Even-hole-free graphs

by

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Abstract

In this thesis we consider the class of simple graphs defined by excluding even holes (i.e. chordless cycles of even length). These graphs are known as even-hole-free graphs. We first prove that every even-hole-free graph has a node whose neighborhood is triangulated. This implies that in an even-hole-free graph, with n nodes and m edges, there are at most $n+2m$ maximal cliques. It also yields a fastest known algorithm for computing a maximum clique in an even-hole-free graph.

Afterwards we prove the main result of this thesis. The result is a decomposition theorem for even-hole-free graphs, that uses star cutsets and 2-joins. This is a significant strengthening of the only other previously known decomposition of even-hole-free graphs, by Conforti, Cornuéjols, Kapoor and Vušković, that uses 2-joins and star, double star and triple star cutsets. It is also analogous to the decomposition of Berge (i.e. perfect) graphs with skew cutsets, 2-joins and their complements, by Chudnovsky, Robertson, Seymour and Thomas. In a graph that does not contain a 4-hole, a skew cutset reduces to a star cutset, and a 2-join in the complement implies a star cutset, so in a way it was expected that even-hole-free graphs can be decomposed with just the star cutsets and 2-joins.

A consequence of this decomposition theorem is an $\mathcal{O}(n^{19})$ recognition algorithm for even-hole-free graphs. The recognition of even-hole-free graphs was first shown to be polynomial by Conforti, Cornuéjols, Kapoor and Vušković. They obtained an algorithm of complexity of about $\mathcal{O}(n^{40})$ by first preprocessing the input graph using a certain "cleaning" procedure, and then constructing a decomposition based recognition algorithm. The cleaning procedure was also the key to constructing a polynomial time recognition algorithm for Berge graphs. At that time it was observed by Chudnovsky and Seymour that once the cleaning is performed, one does not need a decomposition based algorithm, one can instead just look for the "bad structure" directly. Using this idea, as opposed to using the decomposition based approach, one gets significantly faster recognition algorithms for Berge graphs and balanced $0,\pm1$ matrices. However, this approach yields an $\mathcal{O}(n^{31})$ recognition algorithm for even-hole-free graphs. So this is the first example of a decomposition based algorithm being significantly faster than the Chudnovsky/Seymour style algorithm.

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Chapter 1

Introduction

1.1 Overview

We say that a graph G *contains* a graph F, if F is isomorphic to an induced subgraph of G. A graph G is F*-free* if it does not contain F. Let $\mathscr{F}$ be a (possibly infinite) family of graphs. A graph G is $\mathscr{F}$*-free* if it is F-free, for every $F \in \mathscr{F}$.

A *hole* is a chordless cycle of length at least four. A hole is *even* (resp. *odd*) if it contains even (resp. odd) number of nodes. A hole of length n is also called an n*-hole*. In this thesis we are concerned with the class of *even-hole-free* graphs, i.e. graphs that are $\mathscr{F}$-free where $\mathscr{F}$ denotes the family of all even holes.

The main part of this work is a decomposition theorem for even-hole-free graphs using *star cutsets* and *2-joins*. This decomposition is analogous to the decomposition of Berge (i.e. perfect) graphs with *skew cutsets*, 2-joins and their complements, by Chudnovsky, Robertson, Seymour and Thomas [7] (note that in a graph that does not contain a 4-hole, a skew cutset reduces to a star cutset, and a 2-join in the complement implies a star cutset). We also show that the decomposition obtained leads to a fastest known recognition algorithm for even-hole-free graphs. As a second contribution we prove that every even-hole-free graph has a node whose neighborhood is *triangulated*. This implies that in an even-hole-free graph, with n nodes and m edges, there are at most $n+2m$ maximal cliques. As a consequence we obtain an $O(n^2m)$ algorithm that generate all maximal cliques of an even-hole-free graph.

Many interesting classes of graphs can be characterized as being $\mathscr{F}$-free, for some family $\mathscr{F}$. In particular, a question that arises in this domain is to understand to what extent forbidding an induced subgraph impacts the global structure of a given graph. The most famous example in this context is the class of perfect graphs. A graph G is *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$, where $\chi(H)$ denotes the *chromatic number* of H and $\omega(H)$ denotes the size of a largest *clique*. The famous Strong Perfect Graph Theorem (conjectured by Berge [2], and proved by Chudnovsky, Robertson, Seymour and Thomas [7]) states that a graph is perfect if and only if it does not contain an odd hole nor an odd antihole (where an *antihole* is a complement of a hole). The graphs that do not contain an odd hole nor an odd antihole are known as *Berge* graphs.

The structure of even-hole-free graphs was first studied by Conforti, Cornuéjols, Kapoor and Vušković in [13] and [14]. In [13] they obtained a decomposition theorem for even-hole-free graphs that uses 2-joins and star, double star and triple star cutsets (all these cutsets are defined in Section 2.2.1), and in [14] they used it to obtain a polynomial time recognition algorithm for even-hole-free graphs. This is the same paradigm that was used to obtain recognition algorithms for balanced matrices [11, 17]. All these algorithms use "cleaning", a technique first developed by Conforti and Rao [18] to recognize linear balanced matrices. This technique was invented to make use of strong cutsets, such as star cutsets, in a decomposition based recognition algorithm. If one is able to clean the graph for the even-hole-free graph recognition problem, one can then make use of not only star cutsets, but also double star and triple star cutsets, and for that reason all these cutsets were used in the decomposition of even-hole-free graphs in [13]. That decomposition gave the first known recognition algorithm for even-hole-free graphs, but it was always clear that a stronger decomposition theorem was possible. At that time that problem was put aside, since the focus then was on perefect graphs, trying to prove the Strong Perfect Graph Conjecture and obtain a polynomial time recognition algorithm for Berge graphs.

Strong Perfect Graph Conjecture was proved by Chudnovsky, Robertson, Seymour and Thomas in [7], by decomposing Berge graphs using skew cutsets, 2-joins and their complements. Soon after, the recognition of Berge graphs was shown to be polynomial by Chudnovsky, Cornuéjols, Liu, Seymour and Vušković in [4].

Note that by excluding the 4-hole, one also excludes all antiholes of length at least 6. So if we switch parity, the analogous class to even-hole-free graphs are the Berge graphs, rather than just the *odd-hole-free* graphs. As mentioned above, in a graph that does not contain a 4-hole, a skew cutset reduces to a star cutset, and a 2-join in the complement implies the star cutset. The decomposition of Berge graphs with skew cutsets, 2-joins and their complements [7] provided a motivation to believe that it is also possible to decompose even-hole-free graphs with just the star cutsets and 2-joins.

As expected, the key to obtaining a polynomial time recognition algorithm for Berge graphs [4] was the cleaning. What was surprising, as Chudnovsky and Seymour observed, was that once the cleaning is performed, one does not need the decomposition based recognition algorithm, one can simply look for the “bad structure” (in this case an odd hole) directly. So in [4] two recognition algorithms for Berge graphs are given: an $\mathcal{O}(n^9)$ Chudnovsky/Seymour style (that uses the direct method) algorithm, and an $\mathcal{O}(n^{18})$ decomposition based recognition algorithm. (The high complexity of all of these algorithms is primarily due to cleaning). Then Zambelli [40] showed that by using the cleaning with the direct method, the complexity of the recognition algorithm for balanced $0,\pm1$ matrices dramatically drops, in comparison with their original recognition [11] based on the decomposition method.

Another twist in the story is the case of the recognition algorithm for even-hole-free graphs. The original algorithm from [14] is of complexity of about $\mathcal{O}(n^{40})$. In [6] Chudnovsky, Kawarabayashi and Seymour obtain an $\mathcal{O}(n^{31})$ recognition algorithm for even-hole-free graphs, using cleaning with the direct method. In the same paper they sketch another more complicated algorithm that, they claim, runs in time $\mathcal{O}(n^{15})$. This algorithm first needs to test for thetas and prisms in that time (thetas and prisms are defined in Section 2.2). It turns out that testing for thetas can be done in time $\mathcal{O}(n^{11})$ [9]. Detecting a prism is NP-complete in general [28]. In [6] it is claimed that under the assumption that the graph does not contain a theta one can use cleaning to test for prisms in time $\mathcal{O}(n^{15})$. This turns out to be false. Detecting a theta or a prism using the outlined method ends up being of complexity $\mathcal{O}(n^{35})$ [5]. In this work we show that our decomposition of even-hole-free graphs yields an $\mathcal{O}(n^{19})$ time recognition algorithm. So this is the first example in which a decomposition based method performs faster. We note that none of these algorithms are of any practical use, but they are interesting from a theoretical perspective.

The essence of even-hole-free graphs is actually captured by their generalization to signed graphs, called the odd-signable graphs, and in fact the results obtained in this thesis are for the class of graphs that are 4-hole-free odd-signable. We introduce odd-signable graphs in Chapter 2. In Chapter 2 we also review results concerning even-hole-free graphs and outline the decomposition theorem. In Chapter 3 we prove that every even-hole-free graph has a node whose neighborhood is triangulated and show some consequences of this result. The proof for the decompostion theorem is given in in the Chapters 4 and 5. In Chapter 6 we describe the recognition algorithm for even-hole-free graphs.

We now conclude this Chapter with an introduction of relevant concepts, terminology and notation of graph theory that will be used throughout this thesis.

1.2 Graph theory

We first note that all graphs in this work are finite, simple and undirected. We also note that some concepts already mentioned in Section 1.1 will be repeated here. However, now they will be formally defined.

1.2.1 Basic concepts

A *graph* G is an ordered pair $(V(G),E(G))$ consisting of a nonempty *node set* $V(G)$ and *edge set* $E(G)$. Sets $V(G)$ and $E(G)$ are assumed to be finite. We sometimes refer to the nodes of G as *vertices* of G. Because we only consider simple undirected graphs, we define $E(G)$ to be a subset of the set $\{\{u,v\} : u,v \in V(G), u \neq v\}$. For simplicity of notation we denote an edge $\{u,v\}$ by uv. If $uv \in E(G)$, then nodes u and v are said to be *adjacent* (or sometimes u and v are said to be *neighbors*). For $v \in V(G)$, $N(v)$ denotes the set of nodes adjacent to v. The *complement* of G, denoted by $\overline{G}$, is the graph $(V(G), \{uv : uv \notin E(G)\})$.

Two graphs G and H are *isomorphic* if there is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $uv \in E(H)$. For a nonempty set $A \subseteq V(G)$, the *subgraph of G induced by A*, denoted by $G[A]$, is the graph $(A, \{uv : u,v \in A, uv \in E(G)\})$. Such a graph is called an *induced subgraph* of G.

For $S \subseteq V(G)$ and $A \subseteq E(G)$, we denote by $G \setminus (S \cup A)$ the subgraph of G obtained by removing the nodes of S (and all edges with at least one endnode in S) and the edges of A.

For $S \subseteq V(G)$, $N(S)$ denotes the set of nodes in $V(G) \setminus S$ with at least one neighbor in S and $N[S]$ denotes $N(S) \cup S$. For $x \in V(G)$, we also use the following notation: $N(x) = N(\{x\})$ and $N[x] = N[\{x\}]$. For $V' \subseteq V(G)$, $G[V']$ denotes the subgraph of G induced by V'. For $x \in V(G)$, the graph $G[N(x)]$ is called the *neighborhood* of x.

Let $S \subseteq V(G)$ and $x \in V(G)$. Node x is *adjacent* to S, if x is adjacent to some node of S. Node x is *strongly adjacent* to S, if x is adjacent to at least two nodes of S. For an induced subgraph H of G, a node $v \in V(G) \setminus V(H)$ is a *twin* of a node $x \in V(H)$ w.r.t. H, if $N(v) \cap V(H) = N[x] \cap V(H)$.

A *path* P is a sequence of distinct nodes $x_1, \dots, x_n$, $n \geq 1$, such that $x_i x_{i+1}$ is an edge, for all $1 \leq i < n$. These are called the *edges* of a path P. Nodes x_1 and x_n are the *endnodes* of the path. The nodes of $V(P)$ that are not endnodes are called the *intermediate nodes* of P. Let x_i and x_l be two nodes of P, such that $l \geq i$. The path $x_i, x_{i+1}, \dots, x_l$ is called the

x_ix_l-subpath of P. Let Q be the x_ix_l-subpath of P. We write $P = x_1,\dots,x_{i-1},Q,x_{l+1},\dots,x_n$. A *cycle* C is a sequence of nodes $x_1,\dots,x_n,x_1$, $n \geq 3$, such that nodes $x_1,\dots,x_n$ form a path and x_1x_n is an edge. The edges of the of the path $x_1,\dots,x_n$ together with the edge x_1x_n are called the *edges* of C. The *length* of a path P (resp. cycle C) is the number of edges in P (resp. C).

Nodes u and v of G are said to be *connected* if there is a path in G whose endnodes are u and v. Let $V_1,\dots,V_n$ be a partition of the node set $V(G)$ such that two nodes u and v are connected if and only if they belong to the same set V_i. The induced subgraphs $G[V_1],\dots,G[V_n]$ are called the *connected components* (or simply *components* of G). G is *connected* if G has exactly one connected component, otherwise G is said to be *disconnected*.

$S \cup A$ is a *cutset* if $G \setminus (S \cup A)$ contains more connected components than G. For an induced subgraph H of G, we say that a cutset S of G *separates* H if there are nodes of H in different components of $G \setminus S$.

Let A,B be two disjoint node sets such that no node of A is adjacent to a node of B. A path $P = x_1,\ldots,x_n$ *connects A and B* if either $n = 1$ and x_1 has a neighbor in A and B, or $n > 1$ and one of the two endnodes of P is adjacent to at least one node in A and the other is adjacent to at least one node in B. The path P is a *direct connection between A and B* if in $G[V(P) \cup A \cup B]$ no path connecting A and B is shorter than P. The direct connection P is said to be *from A to B* if x_1 is adjacent to a node in A and x_n is adjacent to a node in B.

A *clique* is a graph in which every pair of vertices are adjacent. The size of a largest clique in a graph G is denoted by $\omega(G)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices of G so that no two adjacent vertices receive the same color.

1.2.2 Graph classes and other concepts

Given a path or a cycle Q in a graph G, any edge of G between nodes of Q that is not an edge of Q is called a *chord* of Q. Q is *chordless* if no edge of G is a chord of Q. As mentioned earlier a *hole* is a chordless cycle of length at least 4. It is called a *k-hole* if it has k edges. A k-hole is *even* if k is even, and it is *odd* otherwise.

We say that a graph G *contains* a graph F, if F is isomorphic to an induced subgraph of G. A graph G is *F-free* if it does not contain F. Let $\mathscr{F}$ be a (possibly infinite) family of graphs. A graph G is *$\mathscr{F}$-free* if it is F-free, for every $F \in \mathscr{F}$.

A graph is *even-hole-free* (resp. *odd-hole-free* if it does not contain an even (resp. odd) hole. A graph is *Berge* if it does not contain an odd hole nor the complement of an odd hole. A graph is *triangulated* (also called *chordal*) if it does not contain a hole.

A *tree* is a connected graph that does not contain a cycle. Given a graph G, its line graph $L(G)$ is a graph such that: (i) each vertex of $L(G)$ represents an edge of G, and (ii) two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endnode in G. A graph G is *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$.

In figures, solid lines represent edges and dotted lines represent paths of length at least one.

A note on notation: For a graph G, let $V(G)$ denote its node set. For simplicity of notation we will sometimes write G instead of $V(G)$, when it is clear from the context that we want to refer to the node set of G. We will not distinguish between a node set and the graph induced by that node set. Also a singleton set $\{x\}$ will sometimes be denoted with just x. For example, instead of "$u \in V(G) \setminus \{x\}$", we will write "$u \in G \setminus x$". These simplifications of notation will take place in the proofs, whereas the statements of results will use proper notation.

Chapter 2

Even-hole-free graphs

In the last 15 years a number of classes of graphs defined by excluding a family of induced subgraphs have been studied, perhaps originally motivated by the study of perfect graphs. The kinds of questions this line of research was focused on were whether excluding induced subgraphs affects the global structure of the particular class in a way that can be exploited for putting bounds on parameters such as χ and ω, constructing optimization algorithms (problems such as finding the size of a largest clique or a minimum coloring) and recognition algorithms.

A number of these questions were answered by obtaining a structural characterization of a class through their decomposition. A decomposition theorem elucidates the structure of a class of graphs by showing that every graph in this class has either a prescribed and relatively simple structure (in this case we often say that the graph belong to a *basic* class) or one of prescribed cutset, along with it can be decomposed.

This was the paradigm used in the proof of Strong Perfect Graph Theorem. The idea was to decompose Berge graphs in a way that the basic graphs are perfect and the graphs that are not basic (and hence admit a cutset) cannot be a minimum counterexample to the conjecture. Other classes of graphs in this context, as odd-hole-free graphs and balanced matrices have been studied through decomposition theorems [11, 15, 17].

Recent works include a decomposition of *claw-free graphs* and *bull-free graphs* by Chudnovsky and Seymour (they outline these results in [8]) using a series of cutsets and operations. The decomposition obtained for these classes are “reversible” in the sense that the theorem gives a receipe to build all graphs in the class by gluing basic pieces together.

In [37] Trotignon and Vušković decompose graphs containing no cycle with a unique chord (this class generalizes strongly balanceable graphs, see [16] for a survey). The decompostion obtained also work in both directions: the graph is in the class if and only if it can be constructed by gluing basic graphs along the decompositions. Such structure theorems are less common, but they are stronger and perhaps give a better understanding of the class in a way to construct optimization algorithms (for example note that in [37], as a consequence of the decomposition, a recognition algorithm is obtained as well as algorithms to find an optimal coloring and maximum clique).

The decomposition we prove in this thesis is not reversible, but still provides enough undestanding of the class to lead to a polynomial-time recognition algorithm. We note that other known decompositions for related classes, as the one for Berge graphs and and the one for odd-hole-free graphs are not reversible as well.

2.1 Excluding even-holes

The structure of even-hole-free graphs was first studied by Conforti, Cornuéjols, Kapoor and Vušković in [13] and [14]. They were focused on showing that even-hole-free graphs can be recognized in polynomial time (a problem that at that time was not even known to be in NP), and their primary motivation was to develop techniques which can then be used in the study of perfect graphs. In [13] they obtained a decomposition theorem for even-hole-free graphs that uses 2-joins and star, double star and triple star cutsets (all these cutsets are defined in Section 2.2.1), and in [14] they used it to obtain a polynomial time recognition algorithm for even-hole-free graphs. This algorithm use “cleaning”, a technique first developed by Conforti and Rao [18] to recognize linear balanced matrices. This technique was invented to make use of strong cutsets, such as star cutsets, in a decomposition based recognition algorithm. If one is able to clean the graph for the even-hole-free graph recognition problem, one can then make use of not only star cutsets, but also double star and triple star cutsets, and for that reason all these cutsets were used in the decomposition of even-hole-free graphs in [13]. The complexity of this algorithm is about $\mathcal{O}(n^{40})$. In [6] Chudnovsky, Kawarabayashi and Seymour obtain an $\mathcal{O}(n^{31})$ recognition algorithm for even-hole-free graphs. The algorithm also has a cleaning step, but after this procedure a “direct approach” (looking directly for forbidden structures) is used instead of a decomposition based method. We present in Chapter 6 a new decomposition based algorithm for recognizing even-hole-free graphs. The algorithm is a consequence of the main decomposition obtained in this thesis. The complexity of this new algorithm is $\mathcal{O}(n^{19})$.

One can find a maximum clique of an even-hole-free graph in polynomial time, since as observed by Farber [20] 4-hole-free graphs have $\mathcal{O}(n^2)$ maximal cliques and hence one can list them all in polynomial time. In Chapter 3 we show that every even-hole-free graph contains a vertex whose neighborhood is triangulated (i.e. does not contain a hole). This characterization leads to a faster algorithm for computing a maximum clique in an even-hole-free graph. The complexities of finding a maximum independent set and an optimal coloring are not known for even-hole-free graphs. We note that for odd-hole-free graphs the complexities of finding a maximum independent set, an optimal coloring as well as the recognition problem are also open problems, and that finding a maximum clique for odd-hole-free graphs is NP-complete (follows from 2-subdivision [33]).

More recently, Addario-Berry, Chudnovsky, Havet, Reed and Seymour [1], settle a conjecture of Reed, by proving that every even-hole-free graph contains a *bisimplicial vertex* (a vertex whose set of neighbors induces a graph that is a union of two cliques). This immediately implies that if G is an even-hole-free graph, then $\chi(G) \leq 2\omega(G) - 1$ (observe that if v is a bisimplicial vertex of G, then its degree is at most $2\omega(G) - 2$, and hence G can be colored with at most $2\omega(G) - 1$ colors). It is interesting that this result is also obtained using decomposition, although in [1] not all even-hole-free graphs are decomposed, but enough structures are decomposed using special double star cutsets to obtain the desired result.

Another motivation for the study of even-hole-free graphs is their connection to β-perfect graphs introduced by Markossian, Gasparian and Reed [30]. For a graph G, let $\delta(G)$ be the minimum degree of a vertex in G. Consider the following total order on $V(G)$: order the vertices by repeatedly removing a vertex of minimum degree in the subgraph of vertices not yet chosen and placing it after all the remaining vertices but before all the vertices already removed. Coloring greedily on this order gives the upper bound $\chi(G) \leq \beta(G)$, where $\beta(G) = \max\{\delta(G') + 1 : G'$ is an induced subgraph of $G\}$. A graph is *β-perfect* if for each induced subgraph H of G, $\chi(H) = \beta(H)$. It is easy to see that β-perfect graphs belong to the class of even-hole-free graphs, and that this containment is proper.

A *diamond* is a cycle of length 4 that has exactly one chord. A *cap* is a cycle of length greater than four that has exactly one chord, and this chord forms a triangle with two edges of the cycle. In [30] it is shown that (even-hole, diamond, cap)-free graphs are β-perfect, and in [21] de Figueiredo and Vušković show that (even-hole, diamond, cap-on-6-vertices)-free graphs are β-perfect. Recently these results were extended by Kloks, Müller and Vušković who show in [27] that (even-hole, diamond)-free graphs are β-perfect. This result is obtained by proving that every (even-hole, diamond)-free graph contains a simplicial extreme (where a vertex is *simplicial* if its neighborhood set induces a clique, and it is a *simplicial extreme* if it is either simplicial or of degree 2). And the existence of simplicial extremes is obtained as a consequence of a decomposition of (even-hole, diamond)-free graphs in [27] that uses 2-joins, clique cutsets and bisimplicial cutsets (a special type of a star cutset). We note that the decomposition theorem for even-hole-free graphs in this thesis uses the one in [27] by reducing the problem to the diamond-free case.

Since (even-hole, diamond)-free graph is β-perfect, this class of graphs can be colored in polynomial time by coloring greedily on a particular easily constructable ordering of vertices. Note that for every graph G, there exists an ordering of its vertices on which the greedy coloring will give a $\chi(G)$-coloring of G, the difficulty being in finding this ordering. As mentioned before, complexity of finding an optimal coloring in an even-hole-free graph is an open problem. Also, total characterization of β-perfect graphs remains open, as well as their recognition.

The fact that (even-hole, diamond)-free graphs have simplicial extremes implies that for such a graph G, $\chi(G) \leq \omega(G)+1$ (observe that if v is a simplicial extreme of G, then its degree is at most $\omega(G)$, and hence G can be colored with at most $\omega(G)+1$ colors). So this class of graphs, as well as the class of even-hole-free graphs by the result in [1], belong to the family of χ-bounded graphs, introduced by Gyárfás [26] as a natural extension of the family of perfect graphs: a family of graphs $\mathscr{G}$ is *χ-bounded* with *χ-binding function* f if, for every induced subgraph G' of $G \in \mathscr{G}$, $\chi(G') \leq f(\omega(G'))$. Note that perfect graphs are a χ-bounded family of graphs with the χ-binding function $f(x) = x$.

The essence of even-hole-free graphs is actually captured by their generalization to signed graphs, called the odd-signable graphs, and in fact the decomposition theorem that we prove in this thesis is for the class of graphs that are 4-hole-free odd-signable. Odd-signable graphs are introduced in Section 2.2, and the decomposition theorem is described in Section 2.2.1.

2.2 Odd-signable graphs

We *sign* a graph by assigning $0,1$ weights to its edges. A graph is *odd-signable* if there exists a signing that makes every triangle odd weight and every hole odd weight. To charcterize odd-signable graphs in terms of excluded induced subgraphs, we now introduce two types of *3-path configurations* (3PC's) and even wheels.

Let x,y be two distinct nodes of G. A $3PC(x,y)$ is a graph induced by three chordless

xy-paths, such that any two of them induce a hole. We say that a graph G contains a $3PC(\cdot,\cdot)$ if it contains a $3PC(x,y)$ for some $x,y \in V(G)$. $3PC(\cdot,\cdot)$'s are also known as *thetas*, as in [5].

Let x_1,x_2,x_3,y_1,y_2,y_3 be six distinct nodes of G such that $\{x_1,x_2,x_3\}$ and $\{y_1,y_2,y_3\}$ induce triangles. A $3PC(x_1x_2x_3,y_1y_2y_3)$ is a graph induced by three chordless paths $P_1 = x_1,\dots,y_1$, $P_2 = x_2,\dots,y_2$ and $P_3 = x_3,\dots,y_3$, such that any two of them induce a hole. We say that a graph G contains a $3PC(\Delta,\Delta)$ if it contains a $3PC(x_1x_2x_3,y_1y_2y_3)$ for some $x_1,x_2,x_3,y_1,y_2,y_3 \in V(G)$. $3PC(\Delta,\Delta)$'s are also known as *prisms*, as in [5].

A *wheel*, denoted by (H,x), is a graph induced by a hole H and a node $x \notin V(H)$ having at least three neighbors in H, say $x_1,\dots,x_n$. Such a wheel is also called a *n-wheel*. Node x is the *center* of the wheel. Edges xx_i, for $i \in \{1,\dots,n\}$, are called *spokes* of the wheel. A subpath of H connecting x_i and x_j is a *sector* if it contains no intermediate node x_l, $1 \le l \le n$. A *short sector* is a sector of length 1, and a *long sector* is a sector of length greater than 1. Wheel (H,x) is *even* if it has an even number of sectors. See figure 2.1.

It is easy to see that even wheels, $3PC(\cdot,\cdot)$'s and $3PC(\Delta,\Delta)$'s cannot be contained in even-hole-free graphs. In fact they cannot be contained in odd-signable graphs. The following characterization of odd-signable graphs states that the converse also holds, and it is an easy consequence of a theorem of Truemper [38].

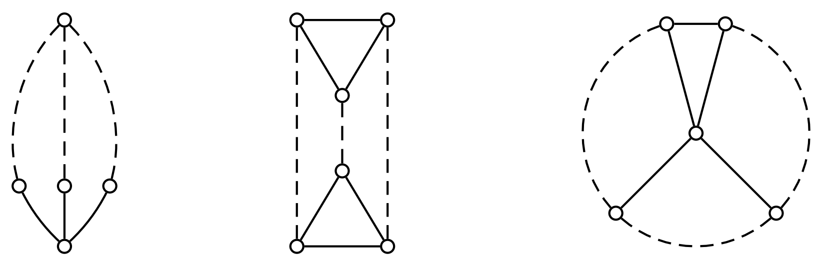


Figure 2.1: $3PC(\cdot,\cdot)$, $3PC(\Delta,\Delta)$ and an even wheel.

Theorem 2.2.1 *[12] A graph is odd-signable if and only if it does not contain an even wheel, a $3PC(\cdot,\cdot)$ nor a $3PC(\Delta,\Delta)$.*

This characterization of odd-signable graphs will be used throughout the thesis.

2.2.1 Decomposition theorem

A node set $S \subseteq V(G)$ is a *k-star cutset* of G if S is comprised of a clique C of size k and nodes with at least one neighbor in C, i.e. $C \subseteq S \subseteq N[C]$. We refer to C as the *center* of S. A 1-star is also refered to as a *star*, a 2-star as a *double star*, and 3-star as a *triple star*. If $S = N[C]$, then S is called a *full k-star*.

A graph G has a *2-join* $V_1|V_2$, with special sets (A_1, A_2, B_1, B_2), if the nodes of G can be partitioned into sets V_1 and V_2 so that the following hold.

(i) For $i = 1, 2$, $A_i \cup B_i \subseteq V_i$, and A_i and B_i are nonempty and disjoint.

(ii) Every node of A_1 is adjacent to every node of A_2, every node of B_1 is adjacent to every node of B_2, and these are the only adjacencies between V_1 and V_2.

(iii) For $i = 1, 2$, the graph induced by V_i, $G[V_i]$, contains a path with one endnode in A_i and the other in B_i. Furthermore, $G[V_i]$ is not a chordless path.

We now introduce two classes of graphs that have no star cutset nor a 2-join.

Let x_1, x_2, x_3, y be four distinct nodes of G such that x_1, x_2, x_3 induce a triangle. A $3PC(x_1x_2x_3, y)$ is a graph induced by three chordless paths $P_{x_1y} = x_1, \ldots, y$, $P_{x_2y} = x_2, \ldots, y$ and $P_{x_3y} = x_3, \ldots, y$, such that any two of them induce a hole. We say that a graph G contains a $3PC(\Delta, \cdot)$ if it contains a $3PC(x_1x_2x_3, y)$ for some $x_1, x_2, x_3, y \in V(G)$. Note that in a $\Sigma = 3PC(\Delta, \cdot)$ at most one of the paths may be of length one. If one of the paths of Σ is of length 1, then Σ is also a wheel that is called a *bug*. If all of the paths of Σ are of length greater than 1, then Σ is a *long* $3PC(\Delta, \cdot)$. $3PC(\Delta, \cdot)$'s are also known as *pyramids*, as in [4]. See Figure 2.2.

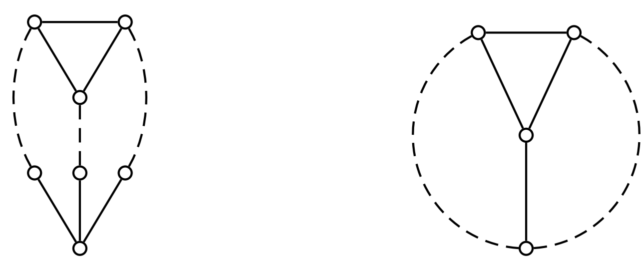


Figure 2.2: A long $3PC(\Delta, \cdot)$ and a bug.

We now define nontrivial basic graphs. Let L be the line graph of a tree. Note that every edge of L belongs to exactly one maximal clique, and every node of L belongs to at most two maximal cliques. The nodes of L that belong to exactly one maximal clique are called *leaf nodes*. A clique of L is *big* if it is of size at least 3. In the graph obtained from L by removing all edges in big cliques, the connected components are chordless paths (possibly of length 0). Such a path P is an *internal segment* if it has its endnodes in distinct big cliques (when P is of length 0, it is called an internal segment when the node of P belongs to two big cliques). The other paths P are called *leaf segments*. Note that one of the endnodes of a leaf segment is a leaf node.

A *nontrivial basic graph* R is defined as follows: R contains two adjacent nodes x and y, called the *special* nodes. The graph L induced by $R \setminus \{x, y\}$ is the line graph of a tree and contains at least two big cliques. In R, each leaf node of L is adjacent to exactly one of the two special nodes, and no other node of L is adjacent to special nodes. The last condition for R is that no two leaf segments of L with leaf nodes adjacent to the same special node have their other endnode in the same big clique. The *internal segments* of R are the internal segments of L, and the *leaf segments* of R are the leaf segments of L together with the node in $\{x,y\}$ to which the leaf segment is adjacent to.

Let G be a graph that contains a nontrivial basic graph R with special nodes x and y. R^* is an *extended nontrivial basic graph* of G if R^* consists of R and all nodes $u \in V(G) \setminus V(R)$ such that for some big clique K of R and for some $z \in \{x,y\}$, $N(u) \cap V(R) = V(K) \cup \{z\}$. We also say that R^* is an *extension* of R. See figure 2.3.

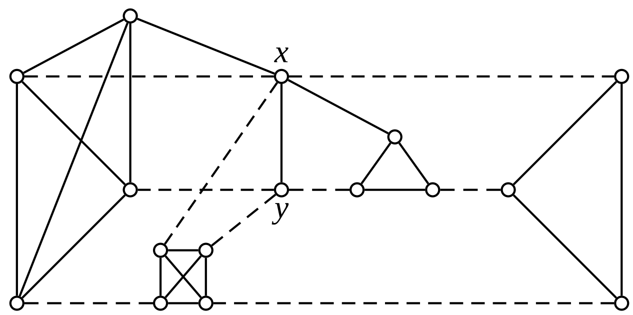


Figure 2.3: An extended nontrivial basic graph.

In [13] even-hole-free graphs are decomposed into cliques, holes, long $3PC(\Delta,\cdot)$ and nontrivial basic graphs using 2-joins and star, double star and triple star cutsets. We obtain the following strengthening of that result.

A graph is *basic* if it is one of the following graphs:

(1) a clique,

(2) a hole,

(3) a long $3PC(\Delta,\cdot)$, or

(4) an extended nontrivial basic graph.

Theorem 2.2.2 (The Main Decomposition Theorem) *A connected 4-hole-free odd-signable graph is either basic, or it has a star cutset or a 2-join.*

Here is a simple restatement of Theorem 2.2.2, that will be used in the recognition algorithm in Chapter 6. A graph is a *clique tree* if each of its maximal 2-connected components is a clique. A graph is an *extended clique tree* if it can be obtained from a clique tree by adding at most two vertices.

Corollary 2.2.3 *A connected even-hole-free graph is either an extended clique tree, or it has a star cutset or a 2-join.*

The key difference in the proof of the decomposition theorem in [13] and the one here, is that in [13] bugs are decomposed with double star cutsets. Since we are using just star cutsets, it is not possible to decompose all bugs, and hence we needed to enlarge the class of basic (undecomposable) graphs to include the extend nontrivial basic graphs.

Proof of Theorem 2.2.2 follows from the following three results.

Theorem 2.2.4 *[27] A connected 4-hole-free odd-signable graph that does not contain a diamond is either basic, or it has a star cutset or a 2-join.*

We note that the star cutsets used in [27] to prove Theorem 2.2.4, are of very special type: they either induce a clique or two cliques with exactly one node in common.

A *connected diamond* is a pair (Σ, Q), where $\Sigma = 3PC(x_1x_2x_3, y)$ and $Q = q_1, \ldots, q_k$, $k \geq 2$, is a chordless path disjoint from Σ such that the only nodes of Q that have a neighbor in Σ are q_1 and q_k. Furthermore $|N(q_1) \cap \Sigma| = |N(q_1) \cap \{x_1, x_2, x_3\}| = 2$, say $N(q_1) \cap \Sigma = \{x_1, x_3\}$, and one of the following holds:

(i) $N(q_k) \cap \Sigma = \{v_1, v_2\}$ where v_1v_2 is an edge of $P_{x_2y} \setminus \{x_2\}$, or

(ii) $N(q_k) \cap \Sigma = \{y, y_1, y_3\}$ where y_1 (resp. y_3) is the neighbor of y in P_{x_1y} (resp. P_{x_3y}), and x_1y and x_3y are not edges.

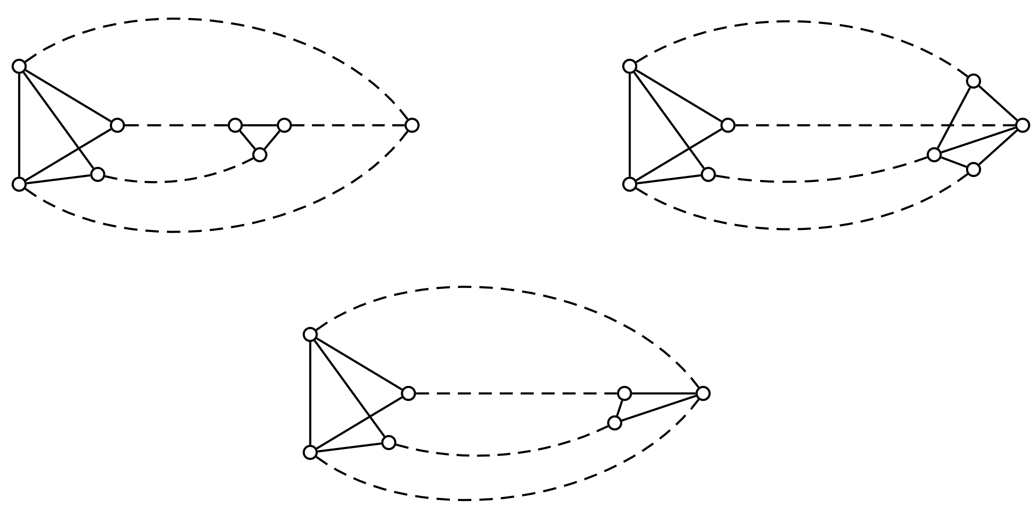


Figure 2.4: Different types of connected diamonds.

Theorem 2.2.5 *Let G be a connected 4-hole-free odd-signable graph. If G contains a diamond, then G has a star cutset or G contains a connected diamond.*

Theorem 2.2.6 *Let G be a connected 4-hole-free odd-signable graph. If G contains a connected diamond, then G has a star cutset or a 2-join.*

Theorem 2.2.5 is proved in Section 4.6 and Theorem 2.2.6 in Section 5.2.

Chapter 3

Triangulated Neighborhoods

The main result of this Chapter is the following structural property of odd-signable graphs that do not contain a 4-hole.

Theorem 3.0.7 *Every 4-hole-free odd-signable graph has a node whose neighborhood is triangulated.*

Parfenoff, Roussel and Rusu in [32] proved exactly the same result for 4-hole-free Berge graphs. Note that 4-hole-free graphs in general need not have this property, see Figure 3.1.

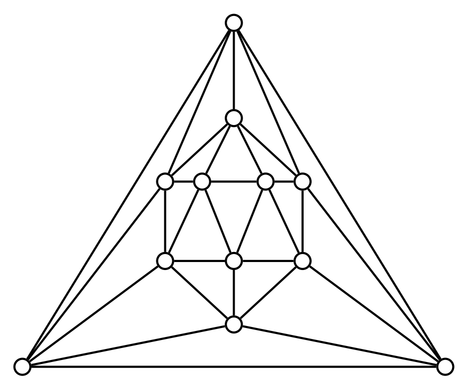


Figure 3.1: A 4-hole-free graph that has no vertex whose neighborhood is triangulated.

A *square*-$3PC(\cdot,\cdot)$ is a graph that consists of three paths between two nodes such that any two of the paths induce a hole, and at least two of the paths are of length 2. In [29] Maffray, Trotignon and Vušković show that every square-$3PC(\cdot,\cdot)$-free even-signable graph has a node whose neighborhood does not contain a long hole (where a *long hole* is a hole of length greater than 4). This result is used in [29] to obtain a combinatorial algorithm of complexity $\mathcal{O}(n^7)$ for finding a clique of maximum weight in square-$3PC(\cdot,\cdot)$-free Berge graphs. Note that this class of graphs generalizes both 4-hole-free Berge graphs and claw-free Berge graphs (where a *claw* is a graph on nodes x,a,b,c with three edges xa,xb,xc). We show in this Chapter that key ideas from [29] extend to 4-hole-free odd-signable graphs.

Using Theorem 3.0.7 one can obtain an efficient algorithm for generating all the maximal cliques in 4-hole-free odd-signable graphs (and in particular even-hole-free graphs). This we describe in Section 3.1. Theorem 3.0.7 is proved in Section 3.2.

As mentioned in Chapter 2, recently Addario-Berry, Chudnovsky, Havet, Reed and Seymour [1] have proved a stronger property of even-hole-free graphs, namely that every even-hole-free graph has a bisimplicial vertex (i.e. a vertex whose neighborhood partitions into two cliques). This result immediately yields that for an even-hole-free graph G, $\chi(G) \leq 2\omega(G)-1$, where $\chi(G)$ is the chromatic number of G and $\omega(G)$ is the size of the largest clique in G (observe that if v is a bisimplicial vertex of G, then its degree is at most $2\omega(G)-2$, and hence G can be colored with at most $2\omega(G)-1$ colors). The two properties of even-hole-free graphs were discovered independently and at about the same time. The proof in [1] is over 40 pages long. Our weaker property is enough to obtain an efficient algorithm for generating all maximal cliques of even-hole-free graphs, and its proof is very short.

3.1 Generating all the maximal cliques of a 4-hole-free odd-signable graph

For a graph G let k denote the number of maximal cliques in G, n the number of nodes in G and m the number of edges of G. Farber [20] shows that there are $\mathcal{O}(n^2)$ maximal cliques in any 4-hole-free graph. Tsukiyama, Ide, Ariyoshi and Shirakawa [39] give an $\mathcal{O}(nmk)$ algorithm for generating all maximal cliques of a graph, and Chiba and Nishizeki [3] improve this complexity to $\mathcal{O}(m^{1.5}k)$. The complexity is further improved for dense graphs by the $\mathcal{O}(M(n)k)$ algorithm of Makino and Uno [31], where $M(n)$ denotes the time needed to multiply two $n\times n$ matrices. Note that Coppersmith and Winograd show that matrix multiplication can be done in $\mathcal{O}(n^{2.376})$ time [19]. So one can generate all the maximal cliques of a 4-hole-free graph in time $\mathcal{O}(m^{1.5}n^2)$ or $\mathcal{O}(n^{4.376})$.

We now show that Theorem 3.0.7 implies that there are at most $n+2m$ maximal cliques in a 4-hole-free odd-signable graph, and it yields an algorithm that generates all the maximal cliques of a 4-hole-free odd-signable graph in time $\mathcal{O}(n^2m)$. In particular, in a weighted graph, a maximum weight clique can be found in time $\mathcal{O}(n^2m)$.

Let $\mathscr{C}$ be any class of graphs closed under taking induced subgraphs, such that for every G in $\mathscr{C}$, G has a node whose neighborhood is triangulated. Consider the following algorithm for generating all maximal cliques of graphs in $\mathscr{C}$.

Find a node x_1 of G whose neighborhood is triangulated (if no such node exists, G is not in $\mathscr{C}$, or in particular, G is not 4-hole-free odd-signable graph by Theorem 3.0.7). Let $G_1 = G[N[x_1]]$ and $G^1 = G \setminus \{x_1\}$. Every maximal clique of G belongs to G_1 or G^1. Recursively construct triangulated graphs $G_1, \dots, G_n$ as follows. For $i \geq 2$, find a node x_i of G^{i-1} whose neighborhood is triangulated and let $G_i = G[N_{G^{i-1}}[x_i]]$ and $G^i = G^{i-1} \setminus \{x_i\} = G \setminus \{x_1, \dots, x_i\}$.

Clearly every maximal clique of G belongs to exactly one of the graphs $G_1, \dots, G_n$. A triangulated graph on n vertices has at most n maximal cliques [22]. So for $i = 1, \dots, n$, graph G_i has at most $1 + d(x_i)$ maximal cliques (where $d(x)$ denotes the degree of vertex x). It follows that the number of maximal cliques of G is at most $\sum_{i=1}^{n}(1 + d(x_i)) = n + 2m$.

Checking whether a graph is triangulated can be done in time $\mathcal{O}(n+m)$ (using lexicographic breadth-first search [34]). So finding a vertex with triangulated neighborhood can be done in time $\mathcal{O}(\sum_{x \in V(G)}(d(x)+m)) = \mathcal{O}(nm)$. Hence constructing the graphs $G_1, \dots, G_n$ takes time $\mathcal{O}(n^2m)$.

Generating all maximal cliques in a triangulated graph can be done in time $\mathcal{O}(n+m)$ (see, for example, [23]). Hence the overall complexity of generating all maximal cliques in a 4-hole-free odd-signable graph is dominated by the construction of the sequence $G_1, \dots, G_n$, i.e. it is $\mathcal{O}(n^2m)$.

Note that this algorithm is *robust* in Spinrad's sense [36]: given any graph G, the algorithm either verifies that G is not in $\mathscr{C}$ (or in particular that G is not a 4-hole-free odd-signable graph) or it generates all the maximal cliques of G. Note that, when G is not in $\mathscr{C}$, the algorithm might still generate all the maximal cliques of G.

3.2 Proof of Theorem 3.0.7

In the next three lemmas we assume that G is a 4-hole-free odd-signable graph, x a node of G that is not adjacent to every other node of G, C_1 a connected component of $G \setminus N[x]$, and H a hole of $N(x)$. Note that H is an odd hole, else (H,x) is an even wheel.

Lemma 3.2.1 *If node u of C_1 has a neighbor in H then u is one of the following two types:*

- *Type 1: u has exactly one neighbor in H.*
- *Type 2: u has exactly two neighbors in H, and they are adjacent.*

Proof: If u has two nonadjecent neighbors a and b in H, then $\{a,b,u,x\}$ induces a 4-hole. $\square$

Let T^3 be a graph on 3 nodes that has exactly one edge.

Lemma 3.2.2 *If H contains a T^3 all of whose nodes have neighbors in C_1, then C_1 contains a path P, of length greater than 0, such that $P \cup H$ induces a $3PC(\Delta,\cdot)$, and the nodes of H that have a neighbor in P induce a T^3.*

Proof: Let C be a smallest subset of C_1 such that $G[C]$ is connected and $H = h_1,\dots,h_n,h_1$ contains a T^3 all of whose nodes have neighbors in C. W.l.o.g. h_1,h_2 and h_i, $3 < i < n$, have neighbors in C. Let $P = p_1,\dots,p_k$ be a shortest path of C such that p_1 is adjacent to h_1 and p_k is adjacent to h_2. Note that no intermediate node of P is adjacent to h_1 or h_2. Also possibly $k = 1$.

Claim 1: No node of $\{h_4,\dots,h_{n-1}\}$ has a neighbor in P.

Proof of Claim 1: Suppose not. Then by minimality of C, h_i has a neighbor in P and w.l.o.g. no node of $\{h_{i+1},\dots,h_{n-1}\}$ has a neighbor in P. By Lemma 3.2.1, $p_1,p_k \notin N(h_i) \cap P$. In particular $k > 1$.

First suppose $N(h_n) \cap P \neq \varnothing$. By Lemma 3.2.1, $h_n p_k$ is not an edge. If $N(h_n) \cap P = p_1$ then $\{x,h_n,h_2,h_1\} \cup P$ induces an even wheel with center h_1. So h_n has a neighbor in $P \setminus \{p_1,p_k\}$. If $h_i h_n$ is not an edge, then since all of h_1,h_n,h_i have neighbors in $P \setminus p_k$, the minimality of C is contradicted. So $h_i h_n$ is an edge of G. But then all of h_i,h_n,h_2 have neighbors in $P \setminus p_1$ and the minimality of C is contradicted. So $N(h_n) \cap P = \varnothing$.

Let p_r be the node of P with highest index adjacent to h_i. Let H' be the hole induced by $\{h_i,\dots,h_n,h_1,h_2,p_k,\dots,p_r\}$. Since (H',x) cannot be an even wheel, it follows that $h_i,\dots,h_n,h_1,h_2$ is an even subpath of H. Let p_s be the node of P with lowest index adjacent to h_i. Then $\{x,h_i,\dots,h_n,h_1,p_1,\dots,p_s\}$ induces an even wheel with center x. This completes the proof of Claim 1.

By Claim 1, h_i is not adjacent to a node of P. But h_i has a neighbor in C, and since C is connected, let $Q = q_1,...,q_l$ be a chordless path in C such that q_1 is adjacent to h_i and q_l has a neighbor in P.

Claim 2: No node of $\{h_4,\dots,h_{n-1}\}$ has a neighbor in $(P\cup Q)\setminus q_1$.

Proof of Claim 2: Suppose that some $h_j \in \{h_4,\dots,h_{n-1}\}$ has a neighbor in $(P\cup Q)\setminus q_1$. Then all of h_1,h_2,h_j have neighbors in $(P\cup Q)\setminus q_1$, contradicting the minimality of C. This completes the proof of Claim 2.

Claim 3: q_1 is of type 1 w.r.t. H.

Proof of Claim 3: By Lemma 3.2.1 q_1 is of type 1 or type 2. Suppose q_1 is of type 2. We now prove that $N(q_1)\cap H$ is either $\{h_3,h_4\}$ or $\{h_{n-1},h_n\}$. Assume not. Then q_1 is adjacent to neither h_3 nor h_n. W.l.o.g. $N(q_1)\cap H = \{h_i,h_{i-1}\}$ and $i \neq 4$. If $N(q_l)\cap P \neq p_1$, then $(P\cup Q)\setminus p_1$ is connected and all of h_i,h_{i-1},h_2 have neighbors in it, contradicting the minimality of C. So $N(q_l)\cap P = p_1$. If $k > 1$, then all of h_i,h_{i-1},h_1 have neighbors in $(P\cup Q)\setminus p_k$, contradicting the minimality of C. So $k = 1$, and hence by Lemma 3.2.1, $N(p_1)\cap H = \{h_1,h_2\}$. Since H is odd, the two subpaths of H, $h_2,\dots,h_{i-1}$ and $h_i,\dots,h_n,h_1$ have different parities. W.l.o.g. $h_2,\dots,h_{i-1}$ is odd, i.e. i is even. By Claim 2, no node of $\{h_4,\dots,h_{n-1}\}$ has a neighbor in $(P\cup Q)\setminus q_1$. If h_3 has no neighbor in Q then $Q\cup P\cup \{h_2,...,h_{i-1},x\}$ contains an even wheel with center x. So h_3 must have a neighbor in Q. But then h_i,h_{i-1},h_3 all have neighbors in Q (note that h_3h_{i-1} is not an edge since $i-1$ is odd greater than 3) contradicting the minimality of C. So $N(q_1)\cap H$ is either $\{h_3,h_4\}$ or $\{h_{n-1},h_n\}$.

W.l.o.g. $N(q_1)\cap H = \{h_3,h_4\}$. If $N(q_l)\cap P \neq p_k$, then since all of h_1,h_3,h_4 have neighbors in $(P\cup Q)\setminus p_k$, the minimality of C is contradicted. So $N(q_l)\cap P = p_k$.

If $N(h_1)\cap Q \neq \varnothing$, then since all of h_1,h_3,h_4 have neighbors in Q, the minimality of C is contradicted. So $N(h_1)\cap Q = \varnothing$.

Now suppose that $N(h_n)\cap Q \neq \varnothing$. If $k > 1$, then since all of h_2,h_3,h_n have neighbors in $(P\cup Q)\setminus p_1$, the minimality of C is contradicted. So $k = 1$. Let q_r be the neighbor of h_n with highest index. If h_2 does not have neighbor in $q_r,q_{r+1},...,q_l$, then $\{q_r,q_{r+1},...,q_l,p_1,h_1,h_2,h_n,x\}$ induces an even wheel with center h_1. So $N(h_2)\cap Q \neq \varnothing$. But then since h_2,h_3,h_n have neighbors in Q, the minimality of C is contradicted. Therefore, $N(h_n) \cap Q = \varnothing$. So, by Claim 2, no node of $h_5, \ldots, h_n, h_1$ has a neighbor in Q.

Suppose $N(h_2) \cap Q \neq \varnothing$. Let q_r be the neighbor of h_2 in Q with lowest index. Then $(H \setminus h_3) \cup \{x, q_1, \ldots, q_r\}$ induces an even wheel with center x. Therefore, $N(h_2) \cap Q = \varnothing$. If $k > 1$, then $Q \cup (H \setminus h_3) \cup \{p_k, x\}$ induces an even wheel with center x. So $k = 1$. Let q_s be the node of Q with highest index adjacent to h_3. Then $\{p_1, q_s, \ldots, q_l, h_1, h_2, h_3, x\}$ induces an even wheel with center h_2. This completes the proof of Claim 3.

Claim 4: $N(q_l) \cap P = p_1$ or p_k.

Proof of Claim 4: Assume not. Then $k > 1$, and both $(P \cup Q) \setminus p_1$ and $(P \cup Q) \setminus p_k$ are connected. $N(h_1) \cap Q = \varnothing$, else all of h_1, h_2, h_i have neighbors in $(P \cup Q) \setminus p_1$, contradicting the minimality of C. Similarly, $N(h_2) \cap Q = \varnothing$.

We now show that h_3 has no neighbor in $P \cup Q$. Suppose it does. Then by Lemma 3.2.1, h_3 has a neighbor in $(P \cup Q) \setminus p_1$. If $i \neq 4$, then since all h_2, h_3, h_i have neighbors in $(P \cup Q) \setminus p_1$, the minimality of C is contradicted. So $i = 4$. If $N(h_3) \cap (P \cup Q) \neq p_k$, then all of h_1, h_3, h_4 have neighbors in $(P \cup Q) \setminus p_k$, contradicting the minimality of C. So $N(h_3) \cap (P \cup Q) = p_k$. But then $P \cup Q \cup \{h_2, h_3, h_4, x\}$ contains an even wheel with center h_3. Therefore, h_3 has no neighbor in $P \cup Q$, and similarly neither does h_n.

By minimality of C, $N(q_l) \cap P$ is either a single vertex or two adjacent vertices of P. If $N(q_l) \cap P = \{a, b\}$, where $ab \in E(G)$, then $P \cup Q \cup \{x, h_1, h_2, h_i\}$ induces a $3PC(q_l ab, x h_1 h_2)$. If $N(q_l) \cap P = \{a\}$, then $P \cup Q \cup \{h_1, h_2, \ldots, h_i\}$ induces a $3PC(a, h_2)$. This completes the proof of Claim 4.

By Claim 4, w.l.o.g. $N(q_l) \cap P = p_k$.

Claim 5: h_1 does not have a neighbor in $(P \cup Q) \setminus p_1$.

Proof of Claim 5: If $k > 1$, the claim follows from the minimality of C. Now suppose $k = 1$ and $N(h_1) \cap Q \neq \varnothing$. If h_2 has a neighbor in Q, then all of h_1, h_2, h_i have a neighbor in Q, contradicting the minimality of C. So h_2 does not have a neighbor in Q.

Suppose h_n has a neighbor in Q. Note that by Claim 3, such a neighbor is in $Q \setminus q_1$. Then h_3 cannot have a neighbor in Q, else all of h_n, h_1, h_3 have neighbors in Q, contradicting the minimality of C. But then $(Q \setminus q_1) \cup (H \setminus h_1) \cup \{x, p_1\}$ contains an even wheel with center x. So h_n does not have a neighbor in Q.

Suppose h_3 has a neighbor in Q. By Claim 3, such a neighbor is in $Q \setminus q_1$. Then $(Q \setminus q_1) \cup (H \setminus h_2) \cup x$ contains an even wheel with center x. So h_3 does not have a neighbor in Q.

Let H' be the hole induced by $\{p_1, h_2, ..., h_i\} \cup Q$, and H'' the hole induced by $\{x, p_1, h_2, h_i\} \cup Q$. Then either (H', h_1) or (H'', h_1) is an even wheel. This completes the proof of Claim 5.

Claim 6: $N(h_n) \cap (P \cup Q) = \varnothing$.

Proof of Claim 6: Assume not. If h_3 has a neighbor in $P \cup Q$ then, by Claim 3, all of h_2, h_3, h_n have a neighbor in $(P \cup Q) \setminus q_1$, contradicting the minimality of C. So $N(h_3) \cap (P \cup Q) = \varnothing$. Let R be a shortest path from h_2 to h_n in the graph induced by $P \cup (Q \setminus q_1) \cup \{h_2, h_n\}$. Then by Claims 2 and 3, $R \cup (H \setminus h_1) \cup x$ induces an even wheel with center x. This completes the proof of Claim 6.

Claim 7: $N(h_3) \cap (P \cup Q) = \varnothing$.

Proof of Claim 7: Assume not. Let R be a shortest path from h_1 to h_3 in the graph induced by $(P \cup Q) \setminus q_1$. Then $R \cup (H \setminus h_2) \cup x$ induces an even wheel with center x. This completes the proof of Claim 7.

If $k > 1$ then the graph induced by $H \cup Q \cup p_k$ contains a $3PC(h_2, h_i)$. So $k = 1$. By symmetry and Claim 5, h_2 does not have a neighbor in Q, and hence $P \cup Q \cup H$ induces a $3PC(\Delta, \cdot)$. □

Lemma 3.2.3 *There exists a node of H that has no neighbor in C_1.*

Proof: Let $H = h_1, ..., h_n, h_1$ and suppose that every node of H has a neighbor in C_1. Recall that since (H, x) cannot be an even wheel, H is of odd length. So H contains a T^3 all of whose nodes have neighbors in C_1. By Lemma 3.2.2, C_1 contains a path $P = p_1, ..., p_k$, $k > 1$, such that $P \cup H$ induces w.l.o.g. a $3PC(h_1 h_2 p_k, h_i)$, $3 < i < n$. If i is odd, then $\{x, h_2, ..., h_i\} \cup P$ induces an even wheel with center x. So i is even.

Let $Q = q_1, ..., q_l$ be a path in C_1 defined as follows: q_1 is adjacent to $h_j \in H \setminus \{h_1, h_2, h_i\}$ where j is odd, q_l is adjacent to a node of P and no proper subpath of Q has this property. We may assume that P and Q are chosen so that $|P \cup Q|$ is minimized.

By the choice of P and Q, $N(q_l) \cap P$ is either one single vertex or two adjacent vertices of P, and h_j has no neighbor in $Q \setminus q_1$. Note that since n is odd, the two subpaths of H, $h_2, \ldots, h_i$ and $h_i, \ldots, h_n, h_1$ are both of even length, so we may assume w.l.o.g. that

$2 < j < i$.

