\delta_{ij}|\}^{1/m}$

$\forall j \;,\; \forall s = j\omega$ (8.66)

where $\hat{W}_r(s) = (m+2)\{\hat{M}_r(s)\}^{(m-1)/m}$,

$\hat{M}_r(s) = \max\{|(\hat{\alpha} + n_{ii}(a_i))\delta_{ij} + \hat{q}_{rij}(s)|, |n_{ij}(a) + \hat{q}_{ij}(s)|\}$

$W_r(s) = (m+2)\{M_r(s)\}^{(m-1)/m}$,

$M_r(s) = \max\{|(\alpha + \hat{n}_{ii}(a_i)\delta_{ij} + q_{rij}(s)|, |n_{ij}(a) + q_{ij}(s)|\}$

Proof:

Application of theorems 5.1 and 6.3 to the modified forms of equations (8.41) and (8.42).

Corollary 8.4

When $N(a) = \text{diag}\{n_i(a_i)\}$, $\hat{N}(a) = \text{diag}\{1/n_i(a_i)\}$, there will be absence of limit cycles in S if one of the following is satisfied:

(1) $$|n_i(a_i) + \hat{\alpha} + \hat{\lambda}_{rj}(s)| > \{\max_j |n_j(a_j) - n_i(a_i) - \hat{\alpha}| + \|\hat{Q}_e(s)\|\}\inf \kappa(P_r(s)),\ \forall j,\ \forall s = j\omega \tag{8.67}$$

$$(\text{resp. } |1/n_i(a_i) + \alpha + \lambda_{rj}(s)| > \{\max_j |1/n_j(a_j) - 1/n_i(a_i) - \alpha| + \|Q_e(s)\|\}\inf \kappa(P_r(s))\ )\quad \forall j,\ \forall s = j\omega .$$

(2) $$|n_i(a_i) + \hat{\alpha} + \hat{\lambda}_{rj}(s)| > \hat{W}_r(s)\{\sum_{j=1}^{m} |n_j(a_j) - n_i(a_i) - \hat{\alpha}| + \sum_{i,j} |\hat{\hat{q}}_{eij}(s)|\}^{1/m}\quad \forall j,\ \forall s = j\omega \tag{8.68}$$

$$(\text{resp. } |1/n_i(a_i) + \alpha + \lambda_{rj}(s)| > W_r(s)\{|1/n_j(a_j) - 1/n_i(a_i) - \alpha| + \sum_{i,j} |q_{eij}(s)|\}^{1/m}\ ) \tag{8.69}$$

Proof:

Follows from theorem 8.12 and using the fact that $\|\text{diag}\{n_j(a_j) - n_i(a_i) - \hat{\alpha}\} + \hat{Q}_e(s)\| \leq \|\text{diag}\{n_j(a_j) - n_i(a_i) - \hat{\alpha}\}\| + \|Q_e(s)\|$ and $\sum_{i,j} |\text{diag}\{n_j(a_j) - n_i(a_i) - \hat{\alpha}\} + \hat{q}_{eij}(s)| \leq \sum_{j=1}^{m} |\text{diag}\{n_j(a_j) - n_i(a_i) - \hat{\alpha}\}| + \sum_{i,j} |\hat{q}_{eij}(s)|$

Corollary 8.5

In the special case, when $N(a) = \text{diag}\{n_c\}$, $\hat{N}(a) = \text{diag}\{1/n_c\}$, S will have no limit cycles for any one of the following conditions:

(1) $$|n_c + \hat{\lambda}_{rj}(s)| > \|\hat{Q}_e(s)\|\inf \kappa(P_r(s))\quad \forall j\ ,\ \forall s = j\omega \tag{8.70}$$

$$(\text{resp. } |1/n_c + \hat{\lambda}_{rj}(s)| > \|Q_e(s)\|\inf \kappa(P_r(s))\ ) \tag{8.71}$$

(2) $|n_c + \hat{\lambda}_{rj}(s)| > \hat{W}_r(s)\{\sum_{i,j} |\hat{q}_{eij}(s)|\}^{1/m}, \forall j \quad , \forall s = j\omega$ (8.72)

resp. $|1/n_c + \lambda_{rj}(s)| > W_r(s)\{\sum_{i,j} |q_{eij}(s)|\}^{1/m}$ (8.73)

Proof:

Follows from corollary 8.4. When $Q_e(s) \to 0$, corollary 8.5 becomes corollary 8.2 as expected.

The graphical interpretation of theorems 8.11 to corollary 8.5 is similar to that shown in figure 8.6, with appropriate changes in the radii of the bands.

8.2.3 Discussions and Comparisons with other Criteria

The proposed criteria (theorems 8.7 to 8.13) have a number of distinct advantages over existing criteria. Mee s' criterion requires diagonal dominance of $\hat{Q}(s) + N(a)$, whereas the new criterion relaxes diagonal dominance, but the width of the band depends on the structure of Q(s) via inf $\kappa(P(s))$ or $Q_e(s)$. In the special case, when Q(s) is normal, for theorems 8.7 to 8.10 the width of the bands is frequency independent. Also, for a nondiagonal N(a), Mee s' criterion requires the nonoverlapping of every corresponding pair of m bands - this can be overcome by transforming to an equivalent N(a), but this would involve m squared bands - whereas the proposed criterion requires the nonoverlapping of every m band with the locus of one chosen diagonal element of N(a), irrespective of the latter's structure. This thus gives a less conservative result and can be computationally cheaper. Further, in Mee s' case, by a modified form of Geshgorin's theorem, it is possible to reduce a bandwidth at the expense of increasing the others; however, with the proposed criterion, the width of all bands can be tuned simultaneously. Also, when the 'nonoverlapping-loci' criterion fails, the '-k critical point' criterion can be used as an alternative. In the special case, when N(a) = diag (n_c), it is possible to reduce the bandwidth to zero with the proposed criterion, but, using Mee's criterion the width of the bands will not reduce to zero.

The RA1 criterion does not offer an easy graphical representation from the designer's point of view, and determining the sign definiteness of the high order matrices when N(a) is nondiagonal would be computationally unattractive. Besides, being of algebraic form, it is not simple to incorporate 'tuning factors' in the expression to obtain less conservative results.

The RA2 criterion is restricted to a 2 input - 2 output system with diagonal N(a) only. Also, it requires all the d.f. and $q_{jj}(s)$ loci and a $\lambda_j(s)$ locus to be plotted, thus causing undue graphical labour.

Due to its flexibility, the proposed criterion gives less conservative results than other criteria in the general case. The attractiveness of applying it when reduced models are used in design has been illustrated. This would be more difficult with the other criteria when absence of limit cycles in S is to be deduced from that in S_r. For example, using Mee s' criterion, when S_r is diagonally dominant, S need not be so, or, using the RA1 criterion, the sign definiteness of the S_r matrices does not necessarily imply sign definiteness of the S matrices, unless restrictive conditions are imposed. With the RA2 criterion, this would be messy, as error bands are required to be drawn around all the loci of the linear system.

Besides limit cycle investigation, the proposed criterion can also be adapted to stability investigation in the Nyquist sense.

Example 8.2.1

Consider the following example[4], with

$$Q(s) = \frac{1}{(s+1)^2(s+6)}\begin{bmatrix} s+2 & -2 \\ -2 & s+5 \end{bmatrix}$$

$Q(s)$ is normal, and, $\lambda_1(s) = 1/(s+6)(s+1)$

$$\lambda_2(s) = 1/(s+1)^2$$

Suppose the nonlinear feedback, $N(t,y)$, is diagonal and is characterised by the describing functions (ideal relay with hysteresis),

$$1/n_1(a_1) = (\pi A_1/4Y_1)\exp\{-j\sin^{-1}(b_1/A_1)\} = j\pi$$

$$1/n_2(a_2) = (\pi A_2/4Y_2)\exp\{-j\sin^{-1}(b_2/A_2)\} = j7\pi/8$$

where $A_1 = A_2 = 1$, $b_1 = b_2 = 1$, $Y_1 = 1/4$, $Y_2 = 1/3.5$

The characteristic loci, $\lambda_1(s)$, $\lambda_2(s)$ are plotted against the inverse describing function loci, $-1/n_1(a_1)$, $-1/n_2(a_2)$, as shown in figure 8.9. Setting $\alpha = 0$, in equation (8.48), the bandwidth is given by $|1/n_1(a_1) - 1/n_2(a_2)| = |(\pi/4)(4 - 3.5)j| = \pi/8 = 0.4, \forall\omega$

As seen in figure 8.9 the bands do not intercept any of the d.f. loci, hence by corollary 8.1 the system cannot sustain a limit cycle.

8.3 Graphical Interpretation for Stability of Nonlinear Systems

8.3.1 The Popov Criterion and Jury -Lee Criterion

Popov[7] gave a stability criterion for the nonlinear system of figure 8.1 where the nonlinearity is slope restricted in the sector $(0,\beta)$ instead of (α,β). The criterion is based on positive realness of a function. Later Jury and Lee[9] extended Popov's criterion to multivariable systems, using the concept of positive real matrices.

It is well known that an $(m \times m)$ matrix $A(s)$ is positive real if and only if[23]

(1) $A(s)$ is holomorphic in $\text{Re}(s) \geq 0$

(2) $A^*(s) = A(s^*)$ for Re (s) > 0

(3) Poles on Re (s) = 0 are simple, and the associated residues $K(s) \geq 0$ and $K(s) = K^H(s)$

(4) $A^H(s) + A(s) \geq 0$ or $\hat{A}^H(s) + \hat{A}(s) \geq 0$ for Re (s) ≥ 0 (8.74)

Alternatively, let $S(s) = (I + A(s))^{-1}(A(s) - I)$. Then A(s) is positive real if and only if,

(1) S(s) is holomorphic in Re (s) ≥ 0

(2) S(s) is real rational

(3) $I - S^H(s)S(s) \geq 0$ or $\hat{S}^H(s)\hat{S}(s) - I \geq 0$ (8.75)

The stability theorem of Falb et al[4] (see Theorem 8.3) can be derived using positive real matrices. From equation (8.4) of theorem 8A,

$$A(s) = (B^{-1} + Q(s))^{-1}(A^{-1} + Q(s))$$

$$\text{i.e.} \quad S(s) = \{(B^{-1} + Q(s))^{-1}(A^{-1} + Q(s)) + I\}^{-1}\{(B^{-1} + Q(s))^{-1}(A^{-1} + Q(s))-I\}$$

$$\text{i.e.} \quad S = (A^{-1} + Q + B^{-1}+Q)^{-1}(A^{-1} + Q - B^{-1} - Q)$$

$$= \tfrac{1}{2}(Q(s) + \tfrac{1}{2}(A^{-1} + B^{-1}))^{-1}(A^{-1} - B^{-1}) \qquad (8.76)$$

If $A = \text{diag}\,\{\max_j a_j\} = aI$, $B = \text{diag}\,\{\max_j b_j\} = bI$, then equation (8.76) can be written as

$$S(s) = \tfrac{1}{2}(1/a - 1/b)I(Q(s) + \tfrac{1}{2}(1/a + 1/b))^{-1} \qquad (8.77)$$

Now consider the expression,

$$\{(1/\rho)(M - kI)\}\{(1/\rho)(M - kI)\}^H - I > 0 \qquad (8.78)$$

where M is an arbitrary complex matrix, ρ a positive constant and k a complex number. If M is normal, then the eigenvalues λ_j of M satisfy $|\lambda_j - k| > \rho$, i.e. λ_j lies outside the critical disc centred at k. From equation (8.77)

$$\hat{S}(s) = (1/\rho)(Q(s) + \tfrac{1}{2}(1/a + 1/b)) \qquad (8.79)$$

where $\rho = \tfrac{1}{2}(1/a - 1/b)$. If $\lambda_j(s)$ is the characteristic loci of Q(s), and if Q(s) is normal, comparing equations (8.78) and (8.79), it is seen that $\hat{S}^H(s)\hat{S}(s) - I > 0$, if and only if $|\lambda_j(s) + \frac{1}{2}(1/a + 1/b)| > \frac{1}{2}(1/a - 1/b)$. Thus by theorem 8A, the system of figure 8.1 is stable i.s.L. if the characteristic loci of normal transfer function matrix do not intercept or embrace the critical disc centred on $(-\frac{1}{2}(1/a + 1/b),0)$ with radius $\frac{1}{2}(1/a - 1/b)$. Here the nonlinear part is slope restricted in the sector (a,b). This is the theorem of Falb, Freedman and Zames[4], rederived by a shorter method. The dual case, using the inverse loci, $\hat{\lambda}_j(s)$ of $\hat{Q}(s)$ can also be rederived, using equation (8.6) of theorem 8A.

Let

$$A(s) = (I + Q(s)A)^{-1}(I + Q(s)B)$$

then

$$\begin{aligned} S(s) &= \{(I + Q(s)A)^{-1}(I + Q(s)B) + I\}^{-1}\{(I + Q(s)A)^{-1}(I + Q(s)B) - I\} \\ &= (2I + Q(s)(A + B))^{-1}(Q(s)(B - A)) \\ &= (\hat{Q}(s) + \tfrac{1}{2}(A + B))^{-1}\tfrac{1}{2}(B - A) \end{aligned}$$

i.e.

$$\hat{S}(s) = (\tfrac{1}{2}(B - A))^{-1}(\hat{Q}(s) + \tfrac{1}{2}(A + B)) \tag{8.80}$$

Following the lines of argument from equations (8.76) to (8.79), it can easily be deduced from equation (8.80), that for stability, the loci $\hat{\lambda}_j(s)$ of $\hat{Q}(s)$ must avoid the critical disc centred on $(-\frac{1}{2}(a + b),0)$, with radius $\frac{1}{2}(b - a)$. It is noted that the above only satisfies equation (8.75) for positive realness test. The other conditions for positive realness of Q(s) are satisfied, if $\lambda_j(s)$ (resp $\hat{\lambda}_j(s)$) does not encircle or intercept the critical line segment, which is the diameter of the critical disc, on the real axis.

Cook's theorem[3] can similarly be rederived, using Gershgorin's theorem and equations (8.76) and (8.80), but it will not be done here.

Returning to the Popov criterion, or the Jury-Lee criterion, mentioned earlier, for later reference, their theorem, regarding figure 8.1, can be stated.

Theorem 8H[9]

Let the nonlinear block of the system in figure 8.1 be slope restricted in the sector $\{a_i, b_i\}$, or equivalently, in the sector $\{0, b_i - a_i\}$. Then the nonlinear system is absolutely stable if

(1) $P(s) + P^H(s) > 0 \quad , \quad \forall s = j\omega$ (8.81)

where $P(s) = (I + sL)Q_c(s) + K^{-1}$ (8.82)

and $L = \text{diag}\{\ell_i\}$ is a real $(m \times m)$ matrix, $K^{-1} = \text{diag}\{1/(b_i - a_i)\}$, $Q_c(s) = (I + Q(s)A)^{-1}Q(s)$ represents the equivalent closed loop transfer function, and $A = \text{diag}\{a_i\}$. Thus for the sector $\{0, b_i\}$, $Q_c(s)$ becomes $Q(s)$. Condition (1), equation (8.81) is equivalent to the fact that $P(s)$ is positive real provided $P(s)$ also satisfies the remaining conditions for positive realness.

Shankar and Atherton[11], gave a graphical interpretation of theorem 8H, using Gershgorin's theorem and the Popov polar plot, and, showed that in certain cases, Cook's result is a special case of the Jury-Lee criterion. They showed that if the area, defined by the circular bands

$$A(s) = q'_{cii}(s) + d_i(s)\exp(j\phi) \quad , \quad \forall \phi \in (0, 2\pi)$$

lies to the right of the line drawn through the point $(-1/(b_i - a_i), 0)$ and of slope $1/\ell_i$, $\forall i$, then the system is absolutely stable. Here

$$\text{Re}\,(q'_{cii}(s)) = \text{Re}\,(q_{cii}(s))$$

$$\text{Im}\,(q'_{cii}(s)) = \omega\text{Im}\,(q_{cii}(s))$$

$$d_i(s) = \tfrac{1}{2}\{|1 + sq_{ii}(s)| \sum_{j\neq i} |q_{jj}(s)| + \tfrac{1}{2}\sum_{j\neq i} (|1 + sq_{jj}(s)||q_{jj}(s)|)\}$$

In this section, a different graphical interpretation will be given to theorem 8H, in terms of characteristic loci and Nyquist polar plots. Consider the nonlinearity be bounded in the sector $\{a_i, b_i\}$. From equations (8.81) and (8.82)

$$T(s) = P(s) + P^H(s) = Q_c(s) + Q_c^H(s) + s(LQ_c(s) - Q_c^H(s)L) + 2\hat{K} \tag{8.83}$$

i.e.
$$Q_c^H(s)T_1(s)Q_c(s) = Q_c^H(s)\left[\hat{Q}_c(s) + \hat{Q}_c^H(s) + s(\hat{Q}_c^H(s)L - L\hat{Q}_c(s)) + 2\hat{Q}_c^H(s)\hat{K}\hat{Q}_c(s)\right]Q_c(s) \tag{8.84}$$

where $\hat{K} = K^{-1}$, $\hat{Q}_c(s) = Q_c^{-1}(s)$, etc and

$$Q_c(s) = (I + Q(s)A)^{-1}Q(s) \tag{8.85}$$

$$\hat{Q}_c(s) = \hat{Q}(s) + A \tag{8.86}$$

$$\hat{Q}_c^H(s) = \hat{Q}^H(s) + A \tag{8.87}$$

assuming $(I + Q(s)A)^{-1}$ exists. Substituting equations (8.86) and (8.87) into equation (8.84).

$$T_1(s) = \hat{Q}(s) + \hat{Q}^H(s) + s(\hat{Q}^H(s)L - L\hat{Q}(s)) + 2A + 2\hat{Q}_c^H(s)\hat{K}\hat{Q}_c(s) \tag{8.88}$$

Writing $T_1(s) = \hat{Q}^H(s)T_2(s)\hat{Q}(s)$, from equation (8.88) $T_2(s)$ can be written, using equation (8.86), as

$$T_2(s) = Q(s) + Q^H(s) + s(LQ(s) - Q^H(s)L) + 2Q^H(s)AQ(s) + 2(I + AQ(s))^H\hat{K}(I + AQ(s)) \tag{8.89}$$

Equations (8.88) and (8.89) can be rewritten as

$$T_1(s) = \hat{Q}(s) + [\hat{\alpha} + 2\hat{\beta}/(b_j - a_j)]I + \hat{W}(s) - [\hat{\alpha} + 2\hat{\beta}/(b_j - a_j)]I \tag{8.90}$$

$$T_2(s) = Q(s) + [\alpha + 2\beta/(b_j - a_j)]I + W(s) - [\alpha + 2\beta/(b_j - a_j)]I \tag{8.91}$$

where

$$\hat{W}(s) = \hat{Q}^H(s) + s(\hat{Q}^H(s)L - L\hat{Q}(s)) + 2A + 2\hat{Q}_c^H(s)\hat{K}\hat{Q}_c(s) \tag{8.92}$$

$$W(s) = Q^H(s) + s(LQ(s) - Q^H(s)L) + 2Q^H(s)AQ(s) + 2(I + AQ(s))^H\hat{K}(I + AQ(s)) \tag{8.93}$$

and α, β, $\hat{\alpha}$, $\hat{\beta}$ are complex numbers. In general $\alpha^{-1} \neq \hat{\alpha}$, $\beta^{-1} \neq \hat{\beta}$, $W(s)^{-1} \neq \hat{W}(s)$.

It is clear that $T(s)$ is positive definite if and only if $T_1(s)$ or $T_2(s)$ is positive definite. The graphical interpretation of theorem 8H can now be stated.

Theorem 8.13

Let $\lambda_k(s)$ (resp $\hat{\lambda}_k(s)$) be the characteristic loci of $Q(s)$ (resp $\hat{Q}(s)$). Then the system of figure 8.1 is absolutely stable if

(1) $\mathrm{Re}\,(\lambda_k(s) + \gamma_j) > 0 \quad , \quad \forall k \quad , \quad \forall s = j\omega$ (8.94)

(resp $\mathrm{Re}\,(\hat{\lambda}_k(s) + \hat{\gamma}_j) > 0$)

(2) $\mathrm{Re}\,(\lambda_k(s) + \gamma_j) > \| W(s) - \gamma_j I \| \inf \kappa(V(s)) \quad , \quad \forall k \quad , \quad \forall s = j\omega$ (8.95)

(resp $\mathrm{Re}\,(\hat{\lambda}_k(s) + \hat{\gamma}_j) > \| \hat{W}(s) - \hat{\gamma}_j I \| \inf \kappa(V(s))$)

where $\gamma_j = \alpha + 2\beta/(b_j - a_j)$, $\hat{\gamma}_j = \hat{\alpha} + 2\hat{\beta}/(b_j - a_j)$ and $V(s)$ diagonalizes $Q(s)$.

Proof:

Conditions (1) and (2) follow from the application of the BF theorem[27] (theorem 5.1) to the r.h.s. of equations (8.90) and (8.91). If the above conditions are satisfied, then $T_1(s)$, $T_2(s)$, hence $T(s)$ is positive definite. Stability follows from theorem 8H.

The graphical interpretation of theorem 8.13 is shown in figure 8.7(a) and (b). If the cicular band defined by $\lambda_j(s) + \|W(s) - \gamma_j I\| \inf \kappa(V(s)) \exp(j\phi)$, $\phi \in \{0, 2\pi\}$, $\forall s = j\omega$ lies to the right of the vertical line through $(-\mathrm{Re}\,(\gamma_j), 0)$, the system is absolutely stable. The same explanation applies to the inverse loci.

In the above theorem, it is assumed that $\mathrm{Re}\,(\lambda_j(0)) < \infty$ and $\mathrm{Re}\,(\hat{\lambda}_j(\infty)) < \infty$, i.e. the real part of the characteristic loci is bounded. If $\mathrm{Re}\,(\lambda_j(s))$ is unbounded, the above theorem may break down. This problem can be overcome by adding and subtracting $D(s) = \mathrm{diag}\,\{d_i(s)\}$ (resp $\hat{D}(s) = \mathrm{diag}\,\{\hat{d}_i(s)\}$) to $T_2(s)$ (resp $T_1(s)$) such that $q_{ii}(s) + d_i(s)$ (resp $\hat{q}_{ii}(s) + \hat{d}_i(s)$) is bounded. The perturbed loci $\mathrm{Re}\,(\lambda_j'(s))$ (resp $\mathrm{Re}\,(\hat{\lambda}_j'(s))$ is thus bounded.

When the nonlinearity is bounded in the sector $\{0, b_i\}$ i.e. $a_i = 0$, hence $A = 0$, the equations (8.92) and (8.93) simplify to

$$\hat{W}(s) = \hat{Q}^H(s) + s(\hat{Q}^H(s)L - L\hat{Q}(s)) + 2\hat{Q}^H(s)\hat{K}\hat{Q}(s) \tag{8.96}$$

$$W(s) = Q^H(s) + s(LQ(s) - Q^H(s)L) + 2\,\mathrm{diag}\,(1/b_j) \tag{8.97}$$

The width of the band is determined greatly by the choice of matrix L and by the scaling factors α and β. There is no simple method to choose L such that the bandwidth is minimum. Trial and error procedure seems to be the best method.

An alternative, sometimes more convenient, method to express stability is via the diagonal elements of Q(s) using Nyquist or Inverse Nyquist loci. Let ζ_j be the eigenvalues of $T(s) = P(s) + P^H(s)$.

Since

$$|\zeta - \gamma_j - q_{ii}(s)| \leq |\zeta - \gamma_j - \lambda_j(s)| + |\lambda_j(s) - q_{ii}(s)| \tag{8.98}$$

i.e. $$\min |\zeta - \gamma_j - q_{ii}(s)| \leq \min |\zeta - \gamma_j - \lambda_j(s)| + |\lambda_j(s) - q_{ii}(s)| \tag{8.99}$$

By Bauer and Fike's theorem,

$$\min |\zeta - \gamma_j - \lambda_j(s)| \leq ||W(s) - \gamma_j I||\inf \kappa(V(s)) \tag{8.100}$$

By Gershgorin's theorem,

$$|\lambda_j(s) - q_{ii}(s)| \leq \sum_{i\neq j} |q_{ij}(s)| \tag{8.101}$$

Equations (8.99) to (8.101) give,

$$\min |\zeta - \gamma_j - q_{ii}(s)| \leq \| W(s) - \gamma_j I\| \inf \kappa(V(s)) + \sum_{i\neq j} |q_{ij}(s)| \tag{8.102}$$

Equation (8.102) means that ζ lies in at least one of the disc centred on $q_{ii}(s)$ and radii given by the r.h.s. expression. Thus

Theorem 8.14

The nonlinear system of figure 8.1 is absolutely stable if

(1) $$\mathrm{Re}\,(q_{ii}(s) + \gamma_j) > \| W(s) - \gamma_j\| \inf \kappa(V(s)) + \sum_{i\neq j} |q_{ij}(s)|, \forall i, \forall s = j\omega \tag{8.103}$$

$$\left(\text{resp. } \mathrm{Re}\,(\hat{q}_{ii}(s) + \hat{\gamma}_j) > \| \hat{W}(s) - \hat{\gamma}_j\| \inf \kappa(V(s)) + \sum_{i\neq j} |\hat{q}_{ij}(s)| \right)$$

Proof:

Follows a similar line of argument as in theorem 8.13.

Theorem 8.14 is illustrated in fig.8.7(c).

The graphical interpretation requires that the band defined by

$$q_{ii}(s) + \{||W(s) - \gamma_j|| \inf \kappa(V(s)) + \sum_{i \neq j} |q_{ij}(s)|\}\exp(j\phi),\ \phi \in \{0,2\pi\},\ \forall s = j\omega$$

lies to the right of the vertical line passing through the point $(-\mathrm{Re}\,(\gamma_j),0)$

Ostrowski's theorem, theorem 6.3, can also be used for theorems 8.13 and 8.14. Thus,

Theorem 8.15

The nonlinear system of figure 8.1 is absolutely stable if any one of the following conditions is satisfied,

(1) $$\mathrm{Re}\,(\lambda_j(s) + \gamma) > (m+2)\{M(s)\}^{(m-1)/m}\{\sum_{i,j} |W_{ij}(s) - \gamma\delta_{ij}|\}^{1/m} = Z \quad \forall j\ ,\ \forall s = j\omega$$

$$\text{resp. } \mathrm{Re}\,(\hat{\lambda}_j(s) + \hat{\gamma}) > (m+2)\{\hat{M}(s)\}^{(m-1)/m}\{\sum_{i,j} |\hat{W}_{ij}(s) - \hat{\gamma}\delta_{ij}|\}^{1/m} = \hat{Z} \quad (8.104)$$

(2) $$\mathrm{Re}\,(q_{ii}(s) + \gamma) > Z + \sum_{i \neq j} |q_{ij}(s)|\ ,\ \forall j\ ,\ \forall s = j\omega \quad (8.105)$$

$$\text{resp. } \mathrm{Re}\,(\hat{q}_{ii}(s) + \hat{\gamma}) > \hat{Z} + \sum_{i \neq j} |\hat{q}_{ij}(s)|$$

where $M(s) = \max\,\{\gamma\delta_{ij} + q_{ij}(s), q_{ij}(s) + W_{ij}(s)\}$,

$\hat{M}(s) = \max\,\{\hat{\gamma}\delta_{ij} + \hat{q}_{ij}(s), \hat{q}_{ij}(s) + \hat{W}_{ij}(s)\}$ and δ_{ij} is the Kronecker delta.

Proof:

Follows similar reasoning as in theorems 8.13 and 8.14.

Theorems 8.13 to 8.15 offer a graphical interpretation, of the Jury-Lee criterion, in terms of the characteristic loci, or Nyquist loci, of the linear part of the transfer function, Q(s). This approach differs from that used by Shankar and Atherton[11], who employed the modified (Popov) polar plot. The usual Nyquist polar plot is preferable, as designers are more familiar with it, to the modified plot. Also, theorems 8.13 and 8.14 are very flexible and tuning factors are incorporated to obtain less conservative results. The width of the bands depends very much on the structure of Q(s). For example, in theorem 8.14 if Q(s) is both normal and diagonal dominant, then sharper results are likely to be obtained.

The "circle criterion" of theorem 8A can also be interpreted graphically as in theorems 8.13 to 8.15. As shown in equation (8.76), for expression (8.4) to be positive real, it is required that $\hat{S}^H(s)\hat{S}(s) - I > 0$ where from equation (8.76),

$$\hat{S}(s) = 2(A^{-1} - B^{-1})^{-1}(Q(s) + \tfrac{1}{2}(A^{-1} + B^{-1}))$$

Suppose $A = \mathrm{diag}(\max\{a_j\}) = aI$, $B = \mathrm{diag}(\max\{b_j\}) = bI$, then

$$\begin{aligned} P(s) &= \hat{S}^H(s)\hat{S}(s) - I \\ &= \theta I(Q^H(s) + \phi)\theta I(Q(s) + \phi) - I \\ &= \theta^2\phi I(\hat{\phi}Q^H(s)Q(s) + \hat{\phi}Q^H(s)\phi + Q(s) + (\phi - \theta^2\hat{\phi})I) \\ &= \theta^2\phi T_2(s) \end{aligned} \tag{8.105a}$$

where $\theta = 2/(1/a - 1/b)$ is a positive constant, $\phi = \tfrac{1}{2}(1/a + 1/b)$ is also positive and $\hat{\theta} = 1/\theta$, $\hat{\phi} = 1/\phi$. Since $P(s) = \theta^2\phi T_2(s)$, $P(s)$ is positive definite if $T_2(s)$ is also positive definite. Writing $T_2(s)$ as

$$T_2(s) = Q(s) + \gamma I + (W(s) - \gamma I) \tag{8.105b}$$

where $W(s) = \hat{\phi}Q^H(s)Q(s) + \hat{\phi}Q^H(s)\phi + (\phi - \theta^2\hat{\phi})I$

it is seen that equation (8.105b) is equivalent in form to equation (8.91). To work with the inverse loci, equation (8.80) can be used.

$$\hat{S}(s) = 2(B - A)^{-1}(\hat{Q}(s) + \tfrac{1}{2}(A + B)) \tag{8.80}$$

$$\begin{aligned} P(s) &= \hat{S}^H(s)\hat{S}(s) - I \\ &= \varepsilon^2\tau I(\hat{\tau}Q^H(s)\hat{Q}(s) + \hat{\tau}Q^H(s)\tau + Q(s) + (\tau - \hat{\varepsilon}^2\hat{\tau})I) \\ &= \varepsilon^2\tau T_1(s) \end{aligned}$$

where $\varepsilon = 2/(b - a)$ is a positive constant, and, $\tau = \tfrac{1}{2}(a + b)$ is also positive, $\hat{\varepsilon} = 1/\varepsilon$ and $\hat{\tau} = 1/\tau$, resp. Writing $T_1(s)$ as

$$T_1(s) = \hat{Q}(s) + \hat{\gamma}I + (\hat{W}(s) - \hat{\gamma}I) \tag{8.105c}$$

where $\hat{W}(s) = \hat{\tau}\hat{Q}^H(s)\hat{Q}(s) + \hat{\tau}Q^H(s)\tau + (\tau - \hat{\varepsilon}^2\hat{\tau})I$,

equation (8.105c) is made equivalent to equation (8.90). Thus,

Theorem 8.16

The system of figure 8.1 is asymptotically stable, i.s.L., in the sense of theorem 8A (expressions 8.4 and 8.6) if either theorems 8.14 or 8.15 are satisfied, with $W(s)$ and $\hat{W}(s)$ given in equations (8.105b) and (8.105c).

Proof:

Follows from theorems 8.14 and 8.15. (see also fig. 8.7(c))

The above theorems assume that $Q(s)$ is real rational and holomorphic in Ω where $\Omega \in \mathrm{Re}\ (s) > 0$.

8.3.2 Applications of reduced models

In design, it is convenient to use a reduced model, $Q_r(s)$ in place of $Q(s)$. The nonlinear block, $N(t,y)$ is assumed intact.

Writing $Q(s) = Q_r(s) + Q_e(s)$, $\hat{Q}(s) = \hat{Q}_r(s) + \hat{Q}_e(s)$, equations (8.90) and (8.91) are modified as

$$T_1(s) = \hat{Q}_r(s) + \hat{\gamma}_j I + \hat{W}(s) + \hat{Q}_e(s) - \hat{\gamma}_j I \tag{8.106}$$

$$T_2(s) = Q_r(s) + \gamma_j I + W(s) + Q_e(s) - \gamma_j I \tag{8.107}$$

Stability results of the original system S, of figure 8.1, can be represented by the reduced system S_r. Theorems 8.13 to 8.15 are modified appropriately in terms of $Q_r(s)$.

Theorem 8.17

Let $\lambda_{rj}(s)$ (resp $\hat{\lambda}_{rj}(s)$) be the characteristic loci of $Q_r(s)$ (resp $\hat{Q}_r(s)$), and $q_{rij}(s)$, $q_{eij}(s)$ be elements of $Q_r(s)$, $Q_e(s)$ respectively. Then S is absolutely stable if any one of the following conditions is satisfied.

(1) $$\mathrm{Re}\ (\lambda_{rj}(s) + \gamma) > \|W(s) + Q_e(s) - \gamma I\| \inf \kappa(V(s)) = X_e,\ \forall j,\ \forall s = j\omega \tag{8.108}$$

$$\left(\text{resp. } \mathrm{Re}\ (\hat{\lambda}_{rj}(s) + \hat{\gamma}) > \|\hat{W}(s) + \hat{Q}_e(s) - \hat{\gamma} I\| \inf \kappa(V(s)) = \hat{X}_e \right)$$

(2) $$\mathrm{Re}\ (q_{rjj}(s) + \gamma) > X_e + \sum_{i \neq j} |q_{rij}(s)| \qquad \forall j,\ \forall s = j\omega \tag{8.109}$$

$$\left(\text{resp. } \mathrm{Re}\ (\hat{q}_{rjj}(s) + \hat{\gamma}) > \hat{X}_e + \sum_{i \neq j} |\hat{q}_{rij}(s)| \right)$$

(3) $\mathrm{Re}\,(\lambda_{rj}(s) + \gamma) > (m+2)\{M_r(s)\}^{(m-1)/m}\{\sum_{i,j} |W_{ij}(s) + q_{eij}(s) - \gamma\delta_{ij}|\}^{1/m}$

$$= Z_e \qquad , \forall j, \forall s = j\omega \tag{8.110}$$

(resp. $\mathrm{Re}(\hat{\lambda}_{rj}(s) + \hat{\gamma}) > (m+2)\{\hat{M}_r(s)\}^{(m-1)/m}\{\sum_{i,j} |\hat{W}_{ij}(s) + \hat{q}_{eij}(s) - \hat{\gamma}\hat{\delta}_{ij}|\}^{1/m}$

$= \hat{Z}_e$)

(4) $$\mathrm{Re}\,(q_{rjj}(s) + \gamma) > Z_e + \sum_{i\neq j} |q_{rij}(s)|, \forall j, \forall s = j\omega \tag{8.111}$$

(resp. $\mathrm{Re}\,(\hat{q}_{rjj}(s) + \hat{\gamma}) > \hat{Z}_e + \sum_{i\neq j} |\hat{q}_{rij}(s)|$)

where $M_r(s) = \max\,\{|\gamma\delta_{ij} + q_{rij}(s)|, |q_{ij}(s) + W_{ij}(s)|\}$,
$\hat{M}_r(s) = \max\,\{|\hat{\gamma}\delta_{ij} + \hat{q}_{rij}(s)|, |\hat{q}_{ij}(s) + \hat{W}_{ij}(s)|\}$

Proof:

Follows directly from theorems 8.13 to 8.15 and considering equations (8.106) and (8.107) in place of equations (8.90) and (8.91). (see also fig.8.7)

Thus, when reduced models are used in stability investigations, the bounds are modified, by having the width of their bands increased accordingly. This is similar to the describing function synthesis method, using reduced models, discussed in section 8.2.

8.3.3 Error estimates for dynamic response

The dynamic response of the nonlinear system of figure 8.1 can be estimated using integral inequalities[22]. Specifically, individual estimates can be made for the reduced system and original system. To determine the effectiveness of using reduced models, in the system, figure 8.1, the error $||x(t) - x_r(t)||$ is estimated for all $t \geq t_o$. In state space form, the system can be described by

$$S \quad : \quad \dot{x} = Ax + Bf(t,x) \tag{8.111a}$$

$$S_{r1} \quad : \quad \dot{x}_{r1} = A_{r1}x_{r1} + B_{r1}f(t,x) \tag{8.111b}$$

where $x = (x_1, x_2, \ldots x_n)^t$, $x_{r1} = (x_{r11}, x_{r12}, \ldots x_{r1r})^t$ are $(n \times 1)$, $(r \times 1)$ vectors respectively. To make the dimensions of x and x_r equal, suppose that different reduced models, with state vectors, x_{r1}, x_{r2}, $x_{r3}, \ldots, x_{rk}$, etc, are obtained for x, such that the response of x_{ri} is very close to some partitioned elements of the state vector x, i.e. $||x_{ri}|| \simeq ||(x_j, x_{j+1}, x_{j+2}, \ldots)^t||$. Equation (8.111b) can be written as

$$S_r \; : \; \dot{x}_r \; = \; A_r x_r + B_r f(t,x) \tag{8.111c}$$

where $x_r = (x_{r1}, x_{r2}, \ldots, x_{rk})^t$ is of dimension (n x 1), A_r = block diag $\{A_{r1}, A_{r2}, \ldots, A_{rk}\}$, $B_r^t = (B_{r1}^t, B_{r2}^t, \ldots, B_{rk}^t)$. The matrices A and A_r, B and B_r are of equal dimensions respectively. From equations (8.111b) and (8.111c),

$$S_e \; : \; \dot{x}_e \; = \; A_e x_e + B_e f(t,x_e) \tag{8.111d}$$

where $A_e = A - A_r$, $B_e = B - B_r$, $x_e = x - x_r$. The solution[21] for equation (8.111d) can be written as

$$x_e(t) \; = \; Y(t_o,t)x_e(t_o) + \int_{t_o}^{t} Y(\tau,t)B_e f(\tau, x_e(\tau))\, d\tau \tag{8.111e}$$

where $Y(\tau,t)$ satisfies

$$\dot{Y}(\tau,t) \; = \; AY(\tau,t) \; , \; Y(\tau,\tau) \; = \; I$$

Before expressing equation (8.111e), using integral estimates, in known parameters A_e, B_e, etc, it is assumed that x_e is defined and continuous over $t \geq t_o$ and

$$||x_e|| \leq H, \text{ a constant,}$$

$$||B_e f(t,x_e|| \leq P(t)||x_e|| + Q(t)||x_e||^{\alpha} \quad , \quad \alpha \geq 0$$

$$||Y(\tau,t)|| \leq \exp\{\int_{\tau}^{t}\gamma(t)\, dt\} \quad , \quad t \geq \tau \geq t_o$$

where P(t), Q(t) and $\gamma(t)$ are non-negative continuous functions on $t > t_o$. The solution x_e for equation (8.111d) then satisfies the following estimate[21].

$$\| x_e(t) ||_i \leq \exp\{\int_{t_o}^{t}(\gamma_i(t) + P(t))\, dt\}\{||x_e(t_o)||_i^{1-\alpha} + (1-\alpha)R(t)\}^{1/(1-\alpha)} \tag{8.111f}$$

$$R(t) \; = \; \int_{t_o}^{t} Q(\tau)\exp\{(\alpha - 1)\int_{t_o}^{\tau}(\gamma_i(\tau) + P(\tau))\, d\tau\}\, d\tau$$

for $t > t_o$, where for i = 1, 2, 3,

$$\gamma_1 = \max_i \{Re\ (a_{eii}) + \sum_{j \neq i} |a_{eij}|\}$$

$$\gamma_2 = \max_j \{Re\ (a_{ejj}) + \sum_{i \neq j} |a_{eij}|\}$$

$$\gamma_3 = \tfrac{1}{2}(A_e + A_e^H)$$

$$||x_e||_1 = \max_i |x_{ei}|$$

$$||x_e||_2 = \sum_i |x_{ei}|$$

$$||x_e||_3 = (\sum_i |x_{ei}|^2)^{\frac{1}{2}}$$

The significance of the estimate in equation (8.111f) lies in its capability of giving estimates for the solutions in terms of given quantities, i.e. a_{eij} of A_e and functions P(t) and Q(t), all of which can be calculated at a given time t.

If the reduced models are very accurate, then $\|x_e\| \simeq 0$, and the r.h.s. of expression (8.111f) should be very small. It can also be shown that expression (8.111f) gives the best estimate for $x_e(t)$ in equation (8.111d).

