Combinatorics of countable ordinal topologies

by

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The candidate confirms that the work submitted is his own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

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Abstract

We study combinatorial properties of ordinals under the order topology, focusing on the subspaces, partition properties and autohomeomorphism groups of countable ordinals.

Our main results concern topological partition relations. Let n be a positive integer, let κ be a cardinal, and write $[X]^n$ for the set of subsets of X of size n. Given an ordinal β and ordinals α_i for all $i \in \kappa$, write $\beta \to_{top} (\alpha_i)^n_{i\in\kappa}$ to mean that for every function $c : [\beta]^n \to \kappa$ (a *colouring*) there is some subspace $X \subseteq \beta$ and some $i \in \kappa$ such that X is homeomorphic to α_i and $[X]^n \subseteq c^{-1}(\{i\})$. We examine the cases $n = 1$ and $n = 2$, defining the *topological pigeonhole number* $P^{top}(\alpha_i)_{i\in\kappa}$ to be the least ordinal β (when one exists) such that $\beta \to_{top} (\alpha_i)^1_{i\in\kappa}$, and the *topological Ramsey number* $R^{top}(\alpha_i)_{i\in\kappa}$ to be the least ordinal β (when one exists) such that $\beta \to_{top} (\alpha_i)^2_{i\in\kappa}$.

We resolve the case $n = 1$ by determining the topological pigeonhole number of an arbitrary sequence of ordinals, including an independence result for one class of cases. In the case $n = 2$, we prove a topological version of the Erdős–Milner theorem, namely that $R^{top}(\alpha, k)$ is countable whenever α is countable and k is finite. More precisely, we prove that $R^{top}(\omega^{\omega^\beta}, k+1) \leq \omega^{\omega^{\beta\cdot k}}$ for all countable ordinals β and all positive integers k. We also provide more careful upper bounds for certain small ordinals, including $R^{top}(\omega + 1, k + 1) = \omega^k + 1$, $R^{top}(\alpha, k) < \omega^\omega$ whenever $\alpha < \omega^2$, $R^{top}(\omega^2, k) \leq \omega^\omega$ and $R^{top}(\omega^2 + 1, k + 2) \leq \omega^{\omega\cdot k} + 1$ for all positive integers k.

Outside the partition calculus, we prove a topological analogue of Hausdorff's theorem on scattered total orderings. This allows us to characterise countable subspaces of ordinals as the order topologies of countable scattered total orderings. As an application, we compute the number of subspaces of an ordinal up to homeomorphism.

Finally, we study the group of autohomeomorphisms of $\omega^n \cdot m + 1$ for finite n and m. We classify the normal subgroups contained in the pointwise stabiliser of the limit points. These subgroups fall naturally into $D(n)$ disjoint sets, each either countable or of size $2^{2^{\aleph_0}}$, where $D(n)$ is the number of $\subseteq$-antichains of $\mathcal{P}(\{1, 2, \ldots, n\})$.

Our techniques span a variety of disciplines, including set theory, general topology and permutation group theory.

Contents

Chapter 1

Introduction

Ordinals were introduced by Georg Cantor in 1883, and have since become a foundational component of modern-day set theory. As with any totally ordered set, we may endow an ordinal with the order topology, generated by open intervals. The resulting ordinal topologies are Hausdorff, totally disconnected and scattered, and in the case of successor ordinals are also compact. They have been studied within general topology, within set theory, and implicitly via Boolean algebras.

Within general topology, ordinals are a familiar source of counterexamples: ω_1 is sequentially compact but not compact; the order topology on the "extended long ray" $\omega_1 \times [0,1) \cup \{\infty\}$ is connected but not path-connected; and the Tychonoff plank $(\omega+1)\times(\omega_1+1)$ is normal but has a non-normal subspace. More general treatment of ordinal topologies appears to have begun with Sierpiński and Mazurkiewicz [MS20], who used the Cantor–Bendixson derivative to show that every countable compact Hausdorff space is homeomorphic to a unique countable ordinal of the form $\omega^\alpha \cdot m + 1$. There have since been various further characterisations of certain ordinals, their subspaces and images, and related properties [Bak72, Mor81, Pur90, vD93, FN99, GP12, Lev13]. This work has typically been done under the general framework of linearly ordered topological spaces (another name for order topologies).

Within set theory, ordinal topologies appear implicitly behind the well-studied notion of a club (closed and unbounded) set. Besides this, ordinal topologies have been studied within the context of topological partition relations. These are defined in the same way as the usual Erdős–Rado partition relations for cardinals [ER56], except that the homogeneous set must have the correct topology rather than the correct cardinality. Preliminary work in this area was performed by Friedman [Fri74], who proved that every stationary subset of ω_1 has a subspace homeomorphic to any given countable ordinal. The study of countable ordinals in this context was begun in earnest by Baumgartner [Bau86], and there have been several recent developments [Sch12, Pn14].

The compact ordinal topologies are closely related to Boolean algebras via Stone duality, so that results in one area (e.g. [MP60]) may sometimes be re-interpreted in the other. In particular, the group of autohomeomorphisms of a compact ordinal is isomorphic to the group of automorphisms of its corresponding Boolean algebra. The automorphism groups of countable Boolean algebras were originally studied by Monk [Mon75], McKenzie [McK77] and Rubin [Rub80], who looked primarily at the problem of reconstructing a Boolean algebra from its automorphism group.

In this thesis we study various combinatorial properties of ordinal topologies, with a focus on countable ordinals. There are three major themes, corresponding loosely to the three perspectives of general topology, set theory and Boolean algebras: subspaces of ordinals, topological partition properties of ordinals, and autohomeomorphism groups of ordinals.

1.1 Subspaces of ordinals

The subspaces of countable ordinals may be characterised topologically in several ways, as follows [KR74, Corollary 3]. We say that a topological space X is *scattered* to mean that every non-empty subspace of X has an isolated point.

Theorem (Knaster–Urbanik–Belnov)**.** *Let X be a topological space. The following are equivalent.*

1. *X is homeomorphic to a subspace of a countable ordinal.*
2. *X is countable, scattered and metrisable.*
3. *X is homeomorphic to a countable complete metric space.*
4. *X is homeomorphic to a subspace of a countable compact Hausdorff space.*

Several basic properties of subspaces of ordinals follow from the general theory of order topologies. Within the field of general topology, a totally ordered set under the order topology is known as a *linearly ordered topological space* (LOTS), and a subspace of a LOTS is known as a *generalised ordered space* (GO-space).

A basic result due to Cech [CFK66, Theorem 17 A.23] states that GO-spaces may be equivalently defined as those whose topology contains the order topology and is generated by a base of order-convex sets. This result may be restricted to well-orderings to obtain the following characterisation of subspaces of ordinals (Theorem 3.1.2). Following Baumgartner [Bau86], we define an *order-homeomorphism* as a bijection that is both an order-isomorphism and a homeomorphism.

Theorem (Cech)**.** *Let X be a well-ordered set endowed with some topology. The following are equivalent.*

1. *X is order-homeomorphic to a subspace of an ordinal.*
2. *The topology on X has a base of the form*

$$\mathcal{B} \cup \{\{y\} : y \in Y\},$$

where $\mathcal{B}$ is the usual base for the order topology on X and $Y \subseteq X$.

Our main result in this area is the following additional characterisation of the subspaces of countable ordinals (Theorem 3.4.1).

Theorem. *A topological space X is homeomorphic to a subspace of a countable ordinal if and only if X is homeomorphic to a countable scattered totally ordered set under the order topology.*

It should be noted that the main substance of this result, that any subspace of a countable ordinal is homeomorphic to a LOTS, follows from a more general result of Purisch [Pur85], which states that any GO-space whose topology is scattered is homeomorphic to a LOTS.

We prove this using topological analogue of the following classical theorem of Hausdorff. We say that a totally ordered set is *scattered* to mean that it has no subset order-isomorphic to $\mathbb{Q}$. We will state this result more precisely and provide a proof later (see Theorem 3.2.4).

Theorem (Hausdorff)**.** *A totally ordered set is scattered if and only if it may be obtained from 1 by repeatedly taking well-ordered sums and backwards orderings.*

Our topological analogue (Theorem 3.3.1) uses a new operation on topological spaces, which we call the "one-point cofinite extension". This is a generalisation of the process used to pass from ω to $\omega + 1$, which we will define later.

Theorem. *A topological space is homeomorphic to a subspace of a countable ordinal if and only if it may be obtained from 1 by taking finite or countable topological disjoint unions and countable "one-point cofinite extensions".*

We prove our main result by combining this topological analogue with Hausdorff's theorem itself. We then use ideas from this result show that if $\alpha < \omega^\omega$ then α has at most countably many subspaces up to homeomorphism. Combined with some simple constructions for $\alpha \geq \omega^\omega$, this allows us to compute the number of subspaces of an ordinal up to homeomorphism (see Theorem 3.6.3).

Theorem. *Let α be an ordinal. Then the number of subspaces of α up to homeomorphism is*

$$\begin{cases} \alpha + 1, & \text{if } \alpha < \omega \\ \aleph_0, & \text{if } \omega \leq \alpha < \omega^\omega \\ 2^{\aleph_0}, & \text{if } \omega^\omega \leq \alpha < \omega_1 \\ 2^\kappa, & \text{if } |\alpha| = \kappa \text{ for some uncountable cardinal } \kappa. \end{cases}$$

1.2 Topological partition properties of ordinals

Motivated by Ramsey's theorem, the partition calculus for cardinals and Rado's arrow notation were introduced by Erdős and Rado in [ER53]. The version for ordinals, where the homogeneous set must have the correct order type, first appears in their seminal paper [ER56]. In the following definition, $[X]^n$ denotes the set of subsets of X of size n.

Definition. Let κ be a cardinal, let n be a positive integer, and let β and all α_i be ordinals for $i \in \kappa$. We write

$$\beta \to (\alpha_i)^n_{i\in\kappa}$$

to mean that for every function $c : [\beta]^n \to \kappa$ (a *colouring*) there exists some subset $X \subseteq \beta$ and some $i \in \kappa$ such that X is *an i-homogeneous copy of* α_i, i.e., $[X]^n \subseteq c^{-1}(\{i\})$ and X is order-isomorphic to α_i.

We write $\beta \to (\alpha)^n_\kappa$ for $\beta \to (\alpha_i)^n_{i\in\kappa}$ when $\alpha_i = \alpha$ for all $i \in \kappa$. For example, Ramsey's theorem may be written as $\omega \to (\omega)^n_k$ for all finite n and k.

With the development of structural Ramsey theory, many variants of this definition have been introduced. We will be concerned with a "topological" version and a closely-related "closed" version. (Jean Larson suggested to call the second version "limit closed".) Although the second version may appear more natural, it is the first that has been considered historically, since it can be defined for arbitrary topological spaces; the second version additionally requires an order structure.

Recall that an order-homeomorphism is a bijection that is both an order-isomorphism and a homeomorphism.

Definition. Let κ be a cardinal, let n be a positive integer, and let β and all α_i be ordinals for $i \in \kappa$.

We write

$$\beta \to_{top} (\alpha_i)^n_{i\in\kappa}$$

to mean that for every function $c : [\beta]^n \to \kappa$ there exists some subspace $X \subseteq \beta$ and some $i \in \kappa$ such that X is an *i-homogeneous topological copy of* α_i, i.e., $[X]^n \subseteq c^{-1}(\{i\})$ and X is homeomorphic to α_i.

We write

$$\beta \to_{cl} (\alpha_i)^n_{i\in\kappa}$$

to mean that for every function $c : [\beta]^n \to \kappa$ there exists some subset $X \subseteq \beta$ and some $i \in \kappa$ such that X is an *i-homogeneous closed copy of* α_i, i.e., $[X]^n \subseteq c^{-1}(\{i\})$ and X is order-homeomorphic to α_i (equivalently, X is both order-isomorphic to α_i and closed in its supremum).

Note that in both cases, c is arbitrary (no continuity or definability is required).

A survey of topological partition relations was carried out by Weiss [Wei90], though this is now slightly outdated. Note that previous authors have written the label *top* inside the parentheses, as in $\beta \to (top\, \alpha_i)^n_{i\in\kappa}$.

The closed partition relation does not appear to have been explicitly distinguished from the topological partition relation before. Extending a result of Baumgartner [Bau86, Theorem 0.2], we will see that the two versions coincide in many cases, in particular for ordinals of the form ω^γ or $\omega^\gamma \cdot m + 1$ with m a positive integer. But in general they may differ; for example, $\omega + 2$ is homeomorphic but not order-homeomorphic to $\omega + 1$, and thus $\omega + 1 \to_{top} (\omega + 2)^1_1$ while $\omega + 1 \not\to_{cl} (\omega + 2)^1_1$.

Our work on these partition relations is cleanly divided into the cases $n = 1$ and $n = 2$ of the above definition.

The topological pigeonhole principle for ordinals

The case $n = 1$ may be viewed as a generalisation of the pigeonhole principle. Thus we define the (*classical*) *pigeonhole number* $P(\alpha_i)_{i\in\kappa}$ to be the least ordinal β (when one exists) such that $\beta \to (\alpha_i)^1_{i\in\kappa}$, and we define the *topological pigeonhole number* $P^{top}(\alpha_i)_{i\in\kappa}$ and the *closed pigeonhole number* $P^{cl}(\alpha_i)_{i\in\kappa}$ in a similar fashion.

The classical pigeonhole numbers were computed by Milner and Rado [MR65]. The pigeonhole number of a finite sequence of ordinals may be written down explicitly using Cantor normal forms. The pigeonhole numbers of arbitrary sequences of ordinals may be described in terms of an algorithm that terminates in a finite number of steps. This algorithm features as a special case the well-known Milner–Rado paradox, which states that if κ is an infinite cardinal then $P(\kappa, \kappa^2, \kappa^3, \dots) = \kappa^+$.

We will compute both the topological and the closed pigeonhole numbers. This will comprise an explicit expression in the topological case (Theorem 4.6.1), and an algorithm that terminates in a finite number of steps in the closed case (Theorem 4.12.1).

We build upon previous work focusing on several key cases. These are covered in Weiss's survey [Wei90].

1. If $\alpha \in \omega_1 \setminus \{0, 1\}$, then $\alpha \to_{top} (\alpha)^1_2$ if and only if $\alpha = \omega^{\omega^\beta}$ for some $\beta \in \omega_1$ [Bau86, Corollary 2.4].

2. $\omega_1 \to_{top} (\alpha)^1_{\aleph_0}$ for all $\alpha \in \omega_1$. This is essentially Friedman's result on stationary sets [Fri74].

3. If $\alpha \in \omega_2$, then $\alpha \not\to_{top} (\omega_1)^1_2$. This follows easily from the fact that ω_1 may be written as a union of two disjoint stationary sets.

4. If $V = L$ then $\alpha \not\to_{top} (\omega_1)^1_2$ for all ordinals α [PS75], but it is equiconsistent with the existence of a Mahlo cardinal that $\omega_2 \to_{top} (\omega_1)^1_2$ [She98].

Note that for an ordinal $\alpha \in \omega_1 \setminus \{0\}$, $\alpha \to (\alpha)^1_2$ if and only if $\alpha = \omega^\beta$ for some $\beta \in \omega_1$. Thus in view of 1 it is natural to ask whether or not there is a link between the classical and topological pigeonhole principles. Our main breakthrough is a full analysis of the topological pigeonhole principle for finite sequences of countable ordinals, where we bring this link to light. Here # denotes the natural sum operation.

Theorem. *Let $\alpha_1, \alpha_2 \ldots \alpha_k \in \omega_1 \setminus \{0\}$.*

1. $P^{top}\left(\omega^{\alpha_1}+1, \omega^{\alpha_2}+1, \ldots, \omega^{\alpha_k}+1\right) = \omega^{\alpha_1 \# \alpha_2 \# \cdots \# \alpha_k} + 1$.
2. $P^{top}\left(\omega^{\alpha_1}, \omega^{\alpha_2}, \ldots, \omega^{\alpha_k}\right) = \omega^{P(\alpha_1, \alpha_2, \ldots, \alpha_k)}$.

We prove this (in Theorems 4.7.4 and 4.7.5) using a result of Weiss [Bau86, Theorem 2.3]. This was published by Baumgartner, who used it to show that $\omega^{\omega^\alpha \cdot (2m+1)} \to_{top} \left(\omega^{\omega^\alpha \cdot (m+1)}\right)^1_2$ for all $m \in \omega$ and all $\alpha \in \omega_1$ [Bau86, Corollary 2.5]. The above theorem greatly generalises this, thereby utilising the full potential of Weiss's result.

The only case in which we do not compute $P^{top}(\alpha_i)_{i\in\kappa}$ or $P^{cl}(\alpha_i)_{i\in\kappa}$ in ZFC is when $1 < \alpha_i \leq \omega_1$ for all $i \in \kappa$ and we have equality in at least two instances. In this case we have an independence result. Prikry and Solovay showed that if $V = L$ then $\alpha \nrightarrow_{top} (\omega_1)^1_2$ for all ordinals α, from which it follows that it is consistent for $P^{top}(\alpha_i)_{i\in\kappa}$ not to exist. On the other hand, we show that $P^{top}(\alpha_i)_{i\in\kappa} \geq \max\{\omega_2, \kappa^+\}$ and deduce from a result of Shelah that it is consistent to have equality in every case, assuming the consistency of the existence of a supercompact cardinal.

It remains open which intermediate values can consistently be taken by these topological pigeonhole numbers, and whether or not Shelah's consistency result can be strengthened to equiconsistency.

Topological ordinal Ramsey numbers

Our work on topological partition relations in the case $n = 2$ is joint with Andrés Caicedo. Here we define the (*classical*) *ordinal Ramsey number* $R(\alpha_i)_{i\in\kappa}$ to be the least ordinal β (when one exists) such that $\beta \to (\alpha_i)^2_{i\in\kappa}$, and we define the *topological ordinal Ramsey number* $R^{top}(\alpha_i)_{i\in\kappa}$ and the *closed ordinal Ramsey number* $R^{cl}(\alpha_i)_{i\in\kappa}$ in a similar fashion.

In the classical setting, a simple argument that goes back to Sierpiński shows that $\beta \nrightarrow (\omega+1, \omega)^2$ for every countable ordinal β [Sie33], which means that if $\alpha > \omega$ and $R(\alpha, \gamma)$ is countable, then γ must be finite (see also [ER56, Theorem 19] and [Spe56, Theorem 4]). On the other hand, Erdős and Milner showed that indeed $R(\alpha, k)$ is countable whenever α is countable and k is finite [EM72]. Much work has been done to compute these countable ordinal Ramsey numbers. In particular, as announced without proof by Haddad and Sabbagh [HS69a, HS69b, HS69c], there are algorithms for computing $R(\alpha, k)$ for several classes of ordinals $\alpha < \omega^\omega$ and all finite k; details are given in [Cai15] for the case $\alpha < \omega^2$ and in [Mil71] for the case $\alpha = \omega^m$ for finite m. See also [Cai14], [Wil77, Chapter 7], [HL10], [Sch10] and [Wei14].

In the topological setting, previous work has tackled uncountable Ramsey numbers. Erdős and Rado [ER56] introduced a "pressing down argument" (described in [Sch12], for example) to show that $\omega_1 \to_{top} (\omega+1)^2_2$. Laver noted in [Lav75] that their argument in fact gives $\omega_1 \to (\text{Stationary}, \text{top}\,\omega+1)^2$, meaning that one can ensure either a 0-homogeneous stationary subset or a 1-homogeneous topological copy of $\omega + 1$ (which is stronger, by Friedman's result on stationary sets). The Erdős–Rado result was later extended by Schipperus using elementary submodel techniques to show that $\omega_1 \to_{top} (\alpha)^2_k$ for all $\alpha \in \omega_1$ and all finite k [Sch12] (the topological Baumgartner–Hajnal theorem). Meanwhile, both $\omega_1 \to (\omega_1, \alpha)^2$ for all $\alpha \in \omega_1$ [Tod83] and $\omega_1 \not\to (\omega_1, \omega+2)^2$ [Haj60] are consistent with ZFC, though the topological version of the former remains unchecked. Finally, it is also known that $\beta \not\to_{top} (\omega+1)^2_{\aleph_0}$ for all ordinals β [Wei90, Theorem 3.4].

We study $R^{top}(\alpha, k)$ and $R^{cl}(\alpha, k)$ when α is a countable ordinal and k is a positive integer, which have not previously been explored. Our main result is a topological version of the Erdős–Milner theorem (Theorem 5.8.2).

Theorem. *Let α and β be countable non-zero ordinals, and let $k > 1$ be a positive integer. If*

$$\omega^{\omega^\alpha} \to_{top} (\omega^\beta, k)^2,$$

then

$$\omega^{\omega^{\alpha\cdot\beta}} \to_{top} (\omega^\beta, k+1)^2.$$

Since trivially $\omega^{\omega^\alpha} \to_{top} (\omega^{\omega^\alpha}, 2)^2$, it follows by induction on k that $R^{top}(\omega^{\omega^\alpha}, k+1) \le \omega^{\omega^{\alpha\cdot k}}$. Hence $R^{top}(\alpha, k)$ (and $R^{cl}(\alpha, k)$, since the two versions coincide when α is a power of ω) are countable for all countable α and all finite k.

We also provide the following more careful bounds for certain small ordinals (see Theorem 5.2.1, Lemmas 5.3.2 and 5.3.3, Theorem 5.6.1, Theorem 5.4.1, Theorem 5.5.1 and Corollary 5.7.2).

Theorem. *Let k be a positive integer.*

1. $R^{top}(\omega+1, k+1) = \omega^k + 1$.
2. $R^{cl}(\omega+2, 3) = \omega^2 \cdot 2 + \omega + 2$.
3. $\omega^2 \cdot 3 \le R^{top}(\omega \cdot 2, 3) \le \omega^3 \cdot 100$.
4. $R^{top}(\alpha, k) < \omega^\omega$ *for all* $\alpha < \omega^2$.
5. $R^{top}(\omega^2, k) \le \omega^\omega$.

6. $R^{top}(\omega^2+1, k+2) \leq \omega^{\omega \cdot k}+1$.

We deduce the last of these from the second last using the following more general result (Theorem 5.7.1).

Theorem. *Let α and β be countable ordinals with $\beta > 0$, let k be a positive integer, and suppose they satisfy a "cofinal version" of*

$$\omega^{\omega^\alpha} \rightarrow_{cl} (\omega^\beta, k+2)^2.$$

Then

$$\omega^{\omega^{\alpha \cdot (k+1)}}+1 \rightarrow_{cl} (\omega^\beta+1, k+2)^2.$$

Moreover, if $\omega^{\omega^\alpha} > \omega^\beta$, then in fact

$$\omega^{\omega^{\alpha \cdot k}}+1 \rightarrow_{cl} (\omega^\beta+1, k+2)^2.$$

We will give the precise meaning of this cofinal partition relation later, and also pose several questions for further research in this area.

Remarking briefly on the case $n > 2$, not much more can be said in the setting of countable ordinals, since Kruse showed that if $n \geq 3$ and β is a countable ordinal, then $\beta \not\rightarrow (\omega+1, n+1)^n$ [Kru65]. As for the ordinal ω_1, [ER56, Theorem 39 (ii)] shows that $\omega_1 \rightarrow (\omega+1)^n_k$ and, in fact, it seems to be a folklore result that $\omega_1 \rightarrow_{top} (\omega+1)^n_k$ for all finite n, k; a proof can be found in [HJW90]. On the other hand, we have the negative relations $\omega_1 \not\rightarrow (n+1)^n_\omega$ for $n \geq 2$ [ER56], $\omega_1 \not\rightarrow (\omega+2, n+1)^n$ for $n \geq 4$ [Kru65], $\omega_1 \not\rightarrow (\omega+2, \omega)^3$ [Jon00] and $\omega_1 \not\rightarrow (\omega_1, 4)^3$ [Haj64]. All that remains to be settled is the conjecture that $\omega_1 \rightarrow (\alpha, k)^3$ for all $\alpha \in \omega_1$ and all finite k, with the strongest result to date being that $\omega_1 \rightarrow (\omega \cdot 2+1, k)^3$ for all finite k [Jon13]. The topological version of this result remains unexplored.

1.3 Autohomeomorphism groups of ordinals

The groups of autohomeomorphisms (homeomorphisms from a space to itself) of ordinals do not appear to have been explicitly studied before. However, they have been studied implicitly from the perspective of Boolean algebras. Under Stone duality, countable successor ordinals correspond to so-called *superatomic* countable Boolean algebras. The automorphism groups of these Boolean algebras were studied by Monk, who gave a description of the automorphism group of the Boolean algebra corresponding to ω^2+1 [Mon75, Theorem 7], and also proved the following basic result, which we have translated into topological language [Mon75, Theorem 6].

Theorem (Monk). *Let α be a countable non-zero ordinal. Then the normal subgroup lattice of the autohomeomorphism group of $\omega^\alpha \cdot 2+1$ contains a chain of order type* $2+\alpha+2$.

We will generalise Monk's description of the autohomeomorphism group of ω^2+1 to ordinals of the form $\omega^n \cdot m+1$ for finite n and m. Our main aim will be to study the normal subgroups of these autohomeomorphism groups. We will vastly improve upon Monk's basic result, finding $2^{2^{\aleph_0}}$ normal subgroups, including chains of size $2^{\aleph_0}$, even within the autohomeomorphism group of ω^2+1. This result is somewhat surprising, since the condition of normality is rather strong in the infinite context. Indeed, this result stands in stark contrast to the following classical result of Schreier and Ulam. We write S_∞ for the group of permutations of a countably infinite set.

Theorem (Schreier–Ulam)**.** *The normal subgroups of S_∞ are exactly: the identity; the group of finitary alternating permutations; the group of finitary permutations; and S_∞.*

Our work may be viewed as a generalisation of this result, since S_∞ is isomorphic to the autohomeomorphism group of $\omega+1$. Indeed, we will make essential use of the following result of Bertram [Ber73], which may be viewed as quantitative version of the statement that the only normal subgroup of S_∞ containing a permutation of infinite support is S_∞ itself.

Theorem (Bertram)**.** *Let $g \in S_\infty$ have infinite support and let $h \in S_\infty$. Then h is the product of 4 conjugates of g.*

Fix two positive integers n and m, and consider only those autohomeomorphisms of $\omega^n \cdot m+1$ lying within the pointwise stabiliser of the set of topological limit points. In this context, we will introduce the notion of a *character* of an autohomeomorphism, which may be thought of as a more refined notion of the size of the support of a permutation. These characters may be defined as subsets of $\mathcal{P}(\{1,2,\ldots,n\})$ in such a way that every character Δ is a *lower set*, meaning that if $t \in \Delta$ and $s \subseteq t$ then $s \in \Delta$, and moreover, every such lower set arises as the character of some autohomeomorphism. We will also introduce the notion of the *flow* of an autohomeomorphism, which measures how much points are "moved towards" topological limit points.

Our main result (Theorem 6.7.7) is a classification of the normal subgroups of the autohomeomorphism group of $\omega^n \cdot m+1$ contained in the pointwise stabiliser of the set topological limit points. Here we write L_Δ for the subgroup of this stabiliser consisting of all autohomeomorphisms whose character is a subset of Δ, and L_Δ^0 for the subgroup of L_Δ consisting of all autohomeomorphisms of "zero flow", which we will define later.

Theorem. *Let N be a normal subgroup of the autohomeomorphism group of $\omega^n \cdot m+1$ contained in the pointwise stabiliser of the set of topological limit points, and assume that N is not the group of alternating permutations of finite support. Then*

$$L_\Delta^0 \leq N \leq L_\Delta$$

for a unique lower set $\Delta \subseteq \mathcal{P}(\{1, 2, \ldots, n\})$.

Thus these normal subgroups fall naturally into $D(n)$ disjoint sets, where $D(n)$ is the number of lower sets in $\mathcal{P}(\{1, 2, \ldots, n\})$ (the nth *Dedekind number*). Moreover, by studying the different possible flows we will show that each of these sets is either countable or of size $2^{2^{\aleph_0}}$. In particular, we will show that there are $2^{2^{\aleph_0}}$ normal subgroups of the autohomeomorphism group of $\omega^n \cdot m + 1$ for all $n \geq 2$.

Some of the groups we will study have appeared before in a different context. Given a positive integer m, consider the autohomeomorphism group of $\omega \cdot m + 1$. By ignoring the action of the group on the set of topological limit points, we may view this group as a transitive subgroup of S_∞. The resulting permutation group has been called the *almost stabiliser* of a partition of a countably infinite set into m infinite pieces [BCP+94], and was shown by Richman [Ric67] to be a maximal proper subgroup of S_∞. However, the corresponding result does not hold for $\omega^2 + 1$ or any larger ordinal.

Our work may also shed some light on whether the countable superatomic Boolean algebras have the small index property, a question which has been studied by Truss and Wencel [TW13].

Chapter 2

Preliminaries

The purpose of this chapter is to provide a self-contained exposition of ordinal topologies from scratch. After covering the basic properties of ordinal topologies, we will provide a short proof of the classification of ordinals up to homeomorphism. We will then briefly recap Stone duality and conclude with some characterisations of the countable compact ordinal topologies.

2.1 Notation

Our notation is standard. Officially, we use the von Neumann definitions of ordinals and cardinals, namely that an ordinal is the set of all smaller ordinals and a cardinal is an initial ordinal. We shall try to mitigate any confusion this may cause by referring explicitly to sets of ordinals where appropriate.

Given a totally ordered class X, we implicitly extend the ordering to include $-\infty$ as a minimum and ∞ as a maximum. We then use interval notation in the usual fashion, so that for example if $a, b \in X$ then $[a, b) = \{x \in X : a \leq x < b\}$ and $(a, \infty) = \{x \in X : a < x\}$.

We denote the cardinal successor of a cardinal κ by κ^+. We denote the cardinality of a set X by $|X|$. We denote the set of subsets of a set X of size n by $[X]^n$. We use the symbol $\cong$ to denote the homeomorphism relation. Unless otherwise stated all arithmetic will be ordinal arithmetic, except for expressions of the form 2^κ where κ is a cardinal.

2.2 Definitions and basic properties

Our starting point is the following generalisation of the Euclidean topology on $\mathbb{R}$.

Definition. Let X be a totally ordered set. The *order topology* on X is the topology generated by

$$\{(a, b) : a \in X \cup \{-\infty\}, b \in X \cup \{\infty\}\},$$

which is easily seen to be a base.

Given an arbitrary set X of ordinals, we may endow X with the subspace topology induced from the order topology on $[0, \alpha)$, where α is any ordinal with $X \subseteq [0, \alpha)$. Note that this does not depend on the choice of α. We refer to this topology simply as the *subspace topology.* Unless otherwise stated we will always assume that X is endowed with this topology.

For example, the subspace topology on $[0, \omega]$ is homeomorphic to the one-point compactification of $\mathbb{N}$, and the subspace topology on $[0, \omega + 1] \setminus \{\omega\}$ is discrete. Thus the subspace topology on a set of ordinals need not coincide with the order topology.

Here are some basic topological properties of ordinals and their subspaces.

Definition. Let X be a topological space.

We say that X is *totally disconnected* to mean that its only non-empty connected subspaces are singletons.

We say that X is *scattered* to mean that every non-empty subspace of X has an isolated point.

Lemma 2.2.1. *Let X be a set of ordinals endowed with the subspace topology. Then X is Hausdorff, totally disconnected and scattered.*

Proof. To see that X is Hausdorff, let $x, y \in X$ with $x < y$. Let $U = X \cap [0, x + 1)$ and $V = X \cap (x, \infty)$. Then U and V are disjoint open sets with $x \in U$ and $y \in V$.

To see that X is totally disconnected, suppose $Y \subseteq X$ and $x, y \in Y$ with $x < y$. Let U and V be as above. Then the sets $Y \cap U$ and $Y \cap V$ partition Y into two disjoint open sets.

To see that X is scattered, suppose $Y \subseteq X$ is non-empty. Let x be the least element of Y, and let U be as above. Then $Y \cap U = \{x\}$, so x is an isolated point of Y. $\square$

The following result provides us with an equivalent definition for the subspace topology on a set of ordinals, which may be familiar from the context of club sets.

Lemma 2.2.2. *Let X be a set of ordinals endowed with the subspace topology and let $Y \subseteq X$. Then Y is closed in X if and only if for every non-empty $Z \subseteq Y$, if $\sup(Z) \in X$ then $\sup(Z) \in Y$.*

Proof. Suppose first that for every non-empty $Z \subseteq Y$, if $\sup(Z) \in X$ then $\sup(Z) \in Y$. We show that $X \setminus Y$ is open. Suppose $x \in X \setminus Y$ and let $Z = Y \cap [0, x)$. If Z is empty then $X \cap [0, x + 1)$ is an open subset of $X \setminus Y$ containing x. Otherwise let $z = \sup(Z)$. Then by assumption $z \notin X \setminus Y$, so $z \neq x$, so $z < x$ since $z \leq x$ by definition of Z. Hence $X \cap (z, x + 1)$ is an open subset of $X \setminus Y$ containing x.

Conversely, suppose Y is closed in X and $Z \subseteq Y$ is non-empty. Let $z = \sup(Z)$ and suppose for contradiction that $z \in X \setminus Y$. Certainly z is a non-zero limit ordinal, or else $z \in Z \subseteq Y$. Now since $X \setminus Y$ is open, there are ordinals x and w with $x < z < w$ such that $X \cap (x, w) \subseteq X \setminus Y$. But then $x + 1 < z$ is an upper bound for Z, contradicting the definition of z. $\square$

Given that sets of ordinals are Hausdorff, there is unsurprisingly a connection between closed and compacts sets, as the following result illustrates.

Lemma 2.2.3. *Let X be a set of ordinals endowed with the subspace topology. Then X is compact if and only if for every non-empty $Z \subseteq X$, $\sup(Z) \in X$.*

Proof. Suppose first that $Z \subseteq X$ is non-empty with $\sup(Z) \notin X$. Let $z = \sup(Z)$. For each $x \in Z$, let $U_x = X \cap [0, x)$, and let $V = X \cap (z, \infty)$. Then $\{U_x : x \in Z\} \cup \{V\}$ is an open cover of X, but if $Y \subseteq Z$ is finite then $\max(Y) \notin \bigcup_{y \in Y} U_y \cup V$. Hence X is not compact.

Conversely, suppose that for every non-empty $Z \subseteq X$, $\sup(Z) \in X$. Given an open cover of X, we choose a finite subcover $V_1, V_2, \ldots, V_n$ recursively as follows. Having chosen $V_1, V_2, \ldots, V_{k-1}$ for some $k \in \omega$, let $Y = X \setminus \bigcup_{i=1}^{k-1} V_i$, a closed set. If Y is empty, then we are done. Otherwise let $\alpha_k = \sup(Y)$. Then $\alpha_k \in X$ by assumption, so $\alpha_k \in Y$ by Lemma 2.2.2. Thus α_k is a maximal element of Y. Take V_k to be any member of the open cover containing α_k. This process must terminate, or else we obtain a strictly decreasing sequence of ordinals $\alpha_1 > \alpha_2 > \alpha_3 > \ldots$. □

In particular, if α is a non-zero ordinal then $[0, \alpha)$ is compact if and only if α is a successor ordinal.

The following definition is from Schipperus [Sch12].

Definition. Let X be a set of ordinals. We say that X is *internally closed* to mean that for every non-empty $Z \subseteq X$, if $\sup(Z) < \sup(X)$ then $\sup(Z) \in X$.

By Lemma 2.2.2, this is equivalent to saying that X is closed in $[0, \alpha)$, where α is the least ordinal such that $X \subseteq [0, \alpha)$. If the context specifies no ambient space, then we may simply say that X is closed to mean that X is internally closed.

By Lemma 2.2.3, every compact set of ordinals is internally closed. However, if α is a non-zero limit ordinal then $[0, \alpha)$ is internally closed but neither compact nor a closed subset of $[0, \alpha]$.

Our final basic result gives another equivalent condition for a set of ordinals to be internally closed.

Proposition 2.2.4. *Let X be a set of ordinals. Then X is internally closed if and only if the subspace topology on X coincides with the order topology.*

The latter condition is equivalent to saying that the order-isomorphism from X to its order type is a homeomorphism.

Proof. Fix an ordinal α with $X \subseteq [0, \alpha)$.

First suppose X is internally closed. Certainly every basic open subset of X in the order topology is open in the subspace topology. For the other way round, suppose $a \in [0, \alpha) \cup \{-\infty\}$ and $b \in [0, \alpha) \cup \{\infty\}$ are such that $X \cap (a, b)$ is non-empty. It is sufficient to find $c \in X \cup \{-\infty\}$ and $d \in X \cup \{\infty\}$ such that $X \cap (c, d) = X \cap (a, b)$. First let

$$d = \begin{cases} \infty, & \text{if } x < b \text{ for all } x \in X \\ \min\left(X \cap [b, \infty)\right), & \text{otherwise.} \end{cases}$$

Next let $Z = X \cap [0, a]$. If Z is empty, then let $c = -\infty$. Otherwise $\sup(Z) \leq a < \sup(X)$ since $X \cap (a, b)$ is non-empty, so $\sup(Z) \in X$ since X is internally closed. In this case let $c = \sup(Z)$. Then c and d are as required.

Conversely, suppose the subspace topology on X coincides with the order topology on X. Let $Z \subseteq X$ be non-empty and suppose $\sup(Z) < \sup(X)$. Let $z = \sup(Z)$. Then $X \cap (z, \infty)$ is non-empty and open in the subspace topology and therefore in the order topology, so we may write

$$X \cap (z, \infty) = \bigcup_{i \in I} X \cap (a_i, b_i)$$

where I is a non-empty set and for all $i \in I$, $a_i \in X \cup \{-\infty\}$, $b_i \in X \cup \{\infty\}$ and $X \cap (a_i, b_i)$ is non-empty. Let $a = \min\{a_i : i \in I\}$. It is then easy to check that

$$X \cap (z, \infty) = X \cap (a, \infty). \tag{$*$}$$

We claim that $a = z$ and hence $z \in X$, as required. To prove the claim, first observe that if $a > z$ then $a \in X \cap (z, \infty)$, contrary to $(*)$. Finally observe that if $a < z$ then $X \cap (a, z] = \emptyset$ by $(*)$ and so a is an upper bound for Z, contrary to the definition of z. □

2.3 The Cantor–Bendixson derivative

The key tool in the analysis of ordinal topologies is the Cantor–Bendixson derivative, which was introduced by Cantor in 1872. In fact, Cantor later introduced ordinals for the very purpose of iterating this operation.

Definition. Let X be a topological space. The *Cantor–Bendixson derivative* X' of X is defined by

$$X' = X \setminus \{x \in X : x \text{ is isolated}\}.$$

The iterated derivatives of X are defined for γ an ordinal by

1. $X^{(0)} = X$,
2. $X^{(\gamma+1)} = \left(X^{(\gamma)}\right)'$, and
3. $X^{(\gamma)} = \bigcap_{\delta<\gamma} X^{(\delta)}$ when γ is a non-zero limit.

For example, if $X = [0, \omega^2]$ then $X' = \{\omega \cdot m : m \in \omega\} \cup \{\omega^2\}$ and $X^{(2)} = \{\omega^2\}$. Here are some basic properties of this operation.

Lemma 2.3.1. *Let X be a topological space, let γ be an ordinal and let $Y \subseteq X$. Then $Y^{(\gamma)} \subseteq X^{(\gamma)}$.*

Proof. First observe that if $y \in Y$ and $\{y\}$ is an open subset of X, then $\{y\} = Y \cap \{y\}$ is also an open subset of Y. Therefore $Y' \subseteq X'$. The result then follows by induction on γ. $\square$

Lemma 2.3.2. *Let X be a topological space and let γ be an ordinal. Then $X^{(\gamma)}$ is closed.*

Proof. First observe that X' is closed since $\{x\}$ is open for all $x \in X \setminus X'$ and $X \setminus X' = \bigcup_{x \in X \setminus X'} \{x\}$. The result then follows by induction on γ. $\square$

We may describe the effect of the Cantor–Bendixson derivative on sets of ordinals of the form $[0, \alpha)$ using the following notion. First recall that if x is an ordinal, then there is a unique sequence of ordinals $\gamma_1 > \gamma_2 > \cdots > \gamma_n$ and a unique sequence of positive integers $m_1, m_2, \ldots, m_n$ such that

$$x = \omega^{\gamma_1} \cdot m_1 + \omega^{\gamma_2} \cdot m_2 + \cdots + \omega^{\gamma_n} \cdot m_n$$

(the *Cantor normal form* of x). (Note that it need not be the case that $\gamma_1 < x$, since for example $\varepsilon_0 = \omega^{\varepsilon_0}$.)

Definition. Let x be an ordinal with the above Cantor normal form. The *Cantor–Bendixson rank* of x is defined by

$$\mathrm{CB}\,(x) = \begin{cases} \gamma_n, & \text{if } x > 0 \\ 0, & \text{if } x = 0. \end{cases}$$

This defines a function $\mathrm{CB} : \left[0, \omega^\beta\right) \to [0, \beta)$ for each non-zero ordinal β.

The relationship between the notions of Cantor–Bendixson derivative and rank is given by the following simple result, which provides us with an alternative definition of Cantor–Bendixson rank.

Proposition 2.3.3. *Let α be an ordinal. If $x \in [0, \alpha)$ then the Cantor–Bendixson rank of x is the greatest ordinal γ such that $x \in [0, \alpha)^{(\gamma)}$.*

In other words, the Cantor–Bendixson rank of an ordinal is the ordinal number of iterated derivatives that can be taken before that point "disappears".

Proof. We prove by induction on γ that for γ an ordinal,

$$[0, \alpha)^{(\gamma)} = \{x \in [0, \alpha) : \mathrm{CB}\,(x) \geq \gamma\}.$$

The case $\gamma = 0$ is trivial.

If γ is a successor ordinal, say $\gamma = \delta + 1$, then $[0,\alpha)^{(\gamma)} = \left([0,\alpha)^{(\delta)}\right)'$, and by the inductive hypothesis $[0,\alpha)^{(\delta)} = \{x \in [0,\alpha) : \mathrm{CB}\,(x) \geq \delta\}$. Let $x \in [0,\alpha)$ be non-zero (the case $x = 0$ is straightforward). If $\mathrm{CB}\,(x) = \delta$, then we may write $x = y + \omega^\delta$ with y a multiple of ω^δ. Then $(y, x+1) \cap [0,\alpha)^{(\delta)} = \{x\}$, so x is isolated in $[0,\alpha)^{(\delta)}$. On the other hand, if $\mathrm{CB}\,(x) > \delta$, then whenever $z < x$ we have $z + \omega^\delta \in [0,\alpha)^{(\delta)}$ and $z + \omega^\delta < x$, so any basic open set containing x contains some point of $[0,\alpha)^{(\delta)}$ and thus x is not isolated in this space.

Finally, if γ is a non-zero limit then $[0,\alpha)^{(\gamma)} = \bigcap_{\delta<\gamma} [0,\alpha)^{(\delta)}$, so by the inductive hypothesis

$$[0,\alpha)^{(\gamma)} = \bigcap_{\delta<\gamma} \{x \in [0,\alpha) : \mathrm{CB}\,(x) \geq \delta\} = \{x \in [0,\alpha) : \mathrm{CB}\,(x) \geq \gamma\}\,. \qquad \square$$

Note that some authors use the term *Cantor–Bendixson rank* to refer to a property of a topological space X (the least ordinal γ such that $X^{(\gamma)} = X^{(\gamma+1)}$), rather than a property of a point in a topological space.

2.4 The classification of ordinal topologies

We shall see in the next chapter that the general problem of classifying the subspace topologies on sets of ordinals is quite complex. The problem of classifying the order topologies on ordinals themselves is nonetheless a simple application of Cantor–Bendixson derivatives. The compact case was stated in terms of Boolean algebras by Mayer and Pierce [MP60, Theorem 4.6] and then directly by Baker [Bak72, Corollary 3]. The general classification was later performed by Flum and Martínez [FM88, Theorem 2.2 and Remark 2.5, part 3] and was independently rediscovered in unpublished work by Kieftenbeld and Löwe [KL06]. We will state this result in a slightly different way to both of those treatments, and provide a short proof based on that of Flum and Martínez.

Our statement of the classification makes use of the following piece of notation. The idea behind it is that the "building blocks" of ordinal topologies have the form 1 or $\omega^\gamma + 1$.

Definition. Let γ be an ordinal and let m be a positive integer. We define

$$\overline{\omega}\,[\gamma, m] = \begin{cases} \omega^\gamma \cdot m + 1, & \text{if } \gamma > 0 \\ m, & \text{if } \gamma = 0. \end{cases}$$

Thus $\overline{\omega}\,[\gamma, m]$ is homeomorphic to the topological disjoint union of m copies of $\overline{\omega}\,[\gamma, 1]$.

Here is the classification.

Theorem 2.4.1 (Flum–Martínez)**.** *Let α be a non-zero ordinal.*

If α is compact, then there is a unique ordinal γ and a unique positive integer m such that

$$\alpha \cong \overline{\omega}\,[\gamma, m]\,.$$

If α is not compact and $\alpha^{(\eta)}$ is either infinite or empty for all ordinals η, then there is a unique non-zero ordinal δ such that

$$\alpha \cong \omega^{\delta}.$$

Otherwise, there are unique non-zero ordinals γ and δ with $\delta \leq \gamma$ and a unique positive integer m such that

$$\alpha \cong \overline{\omega}\,[\gamma, m] + \omega^{\delta}.$$

Moreover, the proof will reveal how to obtain the required ordinals from the Cantor normal form of α.

The existence part of the proof is a straightforward consequence of the following observation.

Lemma 2.4.2. *Let α be a successor ordinal and let β be any ordinal. Then $\alpha + \beta$ is homeomorphic to the topological disjoint union of α and β.*

Proof. Simply write $\alpha = x + 1$ and let $U = [0, x + 1)$ and $V = (x, \alpha + \beta)$. Then U and V are disjoint open subsets of $\alpha+\beta$ with $U \cong \alpha$, $V \cong \beta$ and $U \cup V = \alpha+\beta$. □

In particular, if α and β are both successor ordinals then $\alpha + \beta \cong \beta + \alpha$.

Here is our proof of the classification.

Proof of Theorem 2.4.1. First we prove existence.

If α is compact, then α is a successor ordinal by Lemma 2.2.3. Hence using Cantor normal form we may write

$$\alpha = \overline{\omega}\,[\gamma_1, m_1] + \overline{\omega}\,[\gamma_2, m_2] + \cdots + \overline{\omega}\,[\gamma_n, m_n]$$

with $\gamma_1 > \gamma_2 > \cdots > \gamma_n$ ordinals and $n, m_1, m_2, \ldots, m_n$ positive integers. Let $\zeta = \overline{\omega}\,[\gamma_2, m_2] + \cdots + \overline{\omega}\,[\gamma_n, m_n]$. Then by Lemma 2.4.2, $\alpha = \overline{\omega}\,[\gamma_1, m_1] + \zeta \cong \zeta + \overline{\omega}\,[\gamma_1, m_1] = \overline{\omega}\,[\gamma_1, m_1]$.

If α is not compact, then using Cantor normal form we may write

$$\alpha = \overline{\omega}\,[\gamma_1, m_1] + \overline{\omega}\,[\gamma_2, m_2] + \cdots + \overline{\omega}\,[\gamma_n, m_n] + \omega^{\delta}$$

with $\gamma_1 > \gamma_2 > \cdots > \gamma_n \geq \delta > 0$ ordinals, $m_1, m_2, \ldots, m_n$ positive integers and $n \in \omega$. If $\alpha^{(\eta)}$ is either infinite or empty for all ordinals η, then $n = 0$, or else $\left|\alpha^{(\gamma_1)}\right| = m_1$. Hence α must in fact equal ω^{δ}. Otherwise let $\zeta = \overline{\omega}\,[\gamma_2, m_2] + \cdots + \overline{\omega}\,[\gamma_n, m_n]$. Then by Lemma 2.4.2, $\alpha = \overline{\omega}\,[\gamma_1, m_1] + \zeta + \omega^{\delta} \cong \zeta + \overline{\omega}\,[\gamma_1, m_1] + \omega^{\delta} = \overline{\omega}\,[\gamma_1, m_1] + \omega^{\delta}$.

Now we prove uniqueness by using the fact that $\left|X^{(\gamma)}\right|$ is a topological invariant of a topological space X for any ordinal γ.

For the compact case, let γ be an ordinal and m be a positive integer. Then by Proposition 2.3.3, $\gamma + 1$ is the least ordinal η such that $\varpi\,[\gamma, m]^{(\eta)} = \emptyset$, and $m = \left|\varpi\,[\gamma, m]^{(\gamma)}\right|$. Hence we may recover γ and m from the topology on $\varpi\,[\gamma, m]$.

For the first non-compact case, let δ be a non-zero ordinal. Then by Proposition 2.3.3, δ is the least ordinal η such that $\left(\omega^\delta\right)^{(\eta)} = \emptyset$. Hence we may recover δ from the topology on ω^δ.

For the second non-compact case, let γ and δ be non-zero ordinals with $\delta \leq \gamma$, let m be a positive integer, and let $\alpha = \varpi\,[\gamma, m] + \omega^\delta$. Then by Proposition 2.3.3, $\gamma + 1$ is the least ordinal η such that $\alpha^{(\eta)} = \emptyset$, $m = \left|\alpha^{(\gamma)}\right|$, and δ is the least ordinal ζ such that $\alpha^{(\zeta)}$ is either compact or empty. Hence we may recover γ, m and δ from the topology on α. $\square$

2.5 Stone duality

We now briefly recap the basics of Stone duality applicable to our situation. Further details may be found in [BS69, Chapter 1] for the basic concepts or [Joh86] for a comprehensive treatment.

Definition. A *Stone space* is a topological space that is compact, Hausdorff and totally disconnected.

Write **Stone** for the category of Stone spaces and continuous functions, and **BoolAlg** for the category of Boolean algebras and Boolean algebra homomorphisms.

Given a Stone space X, write $C\,(X)$ for the Boolean algebra of clopen subsets of X under inclusion.

Given a Boolean algebra B, write $S\,(B)$ for the set of ultrafilters of B endowed with the topology generated by the base

$$\{\{U \in S\,(B) : b \in U\} : b \in B\}\,.$$

Theorem 2.5.1 (Stone duality)**.** *The maps C and S extend to functors C : $\mathbf{Stone} \to \mathbf{BoolAlg}^{\mathrm{op}}$ and S : $\mathbf{BoolAlg}^{\mathrm{op}} \to \mathbf{Stone}$ yielding an equivalence of categories.*

This duality allows statements about Boolean algebras to be translated into the language of Stone spaces, and vice versa. For example, quotient algebras correspond to subspaces, and products of algebras correspond to disjoint unions of spaces. In order to state another such correspondence, we require the following notion.

Definition. Let B be a Boolean algebra.

We say that $a \in B$ is an *atom* to mean that a is a minimal element of $B \setminus \{0\}$.

We say that B is *superatomic* to mean that every non-trivial quotient of B has an atom.

Mostowski and Tarski [MT39] gave several equivalent conditions for a Boolean algebra B to be superatomic, for example that every non-trivial subalgebra of B has an atom. See [Roi89] for further details.

Here are some particular correspondences of interest in the context of Stone duality. We say that a topological space X is *second-countable* to mean that X has a countable base.

Lemma 2.5.2. *Let X be a Stone space.*

1. *$C(X)$ is countable if and only if X is second-countable.*
2. *$C(X)$ is superatomic if and only if X is scattered.*

Proof.

1. If $C(X)$ is countable then certainly X has a countable base since $C(X)$ is a base for X. Conversely, suppose that X has a countable base $\mathcal{B}$. We show that each member of $C(X)$ can be written as a finite union of members of $\mathcal{B}$, which suffices. To see this, let $Y \in C(X)$. Then for each $x \in Y$ there exists $U_x \in \mathcal{B}$ with $x \in U_x \subseteq Y$. Then $\{U_x : x \in Y\}$ is an open cover of Y. But Y is a closed subset of a compact space and is therefore compact, so there is a finite subset $S \subseteq Y$ such that $Y = \bigcup_{x \in S} U_x$, as required.

2. First note that under Stone duality, the non-empty subspaces of a Stone space correspond to the non-trivial quotients of its Boolean algebra. Hence it is sufficient to prove that $C(X)$ has an atom if and only if X has an isolated point. Now if $a \in C(X)$ is an atom, then $\{b \in C(X) : a \leq b\}$ is an isolated point of $S(C(X)) \cong X$. Conversely, if U is an isolated point of X, then $\{U\}$ is an atom of $C(X)$. $\square$

The reason that this result is relevant is of course that by Lemma 2.2.1, if X is a compact set of ordinals under the subspace topology, then X is a scattered Stone space.

Note that there are scattered Stone spaces that are not homeomorphic to compact sets of ordinals under the subspace topology. For example, let $X = \omega_1 + 1 + \omega^*$ endowed with the order topology, where ω^* is order-isomorphic to the negative integers and $+$ denotes the ordered sum operation. Then X is a scattered Stone space, but as observed by Baker [Bak72, p. 25], it is not homeomorphic to any set of ordinals under the subspace topology. We will prove this fact in the next chapter (in Proposition 3.4.4).

2.6 Countable compact ordinal topologies

We conclude this chapter with some known characterisations of the countable compact ordinal topologies, since these are a major focus of much of our work.

Theorem 2.6.1. *Let X be a non-empty topological space. The following are equivalent.*

1. *X is countable, compact and Hausdorff.*
2. *X is the Stone space of a countable superatomic Boolean algebra.*
3. *X is homeomorphic to a countable compact set of ordinals under the subspace topology.*
4. *$X \cong \overline{\omega}[\gamma, m]$ for some countable ordinal γ and some positive integer m.*

Note that by Lemma 2.5.2, condition 2 is equivalent to saying that X is a scattered, second-countable Stone space. Note also that by Lemma 2.2.3 and Proposition 2.2.4, condition 3 is equivalent to saying that X is homeomorphic to a countable successor ordinal.

The equivalence of conditions 1, 3 and 4 was proved by Sierpiński and Mazurkiewicz [MS20], and condition 2 was later added by Day [Day67, p. 489]. Here we sketch a short proof of the equivalence of all 4 conditions, using ideas from [Ket78, Section 0].

