Independence and conservativity results for intuitionistic set theory

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The candidate confirms that the work submitted is his own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

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Jointly-authored publication

The author has included as part of his thesis the following jointly-authored publication: Lifschitz' Realizability for Intuitionistic Zermelo-Fraenkel Set Theory.
This is a joint work by my supervisor Michael Rathjen and me. It has been accepted by the Special Issue of the Logic Journal of the IGPL on NON-CLASSICAL MATHEMATICS. This paper mainly represents the final piece of our joint work to differentiate $\mathbf{CT}_0$ from $\mathbf{CT}_0!$ in the context of intuitionistic set theory. Essentially, I executed most of the ideas presented by my supervisor in this research and finalized by him. Based on the same semantics as in this paper, I also independently showed this independence result in Chapter Four and Five in this thesis.

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Abstract

There are two main parts to this thesis. The first part will deal with some independence results. In 1979, Lifschitz in [13] introduced a realizability interpretation for Heyting's arithmetic, **HA**, that could differentiate between Church's thesis with uniqueness condition, $\mathbf{CT}_0!$, and the general form of Church's thesis, $\mathbf{CT}_0$. The objective here is to extend Lifschitz' realizability to intuitionistic Zermelo-Fraenkel set theory with two sorts, $\mathbf{IZF}_N$. In addition to separating Church's thesis with uniqueness condition from its general form in intuitionistic set theory, I also obtain several interesting corollaries. The interpretation repudiates a weak form of countable choice, $\mathbf{AC}^{N2}$, asserting that every countable family of inhabited subsets of $\{0, 1\}$ has a choice function.

The second part will be concerned with Constructive Zermelo-Fraenkel Set Theory and other intuitionistic set theories augmented by various principles, notably choice principles. It will be shown that the addition of these (choice) principles does not change the stock of provable arithmetical theorems.

This type of conservativity result has its roots in a theorem of Goodman [9] who showed that Heyting arithmetic in all finite types augmented by the axiom of choice for all levels is conservative over **HA**. The technique I employ here to obtain such results for intuitionistic set theories, however, owes a lot to a paper by Beeson published in 1979. In [2] he showed how to construe Goodman's Theorem as the composition of two interpretations, namely relativized realizability and forcing. In this thesis, I adopt the same approach and employ it to a plethora of intuitionistic set theories.

Contents

Chapter 1

Introduction

In this chapter we introduce some background knowledge which is related to this thesis. In Section 1.1, I will explain how the structures and notations used in the thesis are treated. Moreover, in Section 1.2, I will informally introduce various intuitionistic set theories and semi-constructive axioms.

Then in Section 1.3, several versions of realizability interpretations for Heyting arithmetic are given; while in Section 1.4, we look at realizability interpretations for intuitionistic set theories.

Furthermore, in Section 1.5, some forcing interpretations for arithmetic and set theory are presented and the approaches to deriving conservativity results for arithmetic and set theory are sketched.

Lastly, in Section 1.6, I summarize our independence and conservativity results and give a sketch how to obtain these results.

There are eight chapters in this thesis.

- Chapter 1: Introduction
  Notations will be fixed and basic facts assumed for this thesis will be introduced. In particular, I will summarize other authors' work on realizability semantics and forcing semantics and compare their work with what is done in this thesis.

- Chapter 2: Formal systems
  I will introduce the languages for our foreground and background theories and some (semi-) constructive formal systems.

- Chapter 3: Applicative structure and universes
  I will start to introduce the framework of our semantics for the formal systems. This mainly consists of building a realizability universe

from an applicative structure. Variants and consequences of the principle of transfinite induction that will be applied in this thesis are also discussed.

After the introduction, I start to build up the two main parts of this thesis: independence results and conservativity results. Chapter 4 and Chapter 5 consist mainly of independence results.

- Chapter 4: Lifschitz' style interpretation
  I will extend Lipschitz' interpretation of Heyting Arithmetic to the context of intuitionistic set theory. All the fundamental results will also be included in this chapter.

- Chapter 5: Lifschitz' style soundness
  I will show the soundness of the axioms of intuitionistic set theory as well as some semi-constructive axioms. Subsequently I will deduce some important independence results from this.

Chapter 6, Chapter 7 and Chapter 8 consist mainly of conservativity results.

- Chapter 6: Relativized realizability
  I will introduce the relativized realizability interpretation for our formal systems and show its soundness.

- Chapter 7: Forcing
  I will introduce the forcing interpretation for our formal systems and show its soundness.

- Chapter 8: Conservativity results
  I will derive all the main conservativity results of this thesis.

1.1 Notations

There are two kinds of notations (Global notations and Local notations) used in this thesis.

Global notations: throughout this thesis, the meaning of these notations remains fixed:

- "$\equiv$ " denotes "abbreviates".
- ":=" denotes "defines as".
- " iff " denotes "if and only if ".

- $\forall x \in y \varphi(x) \equiv \forall x(x \in y \to \varphi(x))$.
- $\exists x \in y \varphi(x) \equiv \exists x(x \in y \wedge \varphi(x))$.
- $\varphi(x_1, x_2, ..., x_n)$ signifies the free variables of φ are among $x_1, x_2, ..., x_n$.
- $\emptyset$ denotes the empty set.
- $\varphi[x/y]$ denotes the new formula after substituting the free variable x in φ by the new variable y.
- $f(x) \downarrow$ means that $f(x)$ is defined and $f(x) \uparrow$ means that $f(x)$ is undefined.
- $x \neq \emptyset$ denotes the formula that x is inhabited, i.e., $\exists y(y \in x)$.
- Other notations are provided in the index.

Moreover, when stating a result we use (**AX**) to indicate that the extra axiom (**AX**) is to be added to the background theory. If a proof or a statement is rather lengthy, we will use $\diamondsuit$ to separate some inferences or statements.

Local notations: The meaning of these notations will be changed. This kind of notation will be used heavily in proofs. The meaning of these notations will be understood from the context or the proof itself. We will not list these notations in the index.

In order to make this thesis as self-contained as possible, we might paraphrase some content or change some notations from other authors' papers or books. This will make it easier for us and our readers to pinpoint the differences between different semantics.

1.2 Background knowledge

In order to facilitate the presentation of set theory, it is convenient to use the usual class notations. First of all, let us use $\mathbb{S}$ to denote the class of all sets and $\mathbb{N}$ to denote the set of all natural numbers. For classes P and Q we use $P \times Q$ to denote the class of pairs $\{(x, y) : x \in P \wedge y \in Q\}$. A class R is a (binary) **relation** if $R \subseteq \mathbb{S} \times \mathbb{S}$. If R satisfies $(x, y) \in R \wedge (x, z) \in R \to y = z$ we call R a **class function**. A **set function** (or just function) is a class function which happens to be a set. We say a binary relation R has **domain** a and **range** b if and only if $R \subseteq a \times b$ and for all $x \in a$, there exists $y \in b$ such that $(x, y) \in R$. Moreover, if a, b are both subsets of $\mathbb{N}$, we call R a

numerical relation. We say that R is a **surjective relation** with domain a and range b if and only if $R \subseteq a \times b$ and for all $x \in a$ there exists $y \in b$ such that $(x, y) \in R$ and for all $y \in b$ there exists $x \in a$ such that $(x, y) \in R$. For any binary relations H, K we say H is a **sub-relation** of K if and only if H, K have the same domain and $H \subseteq K$.

The most important intuitionistic set theories are Intuitionistic Zermelo-Fraenkel Set Theory, **IZF**, and Constructive Zermelo-Fraenkel Set Theory, **CZF**. For both, the underlying logic is Intuitionistic Predicate Logic, **IPL**. The axioms of **IZF** are almost the same as those of **ZF** except that the Foundation Axiom gets replaced by Set Induction and Replacement is swapped with Collection. **CZF** differs from **IZF** in the following ways: the Full Separation Schema is restricted to Bounded Separation, i.e. one has separation only for Δ_0 formulae (where only quantifiers of the form $\forall x \in$ and $\exists y \in$ are allowed). Collection is strengthened to Strong Collection whereas the Powerset Axiom is replaced by Subset Collection.

We will also study systems obtained from **IZF** and **CZF** by adding various other axioms and principles, notably variants of the Axiom of choice, **AC**. Since **AC** conjoined to **CZF** will make the whole system return to **ZFC**, one has to choose strictly weaker versions of **AC**, one being the Presentation Axiom $\mathbf{PA}_X$. $\mathbf{PA}_X$ asserts that for every set a, there exists a surjective set function with a domain b such that $f : b \to a$ and b is a base, where a **base** is a set for which the axiom of choice holds. Another familiar choice axiom is Dependent Choice, **DC** (i.e., for any set relation R with domain set a, range set a and any set $b \in a$ there is a set function $f : \mathbb{N} \to a$ such that $\forall n \in \mathbb{N}(f(0) = b \wedge R(f(n), f(n+1)))$. If a is $\mathbb{N}$, then we call this instance $\mathbf{DC}^{NN}$. A further generalization of **DC** is Relativized Dependent Choice **RDC** (i.e., for any class relation R with domain class K, range class K and any set $b \in K$ there is a class function $f : \mathbb{N} \to K$ such that $\forall n \in \mathbb{N}(f(0) = b \wedge R(f(n), f(n+1)))$. A very useful form of **AC** is the Axiom of Countable Choice $\mathbf{AC}^N$ which postulates that every countable family of inhabited sets has a choice function. An even weaker form of $\mathbf{AC}^N$ is $\mathbf{AC}^{NN}$ (or $\mathbf{AC}_{\omega,\omega}$) where the family consists of subsets of the natural numbers. A still weaker form is $\mathbf{AC}^{N2}$, where the family consists of subsets of $\{0, 1\}$. (Note that $\mathbf{AC}^{NN}$ is provable in **ZF**.)

On the basis of **CZF**, the logical implications between the various choice principles can be summarized as follows:

$$\mathbf{AC} \to \mathbf{PA}_X \to \mathbf{DC} \to \mathbf{AC}^N \to \mathbf{AC}^{NN} \to \mathbf{DC}^{NN} \wedge \mathbf{AC}^{N2}.$$

Intuitionistic logic allows one to adopt several "exotic" axioms that would immediately lead to inconsistency on the basis of classical logic. One such axiom is the so-called Church's thesis, $\mathbf{CT}_0$, which asserts that every total numerical relation has a computable sub-function. Further "exotic" axioms we are going to study include the Uniformity Principle **UP** and a principle called Unzerlegbarkeit (or Indecomposability) **UZ** (c.f. Definition 2.2.7).

A semi-classical axiom we shall consider is Markov's Principle **MP**.

1.3 Realizability for Heyting Arithmetic

Heyting Arithmetic **HA** differs from Peano Arithmetic in the logical axiom. Instead of using classical predicate logic, **HA** uses intuitionistic predicate logic. Realizability for **HA** was introduced by Stephen C. Kleene and nowadays has become a well-known tool for analyzing syntactical systems. There are many variants of realizability. In this introduction, we will simply list those relevant to our research topics. One can view realizability as a particular implementation of the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic logic.

Kleene's 1945 realizability uses codes of partial recursive functions as realizers. His approach lends itself to generalization where one employs realizers from an arbitrary domain of computation known as Applicative Structure.

1.3.1 Kleene's recursive realizability

To describe Kleene's realizability for **HA** we need to introduce some notations. Let $\jmath$ be a pairing function with left-unpairing function $\jmath_0$ and right-unpairing function $\jmath_1$. In the following, we use e_0 to denote $\jmath_0(e)$ and e_1 to denote $\jmath_1(e)$ and $e \cdot n$ denotes the partial recursive function $\{e\}$ applied to n, i.e., $e \cdot n$ denotes $\{e\}(n)$. In addition, $e \cdot n \downarrow$ signifies that $\{e\}(n)$ is defined and $e \Vdash^K \varphi$ abbreviates that e realizes the formula φ. $e \Vdash^K \varphi$ is defined inductively as follows:

- $e \Vdash^K \varphi$ iff $e = 0$ and φ is an atomic true formula.
- $e \Vdash^K \theta \wedge \eta$ iff $e_0 \Vdash^K \theta \wedge e_1 \Vdash^K \eta$.
- $e \Vdash^K \theta \to \eta$ iff $\forall n \in \mathbb{N}[n \Vdash^K \theta \to e \cdot n \downarrow \wedge e \cdot n \Vdash^K \eta]$.
- $e \Vdash^K \forall x \theta(x)$ iff $\forall n \in \mathbb{N}(e \cdot n \downarrow \wedge e \cdot n \Vdash^K \theta[x/n])$.
- $e \Vdash^K \theta \vee \eta$ iff $[e_0 = 0 \wedge e_1 \Vdash^K \theta] \vee [e_0 \neq 0 \wedge e_1 \Vdash^K \eta]$.

- $e \Vdash^K \exists x\theta(x)$ iff $e_1 \Vdash^K \theta[x/e_0]$.

With this interpretation (c.f. Section 82 of Chapter 15 in [11]), he proved that **HA** is sound.

1.3.2 Kleene's relativized realizability

Relativized realizability (c.f. p.356 in [3]) is a generalization of Kleene's recursive realizability. The only difference is that instead of codes of partial recursive functions one uses codes of partial functions that are recursive in a fixed partial function g. All the clauses are then exactly the same as for Kleene's recursive realizability.

1.3.3 Lifschitz' realizability

Dragalin pointed out (c.f. Section 4 of Part 2 in [5]) that there are two formal versions of Church's thesis one could consider adding to Heyting arithmetic **HA**:

$$\mathbf{CT}_0 \qquad \forall x \exists y\, A(x,y) \to \exists z \forall x\, [z \cdot x \downarrow \wedge A(x, z \cdot x)]$$

$$\mathbf{CT}_0! \qquad \forall x \exists ! y\, A(x,y) \to \exists z \forall x\, [z \cdot x \downarrow \wedge A(x, z \cdot x)]$$

(we write $z \cdot x$ for $\{z\}(x)$), and he posed the question whether the latter version is actually weaker than the former. The question was answered affirmatively in 1979 by Vladimir Lifschitz [13]. He introduced a modification of Kleene's realizability that validates $\mathbf{CT}_0!$ but falsifies instances of $\mathbf{CT}_0$. The main idea behind separating $\mathbf{CT}_0$ from $\mathbf{CT}_0!$ is to find a property P of pairs of numbers so that if there is a unique n such that $P(e,n)$ holds then there is an effective procedure to find n from e, while in general there is no such procedure if $\{m \mid P(e,m)\}$ contains more than one element. Lifschitz singled out the property $n \leq e_1 \wedge \forall m \neg T(e_0, n, m)$, where T is Kleene's T-predicate. His interpretation differs from Kleene's in the clause $\exists x \varphi(x)$: (we use $\Vdash^L$ to denote his interpretation for Heyting Arithmetic)

- $m \Vdash^L \exists x \psi(x)$ iff $D_m \neq \varnothing \wedge \forall n \in D_m(n_1 \Vdash^L \psi(n_0))$, where $D_n := \{m \in \mathbb{N} : m \leq n_1 \wedge \forall z \neg T(n_0, m, z)\}$.

This type of realizability has been applied in other contexts as well. For other applications and extensions, the author suggests readers to consult Jaap van Oosten's papers and book, in particular, [19].

1.4 Realizability for set theories

Realizability for set theory is very different from realizability for arithmetic. The main difference lies in the hierarchical buildup of the universe and the fact that the equality and elementhood relation are intimately connected on account of the extensionality axiom. As a result, one has to construct an iterative internal universe (or realizability universe) which forms the basis for the realizability interpretation. Interpretations for **IZF** and **CZF** were given in McCarty's Ph.D. thesis [15] and Michael Rathjen's paper [20], respectively. In this section, I will introduce these interpretations.

1.4.1 KTFBM realizability

This abbreviates Kreisel-Troelstra-Friedman-Beeson-McCarty realizability. McCarty's realizability interpretation for **IZF** has its roots in a realizability interpretation for second order arithmetic by Kreisel and Troelstra [12]. In his Ph.D. thesis [15] (p. 82) he writes (For the definition of APP, please refer to p.82 in [15] or Section 3.1): "Our interpretation is the immediate descendant of the interpretations of Friedman (1973a) and Beeson (1979). The idea of using models of APP came to us from a remark of Solomon Feferman in his paper *A language and axioms for explicit mathematics* (1975)...The form in which our realizability appears was the product of a joint effort expended in Oxford during Michaelmas term 1980. The effort had at least contingent connection with Dana Scott's seminar *Sheaves and logic*. Among the many individuals who made notable contributions, foremost were Guiseppe Rosolini, Simon Thompson and Dana Scott." We will give a rough sketch of this type of realizability. Suppose A is a model of APP. Suppose $|A|$ is its carrier. By using the Powerset operation $\mathcal{P}$ and transfinite recursion, one defines the realizability universe $V(A)$ as follows:

$$V(A)_\alpha := \bigcup_{\beta\in\alpha} \mathcal{P}(|A| \times V(A)_\beta), V(A) := \bigcup_{\alpha\in On} V(A)_\alpha.$$

Let $Atom_{ZF}(V(A))$ and $Form_{ZF}\ (V(A))$ be the collection of all the atomic formulae and the collection of all formulae, respectively, formed in the language of a set theory with parameters from $V(A)$. The realizability relation $\Vdash_M \subseteq |A| \times Form_{ZF}\ (V(A))$ is defined inductively as follows: (for $m \in |A|$ and $\varphi \in Form_{ZF}\ (V(A))$)
If $\varphi \in Atom_{ZF}(V(A))$, the two clauses are as follows:

- $m \Vdash_M a \in b$ iff $\exists c \in V(A)[(m_0, c) \in b \wedge m_1 \Vdash_M a = c]$,

- $m \Vdash_M a = b$ iff $\forall (f,d) \in a(m_0 \cdot f \downarrow \wedge m_0 \cdot f \Vdash_M d \in b) \wedge \forall (f,d) \in b(m_1 \cdot f \downarrow \wedge m_1 \cdot f \Vdash_M d \in a)$.

For φ a compound formula, the clauses are as follows:

- $m \Vdash_M \psi \wedge \eta$ iff $m_0 \Vdash_M \psi \wedge m_1 \Vdash_M \eta$.
- $m \Vdash_M \psi \to \eta$ iff $\forall n \in |A|[n \Vdash_M \psi \to m \cdot n \downarrow \wedge m \cdot n \Vdash_M \eta]$.
- $m \Vdash_M \forall x \psi(x)$ iff $\forall a \in V(A)(m \Vdash_M \psi[x/a])$.
- $m \Vdash_M \exists x \psi(x)$ iff $\exists a \in V(A)(m \Vdash_M \psi[x/a])$.
- $m \Vdash_M \psi \vee \eta$ iff $(m_0 = 0 \wedge m_1 \Vdash_M \psi) \vee (m_0 \neq 0 \wedge m_1 \Vdash_M \eta)$.
- $m \Vdash_M \neg\psi$ iff $\forall n \in |A| \neg(n \Vdash_M \psi)$.
- $V(A) \models_M \varphi$ iff $\exists e \in |A|(e \Vdash_M \psi)$.

The main results (adapted from his paper) McCarty obtained (in the following, Kl is defined in Subsection 3.1.3 and $V(Kl)$ is defined pp.30-31 in [15]) are :

- If **IZF** $\vdash \theta$, then **IZF** $\vdash [V(A) \models_M \theta]$.
- **IZF** $\vdash [V(Kl) \models_M$ $\mathbf{AC}^{\omega\omega} \wedge \mathbf{CT} \wedge \mathbf{ECT} \wedge \mathbf{UP} \wedge \mathbf{UZ}]$.
- **IZF** + **MP** $\vdash [V(Kl) \models_M$ **MP**$]$ and **IZF** + **IP** $\vdash [V(Kl) \models_M$ **IP**$]$.
- **IZF** + **DC** $\vdash [V(Kl) \models_M$ **DC**$]$ and **IZF** + **RDC** $\vdash [V(Kl) \models_M$ **RDC**$]$ and **IZF** + **AC** $\vdash [V(Kl) \models_M$ $\mathbf{PA}_X]$.

1.4.2 Rathjen's realizability for CZF

In 2003 Michael Rathjen in [20] showed that McCarty's realizability can be developed within **CZF** and provides a self-validating semantics for **CZF**. In particular, he introduced bounded formulae into the syntactical system and came up with an interpretation for the bounded quantifier as follows (adapted from his paper [20]): for any $a \in V(A)$ and any $e \in |A|$ (A is a model of APP and $|A|$ is the carrier and $V(A)$ is the realizability universe defined above), he added the following clauses (the other clauses are the same as McCarty's):

- $e \Vdash_{MR} \forall x \in a \varphi(x)$ iff $\forall (f,c) \in a(e \cdot f \Vdash_{MR} \varphi[x/c])$.
- $e \Vdash_{MR} \exists x \in a \varphi(x)$ iff $\exists c \in V(A)((e_0, c) \in a \wedge e_1 \Vdash_{MR} \varphi[x/c])$.

There are another two key features worth mentioning here. First of all, he used a class form of inductive definition to formalize the realizability universe $V(A)$, to overcome the lack of the Powerset axiom in **CZF**. Secondly, he invented an intuitionistic approach to interpreting $\mathbf{PA}_X$ instead of McCarty's approach by using full **AC**. He also showed that this realizability works for the regular extension axiom. His main results can be summarized as follows (adapted from his paper):

- If $\mathbf{CZF} \vdash \theta$, then $\mathbf{CZF} \vdash [V(Kl) \models_{MR} \theta]$.
- $\mathbf{CZF} \vdash [V(Kl) \models_{MR} \mathbf{AC}^{\omega\omega} \wedge \mathbf{CT} \wedge \mathbf{ECT} \wedge \mathbf{UP} \wedge \mathbf{UZ}]$.
- $\mathbf{CZF} + \mathbf{MP} \vdash [V(Kl) \models_{MR} \mathbf{MP}]$ and $\mathbf{CZF} + \mathbf{IP} \vdash [V(Kl) \models_{MR} \mathbf{IP}]$. For the definition of **IP**, please refer to Definition 2.2.16.
- $\mathbf{CZF} + \mathbf{DC} \vdash [V(Kl) \models_{MR} \mathbf{DC}]$ and $\mathbf{CZF} + \mathbf{RDC} \vdash [V(Kl) \models_{MR} \mathbf{RDC}]$ and $\mathbf{CZF} + \mathbf{PA}_X \vdash [V(Kl) \models_{MR} \mathbf{PA}_X]$.

1.4.3 Our realizability for $\mathbf{CZF}_N$

In this thesis, we will modify the above semantics for **CZF** to deal with a version of **CZF**, dubbed $\mathbf{CZF}_N$, that takes the natural numbers as urelements. An important issue here is how to properly set up an interpretation that can accommodate both numbers and sets. This interpretation (denoted by $\Vdash_R$) is tailor-made in a bid to show our conservativity result. The details will be given in Section 6.1

1.4.4 Our realizability for $\mathbf{IZF}_N$

In this thesis, we will extend Lifschitz' interpretation for Heyting Arithmetic to the context of intuitionistic set theory. However, there are still some variations in our approaches. The language we use differs from all other authors. We incorporate the language of arithmetic in our systems explicitly. Then we come up with an interpretation and a universe to accommodate both arithmetic and set theory. For comparison, we call our interpretation Lifschitz' style realizability (notated by $\Vdash_L$). The details will be shown in Section 4.2.

1.5 Conservativity via realizability and forcing

Though both realizability and forcing semantics interpret intuitionistic set theories, they represent quite different perspectives. If one views the process of interpreting a syntactical system as a process of gaining knowledge, then there is a distinction between realizability and forcing semantics. Realizability semantics focuses more on **how** knowledge is gained while forcing semantics focuses more on **when** knowledge is gained. This distinction is nicely summarized in Goodman's paper [9] (pp. 25-26): "Kleene's original notion of recursive realizability...has the great strength that it emphasizes the active aspect of constructive mathematics...However, Kleene's notion has the weakness that it disregards that aspect of constructive mathematics which concerns epistemological change...Precisely that aspect of constructive mathematics which Kleene's notion neglects is emphasized by Kripke's semantics for intuitionistic logic.".

There is a famous result called Goodman's theorem which states that Heyting arithmetic with higher types, $\mathbf{HA}^\omega$, augmented by the axiom of choice for all type levels is conservative over **HA**. In 1979 Michael Beeson published a paper "GOODMAN'S THEOREM AND BEYOND" which simplified Goodman's original proof from [8]. Beeson's proof is in two steps: one step uses realizability, the other step uses forcing. The combination of two well-known tools renders the proof particularly transparent. It should be said, though, that about the same time when Beeson published his paper, Goodman himself gave another fairly simple proof of his theorem in [9] ; his proof also combines ideas related to realizability and forcing.

In this thesis we will adopt Beeson's two-tiered approach–realizability followed by forcing–to the set-theoretic context in order to obtain conservativity results for intuitionistic set theories. To give the reader a flavour of things to come we briefly relate Beeson's analysis of Goodman's theorem.

1.5.1 Forcing for arithmetic

In [2] one finds the following forcing interpretation for Heyting Arithmetic. Let $\mathbb{B}$ be the set of all finite function from $\mathbb{N}$ to $\mathbb{N}$, where a finite function is a function whose domain and range are finite subsets of $\mathbb{N}$. Define a partial order relation $\geq$ on $\mathbb{B}$: $p \geq q$ iff $p \supseteq q$. Let x, y be arbitrary numbers and let φ and θ be arbitrary arithmetical formulae. Beeson then introduced the following clauses (adapted from his paper [2], pp. 3-4):

- $p \Vdash_B x = y$ iff $x = y$.
- $p \Vdash_B \varphi \wedge \theta$ iff $p \Vdash_B \varphi \wedge p \Vdash_B \theta$.
- $p \Vdash_B \varphi \vee \theta$ iff $p \Vdash_B \varphi \vee p \Vdash_B \theta$.
- $p \Vdash_B \varphi \to \theta$ iff $\forall q \geq p[q \Vdash_B \varphi \to \exists r \geq q(r \Vdash_B \theta)]$.

- $p \Vdash_B \exists x \varphi(x)$ iff $\exists n(p \Vdash_B \varphi[x/n])$.
- $p \Vdash_B \forall x \varphi(x)$ iff $\forall n \forall q \geq p \exists r \geq q(r \Vdash_B \varphi[x/n])$.

1.5.2 Forcing for set theories

James Lipton in his paper [14] gave a version of the semantics which interprets **IZF**. Since his interpretation inspires our version of interpretation, we introduce his semantics in this section. Let $(\mathbb{K}, \geq)$ be a partially ordered structure. Let $p \in \mathbb{K}$ and $a, b \in V(\mathbb{K})$ be arbitrary, where

$$V_\alpha(\mathbb{K}) := \bigcup_{\beta \in \alpha} \mathcal{P}(\mathbb{K} \times V_\beta(\mathbb{K})), V(\mathbb{K}) := \bigcup_{\alpha \in On} V_\alpha(\mathbb{K}).$$

Then he inductively defined the following clauses (adapted from his paper):

- $p \Vdash_J a = b$ iff $\forall (q, c) \in a \forall r \geq p, q(r \Vdash_J c \in b) \wedge \forall (q, c) \in b \forall r \geq p, q(r \Vdash_J c \in a)$.
- $p \Vdash_J a \in b$ iff $\exists c \in V(\mathbb{K}) \exists q \leq p((q, c) \in b \wedge p \Vdash_J a = c)$.
- $p \Vdash_J \varphi \wedge \theta$ iff $p \Vdash_J \varphi \wedge p \Vdash_J \theta$.
- $p \Vdash_J \varphi \vee \theta$ iff $p \Vdash_J \varphi \vee p \Vdash_J \theta$.
- $p \Vdash_J \varphi \to \theta$ iff $\forall q \geq p[q \Vdash_J \varphi \to q \Vdash_J \theta]$.
- $p \Vdash_J \exists x \varphi(x)$ iff $\exists a \in V(\mathbb{K})(p \Vdash_J \varphi[a/x])$.
- $p \Vdash_J \forall x \varphi(x)$ iff $\forall a \in V(\mathbb{K}) \forall q \geq p(q \Vdash_J \varphi[a/x])$.
- $p \Vdash_J \varphi(x)$ iff $p \Vdash_J \forall x \varphi(x)$.
- $V(\mathbb{K}) \models_J \varphi$ iff $\forall p \in \mathbb{K}(p \Vdash_J \varphi)$.

1.5.3 Beeson's two-tiered approach

To show

$$\mathbf{HA}^\omega + \mathbf{AC} \vdash \varphi \text{ then } \mathbf{HA} \vdash \varphi$$

for arithmetical φ, [2] combines Kleene's relativized realizability ($e \Vdash^K \varphi$) with forcing as follows:

1. (Theorem 3.1) If $\mathbf{HA}^\omega + \mathbf{AC} \vdash \varphi$ then $\mathbf{HA}^\omega_a \vdash \exists e(e \Vdash^K \varphi)$, where $\mathbf{HA}^\omega_a$ has an additional axiom postulating that a is a partial function from N to N.

2. (Theorem 2.1)
   If $\mathbf{HA}^\omega_a \vdash \exists e(e \Vdash^K \varphi)$ then $\mathbf{HA}^\omega \vdash$ "$\exists e(e \Vdash^K \varphi)$ is forced".

3. (Lemma 4.1) $\mathbf{HA}^\omega \vdash$"($\exists e(e \Vdash^K \varphi) \to \varphi$) is forced".

4. (Lemma 2.1) $\mathbf{HA}^\omega \vdash \varphi$ and thus $\mathbf{HA} \vdash \varphi$.

1.5.4 Our two-tiered approach

We extend Beeson's two steps to the context of intuitionistic set theories, by combining the relativized realizability semantics $\Vdash_R$ with the forcing semantics $\Vdash_{\mathcal{F}}$. The relativized realizability semantics basically is based on Kleene's realizability for arithmetic and KTFBM realizability for set theory, while the forcing semantics is based on Beeson's forcing for arithmetic and Jame's forcing for set theory. The conservativity results obtained in this way are sketched in Subsection 1.6.2 and fully explained from Chapter 6 to Chapter 8.

1.6 Overview of results of this thesis

In the following, I will summarize the main results obtained in the two parts in this thesis: independence results and conservativity results. Before that, I should like to point out several differences between the two parts:

1. Formal systems: the first part is concerned with the syntactical system $\mathbf{IZF}_N$, while the second part is concerned with the two syntactical systems $\mathbf{CZF}_N$ and $\mathbf{IZF}_N$.

2. Realizers: the first part uses recursive realizers while the second uses relativized realizers.

3. Background theories: The first part can be developed in the background theory $\mathbf{IZF}'_N$, i.e., $\mathbf{IZF}_N + \mathbf{MP}_{pr} + \mathbf{B\Sigma^0_2} - \mathbf{MP}$, where $\mathbf{B\Sigma^0_2} - \mathbf{MP}$ is the schema

   $$\neg\neg\exists n \leq m \forall k A(n,k,e) \to \exists n \leq m \forall k A(n,k,e)$$

   with A being primitive recursive.

   The second part uses the background theory $\mathbf{CZF}_N$ for results concerning the syntactical system $\mathbf{CZF}_N$ and $\mathbf{IZF}_N$ for results concerning the syntactical system $\mathbf{IZF}_N$.

1.6.1 The first part: independence results

Here we extend Lifschitz' realizability from **HA** to **IZF** and arrive at the following theorem:

$$\mathbf{IZF}_N + \mathbf{CT}_0! \nvdash \mathbf{CT}_0.$$

As a corollary we obtain additional independence results (cf. Theorem 5.6.5):

$$\mathbf{IZF}_N + \mathbf{CT}_0! \quad \nvdash \quad \mathbf{AC}^{N2} \vee \mathbf{DC} \vee \mathbf{RDC} \vee \mathbf{PA}_X. \tag{1.1}$$

The latter is essentially derived via the following steps.

Step1 One shows that $\mathbf{IZF}_N + \mathbf{CT}_0! \vdash \theta$ implies $\mathbf{IZF}'_N \vdash (V^* \models_L \theta)$. (cf. Theorem 5.4.12)

Step2 As Lifschitz' interpretation $\Vdash^L$ for **HA** can be embedded into our interpretation $\Vdash_L$ for $\mathbf{IZF}_N$ (cf. Theorem 4.4.3), one obtains $\mathbf{IZF}'_N \vdash (V^* \models_L \neg\mathbf{CT}_0)$.

Thus, in view of the above we have shown that $\mathbf{IZF}_N + \mathbf{CT}_0! \nvdash \mathbf{CT}_0$.

Step3 The previous step entails $\mathbf{IZF}_N + \mathbf{CT}_0! \nvdash \mathbf{AC}^{N2}$ (cf. Claim 5.6.2).

Step4 Since $\mathbf{PA}_X \to \mathbf{DC} \to \mathbf{AC}^N \to \mathbf{AC}^{NN} \to \mathbf{AC}^{N2}$, (1.1) follows by the previous steps (cf. Theorem 5.6.5).

Moreover, since Lifschitz' realizability for **IZF** also validates the uniformity principle **UP** and hence Unzerlegbarkeit **UZ**, (1.1) can be strengthened to

$$\mathbf{IZF}_N + \mathbf{CT}_0! + \mathbf{UP} + \mathbf{UZ} \quad \nvdash \quad \mathbf{AC}^{N2} \vee \mathbf{DC} \vee \mathbf{RDC} \vee \mathbf{PA}_X. \tag{1.2}$$

1.6.2 The second part: conservativity results

In this part, we will extend Beeson's two-tiered approach (i.e., relativized realizability and forcing semantics) for **HA** to the context of intuitionistic set theories with two sorts. The first challenge is to find a relativized realizability semantics for intuitionistic set theories with two sorts (numbers and sets). The second challenge is to find a forcing semantics for intuitionistic set theories with two sorts.

In order to describe our results, we use the following abbreviations: (Let T be any of the theories $\mathbf{CZF}_N$ or $\mathbf{IZF}_N$ extended by any combination of the following axioms: $\{\mathbf{DC}, \mathbf{RDC}, \mathbf{PA}_X, \mathbf{MP}\}$)

1. $\mathbf{CZF}_N^\star \equiv \mathbf{CZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ}$;
2. $\mathbf{IZF}_N^\star \equiv \mathbf{IZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ}$;
3. $T^\star \equiv T + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ}$;
4. $T_A \equiv T + PF(A, \mathbb{N}, \mathbb{N})$, where $PF(A, \mathbb{N}, \mathbb{N})$ denotes the axiom: A is a partial function from $\mathbb{N}$ to $\mathbb{N}$.

If one wants to obtain conservativity results, recursive realizability is rather defective in that recursive realizability of a formula (unless it is an almost negative arithmetical one) usually does not entail its truth. Observe that for a disjunctive formula $\eta \vee \delta$ (or existential formulae $\exists x\eta$) there is no effective way to find a recursive realizer to realize $\eta \vee \delta$ (or existential formulae $\exists x\eta$). However, if one employs relativized realizability with an oracle A being a generic partial function from $\mathbb{N}$ to $\mathbb{N}$, then this obstacle can be overcome.

Then next step is to interpret A in the forcing semantics to show that arithmetical formulae are indeed generic self-realizing (i.e., sound and complete in forcing interpretation). By choosing the proper forcing conditions that preserve the self-realizing, one has the following conservativity result: (for any arithmetical sentence θ)

1. (cf. Theorem 8.4.2) If $\mathbf{CZF}_N^\star \vdash \theta$, then $\mathbf{CZF}_N \vdash \theta$;
2. (cf. Theorem 8.4.2) If $\mathbf{IZF}_N^\star \vdash \theta$, then $\mathbf{IZF}_N \vdash \theta$;
3. (cf. Corollary 8.4.4) If $T^\star \vdash \theta$, then $T \vdash \theta$;

Indeed (cf. Corollary 8.4.7) these results can be strengthened by replacing $\mathbf{AC}^{NN}$ with $\mathbf{AC}^{N N^N}$, where $\mathbf{AC}^{N N^N}$ consists of the formulae

$$\forall n \exists f \in N^N \, \varphi(n, f) \;\to\; \exists F : N \to N^N \, \forall n \, \varphi(n, F(n)) \,,$$

with φ arbitrary.

Essentially, to show if $T^\star \vdash \theta$ then $T \vdash \theta$, one needs the following four steps:

Step1 Show $T_A \vdash (V^* \models_R \theta)$. In this step, one shows all the arithmetical theorems of $T^\star$ are interpretable by the relativized realizability semantics with the background theory T_A.

Step2 Show $T \vdash [V^* \models_{\mathcal{F}} (V^* \models_R \theta)]$. In this step, one shows relativized realizability semantics with respect to arithmetical formulae is interpretable by the forcing interpretation.

Step3 Show $T \vdash (V^* \models_{\mathcal{F}} \theta)$. In this step, one shows relativized realizability semantics with respect to arithmetical formulae is generic self-realizing (cf. Lemma 8.3.1).

Step4 Show $T \vdash \theta$. In this step, one shows forcing is absolute (cf. Lemma 8.1.1) with respect to arithmetical formulae.

Moreover, combing the conservativity results with Lifschitz' style semantics yields the following independence result (cf. Corollary 8.4.5):

$$\mathbf{IZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ} + \nvdash \mathbf{CT}_0!.$$

In conclusion, in this chapter, I have informally introduced various intuitionistic set theories and semi-constructive axioms. Furthermore, I have added some background knowledge for both realizability and forcing for arithmetic and set theories. I have also summarized the results of the thesis.

Chapter 2

Formal systems

In Chapter One, we informally introduced intuitionistic set theories. In this chapter, we will use first order language to formalize these systems. There are many ways to set up a formal system. For example, to formalize the natural numbers and all n-ary primitive recursive functions, one can add a constant ω and function symbols f^n to the language and then give their defining axioms; or one can add predicate symbols N and R^{n+1} to the language to do the job. For us, adding predicate symbols rather than constant and function symbols seems to be a more effective approach to introduce our results. We will introduce this language in Section 2.1 to accommodate both arithmetic and set theory.

In Section 2.2, we will introduce the intuitionistic formal systems which include axioms for Heyting arithmetic, intuitionistic set theories and various semi-constructive axioms. Instead of the usual approach which interprets arithmetic in set theory, to reduce the burden of interpretations, we adopt the systems which accommodate both arithmetic and set theory.

2.1 Languages

In addition to our object language $\mathcal{L}$ for formalizing a system with both set theory and arithmetic, we will for the purpose of relativized realizability also introduce a language $\mathcal{L}'$ which has a relation symbol A for the graph of a partial function from $\mathbb{N}$ to $\mathbb{N}$.

2.1.1 The language $\mathcal{L}$

Languages where numbers and sets are regarded as two different kind of objects are familiar from the literature (cf. Michael Beeson's book [3] (p. 164) or Harvey Friedman's paper [7]). $\mathcal{L}$ is defined as follows:
The symbols of the language are:

- Variables (objects): $x_0, x_1, x_2, ..., x_n, ...$
- Constants (number): $\bar{n}$ for all $n \in \mathbb{N}$.
- Predicates: (unary) S, N; (binary) $\in, =$; all other primitive recursive relation symbols: $R_1, R_2, ..., R_n, ...$, in particular, we explicitly use SUC, ADD, MULT for the graph of successor function, addition function and multiplication function.

The collection of all terms, $Term$, consists of the Variables and Constants. The collection of all the atomic formulae, $AtForm$, consists of all strings of symbols of the form $P^n(t_1, t_2, ..., t_n)$, where $t_1, t_2, ..., t_n \in Term$ and P^n is a n-ary predicate symbol.

The collection of all formulae, $Form$, is the smallest class which contains $AtForm$ and is closed under the logical connectives: $\wedge, \vee, \neg, \rightarrow, \exists, \forall$.

2.1.2 Relativized realizability interpretation in a language $\mathcal{L}'$

Relativized realizability differs from recursive realizability by using partial functions recursive in a partial function A from $\mathbb{N}$ to $\mathbb{N}$. To capture this axiomatically, we add an extra constant symbol A to the language $\mathcal{L}$ and denote the resulting language by $\mathcal{L}'$. Then one adds the axiom: A is a partial function from $\mathbb{N}$ to $\mathbb{N}$ to the axiomatic system.

2.2 Axioms

In this section, we will introduce the formal axiomatic systems **CZF** and **IZF**.

We also introduce some (semi-) constructive axioms which one might want to add to these intuitionistic systems.

To facilitate all the descriptions, we use the following abbreviations in the meta-language:

- $\forall n\varphi(n) \equiv \forall x(N(x) \rightarrow \varphi(x))$ and $\exists n\varphi(n) \equiv \exists x(N(x) \wedge \varphi(x))$. When there are more than one quantifiers present, we use $n, m, k, l, i, ...$ for the corresponding abbreviations.

- $\forall nm\varphi(n,m) \equiv \forall n\forall m\varphi(n,m)$.
- $\forall n\exists!m\psi(n,m) \equiv \forall n[\exists m\psi(n,m) \wedge \forall x\forall y(\psi(n,x) \wedge \psi(n,y) \rightarrow x = y)]$.
- $x \notin y \equiv \neg(x \in y)$ and $x \neq y \equiv \neg(x = y)$.
- $x \subseteq y \equiv \forall z(z \in x \rightarrow z \in y)$.
- $\forall x \in y\theta(x) \equiv \forall x[x \in y \rightarrow \theta(x)]$ and $\exists x \in y\theta(x) \equiv \exists x[x \in y \wedge \theta(x)]$.

2.2.1 A1: Axioms on numbers and sets

Numbers and sets will be treated as two different objects. Each object has only one identity and only sets contain elements. These are formalized in this group via the predicates $N(x)$ (x is a number) and $S(x)$ (x is a set) as follows:

1. $\forall x\neg(N(x) \wedge S(x))$.
2. $\forall x\forall y[x \in y \rightarrow S(y)]$.
3. $N(\bar{n})$ for all natural numbers n.

2.2.2 A2: Number-theoretic axioms

These axioms specify the basic operations of arithmetic; in particular, the successor function SUC, addition function ADD (axioms 7,8), multiplication function MULT (axioms 10,11) and mathematical induction.

1. $\mathrm{SUC}(\bar{n}, \overline{n+1})$ for all natural numbers n.
2. $\forall n\exists!m\mathrm{SUC}(n,m)$.
3. $\forall nm(\mathrm{SUC}(n,m) \rightarrow m \neq \bar{0})$.
4. $\forall m(m = \bar{0} \vee \exists n\mathrm{SUC}(n,m))$.
5. $\forall nmk(\mathrm{SUC}(m,n) \wedge \mathrm{SUC}(k,n) \rightarrow m = k)$.
6. $\forall nm\exists!k\mathrm{ADD}(n,m,k)$.
7. $\forall n\mathrm{ADD}(n,\bar{0},n)$.
8. $\forall nkmli[\mathrm{ADD}(n,k,m) \wedge \mathrm{SUC}(k,l) \wedge \mathrm{SUC}(m,i) \rightarrow \mathrm{ADD}(n,l,i)]$.
9. $\forall nm\exists!k\mathrm{MULT}(n,m,k)$.

10. $\forall n \mathrm{MULT}(n, \bar{0}, \bar{0})$.

11. $\forall nkmli[\mathrm{MULT}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{ADD}(m, n, i) \to \mathrm{MULT}(n, l, i)]$.

12. $A(\bar{0}) \wedge \forall nm[A(n) \wedge \mathrm{SUC}(n, m) \to A(m)] \to \forall n A(n)$.

13. Axioms for all other primitive recursive predicates.

2.2.3 A3: Logical axioms for IPL

Intuitionistic Predicate Logic (**IPL**) consists of twelve logical axioms (**LA**), three inference rules and various Identity Axioms (**IA**).

For logical axioms (LA):

(IPL1) $A \to (B \to A)$.

(IPL2) $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$.

(IPL3) $A \to (B \to A \wedge B)$.

(IPL4) $A \wedge B \to A$.

(IPL5) $A \wedge B \to B$.

(IPL6) $A \to A \vee B$.

(IPL7) $B \to A \vee B$.

(IPL8) $(A \vee B) \to [(A \to C) \to ((B \to C) \to C)]$.

(IPL9) $(A \to B) \to ((A \to \neg B) \to \neg A)$.

(IPL10) $A \to (\neg A \to B)$.

(IPL11) $\forall x A(x) \to A[x/y]$, where y is free for x in $A(x)$.

(IPL12) $A[x/y] \to \exists x A(x)$, where y is free for x in $A(x)$.

For Inference Rules: (In the following, we use $FV(C)$ to denote the set of all free variables in C).

(IR1) (Modus Ponens) $\frac{A,\ A \to B}{B}$.

(IR2) (rule $\forall$) $\frac{C \to A(x)}{C \to \forall x A(x)}$, where $x \notin FV(C)$.

(IR3) (rule $\exists$) $\frac{A(x) \to C}{\exists x A(x) \to C}$, where $x \notin FV(C)$.

For the Identity Axioms (IA):

For any $(m+1)$-place atomic predicate P^{m+1}, we use $P^{(j)}(k)$ to denote $P^1(k)$, if $m=0$ and $P^{m+1}(n_1, n_2, ., ., k, .., n_m)$ (i.e., k is placed at j-th arity), if $m > 0$ for all $j \in \{1, 2, ..., m+1\}$.

(**IA1**) $\forall x(x = x)$.

(**IA2**) $\forall x \forall y[x = y \to y = x]$.

(**IA3**) $\forall x \forall y \forall z[x = y \land y = z \to x = z]$.

(**IA4**) $\forall x \forall y \forall z[x = y \land y \in z \to x \in z]$.

(**IA5**) $\forall x \forall y \forall z[x = y \land z \in x \to z \in y]$.

(**IA6**) $\forall n \forall k \forall l[k = l \land \mathrm{SUC}(k, n) \to \mathrm{SUC}(l, n)]$.

(**IA7**) $\forall n \forall k \forall l[k = l \land \mathrm{SUC}(n, k) \to \mathrm{SUC}(n, l)]$.

(**IA8**) $\forall n_1 \forall n_2 \forall k \forall l[k = l \land \mathrm{ADD}^{(i)}(k) \to \mathrm{ADD}^{(i)}(l)]$, for $i \in \{1, 2, 3\}$.

(**IA9**) $\forall n_1 \forall n_2 \forall k \forall l[k = l \land \mathrm{MULT}^{(i)}(k) \to \mathrm{MULT}^{(i)}(l)]$, for $i \in \{1, 2, 3\}$.

(**IA10**) For any $(m+1)$-place primitive recursive relation R^{m+1}, if $m = 0$, then

$$k = l \land R^{(i)}(k) \to R^{(i)}(l);$$

if $m > 0$, then

$$\forall n_1 \forall n_2 ... \forall n_m \forall k \forall l[k = l \land R^{(i)}(k) \to R^{(i)}(l)], \text{ for } i \in \{1, 2, 3, ..., m+1\}.$$

2.2.4 A4.1: Non-logical axioms (CZF with two sorts)

CZF has $\in$-induction rather than the Foundation Axiom, it uses Bounded Separation rather than Full Separation and uses Subset Collection rather than the Powerset Axiom.

1. (Axiom of Extensionality) $\forall x \forall y(S(x) \land S(y) \to [\forall z(z \in x \leftrightarrow z \in y) \to x = y])$.

2. (Pairing Axiom)$\forall x \forall y(\exists u[S(u) \land x \in u \land y \in u])$.

3. (Union Axiom) $\forall x \exists u[S(u) \land \forall z(z \in u \leftrightarrow \exists y(y \in x \land z \in y))]$.

4. (Bounded Separation Schema)*

$$\forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow z \in x \wedge A(z))],$$

where u is not free in $A(z)$ and where $A(z)$ is a bounded formula.

A bounded formula is a formula in which all the occurrences of quantifiers are either in the form $\forall x \in$ or $\exists x \in$. This axiom restricts the full Separation Schema to avoid impredicativity.

5. (Axiom of Infinity) $\exists u(S(u) \wedge \forall z[z \in u \leftrightarrow N(z)])$.

6. (Induction Schema)*

$$\forall x[(\forall y \in x A(y)) \rightarrow A(x)] \rightarrow \forall x A(x).$$

Classically, the Induction Schema is equivalent to **FA**. Since **FA** implies the Principle of Excluded Middle, this axiom becomes an alternative.

7. (Strong Collection)* $\forall x[\forall y \in x \exists z A(y, z)$
$\rightarrow \exists u(S(u) \wedge \forall y \in x \exists z \in u A(y, z) \wedge \forall z \in u \exists y \in x A(y, z))]$.

This axiom strengthens Collection and Replacement Schema.

8. (Subset Collection)* $\forall a \forall b \exists u(S(u) \wedge \forall z[\forall x \in a \exists y \in b A(x, y, z) \rightarrow \exists d \in u(\forall x \in a \exists y \in d A(x, y, z) \wedge \forall y \in d \exists x \in a A(x, y, z))])$.

Since the Powerset Axiom involves impredicativity, this weaker axiom becomes an alternative.

We will use $\mathbf{CZF}_N$ to denote the formal system: $A1 + A2 + A3 + A4.1$.

Remark 2.2.1 ** indicates the differences between* **CZF** *and Zermelo-Fraenkel Set Theory* **ZF**.

2.2.5 A4.11: Non-logical axioms (CZF and A with two sorts)

On top of $\mathbf{CZF}_N$, we add an extra axiom "A is a partial function from N to N", $PF(A, N, N)$:

$$\forall x \in A\, \exists m \exists n(x = (n, m)) \wedge \forall x \forall y \forall z[(x, y) \in A \wedge (x, z) \in A \rightarrow y = z].$$

This extra axiom is not a part of the usual setting for intuitionistic set theory. The reason to add this axiom is to formalize relative computation which will be used in our relativized realizability interpretation. We use $\mathbf{CZF}_{NA}$ to denote the formal system: $\mathbf{CZF}_N + PF(A, N, N)$.

2.2.6 A4.2: Non-logical axioms (IZF with two sorts)

IZF has the following axioms:

1. (Axiom of Extensionality) $\forall x \forall y(S(x) \wedge S(y) \to [\forall z(z \in x \leftrightarrow z \in y) \to x = y])$.
2. (Pairing Axiom) $\forall x \forall y \exists u[S(u) \wedge x \in u \wedge y \in u]$.
3. (Union Axiom) $\forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow \exists y(y \in x \wedge z \in y))]$.
4. (Separation Schema)*

   $$\forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow z \in x \wedge A(z))],$$

   where u is not free in $A(z)$.
5. (Powerset Axiom)* $\forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow (S(z) \wedge z \subseteq x))]$.
6. (Axiom of Infinity) $\exists u(S(u) \wedge \forall z[z \in u \leftrightarrow N(z)])$.
7. (Induction Schema) $\forall x[\forall y(y \in x \wedge A(y)) \to A(x)] \to \forall x A(x)$.
8. (Collection Schema)*

   $$\forall x[\forall y \in x \exists z A(y, z) \to \exists u(S(u) \wedge \forall y \in x \exists z \in u A(y, z))].$$

   This axiom strengthens the Replacement Schema.

We use $\mathbf{IZF}_N$ to denote the formal system: $A1 + A2 + A3 + A4.2$.

Remark 2.2.2 ** indicates the differences between* **CZF** *and* **IZF**.

2.2.7 A5: (Semi-) Constructive axioms

From the constructive point of view, there are some classical axioms, for example, the axiom of choice, which are not intuitionistically justified. Hence we have to consider adding more intuitionistic axioms into our systems and check their consistency. Some of these axioms even contradict the classical axioms and some are simply the weaker versions of the classical axioms. We use n, m, l, p, q to denote meta-variables ranging over naturals and x, y, z, a, b, c, u, v to denote meta-variables ranging over Variables. We also use the following abbreviations:

- $\forall n A(n) \equiv \forall x[N(x) \to A(x)]$;

- $\exists nA(n) \equiv \exists x[N(x) \wedge A(x)]$;
- $\exists! mA(m) \equiv \exists mA(m) \wedge \forall x \forall y[A(x) \wedge A(y) \rightarrow x = y]$;
- $\forall x \in a \exists! yA(x, y) \equiv \forall x[x \in a \rightarrow (\exists yA(x, y) \wedge \forall b \forall c(A(x, b) \wedge A(x, c) \rightarrow b = c))]$;
- $f \subseteq N \times N \equiv \forall x \in f \exists y \exists z[x = (y, z) \wedge (N(y) \wedge N(z))]$;
- $f \subseteq N \times a \equiv \forall x \in f \exists y \exists z[x = (y, z) \wedge (N(y) \wedge z \in a)]$;
- $f \subseteq y \times x \equiv \forall a \in f \exists b \exists c[a = (b, c) \wedge (b \in y \wedge c \in x)]$;
- $Fun(f, N) \equiv \forall x \in f[\exists y \exists z(N(y) \wedge x = (y, z))] \wedge \forall n \exists! z(n, z) \in f$ (i.e., f is a function with domain N);
- $Fun(f, a) \equiv \forall x \in f[\exists y \exists z(y \in a \wedge x = (y, z))] \wedge \forall x \in a \exists! y(x, y) \in f$ (i.e., f is a function with domain a);
- $Fun(f, N, N) \equiv f \subseteq N \times N \wedge \forall n \exists! m(n, m) \in f$ (i.e., f is a function with domain N and range N);
- $Fun(f, N, a) \equiv f \subseteq N \times a \wedge \forall n \exists! x \in a(n, x) \in f$ (i.e., f is a function with domain N and range a);
- $SFun(f, y, x) \equiv Fun(f, y, x) \wedge \forall v \in x \exists u \in y(u, v) \in f$ (i.e., f is a surjective function from y to x);
- $Rel(r, a) \equiv \forall x \in r \exists u \in a \exists v[x = (u, v)] \wedge \forall u \in a \exists v(u, v) \in r$. (i.e., r is a binary relation with domain a);
- $Base(y) \equiv \forall r[Rel(r, y) \rightarrow \exists g(Fun(g, y) \wedge g \subseteq r)]$;
- $A(f(n)) \equiv \exists y[(n, y) \in f \wedge A(y)]$;
- $A(f(n), f(n+1)) \equiv \exists x \exists y[(n, x) \in f \wedge (n+1, y) \in f \wedge A(x, y)]$.

Non-classical axioms:

Definition 2.2.3 (*Church's thesis,* $\mathbf{CT}_0$)

$$\forall n \exists m \varphi(n, m) \rightarrow \exists l \forall n \exists p \exists q(T(l, n, p) \wedge U(p, q) \wedge \varphi(n, q)),$$

where T represents Kleene's T-predicate and U represents result-extraction predicate. If one takes $\varphi(n, m) \equiv (m = 0 \rightarrow \mathcal{A}(n)) \wedge (m \neq 0 \rightarrow \mathcal{B}(n))$, where $\mathcal{A}(n) \equiv \forall l \neg T(a, n, l)$ and $\mathcal{B}(n) \equiv \forall l \neg T(b, n, l)$, then we call this instance $\mathbf{CT}_0^{ab}$.

Definition 2.2.4 (*Church's thesis,* $\mathbf{CT}_0!$)

$$\forall n \exists ! m \varphi(n,m) \to \exists l \forall n \exists p \exists q (T(l,n,p) \wedge U(p,q) \wedge \varphi(n,q)).$$

Definition 2.2.5 (*Extended Church's thesis,* $\mathbf{ECT}_0$)

$$\forall x[N(x) \wedge \eta(x) \to \exists y(N(y) \wedge \varphi(x,y)) \to$$

$$\exists l[N(l) \wedge \forall n(N(n) \wedge \eta(n) \to \exists p,q(N(p) \wedge N(q) \wedge T(l,n,p) \wedge U(p,q) \wedge \varphi(n,q)))],$$

where η is any almost negative formula (c.f. [20]).

Definition 2.2.6 (*Uniformity Principle,* **UP**)

$$\forall x[S(x) \to \exists n A(x,n)] \to \exists n \forall x[S(x) \to A(x,n)].$$

This axiom basically says that the only possible way to label a class of sets by numbers is to give each set the same number.

Definition 2.2.7 (*Unzerlegbarkeit,* **UZ**)

$$[\forall x(S(x) \to B(x) \vee C(x))] \to \forall x(S(x) \to B(x)) \vee \forall x(S(x) \to C(x)).$$

This axiom indeed is a theorem of **UP** by setting $A(x,n)$ to be $(n = 0 \to B(x)) \wedge (n \neq 0 \to C(x))$.
classical axioms:

Definition 2.2.8 (*Axiom of Choice,* $\mathbf{AC}^N$)

$$\forall f[(Fun(f,N) \wedge \forall n \exists y \in f(n)) \to \exists g(Fun(g,N) \wedge \forall n(g(n) \in f(n)))].$$

This axiom restricts **AC** *and says every countable family of nonempty sets has a choice function.*

Definition 2.2.9 (*Axiom of Choice,* $\mathbf{AC}^{N2}$)

$$\forall n \exists y \in \{0,1\} A(n,y) \to \exists g[Fun(g,N,\{0,1\}) \wedge \forall n \exists m((n,m) \in g \wedge A(n,m))].$$

This axiom says every total numerical relation with range $\{0,1\}$ has a sub-function.

Definition 2.2.10 (*Axiom of Choice,* $\mathbf{AC}^{NN}$)

$$\forall n \exists m A(n,m) \to \exists f[Fun(f,N,N) \wedge \forall n \exists m((n,m) \in f \wedge A(n,m))].$$

This axiom says every total numerical relation has a sub-function.

Definition 2.2.11 (*Axiom of Choice,* $\mathbf{AC}^{NN}!$)

$$\forall n \exists ! m A(n,m) \rightarrow \exists f[Fun(f,N,N) \wedge \forall n \exists m((n,m) \in f \wedge A(n,m))].$$

Definition 2.2.12 (*Dependent Choice,* $\mathbf{DC}$)

$$\forall a \forall b[b \in a \wedge \forall x \in a \exists y \in a A(x,y) \rightarrow$$
$$\exists f(Fun(f,N,a) \wedge (0,b) \in f \wedge \forall n A(f(n), f(n+1)))].$$

Definition 2.2.13 (*Dependent Choice,* $\mathbf{DC}^{NN}$)

$$\forall l[\forall n \exists m A(n,m) \rightarrow \exists f(Fun(f,N,N) \wedge (0,l) \in f \wedge \forall n A(f(n), f(n+1)))].$$

Definition 2.2.14 (*Relativized Dependent Choice,* $\mathbf{RDC}$)

$$\forall z[B(z) \wedge \forall x(B(x) \rightarrow \exists y(B(y) \wedge A(x,y)))$$
$$\rightarrow \exists f(Fun(f,N) \wedge (0,z) \in f \wedge \forall n(B(f(n)) \wedge A(f(n), f(n+1))))].$$

Definition 2.2.15 (*Presentation Axiom,* $\mathbf{PA}_X$)

$$\forall x(S(x) \rightarrow \exists y \exists f[Base(y) \wedge SFun(f,y,x)]).$$

Definition 2.2.16 (*Independence of Premises,* $\mathbf{IP}$)

$$(\neg\theta \rightarrow \exists x A(x)) \rightarrow \exists x(\neg\theta \rightarrow A(x)),$$

where θ is any closed formula.

Definition 2.2.17 (*Markov's Principle,* $\mathbf{MP}$)

$$[\forall n(A(n) \vee \neg A(n)) \wedge \neg\neg\exists n(A(n))] \rightarrow \exists n(A(n)).$$

Definition 2.2.18 (*Weak Markov's Principle,* $\mathbf{MP}_{pr}$)

$$\neg\neg\exists n(A(n)) \rightarrow \exists n(A(n)),$$

where A is a primitive recursive formula.

Definition 2.2.19 (*Bounded Markov's Principle,* $\mathbf{B}\sum_{\mathbf{2}}^{\mathbf{0}} - \mathbf{MP}$)

$$\neg\neg\exists n \leq m \forall k A(n,k,e) \rightarrow \exists n \leq m \forall k A(n,k,e),$$

where A is a primitive recursive formula.

In conclusion, we have introduced formal systems to accommodate both Heyting arithmetic, intuitionistic set theories and various semi-constructive axioms. In the later chapters, we will study the properties of these systems by some semantical approaches.

Chapter 3

Applicative structure and universes

This chapter is the preparation for our later chapters. First of all, we mention a theory of computation **APP**. Any model of **APP** is called an applicative structure. In Section 3.1, we introduce the language for this theory and some of its fundamental properties. We then give a model for this theory.

Secondly, in Section 3.2, we study various transfinite inductions and inductive definitions which overcome the lack of the Powerset Axiom $\mathbf{CZF}_N$. These inductions play important roles in the formalization of our semantics.

Thirdly, in Section 3.3, we define our external universe and construct the internal universes for both the realizability and forcing interpretations.

3.1 Language for APP

The language describing **APP** (or $\mathcal{L}_{App}$) has been used in the literature, in particular, McCarty's Ph.D. thesis [15] and Michael Rathjen's paper [20]. We paraphrase this language as follows: First of all, the symbols (for variables, constants and predicates) of the language are defined as follows:

- Variables (Var): $x^1, x^2, x^3, ..., x^n....$
- Constants (Con): $0, k, s, d, \jmath, \jmath_0, \jmath_1$
- Predicates: (unary) N, (binary) $=$, (ternary) App

Now we define the collection of all the terms $Term_{App}$ to be the collection of Var and Con and then define the collection of all the atomic formulae $AutoF_{App}$ to be $\{App(t,u,v)|t,u,v \in Term_{App}\} \cup \{u = v|u,v \in Term_{App}\} \cup$

$\{N(u)|u \in Term_{App}\}$. Finally, we define the collection of all the formulae $Form_{App}$ to be the smallest class which contains $AutoF_{App}$ and is closed under the logical connectives: $\wedge, \vee, \neg, \rightarrow, \exists, \forall$.

In order to define applicative axioms in a more economical way, we have to extend the language at the meta-level and use some abbreviations. First of all, we inductively define the class of all application terms App_{term} as follows:

- If $t \in Term_{App}$, then $t \in App_{term}$;
- If $u, v \in App_{term}$, then $(uv) \in App_{term}$.

To simplify notations, we use the following abbreviations:

- $t^1t^2t^3...t^n \equiv (...((t^1t^2)t^3)...t^n)$, where $t^1, t^2, t^3..., t^n \in App_{term}$.
- $\langle x^i, x^j \rangle \equiv ((\jmath(x^i))x^j)$.
- $x_0 \equiv (\jmath_0 x)$ and $x_1 \equiv (\jmath_1 x)$.
- $x_{\alpha 0} \equiv (\jmath_0 x_\alpha)$, where α is any string of numbers of 0 and 1.
- $x_{\alpha 1} \equiv (\jmath_1 x_\alpha)$, where α is any string of numbers of 0 and 1.

Secondly, we have to define a meta-predicate (application equality) $\simeq$ over $App_{term} \times App_{term}$ and a meta-predicate (defined application) $\downarrow$ over App_{term} as follows:

- (application equality) $s \simeq a$: If $a \in Var$, then define $s \simeq a \equiv s = a$, if $s \in Term_{App}$ and if s is a compound term, i.e., in the form of (uv), then we inductively define $s \simeq a$ as $\exists x \exists y[u \simeq x \wedge v \simeq y \wedge App(x, y, a)]$. If $a \in App_{term} \backslash Var$, then define $s \simeq a \equiv \forall x[s \simeq x \leftrightarrow a \simeq x]$.
- (defined application) $t \downarrow \equiv \exists y(t \simeq y)$.

Remark 3.1.1 *Observe that each meta-formula corresponds to some formal formula in $\mathcal{L}_{App}$ as every meta-formula containing $\simeq$ and $\downarrow$ can be defined by $=$ and App.*

Remark 3.1.2 *We will use $x^i, x^j, x^k, x^h, x^1, x^2, ..., x^n, x^{n+1}, y, z$ to denote the meta-variables ranging over Var.*

3.1.1 Applicative axioms (APP)

A standard setting of the Applicative axioms includes IPL with the Identity Axioms, the arithmetical axioms and the following non-logical Axioms:

(App1) $App(x^i, x^j, x^k) \wedge App(x^i, x^j, x^h) \rightarrow x^k = x^h$.

(App2) $kx^i x^j \downarrow \wedge kx^i x^j \simeq x^i$.

(App3) $sx^i x^j \downarrow \wedge sx^i x^j x^k \simeq (x^i x^k)(x^j x^k)$.

(App4) $\langle x^i, x^j \rangle \downarrow$, $x_0 \downarrow$, $x_1 \downarrow$, $\langle x^i, x^j \rangle_0 \simeq x^i$ and $\langle x^i, x^j \rangle_1 \simeq x^j$.

(App5) $N(x^i) \wedge N(x^j) \wedge x^i = x^j \rightarrow dyzx^i x^j \downarrow \wedge dyzx^i x^j \simeq y$.

(App6) $N(x^i) \wedge N(x^j) \wedge x^i \neq x^j \rightarrow dyzx^i x^j \downarrow \wedge dyzx^i x^j \simeq z$.

3.1.2 Fundamental consequences

Claim 3.1.3 (**APP**) $\simeq$ *is an equivalence relation.*

Proof. (Reflexive) $t \simeq t$: This follows immediately from the definition. (Symmetric) $t^1 \simeq t^2 \leftrightarrow t^2 \simeq t^1$: If $t^2 \in Var$, then $t^1 \simeq t^2 \leftrightarrow t^2 \simeq t^1$ follows immediately from the definition. If $t^2 \in App_{term} \backslash Term_{app}$, then $t^1 \simeq t^2 \leftrightarrow t^2 \simeq t^1$ follows immediately from the definition as well. (Transitive) $t^1 \simeq t^2 \simeq t^3 \rightarrow t^1 \simeq t^3$: It also follows immediately from the definitions. ■

Corollary 3.1.4 (**APP**) $t^1 \simeq t^2 \rightarrow (t^3 t^1) \simeq (t^3 t^2)$, *where* $t^1, t^2, t^3 \in App_{term}$.

Proof. Let x be arbitrary such that $(t^3 t^1) \simeq x$. Then by the definition, $\exists u \exists v (t^3 \simeq u \wedge t^1 \simeq v \wedge App(u, v, x))$. By Claim 3.1.3 and the definition, the result $t^3 t^2 \simeq x$ follows immediately. ■

Claim 3.1.5 (**APP**) $t \simeq x \wedge t \simeq y \rightarrow x = y$, *where* $t \in App_{term}$ *and* $x, y \in Var$.

Proof. One shows this by induction on the complexity of t. If $t \in Var$, then it follows immediately by the definition and the Identity Axiom. Now assume $t = (t^1 t^2)$ and

$$t^1 \simeq u \wedge t^1 \simeq v \rightarrow u = v, t^2 \simeq u \wedge t^2 \simeq v \rightarrow u = v, \tag{3.1}$$

and assume $t \simeq x \wedge t \simeq y$. By the definition we have

$$\exists x^1 \exists x^2 [t^1 \simeq x^1 \wedge t^2 \simeq x^2 \wedge App(x^1, x^2, x)],$$
$$\exists z^1 \exists z^2 [t^1 \simeq z^1 \wedge t^2 \simeq z^2 \wedge App(z^1, z^2, y)].$$

By the assumption (3.1), it follows that $x^1 = z^1 \wedge x^2 = z^2$, i.e., by **IPL** with the Identity Axiom, and (App1) $x = y$. ■

Definition 3.1.6 *For each $x \in Var$, we define an abstraction operator $\lambda x.$: $App_{term} \to App_{term}$ inductively as follows:*

- $\lambda x.x \equiv (sk)k$.
- $\lambda x.y \equiv ky$, if $y \neq x$.
- $\lambda x.(\theta\eta) \equiv s(\lambda x.\theta)(\lambda x.\eta)$.

Definition 3.1.7 $\lambda x^{n+1}.\lambda x^n....\lambda x^2.\lambda x^1.t \equiv \lambda x^{n+1}.(\lambda x^n....\lambda x^2.\lambda x^1.t)$

Definition 3.1.8

$$t[x^1/t^1, ..., x^n/t^n, x^{n+1}/t^{n+1}] \equiv (t(x^1/t^1, ..., x^n/t^n, x^n))[x^{n+1}/t^{n+1}],$$

where x^l/t^l means that the variable x^l is substituted by the term t^l.

Remark 3.1.9 *By mathematical induction, $\lambda x^{n+1}.\lambda x^n....\lambda x^2.\lambda x^1.t \in App_{term}$ and $t[x^1/t^1, x^2/t^2, ..., x^n/t^n] \in App_{term}$ for $\forall n \in \mathbb{N}$, where $t \in App_{term}$.*

Theorem 3.1.10 *Let $x \in Var$ and $t^1, t^2 \in App_{term}$ be arbitrary. Then $\lambda x.t^1 \in App_{term}$ and **APP** $\vdash ((\lambda x.t^1)t^2) \simeq t^1[x/t^2]$.*

Proof. This follows immediately from the definitions and the inductive hypothesis. ■

Corollary 3.1.11 *Let $x^1, x^2, ..., x^n \in Var$ and $t, t^1, t^2, ..., t^n \in App_{term}$. Then*

$$\lambda x^1.\lambda x^2....\lambda x^n.t \in App_{term}$$

and

$$\mathbf{APP} \vdash (\lambda x^1.\lambda x^2....\lambda x^n.t)t^1t^2...t^n \simeq t[x^1/t^1, x^2/t^2, ..., x^n/t^n].$$

Proof. *It follows from Theorem 3.1.10, the definitions and the inductive hypothesis.* ■

Notation 3.1.12 $\Omega \equiv \lambda x.[(\lambda y.(x(yy)))(\lambda y.(x(yy)))]$. *It will be called a* ***fixed-point generator*** *because of the following theorem.*

Theorem 3.1.13 (**APP**) $\Omega \in App_{term} \wedge \forall t \in App_{term}[t(\Omega t) \simeq \Omega t]$.
Proof. *It follows immediately from Theorem 3.1.10.* ■

When we do the transfinite recursion, we will encounter the question: given $t \in App_{term}$, what's the solution for $q \in App_{term}$ such that $(tq) \simeq q$? Moreover, sometimes we do have to apply mutual transfinite inductions and which will involve the following question: given $t, \tilde{t} \in App_{term}$, what's the solution for $u, v \in App_{term}$ such that $(tu)v \simeq u$ and $(\tilde{t}u)v \simeq v$? In the following we will demonstrate how to find the solutions for these questions.