The initial solution $x_e(t_o)$ in expression (8.111f) can be computed from the following estimate.

$$||x_e(t_o)|| \leq (H/M(t))\{1 - (1 - \alpha)N(t)(H/M(t))^{\alpha-1}\}^{1/(1-\alpha)} \tag{8.111g}$$

where $M(t) = \exp\{\max_t \int_{t_o}^{t} (\gamma(t) + P(t))\, dt\}$

$$N(t) = \int_{t_o}^{\infty} Q(\tau) \exp\{(\alpha - 1) \int_{t_o}^{\tau} (\gamma(\tau) + P(\tau))\, d\tau\}\, d\tau$$

Expression (8.111f) and (8.111g) can be used to obtain numerical estimation for $x_e(t)$, $\forall t \geq t_o$

8.4 Model Reference Adaptive Control Systems (MRACS)[16-18]

8.4.1 Reduced Models and Hyperstability Design

Popov has proposed a hyperstability theorem[8] for the nonlinear system, S, of figure 8.1, where this time the nonlinearity N(t,Y) is not slope restricted, but the input signal u(t) is constrained to satisfy

$$\langle u(t),y(t)\rangle_{H_m} \leqslant \delta\| x(0)\| \sup\| x(t)\| \tag{8.112}$$

$$0 \leqslant t \leqslant \tau$$

where H_m is the $L_2^m(0,\tau]$ Hilbert space, $\delta > 0$ is a constant depending on $x(0)$ but independent of time τ. Thus,

Theorem 8I [8]

S is asymptotically hyperstable if

(1) $\| x(t) \| \leqslant k(\| x(0)\| + \delta)$ (8.113)

with $\lim x(t) = 0$ for $t \to \infty$

(2) $Q(s)$ is strictly positive real (8.114)

Theorem 8.1 can be used to assess positive realness of $Q(s)$ in terms of $Q_r(s)$, but this is rather inconvenient in design. A graphical criterion is thus sought in terms of the familiar characteristic loci and Nyquist polar plots.

Theorem 8.18

Let $\lambda_{rj}(s)$ be the usual characteristic loci of $Q_r(s)$, and assume $Q(s)$ and $Q_r(s)$ are real rational and holomorphic in the region Ω, ($\Omega \epsilon \mathrm{Re}(s) > 0$), and that the input/output constraints satisfy expression (8.112). Then S is asymptotically hyperstable if:

(1) $\| x(t) \| \leqslant k(\| x(0) \| + \delta)$ (8.115)

and any one of the following conditions be satisfied;

(2) $\mathrm{Re}\,(\lambda_{rj}(s) + \gamma) > \| Q_r^H + Q_e + Q_e^H - \gamma I\| \inf K(V_r(s)) = X_e,\ \forall j,\ \forall s = j\omega$ (8.116)

(resp. $\mathrm{Re}\,(\hat{\lambda}_{rj}(s) + \hat{\gamma}) > \| \hat{Q}_r^H + \hat{Q}_e + \hat{Q}_e^H - \hat{\gamma} I\| \inf \kappa(V_r(s)) = \hat{X}_e$)

(3) $$\mathrm{Re}\,(\lambda_{rj}(s) + \gamma) > (m+2)\{M_r(s)\}^{(m-1)/m}\{\sum_{i,j} |q_{eij}(s) + q^*_{rij}(s) + q^*_{eij}(s) - \gamma\delta_{ij}|\}^{1/m} = Z_e \tag{8.117}$$

$$\left(\text{resp. } \mathrm{Re}\,(\hat{\lambda}_{rj}(s) + \hat{\gamma}) > (m+2)\{\hat{M}_r(s)\}^{(m-1)/m}\{\sum_{i,j} |\hat{q}_{eij}(s) + \hat{q}^*_{rij}(s) + \hat{q}^*_{eij}(s) - \hat{\gamma}\hat{\delta}_{ij}|\}^{1/m} = \hat{Z}_e,\ \forall j,\ \forall s = j\omega \right)$$

(4) $\mathrm{Re}\,(q_{rjj}(s) + \gamma) > Y_e + \sum_{i\neq j} |q_{rij}(s)|$, $\forall j$, $\forall s = j\omega$ (8.118)

(resp. $\mathrm{Re}\,(\hat{q}_{rjj}(s) + \hat{\gamma}) > \hat{Y}_e + \sum_{i\neq j} |\hat{q}_{rij}(s)|$)

where $Y_e = X_e$ or $Y_e = Z_e$ (resp. $\hat{Y}_e = \hat{X}_e$ or $\hat{Y}_e = \hat{Z}_e$), $V_r(s)$ diagonalizes $Q_r(s)$, $Q_e(s) = Q(s) - Q_r(s)$, $q^*_{rij} = q_{rji}(-s)$, etc; δ_{ij} is the Kronecker delta, and, $M_r(s) = \max\,\{|q_{rij}(s) + \gamma\delta_{ij}|, |q_{ij}(s) + q^*_{ij}(s)|\}$, $\hat{M}_r(s) = \max\,\{|\hat{q}_{rij}(s) + \hat{\gamma}\delta_{ij}|, |\hat{q}_{ij}(s) + \hat{q}^*_{ij}(s)|\}$.

Proof:

Condition (1) is equation (8.113) of theorem 8I.

Conditions (2) to (4) ensure that $Q(s) + Q^H(s) > 0$ in terms of the structural elements of $Q_r(s)$. These conditions are a consequence of theorems 8.13 to 8.17 that are proved earlier. Hyperstability of S follows by theorem 8I. (see also fig.8.7(a) to 8.7(c) for similar graphical interpretations)

In the above $\gamma, \hat{\gamma}$ are suitable complex numbers (tuning factors) incorporated to adjust bandwidth. Thus, graphically, to test that Q(s) is positive real, it is required that the band $\mathrm{Re}\,(\lambda_{rj}(s) + \gamma) + Y_e \exp(j\phi)$, or $\mathrm{Re}\,(q_{rjj}(s) + \gamma) + \{Y_e + \sum_{i\neq j} |q_{rij}(s)|\}\exp(j\phi)$, $\phi\epsilon\{0,2\pi\}$, must lie to the right of the vertical line passing through the point $(-\mathrm{Re}\,(\gamma),0)$.

8.4.2 Reduced models and MRACS

Other than designing multivariable systems, by frequency response or optimal control methods, it is sometimes convenient to design them by Model Reference Adaptive Control (MRAC) approach. The MRAC method is basically, to design a self adaptive controller to stabilize the dynamic characteristics of a feedback control system, under conditions of environmental disturbance and plant parameter changes. Many forms of MRAC design methods exist[16]; the common ones being: (1) Lyapunov design method, (2) Hyperstability design method,[18], (3) Least square error method, and (4) Generalized square error technique with parameter optimization. The purpose of this section is to examine two important methods, viz, Hyperstability and Lyapunov method, with reduced models in design.

Figure 8.8(a) shows a typical MRAC structure, with reference model, adjustable model and plant. The purpose of the adaptation mechanism is to update the nonlinear time dependent gains following a change in output error, such that some performance index is minimized.

The reference model is characterized by

$$\dot{x}_m = A_m x_m + B_m u \qquad (8.119)$$

$$y_m = Cx_m$$

and the adjustable model by

$$\dot{x}_s = A_s(t)x_s + B_s(t)\, u \qquad (8.120)$$

$$y_s = Cx_s$$

and the error by

$$e = x_m - x \qquad (8.121)$$

Equations (8.119) to (8.121) give

$$\dot{e} = A_m e + (A_m - A_s(t))x_s + (B_m - B_s(t))u \qquad (8.122)$$

and a linear compensator, D, which generates the vector v is constructed as

$$v = De \qquad (8.123)$$

Hyperstability design is composed of separating equation (8.122) into the linear and nonlinear parts. Equations (8.122) and (8.123) can now be written as

$$\dot{e} = A_m e - W \qquad (8.124)$$

$$v = De$$

where

$$-W = (A_m - A_s(t))x_s + (B_m - B_s(t))u$$

is defined as a function of $v(t,\tau)$.

$$W = \{\int_o^t \Phi(v(\tau),t)\, dt - A_m + A_s(0)\}y + \{\int_o^t \psi(v(\tau),t)\, dt - B_m + B_s(0)\}u \qquad (8.125)$$

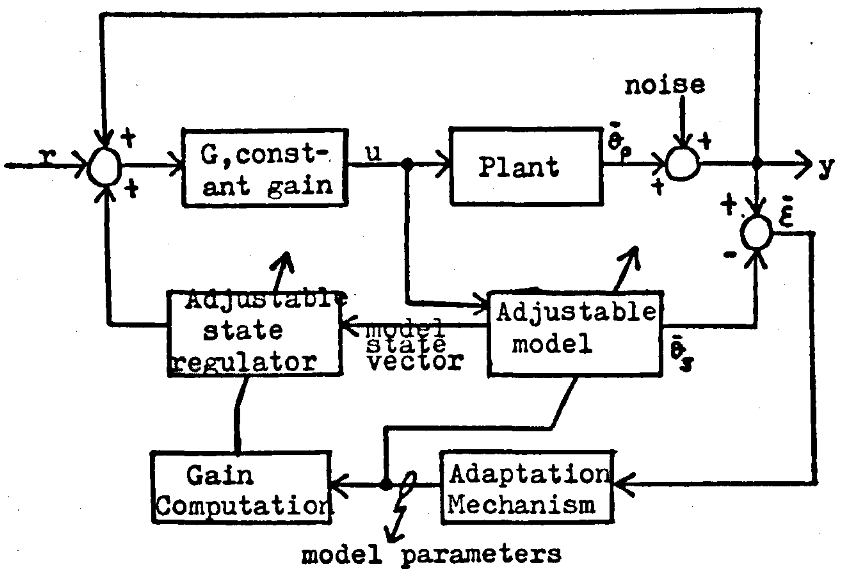


Fig. 8.8(a) Adaptive state regulator with adjustable plant model

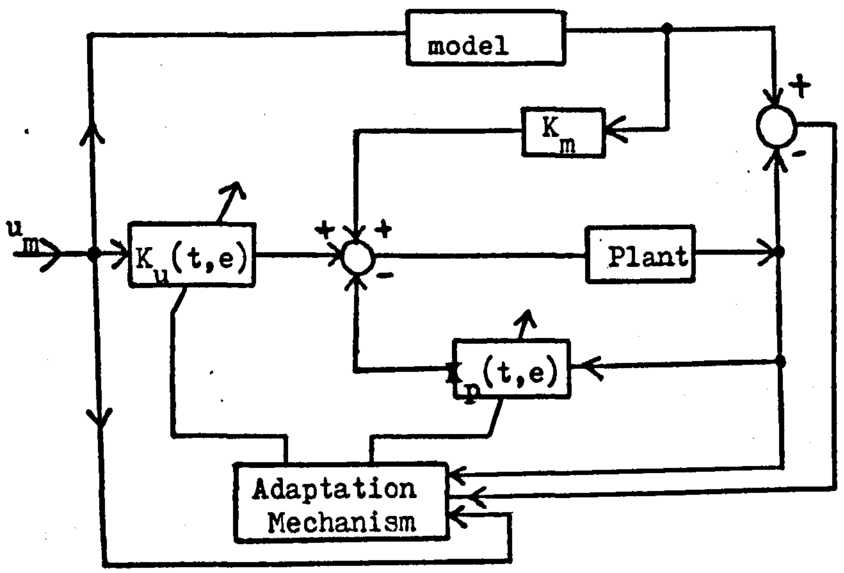


Fig. 8.8(b) Parameter Adaptation

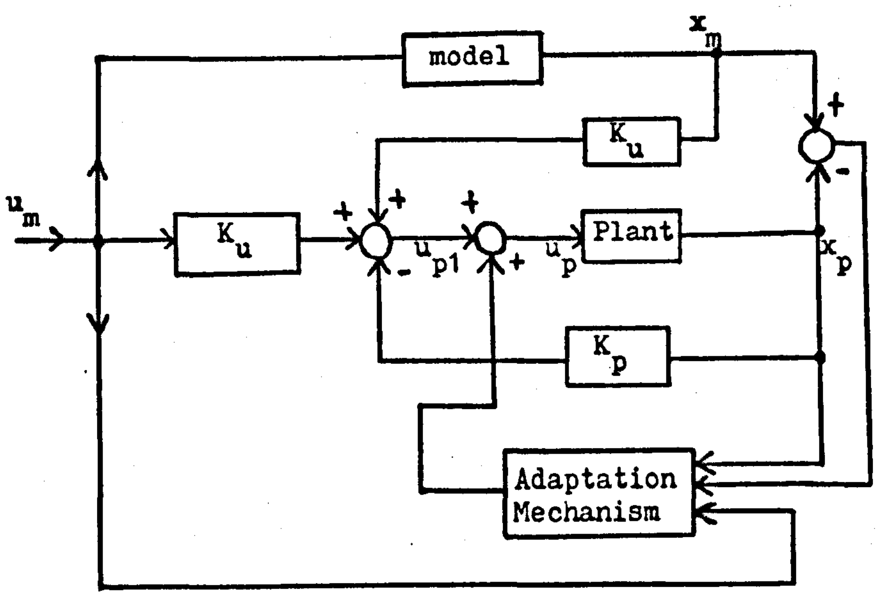


Fig. 8.8(c) Signal Synthesis Adaptation

Equation (8.125) defines the nonlinear part N(t,y) in figure 8.1. In order to satisfy Popov's hyperstability condition[8,18] Φ and ψ must be chosen such that

$$<v(t,\tau),W>_{H_m} \geq -\gamma_o^2 - \gamma_o \sup||x(\tau)|| \qquad (8.126)$$

$$\gamma_o \geq 0, \ 0 \leq \tau \leq t$$

where H_m is the $L_2^m(0,t]$ Hilbert space and m is the number of inputs or outputs of the system. Particular solutions for Φ and ψ, satisfying equation (8.126) are

$$\dot{A}_s(t) = \Phi(v(t),t) = Rvy^t \qquad (8.127)$$

$$\dot{B}_s(t) = \psi(v(t),t) = \tilde{R}vu^t$$

where R and $\tilde{R}$ are positive definite. The system, equation (8.124) is hyperstable with respect to equation (8.125) if there exist positive constants δ and γ such that any solution of the system satisfies

$$\| x(t) \| < \delta(\| x(0)\| + \gamma), \forall t \geq 0, \ \delta > 0, \ \gamma \geq 0 \qquad (8.128)$$

To obtain asymptotic hyperstability ($\lim x(t) = 0$ as $t \to \infty$), D should be designed such that the linear transfer function matrix $Q(s) = D(sI - A_m)^{-1}$ is strictly positive real, i.e.

$$Q(s) + Q^H(s) > 0 \qquad (8.129)$$

By the Kalman-Yocubovitch-Popov (KYP) lemma[19] suppose $S(A_m,I,D)$ is a minimal realization of Q(s), and if Q(s) is positive real, the matices P, K and L satisfy

$$K^tK = 0$$

$$PA_m + A_m^tP = LL^t \qquad (8.130)$$

$$P + K^tL^t = D$$

Now, suppose in design, reduced models A_{rm} and $A_{rs}(t)$ are used in place of A_m and $A_s(t)$. The crucial question is; under what conditions is the original system $S(A_m,A_s(t))$ stable, when the reduced system $S_r(A_{rm},A_{rs}(t))$ is hyperstable? For simplicity, it is assumed that the nonlinear parts of S and S_r satisfy the Popov hyperstability conditions. The remaining condition for stability is that $Q_r(s)$ and Q(s) must be positive real.

From the KYP lemma, equation (8.130), stability can also be assessed by the Lyapunov method, based on the stability of S_r, theorem 5.25.

Proposition 8.1

The original MRACS will be hyperstable given the reduced MRACS is hyperstable if theorems 8.18 or 5.25 be satisfied

Proof:

Requires Q(s) to be positive real

Other than using hyperstability approach, the Lyapunov method can also be used. From equation (8.122), choose a Lyapunov function V in terms of e and the difference between the parameters

$$V = e^tPe + tr\,(A_m - A_s(t))^t - R_1^{-1}(A_m - A_s(t))$$

$$+ tr\,(B_m - B_s(t))^t R_2^{-1}(B_m - B_s(t)) \tag{8.131}$$

from which

$$\dot{V} = e^t(A_m^tP + PA_m)e + 2\,tr\,(A_m - A_s(t))^t(Pey^t - R_1^{-1}\dot{A}_s(t))$$

$$+ 2\,tr\,(B_m - B_s(t))^t(Peu^t - R_s^{-1}\dot{B}_s(t)) \tag{8.132}$$

To assure the stability of the system it is sufficient that the first term is negative definite and the second and third terms vanish. This requires

$$A_m^tP + PA_m = -Q \tag{8.133}$$

where Q > 0, and the adaptation law is chosen as

$$\dot{A}_s(t) = R_1(Pe)y^t \tag{8.134}$$

$$\dot{B}_s(t) = R_s(Pe)u^t$$

Proposition 8.2

In MRACS design, using Lyapunov synthesis, S_r and S will be stable if theorem 5.24 is satisfied.

Proof:

Given in theorem 5.24.

An attractive feature of MRACS is Model Following Adaptive systems[17], MFAS. Here, the dynamics of the plant are made to follow the dynamics of a reference model in the presence of adaptive mechanisms and parameter changes. As opposed to MRACS, the reference model plays a more prominent role in MFAS. The essential features of MFAS are simple control laws and strong stability characteristics. Typical set-ups of MFAS are shown in figures 8.8(b) and 8.8(c), called parameter adaptation and signal synthesis adaptation.

A linear model following system (LMFS) can be described by[17]

$$\begin{aligned} \dot{x}_m &= A_m x_m + B_m u_m \\ \dot{x}_p &= A_p x_p + B_p u_p \\ u_p &= -K_p x_p + K_m x_m + K_u u \end{aligned} \tag{8.135}$$

where the plant and model state vectors, x_p and x_m, are of equal dimensions. The Erzberger conditions for perfect model following, where $\dot{x}_m - \dot{x}_p = 0$, are

$$\begin{aligned} (I - B_p B_p^+) B_m &= 0 \\ (I - B_p B_p^+)(A_m - A_p) &= 0 \end{aligned} \tag{8.136}$$

where B_p^+ is the pseudo inverse of B_p. For the model following adaptive system, with parameter adaptation, the plant input can be written as (figure 8.8(b)),

$$u_p = -K_p(t,e)x_p + K_u(t,e)u_m + K_m x_m \tag{8.137}$$

where $K_p(t,e)$, $K_u(t,e)$ can be further written as,

$$\begin{aligned} K_p(t,e) &= K_p - \Delta K_p(t,e) \\ K_u(t,e) &= K_u + \Delta K_u(t,e) \end{aligned} \tag{8.138}$$

The plant input can be written as

$$u_p = u_{p1} + u_{p2}$$

$$\text{where } u_{p1} = -K_p x_p + K_m x_m + K_u u_m \tag{8.139}$$

$$u_{p2} = \Delta K_p(t,e)x_p + \Delta K_u(t,e)u_m$$

The input u_{p2} is the contribution of the adaptive loop. The adaptive mechanism is decomposed into a linear time invariant part,

$$e = x_m - x_p \tag{8.140}$$

$$v = De$$

and a nonlinear part which generates $\Delta K_p(t,e)$ and $\Delta K_u(t,e)$ as a function of v. Equations (8.135) and (8.140) give

$$\dot{e} = (A_m - B_p K_m)e + B_p W_1 \tag{8.141}$$

$$v = De$$

where

$$-W_1 = (\Delta K_p(t,e) - B_p^+(A_m - A_p) + K_m - K_p)x_p$$

$$+ (\Delta K_u(t,e) - B_p^+ B_m + K_u)u_m \tag{8.142}$$

The nonlinear part is designed in accordance to hyperstability conditions, i.e.

$$\langle v(t),W(t)\rangle_{H_n} \geq -\gamma_o^2, \quad \forall t \geq 0 \tag{8.143}$$

where H_n is the n-dimensional Hilbert space $L_2^n(0,t]$ and n is the order of the system. Particular choices for $\Delta K_u(t,e)$, $\Delta K_p(t,e)$ such that equation (8.143) is satisfied are,

$$\Delta K_p(t,v) = \int_o^t \tilde{L}v(Qx_p)^t\, d\tau + Lv(Qx_p)^t + \Delta K_p(0) \tag{8.144}$$

$$\Delta K_u(t,v) = \int_o^t \tilde{M}v(Ru_m)^t\, d\tau + Mv(Ru_m)^t + \Delta K_u(0)$$

where $\tilde{L}$, Q, $\tilde{M}$, R are positive definite, and, L, M are positive, or non-negative definite. The linear part

$$G(s) = D(sI - A_m + B_p K_m)^{-1} B_p \tag{8.145}$$

must be positive real, for the system to be asymptotic hyperstable. By the KYP lemma, matrices P and H satisfy

$$(A_m - B_p K_m)^t P + P(A_m - B_p K_m) = -H \tag{8.146}$$

$$D = B_p^t P$$

Now suppose reduced models A_{rm} are used in the design.

Proposition 8.3

In a LMFS design with parameter adaptation, the original system S, will be hyperstable if the linear part of S_r satisfies theorems 8.18 or 5.25, assuming the nonlinear parts of S and S_r satisfy equation (8.143).

Proof:

Requires G(s) to be positive real.

Conclusions

The stability bounds expressed in this chapter, for design of multivariable nonlinear systems, using reduced modesls, are closely related to those for linear multivariable systems, given in Chapters 5 and 6. The nonlinear elements above are either sector restricted, functionally approximated or bounded by integral constraints. The bounds imposed on the linear part of the system are thus a natural consequence of those developed in preceding chapters.

Stability bounds are also derived for the original multivariable systems, using describing functions, the Jury Lee criterion and the circle criterion. These bounds are easily modified, with bandwidth widened, when reduced models are used.

As in linear multivariable systems, computer aided graphics are indispensable for the evaluation of these bounds in design work. The linear parts of the system can thus be designed, using the methods described in Chapter 6, bearing in mind the constraints imposed by the nonlinear parts.

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IX

APPLICATION OF REDUCTION TECHNIQUES TO CONTROL SYSTEMS DESIGN - A CASE STUDY OF AN INDUSTRIAL BOILER.

CHAPTER IX

APPLICATION OF REDUCTION TECHNIQUES TO CONTROL SYSTEMS DESIGN - A CASE STUDY OF AN INDUSTRIAL BOILER

Introduction

This chapter is concerned with a case study of an industrial boiler model. The linear model, given in state space form[11], is thirty third order, and, has five inputs and four outputs. From these the transfer function of the model was obtained, and, manipulated into various forms, using algorithmic methods, to meet the requirements of different applications. Such a model was chosen, because, firstly, it is of very high order, hence strengthening the necessity of using reduction techniques and making their applications meaningful, and, secondly, it represents a practical example from a 'real life' process.

The purpose of this chapter is to make a comparison among existing reduction techniques and to investigate the effectiveness of using reduced models, in relation to them, in control systems design. The main interest is to study the advantages and effects of using such models in design, rather, than on obtaining particular control strategies[13] or emphasizing on design sophistication. The latter area is believed to be too broad to be restricted to a single case study. Instead, attention will be focused on the areas of model adequacy, overall system stability and performance deterioration, studied analytically in Chapters V and VI.

Some existing reduction techniques[1-8], together with those developed in Chapters III and IV were applied individually to obtain lower order models. Two different design approaches were considered. The first is the linear optimal regulator design method in the time domain[9], and, the second is the Inverse Nyquist Array design method in the frequency domain[10]. These two design philosophies were used since they are contrasting, hence, different controllers, designed using different models, would be expected. In addition, they would provide logical conclusions on using reduced models, with respect to their reduction techniques, in different design methods.

PART I - Comparison of reduction techniques

9.1 The boiler model and its transfer function

The system considered here, is a mathematical model of a 200 MWatt system generator, used in a power station in West Thurock[11]. The unit is comprised of firing and coal preparation system, air supply system, combustion gases supply system, steam circuit systems, superheaters, attemperators and reheater systems. It is a coal fired installation with a rated load capacity of 1.35 x 10^6 lb/hr and is of the natural circulation radiant type. At full load, the system conditions are 2450 p.s.i.g. pressure and 1050^{o}F temperature at the turbine throttle with 1000^{o}F reheat.

The boiler has four outputs, which are, water level, ℓ, outlet temperature, T_{s7}, outlet pressure, P_{s7}, outlet reheat temperature, T_{s10}, and, four inputs, which are, feed (372.5 lb/sec), spray (2.482 lb/sec), fuel (50.64 lb/sec), damper (0.04, as a fraction), and, steam (375 lb/sec) is considered as a disturbance input. The input and output vectors are thus;

$$(u_1, u_2, u_3, u_4, u_5)^t = (\text{steam, feed, spray, fuel, damper})^t$$

$$(y_1, y_2, y_3, y_4)^t = (\ell, T_{s7}, P_{s7}, T_{s10})^t$$

The schematic diagrams of the boiler and its control loops are shown in figures 9.1(a) and 9.1(b).

The linearized model, obtained by Marshall[11], represented in state space form is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & a_{12}^t \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1^t \\ B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u \end{bmatrix} \tag{9.1}$$

$$y = [\, I_{4\times33} \;\vdots\; 0 \,]x$$

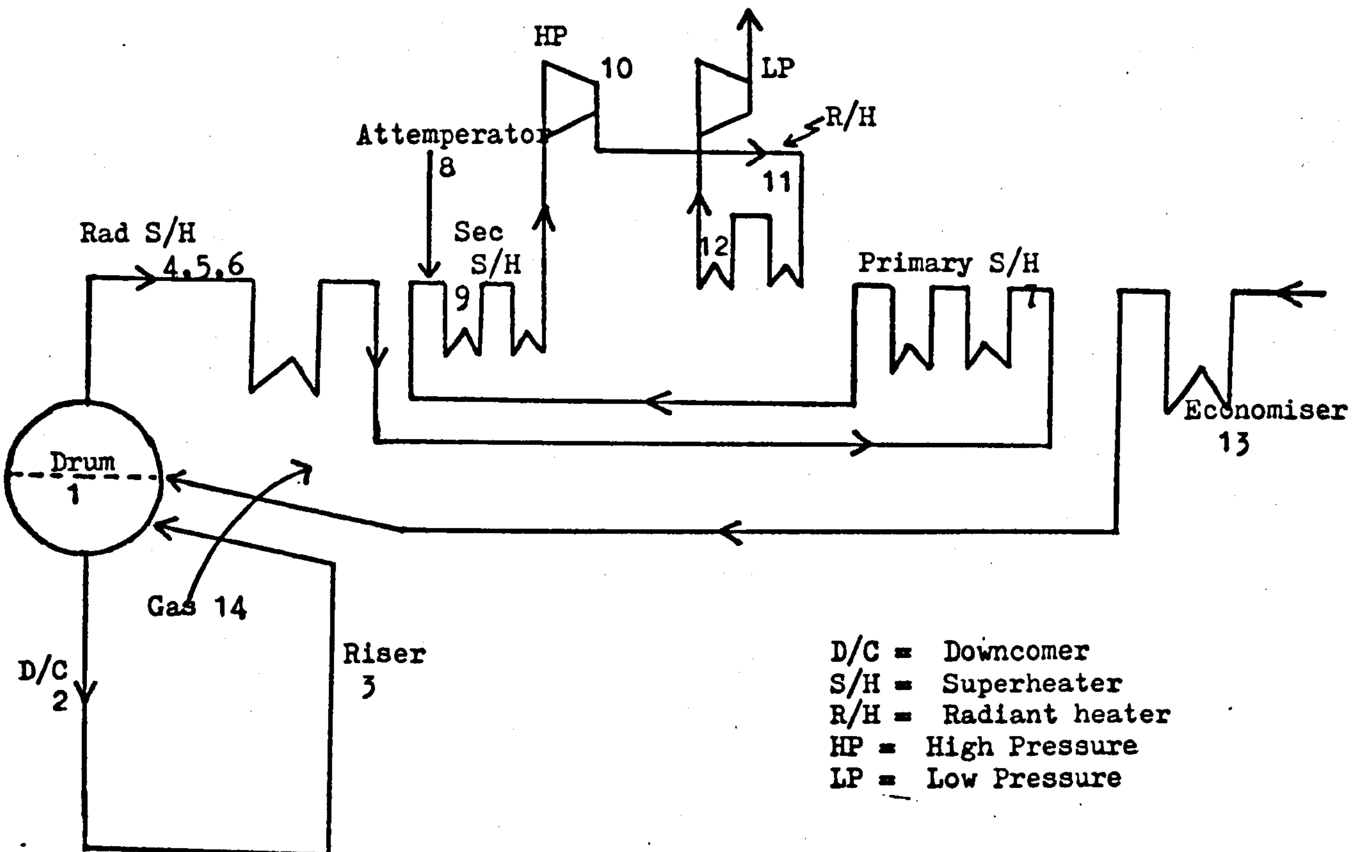
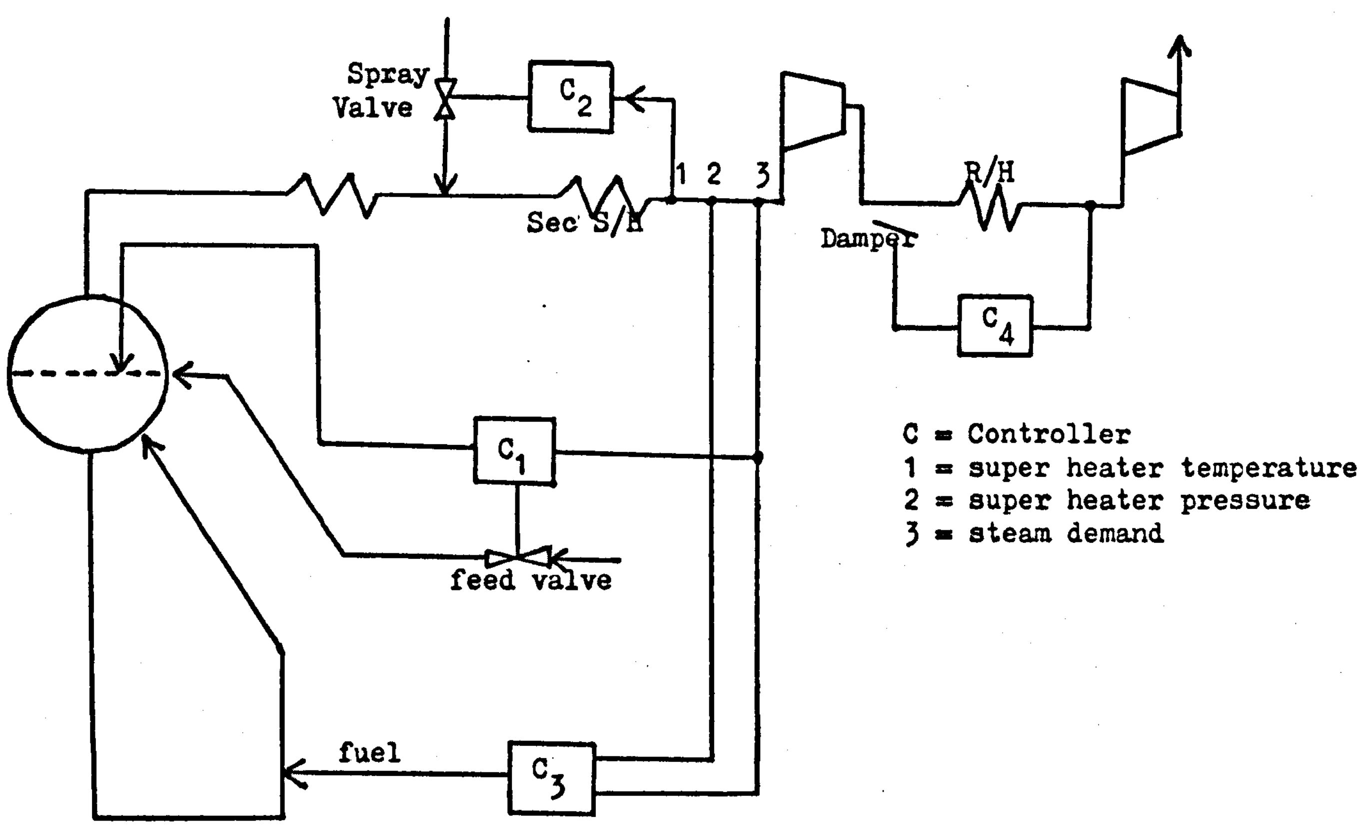


Fig.9.1(a) Simplified Arrangement of Boiler

Fig.9.1(b) Boiler with Control Loops attached



where {A, B, C} are given in Table 9A. The system matrix, A, is of order (33 x 33) and rank 32. This is due to the presence of a zero vector in the first column of A, associated with the water level state, ℓ.

From equation (9.1) the transfer function matrix is,

$$G(s) = (I_{4\times 33} \vdots\ 0)\begin{bmatrix} (a_{12}^{t}(sI - A_{22})^{-1}B_2 + b_1^{t})/s \\ (sI - A_{22})^{-1}B_2 \end{bmatrix} \tag{9.2}$$

where an arbitrary state is included to obtain a square (5 x 5) G(s).

Due to the high order of the model, special numerical techniques must be used to obtain an accurate transfer function.

The normal Faddeev algorithm[15],

$$(s^n - h_1 s^{n-1} - \dots h_{n-1}s - h_n)I = (C_o s^{n-1} + \dots C_{n-2}s + C_{n-1})(sI - A_{22})$$

$$C_o = I,\ h_k = \mathrm{tr}\,(AC_{k-1})/k \qquad C_k = AC_{k-1} - h_k I, \quad k = 1, 2, ..n \tag{9.3}$$

$$C_n = 0$$

and in this case, the reverse Faddeev algorithm[15],

$$(s^n - t_1 s^{n-1} - \dots t_{n-1}s - t_n)I = (T_o s^{n-1} + \dots T_{n-2}s + T_{n-1})(sI - A_{22}^{-1})$$

$$\text{where } h_n = \det A_{22} \neq 0 \tag{9.4}$$

$$h_{n-k} = -t_k h_n,\ C_k = h_n T_{n-k-1} A_{22}^{-1},\ k = 1, 2, \dots n-1$$

were used to evaluate $(sI - A_{22})^{-1}$ in equation (9.2). Use of the reverse algorithm is necessary, as the forward algorithm will give erroneous coefficients (contaminated by rounding errors) of the higher powers of s. With the reverse algorithm, accurate coefficients of higher powers of s will be obtained, while the lower power coefficients will be contaminated by rounding errors. It was found that, using the above algorithms, the two sets of coefficients overlapped reasonably well midway, and, values were taken from the accurate ends of the polynomials. Bosley, et al, used the same numerical technique[15], and obtained good results for very high order models.

The Faddeev algorithm evaluates G(s) in the form

APPENDIX 9A i

Steady state values of model state vector, x, $x_1 = \ell$, ... $x_{33} = T_{ts8}$

- ℓ = 0 ft; position of drum water level.
- T_{s7} = 1055 deg F; steam temp. leaving second superheater.
- P_{s7} = 2450 psig; steam pressure leaving second superheater.
- T_{s10} = 1005 deg F; steam temp. pendant radiant heater.
- T_{te} = 525 deg F; economiser metal temp.
- T_g = 571 deg F: feedwater temp.
- T_1 = 669 deg F; drumwater temp.
- P_{s1} = 2650 psig; drum pressure.
- x = 0.139242 lb/lb; dryness fraction of steam leaving risers.
- T_t = 890 deg F; riser metal temp.
- W_2 = 2897.17 lb/sec; riser mixture flow.
- T_{ts1} = 690 deg F; 1st stage radiant superheater metal temp.
- T_{s2} = 682 deg F; steam temp. leaving 1st stage radiant superheater.
- P_{s2} = 2640 psig; steam pressure leaving 1st stage radiant superheater.
- T_{ts2} = 715 deg F; 2nd stage superheater metal temp.
- T_{s3} = 725° F; steam temp. leaving 2nd stage rad. superheater;
- P_{s3} = 2598 psig; steam pressure leaving 2nd stage rad. superheater.
- T_{ts3} = 742° F; 3rd stage rad. superheater metal temp.
- T_{s4} = 733 deg F; steam temp. leaving 3rd stage rad. superheater.
- P_{s4} = 2583 psig; steam pressure leaving 3rd stage rad. superheater.
- T_{ts4} = 796.5 deg F; primary superheater metal temp.
- T_{s5} = 835 deg F; steam temp. leaving primary superheater.
- P_{25} = 2577 psig; steam pressure leaving primary superheater.
- T_{ts5} = 958.4 deg F; secondary superheater metal temp.
- T_{g2} = 2122 deg F; gas temp. at risers
- T_{g7} = 868.5 deg F; gas temp. at primary superheater.
- T_{g3} = 1653 deg.F; gas temp. at secondary superheater.
- T_{g8} = 580 deg F; gas temp. at economiser.
- T_{g5} = 1112 deg F; gas temp. at horizontal radiant heater.
- T_{g4} = 1503 deg F; gas temp. at pendant radiant heater.
- T_{ts7} = 806.4 deg F; horizontal radiant heater metal temp.
- T_{s9} = 868 deg F; steam temp. leaving horizontal radiant heater
- T_{ts8} = 988.2 deg F; pendant radiant heater metal temp.

Boiler disturbance variable

- W_{steam} = 375 lb/sec; steam demand

Appendix 9A ii

Boiler inputs

- W_{feed} = 372.518 lb/sec; rate of feedwater flow.
- W_{spray} = 2.482 lb/sec; rate of spray water flow.
- W_{fuel} = 50.64 lb/sec; rate of fuel flow.
- K_{damp} = 0.04 (fraction of total gas through bypass); reheater bypass damper position.

TABLE 9A i

System matrix A (33X33)

All elements stored column wise.