Our proof has two main ingredients. The first ingredient is the Cantor–Bendixson theorem, which was originally intended as the "continuum hypothesis for closed sets". We say that a topological space is *perfect* to mean that it has no isolated points.

Theorem 2.6.2 (Cantor–Bendixson)**.** *Let X be a closed subset of a Polish space. Then we may write X as a disjoint union $X = S \cup P$ with S scattered and P perfect in a unique way. Moreover, S is at most countable and P either is empty or has cardinality $2^{\aleph_0}$.*

This theorem may be proved using Cantor–Bendixson derivatives, and was in fact the original result for which the tool was invented. All we will need from it is the fact that a Polish space is countable if and only if it is scattered. Note in particular that any second-countable compact Hausdorff space is a Polish space by Urysohn's metrisation theorem.

The second ingredient of our proof is Vaught's criterion, which may be stated as follows [Pie89, Section 1.2].

Theorem 2.6.3 (Vaught)**.** *Let R be a symmetric binary relation defined on the class of second-countable Stone spaces. Suppose that for all such spaces X and Y,*

- *if $XR\emptyset$ then $X = \emptyset$, and*
- *if XRY and $X = A \cup B$ with $A, B \subseteq X$ disjoint open subsets, then there exist disjoint open subsets $C, D \subseteq Y$ such that ARC and BRD.*

Then for all such spaces X and Y, if XRY then $X \cong Y$.

Sketch of proof of Theorem 2.6.1. The implications $4 \implies 3 \implies 1$ are given by Lemmas 2.2.3 and 2.2.1, so it is enough to prove that $1 \implies 2 \implies 4$.

To see that $1 \implies 2$, let X be countable, compact and Hausdorff. One may show directly that X has a countable base [KT07, Theorem 2.2]. Then X is a Polish space by Urysohn's metrisation theorem, and so X is scattered by the Cantor–Bendixson theorem. This suffices, since any space that is Hausdorff and scattered is automatically totally disconnected.

To see that $2 \implies 4$, let X be a scattered, second-countable Stone space. Then X is a Polish space by Urysohn's metrisation theorem, and so X is countable by the Cantor–Bendixson theorem. Let γ_X be the least ordinal γ such that $X^{(\gamma)} = \emptyset$, which exists since X is scattered and is countable since X is countable. Then since X is compact, γ_X is a successor ordinal, say $\gamma_X = \delta + 1$, and $X^{(\delta)}$ is finite, say $\left|X^{(\delta)}\right| = m_X$. Now define a symmetric binary relation R on second-countable Stone spaces by XRY if and only if X and Y are both scattered, $\gamma_X = \gamma_Y$ and $m_X = m_Y$. One may easily check that this relation satisfies Vaught's criterion, and the result follows. □

Note that the equivalence of conditions 3 and 4 is simply the countable compact case of Theorem 2.4.1, though we have provided an independent proof of this equivalence.

Chapter 3

Subspaces of countable ordinals

In this chapter we define the "one-point cofinite extension" operation on topological spaces and prove our topological analogue of Hausdorff's theorem on scattered total orderings. From this we obtain a characterisation of the countable subspaces of ordinals as the order topologies of countable scattered total orderings. Afterwards we use these ideas to compute the number of subspaces of an ordinal up to homeomorphism.

3.1 Linearly ordered topological spaces

We begin by reviewing some basic ideas from the theory of linearly ordered topological spaces. First recall the following pieces of terminology.

Definition. A *linearly ordered topological space* (LOTS) is a totally ordered set under the order topology.

A *generalised ordered space* (GO-space) is any subset of a LOTS under the induced ordering and the subspace topology.

An *order-homeomorphism* is a bijective function that is both an order-isomorphism and a homeomorphism.

Let X be a totally ordered set. We say that a subset $Y \subseteq X$ is *order-convex* to mean that if $x, y \in Y$ and $z \in X$ with $x < z < y$, then $z \in Y$.

Many basic properties of ordinal topologies follow from results in the general theory of linearly ordered topological spaces, such as the fact that every LOTS is Hausdorff and hereditarily normal (i.e., T_5) [Ste70]. Indeed, several of the results in the previous chapter, such as Lemmas 2.2.2 and 2.2.3, may be viewed as special cases of such results.

Of particular relevance to subspaces of ordinals is the following basic result of Cech [CFK66, 17 Theorem A.23], which provides an equivalent definition for a GO-space.

Proposition 3.1.1 (Cech). *Let X be a totally ordered set endowed with some* *topology. Then X is a GO-space if and only if the topology on X contains the order topology and is generated by a base of order-convex sets.*

We omit the general proof of this result and instead prove the special case in which X is well-ordered.

Theorem 3.1.2 (Cech)**.** *Let X be a well-ordered set endowed with some topology. The following are equivalent.*

1. *X is order-homeomorphic to a set of ordinals under the subspace topology.*

2. *The topology on X has a base of the form*

$$\mathcal{B} \cup \{\{y\} : y \in Y\},$$

where $\mathcal{B}$ is the usual base for the order topology on X and $Y \subseteq X$.

In particular, if X is a set of ordinals under the subspace topology and $f : X \to [0, \alpha)$ is an order-isomorphism, then f is continuous. Of course, f need not be a homeomorphism.

The proof we provide was shown to the author by Robert Leek.

Proof. First let X be a set of ordinals under the subspace topology, say $X \subseteq [0, \alpha)$. Let $Y = \{y \in X : \{y\} \text{ is open}\}$ and let $\mathcal{B}^* = \mathcal{B} \cup \{\{y\} : y \in Y\}$ where $\mathcal{B}$ is the usual base for the order topology on X. To see that $\mathcal{B}^*$ is a base for the topology on X, let $a \in [0, \alpha) \cup \{-\infty\}$, $b \in [0, \alpha) \cup \{\infty\}$ and $x \in (a, b)$. If $x \in Y$, then $\{x\} \in \mathcal{B}^*$ with $x \in \{x\} \subseteq (a, b)$. Otherwise x must be a limit ordinal such that X is cofinal in $[0, x)$, in which case there exists $c \in X$ with $a < c < x$, whence $(c, x] \in \mathcal{B}^*$ with $x \in (c, x] \subseteq (a, b)$.

Conversely, suppose the topology on X has a base of the given form for some $Y \subseteq X$. Let $f : X \to [0, \alpha)$ be an order-isomorphism for some ordinal α. We show that X is order-homeomorphic to a subspace of $[0, 2 \cdot \alpha)$. Define $g : X \to [0, 2 \cdot \alpha)$ by

$$g(x) = \begin{cases} 2 \cdot f(x) + 1, & \text{if } x \in Y \\ 2 \cdot f(x), & \text{if } x \notin Y. \end{cases}$$

Let the image of X under g be Z. As a function $X \to Z$, certainly g is an order-isomorphism, and the image of any basic open set is open. To see that g is continuous, let $a \in [0, 2 \cdot \alpha) \cup \{-\infty\}$ and $b \in [0, 2 \cdot \alpha) \cup \{\infty\}$. We show that $g^{-1}(Z \cap (a, b))$ is open. First let

$$d = \begin{cases} \infty, & \text{if } z < b \text{ for all } z \in Z \\ g^{-1}(\min [Z \cap [b, \infty)]), & \text{otherwise.} \end{cases}$$

Next let $W = Z \cap [0, a]$. If W is empty, then let $c = -\infty$, and if W has a maximal element w then let $c = g^{-1}(w)$. In both cases $g^{-1}(Z \cap (a, b)) = (c, d)$ and we are done. Otherwise by definition of g we must have $a+1=g(c)$ for some $c \in Y$. But then $g^{-1}(Z \cap (a,b)) = (c,d) \cup \{c\}$, and we are done. $\square$

This proof also shows that if X is a set of ordinals of order type α under the subspace topology, then X is homeomorphic to a subspace of $[0, 2 \cdot \alpha)$. In particular, X is homeomorphic to a countable subspace of an ordinal if and only if X is homeomorphic to a subspace of a countable ordinal.

3.2 Hausdorff's theorem on scattered total orderings

In this section we review some basic ideas from the theory of total orderings, following Rosenstein [Ros82, Chapter 5].

First recall some basic terminology.

Definition. Let X be a totally ordered set.

We say that X is *dense* to mean that for all $x, y \in X$ with $x < y$, $(x, y) \neq \emptyset$.

We say that X is *scattered* to mean that X has no densely-ordered subset with more than one point.

Equivalently, X is scattered if and only if no subset of X is order-isomorphic to $\mathbb{Q}$.

Dense and scattered linear orderings play an important role in analysing arbitrary total orderings. For instance, Hausdorff proved that any total ordering may be written as a densely-ordered sum (a notion we shall define shortly) of scattered total orderings [Ros82, Theorem 4.9].

The key tools in the analysis of scattered total orderings are condensation maps, which may be thought of as order-homomorphisms.

Definition. Let X be a totally ordered set.

A *condensation* of X is a partition of X into disjoint non-empty order-convex sets, which we endow with the induced ordering defined by $A < B$ if and only if $a < b$ for all $a \in A$ and all $b \in B$.

A *condensation map* on X is a function $c : X \to \widetilde{X}$ such that $\widetilde{X}$ is a condensation of X and $x \in c(x)$ for all $x \in X$.

The *finite condensation map* c_F is the condensation map defined by $c_F(x) = \{y \in X : (x, y) \text{ is finite}\}$.

For example, the image of $[0, \omega^2]$ under c_F is order-isomorphic to $[0, \omega]$.

Note that $c_F(x)$ is always order-isomorphic to a finite total ordering, $\mathbb{Z}$, $\mathbb{Z}_{>0}$ or $\mathbb{Z}_{<0}$.

The notion of the finite condensation map immediately allows us to make the following simple but useful observation.

Proposition 3.2.1. *Let X be a non-empty scattered totally ordered set. Then there exists $x \in X$ such that $(-\infty, x)$ is either empty or has a maximal element and (x, ∞) is either empty or has a minimal element.*

Proof. Suppose for contradiction that there is no such $x \in X$. Then $|c_F(x)| \leq 2$ for all $x \in X$, and X is certainly infinite. But then it is easy to see that for instance

$$\{\min(c_F(x)) : x \in X\}$$

is a densely-ordered subset of X with more than one element. $\square$

This observation will be easier to use in the following form.

Corollary 3.2.2. *Let X be a scattered totally ordered set with $|X| \geq 3$. Then there exists $x \in X$ such that $(-\infty, x)$ has a maximal element and (x, ∞) has a minimal element.*

Proof. Let $Y = \{x \in X : x \text{ is neither minimal nor maximal in } X\}$. Then $Y \neq \emptyset$ since $|X| \geq 3$, so Y is a non-empty scattered totally ordered set, so by Proposition 3.2.1 there exists $x \in Y$ such that $Y \cap (-\infty, x)$ is either empty or has a maximal element and $Y \cap (x, \infty)$ is either empty or has a minimal element. Then x is as required. $\square$

Here is an easy consequence of our observation, which relates the order-theoretic notion of "scattered" to the topological notion.

Corollary 3.2.3. *Let X be a totally ordered set, and suppose X is scattered. Then the order topology on X is scattered.*

Proof. Suppose $Y \subseteq X$ is non-empty. Then Y is a non-empty scattered totally ordered set, so by Proposition 3.2.1 there exists $y \in Y$ such that $Y \cap (-\infty, y)$ is either empty or has a maximal element and $Y \cap (y, \infty)$ is either empty or has a minimal element. Hence $\{y\}$ is a basic open subset of Y. $\square$

Note that the converse does not hold, in the sense that there is a non-scattered totally ordered set whose order topology is scattered. For example, let $X = \mathbb{Q} \times \mathbb{Z}$ under the lexicographic ordering. Then the order topology on X is discrete.

Let us now state and prove Hausdorff's classical theorem. The precise statement requires the following notions.

Definition. If X is a totally ordered set, then the *backwards ordering* X^* is defined to be X under the relation $\geq$.

If X is a totally ordered set and Y_x is a totally ordered set for all $x \in X$, then the *ordered sum* is defined by

$$\sum_{x \in X} Y_x = \{(x, y) : x \in X, y \in Y_x\}$$

under the lexicographic ordering. For each $x \in X$ we identify the subset $\{(x, y) : y \in Y_x\}$ with Y_x. We also denote $\sum_{x \in \{0,1\}} Y_x$ by $Y_0 + Y_1$, which defines an associative binary operation.

Define $\mathcal{S}$ to be the smallest class of totally ordered sets subject to the following conditions. Here, 1 is a totally ordered set with exactly one element.

- $\mathcal{S}$ is closed under isomorphism.
- $1 \in \mathcal{S}$.
- If $X \in \mathcal{S}$ then $X^* \in \mathcal{S}$.
- If β is a non-zero ordinal and $X_i \in \mathcal{S}$ for all $i \in \beta$, then $\sum_{i \in \beta} X_i \in \mathcal{S}$.

Here is Hausdorff's theorem.

Theorem 3.2.4 (Hausdorff). *$\mathcal{S}$ is the class of non-empty scattered totally ordered sets.*

We use the proof from Rosenstein [Ros82, Theorem 5.26], which requires the finite condensation map to be iterated, in much the same fashion as the Cantor–Bendixson derivative.

Definition. Given a condensation map c defined for every totally ordered set, its iterations are defined for γ an ordinal and $x \in X$ by

1. $c^0(x) = \{x\}$,
2. $c^{\gamma+1}(x) = \{y \in X : c(c^\gamma(x)) = c(c^\gamma(y))\}$, and
3. $c^\gamma(x) = \bigcup_{\delta<\gamma} c^\delta(x)$ when γ is a non-zero limit.

For example if $X = [0, \omega^\beta)$ for some ordinal β, then $c_F^\gamma(0) = [0, \omega^\gamma)$ for all ordinals $\gamma \leq \beta$. Thus the finite condensation map is in some sense analogous to the Cantor–Bendixson derivative.

Here is the proof of Hausdorff's theorem.

Proof of Theorem 3.2.4. First we show by induction on the definition of $\mathcal{S}$ that if $X \in \mathcal{S}$ then X is scattered. Certainly 1 is scattered, and if X is scattered then so is X^*. So suppose β is a non-zero ordinal and X_i is scattered for all $i \in \beta$, and suppose for contradiction that $Y \subseteq \sum_{i \in \beta} X_i$ is a dense subordering with more than one element. Then $Y \cap X_i$ is a dense subordering of X_i and so has at most one element. Let $Z = \{i \in \beta : |Y \cap X_i| = 1\}$, so Z is order-isomorphic to Y. Then Z is a dense subordering of β with more than one element, which is absurd since β has no infinite strictly descending sequence.

Conversely, suppose X is a non-empty scattered totally ordered set. Let η be the least ordinal such that $c_F^{\eta+1}(x) = c_F^\eta(x)$ for all $x \in X$, which exists since

$c_F^\delta(x) \subseteq c_F^\gamma(x)$ for all $x \in X$ whenever δ and γ are all ordinals with $\delta < \gamma$. (η is known as the *F-rank* of X.) Then the condensation $\{c_F^\eta(x) : x \in X\}$ must be dense, so since X is scattered it must consist of a single order-convex set, and thus $c_F^\eta(x) = X$ for all $x \in X$. We now show by induction on $\gamma \leq \eta$ that $c_F^\gamma(x) \in \mathcal{S}$ for all $x \in X$, which suffices.

Certainly $c_F^0(x) = \{x\} \in \mathcal{S}$ for all $x \in X$. If $\gamma = \delta + 1$, then $c_F^\gamma(x)$ is order-isomorphic to an ordered sum of sets of the form $c_F^\delta(y)$ with $y \in c_F^\gamma(x)$ over either a finite totally ordered set, $\mathbb{Z}$, ω or ω^*. Either way it follows by the inductive hypothesis that $c_F^\gamma(x) \in \mathcal{S}$, noting that $\mathbb{Z}$ is order-isomorphic to $\omega^* + \omega$. Finally, suppose γ is a non-zero limit ordinal. Observe that for all $y \in c_F^\gamma(x)$, the least ordinal δ such that $y \in c_F^\delta(x)$ cannot be limit ordinal. Hence

$$c_F^\gamma(x) = \bigcup_{\delta<\gamma} c_F^\delta(x) = \bigcup_{\delta<\gamma} \left(c_F^{\delta+1}(x) \setminus c_F^\delta(x)\right) = \bigcup_{\delta<\gamma} C_\delta^- \cup \bigcup_{\delta<\gamma} C_\delta^+,$$

where

$$C_\delta^- = \left(c_F^{\delta+1}(x) \setminus c_F^\delta(x)\right) \cap (-\infty, x)$$

and

$$C_\delta^+ = \left(c_F^{\delta+1}(x) \setminus c_F^\delta(x)\right) \cap (x, \infty)$$

for each ordinal $\delta < \gamma$. Since $\left(c_F^\delta(x)\right)_{\delta<\gamma}$ forms a nested collection of order-convex sets, it follows that $c_F^\gamma(x)$ is order-isomorphic to

$$\left(\sum_{\delta\in\gamma} \left(C_\delta^-\right)^*\right)^* + \sum_{\delta\in\gamma} C_\delta^+.$$

Now by the inductive hypothesis, for each ordinal $\delta < \gamma$ both C_δ^- and C_δ^+ are suborderings of members of $\mathcal{S}$, which are easily seen by induction to lie in $\mathcal{S}$. Hence $c_F^\gamma(x) \in \mathcal{S}$, as required. □

Since our topological analogue of this result will concern countable spaces, we will only need the special case of this result in which X is countable. We now observe that in this case we in fact do not require ordinals any larger than ω.

Definition. Define $\mathcal{S}_\omega$ to be the smallest class of totally ordered sets subject to the following conditions.

- $\mathcal{S}_\omega$ is closed under isomorphism.
- $1 \in \mathcal{S}_\omega$.
- If $X \in \mathcal{S}_\omega$ then $X^* \in \mathcal{S}_\omega$.
- If $\beta \leq \omega$ is a non-zero ordinal and $X_i \in \mathcal{S}_\omega$ for all $i \in \beta$, then $\sum_{i\in\beta} X_i \in \mathcal{S}_\omega$.

Corollary 3.2.5. *$\mathcal{S}_\omega$ is the class of countable non-empty scattered totally ordered sets.*

We deduce this from Hausdorff's theorem, but it may also be proved directly [Ros82, Theorem 5.24].

Proof. Trivially every member of $\mathcal{S}_\omega$ is countable and $\mathcal{S}_\omega \subseteq \mathcal{S}$, so by Hausdorff's theorem every member of $\mathcal{S}_\omega$ is a countable non-empty scattered totally ordered set.

Conversely, if X is a countable non-empty scattered totally ordered set, then by Hausdorff's theorem $X \in \mathcal{S}$. Since X is countable, all the ordinals used in its construction may be assumed countable. But it is easy to see that every countable ordinal lies in $\mathcal{S}_\omega$, and hence using the same sequence of steps that witnesses this we see that $X \in \mathcal{S}_\omega$. □

3.3 One-point cofinite extensions

In this section we state and prove our topological analogue of Hausdorff's theorem. First we introduce some notation.

Definition. Let κ be a cardinal and let X_i be a topological space for all $i \in \kappa$. We use

$$\bigoplus_{i\in\kappa} X_i$$

to denote the topological disjoint union. We also denote $\bigoplus_{i\in\{0,1\}} X_i$ by $X_0 \oplus X_1$, which defines an associative binary operation. When there is no confusion we identify X_i with the corresponding subspace of the disjoint union.

We now introduce the other operation on topological spaces used in our result.

Definition. Let κ be an infinite cardinal and let X_i be topological spaces for all $i \in \kappa$. Assume for simplicity that the X_i are pairwise disjoint and $* \notin \bigcup_{i\in\kappa} X_i$. We define the *one-point cofinite extension* of $(X_i)_{i\in\kappa}$ by

$$\underset{i\in\kappa}{\circledast} X_i = \bigcup_{i\in\kappa} X_i \cup \{*\}$$

endowed with the topology generated by the base

$$\bigcup_{i\in\kappa} \{U \subseteq X_i : U \text{ is open}\} \cup \left\{ \bigcup_{i\in A} X_i \cup \{*\} : A \subseteq \kappa \text{ is cofinite} \right\}.$$

(The symbol $\circledast$ may be pronounced "starred sum".)

That this set is a base is an easy exercise.

The aim of this definition is to generalise the way in which ordinals are built up topologically. For example, $\circledast_{i\in\omega} \{0\} \cong [0,\omega]$ and $\circledast_{i\in\omega} [0,\omega] \cong [0,\omega^2]$.

Here is our topological analogue of $\mathcal{S}_\omega$, which we have chosen in this way in order for Theorem 3.3.1 to hold. Here, 1 is a one-element topological space.

Definition. Define $\mathcal{A}_\omega$ to be the smallest class of topological spaces subject to the following conditions.

- $\mathcal{A}_\omega$ is closed under homeomorphism.
- $1 \in \mathcal{A}_\omega$.
- If $\kappa \leq \omega$ is a non-zero cardinal and $X_i \in \mathcal{A}_\omega$ for all $i \in \kappa$, then $\bigoplus_{i\in\kappa} X_i \in \mathcal{A}_\omega$.
- If $X_i \in \mathcal{A}_\omega$ for all $i \in \omega$, then $\circledast_{i\in\omega} X_i \in \mathcal{A}_\omega$.

Here is our topological analogue of Hausdorff's theorem, which is ultimately much easier to prove than Hausdorff's theorem once it has been formulated appropriately.

Theorem 3.3.1. *Let X be a topological space. The following are equivalent.*

1. *X is homeomorphic to a countable non-empty set of ordinals under the subspace topology.*
2. *$X \in \mathcal{A}_\omega$.*

Proof. First let X be a countable non-empty set of ordinals under the subspace topology. We show by induction on the order type of X that $X \in \mathcal{A}_\omega$. Let $f : X \to [0, \alpha)$ be an order-isomorphism for some countable ordinal α. Clearly every finite discrete space lies in A_ω, so assume $\alpha \geq \omega$. Write $\alpha = \omega^\gamma \cdot m + \zeta$ with γ a non-zero ordinal, m a positive integer and $\zeta < \omega^\gamma$. If $m > 1$ or $\zeta > 1$ then both $f^{-1}([0, \omega^\gamma])$ and $f^{-1}([\omega^\gamma + 1, \alpha))$ have order type less than α, and by Theorem 3.1.2 X is homeomorphic to their topological disjoint union, so we are done by the inductive hypothesis. So we may assume that $m = 1$ and $\zeta \in \{0, 1\}$. Let $(\eta_i)_{i\in\omega}$ be a strictly increasing cofinal sequence of ordinals less than ω^γ, and write $\eta_{-1} = -1$. By Theorem 3.1.2, if $\zeta = 0$ then $X \cong \bigoplus_{i\in\omega} f^{-1}([\eta_{i-1} + 1, \eta_i])$, whereas if $\zeta = 1$ then either $X \cong \bigoplus_{i\in\omega} f^{-1}([\eta_{i-1} + 1, \eta_i]) \oplus \{*\}$ or $X \cong \circledast_{i\in\omega} f^{-1}([\eta_{i-1} + 1, \eta_i])$. In each case we are done by the inductive hypothesis.

Conversely, we show by induction on the definition of $\mathcal{A}_\omega$ that if $X \in \mathcal{A}_\omega$ then X is homeomorphic to a countable non-empty set of ordinals under the subspace topology. Certainly this is true of 1. Let $\kappa \leq \omega$ be a non-zero cardinal and suppose that for all $i \in \kappa$, X_i is homeomorphic to a countable non-empty set of ordinals under the subspace topology. As we remarked earlier, by the proof of Theorem 3.1.2 we may assume that in fact $X_i \subseteq [0, \alpha_i)$ with α_i a countable ordinal for each $i \in \kappa$. But then $\bigoplus_{i\in\kappa} X_i$ is immediately seen to be homeomorphic to a subspace of $\bigoplus_{i\in\kappa} [0, \alpha_i] \cong \left[0, \sum_{i\in\kappa} (\alpha_i + 1)\right)$. Finally suppose that $\kappa = \omega$. Then $\circledast_{i\in\omega} X_i$ is likewise homeomorphic to a subspace of $\circledast_{i\in\omega} [0, \alpha_i] \cong \left[0, \sum_{i\in\omega} (\alpha_i + 1)\right]$. $\square$

3.4 A characterisation of countable subspaces of ordinals

In this section we prove our main result of this chapter.

Theorem 3.4.1. *A topological space X is homeomorphic to a countable set of ordinals under the subspace topology if and only if X is homeomorphic to a countable scattered totally ordered set under the order topology.*

This result itself does not seem to appear in the literature, though there are several more general results that almost imply it. In the "only if" direction, Purisch [Pur85] showed that any GO-space whose topology is scattered is homeomorphic to a LOTS. A shorter proof for the special case of subspaces of ordinals was given by Gutev [Gut14], and contains some similar ideas to our proof. In the "if" direction, Telgarsky [Tel68, Theorem 9] showed that any scattered metrisable space is homeomorphic to a subspace of an ordinal.

Since the class of spaces homeomorphic to a countable set of ordinals under the subspace topology is itself closed under taking subspaces, we immediately obtain the following.

Corollary 3.4.2. *Every subspace of the order topology on a countable scattered totally ordered set is itself homeomorphic to the order topology on some countable scattered totally ordered set.*

In fact, the proof of Purisch's result [Pur85] should show that the class of scattered spaces homeomorphic to a LOTS is closed under taking subspaces, which would generalise this result.

By Corollary 3.2.5 and Theorem 3.3.1, we may restate Theorem 3.4.1 as follows.

Theorem 3.4.3. *Let X be a topological space. The following are equivalent.*

1. *X is homeomorphic to some member of $\mathcal{S}_\omega$ endowed with the order topology.*
2. *$X \in \mathcal{A}_\omega$.*

We prove this result, and thereby Theorem 3.4.1 and Corollary 3.4.2, by showing that there is a correspondence between the definitions of S_ω and A_ω.

Proof of Theorem 3.4.3. We show by induction on the definition of $\mathcal{S}_\omega$ that if $X \in \mathcal{S}_\omega$ is endowed with the order topology, then $X \in \mathcal{A}_\omega$. Clearly $1 \in \mathcal{A}_\omega$ and if $X \in \mathcal{S}_\omega \cap \mathcal{A}_\omega$ then $X^* \in \mathcal{A}_\omega$. We claim that if $X, Y \in \mathcal{S}_\omega \cap \mathcal{A}_\omega$ then $X + Y \in \mathcal{A}_\omega$. Given the claim, suppose $\beta \leq \omega$ is a non-zero ordinal and $X_i \in \mathcal{S}_\omega \cap \mathcal{A}_\omega$ for all $i \in \beta$. If $\beta < \omega$, then by the claim $\sum_{i \in \beta} X_i \in \mathcal{A}_\omega$ by induction. If $\beta = \omega$, then by the claim we may assume $|X_i| \geq 3$ for all $i \in \omega$ by replacing X_i with $X_{3i} + X_{3i+1} + X_{3i+2}$ for all $i \in \omega$. Then for each $i \in \omega$, pick $x_i \in X_i$ as in Corollary 3.2.2 and let $X_i^- = \{x \in X_i : x \leq x_i\}$ and $X_i^+ = \{x \in X_i : x > x_i\}$. Then

$$X \cong X_0^- \oplus \bigoplus_{i \in \omega} \left(X_i^+ + X_{i+1}^-\right)$$

and we are done.

To prove the claim, suppose $X, Y \in \mathcal{S}_\omega \cap \mathcal{A}_\omega$. If neither X has a maximal element nor Y has a minimal element, or both X has a maximal element and Y has a minimal element, then $X + Y \cong X \oplus Y$ and we are done. So by symmetry, we may assume without loss of generality that X has a maximal element z and Y has no minimal element. Let $\widetilde{X} = \{x \in X : x < z\}$. If $\widetilde{X}$ has a maximal element, then let $X_0 = \widetilde{X}$ and $X_n = \emptyset$ for all $n \in \omega \setminus \{0\}$. Otherwise by repeatedly applying Corollary 3.2.2 we can find a strictly increasing cofinal sequence $(x_n)_{n\in\omega}$ from $\widetilde{X}$ such that $(-\infty, x_n)$ has a maximal element for all $n \in \omega$. Then let $X_0 = (-\infty, x_0)$ and $X_n = [x_{n-1}, x_n)$ for all $n \in \omega \setminus \{0\}$. Likewise we can find a strictly decreasing coinitial sequence $(y_n)_{y\in\omega}$ from Y such that (y_n, ∞) has a minimal element for all $n \in \omega$. Then let $Y_0 = (y_0, \infty)$ and $Y_n = (y_n, y_{n-1}]$ for all $n \in \omega \setminus \{0\}$. Finally let $Z_{2n} = X_n$ and $Z_{2n+1} = Y_n$ for all $n \in \omega$. Then $X + Y \cong \circledast_{n\in\omega} Z_n$ and we are done.

Conversely, we show by induction on the definition of $\mathcal{A}_\omega$ that if $X \in \mathcal{A}_\omega$ then X is homeomorphic to some member of $\mathcal{S}_\omega$ endowed with the order topology. Certainly this is true of 1. Suppose $\kappa \leq \omega$ is a non-zero cardinal and $X_i \in \mathcal{A}_\omega$ is homeomorphic to some member of $\mathcal{S}_\omega$ for all $i \in \kappa$. We claim that for all $i \in \kappa$, X_i is homeomorphic to some member of $\mathcal{S}_\omega$ with a minimal element. Given the claim, we may assume that $X_i \in \mathcal{S}_\omega$ has a minimal element for all $i \in \kappa$. It remains to "match up the ends" of these orderings in such a way as to avoid placing a maximal element before an ordering with no minimal element or a minimal element after an ordering with no maximal element. To do this, write $\{X_i : i \in \kappa\} = \{V_i : i \in \lambda\} \cup \{W_i : i \in \mu\}$ where V_i has a maximal element for all $i \in \lambda$ and W_i has no maximal element for all $i \in \mu$. If λ is finite, then let

$$Z_i = \begin{cases} V_i, & \text{if } i < \lambda \\ W_j, & \text{if } i = \lambda + j \text{ for some even } j < \mu \\ W_j^*, & \text{if } i = \lambda + j \text{ for some odd } j < \mu \end{cases}$$

for all $i \in \kappa$. Otherwise, if μ is finite then let

$$Z_i = \begin{cases} W_i, & \text{if } i < \mu \text{ and } \mu - i \text{ is even} \\ W_i^*, & \text{if } i < \mu \text{ and } \mu - i \text{ is odd} \\ V_j, & \text{if } i = \mu + j \text{ for some } j < \lambda \end{cases}$$

for all $i \in \kappa$. Finally, if $\lambda = \mu = \omega$ then let

$$Z_i = \begin{cases} V_j, & \text{if } i = 3j \text{ for some } j < \omega \\ W_j, & \text{if } i = 3j+1 \text{ for some } j < \omega \\ W_j^*, & \text{if } i = 3j+2 \text{ for some } j < \omega \end{cases}$$

for all $i \in \omega$. Then in each case, $\sum_{i \in \kappa} Z_i \cong \bigoplus_{i \in \kappa} Z_i \cong \bigoplus_{i \in \kappa} X_i$, and if $\kappa = \omega$ then $\left(\sum_{i \in \omega} Z_i\right) + 1 \cong \circledast_{i \in \omega} Z_i \cong \circledast_{i \in \omega} X_i$. Thus we are done.

To prove the claim, let $X \in \mathcal{S}_\omega$ and suppose that X has no minimal element. If X has a maximal element, then $X \cong X^*$ and we are done. Otherwise X has neither a minimal element nor a maximal element. Certainly $|X| \geq 3$, so pick $x \in X$ as in Corollary 3.2.2 and let $X^- = (-\infty, x)$ and $X^+ = [x, \infty)$. Then $X = X^- + X^+ \cong X^- \oplus X^+ \cong X^+ + X^-$ and we are done. □

We conclude this section with the following observation, which was previously made by Baker [Bak72, p. 25]. This demonstrates that there is a scattered total ordering whose order topology is not homeomorphic to a set of ordinals under the subspace topology. Thus in spite of the results of Purisch and others that apply to spaces of arbitrary cardinality, it appears that our techniques are confined to the countable.

Proposition 3.4.4. *$\omega_1 + 1 + \omega^*$ is not homeomorphic to a set of ordinals under the subspace topology.*

Proof. Observe that in $\omega_1 + \{*\} + \omega^*$, $*$ does not have a countable neighbourhood base but does lie in the closure of a disjoint countable set. On the other hand, if X is a set of ordinals and $x \in X$ does not have a countable neighbourhood base then x has uncountable cofinality and hence does not lie in the closure of a disjoint countable set. □

3.5 Subspaces of ordinals less than ω^ω

In this section we use ideas from our topological analogue of Hausdorff's theorem to prove the following result.

Theorem 3.5.1. *Let n be a positive integer. Then there are countably many subspaces of $[0, \omega^n]$ up to homeomorphism.*

This result is slightly less trivial than it first appears, and ultimately boils down to the fact that for each positive integer n, there is a finite set of "building blocks", namely a finite set of topological spaces such that any subspace of $[0, \omega^n]$ is homeomorphic to a finite topological disjoint union of spaces from that set. Our formulation of $\mathcal{A}_\omega$ provides us with the language to describe such a set of topological spaces somewhat explicitly. In order to do this, let us introduce the finitary special cases of the infinitary operations used to define $\mathcal{A}_\omega$.

Definition. Given a finite sequence $X_0, X_2, \ldots, X_{k-1}$ of topological spaces, we write $[X_0, X_1, \ldots, X_{k-1}]_\omega$ for the sequence $(Y_i)_{i\in\omega}$ defined by

$$Y_i = X_{i \bmod k}$$

for all $i \in \omega$, where $i \bmod k$ denotes the reduction of i modulo k. We then write $\bigoplus [X_0, X_1, \ldots, X_{k-1}]_\omega$ for $\bigoplus_{i\in\omega} Y_i$ and $\circledast [X_0, X_1, \ldots, X_{k-1}]_\omega$ for $\circledast_{i\in\omega} Y_i$.

For example, $\circledast [[0,\omega]]_\omega \cong [0,\omega^2]$ and $\circledast [[0,\omega), [0,\omega]]_\omega \cong [0,\omega^2] \setminus \{\omega \cdot (2m+1) : m \in \omega\}$.

By replacing the infinitary operations in the definition of $\mathcal{A}_\omega$ by their finitary special cases, we obtain a subclass of $\mathcal{A}_\omega$. We will soon see that this subclass corresponds to the subspaces of ordinals less than ω^ω.

Definition. Define $\mathcal{A}_{<\omega}$ to be the smallest class of topological spaces subject to the following conditions.

- $\mathcal{A}_{<\omega}$ is closed under homeomorphism.
- $1 \in \mathcal{A}_{<\omega}$.
- If $X_0, \ldots, X_{k-1} \in \mathcal{A}_{<\omega}$, then $\bigoplus_{i\in\{0,1,\ldots,k-1\}} X_i \in \mathcal{A}_{<\omega}$.
- If $X_0, \ldots, X_{k-1} \in \mathcal{A}_{<\omega}$, then $\bigoplus [X_0, \ldots, X_{k-1}]_\omega \in \mathcal{A}_{<\omega}$ and $\circledast [X_0, \ldots, X_{k-1}]_\omega \in \mathcal{A}_{<\omega}$.

We will also use the following closely-related subclass of $\mathcal{A}_\omega$.

Definition. Define $\widehat{\mathcal{A}}_{<\omega}$ in the same way as $\mathcal{A}_{<\omega}$ but without non-empty finite topological disjoint unions. In other words, define $\widehat{\mathcal{A}}_{<\omega} = \bigcup_{n\in\omega} \widehat{\mathcal{A}}_n$, where $\widehat{\mathcal{A}}_0 = \{X : X \cong 1\}$, and for $n \in \omega$,

$$\widehat{\mathcal{A}}_{n+1} = \widehat{\mathcal{A}}_n \cup \Big\{X : X \cong \bigoplus [X_0, \ldots, X_{k-1}]_\omega \text{ or } X \cong \circledast [X_0, \ldots, X_{k-1}]_\omega \text{ for some } X_0, \ldots, X_{k-1} \in \widehat{\mathcal{A}}_n\Big\}.$$

Note that no transfinite recursion is required in this definition, since if $X_0, \ldots, X_{k-1} \in \widehat{\mathcal{A}}_{<\omega}$ then $X_0, \ldots, X_{k-1} \in \widehat{\mathcal{A}}_n$ for some fixed $n \in \omega$.

For each positive integer n, the $\cong$-equivalence classes of $\widehat{\mathcal{A}}_n$ will correspond to our "building blocks". We now make the simple observation that there are only finitely many such equivalence classes.

Lemma 3.5.2. *Let n be a positive integer. Then $\widehat{\mathcal{A}}_n$ has only finitely many $\cong$-equivalence classes.*

Proof. Observe that if $X_0, X_1, \ldots, X_{k-1}$ are topological spaces, then the homeomorphism types of $\bigoplus [X_0, X_1, \ldots, X_{k-1}]_\omega$ and $\circledast [X_0, X_1, \ldots, X_{k-1}]_\omega$ depend only on which homeomorphism types are among $X_0, X_1, \ldots, X_{k-1}$. The result is then clear by induction on n, since if S is a finite set then the set of subsets of S is also finite. $\square$

The following result effectively shows that any subspace of $[0, \omega^n]$ is homeomorphic to a finite topological disjoint union of "building blocks".

Proposition 3.5.3. *Let X be a non-empty topological space. The following are equivalent.*

1. *X is homeomorphic to a finite topological disjoint union of members of $\widehat{\mathcal{A}}_{<\omega}$.*
2. *$X \in \mathcal{A}_{<\omega}$.*
3. *X is homeomorphic to a subspace of $[0, \omega^n]$ for some positive integer n.*

Proof. That $1 \implies 2$ is trivial.

To prove that $2 \implies 3$, we show by induction on the definition of $\mathcal{A}_{<\omega}$ that if $X \in \mathcal{A}_{<\omega}$ then X is homeomorphic to a subspace of $[0, \omega^n]$ for some positive integer n. Certainly this is true of 1, so suppose $X_0, X_1, \ldots, X_{k-1} \in \mathcal{A}_{<\omega}$ are such that X_i is homeomorphic to a subspace of $[0, \omega^n]$ for some positive integer n for all $i \in \{0, 1, \ldots, k-1\}$. Then there is some fixed positive integer n such that X_i is homeomorphic to a subspace of $[0, \omega^n]$ for all $i \in \{0, 1, \ldots, k-1\}$. Hence as in Theorem 3.3.1, $\bigoplus_{i \in \{0,1,\ldots,k-1\}} X_i$ is homeomorphic to a subspace of $[0, \omega^n \cdot k)$, $\bigoplus [X_0, \ldots, X_{k-1}]_\omega$ is homeomorphic to a subspace of $[0, \omega^{n+1})$ and $\circledast [X_0, \ldots, X_{k-1}]_\omega$ is homeomorphic to a subspace of $[0, \omega^{n+1}]$.

To prove that $3 \implies 1$, we show that for a fixed positive integer n, if X is a non-empty subspace of $[0, \omega^n]$, then X is homeomorphic to a finite topological disjoint union of members of $\widehat{\mathcal{A}}_n$. The proof is by induction on n. For the case $n = 1$, simply observe that every subspace of $[0, \omega]$ is homeomorphic to $[0, \omega)$, $[0, \omega]$ or a finite discrete space. For the inductive step, suppose $n > 1$ and let X be a non-empty subspace of $[1, \omega^n] \cong [0, \omega^n]$. For each $i \in \omega$ let $Y_i = X \cap [\omega^{n-1} \cdot i + 1, \omega^{n-1} \cdot (i+1)]$. Then by the inductive hypothesis either $Y_i = \emptyset$ or Y_i is homeomorphic to a finite topological disjoint union of members of $\widehat{\mathcal{A}}_{n-1}$ for all $i \in \omega$. If $Y_i = \emptyset$ for all but finitely many $i \in \omega$, then X is homeomorphic to a finite topological disjoint union of a finite topological disjoint union of members of $\widehat{\mathcal{A}}_{n-1}$ (and possibly the point ω^n), and we are done. Otherwise by dividing up those non-empty Y_i for $i \in \omega$ into members of $\widehat{\mathcal{A}}_{n-1}$ and relabelling if necessary, we may assume $Y_i \in \widehat{\mathcal{A}}_{n-1}$ for all $i \in \omega$. Then by Lemma 3.5.2 there exist finitely many pairwise non-homeomorphic topological spaces $X_0, X_1, \ldots, X_{l-1} \in \widehat{\mathcal{A}}_{n-1}$ such that for all $i \in \omega$, $Y_i \cong X_j$ for some $j \in \{0, 1, \ldots, l-1\}$. Without loss of generality the spaces that are homeomorphic to infinitely many of the Y_i are $X_0, X_1, \dots, X_{k-1}$ for some $k \in \{1, 2, \dots, l\}$. Finally let m be maximal such that Y_{m-1} is not homeomorphic to any of $X_k, \dots, X_{l-1}$. Then

$$X \cong \begin{cases} Y_0 \oplus \cdots \oplus Y_{m-1} \oplus \bigoplus \left[X_0, X_1, \dots, X_{k-1}\right]_\omega, & \text{if } \omega^n \notin X \\ Y_0 \oplus \cdots \oplus Y_{m-1} \oplus \circledast \left[X_0, X_1, \dots, X_{k-1}\right]_\omega, & \text{if } \omega^n \in X \end{cases}$$

and we are done. □

Theorem 3.5.1 now follows immediately.

Proof of Theorem 3.5.1. By the equivalence of 1 and 3 of Proposition 3.5.3, it is enough to prove that $\widehat{\mathcal{A}}_{<\omega}$ has countably many $\cong$-equivalence classes. But this is immediate from Lemma 3.5.2 by definition of $\widehat{\mathcal{A}}_{<\omega}$. □

Note that the proof of Lemma 3.5.2 provides us with an explicit finite upper bound on the number of $\cong$-equivalence classes of $\widehat{\mathcal{A}}_n$. However, our proof makes no attempt to optimise this upper bound. Indeed there are various homeomorphisms between spaces whose explicit definitions differ, such as $\bigoplus \left[[0, \omega)\right]_\omega \cong [0, \omega)$ and $\circledast \left[\{0\}, [0, \omega]\right]_\omega \cong \circledast \left[[0, \omega]\right]_\omega$. More generally, for any $X_0, X_1, \dots, X_{k-1}, Y_0, Y_1, \dots, Y_{l-1} \in \widehat{\mathcal{A}}_{<\omega}$:

- $\bigoplus \left[\bigoplus \left[Y_0, Y_1, \dots, Y_{l-1}\right]_\omega, X_1, \dots, X_{k-1}\right]_\omega \cong \bigoplus \left[Y_0, Y_1, \dots, Y_{l-1}, X_1, \dots, X_{k-1}\right]_\omega$; and

- if Y_0 is used in the explicit definition of X_0, then $\bigoplus \left[Y_0, X_0, X_1, \dots, X_{k-1}\right]_\omega \cong \bigoplus \left[X_0, X_1, \dots, X_{k-1}\right]_\omega$ and $\circledast \left[Y_0, X_0, X_1, \dots, X_{k-1}\right]_\omega \cong \circledast \left[X_0, X_1, \dots, X_{k-1}\right]_\omega$.

It may be an interesting problem to determine the exact number of $\cong$-equivalence classes of $\widehat{\mathcal{A}}_n$. We conjecture that the only homeomorphisms between spaces whose explicit definitions differ are consequences of the above identities.

Note also that we showed in the proof of Proposition 3.5.3 that every subspace of $[0, \omega^n]$ is homeomorphic to a finite topological disjoint union of members of $\widehat{\mathcal{A}}_n$. Thus Lemma 3.5.2 does indeed provide us with a finite set of "building blocks", though there seems to be no reason to think that it is the smallest such set. We therefore propose the related problem of determining the smallest size of such a set of "building blocks".

Question 3.5.4. Let n be a positive integer. What is the smallest size of a set A of topological spaces such that every subspace of $[0, \omega^n]$ is homeomorphic to a finite topological disjoint union of members of A?

3.6 The number of subspaces of an ordinal up to homeomorphism

In this section we will see that, in contrast to Theorem 3.5.1, the number of subspaces of an ordinal greater than or equal to ω^ω up to homeomorphism is as large as it could have been.

Let us illustrate how one may go about constructing non-homeomorphic subspaces of ordinals with a simple example. Let $X = [0, \omega^2] \setminus \{\omega \cdot m : m \in \omega\}$. We show that X is not homeomorphic to $[0, \alpha)$ for any ordinal α. Recall that we say a topological space is *locally compact* to mean that every point has a compact neighbourhood. By Lemma 2.2.3, $[0, \alpha)$ is locally compact for any ordinal α. On the other hand, X is not locally compact, since if U is any neighbourhood of X containing the point ω^2, then $[\omega \cdot m, \omega \cdot (m + 1))$ is a closed subset of U for some $m \in \omega$, but it is not compact.

Putting this idea together with the Cantor–Bendixson derivative gives us a simple way to construct large numbers of pairwise non-homeomorphic subspaces of ordinals.

Theorem 3.6.1. *There is a collection of $2^{\aleph_0}$ pairwise non-homeomorphic subspaces of $[0, \omega^\omega)$.*

Proof. For each $I \subseteq \omega$, let

$$A_I = \bigcup_{i \in I} \{3i, 3i + 1\} \cup \bigcup_{i \in \omega \setminus I} \{3i, 3i + 2\},$$

and let

$$X_I = \{x \in [0, \omega^\omega) : \mathrm{CB}(x) \in A_I\}.$$

Observe that for all $i \in \omega$,

$$X_I^{(2i)} \setminus X_I^{(2i+2)} = \begin{cases} \{x \in [0, \omega^\omega) : \mathrm{CB}(x) \in \{3i, 3i + 1\}\}, & \text{if } i \in I \\ \{x \in [0, \omega^\omega) : \mathrm{CB}(x) \in \{3i, 3i + 2\}\}, & \text{if } i \notin I. \end{cases}$$

Thus $i \in I$ if and only if $X_I^{(2i)} \setminus X_I^{(2i+2)}$ is locally compact. It follows that $(X_I)_{I \subseteq \omega}$ is as required. $\square$

A similar argument works for every uncountable cardinal, which in this context should be thought of as an initial ordinal.

Theorem 3.6.2. *Let κ be an uncountable cardinal. Then there is a collection of 2^κ pairwise non-homeomorphic subspaces of $[0, \kappa)$.*

Proof. First note that since κ is uncountable, $\kappa = \omega^{\omega \cdot \kappa}$ (using ordinal arithmetic).

Now for each $S \subseteq \kappa$, let

$$A_S = \bigcup_{\alpha \in S} [\omega \cdot \alpha, \omega \cdot (\alpha + 1)) \cup \bigcup_{\alpha \in \kappa \setminus S} ([\omega \cdot \alpha, \omega \cdot (\alpha + 1)) \setminus \{\omega \cdot \alpha + 1\}),$$

and let

$$X_S = \{x \in [0, \kappa) : \mathrm{CB}(x) \in A_S\}.$$

Observe that for all $\alpha \in \kappa$,

$$X_S^{(\omega \cdot \alpha)} \setminus X_S^{(\omega \cdot \alpha + 2)} = \begin{cases} \{x \in [0, \kappa) : \mathrm{CB}(x) \in \{\omega \cdot \alpha, \omega \cdot \alpha + 1\}\}, & \text{if } \alpha \in S \\ \{x \in [0, \kappa) : \mathrm{CB}(x) \in \{\omega \cdot \alpha, \omega \cdot \alpha + 2\}\}, & \text{if } \alpha \notin S. \end{cases}$$

Thus $\alpha \in S$ if and only if $X_S^{(\omega \cdot \alpha)} \setminus X_S^{(\omega \cdot \alpha + 2)}$ is locally compact. It follows that $(X_S)_{S \subseteq \kappa}$ is as required. $\square$

With these constructions we have completed the calculation of the number of subspaces of an arbitrary ordinal up to homeomorphism.

Theorem 3.6.3. *Let α be an ordinal. Then the number of $\cong$-equivalence classes of subspaces of $[0, \alpha)$ is*

$$\begin{cases} \alpha + 1, & \textit{if } \alpha < \omega \\ \aleph_0, & \textit{if } \omega \leq \alpha < \omega^\omega \\ 2^{\aleph_0}, & \textit{if } \omega^\omega \leq \alpha < \omega_1 \\ 2^\kappa, & \textit{if } |\alpha| = \kappa \textit{ for some uncountable cardinal } \kappa. \end{cases}$$

Proof. The case $\alpha < \omega$ is trivial, and the remaining cases are Theorems 3.5.1, 3.6.1 and 3.6.2 respectively. $\square$

3.7 One-point F-extensions

We conclude this chapter by generalising the one-point cofinite extension operation on topological spaces to arbitrary filters, as indicated by the following observation. We adopt the convention that a filter on a set is allowed to contain $\emptyset$, and say that a filter F is *proper* to mean that $\emptyset \notin F$.

Lemma 3.7.1. *Let κ be a cardinal and let $(X_i)_{i \in \kappa}$ be a collection of pairwise disjoint topological spaces with $* \notin \bigcup_{i \in \kappa} X_i$. Let $X = \bigcup_{i \in \kappa} X_i \cup \{*\}$. Let F be a collection of subsets of κ and let*

$$\tau = \{U \subseteq X : U \cap X_i \textit{ is open in } X_i \textit{ and if } * \in U \textit{ then } \{i \in \kappa : U \cap X_i = X_i\} \in F\}.$$

Then τ is a topology on X if and only if F is a (not necessarily proper) filter on κ.

Proof. The following easily-checked observations are sufficient.

- $\emptyset \in \tau$.
- $X \in \tau$ if and only if $\kappa \in F$.
- τ is closed under binary intersections if and only if F is closed under binary intersections.
- τ is closed under arbitrary unions if and only if F is upward-closed. □

Out of this this result comes the following definition.

Definition. Let κ be a cardinal and F be a filter on κ.

Given $(X_i)_{i\in\kappa}$ as in the above result, we define the *one-point F-extension* of $(X_i)_{i\in\kappa}$ by

$$\overset{F}{\underset{i\in\kappa}{\circledast}} X_i \;=\; \bigcup_{i\in\kappa} X_i \;\cup\; \{*\}$$

endowed with the topology τ from Lemma 3.7.1.

Thus the one-point cofinite extension $\circledast_{i\in\omega} X_i$ is the one-point F-extension $\circledast^F_{i\in\omega} X_i$ where F is taken to be the cofinite filter on ω.

One-point F-extensions allow for the construction of some unusual topological spaces. We do not have much to say about one-point F-extensions in general, but we do make the curious observation that various properties of topological spaces correspond to properties of filters, in the following sense.

Definition. Given a property P of topological spaces, we say that F *preserves* P to mean that if X_i has property P for all $i \in \kappa$ then $\circledast^F_{i\in\kappa} X_i$ has property P.

We say that a topological space is *extremally disconnected* to mean that the closure of every open set is open.

Proposition 3.7.2. *Let κ be a cardinal and F be a filter on κ.*

1. *F preserves Hausdorffness if and only if F contains the cofinite filter.*
2. *F preserves compactness if and only if F is contained in the cofinite filter.*
3. *F preserves extremal disconnectedness if and only if F is either improper or an ultrafilter.*
4. *F preserves second-countability if and only if κ is countable and F has a countable base.*
5. *F preserves regularity if and only if F contains the cofinite filter.*
6. *F preserves discreteness if and only if F is the improper filter.*

7. *F preserves total disconnectedness if and only if F contains the cofinite filter.*

8. *F preserves scatteredness.*

It is straightforward to extend this list to other properties of topological spaces. We omit the proof of this result since it is a routine rearrangement of definitions.

Chapter 4

The topological pigeonhole principle for ordinals

In this chapter we re-introduce the topological and closed partition relations for ordinals, explore the relationship between them, and discuss other generalities. We then compute the topological and closed pigeonhole numbers of an arbitrary sequence of ordinals.

4.1 Partition relation notation

Let us briefly recap the classical, topological and closed partition relations for ordinals. Recall that we say a subspace X of an ordinal is order-homeomorphic to an ordinal α to mean that there is a bijection $X \to \alpha$ that is both an order-isomorphism and a homeomorphism. By Proposition 2.2.4, this is equivalent to saying that X is both order-isomorphic to α and internally closed, justifying our use of the term "closed".

Definition. Let κ be a cardinal, let n be a positive integer, and let β and all α_i be ordinals for $i \in \kappa$.

We write

$$\beta \to (\alpha_i)^n_{i \in \kappa}$$

to mean that for every function $c : [\beta]^n \to \kappa$ there exists some subset $X \subseteq \beta$ and some $i \in \kappa$ such that X is *an i-homogeneous copy of* α_i, i.e., $[X]^n \subseteq c^{-1}(\{i\})$ and X is order-isomorphic to α_i.

We write

$$\beta \to_{top} (\alpha_i)^n_{i \in \kappa}$$

to mean that for every function $c : [\beta]^n \to \kappa$ there exists some subspace $X \subseteq \beta$ and some $i \in \kappa$ such that X is an *i-homogeneous topological copy of* α_i, i.e., $[X]^n \subseteq c^{-1}(\{i\})$ and $X \cong \alpha_i$.

We write

$$\beta \to_{cl} (\alpha_i)^n_{i \in \kappa}$$

to mean that for every function $c : [\beta]^n \to \kappa$ there exists some subset $X \subseteq \beta$ and some $i \in \kappa$ such that X is an *i-homogeneous closed copy of* α_i, i.e., $[X]^n \subseteq c^{-1}(\{i\})$ and X is order-homeomorphic to α_i.

The function c in these definitions will often be referred to as a *colouring*, and we may say that x *is coloured with* i simply to mean that $c\,(x) = i$. We write $\beta \to (\alpha)^n_\kappa$ for $\beta \to (\alpha_i)^n_{i\in\kappa}$ when $\alpha_i = \alpha$ for all $i \in \kappa$, and similarly for the topological and closed relations.

Although the closed relation may appear more natural than the topological relation, it is the topological one that has been considered historically, since it can be defined for arbitrary topological spaces. Moreover, the two relations are closely related, as we shall see in Section 4.4.