Corollary 3.1.14 (**APP**) *Let* $t \in App_{term}$ *be arbitrary. Then* $tq \simeq q$, *where* q *denotes the term* $(\lambda y.t(yy))(\lambda y.t(yy))$.

Proof. This follows immediately from Theorem 3.1.13. ■

Corollary 3.1.15 *Let* $t, \tilde{t} \in App_{term}$ *be arbitrary. Then* **APP** $\vdash tuv \simeq u \wedge \tilde{t}uv \simeq v$, *where* $u \equiv (\Omega(\lambda x.\langle tx_0x_1, \tilde{t}x_0x_1\rangle))_0 \wedge v \equiv (\Omega(\lambda x.\langle tx_0x_1, \tilde{t}x_0x_1\rangle))_1$.

Proof. Let $l \equiv \lambda x.\langle tx_0x_1, \tilde{t}x_0x_1\rangle \in App_{term}$. Then by Theorem 3.1.13 $l(\Omega l) \simeq \Omega l$, i.e.,

$$\langle t(\Omega l)_0(\Omega l)_1, \tilde{t}(\Omega l)_0(\Omega l)_1\rangle \simeq \Omega l,$$

i.e., by Corollary 3.1.4 and the axiom App4,

$$t(\Omega l)_0(\Omega l)_1 \simeq (\Omega l)_0 \wedge \tilde{t}(\Omega l)_0(\Omega l)_1 \simeq (\Omega l)_1.$$

■

3.1.3 Kl is a model for APP

Let $\mathbb{N}$ be the natural numbers. Each partial recursive function can be effectively associated with one natural number via **Kleene's Normal Form Theorem**: there exists a primitive recursive predicates T and a primitive recursive function U' such that for any partial recursive function f, there is a code (or a Gödel number) k such that $\forall \vec{x}[f(\vec{x}) \simeq U'(\mu y(T(k, \vec{x}, y)))]$, where μ is the minimization operator. For convenience, we will simply write $f^{\#}$ to denote a code of f. Another important theorem is the **S-M-N Theorem**: for every $m+n$ there is a total recursive function $s_{mn} : \mathbb{N}^{m+1} \to \mathbb{N}$ such that for any $(m+n)$-ary partial recursive function $\{e\}$, $\{e\}(\vec{x}, \vec{y}) \simeq \{s_{mn}(e, \vec{x})\}(\vec{y})$.

Because of this theorem, it is sufficient to consider only unary partial recursive functions. Another feature of this theory is that recursively enumerable sets (or r.e. sets) can be identified with Σ_1^0-formulae definable sets. The proofs of these statements can be found in standard textbooks, for example, [11] or [4].

Notation 3.1.16 *PRF ≡the set of all the partial recursive functions.*

Notation 3.1.17 *$PRF^{\#}$ ≡the set of all the Gödel numbers of members of PRF.*

Notation 3.1.18 *$\{n\} \equiv$ the partial recursive function whose Gödel number is n and $n \cdot m \equiv \{n\}(m)$.*

Notation 3.1.19 *$n \cdot m \downarrow \equiv \exists z T(n,m,z)$ and $n \cdot m \uparrow \equiv \forall z \neg T(n,m,z)$.*

Notation 3.1.20 *$n \cdot m \downarrow l \equiv \exists z T(n,m,z) \wedge U(\mu z T(n,m,z), l)$.*

In order to interpret constants in **APP**, we single out the following partial recursive functions:

1. A total recursive function $\underline{k} : \mathbb{N} \to \mathbb{N}$ with $\underline{k}(n) := \mathbf{n}^{\#}$, where $\mathbf{n} : \mathbb{N} \to \mathbb{N}$ with $\mathbf{n}(m) := n$ for all $m \in \mathbb{N}$.

2. A partial recursive function $\underline{s} \equiv s_{21}(s_{21}^{\#}, s_{21}^{\#}, f^{\#})$, where $f(n,m,l) := (n \cdot l) \cdot (m \cdot l)$. Since f is a partial recursive function, by repeatedly applying the S-M-N theorem, we have

   $(n \cdot l) \cdot (m \cdot l) \simeq \{s_{21}(f^{\#}, m, n)\} \cdot l \simeq \{s_{21}(s_{21}^{\#}, f^{\#}, m)\} \cdot n \cdot l \simeq \{s_{21}(s_{21}^{\#}, s_{21}^{\#}, f^{\#})\} \cdot m \cdot n \cdot l$.

3. (disjunction function) $\underline{d} : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $\underline{d}(x,y,u,v) := x$, if $u = v$ and $\underline{d}(x,y,u,v) := y$, if $u \neq v$.

4. (pairing function) $\langle , \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where $\langle , \rangle$ is a bijective primitive recursive function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

5. (left unpairing) $\langle \rangle_0 : \mathbb{N} \to \mathbb{N}$ is the left inverse function of $\langle , \rangle$.

6. (right unpairing) $\langle \rangle_1 : \mathbb{N} \to \mathbb{N}$ is the right inverse function of $\langle , \rangle$.

Notation 3.1.21 *We will use e_0 to denote $\langle \rangle_0(e)$ and e_1 to denote $\langle \rangle_1(e)$.*

Remark 3.1.22 *From now on, we will use* $\mathbf{d}xyuv$ *to denote the disjunction function* $\underline{d}(x, y, u, v)$*. However, if the forms of* x, y *are rather lengthy, then we will use* $\mathbf{d}[x][y]uv$ *to denote* $\mathbf{d}xyuv$*.*

Definition 3.1.23 $Kl \equiv (\mathbb{N}, \underline{k}^{\#}, \underline{s}^{\#}, \underline{d}^{\#}, \langle,\rangle^{\#}, \langle\rangle_0^{\#}, \langle\rangle_1^{\#}, \cdot)$*, where* $n \cdot m$ *is defined to be* $\{n\}(m)$*, if* $\{n\}(m)\downarrow$ *and* $n \cdot m \uparrow$*, if* $\{n\}(m)\uparrow$*.*

Theorem 3.1.24 $(\mathbf{CZF}_N)$ *Kl is a model of* **APP***.*

Proof. If one interprets k, s, d, $\jmath$, $\jmath_0$, $\jmath_1$ to be $\underline{k}^{\#}$, $\underline{s}^{\#}$, $\underline{d}^{\#}$, $\langle,\rangle^{\#}$, $\langle\rangle_0^{\#}$, $\langle\rangle_1^{\#}$ respectively and interprets $N(n)$, $()$ and $App(n, m, l)$ to be $n \in \mathbb{N}$, $\cdot$ and $n \cdot m \downarrow l$ respectively, then the result follows immediately. ■

In order to define λ-terms for Kl, first of all, one extends the set Constants to be $\{0, 1, 2, .., n, ...\}$ and makes the same definitions as in 3.1.6. We call these extended λ-terms. Then one replaces all the constants in the closed extended λ-terms (i.e., the terms consisting only of constants) by their interpretations in Kl. For example, the extended λ-term $\lambda x.(mx)$ (i.e., by the definition, $((s(km))(skk)))$ is replaced by $((\underline{s}^{\#} \cdot (\underline{k}^{\#} \cdot m)) \cdot ((\underline{s}^{\#} \cdot \underline{k}^{\#}) \cdot \underline{k}^{\#}))$. Now one defines the set of all the λ-terms for Kl to be the set of all such replacements of all the closed extended λ-terms. Furthermore, for each λ-term for Kl, we replace every occurrence of the symbol λ in the λ-term with Λ to indicate its code.

Then one can apply the results derived in Subsection 3.1.2. For example, the fixed point generator Ω in $PRF^{\#}$ will be denoted by $\Lambda x.[(\Lambda y.(x \cdot (y \cdot y)) \cdot (\Lambda y.(x \cdot (y \cdot y)))]$. From now on, we will use Ω to denote $\Lambda x.[(\Lambda y.(x \cdot (y \cdot y)) \cdot (\Lambda y.(x \cdot (y \cdot y)))]$.

Remark 3.1.25 *In this thesis we also apply relative computation. All the results are exactly the same if one replaces* T *by* T^A*, where* T^A *is a relativized* T*-predicate and* A *is a partial function from* $\mathbb{N}$ *to* $\mathbb{N}$*.*

3.2 Transfinite induction & inductive definition

This section is based on Section 5 in Peter Aczel and Michael Rathjen's report [1] and Section 3 in Michael Rathjen's paper [20]. Since object inductions and some of its instances and variants are heavily applied, we isolate them in this section to study some of their properties that will be used throughout this thesis. In order to facilitate the whole argument, let us define some predicates in advance. Define $Tran(x)$ (x is transitive) as

$$\forall y \forall z (y \in z \wedge z \in x \rightarrow y \in x).$$

Define $On(x)$ (x is an ordinal) as

$$Tran(x) \wedge \forall y \in x(Tran(y)).$$

Define $TransClos(x, y)$ (y is a transitive closure of x, i.e., y is the least transitive set that contains x) by

$$[S(y) \wedge Tran(y) \wedge x \subseteq y] \wedge \forall z([S(z) \wedge Tran(z) \wedge x \subseteq z] \rightarrow y \subseteq z).$$

Later on we will show for any set x, there is a unique set y such that y is the transitive closure of x and we will use $TC(x)$ to denote this unique y.

Moreover, we use On to denote the class of all the ordinals. Furthermore, we define a rank function rk as follows: for any set a, $rk(a) := \cup\{rk(b) + 1 : b \in a\}$, where $rk(b) + 1 \equiv rk(b) \cup \{rk(b)\}$.

3.2.1 Transfinite inductions

In this thesis, we will apply the following transfinite inductions to construct functions by transfinite recursion, or to build up internal universes or to do the reasoning over the whole universe (external or internal). Different names are assigned to indicate the nature of the inductions. All the quantifiers used in this section range over the class $V \equiv \{b : N(b) \vee S(b)\}$.

- Object Induction: $\forall x[(\forall y \in x\varphi(y)) \rightarrow \varphi(x)] \rightarrow \forall x\varphi(x)$. This provides a proof when one wants to do reasoning about the whole universe of objects. Some of its variants and instances turn out to be very useful.

- Transitive Closure (TC) induction: $\forall x[(\forall y \in TC(x)\varphi(y)) \rightarrow \varphi(x)] \rightarrow \forall x\varphi(x)$. This is a variant (cf. Lemma 3.2.5) of Object Induction.

- $\lhd^2$-induction: $\forall x\forall y[(\forall u\forall v((u, v) \lhd^2 (x, y) \rightarrow A(u, v))) \rightarrow A(x, y)] \rightarrow \forall x\forall yA(x, y)$, where $(u, v) \lhd^2 (x, y) \equiv [(u = x \vee u \in TC(x)) \wedge (v = y \vee v \in TC(y))] \wedge \neg(u = x \wedge v = y)$. This is a theorem of TC-induction (cf. Corollary 3.2.7). This induction will be applied when we want to construct a transfinite recursion function (cf. Claim 3.2.9).

- $\lhd^3$-induction: $\forall x\forall y\forall z[(\forall u\forall v\forall k((u, v, k) \lhd^3 (x, y, z) \rightarrow A(u, v, k))) \rightarrow A(x, y, z)] \rightarrow \forall x\forall y\forall zA(x, y, z)$, where $(u, v, k) \lhd^3 (x, y, z) \equiv [(u = x \vee u \in TC(x)) \wedge (v = y \vee v \in TC(y)) \wedge (k = z \vee k \in TC(z))] \wedge \neg(u = x \wedge v = y \wedge k = z)$. This is a theorem of TC-induction (cf. Claim 3.2.6). This induction will be applied when we want to interpret **IA3** and **IA4**.

- Set Induction: $\forall x[(\forall y \in x(S(y) \to \varphi(y))) \to (S(x) \to \varphi(x))] \to \forall x(S(x) \to \varphi(x))$. This is an instance of Object Induction. If one replaces $\forall y \in x$ with $\forall y \in TC(x)$, then it becomes an instance of TC induction.

- Ordinal Induction: $\forall x[(\forall y \in x(On(y) \to \varphi(y))) \to (On(x) \to \varphi(x))] \to \forall x(On(x) \to \varphi(x))$. This is an instance of Object Induction. If one replaces $\forall y \in x$ with $\forall y \in TC(x)$, then it becomes an instance of TC induction. Since we construct our internal set universe along the ordinals, this induction becomes the main tool used in this thesis whenever we do some reasoning regarding the whole internal set universe.

- Mathematical Induction: $\varphi(\bar{0}) \wedge \forall x \forall y[N(x) \wedge N(y) \wedge \varphi(x) \wedge \mathrm{SUC}(x, y) \to \varphi(y)] \to \forall x[N(x) \to \varphi(x)]$.

Claim 3.2.1 ($\mathbf{CZF}_N$) *$\forall x \exists! y(TransClos(x, y))$.*

Proof. If $x \in \mathbb{N}$, then $TC(x) = \emptyset$. If x is a set, then by Object Induction and Replacement the result follows. ■

Remark 3.2.2 *For each object k, we use $TC(k)$ (or simply k^t_*) to denote the unique transitive closure of k.*

Corollary 3.2.3 ($\mathbf{CZF}_N$) *(1) $b \in a \vee b \subseteq a \vee b \in TC(a) \to TC(b) \subseteq TC(a)$; (2) $\forall a[S(a) \to TC(a) = a \cup (\bigcup_{x \in a} TC(x))]$.*

Proof. (1) Since $b \in a \subseteq TC(a)$, it follows $b \in TC(a)$, i.e., $b \subseteq TC(a)$ and that by Claim 3.2.1 and the definition, the results follow. (2) $a \cup (\bigcup_{x \in a} TC(x)) \subseteq TC(a)$ follows immediately from (1). To show $TC(a) \subseteq a \cup (\bigcup_{x \in a} TC(x))$ it suffices to prove $a \cup (\bigcup_{x \in a} TC(x))$ is a transitive set which contains a. Let c, b be arbitrary such that $c \in b \in a \cup (\bigcup_{x \in a} TC(x))$. Then it follows $c \in \bigcup_{x \in a} TC(x)$. ■

Claim 3.2.4 ($\mathbf{CZF}_N$) *$[\forall x(\varphi[x^t_*] \to \varphi(x))] \to \forall x((\forall z \in x\varphi[z^t_*]) \to \varphi[x^t_*])$, where $\varphi[x^t_*] \equiv \forall y \in TC(x)\varphi(y)$.*

Proof. Assume

$$[\forall x(\varphi[x^t_*] \to \varphi(x))]. \tag{3.2}$$

We have to show $\forall x[(\forall z \in x\varphi[z^t_*] \to \varphi[x^t_*])]$. If x is a number, then it follows that $TC(x) = \emptyset$ and $\varphi[x^t_*]$ and thus $\forall z \in x\varphi[z^t_*] \to \varphi[x^t_*]$. If x is a set, then we have the following inferences: Assume

$$\forall z \in x\varphi[z^t_*]. \tag{3.3}$$

Now we have to show $\varphi[x^t_*]$. Assume $y \in TC(x)$. By Corollary 3.2.3, it follows that $y \in x \vee y \in TC(k)$ for some $k \in x$. If $y \in x$, then from (3.2) and (3.3), it follows that $\varphi(y)$. If $y \in TC(k)$ for some $k \in x$, then by (3.3) and the definition, the result $\varphi(y)$ follows immediately. ■

Lemma 3.2.5 [*TC-induction*] ($\mathbf{CZF}_N$)

$$[\forall x(\varphi[x^t_*] \to \varphi(x))] \to \forall x\varphi(x),$$

where $\varphi[x^t_*] \equiv \forall y \in TC(x)\varphi(y)$.

Proof. Assume $\forall x(\varphi[x^t_*] \to \varphi(x))$. By Claim 3.2.4, it follows that $\forall x((\forall z \in x\varphi[z^t_*]) \to \varphi[x^t_*])$. Then by Object Induction, one has $\forall x\varphi[x^t_*]$. Again by the assumption, the result $\forall x\varphi(x)$ follows. ■

Claim 3.2.6 *TC-induction implies* $\lhd^3$*-induction.*

Proof. Assuming the antecedent of $\lhd^3$, we have to show $\forall x\forall y\forall zA(x, y, z)$. Let $a \in V$ be arbitrary. By TC-induction, it suffices to prove

$$\forall x \in TC(a)B(x) \to B(a),$$

where $B(x) \equiv \forall y\forall zA(x, y, z)$. Assume

$$\forall x \in TC(a)B(x). \tag{3.4}$$

I have to show $B(a)$, i.e., $\forall y\forall zA(a, y, z)$. Now let $b \in V$ be arbitrary. By TC-induction, it suffices to prove $\forall k \in TC(b)\forall zA(a, k, z) \to \forall zA(a, b, z)$. Assume

$$\forall k \in TC(b)\forall zA(a, k, z). \tag{3.5}$$

I have to show $\forall zA(a, b, z)$. Let $c \in V$ be arbitrary. By TC-induction, it suffices to prove $\forall h \in TC(c)A(a, b, h) \to A(a, b, c)$. Assume

$$\forall h \in TC(c)A(a, b, h). \tag{3.6}$$

Now I have to show $A(a, b, c)$. By the antecedent of $\lhd^3$, it suffices to prove $\forall x_1, x_2, x_3[(x_1, x_2, x_3) \lhd^3 (a, b, c) \to A(x_1, x_2, x_3)]$. Now let x_1, x_2 and x_3 be arbitrary such that $(x_1, x_2, x_3) \lhd^3 (a, b, c)$. If $x_1 \in TC(a)$, then by (3.4) $A(x_1, x_2, x_3)$. If $x_1 = a$ and $x_2 \in TC(b)$, then by (3.5), also $A(x_1, x_2, x_3)$. If $x_1 = a$, $x_2 = b$ and $x_3 \in TC(c)$, then by (3.6), it follows that $A(x_1, x_2, x_3)$ as well. ■

Corollary 3.2.7 *TC-induction implies $\lhd^2$-induction:*

Proof. This is the same as above by modifying arity 3 to arity 2. ■

Claim 3.2.8 *$\lhd^2$ is transitive.*

Proof. Let $a, b, c, d, e, f \in V$ be such that $(a, b) \lhd^2 (c, d)$ and $(c, d) \lhd^2 (e, f)$. By the definitions, it follows that $(a = e \vee a \in TC(e)) \wedge (b = f \vee b \in TC(f))$. The only condition that has to be checked is $\neg(a = e \wedge b = f)$. By object induction $\forall x \in V \neg(x \in x)$, and thus $\forall x \in V \neg(x \in TC(x))$. So if we assumes $a = e \wedge b = f$, we have the conclusion $e \in TC(e) \vee f \in TC(f)$ which yields a contradiction. ■

The following claim will be applied to formalize the informal interpretations (cf. Section 4.3). To this end, it suffices to consider the following special non-parameterized version:

Claim 3.2.9 (*transfinite recursion*) ($\mathbf{CZF}_N$)
For any $\mathbf{CZF}_N$-definable class function $\mathcal{H} : V^2 \times V \to V$, there exists uniquely a class function $F : V^2 \to V$ such that

$$\forall (y, z) \in V^2[F(y, z) = \mathcal{H}((y, z), F \restriction_{(y,z)})],$$

where $F \restriction_{(y,z)}$ denotes

$$(\{((y, s), F(y, s)) : s \in TC(z)\}, \{((s, y), F(s, y)) : s \in TC(z)\},$$

$$\{((t, z), F(t, z)) : t \in TC(y)\}),$$

or (formally) $\forall x \subseteq V^2 \exists! f : x \to V$ such that

$$\forall (y, z) \in x[f(y, z) = \mathcal{H}((y, z), f \restriction_{(y,z)})].$$

Proof. We will show this by $\lhd^2$-induction and Replacement. Define

$$(a, b)^{\blacktriangleleft} := \{(c, d) : (c, d) \lhd^2 (a, b) \vee (d, c) \lhd^2 (a, b)\}.$$

Observe that $(c, d) \in (a, b)^{\blacktriangleleft} \leftrightarrow (d, c) \in (a, b)^{\blacktriangleleft}$, $(a, b)^{\blacktriangleleft} = (b, a)^{\blacktriangleleft}$ and

$$(c, d) \lhd^2 (a, b) \to (c, d)^{\blacktriangleleft} \subseteq (a, b)^{\blacktriangleleft}.$$

Let $Fun(f, a, b)$ denote the predicate 'f is a function with domain a and range b'. Assume for all $(c, d) \lhd^2 (a, b) \exists! t$

$$Fun(t, (c, d)^{\blacktriangleleft}, V) \wedge \forall (u, v) \in (c, d)^{\blacktriangleleft}[(t(u, v) = \mathcal{H}((u, v), t \restriction_{(u,v)}))], \quad (3.7)$$

where $t\restriction_{(u,v)}$ denotes

$$(\{((u,s),t(u,s)) : s \in TC(v)\}, \{((s,u),t(s,u)) : s \in TC(v)\},$$

$$\{((k,v),t(k,v)) : k \in TC(u)\}).$$

From this, $\forall (c,d) \in (a,b)^{\blacktriangleleft} \exists! t$

$$Fun(t,(c,d)^{\blacktriangleleft},V) \wedge \forall (u,v) \in (c,d)^{\blacktriangleleft}[(t(u,v) = \mathcal{H}((u,v), t\restriction_{(u,v)}))]. \tag{3.8}$$

Then by Replacement, there is a set function $T : (a,b)^{\blacktriangleleft} \to V$ such that for all $(c,d) \in (a,b)^{\blacktriangleleft}$, $Fun(T(c,d),(c,d)^{\blacktriangleleft},V)$ and

$$\forall (u,v) \in (c,d)^{\blacktriangleleft}[(T(c,d))(u,v) = \mathcal{H}((u,v), T(c,d)\restriction_{(u,v)})]. \tag{3.9}$$

With this and the assumption that $\mathcal{H}$ is a class function, one has

$$\forall (c,d) \in (a,b)^{\blacktriangleleft} \exists! v[\mathcal{H}((c,d), T(c,d)\restriction_{(c,d)}) = v],$$

where $T(c,d)\restriction_{(c,d)}$ denotes

$$(\{((c,s),(T(c,d))(c,s)) : s \in TC(d)\}, \{((s,c),(T(c,d))(s,c)) : s \in TC(d)\},$$

$$\{((k,d),(T(c,d))(k,d)) : k \in TC(c)\}).$$

Then by Replacement, there is a function $\mathcal{F} : (a,b)^{\blacktriangleleft} \to V$ such that

$$\forall (c,d) \in (a,b)^{\blacktriangleleft}[\mathcal{F}(c,d) = \mathcal{H}((c,d), T(c,d)\restriction_{(c,d)})]. \tag{3.10}$$

Now we claim

$$\forall (c,d) \in (a,b)^{\blacktriangleleft}[\mathcal{F}(c,d) = \mathcal{H}((c,d), \mathcal{F}\restriction_{(c,d)})],$$

where $\mathcal{F}\restriction_{(c,d)}$ denotes

$$(\{((c,s),\mathcal{F}(c,s)) : s \in TC(d)\}, \{((s,c),\mathcal{F}(s,c)) : s \in TC(d)\},$$

$$\{((k,d),\mathcal{F}(k,d)) : k \in TC(c)\}).$$

It suffices to prove that $\mathcal{F}\restriction_{(c,d)} = T(c,d)\restriction_{(c,d)}$. Let $s \in V$ be arbitrary such that $s \in TC(d)$. By Claim 3.2.8, it follows that $(c,s) \in (a,b)^{\blacktriangleleft}$ and thus by (3.10)

$$\mathcal{F}(c,s) = \mathcal{H}((c,s), T(c,s)\restriction_{(c,s)}).$$

By (3.8) one knows $T(c,s) \subseteq T(c,d)$ and thus by (3.9)

$$\mathcal{F}(c,s) = \mathcal{H}((c,s), T(c,d)\restriction_{(c,s)}) = (T(c,d))(c,s).$$

Following the same arguments, one reaches the result $\mathcal{F}\restriction_{(c,d)}= T(c,d)\restriction_{(c,d)}$. Hence we have shown that $\mathcal{F}$ is a function with domain $(a,b)^{\blacktriangleleft}$ such that

$$\forall (c,d) \in (a,b)^{\blacktriangleleft} \mathcal{F}(c,d) = \mathcal{H}((c,d), \mathcal{F}\restriction_{(c,d)}).$$

Now we want to show the uniqueness of $\mathcal{F}$. Let $F' : (a,b)^{\blacktriangleleft} \to V$ be another candidate such that

$$\forall (c,d) \in (a,b)^{\blacktriangleleft}[F'(c,d) = \mathcal{H}((c,d), F'\restriction_{(c,d)})]. \tag{3.11}$$

Then by (3.8), it follows that $\forall (u,v) \in (c,d)^{\blacktriangleleft}[F'(u,v) = \mathcal{F}(u,v)]$ and thus by the definition, $F'\restriction_{(c,d)}= \mathcal{F}\restriction_{(c,d)}$, i.e., $F'(c,d) = \mathcal{F}(c,d)$. Hence by $\lhd^2$-induction, we have shown that $\forall x,y \in V \exists! f$

$$[Fun(f,(x,y)^{\blacktriangleleft},V) \wedge \forall (u,v) \in (x,y)^{\blacktriangleleft}(f(u,v) = \mathcal{H}((u,v), f\restriction_{(u,v)}))]. \tag{3.12}$$

Now let K be arbitrary such that $K \subseteq V^2$. Then by the definition it follows that $K \subseteq (\pi_0(K), \pi_1(K))^{\blacktriangleleft}$, where $\pi_0(K) = \{a \in \cup\cup K : \exists b \in \cup\cup K[(a,b) \in K]\}$ and $\pi_1(K) = \{b \in \cup\cup K : \exists a \in \cup\cup K[(a,b) \in K]\}$. Hence the result follows immediately from (3.12). ∎

3.2.2 Inductive definitions

Definition 3.2.10 *$\mathcal{K}$ is an **inductive definition** iff $\mathcal{K}$ is a class of ordered pairs of sets.*

Definition 3.2.11 *Define $\mathcal{K}_1(a) := \{b : (a,b) \in \mathcal{K}\}$ and $\mathcal{K}_1[a] := \bigcup_{b\in a} \mathcal{K}_1(b)$.*

This definition indeed is inspired by proof theory. Each $(a,b) \in \mathcal{K}$ can be viewed as given the premise a, the theorem b follows; while $\mathcal{K}_1(a)$ can be viewed as the collection of all the derivable theorems given the premise a.

Now let's fix an inductive definition $\mathcal{K}$. For any binary class C we use $C_1(a)$ to denote the class $\{b : (a,b) \in C\}$ and $C_1[a]$ to denote the class $\bigcup_{b\in a} C_1(b)$.

Definition 3.2.12 $\Gamma_{\mathcal{K}}(Y) := \{a : \exists x(x \subseteq Y \wedge (x,a) \in \mathcal{K})\}$. *In some sense this collects all the theorems that are derivable by using only some premises in Y or all the premises in Y from $\mathcal{K}$ (a combination of premises and theorems).*

Now define the predicate $BR(g)$ (g is a binary relation) by

$$\forall x \in g \exists y \exists z (x = (y, z)).$$

Define the predicate $Good(g)$ (g is a good set) by

$$S(g) \wedge BR(g) \wedge \forall a[g_1(a) \subseteq \Gamma_{\mathcal{K}}(g_1[a])].$$

This says that, in some sense g is already a saturated combination of premises and theorems, i.e., there are no news theorems given the premises in g. Then one glues all these saturated combinations together as follows:

Definition 3.2.13 $\mathcal{G} \equiv \cup\{g\colon Good(g)\}$.

Definition 3.2.14 *A class Y is $\mathcal{K}$-closed iff $\Gamma_{\mathcal{K}}(Y) \subseteq Y$, i.e., the premise Y cannot produce any new theorem (or Y is saturated).*

Claim 3.2.15 *$\Gamma_{\mathcal{K}}$ is a monotone operator, i.e., $X \subseteq Y$ implies $\Gamma_{\mathcal{K}}(X) \subseteq \Gamma_{\mathcal{K}}(Y)$.*

Proof. This follows immediately from the definition. ■

Lemma 3.2.16 [*Class Inductive Definition*] ($\mathbf{CZF}_N$) *For every inductive definition $\mathcal{K}$, one has $\forall a[S(a) \to \mathcal{G}_1(a) = \Gamma_{\mathcal{K}}(\mathcal{G}_1[a])]$.*

Proof. This is Lemma 5.2 in Peter Aczel and Michael Rathjen's report [1]. ■

Claim 3.2.17 $\forall \alpha, \beta \in On(\alpha \in \beta \vee \alpha \subseteq \beta \to \mathcal{G}_1[\alpha] \subseteq \mathcal{G}_1[\beta])$.

Proof. Let $x \in \mathcal{G}_1[\alpha]$ be arbitrary. Then by the definition, it follows that $\exists \gamma \in \alpha(x \in \mathcal{G}_1(\gamma))$, i.e., $\exists \gamma \in \beta(x \in \mathcal{G}_1(\gamma))$, i.e., $x \in \mathcal{G}_1[\beta]$. ■

3.3 Definitions of universes

3.3.1 External universe: V

Since we adopt two sorts of objects (sets and numbers) in our external universe, we have to define some notations to differentiate them. We call a set in the background set theory "**an external set**"(or simply a **set**) and use $\mathbb{S}$ to denote the class of all the external sets. Similarly, any number constructed by the Infinity Axiom in the background theory will be called "**an external number**"(or simply a **number**). We use $\mathbb{N}$ to denote the set of all the external numbers and V to denote the class of all the **external objects** (i.e., $V \equiv \mathbb{N} \cup \mathbb{S}$).

3.3.2 Internal universe: V^*

The internal universe will serve as the domain of discourse for quantifiers and is constructed from the external universe V via ordinals. In this section, we will mention two internal universes. The first one is the realizability universe and the second one is the forcing universe. For the realizability part, we identify either PRF, the set of all the partial recursive functions, or the set of all the partial A-recursive functions (or Turing machines equipped with an oracle which provides $A(n)$ if it exists) with $\mathbb{N}$. Then we construct the **realizability universe** V^* as follows:

$$\begin{aligned} V_\alpha^{\mathbb{N}} &= \bigcup_{\beta\in\alpha} \mathcal{P}(\mathbb{N} \times (V_\beta^{\mathbb{N}} \cup \mathbb{N})) \\ \mathbb{S}^* &= \bigcup_{\alpha\in On} V_\alpha^{\mathbb{N}} \\ V^* &= \mathbb{N} \cup \mathbb{S}^* \end{aligned}$$

where $\mathcal{P}$ denotes the Powerset operation.

Since our forcing interpretation is a specific one intended to provide conservativity results, we have to consider any arbitrary subset $\mathbb{E}$ of $\mathbb{P}$, where $\mathbb{P}$ is the set of all the finite partial functions from $\mathbb{N}$ to $\mathbb{N}$. We define the **forcing universe** V^* as follows

$$\begin{aligned} V_\alpha^{\mathbb{E}} &= \bigcup_{\beta\in\alpha} \mathcal{P}(\mathbb{E} \times (V_\beta^{\mathbb{E}} \cup \mathbb{N})) \\ \mathbb{S}^* &= \bigcup_{\alpha\in On} V_\alpha^{\mathbb{E}} \\ V^* &= \mathbb{N} \cup \mathbb{S}^* \end{aligned}$$

where $\mathcal{P}$ denotes the Powerset operation. We will call each element in the universe $\mathbb{S}^*$ an **internal set** and each element in V^* an **internal object**. By $\mathbf{CZF}_N$ in the background theory, one can formalize the informal notions of universes as follows:

Claim 3.3.1 ($\mathbf{CZF}_N$) *$\mathbb{S}^*$ and V^* can be formalized in $\mathbf{CZF}_N$ by the class $\bigcup_{\alpha\in On} \mathcal{G}_1[\alpha]$ and $\mathbb{N}\cup(\bigcup_{\alpha\in On} \mathcal{G}_1[\alpha])$ respectively, where $\mathcal{G}_1$ is defined in Subsection 3.2.2.*

Proof. We show this by using an inductive definition. Consider the following inductive definition.

$$\mathcal{K} := \{(x, a) : S(x) \wedge S(a) \wedge a \subseteq \mathbb{N} \times (x \cup \mathbb{N})\}.$$

Then one has

$$\Gamma_{\mathcal{K}}(Y) := \{a : \exists x(x \subseteq Y \wedge (x,a) \in \mathcal{K})\} = \mathcal{P}(\mathbb{N} \times (Y \cup \mathbb{N})).$$

Moreover, by Lemma 3.2.16 it follows that

$$\forall a[S(a) \to \mathcal{G}_1(a) = \Gamma_{\mathcal{K}}(\mathcal{G}_1[a]) = \mathcal{P}(\mathbb{N} \times (\mathcal{G}_1[a] \cup \mathbb{N}))],$$

in particular,

$$\forall \alpha[On(\alpha) \to \mathcal{G}_1(\alpha) = \Gamma_{\mathcal{K}}(\mathcal{G}_1[\alpha]) = \mathcal{P}(\mathbb{N} \times (\mathcal{G}_1[\alpha] \cup \mathbb{N}))]. \tag{3.13}$$

Now one formalizes $V_\alpha^{\mathbb{N}}$ as $\mathcal{G}_1[\alpha]$. Since

$$\mathcal{G}_1[\alpha] = \bigcup_{\beta \in \alpha} \mathcal{G}_1(\beta) = \bigcup_{\beta \in \alpha} \Gamma_{\mathcal{K}}(\mathcal{G}_1[\beta]),$$

by (3.13), it follows that

$$\forall \alpha[\alpha \in On \to V_\alpha^{\mathbb{N}} = \bigcup_{\beta \in \alpha} \mathcal{P}(\mathbb{N} \times (V_\beta^{\mathbb{N}} \cup \mathbb{N}))].$$

■

Though we show the version for the realizability universe, the proof for the forcing universe is exactly the same by replacing $\mathbb{N}$ with $\mathbb{E}$.

Corollary 3.3.2 ($\mathbf{CZF}_N$) *(1) $a \in \mathbb{S}^* \to a \in \mathbb{S}$; (2) $a \in V^* \to (N(a) \vee S(a)) \wedge \neg(N(a) \wedge S(a))$.*

Proof. Both follow immediately from the formalization $a \in \mathbb{S}^* \equiv a \in \bigcup_{\alpha \in On} (\bigcup_{\beta \in \alpha} \mathcal{G}_1(\beta))$ and its background axiom: $\forall x \neg(N(x) \wedge S(x))$. ■

Corollary 3.3.3 ($\mathbf{CZF}_N$) *$\mathbb{N} \cap \mathbb{S}^* = \emptyset$ and $a \in V^* \wedge a \in \mathbb{S} \to a \in \mathbb{S}^*$.*

Proof. Since $\mathbb{S}^*$ is represented by the class $\bigcup_{\alpha \in On} (\bigcup_{\beta \in \alpha} \mathcal{G}_1(\beta))$, each element of which is a set, by the axiom $\forall x \neg(N(x) \wedge S(x))$, the result follows immediately. ■

Claim 3.3.4 $\forall \alpha, \beta \in On[\alpha \in \beta \vee \alpha \subseteq \beta \to V_\alpha^{\mathbb{N}} \subseteq V_\beta^{\mathbb{N}}]$.

Proof. This follows immediately from Claim 3.3.1 and 3.2.17. ■

The following corollary is useful when one needs a bound to be able to apply the Separation Scheme.

Corollary 3.3.5 $\forall a \in \mathbb{S}^*[a \in V^{\mathbb{N}}_{rk(a)+1}]$.

Proof. Let $\alpha \in On$ be arbitrary. Assume $\forall \beta \in \alpha \forall b \in V^{\mathbb{N}}_{\beta}(b \in V^{\mathbb{N}}_{rk(b)+1})$. Now let $a \in V^{\mathbb{N}}_{\alpha}$ be arbitrary. We have to show $a \in V^{\mathbb{N}}_{rk(a)+1}$. Let $f \in \mathbb{N}$ and $d \in V^*$ be arbitrary such that $(f, d) \in a$. Since $d \in TC(a)$, one has $rk(d) + 1 \subseteq rk(a)$. By Claim 3.3.4 and the assumption, it follows that $d \in V^{\mathbb{N}}_{rk(d)+1}$, i.e., $a \subseteq \mathbb{N} \times (V^{\mathbb{N}}_{rk(a)} \cup \mathbb{N})$ and thus $a \in V^{\mathbb{N}}_{rk(a)+1}$. ■

The following claim will be used frequently when we want to demonstrate some classes are indeed internal sets or when we want to construct some internal sets for the interpretation.

Claim 3.3.6 (**CZF**$_N$) *If $a \in \mathbb{S}$ and $a \subseteq \mathbb{N} \times V^*$, then $a \in \mathbb{S}^*$.*

Proof. The main point is to find an ordinal α that will serve as a rank for a. This ordinal α is derived via Strong Collection. By the fact that $\mathbb{N} \times \mathbb{N} \in V^{\mathbb{N}}_{\varnothing} \subseteq \mathbb{S}^*$, one rewrites $a \subseteq \mathbb{N} \times V^*$ explicitly as follows:

$$\forall x \in a \exists \beta[\beta \in On \wedge \exists y \in V^{\mathbb{N}}_{\beta}(\exists n \in \mathbb{N}(x = (n, y)) \vee (y = \mathbb{N} \times \mathbb{N} \wedge x \in y))].$$

By Strong Collection, there exists a set E such that $\forall x \in a \exists \beta \in E$

$$[\beta \in On \wedge \exists y \in V^{\mathbb{N}}_{\beta}(\exists n \in \mathbb{N}(x = (n, y)) \vee (y = \mathbb{N} \times \mathbb{N} \wedge x \in y))], \tag{3.14}$$

and that $\forall \beta \in E \exists x \in a$

$$[\beta \in On \wedge \exists y \in V^{\mathbb{N}}_{\beta}(\exists n \in \mathbb{N}(x = (n, y)) \vee (y = \mathbb{N} \times \mathbb{N} \wedge x \in y))]. \tag{3.15}$$

From (3.15) we have E is a set consisting of ordinals. Thus we can define the ordinal α to be $\alpha = \cup\{\beta + 1 : \beta \in E\}$, where $\beta + 1$ abbreviates $\beta \cup \{\beta\}$. Since $\beta \in E$ implies $\beta \in \alpha$, one rewrites (3.14) as follows: $\forall x \in a \exists \beta \in \alpha$

$$[\beta \in On \wedge \exists y \in V^{\mathbb{N}}_{\beta}(\exists n \in \mathbb{N}(x = (n, y)) \vee (y = \mathbb{N} \times \mathbb{N} \wedge x \in y))],$$

i.e., by Claim 3.3.4,

$$a \subseteq \mathbb{N} \times (V^{\mathbb{N}}_{\alpha} \cup \mathbb{N}),$$

i.e.,

$$a \in V^{\mathbb{N}}_{\alpha+1} \subseteq \mathbb{S}^*.$$

■

Though we have just given the version for the realizability universe, the statement and its proof for the forcing universe are exactly the same by replacing $\mathbb{N}$ with $\mathbb{E}$.

In conclusion, we have introduced the concept and a model of an applicative structure, transfinite inductions, inductive definitions and universes for the formalization and discourse of our realizability and forcing interpretations. This chapter lays the foundations of our later chapters.

Chapter 4

Lifschitz' style interpretation

Lifschitz modified Kleene's recursive realizability and gave an interpretation to separate $\mathbf{CT}_0$ from $\mathbf{CT}_0!$. Though he used an informal argument to show this result, the amount of classical logic needed in his argument was explicitly pointed out by Jaap van Oosten. Based on Lifschitz' interpretation for **HA**, we extend it to the context of $\mathbf{IZF}_N$. In Section 4.1, we introduce Lifschitz' notion of a realizer which uses a criterion to differentiate $\mathbf{CT}_0$ from $\mathbf{CT}_0!$ and then we study the basic operations of these realizers.

In Section 4.2, we come up with and introduce an informal definition of a new interpretation which, in some sense, combines Lifschitz' interpretation with McCarty's interpretation. We call this new interpretation a Lifschitz' style interpretation. Then, in Section 4.3, we give a formal version of this interpretation.

In Section 4.4, we then derive some basic properties, in particular, a faithful extension of Lifschitz' original interpretation for Heyting arithmetic. Most of the notations used in this chapter were defined in Section 3.1.3.

4.1 Lifschitz indices

This section is based on Jaap van Oosten's paper [19] and Lifschitz' paper [13]. In order to fit our purposes, we derive the realizers in explicit forms. Define

$$D_n \equiv |n| := \{m \in \mathbb{N} : m \leq n_1 \wedge \forall k \neg T(n_0, m, k)\},$$

or in abbreviation

$$D_n \equiv \{m \in \mathbb{N} : m \leq n_1 \wedge n_0 \cdot m \uparrow\}.$$

We call D_n a **Lifschitz set** and its subscript n a **Lifschitz index**. These Lifschitz' indices will serve a role of realizers. Though D_n is undecidable, the complement of D_n turns out to be a r.e. (recursively enumerable) set; moreover, the complement of $\cup\{D_n | n \in D_e\}$ is also a r.e. set. Other indices are essentially based on these two properties. The ingenious definition of D_n relies on that the knowledge of the size of the set D_n eventually decides the content of D_n. If we know the size of the set D_n (suppose k) in advance, then we know there are $n_1 - k$ elements of the combination which will halt its operation. Up to this point the remaining ones recover the content of D_n. For any partial recursive function f, we use $f^{\#}$ to indicate its code. We also use the following abbreviations in this section:

Notation 4.1.1 $||D_e|| = 1 \equiv \exists n \forall m[m \in D_e \leftrightarrow m = n]$.

Notation 4.1.2 $D_{\mathfrak{sg}(n)} = \{n\} \equiv \forall m[m \in D_{\mathfrak{sg}(n)} \leftrightarrow m = n]$.

Notation 4.1.3 $D_{\Phi(e,f)} = \{f \cdot g : g \in D_e\} \equiv \forall h[h \in D_{\Phi(e,f)} \leftrightarrow \exists g \in D_e(h = f \cdot g)]$.

Notation 4.1.4 $D_{\mathfrak{un}(e)} = \bigcup_{g \in D_e} D_g \equiv \forall h[h \in D_{\mathfrak{un}(e)} \leftrightarrow \exists g \in D_e(h \in D_g)]$.

Notation 4.1.5 $\mathbf{HA}' \equiv \mathbf{HA} + \mathbf{MP}_{pr} + \mathbf{B}\sum_{\mathbf{2}}^{\mathbf{0}} - \mathbf{MP}$.

Lemma 4.1.6 *There is a total recursive function* $\mathfrak{sg}$ *such that*

$$\mathbf{HA} \vdash \forall n[D_{\mathfrak{sg}(n)} = \{n\}].$$

Proof. Define a partial recursive function by $g(n, m) := 0$, if $n \neq m$ and $g(n, m) \uparrow$, if $n = m$. Then by the S-M-N theorem and the definition one has $|\langle \Lambda m.\{s_{11}(g^{\#}, n)\}(m), n\rangle| = \{n\}$. Now one defines the total recursive function $\mathfrak{sg}(n) := \langle \Lambda m.\{s_{11}(g^{\#}, n)\}(m), n\rangle$. ■

Lemma 4.1.7 *There is a partial recursive function* $\varnothing$ *such that* $\mathbf{HA} + \mathbf{MP}_{pr} \vdash \forall e[||D_e|| = 1 \rightarrow \varnothing(e) \downarrow \wedge \varnothing(e) \in D_e]$.

Proof. Let $e \in \mathbb{N}$ be arbitrary. Let $k \in \mathbb{N}$ be arbitrary such that $k \leq e_1$ and $k \notin D_e$, i.e., $\neg\neg\exists m T(e_0, k, m)$. By $\mathbf{MP}_{pr}$, one has $\forall k \leq e_1(k \notin D_e \rightarrow \exists m T(e_0, k, m))$. Now define $\varnothing(e) := \mu y[y \leq e_1 \wedge \forall l \leq e_1(l \neq y \rightarrow \exists m T(e_0, l, m))]$ and the result follows. ■

Lemma 4.1.8 *There is a partial recursive function* Φ *such that*

$$\mathbf{HA'} \vdash \forall e \forall f[\forall g \in D_e(f \cdot g \downarrow) \to \Phi(e,f) \downarrow \wedge D_{\Phi(e,f)} = \{f \cdot g : g \in D_e\}].$$

Proof. The proof is exactly the same as Lemma 2.4 in Jaap van Oosten's paper [19]. ■

Lemma 4.1.9 *There is a total recursive function* $\mathtt{un}$ *such that*

$$\mathbf{HA'} \vdash \forall e[D_{\mathtt{un}(e)} = \bigcup_{g \in D_e} D_g].$$

Proof. The proof is exactly the same as Lemma 2.5 in Jaap van Oosten's paper [19]. ■

4.2 Informal definitions

Let R be for any arbitrary primitive recursive relation. Let $e \in \mathbb{N}$ (the codes of PRF) and $a, b, c \in V^*$ be arbitrary. To facilitate the description we use the following abbreviations:

- $n \cdot m \downarrow v \equiv \exists k[N(k) \wedge T(n,m,k)] \wedge U(\mu z T(n,m,z), v)$.
- $n \cdot m \Vdash_L \varphi \equiv \exists v(n \cdot m \downarrow v \wedge v \Vdash_L \varphi)$.
- $\forall (f,c) \in a \varphi(f,c) \equiv \forall f \in \mathbb{N} \forall c \in V^*((f,c) \in a \to \varphi(f,c))$.
- $\exists x \in a \varphi(x) \equiv \exists x \in V^*(x \in a \wedge \varphi(x))$.
- $n \in D_e \equiv n \leq e_1 \wedge \forall m \neg T(e_0, n, m)$.
- $D_e \neq \emptyset \equiv \exists n(n \in D_e)$.
- $\forall q \in D_e[q \Vdash_L \varphi] \equiv D_e \neq \emptyset \wedge \forall q \in D_e[q \Vdash_L \varphi]$.
- $e_0 \equiv \jmath_0(e)$, where $\jmath_0 : \mathbb{N} \to \mathbb{N}$ is a left unpairing function; $e_1 \equiv \jmath_1(e)$, where $\jmath_1 : \mathbb{N} \to \mathbb{N}$ is a right unpairing function; $\langle c, d \rangle \equiv \jmath(c,d)$, where $\jmath$ is a pairing function.
- $\forall \vec{a} \in V^* \varphi(\vec{a}) \equiv \forall a_1, a_2, ..., a_n \in V^* \varphi(a_1, a_2, ..., a_n)$.

Recall $\mathcal{L}(V^*)$ is the language for set theory with constants from the internal universe V^*. Let $a, a_1, a_2, ..., a_n, b, c \in V^*$ be arbitrary. Then we inductively define the following clauses on the complexity of $\mathcal{L}(V^*)$:

1. $e \Vdash_L R(a_1, a_2, ..., a_n)$ iff $a_1, a_2, ..., a_n \in \mathbb{N} \wedge R(a_1, a_2, ..., a_n)$.

2. $e \Vdash_L N(a)$ iff $a \in \mathbb{N} \wedge e = a$.

3. $e \Vdash_L S(a)$ iff $a \in \mathbb{S}^*$.

4. $e \Vdash_L a \in b$ iff $D_e \neq \emptyset \wedge \forall d \in D_e \exists c \in V^*[(d_0, c) \in b \wedge d_1 \Vdash_L a = c]$.

5. $e \Vdash_L a = b$ iff $(a, b \in \mathbb{N} \wedge a = b) \vee [D_e \neq \emptyset \wedge a \in \mathbb{S}^* \wedge b \in \mathbb{S}^* \wedge \forall d \in D_e(\forall (f, c) \in a(d_0 \cdot f \Vdash_L c \in b) \wedge \forall (f, c) \in b(d_1 \cdot f \Vdash_L c \in a))]$.

6. $e \Vdash_L A \wedge B$ iff $e_0 \Vdash_L A \wedge e_1 \Vdash_L B$.

7. $e \Vdash_L A \vee B$ iff $D_e \neq \emptyset \wedge \forall d \in D_e[d_0 = 0 \wedge d_1 \Vdash_L A] \vee [d_0 \neq 0 \wedge d_1 \Vdash_L B]$.

8. $e \Vdash_L \neg A$ iff $\forall f \in \mathbb{N} \neg (f \Vdash_L A)$.

9. $e \Vdash_L A \rightarrow B$ iff $\forall f \in \mathbb{N}[f \Vdash_L A \rightarrow e \cdot f \Vdash_L B]$.

10. $e \Vdash_L \forall x A$ iff $D_e \neq \emptyset \wedge \forall d \in D_e \forall c \in V^*(d \Vdash_L A[x/c])$.

11. $e \Vdash_L \exists x A(x)$ iff $D_e \neq \emptyset \wedge \forall d \in D_e \exists c \in V^*(d \Vdash_L A[x/c])$.

Furthermore, one defines $e \Vdash_L A(x)$ iff $e \Vdash_L \forall x A(x)$ and $V^* \models_L B$ iff $\exists e[e \in \mathbb{N} \wedge e \Vdash_L B]$.

4.3 Formal definitions

In correspondence to the informal definitions in Section 4.2, in this section we will show how to provably define these informal definitions in the context of $\mathbf{CZF}_N$. Though in this part it suffices to use the background theory $\mathbf{IZF}_N$, to formalize relativized realizability and the forcing interpretation in Chapter 6 and Chapter 7, we work in $\mathbf{CZF}_N$. If one is interested at formalizing informal interpretations in Section 6.1 and 7.1, then he can consult this section.

To begin with, we use notations of class functions here. Nonetheless, the unique existence of these class functions is provably definable in the context of $\mathbf{CZF}_N$. In the following we show how to formalize the informal definitions for atomic formulae via these class functions.

For each primitive recursive n-ary relation R, by Corollary 3.3.2, one defines a class function $F^n_R : (V^*)^n \rightarrow \{\mathbb{N}, \emptyset\}$ such that $F^n_R(a_1, a_2, ..., a_n) = \mathbb{N}$, if $a_1, a_2, ..., a_n \in \mathbb{N} \wedge R(a_1, a_2, ..., a_n)$ and $F^n_R(a_1, a_2, ..., a_n) = \emptyset$, otherwise. Then the first clause can be formalized as follows:

$1'$. $e \in F_R^n(a_1, a_2, ..., a_n) \leftrightarrow a_1, a_2, ..., a_n \in \mathbb{N} \wedge R(a_1, a_2, ..., a_n)$.

For the predicate N, one defines a class function $F_N : V^* \to \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ such that $F_N(a) = \{a\}$, if $a \in \mathbb{N}$ and $F_N(a) = \emptyset$, otherwise. Then the second clause can be formalized as follows:

$2'$. $e \in F_N(a) \leftrightarrow a \in \mathbb{N} \wedge e = a$.

For the predicate S, by Corollary 3.3.2, one defines a class function $F_S : V^* \to \{\mathbb{N}, \emptyset\}$ such that $F_S(a) = \mathbb{N}$, if $a \in \mathbb{S}^*$ and $F_S(a) = \emptyset$, otherwise. Then the third clause can be formalized as follows:

$3'$. $e \in F_S(a) \leftrightarrow a \in \mathbb{S}^*$.

Now we want to formalize the clauses $e \Vdash_L a = b$ and $e \Vdash_L a \in b$ simultaneously. The idea is to formalize this pair by a class function F with domain V^2 such that

$$F(a, b) = (\{e \in \mathbb{N} : e \Vdash_L a \in b\}, \{e \in \mathbb{N} : e \Vdash_L a = b\}).$$

This definition aims to formalize the interpretations of $\in$ and $=$. In the following we want to show how to provably define this class function F via transfinite recursion in the context of $\mathbf{CZF}_N$. If $a = (b^0, b^1)$, then we use $\rho_i(a)$ to denote b^i and $y \in \rho_i(a)$ to denote the formula $\exists b^0 \exists b^1 [a = (b^0, b^1) \wedge y = b^i]$, where $i \in \{0, 1\}$. Now let us define a class function (based on the interpretations of the predicates $\in$ and $=$) $G : V^3 \to V^2$ as follows: (the first component aims to formalize $\in$; while the second one, $=$)

$$G(x, y, z) := (\{e \in \mathbb{N} : \varphi(x, y, z, e)\}, \{e \in \mathbb{N} : \theta(x, y, z, e)\}),$$

where $\varphi(x, y, z, e) \equiv D_e \neq \emptyset \wedge \forall d \in D_e \exists c \in V^*$

$$(d_0, c) \in y \wedge \exists u \in \rho_0^3(z)[\rho_0(u) = (x, c) \wedge d_1 \in \rho_1(\rho_1(u))],$$

and $\theta(x, y, z, e) \equiv (N(x) \wedge N(y) \wedge x = y) \vee$

$$[S(x) \wedge S(y) \wedge D_e \neq \emptyset \wedge \forall d \in D_e$$
$$\forall (f, c) \in x \exists u \in \rho_2^3(z)(\rho_0(u) = (c, y) \wedge d_0 \cdot f \in \rho_0(\rho_1(u))) \wedge$$
$$\forall (f, c) \in y \exists u \in \rho_1^3(z)(\rho_0(u) = (c, x) \wedge d_1 \cdot f \in \rho_0(\rho_1(u)))],$$

where $u \in \rho_i^3(z)$ denotes $\exists x_0, x_1, x_2[z = (x_0, x_1, x_2) \wedge u \in x_i]$ for $i \in \{0, 1, 2\}$. Then by Claim 3.2.9, there is a unique class function $F : V^2 \to V$ such that

$$F(a, b) = G(a, b, F \restriction_{(a,b)}),$$

where $F\restriction_{(a,b)}$ denotes

$$(\{((a,c),F(a,c)) : c \in TC(b)\}, \{((c,a),F(c,a)) : c \in TC(b)\},$$
$$\{((d,b),F(d,b)) : d \in TC(a)\}).$$

Now one formalizes $e \Vdash_L a \in b$ as $e \in \rho_0(F(a,b))$ and $e \Vdash_L a = b$ as $e \in \rho_1(F(a,b))$. Then clause 4 and 5 are provably defined as follows:

4′. $e \in \rho_0(F(a,b)) \leftrightarrow D_e \neq \emptyset \wedge \forall d \in D_e \exists c \in V^*[(d_0,c) \in b \wedge d_1 \in \rho_1(F(a,c))]$.

5′. $e \in \rho_1(F(a,b)) \leftrightarrow (N(a) \wedge N(b) \wedge a = b) \vee$
$[S(a) \wedge S(b) \wedge D_e \neq \emptyset \wedge \forall d \in D_e$
$(\forall (f,c) \in a(d_0 \cdot f \in \rho_0(F(c,b))) \wedge \forall (f,c) \in b(d_1 \cdot f \in \rho_0(F(c,a)))))]$.

Based on these, the other clauses from 6 to 11 can be defined inductively via the following classes which are defined on the complexity of $\mathcal{L}(V^*)$:

- $F_{A\wedge B} := \{e \in \mathbb{N} : e_0 \in F_A \wedge e_1 \in F_B\}$.
- $F_{A\vee B} := \{e \in \mathbb{N} : D_e \neq 0 \wedge \forall d \in D_e(d_0 = 0 \wedge d_1 \in F_A) \vee (d_0 \neq 0 \wedge d_1 \in F_B)\}$.
- $F_{\neg A} := \{e \in \mathbb{N} : \forall f \in \mathbb{N} \neg(f \in F_A)\}$.
- $F_{A\to B} := \{e \in \mathbb{N} : \forall f \in \mathbb{N}[f \in F_A \to e \cdot f \downarrow \wedge e \cdot f \in F_B]\}$.
- $F_{\forall x A(x)} := \{e \in \mathbb{N} : D_e \neq 0 \wedge \forall c \in V^*[D_e \subseteq F_{A(c)}]\}$.
- $F_{\exists x A(x)} := \{e \in \mathbb{N} : D_e \neq 0 \wedge \forall d \in D_e \exists c \in V^*[d \in F_{A(c)}]\}$.

Now for each formula φ in these clauses, one formalizes $e \Vdash_L \varphi$ as $e \in F_\varphi$. Furthermore, each open formula is formalized as its closure. Finally the clause $V^* \models_L B$ is formalized as $\exists e \in \mathbb{N}(e \in F_B)$.

4.4 Basic properties

Claim 4.4.1 $\langle \Lambda g.\alpha(g), \Lambda k.\beta(k)\rangle \Vdash_L A \vee B \leftrightarrow \exists y[N(y) \wedge ((y = 0 \to A) \wedge (y \neq 0 \to B))]$ *for some partial recursive functions* α *and* β.

Proof. Let $g \in \mathbb{N}$ be arbitrary such that $g \Vdash_L A \vee B$. Then by the definitions, one has $\forall q \in D_g$, if $q_0 = 0$, then

$$\langle \Lambda n.q_1, 0\rangle \Vdash_L (q_0 = 0 \to A) \wedge (q_0 \neq 0 \to B);$$

if $q_0 \neq 0$, then

$$\langle 0, \Lambda n.q_1\rangle \Vdash_L (q_0 = 0 \to A) \wedge (q_0 \neq 0 \to B).$$

Hence by Lemma 4.1.8,

$$\alpha(g) \Vdash_L \exists y[N(y) \wedge ((y = 0 \to A) \wedge (y \neq 0 \to B))],$$

where

$$\alpha(g) := \Phi(g, \Lambda q.\langle q_0, \mathbf{d}[\langle \Lambda n.q_1, 0\rangle][\langle 0, \Lambda n.q_1\rangle]q_0 0\rangle).$$

On the other hand, let $k \in \mathbb{N}$ be arbitrary such that

$$k \Vdash_L \exists y[N(y) \wedge ((y = 0 \to A) \wedge (y \neq 0 \to B))].$$

By the definition, it follows that

$$\forall q \in D_k[q_1 \Vdash (q_0 = 0 \to A) \wedge (q_0 \neq 0 \to B)].$$

Again by the definition, for all $q \in D_k$, if $q_0 = 0$, then

$$q_{10} \cdot q_0 \Vdash_L A;$$

if $q_0 \neq 0$, then

$$q_{11} \cdot q_0 \Vdash_L B.$$

Hence by Lemma 4.1.8,

$$\beta(k) \Vdash_L A \vee B,$$

where $\beta(k) := \Phi(k, \Lambda q.\langle q_0, \mathbf{d}[q_{10} \cdot q_0][q_{11} \cdot q_0]q_0 0\rangle)$. ■

Recall that $\mathbf{IZF}'_N$ is an abbreviation for $\mathbf{IZF}_N + \mathbf{MP}_{pr} + \mathbf{B\Sigma^0_2} - \mathbf{MP}$.

Lemma 4.4.2 *For any formula $A(\vec{x})$ in the language of set theory, one can effectively assign a partial recursive function χ_A such that* $\mathbf{IZF}'_N \vdash \forall e \in \mathbb{N}\forall \vec{a} \in V^*[(D_e \neq \emptyset \wedge \forall d \in D_e(d \Vdash_L A(\vec{a}))) \to \chi_A(e) \Vdash_L A(\vec{a})]$.

Proof. We show this by induction on the complexity of the formulae.
For atomic formulae:
♢ $A(\vec{a}) \equiv N(a)$: By the definition, $||D_e|| = 1$ and thus by Lemma 4.1.7 one assigns χ_A to be $\varnothing$.
♢ $A(\vec{a}) \equiv S(a)$: By the definition, one can simply assign χ_A to be $\Lambda n.0$.
♢ $A(\vec{a}) \equiv a \in b$: By the definition and Lemma 4.1.9 one can simply assign χ_A to be $\Lambda e.\mathfrak{un}(e)$.
♢ $A(\vec{a}) \equiv a = b$: By the definition and Lemma 4.1.9 one can simply assign

χ_A to be $\Lambda e.\mathfrak{un}(e)$.
$\diamondsuit$ $A(\vec{a}) \equiv R(\vec{a})$: By the definition, one can simply assign χ_A to be $\Lambda n.0$.
For compound formulae:
$\diamondsuit$ $A(\vec{a}) \equiv B(\vec{a}) \wedge C(\vec{a})$: By the definition, one has $\forall d \in D_e(d_0 \Vdash_L B(\vec{a}))$ and $\forall d \in D_e(d_1 \Vdash_L C(\vec{a}))$. Now by Lemma 4.1.8 and the inductive hypotheses, one can assign χ_A to be

$$\Lambda e.\langle \chi_B(\Phi(e, \Lambda k.\jmath_0(k))), \chi_C(\Phi(e, \Lambda k.\jmath_1(k)))\rangle.$$

$\diamondsuit$ $A(\vec{a}) \equiv B(\vec{a}) \to C(\vec{a})$: Let f be arbitrary such that $f \Vdash_L B(\vec{a})$. Then by the assumption,

$$\forall d \in D_e(d \cdot f \downarrow \wedge d \cdot f \Vdash_L C(\vec{a})).$$

By Lemma 4.1.8 and the inductive hypothesis, one can assign χ_A to be

$$\Lambda f.\chi_C(\Phi(e, \Lambda d.(d \cdot f))).$$

$\diamondsuit$ $A(\vec{a}) \equiv \forall y B(\vec{a}, y)$: By the definition and Lemma 4.1.9 one can simply assign χ_A to be $\Lambda e.\mathfrak{un}(e)$.
$\diamondsuit$ $A(\vec{a}) \equiv \exists y B(\vec{a}, y)$: By the definition and Lemma 4.1.9 one can simply assign χ_A to be $\Lambda e.\mathfrak{un}(e)$.
$\diamondsuit$ $A(\vec{a}) \equiv \neg B(\vec{a})$: By the definition, one can simply set χ_A to be $\Lambda n.0$.
$\diamondsuit$ $A(\vec{a}) \equiv B(\vec{a}) \vee C(\vec{a})$: By Claim 4.4.1,

$$\forall d \in D_e \alpha(d) \Vdash_L \exists y[N(y) \wedge (y = 0 \to B(\vec{a})) \wedge (y \neq 0 \to C(\vec{a}))],$$

i.e., by the above result and Lemma 4.1.8, it follows that

$$\mathfrak{un}(\Phi(e, \Lambda d.\alpha(d))) \Vdash_L \exists y[N(y) \wedge (y = 0 \to B(\vec{a})) \wedge (y \neq 0 \to C(\vec{a}))]$$

By Claim 4.4.1 again,

$$\beta(\mathfrak{un}(\Phi(e, \Lambda d.\alpha(d)))) \Vdash_L B(\vec{a}) \vee C(\vec{a}),$$

and thus one sets χ_A to be $\Lambda e.\beta(\mathfrak{un}(\Phi(e, \Lambda d.\alpha(d))))$. ∎

We identity each formula η in the language of arithmetic ($\mathcal{L}_a$) with a formula η^* in the extended language of set theory $\mathcal{L}(V^*)$ inductively as follows: For any primitive recursive relation $R(n_1, n_2, ..., n_k) \in \mathcal{L}_a$, one associates $R(n_1, n_2, ..., n_k) \in \mathcal{L}$. For the conjunction $\varphi \wedge \theta \in \mathcal{L}_a$, one associates $\varphi^* \wedge \theta^* \in \mathcal{L}$. For the disjunction $\varphi \vee \theta \in \mathcal{L}_a$, one associates $\varphi^* \vee \theta^* \in \mathcal{L}$.

For the implication $\varphi \to \theta \in \mathcal{L}_a$, one associates $\varphi^* \to \theta^* \in \mathcal{L}$. For the universal formula $\forall x\varphi(x) \in \mathcal{L}_a$, one associates $\forall x[N(x) \to \varphi^*(x)] \in \mathcal{L}$. For the existential formula $\exists x\varphi(x) \in \mathcal{L}_a$, one associates $\exists x[N(x) \wedge \varphi^*(x)] \in \mathcal{L}$. For the negation $\neg\varphi \in \mathcal{L}_a$, one associates $\neg\varphi^* \in \mathcal{L}$. Now we want to show our interpretation faithfully extends Lifschitz' realizability $\Vdash^L$ (cf. Section 1.3.3) for arithmetic in the following sense:

Theorem 4.4.3 *For any formula η in the language of* **HA***, there are partial recursive functions Ψ^η and Ξ^η such that $k \Vdash^L \eta \to \Psi^\eta(k)\downarrow \wedge \Psi^\eta(k) \Vdash_L \eta^*$ and $h \Vdash_L \eta^* \to \Xi^\eta(h)\downarrow \wedge \Xi^\eta(h) \Vdash^L \eta$.*

Proof. We will show this by the induction on the complexity of η.

♢ η is a primitive recursive relation: Assume $k \Vdash^L R(n,m)$. Then by the definitions it follows $R(n,m)$ and thus $k \Vdash_L R(n,m)$. On the other hand, assume $k \Vdash_L R(n,m)$. By the definition, $R(n,m)$ follows and thus $k \Vdash^L R(n,m)$.

♢ $\eta \equiv \varphi \wedge \theta$: Assume $k \Vdash^L \varphi \wedge \theta$ and $h \Vdash_L \varphi^* \wedge \theta^*$. By the definitions and the inductive hypotheses, it follows that

$$\langle \Psi^\varphi(k_0), \Psi^\theta(k_1)\rangle \Vdash_L \varphi^* \wedge \theta^* \wedge \langle \Xi^\varphi(h_0), \Xi^\theta(h_1)\rangle \Vdash^L \varphi \wedge \theta.$$

♢ $\eta \equiv \varphi \to \theta$: Assume $k \Vdash^L \varphi \to \theta$ and $u \Vdash_L \varphi^*$. Then by the inductive hypothesis, $\Xi^\varphi(u) \Vdash^L \varphi$, i.e., by the assumption, $k \cdot \Xi^\varphi(u) \Vdash^L \theta$ and thus

$$\Psi^\theta(k \cdot \Xi^\varphi(u)) \Vdash_L \theta^*.$$

Hence we have shown that $\Lambda u.\Psi^\theta(k \cdot \Xi^\varphi(u)) \Vdash_L \varphi^* \to \theta^*$. On the other hand, assume $h \Vdash_L \varphi^* \to \theta^*$ and $v \Vdash^L \varphi$. Then by the inductive hypothesis, $\Psi^\varphi(v) \Vdash_L \varphi^*$, i.e., by the assumption, $h \cdot \Psi^\varphi(v) \Vdash_L \theta^*$ and thus

$$\Xi^\theta(h \cdot \Psi^\varphi(v)) \Vdash^L \theta.$$

Hence we have shown that $\Lambda v.\Xi^\theta(h \cdot \Psi^\varphi(v)) \Vdash^L \varphi \to \theta$.

♢ $\eta \equiv \forall x\varphi(x)$: Assume $k \Vdash^L \forall x\varphi(x)$. By the definition, $\forall n \in \mathbb{N}[k \cdot n \Vdash^L \varphi(n)]$. By the inductive hypothesis and the definitions, it follows that

$$\forall n \in \mathbb{N}[\Psi^\varphi(k \cdot n) \Vdash_L \varphi^*(n)],$$

and thus by the definition $\mathfrak{sg}(\Lambda n.\Psi^\varphi(k \cdot n)) \Vdash_L \forall x[N(x) \to \varphi^*(x)]$. On the other hand, assume $h \Vdash_L \forall x[N(x) \to \varphi^*(x)]$. By the definition,

$$\forall a \in V^* \forall q \in D_h[q \Vdash_L N(a) \to \varphi^*(a)].$$

Hence it follows that $\forall n \in \mathbb{N} \forall q \in D_h[q \cdot n \Vdash_L \varphi^*(n)]$, i.e, by Lemma 4.1.8 and 4.4.2 , $\forall n \in \mathbb{N}[\chi_{\varphi^*}(\Phi(h, \Lambda q.q \cdot n)) \Vdash_L \varphi^*(n)]$. By the inductive hypothesis and the definition,

$$\Lambda n.\Xi^{\varphi}(\chi_{\varphi^*}(\Phi(h, \Lambda q.q \cdot n))) \Vdash^L \forall x \varphi(x).$$

♢ $\eta \equiv \exists x \varphi(x)$: Assume $k \Vdash^L \exists x \varphi(x)$. Then, by the definition, $\forall q \in D_k[q_1 \Vdash^L \varphi(q_0)]$. By the inductive hypothesis and the definition, it follows that

$$\forall q \in D_k[\langle q_0, \Psi^{\varphi}(q_1)\rangle \Vdash_L N(q_0) \wedge \varphi^*(q_0)].$$

By Lemma 4.1.8, it follows that

$$\Phi(k, \Lambda q.\langle q_0, \Psi^{\varphi}(q_1)\rangle) \Vdash_L \exists x[N(x) \wedge \varphi^*(x)].$$

On the other hand, assume $h \Vdash_L \exists x[N(x) \wedge \varphi^*(x)]$. Then by the definitions, $\forall q \in D_h[q_1 \Vdash_L \varphi^*(q_0)]$. By the inductive hypothesis, it follows that $\forall q \in D_h[\Xi^{\varphi}(q_1) \Vdash^L \varphi(q_0)]$. By Lemma 4.1.8, $\Phi(h, \Lambda q.\langle q_0, \Xi^{\varphi}(q_1)\rangle) \Vdash^L \exists x \varphi(x)$.

♢ $\eta \equiv \neg\varphi$: Assume $k \Vdash^L \neg\varphi$ and assume $\exists n \in \mathbb{N}(n \Vdash_L \varphi^*)$. Then by the inductive hypothesis and the definition, one has a contradiction and thus $k \Vdash_L \neg\varphi^*$ follows. Similarly, if $k \Vdash_L \neg\varphi^*$, then $k \Vdash^L \neg\varphi$. ■

In conclusion, we have introduced a formal and an informal definitions of Lifschitz' style interpretation in the context of both arithmetic and set theory. In the next chapter, we will prove that this definition not only interprets Heyting arithmetic, intuitionistic Zermelo-Fraenkel set theory and various semi-constructive axioms, but also differentiates $\mathbf{CT}_0$ from $\mathbf{CT}_0!$.