Col.1. All elements in first column are zero, i.e. A_{11} to $A_{33,1} = 0.0$

Col.2. 0.0, -1.43, -3.04, A_{42} to $A_{23,2} = 0.0$, 0.029, $A_{25,2}$ to $A_{31,2} = 0.0$, 0.252, 0.0

Col.3. 0.0, 0.451, -3.16, A_{43} to $A_{20,3} = 0.0$, 0.00429, -0.0555, 1.47, $A_{24,3}$ to $A_{33,3} = 0.0$

Col.4. A_{14} to $A_{34} = 0.0$, -4.7, A_{54} to $A_{32,4} = 0.0$, 0.0275

Col.5. A_{15} to $A_{45} = 0.0$, -0.197, 0.142, A_{75} to $A_{27,5} = 0.0$, 22.8, $A_{29,5}$ to $A_{33,5} = 0.0$

Col.6. A_{16} to $A_{46} = 0.0$, 0.189, -0.146, 0.00785, 0.0, 0.0, 0.0, - 0.00455, $A_{12,6}$ to $A_{33,6} = 0.0$

Col.7. -0.000721, A_{27} to $A_{67} = 0.0$, -0.129, 0.518, 0.0000749, 0.0, 15.9, $A_{12,7}$ to $A_{33,7} = 0.0$

Col.8. 0.00988, A_{28} to $A_{68} = 0.0$, 0.00647, - 0.675, 0.000472, 0.0135, -1030.0, 0.0, -0.278, 30.9, $A_{15,8}$ to $A_{33,8} = 0.0$

Col.9. -1.79, A_{29} to $A_{69} = 0.0$, -1.07, -12.7, -0.0814, 0.0, 157.0, $A_{12,9}$ to $A_{33,9} = 0.0$

Col.10. 0.00851, $A_{2,10}$ to $A_{7,10} = 0.0$, 3.23, 0.00104, -0.244, 108.0, $A_{12,10}$ to $A_{24,10} = 0.0$, 0.0654, $A_{26,10}$ to $A_{33,10} = 0.0$

Col.11. -0.904E-04, $A_{2,11}$ to $A_{6,11} = 0.0$, 0.319E-03, -0.216E-02, -0.441E-05, 0.0, -0.92, $A_{12,11}$ to $A_{33,11} = 0.0$

Col.12. $A_{1,12}$ to $A_{10,12} = 0.0$, -151.0, -0.248, 0.505, 4.55, $A_{15,12}$ to $A_{28,12} = 0.0$, 0.0944, $A_{30,12}$ to $A_{33,12} = 0.0$

Col.13. $A_{1,13}$ to $A_{10,13} = 0.0$, 260.0, 0.255, -1.22, -7.85, 0.0, 0.234, 0.755, $A_{18,13}$ to $A_{33,13} = 0.0$

Col.14. -0.0103, $A_{2,14}$ to $A_{7,14} = 0.0$, 0.449, -0.534E-03, 0.0, 1270.0, -0.024, 0.296, -38.5, 0.0, -0.0804, 1.14, $A_{18,14}$ to $A_{33,14} = 0.0$

Col.15. $A_{1,15}$ to $A_{14,15} = 0.0$, -0.13, 0.689, 4.05, $A_{18,15}$ to $A_{24,15} = 0.0$, 0.0142, $A_{26,15}$ to $A_{33,15} = 0.0$

Col.16. $A_{1,16}$ to $A_{14,16} = 0.0$, 0.132, -0.799, -4.41, 0.0, 0.195, 0.48, $A_{21,16}$ to $A_{33,16} = 0.0$

Col.17. $A_{1,17}$ to $A_{10,17} = 0.0$, -242.0, 0.0229, -0.0472, 7.32, -0.0336, 0.255, -4.49, 0.0, -0.096, 8.01, $A_{21,17}$ to $A_{33,17} = 0.0$

Col.18. $A_{1,18}$ to $A_{17,18} = 0.0$, -0.435, 2.66, 14.0, $A_{21,18}$ to $A_{28,18} = 0.0$, 0.113 $A_{30,18}$ to $A_{33,18} = 0.0$

Col.19. $A_{1,19}$ to $A_{17,19} = 0.0$, 0.441, -2.87, -14.7, 0.0, 0.094, 0.0663, $A_{24,19}$ to $A_{33,19} = 0.0$

Table 9A ii

Col.20. $A_{1,20}$ to $A_{14,20} = 0.0$, 0.0332, -0.176, 3.36, -0.308, 1.98, -18.8, 0.0, -0.503, 10.2, $A_{24,20}$ to $A_{33,20} = 0.0$

Col.21. $A_{1,21}$ to $A_{20,21} = 0.0$, -0.0522, 0.599, 2.14, 0.0, 0.0, 32.2, $A_{27,21}$ to $A_{33,21} = 0.0$

Col.22. 0.0, 0.273, 0.231, $A_{4,22}$ to $A_{20,22} = 0.0$, 0.0467, -0.655, -2.2, $A_{24,22}$ to $A_{33,22} = 0.0$

Col.23. 0.0, -0.463, 3.24, $A_{4,23}$ to $A_{17,23} = 0.0$, 0.306, -1.87, 10.9, -0.0041, 0.558, -11.7, $A_{24,23}$ to $A_{33,23} = 0.0$

Col.24. 0.0, 1.26, 2.68, $A_{4,24}$ to $A_{23,24} = 0.0$, -0.0303, 0.0, 0.0, 1.21, $A_{28,24}$ to $A_{33,24} = 0.0$

Col.25. $A_{1,25}$ to $A_{9,25} = 0.0$, 0.0016, 0.0438, 0.0, 0.0, 0.0, 0.00197, $A_{16,25}$ to $A_{24,25} = 0.0$, -1.3, 0.0, 2.45, $A_{28,25}$ to $A_{33,25} = 0.0$

Col.26. $A_{1,26}$ to $A_{20,26} = 0.0$, 0.00614, 0.0, 0.0, 0.0, 0.0, -42.7, 0.0, 5.1, $A_{29,26}$ to $A_{33,26} = 0.0$

Col.27. $A_{1,27}$ to $A_{23,27} = 0.0$, 0.0023, 0.0, 0.0, -4.69, 0.0, 0.0, 4.91, 0.0, 0.0, 0.0

Col.28. $A_{1,28}$ to $A_{4,28} = 0.0$, 0.00782, $A_{6,28}$ to $A_{27,28} = 0.0$, -28.2, $A_{29,28}$ to $A_{33,28} = 0.0$

Col.29. $A_{1,29}$ to $A_{11,29} = 0.0$, 0.00523, $A_{13,29}$ to $A_{17,29} = 0.0$, 0.00104, $A_{19,29}$ to $A_{25,29} = 0.0$, 8.66, 0.0, 0.0, -4.11, 0.0, 0.00238, 0.0, 0.0

Col.30. $A_{1,30}$ to $A_{25,30} = 0.0$, 0.371, 0.0, 0.0, 1.82, -6.53, 0.0, 0.0, 0.00285

Col.31. $A_{1,31}$ to $A_{28,31} = 0.0$, 1.42, 0.0, -0.0176, 1.39, 0.0

Col.32. 0.0, 0.0, 0.0, 1.3, $A_{5,32}$ to $A_{30,32} = 0.0$, 0.0157, 1.66, 0.0

Col.33. 0.0, 0.0, 0.0, 3.41, $A_{5,33}$ to $A_{29,33} = 0.0$, 1.28, 0.0, 0.0, -0.0298

Matrix B (33X5)

Col.1. 0.0, 0.118, -0.705, -0.0762, $B_{5,1}$ to $B_{23,1} = 0.0$, -0.00272, $B_{25,1}$ to $B_{30,1} = 0.0$, -0.00123, -0.029, -0.00272

Col.2. 0.163E-04, B_{22} to $B_{52} = 0.0$, -0.00105, -0.00254, 0.0062, -0.737E-05, 0.0, -0.0393, $B_{12,2}$ to $B_{33,2} = 0.0$

Col.3 0.0, -0.705, -0.595, $B_{4,3}$ to $B_{20,3} = 0.0$, 0.52[illegible]-04, -0.000677, 0.0178, $B_{24,3}$ to $B_{33,3} = 0.0$

Col.4. B_{14} to $B_{44} = 0.0$, 0.0054, B_{64} to $B_{20,4} = 0.0$, 0.0036, 0.0, 0.0, 0.00628, 11.5, 25.2, 15.8, 12.0, 9.99, 7.79, 0.00469, 0.0, 0.011

Col.5. B_{15} to $B_{25,5} = 0.0$, 354.0, 0.0, 0.0, -547.0, 0.0, -0.257, 0.0, 0.0

$$G(s) = \sum_{i=1}^{n} C_{i-1} s^{n-i} \Big/ (s^n + \sum_{i=1}^{n} h_i s^{n-i}) \tag{9.5}$$

where C_i is an (n x n) matrix, n, the order of the system. G(s) can also be expressed in partial fraction form as

$$G(s) = B_n + \sum_{i=1}^{n} \sum_{k=1}^{m} C_{ik} \Big/ (s + p_i)^k = \sum_{i=1}^{n} K_i/(s + p_i) \tag{9.10}$$

as there are no repeated eigenvalues in A. In equation (9.10) K is the residual matrix corresponding to the pole p_i and complex poles and residues occur in conjugate pairs. By using the basic 'cover up rule' a computer program was written to evaluate G(s) in equation (9.10). It is also desirable to find the zeros of G(s). The zeros can be evaluated by finding the roots of the numerator polynomial of each $g_{ij}(s)$, $\forall_{ij}$. This was converted into an 'eigenvalue problem' by finding the eigenvalues of the corresponding companion matrix. The procedure was repeated for every $g_{ij}(s)$.

i.e. $$g_{ij}(s) = K \prod_{n=1}^{n-1} (s - z_k) \Big/ \prod_{k=1}^{n} (s - p_k)$$

Alternatively the zeros of G(s) were found from the state space equations S(A,B,C) using a method due to Davison[16]. A transformation

$$z = Tx \tag{9.11}$$

where

$$T = \begin{bmatrix} I_{k-1} & 0 \\ c_r^t & \\ 0 & I_{n-k} \end{bmatrix}$$

where c_r^t is the rth row of C, was used on equation (9.1), and, the transformed equation was converted into an 'eigenvalue problem' by solving

$$\det(\Lambda - sI) = 0 \tag{9.12}$$

where

Table 9A iii

Matrix $C = [I_{4x33} \vdots 0]$

Eigenvalues of A matrix:

0.0, -0.391E02, -0.427E02, -0.256E02, -0.282E02, -0.653E01, -0.809E01, -0.469E01; -0.472E01, -0.344E01, -0.411E01, -0.25E01 ± j 0.518, -0.1675E01, -0.1146E01, -0.141E01, -0.1303E01, -0.844, -0.635 ± j 0.0523, -0.321, -0.333, -0.1358 ± j 0.456E-03, -0.926E-01, -0.163E-01, -0.223E-01, -0.736E-02, -0.9455E-02, -0.4843E-02 ± j 0.16E-02, -0.2235E-02, -0.2815E-02

Table 9B i

$$G(s) = (C_{32} + C_{31}s + C_{30}s^2 + \ldots C_0 s^{32})/(d_{33}+d_{32}s+d_{31}s^2+\ldots d_1 s^{32}+s^{33})$$

Common denominator of G(s) in ascending powers of s: →
0.0, 0.1183E-09, 0.1851E-06, 0.1197E-03, 0.4205E-01, 0.887E01, 0.1168E04, 0.9668E05, 0.4989E07, 0.1579E09, 0.305E10, 0.3729E11, 0.3008E12, 0.1661E13, 0.648E13, 0.1834E14, 0.3846E14, 0.6073E14, 0.7317E14, 0.6788E14, 0.488E14, 0.2727E14, 0.1186E14, 0.1044E13, 0.2082E12, 0.3137E11, 0.3502E10, 0.2822E09, 0.1586E08, 0.5934E06, 0.1386E05, 0.1809E03, 1.0

Numerator coefficients of elements of G(s) in ascending powers of s: →

$G_{11}(s)$

-0.252E-13, -0.6229E-12, 0.5154E-06, 0.3293E-03, 0.1138, 0.2344E02, 0.2978E04, 0.2334E06, 0.1102E08, 0.3012E09, 0.4614E10, 0.4194E11, 0.2402E12, 0.9024E12, 0.4968E13, -0.4676E10, 0.4928E10, 0.4083E10, 0.2613E10, 0.123E10, 0.4247E09, 0.1063E09, 0.1889E08, 0.231E07, 0.1846E06, 0.8829E04, 0.2159E03, 0.2008E01, 0.302E-06, 0.0, 0.0, 0.0, 0.0

$G_{12}(s)$

0.2328E-13, 0.2576E-13, 0.5968E-07, 0.3813E-04, 0.1317E-01, 0.2713E01, 0.3447E03, 0.2701E05, 0.1275E07, 0.3482E08, 0.5328E09, 0.4838E10, 0.277E11, 0.1041E12, 0.5751E12, -0.5031E11, 0.4425E10, 0.3271E10, 0.2739E10, 0.1766E10, 0.8911E09, 0.352E09, 0.1087E09, 0.2609E08, 0.4828E07, 0.6797E06, 0.7145E05, 0.5465E04, 0.2936E03, 0.1057E02, 0.2391, 0.3032E-02, 0.1633E-04

$G_{13}(s)$

0.252E-13, 0.1052E-12, 0.4351E-06, 0.278E-03, 09606E-01, 0.1979E02, 0.2514E04, 0.197E06, 0.9302E07, 0.2542E09, 0.3894E10, 0.3539E11, 0.2027E12, 0.7612E12, 0.419E13, 0.5345E10, -0.2163E10, -0.1174E10, -0.3952E09, -0.1657E08, 0.5036E08, 0.2525E08, 0.6345E07, 0.9325E06, 0.7818E05, 0.3123E04, 0.1293E02, -0.2126E01, -03468E-01, 0.0, 0.0, 0.0, 0.0

$G_{14}(s)$

0.1472E-16, 0.2071E-11, -0.2282E-06, -0.1458E-03, -05037E-01, -0.1037E02, -0.1317E04, -0.1032E06, -0.4869E07, -0.1329E09, -0.203E10, -0.1837E11, --0.1045E12, -0.3888E12, -0.2155E13, 0.6271E10, 0.1805E10, 0.167E10, 0.1089E10, 0.5414E09, 0.2054E09, 0.5934E08, 0.1297E08, 0.2122E07, 0.2551E06, 0.2202E05, 0.132E04, 0.5274E02, 0.132E01, 0.18565-01, 0.111E-03, 0.0, 0.0

$G_{15}(s)$

0.2409E116, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, -0.1622E12, 0.2908E11, 0.2294E11, 0.1636E11, 0.8761E10, 0.3546E10, 0.108E10, 0.2448E09, 0.4073E08, 0.4855E07, 0.4007E06, 0.2177E05, 0.7213E03, 0.1293E02, 0.9465E-01, 0.0, 0.0, 0.0

Table 9B ii

$G_{21}(s)$

0.0, -0.1399E-09, -0.1568E-06, -0.7142E-04, -0.1717E-01, -0.2377E01, -0.1968E03, -0.9809E04, -0.2866E06, -0.4682E07, -0.4007E08, -0.1104E09, 0.1529E10, -0.1554E12, 0.7476E14, -0.6458E14, 0.2137E13, 0.1022E13, 0.1323E13, 0.1242E13, 0.883E12, 0.4767E12, 0.1953E12, 0.6042E11, 0.1401E11, 0.2402E10, 0.2994E09, 0.2644E08, 0.1599E07, 0.6331E05, 0.1543E04, 0.2079E02, 0.117.

$G_{22}(s)$

0.0, 0.9447E-11, 0.5777E-08, 0.1265E-05, 0.1399E-03, 0.9491E-02, 0.4709, 0.1756E02, 0.4057E03, 0.2775E04, -0.1018E06, -0.2746E07, -0.4937E08, 0.6485E10, -0.2401E13, 0.1693E11, -0.3509E10, -0.3545E10, -0.2954E10, -0.1755E10, -0.7459E09, -L.225E09, -0.4734E08, -0.6732E07, -0.614E06, -0.3274E05, -0.8627E03, -0.8408E01, 0.0, 0.0, 0.0, 0.0, 0.0

$G_{23}(s)$

0.0, -0.3678E-09, -0.4642E-06, -0.246E-03, -0.71E-01, -0.1218E02, -0.1289E04, -0.8537E05, -0.3572E07, -0.9483E08, -0.1612E10, -0.1795E11, -0.1351E12, -0.4955E12, -0.9736E14, 0.3514E15, -0.2175E14, -0.1932E14, -0.2146E14, -0.179E14, -0.1138E14, -0.5526E13, -0.2046E13, -0.5757E12, -0.1222E12, -0.1934E11, -0.7245E10, -0.1864E09, -0.1071E08, -0.4073E06, -0.9626E04, -0.1267E03, -0.7048

$G_{24}(s)$

0.0, 0.1319E-08, 0.1505E-05, 0.697E-03, 0.1707, 0.2426E02, 0.2081E04, 0.1087E06, 0.3428E07, 0.649E08, 0.765E09, 0.5864E10, 0.2671E11, 0.1647E13, -0.6802E15, 0.7002E12, 0.6923E12, 0.7103E12, 0.5455E12, 0.3177E12, 0.1408E12, 0.4748E11, 0.1215E11, 0.2344E10, 0.3376E09, 0.3575E08, 0.2724E07, 0.145E06, 0.5167E04, 0.1159E03, 0.1465E01, 0.7887E-02, 0.0

$G_{25}(s)$

0.0, -0.8514E-08, -0.7843E-05, -0.2845E-02, -0.5382, -0.5872E02, -0.3869E04, -0.1578E06, -0.4032E07, -0.6502E08, -0.6731E09, -0.4602E10, 0.2942E10, -0.9903E13, 0.4232E16, 0.2371E12, -0.8165E10, 0.139E12, 0.2048E12, 0.1659E12, 0.8732E11, 0.3115E11, 0.7551E10, 0.1216E10, 0.1236E09, 0.7084E07, 0.1657E06, -0.1856E04, -0.1493E03, -0.1804E01, 0.0, 0.0, 0.0

$G_{31}(s)$

0.0, -0.3542E-09, -0.4938E-06, -0.2844E-03, -0.8755E-01, -0.1583E02, -0.1743E04, -0.1182E06, -0.4972E07, -0.1304E09, -0.2159E10, -0.231E11, -0.164E12, -0.1001E13, 0.8574E14, 0.368E15, -0.2136E14, -0.1728E14, -0.185E14, -0.1503E14, -0.9432E13, -0.457E13, -0.1705E13, -0.4872E12, -0.1055E12, -0.171E11, -0.2034E10, -0.173E09, -0.1016E08, -0.3934E06, -0.9438E04, -0.1257E03, -0.7053

$G_{32}(s)$

0.0, -0.6762E-10, -0.8127E-07, -0.3692E-04, -0.8593E-02, -0.1125E01, -0.839E02, -0.3362E04, -0.563E05, 0.299E06, 0.2244E08, 0.3119E09, 0.214E10, 0.2972E11, -0.8014E13, -0.7427E11, 0.6485E11, 0.5868E11, 0.4081E11, 0.2079E11, 0.7756E10, 0.2096E10, 0.4022E09, 0.5306E08, 0.4558E07, 0.2323E06, 0.5947E04, 0.5696E02, 0.0, 0.0, 0.0, 0.0, 0.0

$G_{33}(s)$

0.0, 0.3156E-09, 0.4736E-06, 0.2766E-03, 0.851E-01, 0.1531E02, 0.1674E04, 0.1126E06, 0.4677E07, 0.1206E09, 0.1494E10, 0.2016E11, 0.1365E12, 0.7645E12, -0.7121E14, 0.3544E15, -0.1337E13, 0.79E13, 0.6111E13, 0.3205E13, 0.856E12, -0.154E12, -0.2676E12, -0.1342E12, -0.4078E11, -0.833E10, -0.1171E10, -0.1127E09, -0.7245E07, -0.2995E06, -0.7525E04, -0.1036E03, -0.5954

Table 9B iii

$G_{34}(s)$

0.0, 0.2127E-08, 0.2707E-05, 0.1418E-02, 0.3964, 0.644E02, 0.6208E04, 0.3524E06, 0.1155E08, 0.2172E09, 0.2443E10, 0.173E11, 0.7667E11, 0.2054E13, -0.7936E15, 0.406E12, 0.1348E13, 0.1295E13, 0.9604E12, 0.5473E12, 0.2405E12, 0.8147E11, 0.2119E11, 0.4192E10, 0.6224E09, 06804E08, 0.5341E07, 0.2915E06, 0.1059E05, 0.2411E03, 0.3083E01, 0.168E-01, 0.0

$G_{35}(s)$

0.0, -0.3546E-08, -0.2497E-05, -0.735E-03, -0.1293, -0.1666E02, -0.1641E04, -0.1115E06, -0.4752E07, -0.1258E09, -0.2139E10, -0.2353E11, -0.133rE12, -0.1412E14, 0.5677E16, -0.7158E13, -0.6702E13, -0.701E13, -0.5281E13, -0.291E13, -0.1172E13, -0.3417E12, -0.706E11, -0.9953E10, -0.8941E09, -0.4406E08, -0.629E06, 0.3366E05, 0.1429E04, 0.1542E02, 0.0, 0.0, 0.0

$G_{41}(s)$

0.0, -0.1732E-09, -0.1778E-06, -0.7684E-04, -0.1838E-01, -0.2692E01, -0.2475E03, -0.1419E05, -0.503E06, -0.1109E08, -0.1581E09, -0.1517E10, -0.9235E10, -0.4221E12, 0.1646E15, 0.4146E14, -0.178E13, -0.114E13, -0.1288E13, -0.1096E13, -0.7195E12, -0.3642E12, -0.1418E12, -0.422E11, -0.9508E10, -0.1599E10, -0.1968E09, -0.1725E08, -0.1039E07, -0.4106E0t, -0.9996E03, -0.1346E02, -0.7615E-01

$G_{42}(s)$

0.0, 0.6863E-11, 0.2154E-08, 0.2819E-06, 0.2139E-04, 0.1125E-02, 0.4427E-01, 0.1239E01, 0.1905E02, -0.4818E02, -0.8405E04, -0.2895E05, -0.5132E08, 0.1961E11, -0.7549E13, -0.5244E08, -0.8146E08, -0.7828E08, -0.5106E08, -0.2246E08, -0.6557E07, -0.1233E07, -0.1415E06, -0.9004E04, -0.2647E03, -0.275E01, -0.1391E-05, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0

$G_{43}(s)$

0.0, -0.2638E-09, -0.2529E-06, -0.9955E-04, -0.2069E-01, -0.2504E01, -0.1873E03, -0.9E04, -0.286E06, -0.6101E07, -0.88E08, -0.8634E09, -0.6658E10, 0.3572E12, -0.1832E15, -0.1497E12, -0.3903E12, -0.495E12, -0.4633E12, -0.3193E12, -0.1619E12, -0.6019E11, -0.1631E11, -0.3187E10, -0.4422E09, -(.4257E08, -0.2752E07, -0.1145E06, -0.2892E04, -0.3997E02, -0.2305, 0,0, 0.0

$G_{44}(s)$

0.0, 0.1708E-08, 0.1781E-05, 0.7675E-03, 0.1813, 0.2586E02, 0.228E04, 0.1231E06, 0.4E07, 0.777E08, 0.9396E09, 0.7411E10, 0.3164E11, 0.3524E13, -0.149E16, 0.1214E13, 0.1064E13, 0.116E13, 0.9505E12, 0.594E12, 0.2841E12, 0.1039E12, 0.2899E11, 0.6122E10, 0.9677E09, 0.1126E09, 0.9413E07, 0.5465E06, 0.2102E05, 0.5024E03, 0.6677E01, 0.3744E-01, 0.0

$G_{45}(s)$

0.0, -0.2018E-07, -0.2064E-04, -0.8682E-02, -0.195E01, -0.2524E03, -0.1937E05, -0.8973E06, -0.2541E08, -0.4472E09, -0.505E10, -0.3794E11, -0.1414E12, -0.2289E14, 0.9517E16, -0.3181E13, -0.4544E13, -0.4635E13, -0.3535E13, -0.2018E13, -0.8612E12, 10.2735E12, -0.6415E11, -0.1098E11, -0.135E10, -0.1165E09, -0.6859E07, -0.2644E06, -0.6294E04, -0.8313E02, -0.4631, 0.0, 0.0

TABLE 9C i

$$F(s) = \sum_{i=1}^{33} K_i(s+p_i),$$ where K_i is the residue matrix of the pole $-p_i$

-0.213E-03 0.213E-03 0.213E-03 -0.125E-06 0.204E-06

0.0	0.0	0.0	0.0	0.0	
0.0	0.0	0.0	0.0	0.0	residue matrix of
0.0	0.0	0.0	0.0	0.0	1/(s+0.0)
-0.804E-07	-0.923E-06	-0.126E-06	0.312E-07	-0.173E-05	
0.336E-07	0.386E-07	0.527E-07	-0.13E-07	0.723E-06	residue matrix of
-0.217E-06	-0.25E-05	-0.341E-07	0.842E-07	-0.467E-05	1/(s+0.391E02)
0.854E-11	0.981E-10	0.134E-10	-0.331E-11	0.184E-09	
0.107E-13	0.688E-17	0.152E-13	-0.345E-08	-0.717E-07	
-0.933E-11	-0.599E-14	-0.132E-10	0.301E-05	0.625E-04	residue matrix of
0.69E-10	0.443E-13	0.977E-10	-0.222E-04	-0.462E-03	1/(s+0.427E02)
-0.207E-14	-0.133E-17	-0.293E-14	0.668E-09	0.139E-07	
0.509E-06	-0.382E-08	0.706E-06	-0.143E-07	0.616E-07	
0.354E-03	-0.266E-05	0.491E-03	-0.995E-05	0.428E-04	residue matrix of
-0.233E-02	0.167E-04	-0.309E-02	0.627E-04	-0.27E-03	1/(s+0.256E02)
0.232E-06	-0.174E-08	0.322E-06	-0.653E-08	0.281E-07	
0.187E-17	0.139E-18	0.24E-17	-0.483E-12	-0.477E-11	
0.489E-15	0.363E-16	0.627E-15	-0.126E-09	-0.125E-08	residue matrix of
-0.132E-14	-0.98E-16	-0.169E-14	0.34E-09	0.336E-08	1/(s+0.282E02)
-0.54E-15	-0.401E-16	-0.692E-15	0.139E-09	0.138E-08	
-0.456E-05	-0.42E-07	-0.688E-05	0.281E-06	0.12E-04	
0.693E-02	0.639E-04	0.105E-01	0.427E-03	-0.182E-01	residue matrix of
-0.364E-01	-0.336E-03	-0.549E-01	-0.225E-02	0.958E-01	1/(s+0.809E01)
0.105E-03	0.964E-06	0.158E-03	0.645E-05	-0.275E-03	
-0.631E-11	0.593E-14	0.785E-11	-0.859E-07	-0.163E-11	
-0.148E-07	0.139E-09	0.184E-06	-0.201E-02	-0.381E-07	residue matrix of
0.714E-06	-0.67E-09	-0.887E-06	0.971E-02	0.184E-06	1/(s+0.653E01)
-0.129E-05	0.121E-08	0.16E-05	-0.176E-01	10.333E-06	
-0.553E-10	-0.166E-13	-0.167E-10	0.21E-08	0.257E-09	
0.961E-07	0.288E-10	0.289E-07	-0.365E-05	-0.446E-06	residue matrix of,
-0.386E-06	-0.116E-09	-0.116E-06	0.146E-04	0.179E-05	1/(s+0.472E01)
-0.542E-01	-0.163E-04	-0.163E-01	0.206E01	0.252	
-0.818E-09	0.91E-12	0.778E-09	-0.499E-05	-0.602E-09	
0.157E-05	-0.174E-08	-0.149E-05	0.956E-02	0.115E-05	residue matrix of
-0.481E-05	0.535E-08	0.457E-05	-0.293E-01	-0.354E-05	1/(x+0.469E01)
-0.341E-03	0.38E-06	0.325E-03	-0.208E-01	-0.251E-03	
0.427E-08	0.777E-11	-0.816E-08	0.266E-04	-0.277E-03	
-0.111E-05	0.202E-08	0.212E-05	-0.691E-02	0.719E-01	residue matrix of
0.398E-05	-0.725E-08	-0.761E-05	0.248E-01	-0.258	1/(s+0.411E01)
0.62E-05	-0.113E-07	-0.118E-04	0.386E-01	-0.402	

Table 9C ii

0.131E-06	-0.269E-05	-0.182E-06	-0.293E-04	0.762E-04	
0.717E-06	-0.148E-04	-0.998E-06	-0.161E-03	0.418E-03	residue matrix of
0.22E-04	-0.453E-03	-0.307E-04	-0.494E-02	0.128E-01	1/(s+0.333)
0.362E-07	-0.745E-06	-0.504E-07	-0.811E-05	0.211E-04	
0.648E-04	0.496E-04	-0.719E-04	0.111E-03	0.625E-02	
-0.224E-05	0.65E-04	0.24E-05	0.12E-03	0.188E-02	residue matrix of
-0.362E-04	0.666E-02	0.311E-04	0.118E-01	0.207	$s/(s^2+0.272s+$
0.76E-06	0.221E-05	-0.845E-06	0.155E-05	0.124E-03	0.184E-01)
0.913E-05	0.652E-05	-0.101E-04	-0.164E-04	0.865E-03	
-0.825E-07	0.868E-05	0.798E-07	0.154E-04	0.267E-03	residue matrix of
0.183E-04	0.89E-03	-0.215E-04	0.15E-02	0.293E-01	$1/(s^2+0.272s+$
0.113E-06	0.294E-06	-0.125E-06	0.171E-06	0.174E-04	0.184E-01)
0.822E-03	-0.233E-03	-0.868E-03	0.165E-02	-0.858E-03	
0.242E-04	-0.685E-05	-0.256E-04	0.487E-04	-0.253E-04	residue matrix of
0.448E-02	-0.127E-02	-0.473E-02	0.902E-02	-0.468E-02	1/(s+0.926E-01)
0.621E-06	-0.176E-06	-0.655E-06	0.125E-05	-0.648E-06	
0.289E-04	0.364E-06	-0.286E-04	-0.156E-03	0.255E-03	
0.171E-03	0.216E-05	-0.17E-03	-0.926E-03	0.151E-02	residue matrix of
0.146E-02	0.184E-04	-0.145E-02	-0.79E-02	0.129E-01	1/(s+0.223E-01)
-0.159E-04	-0.201E-06	0.158E-04	0.862E-04	-0.141E-03	
0.349E-04	0.482E-06	0.345E-04	0.183E-03	-0.449E-03	
-0.108E-02	-0.15E-04	0.107E-02	0.57E-02	-0.139E-01	residue matrix of
-0.342E-02	-0.472E-04	0.338E-02	0.18E-01	-0.44E-01	1/(s+0.163E-01)
0.835E-04	0.115E-05	-0.826E-04	-0.439E-03	0.107E-02	
-0.123E-07	-0.605E-09	-0.225E-06	0.363E-06	0.434E-06	
-0.396E-05	-0.194E-06	-0.723E-04	0.116E-03	0.139E-01	residue matrix of
0.133E-05	0.653E-07	0.243E-04	-0.392E-04	-0.469E-02	1/(s+0.946E-02)
-0.125E-03	-0.611E-05	-0.228E-02	0.367E-02	0.439	
0.109E-03	-0.554E-05	-0.987E-04	-0.553E-03	0.139E-03	
0.441E-02	-0.225E-03	-0.401E-02	-0.225E-01	0.565E-02	residue matrix of
-0.146E-01	0.746E-03	0.133E-01	0.744E-01	-0.187E-01	1/(s+0.736E-02)
-0.443E-02	0.226E-03	0.403E-02	0.226E-01	-0.568E-02	
0.206E-05	-0.462E-05	0.252E-04	0.355E-04	-0.328E-03	
-0.376E-02	0.554E-05	-0.403E-02	0.216E-01	0.215	residue matrix of
-0.469E-03	0.545E-03	-0.322E-02	-0.289E-02	0.517E-01	$s/(s^2+0.969E-02s+$
0.427E-02	0.943E-03	-0.142E-03	-0.343E-01	-0.201	0.26E-04)
0.374E-07	-0.196E-07	0.138E-06	-0.141E-07	-0.303E-05	
-0.197E-04	-0.173E-05	-0.123E-04	0.131E-03	0.104E-02	residue matrix of
-0.561E-05	0.221E-05	-0.17E-04	0.956E-05	0.421E-03	$1/(s^2+0.969E-02s+$
0.169E-04	0.62E-05	-0.129E-04	-0.161E-03	-0.679E-03	0.26E-04)
-0.267E-04	0.253E-04	-0.724E-04	0.809E-04	-0.231E-02	
0.827E-03	-0.785E-03	0.225E-02	-0.251E-02	0.716E-01	residue matrix of
0.391E-02	-0.371E-02	0.106E-01	-0.119E-01	0.338	1/(s+0.282E-02)
0.etrE-02	-0.336E-02	0.961E-02	-0.107E-01	0.306	
0.195E-04	-0.698E-05	0.368E-04	-0.149E-03	0.212E-02	
-0.285E-02	0.102E-02	-0.565E-02	0.219E-01	-0.311	residue matrix of
-0.329E-02	0.118E-02	-0.653E-02	0.253E-01	-0.36	1/(s+0.223E-02)
-0.613E-02	0.22E-02	-0.122E-01	0.47E-01	-0.669	

Table 9C iii

-0.176E-03	0.271E-06	-0.569E-03	-0.3665-04	0.832E-03	
-0.379E-01	0.583E-04	-0.122	-0.787E-02	0.179	residue matrix of
0.115	-0.177E-03	0.373	0.239E-01	-0.545	1/(s+0.344E01)
0.544E-02	-0.838E-05	0.176E-01	0.113E-02	-0.258E-01	
0.266E-03	-0.616E-06	0.368E-03	0.166E-04	-0.273E-03	
0.144	0.139E-03	-0.556	0.563E-02	-0.122	residue matrix of
-0.65	0.156E-02	-0.981	-0.411E-01	0.671	$s/(s^2+5s+6.53)$
-0.324E-02	-0.101E-03	0.168	0.572E-03	-0.261E-02	
0.546E-03	-0.773E-06	-0.31E-04	0.307-04	-0.534E-03	
0.176	0.125E-02	-0.241E01	-0.853E-03	-0.901E-01	residue matrix of
-0.135E01	0.207E-02	-0.184	-0.769E-01	0.133E01	$1/(s^2+5s+6.53)$
0.207E-01	-0.373E-03	0.548	0.362E-02	-0.392E-01	
-0.351E-(8	0.243E-10	-0.259E-01	0.151E-08	0.652E-01	
-0.211E-05	0.146E-07	-0.156E-04	0.909E-06	0.392E-04	residue matrix of
-0.255E-05	0.177E-07	-0.189E-04	0.11E-05	0.474E-04	1/s+0.167E01)
-0.224E-01	0.155E-03	-0.165	0.964E-02	0.416	
-0.988E-06	-0.173E-06	0.541E-05	-0.154E-03	-0.638E-0e	
-0.672E-03	-0.118E-03	0.368E-02	-0.105	-0.434	residue matrix of
-0.186E-02	-0.326E-03	0.102E-01	-0.29	-0.12E01	1/(s+0.141E01)
-0.244E-03	-0.427E-04	0.134E-02	-0.38E-01	-0.158	
0.601E-05	0.982E-07	-0.205E-04	-0.928E-04	0.188E-03	
0.767E-02	0.125E-03	-0.262E-01	-0.118	0.241	residue matrix of
0.445E-01	0.728E-03	-0.152	-0.688	0.14E01	1/(s+0.115E01)
0.129E-02	0.211E-04	-0.44E-02	-0.199E-01	0.405E-01	
0.579E-08	0.238E-09	-0.256E-07	0.273E-03	0.343E-06	
0.454E-05	0.186E-06	-0.2E-04	0.214	0.269E-03	residue matrix of
0.199E-04	0.815E-06	-0.876E-04	0.936	0.118E-02	1/(s+0.13E01)
0.496E-06	0.203E-07	-0.219E-05	0.234E-01	0.294E-04	
0.408E-04	-0.248E-04	-0.932E-04	0.196E-03	-0.975E-03	
0.233E-04	-0.141E-04	-0.531E-04	0.112E-03	-0.556E-03	residue matrix of
-0.1E-02	0.609E-03	0.228E-02	-0.481E-02	0.239E-01	1/(s+0.844)
0.228E-05	-0.138E-05	-0.521E-05	0.11E-04	-0.545E-0r	
-0.135E-03	-0.421E-04	0.138E-03	0.802E-03	0.141E-02	
0.317E-02	-0.531E-03	-0.9E-02	-0.353E-02	0.123	residue matrix of
-0.433E-01	-0.133E-01	0.45E-01	0.255	0.431	$S/(s^2+1.27s+0.406)$
0.391E-03	-0.213E-04	-0.942E-03	-0.88E-03	0.106E-01	
-0.14E-03	-0.323E-04	0.186E-03	0.717E-03	0.304E-04	
0.313E-03	-0.522E-03	-0.267E-02	0.438E-02	0.604E-01	residue matrix of
-0.444E-01	-0.102E-01	0.593E-01	0.227	0.883E-01	$1/(s^2+1.27s+0.406)$
0.125E-03	-0.271E-04	-0.379E-03	-0.77E-04	0.548E-02	
-0.767E-03	0.49E-04	0.101E-02	-0.198E-02	-0.522E-02	
-0.364E-02	0.232E-03	0.477E-02	-0.94E-02	-0.248E-01	residue matrix of
-0.118	0.752E-02	0.155	-0.304	-0.803	1/s+0.321)
-0.181E-03	0.116E-04	0.238E-03	-0.468E-03	-0.123E-02	

TABLE 9D i

Numerator zeroes of elements of G(s) (close approximation)

$G_{11}(s)$

0.0, -4.627, -1.147, -1.295, -0.006632, -1.721, 0.04159, -0.1203, -0.4699, -0.1428, -24.82, -2.123, -0.7636, 0.007292, -3.999, -0.8932, -0.3274, -39.5, 0.1674, -0.003802, -5.732, -0.6055, -1.529, -0.002951, -0.00693, -0.04, -3.822, -42.52

$G_{12}(s)$

0.0, 0.5982, -0.04045, -41.94, -4.035, -202.2, -3.611, -0.693, -3.723, -2.79, -1.391, -18.63, -1.943, -4.848, -11.59, -4.645, -0.7365, -2.705, -6.446, -0.512, -29.39, -1.992, -0.03421, -0.009386, -0.003443, -0.003996, -0.002109, -0.03907, -0.2419, -0.1521, -1.254, -0.01

$G_{13}(s)$

0.0, -42.64, -4.045, -0.0756, -0.0083, -1.358, -0.4924, -0.1075, -3.054, -28.16, -0.1332, -25.21, -0.414, -0.004372, -4.654, -1.423, -0.3265, -41.75, -6.411, -0.6078, -0.008998, 1.149, -0.5061, -0.1307, -0.004298, -1.203, -1.267, -0.0005168

$G_{14}(s)$

38.46, -4.328, -0.3336, -0.005112, -24.93, -9.3, -1.663, -0.03262, -24.93, -4.715, -0.8222, -35.66, -3.503, -0.8222, -0.005962, -35.66, -3.503, -1.027, -0.1438, -28.42, -1.823, -0.97, -0.007306, -6.826, -1.823, -0.007561, -26.14, -1.844, -0.128, -0.004255

$G_{15}(s)$

-6.645, -0.9839, -0.1384, -0.007463, -10.37, -4.715, -0.9839, 38.63, -6.526, -1.05, -0.02095, -62.32, -4.687, -0.4818, -0.01937, -27.65, -2.312, -1.303, -0.01937, -42.78, -2.312, -1.303, -0.01937, -42.78, -2.312, -0.001043, -25.43, -1.674, -0.3334, 0.006097, -5.261, -0.9839, -0.003448

$G_{21}(s)$

0.0, -41.99, -3.133, -0.05767, -0.01068, -4.877, -0.0008048, -0.1079, -3.146, -27.73, -5.732, -0.04645, -11.62, -1.309, -0.5209, -0.337, -15.9, -1.788, -1.412, -0.1161, -4.217, -6.247, -0.9868, -0.005218, -39.27, -0.8835, 0.1407, -0.008794, -5.732, -1.085, -1.268, 0.002051

$G_{22}(s)$

0.0, -0.1514, -0.19, -0.007068, 1.7, -0.779, -0.138, -4.327, -1.675, -5.114, -0.9935, -0.03359, -3.723, 0.003218, -0.3243, -0.002639, -1.548, -4.73, -0.09487, -11.59, -7.156, 0.02442, -0.0298, -3.153, -3.916, -1.281, -0.005783

$G_{23}(s)$

0.0, -42.66, -1.799, -1.6‥, -0.003291, -0.6409, -0.602, -0.1071, -0.01131, -28.13, -5.215, -0.1293, -6.969, -1.152, -0.8428, -0.3541, -24.43, -0.4153, -0.006946, 0.01769, -3.322, -4.038, -4.717, -0.008976, -39.23, 0.2443, -1.281, -0.02989, -5.961, -5.887, -0.03291, -0.004448

$G_{24}(s)$

0.0, -34.97, -4.833, -0.02846, -0.006325, -3.317, -4.4, -0.3368, -25.12, -1.308, -0.02242, -6.404, -2.666, -0.2897, -0.1247, -28.02, -7.387, -0.9438, -0.1046, -7.339, -4.393, -1.702, -0.009932, -2.485, -0.3592, 0.0239, -0.00674, -4.124, 0.9273, -0.8378, -0.00358

$G_{25}(s)$

0.0, -1.329, -0.1232, -0.005688, -7.61, -0.3304, -4.111, -23.52, -0.9202, -0.03417, -42.67, 0.007853, -0.6578, -0.02286, -28.18, -0.6187, -0.0873, -0.06725, -2.147, -2.376, -0.03157, -25.79, -4.611, -0.7003, -0.001677, -19.02, -0.5653, -1.314, -0.02457.

Table 9D ii

$G_{31}(s)$

0.0, -28.35, -1.541, -0.3308, -0.03008, -1.934, -3.908, -4.702, -1.296, -41.13, 0.1027, -0.0494, -4.08, 2.085, -1.626, -0.046, -2.253, -22.66, -0.7845, -0.1485, -0.06227, -7.55, -0.03506, -0.01368, -42.71, -2.302, -0.676, -0.017, -6.501, -4.88, -0.707, -0.006303

$G_{32}(s)$

0.0, -3.388, -4.154, -0.2338, -0.009999, 0.708E-08, -1.308, -1.327, -1.453, 1.758, -0.3247, -0.005903, -1.281, 0.001692, -3.385, -4.283, -0.07821, -3.723, -4.21, -0.6834, -0.007821, 0.7147, -0.4092, -0.009878, -6.567, -0.752, -0.002475

$G_{33}(s)$

0.0, -41.96, -1.406, -0.1038, -0.003288, 2.171, -1.382, -0.2717, -0.2245, -28.04, -0.5893, -0.08877, -11.62, -1.559, -4.812, -0.3312, -15.9, -6.447, 0.1543, -0.1272, -4.217, -0.6175, -1.334, -0.006425, -39.27, -4.448, -0.7635, -0.003844, -5.732, -4.052, -1.251, -0.0126

$G_{34}(s)$

0.0, -4.124, 1.25, -0.04358, -0.005773, -34.57, -4.336, -0.8507, -3.236, 0.06665, -0.1064, -39.12, -24.86, -1.632, -0.03109, -6.404, -4.453, -0.781, -0.05351, -28.02, -6.474, -0.1641, -0.03375, -7.339, -0.7308, -1.634, -0.0162, -2.485, -5.708, -0.3204, 0.003022

$G_{35}(s)$

0.0, -19.37, -6.503, -1.272, -0.004666, 0.7081E-08, -0.4031, -1.399, -0.06363, -1.453, -6.324, -0.7388, -0.0487, -4.111, -0.05335, -0.0592, -3.301, -4.684, -0.1298, -2.517, -0.5523, -0.01371, -2.147, -0.2463, -0.8996, -0.008908, -0.2891, -0.2987, -0.02512

$G_{41}(s)$

0.0, -39.27, -2.617, -0.1509, -0.004263, -1.427, -5.732, -5.695, -0.02621, -2.185, -42.0, -0.6025, -0.09574, -1.34, -0.396, -0.1212, -3.262, -28.07, -1.297, -0.3161, -11.37, -1.194, -0.8224, -0.04038, -15.9, -6.862, -0.08139, 0.001603, -4.217, -0.2528, -1.284, -0.01068

$G_{42}(s)$

0.0, -1.286, -0.01164, -1.427, -5.413, -3.916, -0.402, 2.36, -0.1514, 0.07835, -4.021, 0.5549, -0.779, -0.03303, -4.327, -0.005451, -3.74, -0.9932, -0.01987, -3.723, -0.5644, -0.3251, -0.04782, -1.155, 0.0101, -0.002929

$G_{43}(s)$

0.0, -5.702, -0.02845, -0.003977, -1.427, -5.961, 0.06283, -0.1039, -42.29, -1.8, -0.1296, -1.019, -5.202, -0.3473, -3.253, -28.12, -0.6174, 0.01152, -13.46, -1.152, -0.8427, -0.02709, -14.47, -4.077, -0.008727, -0.‥513, -88.69, -0.3765, -1.285, 0.002551

$G_{44}(s)$

0.0, -2.482, -3.036, -0.7223, -0.005116, -4.125, -0.9125, -0.05946, -46.45, -3.595, -0.3307, -4.021, 5.443, -1.332, -0.09587, -39.12, -22.11, -0.2619, -0.1193, -6.404, -11.34, -1.189, -0.02516, -28.02, 1.805, -0.3799, 0.00162, -7.338, -0.7031, -0.1232, -0.01117

$G_{45}(s)$

0.0, -2.529, -7.742, -1.298, -0.009371, -31.64, 0.04545, -0.04073, -8.281, -4.181, -0.3246, -6.773, -0.351, -0.1079, -4.111, -3.337, -1.125, -0.1311, -42.67, -25.61, -0.09794, -0.003452, -28.18, -0.2659, -0.8006, -0.02847, -2.147, -1.348, 0.05331, -0.0002369

$$\Lambda = \begin{bmatrix} a_{11}, \dots a_{1,k-1} & \Gamma b_1 & a_{1,k+1}, \dots a_{1n} \\ a_{21}, \dots & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{n1}, \dots a_{n,k-1} & \Gamma b_n & a_{n,k+1}, \dots a_{nn} \end{bmatrix}$$

The zeros of G(s) are the eigenvalues of Λ as $\Gamma \to \infty$. Various values of Γ ranging from 10^8 to 10^{20} were tried, and, it was found that the same spurious roots vary over a large range while the numerator zeros remain sensibly constant.