In this chapter we will be concerned exclusively with the case $n = 1$. In this case we work with β rather than $[\beta]^1$ for simplicity. Clearly if $\beta \to (\alpha_i)^n_{i\in\kappa}$ and $\gamma > \beta$ then $\gamma \to (\alpha_i)^n_{i\in\kappa}$, and similarly for the other relations. Thus it is sensible to make the following definition.

Definition. Let κ be a cardinal and let α_i be an ordinal for each $i \in \kappa$.

We define the (*classical*) *pigeonhole number* $P\,(\alpha_i)_{i\in\kappa}$ to be the least ordinal β such that $\beta \to (\alpha_i)^1_{i\in\kappa}$, the *topological pigeonhole number* $P^{top}\,(\alpha_i)_{i\in\kappa}$ to be the least ordinal β such that $\beta \to_{top} (\alpha_i)^1_{i\in\kappa}$, and the *closed pigeonhole number* $P^{cl}\,(\alpha_i)_{i\in\kappa}$ to be the least ordinal β such that $\beta \to_{cl} (\alpha_i)^1_{i\in\kappa}$.

We extend the usual ordering on the ordinals to include ∞ as a maximum. If there is no ordinal β such that $\beta \to (\alpha_i)^1_{i\in\kappa}$, then we say that $P\,(\alpha_i)_{i\in\kappa}$ does not exist and write $P\,(\alpha_i)_{i\in\kappa} = \infty$, and similarly for the topological and closed pigeonhole numbers.

Thus for example if $n_1, n_2, \ldots, n_k$ are positive integers, then

$$P\,(n_1, n_2, \ldots, n_k) = P^{top}\,(n_1, n_2, \ldots, n_k) = P^{cl}\,(n_1, n_2, \ldots, n_k) = \sum_{i=1}^{k} (n_i - 1) + 1.$$

Note that $P\left((\alpha_i)_{i\in\kappa}, (1)_\lambda\right) = P\,(\alpha_i)_{i\in\kappa}$ for any cardinal λ, and that for fixed κ, $P\,(\alpha_i)_{i\in\kappa}$ is a monotonically increasing function of $(\alpha_i)_{i\in\kappa}$ (pointwise), and similarly for the topological and closed pigeonhole numbers. Note also that the closed partition relation implies the other two, and hence $P\,(\alpha_i)_{i\in\kappa} \leq P^{cl}\,(\alpha_i)_{i\in\kappa}$ and $P^{top}\,(\alpha_i)_{i\in\kappa} \leq P^{cl}\,(\alpha_i)_{i\in\kappa}$.

4.2 The classical pigeonhole principle for ordinals

The classical pigeonhole numbers were computed by Milner and Rado [MR65]. In our calculation of the topological and closed pigeonhole numbers, we will use the special case of their result for a finite sequence of countable ordinals. This uses the natural sum, due to Hessenberg, which may be thought of as the base-ω sum of ordinals in Cantor normal form.

Definition. Let α and β be ordinals. Then we may choose a sequence of ordinals $\gamma_1 > \gamma_2 > \cdots > \gamma_n$ and $l_1, l_2, \ldots, l_n, m_1, m_2, \ldots, m_n \in \omega$ such that

$$\alpha = \omega^{\gamma_1} \cdot l_1 + \omega^{\gamma_2} \cdot l_2 + \cdots + \omega^{\gamma_n} \cdot l_n$$

and

$$\beta = \omega^{\gamma_1} \cdot m_1 + \omega^{\gamma_2} \cdot m_2 + \cdots + \omega^{\gamma_n} \cdot m_n.$$

We define the *natural sum* of α and β by

$$\alpha \# \beta = \omega^{\gamma_1} \cdot (l_1 + m_1) + \omega^{\gamma_2} \cdot (l_2 + m_2) + \cdots + \omega^{\gamma_n} \cdot (l_n + m_n).$$

As part of their computation of the classical pigeonhole numbers, Milner and Rado introduced a related binary operation on ordinals.

Definition. Let α and β be ordinals. Then we define the *Milner–Rado sum* of α and β, denoted by $\alpha \odot \beta$, to be the least ordinal δ such that if $\widetilde{\alpha} < \alpha$ and $\widetilde{\beta} < \beta$ then $\delta \neq \widetilde{\alpha} \# \widetilde{\beta}$.

Milner and Rado [MR65] observed that if $\zeta > \alpha \odot \beta$, $\widetilde{\alpha} < \alpha$ and $\widetilde{\beta} < \beta$, then $\zeta \neq \widetilde{\alpha} \# \widetilde{\beta}$. They also observed that both $\#$ and $\odot$ are commutative and associative, and so brackets may be omitted when three or more ordinals are summed. Notice that if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are ordinals, then $\alpha_1 \odot \alpha_2 \odot \cdots \odot \alpha_k$ is simply the least ordinal δ such that if $\widetilde{\alpha}_i < \alpha_i$ for all $i \in \{1, 2, \ldots, k\}$ then $\delta \neq \widetilde{\alpha}_1 \# \widetilde{\alpha}_2 \# \cdots \# \widetilde{\alpha}_k$.

The relevance of this operation is given by the following result [MR65, Theorem 8].

Theorem 4.2.1 (Milner–Rado)**.** *Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be non-zero ordinals. Then*

$$P(\alpha_1, \alpha_2, \ldots, \alpha_k) = \alpha_1 \odot \alpha_2 \odot \cdots \odot \alpha_k.$$

We omit the proof of this result, but it is essentially the statement that $\alpha_1 \# \alpha_2 \# \cdots \# \alpha_k$ is the largest ordinal that may be written as a disjoint union of k sets of order types $\alpha_1, \alpha_2, \ldots, \alpha_k$ respectively. This follows from the fact that $\omega^\gamma \to (\omega^\gamma)^1_k$ for all ordinals γ, which may be proved by induction on γ.

Milner and Rado also computed the Milner–Rado sum of an arbitrary finite sequence of ordinals in terms of Cantor normal forms [MR65, Theorem 9].

Theorem 4.2.2 (Milner–Rado)**.** *Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be non-zero ordinals. We may choose a sequence of ordinals $\gamma_1 > \gamma_2 > \cdots > \gamma_N$ and, for each $i \in \{1, 2, \ldots, k\}$, $m_{i1}, m_{i2}, \ldots, m_{in_i} \in \omega$ such that for each $i \in \{1, 2, \ldots, k\}$, $m_{in_i} > 0$ and*

$$\alpha_i = \omega^{\gamma_1} \cdot m_{i1} + \omega^{\gamma_2} \cdot m_{i2} + \cdots + \omega^{\gamma_{n_i}} \cdot m_{in_i}.$$

Let $n = \min\{n_1, n_2, \dots, n_k\}$ and let $s_j = \sum_{i=1}^{k} m_{ij}$ for each $j \in \{1, 2, \dots, n\}$. Finally let $t = |\{i \in \{1, 2, \dots, k\} : n_i = n\}|$. Then

$$\alpha_1 \odot \alpha_2 \odot \cdots \odot \alpha_k = \omega^{\gamma_1} \cdot s_1 + \omega^{\gamma_2} \cdot s_2 + \cdots + \omega^{\gamma_{n-1}} \cdot s_{n-1} + \omega^{\gamma_n} \cdot (s_n - t + 1).$$

Together, these results determine the classical pigeonhole number of an arbitrary finite sequence of ordinals in terms of Cantor normal forms. In our calculation of the topological pigeonhole number of a finite sequence of countable ordinals, we will take advantage of this by making the following link between the classical and topological pigeonhole numbers (see Theorems 4.7.4 and 4.7.5).

Theorem 4.2.3. *Let $\alpha_1, \alpha_2, \dots, \alpha_k \in \omega_1 \setminus \{0\}$.*

1. $P^{top}\left(\omega^{\alpha_1} + 1, \omega^{\alpha_2} + 1, \dots, \omega^{\alpha_k} + 1\right) = \omega^{\alpha_1 \# \alpha_2 \# \cdots \# \alpha_k} + 1.$
2. $P^{top}\left(\omega^{\alpha_1}, \omega^{\alpha_2}, \dots, \omega^{\alpha_k}\right) = \omega^{P(\alpha_1, \alpha_2, \dots, \alpha_k)}.$

As we will see in Section 4.4, the corresponding closed pigeonhole numbers are the same in these cases.

4.3 Biembeddability of ordinals

The notion of biembeddability is a weakening of the notion of homeomorphism that is useful for simplifying the calculation of topological pigeonhole numbers.

Definition. Let X and Y be topological spaces. We say that X and Y are *biembeddable*, and write $X \approxeq Y$, if and only if X is homeomorphic to a subspace of Y and Y is homeomorphic to a subspace of X.

Clearly $\approxeq$ is an equivalence relation. Moreover, biembeddable ordinals may be substituted in topological (but not in closed) partition relations, in the sense that if $\beta \approxeq \widetilde{\beta}$ and $\alpha_i \approxeq \widetilde{\alpha}_i$ for all $i \in \kappa$, then $\beta \to_{top} (\alpha_i)^n_{i\in\kappa}$ if and only if $\widetilde{\beta} \to_{top} (\widetilde{\alpha}_i)^n_{i\in\kappa}$.

We will now classify the ordinals up to biembeddability, beginning with a positive result.

Lemma 4.3.1. *Let γ, m and δ be non-zero ordinals with $m \in \omega$ and $\delta < \omega^\gamma$. Then*

$$\omega^\gamma \cdot m + 1 \approxeq \omega^\gamma \cdot m + \delta.$$

Proof. Clearly $\omega^\gamma \cdot m + 1$ is homeomorphic to a subspace of $\omega^\gamma \cdot m + \delta$, so it is enough to show that $\omega^\gamma \cdot m + \delta$ is homeomorphic to a subspace of $\omega^\gamma \cdot m + 1$. In fact, we show that $\omega^\gamma \cdot m + 1 + \delta + 1$ is homeomorphic to $\omega^\gamma \cdot m + 1$, which is sufficient.

Now if α and β are successor ordinals, then $\alpha + \beta \cong \beta + \alpha$ by Lemma 2.4.2. Hence $\omega^\gamma \cdot m + 1 + \delta + 1 \cong \delta + 1 + \omega^\gamma \cdot m + 1 = \omega^\gamma \cdot m + 1$ since $\delta < \omega^\gamma$. $\square$

Because of the substitution property of biembeddable ordinals, this has following immediate consequence, which will be useful.

Proposition 4.3.2. *Let κ be a cardinal and let α_i be an ordinal for each $i \in \kappa$. Suppose that for some non-zero ordinal γ and some positive integer m we have*

$$\omega^\gamma \cdot m + 1 \leq P^{top}\left(\alpha_i\right)_{i\in\kappa} < \omega^\gamma \cdot (m+1).$$

Then in fact

$$P^{top}\left(\alpha_i\right)_{i\in\kappa} = \omega^\gamma \cdot m + 1.$$ □

We now show that Lemma 4.3.1 is best possible.

Proposition 4.3.3. *Let γ, m and δ be non-zero ordinals with $m \in \omega$ and $\delta < \omega^\gamma$. Then*

1. *$\omega^\gamma \cdot m + 1 \not\precsim \omega^\gamma \cdot m$;*

2. *$\omega^\gamma \not\precsim \delta$; and*

3. *$\omega^\gamma \cdot (m+1) \not\precsim \omega^\gamma \cdot m + 1$.*

Proof. 1. By Proposition 2.3.3, we have $\left|(\omega^\gamma \cdot m + 1)^{(\gamma)}\right| = m$ while $\left|(\omega^\gamma \cdot m)^{(\gamma)}\right| = m - 1$. Therefore no subspace of $\omega^\gamma \cdot m$ can be homeomorphic to $\omega^\gamma \cdot m + 1$.

2. If $\gamma = \eta + 1$, then by Proposition 2.3.3, $(\omega^\gamma)^{(\eta)}$ is infinite while $\delta^{(\eta)}$ is finite (or empty). If γ is a limit ordinal, then by Proposition 2.3.3, $(\omega^\gamma)^{(\eta)} \neq \emptyset$ for all $\eta < \gamma$ while $\delta^{(\eta)} = \emptyset$ for some $\eta < \gamma$. In either case no subspace of δ can be homeomorphic to ω^γ.

3. Let $X = \omega^\gamma \cdot (m+1)$. By Proposition 2.3.3, X has the following two properties: firstly, $\left|X^{(\gamma)}\right| = m$; and secondly, X has a closed subset Z with $Z \cap X^{(\gamma)} = \emptyset$ and $Z \cong \omega^\gamma$, namely $Z = [\omega^\gamma \cdot m + 1, \omega^\gamma \cdot (m+1))$.

   Suppose then that Y is a subspace of $\omega^\gamma \cdot m + 1$ with $\left|Y^{(\gamma)}\right| = m$, and that W is a closed subset of Y with $W \cap Y^{(\gamma)} = \emptyset$. We show that $W \not\cong \omega^\gamma$, which suffices. Since $\left|Y^{(\gamma)}\right| = m = \left|(\omega^\gamma \cdot m + 1)^{(\gamma)}\right|$, we must have $Y^{(\gamma)} = (\omega^\gamma \cdot m + 1)^{(\gamma)} = \{\omega^\gamma, \omega^\gamma \cdot 2, \ldots, \omega^\gamma \cdot m\}$. Therefore since W is closed, for each $i \in \{0, \ldots, m-1\}$ there exists $x_i \in [\omega^\gamma \cdot i + 1, \omega^\gamma \cdot (i+1))$ such that $W \cap (x_i, \omega^\gamma \cdot (i+1)) = \emptyset$. It follows that W is homeomorphic to the disjoint union of a finite number of subspaces of ζ for some $\zeta < \omega^\gamma$. The argument of part 2 then shows that $W \not\cong \omega^\gamma$. □

Lemma 4.3.1 and Proposition 4.3.3 may be together restated as follows.

Corollary 4.3.4 (Classification of ordinals up to biembeddability)**.** *Two ordinals $\alpha \leq \beta$ are biembeddable if and only if either $\alpha = \beta = \omega^\gamma \cdot m$ for some ordinal γ and some $m \in \omega$, or $\omega^\gamma \cdot m + 1 \leq \alpha \leq \beta < \omega^\gamma \cdot (m+1)$ for some non-zero ordinal γ and some positive integer m.* □

4.4 Order-reinforcing ordinals

The topological and closed partition relations are closely related, thanks to the following notion.

Definition. Let α be an ordinal. We say that α is *order-reinforcing* if and only if, whenever X is a set of ordinals under the subspace topology with $X \cong \alpha$, there is a subset $Y \subseteq X$ such that Y is order-homeomorphic to α.

Clearly if α_i is order-reinforcing for all $i \in \kappa$, then $P^{top}(\alpha_i)_{i\in\kappa} = P^{cl}(\alpha_i)_{i\in\kappa}$.

Baumgartner [Bau86, Theorem 0.2] showed that every countable ordinal of the form $\omega^\gamma + 1$ or ω^γ is order-reinforcing. We now extend this result.

Theorem 4.4.1. *Let γ be a non-zero ordinal and let m be a positive integer. Then*

1. *$\omega^\gamma \cdot m + 1$ is order-reinforcing; and*
2. *ω^γ is order-reinforcing.*

Baumgartner's proof for ordinals of the form $\omega^\gamma + 1$ is also valid for uncountable ordinals of this form, and our proof of part 1 is almost identical. Baumgartner's proof for ordinals of the form ω^γ is valid for uncountable ordinals of this form providing they have countable cofinality, so we provide a new proof to cover the remaining case.

In the proof, given a topological space A and a subset $B \subseteq A$, we write $\mathrm{cl}_A(B)$ for the closure of B in A.

Proof.
1. Let $\alpha = \omega^\gamma \cdot m + 1$ and let X be a set of ordinals with $X \cong \alpha$. Then X is compact and therefore internally closed, so by Proposition 2.2.4, X is order-homeomorphic to its order type. This order type must be at least α in order for $\left|X^{(\gamma)}\right| = m$. Hence we may take Y to be the initial segment of X of order type α.

2. Let $\alpha = \omega^\gamma$. Baumgartner's proof covers the case in which α has countable cofinality, so assume that α has uncountable cofinality.

   Let X be a set of ordinals with $X \cong \alpha$, and let η be the least ordinal with $X \subseteq \eta$. Then X is not compact and is therefore not a closed subset of the compact space $\eta+1$. So we may let x be the minimal element of $\mathrm{cl}_{\eta+1}(X) \setminus X$. Let $Z = X \cap [0, x)$, so that Z is a closed cofinal subset of $[0, x)$. Then by

Proposition 2.2.4, Z is order-homeomorphic to its order type, say the ordinal β. Observe now that Z is a closed open subset of X but is not compact. We claim that any closed open subset of α that is not compact must be homeomorphic to α. From this it follows that $\beta \cong \alpha$ and hence $\beta \geq \alpha$ by Proposition 4.3.3 part 2. Hence we may take Y to be the initial segment of Z of order type α.

To prove the claim, suppose W is a closed open subset of α that is not compact. Then W and $\alpha \setminus W$ are both closed subsets of α, but they are disjoint and so cannot both be club in α. Now any closed bounded subset of α is compact, so it must be that W is unbounded in α while $\alpha \setminus W$ is bounded. It follows that W has order type α, and so $W \cong \alpha$ by Proposition 2.2.4. This proves the claim, which completes the proof. □

Thus the topological and closed pigeonhole numbers coincide for ordinals of the form ω^γ or $\omega^\gamma \cdot m + 1$ with m a positive integer.

We now show that this result is best possible for infinite ordinals.

Corollary 4.4.2 (Classification of order-reinforcing ordinals)**.** *An ordinal α is order-reinforcing if and only if either α is finite, or $\alpha = \omega^\gamma \cdot m + 1$ for some non-zero ordinal γ and some positive integer m, or $\alpha = \omega^\gamma$ for some non-zero ordinal γ.*

Proof. The "if" statement follows from Theorem 4.4.1 and the fact that every finite ordinal is order-reinforcing.

For the "only if" statement, if α is infinite then we may write $\alpha = \omega^\gamma \cdot m + \delta$ with γ a non-zero ordinal, m a positive integer and $\delta < \omega^\gamma$. Assume that α does not have one of the given forms, so that either $\delta > 1$, or $\delta = 0$ and $m > 1$. If $\delta > 1$, then by Lemma 4.3.1 we may take X to be a subspace of $\omega^\gamma \cdot m + 1$ with $X \cong \alpha$. If $\delta = 0$ and $m > 1$, then we may take $X = (\omega^\gamma \cdot m + 1) \setminus \{\omega^\gamma\}$. In either case X is a witness to the fact that α is not order-reinforcing. □

4.5 The ordinal $\omega + 1$

Before stating the topological pigeonhole principle for ordinals in general, we first prove the following very special case of Theorem 4.2.3 for illustrative purposes.

Proposition 4.5.1. *Let k be a positive integer. Then*

$$P^{top}\left(\omega+1\right)_k = \omega^k + 1.$$

This result may be proved directly by induction on k, much as in our proof of Theorem 5.2.1, which is stronger. We provide an alternative proof in order to illustrate the character of many later proofs. The main idea is the following result, which says that any finite colouring of ω^n is in some sense similar to a colouring which is constant on ordinals of the same Cantor–Bendixson rank.

Lemma 4.5.2. *Let k and n be positive integers and let $c : \omega^n \to k$. Then there is some subset $X \subseteq \omega^n$ such that $X \cong \omega^n$ and c is constant on $X^{(i)} \setminus X^{(i+1)}$ for each $i \in n$.*

Proof. The proof is by induction on n. The case $n = 1$ is simply the ordinary pigeonhole principle $\omega \to_{top} (\omega)^1_k$, so assume $n > 1$. Consider first the restriction of c to $\{\omega \cdot \alpha : \alpha \in \omega^{n-1}\}$. By the inductive hypothesis, passing to a subset we may assume that c is constant on $(\omega^n)^{(i)} \setminus (\omega^n)^{(i+1)}$ for each $i \in n \setminus \{0\}$. By considering the restriction of c to $[\omega^{n-1} \cdot m + 1, \omega^{n-1} \cdot (m+1)]$ for each $m \in \omega$, we may likewise assume that c is constant on $\left(\omega^n \setminus (\omega^n)'\right) \cap [\omega^{n-1} \cdot m + 1, \omega^{n-1} \cdot (m+1)]$ for each $m \in \omega$, taking the value c_m, say. To finish, simply choose an infinite subset $S \subseteq \omega$ such that $c_l = c_m$ for all $l, m \in S$, and take X to be

$$\bigcup_{m \in S} \left[\omega^{n-1} \cdot m + 1, \omega^{n-1} \cdot (m+1)\right]. \qquad \square$$

Proof of Proposition 4.5.1. To see that $\omega^k \nrightarrow_{top} (\omega + 1)^1_k$, simply colour each $x \in \omega^k$ with colour $\mathrm{CB}(x)$, and observe that each colour class is discrete.

To see that $\omega^k + 1 \to_{top} (\omega + 1)^1_k$, let $c : \omega^k + 1 \to k$. Choose $X \subseteq \omega^k$ as in Lemma 4.5.2, and let $Y = X \cup \{\omega^k\}$. Since $Y^{(k)}$ is simply the singleton $\{\omega^k\}$, we in fact have that c is constant on $Y^{(i)} \setminus Y^{(i+1)}$ for each $i \in k + 1$. By the finite pigeonhole principle $k + 1 \to (2)^1_k$, it follows that c is constant on $\left(Y^{(i)} \setminus Y^{(i+1)}\right) \cup \left(Y^{(j)} \setminus Y^{(j+1)}\right)$ for some distinct $i, j \in k + 1$, a set which is easily seen to contain a topological copy of $\omega + 1$. $\square$

The key idea to take from this proof is the importance of colourings of the form $\widetilde{c} \circ \mathrm{CB}$ for some $\widetilde{c} : k \to k$. The negative relation was proved using a counterexample of this form. The positive relation was proved by showing in the Lemma that any colouring must be similar to some colouring of this form, and applying the pigeonhole principle to $k + 1$. The proof of Theorem 4.2.3 will be similar, with this use of the Lemma and the pigeonhole principle replaced by a result of Weiss.

4.6 Statement of the principle

We now state the topological pigeonhole principle for ordinals in full. Although it may not be necessary to go through the details of every case at this stage, they are included here for completeness. Our main breakthrough is given in case 6, and includes Theorem 4.2.3 as a special case. Later we will describe the modifications required to obtain the closed pigeonhole principle for ordinals.

Observe first that if $\alpha_r = 0$ for some $r \in \kappa$, then $P^{top}(\alpha_i)_{i \in \kappa} = 0$, and if $I \subseteq \kappa$ with $\alpha_i = 1$ for all $i \in I$, then $P^{top}(\alpha_i)_{i \in \kappa} = P^{top}(\alpha_i)_{i \in \kappa \setminus I}$. Thus it is sufficient to consider the cases in which $\alpha_i \geq 2$ for all $i \in \kappa$.

Theorem 4.6.1 (The topological pigeonhole principle for ordinals)**.** *Let κ be a cardinal, and let α_i be an ordinal with $\alpha_i \geq 2$ for all $i \in \kappa$.*

1. *If $\alpha_r \geq \omega_1 + 1$ and $\alpha_s \geq \omega + 1$ for some distinct $r, s \in \kappa$, then $P^{top}(\alpha_i)_{i\in\kappa} = \infty$.*
2. *If $\alpha_r \geq \omega_1 + 1$ for some $r \in \kappa$ and $\alpha_i \leq \omega$ for all $i \in \kappa \setminus \{r\}$:*
   - *(a) if $\kappa \geq \aleph_0$:*
     - *i. if α_r is a not a power of ω, then $P^{top}(\alpha_i)_{i\in\kappa} = \alpha_r \cdot \kappa^+$;*
     - *ii. if α_r is a power of ω:*
       - *A. if $\mathrm{cf}(\alpha_r) > \kappa$, then $P^{top}(\alpha_i)_{i\in\kappa} = \alpha_r$;*
       - *B. if $\aleph_0 < \mathrm{cf}(\alpha_r) \leq \kappa$, then $P^{top}(\alpha_i)_{i\in\kappa} = \alpha_r \cdot \kappa^+$;*
       - *C. if $\mathrm{cf}(\alpha_r) = \aleph_0$, then we may write $\alpha_r = \omega^\beta$ and $\beta = \gamma + \omega^\delta$ with δ not a limit ordinal of uncountable cofinality; then*
         - *if $\delta < \kappa^+$, then $P^{top}(\alpha_i)_{i\in\kappa} = \alpha_r \cdot \kappa^+$;*
         - *if $\delta > \kappa^+$, then $P^{top}(\alpha_i)_{i\in\kappa} = \alpha_r$;*
   - *(b) if $\kappa < \aleph_0$ and $\alpha_s = \omega$ for some $s \in \kappa \setminus \{r\}$:*
     - *i. if α_r is a power of ω, then $P^{top}(\alpha_i)_{i\in\kappa} = \alpha_r$;*
     - *ii. if α_r is not a power of ω, then $P^{top}(\alpha_i)_{i\in\kappa} = \alpha_r \cdot \omega$;*
   - *(c) if $\kappa < \aleph_0$ and $\alpha_i < \omega$ for all $i \in \kappa \setminus \{r\}$:*
     - *i. if α_r is a power of ω or $\kappa = 1$, then $P^{top}(\alpha_i)_{i\in\kappa} = \alpha_r$;*
     - *ii. if $\kappa > 1$ and α_r is not a power of ω, then $\omega^\beta \cdot m + 1 \leq \alpha_r \leq \omega^\beta \cdot (m+1)$ for some ordinal β and some positive integer m; then*

$$P^{top}(\alpha_i)_{i\in\kappa} = \omega^\beta \cdot \left(\sum_{i\in\kappa\setminus\{r\}} (\alpha_i - 1) + m \right) + 1.$$

3. *If $\alpha_i \leq \omega_1$ for all $i \in \kappa$ and $\alpha_r, \alpha_s = \omega_1$ for some distinct $r, s \in \kappa$, then the value of $P^{top}(\alpha_i)_{i\in\kappa}$ is independent of* ZFC *in the following sense.*

   Write "$P_\kappa = x$" for the statement, "κ is a cardinal, and for all sequences $(\alpha_i)_{i\in\kappa}$ of ordinals, if $2 \leq \alpha_i \leq \omega_1$ for all $i \in \kappa$ and $\alpha_r, \alpha_s = \omega_1$ for some distinct $r, s \in \kappa$, then $P^{top}(\alpha_i)_{i\in\kappa} = x$". Likewise for "$P_\kappa \geq x$".

   Firstly,

$$\text{“for all cardinals } \kappa \geq 2,\ P_\kappa \geq \max\{\omega_2, \kappa^+\}\text{”}$$

   is a theorem of ZFC*. Secondly, if* ZFC *is consistent, then so is*

$$\mathsf{ZFC} + \text{“for all cardinals } \kappa \geq 2,\ P_\kappa = \infty\text{”}.$$

   Thirdly, if ZFC*+ "there exists a supercompact cardinal" is consistent, then so is*

$$\mathsf{ZFC} + \text{“for all cardinals } \kappa \geq 2,\ P_\kappa = \max\{\omega_2, \kappa^+\}\text{”}.$$

Moreover, some large cardinal assumption is required, since ZFC+ *"there exists a Mahlo cardinal" is consistent if and only if*

$$\mathsf{ZFC} + \text{“}\omega_2 \rightarrow_{top} (\omega_1)^1_2\text{”}$$

is consistent.

4. *If* $\alpha_r = \omega_1$ *for some* $r \in \kappa$ *and* $\alpha_i < \omega_1$ *for all* $i \in \kappa \setminus \{r\}$, *then* $P^{top}(\alpha_i)_{i\in\kappa} = \max\{\omega_1, \kappa^+\}$.
5. *If* $\alpha_i < \omega_1$ *for all* $i \in \kappa$ *and* $\kappa \geq \aleph_0$, *then* $P^{top}(\alpha_i)_{i\in\kappa} = \kappa^+$.
6. *If* $\alpha_i < \omega_1$ *for all* $i \in \kappa$ *and* $\kappa < \aleph_0$:
   (a) *if* $\alpha_i < \omega$ *for all* $i \in \kappa$, *then*

   $$P^{top}(\alpha_i)_{i\in\kappa} = \sum_{i\in\kappa}(\alpha_i - 1) + 1;$$

   (b) *if* α_r *is a power of* ω *for some* $r \in \kappa$, *then*

   $$P^{top}(\alpha_i)_{i\in\kappa} = \omega^{\beta_0\odot\beta_1\odot\cdots\odot\beta_{\kappa-1}},$$

   where for each $i \in \kappa$, β_i *is minimal subject to the condition that* $\alpha_i \leq \omega^{\beta_i}$;

   (c) *if* α_i *is not a power of* ω *for any* $i \in \kappa$ *and* $\alpha_r \geq \omega$ *for some* $r \in \kappa$, *then for each* $i \in \kappa$ *we can find an ordinal* β_i *and a positive integer* m_i *such that either* $\alpha_i = m_i$ *and* $\beta_i = 0$, *or* $\omega^{\beta_i} \cdot m_i + 1 \leq \alpha_i \leq \omega^{\beta_i} \cdot (m_i + 1)$ *and* $\beta_i > 0$; *then:*
      i. *if there exists* $s \in \kappa$ *such that* $\alpha_s = \omega^{\beta_s} \cdot (m_s + 1)$, $\mathrm{CB}(\beta_s) \leq \mathrm{CB}(\beta_i)$ *for all* $i \in \kappa$, *and* $m_i = 1$ *for all* $i \in \kappa \setminus \{s\}$, *then*

      $$P^{top}(\alpha_i)_{i\in\kappa} = \omega^{\beta_0\#\beta_1\#\cdots\#\beta_{\kappa-1}} \cdot (m_s + 1);$$

      ii. *otherwise,*

      $$P^{top}(\alpha_i)_{i\in\kappa} = \omega^{\beta_0\#\beta_1\#\cdots\#\beta_{\kappa-1}} \cdot \left(\sum_{i\in\kappa}(m_i - 1) + 1\right) + 1.$$

We prove this result in a case-by-case fashion, as follows. Case 1 has a simple proof, which we give in Proposition 4.10.1. Case 2 has many subcases, each of which has a relatively straightforward proof; we reformulate these subcases using an elementary argument in Lemma 4.10.2, before proving each one individually in Section 4.11. Case 3 is easy to deduce from results of others, which we do in Section 4.9. Cases 4 and 5 have simple proofs involving stationary sets, which we give in Section 4.8. Finally, case 6 has the most new ideas. We provide the main ingredients for the proof in Section 4.7, before combining them to complete the proof in Section 4.11. We describe the key ideas first, including the proof of Theorem 4.2.3 in Theorems 4.7.4 and 4.7.5.

4.7 Finite sequences of countable ordinals

We will now go through the cases of the principle in roughly reverse order, providing the key ingredients for the proof, before combining them to complete the proof. We begin with case 6 of the principle, including the proof of Theorem 4.2.3. First of all we state Weiss's result, which requires us to introduce some notation.

Definition. Let $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ be ordinals and $S \subseteq \{1, 2, \ldots, n\}$, say $S = \{s_1, s_2, \ldots, s_l\}$ with $s_1 < s_2 < \cdots < s_l$. Then we write

$$\sum_{i \in S} \omega^{\gamma_i} = \omega^{\gamma_{s_1}} + \omega^{\gamma_{s_2}} + \cdots + \omega^{\gamma_{s_l}}$$

and

$$\left(\omega^{\omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}}\right)_S = \begin{cases} \omega^{\sum_{i \in S} \omega^{\gamma_i}}, & \text{if } S \neq \emptyset \\ 0, & \text{if } S = \emptyset. \end{cases}$$

Weiss's result is our key tool for proving positive relations in this section, and was first published by Baumgartner [Bau86, Theorem 2.3].

Theorem 4.7.1 (Weiss). *Let $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ be countable ordinals, let*

$$\beta = \omega^{\omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}},$$

and let $c : \beta \to 2$. Then there exists $S \subseteq \{1, 2, \ldots, n\}$, $X \subseteq c^{-1}(\{0\})$ and $Y \subseteq c^{-1}(\{1\})$ such that $X \cong \beta_S$, $Y \cong \beta_{(\{1,2,\ldots,n\} \setminus S)}$ and X and Y are both either empty or cofinal in β.

A careful reading of Baumgartner's proof reveals that "homeomorphic" can in fact be strengthened to "order-homeomorphic". Furthermore, we will be interested in colourings using more than 2 colours. It will therefore be more convenient to use this result in the following form.

Corollary 4.7.2. *Let β be as in Weiss's theorem, let k be a positive integer and let $c : \beta \to k$. Then there exists a partition of $\{1, 2, \ldots, n\}$ into k pieces $S_0, S_1, \ldots, S_{k-1}$ and for each $i \in k$ a subset $X_i \subseteq c^{-1}(\{i\})$ such that for all $i \in k$, X_i is order-homeomorphic to β_{S_i} and X_i is either empty or cofinal in β.*

Proof. This follows immediately from the "order-homeomorphic" version of Weiss's theorem by induction on k. $\square$

To prove negative relations we will frequently consider colourings based on those of the form $\widetilde{c} \circ \mathrm{CB}$ for some $\widetilde{c} : \beta \to \kappa$, where β is a non-zero ordinal. The following result is our key tool for analysing these colourings.

Proposition 4.7.3. *Let α and η be ordinals. Let Y be a set of ordinals of order type α, and let $X = \{x \in \eta : \mathrm{CB}(x) \in Y\}$. Then $X^{(\alpha)} = \emptyset$.*

Proof. For each $\zeta \leq \alpha$, let Y_ζ be the initial segment of Y of order type ζ and let $X_\zeta = \{x \in \eta : \mathrm{CB}(x) \in Y_\zeta\}$. It is easy to prove by induction on $\zeta \leq \alpha$ that $X^{(\zeta)} = X \setminus X_\zeta$. Hence $X^{(\alpha)} = X \setminus X_\alpha = \emptyset$. □

We can now apply these two tools to prove Theorem 4.2.3, beginning with part 1.

Theorem 4.7.4. *Let $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in \omega_1 \setminus \{0\}$. Then*

$$P^{top}(\omega^{\alpha_0} + 1, \omega^{\alpha_1} + 1, \ldots, \omega^{\alpha_{k-1}} + 1) = \omega^{\alpha_0 \# \alpha_1 \# \cdots \# \alpha_{k-1}} + 1.$$

Proof. Write $\alpha_0 \# \alpha_1 \# \cdots \# \alpha_{k-1} = \delta = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ with $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, and write $\beta = \omega^\delta$.

To see that $\beta \not\to_{top} (\omega^{\alpha_0} + 1, \omega^{\alpha_1} + 1, \ldots, \omega^{\alpha_{k-1}} + 1)^1$, first observe that by definition of the natural sum, there is a partition of $\{1, 2, \ldots, n\}$ into k pieces $S_0, S_1, \ldots, S_{k-1}$ such that for all $i \in k$, $\alpha_i = \sum_{j \in S_i} \omega^{\gamma_j}$. Now define a colouring $c : \beta \to k$ as follows. For each $i \in k$, set $c(x) = i$ if and only if

$$\omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_{j-1}} \leq \mathrm{CB}(x) < \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_j}$$

for some $j \in S_i$. Observe that $c^{-1}(\{i\}) = \{x \in \beta : \mathrm{CB}(x) \in Y_i\}$ for some set Y_i of ordinals of order type α_i. Thus by Proposition 4.7.3, $c^{-1}(\{i\})^{(\alpha_i)} = \emptyset$, whereas $(\omega^{\alpha_i} + 1)^{(\alpha_i)} = \{\omega^{\alpha_i}\}$. Hence $c^{-1}(\{i\})$ cannot contain a topological copy of $\omega^{\alpha_i} + 1$.

To see that $\beta + 1 \to_{top} (\omega^{\alpha_0} + 1, \omega^{\alpha_1} + 1, \ldots, \omega^{\alpha_{k-1}} + 1)^1$, let $c : \beta + 1 \to k$. Choose $S_0, S_1, \ldots, S_{k-1} \subseteq \{1, 2, \ldots, n\}$ and $X_0, X_1, \ldots, X_{k-1} \subseteq \beta$ as in Corollary 4.7.2. If $\beta_{S_i} > \omega^{\alpha_i}$ for some $i \in k$, then we are done. So we may assume $\beta_{S_i} \leq \omega^{\alpha_i}$ for all $i \in k$. But then we must in fact have $\beta_{S_i} = \omega^{\alpha_i}$ for all $i \in k$, or else $\beta < \omega^{\alpha_0 \# \alpha_1 \# \cdots \# \alpha_{k-1}}$. To finish, suppose $c(\beta) = j$. Then since X_j is cofinal in β, $X_j \cup \{\beta\}$ is a topological copy of $\omega^{\alpha_j} + 1$ in colour j. □

The proof of part 2 of Theorem 4.2.3 is similar but a little more complicated as it makes use of the Milner–Rado sum. We make use of the fact that $P(\alpha_0, \alpha_1, \ldots, \alpha_{k-1}) = \alpha_0 \odot \alpha_1 \odot \cdots \odot \alpha_{k-1}$ by using the first expression to prove the negative relation and the second expression to prove the positive relation.

Theorem 4.7.5. *Let $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in \omega_1 \setminus \{0\}$. Then*

$$P^{top}(\omega^{\alpha_0}, \omega^{\alpha_1}, \ldots, \omega^{\alpha_{k-1}}) = \omega^{\alpha_0 \odot \alpha_1 \odot \cdots \odot \alpha_{k-1}}.$$

Proof. First recall from Theorem 4.2.1 that $P(\alpha_0, \alpha_1, \ldots, \alpha_{k-1}) = \alpha_0 \odot \alpha_1 \odot \cdots \odot \alpha_{k-1}$. Write δ for their common value, write $\delta = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ with $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, and write $\beta = \omega^\delta$.

Suppose $\zeta < \beta$. To see that $\zeta \nrightarrow_{top} (\omega^{\alpha_0}, \omega^{\alpha_1}, \ldots, \omega^{\alpha_{k-1}})^1$, first observe that $\zeta \leq \omega^\eta \cdot m + 1$ for some $\eta < \delta$ and some $m \in \omega$, so it is sufficient to consider the case in which $\zeta = \omega^\eta \cdot m + 1$. Since $\eta < P(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$, there is a colouring $\widetilde{c} : \eta \to k$ such that for all $i \in k$, the order type of $\widetilde{c}^{-1}(\{i\})$ is $\widetilde{\alpha}_i < \alpha_i$. Let $c : \zeta \to k$ be a colouring with $c(x) = \widetilde{c}(\mathrm{CB}(x))$ for all $x \in \zeta \setminus \{\omega^\eta, \omega^\eta \cdot 2, \ldots, \omega^\eta \cdot m\}$ (it doesn't matter how the points $\omega^\eta, \omega^\eta \cdot 2, \ldots, \omega^\eta \cdot m$ are coloured). By Proposition 4.7.3, $c^{-1}(\{i\})^{(\widetilde{\alpha}_i)} \subseteq \{\omega^\eta, \omega^\eta \cdot 2, \ldots, \omega^\eta \cdot m\}$ for all $i \in k$, whereas $(\omega^{\alpha_i})^{(\widetilde{\alpha}_i)}$ is infinite since $\widetilde{\alpha}_i < \alpha_i$. Hence $c^{-1}(\{i\})$ cannot contain a topological copy of ω^{α_i}.

To see that $\beta \rightarrow_{top} (\omega^{\alpha_0}, \omega^{\alpha_1}, \ldots, \omega^{\alpha_{k-1}})^1$, let $c : \beta \to k$. Choose $S_0, S_1, \ldots, S_{k-1} \subseteq \{1, 2, \ldots, n\}$ and $X_0, X_1, \ldots, X_{k-1} \subseteq \beta$ as in Corollary 4.7.2. If $\beta_{S_i} \geq \omega^{\alpha_i}$ for some $i \in k$, then we are done, so suppose for contradiction that $\beta_{S_i} < \omega^{\alpha_i}$ for all $i \in k$. Write $\widetilde{\alpha}_i = \sum_{j \in S_i} \omega^{\gamma_j}$, so that $\omega^{\widetilde{\alpha}_i} = \beta_{S_i}$ and $\widetilde{\alpha}_0 \# \widetilde{\alpha}_1 \# \cdots \# \widetilde{\alpha}_{k-1} = \beta$ by definition. Then since $\beta_{S_i} < \omega^{\alpha_i}$ for all $i \in k$ and $\beta = \alpha_0 \odot \alpha_1 \odot \cdots \odot \alpha_{k-1}$, we have $\widetilde{\alpha}_i < \alpha_i$ for all $i \in k$ while $\widetilde{\alpha}_0 \# \widetilde{\alpha}_1 \# \cdots \# \widetilde{\alpha}_{k-1} = \alpha_0 \odot \alpha_1 \odot \cdots \odot \alpha_{k-1}$, contrary to the definition of the Milner–Rado sum. □

This completes the proof of Theorem 4.2.3, which provides us with the topological pigeonhole numbers for finite sequences of countable ordinals when either each ordinal is a power of ω or each ordinal is a power of ω plus 1.

Our next result generalises Theorem 4.7.5 by considering mixtures of such ordinals including at least one power of ω. Using monotonicity, this will provide us with the topological pigeonhole numbers for all finite sequences of countable ordinals in which one of the ordinals is a power of ω, thereby completing case 6b of the principle. The result essentially says that in this case, the topological pigeonhole number is the same as if the other ordinals were "rounded up" to the next largest power of ω.

The proof involves proving two negative relations, the first of which uses ideas from Theorem 4.7.4 and the second of which uses ideas from Theorem 4.7.5.

Theorem 4.7.6. *Let $\alpha_0, \alpha_1, \ldots, \alpha_l, \delta_{l+1}, \delta_{l+2}, \ldots, \delta_{k-1} \in \omega_1 \setminus \{0\}$, where $0 \leq l < k$. Then*

$$P^{top}\left(\omega^{\alpha_0}, \omega^{\alpha_1}, \ldots, \omega^{\alpha_l}, \omega^{\delta_{l+1}} + 1, \omega^{\delta_{l+2}} + 1, \ldots, \omega^{\delta_{k-1}} + 1\right)$$
$$= \omega^{\alpha_0 \odot \alpha_1 \odot \cdots \alpha_l \odot (\delta_{l+1}+1) \odot (\delta_{l+2}+1) \odot \cdots \odot (\delta_{k-1}+1)}.$$

Proof. Write P for the left-hand side and β for the right-hand side. Clearly $P \leq \beta$ by Theorem 4.7.5 and monotonicity, so we prove that $P \geq \beta$.

Suppose first that α_i is a successor ordinal for all $i \in \{0, 1, \ldots, l\}$, say $\alpha_i = \delta_i + 1$.

Then by Theorem 4.2.2, $\beta = \omega^{\delta_0 \# \delta_1 \# \cdots \# \delta_{k-1}+1}$. Suppose $\zeta < \beta$. We will show that

$$\zeta \nrightarrow_{top} \left(\omega^{\delta_0+1}, \omega^{\delta_1}+1, \omega^{\delta_2}+1, \ldots, \omega^{\delta_{k-1}}+1\right)^1,$$

which suffices. Write $\delta = \delta_0 \# \delta_1 \# \cdots \# \delta_{k-1}$, and observe first that $\zeta \leq \omega^\delta \cdot m + 1$ for some $m \in \omega$, so it is sufficient to consider the case in which $\zeta = \omega^\delta \cdot m + 1$. Next recall from the proof of Theorem 4.7.4 that there is a colouring $d : \omega^\delta \to k$ with the property that $d^{-1}(\{i\})^{(\delta_i)} = \emptyset$ for all $i \in k$. Now define a colouring $c : \zeta \to k$ by

$$c(x) = \begin{cases} d(y), & \text{if } x = \omega^\delta \cdot l + y \text{ with } l \in \omega \text{ and } 0 < y < \omega^\delta \\ 0, & \text{if } x \in \left\{0, \omega^\delta, \omega^\delta \cdot 2, \ldots, \omega^\delta \cdot m\right\}. \end{cases}$$

Then for all $i \in \{1, 2, \ldots, k-1\}$, $c^{-1}(\{i\})^{(\delta_i)} = \emptyset$, whereas $\left(\omega^{\delta_i}+1\right)^{(\delta_i)} = \left\{\omega^{\delta_i}\right\}$, so $c^{-1}(\{i\})$ cannot contain a topological copy of $\omega^{\delta_i}+1$. On the other hand, $c^{-1}(\{0\})^{(\delta_0)} \subseteq \left\{\omega^\delta, \omega^\delta \cdot 2, \ldots, \omega^\delta \cdot m\right\}$, whereas $\left(\omega^{\delta_0+1}\right)^{(\delta_0)}$ is infinite, so $c^{-1}(\{0\})$ cannot contain a topological copy of ω^{δ_0+1}. This completes the proof for this case.

Suppose instead that α_j is a limit ordinal for some $j \in \{0, 1, \ldots, l\}$. Write $\beta = \omega^\delta$. Then by Theorem 4.2.2, δ is a limit ordinal. This observation enables us to complete the proof using simpler version of the argument from Theorem 4.7.5. Suppose $\zeta < \beta$. We will show that

$$\zeta \nrightarrow_{top} \left(\omega^{\alpha_0}, \omega^{\alpha_1}, \ldots, \omega^{\alpha_l}, \omega^{\delta_{l+1}}+1, \omega^{\delta_{l+2}}+1, \ldots, \omega^{\delta_{k-1}}+1\right)^1.$$

Observe first that since δ is a limit ordinal, $\zeta \leq \omega^\eta$ for some $\eta < \delta$, so it is sufficient to consider the case in which $\zeta = \omega^\eta$. Write $\alpha_i = \delta_i + 1$ for all $i \in \{l+1, l+2, \ldots, k-1\}$, and recall from Theorem 4.2.1 that $\delta = P(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$. Since $\eta < \delta$, there is a colouring $\widetilde{c} : \eta \to k$ such that for all $i \in k$, the order type of $\widetilde{c}^{-1}(\{i\})$ is $\widetilde{\alpha}_i < \alpha_i$. Define a colouring $c : \zeta \to k$ by $c = \widetilde{c} \circ \mathrm{CB}$. By Proposition 4.7.3, $c^{-1}(\{i\})^{(\widetilde{\alpha}_i)} = \emptyset$ for all $i \in k$. However, $(\omega^{\alpha_i})^{(\widetilde{\alpha}_i)}$ is infinite for all $i \in \{0, 1, \ldots, l\}$, and $\left(\omega^{\delta_i}+1\right)^{(\widetilde{\alpha}_i)} \supseteq \left\{\omega^{\delta_i}\right\}$ for all $i \in \{l+1, l+2, \ldots, k-1\}$. Hence $c^{-1}(\{i\})$ cannot contain a topological copy of ω^{α_i} (if $i \in \{0, 1, \ldots, l\}$) or $\omega^{\delta_i}+1$ (if $i \in \{l+1, l+2, \ldots, k-1\}$). $\square$

Next we move beyond powers of ω and powers of ω plus 1 to consider ordinals of the form $\omega^\alpha \cdot m + 1$ with $\alpha \in \omega_1 \setminus \{0\}$ and m a positive integer. At this point considerations from the finite pigeonhole principle come into play.

At the same time we will also consider finite ordinals, since they behave in a similar fashion: just as $\omega^\alpha \cdot m + 1$ is homeomorphic to the topological disjoint union of m copies of $\omega^\alpha + 1$, so $m \in \omega$ is homeomorphic to the topological disjoint union of m copies of 1. In order to consider both forms of ordinal at the same time we therefore recall the following definition.

Definition. Let α be an ordinal and m be a positive integer. We define

$$\varpi\left[\alpha, m\right] = \begin{cases} \omega^\alpha \cdot m + 1, & \text{if } \alpha > 0 \\ m, & \text{if } \alpha = 0. \end{cases}$$

The following result deals with finite sequences of countable ordinals of the form $\varpi\left[\alpha, m\right]$. It generalises both Theorem 4.7.4 and the finite pigeonhole principle, and the proof essentially combines these two theorems.

Theorem 4.7.7. *Let $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in \omega_1$ and $m_1, m_2, \ldots, m_{k-1}$ be positive integers. Then*

$$P^{top}\left(\varpi\left[\alpha_0, m_0\right], \varpi\left[\alpha_1, m_1\right], \ldots, \varpi\left[\alpha_{k-1}, m_{k-1}\right]\right) = \varpi\left[\alpha, m\right],$$

where $\alpha = \alpha_0 \# \alpha_1 \# \cdots \# \alpha_{k-1}$ and $m = \sum_{i=0}^{k-1}\left(m_i - 1\right) + 1$.

Proof. We assume for simplicity that $\alpha_i > 0$ for all $i \in k$, the other case being no harder. Thus $\varpi\left[\alpha_i, m_i\right] = \omega^{\alpha_i} \cdot m_i + 1$ for all $i \in k$ and $\varpi\left[\alpha, m\right] = \omega^\alpha \cdot m + 1$.

To see that

$$\omega^\alpha \cdot m \nrightarrow_{top} \left(\omega^{\alpha_0} \cdot m_0 + 1, \omega^{\alpha_1} \cdot m_1 + 1, \ldots, \omega^{\alpha_{k-1}} \cdot m_{k-1} + 1\right)^1,$$

recall from the proof of Theorem 4.7.4 that there is a colouring $d : \omega^\alpha \to k$ with the property that $d^{-1}\left(\{i\}\right)^{(\alpha_i)} = \emptyset$ for all $i \in k$. Additionally observe that since $m - 1 \nrightarrow \left(m_0, m_1, \ldots, m_{k-1}\right)^1$, there is a colouring $e : \{1, 2, \ldots, m-1\} \to k$ with the property that $\left|e^{-1}\left(\{i\}\right)\right| \leq m_i - 1$ for all $i \in k$. Now define a colouring $c : \omega^\alpha \cdot m \to k$ by

$$c\left(x\right) = \begin{cases} d\left(y\right), & \text{if } x = \omega^\alpha \cdot l + y \text{ with } l \in \omega \text{ and } 0 < y < \omega^\alpha \\ e\left(l\right), & \text{if } x = \omega^\alpha \cdot l \text{ with } l \in \{1, 2, \ldots, m-1\} \\ 0, & \text{if } x = 0. \end{cases}$$

Then for all $i \in k$,

$$\left|c^{-1}\left(\{i\}\right)^{(\alpha_i)}\right| \leq m_i - 1$$

by construction, whereas

$$\left|\left(\omega^{\alpha_i} \cdot m_i + 1\right)^{(\alpha_i)}\right| = m_i.$$

Hence $c^{-1}\left(\{i\}\right)$ cannot contain a topological copy of $\omega^{\alpha_i} \cdot m_i + 1$.

To see that

$$\omega^\alpha \cdot m + 1 \to_{top} \left(\omega^{\alpha_0} \cdot m_0 + 1, \omega^{\alpha_1} \cdot m_1 + 1, \ldots, \omega^{\alpha_{k-1}} \cdot m_{k-1} + 1\right)^1,$$

let $c : \omega^\alpha \cdot m + 1 \to k$. Observe that for each $j \in m$, $\left[\omega^\alpha \cdot j + 1, \omega^\alpha \cdot (j+1)\right] \cong \omega^\alpha + 1$. Therefore by Theorem 4.7.4 there exists $i_j \in k$ and $X_j \subseteq c^{-1}\left(\{i_j\}\right) \cap$

$[\omega^\alpha \cdot j + 1, \omega^\alpha \cdot (j+1)]$ with $X_j \cong \omega^{\alpha_{i_j}} + 1$. Next observe that by the finite pigeonhole principle, $m \to (m_0, m_1, \ldots, m_{k-1})^1$. Hence there exists $i \in k$ such that $|\{j \in m : i_j = i\}| \geq m_i$, say $S \subseteq \{j \in m : i_j = i\}$ satisfies $|S| = m_i$. But then $\bigcup_{j \in S} X_j$ is a topological copy of $\omega^{\alpha_i} \cdot m_i + 1$ in colour i. □

We conclude this section by considering at last ordinals of the form $\omega^\alpha \cdot (m+1)$ with $\alpha \in \omega_1 \setminus \{0\}$ and m a positive integer. Such an ordinal is homeomorphic to the topological disjoint union of $\omega^\alpha \cdot m + 1$ and ω^α and behaves similarly to $\omega^\alpha \cdot m + 1$, but there are additional complications.

The following simple consequence of Theorem 4.7.5 will be useful for finding extra homeomorphic copies of ω^α in the required colour.

Lemma 4.7.8. *Let $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in \omega_1 \setminus \{0\}$, let $\alpha = \alpha_0 \# \alpha_1 \# \cdots \# \alpha_{k-1}$ and let $c : \omega^\alpha \to k$ be a colouring. Then either $c^{-1}(\{j\})$ contains a topological copy of ω^{α_j+1} for some $j \in k$, or $c^{-1}(\{i\})$ contains a topological copy of ω^{α_i} for all $i \in k$.*

Proof. Fix $i \in k$. It is sufficient to prove that either $c^{-1}(\{i\})$ contains a topological copy of ω^{α_i}, or $c^{-1}(\{j\})$ contains a topological copy of ω^{α_j+1} for some $j \in k \setminus \{i\}$. To see this, simply observe that by Theorem 4.2.2 (or by inspection),

$$(\alpha_0 + 1) \odot \cdots \odot (\alpha_{i-1} + 1) \odot \alpha_i \odot (\alpha_{i+1} + 1) \odot \cdots \odot (\alpha_{k-1} + 1) \leq \alpha$$

and hence by Theorem 4.7.5,

$$\omega^\alpha \to_{top} \left(\omega^{\alpha_0+1}, \ldots, \omega^{\alpha_{i-1}+1}, \omega^{\alpha_i}, \omega^{\alpha_{i+1}+1}, \ldots, \omega^{\alpha_{k-1}+1}\right)^1. \qquad \square$$

In our next result we use this to narrow the topological pigeonhole number down to one of two possibilities.