Claim 1: At least one node of $\{h_2,\dots,h_{j-1}\}$ (resp. $\{h_{j+1},\dots,h_n\}$) has a neighbor in Q.

Proof of Claim 1: First suppose that no node of $H \setminus \{h_1,h_j\}$ has a neighbor in Q. Let p_s be the node of P with highest index adjacent to q_l. If $j > 3$, then $\{x,h_2,\dots,h_j,p_s,\dots,p_k\} \cup Q$ induces an even wheel with center x. So $j = 3$. If $N(h_1) \cap Q = \varnothing$ then $\{x,h_1,h_2,h_3,p_s,\dots,p_k\} \cup Q$ induces an even wheel with center h_2. So $N(h_1) \cap Q \neq \varnothing$. Let q_r be the node of Q with lowest index adjacent to h_1. Then $(H \setminus h_2) \cup \{x,q_1,\dots,q_r\}$ induces an even wheel with center x. So at least one node of $H \setminus \{h_1,h_j\}$ has a neighbor in Q.

Next suppose that no node of $\{h_2,\dots,h_{j-1}\}$ has a neighbor in Q. Let p_s be the node of P with highest index adjacent to q_l. If $j > 3$ then $\{x,h_2,\dots,h_j,p_s,\dots,p_k\} \cup Q$ induces an even wheel with center x. So $j = 3$. Let $h_{j'}$ be the node of $\{h_{j+1},\dots,h_n\}$ with lowest index adjacent to a node of Q. By definition of Q and Lemma 3.2.1, j' is even. Let q_r be the node of Q with lowest index adjacent to $h_{j'}$. If $j' > 4$ then $\{x,h_j,\dots,h_{j'},q_1,\dots,q_r\}$ induces an even wheel with center x. So $j' = 4$. If $N(h_1) \cap Q = \varnothing$ then $\{x,h_1,h_2,h_3,p_s,\dots,p_k\} \cup Q$ induces an even wheel with center h_2. So $N(h_1) \cap Q \neq \varnothing$. In fact, by Lemma 3.2.1, $N(h_1) \cap (Q \setminus q_1) \neq \varnothing$. Suppose $N(h_4) \cap Q \neq q_1$. Let R be a shortest path from h_4 to h_1 in the graph induced by $(Q \setminus q_1) \cup \{h_1,h_4\}$. Then $\{x,h_1,\dots,h_4\} \cup R$ induces an even wheel with center x. So $N(h_4) \cap Q = q_1$. Suppose $N(q_l) \cap P \neq p_1$ or $i > 4$. Then $\{x,h_2,h_3,h_4,p_s,\dots,p_k\} \cup Q$ induces an even wheel with center h_3. So $N(q_l) \cap P = p_1$ and $i = 4$. Let R be a shortest path from p_1 to h_1 in the graph induced by $Q \cup \{p_1,h_1\}$. Then $P \cup R \cup \{h_1,h_4,x\}$ induces a $3PC(p_1,h_1)$. Therefore at least one node of $\{h_2,\dots,h_{j-1}\}$ has a neighbor in Q.

Finally suppose that no node of $\{h_{j+1},\dots,h_n\}$ has a neighbor in Q. Let $h_{j'}$ be a node of $h_2,\dots,h_{j-1}$ such that $N(h_{j'}) \cap Q \neq \varnothing$ and the path from $h_{j'}$ to h_i in the graph induced by $P \cup Q \cup \{h_i,h_{j'}\}$ is minimized. By definition of Q and Lemma 3.2.1, j' is even. Suppose $N(h_1) \cap Q \neq \varnothing$. Let R be a shortest path from h_j to h_1 in the graph induced by $Q \cup \{h_1,h_j\}$. Then $(H \setminus \{h_2,\dots,h_{j-1}\}) \cup R \cup x$ induces an even wheel with center x. So $N(h_1) \cap Q = \varnothing$. Suppose $N(q_l) \cap P \neq p_k$. Let R be a shortest path from h_i to $h_{j'}$ in the graph induced by $P \cup Q \cup \{h_i,h_{j'}\}$. Note that by definition of Q and $h_{j'}$ and by Lemma 3.2.1, no node of $\{h_2,\dots,h_{j'-1}\}$ has a neighbor in R. Then $(H \setminus \{h_{j'+1},\dots,h_{i-1}\}) \cup R \cup x$ induces an even wheel with center x. So $N(q_l) \cap P = p_k$. But then $(H \setminus \{h_2,\dots,h_{j-1}\}) \cup P \cup Q$ induces a $3PC(p_k,h_i)$. This completes the proof of Claim 1.

By Claim 1 at least two nodes, say $h_{j'}$ and $h_{j''}$, of $H \setminus \{h_1, h_j\}$ have a neighbor in Q. Note that by definition of Q and Lemma 3.2.1, j' and j'' are both even. W.l.o.g. $j' < j < j''$. Let $R = r_1, \ldots, r_t$ be a shortest path in the graph induced by Q where $N(h_{j'}) \cap R = r_1$ and $N(h_{j''}) \cap R = r_t$. W.l.o.g and by Lemma 3.2.1 no other node from $H \setminus \{h_1, h_j\}$ has a neighbor in R.

If $N(h_1) \cap R = \emptyset$, then $(H \setminus \{h_{j'+1}, \ldots, h_{j''-1}\}) \cup R \cup x$ induces an even wheel with center x. So $N(h_1) \cap R \neq \emptyset$. Suppose $j' \neq 2$. Let R' be a shortest path from h_1 to $h_{j'}$ in the graph induced by $R \cup \{h_1, h_{j'}\}$. Then $\{x, h_1, \ldots, h'_j\} \cup R'$ induces an even wheel with center x. Therefore $j' = 2$.

Suppose that $N(h_1) \cap R = r_1$. Then by Lemma 3.2.1, $N(r_1) \cap H = \{h_1, h_2\}$. If $r_t = q_1$, then by Lemma 3.2.1, $N(r_t) \cap H = \{h_j, h_{j+1}\}$, and hence $H \cup R$ induces a $3PC(h_1 h_2 r_1, h_{j+1} h_j r_t)$. So $r_t \neq q_1$, and hence $N(r_t) \cap H = \{h_{j''}\}$. Therefore $H \cup R$ induces a $3PC(h_1 h_2 r_1, h_{j''})$. Let R' be a shortest path from q_1 to a node of R in the graph induced by Q. Since $|R \cup R'| < |P \cup Q|$, the choice of P and Q is contradicted.

So $N(h_1) \cap (R \setminus r_1) \neq \emptyset$. Let r_s be the node of R with highest index adjacent to h_1. If h_j has no neighbor in $r_s, \ldots, r_t$, then $\{x, h_1, \ldots, h_{j''}, r_s, \ldots, r_t\}$ induces an even wheel with center x. So h_j does have a neighbor in $r_s, \ldots, r_t$, i.e. $r_t = q_1$. By Lemma 3.2.1, $N(r_t) \cap H = \{h_j, h_{j''}\}$, where $j'' = j + 1$. Note that $i \geq j + 1$ and $r_s \neq q_l$. But then $(H \setminus \{h_2, \ldots, h_j\}) \cup P \cup \{r_s, \ldots, r_t\}$ induces a $3PC(h_1, h_i)$. $\square$

Note that the above lemma does not work if we allow 4-holes. Consider the odd-signable graph in Figure 3.2 (one can see that this graph is odd-signable by assigning 0 to the three bold edges and 1 to all the other edges). Let H be the 5-hole induced by the neighborhood of node x. Then every node of H has a neighbor in the unique connected component obtained by removing $N(x) \cup x$.

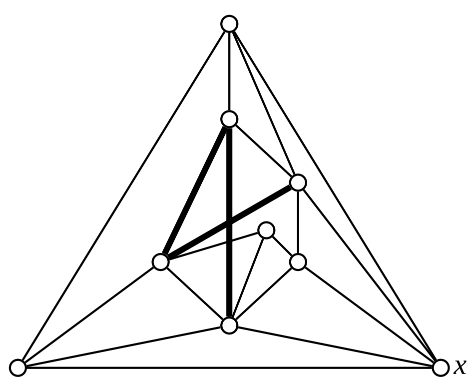


Figure 3.2: An odd-signable graph for which Lemma 3.2.3 does not work.

A class $\mathscr{F}$ of graphs satisfies *property (*) w.r.t. a graph G* if the following holds: for every node x of G such that $G \setminus N[x] \neq \emptyset$, and for every connected component C of $G \setminus N[x]$, if $F \in \mathscr{F}$ is contained in $G[N(x)]$, then there exists a node of F that has no neighbor in C.

The following theorem is proved in [29]. For completeness we include its proof here.

Theorem 3.2.4 (Maffray, Trotignon and Vušković [29]) *Let $\mathscr{F}$ be a class of graphs such that for every $F \in \mathscr{F}$, no node of F is adjacent to all the other nodes of F. If $\mathscr{F}$ satisfies property (*) w.r.t. a graph G, then G has a node whose neighborhood is $\mathscr{F}$-free.*

Proof: Let $\mathscr{F}$ be a class of graphs such that for every $F \in \mathscr{F}$, no node of F is adjacent to all the other nodes of F. Assume that $\mathscr{F}$ satisfies property (*) w.r.t. G, and suppose that for every $x \in V(G)$, $G[N(x)]$ is not $\mathscr{F}$-free. Then G is not a clique (since every graph of $\mathscr{F}$ contains nonadjacent nodes) and hence it contains a node x that is not adjacent to all other nodes of G. Let $C_1, \ldots, C_k$ be the connected components of $G \setminus N[x]$, with $|C_1| \geq \ldots \geq |C_k|$. Choose x so that for every $y \in V(G)$ the following holds: if $C_1^y, \ldots, C_l^y$ are the connected components of $G \setminus N[y]$ with $|C_1^y| \geq \ldots \geq |C_l^y|$, then

- $|C_1| > |C_1^y|$, or
- $|C_1| = |C_1^y|$ and $|C_2| > |C_2^y|$, or
- $\ldots$
- $|C_1| = |C_1^y|, \ldots, |C_{k-1}| = |C_{k-1}^y|$ and $|C_k| > |C_k^y|$, or
- for $i = 1, \ldots, k$, $|C_i| = |C_i^y|$ and $k = l$.

Let $N = N(x)$ and $C = C_1 \cup \ldots \cup C_k$. For $i = 1, \ldots, k$, let N_i be the set of nodes of N that have a neighbor in C_i.

Claim 1: $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_k$ and for every $i = 1, \ldots, k-1$, every node of $(N \setminus N_i) \cup (C_{i+1} \cup \ldots \cup C_k)$ is adjacent to every node of N_i.

Proof of Claim 1: We argue by induction. First we show that every node of $(N \setminus N_1) \cup (C_2 \cup \ldots \cup C_k)$ is adjacent to every node of N_1. Assume not and let $y \in (N \setminus N_1) \cup (C_2 \cup \ldots \cup C_k)$ be such that it is not adjacent to $z \in N_1$. Clearly y has no neighbor in C_1, but z does. So $G \setminus N[y]$ contains a connected component that contains $C_1 \cup z$, contradicting the choice of x.

Now let $i > 1$ and assume that $N_1 \subseteq \ldots \subseteq N_{i-1}$ and every node of $(N \setminus N_{i-1}) \cup (C_i \cup \ldots \cup C_k)$ is adjacent to every node of N_{i-1}. Since every node of C_i is adjacent to every node of N_{i-1}, it follows that $N_{i-1} \subseteq N_i$. Suppose that there exists a node $y \in (N \setminus N_i) \cup (C_{i+1} \cup \ldots \cup C_k)$ that is not adjacent to a node $z \in N_i$. Then $z \in N_i \setminus N_{i-1}$ and z has a neighbor in C_i. Also y is adjacent to all nodes in N_{i-1} and no node of $C_1 \cup \ldots \cup C_i$. So there exist connected components of $G \setminus N[y], C_1^y, \ldots, C_l^y$ such that $C_1 = C_1^y, \ldots, C_{i-1} = C_{i-1}^y$ and $C_i \cup z$ is contained in C_i^y. This contradicts the choice of x. This completes the proof of Claim 1.

Since $G[N]$ is not $\mathscr{F}$-free, it contains $F \in \mathscr{F}$. By property (*), a node y of F has no neighbor in C_k. By Claim 1, y is adjacent to every node of N_k, and no node of $N \setminus N_k$ has a neighbor in C. So (since every node of F has a non-neighbor in F) F must contain another node $z \in N \setminus N_k$, nonadjacent to y. But then $C_1, \ldots, C_k$ are connected components of $G \setminus N[y]$ and z is contained in $(G \setminus N[y]) \setminus C$, so y contradicts the choice of x. $\square$

Proof of Theorem 3.0.7: Let G be a 4-hole-free odd-signable graph. Let $\mathscr{F}$ be the set of all holes. By Lemma 3.2.3, $\mathscr{F}$ satisfies property (*) w.r.t. G. So by Theorem 3.2.4, G has a node whose neighborhood is $\mathscr{F}$-free, i.e. triangulated. $\square$

3.3 Some consequences

In a graph G, for any node x, let $C_1, \ldots, C_k$ be the connected components of $G \setminus N[x]$, with $|C_1| \geq \ldots \geq |C_k|$, and let the numerical vector $(|C_1|, \ldots, |C_k|)$ be associated with x. The nodes of G can thus be ordered according to the lexicographic ordering of the numerical vectors associated with them. Say that a node x is *lex-maximal* if the associated numerical vector is lexicographically maximal over all nodes of G. Theorem 3.2.4 actually shows that for a lex-maximal node x, $N(x)$ is $\mathscr{F}$-free. This implies the following.

Theorem 3.3.1 *Let G be a 4-hole-free odd-signable graph, and let x be a lex-maximal node of G. Then the neighborhood of x is triangulated.*

Possibly a more efficient algorithm for listing all maximal cliques can be constructed by searching for a lex-maximal node.

Lemma 3.2.3 also proves the following decomposition theorem. (H, x) is a *universal wheel* if x is adjacent to all the nodes of H.

Theorem 3.3.2 *Let G be a 4-hole-free odd-signable graph. If G contains a universal wheel, then G has a star cutset.*

Proof: Let (H, x) be a universal wheel of G. If $G = N[x]$, then for any two nonadjacent nodes a and b of H, $N[x] \setminus \{a, b\}$ is a star cutset of G. So assume $G \setminus N[x]$ contains a connected component C_1. By Lemma 3.2.3, a node $a \in H$ has no neighbor in C_1. But then $N[x] \setminus a$ is a star cutset of G that separates a from C_1. $\square$

Chapter 4

Star cutsets

In this Chapter and in the next one we prove the main decomposition theorem in this thesis.

4.1 Appendices to a hole

In this section we assume that G is a 4-hole-free odd-signable graph.

Let H be a hole. A chordless path $P = p_1, \dots, p_k$ in $G \setminus H$ is an *appendix* of H if no node of $P \setminus \{p_1, p_k\}$ has a neighbor in H, and one of the following holds:

(i) $k = 1$ and (H, p_1) is a bug ($N(p_1) \cap V(H) = \{u_1, u_2, u\}$, such that u_1u_2 is an edge), or

(ii) $k > 1$, p_1 has exactly two neighbors u_1 and u_2 in H, u_1u_2 is an edge, p_k has a single neighbor u in H, and $u \notin \{u_1, u_2\}$.

Nodes u_1, u_2, u are called the *attachments* of appendix P to H. We say that u_1u_2 is the *edge-attachment* and u is the *node-attachment*.

Let H'_P (resp. H''_P) be the u_1u-subpath (resp. u_2u-subpath) of H that does not contain u_2 (resp. u_1). H'_P and H''_P are called the *sectors* of H w.r.t. P.

Let Q be another appendix of H, with edge attachment v_1v_2 and node-attachment v. Appendices P and Q are said to be *crossing* if one sector of H w.r.t. P contains v_1 and v_2, say H'_P does, and $v \in V(H''_P) \setminus \{u\}$.

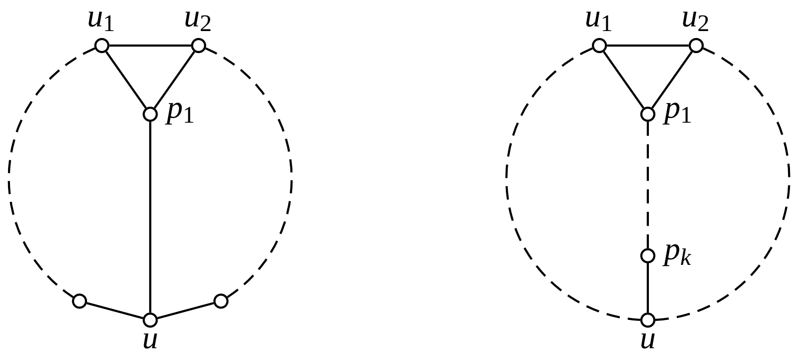


Figure 4.1: An appendix $P = p_1,...,p_k$ of a hole H, with edge-attachment u_1u_2 and node-attachment u.

Lemma 4.1.1 *Let $P = p_1,...,p_k$ be an appendix of a hole H, with edge-attachment u_1u_2 and node-attachment u, where p_1 is adjacent to u_1 and u_2. Let H'_P (resp. H''_P) be the sector of H w.r.t. P that contains u_1 (resp. u_2). Let $Q = q_1,\dots,q_l$ be a chordless path in $G \setminus H$ such that q_1 has a neighbor in H'_P, q_l has a neighbor in H''_P, no node of $Q \setminus \{q_1,q_l\}$ is adjacent to a node of H and one of the following holds:*

(i) *$l = 1$, q_1 is not adjacent to u, and if u_1 (resp. u_2) is the unique neighbor of q_1 in H'_P (resp. H''_P), then q_1 is not adjacent to u_2 (resp. u_1) nor p_1.*

(ii) *$l > 1$, $N(q_1) \cap V(H) \subseteq V(H'_P) \setminus \{u\}$, $N(q_l) \cap V(H) \subseteq V(H''_P) \setminus \{u\}$, q_1 has a neighbor in $H'_P \setminus \{u_1\}$, and q_l has a neighbor in $H''_P \setminus \{u_2\}$.*

Then Q is also an appendix of H and its node-attachment is adjacent to u. Furthermore, no node of P is adjacent to or coincident with a node of Q.

Proof: Let u'_1 (resp. u'_2) be the neighbor of q_1 in H'_P that is closest to u (resp. u_1). Let u''_1 (resp. u''_2) be the neighbor of q_l in H''_P that is closest to u (resp. u_2). Note that either $u'_1 \neq u_1$ or $u''_1 \neq u_2$. Let S'_1 (resp. S'_2) be the u'_1u-subpath (resp. u'_2u_1-subpath) of H'_P, and let S''_1 (resp. S''_2) be the u''_1u-subpath (resp. u''_2u_2-subpath) of H''_P. Let H' (resp. H'') be the hole induced by $H'_P \cup P$ (resp. $H''_P \cup P$).

First suppose that $l = 1$. Note that q_1 cannot be coincident with a node of P. Suppose q_1 has a neighbor in P. Note that q_1 is not adjacent to u, and if q_1 is adjacent to p_1, then $u'_1 \neq u_1$ and $u''_1 \neq u_2$. But then $P \cup S'_1 \cup S''_1 \cup q_1$ contains a $3PC(q_1,u)$. So q_1 has no neighbor in P. Since $H \cup q_1$ cannot induce a $3PC(u'_1,u''_1)$, q_1 has at least three neighbors in H. Since (H,q_1) cannot be an even wheel, w.l.o.g. q_1 has an odd number of neighbors in H'_P and an even number of neighbors in H''_P. Since $H'' \cup q_1$ cannot induce a $3PC(u''_1,u''_2)$ nor an even wheel with center q_1, $u''_1u''_2$ is an edge. Since $H'' \cup S'_2 \cup q_1$ cannot induce an even wheel with center u_2 nor a $3PC(p_1u_1u_2, q_1u_1''u_2'')$, u_2' is adjacent to u, and the lemma holds.

Now suppose that $l > 1$. So $u_1' \neq u_1$ and $u_1'' \neq u_2$. Not both q_1 and q_l can have a single neighbor in H, since otherwise $H \cup Q$ induces a $3PC(u_1', u_1'')$. W.l.o.g. $u_1'' \neq u_2''$.

Suppose that $u_1''u_2''$ is not an edge. A node of P must be adjacent to or coincident with a node of Q, else $H'' \cup Q \cup S_1'$ contains a $3PC(q_l, u)$. Note that no node of $\{q_1, q_l\}$ is coincident with a node of $\{p_1, p_k\}$, and if a node of Q is coincident with a node of P, then a node of Q is also adjacent to a node of P. Let q_i be the node of Q with highest index that has a neighbor in P. (Note that q_i is not coincident with a node of P). Let p_j be the node of P with highest index adjacent to q_i. If $j > 1$ and $i > 1$, then $H \cup \{p_j, \ldots, p_k, q_i, \ldots, q_l\}$ contains a $3PC(q_l, u)$. If $i = 1$, then $S_1' \cup S_1'' \cup Q \cup \{p_j, \ldots, p_k\}$ induces a $3PC(q_1, u)$. So $i > 1$, and hence $j = 1$. If $i < l$, then $S_1'' \cup S_2'' \cup P \cup \{q_i, \ldots, q_l\}$ induces a $3PC(p_1, q_l)$. So $i = l$. Since $H \cup q_l$ cannot induce a $3PC(u_1'', u_2'')$, (H, q_l) is a wheel. But then one of the wheels (H, q_l) or (H'', q_l) must be even. Therefore $u_1''u_2''$ is an edge.

Suppose that $u_1' \neq u_2'$. Then by symmetry, $u_1'u_2'$ is an edge, and hence $H \cup Q$ induces a $3PC(q_1u_1'u_2', q_lu_1''u_2'')$. Therefore $u_1' = u_2'$, i.e. Q is an appendix of H. Note that by definition of Q, $u_1' \notin \{u_1, u\}$.

Suppose that a node of P is adjacent to or coincident with a node of Q. Let q_i be the node of Q with highest index adjacent to a node of P, and let p_j be the node of P with lowest index adjacent to q_i. If $i > 1$ and $j < k$, then $H \cup \{p_1, \ldots, p_j, q_i, \ldots, q_l\}$ induces an even wheel with center u_2 or a $3PC(p_1u_1u_2, q_lu_1''u_2'')$. If $i = 1$, then $P \cup Q \cup S_1' \cup S_1''$ contains a $3PC(q_1, u)$. So $i > 1$, and hence $j = k$.

If p_k has a unique neighbor in Q, then $Q \cup S_1' \cup S_1'' \cup p_k$ induces a $3PC(q_i, u)$. So p_k has more than one neighbor in Q.

Suppose that $k = 1$. Then either $S_2' \cup S_2'' \cup Q \cup p_1$ or $S_1' \cup S_1'' \cup Q \cup p_1$ induces an even wheel with center p_1. So $k > 1$.

Let T' (resp. T'') be the hole induced by $S_1' \cup S_1'' \cup Q$ (resp. $S_2' \cup S_2'' \cup Q$). If both (T', p_k) and (T'', p_k) are wheels, then one of them is even. So p_k has exactly two neighbors in Q. Since $T'' \cup p_k$ cannot induce a $3PC(\cdot, \cdot)$, $N(p_k) \cap Q = \{q_i, q_{i-1}\}$. (Note that q_{i-1} is not coincident with a node of P, since $j = k$). If no node of $P \setminus p_k$ has a neighbor in Q, then $T'' \cup P$ induces a $3PC(p_1u_1u_2, p_kq_iq_{i-1})$. So a node of $P \setminus p_k$ has a neighbor in Q. Let p_t be such a node with lowest index. Let q_s be the node of Q with highest index adjacent to p_t. If $t \neq k-1$ then $H_P'' \cup \{p_1, \ldots, p_t, p_k, q_s, \ldots, q_l\}$ induces an even wheel with center q_l or a $3PC(q_lu_1''u_2'', p_kq_iq_{i-1})$. So $t = k-1$, i.e. p_k and p_{k-1} are the only nodes of P that have a neighbor in Q. If $s \neq 1$ then $(H \setminus S_2'') \cup P \cup \{q_s, \ldots, q_l\}$ induces an even wheel with center p_k. So $s = 1$. If $i > 2$, then $S_1' \cup \{q_1, \ldots, q_{i-1}, p_{k-1}, p_k\}$ induces a $3PC(q_1, p_k)$. So

$i=2$. Since there is no 4-hole, $u_1'u \notin E(G)$. But then $H \cup \{q_1, p_k\}$ induces a $3PC(u_1', u)$.

Therefore, no node of P is adjacent to or coincident with a node of Q. If $u_1'u$ is not an edge, then $(H \setminus S_2'') \cup P \cup Q$ induces a $3PC(u_1', u)$. Therefore $u_1'u$ is an edge. □

Lemma 4.1.2 *Let $P = p_1, \dots, p_k$ be an appendix of a hole H, with edge-attachment u_1u_2 and node-attachment u, with p_1 adjacent to u_1, u_2. Let $Q = q_1, \dots, q_l$ be another appendix of H, with edge-attachment v_1v_2 and node-attachment v, with q_1 adjacent to v_1, v_2. If P and Q are crossing, then one of the following holds:*

(i) uv is an edge,

(ii) $u \in \{v_1, v_2\}$ and q_1 has a neighbor in P, or

(iii) $v \in \{u_1, u_2\}$ and p_1 has a neighbor in Q.

Proof: Let H_P' (resp. H_P'') be the sector of H w.r.t. P that contains u_1 (resp. u_2). W.l.o.g. $\{v_1, v_2\} \subseteq H_P'$ and v_1 is the neighbor of q_1 in H_P' that is closer to u_1. Assume uv is not an edge.

By Lemma 4.1.1 either $v_2 = u$ or $u_2 = v$. W.l.o.g. assume that $v_2 = u$. Let S_1 (resp. S_2) be the uv-subpath (resp. u_2v-subpath) of H_P''. A node of P must be coincident with or adjacent to a node of Q, else $H_P' \cup S_2 \cup P \cup Q$ induces a $3PC(p_1u_1u_2, q_1v_1u)$ or an even wheel with center u_1. Note that no node of $\{q_1, q_l\}$ is coincident with a node of $\{p_1, p_k\}$. Let q_i be the node of Q with lowest index adjacent to P. (So q_i is not coincident with a node of P). Let p_j be the node of P with lowest index adjacent to q_i. If $i = 1$, then (ii) holds. So assume that $i > 1$.

If $j < k$ and $i < l$, then $H \cup \{p_1, \dots, p_j, q_1, \dots, q_i\}$ induces a $3PC(p_1u_1u_2, q_1v_1u)$ or an even wheel with center u_1. So either $j = k$ or $i = l$.

Suppose that $j = k$. If $N(p_k) \cap Q = q_i$, then $S_1 \cup Q \cup p_k$ induces a $3PC(u, q_i)$. So p_k has more than one neighbor in Q. Let T' (resp. T'') be the hole induced by $S_1 \cup Q$ (resp. $(H \setminus (S_1 \setminus v)) \cup Q$). Note that (T', p_k) is a wheel. If (T'', p_k) is also a wheel, then one of these two wheels must be even. So (T'', p_k) is not a wheel, and hence $k > 1$ and p_k has exactly two neighbors in Q. $N(p_k) \cap Q = \{q_i, q_{i+1}\}$, else $T'' \cup p_k$ induces a $3PC(\cdot, \cdot)$. But then $H_P' \cup S_2 \cup Q \cup p_k$ induces a $3PC(q_1v_1u, p_kq_iq_{i+1})$.

So $j < k$, and hence $i = l$. In particular, q_l is the only node of Q that has a neighbor in P. If either $j > 1$ or $v \neq u_2$, then $S_1 \cup Q \cup \{p_j, \dots, p_k\}$ contains a $3PC(u, q_l)$. So $j = 1$ and $v = u_2$, and hence (iii) holds. □

4.2 Proper wheels

A *bug* is a wheel with three sectors, exactly one of which is short. A *twin* wheel is a wheel with exactly two short sectors and one long sector. A *proper* wheel is a wheel that is neither a bug nor a twin wheel. A wheel (H,x) is a *universal* wheel, if x is adjacent to all nodes of H. See figure 4.2.

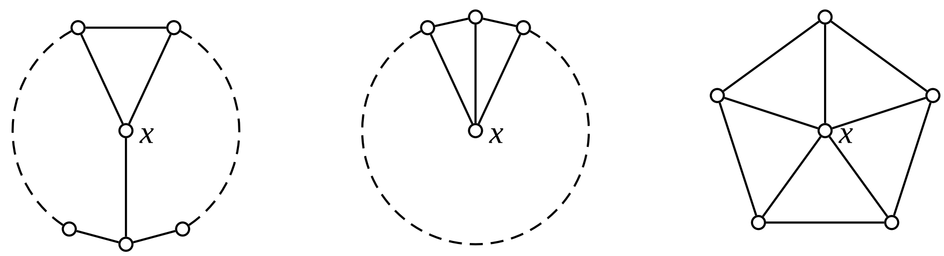


Figure 4.2: A bug, a twin wheel and a universal wheel with center x.

Theorem 4.2.1 *[1] Let G be a 4-hole-free odd-signable graph. If G contains a proper wheel that is not a universal wheel, then G has a star cutset.*

Theorem 4.2.1 was proved by us and in [1] independently and at the same time. Since [1] is about to be published, we do not include our proof of Theorem 4.2.1 here. We also note that in [1], the statement of Theorem 4.2.1 is for even-hole-free graphs, but since in their proof, to obtain the decomposition they only use the exclusion of 4-holes, even-wheels, $3PC(.,.)$'s and $3PC(\Delta,\Delta)$'s, they actually prove the above stated version.

Theorems 3.3.2 and 4.2.1 imply the following result.

Theorem 4.2.2 *Let G be a 4-hole-free odd-signable graph. If G contains a proper wheel, then G has a star cutset.*

4.3 Nodes adjacent to a $3PC(\Delta,\cdot)$ and crossings

Throughout this section Σ denotes a $3PC(x_1x_2x_3,y)$. The three paths of Σ are denoted by P_{x_1y}, P_{x_2y} and P_{x_3y} (where P_{x_iy} is the path that contains x_i). Note that at most one of the paths of Σ is of length 1. For $i=1,2,3$, we denote the neighbor of y in P_{x_iy} by y_i. Also let $X=\{x_1,x_2,x_3\}$.

Lemma 4.3.1 *Let G be a 4-hole-free odd-signable graph that does not contain a proper wheel. If $u \in V(G)\setminus V(\Sigma)$ has a neighbor in Σ, then u is one of the following types.*

pi for i=1,2,3	:	*For some path P of Σ, $N(u) \cap V(\Sigma) \subseteq P$ and $\lvert N(u) \cap V(\Sigma)\rvert = i$. Furthermore, if $i \geq 2$, then u has two adjacent neighbors in Σ.*
crosspath	:	*Node u has exactly three neighbors in Σ. For some $i \in \{1,2,3\}$, u is adjacent to y_i, and the other two neighbors of u in Σ are contained in $P_{x_j y}$, for some $j \in \{1,2,3\} \setminus \{i\}$. Furthermore, $V(P_{x_i y}) \cup V(P_{x_j y}) \cup \{u\}$ induces a bug with center u.*
t2	:	$N(u) \cap V(\Sigma) \subseteq X$ *and* $\lvert N(u) \cap V(\Sigma)\rvert = 2$.
t3	:	$N(u) \cap V(\Sigma) = X$.
d	:	*For some $i, j \in \{1,2,3\}$, $i \neq j$, $N(u) \cap V(\Sigma) = \{y, y_i, y_j\}$.*
pseudo-twin of a node of X	:	*We define a pseudo-twin of x_1: $N(u) \cap V(\Sigma) = \{x_2, x_3, v_1, v_2\}$, where v_1 and v_2 are nodes of $P_{x_1 y}$. Furthermore, if $\{x_1, y\} = \{v_1, v_2\}$ then $x_2 y$ and $x_3 y$ are not edges. Also if $x_1 \notin \{v_1, v_2\}$ then $v_1 v_2$ is an edge, and either $y \notin \{v_1, v_2\}$ or $x_2 y$ and $x_3 y$ are not edges. Pseudo-twins of x_2 and x_3 are defined symmetrically.*
pseudo-twin of y	:	*$N(u) \cap V(\Sigma) = \{y, v_1, v_2, v_3\}$, where for $i = 1,2,3$ v_i is a node of $P_{x_i y} \setminus \{y\}$, at least two of yv_1, yv_2, yv_3 are edges, and $\lvert N(u) \cap X\rvert \leq 1$.*
s1	:	*Σ is a bug, where say $x_i y$ is an edge. Node u is adjacent to x_i, and for some $j \in \{1,2,3\} \setminus \{i\}$, the nodes of $N(u) \cap (V(\Sigma) \setminus \{x_i\})$ are contained in $P_{x_j y} \setminus \{y\}$. Furthermore, $V(P_{x_i y}) \cup V(P_{x_j y}) \cup \{u\}$ induces a twin wheel.*
s2	:	*For distinct $i, j, k \in \{1,2,3\}$, Σ is a bug such that $x_i y$ is an edge, and $N(u) \cap V(\Sigma) = \{x_i, x_j, y, y_k\}$.*

Proof: For $i, j \in \{1,2,3\}$, $i \neq j$, let H_{ij} be the hole induced by $P_{x_i y} \cup P_{x_j y}$. We now consider the following three cases.

Case 1: $|N(u)\cap X|\leq 1$.

If for some $i\in\{1,2,3\}$, $N(u)\cap\Sigma\subseteq P_{x_iy}$, then u is of type p1, p2 or p3, else there is a $3PC(\cdot,\cdot)$ or a proper wheel. So assume w.l.o.g that u has neighbors in both $P_{x_1y}\setminus y$ and $P_{x_2y}\setminus y$, and that it is not adjacent to x_3.

Suppose u is not adjacent to y. Note that P_{x_3y} is an appendix of H_{12}. By Lemma 4.1.1 applied to H_{12}, P_{x_3y} and u, node u is also an appendix of H_{12} and its node-attachment is w.l.o.g. y_1. Furthermore, no node of P_{x_3y} is adjacent to u, and hence u is a crosspath of Σ.

Now assume that u is adjacent to y. Then (H_{12},u) must be a bug or a twin wheel. Suppose (H_{12},u) is a twin wheel. If u has no neighbor in $P_{x_3y}\setminus y$, then u is of type d. So assume u has a neighbor in $P_{x_3y}\setminus y$. Then (H_{23},u) is either a bug or a twin wheel, and hence u is a pseudo-twin of y w.r.t. Σ. Suppose now that (H_{12},u) is a bug. W.l.o.g $N(u)\cap P_{x_1y}=\{y,y_1\}$ and $N(u)\cap P_{x_2y}=\{y,u_1\}$, where yu_1 is not an edge. If u has no neighbor in $P_{x_3y}\setminus y$, then $H_{23}\cup u$ induces a $3PC(y,u_1)$. So u has a neighbor in $P_{x_3y}\setminus y$. If $N(u)\cap P_{x_3y}\neq\{y,y_3\}$, then (H_{23},u) is a proper wheel. So $N(u)\cap P_{x_3y}=\{y,y_3\}$, and hence u is a pseudo-twin of y w.r.t. Σ.

Case 2: $|N(u)\cap X|=2$.

W.l.o.g. $N(u)\cap X=\{x_1,x_2\}$. Assume u is not of type t2. Then u has a neighbor in $\Sigma\setminus X$. First suppose that u does not have a neighbor in $H_{12}\setminus\{x_1,x_2\}$. Then u has a neighbor in $P_{x_3y}\setminus\{x_3,y\}$. Since $H_{13}\cup u$ cannot induce a $3PC(\cdot,\cdot)$, u has at least two neighbors in $P_{x_3y}\setminus\{x_3,y\}$. Then (H_{13},u) is a wheel, and hence it must be a bug, and so u is a pseudo-twin of x_3 w.r.t. Σ.

Now we may assume that u has a neighbor in $H_{12}\setminus\{x_1,x_2\}$. Then (H_{12},u) is a twin wheel or a bug. In particular, $N(u)\cap H_{12}=\{x_1,x_2,u_1\}$. W.l.o.g. assume that $u_1\in P_{x_1y}\setminus x_1$. Suppose $u_1\neq y$. Then u cannot have a neighbor in P_{x_3y}, since otherwise $(\Sigma\setminus\{x_1,x_3\})\cup u$ contains a $3PC(u,y)$. If x_2y is not an edge, then $(\Sigma\setminus x_1)\cup u$ contains a $3PC(x_2,y)$. So x_2y is an edge. If x_1u_1 is not an edge, then $H_{13}\cup u$ induces a $3PC(x_1,u_1)$. So x_1u_1 is an edge, and hence u is of type s1.

We may now assume that $u_1=y$. Note that at least one of x_1y or x_2y is not an edge. W.l.o.g. x_2y is not an edge. Node u must have a neighbor in $P_{x_3y}\setminus y$, else $H_{23}\cup u$ induces a $3PC(x_2,y)$. So (H_{23},u) is a wheel, and hence it must be a bug. In particular, $N(u)\cap P_{x_3y}=\{y,y_3\}$, and so u is of type s2 or it is a pseudo-twin of x_3 w.r.t. Σ.

Case 3: $N(u)\cap X=X$.

Assume u is not of type t3. Then u has a neighbor u_1 in w.l.o.g. $P_{x_1y}\setminus x_1$. So (H_{12},u) is a twin wheel or a bug. Similarly, (H_{13},u) is a twin wheel or a bug. So $N(u)\cap V(\Sigma)=\{x_1,x_2,x_3,u_1\}$. If $u_1\neq y$ or x_2y and x_3y are not edges, then u is a pseudo-twin of x_1 w.r.t.

Σ. So assume that $u_1 = y$ and w.l.o.g. x_2y is an edge. Then u is a pseudo-twin of x_2 w.r.t. Σ. $\square$

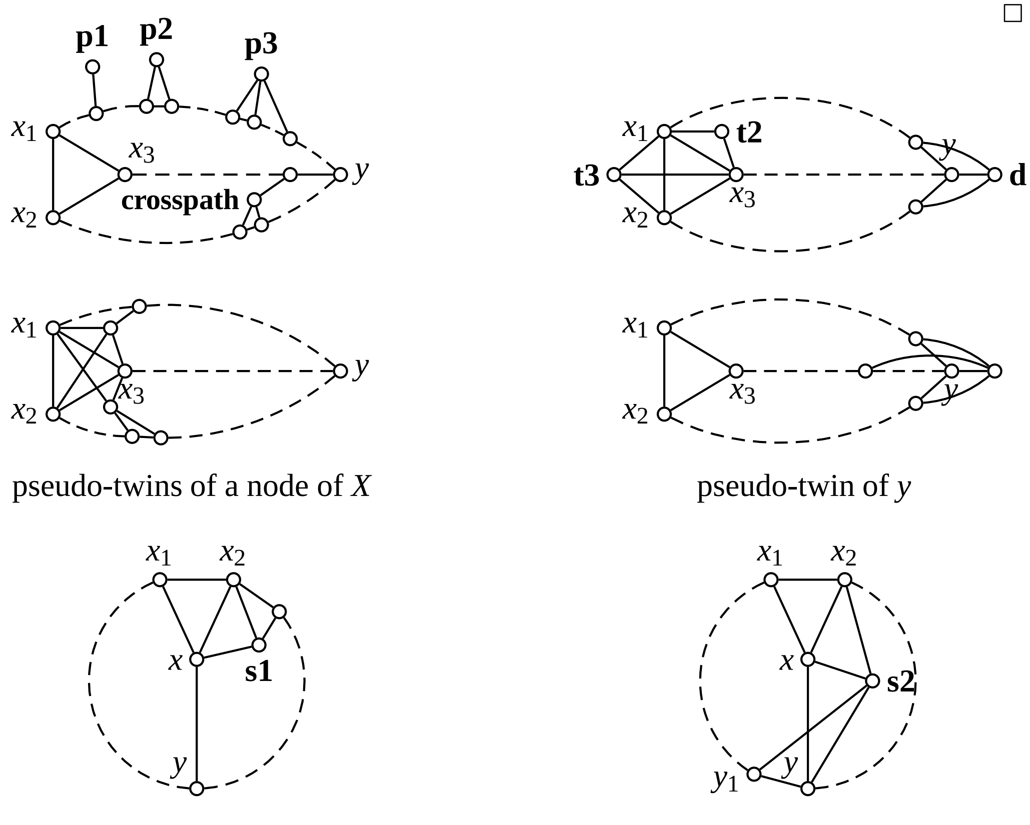


Figure 4.3: Different types of nodes adjacent to a $3PC(x_1x_2x_3,y)$.

Remark 4.3.2 *If a node u is a pseudo-twin of a node of* X, *say* x_1, *w.r.t. a* $\Sigma = 3PC(x_1x_2x_3,y)$, *then* $(\Sigma \setminus \{x_1\}) \cup \{u\}$ *contains a* $\Sigma' = 3PC(ux_2x_3,y)$. *If a node u is a pseudo-twin of y w.r.t.* Σ, *then* $(\Sigma \setminus \{y\}) \cup \{u\}$ *contains a* $\Sigma' = 3PC(x_1x_2x_3,u)$. *If a node u is of type p3 w.r.t.* Σ, *then* $\Sigma \cup \{u\}$ *contains a* $\Sigma' = 3PC(x_1x_2x_3,y)$ *that contains u. We say that in all these cases* Σ' is obtained by substituting u into Σ.

A node u adjacent to Σ is further classified as follows.

Type p : Node u is of type p1, p2 or p3 w.r.t. Σ.

Type p3t : Node u is of type p3 w.r.t. Σ and $N(u) \cap V(\Sigma)$ induces a path of length 2.

Type p3b : Node u is of type p3 w.r.t. Σ and $N(u) \cap V(\Sigma)$ does not induce a path of length 2.

Type dd : Node u is of type d w.r.t. Σ such that if Σ is a bug, then u is not adjacent to its center.

Type dc : Node u is of type d w.r.t. Σ, where Σ is a bug and u is adjacent to its center.

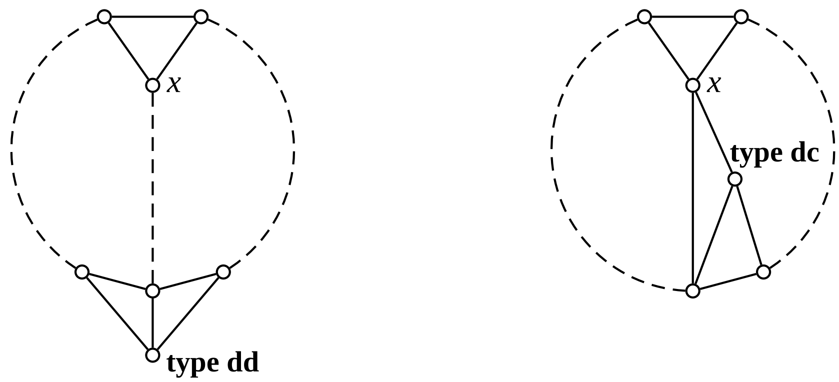


Figure 4.4: Different versions of a type d node w.r.t a $3PC(\Delta,\cdot)$.

A *crossing* of Σ is a chordless path $P = p_1,\dots,p_k$ in $G\setminus\Sigma$ such that either $k=1$ and p_1 is a crosspath w.r.t. Σ; or $k=1$, Σ is a bug and p_1 is of type s1 w.r.t. Σ; or $k>1$ and for some $i,j\in\{1,2,3\}$, $i\neq j$, $N(p_1)\cap V(\Sigma)\subseteq V(P_{x_iy})$, $N(p_k)\cap V(\Sigma)\subseteq V(P_{x_jy})$, p_1 has a neighbor in $V(P_{x_iy})\setminus\{y\}$, p_k has a neighbor in $V(P_{x_jy})\setminus\{y\}$, and no node of $P\setminus\{p_1,p_k\}$ has a neighbor in Σ.

We now define three special types of crossings.

A crossing $P=p_1,\dots,p_k$ of Σ is called a *hat* if $k>1$, p_1 and p_k are both of type p1 w.r.t. Σ adjacent to different nodes of $\{x_1,x_2,x_3\}$.

Let $P=p_1,\dots,p_k$ be a crossing of Σ such that one of the following holds:

(i) $k=1$ and p_1 is a crosspath w.r.t. Σ, say p_1 is adjacent to y_i for some $i\in\{1,2,3\}$, and it has two more neighbors in $P_{x_jy}\setminus\{y\}$, for some $j\in\{1,2,3\}\setminus\{i\}$.

(ii) $k=1$, Σ is a bug and p_1 is of type s1 w.r.t. Σ, such that for some $i\in\{1,2,3\}$ and for some $j\in\{1,2,3\}\setminus\{i\}$, x_iy is an edge and $N(p_1)\cap\{x_1,x_2,x_3\}=\{x_i,x_j\}$.

(iii) $k>1$, p_1 is of type p1 and p_k is of type p2 w.r.t. Σ, for some $i\in\{1,2,3\}$, p_1 is adjacent to y_i, and for some $j\in\{1,2,3\}\setminus\{i\}$, $N(p_k)\cap V(\Sigma)\subseteq V(P_{x_jy})\setminus\{y\}$.

Such a path P is called a *y_i-crosspath* of Σ. We also say that P is a *crosspath* from y_i to $P_{x_j y}$. If say $x_3 y$ is an edge, then Σ induces a bug (H,x), where $x = x_3 = y_3$. In this case, the y_3-crosspath (or x-crosspath) of Σ, is also called the *center-crosspath* of the bug (H,x).

Suppose that Σ is a bug. A crossing P of Σ is an *ear* if $k > 1$, p_1 is of type p1 w.r.t. Σ adjacent to the center of bug Σ, and p_k is of type p2 w.r.t. Σ adjacent to y.

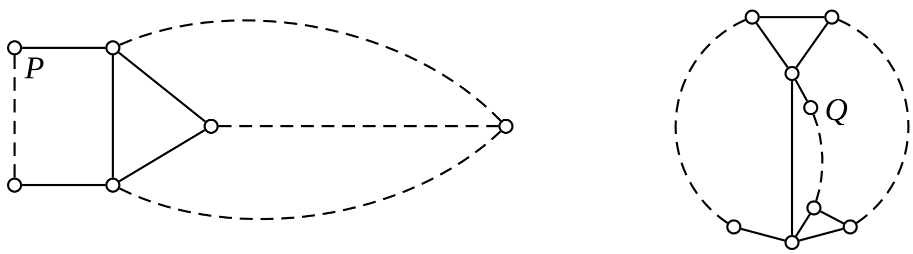


Figure 4.5: A hat P and an ear Q of a $3PC(\Delta,\cdot)$.

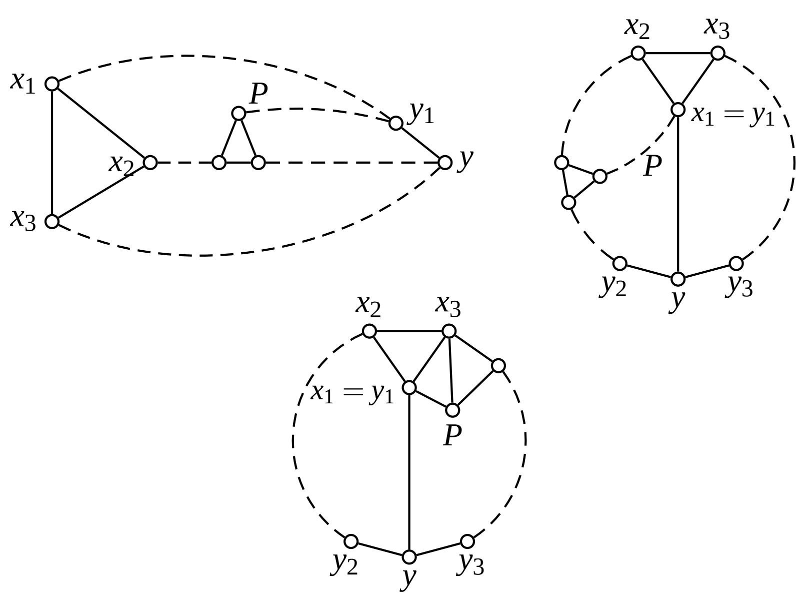


Figure 4.6: A y_1-crosspath P of a $3PC(x_1x_2x_3,y)$. When $x_1 = y_1$, P is also a center-crosspath of a bug.

We next prove the following sequence of decompositions. The order in which these decompositions are obtained is of crucial importance.

Theorem 4.3.3 *Let G be a 4-hole-free odd-signable graph. If G contains a bug with a center-crosspath then G has a star cutset. In particular, if G has no star cutset, then no* *node is of type s1 w.r.t. a* $3PC(\Delta,\cdot)$.

Theorem 4.3.4 *Let G be a 4-hole-free odd-signable graph. If G contains a* $3PC(\Delta,\cdot)$ *with a hat, then G has a star cutset.*

Theorem 4.3.5 *Let G be a 4-hole-free odd-signable graph. If G contains a bug with an ear, then G has a star cutset.*

Theorem 4.3.6 *Let G be a 4-hole-free odd-signable graph. If G contains a bug with a type s2 node, then G has a star cutset.*

We prove Theorems 4.3.3, 4.3.5 and 4.3.6 in Section 4.4. We close this section by proving Theorem 4.3.4. (assuming Theorem 4.3.3 to be true). But first we provea useful lemma about crosspaths.

Lemma 4.3.7 *Let G be a 4-hole-free odd-signable graph that does not contain a proper wheel.* $\Sigma = 3PC(x_1x_2x_3,y)$ *of G can have a crosspath from at most one of the nodes* y_1,y_2,y_3.

Proof: Suppose not and let $P = u_1,\dots,u_n$ be a y_1-crosspath and $Q = v_1,\dots,v_m$ a y_2-crosspath. Let u',u'' (resp. v',v'') be adjacent neighbors of u_n (resp. v_m) in Σ. Note that by definition of a crosspath, y does not coincide with any of the nodes u',u'',v',v''. It suffices to consider the following three cases.

Case 1: $u',u'' \in P_{x_2y}$ and $v',v'' \in P_{x_1y}$.

Note that in this case neither x_1y nor x_2y can be an edge and hence neither u_1 nor v_1 can be of type s1 w.r.t Σ. Let H be the hole induced by $P_{x_1y} \cup P_{x_2y}$. Then P and Q are crossing appendices of H and their node-attachments are not adjacent. So by Lemma 4.1.2, w.l.o.g. $y_1 \in \{v',v''\}$ and v_m has a neighbor in P.

W.l.o.g. u' is the neighbor of u_n in P_{x_2y} that is closer to x_2. Let R' (resp. R'') be the subpath of P_{x_2y} with endnodes u' (resp. u'') and x_2 (resp. y). Since there is no 4-hole, $m > 1$. Node v_m has a unique neighbor in P, else $(P_{x_1y} \setminus y) \cup P \cup R' \cup v_m$ induces a proper wheel with center v_m. The neighbor of v_m in P is u_1, else $P \cup R'' \cup \{y_1, v_m\}$ induces a $3PC(y_1,\cdot)$. But then $P_{x_1y} \cup P_{x_3y} \cup R'' \cup P \cup v_m$ induces an even wheel with center y_1.

Case 2: $u',u'' \in P_{x_3y}$ and $v',v'' \in P_{x_3y}$.

Note that x_3y is not an edge, and at most one of x_1y, x_2y is an edge. Suppose there exists a path from y_1 to y_2 in $P \cup Q \cup (P_{x_3y} \setminus \{x_3,y_3,y\}) \cup \{y_1,y_2\}$, and let R be a shortest such path. Then $P_{x_1y} \cup P_{x_2y} \cup R$ induces a $3PC(y_1,y_2)$. So no such path exists. In particular, no node of P is adjacent or coincident with a node of Q, and x_3y_3 is an edge. In particular, since there is no 4-hole, Σ cannot be a bug. But then $(\Sigma \cup P \cup Q) \setminus y$ induces a proper wheel with center x_3.

Case 3: $u', u'' \in P_{x_3y}$ and $v', v'' \in P_{x_1y}$.

Note that x_1y is not an edge and hence u_1 is not of type s1 w.r.t. Σ. Let H be the hole induced by $P_{x_1y} \cup P_{x_2y}$. Let P' be the shortest path between y_1 and x_3 in $P \cup (P_{x_3y} \setminus y) \cup y_1$. Suppose that v_1 is of type s1 w.r.t. Σ. Then x_2y is an edge. If v_1 has no neighbor in P, then $P' \cup (P_{x_1y} \setminus y) \cup \{x_2, v_1\}$ induces an even wheel with center x_1. So v_1 has a neighbor in P and let u_i be such a neighbor with lowest index. Note that since $\{x_1, y_1, x_2, y\}$ cannot induce a 4-hole, v_1 is not adjacent to y_1. But then $(H \setminus x_1) \cup \{v_1, u_1, \dots, u_i\}$ induces a $3PC(y_1, v_1)$. Therefore v_1 is not of type s1 w.r.t. Σ, and hence P' and Q are crossing appendices of H. Since x_3 does not have a neighbor in Q, by Lemma 4.1.2 applied to H, Q and P', $y_1 \in \{v', v''\}$ and v_m has a neighbor in P. Let H' be the hole induced by $P' \cup P_{x_1y} \setminus y$. Then (H', v_m) is a wheel, and hence it is a twin wheel or a bug. If (H', v_m) is a bug, then $P \cup (P_{x_3y} \setminus x_3) \cup \{y_1, y, v_m\}$ contains a $3PC(y_1, \cdot)$. So (H', v_m) is a twin wheel. In particular, u_1 is the unique neighbor of v_m in P. Since $\{v_m, y_1, y, y_2\}$ cannot induce a 4-hole, $m > 1$. But then $(\Sigma \setminus x_3) \cup P \cup v_m$ contains an even wheel with center y_1. □

Proof of Theorem 4.3.4: Assume G contains a $\Sigma = 3PC(x_1x_2x_3, y)$ with a hat $P = p_1, \dots, p_k$, but G does not have a star cutset. By Theorems 4.2.2 and 4.3.3, G does not contain a proper wheel nor a bug with center-crosspath. For $i = 1, 2, 3$, let x_i' be the neighbor of x_i in P_{x_iy}. W.l.o.g. p_1 is adjacent to x_1 and p_k to x_2. Since $S = N[x_1] \setminus \{p_1, x_1'\}$ is not a star cutset, there exists a direct connection $Q = q_1, \dots, q_l$ from P to $\Sigma \setminus S$ in $G \setminus S$. We may assume w.l.o.g. that P and Q are chosen so that $|P \cup Q|$ is minimized.

By Lemma 4.3.1 and definition of Q, and since G does not contain a bug with a center-crosspath, q_l is of type p, d, s2 or crosspath w.r.t. Σ or it is a pseudo-twin of x_1 or y w.r.t. Σ.

Let p_i (resp. p_j) be the node of P with lowest (resp. highest) index adjacent to q_1. Note that x_1 has no neighbor in Q, q_l has a neighbor in $\Sigma \setminus \{x_1, x_2, x_3\}$, and the only nodes of Σ that may have a neighbor in $Q \setminus q_l$ are x_2 and x_3. If x_2 or x_3 has a neighbor in $Q \setminus q_l$, then let q_t be such a neighbor with lowest index. Let R be a chordless path from x_1 to q_l in $G[(\Sigma \setminus \{x_2, x_3\}) \cup q_l]$ (note that such a path exists since q_l has a neighbor in $\Sigma \setminus \{x_1, x_2, x_3\}$).

Case 1: $i = k$.

Let H be the hole induced by $R\cup P\cup Q$. Since $H\cup x_2$ cannot induce a $3PC(x_1,p_k)$ nor a proper wheel, (H,x_2) must be a bug. In particular, $N(x_2)\cap Q=q_1$ and R does not contain x_2'. Node x_3 cannot have a neighbor in Q, since otherwise $Q\cup P\cup\{x_1,x_2,x_3\}$ would contain a 4-wheel with center x_2. In particular, q_l is not of type s2 w.r.t. Σ nor is it a pseudo-twin of x_1 w.r.t. Σ. If q_l has a neighbor in $P_{x_3y}\setminus y$, then $(P_{x_3y}\setminus y)\cup P\cup Q\cup\{x_1,x_2,x_3\}$ contains a 4-wheel with center x_2. So q_l does not have a neighbor in $P_{x_3y}\setminus y$. In particular, q_l is not a pseudo-twin of y w.r.t. Σ. Suppose that q_l is of type d or crosspath w.r.t. Σ. Then q_l has a neighbor in $P_{x_1y}\setminus y$ and a neighbor in $P_{x_2y}\setminus y$. Hence x_1y is not an edge, since by definition of Q, x_1 cannot be adjacent to q_l. Let R' be the chordless path from q_l to x_3 in $G[(\Sigma\setminus\{x_1,x_1',x_2)\cup q_l]$. Then $P\cup Q\cup R'\cup\{x_1,x_2\}$ induces a proper wheel with center x_2. So q_l is not of type d or crosspath w.r.t. Σ, and hence q_l is of type p w.r.t. Σ.

Suppose that x_1y is an edge. Then the neighbors of q_l in Σ are contained in P_{x_2y}. Since R does not contain x_2', q_l has a neighbor in $P_{x_2y}\setminus\{x_2,x_2'\}$. Let P' be the chordless path from x_2 to y in $G[(P_{x_2y}\setminus x_2')\cup Q]$. Then $P'\cup P_{x_3y}\cup x_1$ induces a bug with center x_1, and P is its center-crosspath, a contradiction. Therefore x_1y is not an edge.