Chapter 5

Lifschitz' style soundness

From Section 5.1 to Section 5.5, we use the interpretation defined in the last chapter to interpret $\mathbf{IZF}_N$ and some semi-constructive axioms. Then we present our main independence results in Section 5.6.

Recall $(\mathbb{N}, \cdot, V^*, \Vdash_L)$ is the truth structure of our Lifschitz' style interpretation. $(\mathbb{N}, \cdot)$ is the truth domain in which $\mathbb{N}$ is the set of all the codes of PRF (the set of all the partial recursive functions from $\mathbb{N}$ to $\mathbb{N}$) and $\cdot$ is the application operation. V^* is the realizability universe defined in Subsection 3.3.2 as follows:

$$\begin{aligned} V_\alpha^{\mathbb{N}} &= \bigcup_{\beta \in \alpha} \mathcal{P}(\mathbb{N} \times (V_\beta^{\mathbb{N}} \cup \mathbb{N})) \\ \mathbb{S}^* &= \bigcup_{\alpha \in On} V_\alpha^{\mathbb{N}} \\ V^* &= \mathbb{N} \cup \mathbb{S}^* \end{aligned}$$

where $\mathcal{P}$ denotes the Powerset operation. $\Vdash_L$ is an interpretation informally defined in Section 4.2 and formally defined in Section 4.3.

In this chapter, we will show that the Lifschitz' style interpretation interprets $\mathbf{IZF}_N$ and some semi-constructive axioms. These will directly give us some independence results. To begin with, let us define the following notations:

Notation 5.0.4 *$\jmath : \mathbb{N}^2 \to \mathbb{N}$ is a pairing function and $\langle e, f \rangle$ denotes $\jmath(e, f)$.*

Notation 5.0.5 *$\jmath_0 : \mathbb{N} \to \mathbb{N}$ is a left unpairing function and e_0 denotes $\jmath_0(e)$.*

Notation 5.0.6 $\jmath_1 : \mathbb{N} \to \mathbb{N}$ *is a right unpairing function and* e_1 *denotes* $\jmath_1(e)$.

Notation 5.0.7 *If* $h = \langle e, \langle f, g\rangle\rangle$, *we use the notation* h_0 *to denote* e *and* h_{10} *to denote* f *and* h_{11} *to denote* g.

Notation 5.0.8 $\mathbf{d}abc_1c_2$ *(or* $\mathbf{d}[a][b]c_1c_2$, *if the form of* a *and* b *are rather lengthy) denotes* $\mathbf{d}$ *is the Gödel number for the partial recursive function such that whenever the condition* $c_1 = c_2$ *holds, then the code* a *is executed; otherwise, the code* b *is executed.*

Notation 5.0.9 $\mathbf{IZF}'_N \equiv \mathbf{IZF}_N + \mathbf{MP}_{pr} + \mathbf{B}\sum^{\mathbf{0}}_{\mathbf{2}} - \mathbf{MP}$.

Notation 5.0.10 $\vec{a} \equiv a_1, a_2, ..., a_n$ *and* $\vec{a} \in V^* \equiv a_1, a_2, ..., a_n \in V^*$.

Notation 5.0.11 $\forall n A(n, \vec{x}) \equiv \forall y[N(y) \to A(y, \vec{x})]$.

Notation 5.0.12 $\exists n A(n, \vec{x}) \equiv \exists y[N(y) \wedge A(y, \vec{x})]$.

Notation 5.0.13 *We will use* $\Lambda e.\xi(e)$ *to denote the code of the partial recursive function* $\lambda e.\xi(e)$ *and* $\Lambda e.\Lambda d.\xi(e, d)$ *to abbreviate* $\Lambda e.(\Lambda d.\xi(e, d))$.

Notation 5.0.14 $\psi_2 \Lambda n.\xi(n) \equiv \psi_2(\Lambda n.\xi(n))$.

Notation 5.0.15 $\psi_2 \Lambda n.\psi_2 \Lambda m.\xi(n, m) \equiv \psi_2(\Lambda n.\psi_2 \Lambda m.\xi(n, m))$.

Notation 5.0.16 $\mathfrak{sg}\langle n, m\rangle \equiv \mathfrak{sg}(\langle n, m\rangle)$ *and* $n \cdot \langle n, m\rangle \equiv n \cdot (\langle n, m\rangle)$.

Notation 5.0.17 *In some cases, if the form of a realizer* ξ *is rather lengthy, we use the notation* $|\xi|$ *to denote* D_ξ *(cf. Section 4.1) and use* $\forall t \in D_e$ *as an abbreviation for* $D_e \neq \emptyset \wedge \forall t(t \in D_e)$.

Lemma 5.0.18 *For each formula* $A(u, \vec{x})$, *there are partial recursive functions* ψ_1 *and* ψ_2 *such that* $\mathbf{IZF}'_N$ *proves* $\forall e \in \mathbb{N} \forall \vec{a} \in V^*$
(i) $e \Vdash_L \forall n A(n, \vec{a}) \to \forall n \in \mathbb{N}[\psi_1(e) \cdot n \Vdash_L A(n, \vec{a})]$,
(ii) $\forall n \in \mathbb{N}[e \cdot n \Vdash_L A(n, \vec{a})] \to \psi_2(e) \Vdash_L \forall n A(n, \vec{a})$,
(iii) $e \Vdash_L \exists n A(n, \vec{a}) \leftrightarrow D_e \neq \emptyset \wedge \forall d \in D_e[d_1 \Vdash_L A(d_0, \vec{a})]$,
where $\psi_1(e) \equiv \Lambda n.\chi_A(\Phi(e, \Lambda q.q \cdot n))$, $\psi_2(e) \equiv \mathfrak{sg}(e)$.

Proof. (i) Assume $e \Vdash_L \forall n A(n, \vec{a})$, i.e., $e \Vdash_L \forall x[N(x) \to A(x, \vec{a})]$. By the definition it follows that $\forall n \in \mathbb{N} \forall q \in D_e[q \cdot n \Vdash_L A(n, \vec{a})]$. By Lemmas 4.1.8 and 4.4.2,

$$\forall n \in \mathbb{N}[\chi_A(\Phi(e, \Lambda q.q \cdot n)) \Vdash_L A(n, \vec{a})].$$

Now set $\psi_1(e) \equiv \Lambda n.\chi_A(\Phi(e, \Lambda q.q \cdot n))$ and the result follows. (ii) Assume $\forall n \in \mathbb{N}[e \cdot n \Vdash_L A(n, \vec{a})]$. Then by the definition it follows that $\mathfrak{sg}(e) \Vdash_L \forall n A(n, \vec{a})$. (iii) Assume $e \Vdash_L \exists n A(n, \vec{a})$, i.e., $\forall d \in D_e \exists c \in V^*[d \Vdash_L N(c) \wedge A(c, \vec{a})]$. The result follows immediately from the definition. On the other hand, assume $D_e \neq \emptyset \wedge \forall d \in D_e[d_1 \Vdash_L A(d_0, \vec{a})]$. Then the result follows immediately from the definition. ■

In the following we will show that this interpretation is sound with respect to our formal system $\mathbf{IZF}_N$ and some semi-constructive axioms. We will not reiterate these formal systems here. The details are all specified in Chapter Two.

5.1 A1: Axioms on numbers and sets

Claim 5.1.1 [**A1** :1] *$\mathfrak{sg}(0) \Vdash_L \forall x \neg(N(x) \wedge S(x))$.*

Proof. Let $c \in V^*$ be arbitrary. Assume $\exists n \in \mathbb{N}$ such that $n \Vdash_L (N(c) \wedge S(c))$. Then by the definition, it follows that $c \in \mathbb{N} \cap \mathbb{S}^*$, but this contradicts Corollary 3.3.3. ■

Claim 5.1.2 [**A1** :2] *$\mathfrak{sg}(\mathfrak{sg}(\Lambda e.0)) \Vdash_L \forall x \forall y[x \in y \to S(y)]$.*

Proof. Let $a, b \in V^*$ and $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_L a \in b$. Then by the definition $D_e \neq \emptyset \wedge \forall d \in D_e \exists c \in V^*(d_0, c) \in b$, i.e., $b \in \mathbb{S}$. Hence, $b \in \mathbb{S}^*$. ■

Claim 5.1.3 [**A1** :3] *$n \Vdash_L N(\bar{n})$ for every natural number n.*

Proof. This follows immediately from the fact that $N(x)$ is an axiom of the background theory. ■

5.2 A2: Number-theoretic axioms

Claim 5.2.1 [**A2** :1] *$0 \Vdash_L \mathrm{SUC}(\bar{n}, \overline{n+1})$ for every natural number n.*

Proof. This follows immediately from the fact that $\mathrm{SUC}(\bar{n}, \overline{n+1})$ is an axiom of the background theory. ■

Claim 5.2.2 [**A2** :2] $r^{22} \Vdash_L \forall n \exists! m \mathrm{SUC}(n,m)$,
where $r^{22} \equiv \psi_2(\Lambda n.\langle \mathfrak{sg}(\langle n+1, 0\rangle), \mathfrak{sg}(\mathfrak{sg}(\Lambda e.0))\rangle)$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. By **A2** :2 in the background theory, there exists a unique number $n+1 \in \mathbb{N}$ such that $\mathrm{SUC}(n, n+1)$. By Lemma 5.0.18 and the definition, it follows that

$$\mathfrak{sg}(\langle n+1, 0\rangle) \Vdash_L \exists m \mathrm{SUC}(n,m). \tag{5.1}$$

Now let $a, b \in V^*$ and $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_L \mathrm{SUC}(n,a) \wedge \mathrm{SUC}(n,b)$. By **A2** :2 again, it follows that $a = b$ and thus

$$\mathfrak{sg}(\mathfrak{sg}(\Lambda e.0)) \Vdash_L \forall x \forall y[\mathrm{SUC}(n,x) \wedge \mathrm{SUC}(n,y) \to x = y]. \tag{5.2}$$

By (5.1), (5.2) and Lemma 5.0.18, the result follows. ■

Claim 5.2.3 [**A2** :3] $r^{23} \Vdash_L \forall n \forall m(\mathrm{SUC}(n,m) \to m \neq \bar{0})$,
where $r^{23} \equiv \psi_2 \Lambda n.\psi_2 \Lambda m.(\Lambda e.0)$.

Proof. Let $n, m, e \in \mathbb{N}$ be arbitrary such that $e \Vdash_L \mathrm{SUC}(n,m)$, i.e., $\mathrm{SUC}(n,m)$. Then by **A2** :3 in the background theory, it follows that $m \neq \bar{0}$ and thus the result follows from Lemma 5.0.18. ■

Claim 5.2.4 [**A2** :4] $r^{24} \Vdash_L \forall m(m = \bar{0} \vee \exists n \mathrm{SUC}(n,m))$,
where $r^{24} \equiv \psi_2 \Lambda m.(\mathfrak{sg}\langle m, \mathbf{d}[0][\mathfrak{sg}(\langle m-1, 0\rangle)]m0\rangle)$.

Proof. Let $m \in \mathbb{N}$ be arbitrary. Then by **A2** :4 in the background theory $m = \bar{0} \vee \exists n \mathrm{SUC}(n,m)$. For the second case, from **A2** : 5 in the background theory, there is a unique number $m-1 \in \mathbb{N}$ such that $\mathrm{SUC}(m-1, m)$. Hence by Lemma 5.0.18, it follows that $\mathfrak{sg}\langle m-1, 0\rangle \Vdash_L \exists n \mathrm{SUC}(n,m)$. Applying the disjunctive realizer $\mathbf{d}$ yields the result. ■

Claim 5.2.5 [**A2** :5] $r^{25} \Vdash_L \forall n \forall m \forall k(\mathrm{SUC}(m,n) \wedge \mathrm{SUC}(k,n) \to m = k)$,
where $r^{25} \equiv \psi_2 \Lambda n.\psi_2 \Lambda m.\psi_2 \Lambda k.(\Lambda e.0)$.

Proof. Let $n, m, k, e \in \mathbb{N}$ be arbitrary such that $e \Vdash_L \mathrm{SUC}(m,n) \wedge \mathrm{SUC}(k,n)$, i.e, $\mathrm{SUC}(m,n) \wedge \mathrm{SUC}(k,n)$. By **A2** :5 in the background theory, $m = k$. ■

Claim 5.2.6 [**A2** :6] $r^{26} \Vdash_L \forall n \forall m \exists! k \mathrm{ADD}(n,m,k)$,
where $r^{26} \equiv \psi_2 \Lambda n.\psi_2 \Lambda m.\langle \mathfrak{sg}(\langle n+m, 0\rangle), \mathfrak{sg}(\mathfrak{sg}(\Lambda e.0))\rangle$.

Proof. Let $n, m \in \mathbb{N}$ be arbitrary. Then by **A2** :6 in the background theory, there is a unique number $n + m \in \mathbb{N}$ such that $\mathrm{ADD}(n, m, n + m)$. Hence by the definitions, $\langle \mathfrak{sg}(\langle n+m, 0\rangle), \mathfrak{sg}(\mathfrak{sg}(\Lambda e.0))\rangle \Vdash_L \exists! k \mathrm{ADD}(n, m, k)$. Then the result follows immediately from Lemma 5.0.18. ■

Claim 5.2.7 [**A2** :7] $\psi_2(\Lambda n.0) \Vdash_L \forall n \mathrm{ADD}(n, \bar{0}, n)$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. Then by **A2** :7 in the background theory, $\mathrm{ADD}(n, \bar{0}, n)$. By the definition the result follows. ■

Claim 5.2.8 [**A2** :8]
$r^{28} \Vdash_L \forall n \forall k \forall m \forall l \forall i[\mathrm{ADD}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{SUC}(m, i) \to \mathrm{ADD}(n, l, i)]$,
where $r^{28} \equiv \psi_2 \Lambda n.\psi_2 \Lambda k.\psi_2 \Lambda m.\psi_2 \Lambda l.\psi_2 \Lambda i.(\Lambda e.0)$.

Proof. Let $n, k, m, l, i, e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_L \mathrm{ADD}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{SUC}(m, i).$$

Applying **A2** :8 in the background theory, $\mathrm{ADD}(n, l, i)$ and then the result follows from the definition. ■

Claim 5.2.9 [**A2** :9] $r^{29} \Vdash_L \forall n \forall m \exists! k \mathrm{MULT}(n, m, k)$,
where $r^{29} \equiv \psi_2 \Lambda n.\psi_2 \Lambda m.\langle \mathfrak{sg}(\langle n \times m, 0\rangle), \mathfrak{sg}(\mathfrak{sg}(\Lambda e.0))\rangle$.

Proof. Let $n, m \in \mathbb{N}$ be arbitrary. Then by **A2** :9 in the background theory, there is a unique number $n \times m \in \mathbb{N}$ such that $\mathrm{MULT}(n, m, n \times m)$. By the definition and Lemma 5.0.18,

$$\langle \mathfrak{sg}(\langle n \times m, 0\rangle), \mathfrak{sg}(\mathfrak{sg}(\Lambda e.0))\rangle \Vdash_L \exists! k \mathrm{MULT}(n, m, k).$$

Then the result follows. ■

Claim 5.2.10 [**A2** :10] $\psi_2(\Lambda n.0) \Vdash_L \forall n \mathrm{MULT}(n, \bar{0}, \bar{0})$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. Then by **A2** :10 in the background theory, $\mathrm{MULT}(n, \bar{0}, \bar{0})$. By the definition the result follows. ■

Claim 5.2.11 [**A2** :11] $r^{211} \Vdash_L$

$\forall n \forall k \forall m \forall l \forall i[\mathrm{MULT}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{ADD}(m, n, i) \to \mathrm{MULT}(n, l, i)]$,

where $r^{211} \equiv \psi_2 \Lambda n.\psi_2 \Lambda k.\psi_2 \Lambda m.\psi_2 \Lambda l.\psi_2 \Lambda i.(\Lambda e.0)$.

Proof. Let $n, k, m, l, i, e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_L \mathrm{MULT}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{ADD}(m, n, i).$$

By **A2** :11 in the background theory, $\mathrm{MULT}(n, l, i)$. Then the result follows from the definition. ■

Claim 5.2.12 [**A2** :12]

$$r^{212} \Vdash_L A(\bar{0}) \wedge \forall n \forall m[A(n) \wedge \mathrm{SUC}(n, m) \rightarrow A(m)] \rightarrow \forall n A(n),$$

where $r^{212} \equiv \Lambda e.\psi_2(f_e^{\#})$.

Proof. Let $e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_L A(\bar{0}) \wedge \forall n \forall m[A(n) \wedge \mathrm{SUC}(n, m) \rightarrow A(m)],$$

i.e., $e_0 \Vdash_L A(\bar{0})$ and

$$e_1 \Vdash_L \forall n \forall m[A(n) \wedge \mathrm{SUC}(n, m) \rightarrow A(m)].$$

By Lemma 5.0.18, one then has

$$\forall n, m \in \mathbb{N}[\delta(e, n) \cdot m \Vdash_L A(n) \wedge \mathrm{SUC}(n, m) \rightarrow A(m)], \tag{5.3}$$

where $\delta(e, n) \equiv \psi_1(\psi_1(e_1) \cdot n)$.

Now define a recursive function $f_e : \mathbb{N} \rightarrow \mathbb{N}$ such that $f_e(0) = e_0$ and $f_e(n + 1) = (\delta(e, n) \cdot (n + 1)) \cdot \langle f_e(n), 0\rangle$ and let $f_e^{\#}$ be its Gödel number. Then by Lemma 5.0.18, the result follows immediately. ■

5.3 A3: Logical axioms for IPL

We will show only nontrivial ones and write down the realizers for the trivial ones.

For logical axioms (LA):

(IPL1) $\Lambda e.\Lambda d.e \Vdash_L A \rightarrow (B \rightarrow A)$.

(IPL2) $\Lambda e.\Lambda d.\Lambda k.((e \cdot k) \cdot (d \cdot k)) \Vdash_L [A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$.

(IPL3) $\Lambda e.\Lambda d.\langle e, d\rangle \Vdash_L A \rightarrow (B \rightarrow A \wedge B)$.

(IPL4) $\Lambda e.e_0 \Vdash_L A \wedge B \rightarrow A$.

(IPL5) $\Lambda e.e_1 \Vdash_L A \wedge B \to B$.

(IPL6) $\Lambda e.\mathfrak{sg}\langle 0, e\rangle \Vdash_L A \to A \vee B$.

(IPL7) $\Lambda e.\mathfrak{sg}\langle 1, e\rangle \Vdash_L B \to A \vee B$.

(IPL8) $r^8 \Vdash_L (A \vee B) \to [(A \to C) \to ((B \to C) \to C)]$,
where $r^8 \equiv \Lambda e.\Lambda s.\Lambda t.\chi_C(\Phi(e, \Lambda q.\mathbf{d}[s \cdot q_1][t \cdot q_1]q_0 0))$.

Proof. Let $e, s, t \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_L A \vee B, s \Vdash_L A \to C, t \Vdash_L B \to C.$$

Then by the definition

$$\forall q \in D_e[(q_0 = 0 \wedge s \cdot q_1 \Vdash_L C) \vee (q_0 \neq 0 \wedge t \cdot q_1 \Vdash_L C)].$$

By applying the disjunctive realizer $\mathbf{d}$, Lemma 4.1.8 and 4.4.2,

$$\chi_C(\Phi(e, \Lambda q.\mathbf{d}[s \cdot q_1][t \cdot q_1]q_0 0)) \Vdash_L C.$$

■

(IPL9) $\Lambda e.\Lambda d.0 \Vdash_L (A \to B) \to ((A \to \neg B) \to \neg A)$.

(IPL10) $\Lambda e.0 \Vdash_L A \to (\neg A \to B)$.

(IPL11) $\Lambda e.\chi_A(e) \Vdash_L \forall x A(x) \to A[x/a]$, where a is a constant in V^*.

(IPL12) $\Lambda e.\mathfrak{sg}(e) \Vdash_L A[x/a] \to \exists x A(x)$, where a is a constant in V^*.

For Inference Rules: (In the following, we use $FV(C)$ to denote the set of all free variables in C).

(IR1) Modus Ponens is preserved, i.e., if $e \Vdash_L A$ and $d \Vdash_L A \to B$, then one can effectively associate a realizer for B via realizers e and d.

Proof. This follows immediately from the definition. ■

(IR2) Rule $\forall$ is preserved, i.e., if $m \Vdash_L C \to A(x)$, then one can find a partial recursive function ξ such that $\xi(m) \Vdash_L C \to \forall x A(x)$, where $x \notin FV(C)$.

Proof. Assume $m \Vdash_L \forall x(C \to A(x))$. Let $n \in \mathbb{N}$ be arbitrary such that $n \Vdash_L C$. Then by the definitions

$$\forall a \in V^* \forall q \in D_m(q \cdot n \Vdash_L A(a)).$$

By Lemma 4.1.8 and the definition, it follows $\Phi(m, \Lambda q.q \cdot n) \Vdash_L \forall x A(x)$. Now one sets $\xi(m) := \Lambda n.\Phi(m, \Lambda q.q \cdot n)$. ∎

(IR3) Rule $\exists$ is preserved, i.e., if $m \Vdash_L A(x) \to C$, then one can find a partial recursive function ξ such that $\xi(m) \Vdash_L \exists x A(x) \to C$, where $x \notin FV(C)$.

Proof. Assume $m \Vdash_L \forall x(A(x) \to C)$. Let $n \in \mathbb{N}$ be arbitrary such that $n \Vdash_L \exists x A(x)$, i.e.,

$$\forall q \in D_n \exists a \in V^*[q \Vdash_L A(a)].$$

By the assumption it follows that

$$\forall u \in D_m \forall q \in D_n[u \cdot q \Vdash_L C].$$

By Lemmas 4.1.8 and 4.1.9 and 4.4.2

$$\chi_C(\mathfrak{un}(\Phi(m, \Lambda u.\delta(u, n)))) \Vdash_L C,$$

where $\delta(u, n) \equiv \Phi(n, \Lambda q.u \cdot q)$. Set

$$\xi(m) := \Lambda n.\chi_C(\mathfrak{un}(\Phi(m, \Lambda u.\delta(u, n)))).$$

∎

For the Identity Axioms (IA): The soundness of **IA** follows immediately from the following claims. These claims will provide some universal realizers, that is, these realizers will depend only on the form of a formula in $\mathcal{L}(V^*)$ (the language $\mathcal{L}$ with parameters from V^*) and are independent of the parameters. Now let $a, b, c \in V^*$ be arbitrary. We have the following claims:

Claim 5.3.1 [**IA1**] $\mathbf{i}_r \Vdash_L a = a$, *where* $\mathbf{i}_r \equiv \Omega(\Lambda y.\xi(y))$ *and where* $\xi(y) \equiv \mathfrak{sg}\langle \Lambda f.\mathfrak{sg}\langle f, y\rangle, \Lambda f.\mathfrak{sg}\langle f, y\rangle\rangle$.

Proof. It suffices to find a realizer $\mathbf{i}_r$ such that $\forall a \in \mathbb{S}^*[\mathbf{i}_r \Vdash_L a = a]$. We show this via ordinal induction and the fixed point theorem. Let $\alpha \in On$ be arbitrary. Assume $\forall \beta \in \alpha \forall b \in V^{\mathbb{N}}_{\beta}[k \Vdash_L b = b]$. Now let $a \in V^{\mathbb{N}}_{\alpha}$ be arbitrary. Let $f \in \mathbb{N}$ and $d \in V^*$ be arbitrary such that $(f, d) \in a$. Then by the inductive hypothesis and the definition, $\mathfrak{sg}\langle f, k\rangle \Vdash_L d \in a$ and thus $\xi(k) \Vdash_L a = a$, where $\xi(k) \equiv \mathfrak{sg}\langle \Lambda f.\mathfrak{sg}\langle f, k\rangle, \Lambda f.\mathfrak{sg}\langle f, k\rangle\rangle$. Applying the fixed point generator Ω, one has the explicit form $k \equiv \Omega(\Lambda y.\xi(y))$. ■

Claim 5.3.2 [**IA2**] $\mathbf{i}_s \Vdash_L a = b \to b = a$, *where* $\mathbf{i}_s \equiv \Lambda e.\Phi(e, \Lambda k.\langle k_1, k_0\rangle)$.

Proof. Let $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_L a = b$. From Lemma 4.1.8, $\{\langle k_1, k_0\rangle : k \in D_e\} = |\Phi(e, \Lambda k.\langle k_1, k_0\rangle)|$ and thus by the definition $\Phi(e, \Lambda k.\langle k_1, k_0\rangle) \Vdash_L b = a$. ■

Claim 5.3.3 [**IA3**] $\mathbf{i}_t \Vdash_L a = b \wedge b = c \to a = c$,
where $\mathbf{i}_t \equiv (\Omega(\lambda x.\langle \partial x_0 x_1, \tilde{\partial} x_0 x_1\rangle))_0$.

Claim 5.3.4 [**IA4**] $\mathbf{i}_0 \Vdash_L a = b \wedge b \in c \to a \in c$,
where $\mathbf{i}_0 \equiv (\Omega(\lambda x.\langle \partial x_0 x_1, \tilde{\partial} x_0 x_1\rangle))_1$.

Proof. For any formulae $\theta_1, \theta_2, ..., \theta_n$, let $\bigwedge_{i=1}^{n} \theta_i$ denote the conjunction $\theta_1 \wedge \theta_2 ... \wedge \theta_n$. We will prove **IA3** and **IA4** simultaneously via $\lhd^3$-induction (cf. Subsection 3.2.1):

$$\forall x_1, x_2, x_3[\forall(y_1, y_2, y_3) \lhd^3 (x_1, x_2, x_3)\varphi(y_1, y_2, y_3) \to \varphi(x_1, x_2, x_3)] \\ \to \forall x, y, z\varphi(x, y, z).$$

and the fixed point theorem (which will produce universal realizers u and v for both) by taking $\varphi(y_1, y_2, y_3)$ to be

$$y_1, y_2, y_3 \in V^* \to \bigwedge_{\substack{i,j,k=1 \\ i,j \neq k \wedge i \neq j}}^{3} \eta(y_i, y_j, y_k),$$

where $\eta(y_i, y_j, y_k)$ denotes

$$u \Vdash_L [y_i = y_j \wedge y_j = y_k \to y_i = y_k] \wedge v \Vdash_L [y_i = y_j \wedge y_j \in y_k \to y_i \in y_k].$$

Let $a^4, a^5, a^6 \in V^*$ and $d^1, d^2, d^3 \in V^*$ be arbitrary such that $(d^1, d^2, d^3) \lhd^3 (a^4, a^5, a^6)$ and

$$u \Vdash_L d^i = d^j \wedge d^j = d^k \to d^i = d^k, \qquad (5.4)$$
$$v \Vdash_L d^i = d^j \wedge d^j \in d^k \to d^i \in d^k,$$

for all $i, j, k \in \{1, 2, 3\}$, where $i, j \neq k$ and $i \neq j$. Now we have to find the forms of the realizers u and v and show that

$$u \Vdash_L a^i = a^j \wedge a^j = a^k \to a^i = a^k,$$
$$v \Vdash_L a^i = a^j \wedge a^j \in a^k \to a^i \in a^k,$$

for all $i, j, k \in \{4, 5, 6\}$, where $i, j \neq k$ and $i \neq j$. Now let $i, j, k \in \{4, 5, 6\}$ be arbitrary such that $i, j \neq k$ and $i \neq j$. Let $g \in \mathbb{N}$ be arbitrary such that $g \Vdash_L a^i = a^j \wedge a^j \in a^k$, i.e.,

$$\forall n \in D_{g_1} \exists q \in V^*[(n_0, q) \in a^k \wedge \langle g_0, n_1 \rangle \Vdash_L a^i = a^j \wedge a^j = q].$$

We want to find a realizer for $a^i \in a^k$. Since $(a^i, a^j, q) \lhd^3 (a^i, a^j, a^k)$, (without loss of generality, suppose $i = 6, j = 4, k = 5$, then $(a^j, q, a^i) \lhd^3 (a^4, a^5, a^6)$) and by the inductive hypothesis (5.4),

$$\forall n \in D_{g_1} \exists q \in V^*[u \cdot \langle g_0, n_1 \rangle \Vdash_L a^i = q].$$

By Lemma 4.1.8, it follows that

$$\{\langle n_0, u \cdot \langle g_0, n_1 \rangle \rangle : n \in D_{g_1}\} = |\sigma(g, u)|,$$

where $\sigma(g, u) \equiv \Phi(g_1, \Lambda n.\langle n_0, u \cdot \langle g_0, n_1 \rangle \rangle)$. Hence by the definition, it follows that $\sigma(g, u) \Vdash_L a^i \in a^k$ and thus

$$\Lambda g.\sigma(g, u) \Vdash_L a^i = a^j \wedge a^j \in a^k \to a^i \in a^k. \tag{5.5}$$

Now let $h \in \mathbb{N}$ be arbitrary such that $h \Vdash_L a^i = a^j \wedge a^j = a^k$. We want to find a realizer for $a^i = a^k$. Let $p, r \in \mathbb{N}$ and $q, w \in V^*$ be arbitrary such that $(p, q) \in a^i$ and $(r, w) \in a^k$. From the assumption $\forall n \in D_{h_0}[n_0 \cdot p \Vdash_L q \in a^j]$ and $\forall n \in D_{h_1}[n_1 \cdot r \Vdash_L w \in a^j]$, i.e.,

$$\forall n \in D_{h_0} \forall m \in D_{n_0 \cdot p} \exists s \in V^*[(m_0, s) \in a^j \wedge m_1 \Vdash_L q = s],$$
$$\forall n \in D_{h_1} \forall m \in D_{n_1 \cdot r} \exists s \in V^*[(m_0, s) \in a^j \wedge m_1 \Vdash_L w = s].$$

By the assumption again, it follows that

$$\forall n \in D_{h_0} \forall m \in D_{n_0 \cdot p} \exists s \in V^* \forall l \in D_{h_1}[l_0 \cdot m_0 \Vdash_L s \in a^k],$$
$$\forall n \in D_{h_1} \forall m \in D_{n_1 \cdot r} \exists s \in V^* \forall l \in D_{h_0}[l_1 \cdot m_0 \Vdash_L s \in a^i],$$

i.e., by Lemmas 4.1.8 and 4.4.2

$$\forall n \in D_{h_0} \forall m \in D_{n_0 \cdot p} \exists s \in V^*[\langle m_1, \varepsilon(h, m) \rangle \Vdash_L q = s \wedge s \in a^k],$$
$$\forall n \in D_{h_1} \forall m \in D_{n_1 \cdot r} \exists s \in V^*[\langle m_1, \tilde{\varepsilon}(h, m) \rangle \Vdash_L w = s \wedge s \in a^i],$$

where $\varepsilon(h,m) \equiv \chi_A(\Phi(h_1, \Lambda l.l_0 \cdot m_0))$, $\tilde{\varepsilon}(h,m) \equiv \chi_A(\Phi(h_0, \Lambda l.l_1 \cdot m_0))$, and $A \equiv x \in y$. Since

$$(q, s, a^k) \lhd^3 (a^i, a^j, a^k), (a^i, s, w) \lhd^3 (a^i, a^j, a^k),$$

by the inductive hypothesis (5.4),

$$\forall n \in D_{h_0} \forall m \in D_{n_0 \cdot p} \rho(h, m, v) \Vdash_L q \in a^k,$$
$$\forall n \in D_{h_1} \forall m \in D_{n_1 \cdot r} \tilde{\rho}(h, m, v) \Vdash_L w \in a^i,$$

where $\rho(h, m, v) \equiv v \cdot \langle m_1, \varepsilon(h,m)\rangle$, and where $\tilde{\rho}(h, m, v) \equiv v \cdot \langle m_1, \tilde{\varepsilon}(h,m)\rangle$. By Lemmas 4.1.8 and 4.1.9,

$$\{n_0 \cdot p : n \in D_{h_0}\} = |\Phi(h_0, \Lambda n.(n_0 \cdot p))|,$$
$$\{n_1 \cdot r : n \in D_{h_1}\} = |\Phi(h_1, \Lambda n.(n_1 \cdot r))|,$$
$$\cup\{D_{n_0 \cdot p} : n \in D_{h_0}\} = |\mathfrak{un}(\Phi(h_0, \Lambda n.(n_0 \cdot p)))|,$$
$$\cup\{D_{n_1 \cdot r} : n \in D_{h_1}\} = |\mathfrak{un}(\Phi(h_1, \Lambda n.(n_1 \cdot r)))|.$$

Thus by Lemmas 4.1.8 and 4.4.2,

$$\beta(p, h, v) \Vdash_L q \in a^k, \tau(r, h, v) \Vdash_L w \in a^i,$$

where

$$\beta(p, h, v) \equiv \chi_A(\Phi(\mathfrak{un}(\Phi(h_0, \Lambda n.(n_0 \cdot p))), \Lambda m.\rho(h, m, v))),$$
$$\tau(r, h, v) \equiv \chi_A(\Phi(\mathfrak{un}(\Phi(h_1, \Lambda n.(n_1 \cdot r))), \Lambda m.\tilde{\rho}(h, m, v))).$$

By the definition it follows that

$$\Lambda h.\pi(h, v) \Vdash_L a^i = a^j \wedge a^j = a^k \to a^i = a^k, \tag{5.6}$$

where $\pi(h, v) \equiv \mathfrak{sg}\langle \Lambda p.\beta(p, h, v), \Lambda r.\tau(r, h, v)\rangle$. From (5.4), (5.5) and (5.6) one finds u and v as follows:

$$u \simeq \Lambda h.\pi(h, v) \wedge v \simeq \Lambda g.\sigma(g, u).$$

Now define $\partial \equiv \Lambda u.\Lambda v.\Lambda h.\pi(h, v)$ and $\tilde{\partial} \equiv \Lambda u.\Lambda v.\Lambda g.\sigma(g, u)$. By Corollary 3.1.15, one has the explicit form as follows: $u \equiv (\Omega(\lambda x.\langle \partial x_0 x_1, \tilde{\partial} x_0 x_1\rangle))_0$ and $v \equiv (\Omega(\lambda x.\langle \partial x_0 x_1, \tilde{\partial} x_0 x_1\rangle))_1$. ■

Claim 5.3.5 [**IA5**]

$$\mathbf{i}_1 \Vdash_L a = b \wedge c \in a \rightarrow c \in b,$$

where $\mathbf{i}_1 \equiv \Lambda g.\mathfrak{un}(\Phi(g_1, \Lambda e.\Phi(g_0, \Lambda d.\mathbf{i}_0 \cdot \langle e_1, d_0 \cdot e_0\rangle)))$

Proof. Let $g \in \mathbb{N}$ be arbitrary such that $g \Vdash_L a = b \wedge c \in a$. Then by the definition, $g_0 \Vdash_L a = b$ and

$$\forall e \in D_{g_1} \exists k \in V^*[(e_0, k) \in a \wedge e_1 \Vdash_L c = k].$$

By the definition and Soundness of **IA4**, it then follows that

$$\forall e \in D_{g_1} \forall d \in D_{g_0}[\mathbf{i}_0 \cdot \langle e_1, d_0 \cdot e_0\rangle \Vdash_L c \in b]. \tag{5.7}$$

By Lemmas 4.1.8, 4.1.9 and 4.4.2, it follows that $\beta(g) \Vdash_L c \in b$, where $\beta(g) \equiv \mathfrak{un}(\Phi(g_1, \Lambda e.\Phi(g_0, \Lambda d.\mathbf{i}_0 \cdot \langle e_1, d_0 \cdot e_0\rangle)))$. ■

Theorem 5.3.6 *There is a partial recursive function* r^{ia} *such that* $r^{ia} \Vdash_L$ **IA**.

Proof. Since **IA6** to **IA10** are all realizable from its background theory and the definition, the result follows immediately from the above claims. ■

Lemma 5.3.7 [*Lif Substitution*] *For any formula* $A(x, \vec{y})$ *in* $\mathcal{L}(V^*)$*, one can inductively define a realizer* $\mathbf{i}_A \in \mathbb{N}$ *such that* $\forall a, \vec{b}, c \in V^*[\mathbf{i}_A \Vdash_L a = c \wedge A(a, \vec{b}) \rightarrow A(c, \vec{b})]$.

Proof. For the atomic formulae, this has been shown in the above claims. For the compound formulae, they all follow immediately via induction over the their complexity. Here we consider the formulae with quantifiers:

◇ $A(a, \vec{b}) \equiv \exists z B(a, \vec{b}, z)$:

Let $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_L a = c \wedge \exists z B(a, \vec{b}, z)$. By the definition, it follows

$$e_0 \Vdash_L a = c \wedge \forall q \in D_{e_1} \exists d \in V^*[q \Vdash_L B(a, \vec{b}, d)].$$

By the inductive hypothesis, it follows that

$$\forall q \in D_{e_1} \exists d \in V^*[\mathbf{i}_B \cdot \langle e_0, q\rangle \Vdash_L B(c, \vec{b}, d)].$$

By Lemma 4.1.8,

$$\Phi(e_1, \Lambda q.\mathbf{i}_B \cdot \langle e_0, q\rangle) \Vdash_L \exists z B(c, \vec{b}, z).$$

Now one defines $\mathbf{i}_A \equiv \Lambda e.\Phi(e_1, \Lambda q.\mathbf{i}_B \cdot \langle e_0, q\rangle)$ and the result follows.

$\diamondsuit$ $A(a, \vec{b}) \equiv \forall z B(a, \vec{b}, z)$: Let $e \in \mathbb{N}$ and $a, b \in V^*$ be arbitrary such that

$$e \Vdash_L a = c \wedge \forall z B(a, \vec{b}, z).$$

Then by the definition, it follows that

$$e_0 \Vdash_L a = c \wedge \forall q \in D_{e_1} \forall d \in V^*[q \Vdash_L B(a, \vec{b}, d)].$$

By the inductive hypothesis, it follows that

$$\forall q \in D_{e_1} \forall d \in V^*[\mathbf{i}_B \cdot \langle e_0, q\rangle \Vdash_L B(c, \vec{b}, d)].$$

By Lemma 4.1.8,

$$\Phi(e_1, \Lambda q.\mathbf{i}_B \cdot \langle e_0, q\rangle) \Vdash_L \forall z B(c, \vec{b}, z),$$

Now one defines $\mathbf{i}_A \equiv \Lambda e.\Phi(e_1, \Lambda q.\mathbf{i}_B \cdot \langle e_0, q\rangle)$ and the result follows. ■

5.4 A4.2: Non-logical axioms (IZF with two sorts)

Lemma 5.4.1 [*Extensionality*] $r^{\boldsymbol{ext}}_{\mathrm{L}} \Vdash_L \forall x \forall y (S(x) \wedge S(y) \rightarrow [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y])$, *where* $r^{\boldsymbol{ext}}_{\mathrm{L}} \equiv \mathfrak{sg}(\mathfrak{sg}(\Lambda e.\Lambda t.\Phi(t, \Lambda m.\langle \xi(m), \delta(m)\rangle))))$.

Proof. Let $a, b \in V^*$ and $e, t \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_L S(a) \wedge S(b) \tag{5.8}$$

and $t \Vdash_L \forall z (z \in a \leftrightarrow z \in b)$, i.e.,

$$\forall m \in D_t \forall c \in V^*[m \Vdash_L c \in a \leftrightarrow c \in b]. \tag{5.9}$$

Observe that from (5.8) and the definitions, $a, b \in \mathbb{S}^*$. Now we want to find a realizer to realize $a = b$. Let $f, g \in \mathbb{N}$ and $d, k \in V^*$ be arbitrary such that $(f, d) \in a$ and $(g, k) \in b$. Then by the soundness of **IA1**, it follows that

$$\mathfrak{sg}\langle f, \mathbf{i}_r\rangle \Vdash_L d \in a \wedge \mathfrak{sg}\langle g, \mathbf{i}_r\rangle \Vdash_L k \in b,$$

i.e., by (5.9)

$$\forall m \in D_t[m_0 \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle \Vdash_L d \in b \wedge m_1 \cdot \mathfrak{sg}\langle g, \mathbf{i}_r\rangle \Vdash_L k \in a]. \tag{5.10}$$

Define $\xi(m) := \Lambda f.m_0 \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle$ and $\delta(m) := \Lambda g.m_1 \cdot \mathfrak{sg}\langle g, \mathbf{i}_r\rangle$. From Lemma 4.1.8, it then follows that

$$\{\langle \xi(m), \delta(m)\rangle : m \in D_t\} = |\Phi(t, \Lambda m.\langle \xi(m), \delta(m)\rangle)|,$$

and thus by the definition

$$\Phi(t, \Lambda m.\langle \xi(m), \delta(m)\rangle) \Vdash_L a = b.$$

Then the result follows immediately from the definition. ■

Lemma 5.4.2 [*Pairing*] $r_{\text{L}}^{\boldsymbol{pair}} \Vdash_L \forall x \forall y \exists u[S(u) \wedge (x \in u \wedge y \in u)]$, *where* $r_{\text{L}}^{\boldsymbol{pair}} \equiv \mathfrak{sg}(\mathfrak{sg}(\mathfrak{sg}\langle 0, \langle \mathfrak{sg}\langle 0, \mathbf{i}_r\rangle, \mathfrak{sg}\langle 1, \mathbf{i}_r\rangle\rangle\rangle))$.

Proof. Let $a, b \in V^*$ be arbitrary. Now define

$$c \equiv \{a, b\}_L := \{(0, a), (1, b)\}.$$

By the Pairing Axiom in the background theory, c is an external set. By Claim 3.3.6, c is also an internal set, i.e., $c \in \mathbb{S}^*$. Then, by the soundness of **IA1**, it follows that

$$\langle \mathfrak{sg}\langle 0, \mathbf{i}_r\rangle, \mathfrak{sg}\langle 1, \mathbf{i}_r\rangle\rangle \Vdash_L a \in c \wedge b \in c.$$

Hence the result follows from the definition. ■

Furthermore, one can also define the internal Cartesian product.

Definition 5.4.3 *For all a and b in V^*,*

$$(a, b)_L := \{(0, \{a, a\}_L), (1, \{a, b\}_L)\}.$$

Since $\{a, a\}_L, \{a, b\}_L \in \mathbb{S}^*$, by Claim 3.3.6, one has $(a, b)_L \in \mathbb{S}^*$.

Corollary 5.4.4 [*Internal Cartesian Product*]

$$r^{prd} \Vdash_L (a, b)_L = (c, d)_L \rightarrow a = c \wedge b = d,$$

where $r^{prd} \equiv \Lambda e.\langle \alpha(e), \rho(e)\rangle$.

Proof. Let $h \in \mathbb{N}$ and $a, c, d \in V^*$ be arbitrary such that $h \Vdash_L \{a, a\}_L = \{c, d\}_L$. Since $(0, c) \in \{c, d\}_L$, by the definitions, Lemma 4.1.8 and the soundness of **IA2**, $\forall q \in D_h[q_1 \cdot 0 \Vdash_L c \in \{a, a\}_L]$ and thus

$$\forall q \in D_h \forall m \in |\Phi(q_1 \cdot 0, \Lambda r.r_1)|[\mathbf{i}_s \cdot m \Vdash_L a = c].$$

By Lemma 4.1.8, 4.1.9 and 4.4.2, one has the following result:

$$\Lambda h.\xi(h) \Vdash_L \{a, a\}_L = \{c, d\}_L \rightarrow a = c, \tag{5.11}$$

where $\xi(h) \equiv \chi_{x=y}(\Phi(\mathfrak{un}(\Phi(h, \Lambda q.\Phi(q_1 \cdot 0, \Lambda r.r_1))), \Lambda m.\mathbf{i}_s \cdot m))$. Now let us find a realizer for $\{a,b\}_L = \{a,c\}_L \to b = c$. Let $n \in \mathbb{N}$ be arbitrary such that $n \Vdash_L \{a,b\}_L = \{a,c\}_L$. Since $(1,b) \in \{a,b\}_L$,

$$\forall q \in D_n \forall p \in D_{q_0 \cdot 1} \exists v \in V^*[(p_0, v) \in \{a,c\}_L \wedge p_1 \Vdash_L b = v]. \tag{5.12}$$

If $v = a$ (i.e., $p_0 = 0$) then by Substitution

$$\eta(p,n) \Vdash_L \{a,a\}_L = \{a,c\}_L,$$

where $\eta(p,n) \equiv \mathbf{i}_B \cdot \langle p_1, n\rangle$ and $B \equiv \{x,y\}_L = \{x,z\}_L$. Since $(1,c) \in \{a,c\}_L$, by the definitions, it follows that

$$\forall r \in D_{\eta(p,n)} \forall t \in D_{r_1 \cdot 1}[t_1 \Vdash_L c = a].$$

By Lemmas 4.1.8 and 4.1.9 and 4.4.2, it follows that

$$\delta(p,n) \Vdash_L c = a, \tag{5.13}$$

where $\delta(p,n) \equiv \chi_{x=y}(\Phi(\mathfrak{un}(\Phi(\eta(p,n), \Lambda r.r_1 \cdot 1)), \Lambda t.t_1))$. From (5.12) and (5.13) and the soundness of **IA2**, **IA3**,

$$\forall q \in D_n \forall p \in D_{q_0 \cdot 1}[p_0 = 0 \to \mathbf{i}_t \cdot \langle p_1, \mathbf{i}_s \cdot \delta(p)\rangle \Vdash_L b = c].$$

Furthermore, if $v = c$ (i.e., $p_0 = 1$), then by (5.12) $p_1 \Vdash_L b = c$. By applying disjunction to both cases, one has the following result:

$$\forall q \in D_n \forall p \in D_{q_0 \cdot 1}\mathbf{d}[\mathbf{i}_t \cdot \langle p_1, \mathbf{i}_s \cdot \delta(p,n)\rangle][p_1]p_0 0 \Vdash_L b = c,$$

and thus by Lemmas 4.1.8 and 4.1.9 and 4.4.2 again, it follows that

$$\Lambda n.\beta(n) \Vdash_L \{a,b\}_L = \{a,c\}_L \to b = c, \tag{5.14}$$

where $\beta(n) \equiv \chi_{x=y}(\mathfrak{un}(\Phi(n, \Lambda n.\Phi(q_0 \cdot 1, \Lambda p.\mathbf{d}[\mathbf{i}_t \cdot \langle p_1, \mathbf{i}_s \cdot \delta(p,n)\rangle][p_1]p_0 0))))$.

Let $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_L (a,b)_L = (c,d)_L$. By the definition, $(0, \{a,a\}_L) \in (a,b)_L$ and thus it follows that $\forall g \in D_e[g_0 \cdot 0 \Vdash_L \{a,a\}_L \in (c,d)_L]$. Hence by (5.11), Lemmas 4.1.8, 4.1.9 and 4.4.2,

$$\alpha(e) \Vdash_L a = c, \tag{5.15}$$

where $\alpha(e) \equiv \chi_{x=y}(\Phi(\mathfrak{un}(\Phi(e, \Lambda g.g_0 \cdot 0)), \Lambda r.\xi(r)))$.

Furthermore, since $(1, \{c,d\}_L) \in (c,d)_L$, by the definition it follows that $\forall g \in D_e[g_1 \cdot 1 \Vdash_L \{c,d\}_L \in (a,b)_L]$, i.e.,

$$\forall g \in D_e \forall r \in D_{g_0 \cdot 1} \exists u \in V^*[(r_0, u) \in (a,b)_L \wedge r_1 \Vdash_L \{c,d\}_L = u].$$

By the definitions it follows that $\forall g \in D_e \forall r \in D_{g_0 \cdot 1}$. If $r_0 = 0$, then $r_1 \Vdash_L \{c, d\}_L = \{a, a\}_L$ and thus by the assumption and (5.14) one can find a realizer θ to realize $b = d$; if $r_0 = 1$, then $r_1 \Vdash_L \{c, d\}_L = \{a, b\}_L$ and thus by (5.15) and (5.14), one can find a realizer π to realize $b = d$. By applying the disjunctive realizer $\mathbf{d}$ it follows that $\forall q \in D_e \forall r \in D_{g_0 \cdot 1} \mathbf{d}[\theta][\pi] r_0 0 \Vdash_L b = d$, i.e., by Lemmas 4.1.8, 4.1.9 and 4.4.2,

$$\rho(e) \Vdash_L b = d, \tag{5.16}$$

where $\rho(e) \equiv \chi_{x=y}(\Phi(\mathfrak{un}(\Phi(e, \Lambda g. g_0 \cdot 1)), \Lambda r. \mathbf{d}[\theta][\pi] r_0 0))$. Then the result follows from (5.15) and (5.16). ∎

Lemma 5.4.5 *[Union]* $r_{\text{L}}^{uni} \Vdash_L \forall x \exists u [S(u) \wedge \forall z (z \in u \leftrightarrow \exists y (y \in x \wedge z \in y))]$, *where* $r_{\text{L}}^{uni} \equiv \mathfrak{sg}(\mathfrak{sg}\langle 0, \mathfrak{sg}\langle \Lambda g. \Phi(g, \Lambda e. \xi(e)), \Lambda l. \sigma(l) \rangle\rangle\rangle)$.

Proof. Let $a \in V^*$ be arbitrary. Define

$$\underline{a} \equiv \{(\langle n, m \rangle, b) : \exists c \in V^* [(n, c) \in a \wedge (m, b) \in c]\}.$$

Then by Separation, the Union and Pairing Axioms in the background theory, one has $\underline{a}$ is an external set. By Claim 3.3.6, $\underline{a}$ is also an internal set, i.e., $\underline{a} \in \mathbb{S}^*$. Now we want to find a realizer which realizes $\forall z (z \in \underline{a} \leftrightarrow \exists y (y \in a \wedge z \in y))$. Let $c \in V^*$ and $g \in \mathbb{N}$ be arbitrary such that $g \Vdash_L c \in \underline{a}$, i.e.,

$$\forall e \in D_g \exists d \in V^* [(e_0, d) \in \underline{a} \wedge e_1 \Vdash_L c = d].$$

By the definition, it follows that $\exists k \in V^* [(e_{00}, k) \in a \wedge (e_{01}, d) \in k]$. Then by the soundness of **IA1** and **IA4**,

$$\forall e \in D_g \exists k \in V^* \xi(e) \Vdash_L k \in a \wedge c \in k,$$

where $\xi(e) \equiv \langle \mathfrak{sg}\langle e_{00}, \mathbf{i}_r \rangle, \mathbf{i}_0 \cdot \langle e_1, \mathfrak{sg}\langle e_{01}, \mathbf{i}_r \rangle\rangle\rangle$. Thus by Lemma 4.1.8 and the definition,

$$\Phi(g, \Lambda e. \xi(e)) \Vdash_L \exists y (y \in a \wedge c \in y). \tag{5.17}$$

On the other hand, let $l \in \mathbb{N}$ be arbitrary such that

$$l \Vdash_L \exists y (y \in a \wedge c \in y),$$

i.e., $\forall t \in D_l \exists u \in V^* [t \Vdash_L u \in a \wedge c \in u]$. From $t_0 \Vdash_L u \in a$,

$$\forall m \in D_{t_0} \exists h \in V^* (m_0, h) \in a \wedge m_1 \Vdash_L u = h. \tag{5.18}$$

From $t_1 \Vdash_L c \in u$ and (5.18) and the soundness of **IA5**, it then follows that $\forall t \in D_l \forall m \in D_{t_0} \exists h \in V^*[(m_0, h) \in a \wedge \mathbf{i}_1 \cdot \langle m_1, t_1 \rangle \Vdash_L c \in h]$, i.e., by the definition,

$$\forall t \in D_l \forall m \in D_{t_0} \forall q \in |\mathbf{i}_1 \cdot \langle m_1, t_1 \rangle| \\ \exists p \in V^*[(\langle m_0, q_0 \rangle, p) \in \underline{a} \wedge q_1 \Vdash_L c = p].$$

By Lemma 4.1.8 and 4.1.9,

$$\{\langle \langle m_0, q_0 \rangle, q_1 \rangle : q \in |\mathbf{i}_1 \cdot \langle m_1, t_1 \rangle|\} = |\rho(m,t)|, \\ \cup\{|\rho(m,t)| : m \in D_{t_0}\} = |\mathfrak{un}(\Phi(t_0, \Lambda m.\rho(m,t)))|, \\ \cup\{|\mathfrak{un}(\Phi(t_0, \Lambda m.\rho(m,t)))| : t \in D_l\} = |\sigma(l)|,$$

where $\rho(m,t) \equiv \Phi(\mathbf{i}_1 \cdot \langle m_1, t_1 \rangle, \Lambda q.\langle \langle m_0, q_0 \rangle, q_1 \rangle)$, and
where $\sigma(l) \equiv \mathfrak{un}(\Phi(l, \Lambda t.\mathfrak{un}(\Phi(t_0, \Lambda m.\rho(m,t)))))$, and thus by Lemma 4.1.9

$$\{\langle \langle u, q_0 \rangle, q_1 \rangle : \exists t \in D_l \exists m \in D_{t_0}[u = m_0 \wedge q \in |\mathbf{i}_1 \cdot \langle m_1, t_1 \rangle|]\} = |\sigma(l)|.$$

By the definition it follows that

$$\sigma(l) \Vdash_L c \in \underline{a}. \tag{5.19}$$

Then the result follows from (5.17) and (5.19). ■

Lemma 5.4.6 [*Separation*] $r_{\text{L}}^{Sep} \Vdash_L \forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow z \in x \wedge A(z))]$, *where* $r_{\text{L}}^{Sep} \equiv$

$$\mathfrak{sg}(\mathfrak{sg}\langle 0, \mathfrak{sg}\langle \Lambda g.\chi_B(\Phi(g, \Lambda e.\xi(e))), \Lambda h.\Phi(h_0, \Lambda l.\langle \langle \mathbf{i}_A \cdot \langle l_1, h_1 \rangle, l_0 \rangle, l_1 \rangle)\rangle\rangle).$$

Proof. Let $a \in V^*$ be arbitrary. Define

$$\underline{a} \equiv \{(\langle f, g \rangle, k) : f, g \in \mathbb{N} \wedge [(g,k) \in a \wedge f \Vdash_L A(k)]\}.$$

Then by Separation, the Union and Pairing Axioms in the background theory, $\underline{a}$ is an external set. By Claim 3.3.6 $\underline{a}$ is also an internal set, i.e., $\underline{a} \in \mathbb{S}^*$. Now we want to find a realizer which realizes $\forall z(z \in \underline{a} \leftrightarrow z \in a \wedge A(z))$.

Let $c \in V^*$ and $g \in \mathbb{N}$ be arbitrary such that $g \Vdash_L c \in \underline{a}$, i.e., $\forall e \in D_g \exists d \in V^*[(e_0, d) \in \underline{a} \wedge e_1 \Vdash_L c = d]$. By the definition, it follows that $(e_{01}, d) \in a \wedge e_{00} \Vdash_L A(d)$ and thus $\mathfrak{sg}\langle e_{01}, \mathbf{i}_r \rangle \Vdash_L d \in a$. Then by the substitution realizer $\mathbf{i}_A$, the soundness of **IA1** and **IA2**, it follows that

$$\forall e \in D_g[\xi(e) \Vdash_L c \in a \wedge A(c)], \tag{5.20}$$

where $\xi(e) \equiv \langle \mathbf{i}_0 \cdot \langle e_1, \mathfrak{sg}\langle e_{01}, \mathbf{i}_r \rangle\rangle, \mathbf{i}_A(\langle \mathbf{i}_s \cdot e_1, e_{00}\rangle)\rangle$. By Lemmas 4.1.8 and 4.4.2,

$$\chi_B(\Phi(g, \Lambda e.\xi(e))) \Vdash_L c \in a \wedge A(c), \tag{5.21}$$

where $B \equiv z \in x \wedge A(z)$.

On the other hand, let $h \in \mathbb{N}$ be arbitrary such that $h \Vdash_L c \in a \wedge A(c)$. Then by the definition and the substitution realizer $\mathbf{i}_A$, it follows that

$$\forall l \in D_{h_0} \exists d \in V^*[(l_0, d) \in a \wedge l_1 \Vdash_L c = d \wedge \mathbf{i}_A \cdot \langle l_1, h_1 \rangle \Vdash_L A(d)],$$

i.e., by the definition $\forall l \in D_{h_0} \exists d \in V^*(\langle \mathbf{i}_A \cdot \langle l_1, h_1\rangle, l_0\rangle, d) \in \underline{a}$ and thus by Lemma 4.1.8

$$\Phi(h_0, \Lambda l.\langle\langle \mathbf{i}_A \cdot \langle l_1, h_1\rangle, l_0\rangle, l_1\rangle) \Vdash_L c \in \underline{a}. \tag{5.22}$$

Then the result follows from (5.21) and (5.22). ■

Claim 5.4.7 *If $a, b \in \mathbb{S}^*$, then $e \Vdash_L b \subseteq a \to \exists b^* \in V^{\mathbb{N}}_{rk(a)+1}[\delta(e) \Vdash_L b = b^*]$, where*

$$\delta(e) \equiv \mathfrak{sg}\langle \Lambda f.\chi_A(\sigma(e, f)), \Lambda g.\mathbf{i}_0 \cdot \langle \mathbf{i}_s \cdot g_{11}, \mathfrak{sg}\langle g_0, \mathbf{i}_r\rangle\rangle\rangle.$$

Proof. Assume $e \Vdash_L b \subseteq a$. By the definition it follows that $\forall q \in D_e \forall (f, d) \in b[q \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle \Vdash_L d \in a]$, i.e., $\forall (f, d) \in b \forall q \in D_e$

$$\forall m \in |q \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle| \exists c \in V^*[(m_0, c) \in a \wedge m_1 \Vdash_L d = c]. \tag{5.23}$$

Define

$$b^* \equiv \{(\langle f, m\rangle, c) : \exists d \in V^*[(f, d) \in b \wedge (m_0, c) \in a \wedge m_1 \Vdash_L d = c]\}.$$

By Pairing and Separation in the background theory, b^* is a set. By Claim 3.3.6 and Corollary 3.3.5 , $b^* \in V^{\mathbb{N}}_{rk(a)+1}$. Now we want to find a realizer which realizes $b = b^*$. Let $f \in \mathbb{N}$ and $d \in V^*$ be arbitrary such that $(f, d) \in b$. Then by (5.23) and soundness of **IA1**, it follows that

$$\forall q \in D_e \forall m \in |q \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle| \exists c \in V^* \mathfrak{sg}\langle\langle f, m\rangle, \mathbf{i}_r\rangle \Vdash_L c \in b^*, m_1 \Vdash_L d = c,$$

i.e., by the soundness of **IA4**,

$$\forall q \in D_e \forall m \in |q \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle|[\xi(m, f) \Vdash_L d \in b^*], \tag{5.24}$$

where $\xi(m, f) \equiv \mathbf{i}_0 \cdot \langle m_1, \mathfrak{sg}\langle\langle f, m\rangle, \mathbf{i}_r\rangle\rangle$. By Lemma 4.1.8 and 4.1.9,

$$\{\xi(m, f) : \exists q \in D_e(m \in |q \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle|)\} = |\sigma(e, f)|,$$

where $\sigma(e,f) = \Phi(\mathfrak{un}(\Phi(e, \Lambda q.(q \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle))), \Lambda m.\xi(m,f))$. Thus by Lemma 4.4.2 and (5.24), it follows that

$$\chi_A(\sigma(e,f)) \Vdash_L d \in b^*, \tag{5.25}$$

where $A \equiv x \in y$. On the other hand, let $g \in \mathbb{N}$ and $k \in V^*$ be arbitrary such that $(g,k) \in b^*$. Then by the definition and the soundness of **IA1** and **IA2** , it follows that

$$\exists d \in V^*[\mathfrak{sg}\langle g_0, \mathbf{i}_r\rangle \Vdash_L d \in b \wedge \mathbf{i}_s \cdot g_{11} \Vdash_L k = d],$$

i.e., by the soundness of **IA4**

$$\mathbf{i}_0 \cdot \langle \mathbf{i}_s \cdot g_{11}, \mathfrak{sg}\langle g_0, \mathbf{i}_r\rangle\rangle \Vdash_L k \in b. \tag{5.26}$$

From (5.25) and (5.26), $\delta(e) \Vdash_L b = b^*$, where

$$\delta(e) \equiv \mathfrak{sg}\langle \Lambda f.\chi_A(\sigma(e,f)), \Lambda g.\mathbf{i}_0 \cdot \langle \mathbf{i}_s \cdot g_{11}, \mathfrak{sg}\langle g_0, \mathbf{i}_r\rangle\rangle\rangle.$$

■

Lemma 5.4.8 *[Power Set]* $r^{pw}_{\mathrm{L}} \Vdash_L \forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow (S(z) \wedge z \subseteq x))]$, *where* $r^{pw} \equiv \mathfrak{sg}(\mathfrak{sg}\langle 0, \mathfrak{sg}\langle \Lambda g.\varepsilon(g), \Lambda h.\mathbf{i}_0 \cdot \langle \delta(h_1), \xi(h)\rangle\rangle\rangle)$.

Proof. Let $a \in V^*$ be arbitrary. Define

$$\underline{a} \equiv \{(g,c) : g \in \mathbb{N} \wedge c \in V^{\mathbb{N}}_{rk(a)+1} \wedge g \Vdash_L c \subseteq a\}.$$

By the Powerset axiom, Pairing and Separation in the background theory, $\underline{a}$ is an external set. By Claim 3.3.6 it also follows that $\underline{a} \in \mathbb{S}^*$. Now we want to find a realizer which realizes $\forall z(z \in \underline{a} \leftrightarrow (S(z) \wedge z \subseteq a))$. Let $k \in V^*$ be arbitrary. Let $g \in \mathbb{N}$ be arbitrary such that $g \Vdash_L k \in \underline{a}$, i.e.,

$$\forall e \in D_g \exists c \in V^{\mathbb{N}}_{rk(a)+1}[e_0 \Vdash_L c \subseteq a \wedge e_1 \Vdash_L k = c].$$

Then by Corollary 3.3.2 and the soundness of **IA2**, it follows that $k \in \mathbb{S}^*$ and thus

$$\forall e \in D_g[\langle 0, \mathbf{i}_A \cdot \langle \mathbf{i}_s \cdot e_1, e_0\rangle\rangle \Vdash_L S(k) \wedge k \subseteq a],$$

where $A \equiv x \subseteq y$. By Lemma 4.1.8 and 4.4.2 one then has

$$\varepsilon(g) \Vdash_L S(k) \wedge k \subseteq a, \tag{5.27}$$

where $\varepsilon(g) \equiv \chi_B(\Phi(g, \Lambda e.\langle 0, \mathbf{i}_A \cdot \langle \mathbf{i}_s \cdot e_1, e_0\rangle\rangle))$ and where $B \equiv S(y) \wedge y \subseteq x$.

Now let $h \in \mathbb{N}$ be arbitrary such that $h \Vdash_L S(k) \wedge k \subseteq a$. Then by Claim 5.4.7, it follows that $\exists k^* \in V^{\mathbb{N}}_{rk(a)+1}$ such that $\delta(h_1) \Vdash_L k = k^*$, where δ is defined in Claim 5.4.7, and thus by Substitution $\mathbf{i}_A \cdot \langle \delta(h_1), h_1 \rangle \Vdash_L k^* \subseteq a$. By the definition of $\underline{a}$, Substitution and the soundness of **IA4**, it then follows that

$$\mathbf{i}_0 \cdot \langle \delta(h_1), \xi(h) \rangle \Vdash_L k \in \underline{a}, \tag{5.28}$$

where $\xi(h) \equiv \mathfrak{sg}\langle \mathbf{i}_A \cdot \langle \delta(h_1), h_1 \rangle, \mathbf{i}_r \rangle$. From (5.27) and (5.28), the result follows. ■

Lemma 5.4.9 [*Infinity*] $r^{\inf}_{\mathrm{L}} \Vdash_L \exists u(S(u) \wedge \forall z[z \in u \leftrightarrow N(z)])$, *where* $r^{\inf}_{\mathrm{L}} \equiv \mathfrak{sg}\langle 0, \mathfrak{sg}\langle \Lambda g.\varnothing(\Phi(g, \Lambda e.e_0)), \Lambda h.\mathfrak{sg}\langle h, \mathbf{i}_r \rangle\rangle\rangle$.

Proof. Define

$$\underline{u} = \{(n, n) : n \in \mathbb{N}\}.$$

Then by the Infinity Axiom and Separation in the background theory, $\underline{u}$ is an external set. Moreover, by the definition, $\underline{u}$ is also an internal set, i.e., $\underline{u} \in \mathbb{S}^*$. Now we want to find a realizer which realizes $\forall z[z \in \underline{u} \leftrightarrow N(z)]$. Let $c \in V^*$ and $g \in \mathbb{N}$ be arbitrary such that $g \Vdash_L c \in \underline{u}$. Then by the definition we have

$$\forall e \in D_g \exists k \in V^*(e_0, k) \in \underline{u} \wedge e_1 \Vdash_L c = k.$$

By the definition, it follows that $\forall e \in D_g[(e_0 = c) \wedge e_0 \Vdash_L N(c)]$. By Lemmas 4.1.7 and 4.1.8 it then follows that $\varnothing(\Phi(g, \Lambda e.e_0)) \Vdash_L N(c)$. On the other hand, assume $h \Vdash_L N(c)$. By the definition it follows that $h = c$, i.e., $(h, c) \in \underline{u}$. Hence by the soundness of **IA1**, it follows that

$$\mathfrak{sg}\langle h, \mathbf{i}_r \rangle \Vdash_L c \in \underline{u}.$$

Hence the result follows from the definitions. ■

Lemma 5.4.10 [*Induction*] $r^{Ind} \Vdash_L \forall x[(\forall y \in xA(y)) \to A(x)] \to \forall xA(x)$, *where* $r^{Ind} \equiv \Lambda g.\Omega(\Lambda k.\chi_A(\Phi(g, \Lambda e.e \cdot \mathfrak{sg}(\Lambda h.\xi(h, k)))))$.

Proof. Let $g \in \mathbb{N}$ be arbitrary such that

$$g \Vdash_L \forall x[\forall y(y \in x \to A(y)) \to A(x)],$$

i.e.,

$$\forall e \in D_g \forall u \in V^*[e \Vdash_L \forall y(y \in u \to A(y)) \to A(u)]. \tag{5.29}$$

Now we want to find a realizer which realizes $\forall x A(x)$. We show this by ordinal induction. Let $\alpha \in On$ be arbitrary. Assume

$$\forall \beta \in \alpha \forall b \in \mathbb{N} \cup V_\beta^{\mathbb{N}}(k \Vdash_L A(b)). \tag{5.30}$$

Now one has to find the explicit form of k via the fixed point theorem such that $\forall a \in \mathbb{N} \cup V_\alpha^{\mathbb{N}}(k \Vdash_L A(a))$. Let $a \in V_\alpha^{\mathbb{N}}$ be arbitrary. We want to find a realizer which realizes $\forall y(y \in a \to A(y))$. Let $c \in V^*$ and $h \in \mathbb{N}$ be arbitrary such that $h \Vdash_L c \in a$. Then by the definition it follows that $\forall t \in D_h \exists d \in V^*(t_0, d) \in a \wedge t_1 \Vdash_L c = d$. Since $d \in \mathbb{N} \cup V_\beta^{\mathbb{N}}$ for some $\beta \in \alpha$, by the inductive hypothesis (5.30) $k \Vdash_L A(d)$. Thus by Substitution, Lemma 4.1.8, Lemma 4.4.2 and the Soundness of **IA2**,

$$\xi(h, k) \Vdash_L A(c),$$

where $\xi(h, k) \equiv \chi_A(\Phi(h, \Lambda t.\mathbf{i}_A\langle \mathbf{i}_s \cdot t_1, k\rangle))$. By the definition, it follows that

$$\mathfrak{sg}(\Lambda h.\xi(h, k)) \Vdash_L \forall y[y \in a \to A(y)]. \tag{5.31}$$

By (5.29) and (5.31)

$$\forall e \in D_g[e \cdot \mathfrak{sg}(\Lambda h.\xi(h, k)) \Vdash_L A(a)].$$

By Lemmas 4.1.8 and Lemma 4.4.2 , it follows that

$$\chi_A(\Phi(g, \Lambda e.e \cdot \mathfrak{sg}(\Lambda h.\xi(h, k)))) \Vdash_L A(a).$$

Let Ω be a fixed point generator. Then one has the explicit form of k: $k \equiv \Omega(\Lambda k.\chi_A(\Phi(g, \Lambda e.e \cdot \mathfrak{sg}(\Lambda h.\xi(h, k)))))$ and this completes the proof. ∎

Lemma 5.4.11 [*Collection*] $r^{co}_{\text{L}} \Vdash_L \forall x[\forall y \in x \exists z A(y, z) \to \exists u(S(u) \wedge \forall y \in x \exists z \in u A(y, z))]$, *where* $r^{co} \equiv \mathfrak{sg}(\Lambda e.\delta(e))$.