Theorem 4.7.9. *Let $\alpha_0, \alpha_1, \ldots, \alpha_l \in \omega_1 \setminus \{0\}$, $\alpha_{l+1}, \alpha_{l+2}, \ldots, \alpha_{k-1} \in \omega_1$ and $m_0, m_1, \ldots, m_{k-1}$ be positive integers, where $0 \leq l < k$. Then*

$$P^{top}\left(\omega^{\alpha_0} \cdot (m_0 + 1), \ldots, \omega^{\alpha_l} \cdot (m_l + 1), \varpi[\alpha_{l+1}, m_{l+1}], \ldots, \varpi[\alpha_{k-1}, m_{k-1}]\right)$$

is equal to either $\omega^\alpha \cdot m + 1$ or $\omega^\alpha \cdot (m+1)$, where $\alpha = \alpha_0 \# \alpha_1 \# \cdots \# \alpha_{k-1}$ and $m = \sum_{i=0}^{k-1} (m_i - 1) + 1$.

Proof. Write P for the topological pigeonhole number in the statement of the theorem. Recall that by Proposition 4.3.2, it is sufficient to prove that $\omega^\alpha \cdot m + 1 \leq P \leq \omega^\alpha \cdot (m+1)$. The first inequality follows immediately from Theorem 4.7.7 and monotonicity since $\omega^{\alpha_i} \cdot (m_0 + 1) > \varpi[\alpha_i, m_i]$ for all $i \in \{0, 1, \ldots, l\}$. The second inequality states that

$$\omega^\alpha \cdot (m+1) \to_{top} \left(\omega^{\alpha_0} \cdot (m_0 + 1), \ldots, \omega^{\alpha_l} \cdot (m_l + 1), \varpi[\alpha_{l+1}, m_{l+1}], \ldots, \varpi[\alpha_{k-1}, m_{k-1}]\right)^1.$$

To see this, let $c : \omega^\alpha \cdot (m+1) \to k$. First note that for $i \in \{0, 1, \ldots, l\}$, $\omega^{\alpha_i} \cdot (m_i + 1)$ is homeomorphic to the topological disjoint union of $\overline{\omega}\,[\alpha_i, m_i] = \omega^{\alpha_i} \cdot m_i + 1$ and ω^{α_i}. Now by Theorem 4.7.7, there exists $i \in k$ and $X \subseteq c^{-1}(\{i\}) \cap (\omega^\alpha \cdot m + 1)$ with $X \cong \overline{\omega}\,[\alpha_i, m_i]$. If $i \in \{l+1, l+2, \ldots, k-1\}$, then we are done, so assume $i \in \{0, 1, \ldots, l\}$. Next consider the restriction of c to $[\omega^\alpha \cdot m + 1, \omega^\alpha \cdot (m+1))$, which is homeomorphic to ω^α. By Lemma 4.7.8, either $c^{-1}(\{j\}) \cap [\omega^\alpha \cdot m + 1, \omega^\alpha \cdot (m+1))$ contains a topological copy of $\omega^{\alpha_j + 1}$ for some $j \in k$, in which case we are done, or there exists $Y \subseteq c^{-1}(\{i\}) \cap [\omega^\alpha \cdot m + 1, \omega^\alpha \cdot (m+1))$ with $Y \cong \omega^{\alpha_i}$, in which case $X \cup Y$ is a topological copy of $\omega^{\alpha_i} \cdot (m_i + 1)$ in colour i. □

Recall that by Lemma 4.3.1 it is enough to consider ordinals of the form $\omega^\alpha \cdot m$ and $\omega^\alpha \cdot m + 1$ with m a positive integer. It follows that Theorems 4.7.7 and 4.7.9 together cover case 6c of the principle. Thus to complete case 6 it remains only to distinguish between the two possibilities presented by Theorem 4.7.9.

In our final result of this section we do this for the case in which $m_i = 1$ for all $i \in k \setminus \{0\}$. In particular this completes case 6(c)i. At this point the Cantor–Bendixson ranks of ordinal *exponents* come into play. They essentially determine whether or not the negative relation can be proved using the type of colouring given in the first half of the proof of Theorem 4.7.4.

Theorem 4.7.10. *Let $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in \omega_1 \setminus \{0\}$, let m_0 be a positive integer and let $0 \le l < k$. Assume without loss of generality that if $m_0 = 1$ then $\mathrm{CB}(\alpha_0) \le \mathrm{CB}(\alpha_i)$ for all $i \in \{1, 2, \ldots, l\}$. Then*

$$\omega^\alpha \cdot m_0 + 1 \to_{top} \left(\omega^{\alpha_0} \cdot (m_0 + 1), \omega^{\alpha_1} \cdot 2, \ldots, \omega^{\alpha_l} \cdot 2, \omega^{\alpha_{l+1}} + 1, \ldots, \omega^{\alpha_{k-1}} + 1\right)^1$$

if and only if $\mathrm{CB}(\alpha_h) < \mathrm{CB}(\alpha_0)$ for some $h \in k$, where $\alpha = \alpha_0 \# \alpha_1 \# \cdots \# \alpha_{k-1}$.

We will prove the "if" part by combining Lemma 4.7.8 with the following result.

Lemma 4.7.11. *Let $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in \omega_1 \setminus \{0\}$, let $\alpha = \alpha_0 \# \alpha_1 \# \cdots \# \alpha_{k-1}$ and let $c : \omega^\alpha + 1 \to k$ be a colouring. Then there exists $j \in k$ such that either $c^{-1}(\{j\})$ contains a topological copy of $\omega^{\alpha_j} \cdot 2$, or $c^{-1}(\{j\})$ contains a topological copy of $\omega^{\alpha_j} + 1$ and $\mathrm{CB}(\alpha_j) \le \mathrm{CB}(\alpha_i)$ for all $i \in k$.*

The proof of this lemma uses ideas from the proof of Weiss's theorem [Bau86, Theorem 2.3]. In particular we will make use of the following result, which was also published by Baumgartner [Bau86, Lemma 2.6].

Lemma 4.7.12 (Weiss)**.** *Let $\alpha \in \omega_1$ not be a power of ω. Write $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ with $\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_n$ and $n > 1$, and let $\delta = \omega^{\omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_{n-1}}}$ and $\varepsilon = \omega^{\omega^{\gamma_n}}$. Suppose $Z \subseteq \{\delta \cdot x : x \in \varepsilon\}$ is order-homeomorphic to ε, say $Z = \{z_\eta : \eta \in \varepsilon\}$. Then for each $\eta \in \varepsilon$ there exists $Y_\eta \subseteq (z_\eta, z_{\eta+1})$ such that Y_η is order-homeomorphic to δ and Y_η is cofinal in $(z_\eta, z_{\eta+1})$.*

Proof of Lemma 4.7.11. Write $\alpha = \omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_n}$ with $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$, and observe that for $j \in k$, the condition that $\mathrm{CB}(\alpha_j) \leq \mathrm{CB}(\alpha_i)$ for all $i \in k$ is equivalent to the condition that $\mathrm{CB}(\alpha_j) = \gamma_n$. The case $k = 1$ is trivial, so assume $k > 1$ (and hence $n > 1$) and let $\delta = \omega^{\omega^{\gamma_1} + \omega^{\gamma_2} + \cdots + \omega^{\gamma_{n-1}}}$ and $\varepsilon = \omega^{\omega^{\gamma_n}}$ as in Weiss's lemma.

First let $c(\omega^\alpha) = j_0$. Next, by Corollary 4.7.2 there exists $j_1 \in k$ and $Z \subseteq c^{-1}(\{j_1\}) \cap \{\delta \cdot x : x \in \varepsilon\}$ such that Z is cofinal in $\{\delta \cdot x : x \in \varepsilon\}$ (and hence in ω^α) and Z is order-homeomorphic to ε, say $Z = \{z_\eta : \eta \in \varepsilon\}$. For each $\eta \in \varepsilon$ choose Y_η as in Weiss's lemma. Then for each $\eta \in \varepsilon$, by Corollary 4.7.2 there exists a partition of $\{1, 2, \ldots, n-1\}$ into k pieces $S_0^\eta, S_1^\eta, \ldots, S_{k-1}^\eta$ and for each $i \in k$ a subset $X_i^\eta \subseteq c^{-1}(\{i\}) \cap Y_\eta$ such that for all $i \in k$, X_i^η is order-homeomorphic to $\delta_{S_i^\eta}$ and X_i^η is either empty or cofinal in Y_η (and hence in $(z_\eta, z_{\eta+1})$). Moreover since $\varepsilon \to (\varepsilon)^1_r$ for all positive integers r (either using Theorem 4.7.5 or simply from the fact that ε is a power of ω), there exists $T \subseteq \varepsilon$ of order type ε and a single partition of $\{1, 2, \ldots, n-1\}$ into k pieces $S_0, S_1, \ldots, S_{k-1}$ such that $S_i^\eta = S_i$ for all $\eta \in T$ and all $i \in k$.

Now if $\delta_{S_j} > \omega^{\alpha_j}$ for some $j \in k$, then we are done. So we may assume $\delta_{S_i} \leq \omega^{\alpha_i}$ for all $i \in k$. But then there must exist $j_2 \in k$ with $\mathrm{CB}(\alpha_{j_2}) = \gamma_n$ such that in fact $\delta_{S_i} = \omega^{\alpha_i}$ for all $i \in k \setminus \{j_2\}$ and $\delta_{S_{j_2} \cup \{n\}} = \omega^{\alpha_{j_2}}$.

There are now three possibilities.

- If $j_1 \neq j_2$, then take $j = j_1$. Pick $\eta_1, \eta_2 \in T$ and take

$$X = X_j^{\eta_1} \cup \{z_{\eta_1+1}\} \cup X_j^{\eta_2}.$$

  Then X is a topological copy of $\omega^{\alpha_j} \cdot 2$ in colour j.

- If $j_0 \neq j_2$, then take $j = j_0$. We now use an argument from the proof of Weiss's theorem. Let $(\eta_r)_{r\in\omega}$ be a strictly increasing cofinal sequence from T and let $(\zeta_r)_{r\in\omega\setminus\{0\}}$ be a strictly increasing cofinal sequence from ω^{α_j}, so that ω^{α_j} is homeomorphic to the topological disjoint union of the collection $(\zeta_r + 1)_{r\in\omega\setminus\{0\}}$. For each $r \in \omega \setminus \{0\}$, choose $W_r \subseteq X_j^{\eta_r}$ with $W_r \cong \zeta_r + 1$, and take

$$X = X_j^{\eta_0} \cup \bigcup_{r\in\omega\setminus\{0\}} W_r \cup \{\omega^\alpha\}.$$

  Then X is a topological copy of $\omega^{\alpha_j} \cdot 2$ in colour j.

- If $j_0 = j_1 = j_2$, then take j to be their common value. We now use another argument from the proof of Weiss's theorem. Let Z_1 be the closure of $\{z_{\eta+1} : \eta \in T\}$ in Z and take

$$X = \bigcup_{\eta\in T} X_j^\eta \cup Z_1 \cup \{\omega^\alpha\}.$$

Then X is a topological copy of $\omega^{\alpha_j}+1$ in colour j, and since $j=j_2$ we have $\mathrm{CB}(\alpha_j)=\gamma_n$. □

Proof of Theorem 4.7.10. Write $\alpha=\omega^{\gamma_1}+\omega^{\gamma_2}+\cdots+\omega^{\gamma_n}$ with $\gamma_1\geq\gamma_2\geq\cdots\geq\gamma_n$, and let $\beta=\omega^{\alpha}\cdot m_0+1$.

Suppose first that $\mathrm{CB}(\alpha_0)\leq\mathrm{CB}(\alpha_i)$ for all $i\in k$. As in the proof of Theorem 4.7.4, observe that by definition of the natural sum, there is a partition of $\{1,2,\ldots,n\}$ into k pieces $S_0,S_1,\ldots,S_{k-1}$ such that for all $i\in k$, $\alpha_i=\sum_{j\in S_i}\omega^{\gamma_j}$. Moreover, since $\mathrm{CB}(\alpha_0)\leq\mathrm{CB}(\alpha_i)$ for all $i\in k$ we may assume that $n\in S_0$. Now define a colouring $c:\beta\to k$ as follows. If $\mathrm{CB}(x)<\alpha$ (i.e., $x\notin\{\omega^{\alpha},\omega^{\alpha}\cdot 2,\ldots,\omega^{\alpha}\cdot m_0\}$), then as in Theorem 4.7.4, for each $i\in k$ set $c(x)=i$ if and only if

$$\omega^{\gamma_1}+\omega^{\gamma_2}+\cdots+\omega^{\gamma_{j-1}}\leq\mathrm{CB}(x)<\omega^{\gamma_1}+\omega^{\gamma_2}+\cdots+\omega^{\gamma_j}$$

for some $j\in S_i$. If $\mathrm{CB}(x)=\alpha$, then set $c(x)=0$. If $i\in k\setminus\{0\}$, then as in Theorem 4.7.4 $c^{-1}(\{i\})$ cannot contain a topological copy of $\omega^{\alpha_i}+1$. To deal with the case $i=0$, let $\eta=\sum_{j\in S_0\setminus\{n\}}\omega^{\gamma_j}$. By the proof of Proposition 4.7.3,

$$c^{-1}(\{0\})^{(\eta)}=\{x\in\beta:\mathrm{CB}(x)\geq\omega^{\gamma_1}+\omega^{\gamma_2}+\cdots+\omega^{\gamma_{n-1}}\}\cong\omega^{\omega^{\gamma_n}}\cdot m_0+1,$$

whereas $(\omega^{\alpha_0}\cdot(m_0+1))^{(\eta)}\cong\omega^{\omega^{\gamma_n}}\cdot(m_0+1)$. It follows by part 3 of Proposition 4.3.3 that $c^{-1}(\{0\})$ cannot contain a topological copy of $\omega^{\alpha_0}\cdot(m_0+1)$.

Suppose instead that $\mathrm{CB}(\alpha_h)<\mathrm{CB}(\alpha_0)$ for some $h\in k$. If $m_0=1$, then by assumption $\mathrm{CB}(\alpha_0)\leq\mathrm{CB}(\alpha_i)$ for all $i\in\{1,2,\ldots,l\}$, so $\mathrm{CB}(\alpha_h)<\mathrm{CB}(\alpha_i)$ for all $i\in\{1,2,\ldots,l\}$ and we are done by Lemma 4.7.11. So assume $m_0>1$. Then for each $p\in m_0$ apply Lemma 4.7.11 to obtain $j_p\in k$ and $X_p\subseteq c^{-1}(\{j_p\})\cap[\omega^{\alpha}\cdot p+1,\omega^{\alpha}\cdot(p+1)]$ such that either $X_p\cong\omega^{\alpha_{j_p}}\cdot 2$, or $X_p\cong\omega^{\alpha_{j_p}}+1$ and $\mathrm{CB}\left(\alpha_{j_p}\right)\leq\mathrm{CB}(\alpha_i)$ for all $i\in k$. If $j_p=0$ for all $p\in m_0$, then $X_p\cong\omega^{\alpha_0}\cdot 2$ for all $p\in m_0$ and so $\bigcup_{p=0}^{m_0-1}X_p$ contains a topological copy of $\omega^{\alpha_0}\cdot(m_0+1)$, and we are done. So assume $j_q\neq 0$ for some $q\in m_0$. Now pick any $r\in m_0\setminus\{q\}$ and apply Lemma 4.7.8 to $[\omega^{\alpha}\cdot r+1,\omega^{\alpha}\cdot(r+1))$. Since we would be done if $c^{-1}(\{j\})$ contained a topological copy of ω^{α_j+1} for some $j\in k$, we may assume that there exists $Y\subseteq c^{-1}(\{j_q\})\cap[\omega^{\alpha}\cdot r+1,\omega^{\alpha}\cdot(r+1))$ with $Y\cong\omega^{\alpha_{j_q}}$. Then $X_q\cup Y$ is a topological copy of $\omega^{\alpha_{j_q}}\cdot 2$ in colour j_q, which suffices. □

We leave the final few considerations pertaining to case 6(c)ii for later.

4.8 Arbitrary sequences of ordinals at most ω_1

We now move on to cases 3, 4 and 5 of the principle, in which no ordinal exceeds ω_1 but either there are infinitely many ordinals or there is at least one ordinal equal to

ω_1. Here the arguments are less combinatorial and more set-theoretical than in the previous section, and stationary sets are ubiquitous.

We will cover cases 4 and 5 in this section and leave the independence results of case 3 to the next section.

To understand the relevance of club sets, recall Proposition 2.2.4. From that result and the fact that ω_1 is order-reinforcing, it follows that given $X \subseteq \omega_1$, X is club if and only if $X \cong \omega_1$.

The essential reason for the ubiquity of stationary sets in this section is the following result of Friedman [Fri74].

Theorem 4.8.1 (Friedman)**.** *Let $S \subseteq \omega_1$ be a stationary set, and let $\alpha \in \omega_1$. Then S has a subset order-homeomorphic to α.*

We will need a slightly more general version of this result. In order to state it we make the following definition.

Definition. Let λ be an uncountable regular cardinal. Define

$$E_\omega^\lambda = \{x \in \lambda : \mathrm{cf}\,(x) = \omega\}\,.$$

Note that E_ω^λ is stationary in λ.

Here is our generalisation of Friedman's theorem.

Theorem 4.8.2. *Let λ be an uncountable regular cardinal, let $S \subseteq E_\omega^\lambda$ be stationary in λ, and let $\alpha \in \omega_1$. Then S has a subset order-homeomorphic to α.*

Proof. The proof is essentially identical to the proof of Friedman's theorem [Fri74]. □

Our final introductory result is a well-known property of stationary sets.

Lemma 4.8.3. *Let λ be an uncountable regular cardinal, let $S \subseteq \lambda$ be stationary, and let $c : S \to \kappa$ for some cardinal $\kappa < \lambda$. Then $c^{-1}(\{i\})$ is stationary in λ for some $i \in \kappa$.*

Proof. This follows easily from the fact that if $C_i \subseteq \lambda$ is club for all $i \in \kappa$ then $\bigcap_{i\in\kappa} C_i$ is also club. □

We are now ready to deal with cases 4 and 5 of the principle. The result for case 5 is an easy consequence of Theorem 4.8.2 and Lemma 4.8.3.

Theorem 4.8.4. *Let $\kappa \geq \aleph_0$ be a cardinal, and let α_i be an ordinal with $2 \leq \alpha_i < \omega_1$ for all $i \in \kappa$. Then*

$$P^{top}\,(\alpha_i)_{i\in\kappa} = \kappa^+.$$

Proof. Clearly if $\zeta < \kappa^+$ then $\zeta \not\rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$ by considering an injection $\zeta \rightarrow \kappa$.

To see that $\kappa^+ \rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, let $c : \kappa^+ \rightarrow \kappa$. Then by Lemma 4.8.3 there exists $i \in \kappa$ such that $c^{-1}(\{i\}) \cap E^{\kappa^+}_\omega$ is stationary in κ^+, which by Theorem 4.8.2 contains a topological copy of α_i. □

The proof for case 4 is a little trickier.

Theorem 4.8.5. *Let κ be a cardinal, let $\alpha_r = \omega_1$ for some $r \in \kappa$, and let α_i be an ordinal with $2 \leq \alpha_i < \omega_1$ for all $i \in \kappa \setminus \{r\}$. Then*

$$P^{top}(\alpha_i)_{i\in\kappa} = \max\left\{\omega_1, \kappa^+\right\}.$$

Proof. As in the proof of Theorem 4.8.4, if $\zeta < \kappa^+$ then $\zeta \not\rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$. Additionally, if $\zeta < \omega_1$ then $\zeta \not\rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$ by considering the constant colouring with colour r.

To see that $\max\{\omega_1, \kappa^+\} \rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, first observe that the case $\kappa < \aleph_0$ follows from the case $\kappa = \aleph_0$. So we may assume $\kappa \geq \aleph_0$, implying that $\max\{\omega_1, \kappa^+\} = \kappa^+$. So let $c : \kappa^+ \rightarrow \kappa$. Then let

$$Z = \left(c^{-1}(\{r\}) \cap E^{\kappa^+}_\omega\right) \cup \left(\kappa^+ \setminus E^{\kappa^+}_\omega\right).$$

Suppose first that Z has a subset C that is club in κ^+. Then $C \cap \{\omega\cdot x : x \in \kappa^+\}$ is also club. Let the initial segment of this set of order type ω_1 be Y, and let $X = Y'$. Then $X \cong \omega_1$ by Proposition 2.2.4, but in addition $X \subseteq E^{\kappa^+}_\omega$ and hence $X \subseteq c^{-1}(\{r\})$ by definition of Z.

Suppose instead that Z has no subset that is club in κ^+. Then $\kappa^+\setminus Z$ is stationary in κ^+. But by definition of Z,

$$\kappa^+ \setminus Z = \bigcup_{i\in\kappa\setminus\{r\}} \left(c^{-1}(\{i\}) \cap E^{\kappa^+}_\omega\right).$$

Hence by Lemma 4.8.3 there exists $i \in \kappa\setminus\{r\}$ such that $c^{-1}(\{i\}) \cap E^{\kappa^+}_\omega$ is stationary in κ^+, which by Theorem 4.8.2 contains a topological copy of α_i. □

4.9 Independence results

We now move on to case 3 of the principle, in which no ordinal exceeds ω_1 and two or more ordinals are equal to ω_1. To begin with we quote the following result, a proof of which can be found in Weiss's article [Wei90, Theorem 2.8]. This follows easily from the fact that ω_1 may be written as a disjoint union of two stationary sets.

Proposition 4.9.1. *If $\beta \in \omega_2$ then $\beta \not\rightarrow_{top} (\omega_1)^1_2$.*

Corollary 4.9.2. *Let κ be a cardinal, and let α_i be an ordinal with $2 \leq \alpha_i \leq \omega_1$ for all $i \in \kappa$. Suppose $\alpha_r, \alpha_s = \omega_1$ for some distinct $r, s \in \kappa$. Then*

$$P^{top}\left(\alpha_i\right)_{i\in\kappa} \geq \max\left\{\omega_2, \kappa^+\right\}.$$

Proof. Clearly if $\zeta < \kappa^+$ then $\zeta \not\to_{top} \left(\alpha_i\right)^1_{i\in\kappa}$, and if $\zeta < \omega_2$ then $\zeta \not\to_{top} \left(\omega_1\right)^1_2$ by Proposition 4.9.1 and hence $\zeta \not\to_{top} \left(\alpha_i\right)^1_{i\in\kappa}$. □

We shall now see that, modulo a large cardinal assumption, this is the strongest ZFC-provable statement applicable to case 3. Recall from the statement of the principle that we write "$P_\kappa = x$" for the statement, "κ is a cardinal, and for all sequences $\left(\alpha_i\right)_{i\in\kappa}$ of ordinals, if $2 \leq \alpha_i \leq \omega_1$ for all $i \in \kappa$ and $\alpha_r, \alpha_s = \omega_1$ for some distinct $r, s \in \kappa$, then $P^{top}\left(\alpha_i\right)_{i\in\kappa} = x$", and likewise for "$P_\kappa \geq x$".

In one direction, we use the following result of Prikry and Solovay [PS75].

Theorem 4.9.3 (Prikry–Solovay)**.** *Suppose $V = L$ and let β be any ordinal. Then*

$$\beta \not\to_{top} \left(\omega_1\right)^1_2.$$

Corollary 4.9.4. *If* ZFC *is consistent, then so is*

$$\mathsf{ZFC} + \text{"for all cardinals } \kappa \geq 2,\ P_\kappa = \infty\text{".}$$ □

Proof. This follows immediately from the Prikry–Solovay theorem and monotonicity of pigeonhole numbers. □

In the other direction, we use a result of Shelah, who introduced the following notation [She98, Chapter X, §7].

Definition (Shelah)**.** Let λ be an uncountable regular cardinal. Write $\mathrm{Fr}^+(\lambda)$ to mean that every subset of E^λ_ω that is stationary in λ has a subset order-homeomorphic to ω_1.

Note the similarity between this notion and Theorem 4.8.2. In fact the letters "Fr" here refer to Friedman, who first asked whether or not there exists an ordinal β with $\beta \to_{top} \left(\omega_1\right)^1_2$ [Fri74].

Here is the result of Shelah [She98, Chapter XI, Theorem 7.6].

Theorem 4.9.5 (Shelah)**.** *If* ZFC+ *"there exists a supercompact cardinal" is consistent, then so is*

$$\mathsf{ZFC} + \text{"}\mathrm{Fr}^+(\lambda) \text{ holds for every regular cardinal } \lambda \geq \aleph_2\text{".}$$

In order to apply Shelah's result to case 3 we make the following observation.

Lemma 4.9.6. *Let $\kappa \geq \aleph_1$ be a cardinal. If* $\mathrm{Fr}^+(\kappa^+)$ *holds, then*

$$\kappa^+ \rightarrow_{top} (\omega_1)^1_\kappa .$$

Proof. Simply apply Lemma 4.8.3. □

Corollary 4.9.7. *If* ZFC+ *"there exists a supercompact cardinal" is consistent, then so is*

$$\mathsf{ZFC} + \text{“for all cardinals } \kappa \geq 2,\ P_\kappa = \max\left\{\omega_2, \kappa^+\right\}\text{”}.$$

Proof. Observe that by Corollary 4.9.2, the following is a theorem of ZFC: "for all cardinals $\kappa \geq 2$, $P_\kappa \geq \max\{\omega_2, \kappa^+\}$". To finish, simply combine Theorem 4.9.5 with Lemma 4.9.6. □

To conclude this section, we address the question of whether a large cardinal assumption is required. To this end we give an equiconsistency result essentially due to Silver and Shelah.

Silver proved the following result by showing that if $\omega_2 \rightarrow_{top} (\omega_1)^1_2$ then $\square_{\omega_1}$ does not hold, a proof of which can be found in Weiss's article [Wei90, Theorem 2.10].

Theorem 4.9.8 (Silver)**.** *If $\omega_2 \rightarrow_{top} (\omega_1)^1_2$ then ω_2 is Mahlo in L.*

Here is the result of Shelah [She98, Chapter XI, Theorem 7.1].

Theorem 4.9.9 (Shelah)**.** *If* ZFC+ *"there exists a Mahlo cardinal" is consistent, then so is* ZFC + *"*$\mathrm{Fr}^+(\aleph_2)$*".*

Corollary 4.9.10. ZFC+ *"there exists a Mahlo cardinal" is consistent if and only if*

$$\mathsf{ZFC} + \text{“}\omega_2 \rightarrow_{top} (\omega_1)^1_2\text{”}$$

is consistent.

Proof. Theorem 4.9.8 gives the "if" statement. The "only if" statement follows by combining Theorem 4.9.9 with Lemma 4.9.6. □

4.10 Sequences including an ordinal larger than ω_1

It remains to cover cases 1 and 2 of the principle, in which one of the ordinals exceeds ω_1. Although this appears to be a very large class of cases, the situation is dramatically simplified by the following elementary argument covering case 1. It is our only result in which the topological pigeonhole number (ZFC-provably) does not exist.

Proposition 4.10.1. *$P^{top}(\omega_1 + 1, \omega + 1) = \infty$.*

Proof. Let β be any ordinal. We show that $\beta \nrightarrow_{top} (\omega_1+1, \omega+1)^1$. First observe that a topological copy of ω_1+1 must contain a point of cofinality ω_1, while a topological copy of $\omega+1$ must contain a point of cofinality ω. The result is then witnessed by the colouring $c : \beta \to 2$ given by

$$c(x) = \begin{cases} 1, & \text{if } \operatorname{cf}(x) \geq \omega_1 \\ 0, & \text{otherwise.} \end{cases}$$ □

We conclude this section by simplifying case 2 using another elementary argument. We leave the rest of the proof for this case for the next section.

Lemma 4.10.2. *Let κ be a cardinal and let α_i be an ordinal for each $i \in \kappa$. Suppose $\alpha_r \geq \omega_1 + 1$ for some $r \in \kappa$ and $2 \leq \alpha_i \leq \omega$ for all $i \in \kappa \setminus \{r\}$, and let*

$$\lambda = \begin{cases} \kappa^+, & \text{if } \kappa \geq \aleph_0 \\ \aleph_0, & \text{if } \kappa < \aleph_0 \text{ and } \alpha_s = \omega \text{ for some } s \in \kappa \setminus \{r\} \\ \sum_{i \in \kappa \setminus \{r\}} (\alpha_i - 1) + 1, & \text{if } \kappa < \aleph_0 \text{ and } \alpha_i < \omega \text{ for all } i \in \kappa \setminus \{r\}. \end{cases}$$

Let β be any ordinal. Then

$$\beta \to_{top} (\alpha_i)^1_{i \in \kappa}$$

if and only if for every subset $A \subseteq \beta$ with $|A| < \lambda$ there exists $X \subseteq \beta \setminus A$ with $X \cong \alpha_r$.

Proof. First suppose that $\beta \to_{top} (\alpha_i)^1_{i \in \kappa}$ and let $A \subseteq \beta$ with $|A| < \lambda$. If $\kappa \geq \aleph_0$, then take $f : A \to \kappa \setminus \{r\}$ to be any injection; if $\kappa < \aleph_0$ and $\alpha_s = \omega$ for some $s \in \kappa \setminus \{r\}$, then take $f : A \to \{s\}$ to be the constant function; and if $\kappa < \aleph_0$ and $\alpha_i < \omega$ for all $i \in \kappa \setminus \{r\}$, then take $f : A \to \kappa \setminus \{r\}$ to be any function with $|f^{-1}(\{i\})| \leq \alpha_i - 1$ for all $i \in \kappa \setminus \{r\}$. Now define a colouring $c : \beta \to \kappa$ by

$$c(x) = \begin{cases} r, & \text{if } x \notin A \\ f(x), & \text{if } x \in A. \end{cases}$$

Then by construction $|c^{-1}(\{i\})| < \alpha_i$ for all $i \in \kappa \setminus \{r\}$, so since $\beta \to_{top} (\alpha_i)^1_{i \in \kappa}$ there exists $X \subseteq c^{-1}(\{r\}) = \beta \setminus A$ with $X \cong \alpha_r$.

Conversely, suppose that for every subset $A \subseteq \beta$ with $|A| < \lambda$ there exists $X \subseteq \beta \setminus A$ with $X \cong \alpha_r$. Let $c : \beta \to \kappa$ be a colouring, and let $A = c^{-1}(\kappa \setminus \{r\})$. If $|A| < \lambda$ then by assumption there exists $X \subseteq \beta \setminus A = c^{-1}(\{r\})$ with $X \cong \alpha_r$ and we are done, so assume $|A| \geq \lambda$. If $\kappa \geq \aleph_0$, then $|c^{-1}(\{j\})| \geq \kappa^+$ for some $j \in \kappa \setminus \{r\}$; if $\kappa < \aleph_0$ and $\alpha_s = \omega$ for some $s \in \kappa \setminus \{r\}$, then $|c^{-1}(\{j\})| \geq \aleph_0$ for some $j \in \kappa \setminus \{r\}$; and if $\kappa < \aleph_0$ and $\alpha_i < \omega$ for all $i \in \kappa \setminus \{r\}$, then by the finite pigeonhole principle $|c^{-1}(\{j\})| \geq \alpha_j$ for some $j \in \kappa \setminus \{r\}$. In every case $|c^{-1}(\{j\})| \geq |\alpha_j|$ and we are done. □

4.11 Proof of the principle

Having provided the key ingredients, we now complete the proof of the principle.

Proof of Theorem 4.6.1. We split into the same cases as in the statement of the theorem.

1. This follows from Proposition 4.10.1.

2. Let λ be as in Lemma 4.10.2, and note first of all that if α_r is a power of ω then $P^{top}\left(\alpha_i\right)_{i\in\kappa} \geq \alpha_r$ by part 2 of Proposition 4.3.3.

   (a) In this case $\lambda = \kappa^+$.

   i. Write $\omega^\beta \cdot m + 1 \leq \alpha_r \leq \omega^\beta \cdot (m+1)$ with β an ordinal and m a positive integer, and note that $\alpha_r \cdot \kappa^+ = \omega^\beta \cdot \kappa^+$.
   Suppose $\zeta < \omega^\beta \cdot \kappa^+$. To see that $\zeta \nrightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, let $A = \zeta \cap \left\{\omega^\beta \cdot \eta : \eta \in \kappa^+ \setminus \{0\}\right\}$, so $|A| < \kappa^+$. Then $(\zeta \setminus A)^{(\beta)} = \emptyset$ whereas $\left|\alpha_r^{(\beta)}\right| = m$, so $\zeta \setminus A$ cannot contain a topological copy of α_r.
   To see that $\omega^\beta \cdot \kappa^+ \rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, let $A \subseteq \omega^\beta \cdot \kappa^+$ with $|A| < \kappa^+$. Then

   $$A \subseteq \left(\bigcup_{\eta\in S} \left[\omega^\beta \cdot \eta + 1, \omega^\beta \cdot (\eta+1)\right]\right) \cup \left\{\omega^{\beta+1} \cdot \eta : \eta \in \kappa^+\right\}$$

   for some $S \subseteq \kappa^+$ with $|S| < \kappa^+$. Let $T \subseteq \kappa^+ \setminus S$ with $|T| = m+1$. Then

   $$\bigcup_{\eta\in T} \left[\omega^\beta \cdot \eta + 1, \omega^\beta \cdot (\eta+1)\right]$$

   is a topological copy of $\omega^\beta \cdot (m+1) + 1$ disjoint from A, which suffices.

   ii. Write $\alpha_r = \omega^\beta$. To see that $\alpha_r \cdot \kappa^+ \rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, simply observe that $\alpha_r \cdot \kappa^+ = (\alpha_r + 1) \cdot \kappa^+ \rightarrow_{top} \left(\alpha_r + 1, (\alpha_i)_{i\in\kappa\setminus\{r\}}\right)^1$ by the previous case and use monotonicity. It remains to show either that $\alpha_r \rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, or that if $\zeta < \alpha_r \cdot \kappa^+$ then $\zeta \nrightarrow_{top} (\alpha_i)^1_{i\in\kappa}$.

   A. To see that $\alpha_r \rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, simply observe that if $A \subseteq \alpha_r$ with $|A| < \kappa^+$, then $\sup A < \alpha_r$ since $\operatorname{cf}(\alpha_r) \geq \kappa^+$, and so $\alpha_r \setminus [0, \sup A] \cong \alpha_r$ since α_r is a power of ω.

   B. Suppose $\zeta < \alpha_r \cdot \kappa^+$. To see that $\zeta \nrightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, let $B \subseteq \alpha_r$ be club with $|B| = \operatorname{cf}(\alpha_r)$, and let

   $$A = \zeta \cap \left\{\alpha_r \cdot \eta + x : \eta \in \kappa^+, x \in B \cup \{0\}\right\}.$$

   Then $|A| < \kappa^+$ since $\operatorname{cf}(\alpha_r) < \kappa^+$. Suppose for contradiction $X \subseteq \zeta \setminus A$ with $X \cong \alpha_r$. Since α_r is a power of ω, using Theorem 4.4.1 and passing to a subspace if necessary, we may assume that

X is order-homeomorphic to α_r. Let $Y = X \cup \{\sup X\} \cong \alpha_r + 1$. Then $Y^{(\beta)} = \{\sup X\}$, so by Proposition 2.3.3 $\sup X = \alpha_r \cdot \eta$ for some $\eta \in \kappa^+ \setminus \{0\}$. It follows using Proposition 2.2.4 that X is club in $\alpha_r \cdot \eta$. But then $\mathrm{cf}\,(\alpha_r \cdot \eta) = \mathrm{cf}\,(\alpha_r) > \aleph_0$ and A is also club in $\alpha_r \cdot \eta$, so $X \cap A \neq \emptyset$, contrary to the definition of X.

C. • Suppose $\zeta < \alpha_r \cdot \kappa^+$. To see that $\zeta \nrightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, let

$$A = \zeta \cap \left\{\alpha_r \cdot \eta + \omega^\gamma \cdot x : \eta \in \kappa^+, x \in \omega^{\omega^\delta}\right\},$$

so $|A| < \kappa^+$ since $\delta < \kappa^+$. Then $(\zeta \setminus A)^{(\gamma)} = \emptyset$ whereas $\alpha_r^{(\gamma)} \cong \omega^{\omega^\delta}$, so $\zeta \setminus A$ cannot contain a topological copy of α_r.

• First note that since $\delta > 0$, either δ is a successor ordinal or $\mathrm{cf}\,(\delta) = \aleph_0$. To see that $\alpha_r \rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, let $A \subseteq \alpha_r$ with $|A| < \kappa^+$. Using the fact that $\delta > \kappa^+$, we now choose a strictly increasing cofinal sequence $(\beta_n)_{n\in\omega}$ from β with $\mathrm{cf}\left(\omega^{\beta_n}\right) = \mathrm{cf}\,(\beta_n) = \kappa^+$ for all $n \in \omega$. If $\delta = \varepsilon + 1$, then take $\beta_n = \gamma + \omega^\varepsilon \cdot n + \kappa^+$ for all $n \in \omega$. If $\mathrm{cf}\,(\delta) = \aleph_0$, then let $(\delta_n)_{n\in\omega}$ be a strictly increasing cofinal sequence from δ with $\delta_n > \kappa^+$ for all $n \in \omega$, and take $\beta_n = \gamma + \omega^{\delta_n} + \kappa^+$ for all $n \in \omega$. Then for each $n \in \omega$, let $x_n = \max\left\{\omega^{\beta_n}, \sup\left(A \cap \omega^{\beta_{n+1}}\right)\right\}$ and let $X_n = \left(x_n, \omega^{\beta_{n+1}}\right)$. Then $X_n \cong \omega^{\beta_{n+1}}$, so there exists $Y_n \subseteq X_n$ with $Y_n \cong \omega^{\beta_n} + 1$. Then $\bigcup_{n\in\omega} Y_n$ is a topological copy of α_r disjoint from A.

(b) In this case $\lambda = \aleph_0$.

i. To see that $\alpha_r \rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, simply observe that if $A \subseteq \alpha_r$ with $|A| < \aleph_0$ then $\alpha_r \setminus [0, \max A] \cong \alpha_r$.

ii. Write $\omega^\beta \cdot m + 1 \leq \alpha_r \leq \omega^\beta \cdot (m+1)$ with β an ordinal and m a positive integer, and note that $\alpha_r \cdot \omega = \omega^{\beta+1}$.

Suppose $\zeta < \omega^{\beta+1}$. To see that $\zeta \nrightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, let $A = \zeta \cap \left\{\omega^\beta \cdot n : n \in \omega \setminus \{0\}\right\}$, which is finite. Then $(\zeta \setminus A)^{(\beta)} = \emptyset$ whereas $\left|\alpha_r^{(\beta)}\right| = m$, so $\zeta \setminus A$ cannot contain a topological copy of α_r.

To see that $\omega^{\beta+1} \rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, simply observe that $\omega^{\beta+1} \rightarrow_{top} \left(\omega^{\beta+1}, (\alpha_i)_{i\in\kappa\setminus\{r\}}\right)^1$ by the previous case and use monotonicity.

(c) In this case $\lambda = \sum_{i\in\kappa\setminus\{r\}} (\alpha_i - 1) + 1$.

i. The result is trivial if $\kappa = 1$, and if α_r is a power of ω then the argument of case 2(b)i suffices.

ii. Suppose $\zeta < \omega^\beta \cdot (\lambda - 1 + m) + 1$. To see that $\zeta \nrightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, let $A = \zeta \cap \left\{\omega^\beta, \omega^\beta \cdot 2, \ldots, \omega^\beta \cdot (\lambda - 1)\right\}$, so $|A| < \lambda$. Then $\left|(\zeta \setminus A)^{(\beta)}\right| \leq m - 1$ whereas $\left|\alpha_r^{(\beta)}\right| = m$, so $\zeta \setminus A$ cannot contain a topological copy of α_r.

To see that $\omega^\beta \cdot (\lambda - 1 + m) + 1 \rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$, suppose $A \subseteq \omega^\beta \cdot (\lambda - 1 + m) + 1$ with $|A| < \lambda$. Then by the argument of case 2(b)i we may assume $A = \left\{\omega^\beta \cdot n : n \in S\right\}$ for some $S \subseteq \{1, 2, \ldots, \lambda - 1 + m\}$ with $S \leq \lambda$. Since $\kappa > 1$ we have $S \neq \emptyset$, say $s \in S$. Then

$$\left(\bigcup_{n\in\{1,2,\ldots,\lambda-1+m\}\setminus S} \left[\omega^\beta \cdot n + 1, \omega^\beta \cdot (n+1)\right]\right) \cup \left[\omega^\beta \cdot s + 1, \omega^\beta \cdot (s+1)\right)$$

is a topological copy of $\omega^\beta \cdot (m+1)$ disjoint from A, which suffices.

3. This is Corollaries 4.9.2, 4.9.4, 4.9.7 and 4.9.10.

4. This is Theorem 4.8.5.

5. This is Theorem 4.8.4.

6. (a) This is the finite pigeonhole principle.

   (b) This follows from Theorems 4.7.5 and 4.7.6 using monotonicity of pigeonhole numbers.

   (c) By Lemma 4.3.1, we may assume that for each $i \in \kappa$, either $\alpha_i = \varpi\left[\beta_i, m_i\right]$ or $\alpha_i = \omega^{\beta_i} \cdot (m_i + 1)$ and $\beta_i > 0$. It follows that one of Theorems 4.7.7 and 4.7.9 applies, and thus $P^{top}\left(\alpha_i\right)_{i\in\kappa}$ is equal to either $\omega^\beta \cdot m + 1$ or $\omega^\beta \cdot (m+1)$, where $\beta = \beta_0 \# \beta_1 \# \cdots \# \beta_{\kappa-1}$ and $m = \sum_{i\in\kappa} (m_i - 1) + 1$. It remains to determine whether or not $\omega^\beta \cdot m + 1 \rightarrow_{top} (\alpha_i)^1_{i\in\kappa}$.

      i. This is the "only if" part of Theorem 4.7.10.

      ii.
      - If there is no $s \in \kappa$ such that $\alpha_s = \omega^{\beta_s} \cdot (m_s + 1)$, then $\alpha_i = \varpi\left[\beta_i, m_i\right]$ for all $i \in \kappa$ and the result is given by Theorem 4.7.7.
      - If there exists $s \in \kappa$ with $\alpha_s = \omega^{\beta_s} \cdot (m_s + 1)$ and $m_i = 1$ for all $i \in \kappa \setminus \{s\}$, then assume without loss of generality that $\mathrm{CB}\left(\beta_s\right)$ is minimal among any $s \in \kappa$ with these properties. By definition of case 6(c)ii, there must still exist $t \in \kappa$ such that $\mathrm{CB}\left(\beta_t\right) < \mathrm{CB}\left(\beta_s\right)$, and so the result is given by the "if" part of Theorem 4.7.10.
      - Otherwise, let $c : \omega^\beta \cdot m + 1 \rightarrow \kappa$ be a colouring, and assume for simplicity that $\beta_i > 0$ for all $i \in \kappa$, the other case being no harder. First note that if $c^{-1}\left(\{j\}\right)$ contains a topological copy of ω^{β_j+1} for some $j \in \kappa$, then we are done. Therefore by Lemma 4.7.8 we may assume that for each $l \in m$ and each $i \in \kappa$, there exists $Y_{i,l} \subseteq c^{-1}\left(\{i\}\right) \cap \left[\omega^\beta \cdot l + 1, \omega^\beta \cdot (l+1)\right)$ with $Y_{i,l} \cong \omega^{\beta_i}$. Now by Theorem 4.7.7, there exists $j \in \kappa$ and $X \subseteq c^{-1}\left(\{j\}\right)$ with

$X \cong \omega^{\beta_j} \cdot m_j + 1$, and moreover by the proof of that theorem we may assume that

$$X \subseteq \bigcup_{l \in S} \left[\omega^{\beta} \cdot l + 1, \omega^{\beta} \cdot (l+1)\right]$$

for some $S \subseteq m$ with $|S| = m_j$. Two possibilities now remain.

– If there exist distinct $s, t \in \kappa$ with $m_s, m_t \geq 2$, then $m > m_i$ for all $i \in \kappa$.

– If there exists $s \in \kappa$ with $\alpha_s = \overline{\omega}\left[\beta_s, m_s\right]$, $m_s \geq 2$ and $m_i = 1$ for all $i \in \kappa \setminus \{s\}$, then $m > m_i$ for all $i \in \kappa \setminus \{s\}$. If $j = s$ then we are done, so we may assume that $j \neq s$.

In either case we have $m > m_j$. Therefore there exists $l \in m \setminus S$, whence $X \cup Y_{j,l}$ is a topological copy of $\omega^{\beta_j} \cdot (m_j + 1)$ in colour j, which suffices. □

4.12 The closed pigeonhole principle for ordinals

We have now completed the proof of the topological pigeonhole principle for ordinals. Using order-reinforcing ordinals and monotonicity, we automatically obtain from this many cases of the closed pigeonhole principle for ordinals. We now examine the remaining cases. This section is joint work with Andrés Caicedo.

Theorem 4.12.1 (The closed pigeonhole principle for ordinals)**.** *Given a cardinal κ and an ordinal $\alpha_i \geq 2$ for each $i \in \kappa$, there is an algorithm to compute $P^{cl}(\alpha_i)_{i \in \kappa}$.*

We use the word “algorithm” here merely as a shorthand for the full statement of the result. Rather than providing this explicitly, as we did for the topological principle, we merely explain within the proof the modifications required to obtain the closed principle from the topological principle.

While the topological principle gave in each case an explicit expression for the topological pigeonhole number in terms of Cantor normal forms, the closed pigeonhole principle will use a recursive expression. However, it will be seen that this yields an explicit expression after a finite number of applications.

Proof. The proof of the topological pigeonhole principle for ordinals shows that $P^{cl}(\alpha_i)_{i \in \kappa} = P^{top}(\alpha_i)_{i \in \kappa}$ except in two cases, which we now examine.

The first case is when κ is finite and greater than 1, $\alpha_r \geq \omega_1 + 1$ for some $r \in \kappa$, α_i is finite for all $i \in \kappa \setminus \{r\}$, and α_r is not a power of ω, say $\alpha_r = \omega^{\beta} \cdot m + 1 + \gamma$ for some ordinal β, some positive integer m and some ordinal $\gamma \leq \omega^{\beta}$ (case 2(c)ii). Then

$$P^{top}(\alpha_i)_{i \in \kappa} = \omega^{\beta} \cdot \left(\sum_{i \in \kappa \setminus \{r\}} (\alpha_i - 1) + m\right) + 1,$$

whereas a very similar argument shows that

$$P^{cl}(\alpha_i)_{i\in\kappa} = \omega^\beta \cdot \left(\sum_{i\in\kappa\setminus\{r\}} (\alpha_i - 1) + m \right) + 1 + \gamma.$$

The second case is when $(\alpha_i)_{i\in\kappa}$ is a finite sequence of countable ordinals (case 6). In this case the following results carry across for $\beta_1, \beta_2, \ldots, \beta_k \in \omega_1 \setminus \{0\}$.

- $P^{cl}(\omega^{\beta_1} + 1, \omega^{\beta_2} + 1, \ldots, \omega^{\beta_k} + 1) = \omega^{\beta_1 \# \beta_2 \# \cdots \# \beta_k} + 1$, where $\#$ denotes the natural sum (Theorem 4.7.4).

- $P^{cl}(\omega^{\beta_1}, \omega^{\beta_2}, \ldots, \omega^{\beta_k}) = \omega^{\beta_1 \odot \beta_2 \odot \cdots \odot \beta_k}$, where $\odot$ denotes the Milner–Rado sum (Theorem 4.7.5).

- Let $\alpha_1, \alpha_2, \ldots, \alpha_k \in \omega_1$ with $\alpha_r = \omega^{\beta_r}$ for some $r \in \{1, 2, \ldots, k\}$. Suppose β_i is minimal subject to the condition that $\alpha_i \leq \omega^{\beta_i}$ for all $i \in \{1, 2, \ldots, k\}$. We then have that $P^{cl}(\alpha_1, \alpha_2, \ldots, \alpha_k) = P^{cl}(\omega^{\beta_1}, \omega^{\beta_2}, \ldots, \omega^{\beta_k})$ (by Theorem 4.7.6).

Thus it remains to compute $P^{cl}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ when $\alpha_i \in \omega_1$ is not a power of ω for any $i \in \{1, 2, \ldots, k\}$. In that case, for all $i \in \{1, 2, \ldots, k\}$ we may write $\alpha_i = \omega^{\beta_i} + 1 + \gamma_i$ for some $\beta_i \in \omega_1 \setminus \{0\}$ and some ordinal $\gamma_i < \omega^{\beta_i + 1}$ (or $\alpha_i = 1 + \gamma_i$ for some ordinal $\gamma_i < \omega$ if α_i is finite).

Let $Q_i = P^{cl}(\alpha_1, \ldots, \alpha_{i-1}, \gamma_i, \alpha_{i+1}, \ldots, \alpha_k)$ for each $i \in \{1, 2, \ldots, k\}$. We claim that

$$P^{cl}(\alpha_1, \alpha_2, \ldots, \alpha_k) = P^{cl}(\omega^{\beta_1} + 1, \omega^{\beta_2} + 1, \ldots, \omega^{\beta_k} + 1) + \max\{Q_1, Q_2, \ldots, Q_k\} \quad (*)$$

(replacing $\omega^{\beta_i} + 1$ with 1 if α_i is finite). That this is large enough is clear. That no smaller ordinal is large enough follows from the existence for each $r \in \{1, 2, \ldots, k\}$ of a colouring $c_r : \omega^{\beta_1 \# \beta_2 \# \cdots \# \beta_k} + 1 \to \{1, 2, \ldots, k\}$ with the property that for each $i \in \{1, 2, \ldots, k\}$, there is a closed copy of $\omega^{\beta_i} + 1$ in colour i if and only if $i = r$, and no closed copy of any ordinal larger than $\omega^{\beta_r} + 1$ in colour r. To obtain such a colouring, simply extend the colouring given in Theorem 4.7.4 by setting $c_r(\omega^{\beta_1 \# \beta_2 \# \cdots \# \beta_k}) = r$. This equation allows $P^{cl}(\alpha_1, \alpha_2, \ldots, \alpha_k)$ to be computed recursively, since it will be reduced to the three cases above after a finite number of steps. □

Let us conclude by illustrating with a simple example of how the recursive expression $(*)$ may be used to compute a closed pigeonhole number. Reusing the definition of $\overline{\omega}\,[\alpha, m]$ from Section 4.7 for clarity, we have

$$\begin{aligned}
&P^{cl}(\omega^2 \cdot 2 + 2, \omega + 2) \\
&= P^{cl}(\overline{\omega}[2,2] + \overline{\omega}[0,1], \overline{\omega}[1,1] + \overline{\omega}[0,1]) \\
&= P^{cl}(\overline{\omega}[2,1], \overline{\omega}[1,1]) \\
&\qquad + \max\{P^{cl}(\overline{\omega}[2,1] + \overline{\omega}[0,1], \overline{\omega}[1,1] + \overline{\omega}[0,1]), P^{cl}(\overline{\omega}[2,2] + \overline{\omega}[0,1], \overline{\omega}[0,1])\} \\
&= \overline{\omega}[3,1] + P^{cl}(\overline{\omega}[2,1] + \overline{\omega}[0,1], \overline{\omega}[1,1] + \overline{\omega}[0,1]) \\
&= \overline{\omega}[3,1] + \overline{\omega}[3,1] + P^{cl}(\overline{\omega}[2,1] + \overline{\omega}[0,1], \overline{\omega}[0,1]) \\
&= \overline{\omega}[3,2] + \overline{\omega}[2,1] + \overline{\omega}[0,1] \\
&= \omega^3 \cdot 2 + \omega^2 + 2.
\end{aligned}$$

Chapter 5

Topological Ramsey theory of countable ordinals

In this chapter we study topological and closed partition relations with superscript 2. We look primarily at upper bounds for topological and closed ordinal Ramsey numbers of countable ordinals, and prove a topological version of the Erdős–Milner theorem, namely that $R^{top}(\alpha, k)$ and $R^{cl}(\alpha, k)$ are countable whenever α is countable and k is finite. The entire chapter is joint work with Andrés Caicedo.

5.1 Ordinal Ramsey numbers

Recall once again the classical, topological and closed partition relations for ordinals. In the previous chapter we solved the case $n = 1$, so we now move on to the case $n = 2$. We therefore make the following definition, which is analogous to definition of the pigeonhole numbers.

Definition. Let κ be a cardinal and let α_i be an ordinal for each $i \in \kappa$.

We define the (*classical*) *Ramsey number* $R\,(\alpha_i)_{i\in\kappa}$ to be the least ordinal β such that $\beta \to (\alpha_i)^2_{i\in\kappa}$, the *topological Ramsey number* $R^{top}\,(\alpha_i)_{i\in\kappa}$ to be the least ordinal β such that $\beta \to_{top} (\alpha_i)^2_{i\in\kappa}$, and the *closed Ramsey number* $R^{cl}\,(\alpha_i)_{i\in\kappa}$ to be the least ordinal β such that $\beta \to_{cl} (\alpha_i)^2_{i\in\kappa}$.

For example, $R(\omega, \omega) = R^{top}(\omega, \omega) = R^{cl}(\omega, \omega) = \omega$ by Ramsey's theorem.

As with the pigeonhole numbers, if there is no ordinal β such that $\beta \to (\alpha_i)^2_{i\in\kappa}$, then we say that $R\,(\alpha_i)_{i\in\kappa}$ does not exist, and similarly for the topological and closed Ramsey numbers. Furthermore, $R(\alpha_i)_{i\in\kappa}$, $R^{top}(\alpha_i)_{i\in\kappa}$ and $R^{cl}(\alpha_i)_{i\in\kappa}$ are also monotonically increasing functions of $(\alpha_i)_{i\in\kappa}$ (pointwise), and $R(\alpha_i)_{i\in\kappa} \leq R^{cl}(\alpha_i)_{i\in\kappa}$ and $R^{top}(\alpha_i)_{i\in\kappa} \leq R^{cl}(\alpha_i)_{i\in\kappa}$ since the closed partition relation implies the other two. Note also that $R((\alpha_i)_{i\in\kappa}, (2)_\lambda) = R(\alpha_i)_{i\in\kappa}$ for any cardinal λ, similarly for the topological and closed Ramsey numbers.

Since our interest is with countable Ramsey numbers, we are restricted by the following simple argument of Specker [Spe56, Theorem 4]. Here and throughout the chapter we identify 0 with the colour red and 1 with the colour blue.