The transfer function, G(s), represented by equations (9. 5), (9.10) and the numerator zeros are given in tables 9B to 9D.

The time response of the model was computed from

$$x\{(k+1)T\} = \Phi(T)x(kT) + \Delta(T)u(kT) \qquad (9.13)$$

$$y\{(k+1)T\} = Cx\{(k+1)T\}$$

$\Phi(T)$ and $\Delta(T)$ were obtained recursively as

$$\Phi(t) = \exp(At) \quad , \quad t = T/2^n \quad , \quad \text{then } \Phi(2kt) = \Phi(kt)\Phi(kt)$$

$k = 1, 2, \dots$ and repeating n times, $\Phi(T)$ is obtained.

$$\Delta'(t) = \int_o^\infty \exp\left[A(t-\tau)\right] d\tau = A^{-1}\left[\Phi(T) - I\right] = t\sum_{n=o}^{\infty} A^n t^n/(n+1)!$$

then $\Delta'(2kt) = A^{-1}\left[\Phi(2kt) - I\right] = \Delta'(kT)\left[\Phi(kt) + I\right]$, $k = 1, 2, \dots$ and repeating n times, $\Delta'(T)$ is obtained. Finally, $\Delta(T) = \Delta'(T)B$.

The above procedure was employed to overcome initial overflow of machine, as T was chosen reasonably large (10 seconds), since the boiler has large time constants, to speed up computation. The time response of the boiler is shown in figures 9.2(a) and 9.2(b), following a 1% step increment to each input, with the other inputs kept constant.

Fig. 9.2(a) Open loop response of boiler model following step input signals

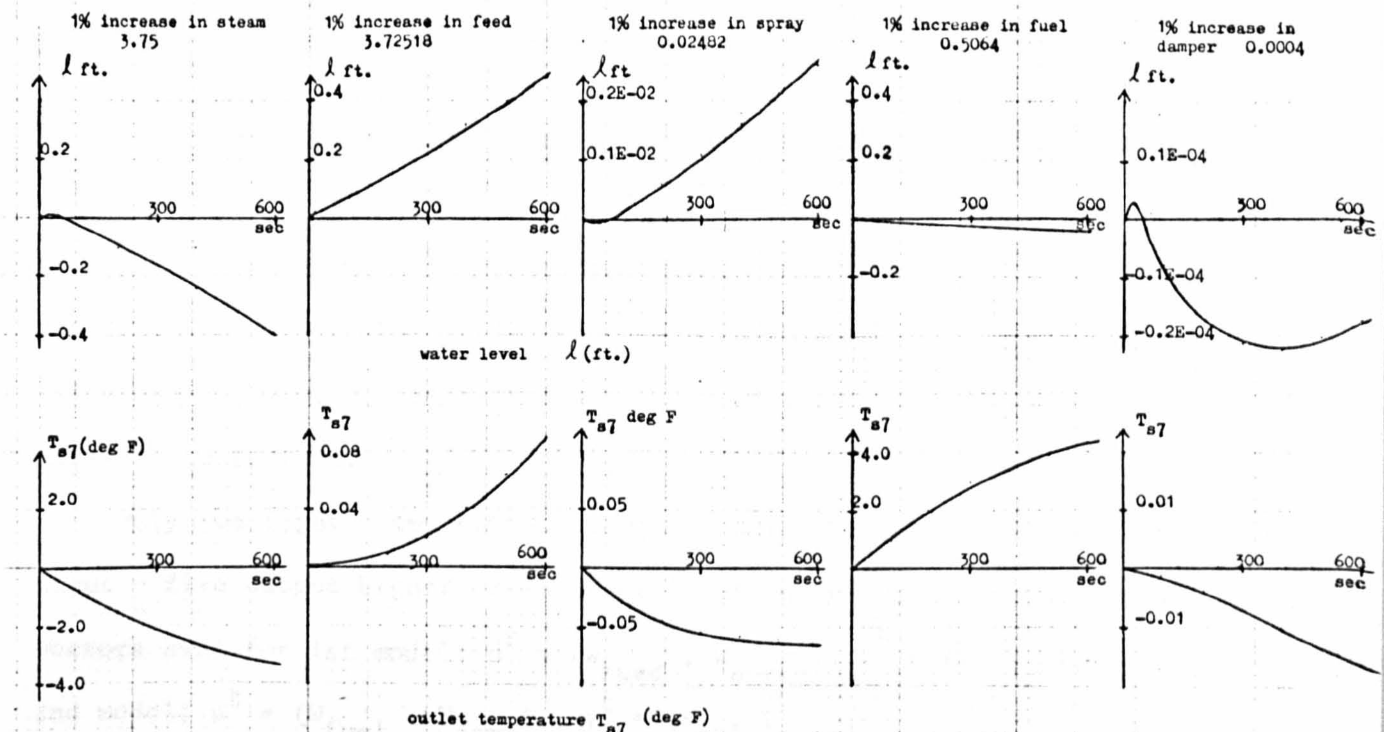
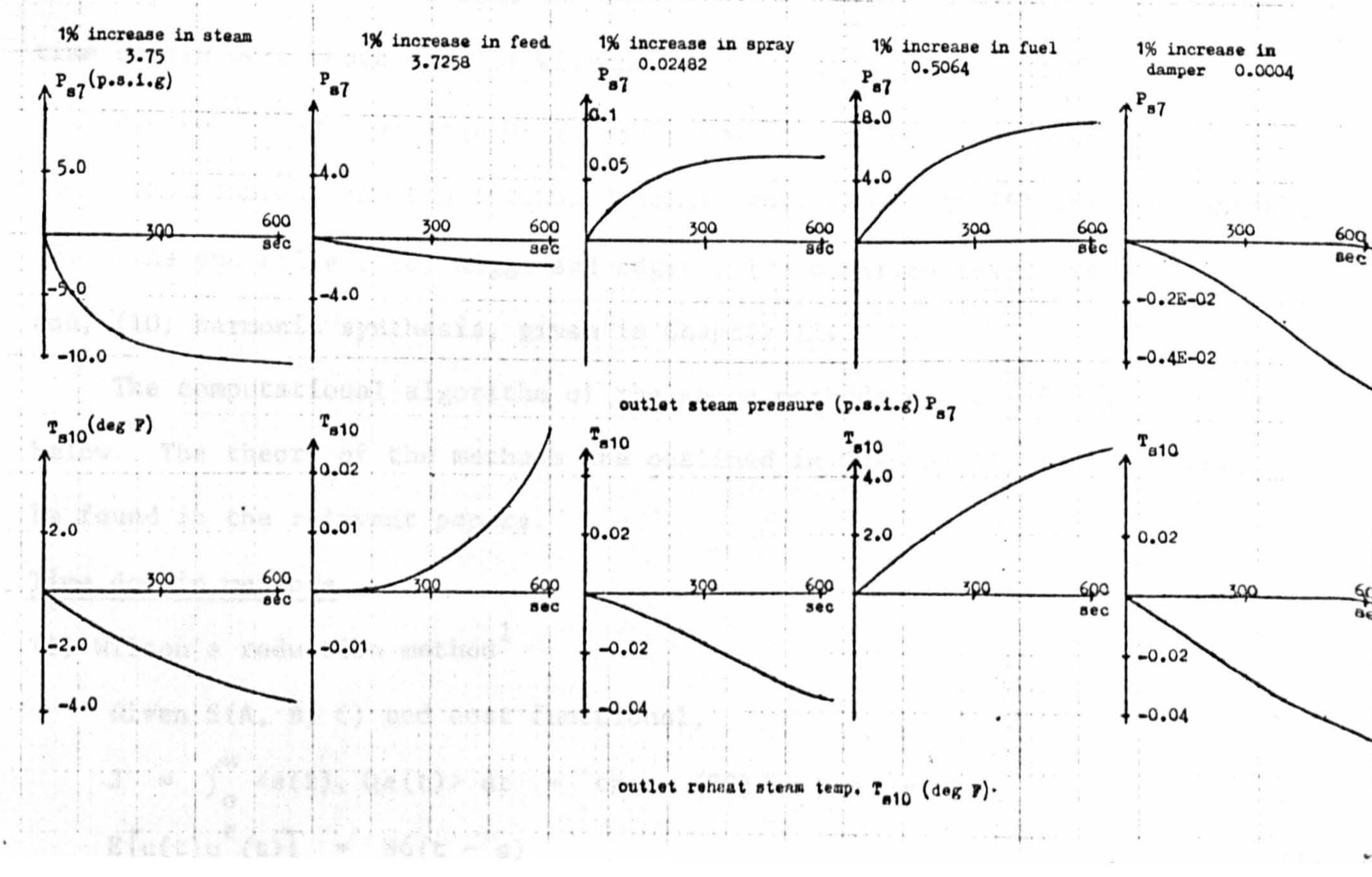


Fig. 9.2(b) Open loop response of boiler model following step input signals



9.2 Application of reduction techniques to obtain lower order models

Eight existing reduction techniques[1-8] plus two reduction techniques proposed by the author[17] were applied to the reduction of the 33rd order boiler model. In view of design, the order of the reduced models was kept as low as possible, in most cases a second order model was found to be both practical and feasible. Obtaining higher order models (say greater than tenth order) would be of little value, although, this may be a good exercise, to compare the accuracies and effectiveness of different reduction techniques. Thus, to facilitate design, stable second order models, where possible, were used.

Only two input - two output reduced models were obtained from the five input - five output higher order model. The corresponding input-output vectors are: for 1st model; $u^t = (W_{feed}, W_{spray})$, $y^t = (\ell, T_{s7})$, 2nd model; $u^t = (W_{fuel}, W_{damp})$, $y^t = (P_{s7}, T_{s10})$.

Out of the ten reduction techniques, five were chosen from the time domain and the other five from the complex or frequency domain. From the time domain were those of: (1) Wilson[1], (s) Mitra[2], (3) Marshall[3], (4) Anderson[4], and, (5) sequential approximation, given in Chapter IV. The methods chosen from the frequency domain were those of: (6) Chen and Shieh[5], (7) Sinha and Pille[6], (8) Riggs and Edgar[7], (9) modified Levy's method[8], and, (10) harmonic synthesis, given in Chapter III.

The computational algorithm of the above methods are briefly given below. The theory of the methods are outlined in Chapter II, and, can also be found in the relevant papers.

9.2.1 Time domain methods

(1) Wilson's reduction method[1]

Given S(A, B, C) and cost functional,

$$J = \int_0^\infty \langle e(t), Qe(t)\rangle \, dt = \text{trace}(PS) = \text{trace}(RM)$$

$$E\left[u(t)u^t(t)\right] = N\delta(t-s) \qquad (9.14)$$

with

$$F^tP + P^tF + M = 0 \tag{9.15}$$

$$FR + RF^t + S = 0 \tag{9.16}$$

where

$$F = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix} \quad M = \begin{bmatrix} C^tC & -C^tC_r \\ -C_r^tC & C_r^tC_r \end{bmatrix} \quad S = \begin{bmatrix} BNB^t & BNB_r^t \\ B_rNB^t & B_rNB_r^t \end{bmatrix}$$

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^t & R_{22} \end{bmatrix} \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^t & P_{22} \end{bmatrix}$$

$S_r(A_r,B_rC_r)$ is given by

$$A_r = \theta_1 A \theta_2 \tag{9.17}$$

$$B_r = \theta_1 B \tag{9.18}$$

$$C_r = C\theta_2 \tag{9.19}$$

where $\theta_1 = -P_{22}^{-1}P_{12}^t, \quad \theta_2 = R_{12}R_{22}^{-1}$

Algorithm

(i) Choose a canonical structure for A_r (eg companion form) with a fixed set of eigenvalues.

(ii) Choose initial matrix $B_r^{(1)}$ with pair $(A_r, B_r^{(1)})$ controllable.

(iii) Solve equation (9.16) for R, and, compute $C_r^{(1)}$ from equation (9.19).

(iv) Use pair $(A_r, C_r^{(1)})$ and solve equation (9.15) for P.

(v) Solve equation (9.18) for $B_r^{(2)}$ using P.

(vi) Go to step (iii) and solve $C_r^{(2)}$ from pair $(A_r, B_r^{(2)})$

(vii) Continue until a pair $(A_r, B_r^{(i)})$ yields a pair $C_r^{(i)}$, and, also the pair $(A_r, C_r^{(i)})$ yields the same $B_r^{(i)}$. The triple $(A_r, B_r^{(i)}, C_r^{(i)})$ is thus obtained, with cost, J, remaining constant.

(viii) Update eigenvalues of A_r using a minimization routine (i.e. hillclimbing)

(ix) If optimum obtained, exit, otherwise go to step (ii).

The models obtained are:

Input $\equiv (W_{feed}, W_{spray})$, Output $\equiv (\ell, T_s)$

$$A_r = \begin{bmatrix} -10^{-7} & 0 \\ 0 & -0.0028 \end{bmatrix} \qquad B_r = \begin{bmatrix} 0.323 & 0.0206 \\ 7.431 & -215.0 \end{bmatrix}$$

$$C_r = \begin{bmatrix} 0.0146 & -5.788E-05 \\ -0.131 & -3.617E-07 \end{bmatrix}$$

Input $\equiv$ (W_{fuel}, W_{damp}), Output $\equiv$ (P_{s7}, T_{s10})

$$A_r = \begin{bmatrix} -0.2235E-02 & 0 \\ 0 & -0.2815E-02 \end{bmatrix} \qquad B_r = \begin{bmatrix} 0.401 & 1.322E-03 \\ 0.198 & 1.371E-03 \end{bmatrix}$$

$$C_r = \begin{bmatrix} 0.561 & -0.6557 \\ 0.138E-03 & -3.73E-04 \end{bmatrix}$$

(2) Mitra's reduction method[2], algorithm

Given S(A, B, C) and

$$W = \int_0^\infty \exp\{A(\infty - t)\}BH^{-1}B^t \exp\{A^t(\infty - t)\}\, dt$$

(i) Solve for W, $AW + WA^t + BH^{-1}B^t = 0$ (9.20)

(ii) Consider m smallest eigenvalues of W, i.e. $\lambda_1 \leq \lambda_2 \leq \dots \lambda_m$ where n - m is the order of the reduced model. From (n x m) matrix $U \triangleq \{u_1, u_2, \dots u_m\}$ where u_i is the eigenvector associated with the eigenvalue λ_i.

Use Gram-Schmidt procedure to construct orthogonal matrix $T \triangleq \{t_1, t_2, \dots t_m\}$ from U, i.e.

$$s_r = u_r - \sum_{k=1}^{r-1} \langle t_k, u_r \rangle t_k \tag{9.21}$$

$$t_r = s_r / \|s_r\|, \text{ with } s_1 = u_1,\ r = 1, \dots m$$

(iii) Form projection matrix, $P = I - T(T^tW^{-1}T)^{-1}T^tW^{-1}$ (9.22)

$$\text{Compute } \hat{A} = PA = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \qquad \hat{B} = PB = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}$$

(iv) Solve non-trivial solutions of equation.

$$(T^tW^{-1}T)^{-1}T^tW^{-1}x = 0 \qquad (9.23)$$

(v) Let any (n - m) independent solutions of x form columns of matrix, F.

$$F = \begin{bmatrix} F_{11} \\ F_{21} \end{bmatrix}$$

where det $F_{11} \neq 0$ is required

det (CF) $\neq 0$ is assumed

(vi) Compute $S_r(A_r, B_r, C_r)$ from

$$A_r = (CF)(F_{11}^{-1}\hat{A}_{11}F_{11} + F_{11}^{-1}\hat{A}_{12}F_{21})(CF)^{-1} \qquad (9.24)$$

$$B_r = (CF)F_{11}^{-1}\hat{B}_1$$

$$C_r = I$$

The reduced models are:

Input $\equiv$ (W_{feed}, W_{spray}), Output $\equiv$ (ℓ, T_{s7})

$$A_r = \begin{bmatrix} -11.26 & 5.63 \\ -16.89 & 8.445 \end{bmatrix} \times 10^{-3}, \quad B_r = \begin{bmatrix} -3.706E-02 & 16.49 \\ -1.667 & -23.62 \end{bmatrix}$$

$$C_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Input $\equiv$ (W_{fuel}, W damp), Output $\equiv$ (P_{s7}, T_{s10})

$$A_r = \begin{bmatrix} -0.7732 & 2.358 \\ -0.1179 & 0.2879 \end{bmatrix} \times 10^{-2}, \quad B_r = \begin{bmatrix} 0.04385 & 61.35 \\ 0.0104 & 9.322 \end{bmatrix}$$

$$C_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Both Wilson and Mitra's methods require the solution of the Lyapunov equation,

$$A^tP + PA = -Q \qquad (9.25)$$

Many methods of solution to equation (9.25) are possible such as eigenvector and state transition matrix methods. However, due to the size of the A matrix, those methods are uneconomical. An iterative scheme[18], with fast convergence was employed to solve equation (9.25).

If A is a stability matrix, then

$$\alpha = (I + A^t)(I - A^t)^{-1} \tag{9.26}$$

is a convergent matrix.

Equation (9.26) transforms (9.25) into

$$P - \alpha P \alpha^t = M \tag{9.27}$$

with
$$M = \tfrac{1}{2}(\alpha + I)Q(\alpha^t + I) \tag{9.28}$$
$$= 2(I + A^t)^{-1}Q(I - A)^{-1}$$

Equation (9.27) can be written as

$$(I - \Psi)P = M$$

such that
$$P = (I - \Psi)^{-1}M = (I + \Psi + \Psi^2 + \ldots)M$$

$$\text{i.e. } P = M + \alpha M \alpha^t + \alpha^2 M(\alpha^t)^2 + \ldots \tag{9.29}$$

The infinite series in equation (9.29) converges, if α converges, i.e. if A is a stability matrix. Further, if λ_i, ζ_i are the eigenvalues of α and A respectively, then by equation (9.26)

$$\lambda_i = (1 + \zeta_i)/(1 - \zeta_i) \tag{9.30}$$

showing the convergence of P in equation (9.29) is fastest when $\zeta_i = -1$ and slowest for $|\zeta_i|$ very small or very large. The following modification, suggested by Smith[19], speeds up the convergence of equation (9.29). Expanding equation (9.28),

$$M = \tfrac{1}{2}(Q + \alpha Q + Q\alpha^t + \alpha Q \alpha^t) \tag{9.31}$$

and substituting into equation (9.29) yields

$$P = T - \tfrac{1}{2}Q + \tfrac{1}{2}(\alpha T + (\alpha T)^t) \tag{9.32}$$

where $T = Q + \alpha Q\alpha^t + \alpha^2 Q(\alpha^t)^2 + \ldots$ (9.33)

Equation (9.33) can be summed by the recursive relation until convergence occurs.

Set $T_1 = Q$

then $T_2 = T_1 + \alpha T_1 \alpha^t$

$$T_3 = T_2 + \alpha^2 T_2 (\alpha^t)^2$$

$$\vdots$$

$$T_{k+1} = T_k + \alpha^j T_k (\alpha^t)^j \qquad (9.34)$$

where $j = 2^k - 1$

In the above, the system matrix A is singular, and to convert A into a stability matrix, the last element in the zero column vector of A was slightly perturbed in the negative direction[16], i.e. from 0.0 to -0.1×10^{-10}. The algorithms of equations (9.32) to (9.34) solved P, in equation (9.25), very effectively, with the new 'value' of A.

(3) Marshall's reduction method[3], algorithm

(i) Rearrange state equation $\dot{x} = Ax + Bu$ such that eigenvalues λ_i of A associated with state x_i are in order of increasing moduli, $\lambda_i \leqslant \lambda_2 \leqslant \ldots \lambda_n$.

(ii) Obtain modal matrix U, and its inverse V, of A.

$$U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}, \quad V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}, \quad UV = I$$

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} = VAU$$

(iii) Partitioned A and B as

$$A = \begin{bmatrix} \overset{m\times m}{A_1} & A_2 \\ A_3 & \underset{(n-m)(n-m)}{A_4} \end{bmatrix}, \quad B = \begin{array}{c} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{array} \begin{array}{l} m\times \ell \\ (n-m)\times \ell \end{array}$$

(iv) Obtain $S_r(A_r, B_r)$ from

$$A_r = U_1\Lambda_1U_1^{-1}, \quad B_r = B_1 - A_2V_4^{-1}\Lambda_2^{-1}(V_3B_1 + V_4B_2)$$

A complex matrix, $U = W + jX$ whose inverse $Y + jZ$ is obtained as follows

$$(W + jX)(Y + jZ) = I$$

i.e. $WY - XZ = I$

$$XY + WZ = 0$$

from which $$Z = (X(W + X)^{-1}(W - X) - W)^{-1}X(W + X)^{-1} \tag{9.35}$$

$$Y = (W + X)^{-1}(I - (W - X)Z) \tag{9.36}$$

where $X(W + X)^{-1}(W - X) - W$ and $(W + X)$ are nonsingular if $(W + jX)$ is nonsingular. If some of the λ_i are real, then X is singular, or, if some λ_i exist in complex conjugate pairs, then W is singular, but W + X is not singular for both cases. Hence a real matrix inversion subroutine is required to obtain Z and Y.

The reduced models are:

Input $\equiv (W_{feed}, W_{spray})$, Output $\equiv (\ell, T_{s7})$

$$A_r = \begin{bmatrix} 0.1655 & -0.2483 \\ 0.331 & -0.4965 \end{bmatrix} x10^{-2}, \quad B_r = \begin{bmatrix} 0.2013 & -406.1 \\ 0.1342 & -492.1 \end{bmatrix} x10^{-5}$$

$$C_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Input $\equiv (W_{fuel}, W_{damp})$, Output $\equiv (P_{s7}, T_{s10})$

$$A_r = \begin{bmatrix} -3.665 & 0.2785 \\ -0.8356 & -2.342 \end{bmatrix} x10^{-3}, \quad B_r = \begin{bmatrix} 0.1303 & 39.17 \\ -0.9037 & 413.1 \end{bmatrix}$$

$$C_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(4) Anderson's reduction method[4], algorithm

(i) From S(A,B) compute , $x\{(k+1)T\} = \Phi(T)x(kT) + \Delta(T)u(kT)$

$$b_q^t = (x_q^{(r)}(T),\ x_q^{(r)}(2T),\ \ldots\ x_q^{(r)}\{(k+1)T\})$$

$$c_q^t = (\phi_{q1},\ \phi_{q2},\ \ldots\ \phi_{qr},\ \Delta_{q1})$$

where $x_q^{(r)}$ is the qth element of the vector, $x^{(r)}$, comprising of the first r elements of x, and, ϕ_{ij}, Δ_{ij} are elements of $\Phi_r(T)$, $\Delta_r(T)$ respectively.

(ii) Form $(k+1)x(r+\ell)$ matrix, $M = (m_1,\ m_2,\ \ldots\ m_r,\ m_{r+1},\ \ldots\ m_{r+\ell})$,

$$\det\ (M^tM) \neq 0$$

where $m_j^t = (x_j^{(r)}(0),\ x_j^{(r)}(T),\ \ldots\ x_j^{(r)}(kT)),\ \forall 1<j<r$

$$m_j^t = (u_j(0),\ u_j(T),\ \ldots\ u_j(kT)),\ \forall r<j<\ell$$

where ℓ is the number of inputs, and, k the number of sampling points.

(iii) Form Grammian, $G = (g_1,\ g_2,\ \ldots\ g_r,\ g_{r+1},\ \ldots\ g_{r+m})$ where

$$g_j^t = (\langle m_1,\ m_j\rangle,\ \langle m_2,\ m_j\rangle,\ \ldots\ \langle m_{r+\ell},\ m_j\rangle),\ \forall 1<j<r$$

$$g_{r+j}^t = (\langle m_1,\ m_{r+j}\rangle,\ \langle m_2,\ m_{r+j}\rangle,\ \ldots\ \langle m_{r+\ell},\ m_{r+j}\rangle),\ \forall r<j<\ell$$

(iv) Compute $v^t = (\langle b_q,\ m_1\rangle,\ \langle b_q,\ m_2\rangle,\ \ldots\ \langle b_q,\ m_{r+\ell}\rangle)$ for $q = 1,\ \ldots\ r$

$$c_i = \det\ (g_1,\ g_2,\ \ldots\ g_{i-1},\ v,\ g_{i+1},\ \ldots g_{r+\ell})/\det(g_1,\ g_2,\ \ldots\ g_{r+\ell})$$

$$i = 1,\ \ldots\ r+\ell$$

(v) Form vector $d_q^t = (c_1,\ c_2,\ \ldots\ c_{r+\ell})$

(vi) Set $q = q+1$, and go to step (iv)

(vii) Compute$\{\Phi_r(T),\Delta_r(T)\} = (I + X)Y = (d_1,\ d_2,\ \ldots\ d_r)^t$

(viii) Obtain $A_r = (1/T)\ln(I + X) = (1/T)\sum_{n=1}^{\infty} (-1)^{n-1}X^n/n$

provided $|\lambda_i(X)| < 1,\ \forall i$

$$B_r = Y(t\sum_{n=0}^{\infty} A^n t^n/(n+1)!)^{-1}$$

$S_r(A_r, B_r, C_r)$: Input $\equiv$ (W_{feed}, W_{spray}), Output $\equiv$ (ℓ, T_{s7})

$$A_r = \begin{bmatrix} -2.681 & 0.8936 \\ -1.34 & 0.4468 \end{bmatrix} x10^{-3} \qquad B_r = \begin{bmatrix} 2.479 & 196.0 \\ 0.292 & -36.27 \end{bmatrix} x10^{-4}$$

$$C_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Input $\equiv$ (W_{fuel}, W_{damp}), Output $\equiv$ (P_{s7}, T_{s10})

$$A_r = \begin{bmatrix} -5.715 & 1.74 \\ -5.8 & 0.665 \end{bmatrix} x10^{-3} \qquad B_r = \begin{bmatrix} 0.1225 & 6.978 \\ 0.08682 & 68.86 \end{bmatrix}$$

$$C_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(5) Sequential approximation method of chapter IV, algorithm

(i) Choose a canonical form for A_r

(ii) For $S(A, B, C)$, $S_r(A_r, B_r, C_r)$, considering each input and output separately,

$$y_j(kT) = \langle c_j, \nabla(kT)b_j \rangle u_j(kT)$$

$$y_j(kT) = \langle c_{rj}, \nabla_r(kT)b_{rj} \rangle u_j(kT)$$

i.e.

$$\langle c_{rj}, \nabla_r(T)b_{rj} \rangle = y(T)$$

$$\langle c_{rj}, \nabla_r(2T)b_{rj} \rangle = y(2T)$$

$$\vdots$$

$$\langle c_{rj}, \nabla_r(kT)b_{rj} \rangle = y(kT)$$

from which

$$b_{rj} = M^{+}q \tag{9.37}$$

(iii) $\langle b_{rj}, \nabla_r^t(T)c_{rj}\rangle = y(T)$

$\langle b_{rj}, \nabla_r^t(2T)c_{rj}\rangle = y(2T)$

$\vdots$

$\langle b_{rj}, \nabla_r^t(kT)c_{rj}\rangle = y(kT)$

from which

$$c_{rj} = N^{+}q \qquad (9.38)$$

(iv) Use pair (b_{rj}, c_{rj}) iteratively, till $\| b_{rj}^{(i)} - b_{rj}^{(i+1)} \| \simeq 0$ and $\| c_{rj}^{(i)} - c_{rj}^{(i+1)} \| \simeq 0$

(v) Set $j = j+1$, till $j = m$, and go to step (ii) till matrix pair (B_r, C_r) is obtained.

(vi) Modify A_r, if necessary and go to step (i) till suitable $S_r(A_r, B_r, C_r)$ is obtained.

$S_r(A_r, B_r, C_r)$: Input $\equiv (W_{feed}, W_{spray})$, Output $\equiv (\ell, T_{s7})$

$$A_r = \begin{bmatrix} -10^{-7} & 0 \\ 0 & -0.002235 \end{bmatrix} \quad B_r = \begin{bmatrix} 0.03891 & -164.2 \\ 0.00146 & 0.3574 \end{bmatrix}$$

$$C_r = \begin{bmatrix} 0.432 & 327.9 \\ 0.03196 & 0.6643 \end{bmatrix} \times 10^{-3}$$

Input $\equiv (W_{fuel}, W_{damp})$, Output $\equiv (P_{s7}, T_{s10})$

$$A_r = \begin{bmatrix} -0.2235 & .0 \\ 0 & -0.2815 \end{bmatrix} \times 10^{-2} \quad B_r = \begin{bmatrix} 0.3198 & -1.157E+04 \\ 0.2061 & -3.956E+03 \end{bmatrix}$$

$$C_r = \begin{bmatrix} 5610.0 & -6557.0 \\ 1.38 & -3.73 \end{bmatrix} \times 10^{-4}$$

9.2.2 Frequency Domain Methods

In frequency domain methods, the transfer function $G_r(s)$ was obtained element by element, i.e., $g_{r11}(s)$, $g_{r12}(s)$, ... $g_{rmm}(s)$; thus single input-single output reduction methods were considered. This was because, with some methods, it was far easier to obtain $G_r(s)$ in that way and also proved to be computationally more feasible.

(6) Chen and Shieh's reduction method[5] algorithm

(i) Consider

$$g_{ij}(s) = (A_{21} + A_{22}s + A_{23}s^2 + \dots A_{2n}s^{n-1}) \Big/ (A_{11} + A_{12}s + A_{13}s^2 + \dots A_{1,n+1}s^n)$$

(ii) Construct Routh table

$$\begin{array}{lllll}
A_{11} & A_{12} & A_{13} & \dots & A_{1,n+1} \\
A_{21} & A_{22} & A_{23} & \dots & A_{2n} \\
A_{31} & A_{32} & \dots & & \\
A_{41} & A_{42} & \dots & & \\
\vdots & & & &
\end{array}$$

$$A_{jk} = A_{j-2,k+1} - (A_{j-2,1}\, A_{j-1,k+1}) \Big/ A_{j-1,1}$$

$$j = 3, 4, \dots 2n+1; \; k = 1, 2, \dots n+1$$

(iii) Compute $h_j = A_{j1}/A_{(j+1)1}$, $j = 1, 2, \dots 2n$

Take h_j, $j = 1, \dots 2r$, i.e. take first $2r$ h_j's if rth order model required.

(iv) Compute $g_{rij}(s)$ from

$$\begin{bmatrix} h_1 & & & \\ 0 & h_2 & & 0 \\ 0 & 1 & h_2h_3 & \\ \vdots & & & \\ \cdot & & \cdot\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} A_{r11} \\ A_{r12} \\ A_{r13} \\ \vdots \end{bmatrix} = \begin{bmatrix} h_1 & & & \\ 1 & h_1h_2 & 0 & \\ 0 & h_1+h_3 & h_1h_2h_3 & \\ \vdots & & & \\ \cdot & \cdot\ \cdot & & \cdot \end{bmatrix} \begin{bmatrix} A_{r21} \\ A_{r22} \\ A_{r23} \\ \vdots \end{bmatrix}$$

For convenience, set $A_{r,1,r+1} = 1$, where $A_{r,1,r+1}$ is the coefficient of s^r.

(v) Go to step (i) for other elements till $G_r(s)$ is generated.

Input $\equiv$ (W_{feed}, W_{spray}); Output $\equiv$ (ℓ, T_{s7})

$g_{r11}(s) = (0.7646E-04s + 0.7694E-07) \,/\, (s^2 + 0.3911E-03s + 0.0)$

$g_{r12}(s) = (0.1853E-03s + 0.1778E-07) \,/\, (s^2 + 0.8347E-04s + 0.0)$

$g_{r21}(s) = (-0.4655E-04s + 0.4211E-06) \,/\, (s^2 + 0.4443E-02s + 0.5273E-05)$

$g_{r22}(s) = (-0.2267E-01s - 0.9635E-04) \,/\, (s^2 + 0.1667E-01s + 0.3099E-04)$

Input $\equiv$ (W_{fuel}, W_{damp}), Output $\equiv$ (P_{s7}, T_{s10})

$g_{r11}(s) = (0.8715E-01s + 0.2805E-03) \,/\, (s^2 + 0.9401E-02s + 0.156E-04)$

$g_{r12}(s) = (0.631E-02s - 0.1537E-03) \,/\, (s^2 + 0.4391E-02s + 0.5127E-05)$

$g_{r21}(s) = (0.2404E-01s + 0.1039E-03) \,/\, (s^2 + 0.5419E-02s + 0.7192E-05)$

$g_{r22}(s) = (-0.1951s - 0.2378E-02) \,/\, (s^2 + 0.8702E-02s + 0.1394E-04)$

(7) Sinha and Pille's reduction method[6], algorithm

(i) From $g_{ij}(z) = (a_o + a_1 z^{-1} + \dots a_m z^{-1})/(1 - b_1 z^{-1} - \dots b_n z^{-n})$

the sampled value of the output at ith instant can be written as

$$y_i = \sum_{j=o}^{m} a_j u_{i-j} + \sum_{j=1}^{n} b_j y_{i-j} \qquad (9.39)$$

where u_{i-j} is a sequence of unit step inputs.

(ii) Form equation (9.39) in matrix form

$$A_k \phi = \hat{y}_k \qquad (9.40)$$

where

$$\hat{y}_k = (y_o, y_1, \dots y_k)^t$$

$$\phi = (a_o, a_1, \dots a_m, b_1, b_2, \dots b_n)^t$$

$$A_k = \begin{bmatrix} u_o, u_{-1}, \dots u_{-m}, y_{-1}, y_{-2}, \dots y_{-n} \\ u_1, u_o, \dots u_{1-m}, y_o, y_1, \dots y_{1-n} \\ \cdot \\ \cdot \\ \cdot \\ u_k, u_{k-1}, \dots u_{k-m}, y_{k-1}, y_{k-2}, \dots y_{k-n} \end{bmatrix}$$

(iii) $\phi = A_k^{+}\hat{y}_k$ (9.41)

A_k^{+} exists if rank $(A_k) = \min\{m,n\}$ where m and n are its number of rows and columns. This condition is assured if the sequence of step inputs, $u_i = 0$ for $k < n$, $u_i = 1$, for $k > n$, Alternatively, to avoid matrix inversion in evaluating A_k^{+}, use the Albert and Sitter recursive algorithm below.

(iv) Let p be the number of equations represented by equation (9.40).

Define

$$A_{k+1} = \begin{bmatrix} A_k \\ a_{k+1}^t \end{bmatrix}, \quad a_{k+1}^t = (u_{k+1}, u_k, \dots u_{k-m+1}, y_k, y_{k-1}, \dots y_{k-n+1})$$

for $k \leqslant p$

$$\phi_{k+1} = \phi_k + \left[Q_k a_{k+1}(y_{k+1} - a_{k+1}^t \phi_k)\right] / a_{k+1}^t Q_k a_{k+1}$$

$$Q_{k+1} = Q_k - \left[Q_k a_{k+1}(Q_k a_{k+1})^t\right] / a_{k+1}^t Q_k a_{k+1}, \text{ with } \phi_o = 0,\ Q_o = I$$

for $k > p$

$$\phi_{k+1} = \phi_k + \left[P_k a_{k+1}(y_{k+1} - a_{k+1}^t \phi_k)\right] / (1 + a_{k+1}^t P_k a_{k+1})$$

$$P_{k+1} = P_k - \left[P_k a_{k+1}(P_k a_{k+1})^t\right] / (1 + a_{k+1}^t P_k a_{k+1}), \text{ with } P_o = 0$$

(v) Obtain $g_r(s) = Z^{-1}\{zg_r(z)/(z - 1)\}$ where Z^{-1} represents the inverse z-transform of $g_r(z)$ preceeded by a sample and zero order hold.

(vi) Go to step (i), and consider other elements till $G_r(s)$ is generated.

Input $\equiv$ (W_{feed}, W_{spray}), Output $\equiv$ (ℓ, T_{s7})

$g_{r11}(s)$ = $(0.2289E\text{-}03s + 0.5528E\text{-}06) / (s^2 + 0.614E\text{-}03s + 0.613E\text{-}09)$

$g_{r12}(s)$ = $(-0.8631E\text{-}05s + 0.4909E\text{-}06) / (s^2 + 0.7081E\text{-}04s + 0.7668E\text{-}10)$

$g_{r21}(s) = (-0.617E{-}04s + 0.6489E{-}06) / (s^2 + 0.735E{-}02s + 0.1325E{-}04)$

$g_{r22}(s) = (-0.2097E{-}01s - 0.8033E{-}04) / (s^2 + 0.1339E{-}01s + 0.2174E{-}04)$

Input $\equiv (W_{fuel}, W_{damp})$, Output $\equiv (P_{s7}, T_{s10})$

$g_{r11}(s) = (0.4516E{-}01s + 0.403E{-}03) / (s^2 + 0.1007E{-}01s + 0.1508E{-}04)$

$g_{r12}(s) = -(0.1718E{-}01s + 0.1353E{-}03)/ (s^2 + 0.5127E{-}02s + 0.6273E{-}05)$

$g_{r21}(s) = (0.3235E{-}01s + 0.5669E{-}04) / (s^2 + 0.503E{-}02s + 0.5959E{-}05)$

$g_{r22}(s) = (-0.4834s - 0.2891E{-}02) / (s^2 + 0.9621E{-}02s + 0.1789E{-}04)$

(8) Riggs and Edgar's reduction method[7]; algorithm

(i) From $y(t) = \sum_{j=1}^{n} r_j \exp(p_j(t))$, $y_r(t) = \sum_{j=1}^{r} r_{rj} \exp(p_{rj}(t))$

$$J = \int_a^b (y(t) - y_r(t))^2\, dt$$

obtain

$$\partial J/\partial p_{ri} = \int_a^b (y(t) - y_r(t))t \exp(p_{ri}t)\, dt = 0 \qquad i = 1, \dots r \tag{9.42}$$

$$\partial J/\partial r_{ri} = \int_a^b (y(t) - y_r(t)) \exp(p_{ri}t)\, dt = 0 \qquad i = 1, \dots r \tag{9.43}$$

(ii) Simplifying equations (9.42) and (9.43)

$$P_1 = S_1R \tag{9.44}$$

$$P_2 = S_2R \tag{9.45}$$

P_1 is a vector with ith component

$$p_{1i} = \sum_{j=1}^{n} (r_j/(p_{rj} + p_j))\left[\exp(p_{ri} + p_j)b - \exp(p_{ri} + p_j)a\right]$$

P_2 is a vector with ith component

$$p_{2i} = \sum_{j=1}^{r} r_j\{\left[\exp(p_j + p_{ri})b/(p_{ri} + p_j)\right]\left[b - 1/(p_{ri} + p_j)\right]$$

$$- \left[\exp(p_{ri} + p_j)a/(p_{ri} + p_j)\right]\left[a - 1/(p_{ri} + p_j)\right]\}$$

S_1 is a matrix, with ijth component,

$$s_{1ij} = (\exp\{(p_{ri} + p_j)b\}/(p_{ri} + p_j))(b - 1/(p_{ri} + p_j))$$

$$- (a - 1/(p_{ri} + p_j))(\exp (p_{ri} + p_j))/(p_{ri} + p_j)$$

S_2 is a matrix with ijth component

$$s_{2ij} = (1/(p_{ri} + p_j))(\exp\{(p_{ri} + p_j)b\} - \exp\{(p_{ri} + p_j)a\})$$

R is a vector of residues $R = (r_{r1}, r_{r2}, \ldots r_{rj})^t$

(iii) Compute $R = S_1^{-1}P_1$ (9.46)

$$P_2 = S_2S_1^{-1}P_1 \qquad (9.47)$$

If poles are specified, equation (9.46) gives a linear equation for R. If poles are unspecified, the nonlinear equation (9.47) gives solution for poles and residues.