Proof. We show this via Collection in the background theory. Let $a \in V^*$ be arbitrary. Let $e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_L \forall y \in a \exists z A(y, z). \tag{5.32}$$

By the definitions and soundness of **IA1**,

$$\forall v \in D_e \forall (f, d) \in a \forall q \in |v \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle| \exists c \eta(q, c, d), \tag{5.33}$$

where $\eta(q, c, d) \equiv c \in V^* \wedge q \Vdash_L A(d, c)$. By Collection in the background theory, it then follows that

$$\forall v \in D_e \forall (f, d) \in a \exists K[S(K) \wedge \forall q \in |v \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle| \exists k \in K \eta(q, k, d)].$$

By Collection again, it follows that

$$\forall v \in D_e \exists \mathbb{K}[S(\mathbb{K}) \wedge \forall (f,d) \in a \exists K \in \mathbb{K} \\ (S(K) \wedge \forall q \in |v \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle| \exists k \in K \eta(q,k,d))],$$

i.e.,

$$\forall v \in D_e \exists \mathbb{K}[S(\mathbb{K}) \wedge \forall (f,d) \in a \exists K \in \mathbb{K}(S(K) \wedge \\ \forall q \in |v \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle| \exists k \in K(k \in V^* \wedge q \Vdash_L A(d,k))]. \tag{5.34}$$

Now define

$$\ddot{\mathbb{K}} := \{0\} \times ((\cup \mathbb{K}) \cap V^*).$$

By Union, Powerset and Separation in the background theory, $\ddot{\mathbb{K}}$ is a set. By Claim 3.3.6, $\ddot{\mathbb{K}}$ is also an internal set, i.e., $\ddot{\mathbb{K}} \in \mathbb{S}^*$. From (5.34), it then follows that

$$\forall v \in D_e \exists \ddot{\mathbb{K}} \in V^*[\forall (f,d) \in a \forall q \in |v \cdot \mathfrak{sg}\langle f, \mathbf{i}_r\rangle| \\ \exists k \in V^*((0,k) \in \ddot{\mathbb{K}} \wedge (q \Vdash_L A(d,k)))]. \tag{5.35}$$

Now we want to find a realizer which realizes $\exists u(S(u) \wedge \forall y \in a \exists z \in uA(y,z))$. Let $l \in \mathbb{N}$ and $b \in V^*$ be arbitrary such that $l \Vdash_L b \in a$, i.e.,

$$\forall t \in D_l \exists c \in V^*[(t_0, c) \in a \wedge t_1 \Vdash_L b = c].$$

By (5.35), it follows that for all v in D_e, there is $\ddot{\mathbb{K}}$ in V^* such that

$$\forall t \in D_l \exists c \in V^* \forall q \in |v \cdot \mathfrak{sg}\langle t_0, \mathbf{i}_r\rangle| \exists k \in V^*((0,k) \in \ddot{\mathbb{K}} \wedge (q \Vdash_L A(c,k))).$$

By the definition and Lemma 4.1.8, it follows that

$$\forall v \in D_e \exists \ddot{\mathbb{K}} \in V^* \forall t \in D_l \exists c \in V^*[\sigma(v,t) \Vdash_L \exists z(z \in \ddot{\mathbb{K}} \wedge A(c,z))],$$

where $\sigma(v,t) = \Phi(|v.\mathfrak{sg}\langle t_0, \mathbf{i}_r\rangle|, \Lambda q.\langle \mathfrak{sg}\langle 0, \mathbf{i}_r\rangle, q\rangle)$.

By Substitution and the soundness of **IA2**, it follows that

$$\forall v \in D_e \exists \ddot{\mathbb{K}} \in V^* \forall t \in D_l[\mathbf{i}_B \cdot \langle \mathbf{i}_s \cdot t_1, \sigma(v,t)\rangle \Vdash_L \exists z(z \in \ddot{\mathbb{K}} \wedge A(b,z))],$$

where $B \equiv \exists z(z \in x \wedge A(y,z))$. Thus by Lemmas 4.1.8 and 4.4.2, it follows that

$$\xi(l,v) \Vdash_L \exists z(z \in \ddot{\mathbb{K}} \wedge A(b,z)),$$

where $\xi(l,v) \equiv \chi_B(\Phi(l, \Lambda t.\mathbf{i}_B \cdot \langle \mathbf{i}_s \cdot t_1, \sigma(v,t)\rangle))$. Hence by the definition

$$\forall v \in D_e \exists \ddot{\mathbb{K}} \in V^* \mathfrak{sg}(\Lambda l.\xi(l,v)) \Vdash_L \forall y \in a \exists z \in \ddot{\mathbb{K}} A(y,z).$$

By Lemma 4.1.8 again, it follows that

$$\delta(e) \Vdash_L \exists u(S(u) \wedge \forall y \in a \exists z \in u A(y, z)),$$

where $\delta(e) = \Phi(e, \Lambda v.\langle 0, \mathfrak{sg}(\Lambda l.\xi(l, v))\rangle)$. ■

Theorem 5.4.12 [*Soundness*] *If* $\mathbf{IZF}_N \vdash \theta$*, then* $\mathbf{IZF}'_N \vdash V^* \models_L \theta$*.*

Proof. Since the logical axioms and non-logical axioms and derivation rules are all realizable, the result follows immediately. ■

5.5 A5: (Semi-) Constructive axioms

Some derivations might be simplified if one applies the following claim. Recall that $\exists! y(N(y) \wedge A(y)) \equiv \exists y(N(y) \wedge A(y)) \wedge \forall x \forall z[A(x) \wedge A(z) \rightarrow x = z]$. Other notations used here are defined in Subsection 2.2.7.

Claim 5.5.1 $[e \Vdash_L \exists! y(N(y) \wedge A(y))]$
$\rightarrow \chi_A(\Phi(e_0, \Lambda k.k_1)) \Vdash_L A(\varnothing(\Phi(e_0, \Lambda k.k_0)))$.

Proof. By the definition

$$\forall q \in D_{e_0}[q_1 \Vdash_L A(q_0)], \tag{5.36}$$

and

$$e_1 \Vdash_L \forall x \forall z[A(x) \wedge A(z) \rightarrow x = z]. \tag{5.37}$$

Now let $q, q' \in D_{e_0}$ be arbitrary. From (5.36), $q_1 \Vdash_L A(q_0)$ and $q'_1 \Vdash_L A(q'_0)$. Then by (5.37), it follows that $q_0 = q'_0$, i.e., the set $\{q_0 : q \in D_{e_0}\}$ has exactly one element and thus by Lemmas 4.1.8 and 4.1.7

$$\forall q \in D_{e_0}[q_1 \Vdash_L A(\varnothing(\Phi(e_0, \Lambda k.k_0)))].$$

Then by Lemmas 4.1.8 and 4.4.2, we have

$$\chi_A(\Phi(e_0, \Lambda k.k_1)) \Vdash_L A(\varnothing(\Phi(e_0, \Lambda k.k_0))).$$

■

Lemma 5.5.2 [$\mathbf{CT_0}!$]

$$r_{\mathrm{L}}^{ct_0!} \Vdash_L \forall n \exists! m A(n, m) \rightarrow \exists l \forall n \exists p \exists q(T(l, n, p) \wedge U(p, q) \wedge A(n, q)),$$

where the explicit form of $r_{\mathrm{L}}^{ct_0!}$ *will depend on the explicit forms of the predicates* T *and* U*.*

Proof. Let $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_L \forall n \exists! m A(n,m)$. By Lemma 5.0.18

$$\forall n \in \mathbb{N}[\psi_1(e) \cdot n \Vdash_L \exists! m A(n,m)].$$

Then by Claim 5.5.1 it follows that for all n in $\mathbb{N}$

$$\delta(e,n) \Vdash_L A(n, \varnothing(\Phi((\psi_1(e) \cdot n)_0, \Lambda k.k_0))),$$

where $\delta(e,n) \equiv \chi_A(\Phi((\psi_1(e) \cdot n)_0, \Lambda k.k_1))$. Now one defines a total recursive function $f_e(n) := \varnothing(\Phi((\psi_1(e) \cdot n)_0, \Lambda k.k_0))$, which can be formalized via the predicates T and U. Then from Lemma 5.0.18, the definitions and explicit forms of predicates T and U, one can find an explicit realizer $r_{\mathrm{L}}^{ct_0!}$ to realize $\mathbf{CT_0}!$. ■

Lemma 5.5.3 $\mathbf{AC}^{NN}!$ *is realizable.*

Proof. $\mathbf{AC}^{NN}!$ is a theorem of $\mathbf{IZF}_N$. ■

Lemma 5.5.4 [**UP**]

$$r_{\mathrm{L}}^{up} \Vdash_L \forall x[S(x) \to \exists n A(x,n)] \to \exists n \forall x[S(x) \to A(x,n)],$$

where $r_{\mathrm{L}}^{up} \equiv \Lambda e.\Phi(\mathfrak{un}(\Phi(e, \Lambda q.q \cdot 0)), \Lambda r.\langle r_0, \mathfrak{sg}(\Lambda n.r_1)\rangle)$.

Proof. Let $e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_L \forall x[S(x) \to \exists n A(x,n)].$$

By Lemma 5.0.18, it follows that $\forall q \in D_e \forall r \in D_{q \cdot 0} \forall a \in \mathbb{S}^*[r_1 \Vdash_L A(a, r_0)]$, i.e.,

$$\forall q \in D_e \forall r \in D_{q \cdot 0}[\mathfrak{sg}(\Lambda n.r_1) \Vdash_L \forall x(S(x) \to A(x, r_0))]. \tag{5.38}$$

By Lemma 4.1.8 and 4.1.9,

$$\begin{aligned}
&\{\langle r_0, \mathfrak{sg}(\Lambda n.r_1)\rangle : \exists q \in D_e(r \in D_{q \cdot 0})\} \\
&= \{\langle r_0, \mathfrak{sg}(\Lambda n.r_1)\rangle : r \in |\mathfrak{un}(\Phi(e, \Lambda q.q \cdot 0))|\} \\
&= |\delta(e)|,
\end{aligned}$$

where $\delta(e) \equiv \Phi(\mathfrak{un}(\Phi(e, \Lambda q.q \cdot 0)), \Lambda r.\langle r_0, \mathfrak{sg}(\Lambda n.r_1)\rangle)$. Hence by Lemma 5.0.18 and (5.38), it follows that

$$\delta(e) \Vdash_L \exists n \forall x[S(x) \to A(x,n)].$$

■

5.6 Independence results

Claim 5.6.1 $\mathbf{IZF}_N \nvdash \mathbf{CT}_0^{ab}$*, where a and b are the codes for any partial recursive functions whose domains are disjoint, recursively inseparable.*

Proof. By the fact that $\mathbf{CT}_0^{ab}$ is not realizable in the language of arithmetic and by Thorem 4.4.3 and the Soundness Theorem 5.4.12, the result follows immediately. ■

Corollary 5.6.2 $\mathbf{IZF}_N \nvdash \mathbf{AC}^{N2}$.

Proof. Since $\mathbf{CT}_0!$ is realized, by Claim 5.6.1, it suffices to prove that $\mathbf{AC}^{N2}$ with $\mathbf{CT}_0!$ proves $\mathbf{CT}_0^{ab}$. Assume

$$\forall n \exists m[(m = 0 \to \{a\}(n)\uparrow) \wedge (m \neq 0 \to \{b\}(n)\uparrow)].$$

Now define $B(n) := \{0 : \{a\}(n)\uparrow\} \cup \{1 : \{b\}(n)\uparrow\}$. Then

$$\forall n \exists m \in \{0,1\}(m \in B(n)),$$

where $m \in B(n) \equiv \exists z \exists x \exists y[x = \{0 : \{a\}(n)\uparrow\} \wedge y = \{1 : \{b\}(n)\uparrow\} \wedge z = x \cup y \wedge (m \in x \to \{a\}(n)\uparrow) \wedge (m \in y \to \{b\}(n)\uparrow)]$. By $\mathbf{AC}^{N2}$, it follows that there exists a function $f : \mathbb{N} \to \{0,1\}$ such that

$$\forall n \in \mathbb{N}[(f(n) = 0 \to \{a\}(n)\uparrow) \wedge (f(n) \neq 0 \to \{b\}(n)\uparrow)].$$

By $\mathbf{CT}_0!$, the result follows immediately. ■

Claim 5.6.3 $\mathbf{AC}^{NN} \to \mathbf{AC}^{N2}$.

Proof. $\mathbf{AC}^{N2}$ is an instance of $\mathbf{AC}^{NN}$. ■

Corollary 5.6.4 $(\mathbf{IZF}'_N)$ $\mathbf{IZF}_N \nvdash \mathbf{AC}^{NN}$.

Proof. This follows immediately from the above claim. ■

Theorem 5.6.5 [*Independence Result*] $\mathbf{IZF}_N \nvdash \mathbf{ECT}_0 \vee \mathbf{CT}_0^{ab} \vee \mathbf{AC}^{N2} \vee \mathbf{DC} \vee \mathbf{RDC} \vee \mathbf{PA}_X$.

Proof.
♢ For $\mathbf{CT}_0^{ab}$, this follows immediately from Claim 5.6.1.
♢ Now we want to show that $\mathbf{ECT}_0$ is independent of $\mathbf{IZF}_N$. Since $\mathbf{ECT}_0$ proves $\mathbf{CT}_0$ and $\mathbf{CT}_0^{ab}$ is an instance of $\mathbf{CT}_0$, the result follows immediately.

♢ Now we want to show that **DC** is independent of $\mathbf{IZF}_N$. By Claim 5.6.3, it suffices to prove that $\mathbf{DC} \to \mathbf{AC}^{NN}$. To begin with, let us show that $\mathbf{AC}^N \to \mathbf{AC}^{NN}$. Assume $\forall n \exists m A(n,m)$. Define $f(n) := \{m \in \mathbb{N} : A(n,m)\}$. By $\mathbf{AC}^N$,

$$\exists g[Fun(g, N, N) \wedge \forall n \in \mathbb{N}(g(n) \in f(n))].$$

Next, let us show that $\mathbf{DC} \to \mathbf{AC}^N$. Let f be an arbitrary function with domain $\mathbb{N}$ such that $\forall n \in \mathbb{N} \exists y \in f(n)$. Since $\forall n \in \mathbb{N} \exists ! a[a = f(n)]$, by Replacement, it follows that there is a set b such that $\forall n \in \mathbb{N}[f(n) \in b]$. Now define a set

$$\mathcal{A} := \{(n,m) : n \in \mathbb{N}, m \in \cup b, m \in f(n)\}.$$

Let F^0, F^1 denote the left and right projection functions, respectively. Then

$$\forall x \in \mathcal{A} \exists y \in \mathcal{A}[F^0(y) = F^0(x) + 1 \wedge F^1(x) \in f(F^0(x)) \wedge F^1(y) \in f(F^0(y))]$$

and then by **DC** it follows that

$$\exists h[Fun(h, N, \mathcal{A}) \wedge h(0) = (0,a) \wedge \forall n \in \mathbb{N}(F^0(h(n)) = n)],$$

where $a \in f(0)$. By defining $g(n) := F^1(h(n))$, by mathematical induction, $\forall n \in \mathbb{N}[g(n) \in f(n)]$.
♢ Lastly, we want to show that **RDC** and $\mathbf{PA}_X$ are all independent of $\mathbf{IZF}_N$. Since **RDC** proves **DC** and so does $\mathbf{PA}_X$, the result follows immediately. ■

Theorem 5.6.6 [*Independence Result*] $\mathbf{IZF}_N + \mathbf{UP} + \mathbf{UZ} + \mathbf{CT}_0! \nvdash \mathbf{PEM}$, *where* **PEM** *denotes the Principle of Excluded Middle.*

Proof. Since $\mathbf{HA} + \mathbf{CT_0}! \vdash \neg\forall n[\{n\}(n) \uparrow \vee \neg\{n\}(n) \uparrow]$, it follows that $V^* \models_L \neg\forall n[\{n\}(n) \uparrow \vee \neg\{n\}(n) \uparrow]$. By the fact that $V^* \models_L \mathbf{IZF}_N + \mathbf{UP} + \mathbf{UZ} + \mathbf{CT}_0!$, the result follows immediately. ■

Corollary 5.6.7 [*Independence Result*] $\mathbf{IZF}_N + \mathbf{UP} + \mathbf{UZ} + \mathbf{CT}_0! \nvdash \mathbf{AC}$.

Proof. Since $\mathbf{AC} \to \mathbf{PEM}$, by Theorem 5.6.6, the result follows immediately. ■

Corollary 5.6.8 [*Independence Result*] $\mathbf{IZF}_N + \mathbf{UP} + \mathbf{UZ} + \mathbf{CT}_0! \nvdash \mathbf{FA}$.

Proof. Since $\mathbf{FA} \to \mathbf{PEM}$, the result follows immediately. ■

In conclusion, by our Lifschitz' style interpretation, we have interpreted Heyting arithmetic, $\mathbf{IZF}_N$ and various semi-constructive axioms. We have also differentiated $\mathbf{CT}_0$ from $\mathbf{CT}_0!$ and proved the independence of a plethora of **AC**-related axioms.

Chapter 6

Relativized realizability

In order to derive our conservativity results, we provide a version of relativized realizability in Section 6.1. We assume that we are given a partial recursive function A from $\mathbb{N}$ to $\mathbb{N}$. The main difference between recursive realizability and relativized realizability is that the realizers are codes for partial A-recursive functions (or Turing machines equipped with an oracle which provides $A(n)$ if it exists) rather than codes for partial recursive functions (or ordinary Turing machines).

In Section 6.2 we use this interpretation to interpret Heyting arithmetic, intuitionistic set theories $\mathbf{CZF}_N$ and $\mathbf{IZF}_N$ and various semi-constructive axioms.

In this chapter, $n \cdot m$ is defined to be the value $\{n\}^A(m)$, where we use $\{n\}^A$ to denote the nth partial A-recursive function.

For relativized realizability, the interpretation structure is $(\mathbb{N}, \cdot, V^*, \Vdash_R)$. Here $\cdot$ is the foregoing partial operation and V^* is defined as follows:

$$\begin{aligned} V_\alpha^{\mathbb{N}} &= \bigcup_{\beta \in \alpha} \mathcal{P}(\mathbb{N} \times (V_\beta^{\mathbb{N}} \cup \mathbb{N})) \\ \mathbb{S}^* &= \bigcup_{\alpha \in On} V_\alpha^{\mathbb{N}} \\ V^* &= \mathbb{N} \cup \mathbb{S}^* \end{aligned}$$

where $\mathcal{P}$ denotes the Powerset operation. $\Vdash_R$ is the realizability relation we are going to define next.

6.1 Definition of relativized realizability

To be specific and to facilitate the description we will use the following abbreviations:

- $T^A \equiv$ relativized T-predicate; $U^A \equiv$ relativized extracting predicate.
- $n \cdot m \simeq v \equiv \exists k[N(k) \wedge T^A(n,m,k)] \wedge U^A(\mu z T^A(n,m,z),v)]$.
- $n \cdot m \Vdash_R \varphi \equiv \exists v(n \cdot m \simeq v \wedge v \Vdash_R \varphi)$.
- $n \cdot m \downarrow \equiv \exists v(T^A(n,m,v))$.
- $\forall (f,c) \in a\varphi(f,c) \equiv \forall f \in \mathbb{N} \forall c \in V^*((f,c) \in a \rightarrow \varphi(f,c))$ and $\exists x \in a\varphi(x) \equiv \exists x \in V^*(x \in a \wedge \varphi(x))$.
- $\langle c,d \rangle \equiv \jmath(c,d)$, where $\jmath : \mathbb{N}^2 \rightarrow \mathbb{N}$ is a pairing function (i.e., a bijective relativized total recursive function).
- $e_0 \equiv \jmath_0(e)$, where $\jmath_0 : \mathbb{N} \rightarrow \mathbb{N}$ is a left unpairing function (i.e., a left inverse function of $\jmath$); $e_1 \equiv \jmath_1(e)$, where $\jmath_1 : \mathbb{N} \rightarrow \mathbb{N}$ is a right unpairing function (i.e., a right inverse function of $\jmath$).

We can now proceed to define the realizability relation $n \Vdash_R A$ for $n \in \mathbb{N}$ and formulae A with parameters from V^* by induction on the complexity of A. Below we assume that $a, a_1, a_2, ..., a_n, b, c \in V^*$. We use $e \cdot f$ to stand for $\{e\}^A(f)$.

1. $e \Vdash_R R(a_1, a_2, ..., a_n)$ iff $a_1, a_2, ..., a_n \in \mathbb{N} \wedge R(a_1, a_2, ..., a_n)$,

   whenever R is a symbol for an n-ary R primitive recursive relation of the language.
2. $e \Vdash_R N(a)$ iff $a \in \mathbb{N} \wedge e = a$.
3. $e \Vdash_R S(a)$ iff $a \in \mathbb{S}^*$.
4. $e \Vdash_R a \in b$ iff $\exists c \in V^*[(e_0, c) \in b \wedge e_1 \Vdash_R a = c]$.
5. $e \Vdash_R a = b$ iff $(a, b \in \mathbb{N} \wedge a = b) \vee [a \in \mathbb{S}^* \wedge b \in \mathbb{S}^* \wedge \forall (f,d) \in a(e_0 \cdot f \Vdash_R d \in b) \wedge \forall (g,k) \in b(e_1 \cdot g \Vdash_R k \in a)]$.
6. $e \Vdash_R A \wedge B$ iff $e_0 \Vdash_R A \wedge e_1 \Vdash_R B$.
7. $e \Vdash_R A \vee B$ iff $[e_0 = 0 \wedge e_1 \Vdash_R A] \vee [e_0 \neq 0 \wedge e_1 \Vdash_R B]$.

8. $e \Vdash_R \neg A$ iff $\forall f \in \mathbb{N} \neg (f \Vdash_R A)$.
9. $e \Vdash_R A \to B$ iff $\forall f \in \mathbb{N}(f \Vdash_R A \to e \cdot f \downarrow \wedge e \cdot f \Vdash_R B)$.
10. $e \Vdash_R \forall x A(x)$ iff $\forall c \in V^*(e \Vdash_R A[x/c])$.
11. $e \Vdash_R \exists x A(x)$ iff $\exists c \in V^*(e \Vdash_R A[x/c])$.

Furthermore, one defines $e \Vdash_R A(x)$ iff $\forall a \in V^*[e \Vdash_R A(a)]$ and $V^* \models_R A$ iff $\exists e[e \in \mathbb{N} \wedge e \Vdash_R A]$.

6.2 Soundness of relativized realizability

We use the following abbreviations and notations:

- If $h = \langle e, \langle f, g \rangle \rangle$, we use the notation h_0 to denote e and h_{10} to denote f and h_{11} to denote g.
- $\mathbf{d}abc_1c_2$ (or $\mathbf{d}[a][b]c_1c_2$, if the form of a and b are too lengthy) denotes $\mathbf{d}$ is the Gödel number for the relativized partial recursive function such that whenever the condition $c_1 = c_2$ holds, then the code a is executed; otherwise, the code b is executed.

Notation 6.2.1 $\forall n A(n, \vec{x}) \equiv \forall y[N(y) \to A(y, \vec{x})]$.

Notation 6.2.2 $\exists n A(n, \vec{x}) \equiv \exists y[N(y) \wedge A(y, \vec{x})]$.

Notation 6.2.3 *We will also use $n \in \mathbb{N}$ for $N(n)$.*

Notation 6.2.4 *We will use $\Lambda e.\xi(e)$ to denote the code of the partial recursive function $\lambda e.\xi(e)$.*

Notation 6.2.5 $\forall \vec{a} \in V^* \varphi(\vec{a}) \equiv \forall a_1, a_2, ..., a_n \in V^* \varphi(a_1, a_2, ..., a_n)$.

Lemma 6.2.6 *For each formula $A(u, \vec{x})$ in the language of set theory,* $\mathbf{CZF}_N$ *proves $\forall e \in \mathbb{N} \forall\ \vec{a} \in V^*$*
(i) $e \Vdash_R \forall n A(n, \vec{a}) \leftrightarrow \forall n \in \mathbb{N}[e \cdot n \Vdash_R A(n, \vec{a})]$,
(ii) $e \Vdash_R \exists n A(n, \vec{a}) \leftrightarrow e_1 \Vdash_R A(e_0, \vec{a})$.

Proof. Both follow immediately from the definitions. ■

6.2.1 A1: Axioms on numbers and sets

Claim 6.2.7 [**A1** :1] $0 \Vdash_R \forall x \neg(N(x) \wedge S(x))$.

Proof. Let $c \in V^*$ be arbitrary. Assume $\exists n \in \mathbb{N}$ such that $n \Vdash_R (N(c) \wedge S(c))$. Then by the definition, it follows that $c \in \mathbb{N} \cap \mathbb{S}^*$, but this contradicts Corollary 3.3.3. ■

Claim 6.2.8 [**A1** :12] $\Lambda e.0 \Vdash_R \forall x \forall y[x \in y \to S(y)]$.

Proof. Let $a, b \in V^*$ and $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R a \in b$. Then by the definition $\exists c \in V^*(e_0, c) \in b$, i.e., $b \in \mathbb{S}$. Hence $b \in \mathbb{S}^*$, thus $0 \Vdash_R S(b)$. ■

Claim 6.2.9 [**A1** :3] $n \Vdash_R N(\bar{n})$ *for all natural numbers* n.

Proof. This follows immediately from the fact that $N(\bar{n})$ is an axiom of the background theory. ■

6.2.2 A2: Number-theoretic axioms

Claim 6.2.10 [**A2** :1] $0 \Vdash_R \mathrm{SUC}(\bar{n}, \overline{n+1})$ *for all natural numbers* n.

Proof. This follows immediately from the fact that $\mathrm{SUC}(\bar{n}, \overline{n+1})$ is an axiom of the background theory. ■

Claim 6.2.11 [**A2** :2] $\Lambda n.\langle\langle n+1, 0\rangle, \Lambda e.0\rangle \Vdash_R \forall n \exists! m \mathrm{SUC}(n, m)$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. By **A2** :2 in the background theory, there exists a unique number $n+1 \in \mathbb{N}$ such that $\mathrm{SUC}(n, n+1)$. Then by Lemma 6.2.6 it follows that $\langle n+1, 0\rangle \Vdash_R \exists m \mathrm{SUC}(n, m)$. Thus the result follows from Lemma 6.2.6. ■

Claim 6.2.12 [**A2** :3] $\Lambda n.\Lambda m.\Lambda e.0 \Vdash_R \forall n \forall m(\mathrm{SUC}(n, m) \to m \neq \bar{0})$.

Proof. Let $n, m, e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R \mathrm{SUC}(n, m)$, i.e., $\mathrm{SUC}(n, m)$. Then by **A2** :3 in the background theory, it follows that $m \neq \bar{0}$ and thus the result follows from Lemma 6.2.6. ■

Claim 6.2.13 [**A2** :4] $r^{24} \Vdash_R \forall m(m = \bar{0} \vee \exists n \mathrm{SUC}(n, m))$,
where $r^{24} \equiv \Lambda m.\ \langle m, \mathbf{d}[0][\langle m-1, 0\rangle]m0\rangle$.

Proof. Let $m \in \mathbb{N}$ be arbitrary. Then by **A2** :4 in the background theory $m = \bar{0} \vee \exists n\mathrm{SUC}(n, m)$. For the second case, by **A2** :5 in the background theory, there exists a unique number $m - 1 \in \mathbb{N}$ such that $\mathrm{SUC}(m-1, m)$. Hence by Lemma 6.2.6, it follows that $\langle m-1, 0\rangle \Vdash_R \exists n\mathrm{SUC}(n, m)$. Applying the disjunctive realizer **d** yields the result. ■

Claim 6.2.14 [**A2** :5] $r^{25} \Vdash_R \forall n\forall m\forall k(\mathrm{SUC}(m, n) \wedge \mathrm{SUC}(k, n) \to m = k)$, *where* $r^{25} \equiv \Lambda n.\Lambda m.\Lambda k.\Lambda e.0$

Proof. Let $n, m, k, e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R \mathrm{SUC}(m, n) \wedge \mathrm{SUC}(k, n)$, i.e., $\mathrm{SUC}(m, n) \wedge \mathrm{SUC}(k, n)$. By **A2** :5 in the background theory, $m = k$. ■

Claim 6.2.15 [**A2** :6] $r^{26} \Vdash_R \forall n\forall m\exists! k\mathrm{ADD}(n, m, k)$, *where* $r^{26} \equiv \Lambda n.\Lambda m.\langle\langle n + m, 0\rangle, \Lambda e.0\rangle$.

Proof. Let $n, m \in \mathbb{N}$ be arbitrary. Then by **A2** :6 in the background theory, there is a unique number $n+m \in \mathbb{N}$ such that $\mathrm{ADD}(n, m, n+m)$. Hence by the definition and Lemma 6.2.6 , $\langle\langle n + m, 0\rangle, \Lambda e.0\rangle \Vdash_R \exists! k\mathrm{ADD}(n, m, k)$. ■

Claim 6.2.16 [**A2** :7] $\Lambda n.0 \Vdash_R \forall n\mathrm{ADD}(n, \bar{0}, n)$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. Then by **A2** :7 in the background theory, $\mathrm{ADD}(n, \bar{0}, n)$. By the definition the result follows. ■

Claim 6.2.17 [**A2** :8]

$r^{28} \Vdash_R \forall n\forall k\forall m\forall l\forall i[\mathrm{ADD}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{SUC}(m, i) \to \mathrm{ADD}(n, l, i)]$,

where $r^{28} \equiv \Lambda n.\Lambda k.\Lambda m.\Lambda l.\Lambda i.(\Lambda e.0)$.

Proof. Let $n, k, m, l, i, e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R \mathrm{ADD}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{SUC}(m, i)$, i.e.,

$$\mathrm{ADD}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{SUC}(m, i).$$

Applying **A2** :8 in the background theory, $\mathrm{ADD}(n, l, i)$ and then the result follows from the definition. ■

Claim 6.2.18 [**A2** :9] $r^{29} \Vdash_R \forall n\forall m\exists! k\mathrm{MULT}(n, m, k)$, *where* $r^{29} \equiv \Lambda n.\Lambda m.\langle\langle n \times m, 0\rangle, \Lambda e.0\rangle$.

Proof. Let $n, m \in \mathbb{N}$ be arbitrary. By **A2** :9 in the background theory, there is a unique number $n \times m \in \mathbb{N}$ such that $\mathrm{MULT}(n, m, n \times m)$. By the definition and Lemma 6.2.6, $\langle\langle n \times m, 0\rangle, \Lambda e.0\rangle \Vdash_R \exists! k\mathrm{MULT}(n, m, k)$. ■

Claim 6.2.19 [**A2** :10] $\Lambda n.0 \Vdash_R \forall n \text{MULT}(n, \bar{0}, \bar{0})$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. Then by **A2** :10 in the background theory, $\text{MULT}(n, \bar{0}, \bar{0})$. By the definition the result follows. ■

Claim 6.2.20 [**A2** :11] $r^{211} \Vdash_R$

$\forall n \forall k \forall m \forall l \forall i[\text{MULT}(n, k, m) \wedge \text{SUC}(k, l) \wedge \text{ADD}(m, n, i) \to \text{MULT}(n, l, i)]$,

where $r^{211} \equiv \Lambda n.\Lambda k.\Lambda m.\Lambda l.\Lambda i.(\Lambda e.0)$.

Proof. Let $n, k, m, l, i, e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R \text{MULT}(n, k, m) \wedge \text{SUC}(k, l) \wedge \text{ADD}(m, n, i)$, i.e.,

$$\text{MULT}(n, k, m) \wedge \text{SUC}(k, l) \wedge \text{ADD}(m, n, i).$$

By **A2** :11 in the background theory, $\text{MULT}(n, l, i)$. Then the result follows from the definition. ■

Claim 6.2.21 [**A2** :12]

$$r^{212} \Vdash_R A(\bar{0}) \wedge \forall n \forall m[A(n) \wedge \text{SUC}(n, m) \to A(m)] \to \forall n A(n),$$

where $r^{212} \equiv \Lambda e.f_e^{\#}$.

Proof. Let $e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_R A(\bar{0}) \wedge \forall n \forall m[A(n) \wedge \text{SUC}(n, m) \to A(m)],$$

i.e., $e_0 \Vdash_R A(\bar{0})$ and

$$e_1 \Vdash_R \forall n \forall m[A(n) \wedge \text{SUC}(n, m) \to A(m)].$$

By Lemma 6.2.6, one then has

$$\forall n, m \in \mathbb{N}[(e_1 \cdot n) \cdot m \Vdash_R A(n) \wedge \text{SUC}(n, m) \to A(m)]. \tag{6.1}$$

Now define a recursive function $f_e : \mathbb{N} \to \mathbb{N}$ such that $f_e(0) = e_0$ and $f_e(n+1) = ((e_1 \cdot n) \cdot (n+1)) \cdot \langle f_e(n), 0 \rangle$ and let $f_e^{\#}$ be its Gödel number. Then the result follows. ■

6.2.3 A3: Logical axioms for IPL

We will show only nontrivial ones and write down the realizers for the trivial ones.

For logical axioms (LA):

(IPL1) $\Lambda e.\Lambda d.e \Vdash_R A \to (B \to A)$.

(IPL2) $\Lambda e.\Lambda d.\Lambda k.((e{\cdot}k){\cdot}(d{\cdot}k)) \Vdash_R [A \to (B \to C)] \to [(A \to B) \to (A \to C)]$.

(IPL3) $\Lambda e.\Lambda d.\langle e, d\rangle \Vdash_R A \to (B \to A \wedge B)$.

(IPL4) $\Lambda e.e_0 \Vdash_R A \wedge B \to A$.

(IPL5) $\Lambda e.e_1 \Vdash_R A \wedge B \to B$.

(IPL6) $\Lambda e.\langle 0, e\rangle \Vdash_R A \to A \vee B$.

(IPL7) $\Lambda e.\langle 1, e\rangle \Vdash_R B \to A \vee B$.

(IPL8) $r^8 \Vdash_R (A \vee B) \to [(A \to C) \to ((B \to C) \to C)]$,
where $r^8 \equiv \Lambda e.\Lambda s.\Lambda t.\mathbf{d}[s \cdot e_1][t \cdot e_1]e_0 0$.

Proof. Let $e, s, t \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_R A \vee B, s \Vdash_R A \to C, t \Vdash_R B \to C.$$

Then by the definition

$$(e_0 = 0 \wedge s \cdot e_1 \Vdash_R C) \vee (e_0 \neq 0 \wedge t \cdot e_1 \Vdash_R C).$$

Applying the disjunctive realizer, one has $\mathbf{d}[s \cdot e_1][t \cdot e_1]e_0 0 \Vdash_R C$. ∎

(IPL9) $\Lambda e.\Lambda d.0 \Vdash_R (A \to B) \to ((A \to \neg B) \to \neg A)$.

(IPL10) $\Lambda e.e \Vdash_R A \to (\neg A \to B)$.

(IPL11) $\Lambda e.e \Vdash_R \forall x A(x) \to A[x/a]$, where a is a constant in V^*.

(IPL12) $\Lambda e.e \Vdash_R A[x/a] \to \exists x A(x)$, where a is a constant in V^*.

For Inference Rules: (In the following, we use $FV(C)$ to denote the set of all free variables in C).

(IR1) Modus Ponens is preserved, i.e., if $e \Vdash_R A$ and $d \Vdash_R A \to B$, then one can effectively find a realizer for B via realizers e and d.

Proof. This follows immediately from the definition. ■

(IR2) Rule $\forall$ is preserved, i.e., if $m \Vdash_R C \to A(x)$, then one can find a partial recursive function ξ such that $\xi(m) \Vdash_R C \to \forall x A(x)$, where $x \notin FV(C)$.

Proof. Assume $m \Vdash_R \forall x(C \to A(x))$. Let $n \in \mathbb{N}$ be arbitrary such that $n \Vdash_R C$. Then by the definitions $m \cdot n \Vdash_R \forall x A(x)$. Now one sets $\xi(m) := \Lambda n.(m \cdot n)$. ■

(IR3) Rule $\exists$ is preserved, i.e., if $m \Vdash_R A(x) \to C$, then one can effectively find a partial recursive function ξ such that $\xi(m) \Vdash_R \exists x A(x) \to C$, where $x \notin FV(C)$.

Proof. Assume $m \Vdash_R \forall x(A(x) \to C)$. Let $n \in \mathbb{N}$ be arbitrary such that $n \Vdash_R \exists x A(x)$, i.e., $\exists a \in V^*[n \Vdash_R A(a)]$ and thus $m \cdot n \Vdash_R C$. Now one sets $\xi(m)$ to be $\Lambda n.(m \cdot n)$. ■

For the Identity Axioms (IA): The soundness of **IA** follows immediately from the following claims. There claims will provide some universal realizers, that is, these realizers will depend only on the form of a formula in $\mathcal{L}(V^*)$ and are independent of the parameters. Now let $a, b, c \in V^*$ be arbitrary. We have the following claims:

Claim 6.2.22 [**IA1**] *$i^r \Vdash_R a = a$, where $i^r \equiv \Omega(\Lambda y.(\langle \Lambda f.\langle f, y\rangle, \Lambda f.\langle f, y\rangle\rangle))$.*

Proof. It suffices to find a realizer i^r such that $\forall a \in \mathbb{S}^*[i^r \Vdash_R a = a]$. We show this via ordinal induction and the fixed point theorem. Let $\alpha \in On$ be arbitrary. Assume $\forall \beta \in \alpha \forall b \in V_\beta^{\mathbb{N}}[k \Vdash_R b = b]$. Now let $a \in V_\alpha^{\mathbb{N}}$ be arbitrary. Let $f \in \mathbb{N}$ and $d \in V^*$ be arbitrary such that $(f, d) \in a$. Then by the inductive hypothesis and the definition, $\langle f, k\rangle \Vdash_R d \in a$, i.e., $\langle \Lambda f.\langle f, k\rangle, \Lambda f.\langle f, k\rangle\rangle \Vdash_R a = a$. Applying the fixed point generator Ω, one has the explicit form of $k \equiv \Omega(\Lambda y.(\langle \Lambda f.\langle f, y\rangle, \Lambda f.\langle f, y\rangle\rangle))$. ■

Claim 6.2.23 [**IA2**] *$i^s \Vdash_R a = b \to b = a]$, where $i^s \equiv \Lambda e.\langle e_1, e_0\rangle$.*

Proof. Let $a, b \in V^*$ and $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R a = b$. Then the result follows from the symmetry of the definition. ■

Claim 6.2.24 [**IA3**]

$$i^t \Vdash_R a = b \wedge b = c \to a = c,$$

where $i^t \equiv (\Omega(\lambda x.\langle t x_0 x_1, \tilde{t} x_0 x_1\rangle))_0$.

Claim 6.2.25 [**IA4**]

$$i^{eb} \Vdash_R a = b \wedge b \in c \rightarrow a \in c,$$

where $i^{eb} \equiv (\Omega(\lambda x.\langle t x_0 x_1, \tilde{t} x_0 x_1\rangle))_1$.

Proof. For any formulae $\theta_1, \theta_2, ..., \theta_n$, let $\bigwedge_{i=1}^{n} \theta_i$ denote the conjunction $\theta_1 \wedge \theta_2 ... \wedge \theta_n$. We will show **IA3** and **IA4** simultaneously via $\lhd^3$-induction (cf. Subsection 3.2.1):

$$\forall x_1, x_2, x_3[\forall (y_1, y_2, y_3) \lhd^3 (x_1, x_2, x_3)\varphi(y_1, y_2, y_3) \rightarrow \varphi(x_1, x_2, x_3)] \\ \rightarrow \forall x, y, z\varphi(x, y, z).$$

and the fixed point theorem (which will produce universal realizers u and v for both) by taking $\varphi(y_1, y_2, y_3)$ to be

$$y_1, y_2, y_3 \in V^* \rightarrow \bigwedge_{\substack{i,j,k=1 \\ i,j \neq k \wedge i \neq j}}^{3} \eta(y_i, y_j, y_k),$$

where $\eta(y_i, y_j, y_k)$ denotes

$$u \Vdash_R [y_i = y_j \wedge y_j = y_k \rightarrow y_i = y_k] \wedge v \Vdash_R [y_i = y_j \wedge y_j \in y_k \rightarrow y_i \in y_k].$$

Let $a^4, a^5, a^6 \in V^*$ and $d^1, d^2, d^3 \in V^*$ be arbitrary such that $(d^1, d^2, d^3) \lhd^3 (a^4, a^5, a^6)$ and

$$u \Vdash_R d^i = d^j \wedge d^j = d^k \rightarrow d^i = d^k, \\ v \Vdash_R d^i = d^j \wedge d^j \in d^k \rightarrow d^i \in d^k, \tag{6.2}$$

for all $i, j, k \in \{1, 2, 3\}$, where $i, j \neq k$ and $i \neq j$. Now we have to find the forms of the realizers u and v and show that

$$u \Vdash_R a^i = a^j \wedge a^j = a^k \rightarrow a^i = a^k, \\ v \Vdash_R a^i = a^j \wedge a^j \in a^k \rightarrow a^i \in a^k,$$

for all $i, j, k \in \{4, 5, 6\}$, where $i, j \neq k$ and $i \neq j$. Now let $i, j, k \in \{4, 5, 6\}$ be arbitrary such that $i, j \neq k$ and $i \neq j$. Let $n \in \mathbb{N}$ be arbitrary such that $n \Vdash_R a^i = a^j \wedge a^j \in a^k$, i.e., $\exists q \in V^*[(n_{10}, q) \in a^k \wedge \langle n_0, n_{11}\rangle \Vdash_R a^i = a^j \wedge a^j = q]$. We want to find a realizer with $a^i \in a^k$. Since $(a^i, a^j, q) \lhd^3 (a^i, a^j, a^k)$, (without loss of generality, suppose $i = 6, j = 4, k = 5$, then

$(a^j, q, a^i) \lhd^3 (a^4, a^5, a^6))$ by the inductive hypothesis (6.2), $u \cdot \langle n_0, n_{11}\rangle \Vdash_R a^i = q$. Hence by the definition, it follows that $\sigma(n,u) \Vdash_R a^i \in a^k$, where $\sigma(n,u) \equiv \langle n_{10}, u \cdot \langle n_0, n_{11}\rangle\rangle$ and thus

$$\Lambda n.\sigma(n,u) \Vdash_R a^i = a^j \wedge a^j \in a^k \to a^i \in a^k. \tag{6.3}$$

Now let $m \in \mathbb{N}$ be arbitrary such that $m \Vdash_R a^i = a^j \wedge a^j = a^k$. Let $p, r \in \mathbb{N}$ and $q, w \in V^*$ be arbitrary such that $(p,q) \in a^i$ and $(r,w) \in a^k$. From the assumption $m_{00} \cdot p \Vdash_R q \in a^j$ and $m_{11} \cdot r \Vdash_R w \in a^j$, i.e.,

$$\exists s \in V^*[((m_{00} \cdot p)_0, s) \in a^j \wedge (m_{00} \cdot p)_1 \Vdash_R q = s],$$
$$\exists s \in V^*[((m_{11} \cdot r)_0, s) \in a^j \wedge (m_{11} \cdot r)_1 \Vdash_R w = s].$$

By the assumption again,

$$m_{10} \cdot (m_{00} \cdot p)_0 \Vdash_R s \in a^k,$$
$$m_{01} \cdot (m_{11} \cdot r)_0 \Vdash_R s \in a^i.$$

Since

$$(q, s, a^k), (a^i, s, w) \lhd^3 (a^i, a^j, a^k),$$

by the inductive hypothesis (6.2),

$$\xi(m,p,v) \Vdash_R q \in a^k,$$
$$\delta(m,r,v) \Vdash_R w \in a^i,$$

where

$$\xi(m,p,v) \equiv v \cdot \langle (m_{00} \cdot p)_1, m_{10} \cdot (m_{00} \cdot p)_0\rangle,$$
$$\delta(m,r,v) \equiv v \cdot \langle (m_{11} \cdot r)_1, m_{01} \cdot (m_{11} \cdot r)_0\rangle.$$

Hence by the definition, it follows that

$$\begin{aligned} &\Lambda m.\langle \Lambda p.\xi(m,p,v), \Lambda r.\delta(m,r,v)\rangle \\ &\qquad \Vdash_R a^i = a^j \wedge a^j = a^k \to a^i = a^k. \end{aligned} \tag{6.4}$$

From (6.3) and (6.4) one finds u and v as follows:

$$u \simeq \Lambda m.\langle \Lambda p.\xi(m,p,v), \Lambda r.\delta(m,r,v)\rangle,$$
$$v \simeq \Lambda n.\langle n_{10}, u \cdot \langle n_0, n_{11}\rangle\rangle.$$

Now define

$$t \equiv \Lambda u.\Lambda v.\Lambda m.\langle \Lambda p.\xi(m,p,v), \Lambda r.\delta(m,r,v)\rangle,$$
$$\tilde{t} \equiv \Lambda u.\Lambda v.\Lambda n.\langle n_{10}, u \cdot \langle n_0, n_{11}\rangle\rangle.$$

By Corollary 3.1.15, one has the explicit forms: $u \equiv (\Omega(\lambda x.\langle tx_0x_1, \tilde{t}x_0x_1\rangle))_0$ and $v \equiv (\Omega(\lambda x.\langle tx_0x_1, \tilde{t}x_0x_1\rangle))_1$. ■

Claim 6.2.26 [**IA5**]

$$i^{be} \Vdash_R a = b \wedge c \in a \to c \in b,$$

where $i^{be} \equiv \Lambda e.\ i^{eb} \cdot \langle e_{11}, e_{00} \cdot e_{10}\rangle$.

Proof. Let $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R a = b \wedge c \in a$, i.e.,

$$\exists k \in V^*[(e_{10}, k) \in a \wedge e_{11} \Vdash_R c = k] \wedge e_0 \Vdash_R a = b.$$

By the definition and **IA4**, it then follows that $i^{eb} \cdot \langle e_{11}, e_{00} \cdot e_{10}\rangle \Vdash_R c \in b$. ■

Remark 6.2.27 **IA6** *to* **IA10** *are all realizable from its background theory and the definition.*

Lemma 6.2.28 [*Substitution*] *For any formula* $A(x, \vec{y})$ *in* $\mathcal{L}(V^*)$*, one can inductively find a realizer* $r_A^{stu} \in \mathbb{N}$ *such that* $\forall a, \vec{b}, c \in V^*[r_A^{stu} \Vdash_R a = c \wedge A(a, \vec{b}) \to A(c, \vec{b})]$.

Proof. For the atomic formulae, they have been given in the above claims. For the compound formulae, they all follow immediately via induction over the complexity of the formulae. Here we show the formulae with quantifiers:
♢ $A(a, \vec{b}) \equiv \exists z B(a, \vec{b}, z)$: Let $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R a = c \wedge \exists z B(a, \vec{b}, z)$. By the definition, it follows that

$$e_0 \Vdash_R a = c \wedge \exists d \in V^*[e_1 \Vdash_R B(a, \vec{b}, d)],$$

i.e., $\exists d \in V^*[e \Vdash_R (a = c \wedge B(a, \vec{b}, d))]$. By the inductive hypothesis, it follows that

$$r_B^{stu} \cdot e \Vdash_R \exists z B(c, \vec{b}, z).$$

Now one defines $r_A^{stu} \equiv \Lambda e.(r_B^{stu} \cdot e)$ and the result follows.
♢ $A(a, \vec{b}) \equiv \forall z B(a, \vec{b}, z)$: Let $e \in \mathbb{N}$ and $a, b \in V^*$ be arbitrary such that

$$e \Vdash_R a = c \wedge \forall z B(a, \vec{b}, z).$$

Then by the definition, it follows that

$$e_0 \Vdash_R a = c \wedge \forall d \in V^*[e_1 \Vdash_R B(a, \vec{b}, d)],$$

i.e., $\forall d \in V^*[e \Vdash_R a = c \wedge B(a, \vec{b}, d)]$. By the inductive hypothesis, it follows that

$$r_B^{stu} \cdot e \Vdash_R \forall z B(a, \vec{b}, z).$$

Now one defines $r_A^{stu} \equiv \Lambda e.(r_B^{stu} \cdot e)$ and the result follows. ■

6.2.4 A4.1: Non-logical axioms (CZF with two sorts)

Lemma 6.2.29 [*Extensionality*] $r^{ext} \Vdash_R \forall x \forall y(S(x) \wedge S(y) \rightarrow [\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y])$, *where* $r^{ext} \equiv \Lambda e.\Lambda t.\langle \Lambda f.t_0 \cdot \langle f, i^r \rangle, \Lambda g.t_1 \cdot \langle g, i^r \rangle \rangle$.

Proof. Let $a, b \in V^*$ and $e, t \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_R S(a) \wedge S(b), \tag{6.5}$$

and $t \Vdash_R \forall z(z \in a \leftrightarrow z \in b)$, i.e.,

$$\forall c \in V^*[t \Vdash_R c \in a \leftrightarrow c \in b]. \tag{6.6}$$

From (6.5) and the definitions, $a, b \in \mathbb{S}^*$. Now we want to find a realizer which realizes $a = b$. Let $f, g \in \mathbb{N}$ and $d, k \in V^*$ be arbitrary such that $(f, d) \in a$ and $(g, k) \in b$. Then by the soundness of **IA1**, it follows that $\langle f, i^r \rangle \Vdash_R d \in a$ and $\langle g, i^r \rangle \Vdash_R k \in b$, i.e., by (6.6)

$$t_0 \cdot \langle f, i^r \rangle \Vdash_R d \in b \wedge t_1 \cdot \langle g, i^r \rangle \Vdash_R k \in a.$$

From the definition, $\langle \Lambda f.(t_0 \cdot \langle f, i^r \rangle), \Lambda g.(t_1 \cdot \langle g, i^r \rangle) \rangle \Vdash_R a = b$. ■

Lemma 6.2.30 [*Pairing*] $r^{pair} \Vdash_R \forall x \forall y \exists u[S(u) \wedge (x \in u \wedge y \in u)]$, *where* $r^{pair} \equiv \langle 0, \langle \langle 0, i^r \rangle, \langle 1, i^r \rangle \rangle \rangle$.

Proof. Let $a, b \in V^*$ be arbitrary. Now define

$$c \equiv \{a, b\}_R \equiv \{(0, a), (1, b)\}.$$

By the Pairing Axiom in the background theory, c is an external set. By Claim 3.3.6, c is also an internal set, i.e., $c \in \mathbb{S}^*$. Then, by the soundness of **IA1**,

$$\langle 0, i^r \rangle \Vdash_R a \in c \wedge \langle 1, i^r \rangle \Vdash_R b \in c.$$

Hence the result follows from the definition. ■

Furthermore, one can also define the internal Cartesian product.

Definition 6.2.31 *For all a, b in V^*,*

$$(a,b)_R := \{(0, \{a,a\}_R), (1, \{a,b\}_R)\}.$$

Since $\{a,b\}_R \in \mathbb{S}^*$, by Claim 3.3.6, $(a,b)_R \in \mathbb{S}^*$.

Corollary 6.2.32 [*Internal Cartesian Product*]

$$r^{prd} \Vdash_R (a,b)_R = (c,d)_R \to a = c \wedge b = d,$$

where $r^{prd} \equiv \Lambda g.\langle i^s \cdot (((g_0 \cdot 0)_{11}) \cdot 0)_1, \mathbf{d}[\theta][\pi](g_1 \cdot 1)_0 0\rangle$.

Proof. By the soundness of **IA1** and **IA2**,

$$\forall h \in \mathbb{N}[h \Vdash_R \{a,a\}_R = \{c,d\}_R \to i^s \cdot (h_1 \cdot 0)_1 \Vdash_R a = c], \tag{6.7}$$

Now let us find a realizer for $\{a,b\}_R = \{a,c\}_R \to b = c$. Let $n \in \mathbb{N}$ be arbitrary such that $n \Vdash_R \{a,b\}_R = \{a,c\}_R$. Since $(1,b) \in \{a,b\}_R$, $n_0 \cdot 1 \Vdash_R b \in \{a,c\}_R$, i.e.,

$$\exists v \in V^*[((n_0 \cdot 1)_0, v) \in \{a,c\}_R \wedge (n_0 \cdot 1)_1 \Vdash_R b = v)]. \tag{6.8}$$

If $v = a$ (i.e., $(n_0 \cdot 1)_0 = 0$), then by Substitution

$$\eta(n) \Vdash_R \{a,a\}_R = \{a,c\}_R,$$

where $A \equiv \{x,y\}_R = \{x,z\}_R$ and $\eta(n) \equiv r_A^{stu} \cdot \langle (n_0 \cdot 1)_1, n\rangle$.

Since $(1,c) \in \{a,c\}_R$, by the definitions, it follows that $((\eta(n))_1 \cdot 1)_1 \Vdash_R c = a$. With this and (6.8) and the soundness of **IA2**, **IA3**,

$$i^t \cdot \langle (n_0 \cdot 1)_1, i^s \cdot ((\eta(n))_1 \cdot 1)_1\rangle \Vdash_R b = c.$$

If $v = c$ (i.e., $(n_0 \cdot 1)_0 = 1$), then $(n_0 \cdot 1)_1 \Vdash_R b = c$. Hence by applying the disjunctive realizer $\mathbf{d}$, we have found a realizer ξ such that

$$\Lambda n.\beta(n) \Vdash_R \{a,b\}_R = \{a,c\}_R \to b = c, \tag{6.9}$$

where $\beta(n) \equiv \Lambda n.\mathbf{d}[i^t \cdot \langle (n_0 \cdot 1)_1, i^s \cdot ((\eta(n))_1 \cdot 1)_1\rangle][(n_0 \cdot 1)_1](n_0 \cdot 1)_0 0$.

Let $g \in \mathbb{N}$ be arbitrary such that $g \Vdash_R (a,b)_R = (c,d)_R$. By the definition, $(0, \{a,a\}_R) \in (a,b)_R$ and thus it follows that $g_0 \cdot 0 \Vdash \{a,a\}_R \in (c,d)_R$. Hence by (6.7)

$$i^s \cdot (((g_0 \cdot 0)_{11}) \cdot 0)_1 \Vdash_R a = c. \tag{6.10}$$

Furthermore, since $(1, \{c,d\}_R) \in (c,d)_R$, by the assumption it follows that $g_1 \cdot 1 \Vdash_R \{c,d\}_R \in (a,b)_R$, i.e.,

$$\exists u \in V^*[((g_1 \cdot 1)_0, u) \in (a,b)_R] \wedge [(g_1 \cdot 1)_1 \Vdash_R \{c,d\}_R = u].$$

If $(g_1 \cdot 1)_0 = 0$, then $(g_1 \cdot 1)_1 \Vdash_R \{c,d\}_R = \{a,a\}_R$ and thus by the assumption and (6.9) one can find a realizer θ to realize $b = d$; if $(g_1 \cdot 1)_0 = 1$, then $(g_1 \cdot 1)_1 \Vdash_R \{c,d\}_R = \{a,b\}_R$ and thus by (6.10) and (6.9), one can find a realizer π to realize $b = d$. By applying the disjunctive realizer $\mathbf{d}$, it follows that

$$\mathbf{d}[\theta][\pi](g_1 \cdot 1)_0 0 \Vdash_R b = d. \tag{6.11}$$

From (6.10) and (6.11), the result follows immediately. ■

Lemma 6.2.33 *[Union]* $r^{uni} \Vdash_R \forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow \exists y(y \in x \wedge z \in y))]$, *where* $r^{uni} \equiv \langle 0, \langle \Lambda e.\xi(e), \Lambda t.\tau(t)\rangle\rangle$.

Proof. Let $a \in V^*$ be arbitrary. Define

$$\underline{a} \equiv \{(\langle n,m\rangle, b) : \exists c \in V^*[(n,c) \in a \wedge (m,b) \in c]\}.$$

Then by Bounded Separation, the Union and Pairing Axioms in the background theory, $\underline{a}$ is an external set. By Claim 3.3.6 , $\underline{a}$ is also an internal set, i.e., $\underline{a} \in \mathbb{S}^*$. Now we want to find a realizer which realizes $\forall z(z \in \underline{a} \leftrightarrow \exists y(y \in a \wedge z \in y))$. Let $c \in V^*$ and $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R c \in \underline{a}$, i.e., $\exists d \in V^*[(e_0, d) \in \underline{a} \wedge e_1 \Vdash_R c = d]$. By the definition, it follows that $\exists k \in V^*[(e_{00}, k) \in a \wedge (e_{01}, d) \in k]$. Hence by the soundness of **IA1** and **IA4**,

$$\xi(e) \Vdash_R \exists y(y \in a \wedge c \in y), \tag{6.12}$$

where $\xi(e) \equiv \langle\langle e_{00}, i^r\rangle, i^{eb} \cdot \langle e_1, \langle e_{01}, i^r\rangle\rangle\rangle$. On the other hand, let $t \in \mathbb{N}$ be arbitrary such that

$$t \Vdash_R \exists y(y \in a \wedge c \in y), \tag{6.13}$$

i.e., $\exists q \in V^*[t \Vdash_R q \in a \wedge c \in q]$. From $t_0 \Vdash_R q \in a$,

$$\exists h \in V^*(t_{00}, h) \in a \wedge t_{01} \Vdash_R q = h. \tag{6.14}$$

From $t_1 \Vdash_R c \in q$ and (6.14) and the soundness of **IA5**, it follows that $\varepsilon(t) \equiv i^{be} \cdot \langle t_{01}, t_1\rangle \Vdash_R c \in h$, i.e.,

$$\exists p \in V^*[((\varepsilon(t))_0, p) \in h \wedge (\varepsilon(t))_1 \Vdash_R c = p]. \tag{6.15}$$

Since $(t_{00}, h) \in a$ and $((\varepsilon(t))_0, p) \in h$, it follows that $(\langle t_{00}, (\varepsilon(t))_0\rangle, p) \in \underline{a}$ and thus

$$\langle\langle t_{00}, (\varepsilon(t))_0\rangle, i^r\rangle \Vdash_R p \in \underline{a}. \tag{6.16}$$

From (6.15) and (6.16) and the soundness of **IA1**, **IA4**, it follows that

$$\tau(t) \Vdash_R c \in \underline{a}, \tag{6.17}$$

where $\tau(t) \equiv i^{eb} \cdot \langle(\varepsilon(t))_1, \langle\langle t_{00}, (\varepsilon(t))_0\rangle, i^r\rangle\rangle$. From (6.12) and (6.17), it follows that

$$\langle\Lambda e.\xi(e), \Lambda t.\tau(t)\rangle \Vdash_R \forall z(z \in \underline{a} \leftrightarrow \exists y(y \in a \wedge z \in y)).$$

■

In order to show the soundness of Bounded Separation, we have the following definitions and claims.

Definition 6.2.34 *We define the collection of the bounded formulae (or Δ_0-formulae), $Form^{\Delta_0}$, with constants from V^* as follows: It consists of the Atomic formulae: $t^1 \in t^2$ and $t^1 = t^2$, and is closed under $\wedge, \vee, \neg, \rightarrow, \forall x \in t$ (an abbreviation for $\forall x[x \in t \rightarrow])$ and $\exists x \in t$ (an abbreviation for $\exists x[x \in t \wedge]$), where $t, t^1, t^2 \in Var \cup V^*$. Let $Form_c^{\Delta_0}$ be the collection of the closed Δ_0-formulae and let $A(b)$ denote a closed Δ_0-formulae with a constant b from V^*.*

Definition 6.2.35 *We define a relation $\Vdash_R^0$ over $\mathbb{N} \times Form_c^{\Delta_0}$ as follows:*

- $e \Vdash_R^0 a \in b$ *iff* $e \Vdash_R a \in b$.
- $e \Vdash_R^0 B(a) \wedge C(a)$ *iff* $e_0 \Vdash_R^0 B(a) \wedge e_1 \Vdash_R^0 C(a)$.
- $e \Vdash_R^0 B(a) \vee C(a)$ *iff* $(e_0 = 0 \wedge e_1 \Vdash_R^0 B(a)) \vee (e_0 \neq 0 \wedge e_1 \Vdash_R^0 C(a))$.
- $e \Vdash_R^0 B(a) \rightarrow C(a)$ *iff* $n \Vdash_R^0 B(a) \rightarrow e \cdot n \downarrow \wedge e \cdot n \Vdash_R^0 C(a)$.
- $e \Vdash_R^0 \neg B(a)$ *iff* $\forall n \in \mathbb{N} \neg n \Vdash_R^0 B(a)$.
- $e \Vdash_R^0 \forall x \in aB(x)$ *iff* $\forall (f, d) \in a[e \cdot f \downarrow \wedge e \cdot f \Vdash_R^0 B(d)]$.
- $e \Vdash_R^0 \exists x \in aB(x)$ *iff* $\exists d \in V^*[(e_0, d) \in a \wedge e_1 \Vdash_R^0 B(d)]$.

Claim 6.2.36 *For all $A(a) \in Form_c^{\Delta_0}$, $\{e \in \mathbb{N} : e \Vdash_R^0 A(a)\}$ is a set.*

Proof. We show by the induction on the complexity of $A(a)$. Let $Z_{B(b)}$ be the class $\{e \in \mathbb{N} : e \Vdash^0_R B(b)\}$ for all $B(b) \in Form^{\Delta_0}_c$. For the closed atomic formula, it can be easily seen to be true. For the compound formulae, by the induction, $Z_{A(a)}$ is a set is validated by the following settings:
$\diamondsuit$ $\{e \in \mathbb{N} : e \Vdash^0_R B(a) \wedge C(a)\} = \{e \in \mathbb{N} : e_0 \in Z_{B(a)} \wedge e_1 \in Z_{C(a)}\}$;
$\diamondsuit$ $\{e \in \mathbb{N} : e \Vdash^0_R B(a) \vee C(a)\} = \{(e_0 \wedge e_1 \in Z_{B(a)}) \vee (e_0 \neq 0 \wedge e_1 \in Z_{C(a)})\}$;
$\diamondsuit$ $\{e \in \mathbb{N} : e \Vdash^0_R B(a) \rightarrow C(a)\} = \{e \in \mathbb{N} : \forall n \in \mathbb{N}[n \in Z_{B(a)} \rightarrow e \cdot n \downarrow \wedge e \cdot n \in Z_{C(a)}]\}$;
$\diamondsuit$ $\{e \in \mathbb{N} : e \Vdash^0_R \neg B(a)\} = \{e \in \mathbb{N} : \forall n \in \mathbb{N} \neg n \in Z_{B(a)}\}$;
$\diamondsuit$ $\{e \in \mathbb{N} : e \Vdash^0_R \forall x \in aB(x)\} = \{e \in \mathbb{N} : \forall (f, d) \in a[e \cdot f \downarrow \wedge e \cdot f \in Z_{B(d)}]\}$;
$\diamondsuit$ $\{e \in \mathbb{N} : e \Vdash^0_R \exists x \in aB(x)\} = \{e \in \mathbb{N} : \exists d \in V^*[(e_0, d) \in a \wedge e_1 \in Z_{B(d)}]\}$.
■

Claim 6.2.37 *For each closed Δ_0-formula $A(a)$ in $Form^{\Delta_0}_c$, there exist partial recursive functions f^A and g^A such that $[e \Vdash_R A(a) \rightarrow f^A(e) \Vdash^0_R A(a)]$ and $[e \Vdash^0_R A(a) \rightarrow g^A(e) \Vdash_R A(a)]$.*

Proof. By the (mutual) induction on the complexity of $A(a)$, we have the following settings (we show only the nontrivial ones in details):
$\diamondsuit$ $e \Vdash_R B(a) \wedge C(a) \rightarrow \langle f^B(e_0), f^C(e_1)\rangle \Vdash^0_R B(a) \wedge C(a)$, $e \Vdash^0_R B(a) \wedge C(a) \rightarrow \langle g^B(e_0), g^C(e_1)\rangle \Vdash_R B(a) \wedge C(a)$;
$\diamondsuit$ $e \Vdash_R B(a) \vee C(a) \rightarrow \langle e_0, \langle \mathbf{d}[f^B(e_1)][f^C(e_1)]e_0 0\rangle\rangle \Vdash^0_R B(a) \vee C(a)$, $e \Vdash^0_R B(a) \vee C(a) \rightarrow \langle e_0, \langle \mathbf{d}[g^B(e_1)][g^C(e_1)]e_0 0\rangle\rangle \Vdash_R B(a) \vee C(a)$;
$\diamondsuit$ Assume $e \Vdash_R B(a) \rightarrow C(a)$. Assume $n \Vdash^0_R B(a)$. By the inductive hypothesis, it follows that $g^B(n) \Vdash_R B(a)$ and thus by the assumption $e \cdot g^B(n) \Vdash_R C(a)$. Again by the inductive hypothesis, it follows that $f^C(e \cdot g^B(n)) \Vdash^0_R C(a)$. Hence we have shown that $[e \Vdash_R B(a) \rightarrow C(a)] \rightarrow \Lambda n.f^C(e \cdot g^B(n)) \Vdash^0_R B(a) \rightarrow C(a)$. By the same argument, one also shows $[e \Vdash^0_R B(a) \rightarrow C(a)] \rightarrow \Lambda n.g^C(e \cdot f^B(n)) \Vdash_R B(a) \rightarrow C(a)$.
$\diamondsuit$ $e \Vdash_R \neg B(a) \rightarrow e \Vdash^0_R \neg B(a)$, $e \Vdash^0_R \neg B(a) \rightarrow e \Vdash_R \neg B(a)$;
$\diamondsuit$ Assume $e \Vdash_R \forall x \in aB(x)$. Let $(h, d) \in a$ be arbitrary. Then by the soundness of **IA1** and the assumption, it follows that $e \cdot \langle h, i^r\rangle \Vdash_R B(d)$ and thus by the inductive hypothesis, $f^B(e \cdot \langle h, i^r\rangle) \Vdash^0_R B(d)$. Hence we have shown that $\Lambda h.f^B(e \cdot \langle h, i^r\rangle) \Vdash^0_R \forall x \in aB(x)$. On the other hand, assume $e \Vdash^0_R \forall x \in aB(x)$. Let $n \Vdash_R b \in a$ be arbitrary, i.e., $\exists c \in V^*[(n_0, c) \in a \wedge n_1 \Vdash_R b = c]$. By the assumption, $e \cdot n_0 \Vdash^0_R B(c)$, which by inductive hypothesis yields $g^B(e \cdot n_0) \Vdash_R B(c)$. By the soundness of **IA2**, Substitution and the definition, we have shown that $\Lambda n.r^{stu}_B \cdot \langle i^s \cdot n_1, g^B(e \cdot n_0)\rangle \Vdash_R \forall x \in aB(x)$.
$\diamondsuit$ Assume $e \Vdash_R \exists x \in aB(x)$, i.e., $\exists d, c \in V^*[((e_{00}, c) \in a \wedge e_{01} \Vdash_R d =$

$c) \wedge e_1 \Vdash_R B(d)]$. By Substitution, it follows that $r_B^{stu} \cdot \langle e_{01}, e_1\rangle \Vdash_R B(c)$, which by the inductive hypothesis yields $f^B(r_B^{stu} \cdot \langle e_{01}, e_1\rangle) \Vdash_R^0 B(c)$. Hence we have shown that $\langle e_{00}, f^B(r_B^{stu} \cdot \langle e_{01}, e_1\rangle)\rangle \Vdash_R^0 \exists x \in aB(x)$. On the other hand, assume $e \Vdash_R^0 \exists x \in aB(x)$, i.e., by the definition and the soundness of **IA1**, $\exists d \in V^*[\langle e_0, i^r\rangle \Vdash_R d \in a \wedge e_1 \Vdash_R^0 B(d)]$. Then by inductive hypothesis and the definition, it follows that $\langle\langle e_0, i^r\rangle, g^B(e_1)\rangle \Vdash_R \exists x \in aB(x)$. ■

Lemma 6.2.38 [*Bounded Separation*]

$$r^{Sep^{\in}} \Vdash_R \forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow z \in x \wedge A(z))],$$

where $A(z)$ is a bounded formula and where $r^{Sep^{\in}} \equiv \langle 0, \langle \Lambda e.\langle \xi(e), \eta(e)\rangle, \Lambda l.\varepsilon(l)\rangle\rangle$.

Proof. Let $a \in V^*$ be arbitrary. Define

$$\underline{a} \equiv \{(\langle f, g\rangle, k) : f, g \in \mathbb{N} \wedge [(g, k) \in a \wedge f \Vdash_R^0 A(k)]\}.$$

Then by Claim 6.2.36, Bounded Separation, the Union and Pairing Axioms in the background theory, one has $\underline{a}$ is an external set. By Claim 3.3.6 , $\underline{a}$ is also an internal set, i.e., $\underline{a} \in \mathbb{S}^*$. Now we want to find a realizer which realizes $\forall z(z \in \underline{a} \leftrightarrow z \in a \wedge A(z))$. Let $c \in V^*$ and $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R c \in \underline{a}$, i.e., $\exists d \in V^*[(e_0, d) \in \underline{a} \wedge e_1 \Vdash_R c = d]$. By the definition, it follows that $(e_{01}, d) \in a \wedge e_{00} \Vdash_R^0 A(d)$, i.e., by Claim 6.2.37, $g^A(e_{00}) \Vdash_R A(d)$. Hence by the soundness of **IA** and **IA4**, $\xi(e) \Vdash_R c \in a$ and, by the soundness of **IA2** and Substitution, $\eta(e) \Vdash_R A(c)$, where

$$\xi(e) \equiv i^{eb} \cdot \langle e_1, \langle e_{01}, i^r\rangle\rangle,$$

$$\eta(e) \equiv r_A^{stu} \cdot \langle i^s \cdot e_1, g^A(e_{00})\rangle.$$

On the other hand, let $l \in \mathbb{N}$ be arbitrary such that $l \Vdash_R c \in a \wedge A(c)$. Then by the definition and Substitution, it follows that

$$\exists d \in V^*[(l_{00}, d) \in a \wedge l_{01} \Vdash_R c = d \wedge \delta(l) \Vdash_R A(d)],$$

where $\delta(l) \equiv r_A^{stu} \cdot \langle l_{01}, l_1\rangle$. By the inductive hypothesis, $f^A(\delta(l)) \Vdash_R^0 A(d)$. Thus by the definition $(\langle f^A(\delta(l)), l_{00}\rangle, d) \in \underline{a}$, which by the soundness of **IA1** and **IA4** yields $\varepsilon(l) \Vdash_R c \in \underline{a}$, where $\varepsilon(l) \equiv i^{eb} \cdot \langle l_{01}, \langle\langle f^A(\delta(l)), l_{00}\rangle, i^r\rangle\rangle$. Hence we have shown that

$$\langle \Lambda e.\langle \xi(e), \eta(e)\rangle, \Lambda l.\varepsilon(l)\rangle \Vdash_R \forall z[z \in \underline{a} \leftrightarrow z \in a \wedge A(z)].$$

■

Lemma 6.2.39 [*Infinity*]

$$r^{Inf} \Vdash_R \exists u(S(u) \wedge \forall z[z \in u \leftrightarrow N(z)]),$$

where $r^{Inf} \equiv \langle 0, \langle \Lambda e.e_0, \Lambda g.\langle g, i^r\rangle\rangle\rangle$.