Theorem 5.1.1 (Specker). *Let β be a countable ordinal. Then*

$$\beta \not\to (\omega+1,\omega)^2.$$

Proof. Since β is countable, we may let $\preccurlyeq$ be an ordering on β of order type ω. Define a colouring $c : [\beta]^2 \to \{\text{red,blue}\}$ as follows. Given $x, y \in \beta$ with $x < y$, set $c(\{x,y\}) = \text{red}$ if and only if $x \preccurlyeq y$. Thus if $X \subseteq \beta$ is red-homogeneous, then the $\preccurlyeq$-order type of X is the same as the $\leq$-order type of X, and if $X \subseteq \beta$ is blue-homogeneous, then the $\preccurlyeq$-order type of X is the same as the $\geq$-order type of X. Hence there is no red-homogeneous copy of $\omega+1$, or else we obtain a set of $\preccurlyeq$-order type $\omega+1$, and there is no blue-homogeneous copy of ω, or else we obtain a set of $\preccurlyeq$-order type ω^*. □

Note that this implies $\beta \not\to_{cl} (\omega+1,\omega)^2$ and $\beta \not\to_{top} (\omega+1,\omega)^2$ for all countable β. It follows that if $\alpha > \omega$ and $R(\alpha,\gamma)$ is countable, then γ must be finite, and similarly for the topological and closed Ramsey numbers. Thus if we wish to work with countable ordinals while avoiding considerations of purely finitary Ramsey theory, then we must look at $R(\alpha,k)$, $R^{top}(\alpha,k)$ and $R^{cl}(\alpha,k)$ with α countable and k finite.

Erdős and Milner [EM72] proved that $R(\alpha,k)$ is indeed countable for all countable α and all finite k. More precisely, they proved the following result, from which it follows by induction on k that $R(\omega^{1+\alpha},2^k) \leq \omega^{1+\alpha\cdot k}$.

Theorem 5.1.2 (Erdos–Milner). *Let α and β be countable non-zero ordinals, and let $k > 1$ be a positive integer. If*

$$\omega^\alpha \to (\omega^{1+\beta},k)^2,$$

then

$$\omega^{\alpha+\beta} \to (\omega^{1+\beta},2k)^2.$$

We will provide a simplified proof of a weak version of this result later, followed by a topological version of this result, which tells us that $R^{top}(\alpha,k)$ and $R^{cl}(\alpha,k)$ are countable for all countable α and all finite k.

Beforehand we study the topological and closed ordinal Ramsey numbers $R^{top}(\alpha,k)$ and $R^{cl}(\alpha,k)$ for certain small values of α. Note that these are closely related, since if α is order-reinforcing then $R^{top}(\alpha,k) = R^{cl}(\alpha,k)$ for all finite k.

Let us conclude this section with a lower bound for Ramsey numbers in terms of pigeonhole numbers, which is obtained by considering a k-partite graph. Although it is very simple, it remains our best lower bound with the exception of a couple of special cases (Lemmas 5.3.3 and 5.6.4), both of which rely on finite combinatorial arguments.

Proposition 5.1.3. *If $\alpha \geq 2$ is an ordinal and k is a positive integer, then*

$$R(\alpha, k+1) \geq P(\alpha)_k,$$

and similarly for the topological and closed relations.

Proof. Suppose $\beta < P(\alpha)_k$. By definition of the pigeonhole number, it follows that there exists a colouring $c : \beta \to k$ such that for all $i \in k$, no subset of $c^{-1}(\{i\})$ is order-isomorphic to α. To see that $\beta \not\to (\alpha, k+1)^2$, simply consider the colouring $d : [\beta]^2 \to \{\text{red,blue}\}$ given by

$$d(\{x,y\}) = \begin{cases} \text{red}, & \text{if } c(x) = c(y) \\ \text{blue}, & \text{if } c(x) \neq c(y). \end{cases}$$

It is straightforward to verify that d does indeed witness $\beta \not\to (\alpha, k+1)^2$.

For the topological and closed relations, simply replace "order-isomorphic" with "homeomorphic" or "order-homeomorphic" as necessary. □

5.2 The ordinal $\omega + 1$

We begin with the simplest non-trivial case of an infinite topological Ramsey number. It may be viewed as a strengthening of Proposition 4.5.1. Later, we will provide a second proof of this result (in Section 5.6).

Theorem 5.2.1. *If k is a positive integer, then*

$$R^{top}(\omega+1, k+1) = R^{cl}(\omega+1, k+1) = \omega^k + 1.$$

Proof. The first equality is immediate from the fact that $\omega+1$ is order-reinforcing, and the fact that $R^{top}(\omega+1, k+1) \geq \omega^k + 1$ follows from Propositions 5.1.3 and 4.5.1.

It remains to prove that $\omega^k + 1 \to_{top} (\omega+1, k+1)^2$, which we do by induction on k. The case $k = 1$ is trivial. For the inductive step, suppose $k \geq 2$ and let $c : [\omega^k + 1]^2 \to \{\text{red}, \text{blue}\}$ be a colouring. For each $m \in \omega$, let $X_m = \left[\omega^{k-1} \cdot m + 1, \omega^{k-1} \cdot (m+1)\right] \cong \omega^{k-1} + 1$. For each $m \in \omega$, we may assume by the inductive hypothesis that X_m has a blue-homogeneous set B_m of size k, or else it has a red-homogeneous closed copy of $\omega + 1$ and we are done. If for any $m \in \omega$ it is the case that $c(\{x, \omega^k\}) = \text{blue}$ for all $x \in B_m$, then $B_m \cup \{\omega^k\}$ is a blue-homogeneous set of size $k+1$ and we are done. Otherwise for each $m \in \omega$ we may choose $x_m \in B_m$ with $c(\{x_m, \omega^k\}) = \text{red}$. Finally, by Ramsey's theorem, within the set $\{x_m : m \in \omega\}$ there is either a blue-homogeneous set of size $k+1$, in which case we are done, or an infinite red-homogeneous set H, in which case $H \cup \{\omega^k\}$ is a red-homogeneous closed copy of $\omega+1$, and we are done as well. □

It is somewhat surprising that we are able to obtain an exact equality here using the lower bound from Proposition 5.1.3. As we shall see, for ordinals larger than $\omega+1$ there is typically a large gap between our upper and lower bounds, and we expect that improvement will usually be possible on both sides.

5.3 Stepping up by one

In this section we consider how to obtain an upper bound for $R^{cl}(\alpha+1,k+1)$, given upper bounds for $R^{cl}(\alpha,k+1)$ and $R^{cl}(\alpha+1,k)$ for a successor ordinal α and a positive integer k. Note that if α is an infinite successor ordinal, then $\alpha+1\cong\alpha$, so trivially $R^{top}(\alpha+1,k+1)=R^{top}(\alpha,k+1)$, which is why we study R^{cl} instead.

First we give two simple upper bounds. The first comes from considering the edges incident to the largest point, as in a standard proof of the existence of the finite Ramsey numbers. The second typically gives worse bounds, but is more similar to the technique we will consider next.

Proposition 5.3.1. *Let α be a successor ordinal, and let $k\geq 2$ be a positive integer.*

1. $R^{cl}(\alpha+1,k+1)\leq P^{cl}(R^{cl}(\alpha,k+1),R^{cl}(\alpha+1,k))+1.$
2. $R^{cl}(\alpha+1,k+1)\leq P^{cl}(R^{cl}(\alpha,k+1))_k+R^{cl}(\alpha+1,k).$

Proof. First note that since α is a successor ordinal, a closed copy of $\alpha+1$ is obtained from any closed copy of α together with any larger point.

1. Let $\beta=P^{cl}(R^{cl}(\alpha,k+1),R^{cl}(\alpha+1,k))$ and let $c:[\beta+1]^2\to\{\text{red},\text{blue}\}$ be a colouring. This induces a colouring $d:\beta\to\{\text{red},\text{blue}\}$ given by $d(x)=c(\{x,\beta\})$. By definition of P^{cl}, there exists $X\subseteq\beta$ such that either $X\subseteq d^{-1}(\{\text{red}\})$ and X is a closed copy of $R^{cl}(\alpha,k+1)$, or $X\subseteq d^{-1}(\{\text{blue}\})$ and X is a closed copy of $R^{cl}(\alpha+1,k)$. In the first case, by definition of R^{cl}, we are either done immediately, or we obtain a red-homogeneous closed copy Y of α, in which case $Y\cup\{\beta\}$ is a red-homogeneous closed copy of $\alpha+1$. The second case is similar.

2. Let $\beta=P^{cl}(R^{cl}(\alpha,k+1))_k$ and let $c:[\beta+R^{cl}(\alpha+1,k)]^2\to\{\text{red},\text{blue}\}$ be a colouring. By definition of R^{cl}, either we are done or there is a blue-homogeneous set of k points among those $\geq\beta$, say $\{x_1,x_2,\ldots,x_k\}$. If for any $y<\beta$ we have $c(\{y,x_i\})=\text{blue}$ for all $i\in\{1,2,\ldots,k\}$, then we are done. Otherwise define a colouring $d:\beta\to k$ by taking $d(y)$ to be some i such that $c(\{y,x_i\})=\text{red}$. Then by definition of P^{cl}, there exists $i\in\{1,2,\ldots,k\}$ and $X\subseteq\beta$ such that $X\subseteq d^{-1}(\{i\})$ and X is a closed copy of $R^{cl}(\alpha,k+1)$. But then X either contains a blue-homogeneous set of $k+1$ points, or a red-homogeneous closed copy Y of α, in which case $Y\cup\{x_i\}$ is a red-homogeneous closed copy of $\alpha+1$, as required. $\square$

Note that $R^{cl}(\alpha, 2) = \alpha$ for any ordinal α. Hence if α is a successor ordinal and we have an upper bound on $R^{cl}(\alpha, k)$ for every positive integer k, then we may easily apply either of these two inequalities recursively to obtain upper bounds on $R^{cl}(\alpha + n, k)$ for all positive integers n and k.

Unfortunately, neither of these techniques appears to generalise well to limit ordinals, since they are "backward-looking" in some sense. Our "forward-looking" technique is a little more complicated and, in the form presented here, only works below ω^2. Before we state the general result, here is an illustrative special case.

Lemma 5.3.2. $R^{cl}(\omega + 2, 3) \leq R^{cl}(\omega + 1, 3) + P^{cl}(\omega + 2)_2 = \omega^2 \cdot 2 + \omega + 2$.

Proof. First note that $R^{cl}(\omega + 1, 3) + P^{cl}(\omega + 2)_2 = (\omega^2 + 1) + (\omega^2 + \omega + 2) = \omega^2 \cdot 2 + \omega + 2$ by Theorem 5.2.1 and Theorem 4.12.1. It remains to prove that $R^{cl}(\omega + 1, 3) + P^{cl}(\omega + 2)_2 \to_{cl} (\omega + 2, 3)^2$.

Let $\beta = R^{cl}(\omega + 1, 3)$, let $c : [\beta + P^{cl}(\omega + 2)_2]^2 \to \{\text{red}, \text{blue}\}$ be a colouring, and suppose for contradiction that there is no red-homogeneous closed copy of $\omega + 2$ and no blue triangle.

By definition of R^{cl}, either there is a blue triangle, in which case we are done, or there exists $X \subseteq \beta$ such that X is a red-homogeneous closed copy of $\omega + 1$. Let x be the largest point in X and let $H = X \setminus \{x\}$.

Let $A_1 = \{y \geq \beta : c(\{x, y\}) = \text{blue}\}$ and let $A_2 = \{y \geq \beta : c(\{h, y\}) = \text{blue}$ for all but finitely many $h \in H\}$.

First of all, we claim that A_1 is red-homogeneous. This is because if $y, z \in A_1$ and $c(\{y, z\}) = \text{blue}$, then $\{x, y, z\}$ is a blue triangle.

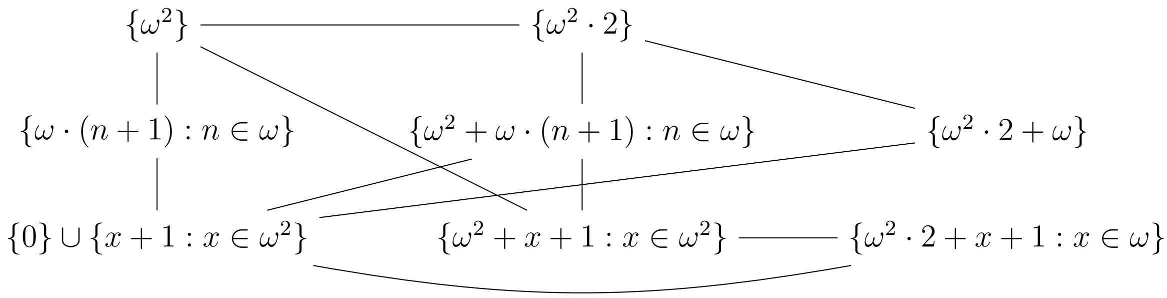
Next, we claim that A_2 is red-homogeneous. This is because if $y, z \in A_2$ and $c(\{y, z\}) = \text{blue}$, then $c(\{h, y\}) = c(\{h, z\}) = \text{blue}$ for all but finitely many $h \in H$, and so $\{h, y, z\}$ is a blue triangle for any such h.

Finally, we claim that if $y \geq \beta$, then $y \in A_1 \cup A_2$. For otherwise we would have $y \geq \beta$ with $c(\{x, y\}) = \text{red}$ and $c(\{h, y\}) = \text{red}$ for all h in some infinite subset $K \subseteq H$, whence $K \cup \{x, y\}$ is a red-homogeneous closed copy of $\omega + 2$. Hence by definition of P^{cl}, either A_1 or A_2 contains a closed copy of $\omega + 2$, which by the above claims must be red-homogeneous, and we are done. $\square$

Digressing briefly, we show that in this particular case the upper bound is optimal, and thus $R^{cl}(\omega + 2, 3) = \omega^2 \cdot 2 + \omega + 2$. The colouring we present was found by Omer Mermelstein.

Lemma 5.3.3. $R^{cl}(\omega + 2, 3) \geq \omega^2 \cdot 2 + \omega + 2$.

Proof. We provide a colouring witnessing $\omega^2 \cdot 2 + \omega + 1 \not\to_{cl} (\omega + 2, 3)^2$. In order to define it, let G be the graph represented by the following diagram.



Define a colouring $c : [\omega^2 \cdot 2+\omega+1]^2 \to \{\text{red}, \text{blue}\}$ by setting $c(\{x, y\}) = \text{blue}$ if and only if x and y lie in distinct, adjacent vertices of G. First note that there is no blue triangle since G is triangle-free. Now suppose X is a closed copy of $\omega + 2$, and write $X = Z \cup \{x\} \cup \{y\}$ with $z < x < y$ for all $z \in Z$. If $Z \cup \{x\}$ is red-homogeneous, then either $x = \omega^2$ and $Z \subseteq \{0\} \cup \{x + 1 : x \in \omega^2\}$, or $x = \omega^2 \cdot 2$ and (discarding a finite initial segment of Z if necessary) we may assume $Z \subseteq \{\omega^2 + x + 1 : x \in \omega^2\}$. In each case either $c(\{x, y\}) = \text{blue}$ or $c(\{z, y\}) = \text{blue}$ for all $z \in Z$. Hence X cannot be red-homogeneous, and we are done. $\square$

Here is the general formulation of our "forward-looking" technique.

Proposition 5.3.4. *Let k, m and n be positive integers with $k \geq 2$. Then*

$$R^{cl}(\omega \cdot m + n + 1, k + 1) \leq R^{cl}(\omega \cdot m + n, k + 1) + P^{cl}(R^{cl}(\omega \cdot m + n + 1, k))_{2m+n-1}.$$

Proof. Let $\beta = R^{cl}(\omega \cdot m+n, k+1)$ and let $c : [\beta+P^{cl}(R^{cl}(\omega \cdot m+n+1, k))_{2m+n-1}]^2 \to \{\text{red}, \text{blue}\}$ be a colouring.

By definition of R^{cl}, either there is a blue-homogeneous set of $k + 1$ points, in which case we are done, or there exists $X \subseteq \beta$ such that X is a red-homogeneous closed copy of $\omega \cdot m + n$. In that case, write

$$X = H_1 \cup \{x_1\} \cup H_2 \cup \{x_2\} \cup \cdots \cup \{x_{m-1}\} \cup H_m \cup \{y_1, y_2, \ldots, y_n\}$$

with $H_1, H_2, \ldots, H_m$ each of order type ω and $h_1 < x_1 < h_2 < x_2 < \cdots < x_{m-1} < h_m < y_1 < y_2 < \cdots < y_n$ whenever $h_i \in H_i$ for all $i \in \{1, 2, \ldots, m\}$.

For each $z \geq \beta$, if every single one of the following $2m + n - 1$ conditions holds, then $X \cup \{z\}$ contains a closed copy of $\omega \cdot m + n + 1$, and we are done.

- $c(\{h, z\}) = \text{red}$ for infinitely many $h \in H_i$ (one condition for each $i \in \{1, 2, \ldots, m\}$)
- $c(\{x_i, z\}) = \text{red}$ (one condition for each $i \in \{1, 2, \ldots, m - 1\}$)
- $c(\{y_i, z\}) = \text{red}$ (one condition for each $i \in \{1, 2, \ldots, n\}$)

Thus we may assume that one of these conditions fails for each $z \geq \beta$. This induces a $2m + n - 1$-colouring of these points, so by definition of P^{cl} there is a closed copy X of $R^{cl}(\omega \cdot m + n + 1, k)$ among these points such that the same condition fails for each $z \in X$.

Finally, by definition of R^{cl}, X contains either a red-homogeneous closed copy of $\omega \cdot m + n + 1$, in which case we are done, or a blue-homogeneous set of k points, in which case using the failed condition we can find a final point with which to construct a blue-homogeneous set of $k + 1$ points. □

Because of Theorem 5.2.1, any one of the inequalities from Propositions 5.3.1 and 5.3.4 is enough for us to obtain upper bounds for $R^{cl}(\omega+n, k)$ for all finite n, k. We conclude this section with an explicit statement of a few of these upper bounds. Curiously, in the second part, we require both Propositions 5.3.1 and 5.3.4 in order to obtain the best bound.

Corollary 5.3.5. *1.* $R^{cl}(\omega + 2, 4) \leq \omega^4 \cdot 3 + \omega^3 + \omega^2 + \omega + 2.$

2. If $k \geq 4$ *is a positive integer, then*

$$R^{cl}(\omega+2, k+1) \leq \omega^r \cdot 3+\omega^{r-1}+\omega^{r-2}+\omega^{r-3}+\omega^{r-4}+\omega^{r-8}+\omega^{r-13}+\omega^{r-19}+\cdots+\omega^k+2,$$

where $r = \frac{k^2+k-4}{2}$.

3. If $n \geq 3$ *is a positive integer, then*

$$R^{cl}(\omega+n, 3) \leq P^{cl}(\omega+n)_n = \omega^n+\omega^{n-1}(n-1)+\omega^{n-2}(n-1)+\cdots+\omega\cdot(n-1)+n.$$

Proof. 1. By Proposition 5.3.4 and Lemma 5.3.2, $R^{cl}(\omega + 2, 4) \leq R^{cl}(\omega + 1, 4) + P^{cl}(R^{cl}(\omega + 2, 3))_2 = \omega^3 + 1 + P^{cl}(\omega^2 \cdot 2 + \omega + 2)_2 = \omega^4 \cdot 3 + \omega^3 + \omega^2 + \omega + 2.$

2. This can be obtained from part 1 by recursively applying the first inequality from Proposition 5.3.1.

3. By Proposition 5.3.4, $R^{cl}(\omega + n, 3) \leq R^{cl}(\omega + (n - 1), 3) + P^{cl}(\omega + n)_n$. Using Lemma 5.3.2 for the base case, it is easy to see by induction that the second term here has the largest power of ω, since $n \geq 3$. Hence $R^{cl}(\omega + n, 3) \leq P^{cl}(\omega + n)_n$, and the equality with the right-hand side is a simple induction exercise. □

Remark 5.3.6. The inequality in part 3 of this result is improved upon by Lemma 5.4.5 from the following section for all $n \geq 9$.

For comparison, the corresponding lower bounds given by Proposition 5.1.3 are as follows. If $k, n \geq 3$ are positive integers, then

$$R^{cl}(\omega + 2, k + 1) \geq P(\omega + 2)_k = \omega^k + \omega^{k-1} + \cdots + \omega + 2$$

and

$$R^{cl}(\omega + n, 3) \geq P(\omega + n)_2 = \omega^2 + \omega \cdot (n-1) + n,$$

but note the latter is already far behind $R^{cl}(\omega+n,3) \geq R^{cl}(\omega+2,3) = \omega^2 \cdot 2+\omega+2$.

5.4 Ordinals less than ω^2

So far we only have upper bounds on $R^{cl}(\alpha, k)$ for $\alpha < \omega \cdot 2$. In this section we extend this to $\alpha < \omega^2$ with the following result.

Theorem 5.4.1. *If k and m are positive integers, then*

$$R^{cl}(\omega \cdot m + 1, k + 1) < \omega^\omega.$$

In fact, in each case the proof gives an explicit upper bound below ω^ω.

Our proof builds upon the stepping up technique from Proposition 5.3.4. We also make use of some classical ordinal Ramsey theory in the form of the following result.

Theorem 5.4.2 (Erdős–Rado)**.** *If k and m are positive integers, then $R(\omega \cdot m, k) < \omega^2$.*

In fact, Erdős and Rado computed the exact values of these Ramsey numbers in terms of a combinatorial property of finite digraphs. More precisely, we consider digraphs for which loops are not allowed, but edges between two vertices pointing in both directions are allowed. The complete digraph on m vertices is denoted by K_m^*. Recall that a *tournament* of order k is a digraph obtained by assigning directions to the edges of the complete (undirected) graph on k vertices, and that a tournament is transitive if and only if these assignments are compatible, that is, if and only if whenever x, y and z are distinct vertices with an edge from x to y and an edge from y to z, then there is also an edge from x to z. The class of transitive tournaments of order k is denoted by L_k.

Using this terminology, Theorem 5.4.2 can be deduced from the following two results of Erdős and Rado (who stated them in a slightly different manner). See [ER56, Theorem 25] and [ER67].

Lemma 5.4.3 (Erdős–Rado)**.** *If k and m are positive integers, then there is a positive integer p such that any digraph on p or more vertices admits either an independent set of size m, or a transitive tournament of order k. We denote the least such p by $R(K_m^*, L_k)$.*

Theorem 5.4.4 (Erdős–Rado)**.** *If $k, m > 1$ are positive integers, then $R(\omega \cdot m, k) = \omega \cdot R(K_m^*, L_k)$.*

Before indicating how to prove Theorem 5.4.1 in general, we first illustrate the key ideas with the following special case. We use the special case of Theorem 5.4.4 that $R(\omega \cdot 2, 3) = \omega \cdot 4$.

Lemma 5.4.5. $R^{cl}(\omega \cdot 2 + 1, 3) \leq \omega^8 \cdot 7 + 1$.

The proof uses a non-principal ultrafilter on ω. This use is superfluous, but makes the idea more transparent, since in some sense the ultrafilter makes many choices for us. The result is actually provable in (a weak sub-theory of) ZF (see Remark 5.4.7).

Proof. To make explicit the reason for using the ordinal $\omega^8 \cdot 7 + 1$, let

- $\beta_1 = P^{cl}(\omega \cdot 2 + 1, \omega \cdot 2 + 1, \omega^2 + 1) = \omega^4 \cdot 3 + 1$,
- $\beta_2 = P^{cl}(\omega \cdot 2 + 1, \omega \cdot 2 + 1, \omega^2 + 1 + \beta_1) = \omega^6 \cdot 5 + 1$,
- $\beta_3 = P^{cl}(\omega \cdot 2 + 1, \omega \cdot 2 + 1, \omega^2 + 1 + \beta_2) = \omega^8 \cdot 7 + 1$ and
- $\beta = \omega^2 + 1 + \beta_3 = \omega^8 \cdot 7 + 1$.

Fix a non-principal ultrafilter $\mathcal{U}$ on ω and let $c : [\beta]^2 \to \{\text{red}, \text{blue}\}$ be a colouring.

Among the first $\omega^2 + 1$ elements of β, we may assume that we have a red-homogeneous closed copy of $\omega + 1$. Let x be its largest point and let H be its subset of order type ω. Write $H = \{h_n : n \in \omega\}$ with $h_0 < h_1 < \dots$.

Now the set of points $\geq \omega^2 + 1$ forms a disjoint union $A_1 \cup A_2 \cup A_3$, where

- $A_1 = \{a \geq \omega^2 + 1 : c(\{x, a\}) = \text{blue}\}$,
- $A_2 = \{a \geq \omega^2 + 1 : c(\{x, a\}) = \text{red but } \{n \in \omega : c(\{h_n, a\}) = \text{blue}\} \in \mathcal{U}\}$ and
- $A_3 = \{a \geq \omega^2 + 1 : c(\{x, a\}) = \text{red and } \{n \in \omega : c(\{h_n, a\}) = \text{red}\} \in \mathcal{U}\}$.

If either A_1 or A_2 contains a closed copy of $\omega \cdot 2 + 1$, then we are done. (For A_2, we use the fact that if $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$.) So by definition of β_3, we may assume that A_3 contains a closed copy X of $\omega^2 + 1 + \beta_2$.

Now we repeat the argument within X. Among the first $\omega^2 + 1$ members of X, we may assume we that have a red-homogeneous closed copy of $\omega + 1$. Let y be its largest point, and let I be its subset of order type ω. Write $I = \{i_n : n \in \omega \dots\}$ with $i_0 < i_1 < \dots$. (Note that at this stage, $\{n \in \omega : c(\{h_n, i\}) = \text{red}\} \in \mathcal{U}$ for all $i \in I$, yet we cannot conclude from this that there are infinite subsets $H' \subseteq H$ and $I' \subseteq I$ such that $H' \cup I'$ is red-homogeneous.)

Just as before, write the subset of X lying above its first $\omega^2 + 1$ members as a disjoint union $B_1 \cup B_2 \cup B_3$, where $b \in B_1$ if and only if $c(\{y, b\}) = \text{blue}$, $b \in B_2$ if and only if $c(\{y, b\}) = \text{red}$ but $\{n \in \omega : c(\{i_n, b\}) = \text{blue}\} \in \mathcal{U}$, and $b \in B_3$ if and only if $c(\{y, b\}) = \text{red}$ and $\{n \in \omega : c(\{i_n, b\}) = \text{red}\} \in \mathcal{U}$. Again we may conclude from the definition of β_2 that B_3 contains a closed copy Y of $\omega^2 + 1 + \beta_1$.

Repeat this argument once again within Y, and then pass to a final red-homogeneous closed copy of $\omega+1$. We obtain a closed set

$$H \cup \{x\} \cup I \cup \{y\} \cup J \cup \{z\} \cup K \cup \{w\}$$

of order type $\omega \cdot 4 + 1$ (with $J = \{j_n : n \in \omega\}$, $j_0 < j_1 < \dots$ and $K = \{k_n : n \in \omega\}$, $k_0 < k_1 < \dots$) such that $H \cup \{x\}$, $I \cup \{y\}$, $J \cup \{z\}$ and $K \cup \{w\}$ are red-homogeneous, $\{x, y, z, w\}$ is red-homogeneous, $c(\{x, i_n\}) = c(\{x, j_n\}) = c(\{x, k_n\}) = c(\{y, j_n\}) = c(\{y, k_n\}) = c(\{z, k_n\}) = \text{red}$ for all $n \in \omega$, and finally for any $a > x$, $b > y$ and $c > z$ in this set, we have $\{n \in \omega : c(\{h_n, a\}) = \text{red}\} \in \mathcal{U}$, $\{n \in \omega : c(\{i_n, b\}) = \text{red}\} \in \mathcal{U}$ and $\{n \in \omega : c(\{j_n, c\}) = \text{red}\} \in \mathcal{U}$.

At last we use the ultrafilter, in which is the crucial step of the argument. Let

- $H' = \{h \in H : c(\{h, y\}) = c(\{h, z\}) = c(\{h, w\}) = \text{red}\}$,
- $I' = \{i \in I : c(\{i, z\}) = c(\{i, w\}) = \text{red}\}$ and
- $J' = \{j \in J : c(\{j, w\}) = \text{red}\}$,

each of which corresponds to some $U \in \mathcal{U}$ and is therefore infinite. (Note that if we had tried to argue directly, without using the ultrafilter or modifying the construction in a substantial way, then we would have been able to deduce that J' is infinite, but it would not have been apparent that H' or I' are.) This ensures that $c(\{a, b\}) = \text{red}$ whenever $a \in \{x, y, z, w\}$ and $b \in H' \cup \{x\} \cup I' \cup \{y\} \cup J' \cup \{z\} \cup K \cup \{w\}$.

To complete the proof, recall that that $\omega \cdot 4 \to (\omega \cdot 2, 3)^2$. It follows that there is either a blue triangle, in which case we are done, or a red-homogeneous subset $M \subseteq H' \cup I' \cup J' \cup K$ of order type $\omega \cdot 2$. Let S be the initial segment of M of order type ω and $T = M \backslash S$, and let $s = \sup(S)$ and $t = \sup(T)$ (so that $s, t \in \{x, y, z, w\}$). Then $S \cup \{s\} \cup T \cup \{t\}$ is a red-homogeneous closed copy of $\omega \cdot 2 + 1$, and we are done. □

We now indicate the modifications required to obtain the general result.

Proof of Theorem 5.4.1. The proof is by induction on k. The case $k = 1$ is trivial. For the inductive step, suppose $k \geq 2$. We can now use the argument of Lemma 5.4.5 with just a couple of changes.

Firstly, we require $\omega^k + 1$ points in order to be able to assume that we have a red-homogeneous closed copy of $\omega + 1$.

Secondly, it is no longer enough for A_1 or A_2 to contain a closed copy of $\omega \cdot m + 1$, but it is enough for one of them to contain a closed copy of $R^{cl}(\omega \cdot m + 1, k)$, which we have an upper bound on by the inductive hypothesis (and likewise for B_1 and B_2, and so on).

Finally, in order to complete the proof using Theorem 5.4.4, we must iterate the argument $R(K_m^*, L_{k+1})$ times.

This argument demonstrates that $R^{cl}(\omega \cdot m+1, k+1) \leq \omega^k + 1 + \beta_{R(K^*_m, L_{k+1})-1}$, where $\beta_0 = 0$ and $\beta_i = P^{cl}(R^{cl}(\omega \cdot m + 1, k), R^{cl}(\omega \cdot m + 1, k), \omega^k + 1 + \beta_{i-1})$ for $i \in \{1, 2, \ldots, R(K^*_m, L_{k+1}) - 1\}$. □

To obtain upper bounds for $R^{cl}(\omega \cdot m + n, k)$ for all finite k, m and n, one can again use any of the three inequalities from Propositions 5.3.1 and 5.3.4, which may give better bounds than simply using the bound on $R^{cl}(\omega \cdot (m+1)+1, k)$ given by Theorem 5.4.1.

Remark 5.4.6. It is perhaps worth pointing out that the classical version of this problem, the precise computation of the numbers $R(\omega \cdot m+n, k)$ for finite k, m and n, was solved more than 40 years ago. It proceeds by reducing the problem to a question about finite graphs that can be effectively, albeit unfeasibly, solved with a computer. This was announced without proof in [HS69a] and [HS69b]. See [Cai15] for further details.

Remark 5.4.7. At the cost of a somewhat more cumbersome approach, we may eliminate the use of the non-principal ultrafilter and any appeal to the axiom of choice throughout this and the next section. Rather than presenting this version of the proof, we mention a simple and well-known absoluteness argument ensuring that choice is indeed not needed.

We present the argument in the context of Lemma 5.4.5; the same approach removes all other uses of choice in this and the next section. Work in ZF. With β as in the proof of Lemma 5.4.5, consider a colouring $c : [\beta]^2 \to 2$, and note that $L[c]$ is a model of choice, and that the definitions of β and of homogeneous closed copies of $\omega \cdot 2+1$ and 3 are absolute between the universe of sets and this inner model. Since $L[c]$ is a model of choice, the argument of Lemma 5.4.5 gives us a homogeneous set as required, with the additional information that it belongs to $L[c]$.

5.5 The ordinal ω^2

In this section we adapt the argument from the previous section to prove the following result.

Theorem 5.5.1. *If k is a positive integer, then $\omega^\omega \to_{cl} (\omega^2, k)$.*

Since ω^2 is order-reinforcing, it follows that $R^{top}(\omega^2, k) = R^{cl}(\omega^2, k) \leq \omega^\omega$.

The ordinal ω^ω appears essentially because $P^{cl}(\omega^\omega)_m = \omega^\omega$ for all finite m, allowing us to iterate the argument of Lemma 5.4.5 infinitely many times.

Again we require a classical ordinal Ramsey result. This one is due to Specker [Spe56, Theorem 1] (see also [HS69b]).

Theorem 5.5.2 (Specker)**.** *If k is a positive integer, then $\omega^2 \to (\omega^2, k)^2$.*

Proof of Theorem 5.5.1. The proof is by induction on k. The cases $k \leq 2$ are trivial. For the inductive step, suppose $k \geq 3$. Fix a non-principal ultrafilter $\mathcal{U}$ on ω and let $c : [\omega^\omega]^2 \to \{\text{red}, \text{blue}\}$ be a colouring.

We argue in much the same way as in the proof of Lemma 5.4.5. Among the first $\omega^{k-1} + 1$ elements of ω^ω, we may assume that we have a red-homogeneous closed copy of $\omega + 1$. Let x_0 be its largest point and let H_0 be its subset of order type ω. Write the set of points $\geq \omega^{k-1} + 1$ as a disjoint union $A_1 \cup A_2 \cup A_3$ as in the proof of Lemma 5.4.5. If either A_1 or A_2 contains a closed copy of ω^ω, then by the inductive hypothesis we may assume it contains a blue-homogeneous set of $k - 1$ points, and we are done by the definitions of A_1 and A_2. But $P^{cl}(\omega^\omega)_3 = \omega^\omega$, so we may assume that A_3 contains a closed copy X_1 of ω^ω.

We can now work within X_1 and iterate this argument infinitely many times to obtain a closed set

$$H_0 \cup \{x_0\} \cup H_1 \cup \{x_1\} \cup \ldots$$

of order type ω^2. For each $i \in \omega$, write $H_i = \{h_{i,n} : n \in \omega\}$ with $h_{i,0} < h_{i,1} < \ldots$. By construction, for all $i, j \in \omega$ with $i < j$,

1. $c(\{x_i, x_j\}) = \text{red}$,
2. $c(\{x_i, h_{j,n}\}) = \text{red}$ for all $n \in \omega$ and
3. $\{n \in \omega : c(\{h_{i,n}, x_j\}) = \text{red}\} \in \mathcal{U}$.

We would like to be able to assume that condition 3 can be strengthened to $c(\{h_{i,n}, x_j\}) = \text{red}$ for all $n \in \omega$ by using the ultrafilter to pass to a subset. However, for each i there are infinitely many $j > i$, and we can only use the ultrafilter to deal with finitely many of these.

In order to overcome this difficulty, we use two new ideas. The first new idea is to modify our construction so as to ensure that for all $i, j \in \omega$ with $i < j$, we also have

4. $c(\{h_{i,n}, x_j\}) = \text{red}$ for all $n < j$.

We can achieve this by modifying the construction of X_j (the closed copy of ω^ω from which we extracted $H_j \cup \{x_j\}$). Explicitly, we now include in our disjoint union one additional set for each pair (i, n) with $i, n < j$, which contains the points y that remain with $c(\{h_{i,n}, y\}) = \text{blue}$. We then extract X_j using the fact that $P^{cl}(\omega^\omega)_{j^2+3} = \omega^\omega$.

This extra condition is enough for us to continue. The second new idea is to pass to a subset of the form

$$H' = \{h_{i,n} : i \in I, n \in N\} \cup \{x_i : i \in I\}$$

for some infinite $I, N \subseteq \omega$, and to build up I and N using a back-and-forth argument. To do this, start with $I = N = \emptyset$ and add an element to I and an element to N

alternately in such a way that $c(\{h_{i,n}, x_j\}) =$ red whenever $n \in N$ and $i, j \in I$ with $i < j$. Condition 4 ensures that we can always add a new element to I simply by taking it to be larger than all other members of I and all members of N so far. Meanwhile, condition 3 ensures there is always some $U \in \mathcal{U}$ from which we may choose any member to add to N: at each stage, there are only finitely many new conditions and so our ultrafilter is enough.

Then $c(\{a, b\}) =$ red whenever $a \in H'$ and $b \in H' \cap \{x_i : i \in \omega\}$. Finally, by Theorem 5.5.2 we may assume that there is a red-homogeneous subset $M \subseteq H' \setminus \{x_i : i \in \omega\}$ of order type ω^2, and then the topological closure of M in H' is a red-homogeneous closed copy of ω^2. □

Remark 5.5.3. We have organised this argument in such a way that the reader may readily verify the following. For any positive integer k and any colouring $c : [\omega^\omega]^2 \to \{\text{red,blue}\}$, there is either a blue-homogeneous set of k points, or a red-homogeneous closed copy of an ordinal *larger* than ω^2, or a red-homogeneous closed copy of ω^2 that is moreover *cofinal* in ω^ω. This strengthening of Theorem 5.5.1 will be useful in Section 5.7.

5.6 The anti-tree partial ordering on ordinals

The techniques from the last few sections enable us to reach ω^2, but do not seem to get us any further without cumbersome machinery. In this section we introduce a new approach, which does not directly help get us beyond ω^2 but does provide a helpful perspective. We use this approach to prove the following result.

Theorem 5.6.1. $\omega^2 \cdot 3 \leq R^{top}(\omega \cdot 2, 3) \leq \omega^3 \cdot 100$.

It is more transparent to describe this new approach in terms of a new partial ordering on ordinals. A variant of this ordering was independently considered by Piña in [Pn14], who identified countable ordinals with families of finite sets. Readers who are familiar with that work may find it helpful to note that for ordinals less than ω^ω, our new relation $\leq^*$ coincides with the superset relation $\supseteq$ under that identification. (Note that none of the results we prove here are used outside this section.)

Definition. Let α and β be ordinals. If $\beta > 0$, then write $\beta = \eta + \omega^\gamma$ with η a multiple of ω^γ. Then we write $\alpha <^* \beta$ to mean that $\beta > 0$ and $\alpha = \eta + \zeta$ for some $0 < \zeta < \omega^\gamma$. We write $\alpha \lhd^* \beta$ to mean that $\alpha <^* \beta$ and there is no ordinal δ with $\alpha <^* \delta <^* \beta$.

Equivalently, $\alpha <^* \beta$ if and only if $\beta = \alpha + \omega^\gamma$ for some $\gamma > \mathrm{CB}(\alpha)$, and $\alpha \lhd^* \beta$ if and only if $\beta = \alpha + \omega^{\mathrm{CB}(\alpha)+1}$. For example, $\omega^3 + \omega <^* \omega^3 \cdot 2$ and $\omega^3 \cdot 2 + 1 <^* \omega^3 \cdot 3$, but $\omega^3 + \omega \not<^* \omega^3 \cdot 3$ and $\omega^3 \cdot 2 + 1 \not<^* \omega^3 \cdot 2$.

Here are some simple properties of these relations.

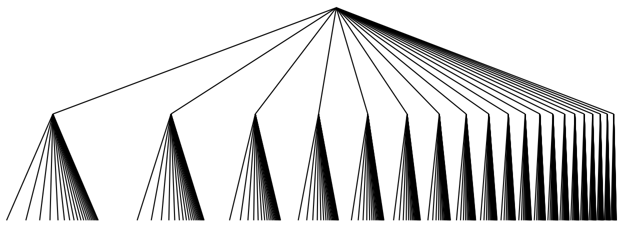
1. $<^*$ is a strict partial ordering.
2. If $\alpha <^* \beta$ then $\alpha < \beta$ and $\mathrm{CB}(\alpha) < \mathrm{CB}(\beta)$.
3. If $\alpha \lhd^* \beta$ then $\mathrm{CB}(\beta) = \mathrm{CB}(\alpha) + 1$.
4. The class of all ordinals forms an "anti-tree" under the relation $<^*$ in the sense that for any ordinal α, the class of ordinals β with $\alpha <^* \beta$ is well-ordered by $<^*$.

By property 4, if k is a positive integer, then $\omega^k + 1$ forms a tree under the relation $>^*$. It fact it is what we will call a *perfect $\aleph_0$-tree of height k*.

Definition. Let k be a positive integer and let X be a single-rooted tree. We say that $x \in X$ *has height k* to mean that x has exactly k predecessors. We say that $x \in X$ is a *leaf of X* to mean that x has no immediate successors, and denote the set of leaves of X by $\ell(X)$. We say that X is a *perfect $\aleph_0$-tree of height k* to mean that every non-leaf of X has $\aleph_0$ immediate successors and every leaf of X has height k.

Let X be a perfect $\aleph_0$-tree of height k. We say that a subset $Y \subseteq X$ is a *full subtree of X* to mean that Y is a perfect $\aleph_0$-tree of height k under the induced relation.

Note that if X is a full subtree of $\omega^k + 1$, then $X \cong \omega^k + 1$. Note also that full subtrees are determined by their leaves.



A perfect $\aleph_0$-tree of height 2, corresponding to the ordinal $\omega^2 + 1$

Here is a simple result about colourings of perfect $\aleph_0$-trees of height k. The proof essentially amounts to k applications of the infinite pigeonhole principle.

Lemma 5.6.2. *Let k be a positive integer, let X be a perfect $\aleph_0$-tree of height k and let $c : \ell(X) \to \{\mathit{red}, \mathit{blue}\}$ be a colouring. Then there exists a full subtree Y of X such that $\ell(Y)$ is monochromatic.*

Proof. The proof is by induction on k. The case $k = 1$ is simply the infinite pigeonhole principle, so assume $k > 1$. Let Z be the set of elements of X of height at most $k - 1$, so Z is a perfect $\aleph_0$-tree of height $k - 1$. Then for each $z \in \ell(Z)$, by the infinite pigeonhole principle again there exists $d(z) \in \{\text{red}, \text{blue}\}$ and an infinite subset Y_z of the successors of z such that $c(x) = d(z)$ for all $x \in Y_z$. This defines a colouring $d : \ell(Z) \to \{\text{red}, \text{blue}\}$, so by the inductive hypothesis there exists a full subtree W of Z such that $\ell(W)$ is monochromatic for d. Finally let

$$Y = W \cup \bigcup_{w \in \ell(W)} Y_w.$$

Then Y is as required. $\square$

Recall Theorem 5.2.1, which states that if k is a positive integer, then $R^{top}(\omega + 1, k+1) = R^{cl}(\omega+1, k+1) \geq \omega^k + 1$. To illustrate the relevance of these notions, we now provide a second proof of this result. This is also the proof that we will mirror when the result is generalised in Theorem 5.7.1.

The crux of the proof is the following result, which says that any $\{\text{red}, \text{blue}\}$-colouring of $\omega^k + 1$ avoiding both a red-homogeneous topological copy of $\omega + 1$ and a blue-homogeneous topological copy of ω is in some sense similar to the $k+1$-partite $\{\text{red}, \text{blue}\}$-colouring that falls out of the proofs of Propositions 5.1.3 and 4.5.1.

Lemma 5.6.3. *Let k be a positive integer and let $c : [\omega^k + 1]^2 \to \{red, blue\}$ be a colouring. Suppose that*

(a) there is no red-homogeneous topological copy of $\omega + 1$, and

(b) there is no blue-homogeneous topological copy of ω.

Under these assumptions, there is a full subtree X of $\omega^k + 1$ such that for all $x, y, z \in X$:

1. *if $x \lhd^* z$ and $y \lhd^* z$ then $c(\{x, y\}) = red$; and*
2. *if $x <^* y$ then $c(\{x, y\}) = blue$.*

The proof makes use of Lemma 5.6.2.

Proof. The proof is by induction on k.

For the base case, suppose $k = 1$. By the infinite Ramsey theorem there exists an infinite homogeneous subset $Y \subseteq \omega$. By condition (b) this must be red-homogeneous. Now by the infinite pigeonhole principle there must exist $i \in \{\text{red}, \text{blue}\}$ and an infinite subset $Z \subseteq Y$ such that $c(\{x, \omega\}) = i$ for all $x \in Z$. By condition (a) we must have $i = \text{blue}$. Then $Z \cup \{\omega\}$ is a full subtree of $\omega + 1$ with the required properties.

For the inductive step, suppose $k > 1$. First apply the inductive hypothesis to obtain a full subtree Y_m of $\left[\omega^k \cdot m + 1, \omega^k \cdot (m+1)\right] \cong \omega^{k-1} + 1$ for each $m \in \omega$, and let $Y = \bigcup_{m \in \omega} Y_m \cup \{\omega^k\}$. Then use the inductive hypothesis again to obtain a full subtree Z of $Y \setminus \ell(Y) \cong \omega^{k-1} + 1$, and let

$$W = Z \cup \{y \in \ell(Y) : y \lhd^* z \text{ for some } z \in Z\}.$$

By our uses of the inductive hypothesis, conditions 1 and 2 hold whenever $x, y, z \in Y_m$ for some $m \in \omega$ or $x, y, z \in Z$. Thus it is sufficient to find a full subtree X of W such that $c(\{x, \omega^k\}) = \text{blue}$ for all $x \in \ell(X)$. To do this, define a colouring $\widetilde{c} : \ell(W) \to \{\text{red}, \text{blue}\}$ by $\widetilde{c}(x) = c(\{x, \omega^k\})$. Then apply Lemma 5.6.2 to obtain $i \in \{\text{red}, \text{blue}\}$ and a full subtree X of W such that $c(\{x, \omega^k\}) = i$ for all $x \in \ell(X)$. Now let V be a cofinal subset of $\ell(X)$ of order type ω. By the infinite Ramsey theorem there exists an infinite homogeneous subset $U \subseteq V$, which by condition (b) must be red-homogeneous. But then $U \cup \{\omega^k\}$ is a topological copy of $\omega + 1$, so by condition (a) we must have $i = \text{blue}$, and we are done. $\square$

Theorem 5.2.1 now follows easily.

Second proof of Theorem 5.2.1. As in the first proof, $R^{top}(\omega + 1, k + 1) = R^{cl}(\omega + 1, k + 1) \geq \omega^k + 1$.

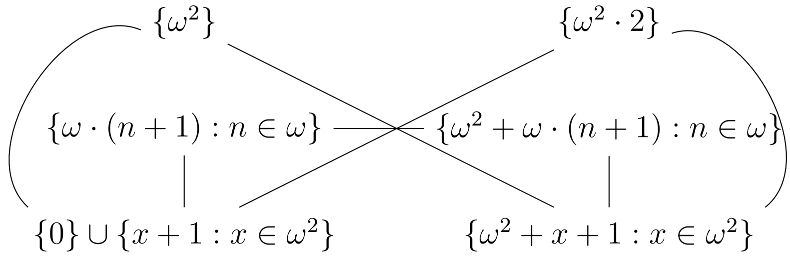
To see that $\omega^k + 1 \to_{top} (\omega + 1, k + 1)^2$, let $c : [\omega^k + 1]^2 \to \{\text{red}, \text{blue}\}$ be a colouring. If there is a red-homogeneous topological copy of $\omega + 1$ or a blue-homogeneous topological copy of ω, then we are done. Otherwise, choose $X \subseteq \omega^k + 1$ as in Lemma 5.6.3. Then any branch (i.e., any maximal chain under $>^*$) of X forms a blue-homogeneous set of $k + 1$ points. $\square$

We conclude this section by proving our bounds on $R^{top}(\omega \cdot 2, 3)$. Before doing this, we remark that it is crucial that we consider here the topological rather than the closed Ramsey number. Since $\omega + n \cong \omega + 1$ for every positive integer n, from a topological perspective, $\omega \cdot 2$ is the simplest ordinal space larger than $\omega + 1$. Moreover, there are sets of ordinals containing a topological copy of $\omega \cdot 2$ but not even a closed copy of $\omega + 2$, such as $(\omega \cdot 2 + 1) \setminus \{\omega\}$. Accordingly, we have only been able to apply the technique we present here to this simplest of cases. Nonetheless, it may still be possible to adapt this technique to obtain upper bounds on closed (as well as topological) Ramsey numbers.

We begin by proving the lower bound. Recall from Theorem 4.12.1 that $P^{top}(\omega \cdot 2)_2 = \omega^2 \cdot 2$. Thus we have indeed improved upon the lower bound given by Proposition 5.1.3. As with the lower bound of Lemma 5.3.3, we provide a simple colouring based on a small finite graph.

Lemma 5.6.4. *$R^{top}(\omega \cdot 2, 3) \geq \omega^2 \cdot 3$.*

Proof. Since any ordinal less than $\omega^2 \cdot 3$ is homeomorphic to a subspace of $\omega^2 \cdot 2 + 1$, it is sufficient to prove that $\omega^2 \cdot 2 + 1 \not\to_{top} (\omega \cdot 2, 3)^2$. To see this, let G be the graph represented by the following diagram.



Now define a colouring $c : [\omega^2 \cdot 2+1]^2 \to \{\text{red}, \text{blue}\}$ by setting $c(\{x, y\}) = \text{blue}$ if and only if x and y lie in distinct, adjacent vertices of G. First note that there is no blue triangle since G is triangle-free. To see that there is no red-homogeneous topological copy of $\omega \cdot 2$, first note that every vertex of G is discrete, and moreover the union of any vertex from the left half of G and any vertex the right half of G is also discrete. Therefore the only maximal red-homogeneous subspaces that are not discrete are $\{\omega \cdot (n+1) : n \in \omega\} \cup \{\omega^2\} \cup \{\omega^2 \cdot 2\}$ and $\{\omega^2 + \omega \cdot (n+1) : n \in \omega\} \cup \{\omega^2\} \cup \{\omega^2 \cdot 2\}$, neither of which contains a topological copy of $\omega \cdot 2$. □

Our upper bound makes use of several classical ordinal Ramsey results.

Firstly, we use the special case of Theorem 5.5.2 that $\omega^2 \to (\omega^2, 3)^2$.

Secondly, we use Theorem 5.4.4. In fact, we essentially prove that $R^{top}(\omega \cdot 2, 3) \leq \omega^3 \cdot R(K_{10}^*, L_3)$. The best known upper bound on $R(K_{10}^*, L_3)$ is due to Larson and Mitchell [LM97].

Theorem 5.6.5 (Larson–Mitchell)**.** *If $n > 1$ is a positive integer, then $R(K_n^*, L_3) \leq n^2$.*

In particular, $R(K_{10}^*, L_3) \leq 100$ and hence $\omega \cdot 100 \to (\omega \cdot 10, 3)^2$ by Theorem 5.4.4.

Finally, we use the following result, which was claimed without proof by Haddad and Sabbagh [HS69b] and has since been proved independently by Weinert [Wei14, Theorem 2.14].

Theorem 5.6.6 (Haddad–Sabbagh; Weinert)**.** *$R(\omega^2 \cdot 2, 3) = \omega^2 \cdot 10$.*

We are now ready to prove our upper bound. The first part of the proof is the following analogue of Lemma 5.6.3. The proof uses in an essential manner that we are looking for a topological rather than a closed copy of $\omega \cdot 2$.

Lemma 5.6.7. *Let $c : [\omega^2 + 1]^2 \to \{red, blue\}$ be a colouring. Suppose that*

(a) there is no red-homogeneous topological copy of $\omega \cdot 2$, and

(b) there is no blue triangle.

Under these assumptions, there is a full subtree X of ω^2+1 such that for all $x, y \in X$:

1. *if $\mathrm{CB}(x) = \mathrm{CB}(y)$ then $c(\{x, y\}) = red$;*

2. *if* $\mathrm{CB}(x) = 0$ *then* $c(\{x, \omega^2\}) =$ *blue; and*

3. *if* $\mathrm{CB}(x) = 1$ *then* $c(\{x, \omega^2\}) =$ *red.*

Proof. First note that $(\omega^2 + 1) \setminus (\omega^2 + 1)'$ has order type ω^2, so by condition (b) it has a red-homogeneous subset W of order type ω^2, since $\omega^2 \to (\omega^2, 3)^2$. Let Y_0 be a full subtree of $\omega^2 + 1$ with $\ell(Y_0) \subseteq W$. By applying the infinite Ramsey theorem to $Y_0' \setminus Y_0^{(2)}$, we may similarly pass to a full subtree Y_1 of Y_0 such that $Y_1' \setminus Y_1^{(2)}$ is red-homogeneous. Thus Y_1 satisfies condition 1.

Next apply Lemma 5.6.2 to obtain $i \in \{\mathrm{red}, \mathrm{blue}\}$ and a full subtree Z of Y_1 such that $c(\{x, \omega^2\}) = i$ for all $x \in \ell(Z)$. If $i = \mathrm{red}$, then $\ell(Z) \cup \{\omega^2\}$ would be red-homogeneous, so by condition (a) $i = \mathrm{blue}$ and Z satisfies condition 2.

Finally apply the infinite pigeonhole principle to $Z' \setminus Z^{(2)}$ to obtain $j \in \{\mathrm{red}, \mathrm{blue}\}$ and a full subtree X of Z such that $c(\{x, \omega^2\}) = j$ for all $x \in X$ with $\mathrm{CB}(x) = 1$. If $j = \mathrm{blue}$, then by condition (b) we would have $c(\{x, y\}) = \mathrm{red}$ for all $x, y \in X$ with $\mathrm{CB}(x) = 0$ and $\mathrm{CB}(y) = 1$, whence $X \setminus \{\omega^2\}$ would be red-homogeneous. Hence by condition (a) $j = \mathrm{red}$ and X is as required. □

We can now complete the proof of our upper bound and hence of Theorem 5.6.1.

Proof of Theorem 5.6.1. By Lemma 5.6.4, it remains only to prove that $R^{top}(\omega \cdot 2, 3) \leq \omega^3 \cdot 100$.

Let $X = \omega^3 \cdot 100$, let $c : [X]^2 \to \{\mathrm{red}, \mathrm{blue}\}$ be a colouring and suppose for contradiction that there is no red-homogeneous topological copy of $\omega \cdot 2$ and no blue triangle.