(iv) Go to step (ii) for other elements of $G_r(s)$.

Input $\equiv$ (W_{feed}, W_{spray}), Output $\equiv$ (ℓ, T_{s7})

$$g_{r11}(s) = (0.2684E\text{-}03s + 0.1316E\text{-}07) / (s^2 + 0.4903E\text{-}04s + 0.5547E\text{-}11)$$

$$g_{r12}(s) = (0.1476E\text{-}03s + 0.2828E\text{-}06) / (s^2 + 0.7376E\text{-}04s + 0.7733E\text{-}11)$$

$$g_{r21}(s) = (0.1477E\text{-}04s + 0.2173E\text{-}06) / (s^2 + 0.4171E\text{-}02s + 0.3543E\text{-}05)$$

$$g_{r22}(s) = (-0.106E\text{-}01s - 0.7593E\text{-}04) / (s^2 + 0.1368E\text{-}01s + 0.2169E\text{-}04)$$

Input $\equiv$ (W_{fuel}, W_{damp}), Output $\equiv$ (P_{s7}, T_{s10})

$$g_{r11}(s) = (0.1089s + 0.545E\text{-}03) / (s^2 + 0.1199E\text{-}01s + 0.2938E\text{-}04)$$

$$g_{r12}(s) = (-0.1525E\text{-}01s - 0.2229E\text{-}03) / (s^2 + 0.6686E\text{-}02s + 0.5858E\text{-}05)$$

$$g_{r21}(s) = (0.4714E\text{-}01s - 0.538E\text{-}04) / (s^2 + 0.2314E\text{-}02s + 0.1338E\text{-}05)$$

$$g_{r22}(s) = (0.8956E\text{-}02s - 0.3026E\text{-}02) / (s^2 + 0.1159E\text{-}01s + 0.1693E\text{-}04)$$

(9) Modified Levy's method;[8] algorithm

(The algorithm of 'modified Levy's method' by Vittal Rao and Lamba is considered).

(i) Consider $T_r(s) = K(1 + c_1 s + \dots c_p s^p)/(1 + d_1 s + \dots d_q s^q)$

$$= K\{(1 \mp a_2\omega^2 + \dots) + j\omega(a_1 - a_3\omega^2 + \dots) /$$
$$(1 - b_2\omega^2 + \dots) + j\omega(b_1 - b_3\omega^2 + \dots)\}$$

$$= K(\alpha + j\omega\beta)/(\sigma + j\omega\tau) \quad = \quad KN_r(\omega)/D_r(\omega)$$

Similarly, $T(s) = K(R + j\omega I)/(G + j\omega L) \quad = \quad KN(\omega)/D(\omega)$

(ii) $e(j\omega) = T_r(j\omega) - T(j\omega) \quad = \quad -K(N(\omega)D_r(\omega) - N_r(\omega)D(\omega))/D(\omega)D_r(\omega)$

$$\min E = \int_\Omega |e(j\omega)D(\omega)D_r(\omega)|^2 \, d\omega$$

$$= \int_\Omega \{(R\sigma - G\alpha - \omega^2 I\tau + \omega^2 L\beta)^2 + (\omega R\tau + \omega I\sigma - \omega G\beta - \omega L\alpha)^2\} \, d\omega$$

obtain
$$\begin{aligned} \partial E/\partial c_1 &= \int_\Omega \{2\omega^2 L(R\sigma - G\alpha - \omega^2 I\tau + \omega^2 L\beta) \\ \vdots \quad & \qquad - 2\omega G(\omega R\tau + \omega I\sigma - \omega G\beta - \omega L\alpha)\} \, d\omega \\ \vdots \quad &= 0 \end{aligned} \tag{9.48}$$

(iii) Simplifying equation (9.48) yields, $MN = P$ (9.49)

where $N^t = (c_1, c_2, c_3, \dots c_p, d_1, d_2, \dots d_q)$,

$$P^t = (U_2 - \lambda_2, T_2 - A_2 - B_4 + V_4, \lambda_4 - U_4, \dots)$$

and A is the $(p + q) \times (p + q)$ matrix given by

$$A = \begin{bmatrix} (T_2 + V_4) & 0 & (T_6 + T_8) & \dots \\ 0 & \cdot & & \\ -T_4 + T_6 & & \cdot & \\ \vdots & & & \cdot \end{bmatrix}$$

where $T_h = \int_\Omega G^2\omega^h \, d\omega$, $V_h = \int_\omega L^2\omega^h \, d\omega$, $A_h = \int GR\omega^h \, d\omega$,

$B_h = \int_\Omega LI\omega^h \, d\omega$, etc

(iv) Compute $N = M^{-1}P$ (9.50)

(v) Go to step (i) and consider other elements of $G_r(s)$

Input $\equiv$ (W_{feed}, W_{spray}), Output $\equiv$ (ℓ, T_{s7})

$g_{r11}(s) = (0.7449E-03s - 0.1874E-05) / (s^2 + 0.338E-03s + 0.276E-07)$

$g_{r12}(s) = (0.5123E-03s - 0.1046E-05) / (s^2 + 0.1091E-03s + 0.1079E-08)$

$g_{r21}(s) = (-0.5053E-04s + 0.5185E-06) / (s^2 + 0.3585E-02s + 0.4084E-06)$

$g_{r22}(s) = (0.1197E+05s - 0.1961E-01) / (s^2 + 0.156s + 0.6084E-02)$

Input $\equiv$ (W_{fuel}, W_{damp}), Output $\equiv$ (P_{s7}, T_{s10})

$g_{r11}(s) = (0.339s + 0.1421E-02) / (s^2 + 0.8355s + 0.3003E-04)$

$g_{r12}(s) = (-0.2225E-01s - 0.1918E-03) / (s^2 + 0.6431E-02s + 0.4665E-05)$

$g_{r21}(s) = (0.3082E-01s + 0.5059E-03) / (s^2 + 0.1365E-01s + 0.4435E-04)$

$g_{r22}(s) = (-0.7002s - 0.1027E-02) / (s^2 + 0.6197E-02s + 0.7945E-05)$

(10) Harmonic synthesis method of Chapter III; algorithm

(i) $$G_r(s) = K_r \prod_{i=1} (s + z_i) \prod_{i=1} (s^2 + 2a_i s + a_i^2 + b_i^2) \Big/ \prod_{j=1} (s + p_j) \prod_{j=1} (s^2 + 2c_j s + c_j^2 + d_j^2)$$

$$y_r(t) = \sum_{k=1} f_{rk} \operatorname{Sin} (\omega_k t + \phi_{rk})$$

$$y(t) = \sum_{k=1} f_k \operatorname{Sin} (\omega_k t + \phi_k)$$

$$f_{rk} = (A_k^2 + B_k^2)^{\frac{1}{2}} |G_r(j\omega_k)|, \ \phi_{rk} = \arg G_r(j\omega_k) + \arg\tan (A_k/B_k)$$

(ii) $$\min E = \int_0^T (y(t) - y_r(t))^2 \, dt$$

$$\partial E/\partial\theta = \int_0^T (y(t) - y_r(t)) \, \partial y_r(t)/\partial\theta = 0$$

$$\partial y_r(t)/\partial\theta = \sum_{k=1}^{h} F_{rk} \operatorname{Sin} (\omega_k t + \phi_{rk} + \psi_{rk}), \ \theta = z_i, a_i, b_i, p_j, c_j, d_j \text{ and } K_r$$

(iii) Specify initial poles and zeros for $G_r(s)$ and solve non-linear equations

$$\sum_{k=1}^{h} \sum_{\substack{j=1 \\ j\neq k}}^{h} V_{kj} F_{rj} \Delta_k \operatorname{Sin}(\zeta_{kj} + \delta_k) + \sum_{k=1}^{h} F_{rk} \Delta_k \varepsilon_k \operatorname{Cos}(\delta_k + \gamma_k) = 0$$

(iv) Go to step (i) for other elements of $G_r(s)$.

Input $\equiv$ (W_{feed}, W_{spray}), Output $\equiv$ (ℓ, T_{s7})

$g_{r11}(s) = (0.3509E\text{-}03s - 0.4047E\text{-}06) / (s^2 + 0.511E\text{-}03s + 0.1032E\text{-}09)$

$g_{r12}(s) = (0.188E\text{-}03s + 0.274E\text{-}06) / (s^2 + 0.2316E\text{-}04s + 0.7242E\text{-}12)$

$g_{r21}(s) = (-0.187E\text{-}04s + 0.6086E\text{-}06) / (s^2 + 0.9056E\text{-}02s + 0.1884E\text{-}04)$

$g_{r22}(s) = (-0.1394E\text{-}02s - 0.3251E\text{-}03) / (s^2 + 0.258E\text{-}01s + 0.8733E\text{-}04)$

Input $\equiv$ (W_{fuel}, W_{damp}), Output $\equiv$ (P_{s7}, T_{s10})

$g_{r11}(s) = (0.489E\text{-}01s + 0.1492E\text{-}03) / (s^2 + 0.7574E\text{-}02s + 0.1119E\text{-}04)$

$g_{r12}(s) = (-0.7174E\text{-}02s - 0.1067E\text{-}03) / (s^2 + 0.5334E\text{-}02s + 0.4772E\text{-}05)$

$g_{r21}(s) = (0.3785E\text{-}01s + 0.1195E\text{-}03) / (s^2 + 0.6894E\text{-}02s + 0.115E\text{-}04)$

$g_{r22}(s) = (-0.4891s - 0.1047E\text{-}02) / (s^2 + 0.642E\text{-}02s + 0.5886E\text{-}05)$

Discussions

The response graphs obtained by state space reduction techniques are shown in figures 9.3 and 9.4. Those obtained by the transfer function reduction techniques are shown in figures 9.5 and 9.6. By inspection the transfer function results are more accurate than the state space results. The chief reason is that the state space graphs are those of second order models while the transfer function graphs are those of eighth order models. Although the individual elements of the transfer function matrices are of second order, application of a minimal realization algorithm yields eighth order state space models for the transfer function matrices. Thus, the transfer function models are of higher order, hence, approximate closer to the original thirty third order response.

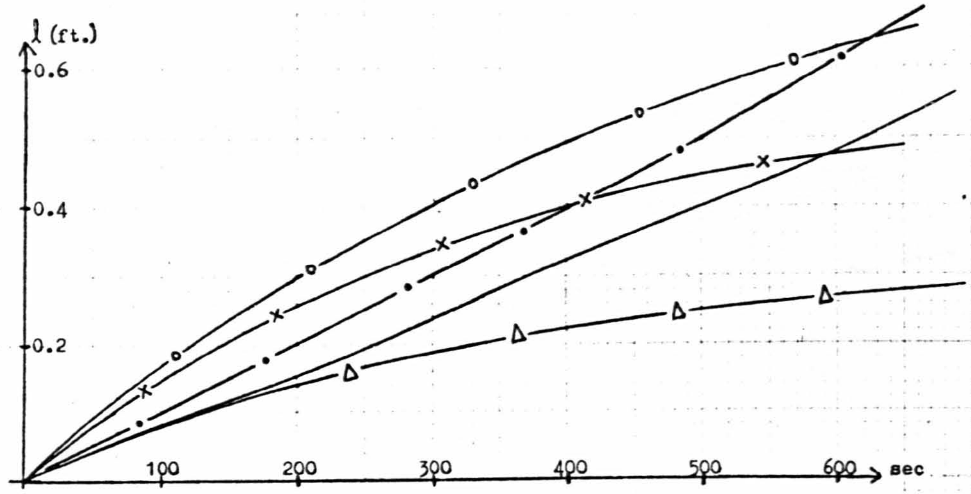


Fig. 9.3(a) Response of output 1 following 1% step increase in feed in input 1

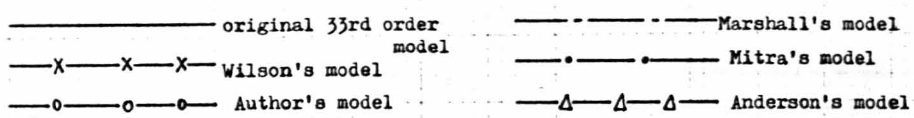


Fig. 9.3(b) Response of output 2 following 1% step increase in feed in input 1

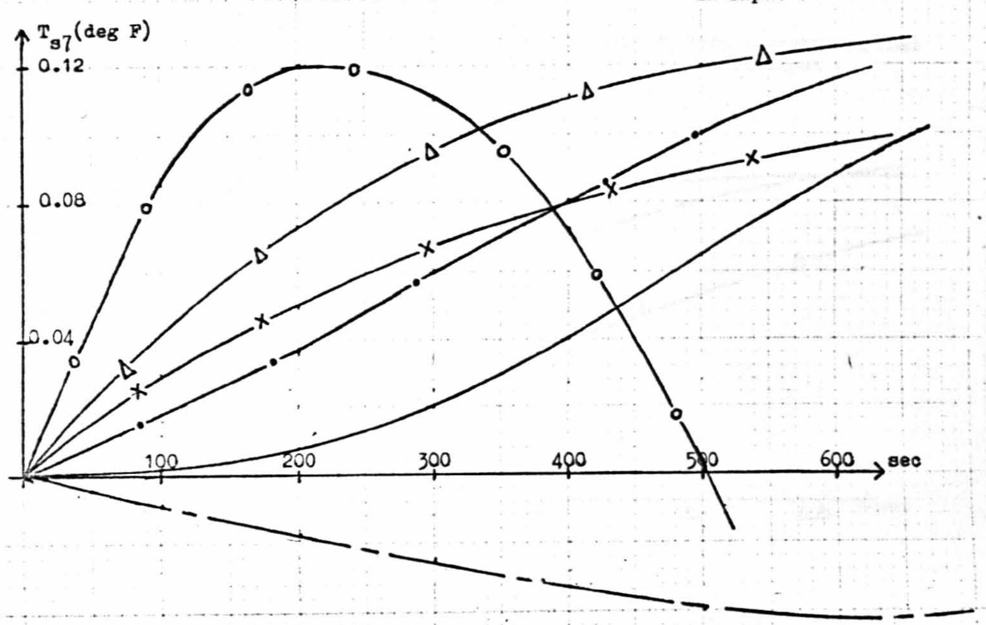
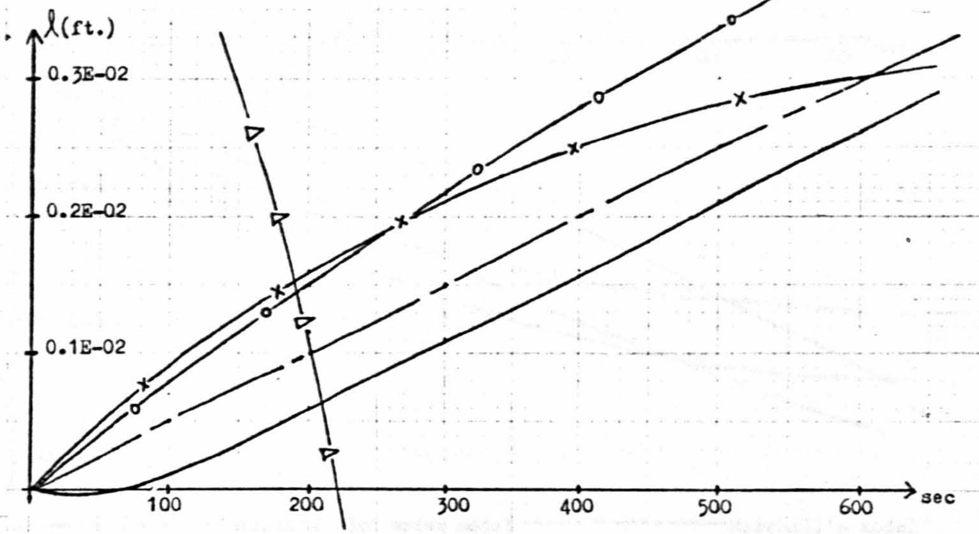


Fig. 9.3(c) Response of output 1 following 1% step increase in spray in input 2



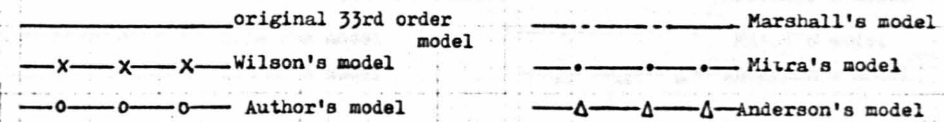
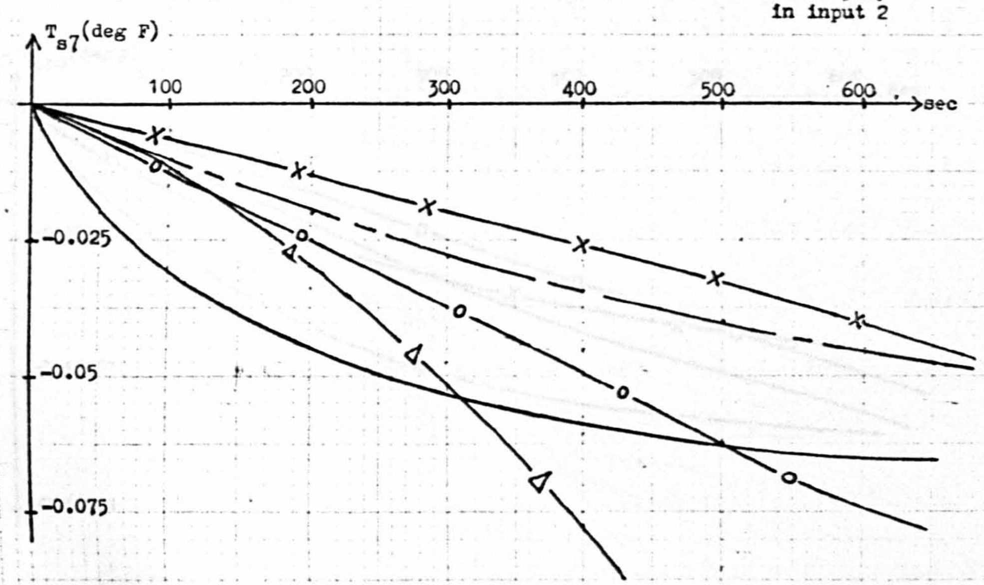


Fig. 9.3(d) Response of output 2 following 1% step increase in spray in input 2



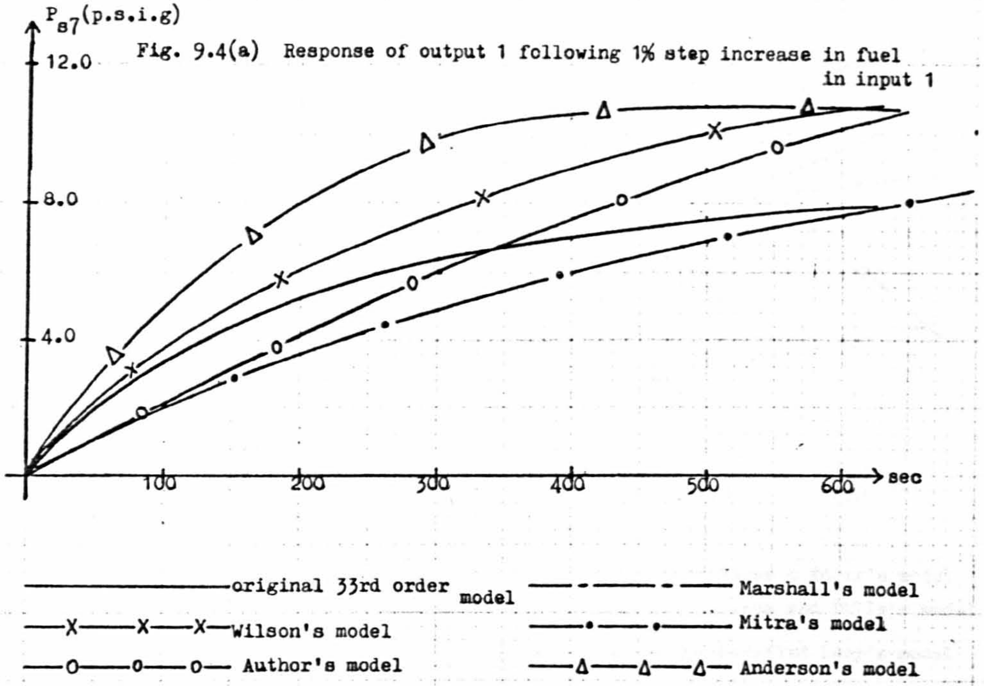


Fig. 9.4(a) Response of output 1 following 1% step increase in fuel in input 1

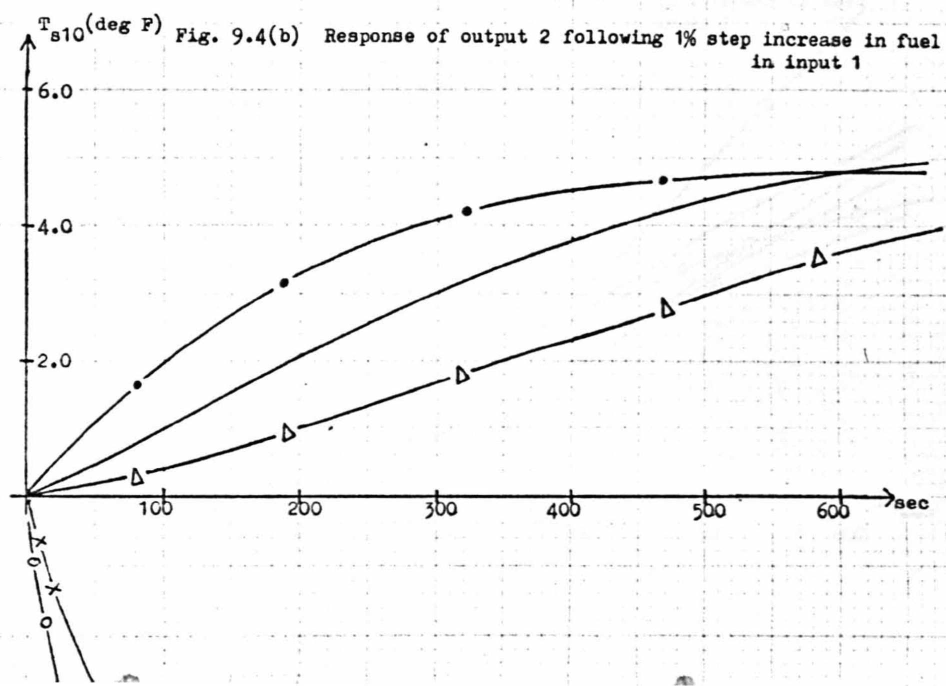


Fig. 9.4(b) Response of output 2 following 1% step increase in fuel in input 1

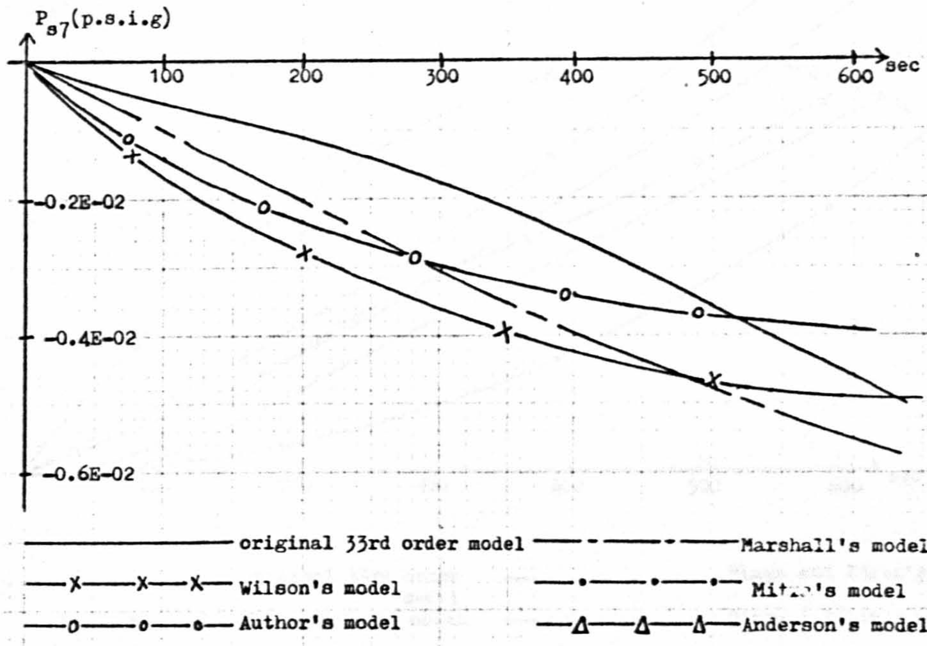


Fig. 9.4(c) Response of output 1 following 1% step increase in damper in input 2

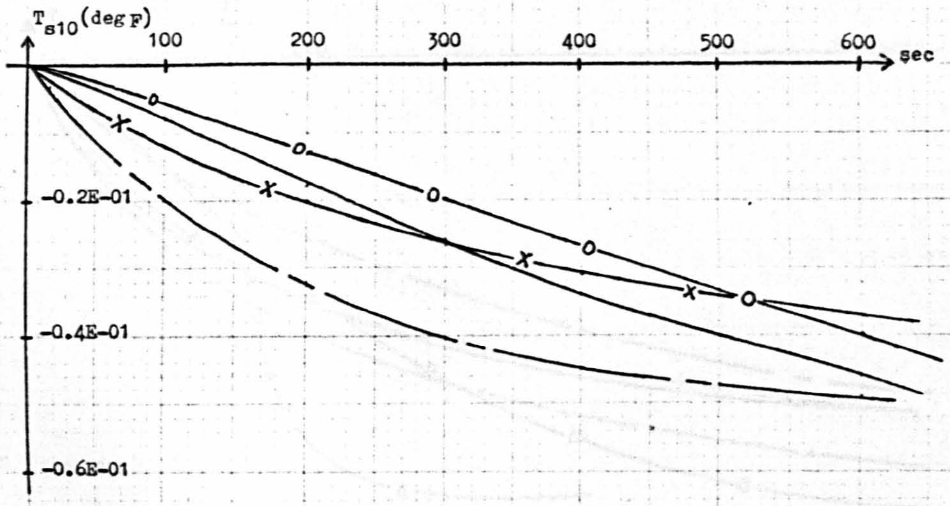


Fig. 9.4(d) Response of output 2 following 1% step increase in damper in input 2

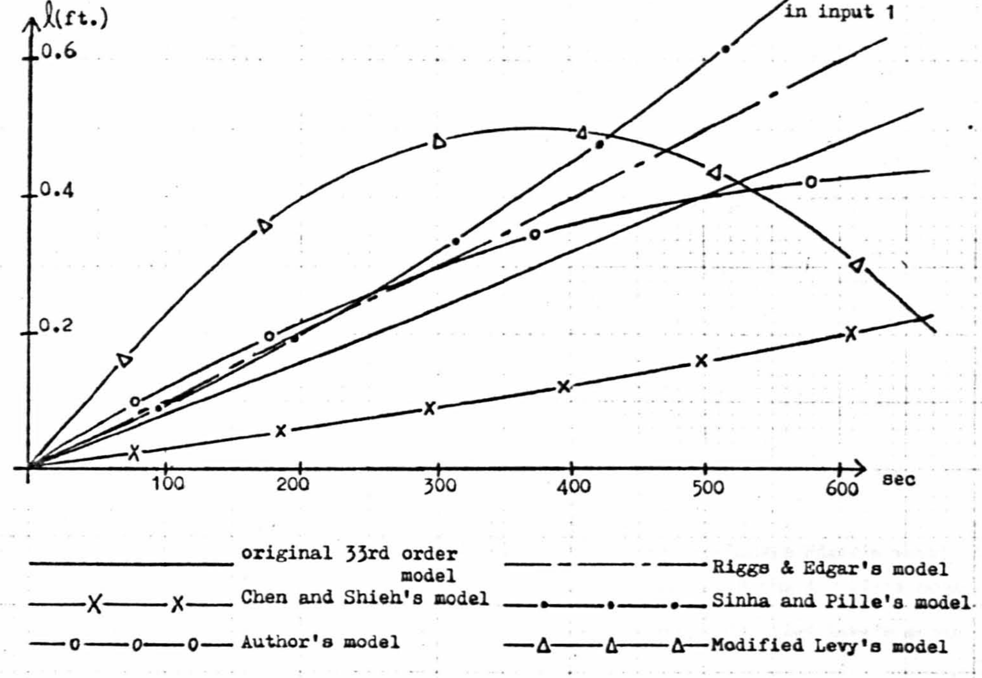


Fig. 9.5(a) Response of output 1 following 1% step increase in feed in input 1

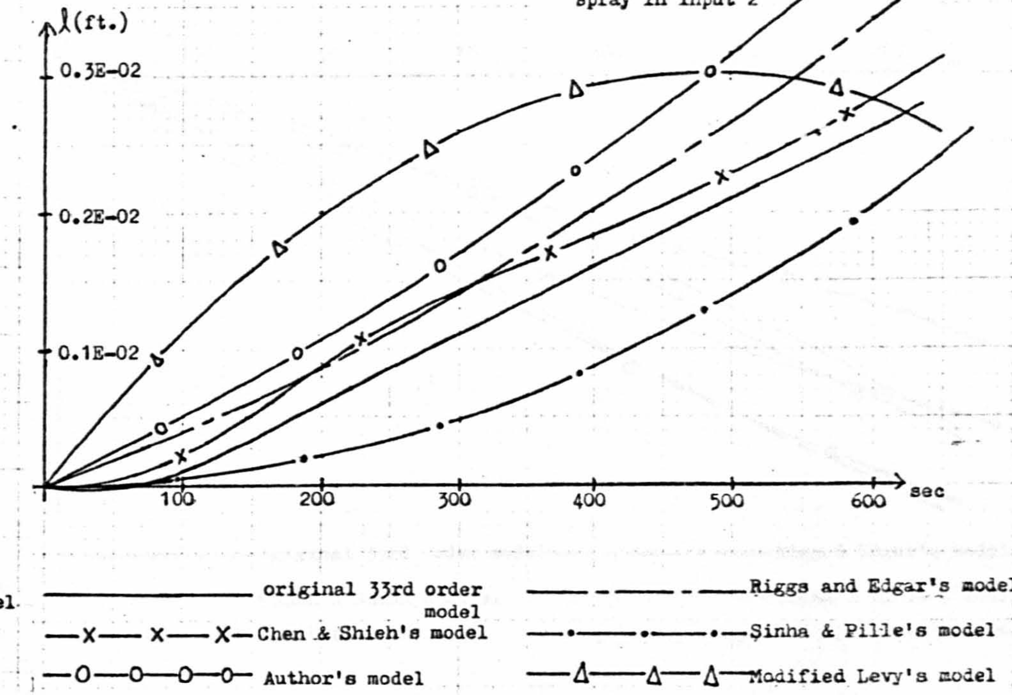


Fig. 9.5(c) Response of output 1 following 1% step increase in spray in input 2

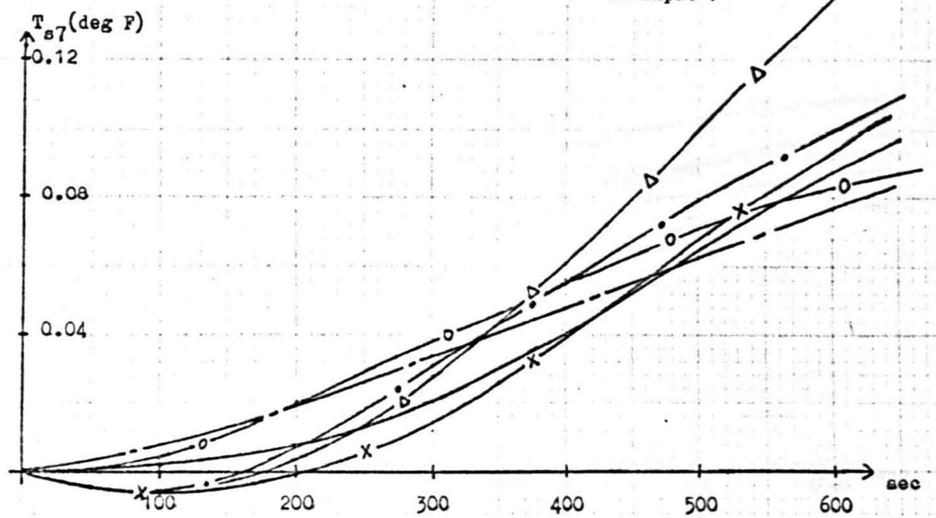


Fig. 9.5(b) Response of output 2 following 1% step increase in feed in input 1

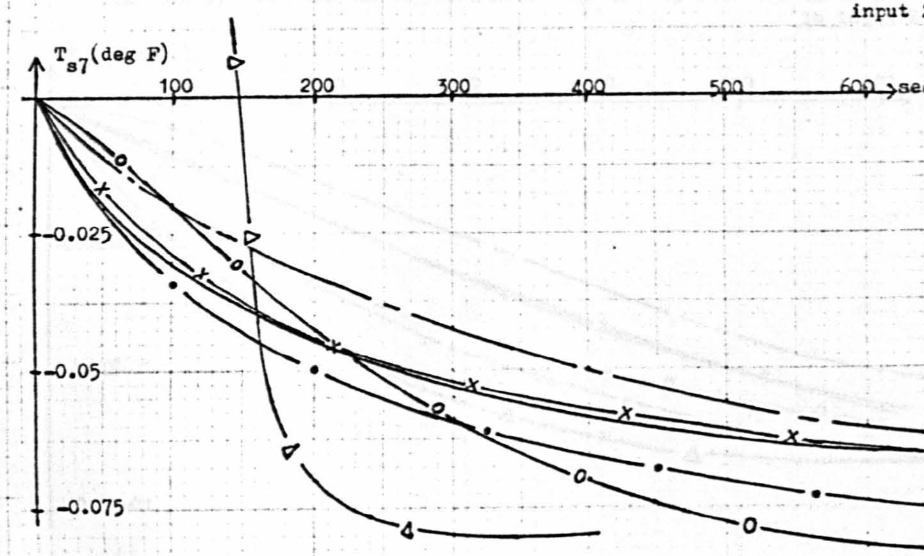
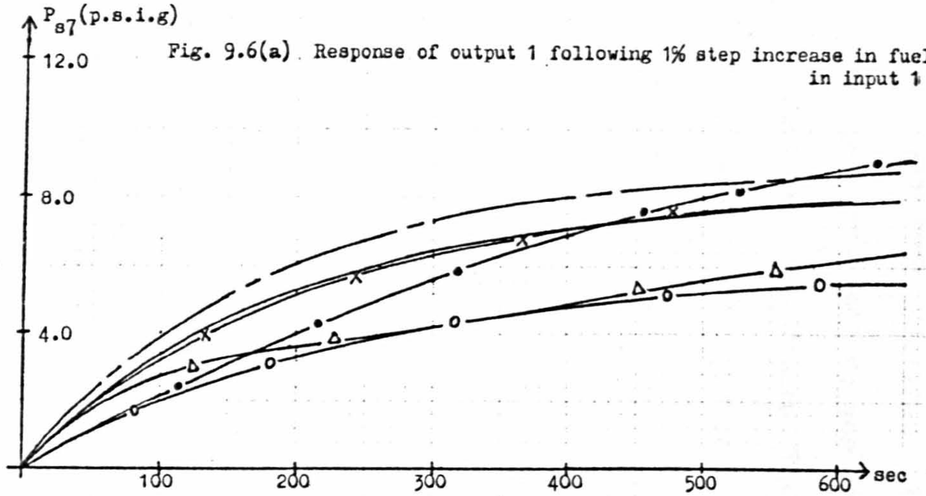


Fig. 9.5(d) Response of output 2 following 1% step increase in spray in input 2

Fig. 9.6(a) Response of output 1 following 1% step increase in fuel in input 1



original 33rd order model
Riggs & Edgar's model
Chen & Shieh's model
Sinha & Pille's model
Author's model
Modified Levy's model

Fig. 9.6(b) Response of output 2 following 1% step increase in fuel in input 1

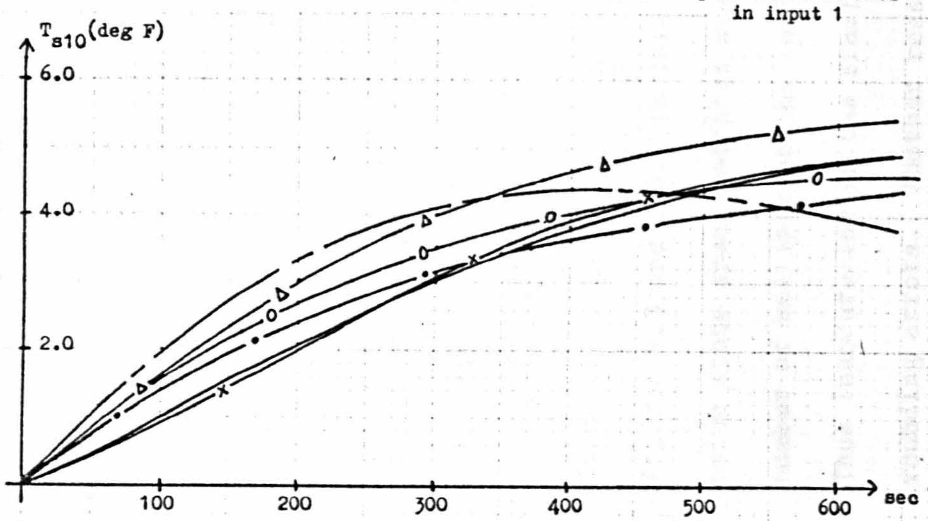
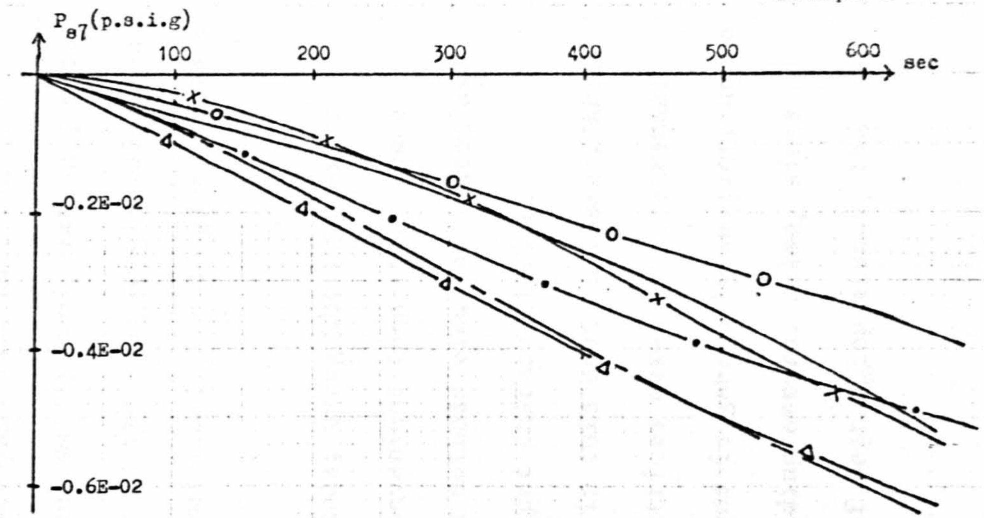
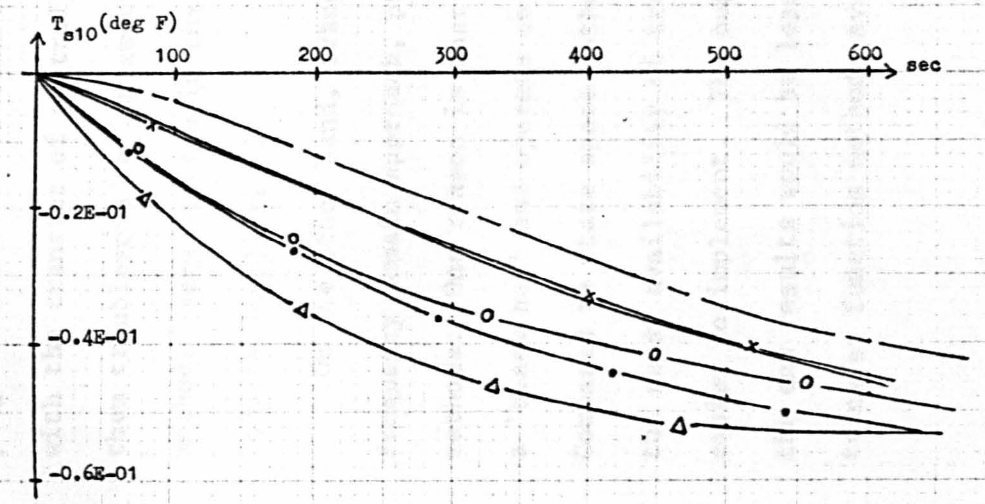


Fig. 9.6(c) Response of output 1 following 1% step increase in damper in input 2



original 33rd order model
Riggs & Edgar's model
Chen & Shieh's model
Sinha & Pille's model
Author's model
Modified Levy's model

Fig. 9.6(d) Response of output 2 following 1% step increase in damper in input 2



Figures 9.3 and 9.5 show the same graph, with W_{feed} and W_{spray} as inputs, and, ℓ and T_{s7} as outputs. Figures 9.4 and 9.6 represent the response, whose input vector is $(W_{fuel}, W_{damp})^t$, and, output vector is $(P_{s7}, T_{s10})^t$. All computations were done on the ICL 1905 computer, and, where necessary, the scientific subroutine packages were used. As the dimensions of the matrices involved were large, the storage problem was facilitated by employing external storing facilities, in the use of magnetic tapes and disc files.