Proof. Define

$$\underline{u} = \{(n,n) : n \in \mathbb{N}\}. \tag{6.18}$$

Then by the Infinity Axiom and Bounded Separation in the background theory, $\underline{u}$ is an external set. Moreover, by the definition, $\underline{u}$ is also an internal set, i.e., $\underline{u} \in \mathbb{S}^*$. Now we want to find a realizer which realizes $\forall z[z \in \underline{u} \leftrightarrow N(z)]$. Let $c \in V^*$ be arbitrary. Let $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R c \in \underline{u}$. Then by the definition we have

$$\exists k \in V^*(e_0, k) \in \underline{u}, \tag{6.19}$$

and

$$e_1 \Vdash_R c = k. \tag{6.20}$$

From (6.18), (6.19) and (6.20), it follows that $k = c = e_0 \in \mathbb{N}$ and thus

$$e_0 \Vdash_R N(c). \tag{6.21}$$

On the other hand, let $g \in \mathbb{N}$ be arbitrary such that $g \Vdash_R N(c)$. Then by the definition it follows that $g = c$, i.e., $(g, c) \in \underline{u}$. Hence by the soundness of **IA1**, it follows that

$$\langle g, i^r\rangle \Vdash_R c \in \underline{u}. \tag{6.22}$$

From (6.21) and (6.22), it follows that

$$\langle \Lambda e.e_0, \Lambda g.\langle g, i^r\rangle\rangle \Vdash_R \forall z[z \in \underline{u} \leftrightarrow N(z)].$$

■

Lemma 6.2.40 [*Induction*] $r^{Ind} \Vdash_R \forall x[(\forall y \in xA(y)) \to A(x)] \to \forall xA(x)$, *where* $r^{Ind} \equiv \Omega(\Lambda k.(e \cdot \Lambda t.r_A^{stu}\langle i^s \cdot t_1, k\rangle))$.

Proof. Let $e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_R \forall x[\forall y(y \in x \to A(y)) \to A(x)],$$

i.e.,

$$\forall u \in V^*[e \Vdash_R (\forall y(y \in u \to A(y)) \to A(u))]. \tag{6.23}$$

Now we want to find a realizer which realizes $\forall x A(x)$. We construct this by ordinal induction. Let $\alpha \in On$ be arbitrary. Assume

$$\forall \beta \in \alpha \forall b \in \mathbb{N} \cup V_{\beta}^{\mathbb{N}}(k \Vdash_R A(b)). \tag{6.24}$$

Now one has to find the explicit form of k via the fixed point theorem such that $\forall a \in \mathbb{N} \cup V_{\alpha}^{\mathbb{N}}(k \Vdash_L A(a))$. Let $a \in \mathbb{N} \cup V_{\alpha}^{\mathbb{N}}$ be arbitrary. We want to find a realizer which realizes $\forall y(y \in a \to A(y))$. Let $t \in \mathbb{N}$ and $c \in V^*$ be arbitrary such that $t \Vdash_R c \in a$. Then by the definition it follows that $\exists d \in V^*(t_0, d) \in a \wedge t_1 \Vdash_R c = d$. Since $d \in \mathbb{N} \cup V_{\beta}^{\mathbb{N}}$ for some $\beta \in \alpha$, i.e., by the inductive hypothesis (6.24) $k \Vdash_R A(d)$, and thus by **IA2** and Substitution $r_A^{stu} \cdot \langle i^s \cdot t_1, k\rangle \Vdash_R A(c)$. By the definition it follows that

$$\Lambda t.r_A^{stu} \cdot \langle i^s \cdot t_1, k\rangle \Vdash_R \forall y[y \in a \to A(y)]. \tag{6.25}$$

By (6.23) and (6.25)

$$e \cdot (\Lambda t.r_A^{stu} \cdot \langle i^s \cdot t_1, k\rangle) \Vdash_R A(a). \tag{6.26}$$

Let Ω be a fixed point generator. Then one has the explicit form of k: $k = \Omega(\Lambda k.(e \cdot \Lambda t.r_A^{stu} \cdot \langle i^s \cdot t_1, k\rangle))$ and this completes the proof. ■

Lemma 6.2.41 [*Strong Collection*] $r^{sc} \Vdash_R$

$$\forall x[\forall y \in x \exists z A(y,z) \to \exists u(S(u) \wedge \forall y \in x \exists z \in u A(y,z) \wedge \forall z \in u \exists y \in x A(y,z))],$$

where $r^{sc} \equiv \Lambda e.\langle 0, \langle \Lambda n.\xi(n,e), \Lambda m.\delta(m,e)\rangle\rangle$.

Proof. We show this via Strong Collection in the background theory. Let $a \in \mathbb{S}^*$ and $e \in \mathbb{N}$ be arbitrary such that a is inhabited and $e \Vdash_R \forall y[y \in a \to \exists z A(y,z)]$, i.e.,

$$\forall b \in V^* \forall n \in \mathbb{N}[(n \Vdash_R b \in a) \to \exists c \in V^*(e \cdot n \Vdash_R A(b,c))].$$

Then one has $\forall x \in a \exists z \eta(x,z)$, where

$$\eta(x,z) \equiv \exists f \in \mathbb{N} \exists d, c \in V^*[x = (f,d) \wedge z = (f,c) \wedge e \cdot \langle f, i^r\rangle \Vdash_R A(d,c)].$$

By Strong Collection in the background theory, there exists a set $\mathbf{D}$ such that

$$\forall x \in a \exists z \in \mathbf{D}\eta(x,z), \tag{6.27}$$

and

$$\forall z \in \mathbf{D} \exists x \in a \eta(x,z). \tag{6.28}$$

From (6.28) $\mathbf{D} \subseteq \mathbb{N} \times V^*$, i.e., by Claim 3.3.6, $\mathbf{D} \in \mathbb{S}^*$. Now we want to find realizer which realizes

$$\forall y[y \in a \to \exists z \in \mathbf{D} A(y,z)] \wedge \forall z[z \in \mathbf{D} \to \exists y \in a A(y,z)].$$

Let $b \in V^*$ be arbitrary. Let $n \in \mathbb{N}$ be arbitrary such that $n \Vdash_R b \in a$. Then by the definition it follows that $\exists k \in V^*[(n_0, k) \in a \wedge n_1 \Vdash_R b = k]$. Hence by (6.27), the soundness of **IA1**, **IA2** and Substitution,

$$\exists c \in V^*[\xi(n,e) \Vdash_R c \in \mathbf{D} \wedge A(b,c)],$$

where $\xi(n,e) \equiv \langle\langle n_0, i^r\rangle, r_A^{stu} \cdot (\langle i^s \cdot n_1, e \cdot \langle n_0, i^r\rangle\rangle)\rangle$. Hence by the definition it follows that

$$\Lambda n.\xi(n,e) \Vdash_R \forall y[y \in a \to \exists z(z \in \mathbf{D} \wedge A(y,z))]. \tag{6.29}$$

Now let $c \in V^*$ be arbitrary. Let $m \in \mathbb{N}$ be arbitrary such that $m \Vdash_R c \in \mathbf{D}$. Then by the definition it follows that $\exists v \in V^*[(m_0, v) \in \mathbf{D} \wedge m_1 \Vdash_R c = v]$. By (6.28)

$$\exists d \in V^*[(m_0, d) \in a \wedge e \cdot \langle m_0, i^r\rangle \Vdash_R A(d,v)],$$

and thus by the soundness of **IA1**, **IA2** and Substitution,

$$\exists d \in V^*[\delta(m,e) \Vdash_R d \in a \wedge A(d,c)],$$

where $\delta(m,e) \equiv \langle\langle m_0, i^r\rangle, r_A^{stu} \cdot (\langle i^s \cdot m_1, e \cdot \langle m_0, i^r\rangle\rangle)\rangle$. Hence by the definition it follows that

$$\Lambda m.\delta(m,e) \Vdash_R \forall z[z \in \mathbf{D} \to \exists y \in a A(y,z)]. \tag{6.30}$$

From (6.29) and (6.30), it follows that

$$\langle \Lambda n.\xi(n,e), \Lambda m.\delta(m,e)\rangle \Vdash_R \forall y \in a \exists z \in \mathbf{D} A(y,z) \wedge \forall z \in \mathbf{D} \exists y \in a A(y,z).$$

For the case $a \in \mathbb{N}$ or a is an empty set, the same realizer also works as follows:

$$\langle \Lambda n.\xi(n,e), \Lambda m.\delta(m,e)\rangle \Vdash_R \forall y \in a \exists z \in \emptyset A(y,z) \wedge \forall z \in \emptyset \exists y \in a A(y,z).$$

■

Lemma 6.2.42 [*Subset Collection*] $r^{sbc} \Vdash_R \forall a \forall b \exists u(S(u) \wedge \forall z[\forall x \in a \exists y \in b A(x,y,z) \to \exists d \in u(\forall x \in a \exists y \in d A(x,y,z) \wedge \forall y \in d \exists x \in a A(x,y,z))])$, *where* $r^{sbc} \equiv \langle 0, \Lambda e.\langle\langle 0, i^r\rangle, \langle \Lambda m.\delta(m,e), \Lambda n.\varepsilon(n,e)\rangle\rangle\rangle$.

Proof. We show this via Subset Collection in the background theory. Let $a \in V^*$ be arbitrary such that a is inhabited (i.e., there is at least one element in a). Let $b \in V^*$ be arbitrary. Let $c \in V^*$ and $e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_R \forall x \in a \exists y \in b A(x, y, c). \tag{6.31}$$

Now we want to find $\underline{u}, \underline{d} \in V^*$ and a realizer which realizes $\underline{d} \in \underline{u} \wedge [\forall x \in a \exists y \in \underline{d} A(x, y, c) \wedge \forall y \in \underline{d} \exists x \in a A(x, y, c)]$. Unravelling (6.31) and applying Substitution yields

$$\forall (f, d) \in a \exists v \in V^*[((e \cdot \langle f, i^r \rangle)_{00}, v) \in b \wedge \xi(f, e) \Vdash_R A(d, v, c)], \tag{6.32}$$

where $\xi(f, e) \equiv r_A^{stu} \cdot \langle (e \cdot \langle f, i^r \rangle)_{01}, (e \cdot \langle f, i^r \rangle)_1 \rangle$. Now define a set

$$\underline{b} \equiv \{(\langle e, f \rangle, v) : e, f \in \mathbb{N} \wedge ((e \cdot \langle f, i^r \rangle)_{00}, v) \in b\}.$$

By Bounded Separation, the Pairing and Union Axioms in the background theory, $\underline{b}$ is an external sets. Moreover, by Claim 3.3.6, also $\underline{b} \in \mathbb{S}^*$. Now one can rewrite (6.32) as follows:

$$\forall u \in a \exists l \in \underline{b} \eta(u, l, c),$$

where $\eta(u, l, c) \equiv \exists f \in \mathbb{N} \exists d, v \in V^*[u = (f, d) \wedge l = (\langle e, f \rangle, v) \wedge ((e \cdot \langle f, i^r \rangle)_{00}, v) \in b \wedge \xi(f, e) \Vdash_R A(d, v, c)]$. Invoking Subset Collection in the background theory yields that there is an external set $\mathbb{D}$ such that $\exists \mathbb{C} \in \mathbb{D}$

$$\forall u \in a \exists l \in \mathbb{C} \eta(u, l, c), \tag{6.33}$$

and

$$\forall l \in \mathbb{C} \exists u \in a \eta(u, l, c). \tag{6.34}$$

Define $\mathbb{D}^* = \{q \cap \underline{b} : q \in \mathbb{D}\}$. By Bounded Separation and Replacement in the background theory, $\mathbb{D}^*$ is an external set. Let $q \cap \underline{b} \in \mathbb{D}^*$ be arbitrary. Then we have $q \cap \underline{b} \subseteq \underline{b} \subseteq \mathbb{N} \times V^*$ and thus by Claim 3.3.6 , $q \cap \underline{b} \in \mathbb{S}^*$, i.e., $\mathbb{D}^* \subseteq \mathbb{S}^*$. Furthermore, by the fact that $\mathbb{C} \in \mathbb{D}$, it follows $\mathbb{C} \cap \underline{b} \in \mathbb{D}^*$ and thus $\mathbb{C} \cap \underline{b} \in \mathbb{S}^*$. Now define

$$\underline{u} := \{(0, d) : d \in \mathbb{D}^*\}.$$

Since $\mathbb{D}^* \subseteq \mathbb{S}^*$, by Claim 3.3.6 , also $\underline{u} \in \mathbb{S}^*$. Moreover, by the soundness of **IA1**, also

$$\langle 0, i^r \rangle \Vdash_R \mathbb{C} \cap \underline{b} \in \underline{u}. \tag{6.35}$$

Now we want to find a realizer which realizes

$$\forall x \in a \exists y \in \mathbb{C} \cap \underline{b} A(x, y, c).$$

Let $m \in \mathbb{N}$ and $\underline{x} \in V^*$ be arbitrary such that $m \Vdash_R \underline{x} \in a$, i.e., $\exists k \in V^*[(m_0, k) \in a \wedge m_1 \Vdash_R \underline{x} = k]$. By (6.33), the soundness of **IA2** and Substitution, it follows that

$$\exists v \in V^*[(\langle e, m_0 \rangle, v) \in \mathbb{C} \cap \underline{b} \wedge r_A^{stu} \cdot (\langle i^s \cdot m_1, \xi(m_0, e) \rangle) \Vdash_R A(\underline{x}, v, c)].$$

By the soundness of **IA1**, it then follows that

$$\delta(m, e) \Vdash_R v \in \mathbb{C} \cap \underline{b} \wedge A(\underline{x}, v, c),$$

where $\delta(m, e) \equiv \langle \langle \langle e, m_0 \rangle, i^r \rangle, r_A^{stu} \cdot (\langle i^s \cdot m_1, \xi(m_0, e) \rangle) \rangle$. Hence we have shown that

$$\Lambda m.\delta(m, e) \Vdash_R \forall x \in a \exists y \in \mathbb{C} \cap \underline{b} A(x, y, c). \tag{6.36}$$

Now we want to find a realizer which realizes

$$\forall y \in \mathbb{C} \cap \underline{b} \exists x \in a A(x, y, c).$$

Let $n \in \mathbb{N}$ and $\underline{y} \in V^*$ be arbitrary such that $n \Vdash_R \underline{y} \in \mathbb{C} \cap \underline{b}$, i.e.,

$$\exists h \in V^*[(n_0, h) \in \mathbb{C} \cap \underline{b} \wedge n_1 \Vdash_R \underline{y} = h].$$

By (6.34), Substitution and the soundness of **IA2**, it follows that

$$\exists d \in V^*[(n_{01}, d) \in a \wedge r_A^{stu} \cdot \langle i^s \cdot n_1, \xi(n_{01}, e) \rangle \Vdash_R A(d, \underline{y}, c)].$$

By the soundness of **IA1**, it then follows that

$$\varepsilon(n, e) \Vdash_R d \in a \wedge A(d, \underline{y}, c),$$

where $\varepsilon(n, e) \equiv \langle \langle n_{01}, i^r \rangle, r_A^{stu} \cdot \langle i^s \cdot n_1, \xi(n_{01}, e) \rangle \rangle$. Hence we have shown that

$$\Lambda n.\varepsilon(n, e) \Vdash_R \forall y \in \mathbb{C} \cap \underline{b} \exists x \in a A(x, y, c). \tag{6.37}$$

From (6.35) and (6.36) and (6.37),

$$\begin{aligned} &\langle \langle 0, i^r \rangle, \langle \Lambda m.\delta(m, e), \Lambda n.\varepsilon(n, e) \rangle \rangle \Vdash_R \\ &\exists d \in \underline{u}[\forall x \in a \exists y \in d A(x, y, c) \wedge \forall y \in d \exists x \in a A(x, y, c)]. \end{aligned}$$

As for the case in which a is the empty set or a is a number, the same realizer also works as follows:

$$\begin{aligned} &\langle \langle 0, i^r \rangle, \langle \Lambda m.\delta(m, e), \Lambda n.\varepsilon(n, e) \rangle \rangle \Vdash_R \\ &\emptyset \in \{(0, \emptyset)\} \wedge [\forall x \in a \exists y \in \emptyset A(x, y, c) \wedge \forall y \in \emptyset \exists x \in a A(x, y, c)]. \end{aligned}$$

■

Theorem 6.2.43 [*Soundness Theorem*]

$$\mathbf{CZF}_N \vdash \varphi \implies \mathbf{CZF}_{NA} \vdash \exists x(N(x) \wedge x \Vdash_R \varphi),$$

for all formulae $\varphi \in \mathcal{L}(V^*)$.

Proof. Since the logical axioms, inference rules and non-logical axioms have shown to be sound, the result follows immediately. ∎

6.2.5 A4.2: Non-logical axioms (IZF with two sorts)

Since $\mathbf{IZF}_N$ and $\mathbf{CZF}_N$ share most of the axioms, there are only a couple left to be checked.

Lemma 6.2.44 [*Separation*] $r^{Sep} \Vdash_R \forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow z \in x \wedge A(z))]$, *where* $r^{Sep} \equiv \langle 0, \langle \Lambda e.\xi(e), \Lambda l.\varepsilon(l)\rangle\rangle$.

Proof. Let $a \in V^*$ be arbitrary. Define

$$\underline{a} \equiv \{(\langle f, g\rangle, k) : f, g \in \mathbb{N} \wedge [(g, k) \in a \wedge f \Vdash_R A(k)]\}.$$

Then by Separation, the Union and Pairing Axioms in the background theory, $\underline{a}$ is an external set. By Claim 3.3.6 , $\underline{a}$ is also an internal set, i.e., $\underline{a} \in \mathbb{S}^*$. Now we want to find a realizer which realizes $\forall z(z \in \underline{a} \leftrightarrow z \in a \wedge A(z))$. Let $c \in V^*$ and $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R c \in \underline{a}$, i.e., $\exists d \in V^*[(e_0, d) \in \underline{a} \wedge e_1 \Vdash_R c = d]$. By the definition, it follows that $(e_{01}, d) \in a \wedge e_{00} \Vdash_R A(d)$ and thus $\langle e_{01}, i^r\rangle \Vdash_R d \in a$. By the soundness of **IA2**, **IA4** and Substitution, it follows that

$$\xi(e) \Vdash_R c \in a \wedge A(c), \tag{6.38}$$

where $\xi(e) \equiv \langle i^{eb} \cdot \langle e_1, \langle e_{01}, i^r\rangle\rangle, r_A^{stu} \cdot \langle i^s \cdot e_1, e_{00}\rangle\rangle$. On the other hand, let $l \in \mathbb{N}$ be arbitrary such that $l \Vdash_R c \in a \wedge A(c)$. Then by the definition and Substitution, it follows that

$$\exists d \in V^*[(l_{00}, d) \in a \wedge l_{01} \Vdash_R c = d \wedge \delta(l) \Vdash_R A(d)],$$

where $\delta(l) \equiv r_A^{stu} \cdot \langle l_{01}, l_1\rangle$. By the definition $(\langle \delta(l), l_{00}\rangle, d) \in \underline{a}$ and thus, by the soundness of **IA1** and **IA4**,

$$\varepsilon(l) \Vdash_R c \in \underline{a}, \tag{6.39}$$

where $\varepsilon(l) \equiv i^{eb} \cdot \langle l_{01}, \langle\langle \delta(l), l_{00}\rangle, i^r\rangle\rangle$. From (6.38) and (6.39),

$$\langle \Lambda e.\xi(e), \Lambda l.\varepsilon(l)\rangle \Vdash_R \forall z[z \in \underline{a} \leftrightarrow z \in a \wedge A(z)].$$

∎

Claim 6.2.45 *If $a, b \in \mathbb{S}^*$, then $e \Vdash_R b \subseteq a \to \exists b^* \in V^{\mathbb{N}}_{rk(a)+1}[\delta(e) \Vdash_R b = b^*]$, where $\delta(e) \equiv \langle \Lambda f.\xi(f,e), \Lambda g.\sigma(g,e)\rangle$.*

Proof. Assume $e \Vdash_R b \subseteq a$, i.e., by the definition $\forall (f,d) \in b[e \cdot \langle f, i^r\rangle \Vdash_R d \in a]$, i.e.,

$$\forall (f,d) \in b \exists c \in V^*[((e \cdot \langle f, i^r\rangle)_0, c) \in a \wedge (e \cdot \langle f, i^r\rangle)_1 \Vdash_R d = c]. \tag{6.40}$$

Define

$$b^* \equiv \{(f,c) : \exists d \in V^*[(f,d) \in b \wedge ((e \cdot \langle f, i^r\rangle)_0, c) \in a \wedge (e \cdot \langle f, i^r\rangle)_1 \Vdash_R d = c]\}.$$

By Pairing and Separation in the background theory, b^* is a set. By Claim 3.3.6 and Corollary 3.3.5, $b^* \in V^{\mathbb{N}}_{rk(a)+1}$. Now we want to find a realizer which realizes $b = b^*$. Let $f \in \mathbb{N}$ and $d \in V^*$ be arbitrary such that $(f,d) \in b$. Then by (6.40) it follows that

$$\exists c \in V^*[\langle f, i^r\rangle \Vdash_R c \in b^*] \wedge [(e \cdot \langle f, i^r\rangle)_1 \Vdash_R d = c],$$

i.e., by the soundness of **IA4**, $\xi(f,e) \Vdash_R d \in b^*$, where

$$\xi(f,e) \equiv i^{eb} \cdot \langle (e \cdot \langle f, i^r\rangle)_1, \langle f, i^r\rangle\rangle.$$

On the other hand, let $g \in \mathbb{N}$ and $k \in V^*$ be arbitrary such that $(g,k) \in b^*$, i.e., by the definition and the soundness of **IA1** and **IA2**, there is $d \in V^*$ such that $\langle g, i^r\rangle \Vdash_R d \in b$ and $i^s \cdot (e \cdot \langle g, i^r\rangle)_1 \Vdash_R k = d$. By the soundness of **IA4**, it follows that $\sigma(g,e) \Vdash_R k \in b$, where

$$\sigma(g,e) \equiv i^{eb} \cdot \langle i^s \cdot (e \cdot \langle g, i^r\rangle)_1, \langle g, i^r\rangle\rangle.$$

Hence, by the definition, $\langle \Lambda f.\xi(f,e), \Lambda g.\sigma(g,e)\rangle \Vdash_R b = b^*$. ■

Lemma 6.2.46 [*Power Set*]

$$r^{pw} \Vdash_R \forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow (S(z) \wedge z \subseteq x))],$$

where $r^{pw} \equiv \langle 0, \langle \Lambda e.\langle 0, r^{stu}_A \cdot \langle i^s \cdot e_1, e_0\rangle\rangle, \Lambda h.\xi(h)\rangle\rangle$.

Proof. Let $a \in V^*$ be arbitrary. Define

$$\underline{a} \equiv \{(g,c) : g \in \mathbb{N} \wedge c \in V^{\mathbb{N}}_{rk(a)+1} \wedge g \Vdash_R c \subseteq a\}.$$

By the Powerset axiom, Pairing and Separation in the background theory, $\underline{a}$ is an external set. By Claim 3.3.6 it also follows that $\underline{a} \in \mathbb{S}^*$. Let $k \in V^*$ be arbitrary. It suffices to find a realizer to realize $k \in \underline{a} \leftrightarrow (S(k) \wedge k \subseteq a)$. Let $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R k \in \underline{a}$, i.e.,

$$\exists c \in V^{\mathbb{N}}_{rk(a)+1}[e_0 \Vdash_R c \subseteq a \wedge e_1 \Vdash_R k = c].$$

Then by Corollary 3.3.2 and the soundness of **IA2**, it follows that $k \in \mathbb{S}^*$ and thus

$$\langle 0, r^{stu}_A \cdot \langle i^s \cdot e_1, e_0\rangle\rangle \Vdash_R S(k) \wedge k \subseteq a, \tag{6.41}$$

where $A \equiv x \subseteq y$.

Now let $h \in \mathbb{N}$ be arbitrary such that $h \Vdash_R S(k) \wedge k \subseteq a$. Then by Claim 6.2.45 and Substitution, it follows that $\exists k^* \in V^{\mathbb{N}}_{rk(a)+1}$ such that $\delta(h_1) \Vdash_R k = k^*$, where $\delta(h_1) \equiv \langle \Lambda f.\xi(f, h_1), \Lambda g.\sigma(g, h_1)\rangle$, and thus $r^{stu}_A \cdot \langle \delta(h_1), h_1\rangle \Vdash_R k^* \subseteq a$. By the definition of $\underline{a}$ and Substitution, it then follows that $\langle r^{stu}_A \cdot \langle \delta(h_1), h_1\rangle, i^r\rangle \Vdash_R k^* \in \underline{a}$ and thus by the soundness of **IA4**

$$\xi(h) \Vdash_R k \in \underline{a}, \tag{6.42}$$

where $\xi(h) \equiv i^{eb} \cdot \langle \delta(h_1), \langle r^{stu}_A \cdot \langle \delta(h_1), h_1\rangle, i^r\rangle\rangle$. From (6.41) and (6.42), the result follows. ■

Lemma 6.2.47 [*Collection*]

$$r^{co} \Vdash_R \forall x[\forall y \in x \exists z A(y, z) \rightarrow \exists u(S(u) \wedge \forall y \in x \exists z \in u A(y, z))],$$

where $r^{co} \equiv \Lambda e.\langle 0, \Lambda n.\xi(n, e)\rangle$.

Proof. We show this via Collection in the background theory. Let $a \in \mathbb{S}^*$ and $e \in \mathbb{N}$ be arbitrary such that a is inhabited (i.e., there is an element in a) and $e \Vdash_R \forall y[y \in a \rightarrow \exists z A(y, z)]$, i.e.,

$$\forall b \in V^* \forall n \in \mathbb{N}[(n \Vdash_R b \in a) \rightarrow \exists c \in V^*(e \cdot n \Vdash_R A(b, c))].$$

Then one has $\forall x \in a \exists z \eta(x, z)$, where

$$\eta(x, z) \equiv \exists f \in \mathbb{N} \exists d, c \in V^*[x = (f, d) \wedge z = (f, c) \wedge e \cdot \langle f, i^r\rangle \Vdash_R A(d, c)].$$

By Collection in the background theory, there exists a set $\mathbf{C}$ such that

$$\forall x \in a \exists z \in \mathbf{C} \eta(x, z). \tag{6.43}$$

Now define

$$\mathbf{D} = \mathbf{C} \cap (\mathbb{N} \times V^*).$$

Since $\mathbf{D} \subseteq \mathbb{N} \times V^*$, by Separation and Claim 3.3.6, $\mathbf{D} \in \mathbb{S}^*$. Now we want to find realizer which realizes $\forall y[y \in a \to \exists z \in \mathbf{D} A(y,z)]$. Let $b \in V^*$ and $n \in \mathbb{N}$ be arbitrary such that $n \Vdash_R b \in a$. Then by the definition it follows that $\exists k \in V^*[(n_0, k) \in a \wedge n_1 \Vdash_R b = k]$. Hence by (6.43) and the soundness of **IA1**, **IA2** and Substitution,

$$\exists c \in V^*[\xi(n,e) \Vdash_R c \in \mathbf{D} \wedge A(b,c)],$$

where $\xi(n,e) \equiv \langle \langle n_0, i^r \rangle, r_A^{stu} \cdot (\langle i^s \cdot n_1, e \cdot \langle n_0, i^r \rangle \rangle) \rangle$. Hence by the definition it follows that

$$\Lambda n.\xi(n,e) \Vdash_R \forall y[y \in a \to \exists z(z \in \mathbf{D} \wedge A(y,z))].$$

As for the case in which a is the empty set or a is a number, by default, the same realizer also works and this completes our proof. ■

Theorem 6.2.48 [*Soundness Theorem*]

$$\mathbf{IZF}_N \vdash \varphi \implies \mathbf{IZF}_{NA} \vdash \exists x(N(x) \wedge x \Vdash_R \varphi),$$

for all formulae $\varphi \in \mathcal{L}(V^*)$.

Proof. This follows immediately from the above lemmata. ■

6.2.6 A5: (Semi-) Constructive axioms of choice

Recall that we use the notation $\forall n A(n)$ to denote $\forall x[N(x) \to A(x)]$ and $\exists n A(n)$ to denote $\exists x[N(x) \wedge A(x)]$. Other notations used here were defined in Subsection 2.2.7.

In addition, each realizer used here denotes a relativized realizer and the background theory in this section is $\mathbf{CZF}_{NA}$.

Lemma 6.2.49 [$\mathbf{AC}^{NN}$] $r^{\mathbf{AC}^{NN}} \Vdash_R$

$$\forall n \exists m A(n,m) \to \exists f[Fun(f,N,N) \wedge \forall n \exists m((n,m) \in f \wedge A(n,m))],$$

where $r^{\mathbf{AC}^{NN}} \equiv \Lambda e.\langle \langle \sigma(e), \rho(e) \rangle, \Lambda n.\xi(n,e) \rangle$.

Proof. Let $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R \forall n \exists m A(n, m)$. Then by Lemma 6.2.6, it follows that

$$\forall n \in \mathbb{N}[(e \cdot n)_1 \Vdash_R A(n, (e \cdot n)_0)]. \tag{6.44}$$

Now define

$$\underline{f} \equiv \{(n, (n, (e \cdot n)_0)_R) : n \in \mathbb{N}\}.$$

By the Infinity Axiom, Bounded Separation, Pairing and Replacement, $\underline{f}$ is a set. By Claim 3.3.6, also $\underline{f} \in \mathbb{S}^*$. With Lemma 6.2.6, Corollary 6.2.32,

$$\sigma(e) \Vdash_R \underline{f} \subseteq N \times N,$$
$$\rho(e) \Vdash_R \forall n \exists! m (n, m) \in \underline{f},$$

where

$$\sigma(e) \equiv \Lambda g.\langle g_1, \langle g_0, (e \cdot g_0)_0 \rangle\rangle, \rho(e) \equiv \Lambda n.\langle\langle (e \cdot n)_0, \langle n, i^r \rangle\rangle, \Lambda k.0\rangle.$$

By the definition, it follows that

$$\langle \sigma(e), \rho(e) \rangle \Vdash_R Fun(\underline{f}, N, N).$$

Let $n \in \mathbb{N}$ be arbitrary. By the soundness of **IA1** and (6.44), it follows that

$$\langle\langle n, i^r \rangle, (e \cdot n)_1 \rangle \Vdash_R (n, (e \cdot n)_0)_R \in \underline{f} \wedge A(n, (e \cdot n)_0),$$

and thus

$$\xi(n, e) \Vdash_R \exists m((n, m) \in f \wedge A(n, m)),$$

where $\xi(n, e) \equiv \langle (e \cdot n)_0, \langle\langle n, i^r \rangle, (e \cdot n)_1 \rangle\rangle$ and this completes the proof. ■

Lemma 6.2.50 [**UP**]

$$r^{up}_{\mathrm{R}} \Vdash_R \forall x[S(x) \to \exists n A(x, n)] \to \exists n \forall x[S(x) \to A(x, n)],$$

where $r^{up}_{\mathrm{R}} \equiv \Lambda e.\langle (e \cdot 0)_0, \Lambda d.(e \cdot 0)_1 \rangle$.

Proof. Let $e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_R \forall x[S(x) \to \exists n A(x, n)].$$

By Lemma 6.2.6, it follows that

$$\forall a \in \mathbb{S}^*[(e \cdot 0)_1 \Vdash_R A(a, (e \cdot 0)_0)],$$

i.e.,

$$\langle (e \cdot 0)_0, \Lambda d.(e \cdot 0)_1 \rangle \Vdash_R \exists n \forall x[S(x) \to A(x, n)].$$

■

Lemma 6.2.51 [**UZ**]

$r^{uz}_{\mathrm{R}} \Vdash_R [\forall x(S(x) \to A(x) \vee B(x))] \to \forall x(S(x) \to A(x)) \vee \forall x(S(x) \to B(x))$,

where $r^{uz}_{\mathrm{R}} \equiv \Lambda e.\langle (e \cdot 0)_0, \Lambda d.(e \cdot 0)_1 \rangle$.

Proof. Since **UZ** follows from **UP** by taking an instance of $A(x, n)$ to be $(n = 0 \wedge A(x)) \vee (n \neq 0 \wedge B(x))$, it is also realizable. Its explicit realizer is found as follows: Let $e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_R \forall x(S(x) \to A(x) \vee B(x)),$$

i.e., by the definition

$$\forall a \in \mathbb{S}^*[(e\cdot 0)_0 = 0 \wedge (e\cdot 0)_1 \Vdash_R A(a)] \vee \forall a \in \mathbb{S}^*[(e\cdot 0)_0 \neq 0 \wedge (e\cdot 0)_1 \Vdash_R B(a)].$$

By the definition, this gives the realizer. ■

Lemma 6.2.52 (**DC**)

$$r^{dc}_{\mathrm{R}} \Vdash_R \forall a \forall b[b \in a \wedge \forall x \in a \exists y \in a A(x, y)$$
$$\to \exists f(Fun(f, N, a) \wedge f(0) = b \wedge \forall n A(f(n), f(n + 1))],$$

where $r^{dc}_{\mathrm{R}} \equiv \Lambda e.\langle\langle \gamma(e), \beta(e)\rangle, \Lambda n.\delta(n, e)\rangle$.

Proof. We show this via Dependent Choice in the background theory. Let $a, b \in V^*$ and $e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_R b \in a \wedge \forall x \in a \exists y \in a A(x, y),$$

i.e., $\exists c \in V^*(e_{00}, c) \in a \wedge e_{01} \Vdash_R b = c$ and

$$e_1 \Vdash_R \forall x \in a \exists y \in a A(x, y). \tag{6.45}$$

Define $\underline{a} \equiv \{(l, k) : (l_{00}, k) \in a\}$. Unravelling (6.45) and applying Substitution, it follows that $\forall x \in \underline{a} \exists y \in \underline{a} B(x, y)$, where

$$B(x, y) \equiv \exists l, m \in \mathbb{N} \exists k, d \in V^*[x = (l, k) \wedge (l_{00}, k) \in a \wedge$$
$$y = (m, d) \wedge m = e_1 \cdot \langle l_{00}, i^r\rangle \wedge r^{stu}_A \cdot \langle m_{01}, m_1\rangle \Vdash_R A(k, d)].$$

Now by **DC**, it follows that there is a function $F : \mathbb{N} \to \underline{a}$ such that

$$F(0) = (e, c) \wedge \forall n \in \mathbb{N}[B(F(n), F(n + 1))],$$

i.e.,

$$\forall n \in \mathbb{N}[F^0(n+1) = e_1 \cdot \langle (F^0(n))_{00}, i^r \rangle \wedge r_A^{stu} \cdot \langle (F^0(n+1))_{01}, (F^0(n+1))_1 \rangle \Vdash_R A(F^1(n), F^1(n+1))], \tag{6.46}$$

where F^0 and F^1 denote the left and right projection function of F, respectively. Since $F^0(0) = e$ and $F^0(n+1) = e_1 \cdot \langle (F^0(n))_{00}, i^r \rangle$, one has that F^0 is a total recursive function. Let $F_e^{\#}$ be its code. Now define

$$\underline{f} \equiv \{(n, (n, F^1(n))_R) : n \in \mathbb{N}\}.$$

By Claim 3.3.6, $\underline{f} \in \mathbb{S}^*$. Now we want to find a realizer which realizes

$$Fun(\underline{f}, N, a) \wedge \underline{f}(0) = b \wedge \forall n A(\underline{f}(n), \underline{f}(n+1)).$$

♢ First of all, let us find a realizer for $Fun(\underline{f}, N, a)$. Let $t \in \mathbb{N}$ and $k \in V^*$ be arbitrary such that $t \Vdash_R k \in \underline{f}$. Then by the definition,

$$t_1 \Vdash_R k = (t_0, F^1(t_0))_R. \tag{6.47}$$

Since $((F^0(t_0))_{00}, F^1(t_0)) \in a$, by the definitions and the soundness of **IA1**, it follows that

$$\xi(t, e) \Vdash_R N(t_0) \wedge F^1(t_0) \in a, \tag{6.48}$$

where $\xi(t, e) \equiv \langle t_0, \langle (F_e^{\#}(t_0))_{00}, i^r \rangle \rangle$. From (6.47) and (6.48)

$$\Lambda t. \langle t_1, \xi(t, e) \rangle \Vdash_R \underline{f} \subseteq N \times a. \tag{6.49}$$

In the meantime, we want to find a realizer for $\forall n \exists! x(n, x) \in \underline{f}$. Let $n, l \in \mathbb{N}$ and $a, b \in V^*$ be arbitrary such that

$$l \Vdash_R (n, a)_R \in \underline{f} \wedge (n, b)_R \in \underline{f}.$$

By the definitions, it follows that

$$l_{01} \Vdash_R (n, a)_R = (l_{00}, F^1(l_{00}))_R, l_{11} \Vdash_R (n, b)_R = (l_{10}, F^1(l_{10}))_R.$$

From Corollary 6.2.32 and the definition, $l_{00} = l_{10}$ and thus by the soundness of **IA2**, **IA3**

$$\sigma(l) \Vdash_R a = b, \tag{6.50}$$

where $\sigma(l) \equiv i^t \langle (r^{prd} \cdot l_{01})_1, i^s \cdot (r^{prd} \cdot l_{11})_1 \rangle$. Moreover, by the definition,

$$\langle n, i^r \rangle \Vdash_R \exists x(n, x) \in \underline{f}. \tag{6.51}$$

By (6.50) and (6.51) and Lemma 6.2.6, we have shown that

$$\Lambda n.\langle\langle n, i^r\rangle, \Lambda l.\sigma(l)\rangle \Vdash_R \forall n \exists! x(n,x) \in \underline{f}. \tag{6.52}$$

Thus by (6.49) and (6.51), it follows that

$$\gamma(e) \Vdash_R Fun(\underline{f}, N, a), \tag{6.53}$$

where $\gamma(e) \equiv \langle \Lambda t.\langle t_1, \xi(t,e)\rangle, \Lambda n.\langle\langle n, i^r\rangle, \Lambda l.\sigma(l)\rangle\rangle$.
♢ Secondly, since $e_{01} \Vdash_R b = F^1(0)$ and $\langle 0, i^r\rangle \Vdash_R (0, F^1(0))_R \in \underline{f}$, by the soundness of **IA2** and Substitution

$$\beta(e) \Vdash_R (0,b)_R \in \underline{f}, \tag{6.54}$$

where $\beta(e) \equiv r_C^{stu} \cdot \langle i^s \cdot e_{01}, \langle 0, i^r\rangle\rangle$, and $C \equiv (0,x)_R \in \underline{f}$.
♢ Lastly, by Lemma 6.2.6 and (6.46), also

$$\Lambda n.\delta(n,e) \Vdash_R \forall n \exists x \exists y[x = F^1(n) \wedge y = F^1(n+1) \wedge A(x,y)], \tag{6.55}$$

where $\delta(n,e) \equiv \langle\langle i^r, i^r\rangle, r_A^{stu} \cdot \langle (F_e^{\#}(n+1))_{01}, (F_e^{\#}(n+1))_1\rangle\rangle$. Hence the result follows immediately from (6.53), (6.54) and (6.55). ■

Lemma 6.2.53 (**RDC**)

$$r_{\mathrm{R}}^{rdc} \Vdash_R \forall z[A(z) \wedge \forall x(A(x) \to \exists y(A(y) \wedge B(x,y))) \to$$
$$\exists f(Fun(f,N) \wedge f(0) = z \wedge \forall n(A(f(n)) \wedge B(f(n), f(n+1))))],$$

where $r_{\mathrm{R}}^{rdc} \equiv \Lambda e.\langle\langle \eta, \langle 0, i^r\rangle\rangle, \xi(e)\rangle$ *and where* $\eta \equiv \langle \Lambda g.g, \Lambda n.\langle\langle n, i^r\rangle, \Lambda l.\sigma(l)\rangle\rangle$.

Proof. We show this via Relativized Dependent Choice in the background theory. Let $c \in V^*$ and $e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_R A(c) \wedge \forall x(A(x) \to \exists y(A(y) \wedge B(x,y))). \tag{6.56}$$

Let us use the following class notation

$$\mathcal{A} \equiv \{(s,d) : s \in \mathbb{N} \wedge d \in V^* \wedge s \Vdash_R A(d)\}.$$

Define $\varphi(z) \equiv z \in \mathcal{A}$, i.e.,

$$\varphi(z) \equiv \exists s \in \mathbb{N} \exists d \in V^*[z = (s,d) \wedge s \Vdash_R A(d)].$$

By (6.56), it then follows that

$$\varphi((e_0, c)) \wedge \forall x(\varphi(x) \to \exists y(\varphi(y) \wedge \theta(x,y))),$$

where $\theta(x,y) \equiv \exists s \in \mathbb{N} \exists d, b \in V^*[x = (s,d) \wedge y = ((e_1 \cdot s)_0, b) \wedge (e_1 \cdot s)_1 \Vdash_R B(d,b)]$. By **RDC**, it follows that there is a function F with domain $\mathbb{N}$ such that $F(0) = (e_0, c)$ and $\forall n \in \mathbb{N}[\varphi(F(n)) \wedge \theta(F(n), F(n+1))]$, i.e., for all n in $\mathbb{N}$

$$F^0(n) \Vdash_R A(F^1(n)) \wedge F^0(n+1) = (e_1 \cdot F^0(n))_0 \\ \wedge (e_1 \cdot F^0(n))_1 \Vdash_R B(F^1(n), F^1(n+1)), \tag{6.57}$$

where F^0 and F^1 denote the left and right projection functions of F, respectively. Since $F^0(0) = e_0$ and $F^0(n+1) = (e_1 \cdot F^0(n))_0$, one has that F^0 is a total recursive function. Let $F_e^{\#}$ be its code. Now define

$$\underline{f} \equiv \{(n, (n, F^1(n))_R) : n \in \mathbb{N}\}.$$

By Claim 3.3.6, $\underline{f} \in \mathbb{S}^*$. Now we want to find a realizer which realizes

$$Fun(\underline{f}, N) \wedge \underline{f}(0) = c \wedge \forall n(A(f(n)) \wedge B(\underline{f}(n), \underline{f}(n+1))).$$

$\diamondsuit$ First of all, let us find a realizer for $Fun(\underline{f}, N)$. Let $g \in \mathbb{N}$ and $a \in V^*$ be arbitrary such that $g \Vdash_R a \in \underline{f}$. By the definition, it follows that $g_1 \Vdash_R a = (g_0, F^1(g_0))_R$ and $g \Vdash_R \exists n \exists z[a = (n,z)]$; thus,

$$\Lambda g.g \Vdash_R \forall x \in \underline{f}[\exists n \exists z(x = (n,z))]. \tag{6.58}$$

Moreover, from (6.52) in the proof of **DC**, one knows that

$$\Lambda n.\langle\langle n, i^r\rangle, \Lambda l.\sigma(l)\rangle \Vdash_R \forall n \exists! z(n,z) \in \underline{f}. \tag{6.59}$$

where $\sigma(l)$ is defined in that proof. Hence from (6.58) and (6.59), we have shown that

$$\eta \Vdash_R Fun(\underline{f}, N), \tag{6.60}$$

where $\eta \equiv \langle \Lambda g.g, \Lambda n.\langle\langle n, i^r\rangle, \Lambda l.\sigma(l)\rangle\rangle$.
$\diamondsuit$ Secondly, since $(0,(0,c)_R) \in \underline{f}$, by the soundness of **IA1** it follows that

$$\langle 0, i^r\rangle \Vdash_R (0,c)_R \in \underline{f}. \tag{6.61}$$

$\diamondsuit$ Lastly, by (6.57) and the soundness of **IA1**, also

$$\delta(n,e) \Vdash_R \exists y(n,y) \in \underline{f} \wedge A(y),$$
$$\varepsilon(n,e) \Vdash_R \exists x \exists y[(n,x) \in \underline{f} \wedge (n+1,y) \in \underline{f} \wedge B(x,y)],$$

where $\delta(n,e) \equiv \langle\langle n, i^r\rangle, F_e^{\#}(n)\rangle$, $\varepsilon(n,e) \equiv \langle\langle\langle n, i^r\rangle, \langle n{+}1, i^r\rangle\rangle, (e_1 \cdot F_e^{\#}(n))_1\rangle$. By Lemma 6.2.6, it then follows that

$$\xi(e) \Vdash_R \forall n[A(\underline{f}(n)) \wedge B(\underline{f}(n), \underline{f}(n+1)))], \tag{6.62}$$

where $\xi(e) \equiv \Lambda n.\langle \delta(n,e), \varepsilon(n,e)\rangle$. Hence the result follows immediately from (6.60), (6.61) and (6.62). ■

Now we want to show that the Presentation Axiom is realizable. The following intuitionistic proof is based on Theorem 10.1 in Michael Rathjen's paper [20]. For any function F with range $D \subseteq V^2$, we will use F^0 to denote the left projection function and F^1 to denote the right projection function of F. We also use x^0 to denote $F^0(x)$ and x^1 to denote $F^1(x)$. Recall that $SFun(f,y,x)$ is the predicate denoting $Fun(f,y,x) \wedge \forall v \in x \exists u \in y (u,v) \in f$ (i.e., f is a surjective function from y to x) and $Base(y)$ is the predicate denoting $\forall r[Rel(r,y) \rightarrow \exists g(Fun(g,y) \wedge g \subseteq r)]$ (i.e., every relation with domain y has a sub-function).

Lemma 6.2.54 ($\mathbf{PA}_X$)

$$r_{\text{R}}^{pa} \Vdash_R \forall x(S(x) \rightarrow \exists y \exists f[Base(y) \wedge SFun(f,y,x)]),$$

where $r_{\text{R}}^{pa} \equiv \Lambda n.\langle \Lambda e.\epsilon(e), \pi\rangle$.

Proof. We show this via the Presentation Axiom in the background theory. Let $a \in \mathbb{S}^*$ be arbitrary. If a is empty, then by default, any code will realize $Base(\emptyset) \wedge SFun(\emptyset,\emptyset,\emptyset)$. Now let a be any arbitrary inhabited internal set. We want to show that there is a realizer that will realize

$$\exists y \exists f[Base(y) \wedge SFun(f,y,a)].$$

By $\mathbf{PA}_X$ in the background theory, externally, there exist a base B and a surjective function $F_B : B \to a$. Now we have to give the internal counterpart for such a base, say, B^* and such a surjective function, say, F^*. The whole idea is to internalize $(V,=,\in)$ and yield an (internal) isomorphic structure $(V^\dagger,=,\in)$, where $V^\dagger \equiv \{u^\dagger : u \in V\}$ and $u^\dagger \equiv \{(0,k^\dagger) : k \in u\}$. By Claim 3.3.6 and set induction, $V^\dagger \subseteq V^*$. Furthermore, from this definition, also

$$u = v \leftrightarrow u^\dagger = v^\dagger, \tag{6.63}$$

and (by mutual $\lhd^2$-induction)

$$u = v \leftrightarrow V^* \models_R u^\dagger = v^\dagger, \tag{6.64}$$
$$u \in v \leftrightarrow V^* \models_R u^\dagger \in v^\dagger.$$

Define $u^* \equiv (F^0_B(u), (F^0_B(u), u^\dagger)_R)$ and $B^* \equiv \{u^* : u \in B\}$. By Claim 3.3.6, $B^* \in \mathbb{S}^*$. From (6.63), $u = v \leftrightarrow u^* = v^*$ and this creates a bijective function between B and B^*.

♢ Now we want to show that, externally, B^* is also a base. Let $R^* \in V$ be any arbitrary binary relation such $Rel(R^*, B^*)$. Then one finds a new relation $R := \{(u, v) : u \in B \wedge (u^*, v) \in R^*\}$. Since $Rel(R, B)$ and B is a base, one finds a function F_R with domain B such that $F_R \subseteq R$. Correspondingly, one finds a new function F_{R^*} with domain B^* such that

$$F_{R^*} := \{(u^*, v) : (u, v) \in F_R\} \subseteq R^*.$$

Hence we have shown that, externally, B^* is also a base.

♢ Now we want to show that, internally, B^* is also a base, i.e., $V^* \models_R Base(B^*)$. Let $r \in V^*$ and $e \in \mathbb{N}$ be arbitrary such that $e \Vdash_R Rel(r, B^*)$. By the definition, it follows that

$$e_1 \Vdash_R \forall u \in B^* \exists v (u, v) \in r.$$

Hence one has

$$\forall x \in B^* \exists y \theta(x, y), \tag{6.65}$$

where $\theta(x, y) \equiv \exists f, g \in \mathbb{N} \exists c, d \in V^*[x = (f, d) \wedge y = (g, c) \wedge g = e_1 \cdot \langle f, i^r \rangle \wedge g \Vdash_R (d, c)_R \in r]$. Now define a new relation

$$K \equiv \{(x, y) : x \in B^* \wedge \theta(x, y)\}. \tag{6.66}$$

Since K is total in B^* and externally B^* is a base, one finds a function F_K (it is total in B^*) such that $F_K \subseteq K$. Moreover, one defines

$$\underline{F_K} \equiv \{(x^0, (x^1, F^1_K(x))_R) : x \in B^*\}.$$

By Claim 3.3.6, $\underline{F_K} \in \mathbb{S}^*$.

♢ Now we want to find a realizer for $Fun(\underline{F_K}, B^*)$. Let $k \in \mathbb{N}$ and $a \in V^*$ be arbitrary such that $k \Vdash_R a \in \underline{F_K}$. Then by the definitions, it follows that

$$\delta \Vdash_R \forall x \in \underline{F_K} \exists u \in B^* \exists v [x = (u, v)], \tag{6.67}$$

where $\delta \equiv \Lambda k.\langle \langle k_0, i^r \rangle, k_1 \rangle$. Now let $a \in V^*$ and $t \in \mathbb{N}$ be arbitrary such that $t \Vdash_R a \in B^*$. Then by the definitions, the soundness of **IA2** and Substitution,

$$\eta(t) \Vdash_R \exists y (a, y) \in \underline{F_K}, \tag{6.68}$$

where $\eta(t) \equiv r_A^{stu} \cdot \langle i^s \cdot t_1, \langle t_0, i^r \rangle \rangle$ and where $A \equiv \exists y(x,y) \in z$. Now let $h \in \mathbb{N}$ and $b, c \in V^*$ be arbitrary such that

$$h \Vdash_R (a,b)_R \in \underline{F_K} \wedge (a,c)_R \in \underline{F_K}.$$

By the definitions, there exist $x, y \in B^*$ such that $x^0 = h_{00}, y^0 = h_{10}$ and

$$h_{01} \Vdash_R (a,b)_R = (x^1, F_K^1(x))_R, h_{11} \Vdash_R (a,c)_R = (y^1, F_K^1(y))_R.$$

By Corollary 6.2.32, it follows that

$$(r^{prd} \cdot h_{01})_0 \Vdash_R a = x^1, (r^{prd} \cdot h_{11})_0 \Vdash_R a = y^1, \tag{6.69}$$

$$(r^{prd} \cdot h_{01})_1 \Vdash_R b = F_K^1(x), (r^{prd} \cdot h_{11})_1 \Vdash_R c = F_K^1(y). \tag{6.70}$$

Since

$$x, y \in B^*, x^0 = h_{00}, y^0 = h_{10},$$

by the definition, $x^1 = (h_{00}, u^\dagger)_R, y^1 = (h_{10}, v^\dagger)_R$ for some $u, v \in V$. Hence by (6.69), Corollary 6.2.32 and (6.64), $x = y$ and thus by (6.70) and the soundness of **IA2**, **IA3**

$$\delta(h) \Vdash_R b = c,$$

where $\delta(h) \equiv i^t \cdot \langle (r^{prd} \cdot h_{01})_1, i^s \cdot (r^{prd} \cdot h_{11})_1 \rangle$. With this and (6.68), we have shown that

$$\langle \Lambda t.\eta(t), \Lambda h.\delta(h) \rangle \Vdash_R Fun(\underline{F_K}, B^*) \tag{6.71}$$

$\diamondsuit$ Now we want to find a realizer for $\underline{F_K} \subseteq r$. Let $s \in \mathbb{N}$ and $c \in V^*$ be arbitrary such that $s \Vdash_R c \in \underline{F_K}$, i.e.,

$$\exists y \in B^*[y^0 = s_0 \wedge (s_0, (y^1, F_K^1(y))_R) \in \underline{F_K} \wedge$$
$$s_1 \Vdash_R c = (y^1, F_K^1(y))_R].$$

By the definition of $\underline{F_K}$, (6.65), (6.66) and the fact that $(y, F_K(y)) \in F_K \subseteq K$, it then follows that $\theta(y, F_K(y))$, i.e.,

$$e_1 \cdot \langle s_0, i^r \rangle \Vdash_R (y^1, F_K^1(y))_R \in r.$$

Hence by the soundness of **IA4** one can find a partial recursive function σ such that $\sigma(s,e) \Vdash_R c \in r$, where $\sigma(s,e) \equiv i^{eb} \cdot \langle s_1, e_1 \cdot \langle s_0, i^r \rangle \rangle$. Therefore, we have shown that

$$\Lambda s.\sigma(s,e) \Vdash_R \underline{F_K} \subseteq r \tag{6.72}$$

From (6.71) and (6.72), it follows that

$$\Lambda e.\epsilon(e) \Vdash_R Base(B^*), \tag{6.73}$$

where $\epsilon(e) \equiv \langle\langle \Lambda t.\eta(t), \Lambda h.\delta(h)\rangle, \Lambda s.\sigma(s, e)\rangle$.
Now define

$$F^* \equiv \{(F^0_B(u), ((F^0_B(u), u^\dagger)_R, F^1_B(u))_R) : u \in B\}.$$

We want to show that $V^* \models_R SFun(F^*, B^*, a)$, where

$$SFun(F^*, B^*, a) \equiv F^* \subseteq B^* \times a \wedge \forall x \in B^* \exists! y \in a (x, y) \in F^* \wedge \\ \forall y \in a \exists x \in B^* (x, y) \in F^*.$$

Let $d \in V^*$ and $k \in \mathbb{N}$ be arbitrary such that $k \Vdash_R d \in F^*$. We have to find a partial recursive function ρ such that $\rho(k) \Vdash_R \exists x \exists y[d = (x, y) \wedge x \in B^* \wedge y \in a]$. From $k \Vdash_R d \in F^*$,

$$\exists u \in B[F^0_B(u) = k_0 \wedge k_1 \Vdash_R d = ((F^0_B(u), u^\dagger)_R, F^1_B(u))_R)].$$

By the definition of B^* and F_B, it then follows that

$$\langle\langle k_0, i^r\rangle, \langle k_0, i^r\rangle\rangle \Vdash_R (F^0_B(u), u^\dagger)_R \in B^* \wedge F^1_B(u) \in a.$$

Hence we have shown that

$$\Lambda k.\rho(k) \Vdash_R F^* \subseteq B^* \times a, \tag{6.74}$$

where $\rho(k) \equiv \langle k_1, \langle\langle k_0, i^r\rangle, \langle k_0, i^r\rangle\rangle\rangle$.

Now let $q \in V^*$ and $m \in \mathbb{N}$ be arbitrary such that $m \Vdash_R q \in B^*$, i.e., by the definitions, there exists $u \in B$ such that $m_0 = F^0_B(u)$ and $m_1 \Vdash_R q = (F^0_B(u), u^\dagger)_R$, and thus by the soundness of **IA2** and Substitution,

$$\langle\langle m_0, i^r\rangle, r^{stu}_E \cdot \langle i^s \cdot m_1, \langle m_0, i^r\rangle\rangle\rangle \Vdash_R \\ F^1_B(u) \in a \wedge (q, F^1_B(u))_R \in F^*,$$

where $E \equiv (x, y)_R \in z$. Moreover, let $a, b, c \in V^*$ and $t \in \mathbb{N}$ be arbitrary such that $t \Vdash_R (a, b)_R \in F^* \wedge (a, c)_R \in F^*$. By the definitions, it follows that there exists $u, v \in B$ such that $F^0_B(u) = t_{00}, F^0_B(v) = t_{10}$ and that

$$t_{01} \Vdash_R (a, b)_R = ((F^0_B(u), u^\dagger)_R, F^1_B(u))_R, \tag{6.75}$$

$$t_{11} \Vdash_R (a, c)_R = ((F^0_B(v), v^\dagger)_R, F^1_B(v))_R. \tag{6.76}$$

With this and Corollary 6.2.32, (6.64) and the soundness of **IA2**, **IA3**, $u = v$ and thus $\theta(t) \Vdash_R b = c$, where $\theta(t) \equiv i^t \cdot \langle (r^{prd} \cdot t_{01})_1, i^s \cdot (r^{prd} \cdot t_{11})_1 \rangle$. Hence we have shown that

$$\Lambda m.\langle\langle\langle m_0, i^r\rangle, r_E^{stu} \cdot \langle i^s \cdot m_1, \langle m_0, i^r\rangle\rangle\rangle, \Lambda t.\theta(t)\rangle \\ \Vdash_R \forall x \in B^* \exists! y \in a(x, y) \in F^*. \tag{6.77}$$

This shows that, internally, F^* is a function with domain B^* and range a. ♢ Now we want to show that, internally, F^* is also surjective. Let $b \in V^*$ and $h \in \mathbb{N}$ be arbitrary such that $h \Vdash_R b \in a$, i.e., by the fact that F is a surjective function from B to a, there is a set u such that $u \in B$ and

$$F_B^0(u) = h_0 \wedge (F_B^0(u), F_B^1(u)) \in a \wedge h_1 \Vdash_R b = F_B^1(u).$$

By the definition of B^* and F^*, it follows that

$$(h_0, (h_0, u^\dagger)_R) \in B^* \wedge (h_0, ((h_0, u^\dagger)_R, F_B^1(u))_R) \in F^*,$$

and thus by Substitution and the soundness of **IA2**,

$$\langle\langle h_0, i^r\rangle, r_E^{stu} \cdot \langle i^s \cdot h_1, \langle h_0, i^r\rangle\rangle\rangle \Vdash_R \\ (h_0, u^\dagger)_R \in B^* \wedge ((h_0, u^\dagger)_R, b)_R \in F^*.$$

Thus we have shown that

$$\Lambda h.\langle\langle h_0, i^r\rangle, r_E^{stu} \cdot \langle i^s \cdot h_1, \langle h_0, i^r\rangle\rangle\rangle \Vdash_R \\ \forall y \in a \exists x \in B^*(x, y) \in F^* \tag{6.78}$$

From (6.74) and (6.77) and (6.78), one finds a realizer π such that $\pi \Vdash_R SFun(F^*, B^*, a)$. With this and (6.73), the result follows immediately. ■

Lemma 6.2.55 (**MP**)

$$r_{\text{R}}^{mp} \Vdash_R [\forall n(A(n) \vee \neg A(n)) \wedge \neg\neg\exists n(A(n))] \to \exists n(A(n)),$$

where $r_{\text{R}}^{mp} \equiv \Lambda e.\langle f^\#(e), (e_0 \cdot f^\#(e))_1\rangle$.

Proof. Let $e \in \mathbb{N}$ be arbitrary such that

$$e \Vdash_R [\forall n(A(n) \vee \neg A(n)) \wedge \neg\neg\exists n(A(n)].$$

Since $e_1 \Vdash_R \neg\neg\exists n A(n)$, by the definition, it follows that

$$\neg\neg\exists m(m \Vdash_R \exists n A(n)). \tag{6.79}$$

Since $e_0 \Vdash_R \forall n(A(n) \vee \neg A(n))$, by Lemma 6.2.6, also

$$\forall n \in \mathbb{N}[e_0 \cdot n \downarrow \wedge e_0 \cdot n \Vdash_R A(n) \vee \neg A(n)],$$

i.e., for all n in $\mathbb{N}$

$$\begin{aligned} &[(e_0 \cdot n)_0 = 0 \wedge (e_0 \cdot n)_1 \Vdash_R A(n)] \vee \\ &[(e_0 \cdot n)_0 \neq 0 \wedge (e_0 \cdot n)_1 \Vdash_R \neg A(n)], \end{aligned} \tag{6.80}$$

and thus by the definition

$$\forall n \in \mathbb{N}[B(n) \vee \neg B(n)], \tag{6.81}$$

where $B(n) \equiv \exists m(m \Vdash_R A(n))$. Now we want to show that $\neg\neg\exists n B(n)$. Assume $\neg\exists n B(n)$, i.e., $\forall n \neg\exists m(m \Vdash_R A(n))$, i.e., by Lemma 6.2.6

$$\forall n \neg\exists m(\langle n, m\rangle \Vdash_R \exists l A(l)). \tag{6.82}$$

But $\exists k(k \Vdash_R \exists n A(n))$ clearly contradicts (6.82), so we deduce that $\neg\exists k(k \Vdash_R \exists n A(n))$, contrary to (6.79) and thus $\neg\neg\exists n B(n)$. With this and (6.81),

$$\forall n[B(n) \vee \neg B(n)] \wedge \neg\neg\exists n B(n)$$

and thus, by **MP** in the background theory, $\exists n \exists m(m \Vdash_R A(n))$, i.e., by (6.80) $\exists n(e_0 \cdot n)_0 = 0$. With this and $\forall n \in \mathbb{N}[(e_0 \cdot n)_0 = 0 \vee (e_0 \cdot n)_0 \neq 0]$, one defines a partial recursive function $f(e) := \mu n[(e_0 \cdot n)_0 = 0]$. Let $f^{\#}$ be the Gödel number of f. Then by (6.80),

$$\langle f^{\#}(e), (e_0 \cdot f^{\#}(e))_1\rangle \Vdash_R \exists n(A(n)).$$

■

In conclusion, we have invented a version of relativized realizability which interprets Heyting arithmetic, $\mathbf{CZF}_N$, $\mathbf{IZF}_N$ and a plethora of semi-constructive axioms. These results play an important role in the inferences of our conservativity results in Chapter 8.

Chapter 7

Forcing

In order to derive our conservativity results, in Section 7.1 we provide a version of a forcing interpretation. Then in Section 7.2 we use this interpretation to interpret Heyting arithmetic, $\mathbf{CZF}_N$ and $\mathbf{IZF}_N$. Let $\mathbb{P}$ be the set of all the finite partial functions from $\mathbb{N}$ to $\mathbb{N}$, i.e.,

$$\mathbb{P} \equiv \{p : p \text{ is a finite function} \wedge dom(p), ran(p) \subset \mathbb{N}\}.$$

The usual order for $\mathbb{P}$ is defined as $p \succeq q$ iff $p \supseteq q$. We also use $f \preceq g$ to denote $g \succeq f$ and use 0 to denote the empty function (i.e., its domain is the empty set).

To obtain our conservativity results, we shall consider various subsets $\mathbb{E}$ of $\mathbb{P}$ as sets of forcing conditions. We will always assume that $0 \in \mathbb{E}$. In other words, the actual form of $\mathbb{E}$ can be anything from $\{0\}$ to $\mathbb{P}$.

The interpretation structure for forcing with $\mathbb{E}$ is $(\mathbb{E}, \succeq, V^*, \Vdash_{\mathcal{F}})$, where V^* is defined as follows (cf. Section 3.3.2):

$$\begin{aligned} V^{\mathbb{E}}_{\alpha} &= \bigcup_{\beta \in \alpha} \mathcal{P}(\mathbb{E} \times (V^{\mathbb{E}}_{\beta} \cup \mathbb{N})) \\ \mathbb{S}^* &= \bigcup_{\alpha \in On} V^{\mathbb{E}}_{\alpha} \\ V^* &= \mathbb{N} \cup \mathbb{S}^* \end{aligned}$$

where $\mathcal{P}$ denotes the Powerset operation. $\Vdash_{\mathcal{F}}$ is an interpretation that will be defined next.

We will use variables f, g, h, p, q, k, n, m to range over $\mathbb{E}$. Though there is a danger that k, n, m might be mistaken for members of $\mathbb{N}$, we think that it will always be sufficiently clear from the context whether we are talking about partial functions or whether we are talking about natural numbers. Besides, we can also view $\mathbb{E}$ as a subset of $\mathbb{N}$ as the members of $\mathbb{E}$ can be coded as naturals.

7.1 Definition of forcing

In what follows we use the abbreviations $\forall f \succeq g\, \varphi(f)$ and $\exists f \succeq g\, \varphi(f)$ for $\forall f{\in}\mathbb{E}[f \succeq g \to \varphi(f)]$ and $\exists f{\in}\mathbb{E}\,[f \succeq g \wedge \varphi(f)]$, respectively. In connection with elements a from $\mathbb{S}^*$, $\forall (f,d){\in}\, a\, \psi(f,d)$ and $\exists (f,d){\in}\, a\, \psi(f,d)$ are short for $\forall f{\in}\, \mathbb{E}\ \forall d{\in}V^*\,[(f,d) \in a \to \psi(f,d)]$ and $\exists f{\in}\mathbb{E}\ \exists d{\in}V^*\,[(f,d) \in a \wedge \psi(f,d)]$, respectively.

The relation $h \Vdash_{\mathcal{F}} \varphi$ with $h \in \mathbb{E}$ and φ being a sentence with parameters from V^* is defined inductively as follows:

- $h \Vdash_{\mathcal{F}} R(a_1, a_2, ..., a_n)$ iff $a_1, a_2, ..., a_n \in \mathbb{N} \wedge R(a_1, a_2, ..., a_n)$.
- $h \Vdash_{\mathcal{F}} N(a)$ iff $a \in \mathbb{N}$.
- $h \Vdash_{\mathcal{F}} S(a)$ iff $a \in \mathbb{S}^*$.
- $h \Vdash_{\mathcal{F}} a \in b$ iff $\exists n \in \mathbb{E} \exists c \in V^*[n \preceq h \wedge (n,c) \in b \wedge h \Vdash_{\mathcal{F}} a = c]$.
- $h \Vdash_{\mathcal{F}} a = b$ iff $a, b \in \mathbb{N} \wedge a = b$ or
$$\begin{aligned} a, b \in \mathbb{S}^* \quad &\wedge \quad \forall (f,d) \in a \forall n \succeq h, f\, \exists m \succeq n(m \Vdash_{\mathcal{F}} d \in b) \\ &\wedge \quad \forall (g,k) \in b \forall s \succeq h, g\, \exists t \succeq s(t \Vdash_{\mathcal{F}} k \in a). \end{aligned}$$
- $h \Vdash_{\mathcal{F}} A \wedge B$ iff $h \Vdash_{\mathcal{F}} A \wedge h \Vdash_{\mathcal{F}} B$.
- $h \Vdash_{\mathcal{F}} A \vee B$ iff $h \Vdash_{\mathcal{F}} A \vee h \Vdash_{\mathcal{F}} B$.
- $h \Vdash_{\mathcal{F}} \neg A$ iff $\forall n \succeq h \neg(n \Vdash_{\mathcal{F}} A)$.
- $h \Vdash_{\mathcal{F}} A \to B$ iff $\forall n \succeq h[n \Vdash_{\mathcal{F}} A \to \exists m \succeq n(m \Vdash_{\mathcal{F}} B)]$.
- $h \Vdash_{\mathcal{F}} \forall x A(x)$ iff $\forall n \succeq h \forall c \in V^* \exists m \succeq n(m \Vdash_{\mathcal{F}} A[x/c])$.
- $h \Vdash_{\mathcal{F}} \exists x A(x)$ iff $\exists c \in V^*(h \Vdash_{\mathcal{F}} A[x/c])$.

  Furthermore, one defines $h \Vdash_{\mathcal{F}} A(x)$ iff $h \Vdash_{\mathcal{F}} \forall x A(x)$ and $V^* \models_{\mathcal{F}} A$ iff $\forall h \in \mathbb{E} \exists h' \succeq h[h' \Vdash_{\mathcal{F}} A]$.

The following Monotonicity Lemma is a key feature of forcing interpretations.

Lemma 7.1.1 [*Monotonicity*] *for any formulae* $\varphi \in \mathcal{L}(V^*)$

$$\forall h, k \in \mathbb{E}[(h \Vdash_{\mathcal{F}} \varphi \wedge k \succeq h) \rightarrow k \Vdash_{\mathcal{F}} \varphi].$$

Proof. Let $h \in \mathbb{E}$ be arbitrary such that $h \Vdash_{\mathcal{F}} \varphi$. Let $k \in \mathbb{E}$ be arbitrary such that $k \succeq h$.
$\diamondsuit$ $\varphi \equiv a = b$: If $a = b \in \mathbb{N}$, then by definition, $\forall n \in \mathbb{E}(n \Vdash_{\mathcal{F}} a = b)$. If $a, b \in \mathbb{S}^*$, then $h \Vdash_{\mathcal{F}} a = b$ iff for all $(f, d) \in a$ and $(g, v) \in b$,

$$\forall r \succeq h, f \exists r' \succeq r[r' \Vdash_{\mathcal{F}} d \in b] \ \wedge \ \forall s \succeq h, g \exists s' \succeq s[s' \Vdash_{\mathcal{F}} v \in a].$$

Now let $f \in \mathbb{E}$ and $d \in V^*$ be arbitrary such that $(f, d) \in a$. Let $p \in \mathbb{E}$ be arbitrary such that $p \succeq k, f$. Since $p \succeq h, f$, by the assumption we have

$$\exists p' \succeq p[p' \Vdash_{\mathcal{F}} d \in b];$$

similarly, we have

$$\forall (g, v) \in b \forall q \succeq k, g \exists q' \succeq q[q' \Vdash_{\mathcal{F}} v \in a].$$

Hence we have shown that

$$(h \Vdash_{\mathcal{F}} a = b) \wedge k \succeq h \rightarrow k \Vdash_{\mathcal{F}} a = b. \tag{7.1}$$

$\diamondsuit$ $\varphi \equiv a \in b$: By the definition we have $h \Vdash_{\mathcal{F}} a \in b$ if and only if

$$\exists n \in \mathbb{E} \exists c \in V^*[n \preceq h \wedge (n, c) \in b \wedge h \Vdash_{\mathcal{F}} a = c].$$

By (7.1), the result follows from the definition immediately.