If $N(q_l)\cap\Sigma=x_1'$, then $P_{x_1y}\cup P_{x_2y}\cup Q$ induces a $3PC(x_1',x_2)$. So q_l has a neighbor in $\Sigma\setminus\{x_1,x_1'\}$. Let P' be the chordless path from q_l to x_3 in $G[(\Sigma\setminus\{x_1,x_2,x_1'\})\cup q_l]$. Then $P\cup P'\cup\{x_1,x_2,x_3\}$ induces a 4-wheel with center x_2.

Case 2: $i<k$.

First note that if $l>1$, then either $i=j$ or $j=i+1$, since otherwise the chordless path from p_1 to p_k in $(P\setminus p_{i+1})\cup q_1$ and $Q\setminus q_1$ contradict the minimality of $|P\cup Q|$. Let H be the hole induced by $R\cup Q\cup\{p_1,\dots,p_i\}$.

Suppose that x_2 has a neighbor in Q. Since $H\cup x_2$ cannot induce a $3PC(\cdot,\cdot)$ nor a proper wheel, (H,x_2) is a bug. In particular, either $l>1$ or $\{x_2,x_2'\}\subseteq N(q_l)\cap\Sigma\subseteq\{x_2,x_2',x_3\}$. If $j=i+1$, then $p_j,\dots,p_k$ is a center-crosspath of (H,x_2). So $j\neq i+1$. If $i=j$, then $P\cup Q\cup\{x_1,x_2\}$ contains a $3PC(x_2,p_i)$. So $j>i+1$. But then $l=1$, and hence $\{x_2,x_2'\}\subseteq N(q_l)\cap\Sigma\subseteq\{x_2,x_2',x_3\}$. By Lemma 4.3.1 and Theorem 4.3.3, $N(q_l)\cap\Sigma=\{x_2,x_2'\}$. If x_1y is not an edge, then $P_{x_2y}\cup P_{x_3y}\cup\{x_1,q_1,p_1,\dots,p_i\}$ induces a 4-wheel with center x_2. So x_1y is an edge. But then Σ is a bug and $p_1,\dots,p_i,q_1$ is its center-crosspath. Therefore x_2 does not have a neighbor in Q. In particular, q_l is not of type s2 w.r.t. Σ, nor a pseudo-twin of x_1 w.r.t. Σ.

Suppose that x_3 has a neighbor in $Q\setminus q_l$. Then paths $p_1,\dots,p_i,q_1,\dots,q_t$ and $q_{t+1},\dots,q_l$ contradict the minimality of $|P\cup Q|$. So x_3 does not have a neighbor in $Q\setminus q_l$.

Suppose that $j = i+1$. If q_l has a neighbor in $\Sigma \setminus \{x_1, x_1', x_2, x_2'\}$, then $(\Sigma \setminus \{x_1', x_2'\}) \cup P \cup Q$ contains a $3PC(q_1 p_i p_{i+1}, x_1 x_2 x_3)$. So q_l does not have a neighbor in $\Sigma \setminus \{x_1, x_1', x_2, x_2'\}$. Since q_l is not adjacent to x_1 nor x_2, $N(q_l) \cap \Sigma \subseteq \{x_1', x_2'\}$. If $N(q_l) \cap \Sigma = x_2'$, then $P_{x_1 y} \cup P_{x_2 y} \cup Q \cup \{p_1, \dots, p_i\}$ induces a $3PC(x_1, x_2')$. If $N(q_l) \cap \Sigma = x_1'$, then $P_{x_1 y} \cup P_{x_2 y} \cup \cup Q\{p_{i+1}, \dots, p_k\}$ induces a $3PC(x_2, x_1')$. So $N(q_l) \cap \Sigma = \{x_1', x_2'\}$. By Lemma 4.3.1, q_l must be of type p2 w.r.t. Σ, and hence either $x_2' = y$ or $x_1' = y$. But then $\{x_1, x_2, x_1', x_2'\}$ induces a 4-hole. So $j \neq i+1$.

Suppose that $i = j$. If q_l has a neighbor in $\Sigma \setminus \{x_1, x_2, x_3, x_1'\}$, then $(\Sigma \setminus \{x_1', x_3\}) \cup P \cup Q$ contains a $3PC(p_i, x_2)$. So q_l is adjacent to x_1' and it does not have a neighbor in $\Sigma \setminus \{x_1, x_2, x_3, x_1'\}$. Since $\{x_1, x_1', x_3, q_l\}$ cannot induce a 4-hole, $N(q_l) \cap \Sigma = x_1'$. If $i \neq 1$, then $P_{x_1 y} \cup P_{x_2 y} \cup Q \cup \{p_i, \dots, p_k\}$ induces a $3PC(x_2, x_1')$. So $i = 1$. But then $P_{x_1 y} \cup P_{x_2 y} \cup P \cup Q$ induces a proper wheel with center x_1. So $i \neq j$. Therefore $j > i+1$, and hence $l = 1$.

If q_1 has a neighbor in $\Sigma \setminus \{x_2, x_2', x_3\}$, then $(\Sigma \setminus \{x_2', x_3\}) \cup \{p_1, \dots, p_i, p_j, \dots, p_k, q_1\}$ contains a $3PC(q_1, x_1)$. So q_1 is adjacent to x_2' and it has no neighbor in $\Sigma \setminus \{x_2', x_3\}$. But then $\{x_1, x_2, x_2', p_1, \dots, p_i, p_j, \dots, p_k, q_1\}$ induces a $3PC(q_1, x_2)$. $\square$

4.4 Bugs

For a bug (H, x) we use the following notation in this section. Let x_1, x_2, y be the neighbors of x in H, such that $x_1 x_2$ is an edge. Let H_1 (resp. H_2) be the sector of (H, x) that contains y and x_1 (resp. x_2). Let y_1 (resp. y_2) be the neighbor of y in H_1 (resp. H_2).

Proof of Theorem 4.3.3: By Theorem 4.2.2 we may assume that G does not contain a proper wheel. Choose a bug (H, x) and its center-crosspath $P = p_1, \dots, p_k$ so that $|H \cup P|$ is minimized.

W.l.o.g. p_1 is adjacent to x, and let u_1, u_2 be the neighbors of p_k in H. W.l.o.g. $u_1, u_2 \in H_2 \setminus y$, and u_1 is the neighbor of p_k in H_2 that is closer to y. We now show that $S = N[x]$ is a star cutset separating H_1 from H_2.

Assume not and let $Q = q_1, \dots, q_l$ be a direct connection from H_1 to H_2 in $G \setminus S$. Note that no node of Q is adjacent to x. So no node of Q is of type t3, s1, s2 nor a pseudo-twin of x_1, x_2, x or y w.r.t. (H, x). Also by Lemma 4.3.7, no node of Q is of type crosspath w.r.t. (H, x). Hence by Lemma 4.3.1, either (i) $l > 1$, and q_1 and q_l are of type p, or (ii) $l = 1$ and q_1 is of type d. Suppose (ii) holds. Note that q_1 cannot be coincident with a node of P. If q_1 does not have a neighbor in P, then $(H \setminus x_2) \cup P \cup \{x, q_1\}$ contains a 4-wheel with center y. So $N(q_1) \cap P \neq \varnothing$. If q_1 has more than one neighbor in P, then $(H_2 \setminus x_2) \cup P \cup \{x, q_1\}$ contains a proper wheel with center q_1. So q_1 has a unique neighbor p_i in P. Since there is no 4-hole, $i > 1$. But then $H_2 \cup \{x, q_1, p_i, ..., p_k\}$ induces either a $3PC(q_1yy_2, p_ku_1u_2)$ or a 4-wheel with center y_2. So (i) holds. Furthermore, q_1 has a neighbor in $H_1 \setminus \{x_1, y\}$ and q_l has a neighbor in $H_2 \setminus \{x_2, y\}$. Also, the only nodes of H that may have a neighbor in $Q \setminus \{q_1, q_l\}$ are x_1, x_2, y. Since there is no 4-hole, every node of $Q \setminus \{q_1, q_l\}$ has a neighbor in at most one of the sets $\{x_1, x_2\}$, $\{y\}$.

Claim 1: *At most one of the sets $\{x_1, x_2\}$ or $\{y\}$ may have a neighbor in $Q \setminus \{q_1, q_l\}$.*

Proof of Claim 1: Assume not. Then there is a subpath Q' of $Q \setminus \{q_1, q_l\}$ such that one endnode of Q' is adjacent to y, the other is adjacent to a node of $\{x_1, x_2\}$, say to x_1, and no intermediate node of Q' has a neighbor in H. Then $H_1 \cup Q' \cup x$ induces a $3PC(x_1, y)$. This completes the proof of Claim 1.

Claim 2: *q_1 is not of type p3b.*

Proof of Claim 2: Assume q_1 is of type p3b, and let H' be the hole of $H \cup q_1$ that contains q_1, x_1, x_2, y. Then (H', x) is a bug. If q_1 is not adjacent to a node of P, then (H', x) and P contradict the minimality of $|H \cup P|$. So q_1 is adjacent to a node of P. Let p_i be the node of P with lowest index adjacent to q_1. Then $H_1 \cup \{x, q_1, p_1, \dots, p_i\}$ contains a $3PC(q_1, x)$. This completes the proof of Claim 2.

Let H_1' (resp. H_2') be the subpath of H_1 (resp. H_2) whose one endnode is x_1 (resp. x_2), the other endnode is adjacent to q_1 (resp. q_l), and no intemediate node of H_1' (resp. H_2') is adjacent to q_1 (resp. q_l). Let v_1 (resp. v_2) be the neighbor of q_1 in H_1 that is closest to x_1 (resp. y).

By Lemma 4.1.1 applied to H, x and Q and Lemma 4.3.7, either y has a neighbor in Q, or a node of $\{x_1, x_2\}$ has a neighbor in $Q \setminus \{q_1, q_l\}$. We now consider the following two cases.

Case 1: No node of $\{x_1, x_2\}$ has a neighbor in $Q \setminus \{q_1, q_l\}$.

Then y has a neighbor in Q. Let q_t be the node of Q with lowest index adjacent to y. By Claim 2, q_1 is of type p1, p2 or p3t. We now consider the following two cases.

Case 1.1: No node of P is adjacent to or coincident with a node of Q.

Let R be a chordless path from q_l to x in $(H_2 \setminus \{x_2, y\}) \cup P \cup \{x, q_l\}$.

First suppose that q_1 is of type p3t. If $t \neq 1$, then $H_1 \cup \{q_1, ..., q_t, x\}$ contains a $3PC(q_1, y)$. So $t = 1$ and consequently $v_2 = y$. Suppose q_1 is the unique node of Q adjacent to y. If $N(q_l) \cap H_2 \neq \{y_2\}$, then q_l has a neighbor in $H_2 \setminus \{x_2, y, y_2\}$ (since x_2y_2 is not an edge, else $\{x, y, x_2, y_2\}$ induces a 4-hole) and hence $Q \cup R \cup H_1' \cup y$ induces a $3PC(q_1, x)$. So

$N(q_l) \cap H_2 = \{y_2\}$. But then $(H \setminus y_1) \cup Q$ induces a $3PC(q_1, y_2)$. So $N(y) \cap (Q \setminus q_1) \neq \emptyset$. If $N(y) \cap (Q \setminus q_1) \neq \{q_2\}$ or $N(q_l) \cap H \subseteq \{y, y_2\}$, then $Q \cup R \cup H'_1 \cup \{x, y\}$ induces a proper wheel with center y. So q_2 is the unique neighbor of y in $Q \setminus q_1$ and $N(q_l) \cap H$ is not contained in the node set $\{y, y_2\}$. But then $Q \cup H'_2 \cup H'_1 \cup \{x, y\}$ induces a $3PC(x_1x_2x, q_1q_2y)$.

So q_1 is of type p1 or p2. Suppose that q_1 is of type p1. Then, $t > 1$. Node v_1 is adjacent to y, else $H_1 \cup \{x, q_1, \ldots, q_t\}$ induces a $3PC(v_1, y)$. But then $H_1 \cup Q \cup R$ induces a proper wheel with center y. Therefore, q_1 must be of type p2.

Suppose that q_1 is adjacent to y. Then $H_1 \cup Q \cup R$ must induce a bug with center y, and hence $y_2 \notin R$ and $N(y) \cap Q = q_1$. In particular, $y_2 \notin H'_2$. But then $H_1 \cup H'_2 \cup Q \cup x$ induces a $3PC(x_1x_2x, q_1yy_1)$. Therefore, q_1 is not adjacent to y.

Since $H'_1 \cup Q \cup R \cup y$ cannot induce a $3PC(x, q_t)$, it must induce a bug, and hence either (i) $y_2 \notin R$ and $N(y) \cap Q = \{q_t, q_{t+1}\}$, or (ii) $y_2 \in R$ and $t = l$. If (i) holds, then $y_2 \notin H'_2$, and hence $H_1 \cup H'_2 \cup Q$ induces a $3PC(yq_tq_{t+1}, q_1v_1v_2)$. So (ii) holds. So q_l is adjacent to y and y_2. Since there is no 4-hole, q_l is not adjacent to x_2. If q_l is of type p3, then there exists a chordless path from q_l to x in $(H_2 \setminus \{x_2, y\}) \cup P \cup \{x, q_l\}$ that does not contain y_2, contradicting the analysis thus far (that shows that $y_2 \in R$). So q_l is of type p2, and hence $H \cup Q$ induces a $3PC(q_1v_1v_2, q_lyy_2)$.

Case 1.2: A node of P is adjacent to or coincident with a node of Q.

Let q_i be the node of Q with lowest index adjacent to a node of P, and let p_j (resp. $p_{j'}$) be the node of P with highest (resp. lowest) index adjacent to q_i. If $i < t$, then by Lemma 4.1.1, $q_1, \ldots, q_i, p_j, \ldots, p_k$ is a crosspath, contradicting Lemma 4.3.7. So $i \geq t$.

Suppose $t = 1$. Then, by Claim 2, q_1 is of type p2 or p3t. Suppose q_1 is of type p2. Since $H_1 \cup \{x, y, q_1, \ldots, q_i, p_1, \ldots, p_{j'}\}$ cannot induce a proper wheel with center y, q_1 is the unique neighbor of y in $q_1, \ldots, q_i$. But then $H \cup \{q_1, \ldots, q_i, p_j, \ldots, p_k\}$ induces a $3PC(\Delta, \Delta)$. So q_1 is of type p3t. If q_1 is the unique neighbor of y in $\{q_1, \ldots, q_i\}$, then $H'_1 \cup \{q_1, ..., q_i, p_1, ..., p_{j'}, y\}$ induces a $3PC(q_1, x)$. So y has a neighbor in $\{q_2, \ldots, q_i\}$, and hence $H'_1 \cup \{q_1, ..., q_i, p_1, ..., p_{j'}, y\}$ induces a bug with center y. In particular $N(y) \cap \{q_1, ..., q_i\} = \{q_1, q_2\}$. Let R be an x_2u_2-subpath of H_2. Since P is a crosspath, yu_2 is not an edge, and hence $H_1 \cup R \cup \{q_1, ..., q_i, p_j, ..., p_k\}$ induces an even wheel with center q_1. So $t > 1$.

$H'_1 \cup \{x, y, q_1, \ldots, q_i, p_1, \ldots, p_{j'}\}$ must induce a bug with center y (since it cannot induce a $3PC(q_t, x)$ nor a proper wheel, and it cannot induce a twin wheel because y is not adjacent to any node of $P \cup x_1$), and hence $y_1 \notin H'_1$ and $N(y) \cap \{q_1, \ldots, q_i\} = \{q_t, q_{t+1}\}$. If q_1 is of type p1 or p3, then $H_1 \cup \{x, q_1, \ldots, q_t\}$ either induces a $3PC(v_1, y)$ or contains a $3PC(q_1, y)$. So q_1 is of type p2. If $i < l$ then $(H \setminus y_2) \cup \{q_1, \ldots, q_i, p_j, \ldots, p_k\}$

contains a $3PC(q_1v_1v_2, yq_tq_{t+1})$ (recall that since P is a crosspath, p_k has a neighbor in $H_2 \setminus \{y, y_2\}$). So $i = l$. If q_l has a neighbor in $H_2 \setminus \{y, y_2\}$, then $(H \setminus y_2) \cup Q$ contains a $3PC(q_1v_1v_2, yq_tq_{t+1})$. So q_l does not have a neighbor in $H_2 \setminus \{y, y_2\}$. Suppose $t+1 = l$. Let H' be the hole induced by $P \cup x$ and the yu_1-subpath of H_2. Since (H', q_l) cannot be a proper wheel, $j' = j$. Since there is no 4-hole, $j > 1$. But then $(H_2 \setminus y_2) \cup P \cup q_l$ contains a $3PC(p_j, x)$. So $t+1 < l$. In particular $N(q_l) \cap H = y_2$.

Suppose $j' = k$ and p_k is adjacent to y_2. If $k = 1$, then $\{x, p_k, y, y_2\}$ induces a 4-hole. So $k > 1$. But then $H_2 \cup \{x, q_{t+1}, ..., q_l, p_k\}$ induces a 4-wheel center y_2. So either $j' \neq k$ or p_k is not adjacent to y_2. But then $\{x, y, y_2, q_{t+1}, \ldots, q_l, p_1, \ldots p_{j'}\}$ induces a $3PC(y, q_l)$.

Case 2: A node of $\{x_1, x_2\}$ has a neighbor in $Q \setminus \{q_1, q_l\}$.

By Claim 1, y has no neighbor in $Q \setminus \{q_1, q_l\}$. Let q_i be the node of $Q \setminus q_1$ with lowest index adjacent to a node of $\{x_1, x_2\}$. Note that $i < l$.

Suppose that q_i is not adjacent to x_1. If q_1 is of type p1 or p3t, then $H \cup \{q_1, \ldots, q_i\}$ either induces a $3PC(x_2, \cdot)$ or contains a $3PC(x_2, q_1)$. So q_1 is of type p2. But then x and $q_1, \ldots, q_i$ are crossing appendices of H, and since x_2y is not an edge and $N(x) \cap Q = \varnothing$, Lemma 4.1.2 is contradicted. Therefore, q_i is adjacent to x_1.

Let q_j be the node of Q with highest index adjacent to x_1. Let R be the chordless path from q_l to y in $H_2 \cup q_l$. Note that R does not contain x_2, since by definition of Q, q_l has a neighbor in $H_2 \setminus \{x_2, y\}$. Let H' be the hole induced by $H_1 \cup R \cup \{q_j, \ldots, q_l\}$. Then $H' \cup x$ induces a $3PC(x_1, y)$. $\square$

Lemma 4.4.1 *Let G be a 4-hole-free odd-signable graph. If G contains a bug (H, x) and has no star cutset, then G has a path $P = p_1, ..., p_k$ disjoint from $V(H) \cup \{x\}$ such that no node of P is adjacent to x, no node of $H \setminus \{y\}$ has a neighbor in $P \setminus \{p_1, p_k\}$, p_1 has a neighbor in $H_1 \setminus \{x_1, y\}$, p_k has a neighbor in $H_2 \setminus \{x_2, y\}$ and P is one of the following types.*

- *A: P and x are crossing appendices of H. Node y is adjacent to the node-attachment of P in H and $N(y) \cap P = \emptyset$.*

- *D: $k = 1$ and p_1 is a node of type dd w.r.t. (H, x).*

- *C: $k > 1$ and one of the following holds.*

  - *(i) P is of type C1: nodes p_1, p_k are of type p2 not adjacent to y, node y has precisely one neighbor in P, and that neighbor lies in $P \setminus \{p_1, p_k\}$.*

  - *(ii) P is of type C2: nodes p_1, p_k are of type p2, exactly one of them, say p_1, is adjacent to y, and $N(y) \cap P = \{p_1, p_2\}$.*

(iii) *P is of type C3: one of $\{p_1, p_k\}$ is of type p3t adjacent to y and the other is of type p2. Say p_1 is of type p3t. Then $N(y) \cap P = p_1$.*

(iv) *P is of type C4: $k = 2$, one of $\{p_1, p_k\}$, is of type p3t and the other is of type p2. Both p_1, p_k are adjacent to y.*

(v) *P is of type C5: $k = 2$; one of $\{p_1, p_k\}$ is of type p3b and the other is of type p2. Both p_1, p_k are adjacent to y, say p_1 is of type p3b. The node-attachment of p_1 in H is y.*

T: Node y has exactly 3 neighbors in P, that are furthermore consecutive in P. Nodes p_1 and p_k are of type p2 or p3 w.r.t. (H, x). If p_1 (resp. p_k) is of type p3, then it is adjacent to y. If p_1 (resp. p_k) is of type p2, then it is not adjacent to y.

Furthermore, any direct connection from H_1 to H_2 in $G \setminus N[x]$ is of type A,D,C or T.

Proof: By Theorems 4.2.2 and 4.3.3 we may assume that G does not contain a proper wheel nor a bug with a center-crosspath. Since $N[x]$ is not a star cutset separating H_1 from H_2, let $P = p_1, \ldots, p_k$ be a direct connection from H_1 to H_2 in $G \setminus N[x]$. So no node of P is adjacent to x and hence no node of P is of type t3, s1, s2, dc w.r.t. (H, x) nor a pseudo-twin of x_1, x_2, x or y w.r.t. (H, x). By Theorem 4.3.3, no node of G is of type s1 w.r.t (H, x). If $k = 1$, then, by Lemma 4.3.1, p_1 is either of type crosspath w.r.t. (H, x) not adjacent to x or of type dd w.r.t. (H, x). So P is either of type A or D w.r.t. (H, x). So assume that $k > 1$.

By Lemma 4.3.1, p_1 and p_k are of type p w.r.t. (H, x). Note that the only nodes of H that may have a neighbor in $P \setminus \{p_1, p_k\}$ are x_1, x_2, y . Also p_1 has a neighbor in $H_1 \setminus \{x_1, y\}$ and p_k has a neighbor in $H_2 \setminus \{x_2, y\}$.

Claim 1: *At most one of the sets $\{x_1, x_2\}$ or $\{y\}$ may have a neighbor in $P \setminus \{p_1, p_k\}$.*

Proof of Claim 1: Assume not and let P' be a shortest subpath of $P \setminus \{p_1, p_k\}$ with the property that one endnode of P' is adjacent to y and the other endnode of P' is adjacent to a node of $\{x_1, x_2\}$. W.l.o.g. x_1 is adjacent to an endnode of P' . Then $H_1 \cup P' \cup x$ induces a $3PC(x_1, y)$. This completes the proof of Claim 1.

Claim 2: *No node of $\{x_1, x_2\}$ has a neighbor in $P \setminus \{p_1, p_k\}$.*

Proof of Claim 2: Assume not. By symmetry, w.l.o.g we may assume that x_2 has a neighbor in $P \setminus \{p_1, p_k\}$. Let p_i be such a neighbor with lowest index. By Claim 1, y does not have a neighbor in $P \setminus \{p_1, p_k\}$. Let R be the subpath of H_1 whose one endnode is y, the other endnode is adjacent to p_1, and no intermediate node of R is adjacent to p_1. Then $H_2 \cup R \cup \{x, p_1, ..., p_i\}$ induces a $3PC(x_2, y)$. This completes the proof of Claim 2.

So by Claim 2, no node of $H \setminus y$ has a neighbor in $P \setminus \{p_1, p_k\}$. If $N(y) \cap P = \varnothing$, then by Lemma 4.1.1, P is of type A. So we may assume that $N(y) \cap P \neq \varnothing$. Let p_i (resp. p_j) be the node of $N(y) \cap P$ with lowest (resp. highest) index. Let v_1 (resp. v_2) be the neighbor of p_1 in H_1 that is closest to x_1 (resp. y). Let v_1' (resp. v_2') be the neighbor of p_k in H_2 that is closest to x_2 (resp. y). Let H_1' (resp. H_2') be the x_1v_1-subpath (resp. x_2v_1'-subpath) of H_1 (resp. H_2). Let H' be the hole induced by $H_1' \cup H_2' \cup P$.

Claim 3: *p_1 and p_k are not of type p1.*

Proof of Claim 3: Suppose p_1 is of type p1. If v_1y is not an edge, then $H_1 \cup \{x, p_1, ..., p_i\}$ induces a $3PC(v_1, y)$. So v_1y is an edge. Suppose $i \neq j$. Since there is no proper wheel and p_1 is of type p1, (H', y) must induce a bug. But then x is its center-crosspath. So $i = j$. Note that $v_1' \neq y$. If $v_1' = y_2$, then (H', y) is either a proper wheel or a bug that has a center-crosspath x. So $v_1' \neq y_2$. But then $H' \cup y$ induces a $3PC(v_1, p_i)$. So p_1 is not of type p1, and by symmetry neither is p_k. This completes the proof of Claim 3.

By Claim 3 it suffices to consider the following two cases.

Case 1: At least one of $\{p_1, p_k\}$ is of type p3.

Assume w.l.o.g. that p_1 is of type p3. If $v_2 \neq y$, then $H_1 \cup \{x, p_1, ..., p_i\}$ contains a $3PC(p_1, y)$. So $v_2 = y$.

Suppose that p_k is not of type p2. So, by Claim 3, p_k is of type p3. Then by symmetry $v_2' = y$. If $k = 2$, then $H_1 \cup H_2' \cup P$ induces a 4-wheel with center p_1. So $k > 2$. If $N(y) \cap (P \setminus \{p_1, p_k\}) = \varnothing$, then $H' \cup y$ induces a $3PC(p_1, p_k)$. So $N(y) \cap (P \setminus \{p_1, p_k\}) \neq \varnothing$. Since there is no proper wheel, (H', y) is either a bug or a twin wheel. If (H', y) is a bug, then x is its center-crosspath. So (H', y) is a twin wheel and hence P is of type T.

So we may assume that p_k is of type p2.

Suppose that p_1 is of type p3b. If $N(y) \cap (P \setminus p_1) = \varnothing$, then (H, p_1) is a bug and $P \setminus p_1$ is its center-crosspath. So $N(y) \cap (P \setminus p_1) \neq \varnothing$. If $k = 2$, then either P is of type C5 or (H, p_1) is a bug with a center-crosspath p_2. So $k > 2$. Since $v_2 = y$ and $N(y) \cap (P \setminus p_1) \neq \varnothing$, y has at least two neighbors in H'. In particular, $j \geq 2$. Suppose $|N(y) \cap H'| = 2$. If $j = 2$, then $H_1' \cup H_2 \cup P$ induces a $3PC(p_1p_2y, v_1'v_2'p_k)$. So $j > 2$. But then $H' \cup y$ induces a $3PC(p_1, p_j)$. So $|N(y) \cap H'| > 2$. Since there is no proper wheel and $k > 2$, (H', y) must be a bug or a twin wheel. If (H', y) is a bug, then x is its center-crosspath. So (H',y) is a twin wheel, and hence P is of type T.

So we may assume that p_1 is of type p3t. Suppose $v_2' = y$. If $k = 2$, then P is of type C4. So assume $k > 2$. Since (H',y) cannot be a proper wheel, (H',y) is a bug. But then x is its center-crosspath. So we may assume that $v_2' \neq y$. If p_1 is the unique neighbor of y in P, then P is of type C3. So we may assume that $j > 1$. If p_j is the unique neighbor of y in $P \setminus p_1$, then either $H' \cup y$ induces a $3PC(p_1, p_j)$ (if $j > 2$) or $H_1' \cup H_2 \cup P$ induces a $3PC(p_1 p_2 y, v_1' v_2' p_k)$ (if $j = 2$). So y has at least three neighbors in H'. Since (H',y) is not a proper wheel nor a bug that has a center-crosspath x, (H',y) is a twin wheel, and hence P is of type T.

Case 2: p_1 and p_k are both of type p2.

Suppose that p_1, p_k are not adjacent to y. So $i \neq 1$ and $j \neq k$. If $i = j$, then P is of type C1. So $i < j$. If $p_i p_j$ is an edge, then $H' \cup \{x,y\}$ induces a $3PC(x_1 x_2 x, p_i p_j y)$. So $p_i p_j$ is not an edge. If p_i, p_j are the only two neighbors of y in P, then $H' \cup y$ induces a $3PC(p_i, p_j)$. So y has at least three neighbors in H'. Since (H',y) cannot be a proper wheel or a bug that has a center-crosspath x, (H',y) is a twin wheel, and hence P is of type T.

Suppose now w.l.o.g that p_1 is adjacent to y. Node p_k is not adjacent to y, since otherwise (H',y) is a proper wheel. If $N(y) \cap P = p_1$, then $H \cup P$ induces a $3PC(v_1 v_2 p_1, v_1' v_2' p_k)$. Therefore, since (H',y) is not a proper wheel nor a bug that has a center-crosspath x, (H',y) is a twin wheel and hence $N(y) \cap P = \{p_1, p_2\}$. So P is of type C2. $\square$

A path described in Lemma 4.4.1 is called a *bridge* of (H,x).

Proof of Theorem 4.3.5: Assume G does not have a star cutset. Then by Theorems 4.2.2, 4.3.3 and 4.3.4, G does not contain a proper wheel, a bug with center-crosspath nor a $3PC(\Delta, \cdot)$ with a hat.

Let (H,x) be a bug and $P = p_1, \ldots, p_k$ its ear. W.l.o.g. $N(p_k) \cap H = \{y, y_2\}$. Let H' be the hole induced by $(H_2 \setminus y) \cup P \cup x$. Then (H',y) is a bug and $H_1 \setminus y$ its ear.

Claim 1: *If u is a node of type p2 or p3 w.r.t. (H,x) such that $\{y\} \subseteq N(u) \cap (H \cup x) \subseteq H_1$, then u does not have a neighbor in P. Furthermore, if $N(u) \cap (H \cup x) = \{y\}$, then u does not have a neighbor in $P \setminus p_k$.*

Proof of Claim 1: Let u be one of the types from the statement of the claim. If u has a neighbor in $P \setminus p_k$, then by Lemma 4.3.1 u must be of type s1 or crosspath w.r.t. (H',y),

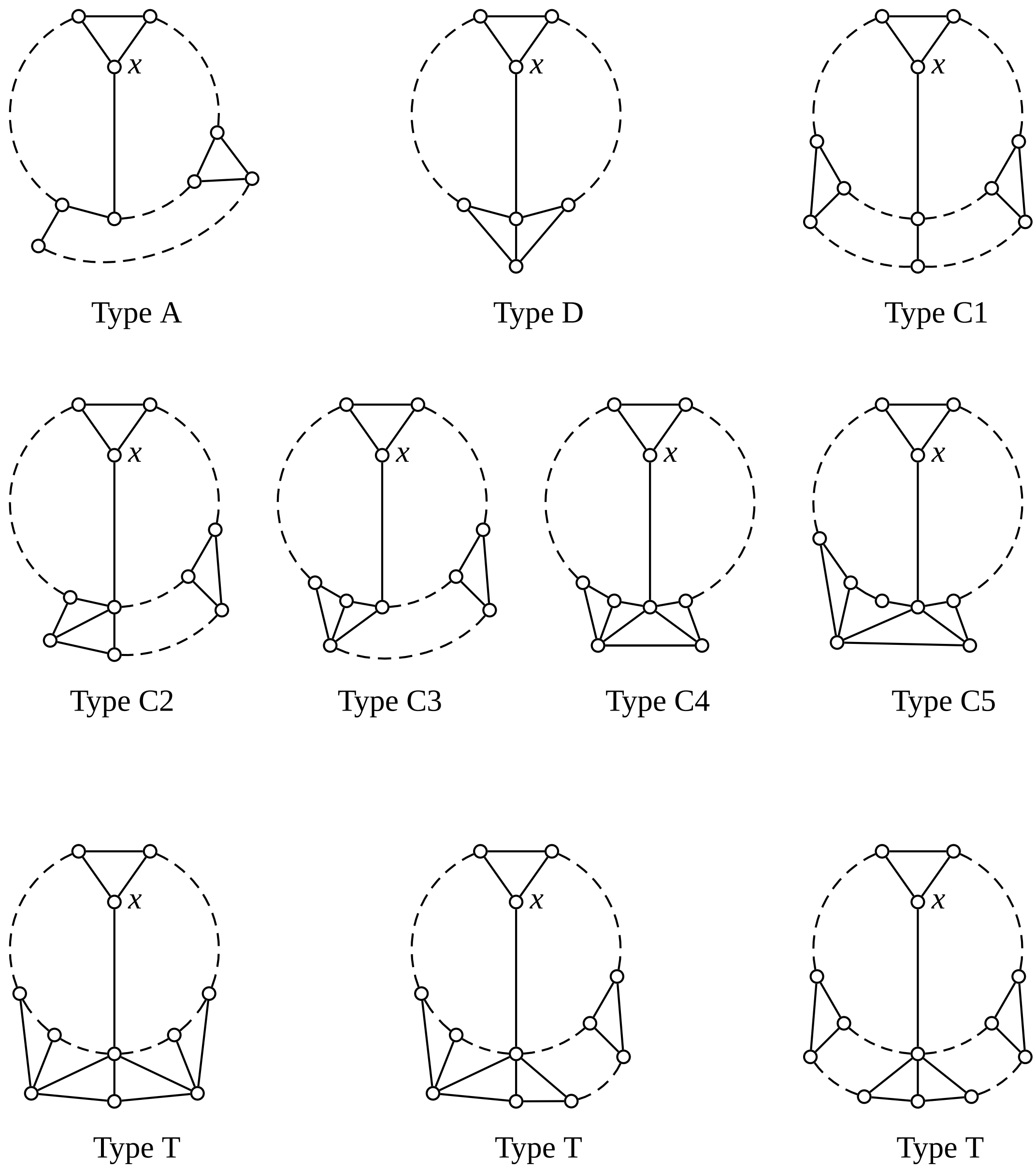


Figure 4.7: Bridges of a bug (H, x).

and hence u is a center-crosspath of (H',y), a contradiction. So u does not have a neighbor in $P \setminus p_k$.

Suppose that u is of type p2 w.r.t. (H,x) such that $N(u) \cap H = \{y, y_1\}$. If u is adjacent to p_k, then $H_1 \cup P \cup \{u,x\}$ induces a 4-wheel with center y. So u cannot have a neighbor in P.

Now suppose that u is of type p3 w.r.t. (H,x) such that $\{y\} \subseteq N(u) \cap (H \cup x) \subseteq H_1$. Suppose u is adjacent to p_k. If u is of type p3t w.r.t. (H,x), then $(H_1 \setminus y_1) \cup P \cup \{u,x\}$ induces a bug with center y, and node y_1 is its center-crosspath. Similarly, if u is of type p3b w.r.t. (H,x) not adjacent to y_1, then $H_1 \cup P \cup \{u,x\}$ induces a bug with center y with a center-crosspath. So we may assume that u is of type p3b w.r.t. (H,x) and u is adjacent to y_1. Then (H,u) is a bug and p_k its center-crosspath. This completes the proof of Claim 1.

Claim 2: *There exists a bridge of type D w.r.t. (H,x).*

Proof of Claim 2: Assume not. Then by Lemma 4.4.1 there exists a bridge $Q = q_1,\dots,q_l$ w.r.t. (H,x) of type A, C or T. W.l.o.g. q_1 has a neighbor in $H_1 \setminus y$ and q_l in $H_2 \setminus y$. Note that the only nodes of p_1, p_k, q_1 and q_l that may coincide are p_k and q_l.

Case 1: *Q is of type A.*

Then $N(y) \cap Q = \varnothing$. First suppose that no node of P is adjacent to or coincident with a node of Q. If $N(q_1) \cap H_1 = y_1$, then $(H \setminus y) \cup P \cup Q \cup x$ induces a $3PC(\Delta,\Delta)$ or a 4-wheel with center x_2. Otherwise, $N(q_l) \cap H_2 = y_2$ and hence $H_1 \cup P \cup Q \cup \{x, y_2\}$ induces a bug with center y with a center-crosspath.

So a node of P is adjacent to or coincident with a node of Q. Let p_i be the node of P with lowest index adjacent to a node of Q, and let q_j be the node of Q with lowest index adjacent to p_i.

Suppose that $i < k$. If $N(q_1) \cap H_1 = y_1$, then $H_1 \cup \{x, p_1, \dots, p_i, q_1, \dots, q_j\}$ induces a $3PC(y_1, x)$. Otherwise $N(q_l) \cap H_2 = y_2$. If $j < l$, then $\{p_1, \dots, p_i, q_1, \dots, q_j\}$ induces a center-crosspath of bug (H,x). So $j = l$. But then q_l and (H',y) contradict Lemma 4.3.1. Therefore $i = k$.

If $N(q_l) \cap H_2 = y_2$, then $(H_1 \setminus y_1) \cup P \cup \{x, q_1, \dots, q_j\}$ contains a $3PC(x, p_k)$. So $N(q_1) \cap H_1 = y_1$. If $j = l$, then $H_2 \cup \{x, p_k, q_l\}$ induces a $3PC(\Delta,\Delta)$ or a 4-wheel with center y_2. So $j < l$. But then $H_1 \cup P \cup \{x, q_1, \dots, q_j\}$ induces a proper wheel with center y.

Case 2: *Q is of type C or T.*

Then y has a neighbor in Q. First suppose that no node of P is adjacent to or coincident with a node of Q. Let R be the chordless path from q_l to y_2 in $(H_2 \setminus \{y, x_2\}) \cup q_l$, and let S

be the chordless path from q_1 to x_1 in $(H_1 \setminus y) \cup q_1$. Then $R \cup S \cup Q \cup P \cup \{x,y\}$ induces a proper wheel with center y.

So a node of P is adjacent to or coincident with a node of Q. Let p_i be the node of P with lowest index adjacent to a node of Q, and let q_j be the node Q with lowest index adjacent to p_i. Let H_1' be the subpath of H_1 whose one endnode is x_1, the other is adjacent to q_1 and no intermediate node of H_1' is adjacent to q_1. We now consider the following 2 cases.

Case 2.1: *q_1 is of type p3 w.r.t. (H,x).*

Then q_1 is adjacent to y. Suppose that $i < k$ and $j < l$. If no node of $q_2,\dots,q_j$ is adjacent to y, then $(H_1 \setminus y_1) \cup \{x, p_1,\dots,p_i,q_1,\dots,q_j\}$ contains a $3PC(x,q_1)$. So y is adjacent to a node of $q_2,\dots,q_j$, and hence Q is a bridge of type T. In particular, $N(y) \cap Q = \{q_1,q_2,q_3\}$. By Claim 1, $j > 3$. But then $H_1' \cup \{x,y,p_1,\dots,p_i,q_1,\dots,q_j\}$ induces a proper wheel with center y. So either $i = k$ or $j = l$.

Suppose that $i = k$. By Claim 1, $j > 1$. But then if $j < l$, $H_1' \cup P \cup \{x,y,q_1,\dots,q_j\}$ induces a proper wheel with center y. So $j = l$. Note that since $j > 1$, p_k and q_l cannot coincide. If q_l is adjacent to y, then $H_1' \cup P \cup Q \cup \{x,y\}$ induces a proper wheel with center y. So q_l is not adjacent to y, and hence it is of type p2 w.r.t. (H,x). But then $H_2 \cup \{x, p_k, q_l\}$ induces a $3PC(\Delta,\Delta)$ or a 4-wheel with center y_2.

So $i < k$, and hence $j = l$. Suppose that q_l is adjacent to y. Then $H_1' \cup Q \cup \{x,y,p_1,\dots,p_i\}$ induces a wheel with center y. This wheel must be a bug. In particular $l = 2$, i.e. Q is a bridge of type C4 or C5, and hence q_l is of type p2 w.r.t. (H,x). Let $P' = p_1,\dots,p_i,q_l$. Then P' is an ear of (H,x) and q_1 is of type p3 w.r.t. (H,x) adjacent to y and a node of P', contradicting Claim 1. So q_l cannot be adjacent to y. But then $|N(y) \cap Q| = 1$ or 3, and hence $H_1' \cup Q \cup \{x,y,p_1,\dots,p_i\}$ induces a $3PC(q_1,x)$ or a proper wheel with center y.

Case 2.2: *q_1 is of type p2 w.r.t. (H,x).*

First suppose that q_1 is not adjacent to y. Suppose that $i < k$ and $j < l$. If no node of $q_2,\dots,q_j$ is adjacent to y, then $\{p_1,\dots,p_i,q_1,\dots,q_j\}$ induces a center-crosspath of (H,x). So a node of $q_2,\dots,q_j$ is adjacent to y. If y has a unique neighbor in $q_2,\dots,q_j$, then $H_1' \cup \{x,y,p_1,\dots,p_i,q_1,\dots,q_j\}$ induces a $3PC(x,\cdot)$. So y has more than one neighbor in $q_2,\dots,q_j$. In particular, Q is a bridge of type T. By Claim 1 y has three neighbors in $q_2,\dots,q_j$ and hence $H_1' \cup \{x,y,p_1,\dots,p_i,q_1,\dots,q_j\}$ induces a proper wheel with center y. Therefore, either $i = k$ or $j = l$.

Suppose that $i = k$ and $j < l$. If no node of $q_2,\dots,q_j$ is adjacent to y, then $H \cup \{p_k,q_1,\dots,q_j\}$ induces a $3PC(\Delta,\Delta)$. So a node of $q_2,\dots,q_j$ is adjacent to y. So $H_1' \cup$

$P \cup \{x, q_1, \dots, q_j\}$ induces a wheel with center y. This wheel must be a bug. But then $H_1 \setminus (H_1' \cup y)$ is a center-crosspath of this bug.

Suppose that $i = k$ and $j = l$. Then p_k and q_l do not coincide. If q_l is not adjacent to y, then q_l is of type p2 w.r.t. (H, x) and hence $H_2 \cup \{x, p_k, q_l\}$ induces a $3PC(\Delta,\Delta)$ or a 4-wheel with center y_2. So q_l is adjacent to y. Then $H_1' \cup P \cup Q \cup \{x, y\}$ induces a wheel with center y, which must be a bug, and hence $H_1 \setminus (H_1' \cup y)$ is its center-crosspath.

Therefore $i < k$ and $j = l$. If q_l is of type p3 w.r.t. (H, x), then q_l is adjacent to y and hence $(H_2 \setminus y_2) \cup \{x, p_1, \dots, p_i, q_l\}$ contains a $3PC(x, q_l)$. So q_l is of type p2 w.r.t. (H, x). If q_l is not adjacent to y, then $p_1, \dots, p_i, q_l$ is a center-crosspath of (H, x). So q_l is adjacent to y, and hence Q is a bridge of type C2. In particular, $N(y) \cap Q = \{q_l, q_{l-1}\}$. But then $H_1 \cup Q \cup \{x, p_1, \dots, p_i\}$ induces a bug with center y with a center-crosspath (namely the path induced by $H_1 \setminus (H_1' \cup y)$).

Finally we may assume that q_1 is adjacent to y. So Q is a bridge of type C2, C4 or C5. By Claim 1, q_1 does not have a neighbor in P and hence $j > 1$. Suppose that q_l is of type p3 w.r.t. (H, x). Then Q is a bridge of type C4 or C5, and in particular $l = 2$ and q_l is adjacent to y. Note that $j = l = 2$, and hence $H_1 \cup Q \cup \{x_1, p_1, \dots, p_i\}$ induces a proper wheel with center y. So q_l must be of type p2 w.r.t. (H, x), and hence Q is a bridge of type C2. In particular, q_l is not adjacent to y and $N(y) \cap Q = \{q_1, q_2\}$. But then $H_1 \cup \{x, p_1, \dots, p_i, q_1, \dots, q_j\}$ induces a proper wheel with center y. This completes the proof of Claim 2.

By Claim 2, let u be a bridge of (H, x) of type D. Then $N(u) \cap (H \cup x) = \{y, y_1, y_2\}$. By analogous argument applied to bug (H', y) and its ear $H_1 \setminus y$, (H', y) has a bridge of type D, say v. So $N(v) \cap (H' \cup y) = \{x, p_1, x_2\}$. Node u must have a neighbor in $P \setminus p_k$, else $H_1 \cup P \cup \{x, y_2, u\}$ contains a proper wheel with center y. By symmetry, v has a neighbor in $H_1 \setminus x_1$. Since $\{x, y, u, v\}$ cannot induce a 4-hole, uv is not an edge. By Lemma 4.3.1, u is a pseudo-twin of p_k w.r.t. (H', y), and hence it has two neighbors in P. But then $(H_1 \setminus x_1) \cup P \cup \{u, v\}$ contains a 4-wheel with center u. $\square$

Proof of Theorem 4.3.6: Assume not. Choose a bug (H, x) and a type s2 node u so that $|H|$ is minimized. W.l.o.g. u is adjacent to x, x_1, y, y_2. By Theorems 4.2.2 and 4.3.3 we may assume that G does not contain a proper wheel nor a bug with a center-crosspath (and in particular no bug with a type s1 node). By Lemma 4.4.1, there is a direct connection $P = p_1, \dots, p_k$ from H_1 to H_2 in $G \setminus N[x]$ of type A, D, C or T w.r.t. (H, x). Let v_1 (resp. v_2) be the node of $N(p_1) \cap H_1$ (resp. $N(p_k) \cap H_2$) that is closest to x_1 (resp. x_2). Let H_1'

(resp. H_2') be the subpath of H_1 (resp. H_2) with endnodes x_1 (resp. x_2) and v_1 (resp. v_2). We now consider the following cases.

Case 1: *P is of type A w.r.t.* (H,x).

Suppose that the node-attachment of P in H is y_1. Suppose that $N(u) \cap P = \varnothing$. Then P and u are crossing appendices of H, and since y_1x_1 cannot be an edge (otherwise there is a 4-hole), Lemma 4.1.2 is contradicted. So $N(u) \cap P \neq \varnothing$. Let p_i be the node of $N(u) \cap P$ with lowest index. Then $H_1 \cup \{p_1,...,p_i,u\}$ induces a $3PC(u,y_1)$. So the node-attachment of P in H is y_2. But then $H_1' \cup P \cup \{x,u,y,y_2\}$ induces a proper wheel with center u.

Case 2: *P is type T w.r.t.* (H,x).

Let p_{i-1}, p_i, p_{i+1} be the neighbors of y in P. Let Σ_1 be the $3PC(xx_1x_2,y)$ induced by $H_1 \cup H_2' \cup \{p_{i+1},...,p_k\}$ and Σ_2 be the $3PC(xx_1x_2,y)$ induced by $H_1' \cup H_2 \cup \{p_1,...,p_{i-1}\}$. Since u is strongly adjacent to Σ_1, by Lemma 4.3.1, $N(u) \cap \{p_{i+1},...,p_k\} = \{p_{i+1}\}$. By Lemma 4.3.1 applied to Σ_2, $N(u) \cap \{p_1,...,p_{i-1}\} = \varnothing$. Let H' be the hole induced by $H_1' \cup H_2' \cup P$. If $up_i \notin E(G)$, then $H' \cup u$ induces a $3PC(x_1,p_{i+1})$. So $up_i \in E(G)$ and hence (H',u) is a bug. If p_k is of type p3t, then $i+1=k$ and y_2 is of type s1 w.r.t. (H',u), a contradiction. Suppose that p_k is of type p3b w.r.t. (H,x). Then $i+1=k$. Let H'' be the hole contained in $(H \setminus y_2) \cup p_k$. Then (H'',x) and u contradict our choice of (H,x) and u. So p_k is not of type p3 w.r.t. (H,x), and hence it is of type p2 w.r.t. (H,x) not adjacent to y. But then $H_2 \setminus (H_2' \cup y)$ induces a center-crosspath of bug (H',u).

Case 3: *P is of type D w.r.t.* (H,x).

So $k=1$ and p_1 is a node of type dd w.r.t. (H,x). If up_1 is not an edge, then $H_1 \cup \{u,p_1,y_2\}$ induces a 4-wheel with center y. So up_1 is an edge.

Since (H,u) is a bug and G does not have a star cutset, by Lemma 4.4.1 there is a path $Q = q_1,...,q_l$ of type A, D, C or T w.r.t. (H,u). W.l.o.g. q_1 has a neighbor in $H_1 \setminus \{x_1,y\}$ and q_l in $H_2 \setminus \{y_2,y\}$. Note that x is of type s2 w.r.t. (H,u). By symmetry and Cases 1 and 2 applied to (H,u) and Q, path Q cannot be of type A or T w.r.t. (H,u).

Suppose that Q is of type D w.r.t. (H,u). If xq_1 is not an edge, then $H_1 \cup \{x,x_2,q_1\}$ induces a 4-wheel with center x_1. So xq_1 is an edge. Since $\{q_1,p_1,x,y\}$ cannot induce a 4-hole, p_1q_1 is not an edge. But then $H_1' \cup \{q_1,p_1,x,u\}$ induces a 4-wheel with center x_1. So Q must be of type C w.r.t. (H,u).

Note that p_1 cannot be coincident with a node of Q. Let H'' be the hole induced by $(H \setminus y) \cup p_1$. By Lemma 4.4.1 applied to (H'',u) and Q, no node of $Q \setminus \{q_1,q_l\}$ can be adjacent to p_1. Let R_1 (resp. R_2) be the subpath of H_1 (resp. H_2) whose one endnode is y, the other endnode of R_1 (resp. R_2) is adjacent to q_1 (resp. q_l), and no intermediate node of R_1 (resp. R_2) is adjacent to q_1 (resp. q_l).

Suppose $N(x) \cap Q = \varnothing$. Suppose that q_l has a neighbor in $H_2 \setminus x_2$. Then q_l must in fact have a neighbor in $H_2 \setminus \{x_2, y, y_2\}$, and hence Q is a direct connection from H_1 to H_2 in $G \setminus N[x]$, and hence by Lemma 4.4.1 applied to (H,x) and Q, nodes x_1 and x_2 do not have a neighbor in $Q \setminus \{q_1, q_l\}$. Since x_1 does not have a neighbor in $Q \setminus \{q_1, q_l\}$, and Q is of type C w.r.t. (H,u), Q must be of type C3, C4 or C5 w.r.t. (H,u). Suppose that Q is of type C4 or C5 w.r.t. (H,u). Since we are assuming that q_l has a neighbor in $H_2 \setminus x_2$, it follows that q_l is of type p3 w.r.t. (H,u) and hence q_1 is of type p2 w.r.t. (H,u), and both q_1 and q_l are adjacent to x_1. But then (H,x) and Q contradict Lemma 4.4.1. Therefore Q must be of type C3 w.r.t. (H,u). If q_l is of type p3t w.r.t. (H,u), then (H,x) and Q contradict Lemma 4.4.1. So q_l is of type p2 w.r.t. (H,u) and q_1 is of type p3t w.r.t. (H,u) adjacent to x_1. But then by Lemma 4.4.1 applied to (H,x) and Q, Q is of type C3 w.r.t. (H,x), q_1 is of type p3t w.r.t. (H,x) and q_1 is adjacent to y. But then $\{x_1, y, x, q_1\}$ induces a 4-hole. So q_l does not have a neighbor in $H_2 \setminus x_2$ and hence Q must be of type C2, C4 or C5 w.r.t. (H,u) and $N(q_l) \cap H = \{x_1, x_2\}$. But then $Q \cup R_1 \cup \{x_1, x_2, x\}$ is a proper wheel with center x_1. So $N(x) \cap Q \neq \varnothing$.

Suppose that Q is of type C1 or C3 w.r.t. (H,u). Let q_i be the neighbor of x_1 in Q. Suppose that x has a unique neighbor in Q. If q_1 is not adjacent to both x and y, then $Q \cup R_1 \cup R_2 \cup x$ induces a $3PC(y,\cdot)$. So q_1 is adjacent to both x and y. If $i < l$, then $H_2 \cup \{x_1, x, q_1, \dots, q_i\}$ induces a 4-wheel with center x. So $i = l$, and hence q_l is of type p3t w.r.t. (H,u) (i.e. q_l is adjacent to x_1, x_2 and the neighbor of x_2 in H_2). But then $H_2 \cup \{q_l, x_1, x\}$ induces a 4-wheel with center x_2. Therefore $|N(x) \cap Q| \geq 2$. If $N(x) \cap \{q_1, \dots, q_i\} \neq \varnothing$, then $R_1 \cup \{q_1, \dots, q_i, x_1, u, x\}$ induces a proper wheel with center x. So $N(x) \cap \{q_1, \dots, q_i\} = \varnothing$, and hence $|N(x) \cap \{q_i, \dots, q_l\}| \geq 2$, But then $(R_2 \setminus y) \cup \{q_i, \dots, q_l, x_1, u, x\}$ induces a proper wheel with center x.

So Q is of type C2, C4 or C5 w.r.t. (H,u). Suppose $N(q_l) \cap H = \{x_1, x_2\}$. If $N(x) \cap Q \neq q_l$, then $Q \cup R_1 \cup R_2 \cup x$ induces a proper wheel with center x. So $N(x) \cap Q = q_l$. Note that p_1 is not adjacent to q_l, else $\{p_1, q_l, x, y\}$ induces a 4-hole. But then $Q \cup \{x_1, x, u, p_1\} \cup (R_1 \setminus y)$ contains a proper wheel with center x_1. So $N(q_l) \cap H \neq \{x_1, x_2\}$, and hence q_l has a neighbor in $H_2 \setminus \{x_2, y\}$ and q_1 is of type p2 w.r.t. (H,u) adjacent to x_1. Let q_i be the neighbor of x in Q with lowest index. Note that p_1 cannot be adjacent to q_1, else $\{p_1, q_1, x_1, u\}$ induces a 4-hole. Also p_1 cannot be adjacent to q_i, else $\{p_1, q_i, x, u\}$ induces a 4-hole. But then $\{q_1, \dots, q_i, x_1, x, u, p_1\} \cup (R_1 \setminus y)$ induces a proper wheel with center x_1.

Case 4: *P is of type C w.r.t. (H,x).*

Suppose that P is either of type C1 or C3. Let p_i be the neighbor of y in P. Let Σ

be the $3PC(x_1x_2x, p_i)$ contained in $H \cup P \cup x$. Note that p_i cannot be adjacent to x_1, else $\{x_1, x, y, p_i\}$ induces a 4-hole. Similarly p_i is not adjacent to x_2. In particular Σ is not a bug. But then since node u is strongly adjacent to Σ, Lemma 4.3.1 is contradicted. So P is of type C2, C4 or C5 w.r.t. (H, x).

Suppose that $N(p_1) \cap H = \{y, y_1\}$ and p_k has a neighbor in $H_2 \setminus \{y, y_2\}$. Let R be the subpath of $H_2 \setminus y$ whose one endnode is y_2, the other endnode of R is adjacent to p_k, and no intermediate node of R is adjacent to p_k (note that possibly $R = y_2$). If $N(u) \cap P = \emptyset$, then $H_1 \cup R \cup P \cup u$ induces a proper wheel with center y. So $N(u) \cap P \neq \emptyset$. Let p_i be the node of $N(u) \cap P$ with lowest index. If $i > 1$, then $H_1 \cup \{u, p_1, \ldots, p_i\}$ induces a 4-wheel with center y. So $i = 1$. If p_1 is the unique neighbor of u in P, then $P \cup R \cup \{y, u\}$ induces a 4-wheel with center y. So $|N(u) \cap P| \geq 2$. Let H' be the hole induced by $H_1' \cup H_2' \cup P$. Since (H', u) cannot be a proper wheel and $y_1 \neq x_1$, (H', u) must be a bug. In particular, $N(u) \cap P = \{p_1, p_2\}$. Suppose that p_k is of type p3b w.r.t. (H, x). Then $k = 2$. Let H'' be the hole contained in $(H \setminus y_2) \cup p_k$. Then (H'', x) and u contradict our choice of (H, x) and u. So p_k is not of type p3b w.r.t. (H, x) and hence it is of type p2 or p3t w.r.t. (H, x). But then R is the center-crosspath of (H', u).

So p_1 has a neighbor in $H_1 \setminus \{y, y_1\}$ and $N(p_k) \cap H = \{y, y_2\}$. If $N(u) \cap P = \emptyset$, then $H_1' \cup P \cup \{u, y, y_2\}$ induces a 4-wheel with center y. So $N(u) \cap P \neq \emptyset$. Let H' be the hole induced by $H_1' \cup H_2' \cup P$. Since (H', u) cannot be a proper wheel and $y_2 \neq x_2$, (H', u) must be a bug. So $N(u) \cap P = \{p_k\}$.

Since (H, u) is a bug, and G has no star cutset, and x is a node of type s2 w.r.t. (H, u), by Lemma 4.4.1 and by symmetry, there is a path $Q = q_1, \ldots, q_l$ of type C2, C4 or C5 w.r.t. (H, u), such that $N(q_l) \cap H = \{x_1, x_2\}$, $N(x) \cap Q = \{q_l\}$, q_1 has a neighbor in $H_1 \setminus \{x_1, x_1'\}$ (where x_1' is the neighbor of x_1 in H_1) and no neighbor in $H_2 \setminus y$. Note that since p_1 is of type p2 or p3 w.r.t. (H, x), p_1 has a neighbor in $H_1 \setminus \{x_1, y\}$. Similarly, q_1 has a neighbor in $H_1 \setminus \{x_1, y\}$. Let R be the shortest path from q_l to p_k in $P \cup Q \cup (H_1 \setminus \{x_1, y\})$. Then $R \cup (H_2 \setminus y) \cup \{x, u\}$ induces a $3PC(q_lx_2x, p_ky_2u)$. $\square$

4.5 Attachments

In the section we use the following notation. Let $\Sigma = 3PC(x_1x_2x_3, y)$. The three paths of Σ are denoted P_{x_1y}, P_{x_2y} and P_{x_3y} (where P_{x_iy} is the path that contains x_i). For $i = 1, 2, 3$, we denote the neighbor of y (resp. x_i) in P_{x_iy} by y_i (resp. x_i'). For $i, j \in \{1, 2, 3\}$, $i \neq j$, let H_{ij} be the hole induced by $P_{x_iy} \cup P_{x_jy}$.