First note that $X^{(2)} \setminus X^{(3)}$ has order type $\omega \cdot 100$, so it has a red-homogeneous subset U of order type $\omega \cdot 10$, since $\omega \cdot 100 \to (\omega \cdot 10, 3)^2$. Next let $V = \{x \in X : x \lhd^* y \text{ for some } y \in U\}$. Note that V has order type $\omega^2 \cdot 10$, and so V has a red-homogeneous subset W of order type $\omega^2 \cdot 2$, since $\omega^2 \cdot 10 \to (\omega^2 \cdot 2, 3)^2$ by Theorem 5.6.6. Finally, let

$$Y = \mathrm{cl}(W) \cup \{x \in X : x \lhd^* y \text{ for some } y \in W\},$$

where cl denotes the topological closure operation. Replacing Y with $Y \setminus \{\max Y\}$ if necessary, we may then assume that $Y \cong \omega^3 \cdot 2$, and by construction both $Y^{(2)} \setminus Y^{(3)}$ and $Y' \setminus Y^{(2)}$ are red-homogeneous.

Assume for notational convenience that $Y = \omega^3 \cdot 2$. By applying Lemma 5.6.7 to the interval $[\omega^2 \cdot \alpha + 1, \omega^2 \cdot (\alpha + 1)]$ for each $\alpha \in \omega \cdot 2$, we may assume that $c(\{\omega^2 \cdot \alpha + \omega \cdot (n + 1), \omega^2 \cdot (\alpha + 1)\}) = \mathrm{red}$ for all $n \in \omega$. By applying Lemma 5.6.7 to $(\omega^3 + 1)'$, we may then assume that $c(\{\omega^2 \cdot (\alpha + 1), \omega^3\}) = \mathrm{red}$ and $c(\{\omega^2 \cdot \alpha + \omega \cdot (n + 1), \omega^3\}) = \mathrm{blue}$ for all $\alpha, n \in \omega$.

Finally by applying the infinite pigeonhole principle to $\{\omega^2 \cdot (\alpha + 1) : \alpha \in [\omega, \omega \cdot 2)\}$, we may assume that $c(\{\omega^3, \omega^2 \cdot (\alpha + 1)\}) = i$ for all $\alpha \in [\omega, \omega \cdot 2)$, where $i \in \{\mathrm{red}, \mathrm{blue}\}$, and then by applying the infinite pigeonhole principle to $\{\omega \cdot (n+1) : n \in \omega\}$, we may assume that $c(\{\omega\cdot(n+1), \omega^3+\omega^2\}) = j$, where $j \in \{\text{red}, \text{blue}\}$. Now if $i = \text{red}$, then $(\omega^3 \cdot 2)^{(2)}$ would be a red-homogeneous topological copy of $\omega \cdot 2$, and if $j = \text{red}$, then $\{\omega \cdot (n+1) : n \in \omega\} \cup \{\omega^3 + \omega \cdot (n+1) : n \in \omega\} \cup \{\omega^3 + \omega^2\}$ would be a red-homogeneous topological copy of $\omega \cdot 2$. So $i = j = \text{blue}$. But then $\{\omega, \omega^3, \omega^3 + \omega^2\}$ is a blue triangle. □

5.7 The ordinal $\omega^2 + 1$

We now use our earlier result on ω^2 together with some of the ideas from the previous section to obtain upper bounds for $\omega^2 + 1$. We deduce these from Theorem 5.5.1 and the following general result.

Theorem 5.7.1. *Let α and β be countable ordinals with $\beta > 0$, let k be a positive integer, and suppose they satisfy a "cofinal version" of*

$$\omega^{\omega^\alpha} \to_{cl} (\omega^\beta, k+2)^2.$$

Then

$$\omega^{\omega^\alpha \cdot (k+1)} + 1 \to_{cl} (\omega^\beta + 1, k+2)^2.$$

Moreover, if $\omega^{\omega^\alpha} > \omega^\beta$, then in fact

$$\omega^{\omega^\alpha \cdot k} + 1 \to_{cl} (\omega^\beta + 1, k+2)^2.$$

The cofinal version of the partition relation requires that for every colouring $c : \left[\omega^{\omega^\alpha}\right]^2 \to \{red, blue\}$,

- *there is a blue-homogeneous set of $k+2$ points, or*
- *there is a red-homogeneous closed copy of ω^β* that is cofinal in ω^{ω^α}, *or*
- *there is already a red-homogeneous closed copy of $\omega^\beta + 1$.*

Before providing the proof, we first deduce our upper bounds for $\omega^2 + 1$. Since $\omega^2 + 1$ is order-reinforcing, it follows that $R^{top}(\omega^2+1, k+2) = R^{cl}(\omega^2+1, k+2) \le \omega^{\omega\cdot k} + 1$.

Corollary 5.7.2. *If k is a positive integer, then $\omega^{\omega\cdot k} + 1 \to_{cl} (\omega^2 + 1, k+2)^2$.*

Proof. By Theorem 5.7.1, since $\omega^\omega > \omega^2$ it is enough to prove the cofinal version of $\omega^\omega \to_{cl} (\omega^2, k+2)^2$. The usual version is precisely Theorem 5.5.1, and the cofinal version is easily obtained from the same proof, as indicated in Remark 5.5.3. □

Observe that by applying Ramsey's theorem instead of Theorem 5.5.1, one obtains yet another proof of Theorem 5.2.1 from the case $\alpha = 0$. Indeed, our proof of Theorem 5.7.1 is similar to our second proof of that result, though we do not explicitly use any of our results on the anti-tree partial ordering.

The bulk of the proof of Theorem 5.7.1 is in the following result, which is our analogue of Lemma 5.6.3. The proof makes detailed use of the topological structure of countable ordinals. In particular, we use two arguments due to Weiss from the proof of [Bau86, Theorem 2.3]. It may be helpful for the reader to first study that proof, and to recall Weiss's lemma [Bau86, Lemma 2.6], which we stated earlier (in Lemma 4.7.12).

Lemma 5.7.3. *Let α, β and k be as in Theorem 5.7.1. Let l be a positive integer and let $c : \left[\omega^{\omega^\alpha \cdot l} + 1\right]^2 \to \{red, blue\}$ be a colouring. Suppose that*

1. *there is no red-homogeneous closed copy of $\omega^\beta + 1$, and*

2. *there is no blue-homogeneous set of $k + 2$ points.*

Under these assumptions, there exists a cofinal subset $X \subseteq \omega^{\omega^\alpha \cdot l}$ such that X is a closed copy of $\omega^{\omega^\alpha \cdot l}$ and $c(\{x, \omega^{\omega^\alpha \cdot l}\}) = blue$ for all $x \in X$.

Proof. The proof is by induction on l.

For the case $l = 1$, since $\omega^{\omega^\alpha} \to_{cl} (\omega^{\omega^\alpha})^1_2$, there exists $X \subseteq \omega^{\omega^\alpha}$ and $i \in \{\text{red}, \text{blue}\}$ such that X is a closed copy of ω^{ω^α} (and therefore X is cofinal in ω^{ω^α}) and $c(\{x, \omega^{\omega^\alpha}\}) = i$ for all $x \in X$. Suppose for contradiction that $i = \text{red}$. By our assumptions together with the definition of the cofinal version of the partition relation, there exists a cofinal subset $Y \subseteq X$ such that Y is a closed copy of ω^β and $[Y]^2 \subseteq c^{-1}(\{\text{red}\})$. But then $Y \cup \{\omega^{\omega^\alpha}\}$ is a red-homogeneous closed copy of $\omega^\beta + 1$, contrary to assumption 1. Hence $i = \text{blue}$ and we are done.

For the inductive step, suppose $l > 1$. Let

$$Z = \left\{\omega^{\omega^\alpha} \cdot \gamma : \gamma \in \omega^{\omega^\alpha \cdot (l-1)} \setminus \{0\}\right\},$$

so Z is a closed copy of $\omega^{\omega^\alpha \cdot (l-1)}$. By the inductive hypothesis, there exists a cofinal subset $Y \subseteq Z$ such that Y is a closed copy of $\omega^{\omega^\alpha \cdot (l-1)}$ and $c(\{x, \omega^{\omega^\alpha \cdot l}\}) = \text{blue}$ for all $x \in Y$. Write $Y = \{y_\delta : \delta \in \omega^{\omega^\alpha \cdot (l-1)}\}$ in increasing order. Then by Weiss's lemma, for each $\delta \in \omega^{\omega^\alpha \cdot (l-1)}$ there exists a cofinal subset $Z_\delta \subseteq (y_\delta, y_{\delta+1})$ such that Z_δ is a closed copy of ω^{ω^α}.

Now since $\omega^{\omega^\alpha} \to_{cl} (\omega^{\omega^\alpha})^1_2$, for each $\delta \in \omega^{\omega^\alpha \cdot (l-1)}$ there exists $X_\delta \subseteq Z_\delta$ and $i_\delta \in \{\text{red}, \text{blue}\}$ such that X_δ is a closed copy of ω^{ω^α} (and therefore X_δ is cofinal in Z_δ) and $c(\{x, \omega^{\omega^\alpha \cdot l}\}) = i_\delta$ for all $x \in X_\delta$. Recall now that $\omega^{\omega^\alpha \cdot (l-1)} \to (\omega^{\omega^\alpha \cdot (l-1)})^1_2$ since $\omega^{\omega^\alpha \cdot (l-1)}$ is a power of ω. It follows that there exists $S \subseteq \omega^{\omega^\alpha \cdot (l-1)}$ of order type $\omega^{\omega^\alpha \cdot (l-1)}$ and $i \in \{\text{red}, \text{blue}\}$ such that $i_\delta = i$ for all $\delta \in S$.

Suppose for contradiction that $i = \text{red}$. We now use an argument from the proof of [Bau86, Theorem 2.3]. Let $(\delta_m)_{m \in \omega}$ be a strictly increasing cofinal sequence from S, and let $(\eta_m)_{m \in \omega}$ be a strictly increasing cofinal sequence from ω^α (or let $\eta_m = 0$ for all $m \in \omega$ if $\alpha = 0$). For each $m \in \omega$, pick $W_m \subseteq X_{\delta_m}$ such that W_m is a closed copy of $\omega^{\eta_m} + 1$, and let $W = \bigcup_{m \in \omega} W_m$. Note that W is cofinal in $\omega^{\omega^\alpha \cdot l}$, W is a closed copy of ω^{ω^α} and $c(\{x, \omega^{\omega^\alpha \cdot l}\}) = \text{red}$ for all $x \in W$. By our assumptions together with the definition of the cofinal version of the partition relation, there exists a cofinal subset $V \subseteq W$ such that V is a closed copy of ω^β and $[V]^2 \subseteq c^{-1}(\{\text{red}\})$. But then $V \cup \{\omega^{\omega^\alpha \cdot l}\}$ is a red-homogeneous closed copy of $\omega^\beta + 1$, contrary to assumption 1.

Therefore $i = \text{blue}$. Finally, let

$$X = \bigcup_{\delta \in S} X_\delta \cup \operatorname{cl}(\{y_{\delta+1} : \delta \in S\}),$$

where cl denotes the topological closure operation. Then the set X is as required. □

We may now deduce Theorem 5.7.1 in the much same way that we deduced Theorem 5.2.1 from Lemma 5.6.3.

Proof of Theorem 5.7.1. First assume that $\omega^{\omega^\alpha} > \omega^\beta$. We prove by induction on l that for all $l \in \{1, 2, \ldots, k\}$,

$$\omega^{\omega^\alpha \cdot l} + 1 \rightarrow_{cl} (\omega^\beta + 1, l + 2)^2.$$

In every case, if either of the assumptions in Lemma 5.7.3 does not hold, then we are done since $l \leq k$. We may therefore choose X as in Lemma 5.7.3.

For the base case $l = 1$, to avoid a blue triangle, X must be red-homogeneous. But X contains a closed copy of $\omega^\beta + 1$ since $\omega^{\omega^\alpha} \geq \omega^\beta + 1$, and so we are done.

For the inductive step, suppose $l \geq 2$. Then X has a closed copy Y of $\omega^{\omega^\alpha \cdot (l-1)} + 1$. By the inductive hypothesis, either Y contains a red-homogeneous closed copy of $\omega^\beta + 1$, in which case we are done, or Y contains a blue-homogeneous set Z of $l + 1$ points. But in that case $Z \cup \{\omega^{\omega^\alpha \cdot l}\}$ is a blue-homogeneous set of $l + 2$ points, and we are done.

Finally, if we cannot assume that $\omega^{\omega^\alpha} > \omega^\beta$, then the base case breaks down. However, we may instead use the base case $\omega^{\omega^\alpha} + 1 \rightarrow_{cl} (\omega^\beta + 1, 2)^2$, which follows from the fact that $\omega^{\omega^\alpha} \geq \omega^\beta$. The inductive step is then identical. □

5.8 The weak topological Erdős–Milner theorem

Finally we reach our main result, which demonstrates that $R^{top}(\alpha, k)$ and $R^{cl}(\alpha, k)$ are countable for all countable α and all finite k.

This is a topological version of a classical result due to Erdős and Milner [EM72]. Before stating it, we first provide a simplified proof of the classical version.

Theorem 5.8.1 (Weak Erdős–Milner). *Let α and β be countable non-zero ordinals,* *and let $k > 1$ be a positive integer. If*

$$\omega^\alpha \to (\omega^{1+\beta}, k)^2,$$

then

$$\omega^{\alpha+\beta} \to (\omega^{1+\beta}, k+1)^2.$$

Since trivially $\omega^{1+\alpha} \to (\omega^{1+\alpha}, 2)^2$, it follows by induction on k that $R(\omega^{1+\alpha}, k+1) \leq \omega^{1+\alpha\cdot k}$ for all countable α and finite k. In fact, Erdős and Milner proved a stronger version of the above theorem, in which $k+1$ is replaced by $2k$, implying that $R(\omega^{1+\alpha}, 2^k) \leq \omega^{1+\alpha\cdot k}$. This is why we use the adjective *weak* here.

Our proof is essentially a simplified version of the original, which can be found in [Wil77, Theorem 7.2.10]. The basic idea is to write $\omega^{\alpha+\beta}$ as a sequence of ω^β intervals, each of order type ω^α, to enumerate these intervals, and to recursively build up a red-homogeneous copy of ω^β consisting of one element from each interval. This would achieve the above theorem with $1+\beta$ weakened to β. To obtain a copy of $\omega^{1+\beta}$, we simply choose infinitely many elements from each interval instead of just one. In the proof we use the fact that $\omega^\alpha \to (\omega^\alpha)^1_m$ for all finite m.

Proof. Let $c : \left[\omega^{\alpha+\beta}\right]^2 \to \{\text{red}, \text{blue}\}$ be a colouring.

For each $x \in \omega^\beta$, let

$$I_x = [\omega^\alpha \cdot x, \omega^\alpha \cdot (x+1)),$$

so I_x has order type ω^α. Let $(x_n)_{n\in\omega}$ be a sequence of points from ω^β in which every member of ω^β appears infinitely many times. We attempt to inductively build a red-homogeneous set $A = \{a_n : n \in \omega\}$ of order-type $\omega^{1+\beta}$ with $a_n \in I_{x_n}$ for every $n \in \omega$.

Suppose that we have chosen $a_1, a_2, \ldots, a_{m-1}$ for some $m \in \omega$. Let

$$P = \{a_n : n \in \{1, 2, \ldots, m-1\}, a_n \in I_{x_m}\},$$

and let

$$J = \begin{cases} I_{x_m}, & \text{if } P = \emptyset \\ I_{x_m} \setminus [0, \max P], & \text{if } P \neq \emptyset, \end{cases}$$

so J has order type ω^α. For each $n \in \{1, 2, \ldots, m-1\}$ let

$$J_n = \{a \in J : c(\{a_n, a\}) = \text{blue}\}.$$

If $\bigcup_{n=1}^{m-1} J_n = J$, then since $\omega^\alpha \to (\omega^\alpha)^1_{m-1}$, J_n has order type ω^α for some $n \in \{1, 2, \ldots, m-1\}$. Then since $\omega^\alpha \to (\omega^{1+\beta}, k)^2$, J_n either has a red-homogeneous subset of order type $\omega^{1+\beta}$, in which case we are done, or a blue-homogeneous subset B of k points, in which case $B \cup \{a_n\}$ is a blue-homogeneous set of $k+1$ points, and we are done. Thus we may assume that $J \setminus \bigcup_{n=1}^{m-1} J_n \neq \emptyset$, and choose a_m to be any member of this set.

Clearly the resulting set A is red-homogeneous. To see that it has order type $\omega^{1+\beta}$, simply observe that by choice of $(x_n)_{n\in\omega}$, $A \cap I_x$ is infinite for all $x \in \omega^\beta$, and that by choice of J, $A \cap I_x$ in fact has order type ω for all $x \in \omega^\beta$. $\square$

Here is our topological version of the weak Erdős–Milner theorem.

Theorem 5.8.2 (The weak topological Erdős–Milner theorem). *Let α and β be countable non-zero ordinals, and let $k > 1$ be a positive integer. If*

$$\omega^{\omega^\alpha} \to_{top} (\omega^\beta, k)^2,$$

then

$$\omega^{\omega^\alpha \cdot \beta} \to_{top} (\omega^\beta, k+1)^2.$$

In particular, as we shall deduce in Corollary 5.8.5, $R^{top}(\alpha, k)$ and $R^{cl}(\alpha, k)$ are countable whenever α is a countable ordinal and k is a positive integer.

Our proof follows the same outline as our proof of the classical version, except that we use intervals in the sense of the anti-tree partial ordering rather than in the usual sense. Furthermore, rather than constructing a closed copy of ω^β directly, we instead construct a larger set and then thin it out. As in the previous section, we make detailed use of the structure of countable ordinals, including an argument from the proof of Weiss's lemma [Bau86, Lemma 2.6]. Note that the proof does not directly use any of our previous results.

Proof. Let $c : \left[\omega^{\omega^\alpha \cdot \beta}\right]^2 \to \{\text{red}, \text{blue}\}$ be a colouring.

First of all, fix a strictly increasing cofinal sequence $(\gamma_n)_{n\in\omega}$ from ω^{ω^α}.

Define an indexing set of pairs

$$S = \left\{(x, y) : x \in \beta, \omega^{\omega^\alpha \cdot (x+1)} \cdot y \in \omega^{\omega^\alpha \cdot \beta}\right\},$$

and for each $(x, y) \in S$, let

$$X_{(x,y)} = \left\{\omega^{\omega^\alpha \cdot (x+1)} \cdot y + \omega^{\omega^\alpha \cdot x} \cdot z : z \in \omega^{\omega^\alpha} \setminus \{0\}\right\},$$

so $X_{(x,y)}$ is a closed copy of ω^{ω^α}. Let $(x_n, y_n)_{n\in\omega}$ be a sequence of pairs from S in which every member of S appears infinitely many times. We attempt to inductively build a red-homogeneous set $A = \{a_n : n \in \omega\}$, which will contain a closed copy of ω^β, with $a_n \in X_{(x_n,y_n)}$ for every $n \in \omega$.

Suppose that we have chosen $a_1, a_2, \ldots, a_{m-1}$ for some $m \in \omega$. Let

$$P = \left\{a_n : n \in \{1, 2, \ldots, m-1\}, a_n \in X_{(x_m,y_m)}\right\},$$

let

$$Q = P \cup \left\{\omega^{\omega^\alpha \cdot (x_m+1)} \cdot y_m + \omega^{\omega^\alpha \cdot x_m} \cdot \gamma_{|P|}\right\},$$

and let

$$Y = X_{(x_m,y_m)} \setminus [0, \max Q],$$

so Y is a closed copy of ω^{ω^α}. For each $n \in \{1, 2, \ldots, m-1\}$ let

$$Y_n = \{a \in Y : c(\{a_n, a\}) = 1\}.$$

If $\bigcup_{n=1}^{m-1} Y_n = Y$, then since $\omega^{\omega^\alpha} \to_{cl} (\omega^{\omega^\alpha})^1_{m-1}$, Y_n contains a closed copy of ω^{ω^α} for some $n \in \{1, 2, \ldots, m-1\}$. Then since $\omega^{\omega^\alpha} \to_{cl} (\omega^\beta, k)^2$, Y_n either contains a red-homogeneous closed copy of ω^β, in which case we are done, or a blue-homogeneous set B of k points, in which case $B \cup \{a_n\}$ is a blue-homogeneous set of $k+1$ points, and we are done. Thus we may assume that $Y \setminus \bigcup_{n=1}^{m-1} Y_n \neq \emptyset$, and choose a_m to be any member of this set.

Clearly the resulting set A is red-homogeneous. To complete the proof, observe that by choice of $(x_n, y_n)_{n\in\omega}$, $A \cap X_{(x,y)}$ is infinite for all $(x, y) \in S$, and that by choice of Q, $A \cap X_{(x,y)}$ is a cofinal subset of $X_{(x,y)}$ of order type ω for all $(x, y) \in S$. We claim that this property is enough to ensure that A contains a closed copy of ω^β.

To prove the claim, for each $\delta \in [1, \beta]$ and each ordinal y with $\omega^{\omega^\alpha\cdot\delta} \cdot y \in \omega^{\omega^\alpha\cdot\beta}$ we find a cofinal subset $C_{\delta,y} \subseteq \left[\omega^{\omega^\alpha\cdot\delta} \cdot y + 1, \omega^{\omega^\alpha\cdot\delta} \cdot (y+1)\right)$ such that $C_{\delta,y} \subseteq A$ and $C_{\delta,y}$ is a closed copy of ω^δ; then $C_{\beta,0} \subseteq A$ is a closed copy of ω^β, as required. We do this by induction on δ.

First suppose $\delta \in [1, \beta]$ is a successor ordinal, say $\delta = x + 1$. Fix an ordinal y with $\omega^{\omega^\alpha\cdot\delta} \cdot y \in \omega^{\omega^\alpha\cdot\beta}$, and observe that $(x, y) \in S$ and that $X_{(x,y)}$ is a cofinal subset of the interval $\left[\omega^{\omega^\alpha\cdot\delta} \cdot y + 1,\ \omega^{\omega^\alpha\cdot\delta} \cdot (y+1)\right)$. Recall now that $A \cap X_{(x,y)}$ is a cofinal subset of $X_{(x,y)}$ of order type ω. Thus if $\delta = 1$ then we may simply take $C_{1,y} = A \cap X_{(0,y)}$, so assume $\delta > 1$, and write $A \cap X_{(x,y)} = \{b_n : n \in \omega\}$ in increasing order. For each $n \in \omega \setminus \{0\}$ we find a cofinal subset $D_n \subseteq (b_{n-1}, b_n)$ such that $D_n \subseteq A$ and D_n is a closed copy of ω^x; then we may take $C_{\delta,y} = \bigcup_{n\in\omega\setminus\{0\}} D_n \cup \{b_n\}$. We do this using an argument essentially taken from the proof of Weiss's lemma [Bau86, Lemma 2.6]. Fix $n \in \omega \setminus \{0\}$ and write $b_n = \omega^{\omega^\alpha\cdot(x+1)} \cdot y + \omega^{\omega^\alpha\cdot x} \cdot z$ with $z \in \omega^{\omega^\alpha} \setminus \{0\}$. Let $v = \omega^{\omega^\alpha} \cdot y + z$ so that $b_n = \omega^{\omega^\alpha\cdot x} \cdot v$. If v is a successor ordinal, say $v = u + 1$, then by the inductive hypothesis we may take $D_n = C_{x,u}$. If v is a limit ordinal, then let $(u_m)_{m\in\omega}$ be a strictly increasing cofinal sequence from v with $\omega^{\omega^\alpha\cdot x} \cdot u_0 \geq b_{n-1}$ and let $(\eta_m)_{m\in\omega}$ be a strictly increasing cofinal sequence from ω^x. By the inductive hypothesis, for each $m \in \omega$ we may choose a subset $E_m \subseteq C_{x,u_m}$ such that E_m is a closed copy of $\eta_m + 1$. Then take $D_n = \bigcup_{m\in\omega} E_m$.

Suppose instead $\delta \in [1, \beta]$ is a limit ordinal. Fix an ordinal y with $\omega^{\omega^\alpha\cdot\delta} \cdot y \in \omega^{\omega^\alpha\cdot\beta}$. Let $(x_n)_{n\in\omega}$ be a strictly increasing cofinal sequence from δ. For each $n \in \omega$, let ζ_n be the ordinal such that $\delta = x_n + 1 + \zeta_n$ and let $y_n = \omega^{\omega^\alpha\cdot\zeta_n} \cdot y + 1$. Then $\omega^{\omega^\alpha\cdot(x_n+1)} \cdot y_n = \omega^{\omega^\alpha\cdot\delta} \cdot y + \omega^{\omega^\alpha\cdot(x_n+1)}$ and so $C_{x_n+1,y_n} \subseteq \left[\omega^{\omega^\alpha\cdot\delta} \cdot y + \omega^{\omega^\alpha\cdot(x_n+1)} + 1, \omega^{\omega^\alpha\cdot\delta} \cdot y + \omega^{\omega^\alpha\cdot(x_n+1)} \cdot 2\right)$. By the inductive hypothesis, for each $n \in \omega$ we may choose a subset $D_n \subseteq C_{x_n+1,y_n}$ such that D_n is a closed copy of $\omega^{x_n} + 1$. Then take $C_{\delta,y} = \bigcup_{n\in\omega} D_n$. $\square$

Remark 5.8.3. We expect that as in the original Erdős–Milner theorem, it should be possible to improve $k+1$ to $2k$ in Theorem 5.8.2. Indeed, the basic argument from [Wil77, Theorem 7.2.10] works in the topological setting. The key technical difficulty appears to be in formulating and proving an appropriate topological version of statement (1) in that account.

The weak topological Erdős–Milner theorem allows us to obtain upper bounds for countable ordinals of the form ω^β. Before describing some of these, we first observe that by very slightly adapting our argument, we may obtain improved bounds for ordinals of other forms. Here is a version for ordinals of the form $\omega^\beta \cdot m + 1$.

Theorem 5.8.4. *Let α and β be countable non-zero ordinals, and let $k > 1$ be a positive integer. If*

$$\omega^{\omega^\alpha} \to_{cl} (\omega^\beta \cdot m + 1, k)^2,$$

then

$$\omega^{\omega^\alpha \cdot \beta} \cdot R(m, k+1) + 1 \to_{top} (\omega^\beta \cdot m + 1, k+1)^2.$$

Proof. Write $\omega^{\omega^\alpha \cdot \beta} \cdot R(m, k+1) + 1$ as a disjoint union $M \cup N$, where

$$N = \left\{\omega^{\omega^\alpha \cdot \beta} \cdot (y+1) : y \in \{0, 1, \ldots, R(m, k+1) - 1\}\right\},$$

and let $c : [M \cup N]^2 \to \{\text{red}, \text{blue}\}$ be a colouring.

First of all, we may assume that N contains a red-homogeneous set of m points, say $a_0, a_1, \ldots, a_{m-1}$. Now continue as in the proof of Theorem 5.8.2 and attempt to build a red-homogeneous set $A = \{a_n : n \in \omega\}$, only start by including $a_0, a_1, \ldots, a_{m-1}$, and then work entirely within M.

If we succeed, then the same proof as the one in Theorem 5.8.2 shows that for each $y \in \{0, 1, \ldots, R(m, k+1) - 1\}$, $A \setminus \{a_0, a_1, \ldots, a_{m-1}\}$ contains a closed copy C_y of ω^β that is a cofinal subset of the interval $\left[\omega^{\omega^\alpha \cdot \beta} \cdot y + 1, \omega^{\omega^\alpha \cdot \beta} \cdot (y+1)\right]$. Writing $a_i = \omega^{\omega^\alpha \cdot \beta} \cdot (y_i + 1)$ for each $i \in \{0, 1, \ldots, m-1\}$, we see that

$$\bigcup_{i=0}^{m-1} C_{y_i} \cup \{a_i\}$$

is a red-homogeneous closed copy of $\omega^\beta \cdot m + 1$, as required. $\square$

We conclude this section with some explicit upper bounds implied by our results. It is easy to verify similar results for ordinals of other forms.

Corollary 5.8.5. *Let α be a countable non-zero ordinal and let k, m and n be positive integers.*

1. $R^{top}(\omega^{\omega^\alpha}, k+1) = R^{cl}(\omega^{\omega^\alpha}, k+1) \leq \omega^{\omega^{\alpha \cdot k}}$.

2. $R^{top}(\omega^{\omega^\alpha}+1, k+1) = R^{cl}(\omega^{\omega^\alpha}+1, k+1) \leq \begin{cases} \omega^{\omega^{\alpha \cdot k}} + 1, & \text{if } \alpha \text{ is infinite} \\ \omega^{\omega^{(n+1)\cdot k-1}} + 1, & \text{if } \alpha = n \text{ is finite.} \end{cases}$

3. $R^{top}(\omega^n \cdot m + 1, k+2) = R^{cl}(\omega^n \cdot m + 1, k+2) \leq \omega^{\omega^{k \cdot n}} \cdot R(m, k+2) + 1$.

Proof. First note that all three equalities are immediate since all ordinals considered are order-reinforcing. It remains to prove the inequalities.

1. This follows immediately from Theorem 5.8.2 by induction on k, the case $k = 1$ being trivial.

2. It follows from part 1 that $\omega^{\omega^{(\alpha+1)\cdot(k-1)}} \to_{cl} (\omega^{\omega^{\alpha+1}}, k)^2$ (the case $k = 1$ again being trivial) and hence $\omega^{\omega^{(\alpha+1)\cdot(k-1)}} \to_{cl} (\omega^{\omega^\alpha} + 1, k)^2$. Therefore $\omega^{\omega^{(\alpha+1)\cdot(k-1)+\alpha}} + 1 \to_{cl} (\omega^{\omega^\alpha} + 1, k+1)^2$ by Theorem 5.8.4. If α is infinite, then $(\alpha+1)\cdot(k-1)+\alpha = \alpha \cdot k$, and if $\alpha = n$ is finite, then $(\alpha+1)\cdot(k-1)+\alpha = (n+1)\cdot k - 1$, as required.

3. It follows from part 1 that $\omega^{\omega^k} \to_{cl} (\omega^\omega, k+1)^2$ and hence $\omega^{\omega^k} \to_{cl} (\omega^n \cdot m + 1, k+1)^2$. The result then follows from Theorem 5.8.4. □

Remark 5.8.6. For the case in which $\alpha = n$ is a positive integer, if a cofinal version of part 1 holds, then we could use Theorem 5.7.1 to improve part 2 to $R^{cl}(\omega^{\omega^n} + 1, k+2) \leq \omega^{\omega^{n\cdot(k+1)\cdot k}} + 1$.

5.9 Questions for further research

We close with a few questions. Firstly, there is typically a large gap between our lower and upper bounds, leaving plenty of room for improvement. In particular, our general lower bound in Proposition 5.1.3 is very simple and yet is still our best bound with the exception of a couple of special cases.

Some further exact equalities could be informative. One of the key reasons why the classical results detailed in [HS69a, HS69b, HS69c, Cai15, Mil71] are more precise than our topological results in Sections 5.3–5.7 is that for various $\alpha < \omega^\omega$, the computation of $R(\alpha, k)$ reduces to a problem in finite combinatorics. Some hint of a topological version of this appears in the proof of the lower bounds of Lemmas 5.3.3 and 5.6.4, though it is far from clear how to obtain an exact equality.

Question 5.9.1. Is it possible to reduce the computation of $R^{top}(\alpha, k)$ or $R^{cl}(\alpha, k)$ to finite combinatorial problems, even for $\alpha < \omega^2$?

A partition ordinal is an ordinal α satisfying $\alpha \to (\alpha, 3)^2$. As we have seen, ω and ω^2 are partition ordinals. Other than these, every countable partition ordinal has the form ω^{ω^β}, and in the other direction, ω^{ω^β} is a partition ordinal if β has the form ω^γ or $\omega^\gamma + \omega^\delta$ [Sch10].

Question 5.9.2. Are there any countable "topological partition" ordinals, satisfying $\alpha \to_{top} (\alpha, 3)^2$, other than ω?

By our lower bound (Proposition 5.1.3), these must all have the form ω^{ω^β}. Since every power of ω is order-reinforcing, this question is equivalent if we replace the topological partition relation by the closed partition relation.

We expect that a strong version of the topological Erdős–Milner theorem should hold, that is, we expect that in Theorem 5.8.2 it should be possible to improve $k+1$ to $2k$.

Question 5.9.3. Let α and β be countable non-zero ordinals, and let $k > 1$ be a positive integer. Is it the case that if

$$\omega^\alpha \to (\omega^{1+\beta}, k)^2,$$

then

$$\omega^{\alpha+\beta} \to (\omega^{1+\beta}, 2k)^2?$$

See Remark 5.8.3 for further details.

Finally, it would be nice to determine whether the existing conjecture $\omega_1 \to (\alpha, k)^3$ holds, and also to discover something about the topological or closed version of this relation.

Question 5.9.4. Is it the case that for all countable ordinals α and all finite k, $\omega_1 \to_{top} (\alpha, k)^3$?

Again, since every power of ω is order-reinforcing, this question is equivalent if we replace the topological partition relation by the closed partition relation. If the answer to this question is yes, then the classical and closed relations are in fact equivalent whenever the ordinal on the left-hand side is ω_1 and the ordinals on the right-hand side are countable.

Chapter 6

Autohomeomorphism groups of countable compact ordinals

We begin this chapter by studying the basic properties of autohomeomorphism groups of compact ordinals in general. We then focus on finding the normal subgroups of autohomeomorphism groups of ordinals of the form $\omega^n \cdot m + 1$ for finite n and m. We manage to find $2^{2^{\aleph_0}}$ normal subgroups for all $n \geq 2$, which is somewhat surprising, since the condition of normality is rather strong in the infinite context. Furthermore, we completely classify those normal subgroups contained in the pointwise stabiliser of the set of topological limit points.

6.1 A criterion for continuity

First we briefly mention a useful way of checking whether or not a function between ordinal topologies is continuous. This generalises the equivalence of the ε-δ definition of continuity for functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition. Let Y and Z be totally ordered sets equipped with the order topology, let $f : Y \to Z$ and let $y \in Y$. We say that f is *continuous at* y if and only if for all $c, d \in Z \cup \{\pm\infty\}$ with $f(y) \in (c, d)$, there exist $a, b \in Y \cup \{\pm\infty\}$ with $y \in (a, b)$ such that for all $x \in Y$, if $x \in (a, b)$ then $f(x) \in (c, d)$.

Lemma 6.1.1. *Let Y and Z be totally ordered sets equipped with the order topology and let $f : Y \to Z$. Then f is continuous if and only if f is continuous at y for all $y \in Y$.*

We omit the proof since it is entirely analogous to the ε-δ definition of continuity for functions from $\mathbb{R}$ to $\mathbb{R}$.

Note that since we will be working with compact Hausdorff topological spaces, it will be enough to check continuity to show that a bijection is a homeomorphism.

6.2 The semidirect product decomposition

Let us introduce some notation for the objects of our study.

Definition. Given an ordinal δ, we write $X_\delta = [1, \delta]$.

We then write H_δ for the group of autohomeomorphisms of X_δ, i.e., the group of homeomorphisms $X_\delta \to X_\delta$.

Given an ordinal β, we also write H_δ^β for the group of autohomeomorphisms of $X_\delta^{(\beta)}$.

Excluding the point 0 from our definition of X_δ makes no essential difference, but makes our results easier to state. Indeed, the notation $\overline{\omega}\,[\gamma, m]$ was introduced in Chapter 2 for related reasons, and for any ordinal γ and any positive integer m we have $X_{\omega^\gamma \cdot m} \cong \overline{\omega}\,[\gamma, m]$.

For the rest of this section, fix a non-zero ordinal δ. Recall that X_δ is compact by Lemma 2.2.3, and so by Theorem 2.4.1 we may as well assume that $\delta = \omega^\alpha \cdot m$ for some ordinal α and some positive integer m.

We begin by introducing the normal subgroups of H_δ identified by Monk [Mon75, Theorem 6], which are the pointwise stabilisers of $X_\delta^{(\beta)}$ for each $\beta \leq \alpha$. These are normal because any autohomeomorphism of X_δ must fix $X_\delta^{(\beta)}$ setwise. We shall see that in fact these normal subgroups yield inner semidirect products of H_δ. For the rest of this section, fix an ordinal $\beta \leq \alpha$.

Definition. Define the map $r_\delta^\beta : H_\delta \to H_\delta^\beta$ to be restriction. This is well-defined since any autohomeomorphism of X_δ must fix $X_\delta^{(\beta)}$ setwise.

Define the "block map" $b_\delta^\beta : H_\delta^\beta \to H_\delta$ as follows. Intuitively, $b_\delta^\beta(f)$ is obtained from f by replacing each isolated point of $X_\delta^{(\beta)}$ by a copy of $\left[1, \omega^\beta\right]$. Formally, let $f \in H_\delta^\beta$ and $x \in X_\delta$. If $x \in X_\delta^{(\beta)}$, then define $b_\delta^\beta(f)(x) = f(x)$. Otherwise $x = \omega^\beta \cdot \eta + \zeta$ for some ordinals η and ζ with $0 < \zeta < \omega^\beta$. Now f must preserve Cantor–Bendixson ranks, and since $\beta \leq \alpha$, we have $\omega^\beta \cdot \eta + \omega^\beta \in X$. Hence $f\left(\omega^\beta \cdot \eta + \omega^\beta\right) = \omega^\beta \cdot \theta + \omega^\beta$ for some ordinal θ. Define $b_\delta^\beta(f)(x) = \omega^\beta \cdot \theta + \zeta$.

It is easy to see that $b_\delta^\beta(f)$ is a bijection, and that $b_\delta^\beta(f)$ is continuous follows easily from Lemma 6.1.1. Thus $b_\delta^\beta(f) \in H_\delta$, in other words, b_δ^β is well-defined.

The key properties of these maps are as follows.

Lemma 6.2.1. *1. r_δ^β and b_δ^β are group homomorphisms.*

2. b_δ^β is an injection.

3. $r_\delta^\beta \circ b_\delta^\beta$ is the identity map on H_δ^β.

Proof. Part 1 is straightforward, and parts 2 and 3 are immediate from the fact that $b_\delta^\beta(f)(x) = f(x)$ whenever $x \in X_\delta^{(\beta)}$. □

From these properties we immediately obtain our semidirect product decomposition of H_δ.

Definition. Define

$$K_\delta^\beta = \operatorname{Ker}\left(r_\delta^\beta\right)$$

and

$$B_\delta^\beta = \operatorname{Im}\left(b_\delta^\beta\right).$$

Thus K_δ^β is the pointwise stabiliser of $X_\delta^{(\beta)}$. Clearly $K_\delta^\beta \trianglelefteq H_\delta$ and $B_\delta^\beta \leq H_\delta$. We may now restate Lemma 6.2.1 as follows. Here $\rtimes$ denotes an inner semidirect product.

Proposition 6.2.2. $H_\delta = K_\delta^\beta \rtimes B_\delta^\beta$. □

In particular, any $f \in H_\delta$ may be written uniquely as $k \circ b$ with $k \in K_\delta^\beta$ and $b \in B_\delta^\beta$. Note moreover that we obtain a distinct decomposition for each ordinal $\beta \leq \alpha$. We will however mainly use this result in the case $\beta = 1$.

6.3 An explicit description of $H_{\omega^n \cdot m}$

Monk [Mon75, Theorem 7] gave an explicit description of which permutations of X_{ω^2} lie in H_{ω^2}. (In fact, he worked with the corresponding Boolean algebra, but it is easy to re-interpret his result in terms of ordinals.) In this section we generalise this to a recursive description of $H_{\omega^n \cdot m}$ for all finite m and n. We achieve this by using the case $\beta = 1$ of Proposition 6.2.2 to describe $H_{\omega^n \cdot m}$ in terms of $H_{\omega^{n-1} \cdot m}$ for all $n > 0$.

Our construction is valid in general, not only for ordinals of the form $\omega^n \cdot m$, so for the rest of this section fix a non-zero ordinal δ, and assume once again that $\delta = \omega^\alpha \cdot m$ with α an ordinal and m a positive integer. We may as well assume that δ is infinite, i.e., $\alpha \geq 1$, for otherwise H_δ is simply a finite symmetric group.

Our description is based upon the following notion.

Definition. A *cofinitary system* on X_δ is a collection of sets $(A_x)_{x \in X_\delta' \setminus X_\delta''}$ with the following property. For each $x \in X_\delta' \setminus X_\delta''$, we may write $x = \eta + \omega$ with η a multiple of ω. Then A_x is a subset of $[\eta + 1, x)$ whose complement is finite and non-empty.

Given such a system, its *complementary set* is

$$X_\delta \setminus \bigcup_{x \in X_\delta' \setminus X_\delta''} A_x \cup \{x\}.$$

A key property of cofinitary systems is the following.

Lemma 6.3.1. *Let $(A_x)_{x \in X_\delta' \setminus X_\delta''}$ be a cofinitary system on X_δ with complementary set A_*. If $\alpha = 1$, then A_* is finite, and if $\alpha \geq 2$, then $A_* \cong X_\delta'$.*

Proof. The case $\alpha = 1$ is easy, so assume $\alpha \geq 2$. In this case, both A_* and X'_δ contain all of X''_δ, so it is sufficient to show the following for each $x \in X''_\delta \setminus X'''_\delta$: writing $x = \eta + \omega^2$ with η a multiple of ω^2, we have that

$$A_* \cap [\eta + 1, x] \cong X'_\delta \cap [\eta + 1, x].$$

To see this, simply observe that the set on the right is homeomorphic to $[1, \omega^2]' \cong [1, \omega]$, while the set on the left is homeomorphic to $[1, \omega]$ by definition of a cofinitary system (being infinite by the condition of non-emptiness). □

In order to state our construction, let δ' be the unique ordinal of the form $\omega^{\alpha'} \cdot m$ such that $X'_\delta \cong X_{\delta'}$. For example, if $\delta = \omega^n \cdot m$ with n and m positive integers then $\delta' = \omega^{n-1} \cdot m$.

Proposition 6.3.2. *Let $f : X_\delta \to X_\delta$ be any function. Then $f \in K^1_\delta$ if and only if there are two cofinitary systems $(A_x)_{x \in X'_\delta \setminus X''_\delta}$ and $(B_x)_{x \in X'_\delta \setminus X''_\delta}$ on X_δ, with respective complementary sets A_* and B_*, such that the following conditions hold.*

1. *$f(x) = x$ for all $x \in X'_\delta \setminus X''_\delta$.*
2. *For each $x \in X'_\delta \setminus X''_\delta$, the restriction of f to A_x is an injection with image B_x.*
3. *The restriction of f to A_* is an injection with image B_*.*
4. *If $\alpha \geq 2$, so that by Lemma 6.3.1 we may view the restriction of f to A_* as a function from $X_{\delta'}$ to $X_{\delta'}$, then this function lies in $K^1_{\delta'}$.*

Note that the cofinitary systems in this result need not be unique. Indeed, for each $x \in X'_\delta \setminus X''_\delta$, one may remove any finite subset from A_x while removing its image under f from B_x.

Proof. We again assume $\alpha \geq 2$ since the case $\alpha = 1$ is similar but easier.

Suppose first that $f \in K^1_\delta$. Condition 1 is immediate. To extract the required cofinitary systems, let $x \in X'_\delta \setminus X''_\delta$, and write $x = \eta + \omega$ with η a multiple of ω. By Lemma 6.1.1, there exist $a, b \in X_\delta$ with $x \in (a, b)$ such that for all $y \in X_\delta$, if $y \in (a, b)$ then $f(y) \in (\eta, x + 1)$. Certainly $a \geq \eta$ since $f(\eta) = \eta$ if $\eta > 0$. Take $A_x = (a + 1, x)$ and $B_x = f(A_x)$. These are clearly both subsets of $[\eta + 1, x)$ whose complements are finite and non-empty.

Conditions 2 and 3 now follow immediately from the fact that f is a bijection. For condition 4, first observe that $f(x) = x$ for all $x \in X''_\delta$ since $f \in K^1_\delta$. Finally, the continuity of the restriction f to A_* follows easily from the continuity of f using Lemma 6.1.1.

Conversely, suppose $(A_x)_{x \in X'_\delta \setminus X''_\delta}$ and $(B_x)_{x \in X'_\delta \setminus X''_\delta}$ are cofinitary systems on X_δ, with respective complementary sets A_* and B_*, such that all 4 conditions hold. The first 3 conditions ensure that f is a bijection, and conditions 1 and 4 together imply that $f(x) = x$ for all $x \in X'_\delta$. It remains only to check continuity, for which we use Lemma 6.1.1.

Continuity at isolated points is trivial, and continuity at a point of X''_δ follows straightforwardly using the continuity of the restriction of f to A_* at that point. To complete the proof, let $x \in X'_\delta \backslash X''_\delta$, and let $c, d \in X_\delta \cup \{\pm\infty\}$ with $x = f(x) \in (c, d)$. Since B_x is cofinite, there exists $e \in (c, x)$ such that $(e, x) \subseteq B_x$. By condition 2, $f^{-1}((e, x))$ is a cofinite subset of A_x, so there exists $a \in A_x$ such that $(a, x) \subseteq f^{-1}((e, x))$. Then for all $y \in X_\delta$, if $y \in (a, x + 1)$ then $f(y) \in (e, x + 1) \subseteq (c, d)$. $\square$

Thus we have described K^1_δ in terms of $K^1_{\delta'}$. We also know that $B^1_\delta \cong H^1_\delta \cong H_{\delta'}$ by part 2 of Lemma 6.2.1. Since any $f \in H_\delta$ can be written uniquely as $k \circ b$ with $k \in K^1_\delta$ and $b \in B^1_\delta$ by Proposition 6.2.2, we therefore have a complete description of H_δ in terms of $H_{\delta'}$.

What is not clear, however, is that $H_{\delta'}$ is any simpler than H_δ. Indeed, δ' must satisfy $\delta = \omega \cdot \delta'$, and hence

$$\delta' = \begin{cases} \omega^{n-1} \cdot m, & \text{if } \alpha = n \text{ is finite} \\ \delta, & \text{otherwise.} \end{cases}$$

Hence we obtain recursively a complete description of $H_{\omega^n \cdot m}$ for all $n \in \omega$ and all positive integers m, but only an infinite regress for H_δ in the case $\delta \geq \omega^\omega$.

6.4 A partial description of H_{ω^ω}

From the next section onwards, we will focus exclusively on the first case from the above dichotomy. But first, let us briefly examine some of the complexities that arise in the second case. In this section, fix $\delta = \omega^\omega$.

In this case X'_δ is order-homeomorphic X_δ, and hence by induction $X^{(n)}_\delta$ is order-homeomorphic to X_δ for all $n \in \omega$. We may therefore make the following definition.

Definition. Let $n \in \omega$ and let $\widetilde{c}_n$ be the unique order-homeomorphism $X^{(n)}_\delta \to X_\delta$. Define a group homomorphism $c^n : H_\delta \to H^n_\delta$ by $c^n(f) = \widetilde{c}_n^{-1} \circ f \circ \widetilde{c}_n$. Then define $d^n = b^n_\delta \circ c^n$, where $b^n_\delta : H^n_\delta \to H_\delta$ is the block map.

Thus if $f \in H_\delta$, then $d^n(f) \in H_\delta$ is obtained by applying f to $X^{(n)}_\delta \cong X_\delta$ and extending this to the whole of X_δ using the block map. One may check that in fact $d^n = (d^1)^n$ for all $n \in \omega$. This map enables us to iteratively decompose any $f \in H_\delta$ using the case $\beta = 1$ of Proposition 6.2.2, as follows.

Definition. Given $f \in H_\delta$, write $f = f_0 = h_0 \circ g_0$ with $g_0 \in K^1_\delta$ and $h_0 \in B^1_\delta$. Since $B^1_\delta = \mathrm{Im}(b^1_\delta)$, we may write $h_0 = d^1(f_1)$ with $f_1 \in H_\delta$. Thus $f = d^1(f_1) \circ g_0$. Iterate this procedure to obtain $f_n \in H_\delta$ and $g_n \in K_\delta^1$ for each $n \in \omega$ such that

$$f = d^n(f_n) \circ d^{n-1}(g_{n-1}) \circ d^{n-2}(g_{n-2}) \circ \cdots \circ d^1(g_1) \circ g_0$$

for all $n \in \omega$. We then define the *block decomposition* of f to be $(g_n)_{n\in\omega}$.

The term "decomposition" here is justified by the following result.

Proposition 6.4.1. *The map taking $f \in H_\delta$ to its block decomposition is an injection.*

Proof. Fix $x \in X_\delta$ and let $f \in H_\delta$ have block decomposition $(g_n)_{n\in\omega}$. Write $f(x)$ in Cantor normal form as

$$f(x) = \omega^N \cdot m_N + \omega^{N-1} \cdot m_{N-1} + \cdots + \omega \cdot m_1 + m_0$$

with $m_n \in \omega$ for all $n \in \{0, 1, \ldots, N\}$, and define $m_n = 0$ for all $n > N$. Thus we may recover $f(x)$ from the sequence $(m_n)_{n\in\omega}$. We claim that for all $n \in \omega$, we may recover $m_0, m_1, \ldots, m_{n-1}$ from $g_0, g_1, \ldots, g_{n-1}$ (without f). It follows that we may recover $f(x)$ from $(g_n)_{n\in\omega}$, and since x is arbitrary the result follows.

To prove the claim, simply recall that for all $n \in \omega$,

$$f = d^n(f_n) \circ d^{n-1}(g_{n-1}) \circ d^{n-2}(g_{n-2}) \circ \cdots \circ d^1(g_1) \circ g_0$$

for some $f_n \in H_\delta$. Observe that $d^n(f_n) \in B_\delta^n$ and so f_n does not affect the values of $m_0, m_1, \ldots, m_{n-1}$. □

Thus H_δ may in some sense be described entirely in terms of its subgroup K_δ^1, though this description is somewhat complicated.

Note that not every sequence $(g_n)_{n\in\omega}$ with $g_n \in K_\delta^1$ for all $n \in \omega$ may be obtained as the block decomposition of some $f \in H_\delta$. For example, let g_n be the transposition $(1\,2)$ for all $n \in \omega$. If we tried to recover $(m_n)_{n\in\omega}$ in the case $x = 1$ as in the above proof, then we would obtain $m_n = 2$ for all $n \in \omega$, which is absurd. There are nonetheless many sequences that may be obtained in this fashion. For example, it is sufficient (but not necessary) for the support of g_n to be subset of $[2, \omega)$ for all $n \in \omega$.

Note also that if $f \in H_\delta$ has block decomposition $(g_n)_{n\in\omega}$, then for all $m \in \omega$, $f \in K_\delta^m$ if and only if g_n is the identity function for all $n \geq m$. In particular, the quotient $H_\delta / \bigcup_{m\in\omega} K_\delta^m$ is non-trivial. It may be interesting to study the normal subgroups of this quotient, since many of the techniques we will present seem to break down in that context. Moreover, the corresponding quotient defined for superatomic Boolean algebras in general is an important notion in the study of uncountable superatomic Boolean algebras [Roi89, Definition 1.7].

6.5 The normal subgroups of $H_{\omega^n \cdot m}$

The remainder of the chapter is devoted to studying the normal subgroups of $H_{\omega^n \cdot m}$ for finite n and m. Fix positive integers n and m and let $\delta = \omega^n \cdot m$.

We have seen the importance of the subgroup K^1_δ, the pointwise stabiliser of X'_δ. Because the more general problem appears to be somewhat harder, our results concern only the normal subgroups of H_δ contained in K^1_δ. After proving these, we will briefly indicate how one may attempt to generalise these results and thereby obtain a complete classification of the normal subgroups of H_δ.

In order to state our main result, we require the notions of the *character* and *flow* of a function in K^1_δ. Note that these are not related to characters in representation theory or flows in topological dynamics. It will take some work to define these precisely, so we begin with some examples in order to make these ideas more transparent.

First let us introduce a useful abbreviation.

Definition. Given a topological space X and an ordinal γ, we write $X^{[\gamma]} = X^{(\gamma)} \setminus X^{(\gamma+1)}$.

For example, $X^{[0]}_\delta = \{x \in X : x \text{ is isolated}\}$.

The character of a function in K^1_δ will be a function of its support, for which we use the following notation.

Definition. Let $f \in H_\delta$. Define the *support* of f by

$$\mathrm{supp}\,(f) = \{x \in X_\delta : f\,(x) \neq x\}\,.$$

For example, if $f \in H_\delta$ then $f \in K^1_\delta$ if and only if $\mathrm{supp}\,(f) \subseteq X^{[0]}_\delta$.

In the very simplest special case, we have $\delta = \omega$. Then X_δ has exactly one limit point, so $K^1_\delta = H_\delta$. Moreover, by ignoring the trivial action of the group on this limit point, we obtain the group S_∞ of permutations of $[1, \omega)$. The Schreier–Ulam theorem states that the normal subgroups of this group are exactly: the identity; the group of alternating permutations of finite support; the group of all permutations of finite support; and S_∞.

Now consider the special case $\delta = \omega^2$. Once again, the following are all normal subgroups of H_δ contained in K^1_δ: the identity; the group of alternating permutations of $X^{[0]}_\delta$ of finite support; the group of all permutations $X^{[0]}_\delta$ of finite support; and K^1_δ. But we now have several new normal subgroups of H_δ contained in K^1_δ, such as:

- the group of all functions in K^1_δ whose support is contained in a proper initial segment of $[1, \omega^2)$;
- the group of all functions in K^1_δ whose support is finite on every proper initial segment of $[1, \omega^2)$; and

- the group of all functions in K_δ^1 whose support is finite on every proper initial segment of $[x, \omega^2)$ for some $x \in [1, \omega^2)$.

The notion of the character of a function in K_δ^1 for arbitrary $\delta < \omega^\omega$ will allow us capture these sorts of characteristics of the support of a permutation that give rise to normal subgroups.

The notion of the flow of a function in K_δ^1 captures the other properties of a permutation that give rise to normal subgroups. To illustrate this, consider the special case $\delta = \omega \cdot 2$. Note that X_δ is homeomorphic to $\mathbb{Z} \cup \{\pm\infty\}$ under the order topology, and moreover the induced action of the group on $\pm\infty$ is determined by its action on $\mathbb{Z}$. We may therefore identify H_δ with the induced group of permutations of $\mathbb{Z}$, which we shall denote by $H_\mathbb{Z}$.