As for the compound formulae, we deal with the following clauses. Others follow immediately from the inductive hypotheses and the definitions.
$\diamondsuit$ $\varphi \equiv \forall x \eta(x)$: By the definition, $h \Vdash_{\mathcal{F}} \forall x \eta(x)$ iff $\forall q \succeq h \forall a \in V^* \exists q' \succeq q[q' \Vdash_{\mathcal{F}} \eta(a)]$. Now we want to show that $k \Vdash_{\mathcal{F}} \forall x \eta(x)$. Let $q \in \mathbb{E}$ be arbitrary such that $q \succeq k$. Let $a \in V^*$ be arbitrary. Since $q \succeq k \succeq h$, by the assumption, one has $\exists q' \succeq q[q' \Vdash_{\mathcal{F}} \eta(a)]$.
$\diamondsuit$ $\varphi \equiv \zeta \rightarrow \eta$: By the definition $h \Vdash_{\mathcal{F}} \zeta \rightarrow \eta$ iff $\forall q \succeq h[q \Vdash_{\mathcal{F}} \zeta \rightarrow \exists r(r \succeq q \wedge r \Vdash_{\mathcal{F}} \eta)]$. Let $s \in \mathbb{E}$ be arbitrary such that $s \succeq k \succeq h \wedge$ $s \Vdash_{\mathcal{F}} \zeta$. By the assumption $\exists r(r \succeq s \wedge r \Vdash_{\mathcal{F}} \eta)$. $\blacksquare$

7.2 Soundness of forcing

In this section we will show that the structure $(\mathbb{E}, \succeq, V^*, \Vdash_{\mathcal{F}})$ interprets all axioms of $\mathbf{CZF}_N$, if one assumes $\mathbf{CZF}_N$ in the background theory. Moreover, it also interprets all axioms of $\mathbf{IZF}_N$, if one assumes $\mathbf{IZF}_N$ in the background theory. In what follows we always argue in one of these background theories.

Notation 7.2.1 $\vec{a} \in V^* \equiv a_1, a_2, ..., a_n \in V^*$.

Notation 7.2.2 $\forall n A(n, \vec{x}) \equiv \forall y[N(y) \to A(y, \vec{x})]$.

Notation 7.2.3 $\exists n A(n, \vec{x}) \equiv \exists y[N(y) \wedge A(y, \vec{x})]$.

Notation 7.2.4 *We will also use* $n \in \mathbb{N}$ *for* $N(n)$.

Lemma 7.2.5 $\forall n\, [\forall k \succeq h \exists k' \succeq k(k' \Vdash_{\mathcal{F}} A(n, \vec{a}))] \leftrightarrow h \Vdash_{\mathcal{F}} \forall n A(n, \vec{a})$, *for all* $h \in \mathbb{E}$.

Proof. To show $h \Vdash_{\mathcal{F}} \forall n A(n, \vec{a})$, it suffices to prove that

$$\forall c \in V^* \forall k \succeq h[k \Vdash_{\mathcal{F}} N(c) \to A(c, \vec{a})].$$

Let $k \in \mathbb{E}$ be arbitrary such that $k \succeq h$. Let $c \in V^*$ and $t \in \mathbb{E}$ be arbitrary such that $t \succeq k$ and $t \Vdash_{\mathcal{F}} N(c)$, i.e., $c \in \mathbb{N}$. Then by the assumption and the definition, $\exists t' \succeq t \succeq k[t' \Vdash_{\mathcal{F}} A(c, \vec{a})]$. On the other hand, assume $h \Vdash_{\mathcal{F}} \forall n A(n, \vec{a})$, i.e.,

$$\forall c \in V^* \forall k \succeq h \exists v \succeq k[v \Vdash_{\mathcal{F}} (N(c) \to A(c, \vec{a}))].$$

Let $n \in \mathbb{N}$ be arbitrary. Let $k \in \mathbb{E}$ be arbitrary such that $k \succeq h$. Then $\exists v \succeq k[v \Vdash_{\mathcal{F}} (N(n) \to A(n, \vec{a}))]$. Since $v \Vdash_{\mathcal{F}} N(n)$, it follows that $\exists v' \succeq v \succeq k[v' \Vdash_{\mathcal{F}} A(n, \vec{a})]$. ■

Corollary 7.2.6 $\forall n \in \mathbb{N}[h \Vdash_{\mathcal{F}} A(n, \vec{a}))] \to h \Vdash_{\mathcal{F}} \forall n A(n, \vec{a})$, *for all* $h \in \mathbb{E}$.

Proof. This follows immediately from the above result and the Monotonicity Lemma. ■

This corollary will be used in the following **A2:**2 to **A2:**11 without explicitly mentioning it.

7.2.1 A1: Axioms on numbers and sets

Claim 7.2.7 [**A1** :1] $V^* \models_{\mathcal{F}} \forall x \neg(N(x) \wedge S(x))$.

Proof. Let $c \in V^*$ be arbitrary. It suffices to prove that $0 \Vdash_{\mathcal{F}} \neg(N(c) \wedge S(c))$. Assume $\exists n \in \mathbb{E}$ such that $n \Vdash_{\mathcal{F}} (N(c) \wedge S(c))$. Then by the definition, it follows that $c \in \mathbb{N} \cap \mathbb{S}^*$, but this contradicts Corollary 3.3.3. ■

Claim 7.2.8 [**A1** :2] $V^* \models_{\mathcal{F}} \forall x \forall y[x \in y \to S(y)]$.

Proof. Let $a, b \in V^*$ be arbitrary. It suffices to prove that $0 \Vdash_{\mathcal{F}} a \in b \to S(b)$. Let $n \in \mathbb{E}$ be arbitrary such that $n \Vdash_{\mathcal{F}} a \in b$, i.e., $\exists m \preceq n \exists c \in V^*(m, c) \in b$, i.e., $b \in \mathbb{S}$. Hence $b \in \mathbb{S}^*$, thus $0 \Vdash_{\mathcal{F}} S(b)$. ■

Claim 7.2.9 [**A1** :3] *$V^* \models_{\mathcal{F}} N(\bar{n})$ for all natural number n.*

Proof. This is obvious as $N(\bar{n})$ is an axiom of the background theory. ■

7.2.2 A2: Number-theoretic axioms

Claim 7.2.10 [**A2** :1] *$V^* \models_{\mathcal{F}} \mathrm{SUC}(\bar{n}, \overline{n+1})$ for all natural number n.*

Proof. Again this follows immediately from the fact that $\mathrm{SUC}(\bar{n}, \overline{n+1})$ is an axiom of the background theory. ■

In the following proofs from **A2:**2 to **A2:**11, we will use Corollary 7.2.6 implicitly.

Claim 7.2.11 [**A2** :2] *$V^* \models_{\mathcal{F}} \forall n \exists! m \mathrm{SUC}(n, m)$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary. Then (in our background theory) there exists a unique number $n + 1 \in \mathbb{N}$ such that $\mathrm{SUC}(n, n + 1)$. Then by unravelling the definition of $\Vdash_{\mathcal{F}}$ it follows that $0 \Vdash_{\mathcal{F}} \exists! m \mathrm{SUC}(n, m)$. ■

Claim 7.2.12 [**A2** :3] *$V^* \models_{\mathcal{F}} \forall n \forall m (\mathrm{SUC}(n, m) \to m \neq \bar{0})$.*

Proof. Let $n, m \in \mathbb{N}$ be arbitrary. It suffices to prove that $0 \Vdash_{\mathcal{F}} \mathrm{SUC}(n, m) \to m \neq \bar{0}$. Again this is obvious. ■

Claim 7.2.13 [**A2** :4] *$V^* \models_{\mathcal{F}} \forall m (m = \bar{0} \vee \exists n \mathrm{SUC}(n, m))$.*

Proof. Let $m \in \mathbb{N}$ be arbitrary. Then $0 \Vdash_{\mathcal{F}} m = \bar{0} \vee \exists n \mathrm{SUC}(n, m)$, so that $0 \Vdash_{\mathcal{F}} \forall m \, (m = \bar{0} \vee \exists n \mathrm{SUC}(n, m))$. ■

Claim 7.2.14 [**A2** :5] *$V^* \models_{\mathcal{F}} \forall n \forall m \forall k (\mathrm{SUC}(m, n) \wedge \mathrm{SUC}(k, n) \to m = k)$.*

Proof. Let $n, m, k \in \mathbb{N}$ and $e \in \mathbb{E}$ be arbitrary such that $e \Vdash_{\mathcal{F}} \mathrm{SUC}(m, n) \wedge \mathrm{SUC}(k, n)$. Then $\mathrm{SUC}(m, n) \wedge \mathrm{SUC}(k, n)$, and hence $m = k$, thus $0 \models_{\mathcal{F}} \forall n \forall m \forall k (\mathrm{SUC}(m, n) \wedge \mathrm{SUC}(k, n) \to m = k)$. ■

Claim 7.2.15 [**A2** :6] *$V^* \models_{\mathcal{F}} \forall n \forall m \exists! k \mathrm{ADD}(n, m, k)$.*

Proof. Let $n, m \in \mathbb{N}$ be arbitrary. Then there is a unique number $n+m \in \mathbb{N}$ such that $\mathrm{ADD}(n, m, n+m)$. Hence $0 \Vdash_{\mathcal{F}} \exists! k \mathrm{ADD}(n, m, k)$. ■

Claim 7.2.16 [**A2** :7] $V^* \models_{\mathcal{F}} \forall n \mathrm{ADD}(n, \bar{0}, n)$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. In the background theory $\mathrm{ADD}(n, \bar{0}, n)$. Hence $0 \Vdash_{\mathcal{F}} \mathrm{ADD}(n, \bar{0}, n)$ from which the claim follows. ■

Claim 7.2.17 [**A2** :8] $V^* \models_{\mathcal{F}}$

$$\forall n \forall k \forall m \forall l \forall i [\mathrm{ADD}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{SUC}(m, i) \ \rightarrow \mathrm{ADD}(n, l, i)].$$

Proof. Let $n, k, m, l, i \in \mathbb{N}$ and $e \in \mathbb{E}$ be arbitrary such that $e \Vdash_{\mathcal{F}} \mathrm{ADD}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{SUC}(m, i)$. The latter entails that $\mathrm{ADD}(n, k, m)$, $\mathrm{SUC}(k, l)$, and $\mathrm{SUC}(m, i)$. Whence $\mathrm{ADD}(n, l, i)$ from which the claim can be easily inferred. ■

Claim 7.2.18 [**A2** :9] $V^* \models_{\mathcal{F}} \forall n \forall m \exists! k \mathrm{MULT}(n, m, k)$.

Proof. Let $n, m \in \mathbb{N}$ be arbitrary. Then there is a unique number $n \times m \in \mathbb{N}$ such that $\mathrm{MULT}(n, m, n \times m)$. Hence by the definition of $\Vdash_{\mathcal{F}}$, $0 \Vdash_{\mathcal{F}} \exists! k \mathrm{MULT}(n, m, k)$. ■

Claim 7.2.19 [**A2** :10] $V^* \models_{\mathcal{F}} \forall n \mathrm{MULT}(n, \bar{0}, \bar{0})$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. Since $\mathrm{MULT}(n, \bar{0}, \bar{0})$ one immediately gets $0 \Vdash_{\mathcal{F}} \mathrm{MULT}(n, \bar{0}, \bar{0})$ from which the desired result follows. ■

Claim 7.2.20 [**A2** :11] $V^* \models_{\mathcal{F}} \forall n \forall k \forall m \forall l \forall i [\mathrm{MULT}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{ADD}(m, n, i) \rightarrow \mathrm{MULT}(n, l, i)]$.

Proof. Let $n, k, m, l, i \in \mathbb{N}$ and $e \in \mathbb{E}$ be arbitrary such that

$$e \Vdash_{\mathcal{F}} \mathrm{MULT}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{ADD}(m, n, i)\,.$$

Then $[\mathrm{MULT}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{ADD}(m, n, i)]$, whence $\mathrm{MULT}(n, l, i)$, from which the desired conclusion follows. ■

Claim 7.2.21 [**A2** :12] $V^* \models_{\mathcal{F}} A(\bar{0}) \wedge \forall n \forall m [A(n) \wedge \mathrm{SUC}(n, m) \rightarrow A(m)] \rightarrow \forall n A(n)$.

Proof. Let $e \in \mathbb{E}$ be arbitrary such that

$$e \Vdash_{\mathcal{F}} A(\bar{0}) \wedge \forall n \forall m[A(n) \wedge \mathrm{SUC}(n, m) \rightarrow A(m)]. \tag{7.2}$$

It suffices to prove that $e \Vdash_{\mathcal{F}} \forall n A(n)$. Now let $t \in \mathbb{E}$ be arbitrary such that $t \succeq e$. Then by Lemma 7.2.5, it suffices to prove that $\forall n \in \mathbb{N} \exists t' \succeq t[t' \Vdash_{\mathcal{F}} A(n)]$. We do this by induction on $n \in \mathbb{N}$. If $n = 0$, then by (7.2) $t \Vdash_{\mathcal{F}} A(\bar{0})$. Let $n \in \mathbb{N}$ be arbitrary such that $\exists k \succeq t[k \Vdash_{\mathcal{F}} A(n)]$. By the definition it follows that

$$k \Vdash_{\mathcal{F}} A(n) \wedge \mathrm{SUC}(n, n+1),$$

and thus by (7.2)

$$\exists k' \succeq k \succeq t[k' \Vdash_{\mathcal{F}} A(n+1)].$$

■

7.2.3 A3: Logical axioms for IPL

For logical axioms (LA): $0 \Vdash_{\mathcal{F}}$ LA**.** Most of them follow immediately from the definition and the Monotonicity Lemma. We just prove the nontrivial ones.

(IPL1) $0 \Vdash_{\mathcal{F}} A \rightarrow (B \rightarrow A)$.

(IPL2) $0 \Vdash_{\mathcal{F}} [A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$.

Proof. Let $m \in \mathbb{E}$ be arbitrary such that

$$m \Vdash_{\mathcal{F}} A \rightarrow (B \rightarrow C). \tag{7.3}$$

It suffices to prove that $m \Vdash_{\mathcal{F}} (A \rightarrow B) \rightarrow (A \rightarrow C)$. Let $n \in \mathbb{E}$ be arbitrary such that $n \succeq m$ and

$$n \Vdash_{\mathcal{F}} (A \rightarrow B). \tag{7.4}$$

We want to show that $n \Vdash_{\mathcal{F}} (A \rightarrow C)$. Let $l \in \mathbb{E}$ be arbitrary such that $l \succeq n$ and $l \Vdash_{\mathcal{F}} A$. Then by (7.3) $\exists l' \succeq l[l' \Vdash_{\mathcal{F}} (B \rightarrow C)]$. Furthermore, by (7.4), $\exists v \succeq l'[v \Vdash_{\mathcal{F}} B]$ and thus $\exists v' \succeq v \succeq l' \succeq l[v' \Vdash_{\mathcal{F}} C]$. ■

(IPL3) $0 \Vdash_{\mathcal{F}} A \rightarrow (B \rightarrow A \wedge B)$.

(IPL4) $0 \Vdash_{\mathcal{F}} A \wedge B \rightarrow A$.

(IPL5) $0 \Vdash_{\mathcal{F}} A \wedge B \rightarrow B$.

(IPL6) $0 \Vdash_{\mathcal{F}} A \to A \vee B$.

(IPL7) $0 \Vdash_{\mathcal{F}} B \to A \vee B$.

(IPL8) $0 \Vdash_{\mathcal{F}} (A \vee B) \to [(A \to C) \to ((B \to C) \to C)]$.

Proof. Let $h, i, j \in \mathbb{E}$ be arbitrary such that

$$h \preceq i \preceq j \wedge h \Vdash_{\mathcal{F}} A \vee B \wedge i \Vdash_{\mathcal{F}} A \to C \wedge j \Vdash_{\mathcal{F}} B \to C.$$

If $h \Vdash_{\mathcal{F}} A$, then by the Monotonicity Lemma, $j \Vdash_{\mathcal{F}} A \wedge (A \to C)$ and thus $\exists j' \succeq j(j' \Vdash_{\mathcal{F}} C)$, i.e., by the definitions

$$i \Vdash_{\mathcal{F}} [(B \to C) \to C] \wedge h \Vdash_{\mathcal{F}} (A \to C) \to ((B \to C) \to C).$$

If $h \Vdash_{\mathcal{F}} B$, again by the Monotonicity Lemma $j \Vdash_{\mathcal{F}} B$ and thus $\exists j' \succeq j($ $j' \Vdash_{\mathcal{F}} C)$, i.e., by the definitions,

$$i \Vdash_{\mathcal{F}} [(B \to C) \to C] \wedge h \Vdash_{\mathcal{F}} (A \to C) \to ((B \to C) \to C).$$

Hence we have shown that

$$0 \Vdash_{\mathcal{F}} (A \vee B) \to [(A \to C) \to ((B \to C) \to C)].$$

■

(IPL9) $0 \Vdash_{\mathcal{F}} (A \to B) \to ((A \to \neg B) \to \neg A)$.

(IPL10) $0 \Vdash_{\mathcal{F}} A \to (\neg A \to B)$.

(IPL11) $0 \Vdash_{\mathcal{F}} \forall x A(x) \to A[x/y]$, where y is free for x in $A(x)$.

(IPL12) $0 \Vdash_{\mathcal{F}} A[x/y] \to \exists x A(x)$, where y is free for x in $A(x)$.

For Inference Rules: (In the following, we use $FV(C)$ to denote the set of all free variables in C).

(IR1) Modus Ponens is preserved, i.e., $[V^* \models_{\mathcal{F}} A \wedge V^* \models_{\mathcal{F}} (A \to B)] \to V^* \models_{\mathcal{F}} B$.

Proof. Let $h \in \mathbb{E}$ be arbitrary. Then by the definitions we have

$$\exists h' \succeq h[h' \Vdash_{\mathcal{F}} A] \wedge \exists h'' \succeq h'[h'' \Vdash_{\mathcal{F}} (A \to B)].$$

By the Monotonicity Lemma, the result follows. ■

(IR2) Rule $\forall$ is preserved, i.e., $V^* \models_{\mathcal{F}} (C \to A(x)) \to V^* \models_{\mathcal{F}} (C \to \forall x A(x))$, where $x \notin FV(C)$.

Proof. It suffices to prove that $0 \Vdash_{\mathcal{F}} C \to \forall x A(x)$. Let $h \in \mathbb{E}$ be arbitrary such that $h \Vdash_{\mathcal{F}} C$. We want to show that $h \Vdash_{\mathcal{F}} \forall x A(x)$. Let $a \in V^*$ be arbitrary. Let $n \in \mathbb{E}$ be arbitrary such that $n \succeq h$. By the assumption, $\exists n' \succeq n[n' \Vdash_{\mathcal{F}} \forall x(C \to A(x))]$ and thus $\exists n'' \succeq n'[n'' \Vdash_{\mathcal{F}} C \to A(a)]$. Hence it follows that $\exists m \succeq n'' \succeq n[m \Vdash_{\mathcal{F}} A(a)]$. ■

(IR3) Rule $\exists$ is preserved, i.e., $V^* \models_{\mathcal{F}} (A(x) \to C) \to V^* \models_{\mathcal{F}} (\exists x A(x) \to C)$, where $x \notin FV(C)$.

Proof. It suffices to prove that $0 \Vdash_{\mathcal{F}} \exists x A(x) \to C$. Let $h \in \mathbb{E}$ be arbitrary such that $h \Vdash_{\mathcal{F}} \exists x A(x)$, i.e., $\exists a \in V^*[h \Vdash_{\mathcal{F}} A(a)]$. By the assumption

$$\exists h' \succeq h[h' \Vdash_{\mathcal{F}} \forall x(A(x) \to C)],$$

and thus by the Monotonicity Lemma, $\exists h'' \succeq h' \succeq h(h'' \Vdash_{\mathcal{F}} C)$. ■

For the Identity Axioms (IA):

The soundness of **IA** follows immediately from the following claims. Let $a, b, c \in V^*$ be arbitrary.

Claim 7.2.22 [**IA1**] $V^* \models_{\mathcal{F}} a = a$.

Proof. Let $a \in V^*$ be arbitrary. It suffices to prove that $0 \Vdash_{\mathcal{F}} a = a$ for all $a \in \mathbb{S}^*$. We show this via Ordinal Induction. Let $\alpha \in On$ be arbitrary. Assume

$$\forall \beta \in \alpha \forall b \in V_\beta^{\mathbb{E}}[0 \Vdash_{\mathcal{F}} b = b]. \tag{7.5}$$

Let $a \in V_\alpha^{\mathbb{E}}$ be arbitrary. Now we have to show that $0 \Vdash_{\mathcal{F}} a = a$, i.e., it is sufficient to show that $\forall (f, d) \in a \forall r \succeq f[r \Vdash_{\mathcal{F}} d \in a]$. Let $f, r \in \mathbb{E}$ and $d \in V^*$ be arbitrary such that

$$(f, d) \in a \wedge r \succeq f. \tag{7.6}$$

By the definition and (7.5), $0 \Vdash_{\mathcal{F}} d = d$. Then by (7.6) and the Monotonicity Lemma, $r \Vdash_{\mathcal{F}} d = d$ and thus $r \Vdash_{\mathcal{F}} d \in a$. ■

Claim 7.2.23 [**IA2**] $V^* \models_{\mathcal{F}} a = b \to b = a$.

Proof. It suffices to prove that $0 \Vdash_{\mathcal{F}} (a = b \to b = a)$ and this follows immediately from the symmetry of the definition. ■

Claim 7.2.24 [**IA3**] $V^* \models_{\mathcal{F}} a = b \wedge b = c \to a = c$.

Claim 7.2.25 [**IA4**] $V^* \models_{\mathcal{F}} a = b \wedge b \in c \to a \in c$.

Proof. For any formulae $\theta_1, \theta_2, ..., \theta_n$, let $\bigwedge_{i=1}^{n} \theta_i$ denote the conjunction $\theta_1 \wedge \theta_2 ... \wedge \theta_n$. We will prove (**IA3**) and (**IA4**) simultaneously via $\lhd^3$-induction (cf. Subsection 3.2.1):

$$\forall x_1, x_2, x_3[\forall (y_1, y_2, y_3) \lhd^3 (x_1, x_2, x_3)\varphi(y_1, y_2, y_3) \to \varphi(x_1, x_2, x_3)] \\ \to \forall x, y, z\varphi(x, y, z).$$

by taking $\varphi(y_1, y_2, y_3)$ to be

$$y_1, y_2, y_3 \in V^* \to \bigwedge_{\substack{i,j,k=1 \\ i,j \neq k \wedge i \neq j}}^{3} \eta(y_i, y_j, y_k),$$

where $\eta(y_i, y_j, y_k)$ denotes

$$0 \Vdash_{\mathcal{F}} [y_i = y_j \wedge y_j = y_k \to y_i = y_k] \wedge 0 \Vdash_{\mathcal{F}} [y_i = y_j \wedge y_j \in y_k \to y_i \in y_k].$$

Let $a^4, a^5, a^6 \in V^*$ and $d^1, d^2, d^3 \in V^*$ be arbitrary such that $(d^1, d^2, d^3) \lhd^3 (a^4, a^5, a^6)$ and

$$\begin{aligned} 0 &\Vdash_{\mathcal{F}} d^i = d^j \wedge d^j = d^k \to d^i = d^k, \\ 0 &\Vdash_{\mathcal{F}} d^i = d^j \wedge d^j \in d^k \to d^i \in d^k, \end{aligned} \tag{7.7}$$

for all $i, j, k \in \{1, 2, 3\}$, where $i, j \neq k$ and $i \neq j$. Now we have to show that

$$\begin{aligned} 0 &\Vdash_{\mathcal{F}} a^i = a^j \wedge a^j = a^k \to a^i = a^k, \\ 0 &\Vdash_{\mathcal{F}} a^i = a^j \wedge a^j \in a^k \to a^i \in a^k, \end{aligned}$$

for all $i, j, k \in \{4, 5, 6\}$, where $i, j \neq k$ and $i \neq j$. Now let $i, j, k \in \{4, 5, 6\}$ be arbitrary such that $i, j \neq k$ and $i \neq j$. Let $n \in \mathbb{E}$ be arbitrary such that $n \Vdash_{\mathcal{F}} a^i = a^j \wedge a^j \in a^k$. Then by the definition it follows that

$$\exists m \preceq n \exists q \in V^*[(m, q) \in a^k \wedge n \Vdash_{\mathcal{F}} a^i = a^j \wedge a^j = q].$$

Since $(a^i, a^j, q) \lhd^3 (a^i, a^j, a^k)$, (without loss of generality, suppose $i = 6, j = 4, k = 5$, then $(a^j, q, a^i) \lhd^3 (a^4, a^5, a^6)$) by the inductive hypothesis (7.7),

$\exists n' \succeq n[n' \Vdash_{\mathcal{F}} a^i = q]$ and thus $n' \Vdash_{\mathcal{F}} a^i \in a^k$. Now let $p \in \mathbb{E}$ be arbitrary such that

$$p \Vdash_{\mathcal{F}} a^i = a^j \wedge a^j = a^k. \tag{7.8}$$

We want to show that $p \Vdash_{\mathcal{F}} a^i = a^k$. Let $u, h \in \mathbb{E}$ and $v \in V^*$ be arbitrary such that $(u, v) \in a^i$ and $h \succeq u, p$. By assumption (7.8), it follows that $\exists h' \succeq h[h' \Vdash_{\mathcal{F}} v \in a^j]$, i.e.,

$$\exists s \preceq h' \exists t \in V^*[(s, t) \in a^j \wedge h' \Vdash_{\mathcal{F}} v = t].$$

Furthermore, let $q \in \mathbb{E}$ be arbitrary such that $q \succeq h'$. Then by the assumption (7.8) again, $\exists q' \succeq q[q' \Vdash_{\mathcal{F}} t \in a^k]$. Since $(v, t, a^k) \lhd^3 (a^i, a^j, a^k)$, by the inductive hypothesis (7.7), it follows that $\exists q'' \succeq q' \succeq h[q'' \Vdash_{\mathcal{F}} v \in a^k]$. Following the same procedure for the other part of the definition, one has the result $p \Vdash_{\mathcal{F}} a^i = a^k$. ■

Claim 7.2.26 [**IA5**] *$V^* \models_{\mathcal{F}} a = b \wedge c \in a \rightarrow c \in b$.*

Proof. It suffices to prove that $0 \Vdash_{\mathcal{F}} [a = b \wedge c \in a \rightarrow c \in b]$. Let $h \in \mathbb{E}$ be arbitrary such that $h \Vdash_{\mathcal{F}} a = b \wedge c \in a$. By the definition it follows that

$$\exists u \preceq h \exists q \in V^*[(u, q) \in a \wedge h \Vdash_{\mathcal{F}} c = q].$$

Hence by the definition it follows that $\exists h' \succeq h[h' \Vdash_{\mathcal{F}} q \in b]$ and thus by (**IA4**), $\exists h'' \succeq h' \succeq h[h'' \Vdash_{\mathcal{F}} c \in b]$. ■

Remark 7.2.27 **IA6** *to* **IA10** *are all realizable from its background theory and the definition.*

Lemma 7.2.28 [*Forcing Substitution*] *For any formula $A(x, \vec{y})$ in $\mathcal{L}(V^*)$, $\forall a, \vec{b}, c \in V^*[V^* \models_{\mathcal{F}} (a = c \wedge A(a, \vec{b})) \rightarrow A(c, \vec{b})]$.*

Proof. For the atomic formulae, this has been shown in the above claims. For the compound formulae, this is shown by induction over the complexity of the formulae. Here we consider the formulae with quantifiers:
♢ $A(a, \vec{b}) \equiv \exists z B(a, \vec{b}, z)$: Let $e \in \mathbb{E}$ be arbitrary such that $e \Vdash_{\mathcal{F}} a = c \wedge \exists z B(a, \vec{b}, z)$. By the definition, it follows that

$$\exists d \in V^*[e \Vdash_{\mathcal{F}} a = c \wedge B(a, \vec{b}, d)].$$

By the inductive hypothesis, it follows that $\exists e' \succeq e[e' \Vdash_{\mathcal{F}} \exists z B(c, \vec{b}, z)]$.
♢ $A(a, \vec{b}) \equiv \forall z B(a, \vec{b}, z)$: Let $e \in \mathbb{E}$ be arbitrary such that $e \Vdash_{\mathcal{F}} [a =$

$c \wedge \forall z B(a, \vec{b}, z)]$. Then by the definition and the Monotonicity Lemma, it follows that

$$\forall d \in V^* \forall h \succeq e \exists q \succeq h[q \Vdash_{\mathcal{F}} a = c \wedge B(a, \vec{b}, d)].$$

By the inductive hypothesis,

$$\forall d \in V^* \forall h \succeq e \exists q' \succeq q \succeq h[q' \Vdash_{\mathcal{F}} B(c, \vec{b}, d)],$$

i.e., $e \Vdash_{\mathcal{F}} \forall z B(c, \vec{b}, z)$. ■

7.2.4 A4.1: Non-logical axioms (CZF with two sorts)

Lemma 7.2.29 [*Extensionality*] $V^* \models_{\mathcal{F}} \forall x \forall y (S(x) \wedge S(y) \to [\forall z (z \in x \leftrightarrow z \in y) \to x = y])$.

Proof. Let $a, b \in V^*$ be arbitrary. It suffices to prove that

$$0 \Vdash_{\mathcal{F}} S(a) \wedge S(b) \to (\forall z (z \in a \leftrightarrow z \in b) \to a = b).$$

Let $k \in \mathbb{E}$ be arbitrary such that

$$k \Vdash_{\mathcal{F}} S(a) \wedge S(b). \tag{7.9}$$

Then it suffices to prove that

$$k \Vdash_{\mathcal{F}} [\forall z (z \in a \leftrightarrow z \in b) \to a = b].$$

Let $t \in \mathbb{E}$ be arbitrary such that $t \succeq k$ and $t \Vdash_{\mathcal{F}} \forall z (z \in a \leftrightarrow z \in b)$, i.e.,

$$\forall c \in V^* \forall l \succeq t \exists m \succeq l (m \Vdash_{\mathcal{F}} c \in a \leftrightarrow c \in b). \tag{7.10}$$

It suffices to prove that $t \Vdash_{\mathcal{F}} a = b$. From (7.9) and the definition, one has $a, b \in \mathbb{S}^*$. Now let $f, g \in \mathbb{E}$ and $d, v \in V^*$ be arbitrary such that $(f, d) \in a$ and $(g, v) \in b$. Let $n, p \in \mathbb{E}$ be arbitrary such that $n \succeq t, f$ and $p \succeq t, g$. Then by the definitions, $n \Vdash_{\mathcal{F}} d \in a$ and $p \Vdash_{\mathcal{F}} v \in b$, i.e., by (7.10) and the Monotonicity Lemma

$$\exists n' \succeq n (n' \Vdash_{\mathcal{F}} d \in b) \wedge \exists p' \succeq p (p' \Vdash_{\mathcal{F}} v \in a). \tag{7.11}$$

Hence by the definition, $t \Vdash_{\mathcal{F}} a = b$. ■

Lemma 7.2.30 [*Pairing*] $V^* \models_{\mathcal{F}} \forall x \forall y \exists u [S(u) \wedge (x \in u \wedge y \in u)]$.

Proof. Let $a, b \in V^*$ be arbitrary. It suffices to prove that there exists $\underline{u} \in \mathbb{S}^*$ such that $0 \Vdash_{\mathcal{F}} a \in \underline{u} \wedge b \in \underline{u}$. Define

$$\underline{u} \equiv \{a, b\}_{\mathcal{F}} \equiv \{(0, a), (0, b)\}.$$

By Pairing in the background theory, $\underline{u}$ is an external set. By Claim 3.3.6, $\underline{u}$ is also an internal set, i.e., $\underline{u} \in \mathbb{S}^*$. By the proof of the soundness of **IA1**, $0 \Vdash_{\mathcal{F}} a = a \wedge b = b$ and thus by the definition, $0 \Vdash_{\mathcal{F}} a \in \underline{u} \wedge b \in \underline{u}$. ■

Furthermore, one can also define the internal Cartesian product.

Definition 7.2.31 *For* $\forall a, b \in V^*$*,*

$$(a, b)_{\mathcal{F}} = \{(0, \{a, a\}_{\mathcal{F}}), (0, \{a, b\}_{\mathcal{F}})\}.$$

Since $\{a, b\}_{\mathcal{F}} \in \mathbb{S}^*$, by Claim 3.3.6, $(a, b)_{\mathcal{F}} \in \mathbb{S}^*$.

Corollary 7.2.32 [*Internal Cartesian Product*]

$$0 \Vdash_{\mathcal{F}} (a, b)_{\mathcal{F}} = (c, d)_{\mathcal{F}} \to a = c \wedge b = d.$$

Proof. By the definitions and the soundness of **IA2**,

$$0 \Vdash_{\mathcal{F}} \{a, a\}_{\mathcal{F}} = \{c, d\}_{\mathcal{F}} \to a = c \wedge b = d. \tag{7.12}$$

Let $g \in \mathbb{E}$ be arbitrary such that $g \Vdash_{\mathcal{F}} (a, b)_{\mathcal{F}} = (c, d)_{\mathcal{F}}$. Since $(0, \{a, a\}_{\mathcal{F}}) \in (a, b)_{\mathcal{F}}$, by the definition, it follows that $\exists g' \succeq g[g' \Vdash_{\mathcal{F}} \{a, a\}_{\mathcal{F}} = \{c, d\}_{\mathcal{F}}]$, i.e., by (7.12) $\exists g'' \succeq g' \succeq g[g'' \Vdash_{\mathcal{F}} a = c]$. The other cases also follow via the similar arguments. ■

Lemma 7.2.33 [*Union*] $V^* \models_{\mathcal{F}} \forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow \exists y(y \in x \wedge z \in y))]$.

Proof. Let $a \in V^*$ be arbitrary. It suffices to prove that there exists $\underline{a} \in \mathbb{S}^*$ such that

$$\forall c \in V^*[0 \Vdash_{\mathcal{F}} (c \in \underline{a} \leftrightarrow \exists y(y \in a \wedge c \in y))].$$

Define

$$\underline{a} \equiv \{(j, k) : j \in \mathbb{E} \wedge \exists (f, d) \in a \exists g \in \mathbb{E}[(g, k) \in d \wedge (j \succeq f, g)]\}.$$

By Bounded Separation, the Union and Pairing Axioms in the background theory, $\underline{a}$ is an external set. By Claim 3.3.6, $\underline{a}$ is also an internal set, i.e., $\underline{a} \in \mathbb{S}^*$. Let $c \in V^*$ be arbitrary. We want to show that

$$0 \Vdash_{\mathcal{F}} c \in \underline{a} \leftrightarrow \exists y(y \in a \wedge c \in y).$$

Let $e \in \mathbb{E}$ be arbitrary such that $e \Vdash_{\mathcal{F}} c \in \underline{a}$, i.e.,

$$\exists k \in V^* \exists j \preceq e[(j,k) \in \underline{a} \wedge e \Vdash_{\mathcal{F}} c = k].$$

By the definition, it follows that $\exists (f,d) \in a \exists g \in \mathbb{E}[(g,k) \in d \wedge (j \succeq f, g)]$ and thus $j \Vdash_{\mathcal{F}} d \in a \wedge k \in d$, i.e., by the soundness of **IA4** $\exists e' \succeq e(e' \Vdash_{\mathcal{F}} d \in a \wedge c \in d)$. On the other hand, let $h \in \mathbb{E}$ be arbitrary such that $h \Vdash_{\mathcal{F}} \exists y(y \in a \wedge c \in y)$, i.e., $\exists b \in V^*(h \Vdash_{\mathcal{F}} b \in a \wedge c \in b)$. By the definition it follows that

$$\exists s \preceq h \exists p \in V^*[(s,p) \in a \wedge h \Vdash_{\mathcal{F}} (b = p \wedge c \in b)],$$

i.e., by the soundness of **IA5** $\exists h' \succeq h(h' \Vdash_{\mathcal{F}} c \in p)$, i.e.,

$$\exists r \preceq h' \exists u \in V^*[(r,u) \in p \wedge h' \Vdash_{\mathcal{F}} c = u].$$

Since $(s,p) \in a$ and $(r,u) \in p$ and $h' \succeq s, r$, by the definition it follows that $(h', u) \in \underline{a}$, i.e., $h' \Vdash_{\mathcal{F}} u \in \underline{a}$. By the soundness of **IA4**, $\exists h'' \succeq h' \succeq h(h'' \Vdash_{\mathcal{F}} c \in \underline{a})$. ■

In order to the soundness of Bounded Separation, we find a new relation over $\mathbb{E} \times Form_c^{\Delta_0}$ (cf. 6.2.34) as follows:

Definition 7.2.34 *We define a relation $\Vdash^0_{\mathcal{F}}$ over $\mathbb{E} \times Form_c^{\Delta_0}$ as follows:*

- $h \Vdash^0_{\mathcal{F}} a \in b$ *iff* $h \Vdash_{\mathcal{F}} a \in b$.
- $h \Vdash^0_{\mathcal{F}} B(a) \wedge C(a)$ *iff* $h \Vdash^0_{\mathcal{F}} B(a) \wedge h \Vdash^0_{\mathcal{F}} C(a)$.
- $h \Vdash^0_{\mathcal{F}} B(a) \vee C(a)$ *iff* $h \Vdash^0_{\mathcal{F}} B(a) \vee h \Vdash^0_{\mathcal{F}} C(a)$.
- $h \Vdash^0_{\mathcal{F}} B(a) \to C(a)$ *iff* $\forall k \succeq h[k \Vdash^0_{\mathcal{F}} B(a) \to \exists k' \succeq k(k' \Vdash^0_{\mathcal{F}} C(a))]$.
- $h \Vdash^0_{\mathcal{F}} \neg B(a)$ *iff* $\forall k \succeq h \neg[k \Vdash^0_{\mathcal{F}} B(a)]$.
- $h \Vdash^0_{\mathcal{F}} \forall x \in aB(x)$ *iff* $\forall (f,d) \in a \forall k \succeq f, h \exists k' \succeq k[k' \Vdash^0_{\mathcal{F}} B(d)]$.
- $h \Vdash^0_{\mathcal{F}} \exists x \in aB(x)$ *iff* $\exists s \preceq h \exists d \in V^*[(s,d) \in a \wedge h \Vdash^0_{\mathcal{F}} B(d)]$.

Claim 7.2.35 *For all $A(a) \in Form_c^{\Delta_0}$, $\{h \in \mathbb{E} : h \Vdash^0_{\mathcal{F}} A(a)\}$ is a set.*

Proof. We show by the induction on the complexity of $A(a)$. Let $Z_{B(b)}$ be the class $\{h \in \mathbb{E} : h \Vdash^0_{\mathcal{F}} B(b)\}$ for all $B(b) \in Form_c^{\Delta_0}$. For the closed atomic formula, it can be easily seen to be true. For the compound formulae, by

the induction, $Z_{A(a)}$ is a set is validated by the following settings:
$\diamondsuit$ $\{h \in \mathbb{E} : h \Vdash^0_{\mathcal{F}} B(a) \wedge C(a)\} = \{h \in \mathbb{E} : h \in Z_{B(a)} \wedge h \in Z_{C(a)}\}$;
$\diamondsuit$ $\{h \in \mathbb{E} : h \Vdash^0_{\mathcal{F}} B(a) \vee C(a)\} = \{h \in \mathbb{E} : h \in Z_{B(a)} \vee h \in Z_{C(a)}\}$;
$\diamondsuit$ $\{h \in \mathbb{E} : h \Vdash^0_{\mathcal{F}} B(a) \to C(a)\} = \{h \in \mathbb{E} : \forall k \succeq h[k \in Z_{B(a)} \to \exists k' \succeq k(k' \in Z_{C(a)})]\}$;
$\diamondsuit$ $\{h \in \mathbb{E} : h \Vdash^0_{\mathcal{F}} \neg B(a)\} = \{h \in \mathbb{E} : \forall k \succeq h \neg k \in Z_{B(a)}\}$;
$\diamondsuit$ $\{h \in \mathbb{E} : h \Vdash^0_{\mathcal{F}} \forall x \in aB(x)\} = \{h \in \mathbb{E} : \forall (f,d) \in a \forall k \succeq h, f \exists k' \succeq k[k' \in Z_{B(d)}]\}$;
$\diamondsuit$ $\{h \in \mathbb{E} : h \Vdash^0_{\mathcal{F}} \exists x \in aB(x)\} = \{h \in \mathbb{E} : \exists s \preceq h \exists d \in V^*[(s,d) \in a \wedge h \in Z_{B(d)}]\}$. ■

Claim 7.2.36 *For each closed Δ_0-formula $A(a)$ in $Form_c^{\Delta_0}$, $h \Vdash_{\mathcal{F}} A(a) \to \exists h' \succeq h[h' \Vdash^0_{\mathcal{F}} A(a)]$ and $h \Vdash^0_{\mathcal{F}} A(a) \to \exists h' \succeq h[h' \Vdash_{\mathcal{F}} A(a)]$.*

Proof. This is shown via the (mutual) induction on the complexity of $A(a)$. We demonstrate the formulae with quantifiers.
$\diamondsuit$ Assume $h \Vdash_{\mathcal{F}} \forall x \in aB(x)$. Let $(u,d) \in a$ and $k \succeq h, u$ be arbitrary. Then by the soundness of **IA1** and the assumption, it follows that $\exists k' \succeq k[k' \Vdash_{\mathcal{F}} B(d)]$, which by the inductive hypothesis yields $\exists k'' \succeq k'[k'' \Vdash^0_{\mathcal{F}} B(d)]$. On the other hand, assume $h \Vdash^0_{\mathcal{F}} \forall x \in aB(x)$. We show $h \Vdash_{\mathcal{F}} \forall x \in aB(x)$. Let $k \succeq h$ and $b \in V^*$ be arbitrary such that $k \Vdash_{\mathcal{F}} b \in a$, i.e., $\exists s \preceq k \exists d \in V^*[(s,d) \in a \wedge k \Vdash_{\mathcal{F}} b = d]$. Then by the assumption it follows that $\exists k' \succeq k[k' \Vdash^0_{\mathcal{F}} B(d)]$, which by the inductive hypothesis yields $\exists k'' \succeq k'[k'' \Vdash_{\mathcal{F}} B(d)]$. Hence by Monotonicity and Substitution, $\exists k''' \succeq k''[k''' \Vdash_{\mathcal{F}} B(b)]$.
$\diamondsuit$ Assume $h \Vdash_{\mathcal{F}} \exists x \in aB(x)$, i.e., $\exists b \in V^*[h \Vdash_{\mathcal{F}} (b \in a \wedge B(b))]$, i.e., $\exists d \in V^* \exists s \preceq h[(s,d) \in a \wedge h \Vdash_{\mathcal{F}} b = d \wedge B(b)]$. Then by Substitution it follows that $\exists h' \succeq h[h' \Vdash_{\mathcal{F}} B(d)]$, which by the inductive hypothesis yields $\exists h'' \succeq h'[h'' \Vdash^0_{\mathcal{F}} B(d)]$ and thus $h'' \Vdash^0_{\mathcal{F}} \exists x \in aB(x)$. On the other hand, assume $h \Vdash^0_{\mathcal{F}} \exists x \in aB(x)$, i.e., $\exists s \preceq h \exists d \in V^*[(s,d) \in a \wedge h \Vdash^0_{\mathcal{F}} B(d)]$. Then by the inductive hypothesis, $\exists h' \succeq h[h' \Vdash_{\mathcal{F}} B(d)]$. Hence by Monotonicity and the soundness of **IA1**, we have shown that $h' \Vdash_{\mathcal{F}} d \in a \wedge B(d)$. ■

Lemma 7.2.37 [*Bounded Separation*] *$V^* \models_{\mathcal{F}} \forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow z \in x \wedge A(z))]$, where $A(z)$ is a bounded formula.*

Proof. Let $a \in V^*$ be arbitrary. It suffices to prove that there exists $\underline{a} \in \mathbb{S}^*$ such that

$$\forall c \in V^*[0 \Vdash_{\mathcal{F}} (c \in \underline{a} \leftrightarrow c \in a \wedge A(c))].$$

Define

$$\underline{a} \equiv \{(j,k) : j \in \mathbb{E} \wedge \exists f, g \in \mathbb{E}[j \succeq f, g \wedge (g,k) \in a \wedge f \Vdash^0_{\mathcal{F}} A(k)]\}.$$

By Claim 7.2.35, Bounded Separation, the Union and Pairing Axioms in the background theory, $\underline{a}$ is an external set. By Claim 3.3.6 , $\underline{a}$ is also an internal set, i.e., $\underline{a} \in \mathbb{S}^*$. Now let $c \in V^*$ be arbitrary. We want to show that

$$0 \Vdash_{\mathcal{F}} c \in \underline{a} \leftrightarrow c \in a \wedge A(c).$$

Let $e \in \mathbb{E}$ be arbitrary such that $e \Vdash_{\mathcal{F}} c \in \underline{a}$, i.e.,

$$\exists d \in V^* \exists j \preceq e[(j,d) \in \underline{a} \wedge e \Vdash_{\mathcal{F}} c = d].$$

By the definition, it follows that

$$\exists f, g \in \mathbb{E}[j \succeq f, g \wedge (g,d) \in a \wedge f \Vdash^0_{\mathcal{F}} A(d) \wedge e \Vdash_{\mathcal{F}} c = d].$$

Hence by Monotonicity and the Claim 7.2.36, it follows that $\exists e' \succeq e[e' \Vdash_{\mathcal{F}} d \in a \wedge A(d)]$ and thus by the soundness of **IA2** and Substitution, $\exists e'' \succeq e'[e'' \Vdash_{\mathcal{F}} (c \in a \wedge A(c))]$.

On the other hand, let $k \in \mathbb{E}$ be arbitrary such that $k \Vdash_{\mathcal{F}} c \in a \wedge A(c)$, i.e.,

$$\exists t \preceq k \exists b \in V^*[(t,b) \in a \wedge k \Vdash_{\mathcal{F}} c = b \wedge k \Vdash_{\mathcal{F}} A(c)].$$

Hence by Substitution and Claim 7.2.36, it follows that $\exists k' \succeq k[k' \Vdash^0_{\mathcal{F}} A(b)]$ and thus $(k', b) \in \underline{a}$, i.e., $k' \Vdash_{\mathcal{F}} b \in \underline{a}$. By the soundness of **IA4**, $\exists k'' \succeq k[k'' \Vdash_{\mathcal{F}} c \in \underline{a}]$. ■

Lemma 7.2.38 [*Infinity*] $V^* \models_{\mathcal{F}} \exists u(S(u) \wedge \forall z[z \in u \leftrightarrow N(z)])$.

Proof. It suffices to prove that there exists $\underline{u} \in \mathbb{S}^*$ such that $\forall c \in V^*[0 \Vdash_{\mathcal{F}} c \in \underline{u} \leftrightarrow N(c)]$. Define

$$\underline{u} = \{(0,n) : n \in \mathbb{N}\}. \tag{7.13}$$

By the Infinity Axiom and Bounded Separation in the background theory, $\underline{u}$ is an external set. By Claim 3.3.6 , $\underline{u}$ is also an internal set, i.e., $\underline{u} \in \mathbb{S}^*$. Now let $c \in V^*$. We want to show that $0 \Vdash_{\mathcal{F}} c \in \underline{u} \leftrightarrow N(c)$. Let $e \in \mathbb{E}$ be arbitrary such that $e \Vdash_{\mathcal{F}} c \in \underline{u}$. Then by the definition,

$$\exists k \in V^* \exists t \preceq e[(t,k) \in \underline{u}], \tag{7.14}$$

and $e \Vdash_{\mathcal{F}} c = k$. From (7.13) and (7.14) it follows that $k = c \in \mathbb{N}$. Hence by the definition,

$$e \Vdash_{\mathcal{F}} N(c).$$

On the other hand, let $g \in \mathbb{E}$ be arbitrary such that $g \Vdash_{\mathcal{F}} N(c)$. Then by the definition it follows that $c \in \mathbb{N}$, i.e., $(0, c) \in \underline{u}$. Hence by the soundness of **IA1**, it follows that

$$g \Vdash_{\mathcal{F}} c \in \underline{u}.$$

■

Lemma 7.2.39 [*Induction*] $V^* \models_{\mathcal{F}} \forall x[(\forall y \in x A(y)) \to A(x)] \to \forall x A(x)$.

Proof. Let $e \in \mathbb{E}$ be arbitrary such that

$$e \Vdash_{\mathcal{F}} \forall x[\forall y(y \in x \to A(y)) \to A(x)],$$

i.e.,

$$\forall b \in V^* \forall l \succeq e \exists l' \succeq l[l' \Vdash_{\mathcal{F}} \forall y(y \in b \to A(y)) \to A(b))]. \tag{7.15}$$

It suffices to prove that $e \Vdash_{\mathcal{F}} \forall x A(x)$, i.e., $\forall a \in V^* \forall l \succeq e \exists l' \succeq l[l' \Vdash_{\mathcal{F}} A(a)]$. We show this by ordinal induction. Let $\alpha \in On$ be arbitrary. Assume

$$\forall \beta \in \alpha \forall b \in \mathbb{N} \cup V_\beta^{\mathbb{E}} \forall l \succeq e \exists l' \succeq l[l' \Vdash_{\mathcal{F}} A(b)]. \tag{7.16}$$

Now we have to show that

$$\forall a \in \mathbb{N} \cup V_\alpha^{\mathbb{E}} \forall l \succeq e \exists l' \succeq l[l' \Vdash_{\mathcal{F}} A(a)].$$

Let $a \in \mathbb{N} \cup V_\alpha^{\mathbb{E}}$ be arbitrary and $l \in \mathbb{E}$ be arbitrary such that $l \succeq e$. We want to show that $l \Vdash_{\mathcal{F}} \forall y(y \in a \to A(y))$. Let $c \in V^*$ be arbitrary. It suffices to prove that $l \Vdash_{\mathcal{F}} c \in a \to A(c)$. Let $t \succeq l$ be arbitrary such that $t \Vdash_{\mathcal{F}} c \in a$. Then by the definition it follows that $\exists d \in V^* \exists v \preceq t$ such that $t \Vdash_{\mathcal{F}} c = d$ and $(v, d) \in a$, i.e., $d \in \mathbb{N} \cup V_\beta^{\mathbb{E}}$ for some $\beta \in \alpha$. Hence, by the inductive hypothesis (7.16), it follows that $\exists t' \succeq t[t' \Vdash_{\mathcal{F}} A(d)]$ and thus by Substitution, $\exists t'' \succeq t' \succeq t[t'' \Vdash_{\mathcal{F}} A(c)]$. By the definition,

$$l \Vdash_{\mathcal{F}} \forall y[y \in a \to A(y)]. \tag{7.17}$$

By (7.15) and (7.17), $\exists l' \succeq l\ [l' \Vdash_{\mathcal{F}} A(a)]$ and this completes the proof. ■

Lemma 7.2.40 [*Strong Collection*]

$$V^* \models_{\mathcal{F}} \forall x[\forall y \in x \exists z A(y, z) \to$$

$$\exists u(S(u) \wedge \forall y \in x \exists z \in u A(y, z) \wedge \forall z \in u \exists y \in x A(y, z))].$$

Proof. We show this via Strong Collection in the background theory. Let $a \in V^*$ and $e \in \mathbb{E}$ be arbitrary such that $e \Vdash_{\mathcal{F}} \forall y[y \in a \to \exists z A(y,z)]$, i.e.,

$$\forall b \in V^* \forall n \succeq e \exists n' \succeq n[n' \Vdash_{\mathcal{F}} (b \in a \to \exists z A(b,z))]. \tag{7.18}$$

It suffices to prove that there exists $\mathbf{D} \in \mathbb{S}^*$ such that

$$e \Vdash_{\mathcal{F}} \forall y \in a \exists z \in \mathbf{D} A(y,z) \wedge \forall z \in \mathbf{D} \exists y \in a A(y,z).$$

Define

$$\underline{a} \equiv \{(r,(l,b)) : r,l \in \mathbb{E} \wedge r \succeq l, e \wedge (l,b) \in a\}.$$

By the Pairing Axiom and Bounded Separation, $\underline{a}$ is a set. Unravelling (7.18), $\forall x \in \underline{a} \exists z \eta(x,z)$, where $\eta(x,z) \equiv$

$$\exists l,r,p \in \mathbb{E} \exists b,c \in V^*[x = (r,(l,b)) \wedge z = (p,c) \wedge p \succeq r \wedge p \Vdash_{\mathcal{F}} A(b,c)].$$

By Strong Collection in the background theory, there exists a set $\mathbf{D}$ such that

$$\forall x \in \underline{a} \exists z \in \mathbf{D} \eta(x,z), \tag{7.19}$$

and

$$\forall z \in \mathbf{D} \exists x \in \underline{a} \eta(x,z). \tag{7.20}$$

From (7.20), $\mathbf{D} \subseteq \mathbb{E} \times V^*$, i.e., by Claim 3.3.6 , $\mathbf{D} \in \mathbb{S}^*$. Now we want to show that

$$e \Vdash_{\mathcal{F}} \forall y[y \in a \to \exists z \in \mathbf{D} A(y,z)].$$

Let $b \in V^*$ be arbitrary. It suffices to prove that $e \Vdash_{\mathcal{F}} b \in a \to \exists z \in \mathbf{D} A(b,z)$. Let $n \in \mathbb{E}$ be arbitrary such that $n \succeq e$ and $n \Vdash_{\mathcal{F}} b \in a$. Then by the definition it follows that $(n,(t,d)) \in \underline{a}$ and $n \Vdash_{\mathcal{F}} b = d$ for some $t \preceq n$ and for some $d \in V^*$. Hence by (7.19)

$$\exists c \in V^* \exists p \succeq n[(p,c) \in \mathbf{D} \wedge p \Vdash_{\mathcal{F}} A(d,c)].$$

Thus by the soundness of **IA1**, **IA2**, Substitution, and the Monotonicity Lemma, $\exists p' \succeq p[p' \Vdash_{\mathcal{F}} c \in \mathbf{D} \wedge A(b,c)]$, i.e.,

$$e \Vdash_{\mathcal{F}} \forall y[y \in a \to \exists z(z \in \mathbf{D} \wedge A(y,z))].$$

Now we want to show that

$$e \Vdash_{\mathcal{F}} \forall z[z \in \mathbf{D} \to \exists y \in a A(y,z)].$$

Let $c \in V^*$ be arbitrary. It suffices to prove that $e \Vdash_{\mathcal{F}} c \in \mathbf{D} \to \exists y \in a A(y,c)$. Let $m \in \mathbb{E}$ be arbitrary such that $m \succeq e \wedge m \Vdash_{\mathcal{F}} c \in \mathbf{D}$. Then by the definition it follows that $\exists v \in V^* \exists q \preceq m[(q,v) \in \mathbf{D} \wedge m \Vdash_{\mathcal{F}} c = v]$. By (7.20),

$$\exists r, l \in \mathbb{E} \exists b \in V^*[(r,(l,b)) \in \underline{a} \wedge q \succeq r \wedge q \Vdash_{\mathcal{F}} A(b,v)].$$

Since $l \preceq r \preceq q \preceq m$, by the soundness of **IA1** and the Monotonicity Lemma, it follows that

$$m \Vdash_{\mathcal{F}} [b \in a \wedge A(b,v)].$$

Since $m \Vdash_{\mathcal{F}} c = v$, applying the soundness of **IA2** and Substitution,

$$\exists m' \succeq m[m' \Vdash_{\mathcal{F}} \exists y(y \in a \wedge A(y,c))],$$

and thus

$$e \Vdash_{\mathcal{F}} \forall z[z \in \mathbf{D} \rightarrow \exists y \in a A(y,z)].$$

■

Lemma 7.2.41 [*Subset Collection*] $V^* \models_{\mathcal{F}} \forall a \forall b \exists u(S(u) \wedge \forall z[\forall x \in a \exists y \in bA(x,y,z) \rightarrow \exists d \in u(\forall x \in a \exists y \in dA(x,y,z) \wedge \forall y \in d \exists x \in aA(x,y,z))])$.

Proof. Let $a, b \in V^*$ be arbitrary. It suffices to prove that there exists $\mathbb{U} \in \mathbb{S}^*$ such that for all c in V^*

$$0 \Vdash_{\mathcal{F}} \forall x \in a \exists y \in bA(x,y,c) \rightarrow \exists d \in \mathbb{U}$$
$$(\forall x \in a \exists y \in dA(x,y,c) \wedge \forall y \in d \exists x \in aA(x,y,c)).$$

Let $c \in V^*$ be arbitrary. Let $e \in \mathbb{E}$ be arbitrary such that

$$e \Vdash_{\mathcal{F}} \forall x \in a \exists y \in bA(x,y,c). \tag{7.21}$$

It suffices to prove that there exists $\mathbb{D} \in V^*$ such that $e \Vdash_{\mathcal{F}} \mathbb{D} \in \mathbb{U}$ and

$$e \Vdash_{\mathcal{F}} \forall x \in a \exists y \in \mathbb{D}A(x,y,c) \wedge \forall y \in \mathbb{D} \exists x \in aA(x,y,c).$$

Define

$$\underline{a} := \{((f,d),j) : f,j \in \mathbb{E}, (f,d) \in a \wedge j \succeq e, f\},$$
$$\underline{b} := \{(i,v) : i \in \mathbb{E} \wedge \exists k \preceq i[(k,v) \in b]\}.$$

By Bounded Separation, the Pairing and Union Axioms in the background theory, both $\underline{a}$ and $\underline{b}$ are external sets. By Claim 3.3.6, also $\underline{b} \in \mathbb{S}^*$. Unravelling (7.21) and applying Substitution yields

$$\forall u \in \underline{a} \exists l \in \underline{b} \eta(u,l,c),$$

where $\eta(u,l,c) \equiv \exists f,j \in \mathbb{E}\exists d,v \in V^*[u = ((f,d),j) \wedge l = (i,v) \wedge i \succeq j \wedge \exists k \preceq i(k,v) \in b \wedge i \Vdash_{\mathcal{F}} A(d,v,c)]$. Invoking Subset Collection in the background theory yields that there is an external set $\mathbb{D}$ such that $\exists \mathbb{C} \in \mathbb{D}$

$$\forall u \in \underline{a}\exists l \in \mathbb{C}\eta(u,l,c), \tag{7.22}$$

and

$$\forall l \in \mathbb{C}\exists u \in \underline{a}\eta(u,l,c). \tag{7.23}$$

Now define

$$\mathbb{D}^* = \{q \cap \underline{b} : q \in \mathbb{D}\}. \tag{7.24}$$

By Bounded Separation and Replacement in the background theory, $\mathbb{D}^*$ is an external set. Let $q \cap \underline{b} \in \mathbb{D}^*$ be arbitrary. Then we have $q \cap \underline{b} \subseteq \underline{b} \subseteq \mathbb{E} \times V^*$ and thus by Claim 3.3.6, $q \cap \underline{b} \in \mathbb{S}^*$, i.e., $\mathbb{D}^* \subseteq \mathbb{S}^*$. Furthermore, by the fact that $\mathbb{C} \in \mathbb{D}$, it follows that $\mathbb{C} \cap \underline{b} \in \mathbb{D}^*$ and thus $\mathbb{C} \cap \underline{b} \in \mathbb{S}^*$. Now define $\mathbb{U} := \{(0,d) : d \in \mathbb{D}^*\}$. Since $\mathbb{D}^* \subseteq \mathbb{S}^*$, by Claim 3.3.6, also $\mathbb{U} \in \mathbb{S}^*$. Moreover, by the soundness of **IA1**, also

$$e \Vdash_{\mathcal{F}} \mathbb{C} \cap \underline{b} \in \mathbb{U}. \tag{7.25}$$

Now we want to show that

$$e \Vdash_{\mathcal{F}} \forall x \in a\exists y \in \mathbb{C} \cap \underline{b}A(x,y,c) \wedge \forall y \in \mathbb{C} \cap \underline{b}\exists x \in aA(x,y,c).$$

It suffices to prove that for all $\underline{x}, \underline{y}$ in V^*

$$e \Vdash_{\mathcal{F}} \underline{x} \in a \rightarrow \exists y(y \in \mathbb{C} \cap \underline{b} \wedge A(\underline{x},y,c)), \tag{7.26}$$

and

$$e \Vdash_{\mathcal{F}} \underline{y} \in \mathbb{C} \cap \underline{b} \rightarrow \exists x(x \in a \wedge A(x,\underline{y},c)). \tag{7.27}$$

Let $m \in \mathbb{E}$ and $\underline{x} \in V^*$ be arbitrary such that $m \succeq e$ and $m \Vdash_{\mathcal{F}} \underline{x} \in a$, i.e.,

$$\exists p \preceq m\exists x' \in V^*[(p,x') \in a \wedge m \Vdash_{\mathcal{F}} \underline{x} = x'].$$

By the definition it follows that $((p,x'),m) \in \underline{a}$. From (7.22), it follows that

$$\exists i \succeq m\exists v \in V^*[(i,v) \in \mathbb{C} \cap \underline{b} \wedge i \Vdash_{\mathcal{F}} A(x',v,c)].$$

By the Monotonicity Lemma, the soundness of **IA1**, **IA2** and Substitution, $\exists i' \succeq i \succeq m[i' \Vdash_{\mathcal{F}} v \in \mathbb{C} \cap \underline{b} \wedge A(\underline{x},v,c)]$. Hence by the definition we have shown that (7.26). Now let $n \succeq e$ and $\underline{y} \in V^*$ be arbitrary such that $n \Vdash_{\mathcal{F}} \underline{y} \in \mathbb{C} \cap \underline{b}$, i.e.,

$$\exists q \preceq n\exists y' \in V^*[(q,y') \in \mathbb{C} \cap \underline{b} \wedge n \Vdash_{\mathcal{F}} \underline{y} = y'].$$

From (7.23), it follows that

$$\exists f, j \in \mathbb{E} \exists d \in V^*[((f,d),j) \in \underline{a} \wedge q \succeq j \wedge q \Vdash_{\mathcal{F}} A(d, y', c)].$$

By the definition it then follows that $q \Vdash_{\mathcal{F}} d \in a \wedge A(d, y', c)$. By the Monotonicity Lemma and Substitution, $\exists n' \succeq n \succeq q[n' \Vdash_{\mathcal{F}} d \in a \wedge A(d, \underline{y}, c)]$. Hence by the definition we have shown that (7.27) and this completes the proof. ■

Theorem 7.2.42 [*Soundness Theorem*]

$$\mathbf{CZF}_N \vdash \varphi \implies \mathbf{CZF}_N \vdash (V^* \models_{\mathcal{F}} \varphi).$$

Proof. Since the logical axioms, inference rules and non-logical axioms have shown to be sound, the result follows immediately. ■

7.2.5 A4.2: Non-logical axioms (IZF with two sorts)

The background theory in this section is $\mathbf{IZF}_N$. Since $\mathbf{CZF}_N$ and $\mathbf{IZF}_N$ share most of the axioms, we only have to check the following:

Lemma 7.2.43 [*Separation*] $V^* \models_{\mathcal{F}} \forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow z \in x \wedge A(z))]$.

Proof. Let $a \in V^*$ be arbitrary. It suffices to prove that there exists $\underline{a} \in \mathbb{S}^*$ such that for all c in V^*

$$0 \Vdash_{\mathcal{F}} [c \in \underline{a} \leftrightarrow c \in a \wedge A(c)].$$

Define

$$\underline{a} \equiv \{(j,k) : j \in \mathbb{E} \wedge \exists f, g \in \mathbb{E}[j \succeq f, g \wedge (g,k) \in a \wedge f \Vdash_{\mathcal{F}} A(k)]\}.$$

By Separation, the Union and Pairing Axioms in the background theory, $\underline{a}$ is an external set. By Claim 3.3.6, $\underline{a}$ is also an internal set, i.e., $\underline{a} \in \mathbb{S}^*$. Now let $c \in V^*$ be arbitrary. We want to show that

$$0 \Vdash_{\mathcal{F}} c \in \underline{a} \leftrightarrow c \in a \wedge A(c).$$

Let $e \in \mathbb{E}$ be arbitrary such that $e \Vdash_{\mathcal{F}} c \in \underline{a}$, i.e.,

$$\exists d \in V^* \exists j \preceq e[(j,d) \in \underline{a} \wedge e \Vdash_{\mathcal{F}} c = d].$$

By the definition, it follows that

$$\exists f, g \in \mathbb{E}[j \succeq f, g \wedge (g,d) \in a \wedge f \Vdash_{\mathcal{F}} A(d) \wedge e \Vdash_{\mathcal{F}} c = d].$$

Hence by the definition it follows that $j \Vdash_{\mathcal{F}} d \in a \wedge A(d)$ and thus by Substitution $\exists e' \succeq e[e' \Vdash_{\mathcal{F}} c \in a \wedge A(c)]$.

On the other hand, let $k \in \mathbb{E}$ be arbitrary such that $k \Vdash_{\mathcal{F}} c \in a \wedge A(c)$, i.e., $\exists t \preceq k \exists b \in V^*[(t,b) \in a \wedge k \Vdash_{\mathcal{F}} c = b]$. Hence by Substitution, it follows that $\exists k' \succeq k[k' \Vdash_{\mathcal{F}} A(b)]$ and thus $(k', b) \in \underline{a}$, i.e., $k' \Vdash_{\mathcal{F}} b \in \underline{a}$. By Substitution, $\exists k'' \succeq k[k'' \Vdash_{\mathcal{F}} c \in \underline{a}]$. ∎

Claim 7.2.44 *If $a, b \in \mathbb{S}^*$, then $e \Vdash_{\mathcal{F}} b \subseteq a \to \exists b^* \in V^{\mathbb{E}}_{rk(a)+1}(e \Vdash_{\mathcal{F}} b = b^*)$.*

Proof. Assume $e \Vdash_{\mathcal{F}} b \subseteq a$. By the definition it follows that $\forall (f,d) \in b \forall r \succeq f, e \exists r' \succeq r[r' \Vdash_{\mathcal{F}} d \in a]$, i.e., $\forall (f,d) \in b \forall r \succeq f, e$

$$\exists r' \succeq r \exists s \preceq r' \exists c \in V^*[(s,c) \in a \wedge r' \Vdash_{\mathcal{F}} d = c]. \tag{7.28}$$

Define

$$b^* \equiv \{(h,c) : h \in \mathbb{E}, \exists f, s \preceq h \exists d \in V^*(f,d) \in b \\ \wedge (s,c) \in a \wedge h \Vdash_{\mathcal{F}} d = c]\}.$$

By Pairing and Separation in the background theory, b^* is a set. By Claim 3.3.6 and Corollary 3.3.5, $b^* \in V^{\mathbb{E}}_{rk(a)+1}$. Now we want to show that $e \Vdash_{\mathcal{F}} b = b^*$. Let $f, r \in \mathbb{E}$ and $d \in V^*$ be arbitrary such that $(f,d) \in b$ and $r \succeq f, e$. Then by (7.28) it follows that there is $r' \succeq r$ and there is $c \in V^*$ such that $r' \Vdash_{\mathcal{F}} c \in b^*$ and $r' \Vdash_{\mathcal{F}} d = c$, i.e., by the soundness of **IA4**,

$$\exists r'' \succeq r' \succeq r[r'' \Vdash_{\mathcal{F}} d \in b^*].$$

On the other hand, let $g, r \in \mathbb{E}$ and $u \in V^*$ be arbitrary such that $(g,u) \in b^*$ and $r \succeq g, e$. By the definition and the soundness of **IA1**, **IA2** and the Monotonicity Lemma, $\exists d \in V^* \exists r' \succeq r$ such that $r' \Vdash_{\mathcal{F}} d \in b \wedge u = d$. By the soundness of **IA4**, it follows that

$$\exists r'' \succeq r' \succeq r[r'' \Vdash_{\mathcal{F}} u \in b].$$

∎

Lemma 7.2.45 [*Power set*] $V^* \models_{\mathcal{F}} \forall x \exists u[S(u) \wedge \forall z(z \in u \leftrightarrow (S(z) \wedge z \subseteq x))]$.

Proof. Let $a \in V^*$ be arbitrary. Define

$$\underline{a} \equiv \{(g,c) : g \in \mathbb{E} \wedge c \in V^{\mathbb{E}}_{rk(a)+1} \wedge g \Vdash_{\mathcal{F}} c \subseteq a\}.$$

By the Powerset Axiom, Pairing and Separation in the background theory, $\underline{a}$ is an external set. By Claim 3.3.6 it also follows that $\underline{a} \in \mathbb{S}^*$. Let $k \in V^*$ be arbitrary. It suffices to prove that $0 \Vdash_{\mathcal{F}} k \in \underline{a} \leftrightarrow S(k) \wedge k \subseteq a$. Let $e \in \mathbb{E}$ be arbitrary such that $e \Vdash_{\mathcal{F}} k \in \underline{a}$, i.e.,

$$\exists c \in V^{\mathbb{E}}_{rk(a)+1}[e \Vdash_{\mathcal{F}} c \subseteq a \wedge e \Vdash_{\mathcal{F}} k = c].$$

Then by the definition and Substitution, it follows that $k \in \mathbb{S}^*$ and thus

$$\exists e' \succeq e[e' \Vdash_{\mathcal{F}} S(k) \wedge k \subseteq a]. \tag{7.29}$$

Now let $h \in \mathbb{E}$ be arbitrary such that $h \Vdash_{\mathcal{F}} S(k) \wedge k \subseteq a$. Then by Claim 7.2.44, it follows that $\exists k^* \in V^{\mathbb{E}}_{rk(a)+1}$ such that $h \Vdash_{\mathcal{F}} k = k^*$. By the definition of $\underline{a}$ and Substitution, it then follows that $\exists h' \succeq h[h' \Vdash_{\mathcal{F}} k^* \in \underline{a}]$ and thus by the soundness of **IA4**

$$\exists h'' \succeq h' \succeq h[h'' \Vdash_{\mathcal{F}} k \in \underline{a}]. \tag{7.30}$$

From (7.29) and (7.30), the result follows. ■

Lemma 7.2.46 [*Collection*] $V^* \models_{\mathcal{F}} \forall x[\forall y \in x \exists z A(y,z) \to \exists u(S(u) \wedge \forall y \in x \exists z \in u A(y,z))]$.

