Universal enveloping algebras of semi-direct products of Lie algebras and their quantum analogues

by

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Declaration of Authorship

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Abstract

The thesis consists of two parts. In the first part (Chapters 3 and 4), we study the universal enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$ of the semi-direct product Lie algebra $\mathfrak{sl}_2 \ltimes V_2$ and its subalgebra $U(\mathfrak{b} \ltimes V_2)$. In the second part (Chapters 5 and 6), we introduce and study the quantum analogues of these two algebras, i.e, the smash product algebra $\mathbb{K}_q[X,Y] \rtimes U_q(\mathfrak{sl}_2)$ and its subalgebra $\mathbb{K}_q[X,Y] \rtimes U_q^{\geqslant 0}(\mathfrak{sl}_2)$. The prime, completely prime, primitive and maximal ideals of these algebras are classified, the generators and inclusions of prime ideals are given explicitly. We also give classifications of *all* the simple weight modules over the algebras $U(\mathfrak{sl}_2 \ltimes V_2)$ and $\mathbb{K}_q[X,Y] \rtimes U_q(\mathfrak{sl}_2)$. In Chapter 4, a central extension of the Lie algebra $\mathfrak{sl}_2 \ltimes V_2$ is also studied, which is called in the literature the *Schrödinger algebra*. It is conjectured that there is no simple *singular* Whittaker module for the Schrödinger algebra. We construct a family of such modules. We also proved that the conjecture holds 'generically'.

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Contents

Chapter 1

Introduction

The thesis consists of two parts. In the first part (Chapters 3 and 4), we study the universal enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$ of the semi-direct product Lie algebra $\mathfrak{sl}_2 \ltimes V_2$ and its subalgebra $U(\mathfrak{b} \ltimes V_2)$. In the second part (Chapters 5 and 6), we introduce and study the quantum analogues of these two algebras, i.e, the smash product algebra $\mathbb{K}_q[X, Y] \rtimes U_q(\mathfrak{sl}_2)$ and its subalgebra $\mathbb{K}_q[X, Y] \rtimes U_q^{\geqslant 0}(\mathfrak{sl}_2)$. The prime, completely prime, primitive and maximal ideals of these algebras are classified, the generators and inclusions of prime ideals are given explicitly. We also give classifications of *all* the simple weight modules over the algebras $U(\mathfrak{sl}_2 \ltimes V_2)$ and $\mathbb{K}_q[X, Y] \rtimes U_q(\mathfrak{sl}_2)$.

The *generalized Weyl algebras*, introduced by V. V. Bavula [5], are a powerful tool in study of the above mentioned algebras. Almost all algebras considered in the thesis contain a chain of subalgebras that are generalized Weyl algebras, or their localizations are generalized Weyl algebras. Moreover, the problem of classification of the weight/torsion simple modules can be reduced to a problem of classification of *all* simple modules but over smaller subalgebras that have close connections with generalized Weyl algebras. These facts enable us to give complete classifications of various classes of simple modules.

Recall that a *prime ideal* in a ring R is any ideal P such that $P \neq R$ and whenever I and J are ideals of R with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. The prime spectrum Spec (R) of a ring R is the set of all its prime ideals. The set of (left) *primitive ideals* of an algebra A is the set of annihilators of simple (left) A-modules and is denoted by Prim (A). Every primitive ideal is a prime ideal but the reverse does not hold, in general. For universal enveloping algebras, the set of left and right primitive ideals coincide and every prime ideal is an intersection of primitive ideals. The classification of prime and primitive ideals is a central theme in this thesis. Our approach is based on using localizations and generalized Weyl algebras.

The classification of simple modules for non-abelian Lie algebras is a very difficult (intractable) problem. The same is true for noncommutative algebras of Gelfand-Kirillov dimension $\geqslant 3$. A reasonable approach is to classify certain families of simple modules such as the weight modules, Whittaker modules, etc. Even so, the problem, in general, is still too difficult, one needs to add more finiteness conditions. In this thesis, we give classifications of *all* the simple weight modules over the algebras $U(\mathfrak{sl}_2 \ltimes V_2)$ and $\mathbb{K}_q[X,Y] \rtimes U_q(\mathfrak{sl}_2)$. Both algebras have Gelfand-Kirillov dimension 5. It seems that it is the first instance where such complete classifications of weight modules are given for algebras of Gelfand-Kirillov dimension larger than or equal to 5.

Let us give a description of the main results of the thesis.

1.1 The universal enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$

Let $\mathbb{K}$ is a field of characteristic zero and $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$. Recall that the Lie algebra $\mathfrak{sl}_2 = \mathbb{K}F \oplus \mathbb{K}H \oplus \mathbb{K}E$ is a simple Lie algebra over $\mathbb{K}$ where the Lie bracket is given by the rule: $[H,E] = 2E$, $[H,F] = -2F$ and $[E,F] = H$. Let $V_2 = \mathbb{K}X \oplus \mathbb{K}Y$ be the 2-dimensional simple $\mathfrak{sl}_2$-module with basis X and Y. Let $\mathfrak{a} := \mathfrak{sl}_2 \ltimes V_2$ be the semi-direct product of Lie algebras where V_2 is viewed as an abelian Lie algebra. In more detail, the Lie algebra $\mathfrak{a}$ admits the basis $\{H, E, F, X, Y\}$ and the Lie bracket is defined as follows

$$\begin{array}{lllll} [H,E] = 2E, & [H,F] = -2F, & [E,F] = H, & [E,X] = 0, & [E,Y] = X, \\ [F,X] = Y, & [F,Y] = 0, & [H,X] = X, & [H,Y] = -Y, & [X,Y] = 0. \end{array}$$