Lemma 4.5.1 *Let G be a 4-hole-free odd-signable graph that does not have a star cutset.*

Let u be a type p1 node w.r.t. Σ adjacent to x_1. Let $P = p_1, \ldots, p_k$ be a chordless path in $G \setminus \Sigma$ such that p_1 is adjacent to u, p_k has a neighbor in $\Sigma \setminus \{x_1, x_2, x_3\}$, no node of $P \setminus \{p_1\}$ is adjacent to u and no node of $P \setminus \{p_k\}$ has a neighbor in Σ. Then p_k is one of the following types:

(i) *p_k is of type p2 with neighbors in P_{x_1y}.*

(ii) *p_k is of type p1 adjacent to x_1'.*

(iii) *p_k is of type d and it has no neighbor in $P_{x_1y} \setminus \{y\}$.*

(iv) *p_k is adjacent to x_1 and it is either of type p3 or d, or it is a pseudo-twin of x_1, x_2, x_3 or y w.r.t. Σ, or it is a crosspath w.r.t. Σ adjacent to x_1, x_1' and a node of $\{y_2, y_3\}$.*

Proof: By Theorems 4.2.2, 4.3.3, 4.3.5 and 4.3.6 we may assume that G does not contain a proper wheel, a bug with a center-crosspath, a bug with an ear nor a $3PC(\Delta, \cdot)$ with a type s1 or s2 node. Since p_k has a neighbor in $\Sigma \setminus \{x_1, x_2, x_3\}$, p_k cannot be of type t2 nor t3 w.r.t. Σ. So, for the node p_k, it sufices to examine the following remaining possibilities of Lemma 4.3.1.

Case 1: p_k is of type p1 w.r.t. Σ.

Let v be the node of $N(p_k) \cap \Sigma$. Note that $v \notin \{x_1, x_2, x_3\}$. If $v \neq x_1'$, then $\Sigma \cup P \cup u$ contains a $3PC(x_1, v)$. So $v = x_1'$ and hence (ii) holds.

Case 2: p_k is of type p2 w.r.t. Σ.

If $N(p_k) \subseteq P_{x_1y}$, then (i) holds. So w.l.o.g. assume that $N(p_k) \subseteq P_{x_2y}$. If x_1y is not an edge, then $H_{23} \cup P \cup u$ induces a $3PC(x_1x_2x_3, \Delta)$ or a 4-wheel with center x_2. So x_1y is an edge. But then u, P is either a center-crosspath or an ear of bug Σ.

Case 3: p_k is of type p3 w.r.t. Σ.

If p_kx_1 is not an edge, then $\Sigma \cup P \cup u$ contais a $3PC(x_1, p_k)$. So p_kx_1 is an edge and hence (iv) holds.

Case 4: p_k is of type crosspath w.r.t Σ.

Let v (resp. v_1v_2) be the node-attachment (resp. edge-attachment) of p_k in an appropriate hole of Σ. Note that since there is no bug with a center-crosspath, $v \notin \{x_1, x_2, x_3\}$. Suppose $v = y_1$. W.l.o.g. v_1v_2 is an edge of P_{x_2y}. Then $H_{23} \cup P \cup \{x_1, u\}$ induces a $3PC(x_1x_2x_3, p_kv_1v_2)$ or a 4-wheel with center x_2. So $v = y_2$ or $v = y_3$. W.l.o.g. let $v = y_2$. Suppose $v_1v_2 \in P_{x_3y}$. Let R be the subpath of P_{x_3y} with one endnode x_3 and the other endnode adjacent to p_k. Then $P_{x_1y} \cup R \cup P \cup \{u, y_2\}$ induces a $3PC(x_1, p_k)$. So $v_1v_2 \in P_{x_1y}$.

Let R be the subpath of P_{x_1y} with one endnode x_1 and the other endnode adjacent to p_k. If p_kx_1 is not an edge, then $(P_{x_2y} \setminus y) \cup R \cup P \cup u$ induces a $3PC(x_1, p_k)$. So p_kx_1 is an edge, and hence (iv) holds.

Case 5: p_k is a pseudo-twin of x_1, x_2 or x_3 w.r.t. Σ.

Suppose that p_k is not adjacent to x_1. Then p_k has two adjacent neighbors in P_{x_1y}. Let R be the subpath of P_{x_1y} with one endnode x_1 and the other endnode is adjacent to p_k. Then $P \cup R \cup \{u, x_2\}$ induces a $3PC(x_1, p_k)$. So p_k is adjacent to x_1, and hence (iv) holds.

Case 6: p_k is of type d w.r.t. Σ, or it is a pseudo-twin of y w.r.t. Σ.

W.l.o.g. p_k has a neighbor in $P_{x_2y} \setminus y$. If p_kx_1 is not an edge and p_k has a neighbor in $P_{x_1y} \setminus y$, then $(\Sigma \setminus P_{x_3y}) \cup P \cup u$ contains a $3PC(x_1, p_k)$. So either p_kx_1 is an edge and hence (iv) holds, or p_k does not have a neighbor in $P_{x_1y} \setminus y$ and hence (iii) holds. □

Lemma 4.5.2 *Let G be a 4-hole-free odd-signable graph that does not have a star cutset. Let u be a type t2 node w.r.t. Σ adjacent to x_2 and x_3. Let $P = p_1, \ldots, p_k$ be a chordless path in $G \setminus \Sigma$ such that p_1 is adjacent to u, p_k has a neighbor in $\Sigma \setminus \{x_1, x_2, x_3\}$, no node of $P \setminus \{p_1\}$ is adjacent to u, and no node of $P \setminus \{p_k\}$ has a neighbor in Σ. Then p_k is one of the following types:*

(i) *p_k is of type p2 w.r.t. Σ and its neighbors in Σ are contained in P_{x_1y}.*

(ii) *x_3y is an edge and p_k is of type p1 w.r.t. Σ adjacent to x_2', or x_2y is an edge and p_k is of type p1 w.r.t. Σ adjacent to x_3'.*

(iii) *p_k is of type p3 w.r.t. Σ, and either p_kx_2 and x_3y are edges, or p_kx_3 and x_2y are edges.*

(iv) *p_k is of type d not adjacent to y_1 and neither x_2y nor x_3y is an edge.*

(v) *p_k is a pseudo-twin of x_1, x_2 or x_3 w.r.t. Σ.*

Proof: By Theorems 4.2.2, 4.3.3 and 4.3.6 we may assume that G does not contain a proper wheel, a bug with a center-crosspath nor a $3PC(\Delta, \cdot)$ with a type s1 or s2 node. Since p_k has a neighbor in $\Sigma \setminus \{x_1, x_2, x_3\}$, p_k cannot be of type t2 nor t3 w.r.t. Σ.

Claim 1: *p_k is not of type crosspath or a pseudo-twin of y w.r.t. Σ.*

Proof of Claim 1: Suppose that p_k is of type crosspath. Let v (resp. v_1v_2) be the node-attachment (resp. edge-attachment) of p_k in an appropriate hole of Σ. Suppose $v = y_1$. W.l.o.g. $\{v_1, v_2\} \subseteq P_{x_3y}$. Then $H_{23} \cup P \cup u$ induces a $3PC(ux_2x_3, p_kv_1v_2)$ or a 4-wheel with center x_3. So $v \neq y_1$. W.l.o.g. $v = y_3$. Note that since p_k cannot be a center-crosspath of bug Σ, $y_3 \neq x_3$. Suppose v_1v_2 is an edge of P_{x_1y}. Let R be the subpath of P_{x_1y} with one endnode x_1 and the other adjacent to p_k. Then $P_{x_2y} \cup R \cup P \cup \{u, y_3\}$ induces a $3PC(x_2, p_k)$. So v_1v_2 is an edge of P_{x_2y}. But then $(P \setminus p_k) \cup u$ is the center-crosspath of the bug (H_{23}, p_k). So p_k is not of type crosspath w.r.t. Σ.

Now suppose that p_k is a pseudo-twin of y w.r.t. Σ. Then either p_kx_2 or p_kx_3 is not an edge. W.l.o.g. p_kx_3 is not an edge. But then $(\Sigma \setminus P_{x_2y}) \cup P \cup u$ contains a $3PC(x_3, p_k)$. This completes the proof of Claim 1.

Suppose that (v) does not hold. Then by Claim 1 and Lemma 4.3.1, p_k is of type p or d w.r.t. Σ.

Suppose that p_k is of type d. Suppose that $p_ky_1 \in E(G)$. So w.l.o.g. $N(p_k) \cap \Sigma = \{y, y_1, y_2\}$. If $x_2y \notin E(G)$, then $(H_{12} \setminus y) \cup P \cup u$ induces a $3PC(x_2, p_k)$. So $x_2y \in E(G)$. But then $(P_{x_1y} \setminus y) \cup P \cup \{u, x_2, x_3\}$ induces a 4-wheel with center x_2. So $p_ky_1 \notin E(G)$. Suppose that one of $\{x_2y, x_3y\}$ is an edge (note that by definition of $3PC(\Delta, \cdot)$, at most one of $\{x_2y, x_3y\}$ can be an edge). W.l.o.g. $x_2y \in E(G)$. But then $H_{12} \cup P \cup \{u, x_3\}$ induces a proper wheel with center x_2. So no one $\{x_2y, x_3y\}$ is an edge, and hence (iv) holds.

Suppose that p_k is of type p1. Let v be the neighbor of p_k in Σ. Note that $v \notin \{x_1, x_2, x_3\}$. If $v \in P_{x_1y}$, then $H_{12} \cup P \cup u$ induces a $3PC(x_2, v)$. So $v \notin P_{x_1y}$. W.l.o.g. $v \in P_{x_2y}$. If $v \neq x_2'$, then $H_{12} \cup P \cup u$ induces a $3PC(x_2, v)$. So $v = x_2'$. If x_3y is not an edge, then $H_{12} \cup P \cup x_3$ induces a 4-wheel with center x_2. So x_3y is an edge and hence (ii) holds.

Suppose that p_k is of type p2. Let v_1, v_2 be the nodes of $N(p_k) \cap \Sigma$. Suppose that v_1v_2 is not an edge of P_{x_1y}. W.l.o.g. v_1v_2 is an edge of P_{x_2y}. Then $H_{23} \cup P \cup u$ induces a $3PC(ux_2x_3, p_kv_1v_2)$ or a 4-wheel with center x_2. So v_1v_2 is an edge of P_{x_1y}, and hence (i) holds.

Suppose that p_k is of type p3. If $N(p_k) \cap \Sigma \subseteq P_{x_1y}$, then $H_{12} \cup P \cup u$ contains a $3PC(x_2, p_k)$. So w.l.o.g. assume $N(p_k) \cap \Sigma \subseteq P_{x_2y}$. If p_kx_2 is not an edge, then $H_{12} \cup P \cup u$ contains a $3PC(x_2, p_k)$. So p_kx_2 is an edge. If x_3y is not an edge, then $H_{12} \cup P \cup \{u, x_3\}$ contains a 4-wheel with center x_2. So x_3y is an edge and hence (iii) holds. $\square$

Lemma 4.5.3 *Let G be a 4-hole-free odd-signable graph that does not have a star cutset. Let u be a type t3 node w.r.t. Σ. Let $P = p_1, \dots, p_k$ be a chordless path in $G \setminus \Sigma$ such that p_1 is adjacent to u, p_k has a neighbor in $\Sigma \setminus \{x_1, x_2, x_3\}$, no node of $P \setminus \{p_1\}$ is adjacent to u, and no node of $P \setminus \{p_k\}$ has a neighbor in Σ. Then p_k is one of the following types:*

(i) p_k is of type p1, p3t, or it is a pseudo-twin of x_1, x_2 or x_3 w.r.t. Σ.

(ii) *p_k is a pseudo-twin of y w.r.t. Σ. Furthermore, if $N(p_k) \cap \Sigma \neq \{y, y_1, y_2, y_3\}$, then p_k is adjacent to a node of $\{x_1, x_2, x_3\}$ and Σ is not a bug.*

(iii) *p_k is of type p3b adjacent to x_i, for some $i \in \{1,2,3\}$, but not to x'_i.*

Proof: By Theorems 4.2.2, 4.3.3 and 4.3.6 we may assume that G does not contain a proper wheel nor a bug with a center-crosspath nor a $3PC(\Delta,\cdot)$ with a type s1 or s2 node. Since p_k has a neighbor in $\Sigma \setminus \{x_1, x_2, x_3\}$, p_k cannot be of type t2 nor t3 w.r.t. Σ.

Claim 1: *p_k is not of type p2, crosspath nor d w.r.t. Σ.*

Proof of Claim 1: Suppose that p_k is of type p2. W.l.o.g. $N(p_k) \cap \Sigma \subseteq P_{x_3y}$. But then $H_{23} \cup P \cup u$ induces a $3PC(\Delta, x_2x_3u)$ or a 4-wheel with center x_3. So p_k is not of type p2 w.r.t. Σ.

Suppose that p_k is of type crosspath. W.l.o.g (H_{23}, p_k) is a bug and y_2 is the node-attachment of p_k in H_{23}. Note that since p_k cannot be a center-crosspath of Σ, $y_2 \neq x_2$. But then $(P \setminus p_k) \cup u$ is a center-crosspath of (H_{23}, p_k). So p_k is not of type crosspath w.r.t. Σ.

Finally suppose that p_k is of type d w.r.t. Σ. W.l.o.g. $N(p_k) \cap \Sigma = \{y, y_1, y_3\}$. But then $H_{23} \cup P \cup u$ induces a $3PC(ux_2x_3, p_kyy_3)$ or a 4-wheel with center x_3. This completes the proof of Claim 1.

Assume (i) does not hold. Then by Claim 1 and Lemma 4.3.1, p_k is of type p3b or it is a pseudo-twin of y w.r.t. Σ. Suppose first that p_k is of type p3b. W.l.o.g. $N(p_k) \cap \Sigma \subseteq P_{x_3y}$. If x_3 is not the node-attachment of p_k in H_{23}, then $(P \setminus p_k) \cup u$ is a center-crosspath of (H_{23}, p_k). So x_3 is the node-attachment of p_k in H_{23}, and hence (iii) holds.

Suppose now that p_k is a pseudo-twin of y w.r.t. Σ. We may assume that $N(p_k) \cap \Sigma \neq \{y, y_1, y_2, y_3\}$, else (ii) holds. W.l.o.g. $N(p_k) \cap \Sigma = \{y, y_1, y_3, v\}$, where v is a node of $P_{x_2y} \setminus \{y, y_2\}$. If $v \neq x_2$, then $(P \setminus p_k) \cup u$ is a center-crosspath of (H_{23}, p_k). So $v = x_2$. Since p_k is a pseudo-twin of y w.r.t. Σ, $|N(p_k) \cap \{x_1, x_2, x_3\}| \leq 1$ and hence Σ cannot be a bug, so (ii) holds. $\square$

4.6 Connected diamonds

In this section we prove Theorem 2.2.5. Recall the definition of a connected diamond (Σ, Q) from Section 2.2.1. Note that if $Q = q_1, \ldots, q_k$, then q_1 is of type t2 w.r.t. Σ and q_k is of type p2 or d w.r.t. Σ.

Lemma 4.6.1 *Let G be a 4-hole-free odd-signable graph. If G contains a $3PC(\Delta,\cdot)$ with a node of type dd, then either G has a star cutset or G contains a connected diamond.*

Proof: Assume not. By Theorems 4.2.2, 4.3.3 and 4.3.6, G does not contain a proper wheel nor a bug with a center-crosspath nor a $3PC(\Delta,\cdot)$ with a type s1 or s2 node. Let u be a type dd node w.r.t. a $\Sigma = 3PC(x_1x_2x_3, y)$ of G, such that w.l.o.g. $N(u) \cap \Sigma = \{y, y_1, y_3\}$. So x_1y and x_3y are not edges.

Since $S = N[y] \setminus \{u, y_2\}$ is not a star cutset separating u from $\Sigma \setminus S$, there is a direct connection $P = p_1, \dots, p_k$ from u to Σ in $G \setminus S$. So p_1 is adjacent to u and p_k has a neighbor in $\Sigma \setminus S$. Note that the only nodes of Σ that may have a neighbor in $P \setminus p_k$ are y_1 and y_3. For $i, j \in \{1,2,3\}$, $i \neq j$, let H_{ij} be the hole induced by $P_{x_iy} \cup P_{x_jy}$. By Lemma 4.3.1 and since p_k is not adjacent to y, p_k is of type p, t2, t3, crosspath or it is a pseudo-twin of x_1, x_2 or x_3 w.r.t. Σ.

Claim 1: At most one of y_1, y_3 has a neighbor in $P \setminus p_k$.

Proof of Claim 1: Suppose both y_1, y_3 have a neighbor in $P \setminus p_k$. Let R be a shortest subpath of $P \setminus p_k$ with one endnode adjacent to y_1 and the other to y_3. Then $H_{13} \cup R$ induces a $3PC(y_1, y_3)$. This completes the proof of Claim 1.

We now consider the following cases.

Case 1: p_k does not have a neighbor in $P_{x_2y} \setminus x_2$.

Case 1.1: No node of $\{y_1, y_3\}$ has a neighbor in $P \setminus p_k$.

Then no node of Σ has a neighbor in $P \setminus p_k$.

Case 1.1.1: p_k is of type crosspath w.r.t. Σ.

Since p_k cannot be a center-crosspath of bug Σ, p_k is not adjacent to x_2. W.l.o.g. $N(p_k) \cap P_{x_1y} = y_1$ and p_k has two adjacent neighbors in P_{x_3y}. If $k = 1$, then $(H_{13} \setminus y) \cup \{u, p_1\}$ induces a 4-wheel with center p_1. So $k > 1$. Let R be the shortest path from u to p_k in $(P_{x_3y} \setminus y) \cup \{u, p_k\}$. Then $P \cup R \cup \{y_1\}$ induces a $3PC(u, p_k)$.

Case 1.1.2: p_k is of type t2, t3 or it is a pseudo-twin of x_1, x_2 or x_3 w.r.t. Σ.

If p_k is of type t2 adjacent to x_1 and x_3, then $\Sigma \cup P \cup u$ induces a connected diamond. Note that since p_k does not have a neighbor in $P_{x_2y} \setminus x_2$, p_k cannot be a pseudo-twin of x_2 w.r.t. Σ. So w.l.o.g. p_k is adjacent to x_1 and x_2 and $N(p_k) \cap (\Sigma \setminus \{x_1, x_2\}) \subseteq P_{x_3y}$. Recall that p_k cannot be adjacent to y. But then $H_{12} \cup P \cup u$ induces a $3PC(uyy_1, x_1x_2p_k)$.

Case 1.1.3: p_k is of type p w.r.t. Σ.

Suppose p_k is of type p1 and let p' be the neighbor of p_k in $\Sigma \setminus S$. If $p' = x_2$, then $\Sigma \cup P \cup u$ induces a connected diamond (Σ', Q), where $\Sigma' = 3PC(yy_1u, x_2)$ and $Q = P_{x_3y} \setminus y$. So $p' \neq x_2$. But then $(H_{13} \setminus y) \cup P \cup u$ induces a $3PC(u, p')$. So p_k is not of type p1. So the neighbors of p_k in $\Sigma \setminus S$ lie in either P_{x_1y} or P_{x_3y}. W.l.o.g. $N(p_k) \cap \Sigma \subseteq P_{x_3y}$. If p_k is of type p2, then $H_{23} \cup P \cup u$ induces either a $3PC(uyy_3, \Delta)$ or a 4-wheel with center y_3. So p_k is of type p3. If $k = 1$, then $(H_{13} \setminus y) \cup \{u, p_1\}$ induces a 4-wheel with center p_1. So $k > 1$. But then $(H_{13} \setminus y) \cup P \cup u$ contains a $3PC(u, p_k)$.

Case 1.2: A node of $\{y_1, y_3\}$ has a neighbor in $P \setminus p_k$.

By Claim 1, exactly one of $\{y_1, y_3\}$ has a neighbor in $P \setminus p_k$. Note that $k > 1$.

Case 1.2.1: p_k is of type p.

If p_k is of type p1 adjacent to x_2, then $\Sigma \cup P$ contains a $3PC(x_2, y_1)$ (if y_1 has a neighbor in $P \setminus p_k$) or a $3PC(x_2, y_3)$ (if y_3 has a neighbor in $P \setminus p_k$). So by symmetry w.l.o.g. $N(p_k) \cap \Sigma \subseteq P_{x_3y} \setminus y$. Let p' (resp. p'') be the node of $N(p_k) \cap P_{x_3y}$ closest to y_3 (resp. x_3). Note that if p_k is of type p1, then $p' \in P_{x_3y} \setminus \{y, y_3\}$. Let R be the subpath of P_{x_3y} between p'' and x_3. Let H be the hole induced by $P_{x_2y} \cup P \cup R \cup u$.

Suppose $N(y_3) \cap (P \setminus p_k) \neq \varnothing$. Since (H, y_3) is not a proper wheel, $|N(y_3) \cap P| = 1$ and $p''y_3$ is not an edge. Let p_i be the unique neighbor of y_3 in P. Note that $i < k$. If p_k is of type p1, then $H_{23} \cup P$ contains a $3PC(y_3, p')$. So p_k is of type p2 or p3. If $N(y_3) \cap P = p_1$, then $P_{x_1y} \cup P \cup R \cup \{y_3, u\}$ induces a 4-wheel with center u. So $i > 1$. If p_k is of type p2, then (H, y_3) is a bug and $P_{x_3y} \setminus (R \cup \{y, y_3\})$ is its center-crosspath. So p_k is of type p3. But then $H_{23} \cup \{p_i, \ldots, p_k\}$ contains a $3PC(y_3, p_k)$.

So $N(y_3) \cap (P \setminus p_k) = \varnothing$. Hence $N(y_1) \cap (P \setminus p_k) \neq \varnothing$. Since (H, y_1) is not a proper wheel, y_1 has a unique neighbor, say p_i, in P. Let R' be the subpath of P_{x_3y} between y_3 and p'. If $i = 1$, then $P \cup R' \cup \{y, y_1, u\}$ induces a 4-wheel with center u. So $i > 1$. But then $P \cup R' \cup \{y_1, u\}$ induces a $3PC(u, p_i)$.

Case 1.2.2: p_k is of type t2, t3 or it is a pseudo-twin of x_1, x_2 or x_3 w.r.t. Σ.

Suppose p_k is of type t2 adjacent to x_1 and x_3. By symmetry w.l.o.g. $N(y_3) \cap P \neq \varnothing$ and $N(y_1) \cap P = \varnothing$. Let H be the hole induced by $P_{x_2y} \cup P \cup \{x_3, u\}$. Since (H, y_3) is not a proper wheel, x_3y_3 is not an edge. But then $H_{23} \cup P$ contains a $3PC(x_3, y_3)$. So p_k is not of type t2 adjacent to x_1 and x_3.

Recall that p_k has no neighbor in $P_{x_2y} \setminus x_2$. So by symmetry w.l.o.g. p_k is adjacent to both x_1 and x_2 and $N(p_k) \cap (\Sigma \setminus \{x_1, x_2\}) \subseteq P_{x_3y} \setminus y$. If $N(y_1) \cap P = \varnothing$, then $H_{12} \cup P \cup u$ induces a $3PC(uyy_1, x_1x_2p_k)$. So $N(y_1) \cap (P \setminus p_k) \neq \varnothing$ and $N(y_3) \cap (P \setminus p_k) = \varnothing$. Let H be the hole induced by $P_{x_2y} \cup P \cup u$. Since (H, y_1) is not a proper wheel y_1 has unique neighbor, say p_i, in P.

Suppose p_k is of type t3. If $i = 1$, then $P_{x_3y} \cup P \cup \{y_1, u\}$ induces a 4-wheel with center u. So $i > 1$. But then $(P_{x_3y} \setminus y) \cup P \cup \{y_1, u\}$ induces a $3PC(p_i, u)$. So p_k is not of type t3.

Suppose p_k is of type t2. If yx_2 is an edge, then since there is no 4-hole y_1x_1 is not an edge. But then $P_{x_3y} \cup \{p_i, \ldots, p_k, y_1, x_2, x_1\}$ induces a 4-wheel center x_2. So yx_2 is not an edge. But then $H_{23} \cup \{p_i, \ldots, p_k, y_1\}$ induces a $3PC(y, x_2)$.

So p_k is a pseudo-twin of x_3 w.r.t. Σ. Let R be the shortest path from p_k to y_3 in $P_{x_3y} \cup p_k$. If $i = 1$, then $P \cup R \cup \{y_1, y, u\}$ induces a 4-wheel with center u. So $i > 1$. But then $P \cup R \cup \{y_1, u\}$ induces a $3PC(u, p_i)$.

Case 1.2.3: p_k is of type crosspath w.r.t. Σ.

Since p_k cannot be a center-crosspath of bug Σ, p_k is not adjacent to x_2.

W.l.o.g. $N(p_k) \cap P_{x_3y} = y_3$ and $N(p_k) \cap (\Sigma \setminus y_3) \subseteq P_{x_1y} \setminus y$. Let p' (resp. p'') be the node of $N(p_k) \cap P_{x_1y}$ closest to y_1 (resp. x_1). Let R' (resp. R'') be the y_1p'-subpath (resp. x_1p''-subpath) of P_{x_1y}. If $N(y_3) \cap (P \setminus p_k) \neq \varnothing$, then $P \cup P_{x_2y} \cup R'' \cup \{u, y_3\}$ induces a proper wheel with center y_3. So $N(y_3) \cap (P \setminus p_k) = \varnothing$ and $N(y_1) \cap (P \setminus p_k) \neq \varnothing$. Let p_i be the node of $N(y_1) \cap P$ with highest index. If $i = 1$, then $P \cup \{y, y_1, y_3, u\}$ induces a 4-wheel with center u. So $i > 1$. Let H be the hole induced by $R'' \cup P_{x_2y} \cup P \cup u$. If $p' = y_1$, then (H, y_1) is a proper wheel. So $p' \neq y_1$, and hence (H, y_1) is a bug. But then $R' \setminus y_1$ is a center-crosspath of (H, y_1).

Case 2: p_k has a neighbor in $P_{x_2y} \setminus x_2$.

Case 2.1: p_k is of type p w.r.t. Σ.

In this case $N(p_k) \cap \Sigma \subseteq P_{x_2y}$.

Suppose that $\{y_1, y_3\}$ have no neighbor in $P \setminus p_k$. If p_k is of type p1, then $\Sigma \cup P$ induces a connected diamond $(\Sigma', P_{x_3y} \setminus y)$ (where Σ' is the $3PC(y_1yu, \cdot)$ induced by $P_{x_1y} \cup P_{x_2y} \cup P$). If p_k is of type p2, then $H_{12} \cup P \cup u$ induces a $3PC(uyy_1, \Delta)$. So p_k is of type p3. Let R be the chordless path from y to x_2 in $P_{x_2y} \cup p_k$ that contains p_k. Then $P_{x_1y} \cup P_{x_3y} \cup P \cup R \cup u$ induces a connected diamond $(\Sigma', P_{x_3y} \setminus y)$ (where Σ' is the $3PC(y_1yu, p_k)$ induced by $P_{x_1y} \cup R \cup P$). So one of $\{y_1, y_3\}$ has a neighbor in $P \setminus p_k$.

Therefore $k > 1$. By Claim 1, w.l.o.g. $N(y_3) \cap (P \setminus p_k) \neq \varnothing$ and $N(y_1) \cap (P \setminus p_k) = \varnothing$. Let R' (resp. R'') be the shortest path in $P_{x_2y} \cup p_k$ between y (resp. x_2) and p_k. Let H be the hole induced by $R' \cup P \cup u$. Since (H, y_3) is not a proper wheel, y_3 has a unique neighbor, say p_i, in P. Note that $i < k$. If p_k is of type p1, then $H_{23} \cup \{p_i, \ldots, p_k\}$ induces a $3PC(y_3, \cdot)$. If p_k is of type p3, then $R' \cup R'' \cup P_{x_3y} \cup \{p_i, \ldots, p_k\}$ induces a $3PC(y_3, p_k)$. So p_k is of type p2. If $i > 1$, then (H, y_3) is a bug and the path induced by $(P_{x_3y} \setminus \{y, y_3\}) \cup (R'' \setminus p_k)$ is its center-crosspath. So $i = 1$. But then $P_{x_1y} \cup P \cup R'' \cup \{y_3, u\}$ induces a 4-wheel with center u.

Case 2.2: p_k is of type t2, t3 or it is a pseudo-twin of x_1, x_2 or x_3 w.r.t. Σ.

Then p_k is a pseudo-twin of x_2 w.r.t. Σ. Let $\Sigma' = 3PC(x_1 p_k x_3, y)$ obtained by substituting p_k into Σ. If no node of $\{y_1, y_3\}$ has a neighbor in $P \setminus p_k$, then $\Sigma' \cup P \cup u$ induces a connected diamond (Σ'', Q), where $\Sigma'' = 3PC(y_1 y u, p_k)$ and $Q = P_{x_3y} \setminus y$. So w.l.o.g. y_3 has a neighbor in $P \setminus p_k$. Let p_i be the node of P with highest index adjacent to y_3. Note that $i < k$. But then $(\Sigma' \setminus (P_{x_1y} \setminus y)) \cup \{p_i, \dots, p_k\}$ induces a $3PC(y_3, p_k)$.

Case 2.3: p_k is of type crosspath w.r.t. Σ.

Suppose $N(p_k) \cap P_{x_2y} = y_2$. W.l.o.g. $N(p_k) \cap (\Sigma \setminus y_2) \subseteq P_{x_3y} \setminus y$ and, in particular, (H_{23}, p_k) is a bug. If $N(y_3) \cap (P \setminus p_k) = \varnothing$, then $(P \setminus p_k) \cup u$ induces a center-crosspath of (H_{23}, p_k). So $N(y_3) \cap (P \setminus p_k) \neq \varnothing$ and consequently $k > 1$. Let p' (resp. p'') be the neighbor of p_k in P_{x_3y} closest to y_3 (resp. x_3). Let R be the subpath of P_{x_3y} between p'' and x_3. Let H be the hole induced by $P \cup \{u, y, y_2\}$. Since (H, y_3) is not a proper wheel, y_3 has a unique neighbor in $P \setminus p_k$ and $p' \neq y_3$. Let p_i be the neighbor of y_3 in P. If $i = 1$, then $P_{x_1y} \cup R \cup P \cup \{y_3, u\}$ induces a 4-wheel with center u. So $i > 1$. But then $(P_{x_1y} \setminus y) \cup P \cup R \cup \{u, y_3\}$ induces a $3PC(u, p_i)$. So $N(p_k) \cap P_{x_2y} \neq y_2$.

W.l.o.g. $N(p_k) \cap P_{x_3y} = y_3$ and p_k has two adjacent neighbors in P_{x_2y}. Let p' (resp. p'') be the node of $N(p_k) \cap P_{x_2y}$ closest to y_2 (resp. x_2). Let R' (resp. R'') be the subpath of P_{x_2y} between y (resp. x_2) and p' (resp. p''). If $k = 1$, then $P_{x_1y} \cup R'' \cup \{p_1, y_3, u\}$ induces a 4-wheel with center u. So $k > 1$. If no node of $\{y_1, y_3\}$ has a neighbor in $P \setminus p_k$, then $(P_{x_1y} \setminus y) \cup P \cup R'' \cup \{u, y_3\}$ induces a $3PC(u, p_k)$. So by Claim 1, exactly one of y_1, y_3 has a neighbor in $P \setminus p_k$. Suppose y_1 has a neighbor in $P \setminus p_k$ and let p_i be the node of $N(y_1) \cap P$ with highest index. Then $H_{13} \cup \{p_i, \dots, p_k\}$ induces a $3PC(y_1, y_3)$. So y_1 does not have a neighbor in $P \setminus p_k$ and hence $N(y_3) \cap (P \setminus p_k) \neq \varnothing$. But then $P \cup R' \cup \{u, y_3\}$ induces a proper wheel with center y_3. $\square$

Lemma 4.6.2 *Let G be a 4-hole-free odd-signable graph. If G contains a bug with a type dc node, then G has a star cutset or G contains a connected diamond.*

Proof: Assume not. By Lemma 4.4.1 every bug (H, x) has a bridge P. Choose a bug (H, x) with a type dc node u, and a bridge $P = p_1, \dots, p_k$ of (H, x) so that the length of P is minimized. Let x_1, x_2, y be the neighbors of x in H such that $x_1 x_2$ is an edge. Let H_1 (resp. H_2) be the sector of (H, x) with endnodes y and x_1 (resp. x_2). Let y_1 (resp. y_2) be the neighbor of y in H_1 (resp. H_2). So u is adjacent to x, y and a node of $\{y_1, y_2\}$. W.l.o.g. p_1 has a neighbor in $H_1 \setminus \{x_1, y\}$ and p_k in $H_2 \setminus \{x_2, y\}$.

By Lemma 4.6.1 G does not contain a $3PC(\Delta,\cdot)$ with a type dd node, and hence P is not a bridge of type D. Let H' be the hole of $(H\setminus y)\cup P$ that contains P. If P is a bridge of type C2, C4, C5 or T, then $H'\cup\{x,y\}$ induces a a union of a $3PC(x_1x_2x,y)$ and a type dd node w.r.t. this $3PC$, a contradiction.

Suppose that P is a bridge of type C3. W.l.o.g. p_1 is adjacent to y, i.e., p_1 is of type p3t w.r.t. (H,x). Note that since $\{x_1,x,y,p_1\}$ cannot induce a 4-hole, p_1x_1 is not an edge. But then $H'\cup\{x,y\}$ induces a $3PC(x_1x_2x,p_1)$ and y_1 is of type dd w.r.t. it, a contradiction.

Suppose that P is a bridge of type C1. Let p_i be the unique neighbor of y in P. Note that $1<i<k$. Let $\Sigma=3PC(x_1x_2x,p_i)$ induced by $H'\cup\{x,y\}$. W.l.o.g. u is adjacent to y_2. If u does not have a neighbor in P, then $(H\setminus\{y_1,x_2\})\cup P\cup\{x,u\}$ contains a 4-wheel with center y. So u has a neighbor in P. By Lemma 4.3.1 applied to Σ and u, $N(u)\cap P=\{p_i\}$, $\{p_{i+1}\}$ or $\{p_{i-1}\}$. Since G does not contain a 4-hole, $N(u)\cap P=\{p_i\}$. Let $H_1'=H'\cap H_1$ and $H_2'=H'\cap H_2$. Let H'' be the hole induced by $H_1\cup H_2'\cup\{p_i,\dots,p_k\}$. Then (H'',x) is a bug, u is of type dc w.r.t. (H'',x) and $P'=p_1,\dots,p_{i-1}$ is a bridge of (H'',x), and hence (H'',x), u and P' contradict our choice of (H,x), u and P.

Therefore P is a bridge of type A. W.l.o.g. $N(p_1)\cap H_1=y_1$ and p_k has two adjacent neighbors in $H_2\setminus y$. First suppose that u is adjacent to y_2. If u does not have a neighbor in P, then $(H\setminus x_2)\cup P\cup\{u,x\}$ contains a 4-wheel with center y. So u has a neighbor in P, and let p_i be such a neighbor with highest index. Since $\{y,y_1,u,p_1\}$ cannot induce a 4-hole, $i>1$. But then $H\cup\{u,p_i,\dots,p_k\}$ induces a $3PC(\Delta,\Delta)$ or a 4-wheel with center y_2.

So u must be adjacent to y_1. If u has a neighbor in P, then $(H_2\setminus y_2)\cup P\cup\{u,y_1,x\}$ contains a proper wheel with center u. So u does not have a neighbor in P. But then $H_2\cup P\cup\{x,y_1\}$ induces a $3PC(\Delta,y)$, and u is of type dd w.r.t. it, a contradiction. □

Lemma 4.6.3 *Let G be a 4-hole-free odd-signable graph. If G contains a* $3PC(\Delta,\cdot)$ *with a node of type d, then either G has a star cutset or G contains a connected diamond.*

Proof: Follows from Lemmas 4.6.1 and 4.6.2. □

For a twin wheel (H,x) we use the following notation. Let x_1, x_2, x_3 be the neighbors of x in H such that x_1x_2 and x_2x_3 are edges. Let x_1' (resp. x_3') be the neighbor of x_1 (resp. x_3) in $H\setminus x_2$. A node $u\in V(G)\setminus(V(H)\cup\{x\})$ is said to be of type d w.r.t. (H,x) if ux is an edge and $N(u)\cap H$ is either $\{x_1,x_1'\}$ or $\{x_3,x_3'\}$.

Lemma 4.6.4 *Let G be a 4-hole-free odd-signable graph. If G contains a twin wheel with a type d node, then either G contains a star cutset or G contains a connected diamond.*

Proof: Assume not. By Theorem 4.2.2, Theorem 4.3.3 and Lemma 4.6.3, G does not contain a proper wheel, a bug with a center-crosspath, nor a $3PC(\Delta,\cdot)$ with a type d node. Let u be a type d node w.r.t. a twin wheel (H,x) in G. Let x_1,x_2,x_3 be the neighbors of x in H such that x_1x_2 and x_2x_3 are edges. Let $P_H = x_3, p_1, \dots, p_k, x_1$ be the long sector of (H,x). Let $P = p_1, \dots, p_k$.

Note that since there is no 4-hole, $k > 1$. W.l.o.g. $N(u) \cap H = \{x_3, p_1\}$. Since $S = N[x] \setminus x_2$ is not a star cutset of G separating x_2 from P, there exists a direct connection $Q = q_1, \dots, q_l$ from x_2 to P in $G \setminus S$. Let p_i (resp. $p_{i'}$) be the node of $N(q_l) \cap P$ with lowest (resp. highest) index. Note that x_1 and x_3 are the only nodes of H that may have a neighbor in $Q \setminus q_l$.

Claim 1: Both u and x_3 have a neighbor in Q.

Proof of Claim 1: $N(u) \cap Q \neq \varnothing$, else $Q \cup \{x, x_2, x_3, u, p_1, \dots, p_i\}$ induces a proper wheel with center x_3. Now suppose $N(x_3) \cap Q = \varnothing$. Let H' be the hole induced by $Q \cup \{x_2, x_3, p_1, \dots, p_i\}$. So (H',u) is a bug or a twin wheel. If (H',u) is a bug, then x is a center-crosspath of (H',u). So (H',u) is a twin wheel, and hence $i = 1$ and $N(u) \cap Q = q_l$. Since $\{u, x, x_1, q_l\}$ cannot induce a 4-hole, x_1q_l is not an edge. Since $\{u, x_3, x_2, q_l\}$ cannot induce a 4-hole, $l > 1$. Suppose $i' = 1$. If $N(x_1) \cap Q = \varnothing$, then $H \cup Q$ induces a $3PC(x_2, p_1)$. So $N(x_1) \cap Q \neq \varnothing$. Let q_s be the node of $N(x_1) \cap Q$ with highest index. Then $\{x, x_1, x_3, p_1, q_s, \dots, q_l, u\}$ induces a 4-wheel with center u. So $i' > 1$. But then $\{u, x_1, x_2, x_3, q_l, p_{i'}, \dots, p_k, x\}$ induces a 4-wheel with center x. So $N(x_3) \cap Q \neq \varnothing$. This completes the proof of Claim 1.

Claim 2: $N(x_1) \cap Q = \varnothing$.

Proof of Claim 2: Suppose x_1 does have a neighbor in Q. By Claim 1, u and x_3 both have neighbors in Q. Let q_s (resp. q_t) be the node of Q with lowest index adjacent to x_3 (resp. u). If $s \leq t$, then $\{x, x_2, x_3, u, q_1, \dots, q_t\}$ induces a proper wheel with center x_3. So $s > t$. In particular, $t < l$ and $s > 1$.

If x_1 has a neighbor in $Q \setminus q_l$, then both x_1 and u (since $t < l$) have a neighbor in $Q \setminus q_l$ and hence $(Q \setminus q_l) \cup P \cup \{x, u, x_1\}$ contains a $3PC(x_1, u)$. So x_1 does not have a neighbor in $Q \setminus q_l$, and hence $N(x_1) \cap Q = \{q_l\}$.

Let H' be the hole induced by $Q \cup \{x_1, x_2\}$. Since $H' \cup x_3$ cannot induce a $3PC(\cdot,\cdot)$, (H', x_3) is a wheel, and hence it is a twin wheel or a bug. Since $s > 1$, (H', x_3) must in fact be a bug. But then x is of type d w.r.t. bug (H', x_3), a contradiction. This completes the proof of Claim 2.

By Claim 1, let q_s (resp. q_t) be the node of Q with lowest index adjacent to x_3 (resp. u). If $s = 1$, then $\{x, x_2, x_3, u, q_1, \dots, q_t\}$ induces a proper wheel with center x_3, a contradiction. So $s > 1$. By Claim 2, the node set $Q \cup \{x_1, x_2, p_{i'}, \dots, p_k\}$ induces a hole, say H'. Node x_3 must have at least two neighbors in Q, else $H' \cup x_3$ induces a $3PC(x_2, q_s)$. So (H', x_3) is a wheel. By our assumption (H', x_3) cannot be a proper wheel, and since $s > 1$ it cannot be a twin wheel, hence it is a bug where x_2 does not belong to the short sector of (H', x_3). But then node x is of type d w.r.t. (H', x_3), a contradiction. □

Proof of Theorem 2.2.5: Suppose not. By Theorems 4.2.2 and 4.3.3 and Lemmas 4.6.3 and 4.6.4 we may assume that G does not contain a proper wheel, a bug with a center-crosspath, a $3PC(\Delta, \cdot)$ with a node of type d, nor a twin wheel with a node of type d.

We may assume that G contains a diamond induced by, say, $\{u, v, a, b\}$, where $ab \notin E(G)$. Let $S = N[u] \setminus \{a, b\}$. Since S cannot be a star cutset separating a from b, there is a direct connection $P = p_1, \dots, p_k$ in $G \setminus S$ from a to b. If v has a neighbor in P, then $P \cup \{a, b, u, v\}$ induces a proper wheel with center v. So $N(v) \cap P = \varnothing$. Let $S' = N[u] \setminus v$. Since S' cannot be a star cutset of G, there is direct connection $Q = q_1, \dots, q_l$ from v to P. Let p_i (resp. $p_{i'}$) be the node of $N(q_l) \cap P$ with lowest (resp. highest) index.

Suppose both a and b have a neighbor in $Q \setminus q_l$. Let R be a shortest path between a and b in the subgraph induced by $(Q \setminus q_l) \cup \{a, b\}$. Then $P \cup R \cup \{a, b, u\}$ induces a $3PC(a, b)$. So one of a, b does not have a neighbor in $Q \setminus q_l$. W.l.o.g. $N(b) \cap (Q \setminus q_l) = \varnothing$.

Claim 1: $N(b) \cap Q = \varnothing$.

Proof of Claim 1: Suppose not. So $N(b) \cap Q = q_l$. Suppose $l = 1$. Since there is no 4-hole, aq_l is not an edge. Since $P \cup \{v, a, b, q_1\}$ cannot induce a proper wheel with center q_1, $i = i'$. If $i = k$, then $P \cup \{a, b, u, v\}$ induces a twin wheel with a node of type d. So $i < k$. But then $\{p_1, \dots, p_i, q_1, a, b, u, v\}$ induces a 4-wheel with center v. So $l > 1$.

Suppose $N(a) \cap Q = \varnothing$. If $i = k$, then $P \cup Q \cup \{a, b, u, v\}$ induces a bug with center b with a node u of type dc. So $i < k$. But then $Q \cup \{p_1, \dots, p_i, a, b, v\}$ induces a $3PC(v, q_l)$. So $N(a) \cap Q \neq \varnothing$.

Suppose a has a unique neighbor, say q_j, in Q. If $j = 1$, then $Q \cup \{a, b, u, v\}$ induces a 4-wheel with center v. So $j > 1$. But then $Q \cup \{a, b, v\}$ induces a $3PC(v, q_j)$. So $|N(a) \cap Q| \geq 2$. Let H be the hole induced by $Q \cup \{v, b\}$. Since there is no proper wheel, (H, a) is either a bug or a twin wheel. If (H, a) is a bug, then u is either its center-crosspath or a node of type dc. So (H, a) is a twin wheel. But then u is a node of type d w.r.t. (H, a).

This completes the proof of Claim 1.

Suppose $N(a) \cap Q = \varnothing$. If $i = i'$, then $P \cup Q \cup \{a,b,v\}$ induces a $3PC(v, p_i)$. So $i' > i$. If $p_i p_{i'}$ is an edge, then $P \cup Q \cup \{a,b,v\}$ induces a $3PC(q_l p_i p_{i'}, v)$ with a node of type dd. So $p_i p_{i'}$ is not an edge. If $l = 1$, then $P \cup \{a,b,v,q_1\}$ induces a proper wheel with center q_1. So $l > 1$. But then $Q \cup \{a,b,v,p_1,\dots,p_i,p_i',\dots,p_k\}$ induces a $3PC(v,q_l)$. So $N(a) \cap Q \neq \varnothing$.

Let H be the hole induced by $Q \cup \{b,v,p_{i'},\dots,p_k\}$. Note that since a has a neighbor in Q, it has at least two neighbors in H. Suppose $|N(a) \cap H| = 2$ and let v' be the neighbor of a in $H \setminus v$. If vv' is an edge, then $H \cup \{a,u\}$ induces a 4-wheel with center v. So vv' is not an edge. But then $H \cup a$ induces a $3PC(v,v')$. Therefore, since (H,a) cannot induce a proper wheel, (H,a) is either a bug or a twin wheel. If (H,a) is a bug, then u is either its center-crosspath or a node of type dc. So (H,a) is a twin wheel, and hence u is a node of type d w.r.t. (H,a). $\square$

Chapter 5

Decomposing Connected Diamonds

5.1 2-joins and blocking sequences

In this section we consider an induced subgraph H of G that contains a 2-join $H_1|H_2$. We say that a 2-join $H_1|H_2$ *extends* to G if there exists a 2-join of G, $H_1'|H_2'$ with $H_1 \subseteq H_1'$ and $H_2 \subseteq H_2'$. We characterize the situation in which the 2-join of H does not extend to a 2-join of G.

Definition 5.1.1 *A* blocking sequence *for a 2-join $H_1|H_2$ of an induced subgraph H of G is a sequence of distinct nodes $x_1,\dots,x_n$ in $G \setminus H$ with the following properties:*

(1) *(i)* *$H_1|H_2 \cup x_1$ is not a 2-join of $H \cup x_1$,*

(ii) *$H_1 \cup x_n|H_2$ is not a 2-join of $H \cup x_n$, and*

(iii) *if $n > 1$ then, for $i = 1,\dots,n-1$, $H_1 \cup x_i|H_2 \cup x_{i+1}$ is not a 2-join of $H \cup \{x_i, x_{i+1}\}$.*

(2) *$x_1,\dots,x_n$ is minimal w.r.t. property (1), in the sense that no sequence $x_{j_1},\dots,x_{j_k}$ with $\{x_{j_1},\dots,x_{j_k}\} \subset \{x_1,\dots,x_n\}$, satisfies (1).*

Blocking sequences for 2-joins were introduced in [13], where the following results are obtained.

Let H be an induced subgraph of G with 2-join $H_1|H_2$ and special sets (A_1,A_2,B_1,B_2). In the following results we let $S = x_1,\dots,x_n$ be a blocking sequence for the 2-join $H_1|H_2$ of an induced subgraph H of G.

Remark 5.1.2 *$H_1|H_2 \cup u$ is a 2-join of $H \cup u$ if and only if $N(u) \cap H_1 = \varnothing, A_1$ or B_1. Similarly, $H_1 \cup u|H_2$ is a 2-join of $H \cup u$ if and only if $N(u) \cap H_2 = \varnothing, A_2$ or B_2.*

Lemma 5.1.3 *If $n > 1$ then, for every node x_j, $j \in \{1, \dots, n-1\}$, $N(x_j) \cap H_2 = \varnothing, A_2$ or B_2, and for every node x_j, $j \in \{2, \dots, n\}$, $N(x_j) \cap H_1 = \varnothing, A_1$ or B_1.*

Lemma 5.1.4 *Assume $n > 1$. Nodes x_i, x_{i+1}, $1 \leq i \leq n-1$, are not adjacent if and only if $N(x_i) \cap H_2 = A_2$ and $N(x_{i+1}) \cap H_1 = A_1$, or $N(x_i) \cap H_2 = B_2$ and $N(x_{i+1}) \cap H_1 = B_1$.*

Theorem 5.1.5 *Let H be an induced subgraph of a graph G that contains a 2-join $H_1|H_2$. The 2-join $H_1|H_2$ of H extends to a 2-join of G if and only if there exists no blocking sequence for $H_1|H_2$ in G.*

Lemma 5.1.6 *For $1 < i < n$, $H_1 \cup \{x_1, \dots, x_{i-1}\}|H_2 \cup \{x_{i+1}, \dots, x_n\}$ is a 2-join of $H \cup (S \setminus \{x_i\})$.*

Lemma 5.1.7 *If $x_i x_k$, $n \geq k > i+1 \geq 2$, is an edge, then either $N(x_i) \cap H_2 = A_2$ and $N(x_k) \cap H_1 = A_1$, or $N(x_i) \cap H_2 = B_2$ and $N(x_k) \cap H_1 = B_1$.*

Lemma 5.1.8 *If x_j is the node of lowest index adjacent to a node of H_2, then $x_1, \dots, x_j$ is a chordless path. Similarly, if x_j is the node of highest index adjacent to a node of H_1, then $x_j, \dots, x_n$ is a chordless path.*

Theorem 5.1.9 *Let G be a graph and H an induced subgraph of G with a 2-join $H_1|H_2$ and special sets (A_1, A_2, B_1, B_2). Let H' be an induced subgraph of G with 2-join $H_1'|H_2$ and special sets (A_1', A_2, B_1', B_2) such that $A_1' \cap A_1 \neq \varnothing$ and $B_1' \cap B_1 \neq \varnothing$. If S is a blocking sequence for $H_1|H_2$ and $H_1' \cap S \neq \varnothing$, then a proper subset of S is a blocking sequence for $H_1'|H_2$.*

5.2 Decomposing connected diamonds

In this section we prove Theorem 2.2.6.

Recall that a connected diamond is a pair (Σ, Q), where $\Sigma = 3PC(x_1x_2x_3, y)$ and $Q = q_1, \dots, q_k$, $k \geq 2$, is a chordless path disjoint from Σ such that the only nodes of Q adjacent to Σ are q_1 and q_k. Furthermore q_1 is of type t2 w.r.t. Σ adjacent to, say x_1 and x_3 and one of the following holds:

(i) q_k is of type p2 such that $N(q_k) \cap V(\Sigma) \subseteq V(P_{x_2y}) \setminus \{x_2\}$, or

(ii) q_k is of type d adjacent to y, y_1, y_3 such that x_1y and x_3y are not edges.

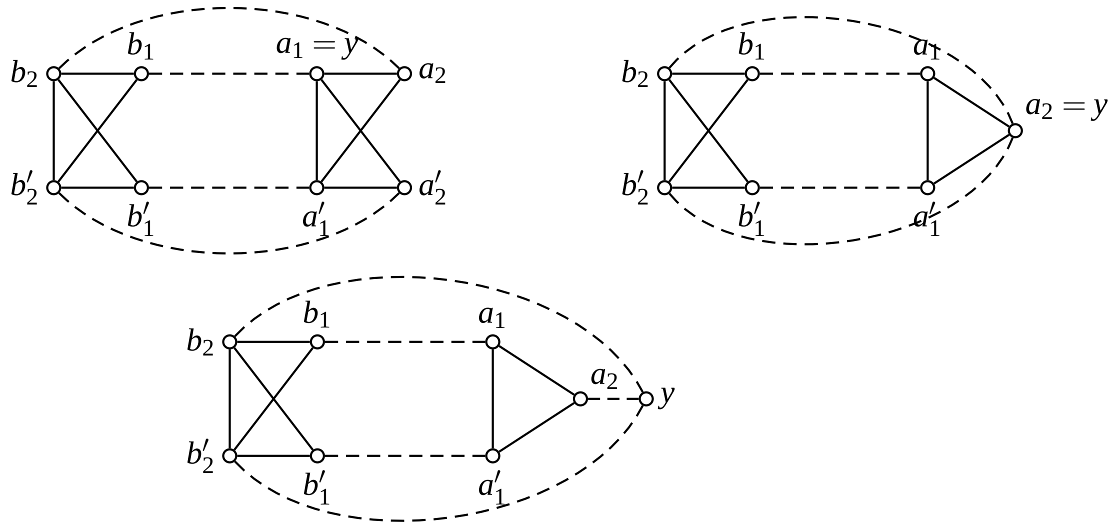


Figure 5.1: Different types of connected diamonds.

We rename some nodes and introduce some additional notation. Let $a_1' = q_k$ and let a_1 be the closest neighbor of a_1' to x_2 in P_{x_2y}. Let $b_1 = x_2$, $b_1' = q_1$, $b_2 = x_1$ and $b_2' = x_3$. Now let $A_1 = \{a_1, a_1'\}$, $A_2 = V(\Sigma) \cap N(a_1') \setminus \{a_1\}$, $B_1 = \{b_1, b_1'\}$ and $B_2 = \{b_2, b_2'\}$. Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. When a_1' is of type d w.r.t. Σ, A_2 has cardinality 2 and let $a_2 = y_1$, $a_2' = y_3$, whereas when a_1' is of type p2, A_2 has cardinality 1 and we let $a_2 = a_2'$ denote its unique node. The connected diamond (Σ, Q) is denoted by $H(A_1, A_2, B_1, B_2)$. Let R be the subpath of P_{x_2y} between a_1 and b_1. Now let $H_1 = R \cup Q$ and $H_2 = H(A_1, A_2, B_1, B_2) \setminus H_1$. Let $P_{a_2b_2}$ be the chordless path from a_2 to b_2 in $H_2 \setminus b_2'$, and define $P_{a_2'b_2'}$ similarly. When $|A_2| = 2$, $P_{a_2b_2}$ and $P_{a_2'b_2'}$ are node-disjoint paths. When $|A_2| = 1$, these two paths are identical between $a_2 = a_2'$ and y. In this case, we refer to the a_2y-subpath of $P_{a_2b_2}$ as P_{a_2y} path, and the b_2y-subpath (resp. $b_2'y$-subpath) of $P_{a_2b_2}$ (resp. $P_{a_2'b_2'}$) as P_{b_2y} (resp. $P_{b_2'y}$) path. Let $P_{a_1b_1}$ be the chordless path from a_1 to b_1 in $H_1 \setminus a_1'$, and define $P_{a_1'b_1'}$ similarly. The two paths $P_{a_1b_1}$ and $P_{a_1'b_1'}$ of H_1 we call the *side-1-paths of* H and the two paths $P_{a_2b_2}$ and $P_{a_2'b_2'}$ of H_2 we call the *side-2-paths of* H. We say that H is *short* if out of all connected diamonds of G, the two side-2-paths of H have as few nodes in common as possible, i.e. there is no connected diamond H' of G such that the side-2-paths of H' have fewer nodes in common that the side-2-paths of H.

We denote by Σ_1 the $3PC(a_1a_1'a_2, b_2)$ induced by $H_1 \cup P_{a_2b_2}$ and by Σ_2 the $3PC(a_1a_1'a_2', b_2')$ induced by $H_1 \cup P_{a_2'b_2'}$. Σ' denotes the $3PC(b_2b_2'b_1', y)$ when $|A_2| = 1$ and the $3PC(b_2b_2'b_1', a_1')$ when $|A_2| = 2$ induced by $H \setminus P_{a_1b_1}$. We denote v_{a_1} (resp. v_{b_1}) the neighbor of a_1 (resp. b_1) in $P_{a_1b_1}$, and we define $v_{a_1'}$, $v_{b_1'}$, v_{b_2}, $v_{b_2'}$ similarly. If $|A_2| = 2$, then we let v_{a_2} (resp. $v_{a_2'}$) be the neighbor of a_2 (resp. a_2') in $P_{a_2b_2}$ (resp. $P_{a_2'b_2'}$). If

$|A_2| = 1$ and $a_2 \neq y$, then we let v_{a_2} be the neighbor of a_2 in $P_{a_2 y}$. Finally, when $|A_2| = 1$, we let y_{b_2}, $y_{b'_2}$ be the neighbor of y in P_{yb_2} and $P_{yb'_2}$ respectively. If $|A_2| = 1$ and $y \neq a_2$, we let y_{a_2} denote the neighbor of y in P_{ya_2}.

A *segment* of H is a path P of H whose endnodes are of degree at least 3, whose intermediate nodes are all of degree 2, and P is not an edge of $G[A]$ or $G[B]$.