Proof. We show this via Collection in the background theory. Let $a \in V^*$ and $e \in \mathbb{E}$ be arbitrary such that $e \Vdash_{\mathcal{F}} \forall y[y \in a \to \exists z A(y,z)]$, i.e.,

$$\forall b \in V^* \forall n \succeq e \exists n' \succeq n[n' \Vdash_{\mathcal{F}} (b \in a \to \exists z A(b,z))]. \tag{7.31}$$

It suffices to prove that there exists $\mathbf{D} \in \mathbb{S}^*$ such that

$$e \Vdash_{\mathcal{F}} \forall y \in a \exists z \in \mathbf{D} A(y,z).$$

Define

$$\underline{a} \equiv \{(r,(l,b)) : r,l \in \mathbb{E} \wedge r \succeq l, e \wedge (l,b) \in a\}.$$

By the Pairing axiom and Separation, $\underline{a}$ is a set. Unravelling (7.31), one has $\forall x \in \underline{a} \exists z \eta(x,z)$, where $\eta(x,z) \equiv$

$$\exists l,r,p \in \mathbb{E} \exists b,c \in V^*[x = (r,(l,b)) \wedge z = (p,c) \wedge p \succeq r \wedge p \Vdash_{\mathcal{F}} A(b,c)].$$

By Collection in the background theory, there exists a set $\mathbf{C}$ such that

$$\forall x \in \underline{a} \exists z \in \mathbf{C} \eta(x,z). \tag{7.32}$$

Now define

$$\mathbf{D} = \mathbf{C} \cap (\mathbb{E} \times V^*).$$

Since $\mathbf{D} \subseteq \mathbb{E} \times V^*$, by Separation and Claim 3.3.6, $\mathbf{D} \in \mathbb{S}^*$. Now we want to show that

$$e \Vdash_{\mathcal{F}} \forall y[y \in a \to \exists z \in \mathbf{D} A(y, z)].$$

Let $b \in V^*$ be arbitrary. It suffices to prove that $e \Vdash_{\mathcal{F}} b \in a \to \exists z \in \mathbf{D} A(b, z)$. Let $n \in \mathbb{E}$ be arbitrary such that $n \succeq e$ and $n \Vdash_{\mathcal{F}} b \in a$. Then by the definition it follows that $(n, (t, d)) \in \underline{a}$ and $n \Vdash_{\mathcal{F}} b = d$ for some $t \preceq n$ and some $d \in V^*$. Hence by (7.32) and Substitution,

$$\exists c \in V^* \exists p \succeq n \exists p' \succeq p[(p, c) \in \mathbf{D} \wedge p' \Vdash_{\mathcal{F}} A(b, c)]$$

and thus $p' \Vdash_{\mathcal{F}} [c \in \mathbf{D} \wedge A(b, c)]$, i.e.,

$$e \Vdash_{\mathcal{F}} \forall y[y \in a \to \exists z(z \in \mathbf{D} \wedge A(y, z))].$$

■

Theorem 7.2.47 [*Soundness Theorem*]

$$\mathbf{IZF}_N \vdash \varphi \implies \mathbf{IZF}_N \vdash (V^* \models_{\mathcal{F}} \varphi),$$

for all formulae $\varphi \in \mathcal{L}(V^*)$.

Proof. This follows immediately from the above lemmas. ■

In conclusion, we have invented a version of a forcing interpretation to interpret Heyting arithmetic, $\mathbf{CZF}_N$ and $\mathbf{IZF}_N$. These results play an important role in the inferences of our conservativity results in Chapter 8.

Chapter 8

Conservativity results

In this chapter, we present the main derivations of our conservativity results. In Section 8.1, we prove that arithmetical formulae are absolute with respect to our forcing interpretation and then in Section 8.2 we construct an internal oracle which is a partial function from $\mathbb{N}$ and $\mathbb{N}$. In Section 8.3 we prove that, by our forcing interpretation, arithmetical formulae are absolute with respect to our relativized realizability. With these and the results from Chapter 6 and Chapter 7, we present several conservativity results in Section 8.4.

Let $\mathcal{C}$ be a collection of closed formulae. For any theory T and any axiom AX, if $T + AX \vdash \varphi$ implies $T \vdash \varphi$ for all $\varphi \in \mathcal{C}$, then we say $T + AX$ is conservative over T with respect to $\mathcal{C}$. In this chapter we will show for several axioms AX that when T is $\mathbf{IZF}_N$ or $\mathbf{CZF}_N$, then $T + AX$ is conservative over T with respect to all arithmetical formulae.

To begin with, we show that forcing is absolute with respect to arithmetical formulae. In a second step we show that relativized realizability is generically self-realizing.

8.1 Absoluteness of forcing for arithmetical formulae

Let us single out the **arithmetical formulae** in the language of set theory. This collection will be denoted by $Form^{\#}$. It is the smallest collection of formulae containing the atoms of the form $R(t_1, t_2, ..., t_k)$ with R being a symbol of a k-ary primitive recursive relation, closed under $\wedge, \vee, \neg, \rightarrow$ and quantifiers $\forall n$ and $\exists n$ (i.e., if $\varphi(x) \in Form^{\#}$, then $\exists x(N(x) \wedge \varphi(x)) \in Form^{\#}$ and $\forall x(N(x) \rightarrow \varphi(x)) \in Form^{\#}$).

Lemma 8.1.1 [***Forcing Absoluteness***] ($\mathbf{CZF}_N$) *Let θ be any arithmetical formula whose free variables are among $x_1, \ldots, x_n$. Then for all $h \in \mathbb{E}$,*

$$N(x_1) \wedge N(x_2) ... \wedge N(x_n) \rightarrow$$
$$[(h \Vdash_{\mathcal{F}} \theta(x_1, x_2, ..., x_n)) \leftrightarrow \theta(x_1, x_2, ..., x_n)].$$

Proof. We will show this by induction on the complexity of θ. If θ is an atomic formula, the result follows immediately from the definition. If θ is a compound formula involving logical connectives $\wedge$ and $\vee$, then the result also follows immediately from the inductive hypotheses and the definitions. The remaining logical connectives, we inspect one by one:

$\diamondsuit$ $\theta(x_1, x_2, ..., x_n) \equiv \eta(x_1, x_2, ..., x_n) \rightarrow \delta(x_1, x_2, ..., x_n)$: Let $h \in \mathbb{E}$ be arbitrary such that $h \Vdash_{\mathcal{F}} \eta(x_1, x_2, ..., x_n) \rightarrow \delta(x_1, x_2, ..., x_n)$. Assume $\eta(x_1, x_2, ..., x_n)$. By the inductive hypothesis $h \Vdash_{\mathcal{F}} \eta(x_1, x_2, ..., x_n)$ and thus $\exists h' \succeq h[h' \Vdash_{\mathcal{F}} \delta(x_1, x_2, ..., x_n)]$. Again, by the inductive hypothesis, the result $\delta(x_1, x_2, ..., x_n)$ follows. On the other hand, assume $\eta(x_1, x_2, ..., x_n) \rightarrow \delta(x_1, x_2, ..., x_n)$. Let $q \in \mathbb{E}$ be arbitrary such that $q \succeq h$ and $q \Vdash_{\mathcal{F}} \eta(x_1, x_2, ..., x_n)$. As a result of the inductive hypothesis, $\eta(x_1, x_2, ..., x_n)$ and thus $\delta(x_1, x_2, ..., x_n)$ and $q \Vdash_{\mathcal{F}} \delta(x_1, x_2, ..., x_n)$.

$\diamondsuit$ $\theta(x_1, x_2, ..., x_n) \equiv \neg\eta(x_1, x_2, ..., x_n)$: Let $h \in \mathbb{E}$ be arbitrary such that $h \Vdash_{\mathcal{F}} \neg\eta(x_1, x_2, ..., x_n)$, i.e., by the definition $\forall q \succeq h \neg[q \Vdash_{\mathcal{F}} \eta(x_1, x_2, ..., x_n)]$, in particular, $\neg[h \Vdash_{\mathcal{F}} \eta(x_1, x_2, ..., x_n)]$, i.e., by the inductive hypothesis, $\neg\eta(x_1, x_2, ..., x_n)$. On the other hand, assume $\neg\eta(x_1, x_2, ..., x_n)$. Assume that there exists $q \in \mathbb{E}$ such that $q \Vdash_{\mathcal{F}} \eta(x_1, x_2, ..., x_n)$. From the assumption and the inductive hypothesis, $\eta(x_1, x_2, ..., x_n)$ and this contradicts our assumption.

$\diamondsuit$ $\theta(x_1, x_2, ..., x_n) \equiv \exists x[N(x) \wedge \eta(x, x_1, x_2, ..., x_n)]$: Let $h \in \mathbb{E}$ be arbitrary such that $h \Vdash_{\mathcal{F}} \exists x[N(x) \wedge \eta(x, x_1, x_2, ..., x_n)]$. By the definition, $\exists a \in V^*[h \Vdash_{\mathcal{F}} (N(a) \wedge \eta(a, x_1, x_2, ..., x_n))]$, i.e., by the inductive hypothesis $N(a) \wedge \eta(a, x_1, x_2, ..., x_n)$. On the other hand, assume $\exists x[N(x) \wedge \eta(x, x_1, x_2, ..., x_n)]$. Then by the inductive hypothesis, it follows immediately that $h \Vdash_{\mathcal{F}} \exists x[N(x) \wedge \eta(x, x_1, x_2, ..., x_n)]$.

$\diamondsuit$ $\theta(x_1, x_2, ..., x_n) \equiv \forall x[N(x) \rightarrow \eta(x, x_1, x_2, ..., x_n)]$: Let $h \in \mathbb{E}$ be arbitrary such that $h \Vdash_{\mathcal{F}} \forall x[N(x) \rightarrow \eta(x, x_1, x_2, ..., x_n)]$, i.e., $\forall a \in V^*[h \Vdash_{\mathcal{F}} (N(a) \rightarrow \eta(a, x_1, x_2, ..., x_n))]$, i.e., $\forall n \in \mathbb{N} \exists h' \succeq h[h' \Vdash_{\mathcal{F}} \eta(n, x_1, x_2, ..., x_n)]$, i.e., by the inductive hypothesis $\forall x[N(x) \rightarrow \eta(x, x_1, x_2, ..., x_n)]$. On the other hand, assume $\forall x(N(x) \rightarrow \eta(x, x_1, x_2, ..., x_n))$. Let $a \in V^*$ be arbitrary. Let $e \in \mathbb{E}$ be arbitrary such that $e \succeq h$ and $e \Vdash_{\mathcal{F}} N(a)$, i.e., $N(a)$. By the assumption, it then follows that $\eta(a, x_1, x_2, ..., x_n)$ and thus by the inductive hypothesis $e \Vdash_{\mathcal{F}} \eta(a, x_1, x_2, ..., x_n)$, i.e., by the definition

$h \Vdash_{\mathcal{F}} \forall x[N(x) \rightarrow \eta(x, x_1, x_2, ..., x_n)]$. ■

8.2 Forcing models for CZF$_{NA}$

Recall that $\mathbf{CZF}_{NA}$ is the formal system $\mathbf{CZF}_N$ with an extra set constant A and corresponding axiom (cf. Subsection 2.2.5) $PF(A, N, N)$ (i.e., "A is a partial function from N to N"):

$$\forall x \in A \exists m \exists n(x = (n, m)) \wedge \forall x \forall y \forall z[(x, y) \in A \wedge (x, z) \in A \rightarrow y = z].$$

In the Soundness Theorem 7.2.42, we have already shown that $\mathbf{CZF}_N$ is sound with respect to forcing, i.e. using the definitions in Section 7.1. To extend this to $\mathbf{CZF}_{NA}$ we need to assign a meaning to the constant A in every forcing universe V^*.

Recall that $\{a, b\}_{\mathcal{F}} \equiv \{(0, a), (0, b)\}$ and $(a, b)_{\mathcal{F}} = \{(0, \{a, a\}_{\mathcal{F}}), (0, \{a, b\}_{\mathcal{F}})\}$. We will interpret A via the following set:

Definition 8.2.1 $A^*_{\mathcal{F}} \equiv \{(f, (m, n)_{\mathcal{F}}) : f \in \mathbb{E} \wedge m, n \in \mathbb{N} \wedge f(m) = n\}$.

Observe that $A^*_{\mathcal{F}}$ is a set and, by Claim 3.3.6, $A^*_{\mathcal{F}} \in \mathbb{S}^*$. Let $\varphi(A)$ be any any formula that contains A as a constant. Forcing for formulae containing A is defined as follows:

$$p \Vdash_{\mathcal{F}} \varphi(A) \quad :\Leftrightarrow \quad p \Vdash_{\mathcal{F}} \varphi[A/A^*_{\mathcal{F}}]. \tag{8.1}$$

Claim 8.2.2 $\forall f \in \mathbb{E} \forall m, n \in \mathbb{N}[f(m) = n \leftrightarrow f \Vdash_{\mathcal{F}} (m, n)_{\mathcal{F}} \in A^*_{\mathcal{F}}]$.

Proof. Let $f \in \mathbb{E}$ and $m, n \in \mathbb{N}$ such that $f(m) = n$. From the definition and the soundness of **IA1**, it follows that $f \Vdash_{\mathcal{F}} (m, n)_{\mathcal{F}} \in A^*_{\mathcal{F}}$. On the other hand, assume $f \Vdash_{\mathcal{F}} (m, n)_{\mathcal{F}} \in A^*_{\mathcal{F}}$. By the definition,

$$\exists g \preceq f \exists (u, v)_{\mathcal{F}} \in V^*[(g, (u, v)_{\mathcal{F}}) \in A^*_{\mathcal{F}} \wedge f \Vdash_{\mathcal{F}} (m, n)_{\mathcal{F}} = (u, v)_{\mathcal{F}}].$$

By Corollary 7.2.32, the soundness of $\mathbf{CZF}_N$ and the definition, it then follows that

$$g(u) = v \wedge m = u \wedge n = v.$$

Since $g \preceq f$, i.e., $g \subseteq f$,

$$f(m) = f(u) = g(u) = v = n.$$

■

Lemma 8.2.3 ($\mathbf{CZF}_N$) $0 \Vdash_{\mathcal{F}} PF(A^*_{\mathcal{F}}, N, N)$.

Proof. It suffices to prove that

$$0 \Vdash_{\mathcal{F}} \forall a \in A^*_{\mathcal{F}} \exists b \exists c [N(b) \wedge N(c) \wedge a = (b,c)_{\mathcal{F}}], \tag{8.2}$$

and

$$0 \Vdash_{\mathcal{F}} \forall x \forall y \forall z [(x,y)_{\mathcal{F}} \in A^*_{\mathcal{F}} \wedge (x,z)_{\mathcal{F}} \in A^*_{\mathcal{F}} \to y = z]. \tag{8.3}$$

Firstly, let us prove (8.2). Let $e \in \mathbb{E}$ and $d \in V^*$ be arbitrary such that $e \Vdash_{\mathcal{F}} d \in A^*_{\mathcal{F}}$. Then by the definition it follows that

$$\exists k \preceq e \exists (u,v)_{\mathcal{F}} \in V^* [(k, (u,v)_{\mathcal{F}}) \in A^*_{\mathcal{F}} \wedge e \Vdash_{\mathcal{F}} d = (u,v)_{\mathcal{F}}].$$

Secondly, let us show (8.3). Let $a, b, c \in V^*$ and $h \in \mathbb{E}$ be arbitrary such that

$$h \Vdash_{\mathcal{F}} [(a,b)_{\mathcal{F}} \in A^*_{\mathcal{F}} \wedge (a,c)_{\mathcal{F}} \in A^*_{\mathcal{F}}]. \tag{8.4}$$

Then, by Lemma 8.2.2, $h(a) = b = c$. ■

Theorem 8.2.4 [*Soundness Theorem*] *For any closed formula $\varphi(A)$ of the language of* $\mathbf{CZF}_{NA}$*,*

$$\mathbf{CZF}_{NA} \vdash \varphi(A) \implies \mathbf{CZF}_N \vdash [V^* \models_{\mathcal{F}} \varphi(A^*_{\mathcal{F}})].$$

Proof. This follows from Lemma 8.2.3 and Soundness Theorem 7.2.42. ■

8.3 Generic self-realizability

We say that a sentence θ is **generically self-realizing** (or **internally self-realizing**) with respect to a notion of forcing, if

$$V^* \models_{\mathcal{F}} [\theta \leftrightarrow \exists e\, e \Vdash_R \theta)],$$

where realizability means that the oracle A is to be interpreted as $A^*_{\mathcal{F}}$.

In this section we will show that for every arithmetical sentence θ, there is a notion of forcing such that θ is generically self-realizing. This can be shown by induction over the complexity of the arithmetical formulae.

In the following we will assume that we have a primitive recursive injective pairing function $\jmath$ and define tuple coding $\langle n_1, \ldots, n_r \rangle$ of tuples of natural numbers $n_1, \ldots, n_r$ for $r \geq 2$ recursively by $\langle n_1, n_2 \rangle = \jmath(n_1, n_2)$ and $\langle n_1, \ldots, n_{r+1} \rangle = \jmath(\langle n_1, \ldots, n_r \rangle, n_{r+1})$ for $r \geq 2$.

We will assume that we have a way of assigning Gödel numbers to formulae of set theory. For a formula ϑ we let $\ulcorner \vartheta \urcorner$ denote its Gödel number.

Lemma 8.3.1 *Fix an arithmetical sentence θ. Then there is a set of forcing conditions $\mathbb{P}^* \subseteq \mathbb{P}$ (definable in $\mathbf{CZF}_N$) such that*

$$\mathbf{CZF}_N \vdash \forall e \in \mathbb{N}\, \forall p \in \mathbb{P}^*\, \exists q \in \mathbb{P}^*\, [q \succeq p \wedge q \Vdash_{\mathcal{F}} [(e \Vdash_R \theta) \to \theta]\,,$$

where we force with the set $\mathbb{P}^*$ and the realizability inside the forcing universe, henceforth called generic realizability*, means again that the oracle constant A is to be interpreted as $A^*_{\mathcal{F}}$.*

Proof. Let $Var = \{v_0, \ldots, v_r\}$ be the variables occurring in θ. For an assignment $\mathfrak{s} : Var \to \mathbb{N}$ we denote by $\langle \mathfrak{s} \rangle$ the number $\langle \mathfrak{s}(v_0), \ldots, \mathfrak{s}(v_r) \rangle$. If $u \in Var$ and $n \in \mathbb{N}$ we denote by $\mathfrak{s}(u|n)$ the assignment $\tau : Var \to \mathbb{N}$ with $\tau(v) = \mathfrak{s}(v)$ whenever $v \in Var$ is a variable different from u and $\tau(v) = n$ if v is u.

If φ is a subformula of θ and $\mathfrak{s} : Var \to \mathbb{N}$ we use $\varphi[\mathfrak{s}]$ to denote the statement resulting from φ by interpreting every free occurrence of v_i in φ by the number $\mathfrak{s}(v_i)$.

We use the abbreviations $\forall v \in N(\ldots)$ and $\exists v \in N(\ldots)$ for $\forall v(N(v) \to \ldots)$ and $\exists v(N(v) \wedge \ldots)$, respectively. The set $\mathbb{P}^*$ consists of all finite functions $f \in \mathbb{P}$ such that for all subformulae ψ, ϑ and $\exists v \in N\, \chi$ of θ (including θ as a subformula of itself) and all assignments $\mathfrak{s} : Var \to \mathbb{N}$ we have with $\mathbf{m}_0 := \langle \ulcorner \exists v \in N\, \chi \urcorner, \langle \mathfrak{s} \rangle \rangle$ and $\mathbf{m}_1 := \langle \ulcorner \psi \vee \vartheta \urcorner, \langle \mathfrak{s} \rangle \rangle$ that

$$(\exists v \in N\, \chi)[\mathfrak{s}] \;\wedge\; f(\mathbf{m}_0)\downarrow \;\;\to\;\; \chi[\mathfrak{s}(v|f(\mathbf{m}_0))] \tag{8.5}$$

$$(\psi[\mathfrak{s}] \;\vee\; \vartheta[\mathfrak{s}]) \;\wedge\; f(\mathbf{m}_1)\downarrow \;\;\to\;\; (f(\mathbf{m}_1) = 0 \;\wedge\; \psi[\mathfrak{s}]) \;\vee\; (f(\mathbf{m}_1) = 1 \;\wedge\; \vartheta[\mathfrak{s}])\,. \tag{8.6}$$

Note that the collection of functions $\mathfrak{s} : Var \to \mathbb{N}$ forms a set in our background theory and that there are only finitely many subformulae of θ. The conditions on f spelled out in (8.5) and (8.6) are expressed via bounded formulae. Thus by bounded separation $\mathbb{P}^*$ is a set in our background theory.

Next we define for each subformula φ of θ a number $\mathfrak{R}_\varphi$ such that the following statements are provable in $\mathbf{CZF}_N$: for all $\mathfrak{s} : Var \to \mathbb{N}$,

$$\forall p \in \mathbb{P}^*\, \exists q \in \mathbb{P}^* (q \succeq p \;\wedge\; q \Vdash_{\mathcal{F}} (\varphi[\mathfrak{s}] \to \{\mathfrak{R}_\varphi\}^A(\langle \mathfrak{s} \rangle) \Vdash_R \varphi[\mathfrak{s}])) \tag{8.7}$$

$$\forall p \in \mathbb{P}^*\, \exists q \in \mathbb{P}^* [q \succeq p \;\wedge\; q \Vdash_{\mathcal{F}} ((\exists e\, e \Vdash_R \varphi[\mathfrak{s}]) \to \varphi[\mathfrak{s}])]\,. \tag{8.8}$$

The numbers $\mathfrak{R}_\varphi$ are defined by induction on the buildup of the subformula φ of θ:

(D0) For atomic φ let $\mathfrak{R}_\varphi$ be an index of the identically zero function.

(D1) Choose $\mathfrak{R}_{\eta\wedge\delta}$ such that $\{\mathfrak{R}_{\eta\wedge\delta}\}^A(n) := \langle\{\mathfrak{R}_\eta\}^A(n), \{\mathfrak{R}_\delta\}^A(n)\rangle$.

(D2) Choose $\mathfrak{R}_{\eta\to\delta}$ such that $\{\mathfrak{R}_{\eta\to\delta}\}^A(n) := \Lambda t.(\{\mathfrak{R}_\delta\}^A(n))$.

(D3) Choose $\mathfrak{R}_{\forall v\in N\,\eta}$ such that

$$\{\mathfrak{R}_{\forall v\in N\,\eta}\}^A(\langle\mathfrak{s}\rangle) \;=\; \Lambda e.\{\mathfrak{R}_\eta\}^A(\langle\mathfrak{s}(v|e)\rangle).$$

(D4) Choose $\mathfrak{R}_{\eta\vee\vartheta}$ such that

$$\{\mathfrak{R}_{\eta\vee\vartheta}\}^A(\langle\mathfrak{s}\rangle) = \begin{cases} \langle 0, \{\mathfrak{R}_\eta\}^A(\langle\mathfrak{s}\rangle)\rangle & \text{if } A(\langle\ulcorner\eta\vee\vartheta\urcorner, \langle\mathfrak{s}\rangle\rangle) = 0 \\ \langle 1, \{\mathfrak{R}_\vartheta\}^A(\langle\mathfrak{s}\rangle)\rangle & \text{if } A(\langle\ulcorner\eta\vee\vartheta\urcorner, \langle\mathfrak{s}\rangle\rangle) = 1\,. \end{cases}$$

(D5) Choose $\mathfrak{R}_{\exists v\in N\,\chi}$ such that

$$\{\mathfrak{R}_{\exists v\in N\,\chi}\}^A(\langle\mathfrak{s}\rangle) \;=\; \langle\mathbf{m}_{\mathfrak{s},\chi,v}, \{\mathfrak{R}_\chi\}^A(\langle\mathfrak{s}(v|\mathbf{m}_{\mathfrak{s},\chi,v})\rangle)\rangle\,,$$

where $\mathbf{m}_{\mathfrak{s},\chi,v} := A(\langle\ulcorner\exists v\in N\,\chi\urcorner, \langle\mathfrak{s}\rangle\rangle)$.

After this definition, we can prove (8.7) and (8.8) by a simultaneous induction on the buildup of φ. If φ is an atom, both are obvious.

Case 1: Let φ be $\eta\wedge\delta$. To show (8.7), let $h\in\mathbb{P}^*$ be arbitrary such that $h\Vdash_{\mathcal{F}}\eta[\mathfrak{s}]\wedge\delta[\mathfrak{s}]$. By the definition and inductive hypotheses, it follows that

$$\exists h'\succeq h\,(h'\Vdash_{\mathcal{F}}\{\mathfrak{R}_\eta\}^A(\langle\mathfrak{s}\rangle)\Vdash_R\eta[\mathfrak{s}]),$$
$$\exists h''\succeq h'\,(h''\Vdash_{\mathcal{F}}\{\mathfrak{R}_\delta\}^A(\langle\mathfrak{s}\rangle)\Vdash_R\delta[\mathfrak{s}]).$$

Hence by the definition of $\Vdash_R$, (**D1**) and the Monotonicity Lemma, it follows that there exists $h'''\succeq h''$ such that

$$h'''\Vdash_{\mathcal{F}}(\{\mathfrak{R}_{\eta\wedge\delta}\}^A(\langle\mathfrak{s}\rangle)\Vdash_R(\eta\wedge\delta)[\mathfrak{s}])\,.$$

To show (8.8), let $p\in\mathbb{P}^*$ be arbitrary. By the inductive hypotheses, we can successively pick $k,q\in\mathbb{P}^*$ such that $q\succeq k\succeq p$ and

$$k\Vdash_{\mathcal{F}}((\exists e\,e\Vdash_R\eta[\mathfrak{s}])\to\eta[\mathfrak{s}])\,, \tag{8.9}$$
$$q\Vdash_{\mathcal{F}}((\exists e\,e\Vdash_R\delta[\mathfrak{s}])\to\delta[\mathfrak{s}])\,. \tag{8.10}$$

Now let $k'\succeq q$ be such that

$$k'\Vdash_{\mathcal{F}}(\exists e\,e\Vdash_R\eta[\mathfrak{s}]\wedge\delta[\mathfrak{s}])\,.$$

Then $k'\Vdash_{\mathcal{F}}(\exists e\,e\Vdash_R\eta[\mathfrak{s}])$ and $k'\Vdash_{\mathcal{F}}(\exists e\,e\Vdash_R\delta[\mathfrak{s}])$. By (8.9), we find $k''\succeq k'$ such $k''\Vdash_{\mathcal{F}}\eta[\mathfrak{s}]$. Since by monotonicity we also have $k''\Vdash_{\mathcal{F}}(\exists e\,e\Vdash_R\delta[\mathfrak{s}])$ and $k''\Vdash_{\mathcal{F}}((\exists e\,e\Vdash_R\delta[\mathfrak{s}])\to\delta[\mathfrak{s}])$, there exists $k'''\succeq k''$ such that

$k''' \Vdash_{\mathcal{F}} \delta[\mathfrak{s}]$. Hence $k''' \Vdash_{\mathcal{F}} (\eta \wedge \delta)[\mathfrak{s}]$. The above chain of arguments shows that $q \Vdash_{\mathcal{F}} ((\exists e\, e \Vdash_R \varphi[\mathfrak{s}]) \to \varphi[\mathfrak{s}])$, confirming (8.8).

Case 2: Let φ be of the form $\eta \to \delta$. Let $p \in \mathbb{P}^*$ be arbitrary. To show (8.7), let $q \succeq p$ and suppose that $q \Vdash_{\mathcal{F}} \varphi[\mathfrak{s}]$. Let $h \succeq q$ such that $h \Vdash_{\mathcal{F}} (t \Vdash_R \eta[\mathfrak{s}])$ for some $t \in \mathbb{N}$. Then, by the induction hypothesis for (8.8) applied to η, there exists $h'' \succeq h$ such that $h'' \Vdash_{\mathcal{F}} \eta[\mathfrak{s}]$. Hence there exists $h''' \succeq h''$ such that $h''' \Vdash_{\mathcal{F}} \delta[\mathfrak{s}]$. By the induction hypothesis for (8.7) applied to δ, we find $h^* \succeq h'''$ such that $h^* \Vdash_{\mathcal{F}} (\{\mathfrak{R}_\delta\}^A(\langle\mathfrak{s}\rangle) \Vdash_R \delta[\mathfrak{s}])$. As a result, for all $t \in \mathbb{N}$,

$$q \Vdash_{\mathcal{F}} (t \Vdash_R \eta[\mathfrak{s}] \to \{\mathfrak{R}_\delta\}^A(\langle\mathfrak{s}\rangle) \Vdash_R \delta[\mathfrak{s}])\,.$$

Therefore we have $q \Vdash_{\mathcal{F}} (\{\mathfrak{R}_{\eta\to\delta}\}^A(\langle\mathfrak{s}\rangle) \Vdash_R \varphi[\mathfrak{s}])$, and consequently

$$p \Vdash_{\mathcal{F}} (\varphi[\mathfrak{s}] \to \{\mathfrak{R}_{\eta\to\delta}\}^A(\langle\mathfrak{s}\rangle) \Vdash_R \varphi[\mathfrak{s}])\,.$$

To show (8.8), let $h \succeq p$ satisfy $h \Vdash_{\mathcal{F}} (V^* \models_R \varphi[\mathfrak{s}])$. Suppose $h' \succeq h$ and $h' \Vdash_{\mathcal{F}} \eta[\mathfrak{s}]$. Then, by the induction hypothesis for (8.7) applied to η, we have $h'' \Vdash_{\mathcal{F}} (V^* \models_R \eta[\mathfrak{s}])$ for some $h'' \succeq h'$, and thus $h''' \Vdash_{\mathcal{F}} (V^* \models_R \delta[\mathfrak{s}])$ for some $h''' \succeq h''$ as $h \Vdash_{\mathcal{F}} (V^* \models_R \varphi[\mathfrak{s}])$. Consequently, by the induction hypothesis for (8.8) applied to δ, there exists $h^* \succeq h'''$ such that $h^* \Vdash_{\mathcal{F}} \delta[\mathfrak{s}]$. As a result, we have shown that $h \Vdash_{\mathcal{F}} \varphi[\mathfrak{s}]$. So the upshot is that $p \Vdash_{\mathcal{F}} ((\exists e\, e \Vdash_R \varphi[\mathfrak{s}]) \to \varphi[\mathfrak{s}])$, as required.

Case 3: φ is of the form $\neg\psi$. This follows immediately from the fact that $\neg\psi$ is logically equivalent to $\psi \to \bar{0} = \bar{1}$ and the result of previous case.

Case 4: φ is of the form $\forall v \in N\, \eta$. Let $p \in \mathbb{P}^*$ be arbitrary. To show (8.7), let $h \in \mathbb{P}^*$ be arbitrary such that $h \succeq p$ and

$$h \Vdash_{\mathcal{F}} \forall v(N(v) \to \eta)[\mathfrak{s}]. \tag{8.11}$$

Then, for all $h' \succeq h$ and $n \in \mathbb{N}$ there exists $h'' \succeq h'$ such that $h'' \Vdash_{\mathcal{F}} \eta[\mathfrak{s}(v|n)]$. Hence, by the inductive assumption for (8.7) applied to η, for all $n \in \mathbb{N}$ there exists $h''' \succeq h''$ such that

$$h''' \Vdash_{\mathcal{F}} \{\mathfrak{R}_\eta\}^A(\langle\mathfrak{s}(v|n)\rangle) \Vdash_R \eta[\mathfrak{s}(v|n)]\,.$$

The latter yields

$$h \Vdash_{\mathcal{F}} \forall n \in N\, \{\mathfrak{R}_\eta\}^A(\langle\mathfrak{s}(v|n)\rangle) \Vdash_R \eta[\mathfrak{s}(v|n)]$$

and hence $h \Vdash_{\mathcal{F}} \{\mathfrak{R}_{\forall v\in N\,\eta}\}^A \Vdash_R (\forall v \in N\, \eta)[\mathfrak{s}]$, by definition of $\{\mathfrak{R}_{\forall v\in N\,\eta}\}^A$. As a result,

$$p \Vdash_{\mathcal{F}} (\forall v(N(v) \to \eta)[\mathfrak{s}] \to \{\mathfrak{R}_{\forall v\in N\,\eta}\}^A \Vdash_R (\forall v \in N\, \eta)[\mathfrak{s}])\,.$$

To verify (8.8) let $h \succeq p$ and $h \Vdash_{\mathcal{F}} \exists e\, e \Vdash_R (\forall v \in N\, \eta)[\mathfrak{s}]$. This implies that for all $n \in \mathbb{N}$ there exists $h' \succeq h$ such that

$$h' \Vdash_{\mathcal{F}} (\exists e\, e \Vdash_R \eta[\mathfrak{s}(v|n)])$$

so that by the inductive hypothesis for (8.7) applied to η, there exists $h'' \succeq h'$ satisfying $h'' \Vdash_{\mathcal{F}} \eta[\mathfrak{s}(v|n)]$. Therefore we obtain

$$h \Vdash_{\mathcal{F}} (\forall v \in N\, \eta)[\mathfrak{s}]\,,$$

and thus $p \Vdash_{\mathcal{F}} ((\exists e\, e \Vdash_R (\forall v \in N\, \eta)[\mathfrak{s}]) \to (\forall v \in N\, \eta)[\mathfrak{s}])$.

Case 5: Let φ be of the form $\eta \vee \delta$. Let $p \in \mathbb{P}^*$. To prove (8.7) let $f \in \mathbb{P}^*$, $f \succeq p$, and $f \Vdash_{\mathcal{F}} \varphi[\mathfrak{s}]$. By the Forcing Absoluteness Lemma 8.1.1, we also know that $\varphi[\mathfrak{s}]$. We have to distinguish cases as to whether f is defined at $\mathbf{m}_1 := \langle \ulcorner\varphi\urcorner, (\langle \mathfrak{s} \rangle)\rangle$ or not.

Let us first assume that $f(\mathbf{m}_1)\downarrow$. Then, since $f \in \mathbb{P}^*$, f satisfies (8.6), and thus $f(\mathbf{m}_1) = 0 \wedge \eta[\mathfrak{s}]$ or $f(\mathbf{m}_1) = 1 \wedge \delta[\mathfrak{s}]$. In the first case, $f \Vdash_{\mathcal{F}} \eta[\mathfrak{s}]$ (by Lemma 8.1.1) and also $f \Vdash_{\mathcal{F}} A(\mathbf{m}_1) = 0$ using Claim 8.2.2 and Lemma 8.2.3. By the induction hypothesis for (8.7) applied to η, there exists $f' \succeq f$ such that $f' \Vdash_{\mathcal{F}} (\{\mathfrak{R}_\eta\}^A(\langle \mathfrak{s} \rangle) \Vdash_R \eta[\mathfrak{s}])$. Hence, by definition of $\{\mathfrak{R}_\varphi\}^A$, $f' \Vdash_{\mathcal{F}} (\{\mathfrak{R}_\varphi\}^A(\langle \mathfrak{s} \rangle) \Vdash_R \varphi[\mathfrak{s}])$. In the second case we proceed similarly and also find $f' \succeq f$ such that $f' \Vdash_{\mathcal{F}} \{\mathfrak{R}_\varphi\}^A(\langle \mathfrak{s} \rangle) \Vdash_R \varphi[\mathfrak{s}]$. Now let us look at the case when $f(\mathbf{m}_1)$ is not defined. If $\eta[\mathfrak{s}]$ holds we can easily extend f to a function $f^* \in \mathbb{P}^*$ such that $f^*(\mathbf{m}_1) = 0$. Likewise, if $\delta[\mathfrak{s}]$ holds we can easily extend f to a function $f^{**} \in \mathbb{P}^*$ such that $f^*(\mathbf{m}_1) = 1$. Subsequently we can proceed as before, so that we find $f' \succeq f$ such that $f' \Vdash_{\mathcal{F}} (\{\mathfrak{R}_\varphi\}^A \Vdash_R \varphi[\mathfrak{s}])$.

The upshot of the foregoing is that

$$p \Vdash_{\mathcal{F}} (\varphi[\mathfrak{s}] \to \{\mathfrak{R}_\varphi\}^A(\langle \mathfrak{s} \rangle) \Vdash_R \varphi[\mathfrak{s}])\,.$$

To prove (8.8), let $f \succeq p$ and $f \Vdash_{\mathcal{F}} \exists e\, e \Vdash_R \varphi[\mathfrak{s}]$. Then $f \Vdash_{\mathcal{F}} \exists e\, e \Vdash_R \eta[\mathfrak{s}]$ or $f \Vdash_{\mathcal{F}} \exists e\, e \Vdash_R \delta[\mathfrak{s}]$, and thus the induction hypothesis for (8.8) supplies us with an $f'' \succeq f$ such that $f'' \Vdash_{\mathcal{F}} \varphi[\mathfrak{s}]$. Hence

$$p \Vdash_{\mathcal{F}} ((\exists e\, e \Vdash_R \varphi[\mathfrak{s}]) \to \varphi[\mathfrak{s}])\,.$$

Case 6: Let φ be of the form $\exists v \in N\eta$. Let $p \in \mathbb{P}^*$. To prove (8.7) let $f \in \mathbb{P}^*$, $f \succeq p$, and $f \Vdash_{\mathcal{F}} \varphi[\mathfrak{s}]$. By the Forcing Absoluteness Lemma 8.1.1, we know that $\varphi[\mathfrak{s}]$ is true, and thus there exists $n \in N$ such that $\eta[\mathfrak{s}(v|n)]$ holds.

If f is defined at $\mathbf{m}_0 := \langle \ulcorner\varphi\urcorner, \langle \mathfrak{s} \rangle\rangle$ we also know, since f satisfies condition (8.5), that $\eta[\mathfrak{s}(v|f(\mathbf{m}_0))]$ holds and thus $f \Vdash_{\mathcal{F}} \eta[\mathfrak{s}(v|f(\mathbf{m}_0))]$ by absoluteness.

If f is not defined at $\mathbf{m}_0$ we can extend f to a function f' by letting $f'(\mathbf{m}_0) = n$ for some $n \in \mathbb{N}$ satisfying $\eta[\mathfrak{s}(v|n)]$. At any rate, we find an extension f' of f in $\mathbb{P}^*$ such that

$$f'(\mathbf{m}_0) \downarrow \; \wedge \; f' \Vdash_{\mathcal{F}} \eta[\mathfrak{s}(v|f(\mathbf{m}_0))] \, .$$

By the induction hypothesis for (8.7) applied to η, there exists $f'' \succeq f'$ in $\mathbb{P}^*$ such that

$$f'' \Vdash_{\mathcal{F}} (\{\mathfrak{R}_\eta\}^A \langle \mathfrak{s}(v|f(\mathbf{m}_0))\rangle \Vdash_R \eta[\mathfrak{s}(v|f(\mathbf{m}_0))]) \, .$$

As $f'' \Vdash_{\mathcal{F}} A(\mathbf{m}_0) = f''(\mathbf{m}_0)$, the latter implies that $f'' \Vdash_{\mathcal{F}} (\{\mathfrak{R}_\varphi\}^A(\langle\mathfrak{s}\rangle) \Vdash_R \varphi[\mathfrak{s}])$, and hence

$$p \Vdash_{\mathcal{F}} (\varphi[\mathfrak{s}] \to (\exists e \, e \Vdash_R \varphi[\mathfrak{s}])) \, .$$

To show (8.8), assume $f \succeq p$ and $f \Vdash_{\mathcal{F}} (\exists e \, e \Vdash_R \varphi[\mathfrak{s}])$. Then $f \Vdash_{\mathcal{F}} (e \Vdash_R \eta[\mathfrak{s}(v|n)])$ for some $e, n \in \mathbb{N}$ and hence by the inductive assumption for (8.8) applied to η, there exists $f' \succeq f$ such that $f' \Vdash_{\mathcal{F}} \eta[\mathfrak{s}(v|n)]$, so that $f' \Vdash_{\mathcal{F}} \varphi[\mathfrak{s}]$. As a result,

$$p \Vdash_{\mathcal{F}} ((\exists e \, e \Vdash_R \varphi[\mathfrak{s}]) \to \varphi[\mathfrak{s}]) \, .$$

■

8.4 Conservativity results

Lemma 8.4.1 *Let θ be an arithmetical sentence. Then*

$$\mathbf{CZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ} \vdash \theta$$

implies

$$\mathbf{CZF}_{NA} \vdash \exists e \, e \Vdash_R \theta \, ,$$

and

$$\mathbf{IZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ} \vdash \theta$$

implies

$$\mathbf{IZF}_{NA} \vdash \exists e \, e \Vdash_R \theta \, .$$

Proof. This follows by combining Lemmas 6.2.49, 6.2.50, 6.2.51, and the Soundness Theorem 6.2.43. ■

Theorem 8.4.2 *Let θ be an arithmetical sentence. If*

$$\mathbf{CZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ} \vdash \theta,$$

then

$$\mathbf{CZF}_N \vdash \theta\,.$$

If

$$\mathbf{IZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ} \vdash \theta,$$

then

$$\mathbf{IZF}_N \vdash \theta\,.$$

Proof. By the previous Lemma 8.4.1, from $\mathbf{CZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ} \vdash \theta$ we infer that $\mathbf{CZF}_{NA} \vdash \exists e\, e \Vdash_R \theta$. Letting $\mathbb{P}^*$ be the set of forcing conditions associated with θ as defined in Lemma 8.3.1, we obtain from the Soundness Theorem 8.2.4 that

$$\mathbf{CZF}_N \vdash \exists p \in \mathbb{P}^*\, p \Vdash_{\mathcal{F}} (\exists e\, e \Vdash_R \theta)\,.$$

It follows thus from Lemma 8.3.1 that

$$\mathbf{CZF}_N \vdash \exists p \in \mathbb{P}^*\, p \Vdash_{\mathcal{F}} \theta\,,$$

and hence, by the Forcing Absoluteness Lemma 8.1.1,

$$\mathbf{CZF}_N \vdash \theta\,.$$

The proof for $\mathbf{IZF}_N$ is similar. ■

Corollary 8.4.3 *Let T be $\mathbf{CZF}_N$ or $\mathbf{IZF}_N$. Let $\mathbf{MP}$ be Markov's principle. For any arithmetical sentence θ, if*

$$T + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ} + \mathbf{MP} \vdash \theta\,,$$

then

$$T + \mathbf{MP} \vdash \theta\,.$$

Proof. From Lemma 6.2.55 we know that $\mathbf{MP}$ is realizable if assumed in the background universe. Hence the results follow by the same inference steps as in the proofs of Lemma 8.4.1 and Theorem 8.4.2. ■

Corollary 8.4.4 *For any arithmetical sentence θ, if*

$$T + \mathbf{UP} + \mathbf{UZ} \vdash \theta\,,$$

then

$$T \vdash \theta\,,$$

where T is any of the theories $\mathbf{CZF}_N$ or $\mathbf{IZF}_N$ extended by any combination of the following axioms: $\{\mathbf{DC}, \mathbf{RDC}, \mathbf{PA}_X, \mathbf{MP}\}$.

Proof. From Theorem 6.2.48 we know that $\mathbf{IZF}_N$ is sound with respect to relativized realizability semantics. We also know that $\mathbf{UP}, \mathbf{UZ}$ are realizable, and, moreover, that any of the axioms $\mathbf{DC}, \mathbf{RDC}, \mathbf{PA}_X$ and $\mathbf{MP}$ is realizable if assumed in the background universe. Hence the results follow by the inference steps employed in the proof of Lemma 8.4.2.
■

If one combines Lifschitz' style semantics (i.e., $\Vdash_L$) with these conservativity results, we get the following independence result:

Corollary 8.4.5

$$\mathbf{IZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ} \nvdash \mathbf{CT}_0!.$$

Proof. Assume

$$\mathbf{IZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ} \vdash \mathbf{CT}_0!.$$

Since $\mathbf{AC}^{NN} + \mathbf{CT}_0! \vdash \mathbf{CT}_0$ and thus

$$\mathbf{IZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ} + \mathbf{CT}_0! \vdash \mathbf{CT}_0,$$

i.e., by the assumption, it follows that

$$\mathbf{IZF}_N + \mathbf{AC}^{NN} + \mathbf{UP} + \mathbf{UZ} \vdash \mathbf{CT}_0^{ab}.$$

By Theorem 8.4.2 $\mathbf{IZF}_N \vdash \mathbf{CT}_0^{ab}$, but this contradicts the fact that all axioms of $\mathbf{IZF}_N$ are Lifschitz realizable while $\mathbf{CT}_0^{ab}$ is not. ■

As pointed out by Michael Rathjen, we could have actually strengthened the foregoing conservativity results by including a choice principle stronger than $\mathbf{AC}^{NN}$ in the statements of Lemma 8.4.1, Theorem 8.4.2, and Corollary 8.4.4, namely $\mathbf{AC}^{N\,N^N}$. Here $\mathbf{AC}^{N\,N^N}$ consists of the formulae

$$\forall n \exists f \in N^N\, \varphi(n, f) \;\to\; \exists F : N \to N^N\, \forall n\, \varphi(n, F(n))\,,$$

with φ arbitrary. The reason for this is that the proof of Lemma 6.2.49 shows more than is stated in the lemma. The proof actually proves that a relativized version of Church's thesis is realized, i.e., the schema

$$(\mathbf{CT}_A) \qquad \forall n \exists m\, \varphi(n,m) \to \exists e \in N\, \forall n\, \varphi(n, \{e\}^A(n))$$

holds in the realizability structure. The combination of $\mathbf{AC}^{NN}$ and $\mathbf{CT}_A$ implies $\mathbf{AC}^{N^{N^N}}$. As a result, we have the following corollaries:

Corollary 8.4.6 *Let θ be an arithmetical sentence. Let T be any of the theories $\mathbf{CZF}_N$, $\mathbf{CZF}_N + \mathbf{MP}$, $\mathbf{IZF}_N$, or $\mathbf{IZF}_N + \mathbf{MP}$. Then*

$$T + \mathbf{AC}^{N^{N^N}} + \mathbf{UP} + \mathbf{UZ} \vdash \theta$$

implies

$$T \vdash \exists e\, e \Vdash_R \theta\,.$$

Corollary 8.4.7 *Let θ be an arithmetical sentence. Let T be any of the theories $\mathbf{CZF}_N$, $\mathbf{CZF}_N + \mathbf{MP}$, $\mathbf{IZF}_N$, or $\mathbf{IZF}_N + \mathbf{MP}$. If*

$$T + \mathbf{AC}^{N^{N^N}} + \mathbf{UP} + \mathbf{UZ} \vdash \theta,$$

then

$$T \vdash \theta\,.$$

In conclusion, we have proved that internally (by the forcing interpretation) arithmetical formulae are absolute with respect to our relativized realizability. Moreover, we have shown that arithmetical formulae are absolute with respect to the forcing interpretation. These results and the results from Chapter 6 and Chapter 7 have led to several interesting conservativity results.

8.5 Conclusion and future work

In this thesis, we have successfully extended independence results and conservativity results from Heyting arithmetic to various intuitionistic set theories. In the future works, we might try to come up with a version that uses only $\mathbf{IZF}_N$ as a background theory for the independence results. Moreover, we will try to include other axioms, for example, the Regular Extension Axiom, into our system. Another question is whether it is possible to extend our conservativity results regarding arithmetical formula to other sets of formulae.

Appendix

Lifschitz' Realizability for Intuitionistic Zermelo-Fraenkel Set Theory

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Abstract

A variant of realizability for Heyting arithmetic which validates Church's thesis with uniqueness condition, but not the general form of Church's thesis, was introduced by V. Lifschitz in [11]. A Lifschitz counterpart to Kleene's realizability for functions (in Baire space) was developed by van Oosten [15]. In that paper he also extended Lifschitz' realizability to second order arithmetic. The objective here is to extend Lifschitz' realizability to intuitionistic Zermelo-Fraenkel set theory, **IZF**. The machinery would also work for extensions of **IZF** with large set axioms. In addition to separating Church's thesis with uniqueness condition from its general form in intuitionistic set theory, we also obtain several interesting corollaries. The interpretation repudiates a weak form of countable choice, $\mathbf{AC}_{\omega,\omega}$, asserting that a countable family of inhabited sets of natural numbers has a choice function. $\mathbf{AC}_{\omega,\omega}$ is validated by ordinary Kleene realizability and is of course provable in **ZF**. On the other hand, a pivotal consequence of $\mathbf{AC}_{\omega,\omega}$, namely that the sets of Cauchy reals and Dedekind reals are isomorphic, remains valid in this interpretation.

MSC:03F50, 03F35

Keywords: Intuitionistic set theory, Lifschitz' realizability, Church's thesis, countable axiom of choice

1 Introduction

In the constructive context, Church's thesis refers to the viewpoint that quantifier combinations $\forall x \exists y$ can be replaced by recursive functions getting y from x. Dragalin pointed out that there are two formal versions of Church's thesis one could consider adding to Heyting arithmetic **HA**:

$$\mathbf{CT}_0 \qquad \forall x \exists y\, A(x,y) \rightarrow \exists z \forall x\, [z \bullet x \downarrow \wedge A(x, z \bullet x)]$$

$$\mathbf{CT}_0! \qquad \forall x \exists ! y\, A(x,y) \rightarrow \exists z \forall x\, [z \bullet x \downarrow \wedge A(x, z \bullet x)]$$

(we write $z \bullet x$ for $\{z\}(x)$), and he posed the question whether the latter version is actually weaker than the former. The question was answered affirmatively in 1979 by Vladimir Lifschitz [11]. He introduced a modification of Kleene's realizability that validates $\mathbf{CT}_0!$ but falsifies instances of $\mathbf{CT}_0$. A Lifschitz counterpart to Kleene's realizability for functions (in Baire space) was developed by van Oosten [15]. In that paper he also extended Lifschitz' realizability to second order arithmetic. The objective here is to extend Lifschitz' realizability to full intuitionistic Zermelo-Fraenkel set theory, **IZF**. In addition to separating Church's thesis with uniqueness condition from its general form in intuitionistic set theory, we also obtain several interesting corollaries. The interpretation repudiates a weak form of countable choice, $\mathbf{AC}_{\omega,\omega}$, asserting that a countable family of inhabited sets of natural numbers has a choice function. $\mathbf{AC}_{\omega,\omega}$ is validated by ordinary Kleene realizability and is of course provable in **ZF**.

Definition: 1.1 Before we can describe the pivotal features of Lifschitz' notion of realizability we need to introduce some terminology. Variables $n, m, l, i, j, k, l, e, d, f, g, p, q$ range over numbers. We assume a bijective primitive recursive pairing function $\jmath : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and inverses $\jmath_1$ and $\jmath_2$. The symbol $\bullet$ denotes partial recursive application, T is Kleene's predicate (so $n \bullet k \downarrow$ iff $\exists m\, \mathrm{T}(n, k, m)$, read $n \bullet k$ is *defined*), and U the result-extracting function. $e \bullet k \simeq l$ stands for $\exists m\, \mathrm{T}(e, k, m)$ and $l = \mathrm{U}(\mu m.\mathrm{T}(n, k, m))$, where μ is the minimalization operator. If X is a set we write $e \bullet k \in X$ instead of $\exists l\, (e \bullet k \simeq l \wedge l \in X)$.

If f is an $n+1$-ary partial recursive function, we use $\lambda x.f(x, k_1, \ldots, k_n)$ to denote an index (usually provided by the S-m-n theorem) of the function $m \mapsto f(m, k_1, \ldots, k_n)$.

The main idea behind separating $\mathbf{CT}_0$ from $\mathbf{CT}_0!$ is to find a property P of pairs of numbers so that if there is a unique n such that $P(e, n)$ holds then there is an effective procedure to find n from e, while in general there is no such procedure if $\{m \mid P(e, m)\}$ contains more than one element. Lifschitz singled out the property $n \leq \jmath_2 e \wedge \forall m \neg \mathrm{T}(\jmath_1 e, n, m)$.

Lemma: 1.2 *Letting*

$$\mathrm{D}_e \;:=\; \{n \leq \jmath_2 e \mid \forall m \neg \mathrm{T}(\jmath_1 e, n, m)\} \tag{1}$$

there is no code g such that, for all e,

$$\mathrm{D}_e \neq \emptyset \;\Rightarrow\; g \bullet e \in \mathrm{D}_e. \tag{2}$$

Proof: This can be seen as follows. Let W_f and W_h be two disjoint, recursively inseparable r.e. sets. (2) would yield the existence of a recursive function F such that

$$\forall n[F(n) \bullet 0 \simeq f \bullet n \wedge F(n) \bullet 1 \simeq h \bullet n].$$

Then always $\mathrm{D}_{\jmath(F(x),1)} \neq \emptyset$, so $g \bullet \jmath(F(x), 1) \in \mathrm{D}_{\jmath(F(x),1)}$ and g would provide a recursive separation of W_f and W_h. If, on the other hand, we know that D_e is a singleton, then we can try to compute $(\jmath_1 e) \bullet 0, (\jmath_1 e) \bullet 1, \ldots, (\jmath_1 e) \bullet (\jmath_2 e)$ simultaneously and as soon as the $(\jmath_2 e) - 1$ many (guaranteed) successes have been recorded we know that the remaining one failure is the unique element of D_e. $\square$

1.1 Realizability for set theories

Realizability semantics for intuitionistic theories were first proposed by Kleene in 1945 [9]. Inspired by Kreisel's and Troelstra's [10] definition of realizability for higher order Heyting arithmetic, realizability was first applied to systems of set theory by Myhill [14] and Friedman [6]. More recently, realizability models of set theory were investigated by Beeson [2, 3] (for non-extensional set theories) and McCarty [12] (directly for extensional set theories). Rathjen [18] adapted realizability to the context of constructive Zermelo-Fraenkel set theory, **CZF**, and developed hybrids [19, 20] which combine realizability for extensional set theory with truth in order to prove metamathematical properties of intuitionistic set theories such as the disjunction and the numerical existence property.

The authors of the present paper had problems making up their mind as to whether to present **IZF** as a pure system of set theory or to opt for a language with urelements as it is done in Friedman's and Beeson's work (cf. [7, 3]). Both approaches have advantages and disadvantages. The disadvantage of pure set theory is that the natural numbers have to be encoded as finite ordinals, rendering the presentation of the basic parts of Lifschitz' realizability for atomic formulas, which are trivial in the arithmetic context, very cumbersome. The disadvantage of having a sorted language with numbers and sets is that realizability for those theories has never been worked out properly in the extensional cases. In the end we went for the latter choice.

1.2 IZF with urelements

We will formalize **IZF** in a similar manner as in [3, chap.viii] by having two unary predicates for natural numbers and for sets. We shall however eschew terms other than variables and constants by avoiding symbols for primitive recursive functions. Instead we will have symbols for primitive recursive relations. This makes the axiomatization of the arithmetic part a bit awkward (albeit still a straightforward affair) but relieves us from the burden of having to deal with complex terms in the realizability interpretation.

1.3 Logic Language

IZF is based on first-order intuitionistic predicate calculus with equality $=$. The language consists of the following. A binary predicate $\in$; unary predicates N and S (for numbers and sets); for each natural number n a constant $\bar{n}$ (but we omit the bar when $n = 0$); a 2-place relation symbol SUC (for the successor relation), two 3-place relation symbols $\mathrm{ADD}, \mathrm{MULT}$ (for the graphs of addition and multiplication), and further relation symbols for all primitive recursive relations.

To alleviate the burden of syntax we shall use variables n, m, k, l, i, j to range over natural numbers, so $\exists n \ldots$ and $\forall n \ldots$ will be abbreviations for $\exists x(\mathrm{N}(x) \wedge \ldots)$ and $\forall x(\mathrm{N}(x) \rightarrow \ldots)$, respectively. $\exists\,!nA(n)$ stands for $\exists x[\mathrm{N}(x) \wedge A(x)] \wedge \forall x \forall y[A(x) \wedge A(y) \rightarrow x = y]$. $x \notin y$ stands for $\neg(x \in y)$. $x \subseteq y$ abbreviates $\forall z(z \in x \rightarrow z \in y)$. We use $\forall x \in y \ldots$ and $\exists x \in y \ldots$ for $\forall x(x \in y \rightarrow \ldots)$ and $\exists x(x \in y \wedge \ldots)$, respectively.

Definition: 1.3 We list the axioms of **IZF** in groups:

A. Axioms on Numbers and Sets

1. $\forall x \neg(\mathrm{N}(x) \wedge \mathrm{S}(x))$
2. $\forall x \forall y(x \in y \rightarrow \mathrm{S}(y))$
3. $\mathrm{N}(\bar{n})$ for all natural numbers n.

B. Number-Theoretic Axioms

1. $\mathrm{SUC}(\bar{n}, \overline{n+1})$ for all naturals n.
2. $\forall n \, \exists ! m \, \mathrm{SUC}(n, m)$
3. $\forall n \forall m[\mathrm{SUC}(n, m) \rightarrow m \neq 0]$
4. $\forall m \, [m=0 \, \vee \, \exists n \, \mathrm{SUC}(n, m)]$
5. $\forall n \forall m \forall k \, (\mathrm{SUC}(m, n) \wedge \mathrm{SUC}(k, n) \rightarrow m=k)$
6. $\forall n \forall m \, \exists ! k \, \mathrm{ADD}(n, m, k)$
7. $\forall n \, \mathrm{ADD}(n, 0, n)$
8. $\forall n \forall k \forall m \forall l \forall i \, [\mathrm{ADD}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{SUC}(m, i) \rightarrow \mathrm{ADD}(n, l, i)]$
9. $\forall n \forall m \, \exists ! k \, \mathrm{MULT}(n, m, k)$
10. $\forall n \, \mathrm{MULT}(n, 0, 0)$
11. $\forall n \forall k \forall m \forall l \forall i \, [\mathrm{MULT}(n, k, m) \wedge \mathrm{SUC}(k, l) \wedge \mathrm{ADD}(m, n, i) \rightarrow \mathrm{MULT}(n, l, i)]$
12. Defining axioms for all symbols of primitive recursive relations R. These are similar to the above. We spare the reader the details.
13. $A(0) \wedge \forall n \forall m[A(n) \wedge \mathrm{SUC}(n, m) \rightarrow A(m)] \rightarrow \forall n A(n)$

C. Set-Theoretic Axioms

1. Extensionality. $\forall x \forall y(\mathrm{S}(x) \wedge \mathrm{S}(y) \rightarrow[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x=y])$
2. Pairing. $\forall x \forall y(\exists u[\mathrm{S}(u) \wedge x \in u \wedge y \in u])$
3. Union. $\forall x \exists u[\mathrm{S}(u) \wedge \forall z(z \in u \leftrightarrow \exists y(y \in x \wedge z \in y))]$
4. Separation. $\forall x \exists u[\mathrm{S}(u) \wedge \forall z(z \in u \leftrightarrow z \in x \wedge A(z))]$
   (u not free in $A(z)$)
5. Power set. $\forall x \exists u[\mathrm{S}(u) \wedge \forall z(z \in u \leftrightarrow(\mathrm{S}(z) \wedge z \subseteq x))]$
6. Infinity. $\exists u(\mathrm{S}(u) \wedge \forall z[z \in u \leftrightarrow \mathrm{N}(u)])$.
7. $\in$-induction. $\forall x[\forall y(y \in x \rightarrow A(y)) \rightarrow A(x)] \rightarrow \forall x A(x)$.

8. Collection. $\forall y \in x \exists z\, A(x,z) \to \exists u[\mathrm{S}(u) \wedge \forall y \in x\, \exists z \in u\, A(y,z)]$

Remark 1.4 The theory **IZF** in [3] comes with the additional axiom $\forall x[\mathrm{N}(x) \vee \mathrm{S}(x)]$. We could have adopted this axiom as well. The reason for not including it is that on the one hand this axioms does not make the theory stronger but on the other hand it would force us to define a more complicated realizability structure in which all objects carry a label which tells one whether it denotes a set or a number. This would have to be done in a hereditary way and would thus burden us with an extra layer of coding. A proof that **IZF**$+\forall x[\mathrm{N}(x)\vee\mathrm{S}(x)]$ can be interpreted in **IZF** using hereditarily labelled sets is sketched in [3, VIII.1]. Moreover, the same techniques can also be used to interpret **IZF** in pure **IZF** without urelements, $\mathbf{IZF}_0$ (cf. [3, VIII.1]). $\mathbf{IZF}_0$ has only the binary predicate $\in$ (no N, no S and no symbols for primitive recursive relations). In **IZF** we define the *pure sets* as those whose transitive closure contains only sets. Let Pure be the class of pure sets. To every formula A of $\mathbf{IZF}_0$ we assign a formula A^{Pure} of **IZF** which is obtained by relativizing all quantifiers to Pure. Then the exact relationship between the two theories is that

$$\mathbf{IZF}_0 \vdash A \Leftrightarrow \mathbf{IZF} \vdash A^{\mathrm{Pure}}.$$

2 The realizability structure

In what follows we shall be arguing informally in a classical set theory with urelements where the urlements are the natural numbers (e.g. **IZF** plus classical logic). The unique set of natural numbers provided by the Infinity axiom will be denoted by $\mathbb{N}$.

Definition: 2.1 Ordinals are transitive sets whose elements are transitive also. We use lower case Greek letters to range over ordinals. By recursion on α define

$$\mathrm{V}_\alpha^{set} = \bigcup_{\beta\in\alpha} \mathcal{P}(\mathbb{N} \times (\mathrm{V}_\beta^{set} \cup \mathbb{N})). \quad (3)$$

$$\mathrm{V}^{set} = \bigcup_{\alpha} \mathrm{V}_\alpha^{set}. \quad (4)$$

$$\mathrm{V}(L) = \mathbb{N} \cup \mathrm{V}^{set} \quad (5)$$

where $\mathcal{P}(x)$ denotes the power set of x.

Lemma: 2.2 *(i) The hierarchy V^{set} is cumulative: if $\alpha \le \beta$ then $\mathrm{V}_\alpha^{set} \subseteq \mathrm{V}_\beta^{set}$.*

(ii) If $x \subseteq \mathrm{V}(L)$ and $\mathrm{S}(x)$ then $x \in \mathrm{V}^{set}$.

(iii) Every $x \in \mathrm{V}^{set}$ is a set, i.e. $\mathrm{S}(x)$ holds.

(iv) $\forall x \in \mathrm{V}(L)\,[\mathrm{N}(x) \vee \mathrm{S}(x)]$.

Proof: (i) is immediate by (3). For (iii) note that if $x \in \mathrm{V}^{set}$ then $x \in \mathcal{P}(\mathbb{N} \times (\mathrm{V}_\beta^{set} \cup \mathbb{N}))$ for some β. So the claim follows from our rendering of the power set axiom which ensures that $\mathcal{P}(y)$ consists only of sets. (iv) follows from (iii).

(ii): If $x \subseteq \mathbb{N} \times \mathrm{V}(L)$ then, using strong collection and (i), there is an α such that $x \subseteq \mathbb{N} \times \mathrm{V}^{set}_{\alpha} \cup \mathbb{N}$, so $x \in \mathrm{V}^{set}_{\alpha+1}$, thus $x \in \mathrm{V}^{set}$. For a more detailed proof see [18, Lemma 3.5]. □

3 Defining Lifschitz' realizability for set theory

We adopt the conventions and notations from Definition 1.1.

Definition: 3.1 Let $a, a_i, b \in \mathrm{V}(L)$ and $e \in \mathbb{N}$. Below R is a symbol for an n-ary primitive recursive relation. Recall that $\mathrm{D}_e = \{n \leq \jmath_2 e \mid \forall m \neg \mathrm{T}(\jmath_1 e, n, m)\}$.

We define a relation $e \Vdash_L B$ between naturals e and sentences of **IZF** with parameters from $\mathrm{V}(L)$. $ef \Vdash_L B$ will be an abbreviation for $\exists k[e \bullet f \simeq k \wedge k \Vdash_L B]$.

$$
\begin{array}{rcl}
e \Vdash_L \mathrm{R}(a_1, \ldots, a_n) & \text{iff} & a_1, \ldots, a_n \in \mathbb{N} \wedge \mathrm{R}(a_1, \ldots, a_n) \\
e \Vdash_L \mathrm{N}(a) & \text{iff} & a \in \mathbb{N} \wedge e = a \\
e \Vdash_L \mathrm{S}(a) & \text{iff} & \mathrm{S}(a) \quad (\text{iff } a \in \mathrm{V}^{set}) \\
e \Vdash_L a \in b & \text{iff} & \mathrm{D}_e \neq \emptyset \wedge (\forall d \in \mathrm{D}_e) \exists c\, [\langle \jmath_1 d, c \rangle \in b \wedge \jmath_2 d \Vdash_L a = c] \\
e \Vdash_L a = b & \text{iff} & (a, b \in \mathbb{N} \wedge a = b) \text{ or } (\mathrm{D}_e \neq \emptyset \wedge \mathrm{S}(a) \wedge \mathrm{S}(b) \wedge \\
 & & (\forall d \in \mathrm{D}_e) \forall f, c\, [\langle f, c \rangle \in a \rightarrow (\jmath_1 d) \bullet f \Vdash_L c \in b] \wedge \\
 & & (\forall d \in \mathrm{D}_e) \forall f, c\, [\langle f, c \rangle \in b \rightarrow (\jmath_2 d) \bullet f \Vdash_L c \in a]) \\
e \Vdash_L A \wedge B & \text{iff} & \jmath_1 e \Vdash_L A \wedge \jmath_2 e \Vdash_L B \\
e \Vdash_L A \vee B & \text{iff} & \mathrm{D}_e \neq \emptyset \wedge (\forall d \in \mathrm{D}_e)([\jmath_1 d = 0 \wedge \jmath_2 d \Vdash_L A] \vee \\
 & & \qquad [\jmath_1 d \neq 0 \wedge \jmath_2 d \Vdash_L B]) \\
e \Vdash \neg A & \text{iff} & (\forall f \in \mathbb{N}) \neg f \Vdash_L A \\
e \Vdash_L A \rightarrow B & \text{iff} & (\forall f \in \mathbb{N}) [f \Vdash_L A \rightarrow e \bullet f \Vdash_L B] \\
e \Vdash_L \forall x A & \text{iff} & \mathrm{D}_e \neq \emptyset \wedge (\forall d \in \mathrm{D}_e)(\forall c \in \mathrm{V}(L))\, d \Vdash_L A[x/c] \\
e \Vdash_L \exists x A & \text{iff} & \mathrm{D}_e \neq \emptyset \wedge (\forall d \in \mathrm{D}_e)(\exists c \in \mathrm{V}(L))\, d \Vdash_L A[x/c] \\
\mathrm{V}(L) \models B & \text{iff} & (\exists e \in \mathbb{N})\, e \Vdash_L B.
\end{array}
$$

Notice that the definitions of $e \Vdash a \in b$ and $e \Vdash_L a = b$ fall under the scope of definition by transfinite recursion.

4 Recursion-theoretic preliminaries

Before we can prove the soundness of Lifschitz' realizability for **IZF** we need to recall some recursion-theoretic facts, mainly Lemmata 1–5 from Lifschitz' paper [11]. Jaap van Oosten has carried out a detailed analysis of these results by singling out the extra amount of classical logic one has to add to intuitionistic first-order arithmetic **HA** to prove them.

$\mathrm{MP_{pr}}$ is Markov's principle for primitive recursive formulas A:

$$\neg\neg\exists n\, A(n) \to \exists n\, A(n).$$

$\mathrm{B}\Sigma^0_2$-MP is Markov's principle for bounded Σ^0_2-formulae:

$$\neg\neg\exists n \leq m\, \forall k\, A(n,k,e) \to \exists n \leq m\, \forall k\, A(n,k,e)$$

for A primitive recursive.

Lemma: 4.1 *There is a total recursive function $\mathfrak{sg}$ such that*

$$\mathbf{HA} \vdash \forall n \forall m (m \in \mathrm{D}_{\mathfrak{sg}(n)} \leftrightarrow m = n).$$

Proof: [11, Lemma 2] and [15, Lemma 2.2]. □

Lemma: 4.2 *There is a partial recursive function ϕ such that*

$$\mathbf{HA} + \mathrm{MP_{pr}} \vdash \forall e [\exists n \forall m (m \in \mathrm{D}_e \leftrightarrow m = n) \to \phi(e)\downarrow \wedge\, \phi(e) \in \mathrm{D}_e].$$

Proof: [11, Lemma 1] and [15, Lemma 2.3]. □

Lemma: 4.3 *There is a partial recursive function Φ such that $\mathbf{HA} + \mathrm{MP_{pr}} + \mathrm{B}\Sigma^0_2\text{-MP}$ proves that for all e and f whenever $(\forall g \in \mathrm{D}_e)\, f \bullet g\downarrow$ then $\Phi(e,f)\downarrow$ and*

$$\forall h [h \in \mathrm{D}_{\Phi(e,f)} \leftrightarrow (\exists g \in \mathrm{D}_e)\, h = f \bullet g].$$

Proof: [11, Lemma 4] and [15, Lemma 2.4]. □

Lemma: 4.4 *There is a total recursive function $\mathfrak{un}$ such that $\mathbf{HA} + \mathrm{MP_{pr}} + \mathrm{B}\Sigma^0_2\text{-MP}$ proves that*

$$\forall e \forall h [h \in \mathrm{D}_{\mathfrak{un}(e)} \leftrightarrow (\exists g \in \mathrm{D}_e)(h \in \mathrm{D}_g)].$$

In other words, $\mathrm{D}_{\mathfrak{un}(e)} = \bigcup_{g \in \mathrm{D}_e} \mathrm{D}_g$.