The graphs, obtained by state space reduction methods, that are out of range with the original thirty third order graphs are not shown, in figures 9.3 and 9.4. In this case of the boiler model, reduction techniques using state space matrices are inferior. This is chiefly due to the large number of multiplications involved in handling high dimensional matrices. Thus inaccuracies in the final result could arise due to computational rounding errors. Another reason is core storage. Efficient computer programmes must be written to economize core storage, and, this together with the transfer of control, involving numerous matrix manipulations, made them troublesome to implement. The algorithms of the time domain reduction methods, were written in the best way that, numerically, would give the least computational errors.

On the other hand, transfer function methods, using the complex or frequency domain approach, prove to be more convenient than state space methods. One reason is that transfer function methods view the system as a 'black box' and operate on its input and output characteristics. Compared to state space methods, core storage in this case is lesser, and, the ready availability of transfer function matrices makes the programmes easier to implement. The number of computations is generally smaller, hence, the end results would be less affected by rounding errors. Also, since transfer function methods avoid the problem of regrouping states, the computer programmes are easier to write than those of state space methods.

Of the ten reduction methods, for this example, the continued fraction method of Chen and Hsieh, for single input - single output systems, is by far the most accurate and the most convenient to implement. It requires modest core storage and is computationally cheap. It does not require prior knowledge of the poles of the reduced models. The accuracy of other reduction methods, that require initial specification of poles, depends on subjective pole specifications and performance indices. In terms of complexity, the reduction methods of Mitra and Wilson are difficult to implement. They are also costly, and so is the method using the modified Levy algorithm. All results that employ 'Hill-climbing' optimization routines are costly, and, in most cases, failures are more likely to be encountered than successes. The chance of success depends very much on the correct initial specification of poles and the scaling of the problem.

Most methods yield a steady state error, except that of Marshall (however, in some cases it is compensated by large transient errors) and the continued fraction method of Chen and Hsieh. In the case where the steady state value is infinite (an 'unstable' model with a pole at the origin) Chen and Hsieh's method is inferior to the other methods. The modified Levy method gives the poorest result compared to other transfer function methods. This may be due to the 'linearization used for approximation' in the method. Of state space methods, the author's method of sequential approximation and Wilson's reduction method seem to give the most consistent results for all outputs. Also they have reasonably small transient errors and seem to give acceptable steady state errors. Mitra's, Anderson's and Marshall's methods fit well in some outputs and give very poor fitting in other outputs. On the other hand, most of the transfer function methods give reasonably small transient and steady state errors.

PART II - Application of reduced models in control systems design

The reduced models obtained above were used to obtain control strategies for the original boiler model. The state space models were used to design sub-optional controllers using a time domain, linear optimal control, technique[12]. The transfer function models, on the other hand, used a frequency domain method, employing the Inverse Nyquist Array[10], to obtain closed loop controllers for the boiler, see figs. 9.7(a) and (b).

9.3 Sub-optimal control design using linear optimal regulator theory[12].

The discrete optimal algorithm was considered here, as it is more suitable to digital computation. In all models, the inputs were considered noise-free and all states were assumed accessible.

Associated with,

$$S \; : \quad x_{k+1} = \Phi x_k + \Delta u_k \tag{9.51}$$

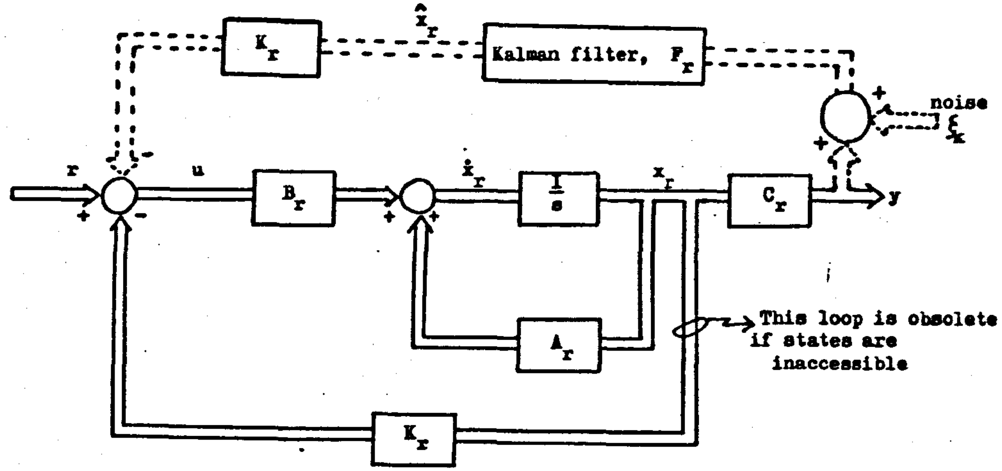
$$S_r \; : \quad x_{rk+1} = \Phi_r \, x_{rk} + \Delta_r u_{rk} \tag{9.52}$$

where S and S_r are controllable, the performance indices are :

$$J = \sum_{k=o}^{N-1} \{ \langle x_k, Qx_k \rangle_E + \langle u_k, Ru_k \rangle_E \} \tag{9.53}$$

$$J_r = \sum_{k=o}^{N-1} \{ \langle x_{rk}, Q_r x_{rk} \rangle_E + \langle u_k, Ru_k \rangle_E \} \tag{9.54}$$

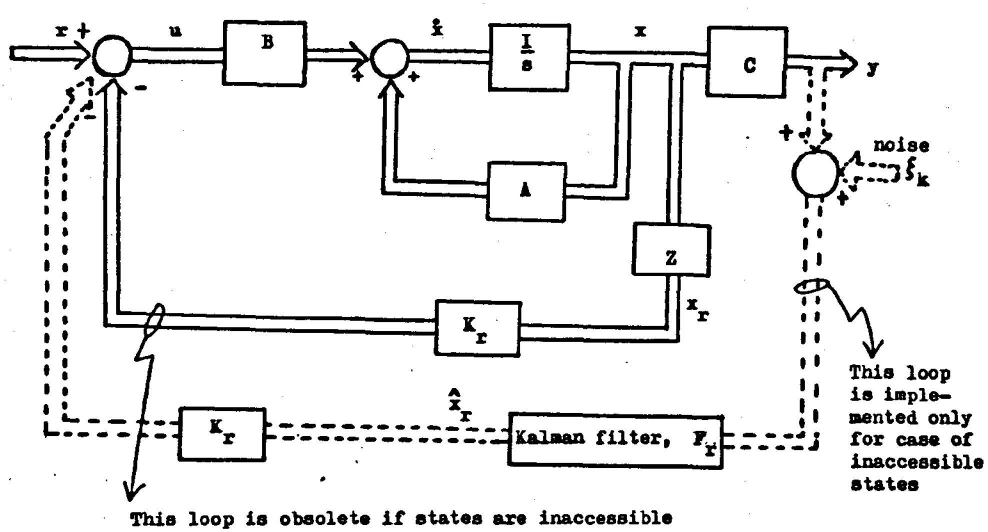
where $x_{rk} = x_r(kT)$, $x_k = x(kT)$, etc. Using dynamic programming, it is well known that the optimal control law for S_r, can be

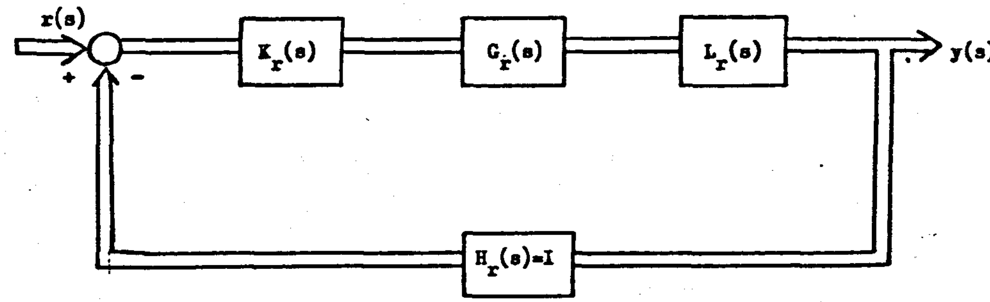


Optimal regulator design using reduced model

Fig.9.7(a) The linear optimal regulator design in the time domain (with on-line state estimation)

Sub optimal regulator performance on original model

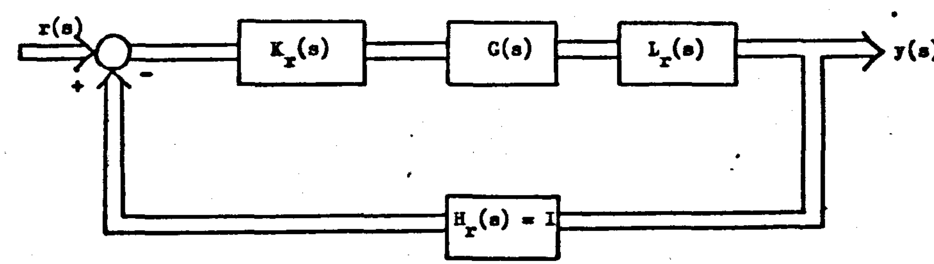




Design using reduced model

Fig.9.7(b) Controller design in the frequency domain

Controller performance on original model



computed from the recursive relationship,

$$u_{rk} = -K_{rk}\, x_{rk} \tag{9.55}$$

where

$$K_{rk} = (R + \Delta_r^t\,(Q_r + P_{r,k+1})\Delta_r)^{-1}\,\Delta_r^t(Q_r + P_{r,k+1})\Phi_r \tag{9.56}$$

$$P_{rk} = \Phi_r^t(I-(Q_r + P_{r,k+1})\Delta_r(R + \Delta_r^t(Q_r + P_{r,k+1})\Delta_r)^{-1}\Delta_r^t)(Q_r+P_{r,k+1})\Phi_r$$

P_{rk} can be solved backwards, starting from $P_{rN} = 0$, in order $k = N-1, N-2, \ldots 2,1,0$, and in considering the infinite time optimal regulator, $K_{rk} \to K_{ro} = K_r$ as $N \to \infty$. The steady state optimal gain, K, associated with S, in eqn. (9.51) can similarly be solved. For convenience the regulator can be converted into a set-point change problem, by adding a reference input in eqn. (9.55) i.e.

$$u_{rk} = -K_r x_{rk} + r \tag{9.56}$$

The closed loop optimal response is given by, from eqns. (9.52) and (9.56),

$$S_r: \quad x_{r,k+1} = (\Phi_r - \Delta_r K_r)x_{rk} + \Delta_r r \tag{9.57}$$

$$y_{r,k+1} = C_r\, x_{r,k+1}$$

Similarly, for S,

$$S: \quad x_{k+1} = (\Phi - \Delta K)\, x_k + \Delta r \tag{9.58}$$

$$y_{k+1} = Cx_{k+1}$$

Sub-optimal control policy is obtained by using K_r in place of K to control S. Using eqn. (9.56),

$$u_{rk} = -K_r Z x_k + r \qquad (9.59)$$

where $x_{rk} = Zx_k$, and eqn (9.51), the closed loop sub-optimal response is given by

$$S_{sub} : \; x_{k+1} = (\Phi - \Delta K_r Z) x_k + \Delta r \qquad (9.60)$$

$$y_{k+1} = C x_{k+1}$$

The weighting matrices Q and Q_r, are related by

$$Q_r = (Z^+)^t \, Q \, Z^+ \qquad (9.61)$$

where $Z^+ = Z^t \, (ZZ^t)^{-1}$, and the aggregation matrix, Z, that relates S to S_r can in general be approximately obtained (least square sense) from the equations

$$A_r Z = ZA$$

$$B_r = ZB \qquad (9.62)$$

$$C_r Z = C$$

as

$$Z = W_r \, W^+ = V_r^{\,+} V$$

where $W = (B, AB, \ldots A^{n-1}B)$, $W_r = (B_r, A_r B_r, \ldots A_r^{\,r-1} B_r)$ are the controllability matrices, $V^t = (C, CA, \ldots CA^{n-1})$, $V_r^t = (C_r, C_r A_r, \ldots C_r A_r^{\,r-1})$ the observality matices of S and S_r, respectively[9].

However, for the reduced state space models above, in most cases, Z was obtained, from $C_r\ Z = C$, since C_r is a square (2 x 2) matrix.

In computing the optimal responses, it was found that K and K_r converged very accurately after N = 30 iterations with a sampling time interval of T = 100 seconds.

In the above analysis, all states are assumed accessible and noise free. In practice, such is not the case, and a filter is required to estimate the states.

Corresponding to eqns. (9.57) and (9.58),

$$S_r \ : \ y_{r,k+1} = C_r\ x_{r,k+1} + \xi_k \tag{9.57}$$

$$S \ : \ y_{k+1} = C\ x_{k+1} + \xi_k \tag{9.58}$$

a zero mean noise vector, ξ_k, of known statistical properties, is added to the output measurements, the other dynamical equation remaining the same. Using a Kalman filter, the zero mean estimated state vector $\hat{x}_{rk}$ can be constructed from the output measurements such that the control law, $\hat{u}_{rk} = -K_{rk}\ \hat{x}_{rk} + r$, minimizes eqn. (9.54). The filter can be computed from the well known recursive algorithm[12], (ξ_k and $\hat{x}_{rk}$ assumed statistically independent)

$$\hat{x}_{r,k+1} = \Phi_r\ \hat{x}_{rk} + \Delta_r\ u_{rk} + W_{r,k+1}(y_{k+1} - C_r(\ \Phi_r\ \hat{x}_{rk} + \Delta_r\ u_{rk}))$$

$$W_{r,k+1} = \Phi_r\ L_{rk}\ \Phi_r^t\ C_r^t\ (C_r\ \Phi_r\ L_{rk}\ \Phi_r^t\ C_r^t + M_k)^{-1} \tag{9.63}$$

$$L_{r,k+1} = (I - W_{r,k+1}\ C_r)\ \Phi_r\ L_{rk}\ \Phi_r^t, \quad k = 0,1,2,..N$$

where $L_{rk} = E\left[(\hat{x}_{rk} - x_{rk})\rangle\langle(\hat{x}_{rk} - x_{rk})\right]$, $M_k = E\left[\xi_k \rangle\langle \xi_k\right]$ are co-variance matrices of state errors and measurement noise, respectively. For unbiassed estimates, $E\left[\hat{x}_{rk}\right] = E\left[x_{rk}\right]$, the initial conditions can be set to $\hat{x}_{ro} = x_{ro} = 0$, and $L_{ro} = L_o$.

A similar filter, dropping the subscript r, in the sets of eqn. (9.63), can be used for the original model, S, such that the control law, $\hat{u}_k = -K\hat{x}_k + r$, minimizes eqn. (9.53).

Assuming

$$\hat{x}_{rk} = Z\,\hat{x}_k \tag{9.64}$$

sub-optimal control policy, corresponding to eqn. (9.59),

$$u_{rk} = -K_r\,Z\hat{x}_k + r \tag{9.65}$$

can be implemented on S, with a 'sub-optimal' filter. The resulting closed loop dynamical equation, see eqn. (9.60), is given by

$$S_{sub}: \quad x_{k+1} = \Phi\,x_k - \Delta(K_r\,\hat{x}_{rk} - r) \tag{9.66}$$
$$y_{k+1} = C\,x_{k+1} + \xi_k$$

The estimated vector, $\hat{x}_{rk}$ can be computed from the filter equations, eqn. (9.63), and K_r computed as before. This would result in considerable computational savings (r<<n) in the sub-optimal control scheme. The respective closed loop optimal response for S and S_r are,

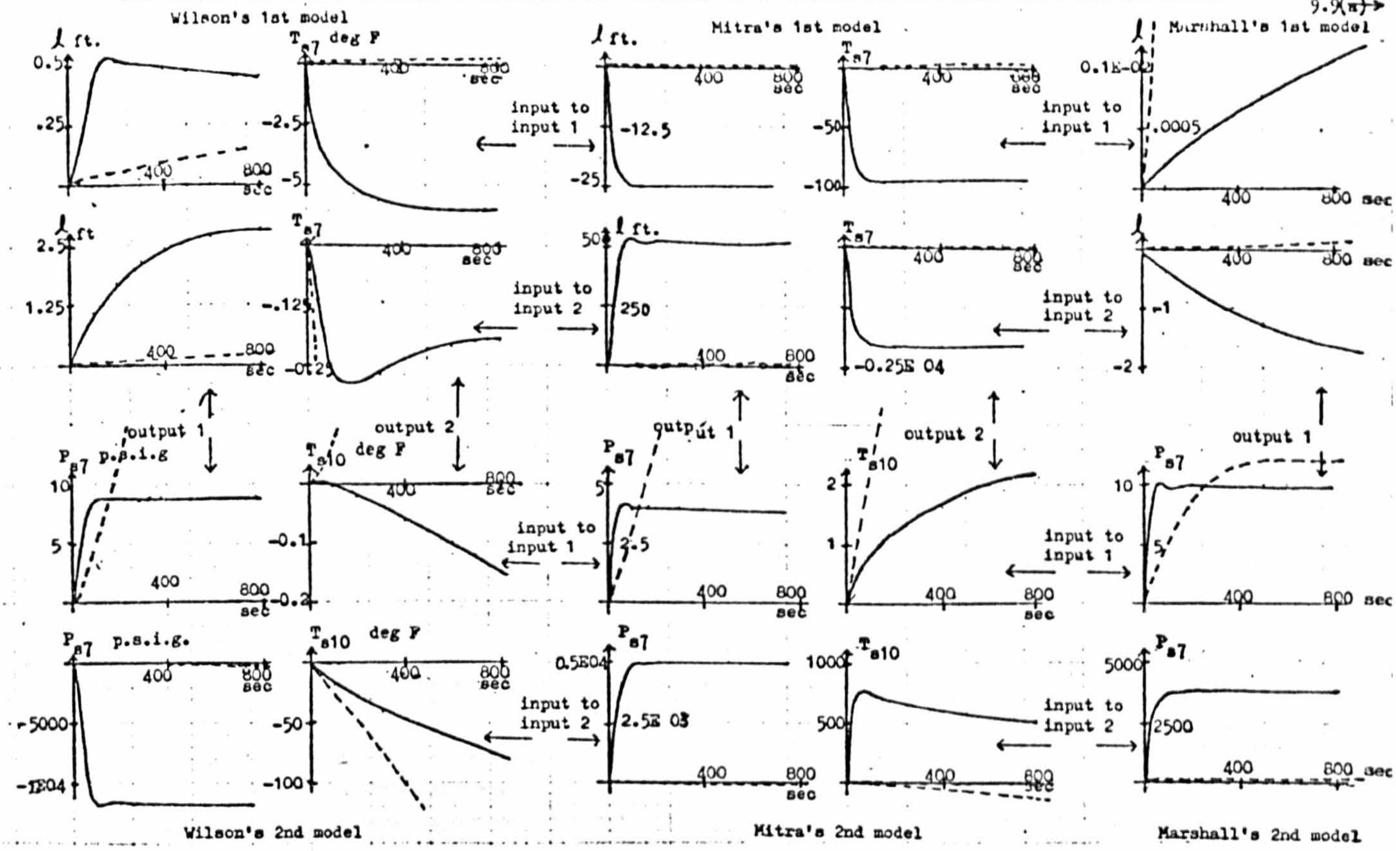
$$S \quad : x_{k+1} = \Phi\,x_k - \Delta\,(K\,\hat{x}_k - r) \tag{9.67}$$
$$y_{k+1} = C\,x_{k+1} + \xi_k$$

$$S_r \quad : x_{r,k+1} = \Phi_r\,x_{rk} - \Delta_r\,(K_r\,\hat{x}_{rk} - r) \tag{9.68}$$
$$y_{r,k+1} = C_r\,x_{r,k+1} + \xi_k$$

The optimal responses of S_r, with and without the Kalman filter, are shown in figs. 9.8(a), (b) and 9.9(a), (b). The corresponding sub-optimal response and optimal resonse of S are shown in figs. 9.10 to 9.11. The matrices R, Q, Q_r, Z, K, K_r associated with the models are given in table 9E.

——— optimal response - - - - - - corresponding sub optimal response

Fig. 9.8(a) Optimal response of reduced models (unit step input to input 1 and zero to input 2, and vice versa)...cont on fig. 9.9(a)



——— optimal response - - - - - - corresponding sub optimal response

cont. from fig. 9.8(a)

Fig. 9.9(a) Optimal response of reduced models (unit step input to input 1 and zero to input 2, and vice versa)

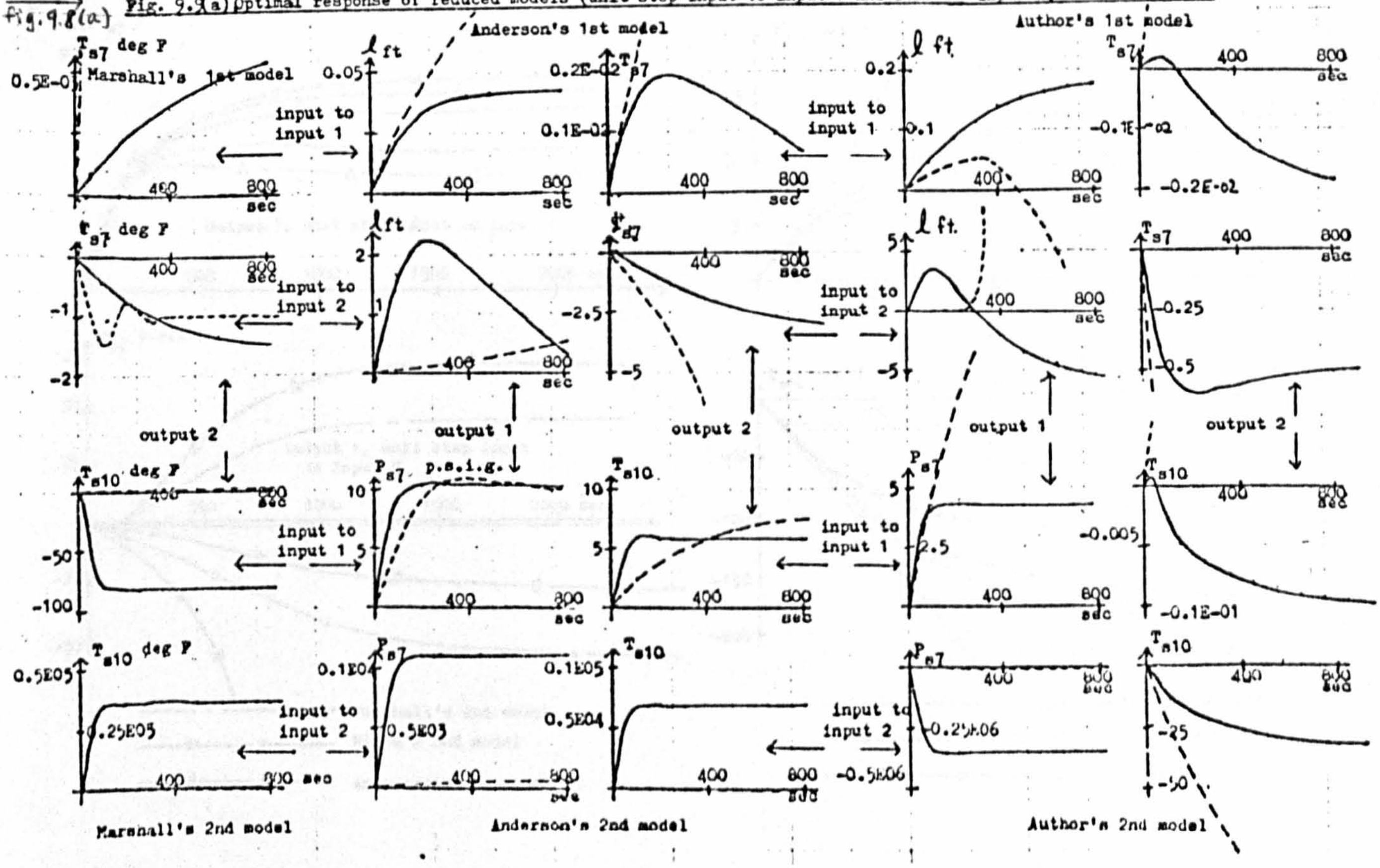


Fig. 9.10(a) Sub-optimal regulator response of reduced models(all states assumed accessible), 1st model

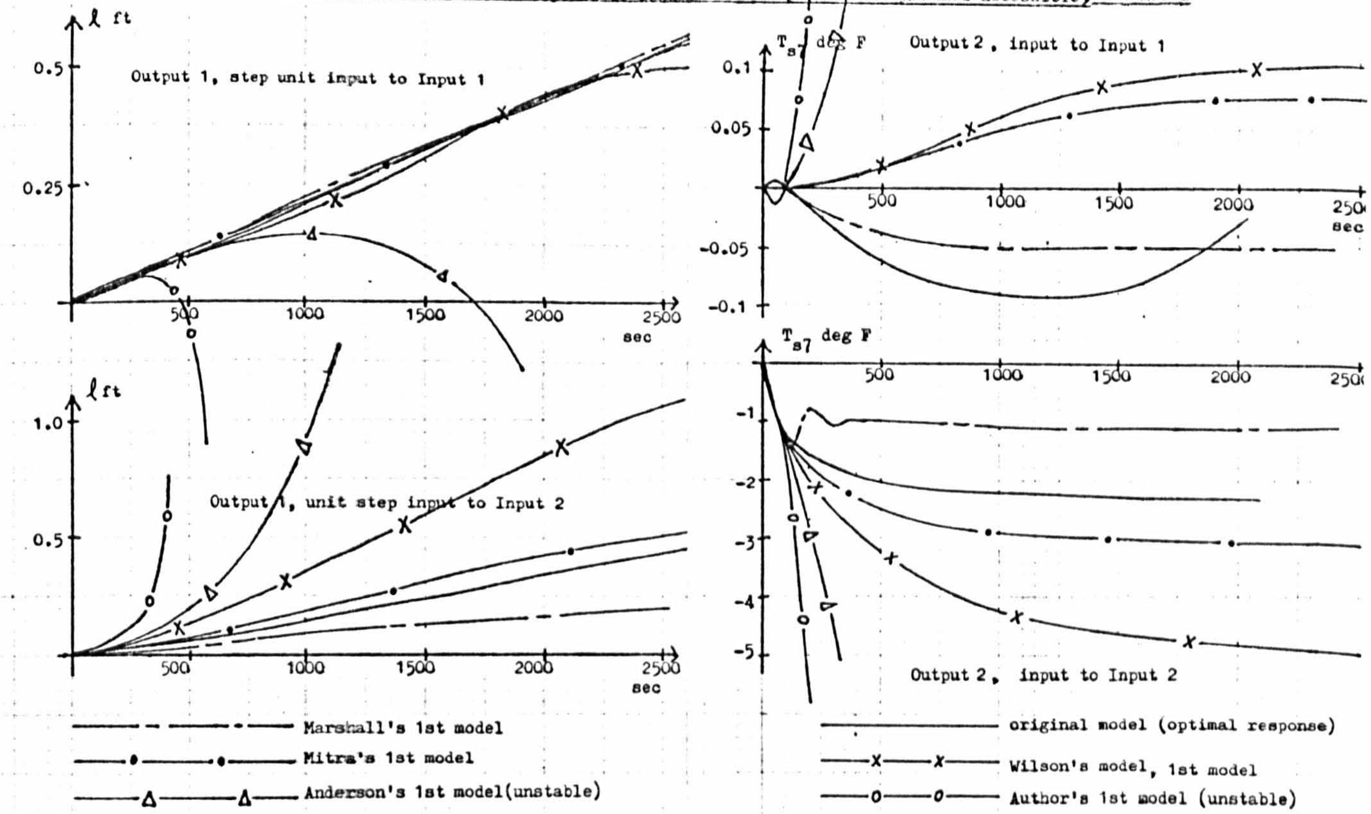
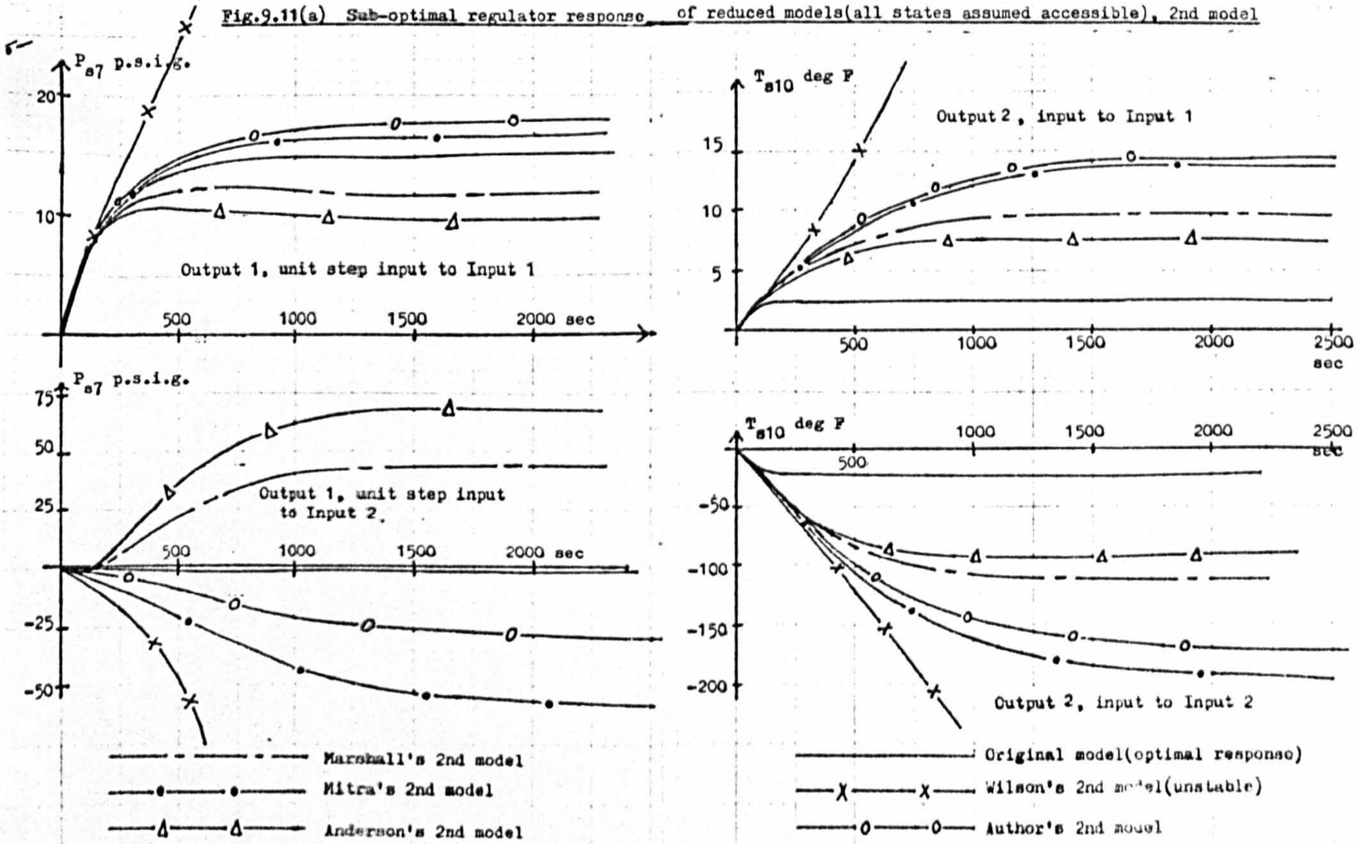


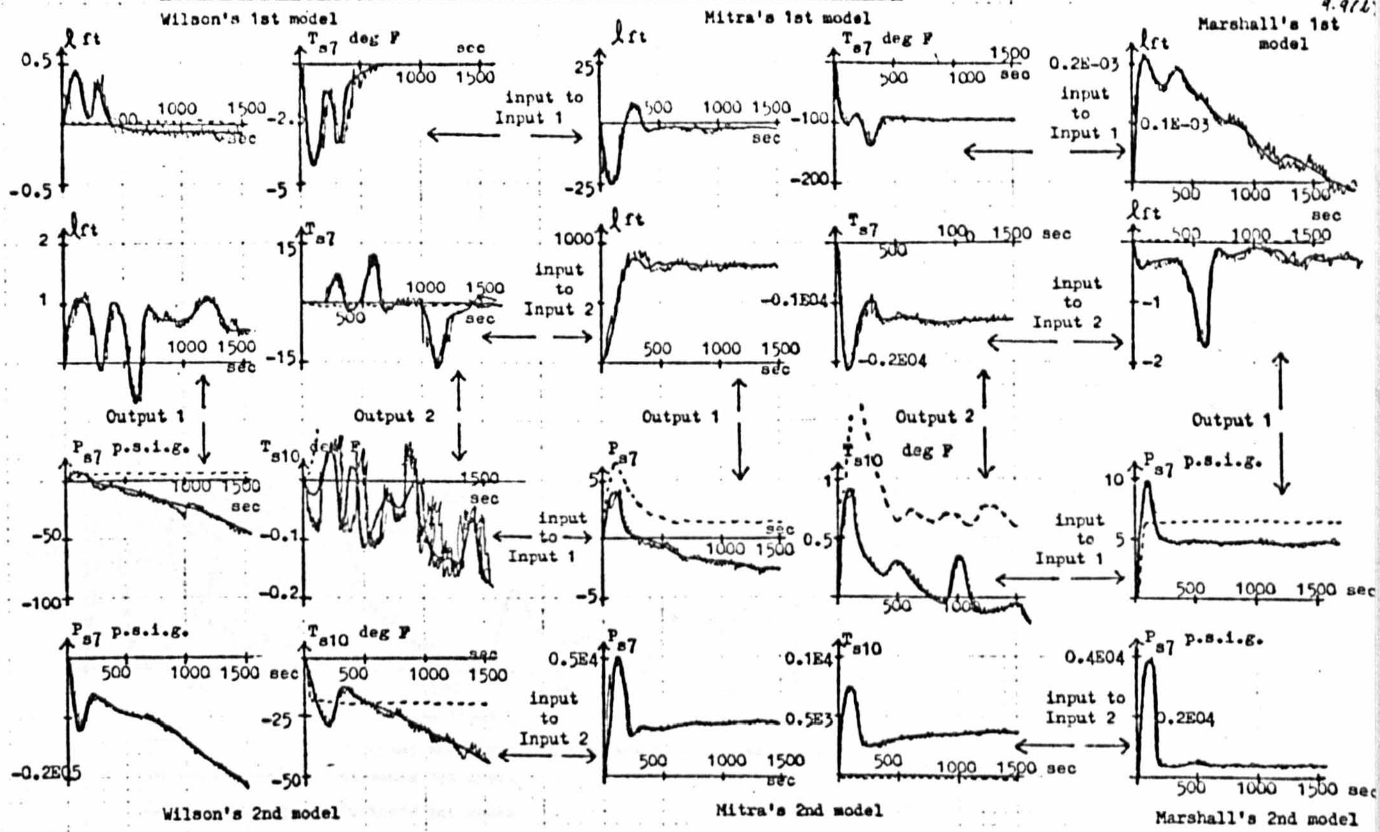
Fig.9.11(a) Sub-optimal regulator response of reduced models(all states assumed accessible), 2nd model



——— optimal response - - - - - - corresponding sub-optimal response

Fig. 9.8(b) Optimal regulator response, with filtering, of reduced models

→ continue on fig. 9.9(b)



——— optimal response - - - - - - corresponding sub-optimal response

→ continued from fig 9.8(b).