Lemma 5.2.1 *Let G be a 4-hole-free odd-signable graph that does not have a proper wheel, a bug with a center-crosspath nor a bug with type s2 node. Let $H(A_1, A_2, B_1, B_2)$ be a short connected diamond of G. A node u of $G \setminus H$ that has a neighbor in H is one of the following types.*

pi, for i=1,2,3	:	*For some segment S of H, $N(u) \cap H \subseteq S$ and $|N(u) \cap H| = i$. Furthermore, if $i \geq 2$, then u has two adjacent neighbors in H. Also if $i = 3$, $|A_2| = 1$ and $S = P_{a_2 y}$, then $N(u) \cap H$ induces a path of length 2.*
A_1	:	$N(u) \cap H = A_1$.
A	:	$N(u) \cap H = A$.
a	:	*$|A_2| = 1$ and u has two neighbors in H, the node of A_2 and one node of A_1.*
B	:	$N(u) \cap H = B$.
B_2	:	$N(u) \cap H = B_2$.
t3	:	*Node u has three neighbors in H: either two nodes of B_2 and one of B_1; or $|A_2| = 2$ and u is adjacent to two nodes of A_1 and one node of A_2.*
d	:	*$|A_2| = 1$ and u has three neighbors in H: if $y = a_2$, then $N(u) \cap H = \{y, y_{b_2}, y_{b'_2}\}$, and otherwise the neighbors of u in H are y and two nodes from $\{y_{a_2}, y_{b_2}, y_{b'_2}\}$.*

Ad	:	$\|A_2\| = 1$, $y = a_2$ and u has four neighbors in H: a_1, a'_1, a_2 and either y_{b_2} or $y_{b'_2}$.
H_1-crossing	:	Either $N(u) \cap H = \{b_1, v_1, v_2\}$ where v_1v_2 is an edge of $P_{a'_1b'_1} \setminus b'_1$ or $N(u) \cap H = \{b'_1, v_1, v_2\}$ where v_1v_2 is an edge of $P_{a_1b_1} \setminus b_1$.
H_2-crossing	:	If $\|A_2\| = 1$, then either $y_{b_2} \neq b_2$ and $N(u) \cap H = \{y_{b_2}, v_1, v_2\}$ where v_1v_2 is an edge of $P_{b'_2y} \setminus y$, or $y_{b'_2} \neq b'_2$ and $N(u) \cap H = \{y_{b'_2}, v_1, v_2\}$ where v_1v_2 is an edge of $P_{b_2y} \setminus y$. If $\|A_2\| = 2$, then $N(u) \cap H = \{a_2, v_1, v_2\}$ where v_1v_2 is an edge of $P_{a'_2b'_2} \setminus a'_2$, or $N(u) \cap H = \{a'_2, v_1, v_2\}$ where v_1v_2 is an edge of $P_{a_2b_2} \setminus a_2$.
pseudo-twin of a node of B_1	:	We define pseudo-twin of b_1: $N(u) \cap H = B_2 \cup \{v_1, v_2\}$, where v_1 and v_2 are nodes of $P_{a_1b_1}$. Furthermore, if $b_1 \notin \{v_1, v_2\}$, then v_1v_2 is an edge. Pseudo-twin of b'_1 is defined symmetrically.
pseudo-twin of a node of B_2	:	We define pseudo-twin of b_2: $N(u) \cap H = B \cup \{v\}$, where if $\|A_2\| = 2$, then v is a node of $P_{a_2b_2} \setminus b_2$, and if $\|A_2\| = 1$, then v is a node of $P_{b_2y} \setminus b_2$ and not both yb'_2 and yu are edges. Pseudo-twin of b'_2 is defined symmetrically.
pseudo-twin of a node of A_1	:	We define pseudo-twin of a_1: $N(u) \cap H = A_2 \cup \{a'_1, v_1, v_2\}$, where v_1 and v_2 are nodes of $P_{a_1b_1}$. Furthermore, if $a_1 \notin \{v_1, v_2\}$, then $\|A_2\| = 1$ and v_1v_2 is an edge. Pseudo-twin of a'_1 is defined symmetrically.
pseudo-twin of a node of A_2	:	We define pseudo-twin of a_2: If $\|A_2\| = 2$, then $N(u) \cap H = A_1 \cup \{v_1, v_2\}$, where v_1 and v_2 are nodes of $P_{a_2b_2}$. Furthermore, if $a_2 \notin \{v_1, v_2\}$, then v_1v_2 is an edge. If $\|A_2\| = 1$ and $a_2 \neq y$, then $N(u) \cap H = A_1 \cup \{a_2, v_{a_2}\}$. If $\|A_2\| = 1$ and $a_2 = y$, then $N(u) \cap H = A_1 \cup \{a_2, v_1, v_2\}$ where $v_1 \in P_{b_2y} \setminus y$, $v_2 \in P_{b'_2y} \setminus y$, at least one of $\{v_1, v_2\}$ is adjacent to y, and u is adjacent to at most one of $\{b_2, b'_2\}$. Pseudo-twin of a'_2 is defined symmetrically.

pseudo-twin of y : *If* $y = a_1$ *or* a_2*, then pseudo-twin of* y *is defined as corresponding pseudo-twins above. So assume* $|A_2| = 1$ *and* $a_2 \neq y$*. Then* $N(u) \cap H = \{y, y_{a_2}, v_1, v_2\}$ *where* $v_1 \in P_{b_2y} \setminus y$*,* $v_2 \in P_{b'_2y} \setminus y$*, at least one of* $\{v_1, v_2\}$ *is adjacent to* y*, and* u *is adjacent to at most one of* $\{b_2, b'_2\}$*.*

s1 : $N(u) \cap H = \{v_1, v_2\}$ *where either* $v_1 \in B_1$ *and* $v_2 \in B_2$*; or* $|A_2| = 2$*,* $v_1 \in A_1$ *and* $v_2 \in A_2$*.*

s2 : $|A_2| = 1$*,* $y \neq a_2$ *and* $N(u) \cap H = \{b_2, b'_2, v_1, v_2\}$ *where* v_1v_2 *is an edge of* P_{a_2y}*. Furthermore, if* $y = v_1$ *or* v_2*, then* yb_2 *and* yb'_2 *are not edges.*

s3 : $|A_2| = 1$ *and either* $N(u) \cap H = B_2 \cup \{a_2, a'_1, b_1\}$ *and* $a_2b'_2$ *is not an edge, or* $N(u) \cap H = B_2 \cup \{a_2, a_1, b'_1\}$ *and* a_2b_2 *is not an edge.*

s4 : $|A_2| = 1$*,* a_2b_2 *and* $a_2b'_2$ *are not edges, and* $N(u) \cap H = A \cup B_2$*.*

Proof: We first prove the following two claims.

Claim 1: If $|A_2| = 1$, then $N(u) \cap H \neq \{y, y_{b_2}, y_{b'_2}, b_1\}$ and $N(u) \cap H \neq \{y, y_{b_2}, y_{b'_2}, b'_1\}$.

Proof of Claim 1: Assume not. By symmetry, w.l.o.g. assume that $N(u) \cap H = \{y, y_{b_2}, y_{b'_2}, b_1\}$. If yb_2 (resp. yb'_2) is an edge, then by definition of a connected diamond yb'_2 (resp. yb_2) is not an edge, $H \setminus P_{a'_1b'_1}$ induces a bug with center b_2 (resp. b'_2) and u is of type s2 w.r.t. this bug, contradicting our assumption.

So yb_2 and yb'_2 are not edges, and hence $y_{b_2} \neq b_2$ and $y_{b'_2} \neq b'_2$. So $(H \setminus P_{a_1b_1}) \cup \{b_1, u\}$ induces a connected diamond $H'(A'_1, A'_2, B_1, B_2)$ where $A'_1 = \{u, y\}$ and $A'_2 = \{y_{b_2}, y_{b'_2}\}$. The two side-2-paths of H' have fewer nodes in common than the two side-2-paths of H, contradicting our assumption. This completes the proof of Claim 1.

Claim 2: If $|N(u) \cap A| \geq 2$ and $|N(u) \cap B| \geq 2$, then u is of type s3 or s4 w.r.t. H.

Proof of Claim 2: Assume that $|N(u) \cap A| \geq 2$ and $|N(u) \cap B| \geq 2$. We first show that $|A_2| = 1$. Assume not. First suppose that $N(u) \cap B_2 = B_2$. Let H' be the hole induced by $P_{a_2b_2} \cup P_{a'_2b'_2} \cup a'_1$. Since (H', u) cannot be a proper wheel, $|N(u) \cap (A_2 \cup a'_1)| \leq 1$.

By symmetry, $|N(u) \cap (A_2 \cup a_1)| \leq 1$. From these two inequalities, and the assumption that $|N(u) \cap A| \geq 2$, it follows that $N(u) \cap A = A_1$. By symmetry $N(u) \cap B = B_2$. In particular, (H', u) is a bug and hence $N(u) \cap H' = \{a'_1, b_2, b'_2\}$. By symmetry, $N(u) \cap (P_{a_1b_1} \cup P_{a_2b_2} \cup b_2) = \{a_1, a'_1, b_2\}$. In particular, $N(u) \cap H = A_1 \cup B_2$. But then Σ and u contradict Lemma 4.3.1. Therefore, $N(u) \cap B_2 \neq B_2$. By symmetry we may assume that $|N(u) \cap B_2| \leq 1$ and $|N(u) \cap A_1| \leq 1$. Since $\{b_2, b_1, b'_1, u\}$ and $\{b'_2, b_1, b'_1, u\}$ cannot induce 4-holes, $|N(u) \cap B_2| \geq 1$, and by symmetry $|N(u) \cap A_1| \geq 1$. Hence $|N(u) \cap B_2| = 1$ and $|N(u) \cap A_1| = 1$. W.l.o.g. $N(u) \cap B_2 = b_2$. By symmetry we may assume that u is adjacent to b_1. Since $\{b'_2, b_1, b'_1, u\}$ cannot induce a 4-hole, $N(u) \cap B = \{b_1, b_2\}$. Suppose that u is adjacent to a_1. Then it is not adjacent to a'_1. By Lemma 4.3.1 applied to Σ and u, $N(u) \cap \Sigma = \{b_1, b_2, a_1, a'_2\}$. But then Σ_2 and u contradict Lemma 4.3.1. So u is not adjacent to a_1, and hence it is adjacent to a'_1. But then Σ' and u contradict Lemma 4.3.1. Therefore $|A_2| = 1$.

Next we show that $N(u) \cap B_2 = B_2$. Assume not, i.e. assume that $|N(u) \cap B_2| \leq 1$. Since $\{b_2, b_1, b'_1, u\}$ and $\{b'_2, b_1, b'_1, u\}$ cannot induce 4-holes, $|N(u) \cap B_2| \geq 1$, and hence $|N(u) \cap B_2| = 1$. W.l.o.g. $N(u) \cap B_2 = b_2$. By symmetry we may assume w.l.o.g. that u is adjacent to b_1. Since $\{b'_2, b_1, b'_1, u\}$ cannot induce a 4-hole, it follows that $N(u) \cap B = \{b_1, b_2\}$. Since $|N(u) \cap A| \geq 2$ and $|A_2| = 1$, u is adjacent to a_1 or a_2. But then Σ and u contradict Lemma 4.3.1 (note that by our assumption G does not contain a bug with a center-crosspath, and so u cannot be of type s1 w.r.t. Σ). Therefore, $N(u) \cap B_2 = B_2$.

Suppose that $N(u) \cap A_1 = A_1$. Since $P_{a_1b_1} \cup P_{a'_1b'_1} \cup \{b_2, u\}$ cannot induce a proper wheel, $N(u) \cap (P_{a_1b_1} \cup P_{a'_1b'_1}) = A_1$. By Lemma 4.3.1 applied to Σ and u, $N(u) \cap \Sigma = \{b_2, b'_2, a_1, a_2\}$. Therefore $N(u) \cap H = B_2 \cup A$. If a_2b_2 is an edge, then Σ is a bug and u is of type s2 w.r.t. Σ, a contradiction. So a_2b_2 is not an edge, and by symmetry neither is $a_2b'_2$, and therefore u is of type s4 w.r.t. H.

Now we may assume that $N(u) \cap A_1 \neq A_1$, and so w.l.o.g. $N(u) \cap A = \{a_1, a_2\}$. By Lemma 4.3.1 applied to Σ and u, $N(u) \cap \Sigma = \{b_2, b'_2, a_1, a_2\}$. By Lemma 4.3.1 applied to Σ' and u, $N(u) \cap \Sigma' = \{b_2, b'_2, b'_1, a_2\}$. Hence $N(u) \cap H = B_2 \cup \{b'_1, a_1, a_2\}$. If a_2b_2 is an edge, then Σ is a bug and u is of type s2 w.r.t. Σ, a contradiction. So a_2b_2 is not an edge and hence u is of type s3 w.r.t. H. This completes the proof of Claim 2.

By Claim 2 we may assume that either $|N(u) \cap A| \leq 1$ or $|N(u) \cap B| \leq 1$. We may assume that $|N(u) \cap H| \geq 2$, since otherwise u is of type p1 w.r.t. H. Suppose that u is not strongly adjacent to Σ nor Σ'. Then u has exactly one neighbor in $P_{a_1b_1}$ and one in $P_{a'_1b'_1}$. By Lemma 4.3.1 applied to Σ_1 and u, $N(u) \cap \Sigma_1 = A_1$, and hence u is of type A_1 w.r.t. H. By symmetry between Σ and Σ' we may now assume that u is strongly adjacent to Σ.

Since G does not contain a bug with center-crosspath, u cannot be of type s1 w.r.t. Σ (nor any other $3PC(\Delta,\cdot)$). So by Lemma 4.3.1 it suffices to consider the following cases.

Case 1: u is of type t3 w.r.t. Σ.

By Lemma 4.3.1 applied to Σ_1, $N(u)\cap H=\{b_1,b_2,b_2'\}$ or B and hence u is of type t3 or B w.r.t. H.

Case 2: u is of type t2 w.r.t. Σ.

Suppose $N(u)\cap\Sigma=\{b_1,b_2\}$ or $\{b_1,b_2'\}$, w.l.o.g. say $N(u)\cap\Sigma=\{b_1,b_2\}$. Since there is no 4-hole, ub_1' is not an edge. Then by Lemma 4.3.1 applied to Σ_1 and u, $N(u)\cap P_{a_1'b_1'}=\varnothing$ and hence u is of type s1 w.r.t. H. Suppose now that $N(u)\cap\Sigma=\{b_2,b_2'\}$. By Lemma 4.3.1 applied to Σ', u is of type B_2, t3 or a pseudo-twin of b_1' w.r.t. H.

Case 3: u is a pseudo-twin of a node of $\{b_1,b_2,b_2'\}$ w.r.t. Σ.

If $|N(u)\cap\{b_1,b_2,b_2'\}|=2$, then let v_1 and v_2 be the two adjacent neighbors of u in $\Sigma\setminus\{b_1,b_2,b_2'\}$. Otherwise let $v_1=v_2$ be the neighbor of u in $\Sigma\setminus\{b_1,b_2,b_2'\}$. Since $|N(u)\cap B|\geq 2$, by our assumption $|N(u)\cap A|\leq 1$.

First suppose that v_1,v_2 are contained in the b_1y-path of Σ. Then $N(u)\cap B_2=B_2$. If $|A_2|=2$, then by Lemma 4.3.1 applied to Σ_1 and u, $N(u)\cap P_{a_1'b_1'}=\varnothing$ and hence u is a pseudo-twin of b_1 w.r.t. H. So we may assume that $|A_2|=1$. Since $|N(u)\cap A|\leq 1$, v_1 and v_2 are contained in either $P_{a_1b_1}$ or in P_{a_2y}. If $\{v_1,v_2\}\subseteq P_{a_1b_1}$, then by Lemma 4.3.1 applied to Σ_1 and u, $N(u)\cap P_{a_1'b_1'}=\varnothing$ and hence u is a pseudo-twin of b_1 w.r.t. H. So assume that $\{v_1,v_2\}\subseteq P_{a_2y}$. Suppose that v_1v_2 is an edge, i.e. $|N(u)\cap\{b_1,b_2,b_2'\}|=2$. By Lemma 4.3.1 applied to Σ_1 and u, $N(u)\cap P_{a_1'b_1'}=\varnothing$. If $y\notin\{v_1,v_2\}$, then u is of type s2 w.r.t. H. So assume w.l.o.g. that $y=v_2$. W.l.o.g. yb_2 is not an edge, and suppose that yb_2' is an edge. Let H' be the hole induced by $P_{a_1b_1}\cup P_{a_2b_2}$. Then (H',b_2') is a bug and u is of type s2 w.r.t. (H',b_2'). So neither yb_2 nor yb_2' is an edge, and hence u is of type s2 w.r.t. H. We may now assume that $v_1=v_2$, i.e. $|N(u)\cap\{b_1,b_2,b_2'\}|=3$. Then ub_1 is an edge. Note that by our assumption, u cannot be adjacent to both a_1' and a_2, and hence by Lemma 4.3.1 applied to Σ' and u, $N(u)\cap P_{a_1'b_1'}=b_1'$. If $v_1\neq y$, then $H_1\cup P_{a_2b_2'}\cup u$ induces a connected diamond $H'(A_1,A_2,B_1,B_2')$ where $B_2'=\{b_2',u\}$, whose side-2-paths have fewer nodes in common than the side-2-paths of H (note that the common nodes of side-2-paths of H are the nodes of P_{a_2y}, and the common nodes of side-2-paths of H' are the nodes of the a_2v_1-subpath of P_{a_2y}), a contradiction. Hence $v_1=y$. W.l.o.g. yb_2' is not an edge, and hence u is a pseudo-twin of b_2 w.r.t. H.

We may now assume that v_1,v_2 are contained in the b_2y-path of Σ or the $b_2'y$-path of Σ. By symmetry we may assume w.l.o.g. that v_1, v_2 are contained in the b_2y-path of Σ. Then

u is adjacent to b_1 and b_2'. First suppose that $|A_2| = 1$. If $|N(u) \cap \{b_1, b_2, b_2'\}| = 2$, then by Lemma 4.3.1 applied to Σ_1 and u, $N(u) \cap P_{a_1'b_1'} = \varnothing$, and hence $(P_{a_2b_2} \setminus v_{b_2}) \cup P_{a_1'b_1'} \cup \{b_1, b_2', u\}$ contains a 4-wheel with center b_2'. So $|N(u) \cap \{b_1, b_2, b_2'\}| = 3$, i.e. $v_1 = v_2$ and ub_2 is an edge. Note that by the argument in the previous paragraph we may assume that $v_1 \neq y$. By Lemma 4.3.1 applied to Σ' and u, $N(u) \cap P_{a_1'b_1'} = b_1'$, and hence u is a pseudo-twin of b_2 w.r.t. H.

We may now assume that $|A_2| = 2$. Since $|N(u) \cap A| \leq 1$, $\{v_1, v_2\} \subseteq P_{a_2b_2}$. If $|N(u) \cap \{b_1, b_2, b_2'\}| = 2$, then by Lemma 4.3.1 applied to Σ_1 and u, $N(u) \cap P_{a_1'b_1'} = \varnothing$, and hence $(P_{a_2b_2} \setminus v_{b_2}) \cup P_{a_1'b_1'} \cup \{b_1, b_2', u\}$ contains a 4-wheel with center b_2'. So $|N(u) \cap \{b_1, b_2, b_2'\}| = 3$, i.e. $v_1 = v_2$ and ub_2 is an edge. Since $v_1 \in P_{a_2b_2}$, by Lemma 4.3.1 applied to Σ' and u, $N(u) \cap P_{a_1'b_1'} = b_1'$, and hence u is a pseudo-twin of b_2 w.r.t. H.

Case 4: u is a pseudo-twin of y w.r.t. Σ.

First suppose that all nodes of $N(u) \cap (\Sigma \setminus y)$ are adjacent to y. If $|A_2| = 2$, then by Lemma 4.3.1 applied to Σ_1, $N(u) \cap P_{a_1'b_1'} = a_1'$ and hence u is a pseudo-twin of a_1 w.r.t. H. So assume that $|A_2| = 1$. W.l.o.g. yb_2 is not an edge. If $a_2 = y$, then by Lemma 4.3.1 applied to Σ_1, $N(u) \cap P_{a_1'b_1'} = a_1'$, and hence u is a pseudo-twin of a_2 w.r.t. H. So we may assume that $a_2 \neq y$. By Lemma 4.3.1 applied to Σ_1, $N(u) \cap P_{a_1'b_1'} = \varnothing$, and hence u is a pseudo-twin of y w.r.t. H.

Now assume that some node of $N(u) \cap (\Sigma \setminus y)$ is not adjacent to y, and let v be such a node. Suppose $|A_2| = 2$. If v is a node of $P_{a_2b_2}$, then by Lemma 4.3.1 applied to Σ_2, $N(u) \cap P_{a_1'b_1'} = a_1'$. But then Lemma 4.3.1 applied to Σ_1 and u is contradicted. So, by symmetry, we may assume that v is a node of $P_{a_1b_1}$. Then by Lemma 4.3.1 applied to Σ_1, $N(u) \cap P_{a_1'b_1'} = a_1'$ and hence u is a pseudo-twin of a_1 w.r.t. H.

Now assume $|A_2| = 1$. If v is a node of $P_{a_1b_1}$, then by Lemma 4.3.1 applied to Σ_1, $v = b_1$ and $N(u) \cap P_{a_1'b_1'} = \varnothing$, contradicting Claim 1. So we may assume w.l.o.g. that v is a node of $P_{a_2b_2}$. Suppose $y = a_2$. Then u is adjacent to a_1. By Lemma 4.3.1 applied to Σ', $N(u) \cap P_{a_1'b_1'} = a_1'$. Since $|N(u) \cap A| \geq 2$, by our assumption $|N(u) \cap B| \leq 1$, and so u cannot be adjacent to both b_2 and b_2'. Hence u is a pseudo-twin of a_2 w.r.t. H. So assume that $y \neq a_2$. By Lemma 4.3.1 applied to Σ_1, $N(u) \cap P_{a_1'b_1'} = \varnothing$. Suppose that u is adjacent to both b_2 and b_2'. Then yb_2' is an edge and $N(u) \cap H = \{b_2, b_2', y, y_{a_2}\}$ (since by definition of connected diamond it is not possible that both yb_2 and yb_2' are edges). But then Σ is a bug, and u is of type s2 w.r.t. it, a contradiction. So u cannot be adjacent to both b_2 and b_2', and hence u is a pseudo-twin of y w.r.t. H.

Case 5: u is of type d w.r.t. Σ.

Suppose $|A_2| = 2$. If $N(u) \cap \Sigma = \{a_1, a_2, v_{a_1}\}$, then by Lemma 4.3.1 applied to Σ_1 and

u, ua'_1 is an edge. But then, since ua'_2 is not an edge, Lemma 4.3.1 applied to Σ_2 and u is contradicted. So $N(u) \cap \Sigma \neq \{a_1, a_2, v_{a_1}\}$. By symmetry $N(u) \cap \Sigma \neq \{a_1, a'_2, v_{a_1}\}$. So $N(u) \cap \Sigma = \{a_1, a_2, a'_2\}$. Then ua'_1 is an edge, else $\{u, a_2, a'_2, a'_1\}$ induces a 4-hole. By Lemma 4.3.1 applied to Σ_2, u has at most two neighbors in $P_{a'_1 b'_1}$. So u is of type A w.r.t. H or it is a pseudo-twin of a'_1 w.r.t. H.

Assume now that $|A_2| = 1$. Suppose u is adjacent to both y_{b_2} and $y_{b'_2}$. So the neighbors of u in Σ are $y, y_{b_2}, y_{b'_2}$. By Lemma 4.3.1 applied to Σ_2, the only node of $P_{a'_1 b'_1}$ that may be adjacent to u is b'_1. Then by Claim 1, ub'_1 is not an edge and hence u is of type d w.r.t. H. So we may assume that u is not adjacent to one node of $\{y_{b_2}, y_{b'_2}\}$. Suppose that $y = a_2$. Suppose u is adjacent to a_1, y, y_{b_2}. By Lemma 4.3.1 applied to Σ_1, ua'_1 is an edge and no other node of $P_{a'_1 b'_1}$ is adjacent to u, and hence u is of type Ad w.r.t. H. Similarly, if u is adjacent to $a_1, y, y_{b'_2}$, then by Lemma 4.3.1 applied to Σ_2, u must be of type Ad w.r.t. H. Assume now that $y \neq a_2$. If u is adjacent to y, y_{a_2}, y_{b_2} (resp. $y, y_{a_2}, y_{b'_2}$), then by Lemma 4.3.1 applied to Σ_1 (resp. Σ_2), u is of type d w.r.t H.

Case 6: u is of type p3t w.r.t. Σ.

Suppose that $N(u) \cap \Sigma$ is contained in $P_{b_1 a_1}$ or $|A_2| = 2$ and it is contained in $P_{a_2 b_2}$ or $P_{a'_2 b'_2}$, or $|A_2| = 1$ and it is contained in $P_{a_2 y}$ or $P_{b_2 y}$ or $P_{b'_2 y}$. Then by Lemma 4.3.1 applied to Σ_1 or Σ_2, $N(u) \cap P_{a'_1 b'_1} = \emptyset$, and hence u is of type p3 w.r.t. H. So we may assume w.l.o.g. that u is adjacent to both a_1 and a_2. Then by Lemma 4.3.1 applied to Σ_1 or Σ_2, $N(u) \cap P_{a'_1 b'_1} = a'_1$, and hence u is a pseudo-twin of a_1 or a_2 w.r.t. H.

Case 7: u is of type p3b w.r.t. Σ.

Let $N(u) \cap \Sigma = \{v, v_1, v_2\}$ such that $v_1 v_2$ is an edge. Suppose that $|A_2| = 2$. If $v_1 v_2 = a_1 a_2$, then by Lemma 4.3.1 applied to Σ_1, $N(u) \cap P_{a'_1 b'_1} = a'_1$, and hence u is a pseudo-twin of a_2 w.r.t. H. Similarly, if $v_1 v_2 = a_1 a'_2$, then u is a pseudo-twin of a'_2 w.r.t. H. If $\{v, v_1, v_2\} \subseteq P_{a_1 b_1}$ or $P_{a_2 b_2}$ or $P_{a'_2 b'_2}$, then by Lemma 4.3.1 applied to Σ_1 or Σ_2 (depending on which path of Σ the neighbors of u are contained in), $N(u) \cap P_{a'_1 b'_1} = \emptyset$ and hence u is of type p3 w.r.t. H. So we may assume w.l.o.g. that $v = a_1$ and $v_1 v_2$ is an edge of $P_{a_2 b_2} \setminus a_2$. By Lemma 4.3.1 applied to Σ_1, $N(u) \cap P_{a'_1 b'_1} = a'_1$, and hence u is a pseudo-twin of a_2 w.r.t. H.

Suppose now that $|A_2| = 1$. If $v_1 v_2 = a_1 a_2$, then by Lemma 4.3.1 applied to Σ_1, $N(u) \cap P_{a'_1 b'_1} = a'_1$. Suppose that v is contained in $P_{a_2 y}$. Note that $va_2 \notin E(G)$. Then $(H \setminus a_2) \cup \{u\}$ contains a connected diamond $H'(A_1, A'_2, B_1, B_2)$ where $A'_2 = \{u\}$. Since va_2 is not an edge, the two side-2-paths of H' have fewer nodes in common than the two side-2-paths of H, contradicting our assumption. So v must be contained in $P_{a_1 b_1}$, and hence u is a pseudo-twin of a_1 w.r.t. H.

So we may assume that $v_1v_2 \neq a_1a_2$. Suppose v is a node of $P_{a_1b_1}$. If v_1v_2 is an edge of $P_{a_1b_1}$, then by Lemma 4.3.1 applied to Σ_1, $N(u) \cap P_{a_1'b_1'} = \varnothing$ and hence u is of type p3 w.r.t. H. Assume now that v_1v_2 is an edge of P_{a_2y}. By Lemma 4.3.1 applied to Σ_1, $v = b_1$ and $N(u) \cap P_{a_1'b_1'} = \varnothing$. Say v_2 is the neighbor of u in P_{a_2y} closer to y. Then $(H \setminus P_{a_1b_1}) \cup \{b_1, u\}$ induces a connected diamond $H'(A_1', A_2', B_1, B_2)$ where $A_1' = \{v_1, u\}$ and $A_2' = \{v_2\}$. The two side-2-paths of H' have fewer nodes in common than the two side-2-paths of H, contradicting our assumption.

We may now assume that v is not in $P_{a_1b_1}$. Suppose that v_1v_2 is in $P_{a_1b_1}$. So $v \in P_{a_2y}$. By Lemma 4.3.1 applied to Σ_1, $v = y$, $yb_2 \in E(G)$ and $N(u) \cap P_{a_1'b_1'} = \varnothing$. Since $yb_2 \in E(G)$, by definition of connected diamonds yb_2' cannot be an edge. Then $P_{a_1b_1} \cup P_{a_1'b_1'} \cup P_{a_2y} \cup \{u, b_2'\}$ induces a $3PC(a_1a_1'a_2, uv_1v_2)$ or a 4-wheel with center a_1. So v_1v_2 is not an edge of $P_{a_1b_1}$, and hence $\{v, v_1, v_2\} \subseteq P$ for some $P \in \{P_{a_2y}, P_{yb_2}, P_{yb_2'}\}$. Then by Lemma 4.3.1 applied to Σ_1 or Σ_2, $N(u) \cap H = \{v, v_1, v_2\}$. If $P = P_{a_2y}$, then $H \cup u$ contains a connected diamond $H'(A_1, A_2, B_1, B_2)$ that contains u and whose side-2-paths have fewer nodes in common than the side-2-paths of H, a contradiction. So $P \in \{P_{yb_2}, P_{yb_2'}\}$, and hence u is of type p3 w.r.t. H.

Case 8: u is of type p2 w.r.t. Σ.

Let v_1v_2 be the edge of $N(u) \cap \Sigma$. Suppose $|A_2| = 2$. If v_1v_2 is an edge of $P_{a_1b_1}$, then by Lemma 4.3.1 applied to Σ_1, u is of type p2 or an H_1-crossing w.r.t. H. Suppose v_1v_2 is an edge of $P_{a_2b_2}$ or $P_{a_2'b_2'}$, w.l.o.g. say v_1v_2 is an edge of $P_{a_2b_2}$. Then by Lemma 4.3.1 applied to Σ_1 and u, b_1' is the only node of $P_{a_1'b_1'}$ that may be adjacent to u. If ub_1' is not an edge, then u is of type p2 w.r.t. H. So assume ub_1' is an edge. If ub_2 is an edge, then u is of type s1 w.r.t. Σ', contradicting our assumption. So ub_2 is not an edge. Hence $H_2 \cup \{u, b_1', a_1\}$ induces a $3PC(b_2b_2'b_1', v_1v_2u)$. We may now assume w.l.o.g. that $N(u) \cap \Sigma = \{a_1, a_2\}$. If u does not have a neighbor in $P_{b_1'a_1'}$, then u is of type s1 w.r.t. H. So assume u does have a neighbor in $P_{b_1'a_1'}$. By Lemma 4.3.1 applied to u and Σ_2, and since u cannot be of type s1 w.r.t. Σ_2, $N(u) \cap P_{a_1'b_1'} = a_1'$, and hence u is of type t3 w.r.t. H.

Now assume that $|A_2| = 1$. If v_1v_2 is an edge of $P_{a_1b_1}$, then by Lemma 4.3.1 applied to Σ_1, u is of type p2 or an H_1-crossing w.r.t. H. Suppose v_1v_2 is an edge of P_{yb_2} or $P_{yb_2'}$, w.l.o.g. say v_1v_2 is an edge of P_{yb_2}. Then by Lemma 4.3.1 applied to Σ' and since u cannot be of type s1 w.r.t. Σ', either $N(u) \cap P_{b_1'a_1'} = \varnothing$, or $y = a_2$ and $N(u) \cap P_{b_1'a_1'} = a_1'$. In the first case u is of type p2 w.r.t. H, and in the second case, by Lemma 4.3.1 applied to Σ_1 and u, node u is of type s1 w.r.t. Σ_1, contradicting our assumption. Now assume that $y \neq a_2$ and v_1v_2 is an edge of P_{a_2y}. By Lemma 4.3.1 applied to Σ_1 and u (and since $N(u) \cap \Sigma = \{v_1, v_2\}$), the only node of $H \setminus \{v_1, v_2\}$ that may be adjacent to u is b_1'. If u is not adjacent to b_1', then u is of type p2 w.r.t. H. Suppose that u is adjacent to b_1'. W.l.o.g. v_2 is closer than v_1 to y on P_{a_2y}. So $(H \setminus P_{a_1'b_1'}) \cup \{b_1', u\}$ induces a connected diamond $H'(A_1', A_2', B_1, B_2)$ where $A_1' = \{v_1, u\}$ and $A_2' = \{v_2\}$. The two side-2-paths of H' have fewer nodes in common than the two side-2-paths of H, contradicting our assumption. Finally suppose that $N(u) \cap \Sigma = \{a_1, a_2\}$. By Lemma 4.3.1 applied to Σ_1, u is of type a, A or a pseudo-twin of a_1' w.r.t. H.

Case 9: u is of type crosspath w.r.t. Σ.

Let $N(u) \cap \Sigma = \{v, v_1, v_2\}$ such that v_1v_2 is an edge. First suppose that $|A_2| = 2$. Note that $v \in \{a_2, a_2', v_{a_1}\}$. Suppose that $v = v_{a_1}$. Then by Lemma 4.3.1 applied to Σ_1 (in the case where v_1v_2 is an edge of $P_{a_2b_2}$) or Σ_2 (in the case where v_1v_2 is an edge of $P_{a_2'b_2'}$), a_1b_1 is an edge. But then u is the center-crosspath of bug Σ. So $v = a_2$ or a_2', w.l.o.g. say $v = a_2$. Suppose v_1v_2 is an edge of $P_{a_1b_1}$. Then by Lemma 4.3.1 applied to Σ_1 and u, either a_2b_2 is an edge and $N(u) \cap P_{a_1'b_1'} = \varnothing$, or $N(u) \cap P_{a_1'b_1'} = a_1'$. In the first case u is a center-crosspath of bug Σ_1, a contradiction. So $N(u) \cap P_{a_1'b_1'} = a_1'$, and hence Σ_2 and u contradict Lemma 4.3.1. So v_1v_2 is an edge of $P_{a_2'b_2'}$. Then by Lemma 4.3.1 applied to Σ', u is an H_2-crossing w.r.t. H.

Now assume that $|A_2| = 1$. Suppose that $v \notin \{y_{b_2}, y_{b_2'}\}$. So w.l.o.g v_1v_2 is an edge of P_{yb_2}. If $y = a_2$, then $v = a_1$ and by Lemma 4.3.1 applied to Σ_1, u is a pseudo-twin of a_2 w.r.t. Σ_1, i.e. $N(u) \cap P_{a_1'b_1'} = a_1'$. Let v_1 be the neighbor of u in $P_{a_2b_2}$ that is closer to b_2, and let P be the b_2v_1-subpath of $P_{a_2b_2}$. Then $P \cup P_{a_1b_1} \cup P_{a_1'b_1'} \cup P_{a_2b_2'} \cup u$ induces a connected diamond $H'(A_1, A_2', B_1, B_2)$, where $A_2' = \{a_2, u\}$. The side-2-paths of H' have fewer nodes in common than the side-2-paths of H, contradicting our choice of H. So $y \neq a_2$. Then $v = y_{a_2}$ and by Lemma 4.3.1 applied to Σ_1, $N(u) \cap H = \{v, v_1, v_2\}$. But then $(H \setminus y_{b_2}) \cup u$ contains a connected diamond whose two side-2-paths have fewer nodes in common than the side-2-paths of H, contradicting our assumption.

So w.l.o.g $v = y_{b_2}$. Since there is no bug with a center-crosspath, yb_2 is not an edge. Suppose that $v_1v_2 = a_1a_2$. Then by Lemma 4.3.1 applied to Σ_1, $N(u) \cap P_{a_1'b_1'} = a_1'$, and hence $N(u) \cap H = \{a_1, a_1', a_2, y_{b_2}\}$. Note that ya_2 is not an edge, else $\{y, a_2, u, y_{b_2}\}$ induces a 4-hole. So $(H \setminus P_{a_2y}) \cup \{y, u\}$ induces a connected diamond $H'(A_1, A_2', B_1, B_2)$ where $A_2' = \{u\}$. Since ya_2 is not an edge, the two side-2-paths of H' have fewer nodes in common that the two side-2-paths of H, contradicting our assumption. So $v_1v_2 \neq a_1a_2$.

Suppose that v_1v_2 is an edge of $P_{a_1b_1}$. Then, by Lemma 4.3.1 applied to Σ_1, $N(u) \cap P_{a_1'b_1'} = \varnothing$ and v is adjacent to b_2. So $y_{b_2}b_2$ is an edge. Node y is not adjacent to b_2', otherwise $\{y, y_{b_2}, b_2, b_2'\}$ induces a 4-hole. But then $P_{a_1b_1} \cup P_{a_1'b_1'} \cup (P_{a_2b_2} \setminus b_2) \cup \{u, b_2'\}$ induces a $3PC(a_1a_1'a_2, uv_1v_2)$ or a 4-wheel with center a_1. So v_1v_2 is not an edge of

$P_{a_1b_1}$. Then by Lemma 4.3.1 applied to Σ', $N(u) \cap H = \{v, v_1, v_2\}$. Note that since neither $\{u, y_{b_2}, y, v_1\}$ nor $\{u, y_{b_2}, y, v_2\}$ can induce a 4-hole, neither v_1y nor v_2y is an edge. If v_1v_2 is an edge of $P_{b'_2y}$, then u is an H_2-crossing w.r.t. H. So assume that v_1v_2 is an edge of P_{a_2y}. Let v_1 be the neighbor of u in P_{a_2y} that is closer to a_2, and let P be the a_2v_1-subpath of P_{a_2y}. Then $P \cup P_{a_1b_1} \cup P_{b_2y} \cup P_{b'_2y} \cup P_{a'_1b'_1} \cup u$ induces a connected diamond $H'(A_1, A_2, B_1, B_2)$. Since v_2y is not an edge, the two side-2-paths of H' have fewer nodes in common than the two side-2-paths of H, contradicting our assumption. $\square$

The following three remarks follow from Lemma 5.2.1.

Remark 5.2.2 *Let $H(A_1, A_2, B_1, B_2)$ be a short connected diamond of G, and let $u \in G \setminus H$. If $|N(u) \cap A| \geq 2$ and $|N(u) \cap B| \geq 2$, then u is of type s3 or s4 w.r.t. H.*

Remark 5.2.3 *Let $H(A_1, A_2, B_1, B_2)$ be a short connected diamond of G. Let $v \in A \cup B \cup \{y\}$ and let u be a pseudo-twin of v w.r.t. H. Then $(H \setminus \{v\}) \cup \{u\}$ contains a short connected diamond H' that contains $((A \cup B \cup \{y\}) \setminus \{v\}) \cup \{u\}$. We say that H'* is obtained by substituting u into H.

Remark 5.2.4 *Let $H(A_1, A_2, B_1, B_2)$ be a short connected diamond of G. If u is of type p3 w.r.t. H, then $H \cup u$ contains a short connected diamond $H'(A_1, A_2, B_1, B_2)$ that contains u. We say that H'* is obtained by substituting u into H.

We first prove a usefull lemma about paths that connect H_1 to H_2, and then show that if there is a node of type s1, s2, s3 or s4 w.r.t. H, then there is a star cutset.

Lemma 5.2.5 *Let G be a 4-hole-free odd-signable graph that does not have a star cutset. Let $H(A_1, A_2, B_1, B_2)$ be a short connected diamond of G. Let $P = p_1, \ldots, p_k$, $k > 1$, be a chordless path in $G \setminus H$ such that $\varnothing \neq N(p_1) \cap H \subseteq H_1$, $\varnothing \neq N(p_k) \cap H \subseteq H_2$, and no intermediate node of P has a neighbor in H. Then P is one of the following types:*

(i) *$N(p_1) \cap H = b_1$ or b'_1, and p_k is of type B_2 w.r.t. H.*

(ii) *p_1 is of type p2 w.r.t. H with neighbors in $P_{a_1b_1}$ or $P_{a'_1b'_1}$, and p_k is of type B_2 w.r.t. H.*

(iii) *p_1 is of type A_1 and p_k is of type p2 w.r.t. H and the following holds. If $|A_1| = 1$, then $a_2 \neq y$ and $N(p_k) \subseteq P_{a_2y}$. If $|A_2| = 2$, then $N(p_k) \subseteq P_{a_2b_2}$ or $P_{a'_2b'_2}$.*

(iv) *p_1 is of type A_1 and $N(p_k) \cap H = a_2$ or a'_2.*

(v) *p_1 is of type A_1 and p_k is of type d w.r.t. H such that $N(p_k) \cap H = \{y, y_{b_2}, y_{b'_2}\}$.*

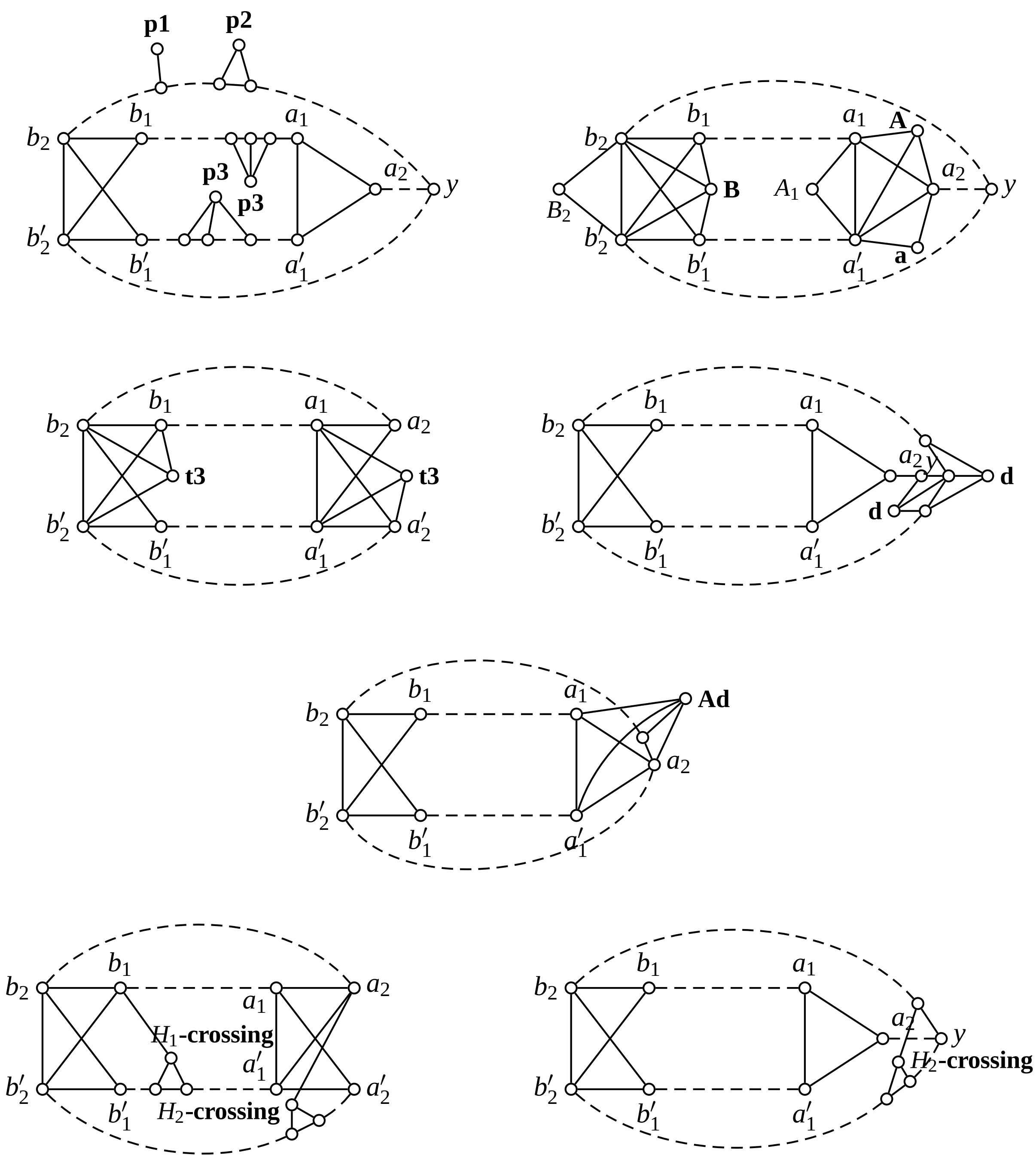
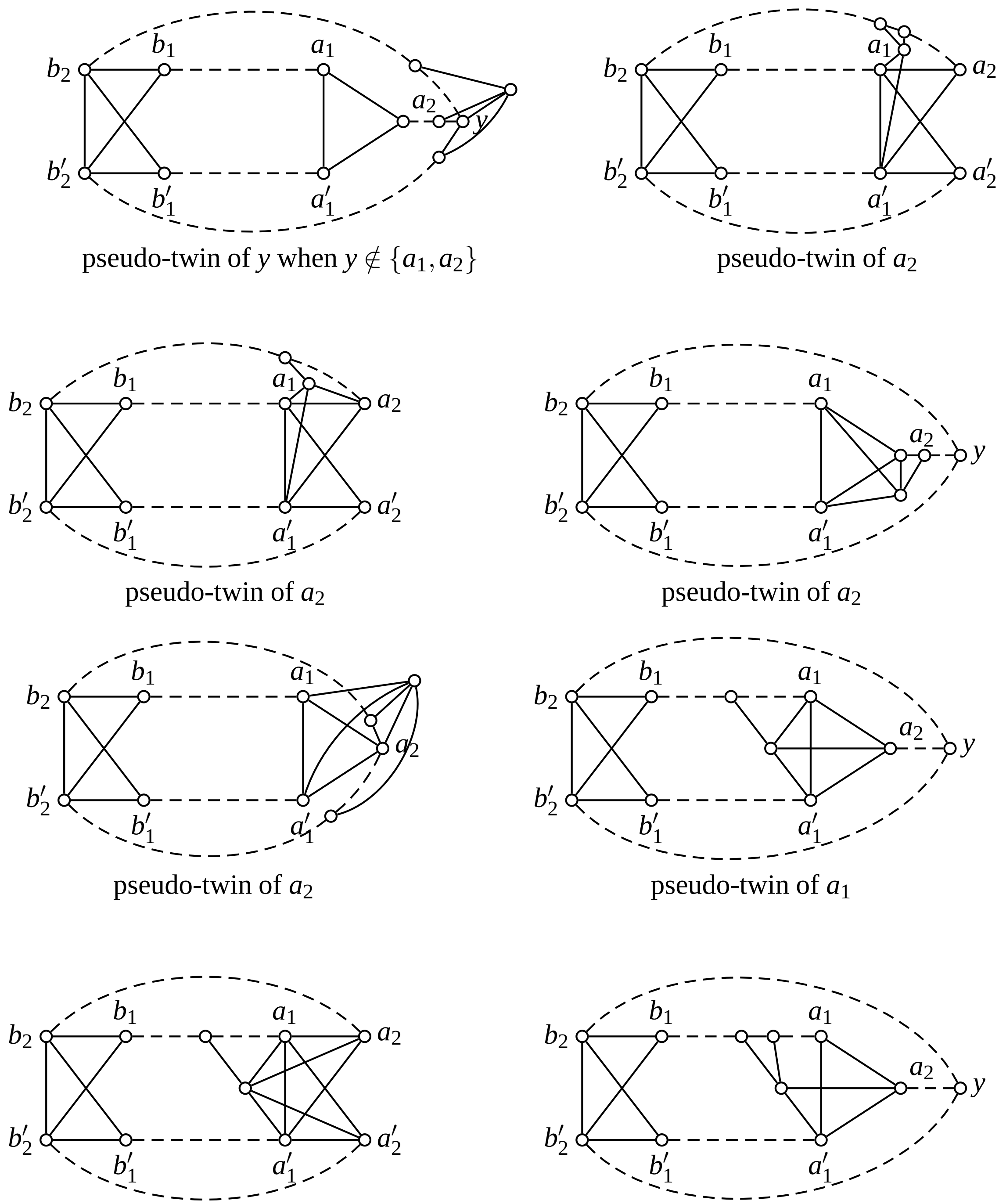


Figure 5.2: Nodes adjacent to a connected diamond.



pseudo-twin of y when $y \notin \{a_1, a_2\}$

pseudo-twin of a_2

pseudo-twin of a_2

pseudo-twin of a_2

pseudo-twin of a_2

pseudo-twin of a_1

pseudo-twin of a_1

pseudo-twin of a_1

Figure 5.3: Pseudo-twins of a node of $A \cup \{y\}$.

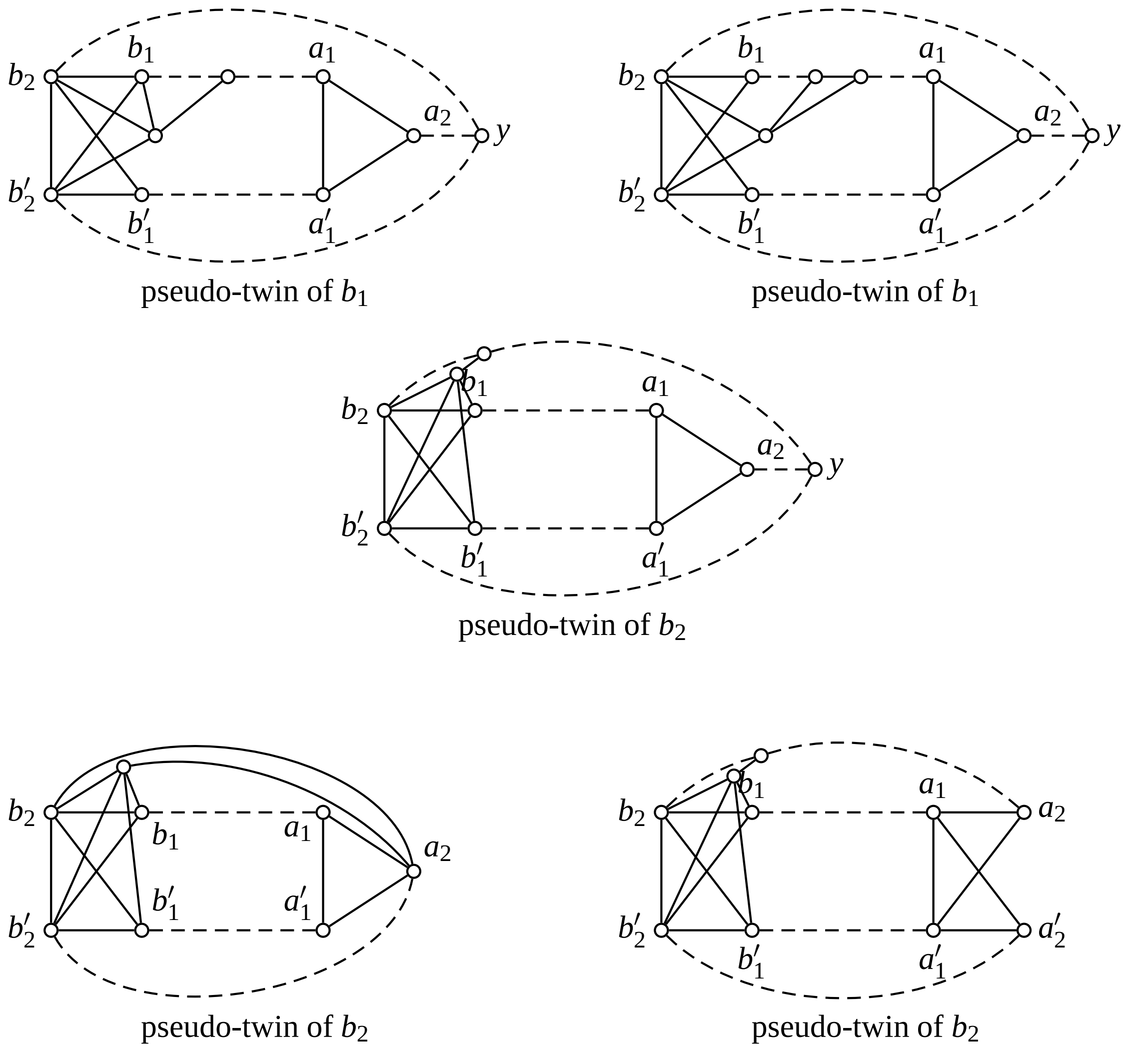


Figure 5.4: Pseudo-twins of a node of B.

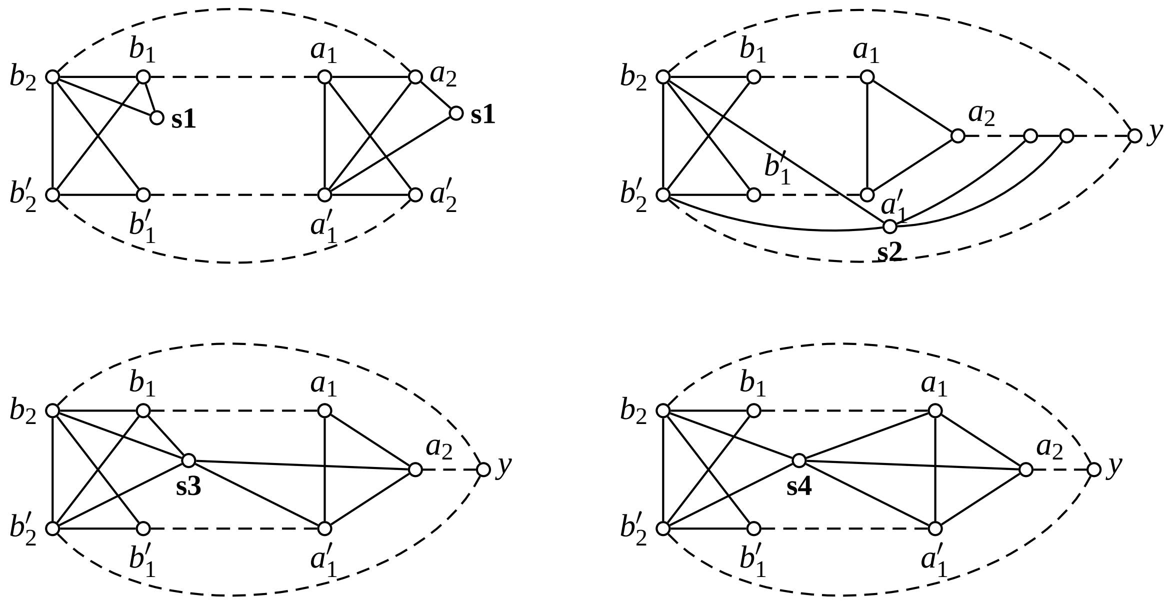


Figure 5.5: Nodes adjacent to a connected diamond that lead to star cutsets.

Proof: Assume G does not have a star cutset. Then by Theorems 4.2.2, 4.3.3, 4.3.4, 4.3.5 and 4.3.6, G does not contain a proper wheel, a bug with a center-crosspath, a $3PC(\Delta,\cdot)$ with a hat, a bug with an ear nor a $3PC(\Delta,\cdot)$ with a type s2 node.

By definition of P and Lemma 5.2.1, the following hold.

(1) p_1 is of type p1, p2, p3, A_1, or H_1-crossing w.r.t. H.

(2) p_k is of type p1, p2, p3, d, B_2, s2 or H_2-crossing w.r.t. H, or $y \notin \{a_1, a_2\}$ and p_k is a pseudo-twin of y w.r.t. H.

By (1) we consider the following cases.

Case 1: p_1 is of type p1 w.r.t. H.

W.l.o.g. p_1 is adjacent to a node v of $P_{a_1b_1}$. Let R_1 (resp. R_2) be the subpath of $P_{a_1b_1}$ with one endnode a_1 (resp. b_1) and the other v.

Suppose that p_k is of type p1 w.r.t. H. W.l.o.g. p_k is adjacent to a node of $P_{a_2b_2}$. Then either P is a hat of Σ_1 (in the case where both p_1a_1 and p_ka_2 are edges), or P is a hat of Σ (in the case where both p_1b_1 and p_kb_2 are edges), or $P \cup P_{a_1b_1} \cup P_{a_2b_2}$ induces a $3PC(\cdot,\cdot)$.

Suppose that p_k is of type p3 w.r.t. H, and let $H'(A_1, A_2, B_1, B_2)$ be the short connected diamond obtained by substituting p_k into H. If $k = 2$, then H' and p_1 contradict Lemma 5.2.1. So $k > 2$, and hence p_{k-1} is of type p1 w.r.t. H' and a contradiction is obtained in the same way as in the previous paragraph.

Suppose that p_k is of type p2 w.r.t. H. W.l.o.g. $N(p_k) \cap H \subseteq P_{a_2 b_2}$. Let H' be the hole induced by $P_{a_2 b_2} \cup P_{a_1 b_1}$. Then P and $P_{a'_1 b'_1}$ are crossing appendices of H', and hence by Lemma 4.1.2, $v = b_1$. If $|A_2| = 2$, then $H_2 \cup P \cup a'_1$ induces a $3PC(\Delta,\Delta)$ or a 4-wheel with center b_2. So $|A_2| = 1$. If $N(p_k) \cap H \subseteq P_{b_2 y}$, then $P_{b_2 y} \cup P_{b'_2 y} \cup P$ induces a $3PC(\Delta,\Delta)$ or a 4-wheel with center b_2. So $N(p_k) \cap H \subseteq P_{a_2 y}$. But then $(H \setminus (P_{a_1 b_1} \setminus b_1)) \cup P$ induces a connected diamond whose side-2-paths have fewer nodes in common than the side-2-paths of H, a contradiction.