Let us now describe $H_\mathbb{Z}$ more explicitly. Given $f \in \operatorname{Sym}(\mathbb{Z})$, say that f *preserves limits* to mean that if $x_i \in \mathbb{Z}$ for all $i \in \omega$ and $\lim_{i\to\infty} x_i = \pm\infty$, then $\lim_{i\to\infty} f(x_i) = \lim_{i\to\infty} x_i$, and say that f *flips limits* to mean that if $x_i \in \mathbb{Z}$ for all $i \in \omega$ and $\lim_{i\to\infty} x_i = \pm\infty$, then $\lim_{i\to\infty} f(x_i) = -\lim_{i\to\infty} x_i$. We then have

$$H_\mathbb{Z} = \{f \in \operatorname{Sym}(\mathbb{Z}) : f \text{ preserves limits or } f \text{ flips limits}\}.$$

Furthermore, let us denote the the subgroup of $H_\mathbb{Z}$ corresponding to $K_{\omega\cdot 2}^1$ under our identification by $K_\mathbb{Z}$. Then

$$K_\mathbb{Z} = \{f \in \operatorname{Sym}(\mathbb{Z}) : f \text{ preserves limits}\}.$$

The flow of $f \in K_\mathbb{Z}$ is intended to capture the amount by which f moves points away from $-\infty$ and towards ∞. It may be defined using disjoint cycle decompositions, so let us first define it for a single cycle. Note that if $f = (\ldots\, x_{-1}\, x_0\, x_1\, \ldots) \in K_\mathbb{Z}$ is an infinite cycle, then $\lim_{i\to-\infty} x_i = \pm\infty$ and $\lim_{i\to\infty} x_i = \pm\infty$ since f preserves limits.

Definition. If $f \in K_\mathbb{Z}$ is a finite cycle, then define $\operatorname{flow}(f) = 0$. If $f = (\ldots\, x_{-1}\, x_0\, x_1\, \ldots) \in K_\mathbb{Z}$ is an infinite cycle, then define

$$\operatorname{flow}(f) = \begin{cases} 0, & \text{if } \lim_{i\to\infty} x_i = \lim_{i\to-\infty} x_i \\ 1, & \text{if } \lim_{i\to-\infty} x_i = -\infty \text{ and } \lim_{i\to\infty} x_i = \infty \\ -1, & \text{if } \lim_{i\to-\infty} x_i = \infty \text{ and } \lim_{i\to\infty} x_i = -\infty. \end{cases}$$

Now if $f \in K_\mathbb{Z}$ is written as a product of disjoint cycles, then only finitely many of these cycles has non-zero flow, since f preserves limits. We may therefore make the following definition.

Definition. Let $f \in K_\mathbb{Z}$. The flow of f, denoted by $\operatorname{flow}(f)$, is defined to be the sum of the flows of each cycle in the disjoint cycle decomposition of f, which converges absolutely.

It is easy to check that an alternative definition of flow for $f \in K_{\mathbb{Z}}$ is given by

$$\text{flow}\,(f) = |\mathbb{N} \setminus f\,(\mathbb{N})| - |f\,(\mathbb{N}) \setminus \mathbb{N}|\,,$$

where both cardinalities in this expression are finite because f preserves limits. We will use a version of this definition later, since it is easier to work with.

It follows from this second definition that flow : $K_{\mathbb{Z}} \to \mathbb{Z}$ is a surjective homomorphism, and moreover $\text{flow}\,(g^{-1} \circ f \circ g) = \text{flow}\,(f)$ for all $f \in K_{\mathbb{Z}}$ and all $g \in H_{\mathbb{Z}}$. Hence $\text{Ker}\,(\text{flow}) \trianglelefteq H_{\mathbb{Z}}$ (it is actually the commutator subgroup of both $K_{\mathbb{Z}}$ and $H_{\mathbb{Z}}$), and we obtain a normal subgroup of $H_{\mathbb{Z}}$ contained in $K_{\mathbb{Z}}$ for each subgroup of $\mathbb{Z}$. In fact, we shall see later that the normal subgroups of $H_{\mathbb{Z}}$ contained in $K_{\mathbb{Z}}$ are exactly: the identity; the group of alternating permutations of $\mathbb{Z}$ of finite support; the group of all permutations of $\mathbb{Z}$ of finite support; and the normal subgroups containing Ker (flow) corresponding to the subgroups of $\mathbb{Z}$.

Our classification result for arbitrary $\delta < \omega^\omega$ essentially says that the normal subgroups of H_δ contained in K_δ^1 are exactly those that emerge via characters and flows. We will give a precise statement of this result once we have fully developed these notions.

6.6 Characters

From now until the end of the chapter, fix positive integers n and m and let $\delta = \omega^n \cdot m$.

In this section we introduce and develop the basic theory of characters. Here is a crucial piece of notation we require.

Definition. Let $l \in \omega$ and suppose $Y \subseteq X_\delta^{[l]}$. Given $k \in \omega$ with $k > l$, we define

$$Y\,[(k)] = \text{cl}\,(Y) \cap X_\delta^{[k]},$$

where cl denotes the topological closure operation relative to X_δ.

Given $k_1, k_2, \ldots, k_r \in \omega$ with $l < k_1 < k_2 < \cdots < k_r$, we inductively define $Y\,[(k_1, \ldots, k_r)] = (Y\,[(k_1, \ldots, k_{r-1})])\,[(k_r)]$. By convention the sequence of length 0 is denoted by $()$ and we take $Y\,[()] = Y$.

For example, suppose $\delta = \omega^2$. If $Y = \{\omega \cdot i + 1 : i \in \omega\}$, then $Y\,[(1)] = \emptyset$, $Y\,[(2)] = \{\omega^2\}$ and $Y\,[(1,2)] = \emptyset$. If $Y = \{\omega \cdot i + j : i \in \omega, j \in \omega \setminus \{0\}\}$, then $Y\,[(1)] = \{\omega \cdot (i+1) : i \in \omega\}$, $Y\,[(2)] = \{\omega^2\}$ and $Y\,[(1,2)] = \{\omega^2\}$.

Here are the key properties of this piece of notation.

Lemma 6.6.1. *Let $l \in \omega$ and suppose $Y, Z \subseteq X_\delta^{[l]}$.*

1. *Let $k \in \omega$ with $k > l$. Then $(Y \cup Z)\,[(k)] = Y\,[(k)] \cup Z\,[(k)]$. In particular, if $Y \subseteq Z$ then $Y\,[(k)] \subseteq Z\,[(k)]$.*

2. *Let* $k_1, k_2 \in \omega$ *with* $l < k_1 < k_2$. *Then* $Y[(k_1, k_2)] \subseteq Y[(k_2)]$.

3. Y *is infinite if and only if* $Y[(k)] \neq \emptyset$ *for some* $k \in \omega$ *with* $k > l$.

Proof. 1. This is immediate from the fact that $\mathrm{cl}(Y \cup Z) = \mathrm{cl}(Y) \cup \mathrm{cl}(Z)$. To see this, observe that the right-hand side is a closed set containing Y and Z.

2. $Y[(k_1, k_2)] = \mathrm{cl}\left(\mathrm{cl}(Y) \cap X_\delta^{[k_1]}\right) \cap X_\delta^{[k_2]} \subseteq \mathrm{cl}(\mathrm{cl}(Y)) \cap X_\delta^{[k_2]} = Y[(k_2)]$.

3. Since $Y \subseteq X_\delta^{[l]}$, Y is discrete in the subspace topology, so Y is compact if and only if Y is finite. Since X_δ is compact and Hausdorff, it follows that Y is closed if and only if Y is finite. But $\mathrm{cl}(Y) \setminus Y \subseteq X_\delta^{(l+1)}$. Hence Y is infinite if and only if $\mathrm{cl}(Y) \setminus Y \neq \emptyset$, if and only if $Y[(k)] \neq \emptyset$ for some $k \in \omega$ with $k > l$. □

We may now use square bracket notation to define characters.

Definition. Define

$$\Gamma_n = \{(k_1, k_2, \ldots, k_r) : r \geq 0, 1 \leq k_1 < k_2 < \cdots < k_r \leq n\}.$$

Thus given $Y \subseteq X_\delta^{[0]}$, $Y[s]$ is defined for all $s \in \Gamma_n$.

We define the *type* of $Y \subseteq X_\delta^{[0]}$ by

$$\mathrm{tp}(Y) = \{s \in \Gamma_n : Y[s] \neq \emptyset\}.$$

We define the *character* of $f \in K_\delta^1$ by

$$\mathrm{char}(f) = \mathrm{tp}(\mathrm{supp}(f)).$$

The following key property of types and hence of characters is inherited from the corresponding property of square bracket notation.

Lemma 6.6.2. *Let* $Y, Z \subseteq X_\delta^{[0]}$. *Then* $\mathrm{tp}(Y \cup Z) = \mathrm{tp}(Y) \cup \mathrm{tp}(Z)$. *In particular, if* $Y \subseteq Z$ *then* $\mathrm{tp}(Y) \subseteq \mathrm{tp}(Z)$.

Proof. Let $s \in \Gamma_n$. Then $(Y \cup Z)[s] = Y[s] \cup Z[s]$ by applying part 1 of Lemma 6.6.1 inductively. Hence $s \in \mathrm{tp}(Y \cup Z)$ if and only if $s \in \mathrm{tp}(Y) \cup \mathrm{tp}(Z)$. □

From this result we immediately obtain several normal subgroups of H_δ contained in K_δ^1.

Definition. Given $\Delta \subseteq \Gamma_n$, we define

$$L_\Delta = \left\{f \in K_\delta^1 : \mathrm{char}(f) \subseteq \Delta\right\}.$$

Proposition 6.6.3. *Let* $\Delta \subseteq \Gamma_n$. *Then* $L_\Delta \trianglelefteq H_\delta$.

Proof. First note that the definition of types is entirely topological, so if $Y \subseteq X_\delta^{[0]}$ and $g \in H_\delta$ then $\operatorname{tp}(g(Y)) = \operatorname{tp}(Y)$. Hence if $f \in L_\Delta$ and $g \in H_\delta$ then $\operatorname{char}(g \circ f \circ g^{-1}) = \operatorname{tp}(\operatorname{supp}(g \circ f \circ g^{-1})) = \operatorname{tp}(g(\operatorname{supp}(f))) = \operatorname{char}(f)$. So it is sufficient to prove that L_Δ is a group.

Clearly the identity function is a member of L_Δ, and if $f \in L_\Delta$ then $\operatorname{supp}(f^{-1}) = \operatorname{supp}(f)$ and so $f^{-1} \in L_\Delta$. Finally, if $f, h \in L_\Delta$ then $\operatorname{supp}(f \circ h) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(h)$ and so $\operatorname{char}(f \circ h) \subseteq \operatorname{char}(f) \cup \operatorname{char}(h) \subseteq \Delta$ by Lemma 6.6.2. □

The other key property of types also follows from the key properties of square bracket notation. This result shows that these normal subgroups need not all be distinct.

Lemma 6.6.4. *Let $Y \subseteq X_\delta^{[0]}$, suppose $(k_1, k_2, \ldots, k_r) \in \operatorname{tp}(Y)$ and let $i \in \{1, 2, \ldots, r\}$. Then $(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_r) \in \operatorname{tp}(Y)$.*

Proof. If $i = r$ then in fact $Y[(k_1, \ldots, k_{r-1})]$ is infinite by part 3 of Lemma 6.6.1, and we are done. Otherwise $Y[(k_1, \ldots, k_{i+1})] \subseteq Y[(k_1, \ldots, k_{i-1}, k_{i+1})]$ by part 2 of Lemma 6.6.1 and hence $Y[(k_1, \ldots, k_r)] \subseteq Y[(k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_r)]$ by part 1 of Lemma 6.6.1, and we are done. □

This result motivates the following definition. It may be helpful to note here that there is a natural way to identify Γ_n with $\mathcal{P}(\{1, 2, \ldots, n\})$ in such a way that subsequences correspond to subsets.

Definition. Let $\Delta, S \subseteq \Gamma_n$. We say that Δ is a *lower set* to mean that if $s \in \Delta$ and t is a subsequence of s, then $t \in \Delta$. We say that S is an *antichain* to mean that if $s, t \in S$ then t is not a subsequence of s.

Given $S \subseteq \Gamma_n$, let $\langle S \rangle = \{t \in \Gamma_n : t \text{ is a subsequence of some } s \in S\}$, or equivalently, $\langle S \rangle$ is the smallest lower set of Γ_n containing S. Abbreviate $\langle \{s_1, s_2, \ldots, s_r\} \rangle$ as $\langle s_1, s_2, \ldots, s_r \rangle$.

Lemma 6.6.4 may now be restated as saying that if $Y \subseteq X_\delta^{[0]}$, then $\operatorname{tp}(Y)$ is a lower set.

The main reason for introducing antichains is that the map

$$\begin{aligned} \{\text{antichains of } \Gamma_n\} &\to \{\text{lower sets of } \Gamma_n\} \\ S &\mapsto \langle S \rangle \end{aligned}$$

is a bijection with inverse given by

$$\Delta \mapsto \{t \in \Delta : t \text{ is not a subsequence of } s \text{ for any } s \in \Delta \setminus \{t\}\}.$$

Thus antichains are simply another way of thinking about lower sets, but they contain no redundant information in the sense that any subset of an antichain is another antichain.

We will show in Section 6.8 that Lemma 6.6.4 is in fact the only restriction on which subsets of Γ_n may be obtained as the character of some $f \in K^1_\delta$. In other words, for each lower set $\Delta \subseteq \Gamma_n$ we will find a witness $f_\Delta \in K^1_\delta$ with $\mathrm{char}\,(f_\Delta) = \Delta$, which we will call the *canonical function with character* Δ. It will follow that Proposition 6.6.3 provides us with exactly one distinct normal subgroup for each lower set $\Delta \subseteq \Gamma_n$. (The exact number of lower sets $\Delta \subseteq \Gamma_n$ is called the nth *Dedekind number* and is denoted by $D\,(n)$.)

Let us conclude this section by illustrating our claim in the special case $\delta = \omega^2$, thereby demonstrating the expressive power of characters. In this case, there are six possible characters of a function $f \in K^1_\delta$, whose corresponding normal subgroups were identified in the previous section:

- $\emptyset$, the character of the identity function;
- $\{()\}$, the character of a function with non-empty finite support;
- $\{(),(1)\} = \langle(1)\rangle$, the character of a function whose support is infinite, but contained in a proper initial segment of $[1, \omega^2)$;
- $\{(),(2)\} = \langle(2)\rangle$, the character of a function whose support is infinite, but finite on every proper initial segment of $[1, \omega^2)$;
- $\{(),(1),(2)\} = \langle(1),(2)\rangle$, the character of a function whose support is neither contained in a proper initial segment of $[1, \omega^2)$ nor finite on every proper initial segment of $[1, \omega^2)$, but is finite on every proper initial segment of $[x, \omega^2)$ for some $x \in [1, \omega^2)$; and
- $\{(),(1),(2),(1,2)\} = \langle(1,2)\rangle$, the character a function whose support is infinite on some proper initial segment of $[x, \omega^2)$ for every $x \in [1, \omega^2)$.

6.7 Flows

In this section we introduce and develop the basic theory of flows, which will at last allow us to state our main result.

Given $f \in K^1_\delta$ and a subset $C \subseteq X_\delta$, we would like to measure the amount by which f moves points "into" C. The following simple result enables us to formalise this when C is clopen (closed and open). Here $C \setminus f\,(C)$ should be thought of as the points "entering" C under f, and $f\,(C) \setminus C$ should be thought of as the points "leaving" C under f.

Lemma 6.7.1. *Let $f \in K^1_\delta$ and $C \subseteq X_\delta$ be clopen. Then $C \setminus f\,(C)$ and $f\,(C) \setminus C$ are finite.*

Proof. Let $Y = C \setminus f\,(C)$. It is enough to show that Y is finite, since $|f\,(C) \setminus C| = |f^{-1}\,(f\,(C) \setminus C)| = |C \setminus f^{-1}\,(C)|$. To see this, simply observe that $Y \subseteq X^{[0]}_\delta$ since

$f \in K^1_\delta$, and that Y is closed since $Y = C \cap f(X_\delta \setminus C)$ and C is clopen. Hence Y is both discrete and compact and therefore finite. □

This result enables us to make the following definition.

Definition. Let $f \in K^1_\delta$ let C be a clopen subset of X_δ. We define the *flow of f into C* by

$$\mathrm{flow}_C(f) = |C \setminus f(C)| - |f(C) \setminus C|.$$

Here are the basic properties of this definition. Let us introduce some standard notation for conjugation: given $f, g \in H_\delta$, write $f^g = g^{-1} \circ f \circ g$.

Lemma 6.7.2. *Let $f \in K^1_\delta$ and let C and D be disjoint clopen subsets of X_δ.*

1. $\mathrm{flow}_{C\cup D}(f) = \mathrm{flow}_C(f) + \mathrm{flow}_D(f)$.
2. $\mathrm{flow}_{X_\delta \setminus C}(f) = -\mathrm{flow}_C(f)$.
3. *If $g \in K^1_\delta$, then* $\mathrm{flow}_C(f \circ g) = \mathrm{flow}_C(f) + \mathrm{flow}_C(g)$.
4. *If $g \in H_\delta$, then* $\mathrm{flow}_C(f^g) = \mathrm{flow}_{g(C)}(f)$.

Proof. These are all simple set-theoretic identities that we leave as exercises. □

The most convenient clopen subsets to use are "blocks", which are defined as follows.

Definition. Let $z \in X'_\delta$. Write $z = \eta + \omega^k$ with η a multiple of ω^k. Define

$$\mathcal{B}(z) = [\eta + 1, z].$$

Note that the image of the block map from Section 6.2 preserves these blocks in the sense that if $z \in X^{[k]}_\delta$ for some $k \in \{1, 2, \ldots, n\}$ and $b \in B^k_\delta$, then $b(\mathcal{B}(z)) = \mathcal{B}(z')$ for some $z' \in X^{[k]}_\delta$.

Clearly $\mathcal{B}(z)$ is clopen for all $z \in X'_\delta$. The collection of all such blocks has the following useful property.

Lemma 6.7.3. *Let C be a clopen subset of X_δ. Then C is a finite Boolean combination of sets of the form $\{x\}$ with $x \in X^{[0]}_\delta$ and $\mathcal{B}(z)$ with $z \in X'_\delta$.*

Proof. Recall that $\delta = \omega^n \cdot m$, and note first of all that the general case follows from the case $m = 1$, since X_δ is a topological disjoint union of m copies of X_{ω^n}. So assume $m = 1$, so $\mathcal{B}(\omega^n) = X_\delta$. The remainder of the proof is by induction on n.

For the base case, suppose $n = 1$. If $\omega \in C$, then $X_\delta \setminus C$ is a finite subset of $X^{[0]}_\delta = [1, \omega)$, and if $\omega \in X_\delta \setminus C$, then C is a finite subset of $X^{[0]}_\delta$. In either case the result follows.

For the inductive step, suppose $n > 1$. Similarly to the case $n = 1$, either C or $X_\delta \setminus C$ (whichever does not contain the point ω^n) is a finite disjoint union $\bigcup_{i=1}^r C_i$, where C_i is a clopen subset of $\mathcal{B}(z_i)$ for some $z_i \in X_\delta^{[n-1]}$ for each $i \in \{1, 2, \ldots, r\}$. Now $X_{\omega^{n-1}}$ is order-homeomorphic to $\mathcal{B}(z_i)$ for all $i \in \{1, 2, \ldots, r\}$, and moreover this order-homeomorphism maps blocks of $X_{\omega^{n-1}}$ to blocks of X_{ω^n}. Hence by the inductive hypothesis, each of $C_1, C_2, \ldots, C_r$ is a finite Boolean combination of sets of the required form, and the result follows. □

We may now define the flow map.

Definition. We define a map flow : $K_\delta^1 \to \mathbb{Z}^{X'_\delta}$ by

$$\text{flow}(f) = \left(\text{flow}_{\mathcal{B}(z)}(f)\right)_{z \in X'_\delta}.$$

The identity $(0)_{z \in X'_\delta}$ of $\mathbb{Z}^{X'_\delta}$ is denoted by $\mathbf{0}$.

It is possible to define this map in terms of disjoint cycles as in Section 6.5, though it is unnecessary and somewhat cumbersome to do so.

We immediately obtain the following important fact from part 3 of Lemma 6.7.2.

Lemma 6.7.4. flow *is a homomorphism.* □

We also obtain the following consequence of Lemma 6.7.3. This result may be interpreted as saying that the flow map encapsulates every possible flow into a clopen subset of X_δ.

Lemma 6.7.5. *Let $f \in K_\delta^1$. Then* flow $(f) = \mathbf{0}$ *if and only if* $\text{flow}_C(f) = 0$ *for every clopen subset $C \subseteq X_\delta$.*

Proof. The "if" statement is clear from the fact that $\mathcal{B}(z)$ is clopen for all $z \in X'_\delta$. For the "only if" statement, suppose flow $(f) = \mathbf{0}$. First observe that $\text{flow}_{\{x\}}(f) = 0$ for every $x \in X_\delta^{[0]}$. It follows by combining Lemma 6.7.3 with parts 1 and 2 of Lemma 6.7.2 that $\text{flow}_C(f) = 0$ for every clopen subset $C \subseteq X_\delta$. □

Trivially $\text{Ker}(\text{flow}) \trianglelefteq K_\delta^1$, but by combining the above result with part 4 of Lemma 6.7.2 we obtain the following.

Lemma 6.7.6. $\text{Ker}(\text{flow}) \trianglelefteq H_\delta$.

Proof. Let $f \in \text{Ker}(\text{flow})$ and $g \in H_\delta$. Then by Lemma 6.7.5 $\text{flow}_C(f) = 0$ for every clopen subset $C \subseteq X_\delta$, so by part 4 of Lemma 6.7.2 $\text{flow}_C(f^g) = \text{flow}_{g(C)}(f) = 0$ for every clopen subset $C \subseteq X_\delta$, so by Lemma 6.7.5 $f^g \in \text{Ker}(\text{flow})$. □

Let us now introduce some notation for some of the relevant groups we have now obtained.

Definition. Let $\Delta \subseteq \Gamma_n$ be a lower set.

Define $\mathbb{Z}_\Delta$ to be the image of L_Δ under flow.

Define $L_\Delta^0 = \text{Ker}(\text{flow}) \cap L_\Delta$.

Thus $L^0_\Delta \trianglelefteq H_\delta$ for every lower set $\Delta \subseteq \Gamma_n$. In fact, it is not too hard to deduce from the proofs of the results in Section 6.9 that L^0_Δ is the commutator subgroup of L_Δ for every lower set $\Delta \subseteq \Gamma_n$.

We may at last state our main result of this chapter, a complete classification of the normal subgroups of H_δ contained in K^1_δ. Note that we obtain in particular our claim about the special case $\delta = \omega \cdot 2$ from Section 6.5.

Theorem 6.7.7. *Let N be a normal subgroup of H_δ contained in K^1_δ other than the group of alternating permutations of finite support. Then*

$$L^0_\Delta \leq N \leq L_\Delta$$

for a unique lower set $\Delta \subseteq \Gamma_n$.

Moreover, given a lower set $\Delta \subseteq \Gamma_n$, the normal subgroups $N \trianglelefteq H_\delta$ with $L^0_\Delta \leq N \leq L_\Delta$ may be characterised in terms of the subgroups of $\mathbb{Z}_\Delta$ to which they correspond. Specifically, since $H_\delta = K^1_\delta \rtimes B^1_\delta$ and the induced conjugation action of K^1_δ on $\mathbb{Z}_\Delta$ is trivial, a subgroup of $\mathbb{Z}_\Delta$ will correspond to a normal subgroup of H_δ if and only if it is invariant under the induced conjugation action of B^1_δ on $\mathbb{Z}_\Delta$. We will come back and analyse these subgroups of $\mathbb{Z}_\Delta$ after proving our main result. In particular, we will find $2^{2^{\aleph_0}}$ such subgroups whenever $(k_1, k_2) \in \Delta$ for some $k_1, k_2 \in \{1, 2, \ldots, n\}$.

6.8 Canonical functions

Recall that if $f \in K^1_\delta$, then $\operatorname{char}(f)$ is a lower set. The purpose of this section is to find for each lower set $\Delta \subseteq \Gamma_n$ a function $f_\Delta \in K^1_\delta$ with $\operatorname{char}(f_\Delta) = \Delta$, which we will call the *canonical function with character* Δ. We will choose these functions carefully, since they will be an important tool in the proof of Theorem 6.7.7. In particular, they will all have zero flow.

Our basic construction uses a product of permutations whose supports are pairwise disjoint, which is defined in the obvious way. Note that if $f_i \in H_\delta$ for all $i \in \omega$ and $\operatorname{supp}(f_i) \cap \operatorname{supp}(f_j) = \emptyset$ for all distinct $i, j \in \omega$, then in general it need not be the case that the infinite product $\prod_{i \in \omega} f_i$ lies H_δ: for example, take f_i to be the transposition $(i+1\ \omega+i+1)$ for all $i \in \omega$. However, the infinite products used in our construction will clearly satisfy the criterion of Proposition 6.3.2 and therefore lie in K^1_δ.

Here is our basic construction.

Definition. For each subset $Z \subseteq X'_\delta$, we define $f_Z \in H_\delta$ as follows.

Let $z \in X'_\delta$. Write $z = \eta + \omega^k$ with η a multiple of ω^k. For each positive integer i, let $a_i = \eta + \omega^{k-1} \cdot i + 1$. Then define $f_{\{z\}}$ to be the cycle $(\ldots\ a_6\ a_4\ a_2\ a_1\ a_3\ a_5\ \ldots)$.

For an arbitrary subset $Z \subseteq X'_\delta$, define $f_Z = \prod_{z\in Z} f_{\{z\}}$. It is easy to check that this is a product of disjoint cycles.

The point of this definition is to obtain the following result.

Lemma 6.8.1. *Let Z be a closed subset of X'_δ. Then*

$$\mathrm{cl}\,(\mathrm{supp}\,(f_Z)) \cap X'_\delta = Z.$$

Note that by definition of the square bracket notation, if $f \in K^1_\delta$ has non-empty support, then $\mathrm{char}\,(f)$ depends only $\mathrm{cl}\,(\mathrm{supp}\,(f)) \cap X'_\delta$, which must be a closed subset of X'_δ. This result shows that in fact any closed subset of X'_δ may be realised in this fashion, thereby reducing the problem of finding canonical functions to finding certain closed subsets of X'_δ.

Proof. First note that by construction, $\mathrm{supp}\left(f_{\{z\}}\right) \subseteq X^{[0]}_\delta$ and $\mathrm{cl}\left(\mathrm{supp}\left(f_{\{z\}}\right)\right) = \mathrm{supp}\left(f_{\{z\}}\right) \cup \{z\}$ for each $z \in Z$, and moreover $\mathrm{supp}\,(f_Z) = \bigcup_{z\in Z} \mathrm{supp}\left(f_{\{z\}}\right)$.

We show that $\mathrm{cl}\,(\mathrm{supp}\,(f_Z)) = \mathrm{supp}\,(f_Z) \cup Z$, which suffices. First of all, $\mathrm{cl}\,(\mathrm{supp}\,(f_Z)) \supseteq \bigcup_{z\in Z} \mathrm{cl}\left(\mathrm{supp}\left(f_{\{z\}}\right)\right) = \mathrm{supp}\,(f_Z) \cup Z$. Conversely, let $z \in \mathrm{cl}\,(\mathrm{supp}\,(f_Z)) \setminus \mathrm{supp}\,(f_Z)$, say $z = \sup\,(V)$ for some $V \subseteq \mathrm{supp}\,(f_Z)$, and suppose for contradiction $z \notin Z$. By definition of f_Z, for all $v \in V$ we may write $v = \eta_v + \omega^{k_v - 1} \cdot i_v + 1$ with i_v a positive integer, η_v a multiple of ω^{k_v} and $\eta_v + \omega^{k_v} \in Z$. Now write $z = \zeta + \omega^l$ with ζ a multiple of ω^l. We may assume without loss of generality that $v \in (\zeta + 1, z)$ for all $v \in V$. But the only members of $(\zeta + 1, z)$ that are multiples of ω^{l-1} plus one lie in $\mathrm{supp}\left(f_{\{z\}}\right)$, so since $z \notin Z$ it follows that $k_v < l$ for all $v \in V$. But then $\eta_v + \omega^{k_v} < z$ for all $v \in V$, so $z = \sup\,(Y)$ where $Y = \left\{\eta_v + \omega^{k_v} : v \in V\right\} \subseteq Z$. Hence $z \in Z$ since Z is closed, and we are done. □

These functions also have the following convenient properties.

Lemma 6.8.2. *Let $Z_1, Z_2, \ldots, Z_r \subseteq X'_\delta$.*

1. $\mathrm{supp}\left(f_{\bigcup_{i=1}^r Z_i}\right) = \bigcup_{i=1}^r \mathrm{supp}\,(f_{Z_i})$.
2. $\mathrm{supp}\,(f_{Z_1} \circ f_{Z_2} \circ \cdots \circ f_{Z_r}) = \bigcup_{i=1}^r \mathrm{supp}\,(f_{Z_i})$.

Proof. 1. This is immediate from the fact that if $Z \subseteq X'_\delta$, then $\mathrm{supp}\,(f_Z) = \bigcup_{z\in Z} \mathrm{supp}\left(f_{\{z\}}\right)$.

2. Let $f = f_{Z_1} \circ f_{Z_2} \circ \cdots \circ f_{Z_r}$ and $Z = \bigcup_{i=1}^r Z_i$. For each $z \in Z$, let $r_z = |\{i \in \{1, 2, \ldots, r\} : z \in Z_i\}|$. Then $f = \prod_{z\in Z} f^{r_z}_{\{z\}}$, a product of disjoint permutations. Moreover, for each $z \in Z$, since $f_{\{z\}}$ is an infinite cycle, $\mathrm{supp}\left(f^{r_z}_{\{z\}}\right) = \mathrm{supp}\left(f_{\{z\}}\right)$. Hence $\mathrm{supp}\,(f) = \mathrm{supp}\left(\prod_{z\in Z} f_{\{z\}}\right) = \bigcup_{i=1}^r \mathrm{supp}\,(f_{Z_i})$. □

It will also be important for the proof of Theorem 6.7.7 that these functions have zero flow.

Lemma 6.8.3. *Let Z be a closed subset of X'_δ. Then* $\operatorname{flow}(f_Z) = \mathbf{0}$.

Proof. Let $z \in Z$ and $y \in X'_\delta$. Note that $\mathcal{B}(y)$ and $\mathcal{B}(z)$ are either nested or disjoint. If $z \notin \mathcal{B}(y)$, then $\operatorname{supp}\left(f_{\{z\}}\right) \cap \mathcal{B}(y)$ is either empty (if $\mathcal{B}(z) \cap \mathcal{B}(y) = \emptyset$) or finite (if $\mathcal{B}(y) \subsetneq \mathcal{B}(z)$). On the other hand if $z \in \mathcal{B}(y)$, then $\operatorname{supp}\left(f_{\{z\}}\right) \subseteq \mathcal{B}(z) \subseteq \mathcal{B}(y)$. In either case we see that $\operatorname{flow}_{\mathcal{B}(y)}\left(f_{\{z\}}\right) = 0$. Moreover, there are at most finitely many $z \in Z$ with $\mathcal{B}(y) \subsetneq \mathcal{B}(z)$, and hence $\operatorname{flow}_{\mathcal{B}(y)}(f_Z) = 0$ for all $y \in X'_\delta$, as required. □

Suppose we were to extend the definition of the square bracket notation and of types to include all subsets of X_δ, rather than only subsets of $X_\delta^{[0]}$. Lemma 6.8.1 would then imply that $Z[s] = \operatorname{supp}(f_Z)[s]$ whenever Z is a closed subset of X'_δ and $s \in \Gamma_n \setminus \{()\}$. Hence if $\Delta \subseteq \Gamma_n$ is a lower set, then in order to find $f \in K_\delta^1$ with $\operatorname{char}(f) = \Delta$, it is sufficient to find a closed subset $Z \subseteq X'_\delta$ with $\operatorname{tp}(Z) = \Delta$ in this sense.

We now construct such closed subsets of X'_δ, beginning with the case in which Δ has the form $\langle s \rangle$ with $s \in \Gamma_n$.

Definition. Define $\mathcal{Z}(\langle s \rangle)$ for each $s \in \Gamma_n$ inductively by $\mathcal{Z}(\langle () \rangle) = \emptyset$ and

$$\mathcal{Z}(\langle (k_1, k_2, \ldots, k_r) \rangle) = \left\{\omega^{k_r - 1} \cdot 2i + z : i \in \omega, z \in \mathcal{Z}(\langle (k_1, k_2, \ldots, k_{r-1}) \rangle)\right\} \cup \left\{\omega^{k_r}\right\}$$

for $r \geq 1$.

For example, when $\delta = \omega^3$, $\mathcal{Z}(\langle (1,3) \rangle) = \{\omega^2 \cdot 2i + \omega : i \in \omega\} \cup \{\omega^3\}$.

We now give the key properties of these sets, showing in particular that they are as we desired.

Lemma 6.8.4. *Let $(k_1, k_2, \ldots, k_r) \in \Gamma_n$ for some $r \geq 1$.*

1. *$\mathcal{Z}(\langle (k_1, k_2, \ldots, k_r) \rangle)$ is a closed subset of $X'_\delta \cap \left[1, \omega^{k_r}\right]$.*

2. *$\mathcal{Z}(\langle (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_r) \rangle) \subseteq \mathcal{Z}(\langle (k_1, k_2, \ldots, k_r) \rangle)$ for all $i \in \{1, 2, \ldots, r\}$.*

3. *$\operatorname{char}\left(f_{\mathcal{Z}(\langle (k_1, k_2, \ldots, k_r) \rangle)}\right) = \langle (k_1, k_2, \ldots, k_r) \rangle$.*

Proof. 1. This is immediate by induction on r.

2. Let $i \in \{1, 2, \ldots, r\}$. First note that $\mathcal{Z}(\langle (k_1, \ldots, k_{i-1}) \rangle) \subseteq \mathcal{Z}(\langle (k_1, k_2, \ldots, k_i) \rangle)$. It follows by induction on j that $\mathcal{Z}(\langle (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_j) \rangle) \subseteq \mathcal{Z}(\langle (k_1, k_2, \ldots, k_j) \rangle)$ for all $j \in \{i+1, i+2, \ldots, r\}$, as required.

3. Let $Z = \mathcal{Z}(\langle (k_1, k_2, \ldots, k_r) \rangle)$. Extend the definition of the square bracket notation and of types to include all subsets of X_δ. As we remarked above, by Lemma 6.8.1 it is sufficient to prove that $\operatorname{tp}(Z) = \langle (k_1, k_2, \ldots, k_r) \rangle$.

The proof is by induction on r. For the case $r = 1$, $\mathcal{Z}(\langle(k_1)\rangle) = \{\omega^{k_1}\}$ and so $\operatorname{tp}(\mathcal{Z}(\langle(k_1)\rangle)) = \{(), (k_1)\}$, as required. So assume that $r > 1$ and let $W = \mathcal{Z}(\langle(k_1, k_2, \ldots, k_{r-1})\rangle)$. By Lemma 6.6.4 it is sufficient to prove that $(k_1, k_2, \ldots, k_r) \in \operatorname{tp}(Z)$ and if $k \in \{1, 2, \ldots, n\} \setminus \{k_1, k_2, \ldots, k_r\}$ then $(k) \notin \operatorname{tp}(Z)$.

For the first part, let $V = W[(k_1, k_2, \ldots, k_{r-1})]$. By the inductive hypothesis, $V \neq \emptyset$, so let $x \in V$, and note that $x \in [1, \omega^{k_{r-1}}]$ by part 1. Then $\{\omega^{k_r-1} \cdot 2i + x : i \in \omega\} \subseteq Z[(k_1, k_2, \ldots, k_{r-1})]$ and so $\omega^{k_r} \in \operatorname{cl}(Z[(k_1, k_2, \ldots, k_{r-1})]) \cap X_\delta^{[k_r]} = Z[(k_1, k_2, \ldots, k_r)]$. Hence $(k_1, k_2, \ldots, k_r) \in \operatorname{tp}(Z)$, as required.

For the second part, suppose $k \in \{1, 2, \ldots, n\} \setminus \{k_1, k_2, \ldots, k_r\}$. Then $\operatorname{cl}(W) \cap X_\delta^{[k]} = \emptyset$ by the inductive hypothesis. But $\operatorname{cl}(Z) = \{\omega^{k_r-1} \cdot 2i + z : i \in \omega, z \in \operatorname{cl}(W)\} \cup \{\omega^{k_r}\}$, and hence $\operatorname{cl}(Z) \cap X_\delta^{[k]} = \emptyset$. In other words, $(k) \notin \operatorname{tp}(Z)$, as required. □

Our definition of $\mathcal{Z}(\langle s\rangle)$ for $s \in \Gamma_n$ may now be extended to a definition of $\mathcal{Z}(\Delta)$ for every lower set $\Delta \subseteq \Gamma_n$. Part 2 of Lemma 6.8.4 ensures that our two definitions agree when $\Delta = \langle s\rangle$ for some $s \in \Gamma_n$.

Definition. Let $\Delta \subseteq \Gamma_n$ be a lower set. Define

$$\mathcal{Z}(\Delta) = \bigcup_{s \in \Delta} \mathcal{Z}(\langle s\rangle).$$

By part 2 of Lemma 6.8.4, we may equivalently define $\mathcal{Z}(\Delta) = \bigcup_{s \in S} \mathcal{Z}(\langle s\rangle)$, where S is the antichain of Γ_n with $\langle S\rangle = \Delta$.

We now have all the ingredients ready for us to define canonical functions. We need to distinguish the case $\Delta = \{()\}$ because $\mathcal{Z}(\{()\}) = \mathcal{Z}(\emptyset) = \emptyset$. Indeed, if $f \in K_\delta^1$ has finite support, then $\operatorname{cl}(\operatorname{supp}(f)) \cap X_\delta' = \emptyset$ whether or not $\operatorname{char}(f) = \{()\}$ or $\operatorname{char}(f) = \emptyset$.

Definition. Let $\Delta \subseteq \Gamma_n$ be a lower set. We define the *canonical function with character* Δ by

$$f_\Delta = \begin{cases} (1\,2), & \text{if } \Delta = \{()\} \\ f_{\mathcal{Z}(\Delta)}, & \text{otherwise.} \end{cases}$$

By convention, the empty product $f_\emptyset$ is the identity function.

It is now easy for us to check that these functions are as we desired.

Proposition 6.8.5. *Let $\Delta \subseteq \Gamma_n$ be a lower set. Then* $\operatorname{char}(f_\Delta) = \Delta$.

Proof. The result is clear if $\Delta = \{()\}$. Otherwise, $\operatorname{char}(f_\Delta) = \operatorname{tp}\left(\bigcup_{s \in \Delta} \operatorname{supp}\left(f_{\mathcal{Z}(\langle s\rangle)}\right)\right) = \bigcup_{s \in \Delta} \operatorname{char}\left(f_{\mathcal{Z}(\langle s\rangle)}\right) = \bigcup_{s \in \Delta} \langle s\rangle = \Delta$ by, respectively, part 1 of Lemma 6.8.2, Lemma 6.6.2, part 3 of Lemma 6.8.4 and the fact that Δ is a lower set. □

Combined with Lemma 6.6.4, this result finally shows that if $\Delta \subseteq \Gamma_n$, then there exists $f \in K_\delta^1$ with $\operatorname{char}(f) = \Delta$ if and only if Δ is a lower set.

Because of how we chose our canonical functions, we may use part 2 of Lemma 6.8.2 to generalise this result.

Proposition 6.8.6. *Let $\Delta_1, \Delta_2, \ldots, \Delta_r \subseteq \Gamma_n$ be lower sets with $\Delta_i \neq \{()\}$ for all $i \in \{1, 2, \ldots, r\}$. Then*

$$\operatorname{char}(f_{\Delta_1} \circ f_{\Delta_2} \circ \cdots \circ f_{\Delta_r}) = \bigcup_{i=1}^{r} \Delta_i.$$

Proof. By definition $f_{\Delta_i} = f_{\mathcal{Z}(\Delta_i)}$ for all $i \in \{1, 2, \ldots, r\}$. Hence $\operatorname{char}(f_{\Delta_1} \circ f_{\Delta_2} \circ \cdots \circ f_{\Delta_r}) = \operatorname{tp}\left(\bigcup_{i=1}^{r} \operatorname{supp}\left(f_{\mathcal{Z}(\Delta_i)}\right)\right) = \bigcup_{i=1}^{r} \operatorname{char}\left(f_{\mathcal{Z}(\Delta_i)}\right) = \bigcup_{i=1}^{r} \Delta_i$ by, respectively, part 2 of Lemma 6.8.2, Lemma 6.6.2 and Proposition 6.8.5. □

We now use canonical functions to reduce our main result, Theorem 6.7.7, to a pair of statements about conjugate closures, which may be viewed as roughly converse to one another.

Theorem 6.8.7. *Let $h \in K_\delta^1$ have infinite support and let $\Delta = \operatorname{char}(h)$.*

1. *If $\{()\} \subsetneq \Delta' \subseteq \Delta$ is a lower set, then $f_{\Delta'}$ is the product of finitely many conjugates of h or h^{-1}.*

2. *If $\operatorname{flow}(h) = \mathbf{0}$ then h is the product of finitely many conjugates of f_Δ or f_Δ^{-1}.*

It is straightforward to deduce Theorem 6.7.7 from this result. The crucial claim in our proof uses Proposition 6.8.6.

Proof of Theorem 6.7.7 from Theorem 6.8.7. Let N be a normal subgroup of H_δ contained in K_δ^1. If every member of N has finite support, then by the Schreier–Ulam theorem, N is either the identity, or the group of alternating permutations of $X_\delta^{[0]}$ of finite support, or the group of all permutations of $X_\delta^{[0]}$ of finite support, and we are done. Assume instead that N has an element of infinite support. We show that $L_\Delta^0 \leq N \leq L_\Delta$, where $\Delta = \bigcup_{h \in N} \operatorname{char}(h)$. Uniqueness is then immediate from the fact that $f_{\Delta'} \in L_{\Delta'}^0$ and $\operatorname{char}(f_{\Delta'}) = \Delta'$ for every lower set $\Delta' \subseteq \Gamma_n$.

Clearly $N \leq L_\Delta$. To see that $L_\Delta^0 \leq N$, let $g \in L_\Delta^0$. It is sufficient to prove that g is the product of finitely many conjugates of members of N. If g has finite support, then this is immediate from the Schreier–Ulam theorem. Otherwise let $\Delta' = \operatorname{char}(g)$, so Δ' is a lower set with $\{()\} \subsetneq \Delta' \subseteq \Delta$. We claim that there is a product h of finitely many conjugates of members of N with $\operatorname{char}(h) = \Delta$. Given the claim, $f_{\Delta'}$ is the product of finitely many conjugates of h or h^{-1} by part 1 of Theorem 6.8.7, and g is the product of finitely many conjugates of $f_{\Delta'}$ or $f_{\Delta'}^{-1}$ by part 2 of Theorem 6.8.7, which completes the proof.

It remains to prove the claim. Since Γ_n is finite and $()$ is a member of every non-empty lower set, there exist $h_1, h_2, \ldots, h_r \in N$ such that $\bigcup_{i=1}^{r} \mathrm{char}\,(h_i) = \bigcup_{h \in N} \mathrm{char}\,(h) = \Delta$ and $\mathrm{char}\,(h_i) \supsetneq \{()\}$ for all $i \in \{1, 2, \ldots, r\}$. For each $i \in \{1, 2, \ldots, r\}$, let $\Delta_i = \mathrm{char}\,(h_i)$. Then by part 1 of Theorem 6.8.7, f_{Δ_i} is the product of finitely many conjugates of h_i or h_i^{-1} for all $i \in \{1, 2, \ldots, r\}$. Take $h = f_{\Delta_1} \circ f_{\Delta_2} \circ \cdots \circ f_{\Delta_r}$. Then $\mathrm{char}\,(h) = \Delta$ by Proposition 6.8.6, as required. □

6.9 Proof of the classification

In this section we prove Theorem 6.8.7, thereby completing the proof of our main result, Theorem 6.7.7.

A crucial ingredient in our proof is the following result of Bertram [Ber73]. It is essentially the corresponding result for S_∞, the group of permutations of $[1, \omega)$.

Theorem 6.9.1 (Bertram). *Let $g \in S_\infty$ have infinite support and let $h \in S_\infty$. Then h is the product of 4 conjugates of g.*

The number 4 is best possible here in general, though this result may be further refined using case distinctions [Dro83, Dro85, Dro87, Mor89]. However, we will not be unduly concerned with careful numerical bounds, and so Bertram's formulation is the most convenient for our purposes.

As in the previous section, we make frequent use of infinite products of disjoint permutations. In each case it will again be easy to check these products lie in H_δ, either using Lemma 6.1.1, or using Proposition 6.3.2 for functions lying in K_δ^1.

We begin by proving the first part of Theorem 6.8.7, which states that we may obtain a canonical function as a product of finitely many conjugates of a suitable given function and its inverse. Our proof has two steps. In the first step, we obtain a function of the form f_Z.

Proposition 6.9.2. *Let $h \in K_\delta^1$ and let $Z \subseteq \mathrm{cl}\,(\mathrm{supp}\,(h)) \cap X'_\delta$. Then f_Z is the product of 8 conjugates of h or h^{-1}.*

The first step in our proof of the first part of Theorem 6.8.7 essentially amounts to the case $Z = \mathrm{cl}\,(\mathrm{supp}\,(h)) \cap X'_\delta$. However, our more general statement makes this result a useful tool at several other points in the proof of Theorem 6.8.7.

Proof. The idea of the proof is straightforward, though the details are somewhat technical. Let $\widetilde{Z} = \mathrm{cl}\,(\mathrm{supp}\,(h)) \cap X'_\delta$. We aim to construct a product $r \in K_\delta^1$ of 2 conjugates of h or h^{-1} such that r is an infinite product $\prod_{z \in \widetilde{Z}} r_z$ of disjoint permutations with $\mathrm{supp}\,(r_z) \subseteq \mathrm{supp}\left(f_{\{z\}}\right)$ for all $z \in \widetilde{Z}$. We may then simultaneously apply Bertram's theorem to r_z for each $z \in \widetilde{Z}$ and obtain f_Z as the product of 4 conjugates of r. In order to obtain r, we first construct $g \in K_\delta^1$ so that the commutator $p = g^{-1} \circ h \circ g \circ h^{-1}$ may be written as an infinite product

$\prod_{z\in\widetilde{Z}} p_z$ of disjoint permutations with $\mathrm{cl}\left(\mathrm{supp}\left(p_z\right)\right) \cap X'_\delta = \{z\}$ for all $z \in \widetilde{Z}$ (note that in particular, p has zero flow). We then construct $q \in K^1_\delta$ and obtain r as the conjugate $q^{-1} \circ p \circ q$.

First we construct $g \in K^1_\delta$ as the product $\prod_{z\in\widetilde{Z}} g_z$ of disjoint permutations. For each $k \in \{1, 2, \ldots, n\}$, let us simultaneously define g_z for each $z \in \widetilde{Z} \cap X^{[k]}_\delta$ by backwards induction on k. Fix $k \in \{1, 2, \ldots, n\}$ and assume that we have defined g_z for each $z \in \widetilde{Z} \cap X^{(k+1)}_\delta$ in such a way that $\mathrm{cl}\left(\mathrm{supp}\left(\prod_{z\in\widetilde{Z}\cap X^{(k+1)}_\delta} g_z\right)\right) \cap X'_\delta \subseteq X^{(k+1)}_\delta$. Now fix $z \in \widetilde{Z} \cap X^{[k]}_\delta$ and write $z = \eta + \omega^k$ with η a multiple of ω^k. Then there is a strictly increasing sequence $(b_i)_{i\in\omega}$ with $b_i \in \mathrm{supp}\,(h)$ for all $i \in \omega$ and $z = \sup\left(\{b_i : i \in \omega\}\right)$. By passing to a subsequence if necessary, we may assume $\left(h\left(b_i\right)\right)_{i\in\omega}$ is monotonic and therefore strictly increasing. Then $z = \sup\left(\{h\left(b_i\right) : i \in \omega\}\right)$ since $h\left(z\right) = z$ and h is continuous at z, so by passing to another subsequence if necessary we may assume both b_{i+1} and $h\left(b_{i+1}\right)$ are larger than $\max\left(\{b_i, h\left(b_i\right)\}\right)$ for all $i \in \omega$. Finally, by the inductive hypothesis, $\mathrm{supp}\left(\prod_{z\in\widetilde{Z}\cap X^{(k+1)}_\delta} g_z\right) \cap [\eta + 1, z)$ is finite, so by passing to yet another subsequence we may assume both b_i and $h\left(b_i\right)$ are larger than every member of this set for all $i \in \omega$. Then take g_z to be the product $\prod_{i\in\omega} \left(b_{2i}\, b_{2i+1}\right)$ of disjoint cycles. Observe that $g_z^{-1} \circ h \circ g_z \circ h^{-1} = \prod_{i\in\omega} \left(b_{2i}\, b_{2i+1}\right) \left(h\left(b_{2i}\right)\, h\left(b_{2i+1}\right)\right)$. Note also that $\mathrm{cl}\left(\mathrm{supp}\left(g_z\right)\right) \cap X'_\delta = \{z\} \in X^{[k]}_\delta$ as required for the inductive hypothesis. This completes our construction of g.

Let $p = g^{-1} \circ h^{-1} \circ g \circ h$ and write $p_z = g_z^{-1} \circ h^{-1} \circ g_z \circ h$ for each $z \in \widetilde{Z}$, so that $p = \prod_{z\in\widetilde{Z}} p_z$. Next we construct $q \in K^1_\delta$ as the product $\prod_{z\in\widetilde{Z}} q_z$ of disjoint permutations. Fix $z \in \widetilde{Z}$, write $z = \eta + \omega^k$ with η a multiple of ω^k and write $g_z = \prod_{i\in\omega} \left(b_{2i}\, b_{2i+1}\right)$ as above. For each positive integer i, let $a_i = \eta + \omega^{k-1} \cdot i + 1$ (so that $f_{\{z\}}$ is the cycle $\left(\ldots\, a_6\, a_4\, a_2\, a_1\, a_3\, a_5\, \ldots\right)$). Pick a positive integer j such that $a_j \geq b_0$, which ensures that $\{a_j, a_{j+1}, \ldots\}$ is disjoint from $\mathrm{supp}\left(p_z\right)$ for each $z \in \widetilde{Z} \cap X^{(k+1)}_\delta$. Then take q_z to be the product $\prod_{i\in\omega} \left(a_{j+4i}\, b_{2i}\right) \left(a_{j+4i+1}\, b_{2i+1}\right) \left(a_{j+4i+2}\, h\left(b_{2i}\right)\right) \left(a_{j+4i+3}\, h\left(b_{2i+1}\right)\right)$ of disjoint permutations. Observe that $q_z^{-1} \circ p_z \circ q_z = \prod_{i\in\omega} \left(a_{j+4i}\, a_{j+4i+1}\right) \left(a_{j+4i+2}, a_{j+4i+3}\right)$. This completes our construction of q.

Finally let $r = q^{-1} \circ p \circ q$ and write $r_z = q_z^{-1} \circ p_z \circ q_z$ for each $z \in \widetilde{Z}$, so that $r = \prod_{z\in\widetilde{Z}} r_z$. By Bertram's theorem, $f_{\{z\}}$ is the product of 4 conjugates of r_z for each $z \in \widetilde{Z}$, and the identity is the product of 4 conjugates of r_z for each $z \in Z \setminus \widetilde{Z}$. By taking the infinite product of these conjugands in much the same way as above, it follows that $f_Z = \prod_{z\in Z} f_{\{z\}}$ is the product of 4 conjugates of r and hence 8 conjugates of h or h^{-1}. $\square$

The second step of our proof of the first part Theorem 6.8.7 is to obtain a canonical function from a function of the form f_Z.

Proposition 6.9.3. *Let Z be a closed subset of X'_δ and let $\Delta = \mathrm{char}\left(f_Z\right)$. Then f_Δ is the product of finitely many conjugates of f_Z or f_Z^{-1}.*

The basic idea of our proof is that by using Proposition 6.9.2, it is in some sense sufficient to build up for each $s \in \Delta$ a copy of $\mathcal{Z}(\langle s \rangle)$ inside a conjugate of f_Z. We do this inductively by mirroring the definition of $\mathcal{Z}(\langle s \rangle)$, which it may be helpful to recall at this stage. The fact that we used even numbers in that definition makes our construction cleaner. In order to build up the copy of $\mathcal{Z}(\langle s \rangle)$, we conjugate by functions obtained from block maps. This is to be contrasted with the proof of Proposition 6.9.2, where we only conjugated by functions from K_δ^1.

Proof. The result is trivial if $\Delta = \emptyset$, so assume $\Delta \neq \emptyset$ and note that $\Delta \neq \{()\}$. Let S be the antichain of Γ_n with $\langle S \rangle = \Delta$. We claim that for all $s \in S$ there is a conjugate g of f_Z such that $\mathcal{Z}(\langle s \rangle) \subseteq \mathrm{cl}(\mathrm{supp}(g)) \cap X'_\delta$. Given the claim, it follows from Proposition 6.9.2 that $f_{\mathcal{Z}(\langle s \rangle)}$ is the product of 8 conjugates of f_Z or f_Z^{-1} for all $s \in S$. Now let $S = \{s_1, s_2, \dots, s_r\}$ and let $h = f_{\mathcal{Z}(\langle s_1 \rangle)} \circ f_{\mathcal{Z}(\langle s_2 \rangle)} \circ \dots \circ f_{\mathcal{Z}(\langle s_r \rangle)}$. Then $\mathrm{supp}(h) = \bigcup_{i=1}^r \mathrm{supp}\left(f_{\mathcal{Z}(\langle s_i \rangle)}\right)$ by part 2 of Lemma 6.8.2, so $\mathrm{cl}(\mathrm{supp}(h)) \cap X'_\delta = \bigcup_{i=1}^r \mathcal{Z}(\langle s_i \rangle) = \mathcal{Z}(\Delta)$ by Lemma 6.8.1. Hence $f_\Delta = f_{\mathcal{Z}(\Delta)}$ is the product of 8 conjugates of h or h^{-1} by Proposition 6.9.2, and therefore the product of $64r$ conjugates of f_Z or f_Z^{-1}, as required.