Fig. 9.9(b) Optimal regulator response, with filtering, of reduced models

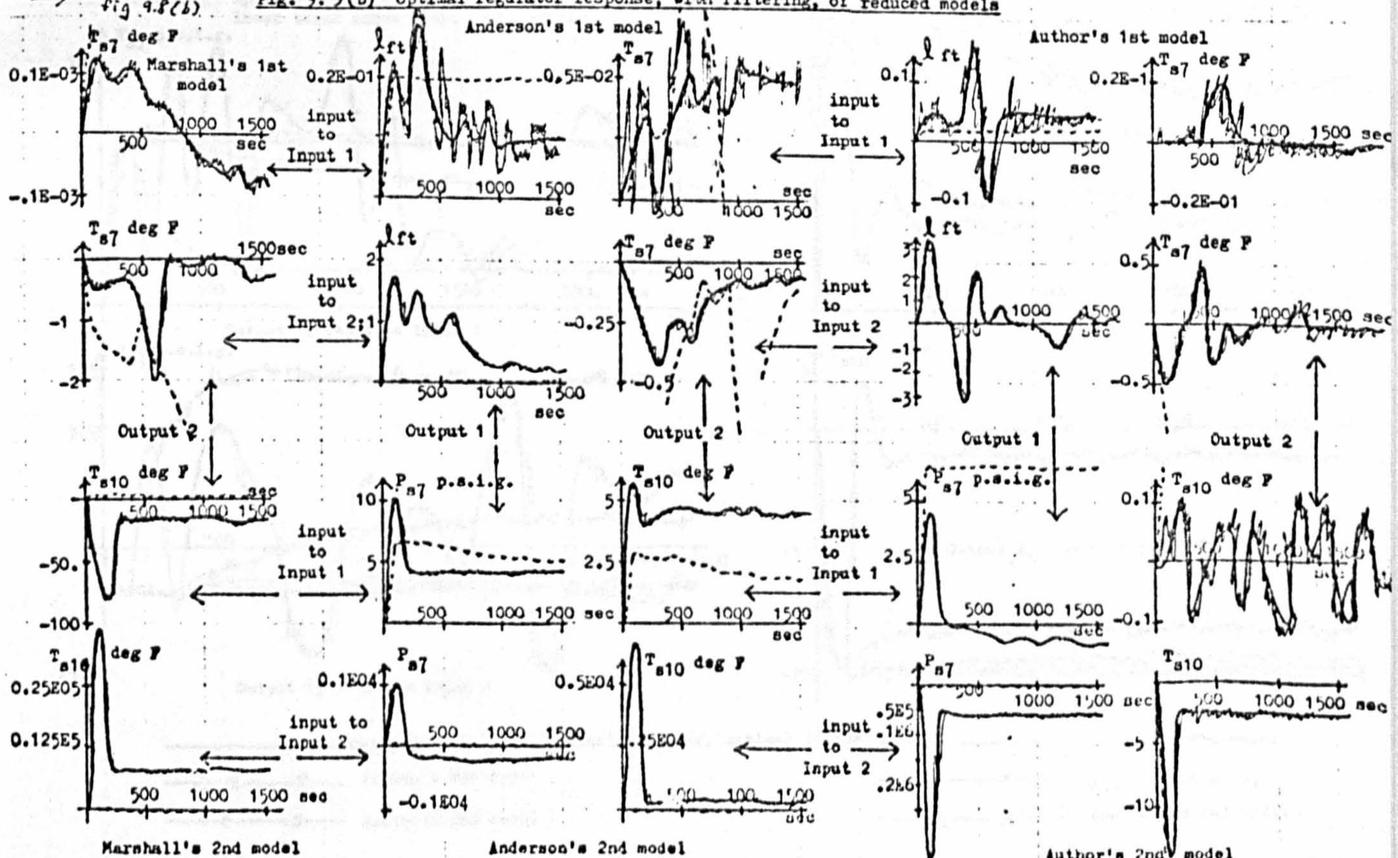


Fig. 9.10(b) Sub-optimal regulator response of 1st model(with sub-optimal filtering)

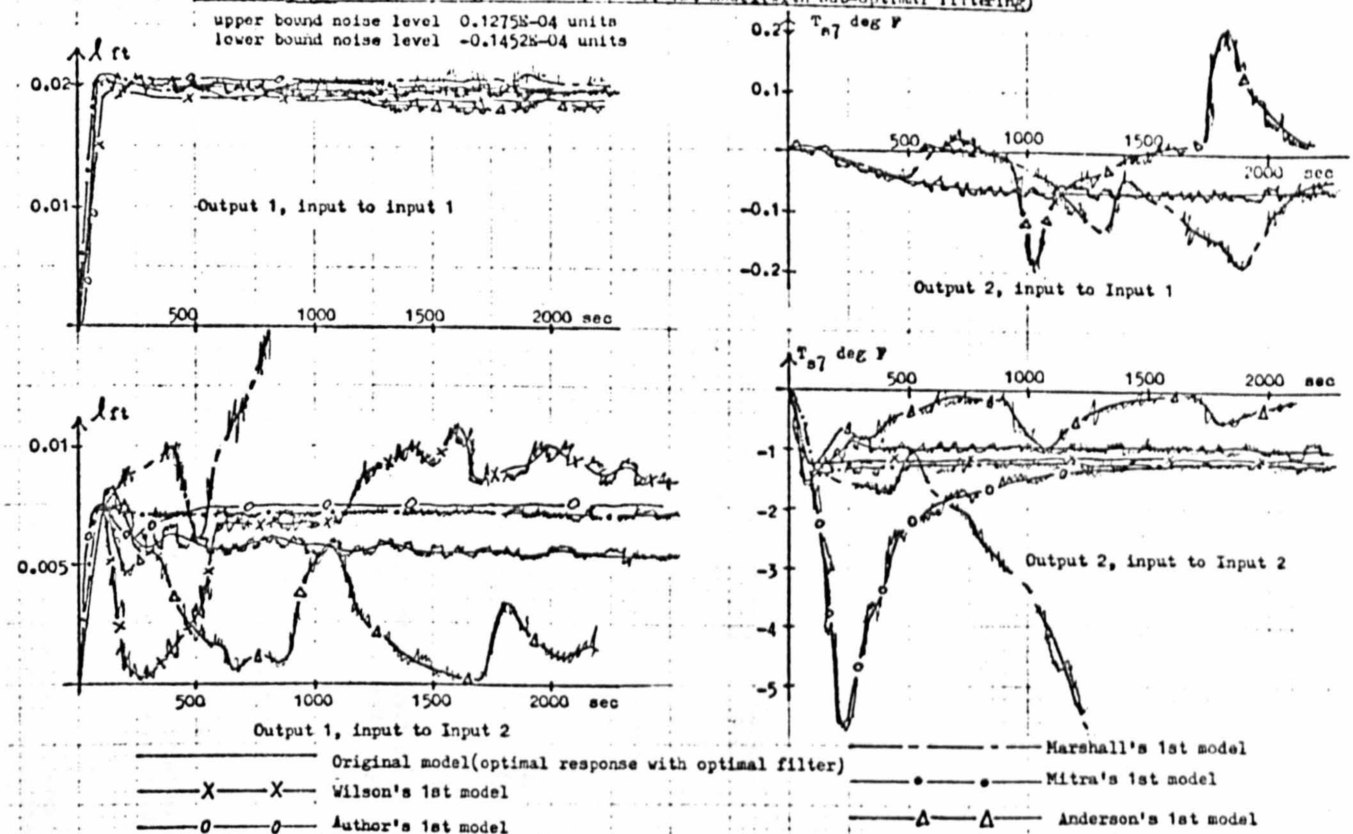


Fig. 9.11(b) Sub-optimal regulator response of 2nd model(with sub-optimal filtering)

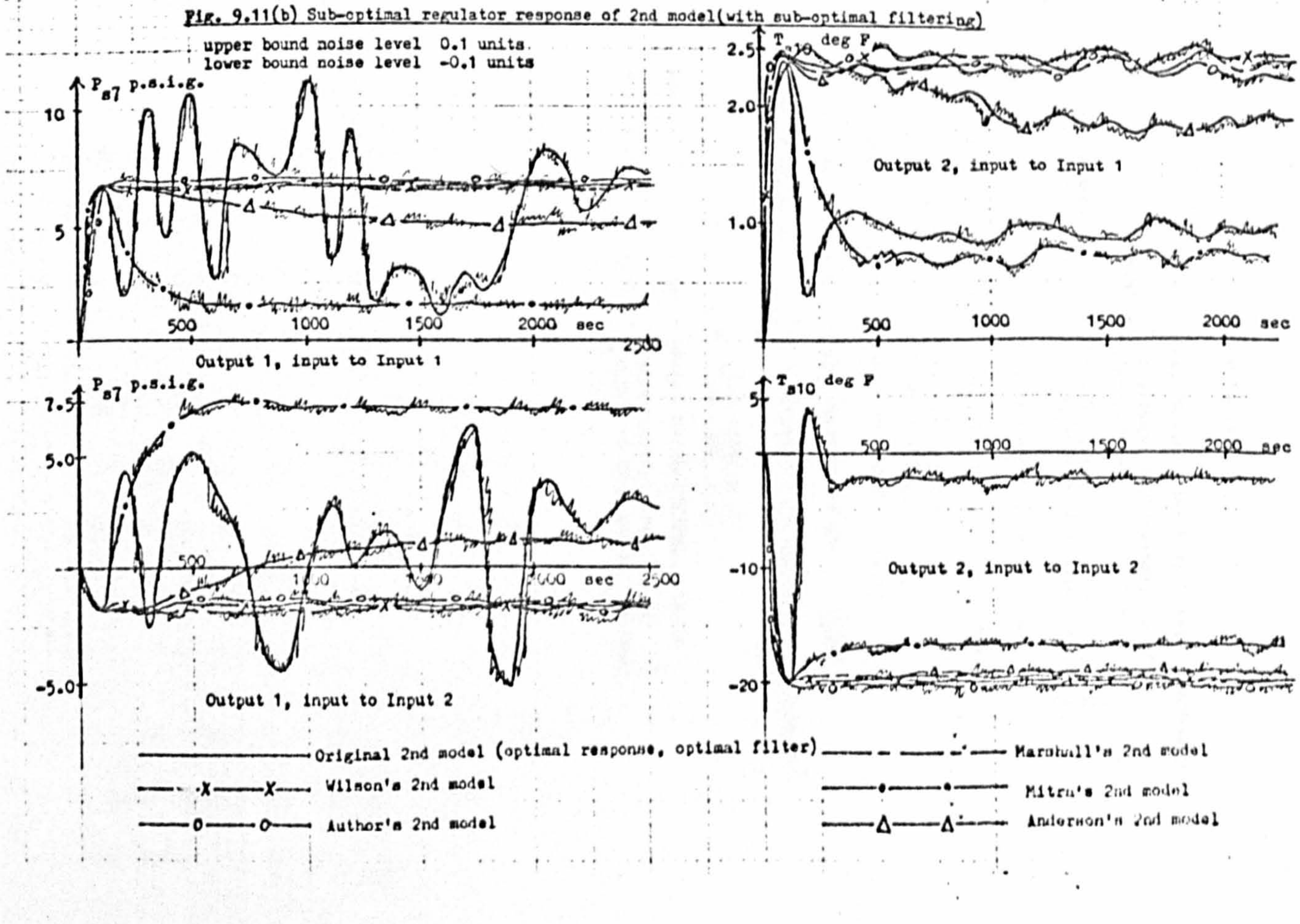


TABLE 9E i

Original model:

1st $Q = \mathrm{diag}(0,0.1E\text{-}03,0.1E\text{-}03,0,\ldots0)$; $R = \mathrm{dia}(0.1E\text{-}02,0.1E\text{-}02)$

2nd $Q = \mathrm{diag}(0,0,0,0.1E\text{-}03,0.1E\text{-}03,0,\ldots0)$; $R = \mathrm{diag}(0.1E\text{-}02,0.1E\text{-}02)$

Reduced model:

Wilson 1

$$Q_r = \begin{bmatrix} 0.1737E\text{-}05 & -0.7977E\text{-}10 \\ -0.7977E\text{-}10 & 0.335E\text{-}12 \end{bmatrix},\quad K_r = \begin{bmatrix} 0.2209E\text{-}01 & -0.1127E\text{-}06 \\ 0.1688E\text{-}02 & -0.7279E\text{-}05 \end{bmatrix}$$

$$Z = \left[\begin{array}{ccc|c} 0, & 0.4767E\text{-}01 & -0.7629E01 & 0 \\ 0, & -0.1727E05 & -0.1924E04 & \end{array}\right]$$

Wilson 2

$$Q_r = \begin{bmatrix} 0.3148E\text{-}04 & -0.3679E\text{-}04 \\ -0.3679E\text{-}04 & 0.4301E\text{-}04 \end{bmatrix},\quad K_r = \begin{bmatrix} 0.4036E\text{-}03, & -0.3766E\text{-}03 \\ -0.3742E\text{-}04, & 0.4133E\text{-}04 \end{bmatrix}$$

$$Z = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 3.141 & -5521.0 & 0 \\ 0 & 0 & 0 & 1.162 & -4724.0 & \end{array}\right]$$

Mitra 1

$$Q_r = \begin{bmatrix} 0.1E\text{-}03 & 0 \\ 0 & 0.1E\text{-}03 \end{bmatrix},\quad K_r = \begin{bmatrix} -0.5959E\text{-}03 & -0.4829E\text{-}02 \\ 0.2826E\text{-}03 & 0.1031E\text{-}03 \end{bmatrix}$$

$$Z = \left[\begin{array}{ccc|c} 0, & 1.0, & 0 & 0 \\ 0, & 0, & 1.0 & \end{array}\right]$$

Mitra 2

$$Q_r = \begin{bmatrix} 0.1E\text{-}03 & 0 \\ 0 & 0.1E\text{-}03 \end{bmatrix},\quad K_r = \begin{bmatrix} -0.2615E\text{-}01 & 0.1663E00 \\ 0.919E\text{-}04 & 0.2557E\text{-}03 \end{bmatrix}$$

$$Z = \left[\begin{array}{ccccc|c} 0, & 0, & 0, & 1.0, & 0 & 0 \\ 0, & 0, & 0, & 0, & 1.0 & \end{array}\right]$$

Marshall 1

$$Q_r = \begin{bmatrix} 0.1E\text{-}03 & 0 \\ 0 & 0.1E\text{-}03 \end{bmatrix},\quad K_r = \begin{bmatrix} 0.3246E\text{-}03 & -0.1274E\text{-}03 \\ -0.4441 & 0.135 \end{bmatrix}$$

$$Z = \left[\begin{array}{ccc|c} 0, & 1, & 0 & 0 \\ 0, & 0, & 1 & \end{array}\right]$$

Marshall 2

$$Q_r = \begin{bmatrix} 0.1E\text{-}03 & 0 \\ 0 & 0.1E\text{-}03 \end{bmatrix},\quad K_r = \begin{bmatrix} 0.3736E\text{-}01 & -0.3218E\text{-}02 \\ 0.8085E\text{-}04 & 0.1444E\text{-}04 \end{bmatrix}$$

$$Z = \left[\begin{array}{ccccc|c} 0, & 0, & 0, & 1, & 0 & 0 \\ 0, & 0, & 0, & 0, & 1 & \end{array}\right]$$

Anderson 1

$$Q_r = \begin{bmatrix} 0.1E\text{-}03 & 0 \\ 0 & 0.1E\text{-}03 \end{bmatrix},\quad K_r = \begin{bmatrix} 0.1703E\text{-}02 & 0.2251E\text{-}02 \\ 0.1695 & -0.3042 \end{bmatrix}$$

$$Z = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \end{array}\right]$$

Anderson 2

$$Q_r = \begin{bmatrix} 0\text{-}1E\text{-}03 & 0 \\ 0 & 0.1E\text{-}3 \end{bmatrix},\quad K_r = \begin{bmatrix} 0.595E\text{-}01 & -0.1895E\text{-}02 \\ -0.1151E\text{-}03 & 0.1513E\text{-}03 \end{bmatrix}$$

$$Z = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \end{array}\right]$$

Author 1

$$Q_r = \begin{bmatrix} 0.1877E\text{-}10 & 0.1417E\text{-}07 \\ 0.1417E\text{-}07 & 0.1075E\text{-}04 \end{bmatrix},\quad K_r = \begin{bmatrix} 0.7818E\text{-}05 & 0.3166E\text{-}02 \\ -0.6626E\text{-}04 & -0.8063E\text{-}02 \end{bmatrix}$$

$$Z = \left[\begin{array}{ccc|c} 0 & -65.17 & 0.3217E05 & 0 \\ 0 & 3.136 & -42.38 & \end{array}\right]$$

Author 2

$$Q_r = \begin{bmatrix} 0.3148E\text{-}04 & -0.3679E\text{-}04 \\ -0.3679E\text{-}04 & 0.4301E\text{-}04 \end{bmatrix},\quad K_r = \begin{bmatrix} -0.5686E\text{-}07 & 0.1614E\text{-}06 \\ -0.1262E\text{-}05 & 0.1392E\text{-}05 \end{bmatrix}$$

$$Z = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 3.141 & -5520.0 & 0 \\ 0 & 0 & 0 & 1.162 & -4723.6 & \end{array}\right]$$

For frequency response models, $L(s)$ or $L_r(s) = I$, unless otherwise stated.

Table 9E ii

Original model:

1st $K(s) = \mathrm{diag}\ (0.118(0.3s+1)/(0.05s+1),\ -0.01667)$

2nd
$$K(s) = \begin{bmatrix} (0.4225s+1.408)/(s+20), & 0.07 \\ -(3s+10)/(s+20), & 0.5 \end{bmatrix}$$

Reduced model:

Chen & Shieh 1
$$K_r(s) = \begin{bmatrix} (6s+20)/(s+20), & 0 \\ (-0.0225s-0.075)/(s+20), & -0.005 \end{bmatrix}$$

Chen & Shieh 2 $K_r(s) = \mathrm{diag}\ \{(6s+20)/(s+20), -0.5\}$

Sinha & Pille 1 $K_r(s) = \mathrm{diag}\ \{(6s+20)/(s+20, -0.005\}$

Sinha & Pille 2
$$K_r(s) = \begin{bmatrix} (6s+20)/(s+20) & 0 \\ (0.6s+2)/(s+20) & -0.1 \end{bmatrix}$$

Mod. Levy 2 $K_r(s) = \mathrm{diag}\ \{(0.3s+1)/(0.05s+1), -1\}$

Riggs & Edgar 1; $K_r(s) = \mathrm{diag}\ \{(6s+20)/(s+20), -0.02\}$

Riggs & Edgar 2;
$$K_r(s) = \begin{bmatrix} (9.333E\text{-}06s+2.667E\text{-}05)/(s+20), & 1.333E\text{-}06 \\ 0 & 1.0E\text{-}05 \end{bmatrix}$$

$$L_r(s) = \begin{bmatrix} 0.4348, & 0 \\ -0.4348, & 1.0 \end{bmatrix}$$

Author 1 $K_r(s) = \mathrm{diag}\ \{(6s+20)/(s+20), -1.0\}$

Author 2
$$K_r(s) = \begin{bmatrix} (0.06s+0.2)/(s+20), & 0 \\ (0.006s+0.02)/(s+20), & -0.002 \end{bmatrix}$$

9.3.1 Discussions

It can be seen from the graphs, figs. 9.10(a) and 9.11(a), noise free case, that some sub-optimal responses do not guarantee stability, unlike all optimal responses. A prominent feature of the sub-optimal responses, for both models one and two, is that they have very close transient periods compared to the relatively large offsets in steady state values. The open loop responses of the reduced models have small transient errors compared to their steady state errors, and in general, no hard and fast rules exist that relate the open loop performance of S and S_r to the sub-optimal and optimal response of S. The optimal responses of S_r, on the other hand, guarantee stability, figs. 9.8(a) and 9.9(a), but the errors, transient and steady state, between S_r and, the sub-optimal response, S_{sub} are relatively large compared to those between S and S_{sub}. All the responses computed correspond to a given set of weighting matrices, Q and R, only.

The responses involving inaccessible states and the use of the Kalman filter are shown in figs. 9.8(b) and 9.9(b) and those involving the sub-optimal filter in figs. 9.10(b) and 9.11(b). For both models, one and two, the signal to noise ratio was kept large, by introducing a zero-mean gaussion noise generator of small magnitude. This was to ensure, sufficient for the purpose of this experiment, that the measurements were not overwhelmingly contaminated by noise. Again, as can be seen from the graphs, the errors of the filter responses between S_r and S_{sub} are larger than those between S and S_{sub}. As expected, the filter responses are 'irregular', due to the nature of the state estimator, compared to the smoother responses without state estimation. The filter responses of S and S_{sub} of the first model, in general, have small transient times compared to those of the second model. Of particular interest, the first output, y, of S, of the second model has very large transient times, while the output y_2, of S, has small transient times, characterised by peak magnitudes, and small steady state values. A common feature of the sub-optimal response, S_{sub}, is that they all settle and yield small steady state errors with the optimal response, S. This is more noticeable in the first model than in the second. Like the sub-optimal regulator with accessible states, the sub-optimal inaccessible state case does not guarantee stability, while the optimal filters of S and S_r are all stable.

9.4 Frequency response design using the Inverse Nyquist Array (INA)[10].

The design here was done using the CAD package on the PDP-10 computer at UMIST. The INA design method, by Rosenbrock, was used to design compensators for the reduced transfer function matrix models, obtained in section 9.2.2.

Algorithm for INA design:

The controller $K_r(s) = K_a K_b(s) K_c(s)$ is designed as follows. $\hat{K}(s) = \hat{K}_c \hat{K}_b(s) \hat{K}_a$ where $\hat{K}_a$ is a permutation matrix for preliminary renumbering of inputs to G(s), so that the i^{th} input is regulated by the i^{th} output. $\hat{K}_b(s) = \hat{K}_{bn}(s)\ldots\hat{K}_{b1}(s)$ is chosen to make $\hat{Q}(s) = \hat{K}_c \hat{K}_b(s) \hat{G}(s)$ diagonal dominant, and, it consists of elementary row or column operations. $\hat{K}_c(s)$ is diagonal, and is designed at the last stage using single loop approach.

(i) Obtain $\hat{G}(s)$ (i.e. $G^{-1}(s)$)

(ii) Choose $\hat{K}_a$ as desired

(iii) Design $\hat{K}_b(s) = \hat{K}_{bn}(s)\ldots\hat{K}_{b1}(s)$ such that interaction is reduced, $\forall s = j\omega \in \Omega$, choosing $\hat{K}_{b1} = G(0)$ giving $\hat{Q}_1(s) = \hat{K}_{b1}(s)\ \hat{G}(s)$; final form $\hat{Q}_n(s) = \hat{K}_{nr}(s)\ \hat{Q}_{n-1}(s)$.

Check stability from INA theorems, (or from characteristic Loci criterion, to give sufficient and necessary conditions).

(iv) Choose $K_c(s)$ to be simple, example first order lag or simple proportional controllers, $\hat{Q}(s) = \hat{K}_c(s)\ \hat{Q}_n(s)$; check stability of overall system from INA theorems. Go to step (iii) if necessary.

(v) $\hat{K}_r(s) = \hat{K}_c(s)\ \hat{K}_b(s)\ \hat{K}_a$; $K_r(s) = (\hat{K}(s))^{-1}$.

(vi) Simulate closed loop step response for $R(s) = (I + Q(s))^{-1}\ Q(s)$.

The post compensator $L_r(s)$ can be designed, if necessary, in the same way using column operation.

9.4.1 Design examples

Two design examples are chosen here to illustrate the INA method.

(a) Author's 2nd model; $u = $ (fuel, damper); $y = (P_{s7}, T_{s10})$

1st stage:

Set $\hat{K}_a = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and $\hat{L} = I$, since fuel output was chosen to regulate output, P_{s7}, and damper input chosen to regulate output, T_{s10}. INA diagram was plotted for frequency range 0 to 10 radians, increment 0.1 radian.

2nd stage:

Choose $\hat{K}_{b1} = \begin{bmatrix}1 & 0\\0 & 5\end{bmatrix}$ such that element $\hat{q}_{22}(s)$ is scaled 5 times (see first diagram of fig. 9.12(a)). Interaction is heavy in second loop and system possibly closed loop unstable.

Choose $\hat{K}_{b2} = \begin{bmatrix} 1 & 0 \\ 0.5 & -1 \end{bmatrix}$, with view to reduce interaction in loop 2 and ensure closed loop stability by inverting element $\hat{q}_{22}(s)$. (See 2nd diagram of fig. 9.12(a)).

3rd stage:

Interaction reduction satisfactory. Phase-advance compensator $\hat{K}_3(s) = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} \frac{0.05s+1}{0.3s+1} & 0 \\ 0 & 1 \end{bmatrix}$ was added to the first loop to improve performance. Resulting INA diagram found satisfactory and stable (3rd diagram of fig. 9.12(a)).

Final controller $\hat{K}_r(s) = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} \frac{0.05s+1}{0.3S+1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0.5 & -1 \end{bmatrix}$

$$\therefore \quad K_r(s) = \begin{bmatrix} \frac{0.06s+0.2}{s+20} & 0 \\ \frac{0.006s+0.02}{s+20} & -0.002 \end{bmatrix}$$

The closed-loop response of the above design is shown in fig. 9.12(b). Both loops have low interactions to unit step inputs.

(b) Author's 1st = model; u = (feed, spray); y = (ℓ, T_{s7})

1st stage:

Set $\hat{K}_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\hat{L} = I$, as first input chosen to regulate first output, and second input chosen to regulate second output.

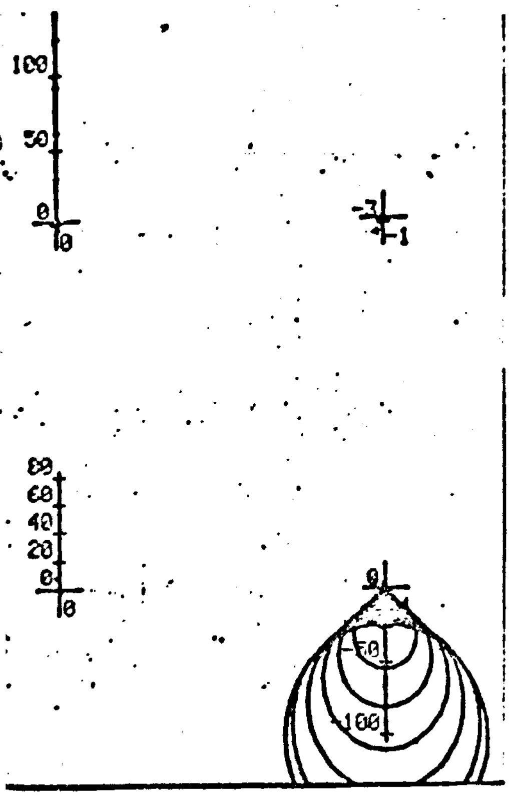
Choose $\hat{K}_{b1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to invert $\hat{q}_{22}(s)$, in view of stability.

2nd stage:

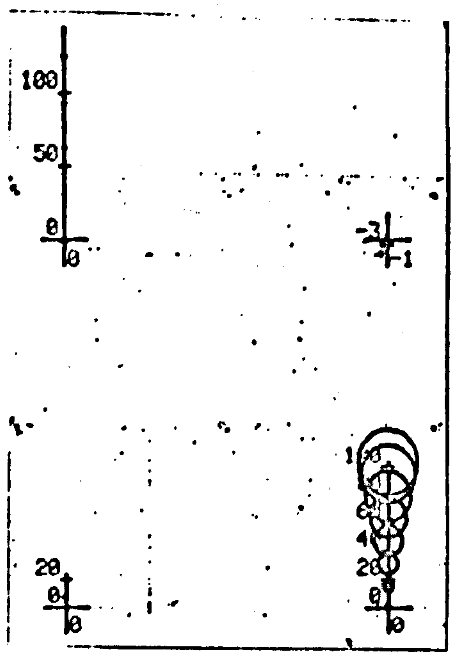
In this example, not much emphasis was laid on interaction, however, interaction was sufficiently reduced in both loops. Again phase-advance compensator $\hat{K}_3(s) = \begin{bmatrix} (0.05s+1)/(0.3s+1) & 0 \\ 0 & 1 \end{bmatrix}$ was introduced in first loop.

Final INA diagram shown in fig. 9.13(a) and time response shown in fig. 9.13(b). Final controller $K_r(s) = \begin{bmatrix} (6s+20)/(s+20) & 0 \\ 0 & -1 \end{bmatrix}$

Fig.9.12(a) Inverse Nyquist Array(INA) diagrams of Author's 2nd model(illustration)

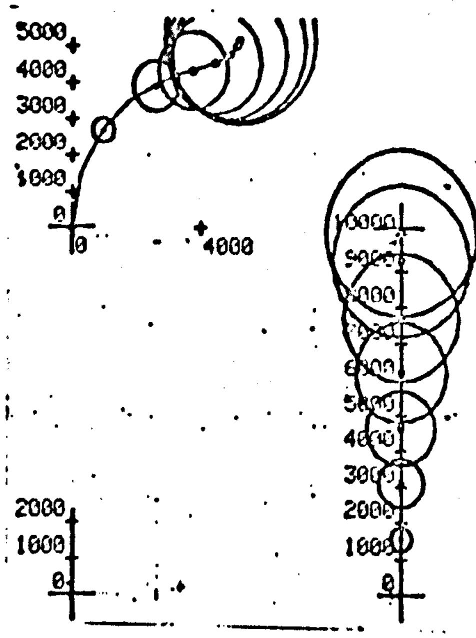


$\hat{K}_r = \begin{bmatrix}1 & 0\\ 0 & 5\end{bmatrix}$ GCR circles



$\hat{K}_r = \begin{bmatrix}1 & 0\\ 0.5 & -1\end{bmatrix}$ GCR circles

$K_r(s) = \begin{bmatrix}\frac{0.06s+0.2}{s+20} & 0\\ \frac{0.006s+0.02}{s+20} & -0.002\end{bmatrix}$



$\hat{K}_r(s) = \begin{bmatrix}100 & 0\\ 0 & 100\end{bmatrix}\begin{bmatrix}\frac{0.05s+1}{0.3s+1} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0.5 & -1\end{bmatrix}$

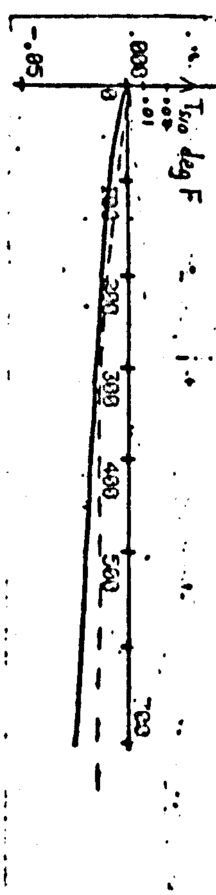
$L_r(s) = I$ GCC & GCR circles

Output responses y_1 and y_2 following unit step input to Input 2

—— K_r on reduced model ------ K_r on original model

Fig. 9.12(b) Closed loop response of Author's 2nd model

Output responses y_1 and y_2 following unit step input to Input 1



The other reduced models were used in design like in the above examples. Their INA diagrams are shown in figs. 9.14(a) to 9.20(a) and their time responses shown in figs. 9.14(b) to 9.20(b). Controllers were also designed independently by using the two original models. This is shown in figs. 9.21 and 9.22. The responses of the original models and that of the reduced models' controllers on the original models are given in figs. 9.23 and 9.24.

9.4.2 Discussions

The simulation for the closed loop responses of the original models was done 'off-line' using the ICL digital computer, since their orders were too large for the PDP-10 interactive CAD package to handle.

If $S_1(A_1, B_1, C_1, D_1)$ represents the minimal realization of the pre-compensator $K_r(s)$ or $K(s)$ and S_2 (A_2, B_2, C_2) represents the state space description of the model, $G(s)$, then S (A, B, C) represents the tandem connection $Q(s) = G(s)\, K_r(s)$, where

$$A = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}, \quad C = (0 \;\; C_2)$$

The above representation can also be used for the post compensator, $L_r(s)$, and $G(s)$ by interchanging S_1 and S_2. By introducing unity feedback, the state space description for $R(s) = (I + Q(s))^{-1}\, Q(s)$ is given by $S_f(A-BC, B, C)$. The standard discrete transition matrix method can be used to compute the response of S_f.

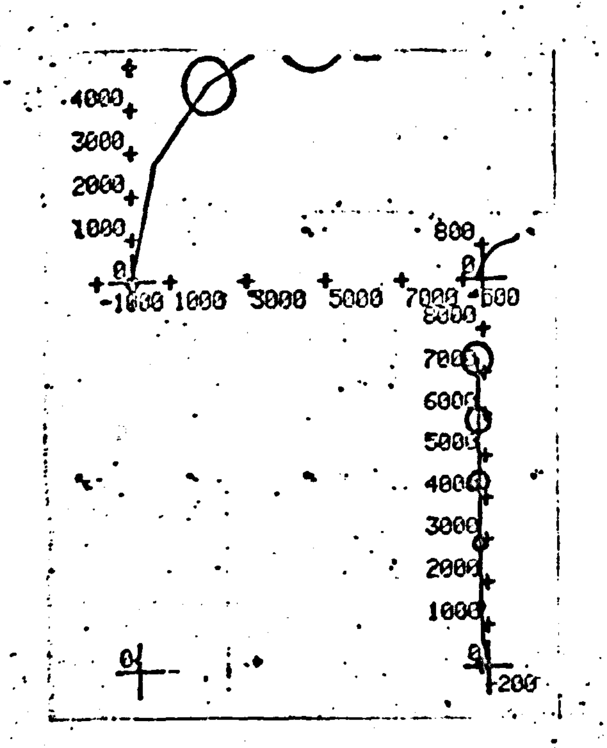


Fig.9.13(a) INA diagram of Author's 1st model

GCR circles

$$K_r(s) = \begin{pmatrix} \dfrac{6s+20}{s+20} & 0 \\ 0 & -1.0 \end{pmatrix} \qquad L_r(s) = I$$

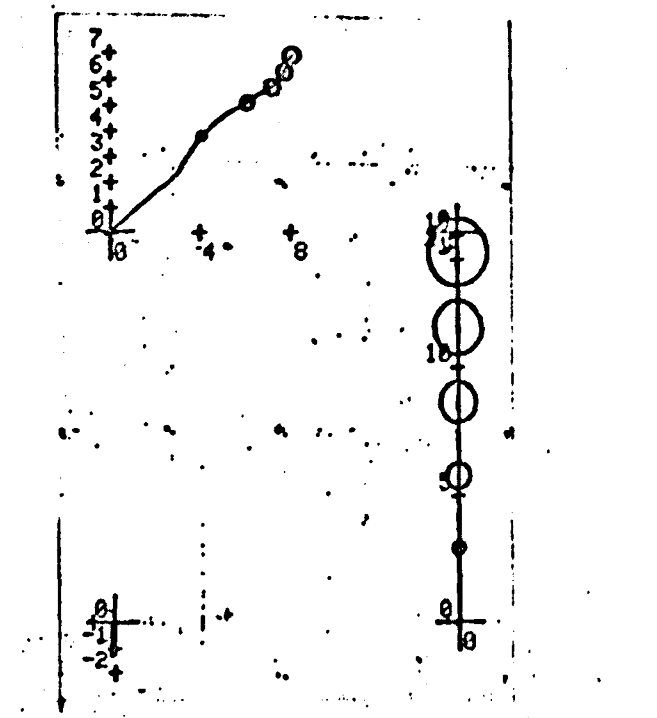


Fig.9.14(a) INA diagram of Modified Levy's 2nd model

GCR circles

$$K_r(s) = \begin{pmatrix} \dfrac{0.3s+1}{0.05s+1} & 0 \\ 0 & -1.0 \end{pmatrix} \qquad L_r(s) = I$$

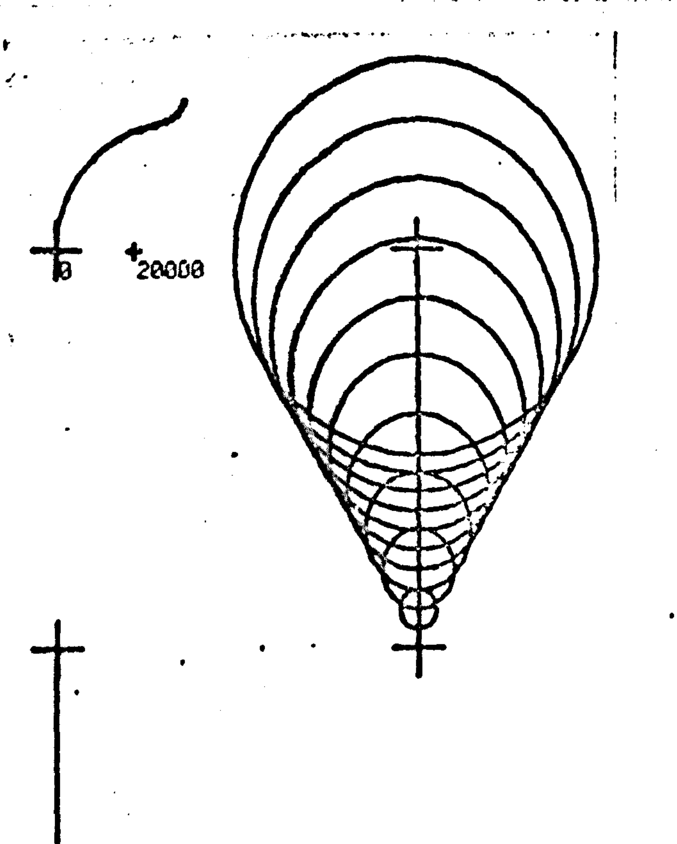


Fig.9.15(a) INA diagram of Chen&Shieh's 1st model

GCR circles

$$K_r(s) = \begin{pmatrix} \dfrac{6s+20}{s+20} & 0 \\ \dfrac{-0.0225s-0.075}{s+20} & -0.005 \end{pmatrix} \qquad L_r(s) = I$$

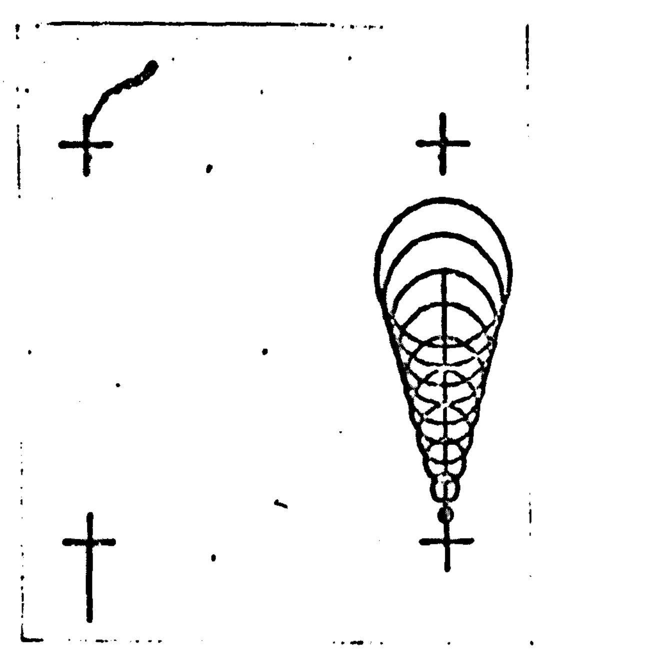
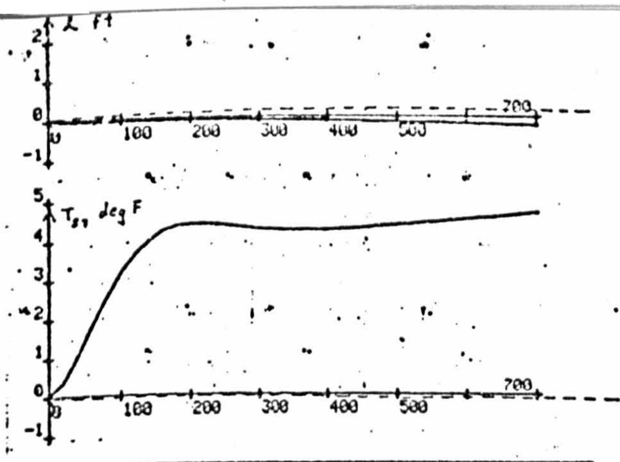


Fig.9.16(a) INA diagram of Chen&[illegible]n's 2nd model

GCR circles

$$K_r(s) = \begin{pmatrix} \dfrac{6s+20}{s+20} & 0 \\ 0 & -0.5 \end{pmatrix} \qquad L_r(s) = I$$

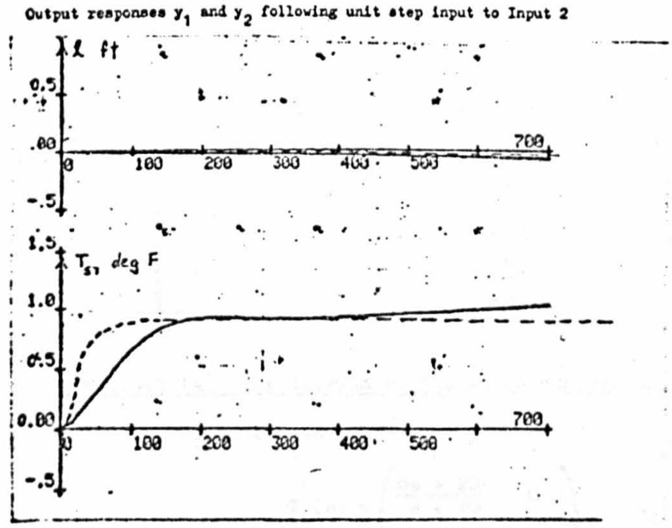


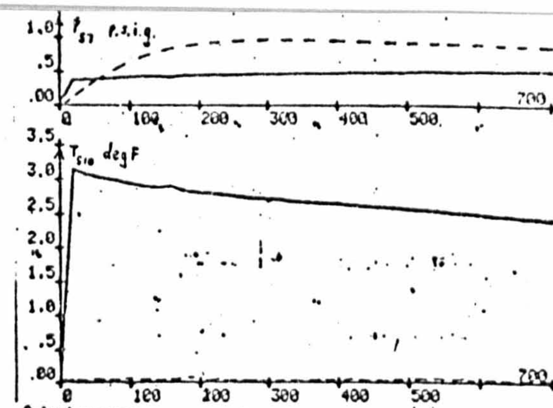
Output responses y_1 and y_2 following unit step input to Input 1

Fig. 9.13(b) Closed loop response of Author's 1st model

——————— K_r on reduced model - - - - - - - K_r on original model

Output responses y_1 and y_2 following unit step input to Input 2



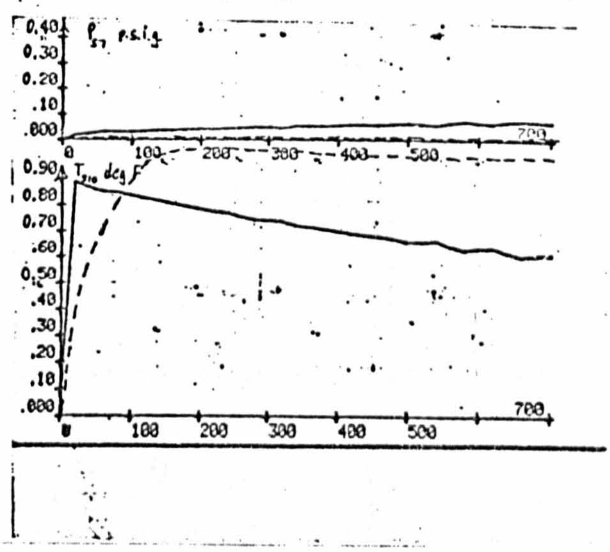


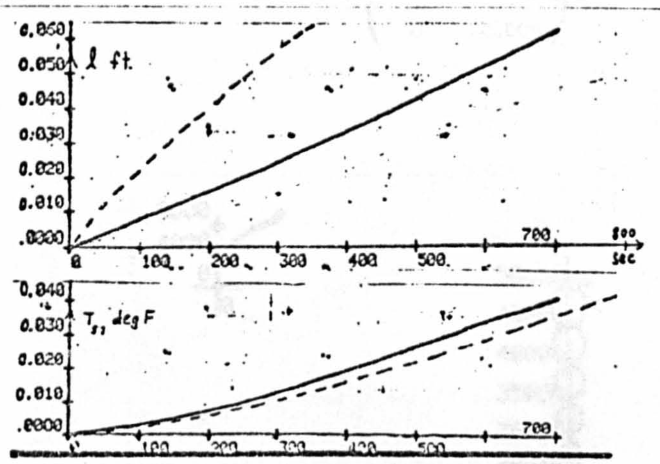
Output responses y_1 and y_2 following unit step input to Input 1