Suppose that p_k is of type d w.r.t. H. So $|A_2| = 1$. Suppose $N(p_k) \cap H = \{y, y_{b_2}, y_{b'_2}\}$. Let H' be the hole induced by $P_{a_1 b_1} \cup P_{a_2 b_2}$. Then P and $P_{a'_1 b'_1}$ are crossing appendices of H', and hence by Lemma 4.1.2, $v = b_1$. Suppose one of $\{yb_2, yb'_2\}$ is an edge, w.l.o.g. say $yb_2 \in E(G)$. Then $P \cup P_{a_2 b_2} \cup P_{a'_1 b'_1} \cup \{b_1, b'_2\}$ induces a proper wheel with center b_2. So both yb_2 and yb'_2 are not edges. But then $P \cup H_2 \cup P_{a'_1 b'_1} \cup b_1$ induces a connected diamond $H'(A'_1, A'_2, B_1, B_2)$, where $A'_1 = \{p_k, y\}$, and $A'_2 = \{y_{b_2}, y_{b'_2}\}$, and the two side-2-paths of H' have fewer nodes in common than the two side-2-paths of H, contradicting our assumption. So w.l.o.g. $N(p_k) \cap H = \{y, y_{a_2}, y_{b_2}\}$. But then $P \cup P_{a_1 b_1} \cup (P_{a_2 b_2} \setminus y)$ induces a $3PC(p_k, v)$.

Suppose that p_k is of type s2 w.r.t. H or $y \notin \{a_1, a_2\}$ and p_k is pseudo-twin of y w.r.t. H. Then p_k has two nonadjacent neighbors in $P_{a_2 b_2}$. But then $P_{a_1 b_1} \cup P_{a_2 b_2} \cup P$ contains a $3PC(p_k, v)$.

Suppose that p_k is an H_2-crossing w.r.t. H. First assume that $|A_2| = 2$. W.l.o.g. p_k is adjacent to a_2. Let v' be the neighbor of p_k in $P_{a'_2 b'_2}$ that is closer to a'_2, and let R be the $v'a'_2$-subpath of $P_{a'_2 b'_2}$. Then $R \cup P \cup R_1 \cup a_2$ induces a $3PC(p_k, a_1)$. So $|A_2| = 1$. Let H' be the hole induced by $P_{y b_2} \cup P_{y b'_2}$. If either $v \neq a_1$ or $y \neq a_2$, then (H', p_k) is a bug and $R_2 \cup (P \setminus p_k)$ induces its center-crosspath or an ear, contradiction our assumption. So $v = a_1$ and $y = a_2$. W.l.o.g. $p_k y_{b_2}$ is an edge, and hence $P_{y b_2} \cup P_{a_1 b_1} \cup P$ induces a $3PC(v, y_{b_2})$.

So p_k must be of type B_2 w.r.t. H. If $v \neq b_1$, then Σ, p_k and $p_1, \ldots, p_{k-1}$ contradict Lemma 4.5.2. So $v = b_1$, and hence (i) holds.

Case 2: p_1 is an H_1-crossing w.r.t. H.

W.l.o.g. p_1 is adjacent to b'_1. Let R be the shortest subpath of $P_{a_1 b_1}$ with one endnode b_1 and the other adjacent to p_1. If p_k is adjacent to b_2, then $P \cup R \cup \{b_2, b'_1\}$ induces a $3PC(p_1, b_2)$. If p_k is adjacent to b'_2, then $P \cup R \cup \{b'_2, b'_1\}$ induces a $3PC(p_1, b'_2)$. So neither $p_k b_2$ nor $p_k b'_2$ is an edge, and hence p_k has a neighbor in $H_2 \setminus \{b_2, b'_2\}$. By Lemma 4.5.1 applied to Σ', p_1 and $P \setminus p_1$, $|A_2| = 1$ and the following holds. Node p_k is either of type p2 w.r.t. H with neighbors contained in $P_{a_2 y}$ or of type d adjacent to $\{y, y_{b_2}, y_{b'_2}\}$. But then in both cases $P_{a_1b_1} \cup P_{a_2b_2} \cup P$ induces a $3PC(\Delta,\Delta)$.

Case 3: p_1 is of type A_1 w.r.t. H.

Note that if $|A_2| = 2$, then p_k cannot be adjacent to both a_2 and a'_2 (else $\{p_k, a_2, a'_2, a'_1\}$ induces a 4-hole). Supose (iv) does not hold. Then p_k has a neighbor in $H_2 \setminus \{a_2, a'_2\}$. By symmetry, w.l.o.g. $N(p_k) \cap (P_{a_2b_2} \setminus a_2) \neq \varnothing$. By Lemma 4.5.2 applied to Σ_1, p_1 and $P \setminus p_1$, p_k is of type p2 w.r.t. Σ_1 with neighbors in $P_{a_2b_2}$. So by (2), p_k is of type p2 or d w.r.t. H or $|A_2| = 1$ and p_k is an H_2-crossing w.r.t. H. If p_k is an H_2-crossing w.r.t. H, then Σ_2, p_1 and $P \setminus p_1$ contradict Lemma 4.5.2. Suppose that p_k is of type d w.r.t. H. By Lemma 4.5.2 applied to Σ_2, p_1 and $P \setminus p_1$, p_k is of type p2 w.r.t. Σ_2. Hence $N(p_k) \cap H = \{y, y_{b_2}, y_{b'_2}\}$ and so (v) holds. Finally suppose that p_k is of type p2 w.r.t. H. If $|A_2| = 2$, then (iii) holds. So assume that $|A_2| = 1$. Suppose that $y = a_2$. If p_k is not adjacent to y, then $(H \setminus y_{b_2}) \cup P$ contains a connected diamond $H'(A_1, A'_2, B_1, B_2)$, where $A'_2 = \{a_2, p_1\}$, and the side-2-paths of H' have fewer nodes is common than the side-2-paths of H, contradicting our assumption. So p_k is adjacent to y and hence $P_{a_1b_1} \cup P_{a_2b_2} \cup P$ induces a bug with center a_2, and $P_{a_2b'_2} \setminus a_2$ is its center-crosspath. So $y \neq a_2$. Suppose that $N(p_k) \cap H \subseteq P_{b_2y}$. If p_k is adjacent to y, then Σ_2 and P contradict Lemma 4.5.2. So p_k is not adjacent to y. Then $(H \setminus y_{b_2}) \cup P$ contains a connected diamond $H'(A_1, A'_2, B_1, B_2)$, where $A'_2 = \{a_2, p_1\}$, and the side-2-paths of H' have fewer nodes in common than the side-2-paths of H, contradicting our assumption. So $N(p_k) \cap H \subseteq P_{a_2y}$ and hence (iii) holds.

Case 4: p_1 is of type p2 w.r.t. H.

W.l.o.g. $N(p_1) \cap H \subseteq P_{a_1b_1}$.

Suppose that p_k is of type p1, p2 or p3 w.r.t. H. Then w.l.o.g. $N(p_k) \cap H \subseteq P_{a_2b_2}$. Let H' be the hole induced by $P_{a_1b_1} \cup P_{a_2b_2}$. Note that $P_{a'_1b'_1}$ is an appendix of H' with node-attachment b_2 and edge-attachment a_1a_2. By Lemma 4.1.1 applied to H', $P_{a'_1b'_1}$ and P, one of the following must hold: p_k is adjacent to b_2 or $N(p_k) \cap H = a_2$ or $N(p_k) \cap H = v_{b_2}$. If $N(p_k) \cap H = a_2$, then Σ_1, p_k and $P \setminus p_k$ contradict Lemma 4.5.1. Suppose that $N(p_k) \cap H = v_{b_2}$. Let R be a shortest subpath of $P_{a_1b_1}$ whose one endnode is b_1 and the other is a neighbor of p_1 in $P_{a_1b_1}$. If $|A_2| = 2$, or $|A_2| = 1$ and yb'_2 is not an edge, then $P_{a_2b_2} \cup P_{a'_1b'_1} \cup P \cup R \cup b'_2$ induces a 4-wheel with center b_2. So $|A_2| = 1$ and yb'_2 is an edge. Then yb_2 is not an edge, i.e. $v_{b_2} \neq y$, and since $\{b_2, b'_2, y, v_{b_2}\}$ cannot induce a 4-hole, $v_{b_2}y$ is not an edge. But then $P_{a_2b_2} \cup (P_{a_1b_1} \setminus b_1) \cup P \cup b'_2$ contains a $3PC(v_{b_2}, y)$. Therefore p_k must be adjacent to b_2. If p_k is of type p1 w.r.t. H, then Σ, p_k and $P \setminus p_k$ contradict Lemma 4.5.1. If p_k is of type p2 w.r.t. H, then $H' \cup P$ induces a $3PC(\Delta,\Delta)$. So p_k is of type p3 w.r.t. H. Let $H'(A_1, A_2, B_1, B_2)$ be the short connected diamond obtained by substituing

p_k into H. By Lemma 5.2.1 applied to H' and p_1, $k > 2$. But now $P \setminus p_k$ is a path such that p_k is of type p2 w.r.t. H', p_{k-1} is of type p1 w.r.t. H', and we have already shown that this cannot happen. So p_k cannot be of type p1, p2 nor p3 w.r.t. H.

Suppose that p_k is of type d w.r.t. H. W.l.o.g. p_k is adjacent to $y_{b_2'}$, and hence $P \cup P_{a_1b_1} \cup Pa_2b_2$ induces a $3PC(\Delta,\Delta)$. So p_k cannot be of type d w.r.t. H.

Suppose that $y \notin \{a_1, a_2\}$ and p_k is a pseudo-twin of y w.r.t. H. Then w.l.o.g. p_k is not adjacent to b_2. Let H' be the hole contained in $P_{a_1b_1} \cup (P_{a_2b_2} \setminus y) \cup p_k$. Then H', $P_{a_1'b_1'}$ and $P \setminus p_k$ contradict Lemma 4.1.2. So p_k cannot be a pseudo-twin of y w.r.t. H.

If p_k is of type s2 w.r.t. H, then (H', p_k) is a bug, where H' is the hole induced by $P_{a_1b_1} \cup P_{a_2b_2}$, and $P \setminus p_k$ is its center-crosspath, a contradiction. So p_k cannot be of type s2 w.r.t. H.

Suppose that p_k is an H_2-crossing w.r.t. H. If $|A_2| = 2$, then w.l.o.g. p_k is adjacent to a_2, and hence Σ_1, p_k and $P \setminus p_k$ contradict Lemma 4.5.1. So $|A_2| = 1$. Let H' be the hole induced by $P_{yb_2} \cup P_{yb_2'}$. Then (H', p_k) is a bug, and the path from p_{k-1} to b_1 in the graph induced by $(P \setminus p_k) \cup (P_{a_1b_1} \setminus a_1)$ is its center-crosspath or ear, a contradiction. So p_k cannot be an H_2-crossing w.r.t. H. Therefore by (2), p_k is of type B_2 w.r.t. H, and hence (ii) holds.

Case 5: p_1 is of type p3 w.r.t. H .

Let $H'(A_1, A_2, B_1, B_2)$ be the short connected diamond obtained by substituting p_1 into H. If $k > 2$, then p_2 is of type p1 w.r.t. H' and it is not adjacent to b_1 nor b_1', and we obtain a contradiction as in Case 1. So $k = 2$. But then by (2), p_2 and H' contradict Lemma 5.2.1. $\square$

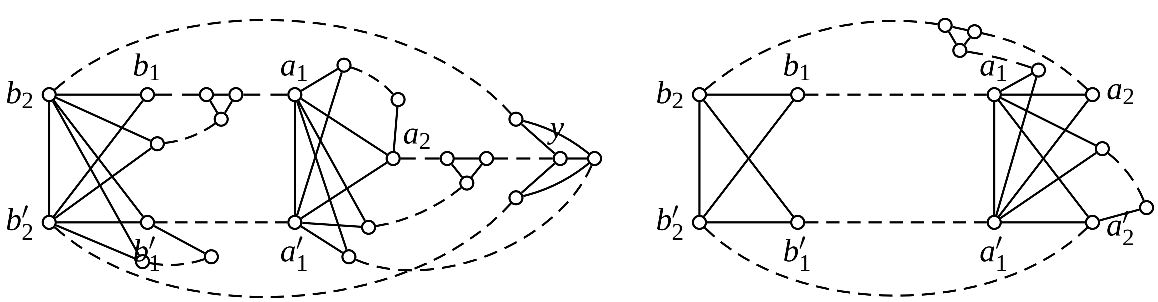


Figure 5.6: Paths from Lemma 5.2.5.

Lemma 5.2.6 *Let G be a 4-hole-free odd-signable graph that does not have a star cutset. Let $H(A_1, A_2, B_1, B_2)$ be a short connected diamond of G. Then no node of $G \setminus H$ is of type s1 w.r.t. H.*

Proof: Assume G does not have a star cutset. Then by Theorems 4.2.2, 4.3.3, 4.3.4, 4.3.5 and 4.3.6 G does not contain a proper wheel, a bug with a center-crosspath, a $3PC(\Delta,\cdot)$ with a hat, a bug with an ear, nor a $3PC(\Delta,\cdot)$ with a type s2 node.

Assume that the lemma does not hold. By symmetry we may assume that there is a node u that is of type s1 w.r.t. H, adjacent to b_2'. Then the second neighbor of u in H is either b_1 or b_1'. Let $S = N[b_2] \setminus v_{b_2}$. Since S is not a star cutset, there exists a direct connection $P = p_1, \dots, p_k$ in $G \setminus S$ from u to $H \setminus S$. We may assume w.l.o.g. that H, u and P are chosen so that $|P|$ is minimized. Note that p_k has a neighbor in $H \setminus S$ and the only nodes of H that may have a neighbor in $P \setminus p_k$ are b_1, b_2' and b_1'.

So if a node of $P \setminus p_k$ has a neighbor in H, then it is either not strongly adjacent to H or by Lemma 5.2.1 it is of type s1 w.r.t. H adjacent to b_2'. In fact, by the choice of H, u and P, no node of $P \setminus p_k$ can be of type s1 w.r.t. H. So nodes of $P \setminus p_k$ are not strongly adjacent to H.

We may assume w.l.o.g. that $N(u) \cap H = \{b_2', b_1'\}$.

Claim 1: *p_k is of type p1, p2, A_1, A, a, s1 (with neighbors in A), t3 (with neighbors in A), d, Ad, H_1-crossing or H_2-crossing w.r.t. H.*

Proof of Claim 1: Since p_k has a neighbor in $H \setminus S$, it cannot be of type s1 w.r.t. H with neighbors in B. Since p_k is not adjacent to b_2, node p_k cannot be of type B, B2, t3 (with neighbors in B), s2, s3 nor s4 w.r.t. H, nor a pseudo-twin of a node of B w.r.t. H.

Suppose that p_k is of type p3 w.r.t. H, and let H' be the short connected diamond obtained by substituting p_k into H. By Lemma 5.2.1 applied to H' and u, $k > 1$, and hence H', u and $P \setminus p_k$ contradict our choice of H, u and P. So p_k is not of type p3 w.r.t. H.

Suppose that p_k is a pseudo-twin of a node of $A \cup y$ w.r.t. H, and let H' be the short connected diamond obtained by substituting p_k into H. By Lemma 5.2.1 applied to H' and u, $k > 1$, and hence H', u and $P \setminus p_k$ contradict our choice of H, u and P. So p_k is not a pseudo-twin of a node of $A \cup y$ w.r.t. H. Now by Lemma 5.2.1, the proof of Claim 1 is complete.

We now consider the following two cases.

Case 1: A node of $P \setminus p_k$ has a neighbor in H.

Recall that for $i < k$, $N(p_i) \cap H \subseteq \{b_1, b_1', b_2'\}$ and $|N(p_i) \cap H| \leq 1$. Let p_i (resp. p_j) be a node of $P \setminus p_k$ with lowest (resp. highest) index that has a neighbor in H. Node p_i is not adjacent to b_1, since otherwise $u, p_1, \dots, p_i$ is a hat of Σ. So p_i is adjacent to b_1' or b_2'.

If there are two distinct nodes of $\{b_1, b_2', b_1'\}$ that have a neighbor in $P \setminus p_k$, then a subpath of $P \setminus p_k$ is a hat of Σ or Σ'. So either b_1' or b_2' is the only node of H that has a neighbor in $P \setminus p_k$.

Case 1.1: b_1' is the only node of H that has a neighbor in $P \setminus p_k$.

By definition of S and Lemma 5.2.5 applied to H and $p_j, \ldots, p_k$, node p_k must have a neighbor in H_1. In particular, p_k cannot be of type d nor an H_2-crossing w.r.t. H.

Suppose that p_k is an H_1-crossing w.r.t. H. If p_k is adjacent to b_1' then $(P_{a_1b_1} \setminus a_1) \cup P \cup \{u, b_1', b_2'\}$ contains a proper wheel with center b_1'. So p_k is adjacent to b_1. But then $(P_{a_1'b_1'} \setminus a_1') \cup \{b_2', b_1, p_j, \ldots, p_k\}$ contains a $3PC(b_1', p_k)$. So p_k is not an H_1-crossing w.r.t. H.

If p_k is of type A or A_1 w.r.t. H, then Σ, u and P contradict Lemma 4.5.1.

If p_k is of type a w.r.t. H, then by Lemma 4.5.1 applied to Σ, u and P, $N(p_k) \cap H = \{a_1', a_2\}$, $y = a_2$ and yb_2' is an edge. But then Σ_1, p_k and $p_j, \ldots, p_{k-1}$ contradict Lemma 4.5.2.

If p_k is of type s1 w.r.t. H, then Σ, b_1' and $p_j, \ldots, p_k$ contradict Lemma 4.5.2.

Suppose that p_k is of type t3 w.r.t. H. If $N(p_k) \cap H = \{a_1, a_1', a_2'\}$ then Σ', p_j and $p_{j+1}, \ldots, p_k$ contradict Lemma 4.5.1. So $N(p_k) \cap H = \{a_1, a_1', a_2\}$, and hence Σ, u and P contradict Lemma 4.5.1. Therefore p_k is not of type t3 w.r.t. H.

If p_k is of type Ad w.r.t. H, then Σ', p_j and $p_{j+1}, \ldots, p_k$ contradict Lemma 4.5.1.

So by Claim 1, p_k is of type p1 or p2 w.r.t. H, and since p_k must have a neighbor in H_1, $N(p_k) \cap H \subseteq H_1$. If $N(p_k) \cap H \subseteq P_{a_1b_1}$, then Σ, u and P contradict Lemma 4.5.1. So $N(p_k) \cap H \subseteq P_{a_1'b_1'}$. If $|A_2| = 2$, then $P_{a_2b_2} \cup P_{a_1'b_1'} \cup P \cup \{u, b_2'\}$ contains a proper wheel with center b_1'. So $|A_2| = 1$. Let R be the chordless path from p_1 to a_1' in $P \cup (P_{a_1'b_1'} \setminus b_1')$. Then Σ, u and R contradict Lemma 4.5.1.

Case 1.2: b_2' is the only node of H that has a neighbor in $P \setminus p_k$.

By Lemma 5.2.5 applied to H and $p_j, \ldots, p_k$, node p_k must have a neighbor in H_2. In particular, p_k is not an H_1-crossing w.r.t. H.

If p_k is of type t3, A_1, A, s1 (adjacent to a_1) or a (adjacent to a_1) w.r.t. H, then $P_{a_1b_1} \cup P \cup \{u, b_2, b_1', b_2'\}$ induces a proper wheel with center b_2'. If p_k is adjacent to a_1' and it is of type a or s1 w.r.t. H, then $P_{a_1b_1} \cup P_{a_1'b_1'} \cup \{b_2', p_j, \ldots, p_k\}$ induces a $3PC(b_2', a_1')$. So p_k is not of type t3, A_1, A, s1 nor a w.r.t. H.

Suppose that p_k is of type Ad w.r.t. H. If p_k is adjacent to $y_{b_2'}$ and $y_{b_2'} \neq b_2'$, then Σ, p_j and $p_{j+1}, \ldots, p_k$ contradict Lemma 4.5.1. If p_k is adjacent to $y_{b_2'}$ and $y_{b_2'} = b_2'$, then $P_{a_1'b_1'} \cup P \cup \{b_2', u\}$ induces a proper wheel with center b_2'. So p_k is adjacent to y_{b_2}. Note that by definition of S, p_k is not adjacent to b_2. But then $P_{a_1b_1} \cup P \cup \{u, b_2, b_1', b_2'\}$ contains a proper wheel with center b'_2. So p_k is not of type Ad w.r.t. H.

If p_k is of type d w.r.t. H, then by Lemma 4.5.1 applied to Σ, p_j and $p_{j+1}, \dots, p_k$, either $N(p_k) \cap H = \{y, y_{a_2}, y_{b_2}\}$ or p_k is adjacent to b'_2. In the first case $P \cup (P_{b_2y} \setminus y) \cup \{u, b'_1, b'_2\}$ induces a proper wheel with center b'_2. So p_k is adjacent to b'_2, and hence $P \cup P_{b_2y} \cup \{u, b'_1, b_2\}$ induces a proper wheel with center b'_2. Similarly, if p_k is an H_2-crossing w.r.t. H, then either $P \cup (P_{b_2y} \setminus y) \cup \{u, b'_1, b'_2\}$ (if $|A_2| = 1$) or $P \cup P_{a_2b_2} \cup \{u, b'_1, b'_2\}$ (if $|A_2| = 2$) contains a proper wheel with center b'_2.

So by Claim 1, p_k is of type p1 or p2 w.r.t. H, and since p_k must have a neighbor in H_2, $N(p_k) \cap H \subseteq H_2$.

By Lemma 4.5.1 applied to Σ, p_j and $p_{j+1}, \dots, p_k$, if $|A_2| = 2$, then $N(p_k) \cap H \subseteq P_{a'_2b'_2}$, and if $|A_2| = 1$, then $N(p_k) \cap H \subseteq P_{b'_2y}$. If $|A_2| = 2$, then $P_{a_1b_1} \cup P_{a'_2b'_2} \cup P \cup \{b'_1, b_2, u\}$ contains a proper wheel with center b'_2, and if $|A_2| = 1$, then $P_{b_2y} \cup P_{b'_2y} \cup P \cup \{u, b'_1\}$ contains a proper wheel with center b'_2.

Case 2: No node of $P \setminus p_k$ has a neighbor in H.

Suppose p_k is an H_1-crossing w.r.t. H. If p_k is adjacent to b_1, then P is hat of Σ. So p_k is adjacent to b'_1. But then Σ, u and P contradict Lemma 4.5.1. So p_k is not an H_1-crossing w.r.t. H.

If p_k is of type A_1, t3, A, or Ad w.r.t. H, then $P_{a_1b_1} \cup P \cup \{u, b_2, b'_1, b'_2\}$ induces a proper wheel with center b'_2 (recall that by definition of S, p_k is not adjacent to b_2).

If p_k is of type a w.r.t. H, then Σ', u and P contradict Lemma 4.5.2. So p_k is not of type a w.r.t. H.

Suppose that p_k is of type s1 w.r.t H. If p_k is adjacent to a_1, then $P_{a_1b_1} \cup P \cup \{u, b_2, b'_1, b'_2\}$ induces a 4-wheel with center b'_2. So p_k is adjacent to a'_1. By Lemma 4.5.1 applied to Σ, u and P, $N(p_k) \cap H = \{a'_1, a'_2\}$. But then Σ', u and P contradict Lemma 4.5.2. So p_k is not of type s1 w.r.t. H.

Suppose that p_k is of type d w.r.t. H. By Lemma 4.5.2 applied to Σ', u and P, $N(p_k) \cap H = \{y, y_{a_2}, y_{b'_2}\}$ and $y_{b'_2} \neq b'_2$. But then Σ, u and P contradict Lemma 4.5.1. So p_k is not of type d w.r.t. H.

If p_k is an H_2-crossing w.r.t. H, then Σ', u and P contradict Lemma 4.5.2.

So by Claim 1, p_k is of type p1 or p2 w.r.t. H. If $N(p_k) \cap H \subseteq P_{a_1b_1}$, then Σ, u and P contradict Lemma 4.5.1. If $N(p_k) \cap H \subseteq P_{a'_1b'_1}$, then Σ, u and R contradict Lemma 4.5.1, where R is the chordless path from p_1 to a'_1 in $P \cup (P_{a'_1b'_1} \setminus b'_1)$. So $N(p_k) \cap H \subseteq H_2$. If $|A_2| = 2$, then by Lemma 4.5.1 applied to Σ, u and P, $N(p_k) \cap H \subseteq P_{a'_2b'_2}$, and hence $P_{a_1b_1} \cup P_{a'_2b'_2} \cup P \cup \{u, b_2, b'_1\}$ contains a proper wheel with center b'_2. So $|A_2| = 1$. By Lemma 4.5.1 applied to Σ, u and P, $N(p_k) \cap H \subseteq P_{b'_2y}$. But then Σ', u and P contradict

Lemma 4.5.2. □

Lemma 5.2.7 *Let G be a 4-hole-free odd-signable graph that does not have a star cutset. Let $H(A_1,A_2,B_1,B_2)$ be a short connected diamond of G. Then no node of $G \setminus H$ is of type s2 w.r.t. H.*

Proof: Assume that G does not have a star cutset. Then by Theorems 4.2.2, 4.3.3, 4.3.4, 4.3.5 and 4.3.6 G does not contain a proper wheel, a bug with a center-crosspath, a $3PC(\Delta,\cdot)$ with a hat, a bug with an ear nor a $3PC(\Delta,\cdot)$ with a type s2 node.

Assume that G has a node u of type s2 w.r.t. H. Let v_1 and v_2 be the neighbors of u in P_{a_2y}, so that v_1 is closer to a_2 on P_{a_2y}. Let P_{v_2y} (resp. $P_{a_2v_1}$) be the v_2y-subpath (resp. a_2v_1-subpath) of P_{a_2y}. We choose H and such a node u so that the length of P_{v_2y} is shortest possible. Note that since u is of type s2 w.r.t. H, $|A_2|=1$ and if $y=v_2$, then yb_2 and yb_2' are not edges.

Let $S=N[u]\setminus v_1$, and let $P=p_1,\dots,p_k$ be a direct connection from $H_1\cup P_{a_2v_1}$ to $H_2\setminus(P_{a_2v_1}\cup\{v_2,b_2,b_2'\})$ in $G\setminus S$. So p_1 has a neighbor in $H_1\cup P_{a_2v_1}$, p_k in $H_2\setminus(P_{a_2v_1}\cup\{v_2,b_2,b_2'\})$, and the only nodes of H that may have a neighbor in $P\setminus\{p_1,p_k\}$ are v_2,b_2 and b_2'. Subject to the previous choice of H and u, we choose H, u and P so that $|P|$ is minimized.

Claim 1: *Node p_1 is of type p1, p2, B, A, a, t3 (with neighbors in B), s2 (with neighbors contained in $B_2\cup(P_{a_2v_1}\setminus v_1)$), s3 or s4 w.r.t. H. Node p_k is of type p1, p2, d or an H_2-crossing w.r.t. H. Furthermore if p_k is of type d w.r.t. H, then p_k is not adjacent to v_1. In particular, $N(p_1)\cap H=\{v_1,v_2\}$ or $N(p_1)\cap H\subseteq H_1\cup P_{a_2v_1}\cup B_2$, $N(p_k)\cap H\subseteq H_2\setminus P_{a_2v_1}$ and $k>1$.*

Proof of Claim 1: Since $|A_2|=1$, no node of G is of type t3 (with neighbors in A) w.r.t. H. Since $y\neq a_2$, no node is of type Ad w.r.t. H. By Lemma 5.2.6 no node is of type s1 w.r.t. H.

Suppose that p_1 is a pseudo-twin of a node of B_1, and let H' be the short connected diamond obtained by substituting p_1 into H. Then H',u and $P\setminus p_1$ contradict our choice of H, u and P. So no node of P is a pseudo-twin of a node of B_1 w.r.t. H. By analogous argument no node of P is a pseudo-twin of a node of A_1 w.r.t. H.

Suppose that p_1 is a pseudo-twin of a node of B_2 w.r.t. H, and let H' be the short connected diamond obtained by substituting p_1 into H. Recall that if $v_2=y$, then yb_2 and yb_2' are not edges, and hence u cannot be of type d w.r.t. H'. So H' and u contradict Lemma 5.2.1. So no node of P is a pseudo-twin of a node of B_2 w.r.t. H.

Suppose that p_i, $i \in \{1,k\}$, is of type p3 w.r.t. H, and let H' be the short connected diamond obtained by substituting p_i into H. If $N(p_i) \cap H \subseteq H_1 \cup P_{a_2v_1}$, then $i = 1$ and hence H', u and $P \setminus p_1$ contradict our choice of H, u and P. A contradiction is obtained by analogous argument if $N(p_i) \cap H \subseteq P_{b_2y} \cup P_{b_2'y} \cup P_{v_2y}$. So $N(p_i) \cap H \subseteq P_{a_2y}$ and p_i has a neighbor in both $P_{a_2v_1}$ and P_{v_2y}. Hence $N(p_i) \cap H$ induces a path of length 2, i.e. p_i is a twin w.r.t. H of a node $v \in P_{a_2y}$. Since p_i has a neighbor in both $P_{a_2v_1}$ and P_{v_2y}, $v \in \{v_1, v_2\}$, and hence H' and u contradict Lemma 5.2.1 (recall that by definition of S, p_i is not adjacent to u). Therefore no node of P is of type p3 w.r.t. H.

Suppose that p_1 is a pseudo-twin of a_2 w.r.t. H, and let H' be the short connected diamond obtained by substituting p_1 into H. Note that since $a_2 \neq y$, $N(p_1) \cap H = A \cup v_{a_2}$. If $v_1 \neq a_2$, then H', u and $P \setminus p_1$ contradict our choice of H, u and P. So $v_1 = a_2$, and hence H' and u contradict Lemma 5.2.1. So no node of P is a pseudo-twin of a_2 w.r.t. H.

Suppose that p_k is a pseudo-twin of y w.r.t. H. Note that p_k is adjacent to y_{a_2}. Let H' be the short connected diamond obtained by substituting p_k into H. If $v_1 \neq y_{a_2}$, then $k > 1$ and hence H', u and $P \setminus p_k$ contradict our choice of H, u and P. So $v_1 = y_{a_2}$, and hence H' and u contradict Lemma 5.2.1. So no node of P is a pseudo-twin of y w.r.t. H.

Suppose that p_1 is of type A_1 or H_1-crossing w.r.t. H. Let p_i be the node of $P \setminus p_1$ with lowest index adjacent to a node of H_2. Note that $N(p_1) \cap H \subseteq H_1$ and $N(p_i) \cap H \subseteq H_2$. By Lemma 5.2.5 applied to H and $p_1, \ldots, p_i$, node p_1 is of type A_1 w.r.t. H and p_i is either of type p2 w.r.t. H and $N(p_i) \cap H \subseteq P_{a_2y}$, or of type d w.r.t. H such that $N(p_i) \cap H = \{y, y_{b_2}, y_{b_2'}\}$. In fact, since $i \neq 1$, $i = k$ and hence $N(p_k) \cap H \subseteq P_{v_2y} \cup \{y_{b_2}, y_{b_2'}\}$. In particular, no node of H has a neighbor in $P \setminus \{p_1, p_k\}$. Let H' be the hole induced by $P_{a_1b_1} \cup P_{a_2b_2}$. Note that u and P are appendices of H' that contradict Lemma 4.1.1. So no node of P is of type A_1 nor H_1-crossing w.r.t. H.

So by Lemma 5.2.1, nodes of P are of type p1, p2, A, B, B_2, a, d, t3 (with neighbors in B), s2, s3, s4 or H_2-crossing w.r.t. H. By definition of P, p_1 and p_k are not of type B_2 w.r.t. H. Suppose that a node p_i of P is of type s2 w.r.t. H. Then by the choice of u, $N(p_i) \cap P_{a_2y} \subseteq P_{a_2v_1} \cup v_2$. Since $\{u, p_i, b_2, v_1\}$ and $\{u, p_i, b_2, v_2\}$ cannot induce 4-holes, $N(p_i) \cap P_{a_2y} \subseteq P_{a_2v_1} \setminus v_1$. In particular, $i = 1$ and $k > 1$. Suppose that p_i is of type d w.r.t. H. Then $i = k$. If p_k is adjacent to v_1, then $v_2 = y$ and w.l.o.g. $N(p_k) \cap H = \{y, y_{a_2}, y_{b_2}\}$, and hence $P_{b_2y} \cup \{u, y_{a_2}, p_k\}$ induces a 4-wheel with center y. So p_k is not adjacent to v_1, and hence $k > 1$. This completes the proof of Claim 1.

Claim 2: *Node v_2 does not have a neighbor in $P \setminus \{p_1, p_k\}$. In particular, for $i = 2, .., k-1$, $N(p_i) \cap H \subseteq B_2$.*

Proof of Claim 2: Suppose that v_2 has neighbor in $P \setminus \{p_1, p_k\}$. We first show that no node of B_2 has a neighbor in $P \setminus \{p_1, p_k\}$. Assume it does. Then there is a minimal subpath P' of $P \setminus \{p_1, p_k\}$ such that one endnode of P' is adjacent to v_2 and the other to a node of B_2. W.l.o.g. b_2 is adjacent to an endnode of P'. By minimality of P', b_2, P', v_2 is a chordless path, and hence $P_{b_2y} \cup P_{v_2y} \cup P' \cup u$ induces a $3PC(b_2, v_2)$ (recall that if $y = v_2$, then yb_2 is not an edge). So no node of B_2 has a neighbor in $P \setminus \{p_1, p_k\}$.

Let p_i be the node of $P \setminus \{p_1, p_k\}$ with lowest index adjacent to v_2. If $N(p_1) \cap H \subseteq H_1$, then H and $p_1, \dots, p_i$ contradict Lemma 5.2.5. So p_1 has a neighbor in $P_{a_2v_1}$. Let H' be the hole induced by $P_{a_2b_2} \cup P_{a_1b_1}$. Then (H', u) is a bug. If $N(p_1) \cap H = v_1$, then $p_1, \dots, p_i$ is a hat of (H', u). So $N(p_1) \cap H \neq v_1$.

Suppose that $N(p_1) \cap H = \{v_1, v_2\}$. By Claim 1 and definition of P, w.l.o.g. p_k has a neighbor in $(P_{v_2y} \cup P_{b_2y}) \setminus v_2$. Let P' be the chordless path from p_k to b_2 in $((P_{v_2y} \cup P_{b_2y}) \setminus v_2) \cup p_k$. Note that by Claim 1, p_k is not adjacent to v_1, and hence $P' \cup P \cup \{u, v_1, v_2\}$ induces a proper wheel with center v_2. So $N(p_1) \cap H \neq \{v_1, v_2\}$.

Therefore p_1 has a neighbor in $H_1 \cup (P_{a_2v_1} \setminus v_1)$. W.l.o.g. p_1 has a neighbor in $P_{a_1b_1} \cup (P_{a_2v_1} \setminus v_1)$ and if p_1 is of type t3 w.r.t. H, then it is adjacent to b_1. Let H' be the hole induced by $P_{a_1b_1} \cup P_{a_2b_2}$. Then (H', u) is a bug, and by Claim 1, (H', u), p_i and $p_1, \dots, p_{i-1}$ contradict Lemma 4.5.1. This completes the proof of Claim 2.

We now consider the following cases.

Case 1: A node of H has a neighbor in $P \setminus \{p_1, p_k\}$.

Let p_i be such a neighbor with highest index. By Claim 2, $N(p_i) \cap H \subseteq B_2$. W.l.o.g. it suffices to consider the following two cases.

Case 1.1: p_i is of type B_2 w.r.t. H.

Note that by definition of P, p_k has a neighbor in $\Sigma \setminus \{b_2, b'_2, b_1\}$. By Claim 1 and Lemma 4.5.2 applied to Σ, p_i and $p_{i+1}, \dots, p_k$ one of the following holds:

(a) p_k is of type d w.r.t. H , $N(p_k) \cap H = \{y, y_{b_2}, y_{b'_2}\}$, $y_{b_2} \neq b_2$ and $y_{b'_2} \neq b'_2$,

(b) w.l.o.g. yb_2 is an edge and $N(p_k) \cap H = v_{b'_2}$, or

(c) p_k is of type p2 w.r.t. H and $N(p_k) \cap H \subseteq P_{v_2y}$.

If (a) or (c) holds, then $(H \setminus P_{a_1b_1}) \cup \{p_i, \dots, p_k\}$ induces a connected diamond whose side-2-paths have fewer nodes in common than the side-2-paths of H, contradicting our choice of H. So (b) must hold, and hence yb'_2 and yu are not edges. Let P' be a chordless path from p_1 to y in $H_1 \cup P_{a_2y} \cup p_1$, and let H' be the hole induced by $P' \cup P \cup (P_{b'_2y} \setminus b'_2)$.

Since $H' \cup b_2'$ cannot induce a $3PC(p_i, v_{b_2'})$, (H', b_2') is a wheel. Since $v_{b_2'} p_i$ is not an edge, (H', b_2') cannot be a twin wheel, and hence it is a bug. If H' contains both v_1 and v_2, then u is a center-crosspath of (H', b_2'). So H' does not contain both v_1 and v_2. By Claim 1 and definition of P it follows that $N(p_1) \cap H = \{v_1, v_2\}$. But then $P_{a_1' b_1'} \cup P_{a_2 v_1}$ is a center-crosspath of (H', b_2').

Case 1.2: $N(p_i) \cap H = b_2'$.

As before, p_k has a neighbor in $\Sigma \setminus \{b_2, b_2', b_1\}$. By Claim 1 and Lemma 4.5.1 applied to Σ, p_i and $p_{i+1}, \dots, p_k$ one of the following holds:

(a) $N(p_k) \cap H = v_{b_2'}$,

(b) p_k is of type p2 w.r.t. H and $N(p_k) \cap H \subseteq P_{b_2' y}$,

(c) p_k is of type d w.r.t. H and either $N(p_k) \cap H = \{y, y_{b_2}, y_{a_2}\}$ or p_k is adjacent to b_2', or

(d) p_k is an H_2-crossing w.r.t. H and $N(p_k) \cap H = \{b_2', v_{b_2'}, y_{b_2}\}$.

Let P' be a chordless path from p_1 to y in $H_1 \cup P_{a_2 y} \cup p_1$. Suppose that (a) holds. Let H' be the hole induced by $P' \cup P \cup (P_{b_2' y} \setminus b_2')$. Since $H' \cup b_2'$ cannot induce a $3PC(v_{b_2'}, p_i)$, (H', b_2') is a wheel, and hence it must be a bug. If H' contains both v_1 and v_2, then u is a center-crosspath of (H', b_2'). So H' does not contain both v_1 and v_2. By Claim 1 and definition of P it follows that $N(p_1) \cap H = \{v_1, v_2\}$. But then $P_{a_1' b_1'} \cup P_{a_2 v_1}$ is a center-crosspath of (H', b_2').

Suppose that (b) holds. If p_k is not adjacent to b_2', then $(H \setminus v_{b_2'}) \cup \{p_i, \dots, p_k\}$ contains a short connected diamond $H'(A_1, A_2, B_1, B_2)$ and H', u and $p_1, \dots, p_{i-1}$ contradict our choice of H', u and P. So p_k is adjacent to b_2'. Let H' be the hole induced by $P' \cup P \cup (P_{b_2' y} \setminus b_2')$. Since (H', b_2') cannot be a proper wheel, $N(b_2') \cap H' = \{p_i, p_k, v_{b_2'}\}$. In particular, b_2' is not adjacent to p_1, and hence by Claim 1, b_2 is not adjacent to p_1. Also H' does not contain b_1 nor b_1'. If b_2 has a neighbor in $P \setminus \{p_1, p_k\}$, then a subpath of $P \setminus \{p_1, p_k\}$ is a hat of Σ. So b_2 has no neighbor in P. Since b_2 and b_2' are not adjacent to p_1, by Claim 1, p_1 is of type p1, p2, A or a w.r.t. H. Since H' does not contain b_1 nor b_1', $N(p_1) \cap H \neq b_1$ nor b_1'. In particular p_1 has a neighbor in w.l.o.g. $\Sigma \setminus \{b_2, b_2', b_1\}$. But then Σ, p_i and $p_1, \dots, p_{i-1}$ contradict Lemma 4.5.1.

Suppose that (c) holds. First assume that $N(p_k) \cap H = \{y, y_{b_2}, y_{a_2}\}$. Then $(H \setminus (P_{b_2' y} \setminus b_2')) \cup \{p_i, \dots, p_k\}$ induces a short connected diamond $H'(A_1, A_2, B_1, B_2)$. By Claim 1, u is of type s2 w.r.t. H', and hence H', u and $p_1, \dots, p_{i-1}$ contradict our choice of H, u and P. So p_k must be adjacent to b_2', so $y b_2'$ is an edge. Suppose that $N(p_k) \cap H = \{y, b_2', y_{b_2}\}$.

Let H' be the hole induced by $P' \cup P$. Since $\{y, p_k, p_i\} \subseteq N(b_2') \cap H'$, (H', b_2') is a twin wheel or a bug, i.e. $N(b_2') \cap H' = \{y, p_k, p_i\}$. In particular, b_2' is not adjacent to p_1, and hence by Claim 1, b_2 is not adjacent to p_1. Also H' does not contain b_1 nor b_1'. If b_2 has a neighbor in $P \setminus \{p_1, p_k\}$, then a subpath of $P \setminus \{p_1, p_k\}$ is a hat of Σ. So b_2 has no neighbor in P. Since b_2 and b_2' are not adjacent to p_1, by Claim 1, p_1 is of type p1, p2, A or a w.r.t. H. Since H' does not contain b_1 nor b_1', $N(p_1) \cap H \neq b_1$ nor b_1'. In particular, p_1 has a neighbor in w.l.o.g. $\Sigma \setminus \{b_2, b_2', b_1\}$. But then Σ, p_i and $p_1, ..., p_{i-1}$ contradicts Lemma 4.5.1. Therefore $N(p_k) \cap H = \{y, b_2', y_{a_2}\}$. Since yb_2' is an edge, yb_2 is not. Suppose that $N(p_1) \cap H$ is not contained in $\{v_1, v_2\}$. Then by Claim 1, p_1 is not adjacent to v_2 and p_1 has a neighbor in $H_1 \cup (P_{a_2v_1} \setminus v_1)$. Let P'' be a chordless path from p_i to b_2 in $H_1 \cup (P_{a_2v_1} \setminus v_1) \cup \{p_1, ..., p_i, b_2\}$, and let H'' be the hole induced by $P'' \cup (P_{v_2y} \setminus y) \cup \{u, p_{i+1}, \dots, p_k\}$. Note that b_2' is adjacent to b_2, u, p_i and p_k, and hence (H'', b_2') is a proper wheel, a contradiction. Therefore $N(p_1) \cap H \subseteq \{v_1, v_2\}$, and hence p_1 is adjacent to v_1. But then $P_{a_1'b_1'} \cup P_{a_2v_1} \cup \{u, p_1, \dots, p_i, b_2'\}$ contains a $3PC(b_2', v_1)$.

So (d) must hold. Then $y_{b_2} \neq b_2$ and $v_{b_2'} \neq y$, and hence $P' \cup P \cup (P_{b_2'y} \setminus b_2') \cup y_{b_2}$ induces a $3PC(p_k, y)$.

Case 2: No node of H has a neighbor in $P \setminus \{p_1, p_k\}$.

By Claim 1 it suffices to consider the following cases.

Case 2.1: p_1 is of type p1 or p2 w.r.t. H.

By Claim 1, $N(p_k) \cap H \subseteq H_2$. If $N(p_1) \cap H \subseteq H_1$, then H and P contradict Lemma 5.2.5. So $N(p_1) \cap H \subseteq P_{a_2v_1} \cup v_2$.

First suppose that p_1 is not strongly adjacent to H, and let v be its neighbor in H. By definition of P, $v \in P_{a_2v_1}$. Note that by Claim 1, p_k is not adjacent to v_1. W.l.o.g. p_k has a neighbor in $P_{b_2y} \cup (P_{v_2y} \setminus v_2)$. Let P' be the chordless path from p_k to b_2 in $P_{b_2y} \cup (P_{v_2y} \setminus v_2) \cup p_k$. Then $P' \cup P \cup P_{a_1b_1} \cup P_{a_2v_1} \cup u$ induces a $3PC(b_2, v)$. Therefore p_1 is of type p2 w.r.t. H.

Let H' (resp. H'') be the hole induced by $P_{a_2b_2} \cup P_{a_1b_1}$ (resp. $P_{a_2b_2'} \cup P_{a_1'b_1'}$). If p_k is of type p2, d or H_2-crossing w.r.t. H, then either $H' \cup P$ or $H'' \cup P$ induces a $3PC(\Delta, \Delta)$ or a 4-wheel with center v_2. So by Claim 1, p_k is not strongly adjacent to H. Let v be the neighbor of p_k in H. W.l.o.g. $v \in (P_{b_2y} \cup P_{v_2y}) \setminus \{b_2, v_2\}$. Recall that if $y = v_2$ then yb_2 and yb_2' are not edges, and hence (H', u) is a bug. If $N(p_1) \cap H = \{v_1, v_2\}$, then bug (H', u), p_1 and $P \setminus p_1$ contradict Lemma 4.5.2. So $N(p_1) \cap H \subseteq P_{a_2v_1}$. By Lemma 4.1.1 applied to H', u and P, $v = v_{b_2}$. By Lemma 4.1.1 applied to H'', u and $P \cup (P_{b_2y} \setminus b_2)$, yb_2' is an edge. Hence $v_{b_2} \neq y$ and since $\{b_2, b_2', y, v_{b_2'}\}$ cannot induce a 4-hole, $v_{b_2}y$ is not an edge. But then $(P_{a_2b_2} \cup P_{a_2b_2'} \cup P) \setminus a_2$ contains a $3PC(v_{b_2}, y)$.

Case 2.2: p_1 is of type B or t3 w.r.t. H.

W.l.o.g. p_1 is adjacent to b_1. By definition of P, p_k has a neighbor in $\Sigma \setminus \{b_2, b_2', b_1\}$, and by Claim 1, p_k is of type p1, p2, d or crosspath (in the case where p_k is an H_2-crossing w.r.t. H) w.r.t. Σ. By Lemma 4.5.3 applied to Σ, p_1 and $P \setminus p_1$, it follows that p_k is not strongly adjacent to Σ, and hence it is not strongly adjacent to H. Let v be the neighbor of p_k in H.

Suppose that $v \in P_{b_2'y} \setminus b_2'$. If b_2y is not an edge, then $P_{b_2y} \cup P_{v_2y} \cup (P_{b_2'y} \setminus b_2') \cup P \cup u$ contains a $3PC(b_2, y)$. So b_2y is an edge and hence $v_2 \neq y$. Let H' be the hole contained in $P_{a_1b_1} \cup (P_{a_2b_2'} \setminus b_2') \cup P$ that contains $P_{a_1b_1} \cup P$. Then (H', b_2) is a bug and u is its center-crosspath. So $v \notin P_{b_2'y} \setminus b_2'$.

Suppose that $v \in P_{b_2y} \setminus \{b_2, y\}$. Let H' be the hole induced by $P_{a_1b_1} \cup P_{a_2y} \cup P$ together with the vy-subpath of P_{b_2y}. If b_2v is not an edge, then $H' \cup P_{a_1'b_1'}$ induces a $3PC(b_2b_1p_1, a_1'a_1a_2)$. So b_2v is an edge, and hence (H', b_2) is a bug and $P_{a_1'b_1'}$ its center-crosspath, a contradiction.

Therefore $v \in P_{v_2y} \setminus \{v_2, y\}$. But then $P_{a_1b_1} \cup P \cup u$ together with the a_2v-subpath of P_{a_2y} induces a $3PC(b_1b_2p_1, v_1uv_2)$.

Case 2.3: p_1 is of type A or a w.r.t. H.

W.l.o.g. p_1 is adjacent to a_1'. If p_1 is not adjacent to a_1, then by Claim 1, either Σ_1, p_1 and $P \setminus p_1$ or Σ_2, p_1 and $P \setminus p_1$ contradict Lemma 4.5.2. So p_1 is adjacent to a_1. W.l.o.g. p_k has a neighbor in $(P_{v_2y} \cup P_{b_2'y}) \setminus \{b_2', v_2\}$. By Claim 1 and Lemma 4.5.3 applied to Σ_2, p_1 and $P \setminus p_1$, node p_k is not strongly adjacent to Σ_2. Let v be the unique neighbor of p_k in Σ_2. By our assumption $v \in (P_{v_2y} \cup P_{b_2'y}) \setminus \{b_2', v_2\}$. If vb_2' is not an edge, then the hole induced by $P_{a_1'b_1'} \cup P_{a_2b_2'}$ and paths u and P contradict Lemma 4.1.1. So vb_2' is an edge. Since $\{b_2, b_2', p_k, v\}$ cannot induce a 4-hole, p_k is not adjacent to b_2. If yb_2 is not an edge, then $(P_{a_2b_2'} \setminus b_2') \cup P_{a_1b_1} \cup P \cup \{u, b_2\}$ induces a $3PC(uv_1v_2, a_1a_2p_1)$ or a 4-wheel with center a_2. So yb_2 is an edge, and hence yb_2' is not. Since $\{b_2, b_2', v, y\}$ cannot induce a 4-hole, vy is not an edge. If follows by Claim 1 that $N(p_k) \cap H = v$, and hence $H_2 \cup P$ induces a $3PC(v, y)$.

Case 2.4: p_1 is of type s2, s3 or s4 w.r.t. H.

If p_1 is of type s3 we may assume w.l.o.g. that p_1 is adjacent to a_1'. Let H' be the hole induced by $P_{a_1'b_1'} \cup P_{a_2b_2'}$. Then (H', p_1) is a bug such that b_2' is the node-attachment of p_1 to H'.

Suppose that p_k is not strongly adjacent to H, and let v be its neighbor in H. Then $v \in (P_{b_2y} \cup P_{b_2'y} \cup P_{v_2y}) \setminus \{b_2, b_2', v_2\}$. If $v \in (P_{b_2'y} \cup P_{v_2y}) \setminus \{b_2', v_2\}$, then $P_{b_2'y} \cup P_{a_2y} \cup P$ contains a $3PC(p_1, v)$. So $v \in P_{b_2y} \setminus \{b_2, y\}$, and hence the vy-subpath of P_{b_2y} together with $P_{a_2y} \cup P_{b'_2y} \cup P$ contains a $3PC(p_1, y)$. Therefore, p_k must be strongly adjacent to H.

Suppose that p_k is of type p2 w.r.t. H. If $N(p_k) \cap H \subseteq P_{v_2y} \cup (P_{b'_2y} \setminus b'_2)$, then $p_2, \ldots, p_k$ is a center-crosspath of (H', p_1). If p_k is adjacent to b'_2, then $P_{b_2y} \cup P_{b'_2y} \cup P$ induces a 4-wheel with center b'_2. So p_k is not adjacent to b'_2, and hence $N(p_k) \cap H \subseteq P_{b_2y}$. Note that p_1 is not adjacent to y, and hence $(H \setminus (H_1 \cup b_2)) \cup P$ contains a $3PC(p_1, y)$. So p_k is not of type p2 w.r.t. H.

Suppose that p_k is of type d w.r.t. H. First suppose that p_k is not adjacent to b'_2. Then $N(p_k) \cap H = \{y, y_{a_2}, y_{b'_2}\}$, else $p_2, \ldots, p_k$ is a center-crosspath of (H', p_1). If $k > 2$, then $P \cup (H \setminus (H_1 \cup P_{b_2y}))$ contains a $3PC(p_1, p_k)$. So $k = 2$, and hence $(H' \setminus y) \cup P$ induces a a 4-wheel with center p_1. Therefore p_k is adjacent to b'_2. If p_k is not adjacent y_{b_2}, then $P_{b_2y} \cup P_{b'_2y} \cup P$ induces a 4-wheel with center b'_2. So p_k is adjacent to y_{b_2}. Since yb'_2 is an edge, yb_2 is not an edge, i.e. $y_{b_2} \neq b_2$. So $P_{a_1b_1} \cup P_{a_2b_2} \cup p_1$ induces a bug with center p_1 and $P \setminus p_1$ is its center-crosspath. Therefore, p_k is not of type d w.r.t. H.

So by Claim 1, p_k is an H_2-crossing w.r.t. H. First suppose that $|N(p_k) \cap P_{b'_2y}| = 2$. Then $p_k y_{b_2}$ is an edge and $y_{b_2} \neq b_2$. If either $k > 2$ or $p_k b'_2$ is not an edge, then $P \setminus p_1$ is either a center-crosspath or an ear of (H', p_1). So $k = 2$ and $p_k b'_2$ is an edge. But then $P_{a_2b_2} \cup P$ contains a $3PC(p_1, y_{b_2})$. Therefore $|N(p_k) \cap P_{b'_2y}| = 1$, and hence $p_k y_{b'_2}$ is an edge, $y_{b'_2} \neq b'_2$ and $|N(p_k) \cap P_{b_2y}| = 2$. But then $P_{a_2b'_2} \cup P$ contains a $3PC(p_1, y_{b'_2})$. $\square$

Lemma 5.2.8 *Let G be a 4-hole-free odd-signable graph that does not have a star cutset. Let $H(A_1, A_2, B_1, B_2)$ be a short connected diamond of G. Then no node of $G \setminus H$ is of type s3 or s4 w.r.t. H.*

Proof: Assume that G does not have a star cutset. Then by Theorems 4.2.2, 4.3.3, 4.3.4, 4.3.5 and 4.3.6 G does not contain a proper wheel, a bug with a center-crosspath, a $3PC(\Delta, \cdot)$ with a hat, a bug with an ear nor a $3PC(\Delta, \cdot)$ with a type s2 node.

Assume that G has a node u of type s3 or s4 w.r.t. H. Then $|A_2| = 1$, and if u is of type s4, then a_2b_2 and $a_2b'_2$ are not edges. Let $S = N[u] \setminus (A_1 \cup B_1)$. Since S is not a star cutset, there exists a direct connection $P = p_1, \ldots, p_k$ from H_1 to $H_2 \setminus \{a_2, b_2, b'_2\}$ in $G \setminus S$. So p_1 has a neighbor in H_1, p_k in $H_2 \setminus \{a_2, b_2, b'_2\}$, and the only nodes of H that may have a neighbor in $P \setminus \{p_1, p_k\}$ are a_2, b_2 and b'_2. We choose H, u and P so that $|P|$ is minimized.

Claim 1: *No node of P is of type Ad w.r.t. H, nor a pseudo-twin w.r.t. H of a node of $B_2 \cup a_2$. In particular, $k > 1$.*

Proof of Claim 1: By Lemma 5.2.1, $k = 1$ if and only if p_1 is of type Ad w.r.t. H, or it is a pseudo-twin w.r.t. H of a node of $B_2 \cup a_2$. We now show that none of these types of nodes can occur.

Suppose that p_1 is of type Ad w.r.t. H. Then $a_2 = y$ and w.l.o.g. $p_1 y_{b'_2}$ is an edge. If u is adjacent to a_1, then $P_{a_2 b'_2} \cup \{u, a_1, p_1\}$ induces a 4-wheel with center a_2. So u is not adjacent to a_1, and hence $N(u) \cap H = \{b_1, b_2, b'_2, a'_1, a_2\}$. But then $P_{a_2 b'_2} \cup \{u, a'_1, p_1\}$ induces a 4-wheel with center a_2.

Suppose that p_1 is a pseudo-twin of a node of B_2 w.r.t. H. W.l.o.g. p_1 is a pseudo-twin of b_2. Let H' be the short connected diamond obtained by substituting p_1 into H. Since u is not adjacent to p_1, u cannot be of type s3 or s4 w.r.t. H', so by Remark 5.2.2 (applied to H' and u), $|N(u) \cap \{b_1, b'_1, b'_2, p_1\}| \leq 1$. So u is of type s4 w.r.t. H, and hence $a_2 b_2$ and $a_2 b'_2$ are not edges. But then H' and u contradict Lemma 5.2.1.

Finally suppose that p_1 is a pseudo-twin of a_2 w.r.t. H, and let H' be the short connected diamond obtained by substituting p_1 into H. Since u is not adjacent to p_1, it follows that H' and u contradict Lemma 5.2.1. This completes the proof of Claim 1.

Claim 2: *Node p_1 is of type p1, p2, B, A_1, A, a, t3 (with neighbors in B) or H_1-crossing w.r.t. H, and p_k is of type p1, p2, d or H_2-crossing w.r.t. H.*

Proof of Claim 2: By Lemmas 5.2.6 and 5.2.7 no node is of type s1 nor s2 w.r.t. H. Since $\{a_2, b_2, u, p_i\}$ cannot induce a 4-hole, no node of P is of type s3 nor s4 w.r.t. H. Since $|A_2| = 1$, no node is of type t3 (with neighbors in A) w.r.t. H.

Suppose that p_k is a pseudo-twin of y w.r.t. H in the case $a_2 \neq y$, and let H' be the short connected diamond obtained by substituting p_k into H. Note that u is of the same type w.r.t. H' as it is w.r.t. H, and hence H', u and $P \setminus p_k$ contradict our choice of H, u and P. So no node of P is a pseudo-twin of y w.r.t. H in the case $a_2 \neq y$.

By analogous argument, no node of P is of type p3 w.r.t. H.

Suppose that p_1 is a pseudo-twin w.r.t. H of a node of $A_1 \cup B_1$ and let H' be the short connected diamond obtained by substituting p_1 into H. By Lemma 5.2.1 u is of the same type w.r.t. H' as it is w.r.t. H, and hence H', u and $P \setminus p_1$ contradict our choice of H, u and P. So no node of P is a pseudo-twin w.r.t. H of a node of $A_1 \cup B_1$.

By Claim 1, no node of P is a pseudo-twin w.r.t. H of a node of $B_2 \cup a_2$, nor of type Ad w.r.t. H. By definition of P, p_1 and p_k cannot be of type B_2 w.r.t. H. By Lemma 5.2.1, the proof of Claim 2 is complete.