It remains to prove the claim. To do this, fix $s \in S$ and write $s = (k_1, k_2, \dots, k_r)$ with $r \geq 1$. We show that for all $j \in \{1, 2, \dots, r\}$ there is a conjugate g_j of f_Z such that $\mathrm{supp}(g_j)[(k_1, k_2, \dots, k_j)] = \mathrm{supp}(f_Z)[(k_1, k_2, \dots, k_j)]$, and for all $z \in \mathrm{supp}(g_j)[(k_1, k_2, \dots, k_j)]$, writing $z = \eta + \omega^{k_j}$ with η a multiple of ω^{k_j},

$$\{\eta + x : x \in \mathcal{Z}(\langle (k_1, k_2, \dots, k_j) \rangle)\} \subseteq \mathrm{cl}(\mathrm{supp}(g_j)) \cap X'_\delta.$$

Once we have done this, we may choose $z \in \mathrm{supp}(g_r)[(k_1, k_2, \dots, k_r)]$ and take g to be the conjugate of g_r by $b_\delta^{k_r}\left(\left(z\ \omega^{k_r}\right)\right)$, where $b_\delta^{k_r}$ is the block map and $\left(z\ \omega^{k_r}\right) \in H_\delta^{k_r}$ is a transposition. Then g is as required since $\mathcal{Z}(\langle s \rangle) \subseteq \left[1, \omega^{k_r}\right]$.

We prove this by induction on j. For the case $j = 1$ we may simply take g_1 to be f_Z since $\mathcal{Z}(\langle (k_1) \rangle) = \left\{\omega^{k_1}\right\}$. So suppose $j > 1$. Fix $z \in \mathrm{supp}(g_{j-1})[(k_1, k_2, \dots, k_j)]$ and write $z = \eta + \omega^{k_j}$ with η a multiple of ω^{k_j}. Then there is a strictly increasing sequence $(a_i)_{i \in \omega}$ such that $a_0 > \eta$, $z = \sup(\{a_i : i \in \omega\})$ and $a_i \in \mathrm{supp}(g_{j-1})[(k_1, k_2, \dots, k_{j-1})]$ for all $i \in \omega$. For each $i \in \omega$ write $a_i = \eta + \omega^{k_j - 1} \cdot c_i + \zeta_i + \omega^{k_{j-1}}$ with $c_i \in \omega$ and $\zeta_i < \omega^{k_j - 1}$ a multiple of $\omega^{k_{j-1}}$ (it is possible that $k_{j-1} = k_j - 1$, in which case $\zeta_i = 0$). By passing to a subsequence if necessary, we may assume $c_{i+1} \geq c_i + 2$ for all $i \in \omega$. Then there is a permutation σ of ω such that $\sigma(c_i) = 2i$ for all $i \in \omega$, which induces a function $p_z \in H_\delta^{k_j - 1}$ such that $p_z\left(\eta + \omega^{k_j - 1} \cdot (c_i + 1)\right) = \eta + \omega^{k_j - 1} \cdot (2i + 1)$ for all $i \in \omega$. Let $q_z = \prod_{i \in \omega}\left(a_i\ \ \eta + \omega^{k_j - 1} \cdot c_i + \omega^{k_{j-1}}\right)$, a product of disjoint transpositions lying in $H_\delta^{k_{j-1}}$. Now let $t_z = b_\delta^{k_j - 1}(p_z) \circ b_\delta^{k_{j-1}}(q_z)$. Then $t_z(z) = z$ and $t_z(a_i) = \eta + \omega^{k_j - 1} \cdot 2i + \omega^{k_{j-1}}$ for all $i \in \omega$. It follows by the inductive hypothesis that $\{\eta + x : x \in \mathcal{Z}(\langle (k_1, k_2, \dots, k_j) \rangle)\} \subseteq \mathrm{cl}(\mathrm{supp}(t_z \circ g_{j-1} \circ t_z^{-1})) \cap X'_\delta$. Thus we may let $t = b_\delta^{k_j - 1}\left(\prod_{z \in Z} p_z\right) \circ b_\delta^{k_{j-1}}\left(\prod_{z \in Z} q_z\right)$ and take g_j to be $t \circ g_{j-1} \circ t^{-1}$. This completes the proof. $\square$

We will not keep any further track of how many conjugates we use, but in this case we used $64\,|S|$ conjugates, where S is the antichain of Γ_n with $\langle S\rangle = \Delta$. By Sperner's theorem, this is at most $64\binom{n}{\lfloor n/2\rfloor}$. It may be interesting to determine whether or not this upper bound may be reduced to a constant (independent of n).

Let us now put Propositions 6.9.2 and 6.9.3 together to complete the proof of the first part of Theorem 6.8.7.

Proof of part 1 of Theorem 6.8.7. Recall that $h \in K_\delta^1$ has infinite support and $\Delta = \operatorname{char}(h)$. Let $\{()\} \subsetneq \Delta' \subseteq \Delta$ be a lower set. We are required to prove that $f_{\Delta'}$ is the product of finitely many conjugates of h or h^{-1}.

First we show that f_Δ is the product of finitely many conjugates of h or h^{-1}. To see this, let $Z = \operatorname{cl}(\operatorname{supp}(h)) \cap X'_\delta$. Then $\operatorname{char}(f_Z) = \Delta$. Hence f_Z is the product of 8 conjugates of h or h^{-1} by Proposition 6.9.2, and f_Δ is the product of finitely many conjugates of f_Z or f_Z^{-1} by Proposition 6.9.3.

To complete the proof, simply observe that $\mathcal{Z}(\Delta') \subseteq \mathcal{Z}(\Delta) = \operatorname{cl}\left(\operatorname{supp}\left(f_{\mathcal{Z}(\Delta)}\right)\right) \cap X'_\delta$, and so by Proposition 6.9.2, $f_{\mathcal{Z}(\Delta')} = f_{\Delta'}$ is the product of 8 conjugates of $f_{\mathcal{Z}(\Delta)} = f_\Delta$ or f_Δ^{-1}. $\square$

We now prove the second part of Theorem 6.8.7, which states that we may obtain an arbitrary function in K_δ^1 of zero flow as a product of finitely many conjugates of an appropriate canonical function and its inverse. Like the first part, our proof has two steps. In the first step, we obtain an appropriate function of the form f_Z. This may be viewed as the converse of Proposition 6.9.3.

Proposition 6.9.4. *Let Z be a closed subset of X'_δ and let $\Delta = \operatorname{char}(f_Z)$. Then f_Z is the product of finitely many conjugates of f_Δ or f_Δ^{-1}.*

The proof of this result is the most technically challenging part of this chapter, essentially because of the lack of control we have over Z. In order to address this, we use the following notion.

Definition. Let $Y \subseteq X_\delta^{[0]}$ and $\Delta = \operatorname{tp}(Y)$. We say that Y is *primitive* to mean that $\Delta = \langle s\rangle$ for some $s \in \Gamma_n$, and $Y[(k_1, k_2, \ldots, k_j)] = Y[(k_j)]$ for all $j \in \{1, 2, \ldots, r\}$, where $s = (k_1, k_2, \ldots, k_r)$.

For example, let $n = 3$, $Z = \mathcal{Z}(\langle(2)\rangle) \cup \{\omega^2 + z : z \in \mathcal{Z}(\langle(1,2)\rangle)\}$ and $Y = \operatorname{supp}(f_Z)$. Then $\operatorname{tp}(Y) = \langle(1,2)\rangle$ but Y is not primitive since $Y[(2)] \setminus Y[(1,2)] = \mathcal{Z}(\langle(2)\rangle) \neq \emptyset$.

Let $Y \subseteq X_\delta^{[0]}$ and $\Delta = \operatorname{tp}(Y)$, and suppose $\Delta = \langle s\rangle$ with $s = (k_1, k_2, \ldots, k_r)$. By Lemma 6.6.1, if $j \in \{1, 2, \ldots, r\}$ and t is any subsequence of $(k_1, k_2, \ldots, k_j)$ such that the last term of t is k_j, then $Y[(k_1, k_2, \ldots, k_j)] \subseteq Y[t] \subseteq Y[(k_j)]$, so if Y is primitive then $Y[t] = Y[(k_1, k_2, \ldots, k_j)]$.

It is easy to check that if $s \in \Gamma_n \setminus \{()\}$ then $\operatorname{supp}\left(f_{\mathcal{Z}(\langle s \rangle)}\right)$ is primitive. Thus by definition, sets of the form $\operatorname{supp}\left(f_{\mathcal{Z}(\Delta)}\right)$ with $\{()\} \neq \Delta \subseteq \Gamma_n$ may be written as a union of primitive sets. The following result shows that this in fact holds for any subset of $X_\delta^{[0]}$, and moreover that this union may be taken to be disjoint, as we indicate here with the word "partition". This powerful result will allow us to prove Proposition 6.9.3, though strictly speaking the disjointness condition will not be required when we apply this result.

Lemma 6.9.5. *Let $Y \subseteq X_\delta^{[0]}$ and let $\Delta = \operatorname{tp}(Y)$. Then there is a partition $Y = \bigcup_{s \in \Delta} Y_s$ such that Y_s is primitive with $\operatorname{tp}(Y_s) = \langle s \rangle$ for all $s \in \Delta$.*

Proof. If $|\Delta| \leq 1$ then the result is trivial, so assume $|\Delta| > 1$, let S be the antichain of Γ_n with $\langle S \rangle = \Delta$ and let $s \in S$. We show that there is a partition $Y = A \cup B$ such that A is primitive with $\operatorname{tp}(A) = \langle s \rangle$ and $\operatorname{tp}(B) = \Delta \setminus \{s\}$. The result then follows by induction on $|\Delta|$.

Let $s = (k_1, k_2, \ldots, k_r)$. For each $j \in \{r, r-1, \ldots, 0\}$ we construct a partition $Y[(k_1, k_2, \ldots, k_j)] = A_j \cup B_j$. First of all let $A_r = Y[(k_1, k_2, \ldots, k_r)]$ and $B_r = \emptyset$. For each $j \in \{r, r-1, \ldots, 0\}$, let $D_j = \bigcup_t Y[t]$, with the union ranging over all $t \in \Delta$ such that the last term of t is k_j and $(k_1, k_2, \ldots, k_j)$ is a proper subsequence of t. Note that $D_r = \emptyset$ since $s \in S$. Now for each $j \in \{r-1, r-2, \ldots, 0\}$, we construct A_j and B_j inductively in such a way that for all $k \in \{k_j + 1, k_j + 2, \ldots, n\}$ (writing $k_0 = 0$):

1. $A_j[(k)] = \begin{cases} A_{j'}, & \text{if } k = k_{j'} \text{ for some } j' \in \{j+1, j+2, \ldots, r\} \\ \emptyset, & \text{otherwise;} \end{cases}$

2. $B_j[(k)] = \begin{cases} B_{j+1}, & \text{if } k = k_{j+1} \\ Y[(k_1, \ldots, k_j, k)], & \text{otherwise;} \end{cases}$

3. $A_j \cap D_j = \emptyset$; and

4. $B_{r-1} \neq \emptyset$.

This is sufficient to complete the proof since we may then take $A = A_0$ and $B = B_0$.

To do this, suppose A_{j+1} and B_{j+1} have been defined for some $j \in \{r-1, r-2, \ldots, 0\}$. Fix $x \in A_{j+1}$ and write $x = \eta + \omega^{k_{j+1}}$ with η a multiple of $\omega^{k_{j+1}}$. Then there is a strictly increasing sequence $(a_i)_{i \in \omega}$ such that $a_0 > \eta$, $x = \sup(\{a_i : i \in \omega\})$ and $a_i \in Y[(k_1, k_2, \ldots, k_j)]$ for all $i \in \omega$. Now by the inductive hypothesis (or the fact that $D_r = \emptyset$), $x \notin D_{j+1}$. Therefore we may choose $(a_i)_{i \in \omega}$ in such a way that $a_i \notin D_j$ for all $i \in \omega$, and also in such a way that $Y[(k_1, k_2, \ldots, k_j)] \cap [a_0, x) = \{a_i : i \in \omega\}$, or else $x \in Y[(k_1, \ldots, k_j, k, k_{j+1})]$ for some $k \in \{k_j + 1, k_j + 2, \ldots, k_{j+1} - 1\}$. Let $c_x = a_0$ and $C_x = \{a_i : i \in \omega \setminus \{0\}\}$ (it is important that a_0 is excluded here). Finally take $A_j = \bigcup_{x \in A_{j+1}} C_x$ and $B_j = Y[(k_1, k_2, \ldots, k_j)] \setminus A_j$. We now show that A_j and B_j are as claimed.

1. Firstly let $j' \in \{j+1, j+2, \ldots, r\}$. Let $x \in X_\delta^{[k_{j'}]}$ and write $x = \eta + \omega^{k_{j'}}$ with η a multiple of $\omega^{k_{j'}}$. If $x \notin A_{j'}$, then it is clear inductively that $A_{j'-1}, A_{j'-2}, \ldots, A_j$ must each be disjoint from $[\eta+1, x)$, and so $x \notin A_j\,[(k_{j'})]$. Conversely, suppose $x \in A_{j'}$. If $j' > j+1$ then $A_{j'} \subseteq A_{j+1}\,[(k_{j'})]$ by the inductive hypothesis and $A_j\,[(k_{j+1}, k_{j'})] \subseteq A_j\,[(k_{j'})]$ by part 2 of Lemma 6.6.1, so it is sufficient to prove that $A_{j+1} \subseteq A_j\,[(k_{j+1})]$. In other words, it is enough to consider the case $j' = j+1$. But then $x \in \mathrm{cl}\,(C_x)$, so $x \in A_j\,[(k_{j+1})]$. Hence $A_j\,[(k_{j'})] = A_{j'}$.

   Suppose instead $k \in \{k_j+1, \ldots, n\} \setminus \{k_{j+1}, \ldots, k_r\}$. Let $x \in X_\delta^{[k]}$ and write $x = \eta + \omega^k$ with η a multiple of ω^k. If $k < k_{j+1}$ then $[\eta+1, x)$ intersects (c_y, y) for at most one $y \in A_{j+1}$, so $A_j \cap [\eta+1, x)$ is finite and $x \notin A_j\,[(k)]$. So assume $k > k_{j+1}$ and suppose for contradiction $x \in A_j\,[(k)]$. Then there is a strictly increasing sequence $(y_i)_{i\in\omega}$ with $y_i \in A_{j+1}$ for all $i \in \omega$ and for each $i \in \omega$ a member $c_i \in C_{y_i}$ such that $x = \sup(\{c_i : i \in \omega\})$. But then $x = \sup(\{y_i : i \in \omega\})$, so $x \in A_{j+1}\,[(k)]$, contrary to the inductive hypothesis.

2. Firstly we show that $B_j\,[(k_{j+1})] = B_{j+1}$. Let $x \in Y\,[(k_1, k_2, \ldots, k_{j+1})]$ and write $x = \eta + \omega^{k_{j+1}}$ with η a multiple of $\omega^{k_{j+1}}$. If $x \in B_{j+1}$ then A_j is disjoint from $[\eta+1, x)$, so $x \in B_j\,[(k_{j+1})]$. Conversely, if $x \in A_{j+1}$ then B_j is disjoint from (c_x, x), so $x \notin B_j\,[(k_{j+1})]$.

   Suppose now $k \in \{k_j+1, \ldots, n\} \setminus \{k_{j+1}\}$ and let $x \in Y\,[(k_1, \ldots, k_j, k)]$. Then there is a strictly increasing sequence $(c_i)_{i\in\omega}$ with $c_i \in Y\,[(k_1, k_2, \ldots, k_j)]$ for all $i \in \omega$ and $x = \sup(\{c_i : i \in \omega\})$. If $c_i \in B_j$ for infinitely many $i \in \omega$ then we are done, so we may assume $c_i \in A_j$ for all $i \in \omega$. But since $k \neq k_{j+1}$ we may also assume there is a strictly increasing sequence $(y_i)_{i\in\omega}$ with $c_i \in C_{y_i}$ for all $i \in \omega$. But then $c_{y_i} \in B_j$ for all $i \in \omega$ and $x = \sup(\{c_{y_i} : i \in \omega\})$, and we are done.

3. This is clear by construction.

4. This is immediate from the fact that $c_x \in B_{r-1}$ for all $x \in A_r$. □

We are now ready to prove Proposition 6.9.4, thereby completing the first step in our proof of the second part of Theorem 6.8.7. We do this by using Lemma 6.9.5 to reduce to the case in which $\mathrm{supp}\,(f_Z)$ is primitive. This makes Z simple enough that we are almost able to build up a copy of $\mathcal{Z}\,(\Delta)$ inside it, in a similar sense to the proof of Proposition 6.9.3. However, for technical reasons we need to take the product of a pair of conjugates at each stage rather than using a single conjugate as in the proof of Proposition 6.9.3.

Proof of Proposition 6.9.4. The result is trivial if $Z = \emptyset$, so assume $Z \neq \emptyset$ (and hence $\{()\} \subsetneq \Delta$).

We claim that it is sufficient to prove the result for the case in which $\operatorname{supp}(f_Z)$ is primitive. To see this, suppose the result holds for this case. Let $Y = \operatorname{supp}(f_Z)$ and let $(Y_s)_{s\in\Delta}$ be as in Lemma 6.9.5. For each $s \in \Delta \setminus \{()\}$, let $Z_s = \operatorname{cl}(Y_s) \cap X'_\delta$, so $\operatorname{supp}(f_{Z_s})$ is primitive, $\operatorname{char}(f_{Z_s}) = \langle s\rangle$ and $Z = \bigcup_{s\in\Delta\setminus\{()\}} Z_s$. Then for each $s \in \Delta \setminus \{()\}$, by assumption f_{Z_s} is the product of finitely many conjugates of $f_{\langle s\rangle}$ or $f_{\langle s\rangle}^{-1}$, and furthermore $f_{\langle s\rangle}$ is the product of 8 conjugates of f_Δ or f_Δ^{-1} by Proposition 6.9.2, since $\mathcal{Z}(\langle s\rangle) \subseteq \mathcal{Z}(\Delta) = \operatorname{cl}(\operatorname{supp}(f_\Delta)) \cap X'_\delta$. Hence writing $\Delta \setminus \{()\} = \{s_1, s_2, \ldots, s_r\}$, $f_{Z_{s_1}} \circ f_{Z_{s_2}} \circ \cdots \circ f_{Z_{s_r}}$ is the product of finitely many conjugates of f_Δ or f_Δ^{-1}. But $\operatorname{cl}\left(\operatorname{supp}\left(f_{Z_{s_1}} \circ f_{Z_{s_2}} \circ \cdots \circ f_{Z_{s_r}}\right)\right) \cap X'_\delta = \bigcup_{i=1}^r \operatorname{cl}\left(\operatorname{supp}\left(f_{Z_{s_i}}\right)\right) \cap X'_\delta = \bigcup_{s\in\Delta\setminus\{()\}} Z_s = Z$ by Lemma 6.8.2, so using Proposition 6.9.2 again, f_Z is the product of finitely many conjugates of f_Δ or f_Δ^{-1}, as required.

So assume $\operatorname{supp}(f_Z)$ is primitive and write $\Delta = \langle(k_1, k_2, \ldots, k_r)\rangle$. Let $Y = \operatorname{supp}(f_Z)$. We claim that it is sufficient to construct for each $j \in \{r, r-1, \ldots, 1\}$ a product f_j of finitely many conjugates of f_Δ in such a way that

$$\operatorname{supp}(f_j)[(k_1, k_2, \ldots, k_j)] = Y[(k_1, k_2, \ldots, k_j)]$$

and $\operatorname{supp}(f_j)$ is primitive. To see this, suppose we have constructed such a collection. Then for all $j \in \{1, 2, \ldots, r\}$, $\operatorname{supp}(f_1)[(k_j)] = \operatorname{supp}(f_1)[(k_1, k_2, \ldots, k_j)] = Y[(k_1, k_2, \ldots, k_j)] = Y[(k_j)]$ since $\operatorname{supp}(f_1)$ and Y are primitive. Furthermore, $\operatorname{supp}(f_1)[(k)] = \emptyset$ for all $k \in \{1, 2, \ldots, n\} \setminus \{k_1, k_2, \ldots, k_r\}$ since $f_1 \in L_\Delta$. Thus $\operatorname{supp}(f_1)[(k)] = Y[(k)]$ for all $k \in \{1, 2, \ldots, n\}$ and so $\operatorname{cl}(\operatorname{supp}(f_1)) \cap X'_\delta = \operatorname{cl}(Y) \cap X'_\delta = Z$. Hence f_Z is the product of 8 conjugates of f_1 and f_1^{-1} by Proposition 6.9.2, and we are done.

It remains to construct f_j for each $j \in \{r, r-1, \ldots, 1\}$, which we do inductively. First of all, since $Y[(k_1, k_2, \ldots, k_r)]$ is finite, write $Y[(k_1, k_2, \ldots, k_r)] = \{x_1, x_2, \ldots, x_l\}$ and take $f_r = f_\Delta^{b_\delta^{k_r}((\omega^{k_r} x_1))} \circ f_\Delta^{b_\delta^{k_r}((\omega^{k_r} x_2))} \circ \cdots \circ f_\Delta^{b_\delta^{k_r}((\omega^{k_r} x_l))}$, which is as required. Now suppose f_{j+1} has been constructed for some $j \in \{r-1, r-2, \ldots, 0\}$. For each $z \in Y[(k_1, k_2, \ldots, k_{j+1})]$, write $z = \eta + \omega^{k_{j+1}}$ with η a multiple of $\omega^{k_{j+1}}$, and let

$$F(z) = \begin{cases} [\sup(Y[(k_1, k_2, \ldots, k_{j+1})]), \delta] \cup [1, \eta] & \text{if } z = \min(Y[(k_1, k_2, \ldots, k_{j+1})]) \\ [\sup(Y[(k_1, k_2, \ldots, k_{j+1})] \cap [1, z)), \eta], & \text{otherwise.} \end{cases}$$

Thus $F(z)$ is the "gap" between $\mathcal{B}(z)$ and the previous member of $Y[(k_1, k_2, \ldots, k_{j+1})]$, and $X_\delta = \bigcup_{z\in Y[(k_1,k_2,\ldots,k_{j+1})]} \mathcal{B}(z) \cup F(z)$. Now fix $z \in Y[(k_1, k_2, \ldots, k_{j+1})]$. Then $Y[(k_1, k_2, \ldots, k_j)] \cap F(z)$ is finite since Y is primitive, say $Y[(k_1, k_2, \ldots, k_j)] \cap F(z) = \{x_0, x_1, \ldots, x_{l-1}\}$, and $Y[(k_1, k_2, \ldots, k_j)] \cap \mathcal{B}(z)$ is an infinite set of order type ω, say $Y[(k_1, k_2, \ldots, k_j)] \cap \mathcal{B}(z) = \{x_l, x_{l+1}, \ldots\}$ with $x_l < x_{l+1} < \ldots$. Also, $\operatorname{supp}(f_{j+1})[(k_1, k_2, \ldots, k_j)] \cap \mathcal{B}(z) = \left\{\eta + \omega^{k_j} \cdot (2i+1) : i \in \omega\right\}$ because of how f_{j+1} was constructed from the canonical function f_Δ. Let $G(z) =$

$\{x_0, x_1, x_2, \ldots, x_{l-1}\} \cup \left(X_\delta^{(k_j)} \cap \mathcal{B}(z)\right)$, a clopen subset of $X_\delta^{(k_j)}$. We would like to be able to find an autohomeomorphism g_z of $G(z)$ such that $g_z(x_i) = \eta + \omega^{k_j} \cdot (2i+1)$ for all $i \in \omega$. However, this may not be possible since if $k_{j+1} = k_j + 1$, then $\{x_i : i \in \omega\}$ may be a cofinite subset of $X_\delta^{[k_j]} \cap G(z) = \{x_0, x_1, \ldots, x_{l-1}\} \cup \{\eta + \omega^{k_j} \cdot (i+1) : i \in \omega\}$. Instead, we construct two autohomeomorphisms g_z and h_z of $G(z)$ such that $g_z(x_{2i}) = \eta + \omega^{k_j} \cdot (2i+1)$ for all $i \in \omega$ and $h_z(x_{2i+1}) = \eta + \omega^{k_j} \cdot (2i+1)$ for all $i \in \omega$. To construct g_z, observe that if $k_{j+1} = k_j + 1$, then both $\{x_{2i} : i \in \omega\}$ and $\{\eta + \omega^{k_j} \cdot (2i+1) : i \in \omega\}$ are infinite-coinfinite subsets of $X_\delta^{[k_j]} \cap G(z) = \{x_0, x_1, \ldots, x_{l-1}\} \cup \{\eta + \omega^{k_j} \cdot (i+1) : i \in \omega\}$. Hence we can simply take $g_z(x_{2i}) = \eta + \omega^{k_j} \cdot (2i+1)$ for all $i \in \omega$ and extend g_z using the order-isomorphism $G(z) \setminus \{x_{2i} : i \in \omega\} \to G(z) \setminus \{\eta + \omega^{k_j} \cdot (2i+1) : i \in \omega\}$ (using the usual ordering but with $x_0, x_1, \ldots, x_{l-1}$ considered less than everything else if they are not already). h_z may be constructed similarly. By extending them to act as the identity, we may view g_z and h_z as members of $H_\delta^{k_j}$. Finally define $g = \prod_{z \in Y[(k_1, k_2, \ldots, k_{j+1})]} b_\delta^{k_j}(g_z)$ and $h = \prod_{z \in Y[(k_1, k_2, \ldots, k_{j+1})]} b_\delta^{k_j}(h_z)$ and take $f_j = f_{j+1}^g \circ f_{j+1}^h$. Then $\operatorname{supp}(f_j)[(k_1, k_2, \ldots, k_j)] = Y[(k_1, k_2, \ldots, k_j)]$ using the argument from the proof of part 2 of Lemma 6.8.2, and $\operatorname{supp}(f_j)$ is easily seen to be primitive, as required. $\square$

The second and final step in our proof of the second part Theorem 6.8.7 is to obtain an arbitrary function from K_δ^1 of zero flow from an appropriate function of the form f_Z.

Proposition 6.9.6. *Let $f \in \operatorname{Ker}(\text{flow})$ and let $Z = \operatorname{cl}(\operatorname{supp}(f)) \cap X'_\delta$. If $Z \neq \emptyset$ then f is the product of finitely many conjugates of f_Z.*

We prove this using an inductive argument, dealing with $Z \cap X_\delta^{[i]}$ during the ith stage for each $i \in \{1, 2, \ldots, n\}$. For simplicity we include details only for the first stage. The basic idea is to apply Proposition 6.3.2. Together with the fact that f has zero flow, this allows us to make a small modification to f and to decompose the result as a product of disjoint permutations. We may then apply Bertram's theorem to each of these permutations simultaneously, much as in the proof of Proposition 6.9.2.

Proof. First of all, let $(A_x)_{x \in X'_\delta \setminus X''_\delta}$ and $(B_x)_{x \in X'_\delta \setminus X''_\delta}$ be the cofinitary systems on X_δ given by Proposition 6.3.2, with respective complementary sets A_* and B_*. For each $x \in X'_\delta \setminus X''_\delta$, write $x = \eta + \omega$ with η a multiple of ω. If $x \notin Z$, then we may assume without loss of generality that f fixes A_x pointwise. Suppose instead $x \in Z$, and assume without loss of generality that $\eta + 1 \notin A_x \cup B_x$. Since $f(A_x) = B_x$ and $\operatorname{flow}_{A_x}(f) = 0$ by Lemma 6.7.5 we have $|A_x \setminus B_x| - |B_x \setminus A_x| = 0$ and hence the finite sets $(\eta + 1, x) \setminus A_x$ and $(\eta + 1, x) \setminus B_x$ have the same size. We may therefore choose a bijection $(\eta + 1, x) \setminus A_x \to (\eta + 1, x) \setminus B_x$ and let t_x be a permutation of $(\eta + 1, x) \setminus (A_x \cap B_x)$ extending this bijection. Now let $t = \prod_{x \in Z \setminus X''_\delta} t_x$, a product of disjoint permutations. Then $f \circ t^{-1}$ stabilises A_x setwise for all $x \in Z \setminus X''_\delta$, so we may write $f \circ t^{-1} = \prod_{x \in Z \setminus X''_\delta} g_x \circ g_*$, where g_x is a permutation of A_x for each $x \in Z \setminus X''_\delta$ and g_* is a permutation of A_*.

Take g_z to be the identity function for all $z \in Z \cap X''_\delta$ so that $f \circ t^{-1} = \prod_{z \in Z} g_z \circ g_*$. Then by Bertram's theorem, for each $z \in Z$ we may write g_z as a product of 4 conjugates of $f_{\{z\}}$, and hence we may write $\prod_{z \in Z} g_z$ as a product of 4 conjugates of f_Z. Therefore it is sufficient to write $g_* \circ t$ as a product of finitely many conjugates of f_Z.

Now $\text{flow}\,(t) = \text{flow}\left(\prod_{z \in Z} g_z\right) = \mathbf{0}$ and hence $\text{flow}\,(g_*) = \text{flow}\,(g_* \circ t) = \mathbf{0}$. Morevover, $\text{cl}\,(\text{supp}\,(g_* \circ t)) \cap X'_\delta \subseteq Z \cap X''_\delta$ by construction. Hence there exists $f_1 \in \text{Ker}\,(\text{flow})$ conjugate to $g_* \circ t$ such that $\text{cl}\,(\text{supp}\,(f_1)) \cap X'_\delta \subseteq Z \cap X''_\delta$ and, for all $x \in X^{[1]}_\delta$, writing $x = \eta + \omega$ with η a multiple of ω, $\text{supp}\,(f_1) \cap (\eta + 1, x) = \emptyset$. It is now sufficient to write f_1 as a product of finitely many conjugates of f_Z.

We may now repeat our entire argument so far with f replaced by f_1 to obtain some $f_2 \in \text{Ker}\,(\text{flow})$ with $\text{cl}\,(\text{supp}\,(f_2)) \cap X'_\delta \subseteq X^{(3)}_\delta$ and, for all $x \in X^{[2]}_\delta$, writing $x = \eta + \omega^2$ with η a multiple of ω^2, $\text{supp}\,(f_2) \cap (\eta + 1, x) = \emptyset$. It is now sufficient to write f_2 as a product of finitely many conjugates of f_Z.

Continuing in this fashion, we eventually obtain $f_n \in \text{Ker}\,(\text{flow})$ with $\text{cl}\,(\text{supp}\,(f_n)) \cap X'_\delta \subseteq X^{(n+1)}_\delta = \emptyset$, such that it is sufficient to write f_n as a product of finitely many conjugates of f_Z. But f_n has finite support and may therefore be written as a product of finitely many conjugates of f_Z, either by Bertram's theorem or by writing f_n as a product of transpositions and checking the result directly for a single transposition. □

Let us now put Propositions 6.9.4 and 6.9.6 together to complete the proof of the second part of Theorem 6.8.7. This completes the proof of Theorem 6.8.7 and hence of our main result, Theorem 6.7.7.

Proof of part 2 of Theorem 6.8.7. Recall that $h \in K^1_\delta$ has infinite support and $\Delta = \text{char}\,(h)$. Assume that $\text{flow}\,(h) = \mathbf{0}$. We are required to prove that h is the product of finitely many conjugates of f_Δ and f_Δ^{-1}. Let $Z = \text{cl}\,(\text{supp}\,(h)) \cap X'_\delta$. Then Z is a non-empty closed subset of X'_δ and $\Delta = \text{char}\,(f_Z)$. Hence f_Z is the product of finitely many conjugates of f_Δ or f_Δ^{-1} by Proposition 6.9.4, and h is the product of finitely many conjugates of f_Z by Proposition 6.9.6. □

6.10 The number of normal subgroups of $H_{\omega^n \cdot m}$

We have now completed the proof of our main result, which states that if N is a normal subgroup of H_δ contained in K^1_δ other than the group of alternating permutations of finite support, then $L^0_\Delta \leq N \leq L_\Delta$ for a unique lower set $\Delta \subseteq \Gamma_n$. In this section we study which normal subgroups $N \trianglelefteq H_\delta$ satisfying $L^0_\Delta \leq N \leq L_\Delta$

actually arise for each lower set $\Delta \subseteq \Gamma_n$. In particular, we show that if $n \geq 2$ then H_δ has $2^{2^{\aleph_0}}$ normal subgroups.

We begin by describing which $\mathbf{x} \in \mathbb{Z}^{X'_\delta}$ arise as flow (f) for some $f \in K^1_\delta$. Let us first introduce some coordinate notation.

Definition. Given $\mathbf{x} \in \mathbb{Z}^{X'_\delta}$ we write $\mathbf{x} = (\mathbf{x}(z))_{z \in X'_\delta}$.

The following result shows that there is just one simple restriction on which $\mathbf{x} \in \mathbb{Z}^{X'_\delta}$ arise as flow (f) for some $f \in K^1_\delta$.

Proposition 6.10.1. *The image of K^1_δ under* flow *is*

$$\left\{ \mathbf{x} \in \mathbb{Z}^{X'_\delta} : \sum_{z \in X^{[n]}_\delta} \mathbf{x}(z) = 0 \right\}.$$

Proof. First note that if $f \in K^1_\delta$ then $\sum_{z \in X^{[n]}_\delta} \mathrm{flow}_{\mathcal{B}(z)}(f) = \mathrm{flow}_{\bigcup_{z \in X^{[n]}_\delta} \mathcal{B}(z)}(f) = \mathrm{flow}_{X_\delta}(f) = 0$. Conversely, let $\mathbf{x} \in \mathbb{Z}^{X'_\delta}$ with $\sum_{z \in X^{[n]}_\delta} \mathbf{x}(z) = 0$. We construct $f \in K^1_\delta$ with $\mathrm{flow}(f) = \mathbf{x}$.

Our construction is analogous to our construction of f_Z, which is defined in terms of $f_{\{z\}}$ for $z \in X'_\delta$. First we define $f_{z_1 \to z_2} \in K^1_\delta$ for each $z_1, z_2 \in X'_\delta$ in such a way that if C is a clopen subset of X_δ, then

$$\mathrm{flow}_C(f_{z_1 \to z_2}) = \begin{cases} 1, & \text{if } z_1 \notin C \text{ and } z_2 \in C \\ -1, & \text{if } z_1 \in C \text{ and } z_2 \notin C \\ 0, & \text{otherwise.} \end{cases}$$

To do this, given $z_1, z_2 \in X'_\delta$, write $z_1 = \eta_1 + \omega^{k_1}$ with η_1 a multiple of ω^{k_2} and $z_2 = \eta_2 + \omega^{k_2}$ with η_2 a multiple of ω^{k_2}. For each positive integer i, let $a_i = \eta_1 + \omega^{k_1 - 1} \cdot i + 1$ and $b_i = \eta_2 + \omega^{k_2 - 1} \cdot i + 1$. Then let $f_{z_1 \to z_2}$ be the cycle $(\dots a_2\, a_1\, b_1\, b_2 \dots)$. (This should be compared with the definition of $f_{\{z\}}$ for $z \in X'_\delta$.) It is easy to see that $f_{z_1 \to z_2}$ is as required.

We may now combine infinite sequences of such functions to obtain the desired flow into $\mathcal{B}(z)$ for each $z \in X'_\delta \setminus X^{[n]}_\delta$. To do this, given $z \in X^{(2)}_\delta$, write $z = \eta + \omega^k$ with η a multiple of ω^k, let $z_i = \eta + \omega^{k-1} \cdot (i+1)$ for all $i \in \omega$ and define

$$f_z = f^{\mathbf{x}(z_0)}_{z_1 \to z_0} \circ f^{\mathbf{x}(z_0) + \mathbf{x}(z_1)}_{z_2 \to z_1} \circ f^{\mathbf{x}(z_0) + \mathbf{x}(z_1) + \mathbf{x}(z_2)}_{z_3 \to z_2} \circ \cdots.$$

This "infinite composition of functions" makes sense because each member of X_δ is in the support of only finitely many of these functions, and it is easy to see that it lies in K^1_δ. Moreover $\mathrm{flow}_{\mathcal{B}(z_i)}(f_z) = \sum_{j=0}^{i} \mathbf{x}(z_j) - \sum_{j=0}^{i-1} \mathbf{x}(z_j) = \mathbf{x}(z_i)$ for all $i \in \omega$.

To obtain the desired flow into $\mathcal{B}(z)$ for each $z \in X^{[n]}_\delta$, let $y_i = \omega^n \cdot (i+1)$ for each $i \in \{0, 1, \ldots, m-1\}$ and define

$$g = f_{y_1 \to y_0}^{\mathbf{x}(y_0)} \circ f_{y_2 \to y_1}^{\mathbf{x}(y_0)+\mathbf{x}(y_1)} \circ \cdots \circ f_{y_{m-1} \to y_{m-2}}^{\sum_{i=0}^{m-2} \mathbf{x}(y_i)}.$$

Then $\mathrm{flow}_{\mathcal{B}(y_i)}(g) = \mathbf{x}(y_i)$ for all $i \in \{0, 1, \ldots, m-1\}$ by a similar argument, using the assumption that $\sum_{i=0}^{m-1} \mathbf{x}(y_i) = 0$.

Finally take

$$f = g \prod_{z \in X_\delta^{(2)}} f_z,$$

a product of disjoint permutations lying in K_δ^1. Then f is as required. □

Let $\Delta \subseteq \Gamma_n$ be a lower set. Recall that $\mathbb{Z}_\Delta$ is defined to be the image of L_Δ under flow. In order to find the normal subgroups $N \trianglelefteq H_\delta$ with $L_\Delta^0 \leq N \leq L_\Delta$, we must consider the conjugation action of H_δ on L_Δ, which induces an action of H_δ on $\mathbb{Z}_\Delta$. Now the induced action of K_δ^1 on $\mathbb{Z}_\Delta$ is trivial. So since $H_\delta = K_\delta^1 \rtimes B_\delta^1$ by Proposition 6.2.2, it is enough to consider the induced action of B_δ^1 on $\mathbb{Z}_\Delta$. This motivates the following definition.

Definition. Given $\mathbf{x} \in \mathbb{Z}_{\Gamma_n}$ and $g \in H_\delta^1$, define $\mathbf{x}^g = \mathrm{flow}\left(f^{b_\delta^1(g)}\right)$, where $f \in K_\delta^1$ is any function with $\mathrm{flow}(f) = \mathbf{x}$. (Here b_δ^1 is the block map defined in Section 6.2, though by Lemma 6.7.4 we may equivalently replace $b_\delta^1(g)$ by any $h \in H_\delta$ with $r_\delta^1(h) = g$.) This is well-defined because if $f_1, f_2 \in K_\delta^1$ with $\mathrm{flow}(f_1) = \mathrm{flow}(f_2)$ and $h \in H_\delta$, then $\mathrm{flow}\left(f_1^h\right) = \mathrm{flow}\left(f_2^h\right)$ by Lemma 6.7.6.

For example, it is easy to show using part 4 of Lemma 6.7.2 that if $\mathbf{x} \in \mathbb{Z}_{\Gamma_n}$, $g \in H_\delta^1$ and $z \in X_\delta^{[1]}$, then $(\mathbf{x}^g)(z) = \mathbf{x}(g(z))$. Note that this simple formula need not hold if we allow $z \in X_\delta^{(2)}$.

Let $\Delta \subseteq \Gamma_n$ be a lower set, suppose $L_\Delta^0 \leq N \leq L_\Delta$ and let M be the image of N/L_Δ^0 under the natural isomorphism $L_\Delta/L_\Delta^0 \to \mathbb{Z}_\Delta$. Clearly N is a normal subgroup of H_δ if and only if whenever $\mathbf{x} \in M$ and $g \in H_\delta^1$ we have $\mathbf{x}^g \in M$.

When $\delta = \omega^2$ and $\Delta = \Gamma_2$, $\mathbb{Z}_\Delta$ is isomorphic to the countable direct product $\mathbb{Z}^\omega$, which is known as the Baer–Specker group. We now show that H_{ω^2} has $2^{2^{\aleph_0}}$ normal subgroups lying between $L_{\Gamma_2}^0 = \mathrm{Ker}(\mathrm{flow})$ and $L_{\Gamma_2} = K_\delta^1$, stating our result in terms of $\mathbb{Z}^\omega$. Here, given $\mathbf{x} \in \mathbb{Z}^\omega$ and $g \in \mathrm{Sym}(\omega)$, we write $\mathbf{x}^g = (\mathbf{x}(g(z)))_{z \in \omega}$.

Proposition 6.10.2. *There are $2^{2^{\aleph_0}}$ subgroups $M \leq \mathbb{Z}^\omega$ with the property that whenever $\mathbf{x} \in M$ and $g \in \mathrm{Sym}(\omega)$ we have $\mathbf{x}^g \in M$.*

Proof. Let P be the set of primes. By a standard argument we may let $\mathcal{Q}$ be a collection of $2^{\aleph_0}$ infinite subsets of P with the property that $Q_1 \cap Q_2$ is finite for all $Q_1, Q_2 \in \mathcal{Q}$. Let $\mathcal{Q}^C = \{P \setminus Q : Q \in \mathcal{Q}\}$. For each $R \in \mathcal{Q}^C$, let $\mathbf{e}_R = (1, p_0, p_0 p_1, \ldots) \in \mathbb{Z}^\omega$, where $R = \{p_i : i \in \omega\}$ and $p_0 < p_1 < \ldots$. Finally, for each subset $\mathcal{A} \subseteq \mathcal{Q}^C$, let

$$M_{\mathcal{A}} = \left\{ \pm \sum_{i=1}^{r} \mathbf{e}_{R_i}^{g_i} : r \in \omega, R_1, R_2, \ldots, R_r \in \mathcal{A}, g_1, g_2, \ldots, g_r \in \mathrm{Sym}\,(\omega) \right\}.$$

Certainly there are $2^{2^{\aleph_0}}$ subsets of $\mathcal{Q}^C$, and $M_{\mathcal{A}} \leq \mathbb{Z}^\omega$ is a subgroup such that whenever $\mathbf{x} \in M_{\mathcal{A}}$ and $g \in \mathrm{Sym}\,(\omega)$ we have $\mathbf{x}^g \in M_{\mathcal{A}}$ for every subset $\mathcal{A} \subseteq \mathcal{Q}^C$. So it is sufficient to prove that for all $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{Q}^C$, if $\mathcal{A}_1 \neq \mathcal{A}_2$ then $M_{\mathcal{A}_1} \neq M_{\mathcal{A}_2}$.

So suppose $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{Q}^C$ with $\mathcal{A}_1 \neq \mathcal{A}_2$. Without loss of generality there exists $R \in \mathcal{A}_1 \setminus \mathcal{A}_2$. Then $\mathbf{e}_R \in \mathcal{A}_1$ has infinitely many non-zero entries modulo p for all $p \in P \setminus R$. We claim that on the other hand, for all $\mathbf{x} \in \mathcal{A}_2$ there exists $p \in P \setminus R$ such that $\mathbf{x}$ has only finitely many non-zero entries modulo p. To see this, let $\mathbf{x} \in \mathcal{A}_2$ and write $\mathbf{x} = \pm \sum_{i=1}^r \mathbf{e}_{R_i}^{g_i}$ with $R_1, R_2, \ldots, R_r \in \mathcal{A}_2$ and $g_1, g_2, \ldots, g_r \in \mathrm{Sym}\,(\omega)$. Let $Q = \bigcup_{i=1}^r P \setminus R_i$. Then $Q \cap (P \setminus R)$ is finite by definition of $\mathcal{Q}$, so there exists $p \in (P \setminus Q) \cap (P \setminus R)$ since $P \setminus R$ is infinite. But then $p \in R_i$ for all $i \in \{1, 2, \ldots, r\}$, so $\mathbf{e}_{R_i}$ has only finitely many non-zero entries modulo p for all $i \in \{1, 2, \ldots, r\}$, so $\mathbf{x}$ has only finitely many non-zero entries modulo p, as required. □

Essentially the same argument works for any lower set $\Delta \subseteq \Gamma_n$ such that $(k_1, k_2) \in \Delta$ for some $k_1, k_2 \in \{1, 2, \ldots, n\}$.

Corollary 6.10.3. *Let $\Delta \subseteq \Gamma_n$ be a lower set and suppose $(k_1, k_2) \in \Delta$ for some $k_1, k_2 \in \{1, 2, \ldots, n\}$. Then there are $2^{2^{\aleph_0}}$ subgroups $M \leq \mathbb{Z}_\Delta$ with the property that whenever $\mathbf{x} \in M$ and $g \in H_\delta^1$ we have $\mathbf{x}^g \in M$.*

Proof. Let $z = \omega^{k_2}$, let $z_i = \omega^{k_2 - 1} \cdot i + \omega^{k_1}$ for each $i \in \omega$ and let $N = \{z_i : i \in \omega\}$. Since $(k_1, k_2) \in \Delta$, if $f \in K_\delta^1$ is such that $\mathrm{cl}\,(\mathrm{supp}\,(f)) \cap X'_\delta = N \cup \{z\}$, then $f \in L_\Delta$. Hence by the construction given in Proposition 6.10.1, $\mathbb{Z}^N \times (0)_{z \in X'_\delta \setminus N} \leq \mathbb{Z}_\Delta$. Moreover, if $g \in B_\delta^{k_1}$ and $f \in L_\Delta$ then by part 4 of Lemma 6.7.2, $\mathrm{flow}_{\mathcal{B}(z)}\,(f^g) = \mathrm{flow}_{g(\mathcal{B}(z))}\,(f) = \mathrm{flow}_{\mathcal{B}(g(z))}\,(f)$ for all $z \in X_\delta^{[k_1]}$. Now copy the argument of Proposition 6.10.2 with ω replaced by N and $\mathrm{Sym}\,(\omega)$ replaced by $r_\delta^1\left(B_\delta^{k_1}\right)$ to obtain the result. □

The assumption that $(k_1, k_2) \in \Delta$ for some $k_1, k_2 \in \{1, 2, \ldots, n\}$ is necessary here, because otherwise $\mathbb{Z}_\Delta$ is not large enough to have $2^{2^{\aleph_0}}$ subgroups, as we now observe.

Proposition 6.10.4. *Let $\Delta \subseteq \Gamma_n$ be a lower set such that every member of Δ has length at most 1. Then $\{z \in X'_\delta : \mathbf{x}\,(z) \neq 0\}$ is finite for all $\mathbf{x} \in \mathbb{Z}_\Delta$. In particular, $\mathbb{Z}_\Delta$ is countable.*

Proof. Let $f \in L_\Delta$ and let $\mathbf{x} = \mathrm{flow}\,(f)$. Since every member of Δ has length at most 1, it follows from part 3 of Lemma 6.6.1 that $\mathrm{cl}\,(\mathrm{supp}\,(f)) \cap X'_\delta$ is finite. But for each $y \in X'_\delta$, there are only finitely many $z \in X'_\delta$ such that $y \in \mathcal{B}\,(z)$. Hence there are only finitely many $z \in X'_\delta$ such that $\mathrm{cl}\,(\mathrm{supp}\,(f)) \cap X'_\delta \cap \mathcal{B}\,(z) \neq \emptyset$. We claim that for any clopen subset $C \subseteq X_\delta$, if $\mathrm{flow}_C\,(f) \neq 0$ then $\mathrm{cl}\,(\mathrm{supp}\,(f)) \cap X'_\delta \cap C \neq \emptyset$. Given the claim, there must be only finitely many $z \in X'_\delta$ with $\mathbf{x}\,(z) \neq 0$, as required.

To prove the claim, let C be a clopen subset of X_δ. If $\mathrm{cl}\,(\mathrm{supp}\,(f)) \cap X'_\delta \cap C = \emptyset$, then $\mathrm{supp}\,(f) \cap C$ is finite, so by part 1 of Lemma 6.7.2, $\mathrm{flow}_C\,(f) = \mathrm{flow}_{C \setminus \mathrm{supp}(f)}\,(f) - \sum_{x \in \mathrm{supp}(f)} \mathrm{flow}_{\{x\}}\,(f) = \mathrm{flow}_{C \setminus \mathrm{supp}(f)}\,(f) = 0$. $\square$

It is not much harder to show that if $\Delta \subseteq \Gamma_n$ is a lower set such that every member of Δ has length at most 1, then in fact $\mathbb{Z}_\Delta$ is isomorphic to

$$\begin{cases} \mathbb{Z}^{<\omega}, & \text{if } (k) \in \Delta \text{ for some } k < n \\ \mathbb{Z}^{m-1}, & \text{if } \Delta = \{(n)\} \\ \{0\}, & \text{if } \Delta = \emptyset \text{ or } \Delta = \{()\}, \end{cases}$$

where $\mathbb{Z}^{<\omega} = \{\mathbf{x} \in \mathbb{Z}^\omega : \{z \in \omega : \mathbf{x}\,(z) \neq 0\} \text{ is finite}\}$.

Moreover, if $\mathbb{Z}_\Delta \neq \{\mathbf{0}\}$, then in this case there are exactly $\aleph_0$ subgroups $M \leq \mathbb{Z}_\Delta$ with the property that whenever $\mathbf{x} \in M$ and $g \in H^1_\delta$ we have $\mathbf{x}^g \in M$. This may be contrasted with Corollary 6.10.3. Rather than proving this claim in general, we deal only with the special case in which $\delta = \omega^2$ and $\Delta = \langle (1), (2) \rangle$, stating our result in terms of $\mathbb{Z}^{<\omega}$. Again, given $\mathbf{x} \in \mathbb{Z}^{<\omega}$ and $g \in \mathrm{Sym}\,(\omega)$, we write $\mathbf{x}^g = (\mathbf{x}\,(g\,(z)))_{z \in \omega}$.

The following result was discovered in discussion with Lovkush Agarwal.

Proposition 6.10.5. *Suppose $M \leq \mathbb{Z}^{<\omega}$ is a subgroup with the property that whenever $\mathbf{x} \in M$ and $g \in \mathrm{Sym}\,(\omega)$ we have $\mathbf{x}^g \in M$. Then there exist $a, b \in \mathbb{Z}_{\geq 0}$ such that either*

$$M = \left\{ a\mathbf{x} : \mathbf{x} \in \mathbb{Z}^{<\omega}, \sum_{z \in \omega} \mathbf{x}\,(z) = 0 \right\},$$

or $b > 0$ and

$$M = \left\{ a\mathbf{x} : \mathbf{x} \in \mathbb{Z}^{<\omega}, \sum_{z \in \omega} \mathbf{x}\,(z) \text{ is divisible by } b \right\}.$$

In particular, there are exactly $\aleph_0$ such subgroups.

Proof. Assume $M \neq \{\mathbf{0}\}$, since this is covered by the case $a = 0$. Let a be the minimum absolute value of any non-zero coordinate of any member of M. Then every coordinate of every member of M must be a multiple of a, else we could use the Euclidean algorithm to contradict the minimality of a. Next, we may choose $\mathbf{x} \in M$ with $\mathbf{x}\,(0) = a$ and $\mathbf{x}\,(1) = 0$, whence $(-a, a, 0, 0, \dots) = \mathbf{x}^{(0\,1)} - \mathbf{x} \in M$. Taking sums of permutations of this, we see that $\left\{ a\mathbf{x} : \mathbf{x} \in \mathbb{Z}^{<\omega}, \sum_{z \in \omega} \mathbf{x}\,(z) = 0 \right\} \leq M$.

If we have equality here, then take $b = 0$ and we are done. Otherwise, let $ab = \min\left(\left\{ \sum_{z \in \omega} \mathbf{x}\,(z) : \mathbf{x} \in M, \sum_{z \in \omega} \mathbf{x}\,(z) > 0 \right\}\right) > 0$. We show that for all $\mathbf{y} \in \mathbb{Z}^{<\omega}$, $a\mathbf{y} \in M$ if and only if $\sum_{z \in \omega} \mathbf{y}\,(z)$ is divisible by b, which suffices. The "only if" statement follows because otherwise we could use the Euclidean algorithm again to contradict the minimality of b. For the "if" statement, let $\mathbf{w} \in M$ witness the minimality of b. If $\mathbf{y} \in \mathbb{Z}^{<\omega}$ and $\sum_{z \in \omega} \mathbf{y}(z) = kb$ for some $k \in \mathbb{Z}$, then $k\mathbf{w} - a\mathbf{y} \in \left\{a\mathbf{x} : \mathbf{x} \in \mathbb{Z}^{<\omega}, \sum_{z \in \omega} \mathbf{x}(z) = 0\right\} \leq M$, and hence $a\mathbf{y} \in M$, as required. $\square$

This completes our analysis of the normal subgroups of H_δ contained in K_δ^1. Let us now conclude this chapter by briefly indicating how one may attempt to generalise our results and thereby obtain a complete classification of the normal subgroups of H_δ.

First observe that by Proposition 6.2.2, $H_\delta / K_\delta^1 \cong B_\delta^1 \cong H_\delta^1 \cong H_{\omega^{(n-1)} \cdot m}$. Hence the normal subgroups of H_δ containing K_δ^1 correspond to the normal subgroups of $H_{\omega^{(n-1)} \cdot m}$, many of which we have already analysed. There may nonetheless be many normal subgroups of H_δ that neither are contained in nor contain K_δ^1. For example, the group of all functions in H_{ω^2} whose support is contained in a proper initial segment of $[1, \omega^2)$ is a normal subgroup of H_{ω^2} that neither is contained in nor contains $K_{\omega^2}^1$.

It may be possible to extend our analysis by generalising our notions of characters and flows to the whole of H_δ. For example, one may define the *generalised character* of an arbitrary function $f \in H_\delta$ to be

$$\left(\operatorname{tp}\left(\operatorname{supp}(f) \cap X_\delta^{[0]}\right), \operatorname{tp}\left(\operatorname{supp}(f) \cap X_\delta^{[1]}\right), \dots, \operatorname{tp}\left(\operatorname{supp}(f) \cap X_\delta^{[n-1]}\right)\right),$$

where we have extended the definition of types to allow subsets of $X_\delta^{[k]}$ for all $k \in \{0, 1, \dots, n-1\}$. This generalised character must satisfy certain conditions. For example, if (Δ_0, Δ_1) is the generalised character of a function in H_{ω^2} and $\Delta_1 \neq \emptyset$, then $(1) \in \Delta_0$. Many other details remain to be checked, and there is certainly plenty of room for further research.

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