Proof: [11, Lemma 3] and [15, Lemma 2.5]. □

Lemma: 4.5 *Let $\vec{x} = x_1, \ldots, x_r$ and $\vec{a} = a_1, \ldots, a_r$. To each formula $A(\vec{x})$ of $\mathbf{IZF}$ (with all free variables among $\vec{x}$) we can effectively assign (a code of) a partial recursive function χ_A such that, letting $\mathbf{IZF'} := \mathbf{IZF} + \mathrm{MP_{pr}} + \mathrm{B}\Sigma^0_2\text{-MP}$,*

$$\mathbf{IZF'} \vdash (\forall e \in \mathbb{N})(\forall \vec{a} \in \mathrm{V}(L))[\mathrm{D}_e \neq \emptyset \wedge ((\forall d \in \mathrm{D}_e)\, d \Vdash_L A(\vec{a})) \to \chi_A(e) \Vdash_L A(\vec{a})].$$

Proof: This is similar to [11, Lemma 5] and [15, Lemma 2.6]. However, due to the vastly more complicated setting we are dealing with here we provide a detailed proof. We use induction on the build-up of A.

If $A(\vec{x})$ is of the form $\mathrm{N}(x_i)$, define $\chi_A(e) := \phi(e)$, where ϕ is from Lemma 4.2. To see that this works note that $\mathrm{D}_e \neq \emptyset$ and for all $(\forall d \in \mathrm{D}_e)\, d \Vdash_L \mathrm{N}(a_i)$ entails that $\mathrm{N}(a_i)$ and $\mathrm{D}_e = \{a_i\}$, thus $\phi(e) = a_i$ and $\phi(e) \Vdash_L \mathrm{N}(a_i)$ follow by Lemma 4.2.

If $A(\vec{x})$ is of either form $\mathrm{S}(x_j)$ or $R(\vec{t})$ let $\chi_A(e) := 0$.

If $A(\vec{x})$ is of the form $x_i = x_j$ let $\chi_A(e) := \mathfrak{un}(e)$, where $\mathfrak{un}$ stems from Lemma 4.4. Note that $\mathfrak{un}$ is a total recursive function. To see that this works assume that $\mathrm{D}_e \neq \emptyset$ and for all $(\forall d \in \mathrm{D}_e)\, d \Vdash_L a_i = a_j$. Now, either $a_i, a_j \in \mathbb{N}$ or a_i and a_j are both sets. In the former case we then have $a_i = a_j$ and for any $n \in \mathbb{N}$, $n \Vdash_L a_i = a_j$, so in particular $\mathfrak{un}(e) \Vdash_L a_i = a_j$. If both a_i and a_j are sets, then $\mathfrak{un}(e) \Vdash_L a_i = a_j$ holds owing to Lemma 4.4 and the definition of realizability in this case.

Let $A(\vec{x})$ be $B(\vec{x}) \wedge C(\vec{x})$ and χ_B and χ_C be already defined. Let $\jmath_1^*$ and $\jmath_2^*$ be indices for $\jmath_1$ and $\jmath_2$, respectively. Consider the set $\mathrm{D}_{\Phi(\jmath_1^*, e)} = \{\jmath_1 n \mid n \in \mathrm{D}_e\}$ with Φ as in Lemma 4.3. If D_e is non-empty then so is $\mathrm{D}_{\Phi(\jmath_1^*, e)}$. If every element of D_e realizes $A(\vec{a})$ then every element of $\mathrm{D}_{\Phi(\jmath_1^*, e)}$ realizes $B(\vec{a})$. Hence under these assumptions $\chi_B(\Phi(\jmath_1^*, e))$ realizes $B(\vec{a})$. Similarly, $\chi_C(\Phi(\jmath_2^*, e))$ realizes $C(\vec{a})$. Hence the claim follows with $\chi_A(e) := \jmath(\chi_B(\Phi(\jmath_1^*, e)), \chi_C(\Phi(\jmath_2^*, e)))$.

Let $A(\vec{x})$ be $B(\vec{x}) \to C(\vec{x})$ and χ_B and χ_C be already defined. Let θ be a partial recursive function such that $\{\theta(m)\}(k) \simeq k \bullet m$. Assume that $\mathrm{D}_e \neq \emptyset$. Suppose $m \Vdash_L B(\vec{a})$. Then $d \bullet m \downarrow$ and $d \bullet m \Vdash_L C(\vec{a})$ for all $d \in \mathrm{D}_e$. Thus, by Lemma 4.5, we have $\mathrm{D}_{\Phi(\theta(m), e)} = \{d \bullet m \mid d \in \mathrm{D}_e\}$. Moreover, $\mathrm{D}_{\Phi(\theta(m), e)}$ is non-empty and every of its elements realizes $C(\vec{a})$, hence, by the inductive assumption, $\chi_C(\Phi(\theta(m), e))$ realizes $C(\vec{a})$. Thus we may define $\chi_A(e) := \lambda m.\chi_C(\Phi(\theta(m), e))$.

In all the remaining cases $\chi_A(e) := \mathfrak{un}(e)$ will work owing to Lemma 4.4 and the definition of realizability in these cases. □

The next result shows that our definition of realizability for arithmetic formulae coincides with the one given by Lifschitz [11].

Lemma: 4.6 *For every formula $A(u, \vec{x})$ there are partial recursive functions ψ_1 and ψ_2 such that provably in* $\mathbf{IZF}'$ *we have for all $e \in \mathbb{N}$ and $\vec{a} \in \mathrm{V}(L)$:*

(i) $e \Vdash_L \forall x[\mathrm{N}(x) \to A(x, \vec{a})] \;\to\; \forall n\, \psi_1(e) \bullet n \Vdash_L A(n, \vec{a})$,

(ii) $\forall n\, e \bullet n \Vdash_L A(n, \vec{a}) \;\to\; \psi_2(e) \Vdash_L \forall x[\mathrm{N}(x) \to A(x, \vec{a})]$.

(iii) $e \Vdash_L \exists x[\mathrm{N}(x) \wedge A(x, \vec{a})] \;\leftrightarrow\; \mathrm{D}_e \neq \emptyset \wedge (\forall d \in \mathrm{D}_e) \jmath_2 d \Vdash_L A(\jmath_1 d, \vec{a})$.

Proof: (i). Suppose $e \Vdash_L \forall x[\mathrm{N}(x) \to A(x, \vec{a})]$. Then $\mathrm{D}_e \neq \emptyset$ and for all $d \in \mathrm{D}_e$ and $n \in \mathbb{N}$, $d \bullet n \Vdash_L A(n, \vec{a})$. Thus, if we define f_n such that $f_n \bullet d \simeq d \bullet n$, we conclude with the aid of Lemma 4.3 that for all $n \in \mathbb{N}$ and $h \in \mathrm{D}_{\Phi(e, f_n)}$, $h \Vdash_L A(n, \vec{a})$. Hence, by Lemma 4.5, $(\forall n \in \mathbb{N})\, \chi_A(\Phi(e, f_n)) \Vdash_L A(n, \vec{a})$. So we can define ψ_1 by letting $\psi_1(e) := \lambda n.\chi_A(\Phi(e, f_n))$.

(ii). Suppose $\forall n\, e \bullet n \Vdash_L A(n,\vec{a})$. Then $e \Vdash_L \mathrm{N}(x) \to A(x,\vec{a})$ for all $x \in \mathrm{V}(L)$, hence $\mathfrak{sg}(e) \Vdash_L \forall x[\mathrm{N}(x) \to A(x,\vec{a})]$, so $\psi_2(n) := \mathfrak{sg}(n)$ will work.

(iii). Suppose $e \Vdash_L \exists x[\mathrm{N}(x) \wedge A(x,\vec{a})]$. Then $\mathrm{D}_e \neq \emptyset$ and for all $d \in \mathrm{D}_e$ there exists $c \in \mathrm{V}(L)$ such that $\jmath_1 d \Vdash_L \mathrm{N}(c)$ and $\jmath_2 d \Vdash_L A(c,\vec{a})$. But $\jmath_1 d \Vdash_L \mathrm{N}(c)$ entails that $c = \jmath_1 d$, thus $\jmath_2 d \Vdash_L A(\jmath_1 d,\vec{a})$. The converse is obvious. □

4.1 The soundness theorem for intuitionistic predicate logic with equality

Lemma: 4.7 *There are $\mathbf{i_r}, \mathbf{i_s}, \mathbf{i_t}, \mathbf{i_0}, \mathbf{i_1} \in \mathbb{N}$ such that for all $x, y, z \in \mathrm{V}(L)$,*

1. $\mathbf{i}_r \Vdash_L x = x$.
2. $\mathbf{i}_s \Vdash_L x = y \to y = x$.
3. $\mathbf{i}_t \Vdash_L (x = y \wedge y = z) \to x = z$.
4. $\mathbf{i}_0 \Vdash_L (x = y \wedge y \in z) \to x \in z$.
5. $\mathbf{i}_1 \Vdash_L (x = y \wedge z \in x) \to z \in y$.
6. *Moreover, for each formula $A(v, u_1, \ldots, u_r)$ of **IZF** all of whose free variables are among $v, u_1, \ldots, u_r$ there exists $\mathbf{i}_A \in \mathbb{N}$ such that for all $x, y, z_1, \ldots, z_r \in \mathrm{V}(L)$,*

$$\mathbf{i}_A \Vdash_L x = y \wedge A(x,\vec{z}) \to A(y,\vec{z}),$$

where $\vec{z} = z_1, \ldots, z_r$.

Proof: (1) Note that $n \Vdash_L x = x$ holds for all $n, x \in \mathbb{N}$. Let $x \in \mathbb{N}$ and $a \in \mathrm{V}^{set}_\alpha$. Suppose $e \bullet 0 \downarrow$ and $e \bullet 0 \Vdash_L b = b$ holds for all $b \in \mathbb{N} \cup \bigcup_{\beta\in\alpha} \mathrm{V}^{set}_\beta$. Then we have $(\forall \langle f, b\rangle \in a)\, \mathfrak{sg}(\jmath(f, e\bullet 0)) \Vdash_L b \in a$. There is a recursive function ℓ such that $(\ell(e\bullet 0))\bullet f \simeq \mathfrak{sg}(\jmath(f, e\bullet 0))$, and hence, by the foregoing,

$$(\forall \langle f, b\rangle \in a)\, (\jmath_1 d) \bullet f \Vdash_L b \in a$$

with $d = \jmath(\ell(e\bullet 0), \ell(e\bullet 0))$. As a result, $\mathfrak{sg}(\jmath(\ell(e\bullet 0), \ell(e\bullet 0))) \Vdash_L a = a$. By the recursion theorem there exists an e^* such that

$$e^* \bullet 0 \simeq \mathfrak{sg}(\jmath(\ell(e^* \bullet 0), \ell(e^* \bullet 0))).$$

By induction on α it therefore follows that $e^* \bullet 0 \Vdash_L a = a$ holds for all $a \in \mathrm{V}^{set}$. So we may put $\mathbf{i}_r := e^* \bullet 0$. As $\mathbf{i}_r \Vdash_L n = n$ (trivially) holds for all $n \in \mathbb{N}$, too, we get $\mathbf{i}_r \Vdash_L z = z$ for all $z \in \mathrm{V}(L)$.

(2): It is routine to check that

$$\mathbf{i}_s := \lambda e.\Phi(e, \lambda d.\jmath(\jmath_2 d, \jmath_1 d)) \Vdash_L x = y \to y = x,$$

with Φ from Lemma 4.3.

(3) and (4): We prove these simultaneously. Let $\mathbf{TC}(a)$ denote the transitive closure of a. We employ (transfinite) induction on the ordering $\lhd$ which is the transitive closure of the ordering $\lhd_1$ on ordered triples:

$$\langle x,y,z\rangle \lhd_1 \langle a,b,c\rangle \quad \text{iff} \quad (x=a \wedge y=b \wedge z\in \mathbf{TC}(c)) \vee (x=a \wedge y\in \mathbf{TC}(b) \wedge z=c) \\ \vee (x\in \mathbf{TC}(a) \wedge y=b \wedge z=c).$$

$\lhd$-induction follows from the usual $\in$-induction.

Now suppose $a,b,c\in \mathrm{V}(L)$ and inductively assume that for all $\langle x,y,z\rangle \lhd \langle a,b,c\rangle$,

$$e^{\#}\bullet 0 \quad \Vdash_L \quad (x=y \wedge y=z)\rightarrow x=z \tag{6}$$
$$e^{\#}\bullet 1 \quad \Vdash_L \quad (x=y \wedge y\in z)\rightarrow x\in z. \tag{7}$$

Suppose $e\Vdash_L a=b \wedge b=c$. Then $\jmath_1 e\Vdash_L a=b$ and $\jmath_2 e\Vdash_L b=c$. Then either $a,b,c\in\mathbb{N}$ and for any $n\in\mathbb{N}$ we have $n\Vdash_L b=c$, or $a,b,c\in \mathrm{V}^{set}_\alpha$. So let's assume $a,b,c\in\mathrm{V}^{set}$. Let $d\in \mathrm{D}_{\jmath_1 e}$ and $d'\in\mathrm{D}_{\jmath_2 e}$. If $\langle f,u\rangle\in a$, then $(\jmath_1 d)\bullet f\Vdash_L u\in b$, and hence, for all $g\in \mathrm{D}_{(\jmath_1 d)\bullet f}$ there exists v such that $\langle \jmath_1 g, v\rangle\in b$ and $\jmath_2 g\Vdash_L u=v$. Moreover, $(\jmath_1 d')\bullet(\jmath_1 g)\Vdash_L v\in c$. As $\langle u,v,c\rangle\lhd\langle a,b,c\rangle$ we can employ (7) to conclude that

$$\ell_1(e^{\#},g,d') := (e^{\#}\bullet 1)\bullet \jmath(\jmath_2 g,(\jmath_1 d')\bullet(\jmath_1 g))\Vdash_L u\in c.$$

Using Lemmata 4.3 and 4.5 repeatedly we get

$$\ell_2(e^{\#},f,d,d') := \chi_1(\Phi((\jmath_1 d)\bullet f,\lambda g.\ell_1(e^{\#},g,d'))) \quad \Vdash_L \quad u\in c$$
$$\ell_3(e^{\#},f,d) := \chi_2(\Phi(\jmath_2 e,\lambda d'.\ell_2(e^{\#},f,d,d'))) \quad \Vdash_L \quad u\in c$$
$$\ell^*(e^{\#},e,f) := \chi_3(\Phi(\jmath_2 e,\lambda d.\ell_3(e^{\#},f,d))) \quad \Vdash_L \quad u\in c \tag{8}$$

for appropriate partial recursive functions χ_i.

Similarly one distills a partial recursive function ℓ^{**} such that for $\langle f,u\rangle\in c$,

$$\ell^{**}(e^{\#},e,f) := \chi_3(\Phi(\jmath_2 e,\lambda d.\ell_3(e^{\#},f,d))) \quad \Vdash_L \quad u\in a. \tag{9}$$

As a result of (8) and (9) we have with

$$\wp_1(e^{\#}) \quad := \quad (\lambda e.\mathfrak{sg}(\jmath(\lambda f.\ell^*(e^{\#},e,f),\lambda f.\ell^{**}(e^{\#},e,f))),$$
$$\wp_1(e^{\#}) \quad \Vdash_L \quad a=b \wedge b=c\rightarrow a=c. \tag{10}$$

Next suppose $e\Vdash_L a=b \wedge b\in c$. Then $\jmath_1 e\Vdash_L a=b$ and $\jmath_2 e\Vdash_L b\in c$. Hence $\mathrm{D}_{\jmath_2 e}\neq\emptyset$ and for all $d\in\mathrm{D}_{\jmath_2 e}$ there exists v such that $\langle \jmath_1 d,v\rangle\in c$ and $\jmath_2 d\Vdash_L b=v$, thus $\jmath(\jmath_1 e,\jmath_2 d)\Vdash_L a=b\wedge b=v$. As $\langle a,b,v\rangle\lhd\langle a,b,c\rangle$ we can employ (6) to conclude

$$\ell_4(e^{\#},e,d) := (e^{\#}\bullet 0)\bullet\jmath(\jmath_1 e,\jmath_2 d) \Vdash_L a=v.$$

Letting $\ell_4(e^{\#},e,d) := \jmath(\jmath_1 d,(e^{\#}\bullet 0)\bullet\jmath(\jmath_1 e,\jmath_2 d))$, we thus have $\langle \jmath_1(\ell_4(e^{\#},e,d)),v\rangle\in c$ and $\jmath_2(\ell_4(e^{\#},e,d))\Vdash_L a=v$. Hence, by Lemma 4.3, $\Phi(\jmath_1 e,\lambda d.\ell_4(e^{\#},e,d))\Vdash_L a\in c$. So the upshot is that

$$\wp_2(e^{\#}) := \Phi(\jmath_1 e,\lambda d.\ell_4(e^{\#},e,d)) \quad \Vdash_L \quad a=b\wedge b\in c\rightarrow a\in c. \tag{11}$$

Finally we use the recursion theorem to find an index $e^{\#}$ such that

$$\begin{array}{rcl} e^{\#} \bullet 0 & \simeq & \wp_1(e^{\#}) \\ e^{\#} \bullet 1 & \simeq & \wp_2(e^{\#}). \end{array}$$

With $\mathbf{i}_t := e^{\#} \bullet 0$ and $\mathbf{i}_0 := e^{\#} \bullet 1$ the above shows that (3) and (4) are satisfied.

(5). Suppose $e \Vdash_L a = b \wedge c \in a$. Then $\jmath_1 e \Vdash_L a = b$ and $\jmath_2 e \Vdash_L c \in a$. From the latter we get that $\mathrm{D}_{\jmath_2 e} \neq \emptyset$ and for all $d \in \mathrm{D}_{\jmath_2 e}$ there exists v such that $\langle \jmath_1 d, v \rangle \in a$ and $\jmath_2 d \Vdash_L c = v$. Thus, $\mathrm{D}_{\jmath_1 e} \neq \emptyset$ and since $\jmath_1 e \Vdash_L a = b$, it follows that for all $h \in \mathrm{D}_{\jmath_1 e}$, $(\jmath_1 h) \bullet (\jmath_1 d) \Vdash_L v \in b$, so that by (4),

$$\ell_5(d,h) := \mathbf{i}_0(\jmath(\jmath_2 d, (\jmath_1 h) \bullet (\jmath_1 d)) \Vdash_L c \in b.$$

Using Lemmata 4.3 and 4.5 repeatedly we get

$$\begin{array}{rcl} \ell_6(e,d) := \chi_3(\Phi(\jmath_1 e, \lambda h.\ell_5(d,h)) & \Vdash_L & c \in b \\ \ell_7(e) := \chi_4(\Phi(\jmath_2 e, \lambda d.\ell_6(e,d))) & \Vdash_L & c \in b \end{array}$$

for appropriate partial recursive functions χ_i. So we may put $\mathbf{i}_1 := \lambda e.\ell_7(e)$.

(6). This is shown by a routine induction on the complexity of A, the non-trivial atomic cases being provided (2)-(5). □

Corollary: 4.8 *There is a total recursive function θ such that for all $a \in \mathrm{V}(L)$,*

$$(\forall \langle f, u \rangle \in a)\, \theta(f) \Vdash_L u \in a.$$

Proof: Let

$$\theta(f) \quad := \quad \mathfrak{sg}(\jmath(f, \mathbf{i}_r)). \tag{12}$$

□

Theorem: 4.9 *Let $\mathcal{D}$ be a proof in intuitionistic predicate logic with equality of a formula $A(u_1, \ldots, u_r)$ of* **IZF** *all of whose free variables are among $u_1, \ldots, u_r$. Then there is $e_{\mathcal{D}} \in \mathbb{N}$ such that* **IZF′** *proves*

$$e_{\mathcal{D}} \Vdash_L \forall u_1 \ldots \forall u_r\, A(u_1, \ldots, u_r).$$

Proof: We use a standard Hilbert-type systems for intuitionistic predicate logic. The proof proceeds by induction on the derivation. The correctness of axioms and rules pertaining to the connectives $\wedge, \neg, \rightarrow$ is exactly the same as for Kleene's realizability. We have also shown realizability of the equality axioms in Lemma 4.7. So it remains to address the axioms and rules for $\vee, \forall, \exists$.

Axioms for $\vee$:

$A \to A \vee B$ or $A \to B \vee A$. Suppose $e \Vdash_L A$. As $\mathrm{D}_{\mathfrak{sg}(\jmath(0,e))} = \{\jmath(0,e)\}$ by Lemma 4.1, it follows that $\mathfrak{sg}(\jmath(0,e)) \Vdash_L A \vee B$ and hence $\lambda e.\mathfrak{sg}(\jmath(0,e)) \Vdash_L A \to A \vee B$. Similarly, $\lambda e.\mathfrak{sg}(\jmath(1,e)) \Vdash_L A \to B \vee A$.

$A \vee B \to ((A \to C) \to ((B \to C) \to C))$. Suppose $e \Vdash_L A \vee B$. Then $\mathrm{D}_e \neq \emptyset$. Let $d \in \mathrm{D}_e$. Then $\jmath_1 d = 0 \wedge \jmath_2 d \Vdash_L A$ or $\jmath_1 d \neq 0 \wedge \jmath_2 d \Vdash_L B$. Suppose $f \Vdash_L A \to C$ and $g \Vdash_L B \to C$. Define a partial recursive function $\mathfrak{f}$ by

$$\mathfrak{f}(d, f', g') = \begin{cases} f' \bullet (\jmath_2 d) & \text{if } \jmath_1 d = 0 \\ g' \bullet (\jmath_2 d) & \text{if } \jmath_1 d \neq 0 \end{cases}$$

Then $\mathfrak{f}(d,f,g) \Vdash_L C$ and hence $\lambda f.\lambda g.\mathfrak{f}(d,f,g) \Vdash_L (A \to C) \to ((B \to C) \to C)$. With the aid of Lemma 4.3 we can thus conclude that $\mathrm{D}_{\Phi(e,\lambda d.\lambda f.\lambda g.\mathfrak{f}(d,f,g))} \neq \emptyset$ and for all $p \in \mathrm{D}_{\Phi(e,\lambda d.\lambda f.\lambda g.\mathfrak{f}(d,f,g))}$ we have

$$p \Vdash_L (A \to C) \to ((B \to C) \to C).$$

Let $E := (A \to C) \to ((B \to C) \to C)$. By Lemma 4.5 we can therefore conclude that

$$\chi_E(\Phi(e, \lambda d.\lambda f.\lambda g.\mathfrak{f}(d,f,g))) \Vdash_L E.$$

As a result, $\lambda e.\chi_E(\Phi(e, \lambda d.\lambda f.\lambda g.\mathfrak{f}(d,f,g))) \Vdash_L A \vee B \to E$.

Axioms and Rules for $\forall$:

If $e \Vdash_L \forall x A(x, \vec{a})$, then $\mathrm{D}_e \neq \emptyset$ and $(\forall b \in \mathrm{V}(L))(\forall d \in \mathrm{D}_e)\, d \Vdash_L A(b, \vec{a})$, and hence, by Lemma 4.5, $\chi_A(e) \Vdash_L A(b, \vec{a})$ for all $b \in \mathrm{V}(L)$. Consequently,

$$\lambda e.\chi_A(e) \Vdash_L \forall x A(x, \vec{a}) \to A(b, \vec{a})$$

for all $b, \vec{a} \in \mathrm{V}(L)$.

We also have the rule from $B(\vec{u}) \to A(x, \vec{u})$ infer $B(\vec{u}) \to \forall x A(x, \vec{u})$ if x is not free in $B(\vec{u})$. Inductively we have a realizer $\mathfrak{h}$ such that for all $b, \vec{a} \in \mathrm{V}(L)$,

$$\mathfrak{h} \Vdash_L B(\vec{a}) \to A(b, \vec{a}).$$

Suppose $d \Vdash_L B(\vec{a})$. then $\mathfrak{h} \bullet d \Vdash_L A(b, \vec{a})$ holds for all $b \in \mathrm{V}(L)$, whence $\mathfrak{sg}(\mathfrak{h} \bullet d) \Vdash_L \forall x A(x, \vec{a})$. As a result,

$$\lambda d.\mathfrak{sg}(\mathfrak{h} \bullet d) \Vdash_L B(\vec{a}) \to \forall x A(x, \vec{a})$$

for all $\vec{a} \in \mathrm{V}(L)$.

Axioms and Rules for $\exists$:

If $e \Vdash_L A(a)$ then $\mathfrak{sg}(e) \Vdash_L \exists x A(x)$, thus $\lambda e.\mathfrak{sg}(e) \Vdash_L A(a) \to \exists x A(x)$ for all $a \in \mathrm{V}(L)$.

Finally we have the rule, from $A(x, \vec{u}) \to B(\vec{u})$ infer $\exists x\, A(x, \vec{u}) \to B(\vec{u})$ if x is not free in $B(\vec{u})$. Inductively we have a realizer $\mathfrak{g}$ such that for all $b, \vec{a} \in \mathrm{V}(L)$,

$$\mathfrak{g} \Vdash_L A(b, \vec{a}) \to B(\vec{a}).$$

Suppose $e \Vdash_L \exists x A(x, \vec{a})$. Then $\mathrm{D}_e \neq \emptyset$ and for all $d \in \mathrm{D}_e$ exists $c \in \mathrm{V}(L)$ such that $d \Vdash_L A(c, \vec{a})$. Consequently, $(\forall d \in \mathrm{D}_e)\, \mathfrak{g} \bullet d \Vdash_L B(\vec{a})$. By Lemma 4.3 we then have $\mathrm{D}_{\Phi(e,\mathfrak{g})} \neq \emptyset$ and $(\forall g \in \mathrm{D}_{\Phi(e,\mathfrak{g})})\, g \Vdash_L B(\vec{a})$. Using Lemma 4.5 we arrive at $\chi_B(\Phi(e, \mathfrak{g})) \Vdash_L B(\vec{a})$; whence $\lambda e.\chi_B(\Phi(e, \mathfrak{g})) \Vdash_L \exists x A(x, \vec{a}) \to B(\vec{a})$. $\square$

5 The soundness theorem for IZF

Lemma: 5.1 *There is a partial recursive function* $\mathfrak{sub}$ *such that for all* α, $a \in \mathrm{V}^{set}_{\alpha}$ *and* $b \in \mathrm{V}^{set}$,

$$e \Vdash_L b \subseteq a \to \exists b^* \in \mathrm{V}^{set}_{\alpha}\, \mathfrak{sub}(e) \Vdash_L b = b^*.$$

Proof: Suppose $a \in \mathrm{V}^{set}_{\alpha}$, $b \in \mathrm{V}^{set}$, and $e \Vdash_L b \subseteq a$. Then $\mathrm{D}_e \neq \emptyset$. Let

$$b^* := \{\langle \jmath(f', g'), u\rangle \mid (\exists \langle f', x\rangle \in b)[\langle \jmath_1 g', u\rangle \in a \wedge \jmath_2 g' \Vdash_L x = u]\}.$$

Clearly, $b^* \in \mathrm{V}^{set}_{\alpha}$. We have

$$\begin{array}{rcl}
\langle f, x\rangle \in b & \to & \theta(f) \Vdash_L x \in b \\
& \to & (\forall d \in \mathrm{D}_e)\ d \bullet \theta(f) \Vdash_L x \in a \\
& \to & (\forall d \in \mathrm{D}_e)\ (\mathrm{D}_{d\bullet\theta(f)} \neq \emptyset\ \wedge \\
& & \quad (\forall d' \in \mathrm{D}_{d\bullet\theta(f)})\, \exists u\, [\langle \jmath_1 d', u\rangle \in a\ \wedge\ \jmath_2 d' \Vdash_L x = u]) \\
& \to & (\forall d \in \mathrm{D}_e)\ (\forall d' \in \mathrm{D}_{d\bullet\theta(f)})\, \exists u\, [\langle \jmath(f, d'), u\rangle \in b^*\ \wedge\ \jmath_2 d' \Vdash_L x = u] \\
& \to & (\forall d \in \mathrm{D}_e)(\forall h \in \mathrm{D}_{\Phi(d\bullet\theta(f), \lambda d'.\jmath(\jmath(f,d'), \jmath_2 d'))}) \\
& & \quad \exists u\, [\langle \jmath_1 h, u\rangle \in b^*\ \wedge\ \jmath_2 h \Vdash_L x = u] \\
& \to & (\forall d \in \mathrm{D}_e)\ \Phi(d \bullet \theta(f), \lambda d'.\jmath(\jmath(f, d'), \jmath_2 d')) \Vdash_L x \in b^* \\
& \to & (\forall g \in \mathrm{D}_{\Phi(e, \lambda d.\Phi(d\bullet\theta(f), \lambda d'.\jmath(\jmath(f,d'), \jmath_2 d')))})\ g \Vdash_L x \in b^* \\
& \to & \chi_A(\Phi(e, \lambda d.\Phi(d \bullet \theta(f), \lambda d'.\jmath(\jmath(f, d'), \jmath_2 d')))) \Vdash_L x \in b^*
\end{array}$$

where the fifth and seventh arrow are justified by Lemma 4.3 and the last arrow follows by Lemma 4.5 with $A \equiv x_1 \in x_2$.

Conversely, we have

$$\begin{array}{rcl}
\langle h, u\rangle \in b^* & \to & \exists x\, [\langle \jmath_1 h, x\rangle \in b\ \wedge\ \langle \jmath_1(\jmath_2 h), u\rangle \in a\ \wedge\ \mathbf{i}_s((\jmath_2(\jmath_2 h))) \Vdash_L u = x] \\
& \to & \mathfrak{sg}(\jmath(\jmath_1 h, \mathbf{i}_s((\jmath_2(\jmath_2 h))))) \Vdash_L u \in b.
\end{array}$$

with $\mathbf{i}_s$ from Lemma 4.7. The upshot of the foregoing is that with

$$\begin{array}{rcl}
\nu(e, f) & := & \chi_A(\Phi(e, \lambda d.\Phi(d \bullet \theta(f), \lambda d'.\jmath(\jmath(f, d'), \jmath_2 d')))), \\
\mu(h) & := & \mathfrak{sg}(\jmath(\jmath_1 h, \mathbf{i}_s((\jmath_2(\jmath_2 h))))), \\
\mathfrak{sub}(e) & := & \mathfrak{sg}(\jmath(\lambda f.\nu(e, f), \lambda h.\mu(h)))
\end{array}$$

we have $\mathfrak{sub}(e) \Vdash_L b = b^*$. □

Theorem: 5.2 *For every axiom* A *of* **IZF**, *one can effectively construct an index* e *such that*

$$\mathbf{IZF}' \vdash\ (\bar{e} \Vdash_L A).$$

Proof: We treat the axioms one after the other.

(**Arithmetic axioms**): There are several and they are very boring to validate. In view of Lemma 4.6 it's also obvious how to realize them. We do one case study. $0 \Vdash_L \mathrm{SUC}(n, n{+}1)$ holds for all $n \in \mathbb{N}$. Hence $\jmath(n+1, 0) \Vdash_L \mathrm{N}(n+1) \wedge \mathrm{SUC}(n, n+1)$, thus

$$\mathfrak{sg}(\jmath(n+1,0)) \Vdash_L \exists k\, \mathrm{SUC}(n,k),$$

so $\forall n\ e^* \bullet n \Vdash_L \exists k\, \mathrm{SUC}(n,k)$ with e^* is chosen such that $e^* \bullet n = \mathfrak{sg}(\jmath(n+1,0))$. By Lemma 4.6 we then have

$$\psi_2(e^*) \quad \Vdash_L \quad \forall n \exists k\, \mathrm{SUC}(n,k). \tag{13}$$

Now suppose $e \Vdash_L \mathrm{SUC}(c,a) \wedge \mathrm{SUC}(c,b)$. Then $c, a, b \in \mathbb{N}$ and $c+1 = a = b$, thus $0 \Vdash_L a = b$ and hence

$$\mathfrak{sg}(\mathfrak{sg}(\mathfrak{sg}(\lambda u.0))) \quad \Vdash_L \quad \forall x \forall y \forall z\, [\mathrm{SUC}(x,y) \wedge \mathrm{SUC}(x,z) \to y = z]. \tag{14}$$

From (13) and (14) we obtain a realizer for the first number-theoretic axiom.

(**Induction on** $\mathbb{N}$): Suppose

$$e \Vdash_L A(0) \wedge \forall x \forall y [\mathrm{N}(x) \wedge \mathrm{N}(y) \wedge A(x) \wedge \mathrm{SUC}(x,y) \to A(y)].$$

Then $\mathrm{D}_{\jmath_2 e} \neq \emptyset$ and $(\forall d \in \mathrm{D}_{\jmath_2 e})\, \mathrm{D}_d \neq \emptyset$. Moreover, if $d \in \mathrm{D}_{\jmath_2 e}$ then for all $h \in \mathrm{D}_d$, $h \Vdash_L \mathrm{N}(x) \wedge \mathrm{N}(y) \wedge A(x) \wedge \mathrm{SUC}(x,y) \to A(y)$ for all $x, y \in \mathrm{V}(L)$. Thus for all $h \in \mathrm{D}_{\mathfrak{un}(\jmath_2 e)}$ (with $\mathfrak{un}$ from Lemma 4.4) and all $x, y \in \mathrm{V}(L)$ we have

$$h \Vdash_L \mathrm{N}(x) \wedge \mathrm{N}(y) \wedge A(x) \wedge \mathrm{SUC}(x,y) \to A(y). \tag{15}$$

Clearly, $\jmath_1 e \Vdash_L A(0)$. Now suppose $n \in \mathbb{N}$ and $\mathrm{SUC}(n,m)$ and we have an index e^* such that

$$(\forall h \in \mathrm{D}_{\mathfrak{un}(\jmath_2 e)}\, e^* \bullet \jmath(h,n) \Vdash_L A(n).$$

Then $\jmath(n,m) \Vdash_L \mathrm{N}(n) \wedge \mathrm{N}(m)$, so $\jmath(\jmath(n,m), e^* \bullet \jmath(h,n) \Vdash_L (\mathrm{N}(n) \wedge \mathrm{N}(m)) \wedge A(n)$, and finally $\jmath(\jmath(\jmath(n,m), e^* \bullet \jmath(h,n), 0) \Vdash_L ((\mathrm{N}(n) \wedge \mathrm{N}(m)) \wedge A(n)) \wedge \mathrm{SUC}(n,m)$. From the latter we get

$$\mathfrak{l}^{\#}(e^*, n, h) := h \bullet \jmath(\jmath(\jmath(n,m), e^* \bullet \jmath(h,n)), 0) \Vdash_L A(m).$$

We suppressed m in $\mathfrak{l}^{\#}$ since m is computable from n ($m = n+1$). Now choose e^* by the recursion theorem in such a way that $e^* \bullet \jmath(h,0) = \jmath_1 e$ and

$$e^* \bullet \jmath(h, k+1) \simeq \mathfrak{l}^{\#}(e^*, k, h).$$

If we inductively assume that $e^* \bullet \jmath(h,n) \downarrow$ for all $h \in \mathrm{D}_{\mathfrak{un}(e)}$ then the foregoing showed that $e^* \bullet \jmath(h,m) \downarrow$ for all $h \in \mathrm{D}_{\mathfrak{un}(e)}$. Hence $(\forall g \in \mathrm{D}_{\Phi(\mathfrak{un}(e), \lambda h. e^* \bullet \jmath(h,m))})\, g \Vdash_L A(m)$ by Lemma 4.3 and thus with

$$\mathfrak{l}^{\diamond}(e,m) = \begin{cases} \jmath_1 e & \text{if } m = 0 \\ \chi_A(\Phi(\mathfrak{un}(e), \lambda h. e^* \bullet \jmath(h,m))) & \text{if } m \neq 0 \end{cases}$$

(using Lemma 4.5) we have $\mathfrak{l}^{\diamond}(e,m) \Vdash_L A(m)$ for all $m \in \mathbb{N}$. As a result,

$$\lambda m.\mathfrak{l}^{\diamond}(e,m) \Vdash_L \mathrm{N}(a) \to A(a)$$

holds for all $a \in \mathrm{V}(L)$ since $d \Vdash_L \mathrm{N}(a)$ implies $d = a$. Thus

$$\mathfrak{sg}(\lambda m.\mathfrak{l}^{\diamond}(e,m)) \Vdash_L \forall x(\mathrm{N}(x) \to A(x)),$$

and hence

$$\lambda e.\mathfrak{sg}(\lambda m.\mathfrak{l}^{\diamond}(e,m)) \Vdash_L A(0) \wedge \forall n \forall m[A(n) \wedge \mathrm{SUC}(n,m) \to A(m)] \to \forall n A(n).$$

(**Extensionality**): Let $a, b \in \mathrm{V}(L)$. Also suppose that $\mathrm{S}(a) \wedge \mathrm{S}(b)$ and

$$e \Vdash_L \forall x(x \in a \leftrightarrow x \in b).$$

Then $(\forall d \in \mathrm{D}_e)(\forall u \in \mathrm{V}(L))\, d \Vdash_L (u \in a \leftrightarrow u \in b)$. Thus for all $d \in \mathrm{D}_e$ we have

$$(\forall \langle f, y\rangle \in a)\ (\jmath_1 d) \bullet \theta(f) \Vdash_L y \in b$$
$$(\forall \langle f, y\rangle \in b)\ (\jmath_2 d) \bullet \theta(f) \Vdash_L y \in a$$

with θ defined as in Corollary 4.8. Letting $\psi(d) := \jmath(\lambda f.(\jmath_1 d) \bullet \theta(f), \lambda f.(\jmath_2 d) \bullet \theta(f))$ we therefore have

$$(\forall \langle f, y\rangle \in a)\ (\jmath_1(\psi(d))) \bullet f \Vdash_L y \in b$$
$$(\forall \langle f, y\rangle \in b)\ (\jmath_2(\psi(d))) \bullet f \Vdash_L y \in a.$$

Thus, by Lemma 4.3, $\Phi(e, \lambda x.\psi(x))\downarrow$, $\mathrm{D}_{\Phi(e,\lambda x.\psi(x))} \neq \emptyset$ and every $h \in \mathrm{D}_{\Phi(e,\lambda x.\psi(x))}$ is of the form $(\lambda x.\psi(x)) \bullet d = \psi(d)$ for some $d \in \mathrm{D}_e$. Thus $\Phi(e, \lambda x.\psi(x)) \Vdash_L a = b$. Furthermore,

$$\lambda f.\lambda e.\Phi(e, \lambda x.\psi(x)) \Vdash_L \mathrm{S}(a) \wedge \mathrm{S}(b) \to (\forall x(x \in a \leftrightarrow x \in b) \to a = b)$$

and hence

$$\mathfrak{sg}(\mathfrak{sg}(\lambda f.\lambda e.\Phi(e, \lambda x.\psi(x)))) \Vdash_L \forall u \forall y[\mathrm{S}(u) \wedge \mathrm{S}(y) \to (\forall x(x \in u \leftrightarrow x \in y) \to u = y)].$$

(**Pair**): Let $u, v \in \mathrm{V}(L)$. Put $a = \{\langle 0, u\rangle, \langle 0, v\rangle\}$. Then $a \in \mathrm{V}^{set}$ and $\theta(0) \Vdash_L u \in a$ and $\theta(0) \Vdash_L v \in a$, whence $\jmath(0, \jmath(\theta(0), \theta(0))) \Vdash_L \mathrm{S}(a) \wedge u \in a \wedge v \in a$, so $\mathfrak{sg}(\jmath(0, \jmath(\theta(0), \theta(0)))) \Vdash_L \exists y[\mathrm{S}(y) \wedge u \in y \wedge v \in y]$.

(**Union**): For each $u \in \mathrm{V}(L)$, put

$$\mathrm{Un}(u) \quad = \quad \{\langle \jmath(f,h), y\rangle \mid \exists x(\langle f, x\rangle \in u \wedge \langle h, y\rangle \in x)\}.$$

Then $\mathrm{Un}(u) \in \mathrm{V}^{set}$. Suppose

$$e \Vdash_L \exists x(x \in u \wedge z \in x).$$

Then

$$(\forall d \in \mathrm{D}_e)(\exists x \in \mathrm{V}(L))\,[\jmath_1 d \Vdash_L x \in u \,\wedge\, \jmath_2 d \Vdash_L z \in x].$$

Fix $d \in \mathrm{D}_e$ and $x \in \mathrm{V}(L)$ such that $\jmath_1 d \Vdash_L x \in u \wedge \jmath_2 d \Vdash_L z \in x$. Then $(\forall f \in \mathrm{D}_{\jmath_1 d})\exists w\,[\langle \jmath_1 f, w\rangle \in u \,\wedge\, \jmath_2 f \Vdash_L x = w]$. Letting $\mathfrak{q}(f,d) := \mathbf{i}_1 \bullet \jmath(\jmath_2 f, \jmath_2 d)$ with $\mathbf{i}_1$ from Lemma 4.7 we get

$$(\forall f \in \mathrm{D}_{\jmath_1 d})\exists w\,[\langle \jmath_1 f, w\rangle \in u \,\wedge\, \mathfrak{q}(f,d) \Vdash_L z \in w]$$

and hence

$$(\forall f \in \mathrm{D}_{\jmath_1 d})\exists w\,[\langle \jmath_1 f, w\rangle \in u \,\wedge\, \exists v(\langle \jmath_1(\mathfrak{q}(f,d)), v\rangle \in w \wedge \Vdash_L \jmath_2(\mathfrak{q}(f,d))z = v].$$

Since $\langle \jmath(\jmath_1 f, \jmath_2(\mathfrak{q}(f,d))), v\rangle \in \mathrm{Un}(u)$, we arrive at

$$(\forall f \in V_{\jmath_1 d})\, \mathfrak{l}(f,d) \Vdash_L z \in \mathrm{Un}(u),$$

where $\mathfrak{l}(f,d) := \mathfrak{sg}(\jmath(\jmath(\jmath_1 f, \jmath_1(\mathfrak{q}(f,d))), \jmath_2(\mathfrak{q}(f,d))))$. As a result,

$$(\forall h \in \mathrm{D}_{\Phi(\jmath_1 d, \lambda f.\mathfrak{l}(f,d))} \Vdash_L z \in \mathrm{Un}(u),$$

hence

$$\chi_A(\Phi(\jmath_1 d, \lambda f.\mathfrak{l}(f,d))) \Vdash_L z \in \mathrm{Un}(u)$$

where A is the formula $x_0 \in x_1$. Since the latter holds for all $d \in \mathrm{D}_e$ we get

$$(\forall g \in \mathrm{D}_{\Phi(e,\lambda d.\chi_A(\Phi(\jmath_1 d,\lambda f.\mathfrak{l}(f,d))))}) \Vdash_L z \in \mathrm{Un}(u)$$

so

$$\chi_A(\Phi(e, \lambda d.\chi_A(\Phi(\jmath_1 d, \lambda f.\mathfrak{l}(f,d))))) \Vdash_L z \in \mathrm{Un}(u).$$

The upshot is that $\mathfrak{sg}(\jmath(0, \mathfrak{sg}(\lambda e.\chi_A(\Phi(e, \lambda d.\chi_A(\Phi(\jmath_1 d, \lambda f.\mathfrak{l}(f,d))))))))$ realizes $\exists w[\mathrm{S}(w) \wedge \forall z(\exists x(x \in u \wedge z \in x) \to z \in w)]$ from which one gets a realizer for the union axiom via realizers for the separation axioms.

(**Infinity**): Let $M := \{\langle n, n\rangle \mid n \in \mathbb{N}\}$. Then $M \in \mathrm{V}^{set}$ and $\mathrm{S}(M)$. Suppose $e \Vdash_L z \in M$. Then $\mathrm{D}_e \neq \emptyset$ and

$$(\forall d \in \mathrm{D}_e)\exists n[\langle \jmath_1 d, n\rangle \in M \,\wedge\, \jmath_2 d \Vdash_L z = n].$$

Note that $\langle \jmath_1 d, n\rangle \in M$ and $\jmath_2 d \Vdash_L z = n$ with $n \in \mathbb{N}$ entail that $\jmath_1 d = z = n$. We then also (trivially) have $\jmath_1 d \Vdash_L N(z)$. Invoking Lemma 4.3 we have $\mathrm{D}_{\Phi(e,\lambda d.\jmath_1 d)} = \{n\}$. Thus, by Lemma 4.2, $\phi(\Phi(e, \lambda d.\jmath_1 d))\downarrow$ and $\phi(\Phi(e, \lambda d.\jmath_1 d)) \Vdash_L \mathrm{N}(z)$.

Conversely, if $e \Vdash_L \mathrm{N}(z)$, then $e = z \wedge \langle z, z\rangle \in M$, so $\theta(e) \Vdash_L z \in M$.

The upshot is that $\mathfrak{sg}(\jmath(\lambda x.\phi(\Phi(x, \lambda d.\jmath_1 d)), \lambda x.\theta(x))) \Vdash_L \forall u(u \in M \leftrightarrow \mathrm{N}(u))$. Hence

$$\mathfrak{sg}(\jmath(0, \mathfrak{sg}(\jmath(\lambda x.\phi(\Phi(x, \lambda d.\jmath_1 d)), \lambda x.\theta(x))))) \Vdash_L \exists z[\mathrm{S}(z) \,\wedge\, \forall u(u \in z \leftrightarrow \mathrm{N}(u))].$$

(**Powerset**): Let $a \in \mathrm{V}^{set}_\alpha$. It suffices to find a realizer for the formula

$$\exists y[\mathrm{S}(y) \,\wedge\, \forall x(\mathrm{S}(x) \,\wedge\, x \subseteq a \to x \in y)]$$

since realizability of the power set axiom follows then with the help of Separation. Define

$$\mathfrak{V}_\alpha := \{\langle q, b\rangle \mid b \in \mathrm{V}^{set}_\alpha \wedge q \in \mathbb{N} \wedge q \Vdash_L \forall x(x \in b \to x \in a)\}.$$

Then $\mathfrak{V}_\alpha \in \mathrm{V}^{set}$. Suppose $b \in \mathrm{V}^{set}$ and $e \Vdash_L b \subseteq a$. Then $\mathfrak{sub}(e) \Vdash_L b = b^*$ for some $b^* \in \mathrm{V}^{set}_\alpha$ by Lemma 5.1. Thus, as $\langle \mathfrak{sub}(e), b^*\rangle \in \mathfrak{V}_\alpha$, we have

$$\mathfrak{sg}(\jmath(\mathfrak{sub}(e), \mathfrak{sub}(e))) \Vdash_L b \in \mathfrak{V}_\alpha.$$

Thus $\mathfrak{sg}(\lambda f.\mathfrak{sg}(\jmath(\mathfrak{sub}(\jmath_2 f), \mathfrak{sub}(\jmath_2 f)))) \Vdash_L \forall x(\mathrm{S}(x) \wedge x \subseteq a \to x \in y)$ and consequently

$$\mathfrak{sg}(\jmath(0, \mathfrak{sg}(\lambda f.\mathfrak{sg}(\jmath(\mathfrak{sub}(\jmath_2 f), \mathfrak{sub}(\jmath_2 f)))))) \Vdash_L \exists y[\mathrm{S}(y) \wedge \forall x(\mathrm{S}(x) \wedge x \subseteq a \to x \in y)].$$

(**Set Induction**): Suppose $\bar{e} \Vdash_L \forall x[\forall y(y \in x \to A(y)) \to A(x)]$. Then $\mathrm{D}_{\bar{e}} \neq \emptyset$ and

$$(\forall d \in \mathrm{D}_{\bar{e}})(\forall x \in \mathrm{V}(L))\, d \Vdash_L \forall y(y \in x \to A(y)) \to A(x). \tag{16}$$

Let $a \in \mathrm{V}(L)$. Suppose we have an index e^* such that for all $d \in \mathrm{D}_{\bar{e}}$, $e^* \bullet d\downarrow$ and for all $\langle f', b\rangle \in a$, $e^* \bullet d \Vdash_L A(b)$. Assume $f \Vdash_L y \in b$. Then there exists b such that $\langle \jmath_1 f, b\rangle \in a$ and $\jmath_2 f \Vdash_L y = b$. Hence, by Lemma 4.5, $\mathbf{i}_{A'}(\jmath(e^* \bullet d, \jmath_2 f)) \Vdash_L A(y)$ for an appropriate formula A'. As a consequence we have

$$\lambda f.\mathbf{i}_{A'}(\jmath(e^* \bullet d, \jmath_2 f)) \Vdash_L y \in a \to A(y),$$

so

$$\mathfrak{l}^*(e^*, d) := \mathfrak{sg}(\lambda f.\mathbf{i}_{A'}(\jmath(e^* \bullet d, \jmath_2 f))) \Vdash_L \forall y[y \in a \to A(y)],$$

so

$$(\forall d \in V_{\bar{e}})\, d \bullet \mathfrak{l}^*(e^*, d) \Vdash_L A(a). \tag{17}$$

By the recursion theorem there exists an index e^* such that

$$e^* \bullet n \simeq \mathfrak{l}^*(e^*, n)$$

for all $n \in \mathbb{N}$. In view of the foregoing, it follows by set induction on $a \in \mathrm{V}(L)$ that for all $d \in \mathrm{D}_e$, $e^* \bullet d\downarrow$ and $e^* \bullet d \Vdash_L A(a)$. Hence, by Lemma 4.3, $V_{\Phi(e,\lambda d.e^* \bullet d)} \neq \emptyset$ and for all $h \in V_{\Phi(e,\lambda d.e^* \bullet d)}$ and all $a \in \mathrm{V}(L)$ we have $h \Vdash_L A(a)$. Thus $\Phi(e, \lambda d.e^* \bullet d) \Vdash_L \forall x A(x)$. Hence

$$\lambda e.\Phi(e, \lambda d.e^* \bullet d) \Vdash_L \forall x[\forall y(y \in x \to A(y)) \to A(x)] \to \forall x A(x).$$

(**Separation**): Given $a \in \mathrm{V}(L)$ we seek a realizer $\mathfrak{e}$ such that

$$\mathfrak{e} \Vdash_L \exists z[\mathrm{S}(z) \wedge \forall u(u \in z \to u \in a \wedge A(a)) \wedge \forall u(u \in a \wedge A(u) \to u \in z)]. \tag{18}$$

$\mathfrak{e}$ will not depend on a nor on other parameters occurring in A. Let

$$b = \{\langle \jmath(f, g), x\rangle \mid \langle f, x\rangle \in a \wedge g \Vdash_L A(x)\}. \tag{19}$$

Then b is a set by separation in the background universe, and also $b \in \mathrm{V}^{set}$.

Assume $e \Vdash_L u \in b$. Then $\mathrm{D}_e \neq \emptyset$ and for every $d \in \mathrm{D}_e$ there exists x such that $\langle \jmath_1 d, x\rangle \in b \wedge \jmath_2 d \Vdash_L u = x$. By definition of b, $\jmath_1 d = \jmath(f, g)$ for some $f, g \in \mathbb{N}$ such that $\langle f, x\rangle \in a$ and $g \Vdash_L A(x)$. From $\jmath_2 d \Vdash_L u = x$ and $\theta(f) \Vdash_L x \in a$ we deduce $\mathfrak{q}(d, f) := \mathbf{i}_0(\jmath(\jmath_2 d, \theta(f))) \Vdash_L u \in a$ with the help of Corollary 12 and Lemma 4.7(4). As $g \Vdash_L A(x)$ we get $\mathfrak{p}(d, g) := \mathbf{i}_{A'}(\jmath(\jmath_2 d, g)) \Vdash_L A(u)$ from Lemma 4.7, where A' is obtained from A by replacing parameters from $\mathrm{V}(L)$ with free variables. Thus, from the above we conclude that

$$\jmath(\mathfrak{q}(d, f), \mathfrak{p}(d, g)) \Vdash_L u \in a \wedge A(u). \tag{20}$$

We can write $\mathfrak{l}(d) := \jmath(\mathfrak{q}(d, f), \mathfrak{p}(d, g))$ solely as a partial recursive function of d since $f = \jmath_1(\jmath_1 d))$ and $g = \jmath_2(\jmath_1 d))$. Thus (20) yields $(\forall d \in \mathrm{D}_e)\, \mathfrak{l}(d) \Vdash_L u \in a \wedge A(u)$, whence $(\forall h \in \mathrm{D}_{\Phi(e,\lambda d.\mathfrak{l}(d))}) \Vdash_L u \in a \wedge A(u)$ by Lemma 4.3, so

$$\chi_B(\Phi(e, \lambda d.\mathfrak{l}(d))) \Vdash_L u \in a \wedge A(u) \tag{21}$$

by Lemma 4.5 for an appropriate formula B. (21) yields

$$e^* := \mathfrak{sg}(\lambda e.\chi_B(\Phi(e, \lambda d.\mathfrak{l}(d)))) \Vdash_L \forall u(u \in b \to u \in a \wedge A(u)). \tag{22}$$

Conversely, assume $e \Vdash_L u \in a \wedge A(u)$. Then $\jmath_1 e \Vdash_L u \in a$ and $\jmath_2 e \Vdash_L A(u)$. Thus, for all $d \in \mathrm{D}_{\jmath_1 e}$ there exists x such that $\langle \jmath_1 d, x\rangle \in$ and $\jmath_2 d \Vdash_L u = x$. Then, by Lemma 4.7, $\mathfrak{l}_1(d, e) := \mathbf{i}_{A_0}(\jmath(\jmath_2 d, \jmath_2 e)) \Vdash_L A(x)$ for a suitable formula A_0. So $\langle \jmath(\jmath_1 d, lf_1(d, e)), x\rangle \in b$, which together with $\jmath_2(d) \Vdash_L u = x$ yields

$$\mathfrak{l}_2(d, e) := \jmath(\jmath(\jmath_1 d, lf_1(d, e)), \jmath_2 d) \Vdash_L u \in b.$$

Consequently, by Lemma 4.3,

$$(\forall h \in \mathrm{D}_{\Phi(\jmath_1 e, \lambda d.\mathfrak{l}_2(d,e))})\, h \Vdash_L u \in b,$$

thus $\chi_C(\Phi(\jmath_1 e, \lambda d.\mathfrak{l}_2(d, e))) \Vdash_L u \in b$ by Lemma 4.5, where $C \equiv x_1 \in x_2$. Hence

$$e^{**} := \mathfrak{sg}(\lambda e.\chi_C(\Phi(\jmath_1 e, \lambda d.\mathfrak{l}_2(d, e)))) \Vdash_L \forall u[u \in a \wedge A(u) \to u \in b]. \tag{23}$$

Finally, by (22) and (23), we arrive at (18) with $\mathfrak{e} := \mathfrak{sg}(\jmath(0, \jmath(e^*, e^{**})))$.

(**Collection**): Suppose

$$e \Vdash_L \forall u(u \in a \to \exists y\, B(u, y)). \tag{24}$$

Then $\mathrm{D}_e \neq \emptyset$ and

$$(\forall d \in \mathrm{D}_e)(\forall u \in \mathrm{V}(L))\, d \Vdash_L (u \in a \to \exists y\, B(u, y)). \tag{25}$$

Fix $d \in \mathrm{D}_e$. If $\langle f, x\rangle \in a$ then $\theta(f) \Vdash_L x \in a$, so $d \bullet \theta(f) \Vdash_L \exists y B(x, y)$. Consequently, $(\forall h \in \mathrm{D}_{d\bullet\theta(f)})(\exists y{\in}\mathrm{V}(L))\, h \Vdash_L B(x, y)$. Therefore, using Collection in the background universe, there exists a set $C \subseteq \mathrm{V}(L)$ such that

$$(\forall d \in \mathrm{D}_e)(\forall \langle f, x\rangle \in a)(\forall h \in \mathrm{D}_{d\bullet\theta(f)})(\exists y \in C)\, h \Vdash_L B(x, y). \tag{26}$$

Let

$$C^* \quad = \quad \{\langle \jmath(\jmath(d,f),h),y\rangle \mid d \in \mathrm{D}_e \,\wedge\, y \in C \,\wedge\, \exists x(\langle f,x\rangle \in a \,\wedge\, h \Vdash_L B(x,y))\}. \quad (27)$$

C^* is a set by Separation. Also $C^* \in \mathrm{V}^{set}$. Now assume that $d \in \mathrm{D}_e$ and $e' \Vdash_L u \in a$. Then, for all $d' \in \mathrm{D}_{e'}$ there exists x such that $\langle \jmath_1 d', x\rangle \in a$ and $\jmath_2 d' \Vdash_L u = x$. Moreover, by (25), for all $h \in \mathrm{D}_{d \bullet \theta(\jmath_1 d')}$ there exists $y \in C$ such that $h \Vdash_L B(x,y)$. Whence $\langle \mathfrak{l}_3(d,d',h), y\rangle \in C^*$, where $\mathfrak{l}_3(d,d',h) := \jmath(\jmath(d,\jmath_1 d'),h)$. From $\jmath_2 d' \Vdash_L u = x$ and $h \Vdash_L B(x,y)$ we also obtain $\mathbf{i}_{B'}(\jmath(\jmath_2 d',h)) \Vdash_L B(u,y)$ by Lemma 4.7 for an appropriate formula B'. Since $\theta(\mathfrak{l}_3(d,d',h)) \Vdash_L y \in C^*$, we have

$$\mathfrak{l}_4(d,d',h) := \jmath(\theta(\mathfrak{l}_3(d,d',h)), \mathbf{i}_{B'}(\jmath(\jmath_2 d',h))) \Vdash_L y \in C^* \,\wedge\, B(u,y), \qquad (28)$$

so $\mathfrak{sg}(\mathfrak{l}_4(d,d',h)) \Vdash_L \exists y(y \in C^* \,\wedge\, B(u,y))$, hence, using Lemmata 4.3, 4.4 and 4.5 repeatedly with appropriate formulas D and E,

$$\begin{aligned} \mathfrak{l}_5(d,d') &:= \chi_D(\Phi(d \bullet \theta(\jmath_1 d'), \lambda h.\mathfrak{sg}(\mathfrak{l}_4(d,d',h)))) \Vdash_L \exists y(y \in C^* \,\wedge\, B(u,y)), \\ \mathfrak{l}_6(d,e') &:= \chi_E(\Phi(e', \lambda d'.\mathfrak{l}_5(d,d'))) \Vdash_L \exists y(y \in C^* \,\wedge\, B(u,y)). \end{aligned} \qquad (29)$$

As we established (29) under the assumption $e' \Vdash_L u \in a$, we get

$$\lambda e'.\mathfrak{l}_6(d,e') \Vdash_L u \in a \rightarrow \exists y(y \in C^* \,\wedge\, B(u,y)).$$

Thus, by Lemmata 4.3 and 4.5, we have

$$\mathfrak{l}_7(e) := \chi_F(\Phi(e, \lambda d.\lambda e'.\mathfrak{l}_6(d,e'))) \Vdash_L u \in a \rightarrow \exists y(y \in C^* \,\wedge\, B(u,y)) \qquad (30)$$

for an appropriate formula F. Finally, by repeatedly applying Lemma 4.1, we see that

$$\begin{aligned} \mathfrak{sg}(\mathfrak{l}_7(e)) \quad &\Vdash_L \quad \forall u[u \in a \rightarrow \exists y(y \in C^* \,\wedge\, B(u,y))] \\ \mathfrak{sg}(\jmath(0, \mathfrak{sg}(\mathfrak{l}_7(e)))) \quad &\Vdash_L \quad \exists z(\mathrm{S}(z) \,\wedge\, \forall u[u \in a \rightarrow \exists y(y \in z \,\wedge\, B(u,y))]) \\ \lambda e.\mathfrak{sg}(\jmath(0, \mathfrak{sg}(\mathfrak{l}_7(e)))) \quad &\Vdash_L \quad \forall u[u \in a \rightarrow \exists y\, B(u,y)] \rightarrow \\ &\qquad\qquad \exists z(\mathrm{S}(z) \,\wedge\, \forall u[u \in a \rightarrow \exists y(y \in C^* \,\wedge\, B(u,y))]). \end{aligned}$$

□

6 Church's thesis in V(L)

Lemma: 6.1 (**IZF′**) $\mathrm{V}(L) \models \mathbf{CT}_0!$.

Proof: Note that according to Lemma 4.6 our realizability for arithmetic formulae is the same as in [11]. As a result, the same proof as in [11, Lemma 3] will do. □

Lemma: 6.2 $\mathrm{V}(L) \not\models \mathbf{CT}_0$. *More precisely, let $\bar{e}, \tilde{e} \in \mathbb{N}$ be indices of two disjoint recursively inseparable r.e. sets, i.e. $X = \{m \mid \exists m\, \mathrm{T}(\bar{e}, n, m)\}$ and $Y = \{m \mid \exists m\, \mathrm{T}(\tilde{e}, n, m)\}$ are disjoint and recursively inseparable. Let $A(n) := \forall m\, \neg \mathrm{T}(\bar{e}, n, m)$, $B(n) := \forall m\, \neg \mathrm{T}(\tilde{e}, n, m)$ and $C(n,k) := (A(n) \wedge k = 0) \vee (B(n) \wedge k = 1)$. Then*

$$\mathrm{V}(L) \not\models \forall n \exists k\, C(n,k) \to \exists d \forall n\, C(n, d \bullet n).$$

Proof: The proof is the same as in [11, section 4]. First one shows that $\mathrm{V}(L) \models \forall n \exists k\, C(n,k)$. Next one shows that from $e^* \Vdash_L \exists d \forall n\, C(n, d \bullet n)$ one would be able to engineer a recursive separation of X and Y above, which is impossible. □

The foregoing Lemmata also show that a "binary" version of number choice is not provable in **IZF**. Let $\mathbf{AC}_{\omega,2}$ be the statement that whenever $(A_i)_{i \in \mathbb{N}}$ is family of inhabited sets A_i with $A_i \subseteq \{0,1\}$, then there exists a function $F : \mathbb{N} \to \bigcup_{i \in \mathbb{N}} A_i$ such that $\forall i\, F(i) \in A_i$.

Corollary: 6.3 $\mathrm{V}(L) \not\models \mathbf{AC}_{\omega,2}$. *In particular,* **IZF** *does not prove* $\mathbf{AC}_{\omega,2}$.

Proof: We argue in $\mathrm{V}(L)$. We have $\forall n \exists k\, C(n,k)$ with C as in the proof of Lemma 6.2. Then with $A_n := \{k \in \{0,1\} \mid C(n,k)\}$, $A_n \subseteq \{0,1\}$ and A_n is inhabited. Thus if $\mathbf{AC}_{\omega,2}$ were to hold in $\mathrm{V}(L)$ we would get a function $F : \mathbb{N} \to \bigcup_{n \in \mathbb{N}} A_n$ such that $\forall n\, F(n) \in A_n$. Since $\forall n \exists! k F(n) = k$, $\mathbf{CT}_0!$ implies the existence of an index d such that $\forall n\, F(n) = d \bullet n$, and hence $\exists d \forall n\, C(n, d \bullet n)$. This contradicts Lemma 6.2. □

7 Some classical and non-classical principles that hold in V(L)

The next definitions lists several interesting principles that are validated in $\mathrm{V}(L)$.

Definition: 7.1 1. UP, the *Uniformity Principle*, is expressed by the schema:

$$\forall x\, [\mathrm{S}(x) \to \exists n\, A(x,n)] \;\to\; \exists n\, \forall x [\mathrm{S}(x) \to A(n,x)].$$

2. *Unzerlegbarkeit*, UZ, is the schema

$$\forall x [\mathrm{S}(x) \to (A(x) \vee B(x))] \;\to\; \forall x (\mathrm{S}(x) \to A(x)) \;\vee\; \forall x (\mathrm{S}(x) \to B(x))$$

for all formulas A, B.

Lemma: 7.2 UP *and* UZ *are Lifschitz realizable.*

Proof: Suppose $e \Vdash_L \forall x\,[\mathrm{S}(x) \to \exists n\, A(x,n)]$. Then $\mathrm{D}_e \neq \emptyset$. Since $0 \Vdash_L \mathrm{S}(a)$ holds for all $a \in \mathrm{V}^{set}$, we have

$$(\forall d \in \mathrm{D}_e)(\forall a \in \mathrm{V}^{set})\jmath_1 d \bullet 0 \Vdash_L \exists y[\mathrm{N}(y) \wedge A(a,y)].$$

Let $d \in \mathrm{D}_e$ and $a \in \mathrm{V}^{set}$. If $f \in \mathrm{D}_{\jmath_1 d}$ then there exists $y {\in} \mathrm{V}(L)$ such that $f \Vdash_L \mathrm{N}(y) \wedge A(a,y)$, thus $\jmath_1 f = y$ and so $\jmath_2 f \Vdash_L A(a, \jmath_1 f)$. Hence

$$(\forall f \in \mathrm{D}_{\jmath_1 d})(\forall a \in \mathrm{V}^{set})\jmath_2 f \Vdash_L A(a, \jmath_1 f),$$

and so

$$\begin{array}{rcl} (\forall f \in \mathrm{D}_{\jmath_1 d})\lambda x.\jmath_2 f & \Vdash_L & \forall x[\mathrm{S}(x) \to A(x, \jmath_1 f)] \\ (\forall f \in \mathrm{D}_{\jmath_1 d})\jmath(\jmath_1 f, \lambda x.\jmath_2 f) & \Vdash_L & \mathrm{N}(\jmath_1 f) \wedge \forall x(\mathrm{S}(x) \to A(x, \jmath_1 f)), \\ \mathfrak{l}(d) := \Phi(\jmath_1 d, \lambda f.\jmath(\jmath_1 f, \lambda x.\jmath_2 f)) & \Vdash_L & \exists y[\mathrm{N}(y) \wedge \forall x(\mathrm{S}(x) \to A(x,y))], \end{array}$$

where we used Lemma 4.3 in the last step. Finally, by applying Lemmata 4.3 and 4.5 we arrive at

$$\chi_{A'}(\Phi(e, \lambda d.\mathfrak{l}(d))) \Vdash_L \exists y[\mathrm{N}(y) \wedge \forall x(\mathrm{S}(x) \to A(x,y))]$$

for an appropriate formula A'. Hence, with $e^* := \lambda e.\chi_{A'}(\Phi(e, \lambda d.\mathfrak{l}(d)))$,

$$e^* \Vdash_L \forall x\,[\mathrm{S}(x) \to \exists n\, A(x,n)] \to \exists y[\mathrm{N}(y) \wedge \forall x(\mathrm{S}(x) \to A(x,y))].$$

As to Lifschitz realizability of UZ, note that $\forall x[\mathrm{S}(x) \to (A(x) \vee B(x))]$ implies $\forall x[\mathrm{S}(x) \to \exists n[(n = 0 \wedge A(x)) \vee (n \neq 0 \wedge B(x))]$. The latter yields

$$\exists n\, \forall x[\mathrm{S}(x) \to [(n = 0 \wedge A(x)) \vee (n \neq 0 \wedge B(x))]$$

via UP, and hence $\forall x(\mathrm{S}(x) \to A(x)) \;\vee\; \forall x(\mathrm{S}(x) \to B(x))$. Thus UZ is a consequence of UP. Therefore $\mathrm{V}(L) \models \mathrm{UZ}$. □

The principle $\mathrm{MP_{pr}}$ holds in $\mathrm{V}(L)$. Another classically valid principle considered in connection with intuitionistic theories is the *Principle of Independence of Premisses*, IP, which is expressed by the schema

$$(\neg A \to \exists x\, B(x)) \;\to\; \exists x(\neg A \to B(x)),$$

where A is assumed to be closed.

Theorem: 7.3 *1.* (**IZF′**) $\mathrm{V}(L) \models \mathrm{MP_{pr}}$.

2. Assuming classical logic in V, $\mathrm{V}(L) \models \mathrm{IP}$.

Proof: (1). Assume $e \Vdash_L \neg\neg\exists n\, A(n)$ where $A(n)$ is of the form $R(n, \vec{k})$ with R primitive recursive and $\vec{k} \in \mathbb{N}$. Then $\neg\neg\exists f\, f \Vdash_L \exists n\, A(n)$, and thus by Lemma 4.6, $\neg\neg\exists f \Vdash_L A(\jmath_1 f)$, thus $\neg\neg\exists f\, A(\jmath_1 f)$. Using $\mathrm{MP_{pr}}$ in the background universe we have $\exists n\, A(n)$. Then, with

$r := \mu n.A(n)$, we have $\mathfrak{sg}(\jmath(r,0)) \Vdash_L \exists n\, A(n)$. Whence $\lambda e.\mathfrak{sg}(\jmath(r,0))$ realizes this instance of $\mathrm{MP_{pr}}$.

(2). Assume that $e \Vdash_L \neg A \to \exists x\, B(x)$. Then, if $g \Vdash_L \neg A$, $0 \Vdash_L \neg A$ and $e \bullet 0 \Vdash_L \exists x\, B(x)$. Therefore, $\mathrm{D}_{e\bullet 0} \neq \emptyset$ and for all $d \in \mathrm{D}_{e\bullet 0}$ there is an $a \in \mathrm{V}(L)$ such that $d \Vdash_L B(a)$, and therefore $\lambda u.d \Vdash_L \neg A \to B(a)$. Hence, if A is not realized,

$$\Phi(e \bullet 0, \lambda d.\lambda u.d) \Vdash_L \exists x(\neg A \to B(x)).$$

On the other hand, should A be realized, then $\neg A$ is never realized, so $\lambda u.u$ would realize this instance of IP. □

8 The reals in $\mathrm{V}(L)$

By Lemma 6.1 the Cauchy reals in $\mathrm{V}(L)$ are the recursive reals. A well-known consequence of $\mathbf{AC}_{\omega,2}$ is that the sets of Cauchy reals and Dedekind reals are isomorphic. As it turns out, in $\mathrm{V}(L)$ the notions of Cauchy real and Dedekind real coincide in $\mathrm{V}(L)$ despite the failure of $\mathbf{AC}_{\omega,2}$.

Lemma: 8.1 *In* $\mathrm{V}(L)$ *the set of Cauchy reals is order-isomorphic to the set of Dedekind reals.*

Proof: This was proved by van Oosten to hold in the Lifschitz topos [16, IV. Proposition 2.5]. The proof utilizes [21, Ch.5 Proposition 5.10] saying that the collection of strong Dedekind reals is order-isomorphic to the collection of Cauchy reals. It is shown in [16, IV. Proposition 2.5] that in the Lifschitz topos every Dedekind real is a strong Dedekind real. As the proof carries over to $\mathrm{V}(L)$, we do not repeat it here. □

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