Claim 3: *At most one of the node sets B_2 or $\{a_2\}$ may have a neighbor in $P \setminus \{p_1, p_k\}$. So, if a node $p_i \in P \setminus \{p_1, p_k\}$ has a neighbor in H, then either p_i is of type B_2 w.r.t. H or it is not strongly adjacent to H with a neighbor in $\{b_2, b'_2, a_2\}$.*

Proof of Claim 3: Since b_2, b'_2 and a_2 are the only nodes of H that may have a neighbor in

$P \setminus \{p_1, p_k\}$, by Lemma 5.2.1 if $p_i \in P \setminus \{p_1, p_k\}$ has a neighbor in H, then p_i is either of type B_2 w.r.t. H or it is not strongly adjacent to H with a neighbor in $\{b_2, b'_2, a_2\}$. Suppose that both a_2 and a node of B_2 have a neighbor in $P \setminus \{p_1, p_k\}$. Then there is a subpath P' of $P \setminus \{p_1, p_k\}$ of length at least 1, whose one endnode is adjacent to a_2, the other to a node of B_2, w.l.o.g. say to b_2, and no intermediate node of P' has a neighbor in H. If a_2b_2 is not an edge, then $P_{a_1b_1} \cup P' \cup P_{a_2b_2}$ induces a $3PC(a_2, b_2)$. So a_2b_2 is an edge, and hence by definition of type s3 and s4 nodes w.r.t. H, $N(u) \cap H = B_2 \cup \{a_2, a'_1, b_1\}$. Then $a_2b'_2$ is not an edge.

Suppose that b'_2 has a neighbor in $P \setminus \{p_1, p_k\}$. Then there exists a minimal subpath P'' of $P \setminus \{p_1, p_k\}$ such that one endnode of P'' is adjacent to a_2, the other to b'_2 and no intermediate node of P'' has a neighbor in $H \setminus b_2$. But then $P_{a_1b_1} \cup P_{a_2b'_2} \cup P''$ induces a $3PC(a_2, b'_2)$. So b'_2 has no neighbor in $P \setminus \{p_1, p_k\}$.

Since a_2b_2 is an edge, p_k cannot be an H_2-crossing w.r.t. H. So by Claim 2, p_k is of type p1, p2 or d w.r.t. H. Note that since $a_2 = y$ if p_k is of type d w.r.t. H, $N(p_k) \cap H = \{b_2, y, y_{b'_2}\}$. By definition of P, if p_k is of type p1 or p2 w.r.t. H, then $N(p_k) \cap H \subseteq P_{a_2b'_2}$ and p_k has a neighbor in the interior of $P_{a_2b'_2}$.

Let p_i (resp. p_j) be the node of $P \setminus \{p_1, p_k\}$ with highest (resp. lowest) index adjacent to a node of H. Suppose that p_k is of type d w.r.t. H, i.e. $N(p_k) \cap H = \{b_2, y, y_{b'_2}\}$. If p_1 is of type B or t3 w.r.t. H, then $(P_{a_2b'_2} \setminus a_2) \cup P \cup b_2$ induces a proper wheel with center b_2. If p_1 is of type A_1, A or a w.r.t. H, then either $P_{a'_1b'_1} \cup P_{a_2b'_2} \cup P$ (if p_1 is adjacent to a'_1) or $P_{a_1b_1} \cup P_{a_2b'_2} \cup P$ (if p_1 is not adjacent to a'_1) induces a proper wheel with center a_2. So by Claim 1, p_1 must be of type p1, p2 or H_1-crossing w.r.t. H. Then $p_1, \ldots, p_j$ contradicts Lemma 5.2.5. Therefore p_k cannot be of type d w.r.t. H.

So by Claim 2, p_k is of type p1 or p2 w.r.t. H, and hence by definition of P, $N(p_k) \cap H \subseteq P_{a_2b'_2}$ and p_k has a neighbor in $P_{a_2b'_2} \setminus \{a_2, b'_2\}$. Let v_1 (resp. v_2) be the neighbor of p_k in $P_{a_2b'_2}$ that is closer to b'_2 (resp. a_2). Let $P_{b'_2v_1}$ (resp. $P_{v_2a_2}$) be the b'_2v_1-subpath (resp. v_2a_2-subpath) of $P_{a_2b'_2}$. If p_i is adjacent to b_2, then Σ, p_i and $p_{i+1}, \ldots, p_k$ contradict Lemma 4.5.1. So p_i is adjacent to a_2.

Suppose that $N(p_1) \cap H \subseteq H_1$. Then by Lemma 5.2.5 applied to H and $p_1, \ldots, p_j$, node p_1 is of type A_1 w.r.t. H and p_j is adjacent to a_2. In particular, a_2 has at least two neighbors in $P \setminus \{p_1, p_k\}$. Note that since b_2 has a neighbor in $P \setminus \{p_1, p_k\}$, $j \neq i$ and $j \neq i+1$. But then $P_{a'_1b'_1} \cup P_{b'_2v_1} \cup P \cup a_2$ induces a proper wheel with center a_2. Therefore $N(p_1) \cap H$ is not contained in H_1.

Suppose that p_1 is of type A or a w.r.t. H. If p_1 is not adjacent to a'_1, then $P_{a'_1b'_1} \cup P_{b'_2v_1} \cup P \cup \{a_1, a_2\}$ induces a proper wheel with center a_2. So p_1 is adjacent to a'_1, and $P_{a'_1b'_1} \cup P_{b'_2v_1} \cup P \cup a_2$ induces a wheel with center a_2, and hence a_2 has exactly one neighbor in $P \setminus \{p_1, p_k\}$ and a_2 does not have a neighbor in $P_{b_2'v_1}$. Let p_l be the neighbor of b_2 in $P \setminus \{p_1, p_k\}$ with highest index. Then $P_{b_2'v_1} \cup \{p_l, ..., p_k, a_2, b_2\}$ induces a $3PC(b_2, p_i)$. Therefore, p_1 is not of type A nor a w.r.t. H.

So by Claim 2, p_1 is of type B or t3 w.r.t. H. $P \cup P_{b_2'v_1} \cup b_2$ induces a wheel with center b_2, and hence (since this wheel cannot be proper) $N(b_2) \cap P = \{p_1, p_l\}$. Let $p_{i'}$ be the neighbor of a_2 in $\{p_{l+1}, \ldots, p_i\}$ with lowest index. If a_2 has no neighbor in $\{p_2, \ldots, p_{l-1}\}$, then $P_{a_2b_2'} \cup \{b_2, p_1, \ldots, p_{i'}\}$ induces a proper wheel with center b_2. So a_2 has a neighbor in $\{p_2, \ldots, p_{l-1}\}$, and let $p_{j'}$ be such a neighbor with highest index. Then $\{p_{j'}, \ldots, p_{i'}, a_2, b_2\}$ induces a $3PC(p_l, a_2)$. This completes the proof of Claim 3.

By Claim 2, it suffices to consider the following cases.

Case 1: p_1 is of type p1, p2, A_1 or H_1-crossing w.r.t. H.

Then $N(p_1) \cap H \subseteq H_1$. Let p_i be the node of P with lowest index that has a neighbor in H_2. By Claim 2 $N(p_i) \cap H \subseteq H_2$ and no node of $\{p_2, ..., p_{i-1}\}$ has a neighbor in H. By Lemma 5.2.5 applied to H and $p_1, ..., p_i$, and by symmetry w.l.o.g. one of the following holds:

(a) $N(p_1) \cap H = A_1$ and p_i is either of type p2 w.r.t. H with neighbors in P_{a_2y} or $N(p_i) \cap H = \{y, y_{b_2}, y_{b_2'}\}$,

(b) $N(p_1) \cap H = A_1$ and $N(p_i) \cap H = a_2$,

(c) $N(p_i) \cap H = B_2$ and p_1 is of type p2 w.r.t. H with neighbors in $P_{a_1b_1}$, or

(d) $N(p_i) \cap H = B_2$ and $N(p_1) \cap H = b_1'$.

Suppose that (a) holds. W.l.o.g. u is adjacent to a_1'. Then $P_{a_1'b_1'} \cup (P_{a_2b_2'} \setminus a_2) \cup P \cup u$ contains a $3PC(b_2', a_1')$.

Suppose that (c) holds. Then $(H \setminus b_1) \cup \{p_1, ..., p_i\}$ contains a short connected diamond $H'(A_1, A_2, B_1', B_2)$ where $B_1' = \{b_1', p_i\}$. By Lemma 5.2.1, u is of type s3 or s4 w.r.t. H', and hence H', u and $p_{i+1}, ..., p_k$ contradict our choice of H, u and P.

Suppose that (d) holds. By Claim 3, a_2 does not have a neighbor in $P \setminus p_k$. Let P' be a chordless path from p_k to a_2 in $(H_2 \setminus B_2) \cup p_k$, and let H' be the hole induced by $P' \cup P_{a_1'b_1'} \cup P$. Since $H' \cup b_2'$ cannot induce a $3PC(b_1', p_i)$, (H', b_2') is a bug. If u is adjacent to a_1', then u is a center-crosspath of (H', b_2'). So u is not adjacent to a_1', and hence it is adjacent to b_1'. But then $H' \cup u$ induces a $3PC(a_2, b_1')$.

So (b) must hold. By Claim 3, b_2 and b_2' do not have neighbors in $P \setminus p_k$. W.l.o.g. u is adjacent to a_1. If p_k and b_2 are connected in $G[(H_2 \setminus \{a_2, b_2'\}) \cup p_k]$, then let P' be a chordless path from p_k to b_2 in $G[(H_2 \setminus \{a_2, b_2'\}) \cup p_k]$. Then $P_{a_1b_1} \cup P \cup P' \cup u$ induces a $3PC(a_1, b_2)$. So p_k and b_2 are not connected in $G[(H_2 \setminus \{a_2, b_2'\}) \cup p_k]$, i.e. $a_2 = y$ and $N(p_k) \cap H \subseteq P_{a_2b_2'}$. Let P' be a chordless path from p_k to b_2' in $G[(P_{a_2b_2'} \setminus a_2) \cup p_k]$. Then $P_{a_1b_1} \cup P \cup P' \cup u$ induces a $3PC(a_1, b_2')$.

Case 2: p_1 is of type A or a w.r.t. H.

W.l.o.g. we may assume that p_1 is adjacent to a_1 and a_2. First we show that b_2 and b_2' cannot have a neighbor in $P \setminus p_k$. Assume otherwise, and let p_i be the node of P with lowest index adjacent to a node of B_2. By Claim 3, a_2 does not have a neighbor in $P \setminus \{p_1, p_k\}$. If p_i is not of type B_2, then Σ and $p_1, \ldots, p_i$ contradict Lemma 4.5.1. So $N(p_i) \cap H = B_2$, and hence by Lemma 4.5.2 applied to Σ' and $p_1, \ldots, p_i$, $N(p_1) \cap H = A$. Let $H'(A_1', A_2, B_1', B_2)$ where $A_1' = \{p_1, a_1'\}$ and $B_1' = \{b_1', p_i\}$, be the short connected diamond induced by $(H \setminus P_{a_1b_1}) \cup \{p_1, \ldots, p_i\}$. Then H' and u contradict Lemma 5.2.1. Therefore, no node of B_2 has a neighbor in $P \setminus p_k$.

First suppose that either $a_2 \neq y$, or $a_2 = y$ and p_k has a neighbor in $P_{a_2b_2} \setminus a_2$. Let P' be the chordless path from p_k to b_2 in $(H_2 \setminus \{b_2', a_2\}) \cup p_k$. If u is adjacent to a_1, then $P_{a_1b_1} \cup P' \cup P \cup u$ induces a $3PC(b_2, a_1)$. So u is not adjacent to a_1, and hence $N(u) \cap H = \{b_1, b_2, b_2', a_1', a_2\}$. If p_1 is not adjacent to a_1', then $P' \cup P \cup A \cup u$ induces a proper wheel whith center a_2. So p_1 is adjacent to a_1'. But then $P_{a_1b_1} \cup P \cup P' \cup \{a_1', u\}$ induces a $3PC(ub_1b_2, a_1'a_1p_1)$. Therefore $a_2 = y$ and p_k does not have a neighbor in $P_{a_2b_2} \setminus a_2$. So by Claim 2, p_k is of type p1 or p2 w.r.t. H and $N(p_k) \cap H \subseteq P_{a_2b_2'}$. In particular, a_2b_2' is not an edge. If p_1 is not adjacent to a_1' then Σ_2, p_1 and $P \setminus p_1$ contradict Lemma 4.5.2. So p_1 is adjacent to a_1', and hence $(H \setminus a_2) \cup P$ contains a short connected diamond $H'(A_1, A_2', B_1, B_2)$ where $A_2' = \{p_1\}$. But then H' and u contradict Lemma 5.2.1.

Case 3: p_1 is of type B or t3 (with neighbors in B) w.r.t. H.

W.l.o.g. we may assume that p_1 is adjacent to b_1. Suppose that a_2 has a neighbor in $P \setminus p_k$, and let p_i be such a neighbor with lowest index. By Claim 3, b_2 and b_2' do not have neighbors in $P \setminus \{p_1, p_k\}$. If a_2b_2 is not an edge, then $P_{a_2b_2} \cup \{u, p_1, \ldots, p_i\}$ induces a $3PC(a_2, b_2)$. So a_2b_2 is an edge, and hence a_2b_2' is not. But then $P_{a_2b_2'} \cup \{u, p_1, \ldots, p_i\}$ induces a $3PC(a_2, b_2')$. Therefore, a_2 does not have a neighbor in $P \setminus p_k$.

Suppose that a node of B_2 has a neighbor in $P \setminus \{p_1, p_k\}$, and let p_i be such a neighbor with highest index. W.l.o.g. p_i is adjacent to b_2. Let P' be the chordless path from p_k to a_2 in $(H_2 \setminus B_2) \cup p_k$ and let H' be the hole induced by $P' \cup P \cup P_{a_1b_1}$. Then (H', b_2) is a twin wheel or a bug. In particular, p_k is not adjacent to b_2, a_2b_2 is not an edge and H' does not contain v_{b_2}, i.e. p_k has a neighbor in $H_2 \setminus (B_2 \cup v_{b_2})$.

Suppose that p_i is of type B_2 w.r.t. H. Then by symmetry, a_2b_2' is not an edge, H'

does not contain $v_{b'_2}$, i.e. p_k has a neighbor in $H_2 \setminus (B_2 \cup \{v_{b_2}, v_{b'_2}\})$. So by Claim 3 and Lemma 4.5.2 applied to Σ, p_i and $p_{i+1}, \ldots, p_k$, node p_k is either of type p2 w.r.t. H with neihgbors contained in P_{a_2y}, or p_k is of type d w.r.t. H adjacent to $y, y_{b_2}, y_{b'_2}$. In both cases $(H \setminus P_{a_1b_1}) \cup \{p_i, \ldots, p_k\}$ induces a connected diamond whose side-2-paths have fewer nodes in common than the side-2-paths of H.

Therefore $N(p_i) \cap H = b_2$. Since p_k is not adjacent to b_2, and it has a neighbor in $H_2 \setminus (B_2 \cup v_{b_2})$, by Claim 2 and by Lemma 4.5.1 applied to Σ, p_i and $p_{i+1}, \ldots, p_k$, it follows that either p_k is of type p2 w.r.t. H and $N(p_k) \cap H \subseteq P_{b_2y} \setminus b_2$, or p_k is of type d w.r.t. H and $N(p_k) \cap H = \{y, y_{a_2}, y_{b'_2}\}$ (in particular $a_2 \neq y$). In both cases $(H \setminus v_{b_2}) \cup \{p_i, \ldots, p_k\}$ contains a short connected diamond $H'(A_1, A_2, B_1, B_2)$ that contains $p_i, \ldots, p_k$. But then H', u and $p_1, \ldots, p_{i-1}$ contradict our choice of H, u and P.

Therefore no node of H has a neighbor in $P \setminus \{p_1, p_k\}$. Note that by definition of P, p_k has a neighbor in $\Sigma \setminus \{b_2, b'_2, b_1\}$. By Lemma 4.5.3 applied to Σ, p_1 and $P \setminus p_1$, node p_k cannot be of type p2, d nor H_2-crossing w.r.t. H. Hence by Claim 2, p_k is not strongly adjacent to H. Let v be the neighbor of p_k in H.

Suppose that $p_1b'_1$ is not an edge. Then by Lemma 4.5.2 applied to Σ', p_1 and $P \setminus p_1$, either a_2b_2 is an edge and $v = v_{b'_2}$, or $a_2b'_2$ is an edge and $v = v_{b_2}$. In the first case $P_{a_1b_1} \cup P_{a_2b'_2} \cup P$ induces a bug with center b'_2 and $P_{a'_1b'_1}$ is its center-crosspath. In the second case $P_{a_1b_1} \cup P_{a_2b_2} \cup P$ induces a bug with center b_2 and $P_{a'_1b'_1}$ is its center-crosspath. Therefore $p_1b'_1$ is an edge.

W.l.o.g. u is adjacent to a_1, and hence by definition of type s3 and s4 nodes w.r.t. H it is not adjacent to b_1 and a_2b_2 is not an edge. Let P' be the chordless path from p_k to a_2 in $(H_2 \setminus B_2) \cup p_k$. If $v \neq v_{b_2}$, then $P' \cup P \cup P_{a_1b_1} \cup \{u, b_2\}$ induces a $3PC(b_1b_2p_1, a_1ua_2)$. So $v = v_{b_2}$. Let H' be the hole induced by $(P_{a_2b_2} \setminus b_2) \cup P_{a_1b_1} \cup P$. Then (H', b_2) is a bug and u its center-crosspath. $\square$

Lemma 5.2.9 *Let G be a 4-hole-free odd-signable graph that does not have a star cutset. Let $H(A_1, A_2, B_1, B_2)$ be a short connected diamond of G. If a node u is of type a, t3, p3 w.r.t. H or it is a pseudo-twin of a node of $B \cup A_1$ w.r.t. H, or a pseudo-twin of y w.r.t. H when $y \notin \{a_1, a_2\}$, or it is a pseudo-twin of a node of A_2 w.r.t. H when $|A_2| = 2$, then there exists a short connected diamond H' such that one of the following holds:*

(i) *$H_2 \subseteq H'$, $u \in H'_1 = H' \setminus H_2$, $H'_1|H_2$ is a 2-join of H' with special sets A'_1, A_2, B'_1, B_2 such that $A'_1 \cap A_1 \neq \varnothing$ and $B'_1 \cap B_1 \neq \varnothing$.*

(ii) *$H_1 \subseteq H'$ and $u \in H'_2 = H' \setminus H_1$, $H_1|H'_2$ is a 2-join of H' with special sets A_1, A'_2, B_1, B'_2 such that $A'_2 \cap A_2 \neq \varnothing$ and $B'_2 \cap B_2 \neq \varnothing$.*

Proof: Assume that G does not have a star cutset. Then by Theorems 4.2.2, 4.3.3, 4.3.4, 4.3.5 and 4.3.6 G does not contain a proper wheel, a bug with a center-crosspath, a $3PC(\Delta,\cdot)$ with a hat, a bug with an ear nor a $3PC(\Delta,\cdot)$ with a type s2 node. We consider the following cases.

Case 1: u is of type p3 w.r.t. H or it is a pseudo-twin w.r.t. H as in the statement of the lemma.

Let H' be the short connected diamond obtained by substituting u into H. Then clearly H' satisfies (i) or (ii).

Case 2: Node u is of type a w.r.t. H.

Then $|A_2| = 1$ and w.l.o.g. $N(u) \cap H = \{a_1, a_2\}$. Let $S = (N[a_2] \setminus (H \cup u)) \cup A$. Since S cannot be a star cutset, there exists a direct connection $P = p_1, \dots, p_k$ from u to $H \setminus S$ in $G \setminus S$. So p_1 is adjacent to u, p_k to a node of $H \setminus S$, and a_1 and a_1' are the only nodes of H that may have a neighbor in $P \setminus p_k$.

(1) p_k is of type p1, p2, p3, d, B, B_2, t3 (with neighbors in B), H_1-crossing or H_2-crossing w.r.t. H, or it is a pseudo-twin w.r.t. H of a node of B, or y when $y \neq a_2$. In particular, p_k is adjacent to at most one node of A.

Proof of (1): By Lemmas 5.2.6, 5.2.7 and 5.2.8, no node is of type s1, s2, s3 nor s4 w.r.t. H. Since $|A_2| = 1$, p_k is not adjacent to a_2 and it has a neighbor in $H \setminus S$, p_k cannot be of type A_1, A, a, t3 (with neighbors in A), Ad nor a pseudo-twin of a node of A w.r.t. H. So the result follows by Lemma 5.2.1. This proves (1).

(2) a_1 cannot have a neighbor in $P \setminus p_k$.

Proof of (2): Suppose it does. Let R be a chordless path from p_k to a_2 in $(H \setminus A_1) \cup p_k$, and let H' be the hole induced by $R \cup P \cup u$. Since (H', a_1) cannot be a proper wheel, a_1 has exactly one neighbor p_j in P and $j < k$.

Suppose that a_1' does not have a neighbor in $P \setminus p_k$. By Lemma 5.2.5 applied to H and $p_j, \dots, p_k$, node p_k must have a neighbor in H_1. So by (1), p_k has a neighbor in $H_1 \setminus A_1$. Recall that by definition of a connected diamond at least one of a_2b_2, a_2b_2' is not an edge. W.l.o.g. assume that a_2b_2' is not an edge. Let T be a chordless path from p_k to a_1' in $(H_1 \setminus a_1) \cup \{p_k, b_2'\}$. Recall that no node of P is adjacent to a_2 and hence $T \cup P \cup \{a_1, a_2, u\}$ induces a proper wheel with center a_1. So a_1' has a neighbor in $P \setminus p_k$.

If a_1' is not adjacent to p_j, then a subpath of $P \setminus p_k$ is a hat of Σ_1, a contradiction. So a_1' is adjacent to p_j. If a_1' does not have a neighbor in $p_1, \dots, p_{j-1}$, then

$\{p_1,...,p_j,u,a_1,a_2,a'_1\}$ induces a proper wheel with center a_1. So a'_1 has a neighbor in $p_1,...,p_{j-1}$. So (H',a_1) and (H',a'_1) are both bugs. In particular, $N(a_1)\cap P = p_j$ and $N(a'_1)\cap P = \{p_j,p_{j-1}\}$.

Suppose that $N(p_k)\cap H \subseteq H_2$. Then by Lemma 5.2.5 applied to H and $p_j,...,p_k$, node p_k is either of type p2 w.r.t. H with neighbors in P_{a_2y} or of type d w.r.t. H such that $N(p_k)\cap H = \{y,y_{b_2},y_{b'_2}\}$. In both cases $P_{a_1b_1}\cup P_{a_2b_2}\cup P\cup u$ induces a bug (H',a_1) with a center-crosspath, a contradiction.

So p_k has a neighbor in H_1, and hence by (1), it has a neighbor in $H_1\setminus A_1$. By (1) p_k has at most one neighbor in A and hence by Lemma 4.5.2 applied to Σ_1, p_j and $p_{j+1},...,p_k$, $N(p_k)\cap\Sigma_1 = \{b_2,b_1,b'_1\}$. But then $P_{a_1b_1}\cup P_{a_2b_2}\cup P\cup u$ induces a bug (H',a_1) with center-crosspath $P_{a_1b_1}\setminus a_1$, a contradiction. This proves (2).

We now consider the following two cases.

Case 2.1: a'_1 has a neighbor in $P\setminus p_k$.

Let p_j be such a neighbor with highest index. If p_k is of type d, B_2, B, H_2-crossing, a pseudo-twin of y when $y\neq a_2$, or a pseudo-twin of a node of $B_2\cup b_1$ w.r.t. H, then Σ_1, p_j and $p_{j+1},...,p_k$ contradict Lemma 4.5.1.

Suppose that p_k is a pseudo-twin of b'_1 w.r.t. H. Then by (2), $H_2\cup P_{a_1b_1}\cup P\cup u$ induces a short connected diamond $H'(A'_1,A_2,B'_1,B_2)$ where $A'_1 = \{a_1,u\}$ and $B'_1 = \{b_1,p_k\}$ and H' satisfies (i). So we may assume that p_k is not a pseudo-twin of b'_1 w.r.t. H.

If p_k is an H_1-crossing w.r.t. H, then by Lemma 4.5.1 applied to Σ_1, p_j and $p_{j+1},...,p_k$, node p_k is adjacent to b_1 and a'_1, and hence $P_{a'_1b'_1}\cup P_{a_2b'_2}\cup P\cup u$ induces a proper wheel with center a'_1.

So by (1), p_k is of type p1, p2, p3 or t3 (with neighbors in B) w.r.t. H. If $N(p_k)\cap H\subseteq P_{a'_1b'_1}$, then by (2), $(H\setminus a'_1)\cup(P\cup u)$ contains a short connected diamond $H'(A'_1,A_2,B_1,B_2)$, where $A'_1 = \{a_1,u\}$, that satisfies (i). So we may assume that p_k has a neighbor in $H\setminus P_{a'_1b'_1}$. But then by Lemma 4.5.1 applied to p_j, path $p_{j+1},...,p_k$ and either Σ_1 or Σ_2, node p_k must be of type t3 w.r.t. H such that $N(p_k)\cap H = \{b'_1,b_2,b'_2\}$. But then by (2), $H_2\cup P_{a_1b_1}\cup P\cup u$ induces a short connected diamond $H'(A'_1,A_2,B'_1,B_2)$, where $A'_1 = \{a_1,u\}$ and $B'_1 = \{b_1,p_k\}$, and hence (i) holds.

Case 2.2: a'_1 does not have a neighbor in $P\setminus p_k$.

So by (2), no node of H has a neighbor in $P\setminus p_k$. If p_k does not have a neighbor in $\Sigma_1\setminus\{a_1,a'_1,a_2\}$, then it has a neighbor in $\Sigma_2\setminus\{a_1,a'_1,a_2\}$ and hence (since p_k is adjacent to at most one node of $\{a_1,a'_1,a_2\}$ by (1)) Σ_2, u and P contradict Lemma 4.5.2. So p_k has a neighbor in $\Sigma_1\setminus\{a_1,a'_1,a_2\}$. By Lemma 4.5.2 applied to Σ_1, u and P, and since by (1) p_k is adjacent to at most one node of $\{a_1,a'_1,a_2\}$, one of the following holds:

(a) $N(p_k) \cap \Sigma_1 = \{b_2, b'_1\}$.

(b) $N(p_k) \cap \Sigma_1 = \{v_1, v_2\}$ where v_1v_2 is an edge of $P_{a'_1b'_1}$.

(c) $N(p_k) \cap \Sigma_1 = \{b_1, b_2, v_{b_2}\}$.

(d) a_2b_2 is an edge and $N(p_k) \cap \Sigma_1 = \{v_{a_1}\}$.

(e) a_2b_2 is an edge, p_k is of type p3 w.r.t. Σ_1 and p_k is adjacent to a_1.

By (1) in fact (c) cannot happen. Suppose that (b) holds. Then by (1), p_k is of type p2 w.r.t. H, and hence $(H \setminus a'_1) \cup P \cup u$ contains a short connected diamond $H'(A'_1, A_2, B_1, B_2)$, where $A'_1 = \{u, a_1\}$, that satisfies (i).

Suppose that (a) holds. By Lemma 4.5.2 applied to Σ_2, u and P, and since by (1) p_k is adjacent to at most one of $\{a_1, a'_1, a_2\}$, $N(p_k) \cap \Sigma_2 = \{b'_2, b'_1\}$. So $N(p_k) \cap H = \{b'_1, b_2, b'_2\}$ and hence $H_2 \cup P_{a_1b_1} \cup P \cup u$ induces a connected diamond $H'(A'_1, A_2, B'_1, B_2)$, where $A'_1 = \{u, a_1\}$ and $B'_1 = \{b_1, p_k\}$, that satisfies (i).

Suppose that (d) holds. Then by (1), $N(p_k) \cap H = \{v_{a_1}\}$. Since a_2b_2 is an edge, $a_2b'_2$ is not an edge, and hence $H_1 \cup P \cup \{a_2, b'_2, u\}$ induces a 4-wheel with center a_1.

Suppose that (e) holds. Then by (1), p_k is of type p3 w.r.t. H. Since a_2b_2 is an edge, $a_2b'_2$ is not an edge, and hence $(H_1 \setminus v_{a_1}) \cup P \cup \{a_2, b'_2, u\}$ induces a 4-wheel with center a_1.

Case 3: Node u is of type t3 w.r.t H.

W.l.o.g. we may assume that $N(u) \cap H = \{b_1, b_2, b'_2\}$. Assume that the result does not hold.

(1) *Let $S_1 = (N[b_2] \setminus (H \cup u)) \cup B$, and let $P = p_1, \ldots, p_k$ be a direct connection from u to $H \setminus S_1$ in $G \setminus S_1$. Then $k = 1$ and p_1 is an H_1-crossing w.r.t. H adjacent to b_1. In particular, there exists a node that is an H_1-crossing w.r.t. H adjacent to b_1 and u.*

Proof of (1): Since G does not have a star cutset, there exists a direct connection P as in statement of (1), so we just need to show that $k = 1$ and p_1 is an H_1-crossing w.r.t. H adjacent to b_1. By definition of P, node p_1 is adjacent to u, p_k to a node of $H \setminus S_1$, and the only nodes of H that may have a neighbor in $P \setminus p_k$ are b_1, b'_2 and b'_1.

(1.1) *p_k is of type p1, p2, p3, A_1, A, a, d, Ad, t3 (with neighbors in A), H_1-crossing, H_2-crossing w.r.t. H or a pseudo-twin of a node of $A \cup y$ w.r.t. H. In particular, p_k is adjacent to at most one node of B.*

Proof of (1.1): By Lemmas 5.2.6, 5.2.7 and 5.2.8, p_k cannot be of type s1, s2, s3 nor s4 w.r.t. H. Since p_k is not adjacent to b_2, it cannot be of type B, B_2, t3 (with neighbors in B) nor a pseudo-twin of a node of B w.r.t. H. By Lemma 5.2.1, the proof of (1.1) is complete.

(1.2) *No node of $H \setminus \{b_1, b'_1, b'_2\}$ has a neighbor in $P \setminus p_k$ and at most one node of $\{b_1, b'_1, b'_2\}$ has a neighbor in $P \setminus p_k$.*

Proof of (1.2): We have already established that no node of $H \setminus \{b_1, b'_1, b'_2\}$ has a neighbor in $P \setminus p_k$. By Lemma 5.2.1 and Lemma 5.2.6, no node of $P \setminus p_k$ is adjacent to more than one node of $\{b_1, b'_1, b'_2\}$. If at least two nodes of $\{b_1, b'_1, b'_2\}$ have a neighbor in $P \setminus p_k$, then a subpath of $P \setminus p_k$ is a hat of Σ or Σ', a contradiction. This proves (1.2).

If a node of $\{b_1, b'_1, b'_2\}$ has a neighbor in $P \setminus p_k$, then let p_j (resp. p_i) be such a neighbor with highest (resp. lowest) index.

(1.3) *b'_1 does not have a neighbor in $P \setminus p_k$.*

Proof of (1.3): Assume it does. Then by (1.2) $H_1 \cup \{u, p_1, \dots, p_i, b_2\}$ induces a bug with center b_2, and $P_{a_2 b_2} \setminus b_2$ is its center-crosspath, a contradiction. This proves (1.3).

(1.4) *b_1 does not have a neighbor in $P \setminus p_k$.*

Proof of (1.4): Assume it does. By (1.2) no node of $H \setminus b_1$ has a neighbor in $P \setminus p_k$. By (1.1) p_k is adjacent to at most one node of B, and hence if $N(p_k) \cap H \subseteq H_2$, then H and $p_j, \dots, p_k$ contradict Lemma 5.2.5. So p_k has a neighbor in H_1. In particular, p_k is not of type d, H_2-crossing nor a pseudo-twin of y when $y \notin \{a_1, a_2\}$ w.r.t H.

Suppose that p_k is of type A_1 w.r.t. H. By Lemma 4.5.1 applied to Σ, p_j and $p_{j+1}, \dots, p_k$, $a_1 b_1$ is an edge. But then $P_{a_1 b_1} \cup P_{a_2 b_2} \cup P \cup u$ induces a proper wheel with center b_1. So p_k is not of type A_1 w.r.t. H.

Suppose p_k is of type a w.r.t. H. So $|A_2| = 1$ and $N(p_k) \cap H = \{a_2, a'_1\}$ or $\{a_2, a_1\}$. In the first case Σ, p_j and $p_{j+1}, \dots, p_k$ contradict Lemma 4.5.1, and in the second case Σ', u and P contradict Lemma 4.5.2. So p_k is not of type a w.r.t. H.

Suppose that p_k is of type A or it is a pseudo-twin of a node of A_1 w.r.t. H. If p_k has a neighbor in $P_{a'_1 b'_1} \setminus a'_1$, then Σ', u and P contradict Lemma 4.5.2. So $N(p_k) \cap H \subseteq A \cup P_{a_1 b_1}$. But then $(H \setminus P_{a_1 b_1}) \cup P \cup u$ induces a short connected diamond $H'(A'_1, A_2, B'_1, B_2)$ where $A'_1 = \{a'_1, p_k\}$ and $B'_1 = \{b'_1, u\}$, and H' satisfies (i), contradicting our assumption. So p_k is not of type A nor a pseudo-twin of a node of A_1 w.r.t. H.

Suppose that p_k is of type t3 w.r.t. H. Then by (1.1) $|A_2| = 2$ and $N(p_k) \cap H = \{a_1, a_1', a_2'\}$ or $\{a_1, a_1', a_2\}$. In the first case Σ, p_j and $p_{j+1}, \dots, p_k$ contradict Lemma 4.5.1, and in the second case Σ', u and P contradict Lemma 4.5.2. So p_k is not of type t3 w.r.t. H.

Node p_k is not of type Ad nor a pseudo-twin of a node of A_2 w.r.t. H, since otherwise Σ, p_j and $p_{j+1}, \dots, p_k$ contradict Lemma 4.5.1.

Suppose that p_k is an H_1-crossing w.r.t. H. If p_k is adjacent to b_1', then $(P_{a_1b_1} \setminus a_1) \cup \{b_1', b_2', p_j, \dots, p_k\}$ contains a $3PC(b_1, p_k)$. So p_k is adjacent to b_1. But then $(P_{a_1'b_1'} \setminus a_1') \cup P \cup \{b_2', b_1, u\}$ contains a proper wheel with center b_1. So p_k is not an H_1-crossing w.r.t H.

By (1.1) p_k is of type p1, p2 or p3 w.r.t. H. Since p_k has a neighbor in H_1, it follows that $N(p_k) \cap H \subseteq P_{a_1b_1}$ or $P_{a_1'b_1'}$. By definition of P, p_k has a neighbor in $H_1 \setminus \{b_1, b_1'\}$. If $N(p_k) \cap H \subseteq P_{a_1'b_1'}$, then Σ, p_j and $p_{j+1}, \dots, p_k$ contradict Lemma 4.5.1. So $N(p_k) \cap H \subseteq P_{a_1b_1}$. But then $(H \setminus b_1) \cup P \cup u$ contains a short connected diamond $H'(A_1, A_2, B_1', B_2)$ where $B_1' = \{u, b_1'\}$, and H' satisfies (i), contradicting our assumption. This proves (1.4).

(1.5) *b_2' does not have a neighbor in $P \setminus p_k$.*

Proof of (1.5): Assume it does. By (1.2) no node of $H \setminus b_2'$ has a neighbor in $P \setminus p_k$. If $N(p_k) \cap H \subseteq H_1$, then H and $p_j, \dots, p_k$ contradict Lemma 5.2.5. So p_k has a neighbor in H_2. In particular, p_k is not of type A_1 nor H_1-crossing w.r.t. H.

Node p_k is not of type A nor a pseudo-twin of a node of A_1 w.r.t. H, since otherwise Σ', p_j and $p_{j+1}, \dots, p_k$ contradict Lemma 4.5.1.

Suppose that p_k is of type a w.r.t. H. Then by Lemma 4.5.1 applied to Σ', p_j and $p_{j+1}, \dots, p_k$, $y = a_2$ and yb_2' is an edge. But then $P_{a_2b_2} \cup P \cup \{u, b_2'\}$ induces a proper wheel with center b_2'. So p_k is not of type a w.r.t. H.

Suppose that p_k is of type t3 (with neighbors in A), Ad or a pseudo-twin of a node of A_2 w.r.t. H. So $N(p_k) \cap H_1 = \{a_1, a_1'\}$. By definition of P, p_k is not adjacent to b_2, and hence $H_1 \cup P \cup \{u, b_2\}$ induces a $3PC(b_1b_2u, a_1a_1'p_k)$. So p_k is not of type type t3 (with neighbors in A), Ad nor a pseudo-twin of a node of A_2 w.r.t. H.

Suppose that p_k is of type d or a pseudo-twin of y when $y \notin \{a_1, a_2\}$ w.r.t. H. Let H' be the hole contained in $P_{a_1b_1} \cup P_{a_2y} \cup P \cup u$ that contains $P_{a_1b_1} \cup P \cup u$. Note that if H' contains y, then p_k has a neighbor in $P_{b_2y} \setminus y$. Since by definition of P, b_2 is not adjacent to any node of P, it follows that $N(b_2) \cap H' = \{u, b_1\}$. But then $H' \cup P_{a_1'b_1'}$ induces a $3PC(b_1b_2u, a_1a_1'a_2)$. So p_k is not of type d nor a pseudo-twin of y when $y \notin \{a_1, a_2\}$ w.r.t. H.

Suppose that p_k is an H_2-crossing w.r.t. H. By Lemma 4.5.1 applied to Σ', p_j and $p_{j+1},\dots,p_k$, node p_k is adjacent to b'_2. Let H' be the hole contained in $P_{a_2b_2}\cup P\cup u$ that contains $P\cup\{u,b_2\}$. Then (H',b'_2) is a proper wheel. So p_k is not an H_2-crossing w.r.t. H.

So by (1.1) and since p_k has a neighbor in H_2, $N(p_k)\cap H\subseteq H_2$ and p_k is of type p1, p2 or p3 w.r.t. H. By definition of P, p_k has a neighbor in $H_2\setminus\{b_2,b'_2\}$. By Lemma 4.5.1 applied to Σ', p_j and $p_{j+1},\dots,p_k$, either $|A_2| = 2$ and $N(p_k)\cap H\subseteq P_{a'_2b'_2}$, or $|A_2| = 1$ and $N(p_k)\cap H\subseteq P_{b'_2y}$. If $|A_2| = 2$, then $H_1\cup(P_{a'_2b'_2}\setminus b'_2)\cup P\cup\{u,b_2\}$ contains a $3PC(b_1b_2u, a_1a'_1a'_2)$. So $|A_2| = 1$. Let H' be the hole contained in $P_{a_1b_1}\cup(P_{a_2b'_2}\setminus b'_2)\cup P\cup u$ that contains $P_{a_1b_1}\cup P\cup u$. If yb_2 is not an edge, then $H'\cup P_{a'_1b'_1}\cup b_2$ induces a $3PC(b_1b_2u, a_1a'_1a_2)$. So yb_2 is an edge, and hence (H',b_2) is a bug. But then $P_{a'_1b'_1}$ is either a center-crosspath or an ear of (H',b_2). This proves (1.5).

By (1.2), (1.3), (1.4) and (1.5), no node of H has a neighbor in $P\setminus p_k$.

Node p_k cannot be of type A_1, A, t3 (with neighbors in A), Ad nor a pseudo-twin of a node of A_2 w.r.t. H, since otherwise $N(p_k)\cap H_1 = A_1$ and since p_k is not adjacent to b_2, $H_1\cup P\cup\{u,b_2\}$ induces a $3PC(b_1b_2u, a_1a'_1p_k)$.

Suppose that p_k is of type a or a pseudo-twin of a node of A_1 w.r.t. H. If p_k is adjacent to a_1 and a_2, and it does not have a neighbor in $P_{a_1b_1}\setminus a_1$, then $P_{a_2b_2}\cup P_{a_1b_1}\cup P\cup u$ induces a $3PC(b_1b_2u, a_1a_2p_k)$. Otherwise $(H\setminus P_{a_1b_1})\cup P\cup u$ induces a short connected diamond $H'(A'_1,A_2,B'_1,B_2)$ where $A'_1 = \{a'_1,p_k\}$ and $B'_1 = \{u,b'_1\}$, and satisfies (i), contradicting our assumption. So p_k is not of type a nor a pseudo-twin of a node of A_1 w.r.t. H.

Suppose that p_k is of type d w.r.t. H. By Lemma 4.5.2 applied to Σ', u and P, $N(p_k)\cap H = \{y, y_{b_2}, y_{b'_2}\}$, $y_{b_2}\neq b_2$ and $y_{b'_2}\neq b'_2$. But then $(H\setminus P_{a_1b_1})\cup P\cup u$ induces a connected diamond whose side-2-paths have fewer nodes in common than the side-2-paths of H, a contradiction. So p_k is not of type d w.r.t. H.

Node p_k cannot be an H_2-crossing nor a pseudo-twin of y when $y\notin\{a_1,a_2\}$ w.r.t. H, since otherwise Σ', u and P contradict Lemma 4.5.2.

Suppose that p_k is of type p1, p2 or p3 w.r.t. H. Note that by definition of P, p_k has a neighbor in $H\setminus B$. If $N(p_k)\cap H\subseteq P_{a_1b_1}$ then $(H\setminus b_1)\cup P\cup u$ contains a short connected diamond $H'(A_1,A_2,B'_1,B_2)$ where $B'_1 = \{u,b'_1\}$, that contains $H_2\cup P_{a'_1b'_1}$, and H' satisfies (i), contradicting our assumption. So p_k has a neighbor in $\Sigma'\setminus B$. By Lemma 4.5.2 applied to Σ', u and P w.l.o.g. one of the following holds: (a) $|A_2| = 1$, b_2y is an edge, and either $N(p_k)\cap H = \{v_{b'_2}\}$ or p_k is of type p3 w.r.t. H adjacent to b'_2, (b) p_k is of type p2 w.r.t. H and its neighbors are contained in $P_{a'_1b'_1}$, or (c) $|A_2| = 1$, p_k is of type p2 w.r.t. H, and $N(p_k)\cap H\subseteq P_{a_2y}$. If (a) holds, then $P_{a_1b_1}\cup P_{a_2b'_2}\cup P\cup u$ contains a bug with center b'_2, and $P_{a'_1 b'_1}$ is its center-crosspath or an ear. If (b) holds, then $H_1 \cup P \cup \{u, b_2\}$ induces a $3PC(b_1 b_2 u, \Delta)$. So (c) holds. But then Σ, u and P contradict Lemma 4.5.3. So p_k is not of type p1, p2 or p3 w.r.t. H.

Therefore, by (1.1) p_k is an H_1-crossing w.r.t. H. By Lemma 4.5.3 applied to Σ, u and P, node p_k must be adjacent to b_1. If $k > 1$, then $H_1 \cup P \cup \{u, b_2\}$ induces a bug with center p_k with an ear. So $k = 1$. This proves (1).

Let $S_2 = (N[b_1] \setminus (H \cup u)) \cup \{b_1, b_2, b'_2\}$. Since S_2 cannot be a star cutset, there exists a direct connection $P = p_1, \ldots, p_k$ from u to $H \setminus S_2$ in $G \setminus S_2$. So p_1 is adjacent to u, p_k to a node of $H \setminus S_2$, and the only nodes of H that may have a neighbor in $P \setminus p_k$ are b_2 and b'_2. By (1) there exists a node v adjacent to u that is an H_1-crossing w.r.t. H adjacent to b_1.

(2) *p_k has a neighbor in $H \setminus B$.*

Proof of (2): Suppose that $N(p_k) \cap H \subseteq B$. By definition of P, p_k must be adjacent to b'_1. By Lemma 5.2.6, p_k cannot be of type s1 w.r.t. H. $N(p_k) \cap H \neq \{b'_1\}$ nor $\{b'_1, b_2, b'_2\}$, since otherwise $H_1 \cup P \cup \{u, v\}$ induces a proper wheel with center v. Since p_k is not adjacent to b_1 and it is adjacent to b'_1, it follows that p_k cannot be of type B_2 nor B w.r.t. H, and if it is of type t3 w.r.t. H then its neighbors in H are contained in A. Hence, p_k has a neighbor in $H \setminus B$. This proves (2).

(3) *p_k is either not strongly adjacent to H or it is of type p1, p2, p3, A_1, A, a, d, Ad, t3 (with neighbors in A), H_1-crossing (adjacent to b'_1), H_2-crossing or a pseudo-twin of a node of $A \cup B_1 \cup y$ w.r.t. H.*

Proof of (3): By Lemmas 5.2.6, 5.2.7 and 5.2.8, p_k cannot be of type s1, s2, s3 nor s4 w.r.t H. By (2) p_k cannot be of type B_2 nor B w.r.t H, and if it is of type t3 w.r.t. H, then its neighbors in H are contained in A. Since p_k is not adjacent to b_1, it cannot be a pseudo-twin of a node of B_2 w.r.t. H, and if it is an H_1-crossing w.r.t. H, then it is adjacent to b'_1. The result follows from Lemma 5.2.1. This proves (3).

(4) *If b_2 does not have a neighbor in $P \setminus p_k$, then p_k is adjacent to b_2 and it is of type p2, p3, d, Ad, H_2-crossing, a pseudo-twin of a node of $B_1 \cup A_2$ or a pseudo-twin of y when $y \notin \{a_1, a_2\}$ w.r.t. H.*

Proof of (4): Assume that b_2 does not have a neighbor in $P \setminus p_k$. By (2) p_k has a neighbor in $H \setminus B$. If p_k is not adjacent to b_2, then P is a direct connection from u to $H \setminus S_1$ in $G \setminus S_1$, and hence by (1) p_k is adjacent to b_1, a contradiction. So p_k is adjacent to b_2. In particular,

p_k cannot be of type A_1, A, a, t3 (with neighbors in A), H_1-crossing nor a pseudo-twin of a node of A_1 w.r.t. H. Also since p_k is adjacent to b_2 and it has a neighbor in $H \setminus S_2$, p_k must be strongly adjacent to H. The result now follows from (3). This proves (4).

(5) *b_2 does not have a neighbor in $P \setminus p_k$.*

Proof of (5): Assume it does. Let p_j be the node of $P \setminus p_k$ with highest index adjacent to a node of H. By (2), p_k has a neighbor in $H \setminus B$ and hence in the graph induced by $(H \setminus B) \cup \{b_1, p_k\}$ there is a chordless path from b_1 to p_k, and this path together with $P \cup u$ induces a hole H'. Since b_2 has at least three neighbors in H', (H', b_2) must be a twin wheel or a bug, i.e. b_2 has a unique neighbor in P and this neighbor is contained in $P \setminus p_k$. Since (H', b_2') cannot be a proper wheel, b_2' has at most one neighbor in P. If p_j is not adjacent to b_2, then a subpath of $P \setminus p_k$ is a hat of Σ. So p_j is adjacent to b_2. Also $N(b_2') \cap P \subseteq \{p_j, p_k\}$, else a subpath of $P \setminus p_k$ is a hat of Σ.

Next we show that v does not have a neighbor in P. Assume it does. Then (H', v) is a wheel, and hence it must be a twin wheel or a bug. In particular, v has exactly one neighbor p_i in P. Let H'' be the hole induced by the $p_i p_j$-subpath of P together with b_1, b_2 and v. If $i = 1$ or $j = 1$ then (H'', u) is a proper wheel. So $i \neq 1$ and $j \neq 1$. But then $(H'' \setminus b_1) \cup \{u, p_1, ..., p_i\}$ induces a $3PC(u, p_i)$ if $i < j$ and a $3PC(u, p_j)$ otherwise. Therefore, v does not have a neighbor in P.

Next we show that p_k does not have a neighbor in H_1. Assume it does. Suppose that $N(p_k) \cap H_1 = v_{b_1}$. Then by (3), $N(p_k) \cap (H_1 \cup b_2) = v_{b_1}$, and hence $H_1 \cup \{b_2, p_j, ..., p_k\}$ induces a $3PC(b_2, v_{b_1})$. So p_k has a neighbor in $H_1 \setminus v_{b_1}$, and hence by (2) and (3) and since p_k is not adjacent to b_1, p_k has a neighbor in $H_1 \setminus \{v_{b_1}, b_1, b_1'\}$. Let P' be a chordless path from p_k to v in $(H_1 \setminus \{b_1, b_1', v_{b_1}\}) \cup \{v, p_k\}$. If $j \neq 1$, then $P \cup P' \cup \{u, b_2\}$ induces a $3PC(u, p_j)$. So $j = 1$. But then $P \cup P' \cup \{u, b_1, b_2\}$ induces a proper wheel with center u. Therefore p_k does not have a neighbor in H_1.

If $N(p_k) \cap H = v_{b_2}$, then $P_{a_1 b_1} \cup P_{a_2 b_2} \cup P \cup u$ induces a proper wheel with center b_2. So p_k has a neighbor in $H \setminus v_{b_2}$. It follows, by (2) and since p_k does not have a neighbor in $H_1 \cup b_2$, that p_k has a neighbor in $H_2 \setminus \{v_{b_2}, b_2, b_2'\}$. Let P' be a chordless path from p_k to v in $(H_2 \setminus \{v_{b_2}, b_2, b_2'\}) \cup (P_{a_1' b_1'} \setminus b_1') \cup \{v, p_k\}$. If $j \neq 1$, then $P' \cup P \cup \{u, b_2\}$ induces a $3PC(u, p_j)$. So $j = 1$. But then $P' \cup P \cup \{b_1, b_2\}$ induces a 4-wheel with center u. This proves (5).

(6) *b_2' does not have a neighbor in $P \setminus p_k$.*

Proof of (6): Assume it does. Let p_j be the node of $P \setminus p_k$ with highest index adjacent to b_2'. By (5) no node of $H \setminus b_2'$ has a neighbor in $P \setminus p_k$. By (4) p_k is adjacent to b_2. Since $P \cup \{u, b_2, b_2'\}$ cannot induce a proper wheel with center b_2', $N(b_2') \cap P = p_j$.

Next we show that v does not have a neighbor in P. Assume it does. By (2) p_k has a neighbor in $H\setminus B$ and hence in $(H\setminus B)\cup\{b_1,p_k\}$ there is a chordless path from b_1 to p_k, and this path together with $P\cup u$ induces a hole H'. Since (H',v) cannot be a proper wheel, $N(v)\cap P=p_i$ for some $i\in\{1,\dots,k\}$. Let H'' be the hole induced by the p_ip_j-subpath of P together with b_1,b_2' and v. Since (H'',u) cannot be a 4-wheel, $i\neq 1$ and $j\neq 1$. But then $(H''\setminus b_1)\cup\{u,p_1,...,p_i\}$ induces a $3PC(u,p_i)$ if $i<j$ or $3PC(u,p_j)$ otherwise. Therefore v does not have a neighbor in P.

Suppose that p_k has a neighbor in $H\setminus(B\cup v_{b_2})$. Let P' be a chordless path from p_k to v in $(H\setminus(B\cup v_{b_2}))\cup\{p_k,v\}$. Then $P'\cup P\cup\{u,b_2\}$ induces a $3PC(p_k,u)$. Therefore $N(p_k)\cap H\subseteq B\cup v_{b_2}$, and hence by (2) p_k is adjacent to v_{b_2}. But then $P_{a_1b_1}\cup P_{a_2b_2}\cup P\cup u$ induces a 4-wheel with center b_2. This proves (6).

By (5) and (6) no node of H has a neighbor in $P\setminus p_k$. By (4) p_k is adjacent to b_2.

Suppose p_k is of type p2, d, Ad, H_2-crossing or a pseudo-twin of a node of A_2 or y when $y\notin\{a_1,a_2\}$ w.r.t. H. Since p_k is adjacent to b_2, it follows that Σ',u and P contradict Lemma 4.5.2. Therefore p_k cannot be any of these types, and hence by (4) p_k is either of type p3 w.r.t. H or it is a pseudo-twin of a node of B_1 w.r.t. H.

Suppose that p_k is of type p3 w.r.t. H. Since p_k is adjacent to b_2, by Lemma 4.5.2 applied to Σ',u and P, it follows that $|A_2|=1$ and $b_2'y$ is an edge. Let w be the neighbor of p_k in P_{b_2y} that is closest to y. Let P' be the wy-subpath of P_{b_2y}, and let H' be the hole induced by $P\cup P'\cup P_{a_2y}\cup P_{a_1b_1}\cup u$. Then (H',b_2') is a bug and $P_{a_1'b_1'}$ its center-crosspath or ear, a contradiction.

So p_k is a pseudo-twin of a node of B_2 w.r.t. H. Suppose that p_k is not adjacent to a node of B_1. If $k\neq 1$, then $H_1\cup P\cup\{u,b_2'\}$ induces a bug with center p_k with an ear (where the ear is the path induced by $(P\setminus p_k)\cup u$). So $k=1$. Since $\{p_1,v,b_1,b_2\}$ cannot induce a 4-hole, p_1v is not an edge. Note that both p_1 and v have a neighbor in $H_1\setminus\{b_1,b_1',v_{b_1}\}$. Let P' be a chordless path from p_1 to v in $(H_1\setminus\{b_1,b_1',v_{b_1}\})\cup\{p_1,v\}$. Then $P'\cup\{u,v,b_1,b_2\}$ induces a 4-wheel with center u. So p_k must be adjacent to a node of B_1.

By definition of P, p_k is not adjacent to b_1, and hence it is adjacent to b_1'. Therefore, p_k is a pseudo-twin of b_1' w.r.t. H. Suppose that v does not have a neighbor in P. Let P' be the path from p_k to v in $(P_{a_1'b_1'}\setminus b_1')\cup\{p_k,v\}$. If $k>1$, then $P'\cup P\cup\{u,b_2'\}$ induces a $3PC(p_k,u)$. So $k=1$, and hence $P'\cup P\cup\{u,b_1,b_2'\}$ induces a 4-wheel with center u. Therefore v has a neighbor in P. Let P' be the chordless path from p_k to b_1 in $(H_1\setminus b_1')\cup p_k$. Since $P'\cup P\cup\{b_1,u,v\}$ cannot induce a proper wheel with center v, $N(v)\cap(P'\cup P)=p_i$ for some $i\in\{1,...,k\}$. But then $P'\cup\{p_i,...,p_k,b_2,v\}$ induces $3PC(b_1,p_k)$. □

Proof of Theorem 2.2.6: Assume G does not have a star cutset. Then by Theorems 4.2.2, 4.3.3, 4.3.4, 4.3.5 and 4.3.6 G does not contain a proper wheel, a bug with a center-crosspath, a $3PC(\Delta,\cdot)$ with a hat, a bug with an ear nor a $3PC(\Delta,\cdot)$ with a type s2 node. We prove that for some connected diamond H of G, the 2-join $H_1|H_2$ of H extends to a 2-join of G. Assume not. Then by Theorem 5.1.5 every connected diamond H of G has a blocking sequence for $H_1|H_2$. Consider all short connected diamonds H, and amongst them choose an H with a shortest blocking sequence $S = x_1,\dots,x_n$ for $H_1|H_2$.

By Lemmas 5.2.1, 5.2.6, 5.2.7 and 5.2.8 the following holds:

(1) If a node of $G\setminus H$ has a neighbor in H, then it is of type p1, p2, p3, A_1, A, B, B_2, a, t3, d, Ad, H_1-crossing, H_2-crossing w.r.t. H or it is a pseudo-twin of a node of $A\cup B\cup y$ w.r.t. H.

By (1), Lemma 5.2.9, Theorem 5.1.9 and our choice of H and S, the following holds:

(2) If a node of S has a neighbor in H, then it is of type p1, p2, A_1, A, B, B_2, d, Ad, H_1-crossing or H_2-crossing w.r.t. H, or $|A_2| = 1$ and it is a pseudo-twin of a_2 w.r.t. H.

So by Remark 5.1.2 and since neither $H_1|H_2\cup x_1$ nor $H_1\cup x_n|H_2$ is a 2-join, $N(x_1)\cap H_1 \neq \varnothing, A_1, B_1$ and $N(x_n)\cap H_2 \neq \varnothing, A_2, B_2$ and hence by (2) the following hold:

(3) $n > 1$.

(4) x_1 has a neighbor in H_1, and it is of type p1, p2 or H_1-crossing w.r.t. H.

(5) x_n has a neighbor in H_2, and it is of type p1, p2, d, Ad, H_2-crossing w.r.t. H, or it is a pseudo-twin of a_2 w.r.t. H when $|A_2| = 1$.

Let x_l be the node of S with lowest index adjacent to a node of H_2. By (4), $N(x_1)\cap H \subseteq H_1$ and hence $l > 1$. By Lemma 5.1.8, $x_1,\dots,x_l$ is a chordless path. Let x_j be the node of $S\setminus x_1$ with lowest index that has a neighbor in H. Clearly $j \leq l$ and hence $x_1,\dots,x_j$ is a chordless path. Note that nodes $x_2,\dots,x_{j-1}$ have no neighbors in H. Furthermore by (2), (5) and Lemma 5.1.3, the following holds:

(6) Either $j = n$ and x_j is one of the types in (5), or $j < n$ and x_j is of type A_1, A, B or B_2 w.r.t. H.

Let C (resp. C') be the hole induced by $P_{a_1b_1}\cup P_{a'_1b'_1}\cup b_2$ (resp. $P_{a_1b_1}\cup P_{a'_1b'_1}\cup b'_2$).