Fig. 9.14(b) Closed loop response of Modified Levy's 2nd model

——————— K_r on reduced model(unstable) - - - - - - - K_r on original model

Output responses y_1 and y_2 following unit step input to Input 2



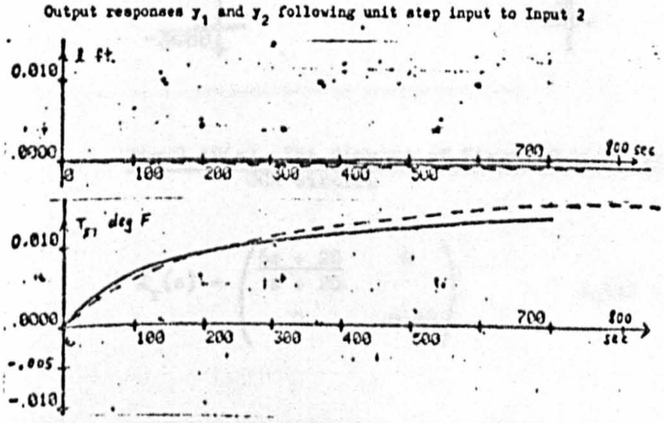


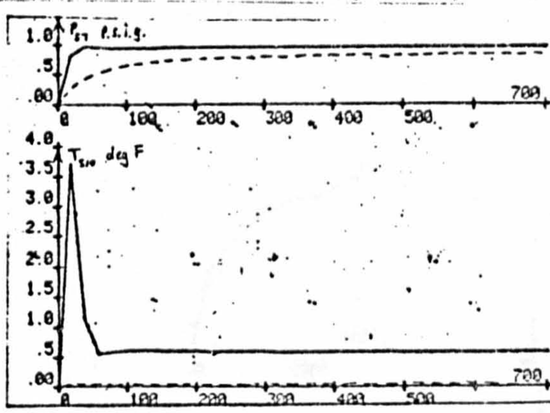
Output responses y_1 and y_2 following unit step input to Input 1

Fig.9.15(b) Closed loop response of Chen & Shieh's 1st model

——————— K_r on reduced model - - - - - - - K_r on original model

Output responses y_1 and y_2 following unit step input to Input 2



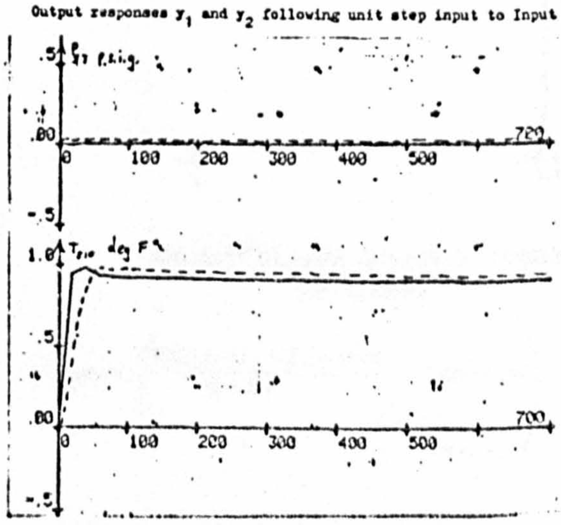


Output responses y_1 and y_2 following unit step input to Input 1

Fig.9.16(b) Closed loop response of Chen & Shieh's 2nd model

——————— K_r on reduced model - - - - - - - - K_r on original model

Output responses y_1 and y_2 following unit step input to Input 2



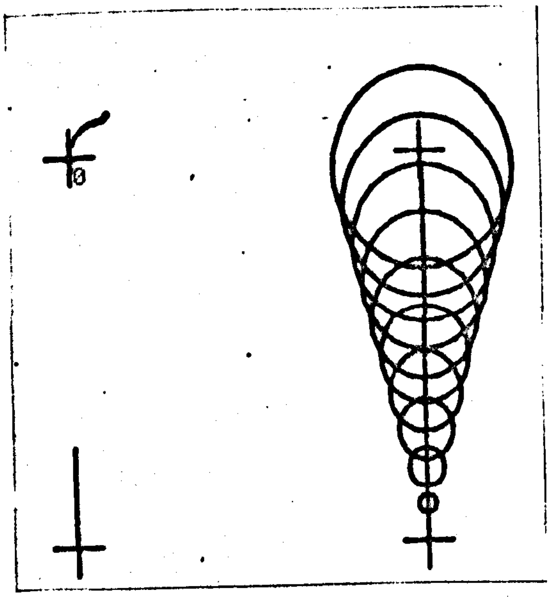


Fig.9.17(a) INA diagram of Sinha&Pille's 1st model

GCR circles

$$K_r(s) = \begin{pmatrix} \frac{6s + 20}{s + 20} & 0 \\ 0 & -0.005 \end{pmatrix} \qquad L_r(s) = I$$

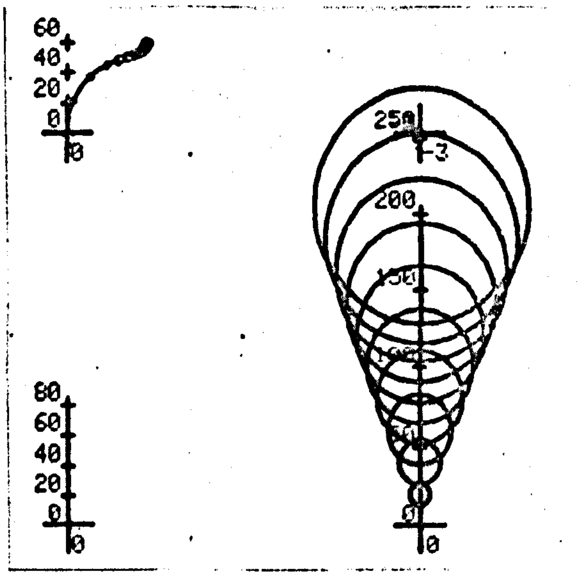


Fig.9.18(a) INA diagram of Sinha&Pille's 2nd model

GCR circles

$$K_r(s) = \begin{pmatrix} \frac{6s + 20}{s + 20} & 0 \\ \frac{0.6s + 2}{s + 20} & -0.1 \end{pmatrix} \qquad L_r(s) = I$$

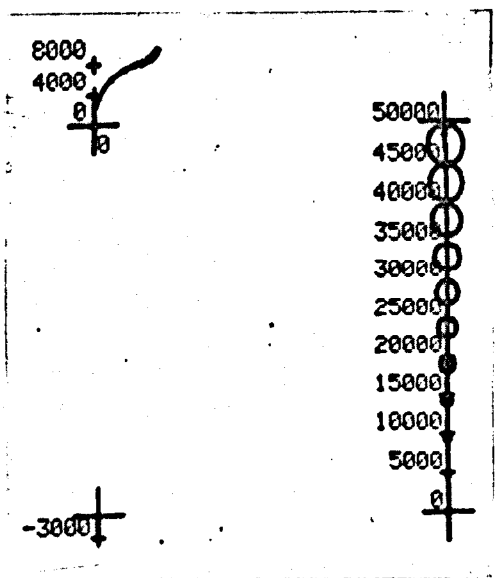


Fig.9.19(a) INA diagram of Riggs&Edgar's 1st model
GCR circles

$$K_r(s) = \begin{pmatrix} \frac{6s + 20}{s + 20} & 0 \\ 0 & -0.02 \end{pmatrix} \qquad L_r(s) = I$$

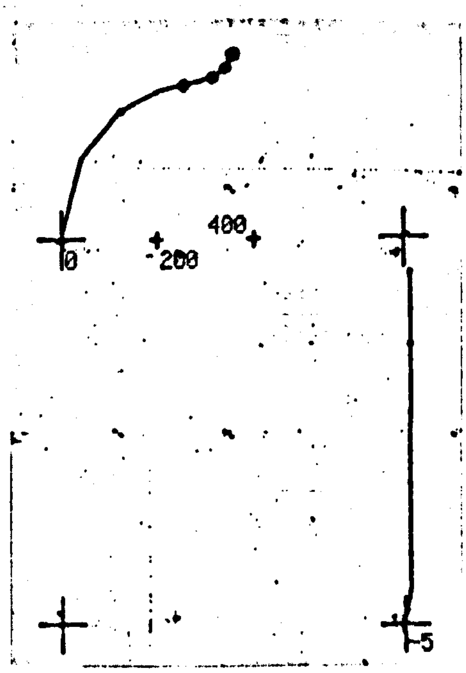
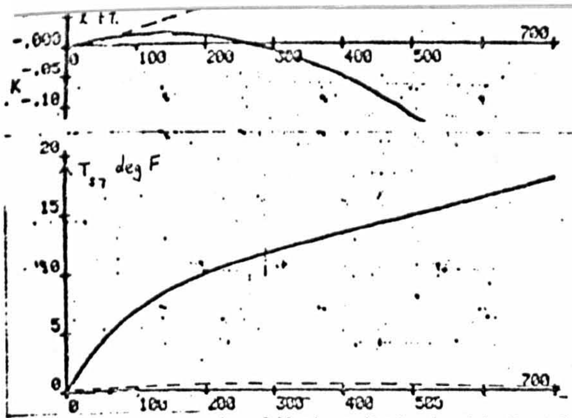


Fig.9.20(a) INA diagram of Riggs&Edgar's 2nd model

GCR circles

$$K_r(s) = \begin{pmatrix} \frac{9.333E-06\,s + 2.666E-05}{s + 20} & 1.333E-06 \\ 0 & 1.0E-05 \end{pmatrix}, \quad L_r(s) = \begin{pmatrix} 0.4348, & 0 \\ -0.4348, & 1.0 \end{pmatrix}$$

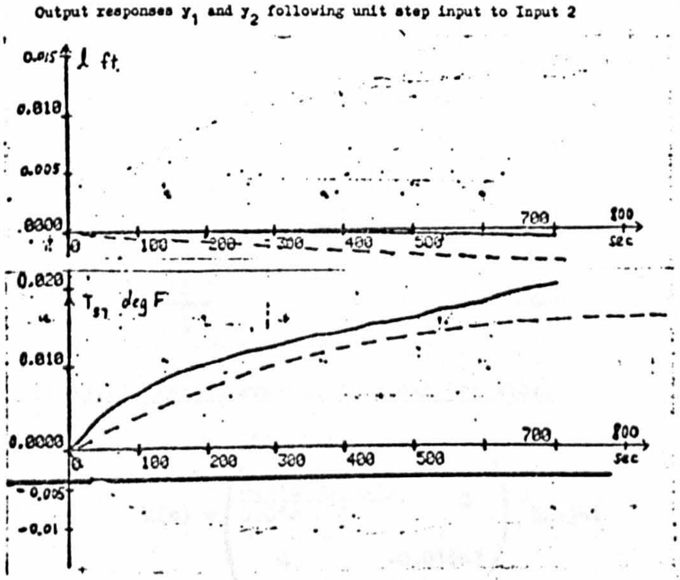


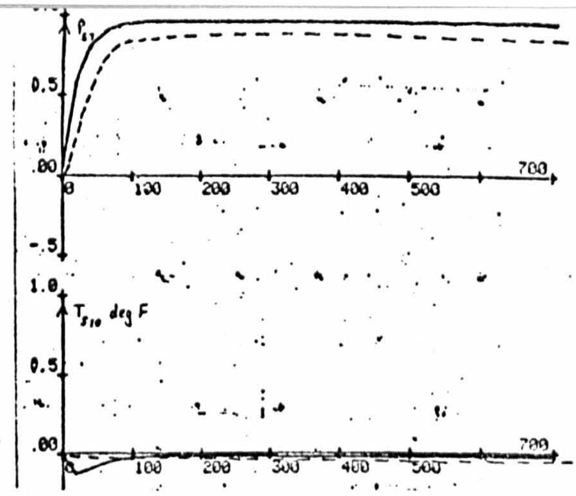
Output responses y_1 and y_2 following unit step input to Input 1

Fig.9.17(b) Closed loop response of Sinha & Pille's 1st model

——— K_r on reduced model - - - - - K_r on original model

Output responses y_1 and y_2 following unit step input to Input 2



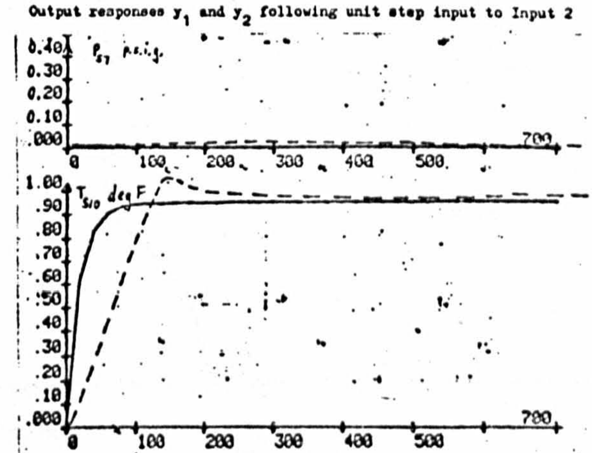


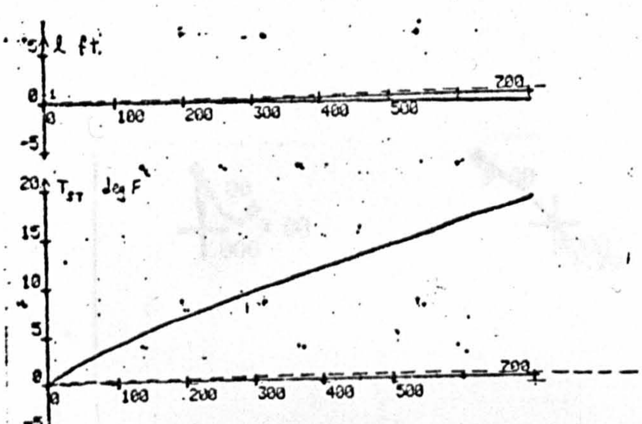
Output responses y_1 and y_2 following unit step input to Input 1

Fig.9.18(b) Closed loop response of Sinha & Pille's 2nd model

——— K_r on reduced model - - - - - K_r on original model

Output responses y_1 and y_2 following unit step input to Input 2



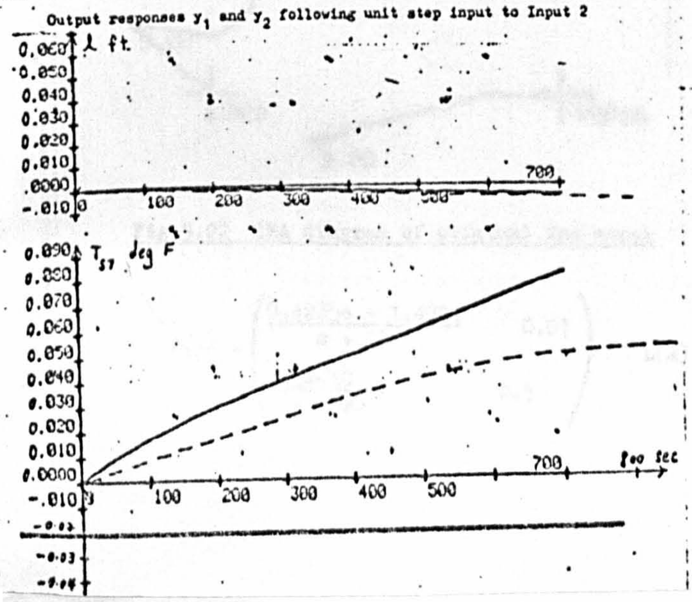


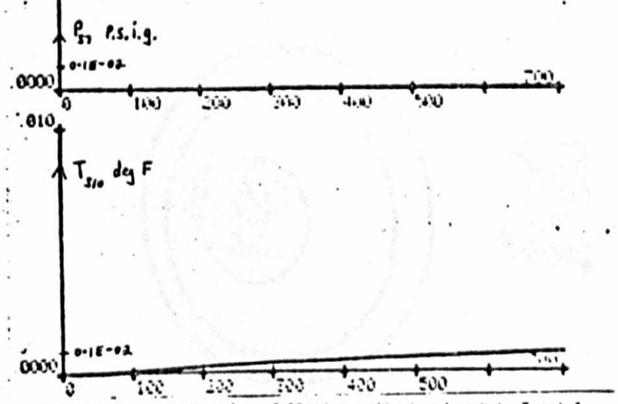
Output responses y_1 and y_2 following unit step input to Input 1

Fig.9.19(b) Closed loop response of Riggs & Edgar's 1st model

——— K_r on reduced model - - - - - K_r on original model

Output responses y_1 and y_2 following unit step input to Input 2



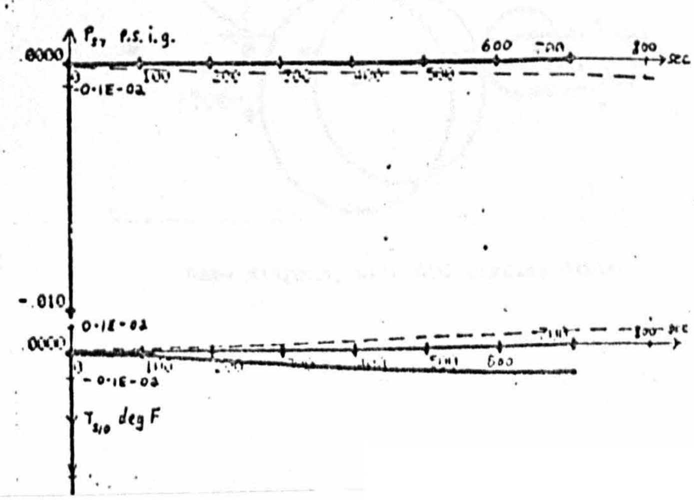


Output responses y_1 and y_2 following unit step input to Input 1

Fig.9.20(b) Closed loop response of Riggs & Edgar's 2nd model

——— K_r on reduced model - - - - - K_r on original model

Output responses y_1 and y_2 following unit step input to Input 2



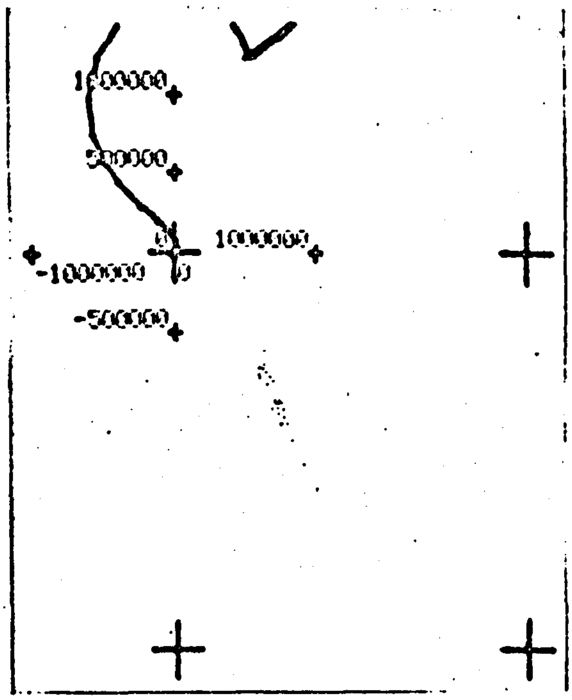
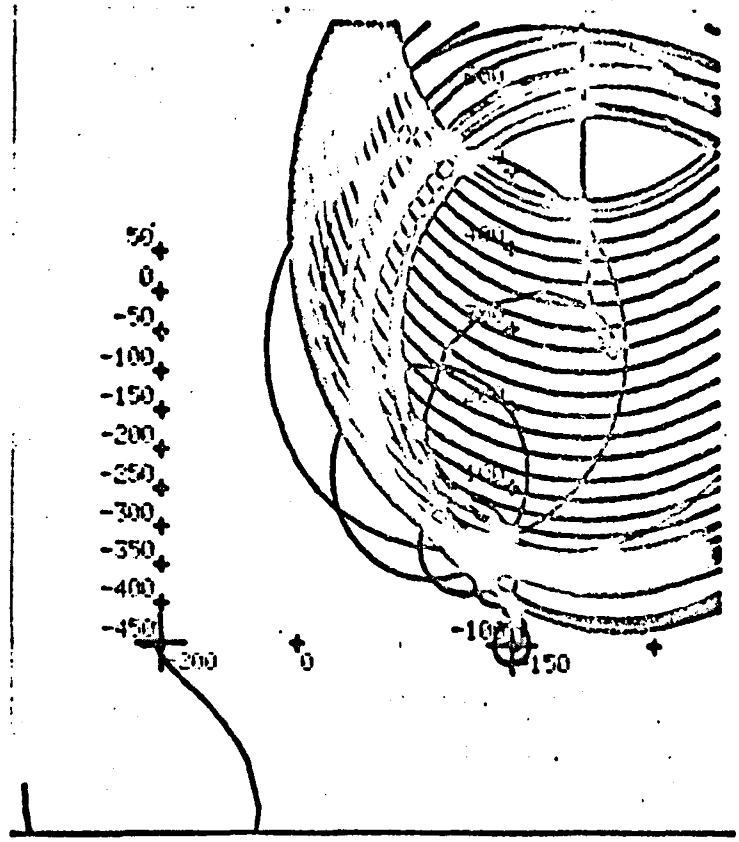


Fig.9.21 INA diagram of original 1st model

GCR circles

$$K(s) = \begin{pmatrix} \dfrac{0.118(0.3s+1)}{0.05s+1} & 0 \\ 0 & -0.01667 \end{pmatrix}, \quad L(s) = I$$



Second row of INA diagram rescaled 500 times
GCR circles

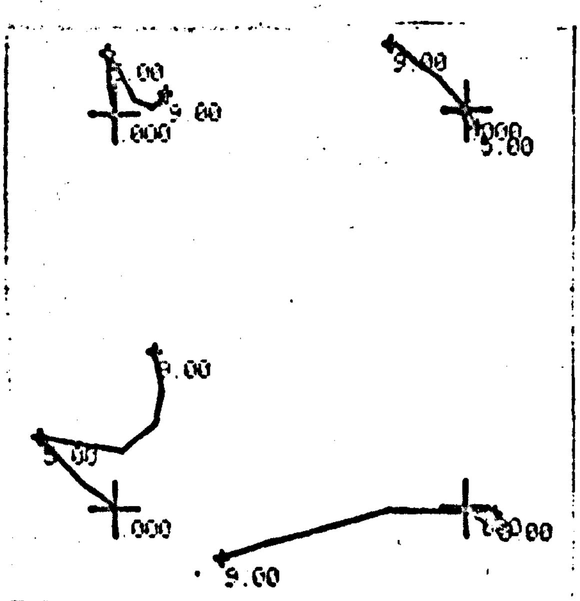
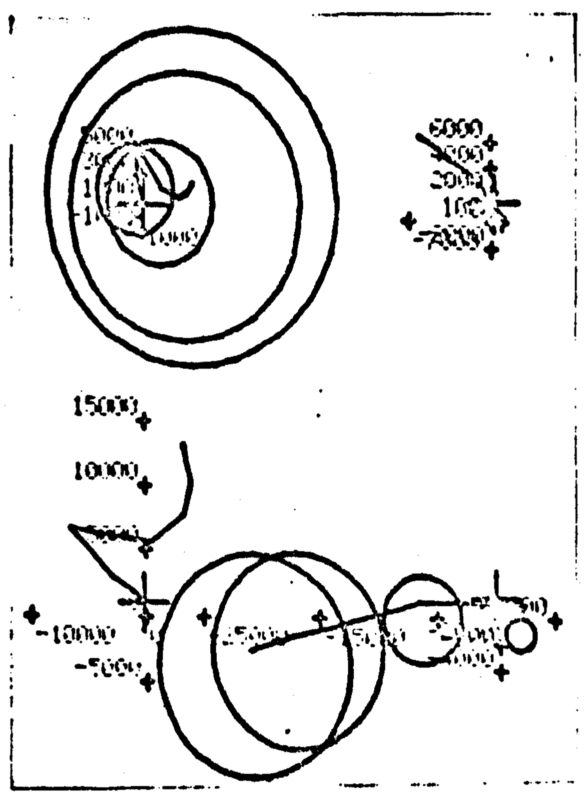


Fig.9.22 INA diagram of original 2nd model

$$K(s) = \begin{pmatrix} \dfrac{0.42[illegible]s + 1.4084}{s+20} & 0.07 \\ \dfrac{-3s-10}{s+20} & 0.5 \end{pmatrix} \quad L(s) = I$$



same diagram, with GCC circles drawn

Fig. 9.23 Comparison of reduced models Controller Action on Original model

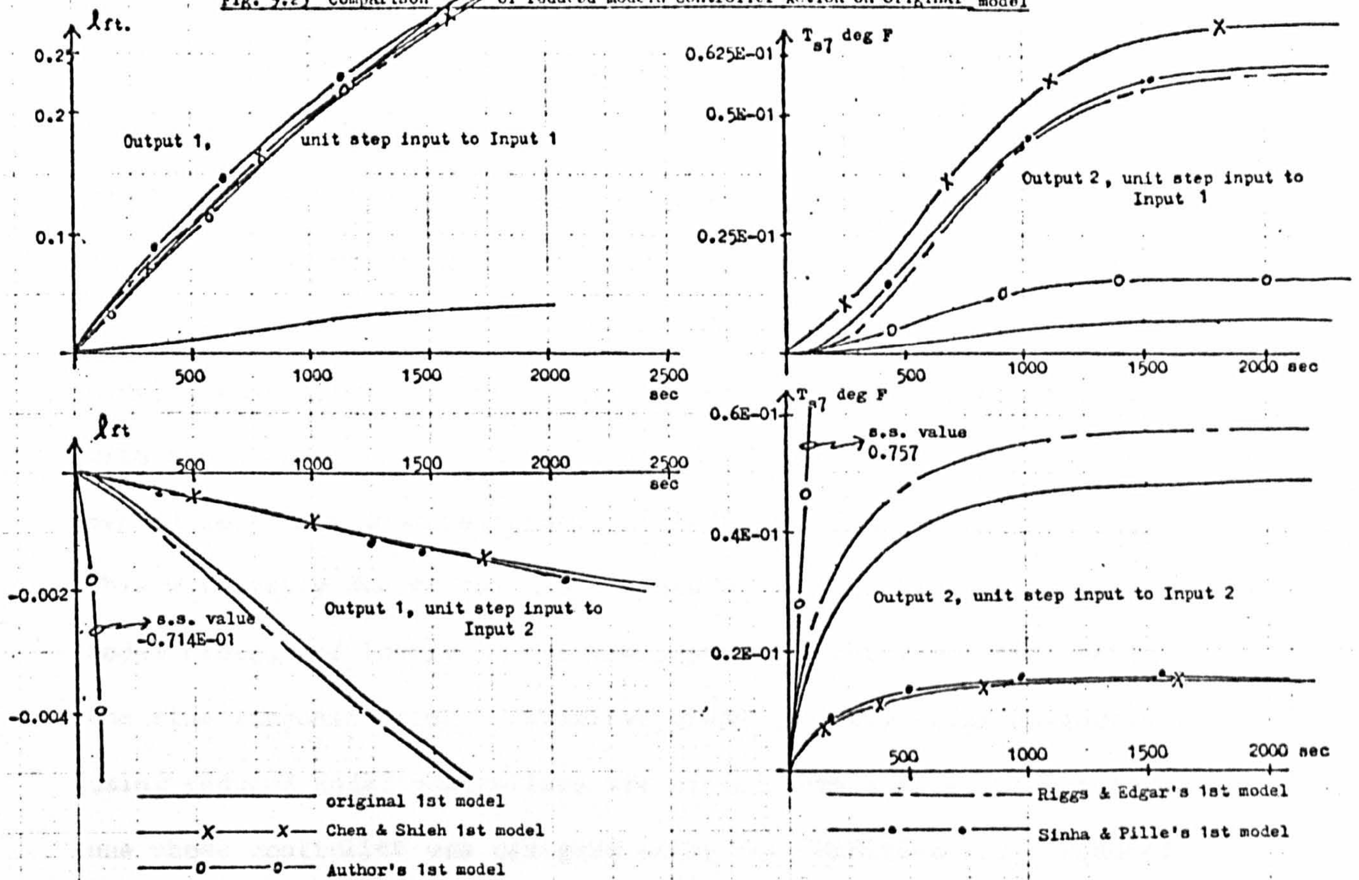
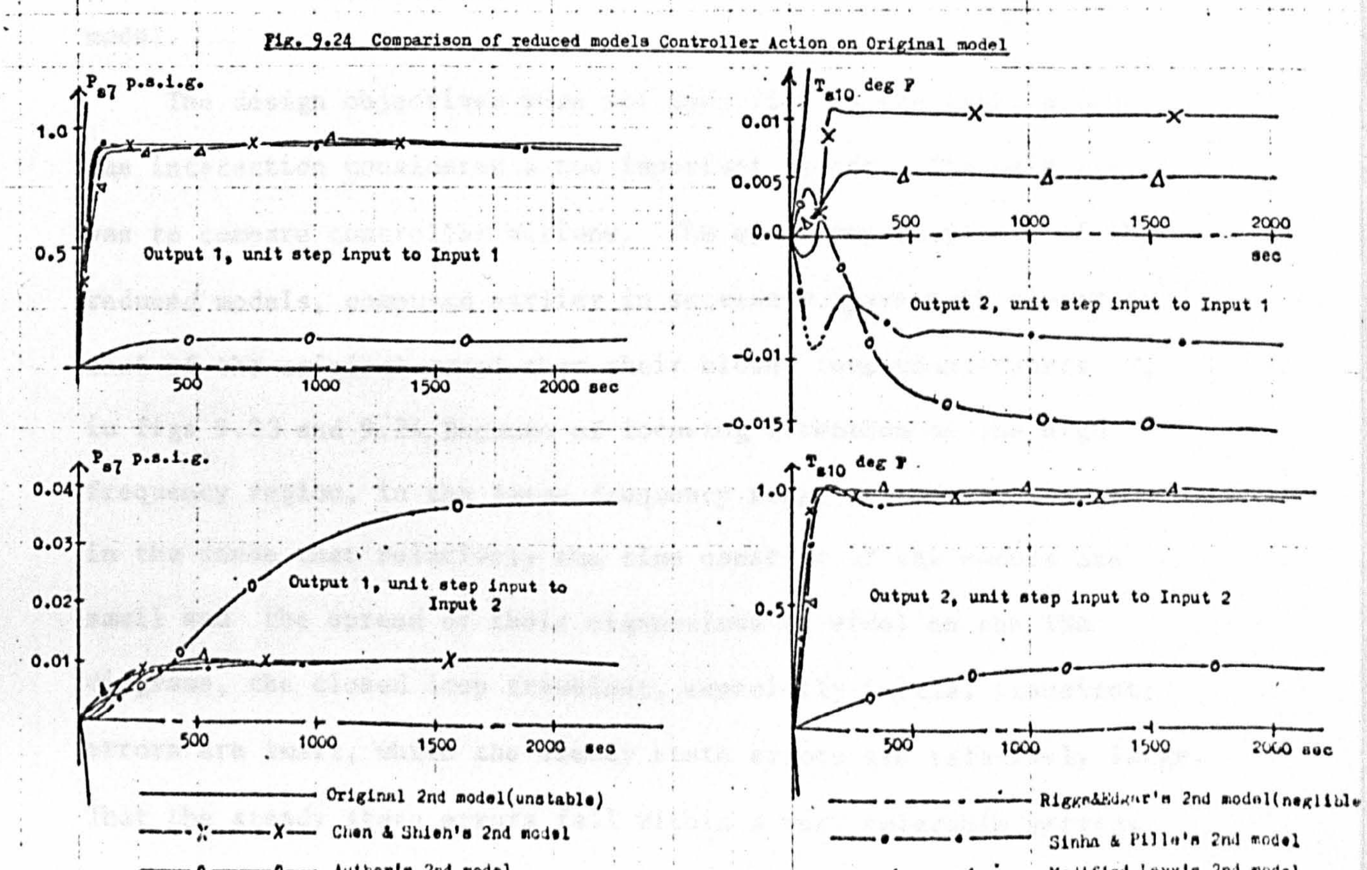


Fig. 9.24 Comparison of reduced models Controller Action on Original model



Looking at figs. 9.23 and 9.24, it can be said that the performances of the reduced models' controllers are fairly close to each other. The transient errors are small in relation to the apparently large steady state errors. In particular, the interaction structure of the system is not violently disrupted, in fact, it is more or less preserved as all loops have low interactions. The original design on the first model yields a low interacting system in both loops, in conformity with the designs using reduced models. The original design on the second model was unsatisfactory, as it produced an unstable model. This was partly due to the tediousness in design, because of the model order, and largely due to the storage facility for simulating the time response using interactive graphics. All original models using reduced model controllers are closed loop stable, except for one whose controller was designed using the "Modified Levy" reduced model.

The design objectives were not specified in the task, neither was interaction considered a too important factor. The main aim was to compare controller actions. The open loop responses of the reduced models, computed earlier in section 9.2, are much closer to that of the original model than their closed loop counterparts , in figs 9.23 and 9.24. Because of focusing attention on the high frequency region, in the large frequency range specified, (large in the sense that relatively the time constant of the models are small and the spread of their eigenvalues is wide) on the INA diagrams, the closed loop transient, especially initial transient, errors are small, while the steady state errors are relatively large. That the steady state errors fall within a very tolerable margin, may be due to the close proximity, of the steady state open loop response, of the reduced models, to that of the original model.

9.5 Conclusions

The use of reduced models in design, in the above sections, is, in general successful. The advantages and disadvantages of the various reduction methods have been discussed in section 9.2. The main justification in using reduced models lies in the success and economy of the final design or in the effectiveness on which reduced models are put to use. This is balanced by the cost and complexity of reduction, the accuracy afforded when using the original model, and the order of the original model. For the thirty third order boiler model discussed above, the application of reduced models is justified in this direction, when the reduction method is fast, economical and reliable.

A comparison of the cost for reduction and application of the models used is roughly summarized in the table below. The units given correspond to computing time and are proportional to the unit time of one of the author's reduction method (time domain).

Model	Reduction Cost	Optimal control policy cost*		Sub-optimal policy cost*		Frequency design & simulation cost
1. Original	-	8.0,	12.0	-	-	150
2. Wilson	3.0	1.1,	1.5	1.4,	1.8	-
3. Mitra	3.5	1.1,	1.5	1.4,	1.8	-
4. Marshall	2.2	1.1,	1.5	1.4,	1.8	-
5. Anderson	1.1	1.1,	1.5	1.4,	1.8	-
6. Author (time)	1.0	1.1,	1.5	1.4,	1.8	-
7. Chen & Shieh	0.5	-	-	-	-	3
8. Sinha & Pille	1.2	-	-	-	-	3
9. Mod. Levy	1.8	-	-	-	-	4.5
10. Riggs & Edgar	0.9	-	-	-	-	3.5
11. Author(frequency)	1.2	-	-	-	-	3

* first and second column corresponds to the case of filtering and non filtering, respectively.

The above cost, in general, corresponds roughly to the size and complexity of the computer programmes. In the applications, the design objectives were not critically specified, the chief intention was to compare reduced model controller performance on original model. For this reason, the reduced models obtained were assumed 'normalized', in the sense that unit input to the models would correspond to the desired input to the plant. In reality, because different states have different static levels under quiescent conditions, it can only be meaningful to talk in terms of unit percentage inputs. The reason for assuming unit inputs was that it was more convenient for studying the effects of interaction, especially in the frequency design method. In the general setting of the problem, it would not invalidate the accuracy of using reduced models.

The design using the linear optimal regulator is more algorithmic in nature, compared to the INA method. The former method does not study the effects of interaction; it relies on the dynamic properties of the states, and, if the latter is inaccessible, an estimator is required for on-line application. Thus in the case of the optimal regulator, the interaction pattern between the reduced and original models need not necessarily be preserved, to obtain a good sub-optimal controller. It depends on which dynamic variables are more dominant than others, their overall contributions, and, the nature of the aggregation matrix, that relates the states of the reduced and original models.

The frequency INA method, on the other hand, stresses that the interaction structure is important. It thus follows that the interaction pattern between the original and reduced models must roughly be preserved to produce a final controller that works for both models. As with most frequency response methods the INA is more tolerable to model inaccuracies, and, design experience is required. The important thing is that the frequency response curves must fit closely. In general, this would correspond to good fits in the time response curves as well, but, there are specific cases where this rule can be violated. However, due to the flexibility of frequency response methods, and, the permissible inclusion of engineering constraints during design, modelling errors will be less penalised. This is in contrast to time domain methods where they are usually more sensitive to modelling errors. For all the reduced models used above, both their open loop frequency and time response curves fit admirably well, with those of the original model. This is chiefly due to the fact that the models are dominated by very large time constants.

During design on the reduced models it was difficult to predict the state of stability or performance degradation of the original models. This, however, can be overcome by incorporating the facilities for pre-determining stability, developed in Chapters V and VI, into a computer interactive graphics software.

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X

CONCLUSIONS OF THE THESIS.

CHAPTER X.

CONCLUSIONS OF THE THESIS.

10.1 Conclusions.

Linear systems reduction has been reviewed, analysed, and, its application and subsequent effects, studied in the thesis. This is a fairly wide area in research, and, during the study, some related areas of interest have also been uncovered.

Two eminent questions regarding model reduction are: (i) are reduction techniques now considered saturated and should future efforts be concentrated on them? (ii) how valid must a technique be, for it to be regarded as a useful tool rather than a toy?

Any novel techniques of reduction are worthwhile investigating, other than satisfying the basic theme of reduction, it must also be feasible, reliable in general and computationally cheap. Sophistication in reduction takes second place to the simple and direct approach method, if end results are considered important. Even to now there are many more studies being made on model reduction alone. Perhaps, if studies are still to be made they should concentrate on improving existing techniques to widen their scope to cater for various classes of models and inputs. (Some latest techniques examine a number of related Pade approximation methods and as to how they are linked up.) Chiefly, the heart of the problem should be looked at, to obtain more stable and accurate open loop response, to yield less complicated, yet faster computer programmes. Any new techniques produced that do not have these qualities would make reduction unattractive, for the cost of reduction may not justify its application. It would be much better off to do with the original model. Perhaps it could be said that a reduced model is only considered 'valid', apart from it being accurate and open loop stable, if it is only used successfully in applications; otherwise it or its reduction method may be considered as a toy rather than a useful tool. An example is that, the original model under the influence of a controller, designed using a reduced model, may not be stable under the same range of gain (or may be totally unstable) as the reduced model would be under the same controller. In this particular situation, it is difficult to judge the 'validity' of the model or its reduction technique, for different models, derived by various techniques, would yield different results in similar circumstances.

Therefore, in general, it is only fair to say that a model and its associated technique is 'successful' if it is reasonably successful in a wide class of applications. For example, in feedback design, it is desirable that closed loop behaviour of the original system can be easily predicted when using the reduced model. This would not be a fair test if the model is used for open loop simulation purposes. Ironically, open loop unstable reduced models can also be used in design which can result in a closed loop stable system. (The unstable model may be obtained by a reduction method, where the Nyquist plots fit well, although the time responses do not.) In general, there are no hard and fast rules as to how a reduced model should be obtained, such that the result of its application is deemed good. The model is good, only if the error bound between reduced and original model simulation, is small.

Hence it is important that error bounds be formulated in applications to estimate the response of the original system in terms of that of the reduced system. The width of the bounds should depend on the difference between the models, among other factors; thus the closer the models, the narrower the bounds. All the error bounds derived for reduced model applications in the thesis have this property. In the limit as the models coincide, they vanish.

The bounds obtained are expressed in various simple analytical forms, suitable for different design applications and computation. They are also very flexible, in the sense that their sharpness can be adjusted, by 'tuning factors', according to the situation. To find the optimum width of a bound analytically is normally a difficult task; for it is expressed in terms of the structure of transfer function matrices, and, is frequency dependent. A nonlinear expression would result whose solution is required. Therefore, trial and error procedures, with the aid of interactive graphics, would provide the best width. Besides, in the latter methods, engineering constraints can also be incorporated. Although the bounds are general in their formulation, and are applicable to a general class of models, most of them impose only sufficient conditions. This is the only restriction. For example, in the case of stability, they do not express necessary and sufficient conditions, but only sufficient conditions, for the system to be stable. This limitation can be overcome by posing stability conditions on the original model, instead of on the reduced model, but, this would defeat the purpose, as it would not give a stability relationship between the two models.

The main results obtained in the thesis are analytical in nature, and they also have theoretical interests, other than practical utility. Due to the duality between model reduction and inexact modelling they can be used in dual ways. In the study of reduced model applications, new areas are also uncovered; example some modified bounds results are derived for both the original linear and nonlinear multivariable system. This could lead to further areas of investigation.

10.2 Suggestions for further work.

It would be useful to develop an interactive graphics package, for computer aided design purpose, for those bounds given in the preceeding chapters, especially in the areas of stability and performance. The package can be adapted to any useful multivariable design or simulation methods, where reduced models are made use of. If design is done without using a reduced model, the bounds would still be useful, for they can be used to provide a valuable estimate of the likely response error in the system, if the system model is inexact due to crude modelling, or, if the system undergoes parameter changes during or after design.

If future efforts are still to be devoted to model reduction, in addition, it would be advantageous, if possible,to obtain necessary and sufficient conditions for original system stability in terms of that of the reduced system.

Further theoretical investigations, seems worthwhile, in the area of nonlinear systems, where reduced models are concerned. This is a new field, and, it is interesting to consider the situation when the nonlinear part of the system is designed using a 'reduced nonlinear system'.