Claim 1: x_1 *is not an* H_1*-crossing w.r.t.* H.

Proof of Claim 1: Assume it is. W.l.o.g. x_1 is adjacent to b_1. Then (C,x_1) and (C',x_1) are both bugs. If x_j is of type A_1, A, Ad or a pseudo-twin of a_2 when $|A_2| = 1$ w.r.t. H, then x_j is not adjacent to at least one of b_2, b'_2 and hence $x_2,\dots,x_j$ is a center-crosspath of (C,x_1) or (C',x_1), a contradiction. If x_j is of type B_2 w.r.t. H, then $(C \setminus A_1) \cup \{x_1,\dots,x_j\}$ contains a $3PC(b_2,x_1)$.

Suppose that x_j is of type B w.r.t. H. If $j = 2$, then bug (C,x_1) and x_2 contradict Lemma 4.3.1. So $j > 2$ and hence $(C \setminus A_1) \cup \{x_1,\dots,x_j\}$ contains a $3PC(x_1,x_j)$. So by (6), x_j has a neighbor in H_2 and it is of type p1, p2, d or H_2-crossing w.r.t. H. In particular, $N(x_1) \cap H \subseteq H_1$ and $N(x_j) \cap H \subseteq H_2$, and hence H and $x_1,\dots,x_j$ contradict Lemma 5.2.5. This completes the proof of Claim 1.

Claim 2: *x_1 is not of type p2 w.r.t. H.*

Proof of Claim 2: Assume it is. W.l.o.g. the neighbors of x_1 in H are contained in $P_{a_1b_1}$. If x_j is of type A_1, A, Ad or a pseudo-twin of a_2 when $|A_2| = 1$ w.r.t. H, then x_j is not adjacent to at least one of b_2, b'_2, and hence either $C \cup \{x_1,\dots,x_j\}$ or $C' \cup \{x_1,\dots,x_j\}$ induces a $3PC(\Delta,\Delta)$ or a 4-wheel with center a_1.

Node x_j cannot be of type B, p2, d nor H_2-crossing w.r.t. H, since otherwise either $P_{a_1b_1} \cup P_{a'_2b'_2}$ or $P_{a_1b_1} \cup P_{a_2b_2}$ induces a $3PC(\Delta,\Delta)$ or a 4-wheel with center b_1.

Suppose that x_j is of type B_2 w.r.t. H. Let P be the chordless path from x_j to a_1 in $G[P_{a_1b_1} \cup \{x_1,\dots,x_j\}]$. Let H' be the short connected diamond induced by $P \cup P_{a'_1b'_1} \cup H_2$. Then by Theorem 5.1.9 applied to H' and S, our choice of H is contradicted.

So by (6), $N(x_j) \cap H = r$ and $r \in H_2$. But then H and $x_1,\dots,x_j$ contradict Lemma 5.2.5. This completes the proof of Claim 2.

Claim 3: *If $N(x_1) \cap H = b_1$, then there exists a chordless path $P = p_1,\dots,p_k$ in $G \setminus H$ such that p_1 is adjacent to x_1, no node of $P \setminus p_1$ is adjacent to x_1, no node of $P \setminus p_k$ has a neighbor in H and one of the following holds:*

(i) $N(p_k) \cap H = v_{b_1}$, *or*

(ii) p_k *is of type p2 w.r.t. H and its neighbors in H are contained in $P_{a'_1b'_1}$.*

Proof of Claim 3: Let $S = N[b_1] \setminus \{x_1, v_{b_1}\}$. Since S cannot be a star cutset, there exists a direct connection $P = p_1,\dots,p_k$ from x_1 to H in $G \setminus S$. So p_1 is adjacent to x_1, no node of $P \setminus p_1$ is adjacent to x_1, p_k has a neighbor in $H \setminus \{b_1,b_2,b'_2\}$ and it is not adjacent to b_1, and the only nodes of H that may have a neighbor in $P \setminus p_k$ are b_2 and b'_2.

Case 1: b_2 and b'_2 do not have neighbors in $P \setminus p_k$.

Case 1.1: p_k has a neighbor in $\Sigma \setminus \{b_2, b_2'\}$.

By Lemma 4.5.1 applied to Σ, x_1 and P, and since no node of P is adjacent to b_1, one of the following holds: (a) $N(p_k) \cap \Sigma = v_{b_1}$; (b) p_k is of type p2 w.r.t. Σ with neighbors in $P_{b_1 y}$ path of Σ; or (c) p_k is of type d w.r.t. Σ and it has no neighbor in $P_{b_1 y} \setminus y$.

Suppose that (a) holds. By (1) either $N(p_k) \cap H = v_{b_1}$ and hence (i) holds, or $a_1 b_1$ is an edge and $N(p_k) \cap H = \{a_1, a_1'\}$. The second case cannot hold, since then $P_{a_1 b_1} \cup P_{a_2 b_2} \cup P \cup \{x_1, a_1'\}$ induces a 4-wheel with center a_1.

Suppose that (b) holds. First suppose that $N(p_k) \cap \Sigma \subseteq P_{a_1 b_1}$. Then by (1), p_k is of type p2 or H_1-crossing w.r.t. H. If p_k is an H_1-crossing w.r.t. H, then $(P_{a_1 b_1} \setminus a_1) \cup P \cup \{x_1, b_2, b_1'\}$ contains a $3PC(b_1, p_k)$. So p_k is of type p2 w.r.t. H. Note that p_k is not adjacent to b_1, and hence $(H \setminus v_{b_1}) \cup P \cup x_1$ contains a short connected diamond $H'(A_1, A_2, B_1, B_2)$ that contains x_1, and hence by Theorem 5.1.9 our choice of H and S is contradicted. Therefore $N(p_k) \cap \Sigma$ is not contained in $P_{a_1 b_1}$, and hence $|A_2| = 1$. Suppose that $N(p_k) \cap \Sigma \subseteq P_{a_2 y}$. So by (1), p_k is of type p2 w.r.t. H. But then $(H \setminus (P_{a_1 b_1} \setminus b_1)) \cup P \cup x_1$ contains a connected diamond whose side-2-paths have fewer nodes in common than the side-2-paths of H, contradicting our choice of H. Therefore $N(p_k) \cap \Sigma = \{a_1, a_2\}$. By (1) p_k is of type a, A or it is a pseudo-twin of a_1' w.r.t. H. By Lemma 4.5.2 applied to Σ', b_1 and path x_1, P, node p_k must in fact be of type A w.r.t. H. But then $(H \setminus (P_{a_1 b_1} \setminus b_1)) \cup P \cup x_1$ induces a short connected diamond $H'(A_1', A_2, B_1, B_2)$ where $A_1' = \{a_1', p_k\}$ that contains x_1. But then by Theorem 5.1.9 our choice of H and S is contradicted.

So we may now assume that (c) holds. Suppose that $|A_2| = 2$. Then $N(p_k) \cap \Sigma = \{a_1, a_2, a_2'\}$ and so by (1) p_k is of type A or it is a pseudo-twin of a_1' w.r.t. H. If p_k is a pseudo-twin of a_1' w.r.t. H, then $P_{a_1 b_1} \cup (P_{a_1' b_1'} \setminus a_1') \cup P \cup \{x_1, b_2'\}$ contains a $3PC(b_1, p_k)$. So $N(p_k) \cap H = A$. Let H' be the short connected diamond induced by $P_{a_1' b_1'} \cup P \cup H_2 \cup \{x_1, b_1\}$. Then by Theorem 5.1.9 applied to H' and S, our choice of H is contradicted. So $|A_2| = 1$, and hence $N(p_k) \cap \Sigma = \{y, y_{b_2}, y_{b_2'}\}$. By (1), $N(p_k) \cap H = \{y, y_{b_2}, y_{b_2'}\}$. Suppose that p_k is not adjacent to a node of B_2. Let H' be the connected diamond induced by $(H \setminus (P_{a_1 b_1} \setminus b_1)) \cup P \cup x_1$. Then the two side-2-paths of H' have fewer nodes in common than the two side-2-paths of H, contradicting our choice of H. So p_k is adjacent to a node of B_2, w.l.o.g. say it is adjacent to b_2. Then $b_2 y$ is an edge, and hence $b_2' y$ is not an edge. But then $P \cup P_{a_1' b_1'} \cup P_{a_2 y} \cup \{x_1, b_2, b_2'\}$ induces a proper wheel with center b_2.

Case 1.2: p_k has no neighbor in $\Sigma \setminus \{b_2, b_2'\}$.

Then $N(p_k) \cap H \subseteq P_{a_1' b_1'} \cup B_2$. So by (1) either $N(p_k) \cap H \subseteq P_{a_1' b_1'}$ or p_k is of type t3 w.r.t. H (adjacent to b_1') or p_k is a pseudo-twin of b_1' w.r.t. H. If p_k is a pseudo-twin of b'_1 w.r.t. H, then $P_{a_1b_1} \cup (P_{a'_1b'_1} \setminus b'_1) \cup P \cup \{x_1, b_2\}$ contains a $3PC(b_1, p_k)$. If p_k is of type t3 w.r.t. H, then $H_1 \cup P \cup \{x_1, b_2\}$ induces a bug with center b_2, and $P_{a_2b_2} \setminus b_2$ is its center-crosspath. So $N(p_k) \cap H \subseteq P_{a'_1b'_1}$. If $N(p_k) \cap H = b'_1$, then $C \cup P \cup x_1$, induces a $3PC(b_1, b'_1)$. So p_k has a neighbor in $\Sigma' \setminus \{b_2, b'_2, b'_1\}$. Note that b_1 is of type t2 w.r.t. Σ'. By Lemma 4.5.2 applied to Σ', b_1 and P, (ii) holds.

Case 2: b_2 or b'_2 has a neighbor in $P \setminus p_k$.

Let p_i be the node of $P \setminus p_k$ with highest index that has a neighbor in $\{b_2, b'_2\}$. W.l.o.g. we may assume that p_i is adjacent to b_2.

Suppose that p_k does not have a neighbor in $\Sigma \setminus \{b_2, b'_2\}$. Then p_k has a neighbor in $P_{a'_1b'_1}$. Let C be the hole contained in $H_1 \cup P \cup x_1$ that contains $P_{a_1b_1} \cup P \cup x_1$. Since $C \cup b_2$ cannot induce a $3PC(b_1, p_i)$, (C, b_2) is a wheel and hence it must be a bug. But then $P_{a_2b_2} \setminus b_2$ is its center-crosspath. Therefore p_k has a neighbor in $\Sigma \setminus \{b_2, b'_2\}$. We now consider the following cases.

Case 2.1: $N(p_i) \cap H = b_2$.

Since p_k is not adjacent to b_1 and it has a neighbor in $\Sigma \setminus \{b_2, b'_2\}$, it cannot be of type B, B_2 nor a pseudo-twin of a node of $B_2 \cup b'_1$ w.r.t. H. If p_k is of type A_1, A, a, H_1-crossing, a pseudo-twin of a node of A_1 w.r.t. H or a pseudo-twin of a'_2 when $|A_2| = 2$ w.r.t. H, then Σ, p_i and $p_{i+1}, \ldots, p_k$ contradict Lemma 4.5.1.

Suppose that p_k is of type d or it is a pseudo-twin of y when $y \notin \{a_1, a_2\}$ w.r.t. H. Note that $|A_2| = 1$. By Lemma 4.5.1 applied to Σ, p_i and $p_{i+1}, \ldots, p_k$, node p_k is either adjacent to b_2 or $N(p_k) \cap H = \{y, y_{b'_2}, y_{a_2}\}$. Let P' be the chordless path from p_k to a_2 in $G[P_{a_2y} \cup p_k]$ and let C be the hole induced by $P' \cup P \cup P_{a_1b_1} \cup x_1$. Since $C \cup b_2$ cannot induce a $3PC(b_1, p_i)$, (C, b_2) is a wheel, and hence it is a bug. But then $P_{a'_1b'_1}$ is a center-crosspath of bug (C, b_2).

Suppose that p_k is of type t3, Ad or it is a pseudo-twin of a_2 w.r.t. H. Note that if p_k is of type t3 w.r.t. H, then since p_k has a neighbor in $\Sigma \setminus \{b_2, b'_2\}$, $N(p_k) \cap H \subseteq A$. So in all three cases, $N(p_k) \cap H_1 = A_1$. Let C be the hole induced by $P_{a_1b_1} \cup P \cup x_1$. Since $C \cup b_2$ cannot induce a $3PC(b_1, p_i)$, (C, b_2) is a wheel, and hence it is a bug. But then $P_{a'_1b'_1}$ is a center-crosspath of bug (C, b_2).

Suppose that p_k is an H_2-crossing w.r.t. H. First suppose that $|A_2| = 2$. If p_k is adjacent to a_2 (resp. a'_2), then let C be the hole induced by $P_{a_1b_1} \cup P \cup \{a_2, x_1\}$ (resp. $P_{a_1b_1} \cup P \cup \{a'_2, x_1\}$). Since $C \cup b_2$ cannot induce a $3PC(p_i, b_1)$, (C, b_2) is a wheel and hence it must be a bug. But then $P_{a'_1b'_1}$ is its center-crosspath. So $|A_2| = 1$. Let P' be the chordless path from p_k to a_2 in $G[(P_{a_2b_2} \setminus b_2) \cup p_k]$, and let C be the hole induced by $P' \cup P \cup x_1$. Then again (C, b_2) is a bug and $P_{a'_1b'_1}$ is its center-crosspath.

Suppose that p_k is a pseudo-twin of b_1 w.r.t. H. Since p_k is not adjacent to b_1, $N(p_k)\cap H = \{b_2, b_2', v_1, v_2\}$ where v_1v_2 is an edge of $P_{a_1b_1}\setminus b_1$. Let P' be the chordless path from p_k to b_1 in $G[P_{a_1b_1}\cup p_k]$, and let C be the hole induced by $P'\cup P\cup x_1$. Then (C, b_2) must be a bug, and hence $H_1\cup P\cup\{b_2, x_1\}$ induces a bug (C, b_2) and its center-crosspath.

Therefore by (1), p_k is of type p1, p2 or p3 w.r.t. H. By Lemma 4.5.1 applied to Σ, p_i and $p_{i+1},\dots,p_k$, $N(p_k)\cap H\subseteq P_{a_2b_2}$. Let P' be the chordless path from p_k to a_2 in $G[(P_{a_2b_2}\setminus b_2)\cup p_k]$, and let C be the hole induced by $P'\cup P\cup x_1$. Since $C\cup b_2$ cannot be a $3PC(b_1, p_i)$, (C, b_2) must be a bug, and hence $P_{a_1'b_1'}$ is its center-crosspath.

Case 2.2: $N(p_i)\cap H = \{b_2, b_2'\}$.

Since p_k is not adjacent to b_1 and it has a neighbor in $\Sigma\setminus\{b_2, b_2'\}$, it cannot be of type B, B_2 nor a pseudo-twin of a node of $B_2\cup b_1'$. If p_k is of type A_1, Ad, H_2-crossing or a pseudo-twin of a node of $A_2\cup\{a_1, y\}$ w.r.t. H, then Σ, p_i and $p_{i+1},\dots,p_k$ contradict Lemma 4.5.2.

Suppose that p_k is of type A w.r.t. H. Let C be the hole induced by $P_{a_1b_1}\cup P\cup x_1$. Since $C\cup b_2$ cannot induce a $3PC(b_1, p_i)$, (C, b_2) is a wheel, and hence it is a bug. But then $P_{a_1'b_1'}$ is its center-crosspath.

If p_k is of type a w.r.t. H, then by Lemma 4.5.2 applied to Σ, p_i and $p_{i+1},\dots,p_k$, $N(p_k)\cap H = \{a_1, a_2\}$. But then $H_1\cup\{p_i,\dots,p_k, b_2\}$ induces a $3PC(a_1, b_2)$.

Suppose that p_k is of type t3 w.r.t. H. Since p_k is not adjacent to b_1 and it has a neighbor in $\Sigma\setminus\{b_2, b_2'\}$, $N(p_k)\cap H\subseteq A$. But then Σ, p_i and $p_{i+1},\dots,p_k$ contradict Lemma 4.5.2.

Suppose that p_k if of type d w.r.t. H. By Lemma 4.5.2 applied to Σ, p_i and $p_{i+1},\dots,p_k$, $N(p_k)\cap H = \{y, y_{b_2}, y_{b_2'}\}$ and p_k is not adjacent to b_2 and b_2'. But then $(H\setminus P_{a_1b_1})\cup\{p_i,\dots,p_k\}$ induces a connected diamond whose side-2-paths have fewer nodes in common than the side-2-paths of H, contradicting our choice of H.

If p_k is an H_1-crossing w.r.t. H, then it must be adjacent to b_1', and hence $(P_{a_1b_1}\setminus a_1)\cup\{p_i,\dots p_k, b_1', b_2\}$ contains a $3PC(b_2, p_k)$.

If p_k is a pseudo-twin of a_1' w.r.t. H, then $(H_1\setminus a_1')\cup\{p_i,\dots,p_k, b_2\}$ contains a $3PC(b_2, p_k)$.

Suppose that p_k is of type p1 w.r.t. H. By Lemma 4.5.2 applied to Σ, p_i and $p_{i+1},\dots,p_k$, $|A_2| = 1$ and either yb_2 is an edge and p_k is adjacent to $v_{b_2'}$, or yb_2' is an edge and p_k is adjacent to v_{b_2}. In the first case $(H\setminus(P_{a_1'b_1'}\cup b_2'))\cup P\cup x_1$ induces a proper wheel with center b_2. In the second case, $P_{a_1b_1}\cup P_{a_2b_2}\cup P\cup x_1$ induces a proper wheel with center b_2.

Suppose that p_k is a pseudo-twin of b_1 w.r.t. H. Since p_k is not adjacent to b_1, $N(p_k) \cap H = \{b_2, b_2', v_1, v_2\}$ where v_1v_2 is an edge of $P_{a_1b_1} \setminus b_1$. Let P' be the chordless path from p_k to b_1 in $G[P_{a_1b_1} \cup p_k]$, and let C be the hole induced by $P' \cup P \cup x_1$. Then (C, b_2) must be a bug, and hence $H_1 \cup P \cup \{b_2, x_1\}$ induces a bug (C, b_2) and its center-crosspath.

Suppose that p_k is of type p3 w.r.t. H. By Lemma 4.5.2 applied to Σ, p_i and $p_{i+1}, \ldots, p_k$, $|A_2| = 1$ and p_k is adjacent to b_2 or b_2', w.l.o.g. say to b_2. Let P' be the chordless path from p_k to y in $G[(P_{b_2y} \setminus b_2) \cup p_k]$, and let C be the hole induced by $P' \cup P \cup P_{a_2y} \cup P_{a_1b_1} \cup x_1$. Then (C, b_2) must be a bug and $P_{a_1'b_1'}$ is its center-crosspath.

Therefore by (1), p_k is of type p2 w.r.t. H. By Lemma 4.5.2 applied to Σ, p_i and $p_{i+1}, \ldots, p_k$, either $N(p_k) \cap H \subseteq P_{a_1b_1}$, or $|A_2| = 1$ and $N(p_k) \cap H \subseteq P_{a_2y}$. Let P' be the chordless path from p_k to b_1 in $G[P_{a_1b_1} \cup (P_{a_2b_2} \setminus b_2) \cup p_k]$, and let C be the hole induced by $P' \cup P \cup x_1$. Since $C \cup b_2$ cannot induce a $3PC(b_1, p_i)$, (C, b_2) is a wheel, and hence it is a bug. If $N(p_k) \cap H \subseteq P_{a_2y}$, then $P_{a_1'b_1'}$ is a center-crosspath of (C, b_2). So $N(p_k) \cap H \subseteq P_{a_1b_1}$. But then $H_1 \cup P \cup \{b_2, x_1\}$ induces a bug (C, b_2) and its center-crosspath. This completes the proof of Claim 3.

Claim 4: *If $N(x_1) \cap H = a_1$, then there exists a chordless path $P = p_1, \ldots, p_k$ in $G \setminus H$ such that p_1 is adjacent to x_1, no node of $P \setminus p_1$ is adjacent to x_1, no node of $P \setminus p_k$ has a neighbor in H and $N(p_k) \cap H = v_{a_1}$.*

Proof of Claim 4: Let $S = N[a_1] \setminus \{x_1, v_{a_1}\}$. Since S cannot be a star cutset, there exists a direct connection $P = p_1, \ldots, p_k$ from x_1 to H in $G \setminus S$. So p_1 is adjacent to x_1, no node of $P \setminus p_1$ is adjacent to x_1, p_k has a neighbor in $H \setminus A$ and it is not adjacent to a_1, and the only nodes of H that may have a neighbor in $P \setminus p_k$ are a_2, a_2' and a_1'.

Since p_k is not adjacent to a_1 and it has a neighbor in $H \setminus A$, p_k cannot be of type A_1, A, a, Ad, t3 (with neighbors in A), nor a pseudo-twin of a node of $A_2 \cup a_1'$ w.r.t. H. So by (1) the following holds.

(4.1) p_k is not adjacent to a_1, and it is of type p1, p2, p3, B, B_2, t3 (with neighbors in B), d, H_1-crossing, H_2-crossing or a pseudo-twin of $B \cup a_1$ or y when $y \notin \{a_1, a_2\}$ w.r.t. H.

Case 1: a_2 and a_1' do not have a neighbor in $P \setminus p_k$.

Then a_2' is the only node of H that may have a neighbor in $P \setminus p_k$. If a_2' has a neighbor in $P \setminus p_k$, then $(P \setminus p_k) \cup x_1$ contains a hat of Σ_2, a contradiction. So no node of H has a neighbor in $P \setminus p_k$.

If p_k is of type B_2, B, d, H_1-crossing, H_2-crossing or it is a pseudo-twin of a node of $B \cup a_1$ or y when $y \notin \{a_1, a_2\}$ w.r.t. H, then since p_k is not adjacent to a_1, Lemma 4.5.1 applied to Σ_1, x_1 and P is contradicted.

Suppose that p_k is an H_2-crossing w.r.t. H. If $|A_2| = 1$ or p_k is adjacent to a_2', then Σ, x_1 and P contradict Lemma 4.5.1. So $|A_2| = 2$ and p_k is adjacent to a_2. But then x_1, P is a hat of Σ_1.

Suppose that p_k is of type t3 (with neighbors in B) w.r.t. H. By Lemma 4.5.1 applied to Σ_1, x_1 and P, $N(p_k) \cap H = \{b_2, b_2', b_1\}$. But then $H \setminus (P_{a_1 b_1} \setminus a_1) \cup P \cup x_1$ induces a short connected diamond $H'(A_1, A_2, B_1', B_2)$ where $B_1' = \{p_k, b_1'\}$, which by Theorem 5.1.9 contradicts our choice of H.

So by (4.1), p_k is of type p1, p2 or p3 w.r.t. H. W.l.o.g. $N(p_k) \cap H \subseteq \Sigma_1$. By Lemma 4.5.1 applied to Σ_1, x_1 and P, $N(p_k) \cap H = v_{a_1}$, or p_k is of type p2 w.r.t. H and $N(p_k) \cap H \subseteq P_{a_1 b_1}$. Suppose that p_k is of type p2 w.r.t. H. Then, since p_k is not adjacent to a_1, $(H \setminus v_{a_1}) \cup P \cup x_1$ contains a short connected diamond $H'(A_1, A_2, B_1, B_2)$ that contains x_1, and hence by Theorem 5.1.9 our choice of H is contradicted. So $N(p_k) \cap H = v_{a_1}$ and the result holds.

Case 2: a_2 or a_1' has a neighbor in $P \setminus p_k$.

Let p_i (resp. p_l) be the node of $P \setminus p_k$ with lowest (resp. highest) index adjacent to a node of $\{a_2, a_1'\}$. Since $x_1, p_1, \ldots, p_i$ cannot be a hat of Σ_1, p_i is adjacent to both a_2 and a_1'. Then by (1), p_i is of type a w.r.t. H. In particular, $|A_2| = 1$. W.l.o.g. p_k has a neighbor in $\Sigma_1 \setminus A$.

First suppose that p_l is adjacent to a_2 but not a_1'. Then $l > i$. By Lemma 4.5.1 applied to Σ_1, p_l and $p_{l+1}, \ldots, p_k$, node p_k has a neighbor in $(P_{a_1 b_2} \cup P_{a_2 b_2}) \setminus \{a_1, a_2\}$. Let P' be a chordless path from p_k to a_1 in $G[P_{a_1 b_1} \cup (P_{a_2 b_2} \setminus a_2) \cup p_k]$, and let C be the hole induced by $P' \cup P \cup x_1$. Then (C, a_2) is a wheel, and hence it must be a bug, i.e. $l = i + 1$. So p_k is not adjacent to a_2. If p_k is adjacent to a_1', then by (4.1), p_k is an H_1-crossing w.r.t. H adjacent to b_1 or a pseudo-twin of b_1' w.r.t. H. But then Σ_1, p_l and $p_{l+1}, \ldots, p_k$ contradict Lemma 4.5.1. So p_k is not adjacent to a_1', and hence $C \cup a_1'$ induces $3PC(a_1, p_i)$.

Now suppose that p_l is adjacent to a_1', but not a_2. Then $l > i$. By Lemma 4.5.1 applied to Σ_1, p_l and $p_{l+1}, \ldots, p_k$, node p_k has a neighbor in $((P_{a_1 b_1} \cup P_{a_1' b_1'}) \setminus \{a_1, a_1'\}) \cup b_2$. Let P' be a chordless path from p_k to a_1 in $G[P_{a_1 b_1} \cup (P_{a_1' b_1'} \setminus a_1') \cup \{p_k, b_2\}]$, and let C be the hole induced by $P' \cup P \cup x_1$. Then (C, a_1') is a wheel, and hence it must be a bug, i.e. $l = i + 1$. So p_k is not adjacent to a_1'. If p_k is adjacent to a_2, then by (4.1), p_k is of type d w.r.t. H or it is a pseudo-twin of a node of B_2 or y when $y \notin \{a_1, a_2\}$ w.r.t. H. But then Σ_1, p_l and $p_{l+1}, \ldots, p_k$ contradict Lemma 4.5.1. So p_k is not adjacent to a_2, and hence $C \cup a_2$ induces a $3PC(a_1, p_i)$.

Therefore, p_l must be adjacent to both a_2 and a'_1, and hence p_l is of type t2 w.r.t. Σ_1. If p_k is of type B_2, B, d, H_1-crossing, H_2-crossing or a pseudo-twin of a node of $B_2 \cup b_1$ or y when $y \notin \{a_1, a_2\}$ w.r.t. H, then Σ_1, p_l and $p_{l+1}, ..., p_k$ contradict Lemma 4.5.2.

Suppose that p_k is of type p3 w.r.t. H. By Lemma 4.5.2 applied to Σ_1, p_l and $p_{l+1}, \ldots, p_k$, a_2b_2 is an edge and p_k is adjacent to a'_1. Then $a_2b'_2$ is not an edge. Let P' be the chordless path from p_k to b'_1 in $G[(P_{a'_1b'_1} \setminus a'_1) \cup p_k]$, and let C be the hole induced by $P' \cup P_{a_1b_1} \cup \{b'_2, a_2, p_l, \ldots, p_k\}$. Then (C, a'_1) is a 4-wheel.

If p_k is of type t3 w.r.t. H with neighbors in B, then by Lemma 4.5.1 applied to Σ_1, p_l and $p_{l+1}, ..., p_k$, $N(p_k) \cap H = \{b_2, b'_2, b_1\}$. If p_k is of type p2 w.r.t. H, then by Lemma 4.5.2 applied to Σ_1, p_l and $p_{l+1}, ..., p_k$, $N(p_k) \cap H \subseteq P_{a_1b_1}$. In both cases let P' be the chordless path from p_k to a_1 in $G[P_{a_1b_1} \cup p_k]$, and let C be the hole induced by $P' \cup P \cup x_1$. Since $C \cup a'_1$ cannot induce a $3PC(a_1, p_l)$, (C, a'_1) is a wheel and hence it must be a bug. But then $H_1 \cup P \cup \{x_1, b_2\}$ induces a bug (C, a'_1) with its center-crosspath. Therefore p_k cannot be of type p2 nor t3 (with neighbors in B) w.r.t. H.

Suppose that p_k is a pseudo-twin of b'_1 w.r.t. H. By Lemma 4.5.2 applied to Σ_1, p_l and $p_{l+1}, ..., p_k$, node p_k is adjacent to a'_1. Let C be the hole induced by $P_{a_1b_1} \cup P \cup \{x_1, b_2\}$. Then (C, a'_1) must be a bug, and hence $i = l$ and $k = l + 1$. But then $C \cup a_2$ induces a $3PC(a_1, p_l)$, or a proper wheel with center a_2 (in the case when a_2b_2 is an edge).

Suppose p_k is a pseudo-twin of a_1 w.r.t. H. Note that since p_k is not adjacent to a_1, $N(p_k) \cap H = \{a_2, a'_1, v_1, v_2\}$ where v_1v_2 is an edge of $P_{a_1b_1} \setminus a_1$. Let C be the hole contained in $(P_{a_1b_1} \setminus b_1) \cup P \cup x_1$. Then (C, a'_1) must be a bug, and hence $H_1 \cup P \cup \{b_2, x_1\}$ induces a bug (C, a'_1) and its center-crosspath.

Therefore by (4.1), p_k is of type p1 w.r.t. H. By Lemma 4.5.2 applied to Σ_1, p_l and $p_{l+1}, ..., p_k$, a_2b_2 is an edge and $N(p_k) \cap H = v_{a'_1}$. But then $H_1 \cup P \cup \{b_2, x_1\}$ induces a proper wheel with center a'_1. This completes the proof of Claim 4.

By (4) and Claims 1 and 2, $N(x_1) \cap H = r$ where $r \in H_1$. W.l.o.g. $r \in P_{a_1b_1}$. By (6) it suffices to consider the following cases.

Case 1: x_j is of type p1, p2, d or H_2-crossing w.r.t. H.

Then $N(x_j) \cap H \subseteq H_2$, and H and $x_1, ..., x_j$ contradict Lemma 5.2.5.

Case 2: x_j is of type Ad or a pseudo-twin of a_2 when $|A_2| = 1$ w.r.t. H.

Suppose that $r \neq a_1$. If x_j has a neighbor in $P_{a_2b_2} \setminus a_2$, then $(P_{a_2b_2} \setminus a_2) \cup P_{a_1b_1} \cup \{x_1, ..., x_j\}$ contains a $3PC(r, x_j)$. Otherwise $(P_{a'_2b'_2} \setminus a'_2) \cup P_{a'_1b'_1} \cup \{x_1, ..., x_j\}$ contains a $3PC(r, x_j)$. So $r = a_1$.

Let P be the path from Claim 4. If no node of P is adjacent to or coincident with a node of $\{x_2,...,x_j\}$, then $P_{a_1b_1} \cup P_{a'_1b'_1} \cup P \cup \{x_1,...,x_j\}$ together with either b_2 or b'_2 induces a 4-wheel with center a_1. So a node of P is adjacent to or coincident with a node of $\{x_2,...,x_j\}$. Let p_i be the node of P with highest index that has a neighbor in $\{x_2,...,x_j\}$, and let x_l be the node of $\{x_2,...,x_j\}$ with highest index adjacent to p_i. If x_j has a neighbor in $P_{a_2b_2} \setminus a_2$, then $P_{a_1b_1} \cup (P_{a_2b_2} \setminus a_2) \cup \{p_i,...,p_k,x_l,...,x_j\}$ contains a $3PC(v_{a_1},x_j)$. So x_j does not have a neighbor in $P_{a_2b_2} \setminus a_2$, and hence x_j is of type Ad w.r.t. H, $|A_2| = 1$, $y = a_2$ and $N(x_j) \cap H = \{a'_1,a_1,a_2,y_{b'_2}\}$. But then $P_{a_1b_1} \cup (P_{a_2b'_2} \setminus a_2) \cup \{p_i,...,p_k,x_l,...,x_j\}$ contains a $3PC(v_{a_1},x_j)$.

Case 3: x_j is of type A_1 w.r.t. H.

If $r \neq a_1$, then Σ_1, x_j and $x_1,...,x_{j-1}$ contradict Lemma 4.5.2. So $r = a_1$. Let P be the path from Claim 4. Then $P_{a_1b_1} \cup P_{a_2b_2} \cup P \cup \{x_1,...,x_j\}$ contains a proper wheel with center a_1.

Case 4: x_j is of type A w.r.t. H.

First suppose that $r \neq a_1$. Let P be the chordless path from x_j to b_1 in $G[(P_{a_1b_1} \setminus a_1) \cup \{x_1,...,x_j\}]$. Then $H_2 \cup P \cup P_{a'_1b'_1}$ induces a short connected diamond H' which by Theorem 5.1.9 contradicts our choice of H. So $r = a_1$. Let P be the path from Claim 4. Let P' be the chordless path from x_j to b_1 in $G[(P_{a_1b_1} \setminus a_1) \cup P \cup \{x_1,...,x_j\}]$. Then $H_2 \cup P' \cup P_{a'_1b'_1}$ induces a short connected diamond H' which by Theorem 5.1.9 contradicts our choice of H.

Case 5: x_j is of type B_2 w.r.t. H.

By Lemma 5.2.5 applied to H and $x_1,...,x_j$, $r = b_1$. Let P be the path from Claim 3.

Suppose that P satisfies (i) of Claim 3. Let P' be a chordless path from x_j to a_1 in $G[(P_{a_1b_1} \setminus b_1) \cup P \cup \{x_1,...,x_j\}]$. Then $H_2 \cup P' \cup P_{a'_1b'_1}$ induces a short connected diamond H' which by Theorem 5.1.9 contradicts our choice of H.

So P satisfies (ii) of Claim 3. If no node of P is adjacent to or coincident with a node of $\{x_2,...,x_j\}$, then $(P_{a'_1b'_1} \setminus a'_1) \cup P \cup \{b_1,b'_2,x_1,...,x_j\}$ contains a $3PC(b'_2,x_1)$. Otherwise, there exists a chordless path P' from x_j to a'_1 in $G[(P_{a'_1b'_1} \setminus b'_1) \cup P \cup \{x_2,...,x_j\}]$, and hence $H_2 \cup P' \cup P_{a_1b_1}$ induces a short connected diamond H' which by Theorem 5.1.9 contradicts our choice of H.

Case 6: x_j is of type B w.r.t. H.

If $r \neq b_1$, then $P_{a_1b_1} \cup P_{a'_1b'_1} \cup \{x_1,...,x_j\}$ induces a $3PC(r,x_j)$. So $r = b_1$. Let P be the path from Claim 3. Suppose that P satisfies (i) of Claim 3. If no node of P is adjacent to or coincident with a node of $\{x_2,...,x_j\}$, then $P_{a_1b_1} \cup P_{a_2b_2} \cup P \cup \{x_1,...,x_j\}$ induces a

4-wheel with center b_1. Otherwise, $P_{a_1b_1} \cup P_{a'_1b'_1} \cup P \cup \{x_2, \ldots, x_j\}$ contains a $3PC(x_j, v_{b_1})$. So P must satisfy (ii) of Claim 3.

If a node of P is adjacent to or coincident with a node of $\{x_2, \ldots, x_j\}$, then $P_{a_1b_1} \cup (P_{a'_1b'_1} \setminus b'_1) \cup P_{a_2b_2} \cup P \cup \{x_2, \ldots, x_j\}$ contains a $3PC(x_jb_1b_2, a_1a'_1a_2)$. So no node of P is adjacent to or coincident with a node of $\{x_2, \ldots, x_j\}$. If $j = 2$, then $(P_{a'_1b'_1} \setminus a'_1) \cup P \cup \{b_1, b_2, x_1, \ldots, x_j\}$ contains a 4-wheel with center x_j. So $j > 2$. But then $(P_{a'_1b'_1} \setminus a'_1) \cup P \cup \{b_1, x_1, \ldots, x_j\}$ contains a $3PC(x_1, x_j)$. $\square$

Chapter 6

Recognition Algorithm

In this section we give a new recognition algorithm for even-hole-free graphs. As already discussed in Sections 1.1 and 2.1, two different recognition algorithms are given in [14] and [6].

6.1 Cleaning algorithm

Let H be a hole, and $v \in V(G) \setminus V(H)$. We say that v is *major* w.r.t. H if there exist three of its neighbors in H that are parwise nonadjacent. This is the terminology from [6].

Let H be a smallest even hole of a graph G. We say that H is *clean* if no vertex of G is major w.r.t. H.

Let H be a smallest even hole of G. Let $u \in G \setminus H$. We say that u is of type gi, for $i = 1, 2, 3$, if $|N(u) \cap V(H)| = i$ and $N(u) \cap V(H)$ induces a path on i nodes. We say that u is of type b1 if $V(H) \cup \{u\}$ induces a $3PC(\cdot, \cdot)$; u is of type b2 if (H, u) is a 4-wheel that has exactly two long sectors and these two long sectors do not have a node in common; and u is of type b3 if (H, u) is a 4-wheel that has exactly two long sectors and these two long sectors have a node in common. This is the terminology from [14].

Let H be a smallest even hole of G. Let u be a type g3 node w.r.t. H, with neighbors u_1, u_2, u_3 in H such that u_1u_2 and u_2u_3 are edges. Let H' be the hole induced by $(V(H) \setminus \{u_2\}) \cup \{u\}$. We say that H' is obtained from H by a *type-g3-node-substitution*. Let $\mathscr{C}_G(H)$ be the set of all holes obtained from H through a sequence of type-g3-nodesubstitutions.

A graph G is *clean* if it is either even-hole-free or it contains a smallest even hole H such that all holes of $\mathcal{C}_G(H)$ are clean.

A *short 4-wheel* is a 4-wheel (H,x) such that either exactly three of the four sector are of length 1, or exactly two of the four sectors are of length 1 and they do not have a common endnode and one of the sectors is of length 3.

In both [14] and [6] a "cleaning procedure" is given, that takes an input graph G and produces a clean graph G' that is even-hole-free if and only if G is even-hole-free. In [14] a smallest even hole is "cleaned" in the sense that all major nodes are eliminated but also the type b1, b2 and b3 nodes. Here we give the cleaning from [6] that cleans just the major nodes, and hence has better complexity.

Theorem 6.1.1 *[6] There exists an algorithm with following specifications:*

Input	:	*A graph G.*
Output	:	*A sequence of subsets* $X_1,\dots,X_r$ *of* $V(G)$ *with* $r \leq |V(G)|^9$ *such that for every smallest even hole* H *of* G*, one of* $X_1,\dots,X_r$ *is disjoint from* $V(H)$ *and includes all major vertices for* H.
Running Time	:	$\mathcal{O}(|V(G)|^{10})$.

Lemma 6.1.2 *Let* H *be a smallest even hole of* G*. If* $x \in V(G) \setminus V(H)$ *has an odd number of neighbors in* H*, then* x *is of type g1 or g3 w.r.t.* H.

Proof: Assume that x has an odd number of neighbors in H, and that it is not of type g1 or g3 w.r.t. H. Then (H,x) is a wheel. If S is any sector of (H,x), then $V(S) \cup \{x\}$ induces either a triangle or a hole that is of length smaller than H. So every sector of (H,x) is of odd length, and since (H,x) has an odd number of sectors, it follows that H is of odd length, a contradiction. □

Lemma 6.1.3 *Assume that* G *does not contain a short 4-wheel nor a smallest even hole with a type b3 node. Let* H *be a smallest even hole of* G*. If* H *is clean, then all holes in* $\mathcal{C}_G(H)$ *are clean.*

Proof: Assume that H is clean. Let u be a node that is of type g3 w.r.t. H, with neighbors u_1, u_2, u_3 in H such that u_1u_2 and u_2u_3 are edges. Let H' be the hole induced by $(V(H) \setminus \{u_2\}) \cup \{u\}$. To prove the result, it suffices to show that H' is clean.

Suppose that there exists a vertex v that is major w.r.t. H'. Since v cannot be major w.r.t. H, it follows that v is adjacent to u, it has at least two nonadjacent neighbors in H, and it is not adjacent to u_2.

Since v is major w.r.t. H', by Lemma 6.1.2 v has an even number of neighbors in H'. So v has an odd number of neighbors in H. Since v has at least two neighbors in H, by Lemma 6.1.2, v is of type g3 w.r.t. H. But then either (H', v) is a short 4-wheel or v is of type b3 w.r.t. H', a contradiction. □

Lemma 6.1.4 *[14] Let G be a graph that does not contain a 4-hole nor a short 4-wheel. Let H be a smallest even hole of G, and suppose that node u is of type b3 w.r.t. H. Let* $N(u) \cap V(H) = \{u_1, u_2, u_3, u_4\}$ *such that* u_1u_2 *and* u_2u_3 *are edges. If v is major w.r.t. H, then* $N(v) \cap \{u_2, u_4, u\} \neq \varnothing$.

Theorem 6.1.5 *There exists an algorithm with following specifications:*

Input	:	*A graph G that does not contain a 4-hole, nor a short 4-wheel.*
Output	:	*A family* $\mathscr{L}$ *of induced subgraphs of G such that if G contains an even hole, then for some smallest even hole H of G and some* $G' \in \mathscr{L}$, G' *contains H and all holes in* $\mathscr{C}_{G'}(H)$ *are clean. Furthermore,* $|\mathscr{L}|$ *is* $\mathscr{O}(|V(G)|^9)$.
Running Time	:	$\mathscr{O}(|V(G)|^{10})$.

Proof: Consider the following algorithm:

Step 1: Set $\mathscr{L} = \{G\}$.

Step 2: For every (P_1, P_2, u), where $P_1 = x_1, x_2, x_3$ and $P_2 = y_1, y_2, y_3$ are disjoint chordless paths in G and $u \in N(x_2) \cap N(y_2)$, add to $\mathscr{L}$ the graph obtained from G by removing the node set $N(\{x_2, y_2, u\}) \setminus (V(P_1) \cup V(P_2))$.

Step 3: Apply the algorithm from Theorem 6.1.1 to G, and let $X_1, \ldots, X_r$ be the output sequence of subsets of $V(G)$. For $i = 1, \ldots, r$ add to $\mathscr{L}$ the graph obtained from G by removing X_i.

Clearly this algorithm runs in time $\mathscr{O}(|V(G)|^{10})$, and $|\mathscr{L}|$ is $\mathscr{O}(|V(G)|^9)$. Suppose that G contains an even hole.

First suppose that G contains a smallest even hole H with a type b3 node u. Let $N(u) \cap V(H) = \{u_1, u_2, u_3, u_4\}$ such that u_1u_2 and u_2u_3 are edges. Let u'_3 (resp. u'_1) be the neighbor of u_4 in the sector of wheel (H,u) whose endnodes are u_4 and u_3 (resp. u_1). Let G' be the graph obtained from G by removing the node set $N(\{u_2,u_4,u\}) \setminus V(H)$. Clearly G' contains H and is one of the graphs added to $\mathscr{L}$ in Step 2. Let H' be any hole of $\mathscr{C}_{G'}(H)$. By construction of G' and since G does not contain a 4-hole, H' contains $u_1,u_2,u_3,u'_3,u_4,u'_1$ and hence u is of type b3 w.r.t. H'. So by Lemma 6.1.4 and since no node of G' is adjacent to any of the nodes of $\{u_2,u_4,u\}$, it follow that no node of G' is major w.r.t. H'. Therefore $\mathscr{C}_{G'}(H)$ is clean, proving the theorem.

Now we may assume that G does not contain a smallest even hole with a type b3 node. Let H be any smallest even hole of G. By Theorem 6.1.1, for some graph G' added to $\mathscr{L}$ in Step 3, G' contains H and H is clean in G'. By Lemma 6.1.3, all holes in $\mathscr{C}_{G'}(H)$ are clean, and the theorem holds. □

6.2 Star decomposition

In this section we decompose clean graphs with star cutsets.

Let $S = N[x]$ be a full star cutset of a graph G, and let $C_1,\dots,C_n$ be the connected components of $G \setminus S$. The *blocks of decomposition* of G by S are graphs $G_1,\dots,G_n$, where G_i is the subgraph of G induced by $V(C_i) \cup S$.

Lemma 6.2.1 *Assume that G is a graph that does not contain a theta, a short 4-wheel nor a 4-hole. If H^* is a smallest even hole of G and it is clean, then H^* contains two nodes that are at distance at least 3 in G.*

Proof: Since G does not contain a 4-hole, H^* is of length at least 6, and hence it contains two nodes u and v that are at distance 3 in H^*. Suppose that u and v are not at distance 3 in G. Then there exists a node $w \in G \setminus H^*$ that is adjacent to both u and v. Since G does not contain a theta, w has at least 3 neighbors in H^*. By Lemma 6.1.2, w has at least 4 neighbors in H^*. Since G does not contain a 4-hole nor a short 4-wheel, it follows that w is major w.r.t. H^*, contradicting the assumption that H^* is clean. □

We note that for the result of the above lemma to hold it is not neccessary to exclude thetas, there is a way to just deal with type b1 nodes as in [14], but since thetas can be recognized in time $\mathscr{O}(|V(G)|^{11})$ [9], for simplicity of the argument we exclude them here.

We say that u is *dominated* by v if u is adjacent to v and $N(u) \subseteq N[v]$.

Lemma 6.2.2 *Let G be a clean graph such that for some smallest even hole H^* of G, all holes of $\mathscr{C}_G(H^*)$ are clean. Assume that G does not contain a short 4-wheel. If node u is dominated by node v, then $G \setminus \{u\}$ contains a hole of $\mathscr{C}_G(H^*)$.*

Proof: Assume that H^* contains u, and let u_1 and u_2 be the neighbors of u in H^*. Since u is dominated by v, node v is adjacent to u_1, u_2 and u. Since H^* is clean and G does not contain a short 4-wheel, v is of type g3 w.r.t. H^*. But then $(H^* \setminus u) \cup v$ is in $\mathscr{C}_G(H^*)$ and in $G \setminus u$. $\square$

A 4-wheel (H,x) is *decomposition detectable* w.r.t. a full star cutset S if $S = N[x]$, x is of type b2 w.r.t. H and the interior nodes of the two long sectors of (H,x) are contained in different connected components of $G \setminus S$.

Lemma 6.2.3 *Let G be a clean graph such that for some smallest even hole H^* of G, all holes of $\mathscr{C}_G(H^*)$ are clean. Assume that G does not contain a short 4-wheel nor a theta. When decomposing G with a full star cutset S, then either some hole in $\mathscr{C}_G(H^*)$ is entirely contained in one of the blocks of decomposition, or there exists a decomposition detectable 4-wheel w.r.t. S.*

Proof: Let $S = N[x]$ and suppose that nodes of H^* are contained in different connected components of $G \setminus S$. Then $x \notin H^*$ and x has at least two nonadjacent neighbors in H^*. Since G does not contain a theta, x has at least three neighbors in H^*.

First suppose that x has an odd number of neighbors in H^*. Then by Lemma 6.1.2, x is of type g3 w.r.t. H^*. Let H be the hole obtained by substituting x into H^*. Then H is contained in $\mathscr{C}_G(H^*)$ and in one of the blocks of decomposition by S.

So we may now assume that x has an even number of neighbors in H^*, and hence $|N(x) \cap H^*| \geq 4$. Since G does not contain a short 4-wheel, and x cannot be major w.r.t. H^*, it follows that x is of type b2 w.r.t. H^*. But then (H^*,x) is a decomposition detectable 4-wheel w.r.t. S. $\square$

Theorem 6.2.4 *There exists an algorithm with the following specifications:*

Input : *A connected graph G that does not contain a short 4-wheel, a theta, nor a 4-hole.*

Output : *Either G is identified as not being even-hole-free, or a list $\mathscr{L}$ of induced subgraphs of G with the following properties.*

(1) The graphs in $\mathscr{L}$ do not have a star cutset.

(2) If G contains a smallest even hole H^* such that all holes of $\mathscr{C}_G(H^*)$ are clean, then one of the graphs in $\mathscr{L}$ contains a hole in $\mathscr{C}_G(H^*)$.

(3) The number of graphs in $\mathscr{L}$ is $\mathscr{O}(|V(G)|^2)$.

Running Time : $\mathscr{O}(|V(G)|^{10})$.

Proof: The algorithm is as follows. Initialize $\mathscr{L} = \varnothing$ and $\mathscr{L}' = \{G\}$, and perform the following iterative step. If $\mathscr{L}' = \varnothing$, then stop. Otherwise, remove a graph F from $\mathscr{L}'$. If the distance between every pair of vertices of F is strictly less than 3 in G, then discard F and iterate. If F contains a dominated node u, then add $F \setminus u$ to $\mathscr{L}'$ and iterate. If F does not have a full star cutset, then add F to $\mathscr{L}$ and iterate. Otherwise, let S be a full star cutset of F. If there is a decomposition detectable 4-wheel w.r.t. S, then output that G is not even-hole-free and stop. Otherwise construct the blocks of decomposition by S, add them to $\mathscr{L}'$ and iterate.

Note that if a 4-wheel is found, then clearly G is not even-hole-free. (1) holds by the construction of the algorithm (note that, as was first observed by Chvátal [10], a graph has a star cutset if and only if it has a dominated node or a full star cutset). (2) holds by Lemma 6.2.1, 6.2.2 and 6.2.3.

We prove (3) by showing that the number of graphs in $\mathscr{L}$ is bounded by the number of pairs of vertices at distance at least 3 in G. Let S be a full star cutset of a graph F, and let $F_1,\dots,F_m$ be the blocks of decomposition. Let u and v be two vertices of F that are at distance at least 3 in G (and hence in F). The pair of vertices $\{u,v\}$ cannot be contained in two different blocks of decompostion, since otherwise they would both have to be in S, but since S is a star, all vertices of S are at distance at most 2. Therefore, no pair of vertices that are at distance at least 3 in G can be contained in different graphs in $\mathscr{L}$.

Finding a dominated node, or finding a full star cutset and construting blocks of decomposition can be done in time $\mathscr{O}(|V(G)|^3)$. For a given full star cutset $S = N[x]$, checking whether there exists a decomposition detectable 4-wheel can be done in time $\mathscr{O}(|V(G)|^8)$ as follows: let $C_1,\dots,C_k$ be the connected components of $G \setminus S$; for every 4-tuple (x_1,x_2,x_3,x_4), where $\{x_1,x_2,x_3,x_4\} \subseteq N(x)$ and $G[\{x_1,x_2,x_3,x_4\}]$ consists of exactly two edges, x_1x_2 and x_3x_4; and for every 2-tuple (C_i, C_j), where $i, j \in \{1, \ldots, k\}$ and $i \neq j$; check whether x_1 and x_4 both have a neighbor in the same connected component of $C_i \setminus (N(x_2) \cup N(x_3))$, and whether x_2 and x_3 both have a neighbor in the same connected component of $C_i \setminus (N(x_1) \cup N(x_4))$. All this is performed at most $\mathcal{O}(|V(G)|^2)$ times, giving $\mathcal{O}(|V(G)|^{10})$ time complexity. □

6.3 2-join decomposition

In this section we decompose a clean graph that has no star cutset using 2-join decompositions, without creating any new star cutsets.

Let $V_1|V_2$ be a 2-join with special sets (A_1, A_2, B_1, B_2). For $i = 1, 2$, let $\mathscr{P}_i$ be the family of chordless paths $P = x_1, \ldots, x_n$ where $x_1 \in A_i$, $x_n \in B_i$ and $x_j \in V_i \setminus (A_i \cup B_i)$ for $2 \leq j \leq n-1$.

The *blocks of a 2-join decomposition* are graphs G_1 and G_2 defined as follows. Block G_1 consists of the subgraph of G induced by node set V_1 plus a *marker path* $P_2 = a_2, \ldots, b_2$ that is chordless and satisfies the following properties. Node a_2 is adjacent to all nodes in A_1, node b_2 is adjacent to all nodes in B_1 and these are the only adjacencies between P_2 and the nodes of V_1. Furthermore, let $Q \in \mathscr{P}_2$. The marker path P_2 has length 3 if Q is of odd length, and length 4 otherwise. Block G_2 is defined similarly.

Theorem 6.3.1 *[14] Let G be a graph that does not contain a 4-hole. Let G_1 and G_2 be the blocks of a 2-join decomposition of G. G is even-hole-free if and only if G_1 and G_2 are even-hole-free. Furthermore, if G does not have a star cutset, then neither do G_1 and G_2.*

Theorem 6.3.2 *There exists an algorithm with the following specifications:*

Input : *A connected graph G that does not have a 4-hole nor a star cutset.*

Output : *Either an even hole of G, or a list $\mathscr{L}$ of graphs with the following properties:*

(1) The graphs in $\mathscr{L}$ do not contain a 4-hole, a star cutset nor a 2-join.

(2) G is even-hole-free if and only if all graphs in $\mathscr{L}$ are even-hole-free.

(3) The number of graphs in $\mathscr{L}$ is $\mathcal{O}(|V(G)|)$.

Running Time : $\mathcal{O}(|V(G)|^8)$.

Proof: The algorithm is as follows. Initialize $\mathcal{L} = \varnothing$ and $\mathcal{L}' = \{G\}$, and perform the following iterative step. If $\mathcal{L}' = \varnothing$, then stop. Otherwise, remove a graph F from $\mathcal{L}'$. If F does not have a 2-join, then add F to $\mathcal{L}$ and iterate. Otherwise, let $V_1|V_2$ be a 2-join of F. Construct the blocks of the 2-join decomposition of F, say F_1 and F_2. For $i = 1, 2$, if $|V_i| \leq 7$, then check directly whether F_i contains an even hole. If it does, output this result and stop, and otherwise discard F_i. If $|V_i| > 7$, add F_i to $\mathcal{L}'$, and iterate.

By constructing blocks of decomposition we do not create any 4-holes, and by Theorem 6.3.1 we do not create any star cutsets. So by the construction of the algorithm, (1) holds. (2) holds by Theorem 6.3.1.

In [4] and [14] it is shown how with this construction of the algorithm (3) holds.

Finding a 2-join takes time $\mathcal{O}(|V(G)|^7)$ using the crude implementation in [14], and this algorithm is applied at most $\mathcal{O}(|V(G)|)$ times, yielding an overall complexity of $\mathcal{O}(|V(G)|^8)$. □

6.4 Recognition algorithm for even-hole-free graphs

Theorem 6.4.1 *There exists an algorithm with the following specifications:*

Input : *A graph G.*

Output : *EVEN-HOLE-FREE when G is even-hole-free, and NOT EVEN-HOLE-FREE otherwise.*

Running Time : $\mathcal{O}(|V(G)|^{19})$.

Proof: Consider the following algorithm:

Step 1: Test whether G contains a short 4-wheel, a theta, or a 4-hole. If it does, then output NOT EVEN-HOLE-FREE and stop.

Step 2: Apply algorithm from Theorem 6.1.5, and let $\mathcal{L}_1$ be the output family of graphs.

Step 3: Let $\mathscr{L}_2 = \varnothing$. For every graph in $\mathscr{L}_1$, apply the algorithm from Theorem 6.2.4. If the graph is identified as not being even-hole-free, then output the same and stop. Otherwise merge the output family of graphs with $\mathscr{L}_2$.

Step 4: Let $\mathscr{L}_3 = \varnothing$. For every graph in $\mathscr{L}_2$, apply the algorithm from Theorem 6.3.2. If the graph is identified as not being even-hole-free, then output the same and stop. Otherwise merge the output family of graphs with $\mathscr{L}_3$.

Step 5: Check whether every graph in $\mathscr{L}_3$ is an extended clique tree. If some is not then output NOT EVEN-HOLE-FREE. Otherwise, for each graph in $\mathscr{L}_3$ check whether it contains an even hole. If some does, then output NOT EVEN-HOLE-FREE, and otherwise output EVEN-HOLE-FREE.

The correctness of the algorithm follows from Corollary 2.2.3. Testing whether a graph contain a short 4-wheel or a 4-hole can be done by brute force in time $\mathscr{O}(|V(G)|^9)$. Testing whether a graph contains a theta can be done in time $\mathscr{O}(|V(G)|^{11})$ [9]. So Step 1 can be implemented to run in time $\mathscr{O}(|V(G)|^{11})$.

By Theorem 6.1.5, Step 2 can be implemented to run in time $\mathscr{O}(|V(G)|^{10})$ and $|\mathscr{L}_1| = \mathscr{O}(|V(G)|^9)$. By Theorem 6.2.4 and since $|\mathscr{L}_1| = \mathscr{O}(|V(G)|^9)$, Step 3 can be implemented to run in time $\mathscr{O}(|V(G)|^{19})$ and $|\mathscr{L}_2| = \mathscr{O}(|V(G)|^{11})$. By Theorem 6.3.2 and since $|\mathscr{L}_2| = \mathscr{O}(|V(G)|^{11})$ Step 4 can be implemented to run in time $\mathscr{O}(|V(G)|^{19})$ and $|\mathscr{L}_3| = \mathscr{O}(|V(G)|^{12})$.

It is easy to see that in a clique tree there is at most one chordless path between any pair of vertices. So if $G \setminus x$ is a clique tree, then to determine whether G contains an even hole we need only test for every pair of neighbors of x whether the chordless path between them in $G \setminus x$ contains no other neighbor of x and is of even length. Similarly one can test whether an extended clique tree contains an even hole. So, since $|\mathscr{L}_3| = \mathscr{O}(|V(G)|^{12})$, Step 5 can be implemented to run in time $\mathscr{O}(|V(G)|^{17})$. Therefore the overall running time is $\mathscr{O}(|V(G)|^{19})$. $\square$

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