Let $A = U(\mathfrak{a})$ be the enveloping algebra of the Lie algebra $\mathfrak{a}$. Briefly,

(i) in Chapter 4, we give a complete classification of *all* simple weight A-modules,

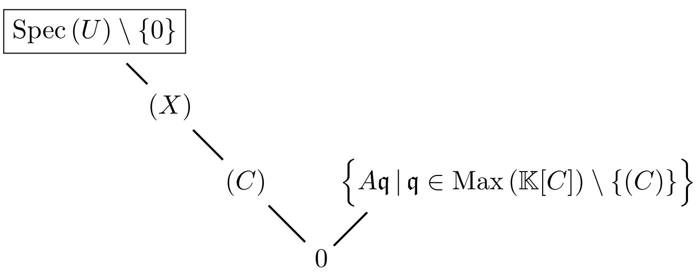
(ii) explicit descriptions of the prime, primitive, completely prime and maximal spectra of A are given,

(iii) explicit generators and defining relations for the centralizer $C_A(H)$ are found and simple $C_A(H)$-modules are classified.

The centre of the algebra A is a polynomial algebra, $Z(A) = \mathbb{K}[C]$ where $C = FX^2 - HXY - EY^2$ (Lemma 4.1). Let us give some more details.

We give an explicit description of the set Spec (A) of prime ideals of the algebra A. The universal enveloping algebra $U := U(\mathfrak{sl}_2)$ is a factor algebra $A/(X)$. Hence, Spec $(U) \subseteq$ Spec (A) is an inclusion of partially ordered sets (with respect to $\subseteq$).

Theorem 1.1. *(Theorem 4.6) The prime spectrum of the algebra A is a disjoint union* $\mathrm{Spec}(A) = \{(X, \mathfrak{p}) \,|\, \mathfrak{p} \in \mathrm{Spec}(U)\} \sqcup \{A\mathfrak{q} \,|\, \mathfrak{q} \in \mathrm{Spec}(\mathbb{K}[C])\}$ *where $U = U(\mathfrak{sl}_2)$. Furthermore, all the inclusions of prime ideals are given in the following diagram (lines represent inclusions of primes).*



The annihilator of a simple module is called a *primitive ideal*. The next theorem is a description of the set Prim(A) of primitive ideals of the algebra A.

Theorem 1.2. *(Theorem 4.8)* $\mathrm{Prim}\,(A) = \big\{(X,\mathfrak{p}) \,|\, \mathfrak{p} \in \mathrm{Spec}\,(U)\setminus\{0\}\big\} \sqcup \big\{A\mathfrak{q} \,|\, \mathfrak{q} \in \mathrm{Spec}\,\mathbb{K}[C]\setminus\{0\}\big\}$.

In the second part of Chapter 4, we give a classification of simple weight A-modules. An A-module M is called a *weight* module if $M = \oplus_{\mu\in\mathbb{K}} M_\mu$ where $M_\mu = \{m \in M \,|\, Hm = \mu m\}$. Let $C_A(H) := \{a \in A \,|\, aH = Ha\}$ be the centralizer of the element H in A. Each nonzero weight component M_μ of M is a $C_A(H)$-module. If, in addition, the weight A-module M is simple then all nonzero weight components M_μ are *simple* $C_A(H)$-modules. So, the problem of classification of *simple weight* A-modules is closely related to the problem of classification of *all simple* $C_A(H)$-modules, which can be seen as the first, the more difficult, of two steps. The second one is about how 'to assemble' some of the simple $C_A(H)$-modules into a simple A-module. The difficulty of the first step stems from the fact that the algebra $C_A(H)$ is of comparable size to the algebra A itself ($\mathrm{GK}\,(C_A(H)) = 4$ and $\mathrm{GK}\,(A) = 5$ where GK stands for the Gelfand-Kirillov dimension) and the defining relations of the algebra $C_A(H)$ are much more complex than the defining relations of the algebra A, as the following theorem shows.

Theorem 1.3. *(Corollary 4.15) Let $t := YX$, $\phi := EY^2$ and $\Theta := FE$. Then the algebra $C_A(H)$ is generated by the elements C, H, t, ϕ and Θ subject to the defining relations (where C and H are central in the algebra $C_A(H)$):*

$$\begin{aligned}
[\phi, t] &= t^2,\\
[\Theta, t] &= 2\phi + (H-2)t + C,\\
[\Theta, \phi] &= 2\Theta t + (-\phi + 2t)H,\\
\Theta t^2 &= (\phi + Ht + C)\phi.
\end{aligned}$$

Furthermore, $Z\big(C_A(H)\big) = \mathbb{K}[C,H]$.

For an algebraically closed field $\mathbb{K}$, the problem of classification of simple $C_A(H)$-modules is equivalent to the same problem but for all the factor algebras $C^{\lambda,\mu} := C_A(H)/(C-\lambda, H-\mu)$ where $\lambda, \mu \in \mathbb{K}$. We assume that the field $\mathbb{K}$ is algebraically closed. There are two distinct cases: $\lambda \neq 0$ and $\lambda = 0$. They require different approaches. The common feature is a discovery of the fact that in order to study simple modules over the algebras $C^{\lambda,\mu}$ we embed them into larger algebras for which classifications of simple modules are known. A surprise is that the sets of simple modules of the algebras $C^{\lambda,\mu}$ and their over-algebras are tightly connected. In the case $\lambda \neq 0$, such an algebra is the first Weyl algebra, but in the second case when $\lambda = 0$, it is a skew polynomial algebra $\mathbb{K}[h][t;\sigma]$ where $\sigma(h) = h-1$. For $\lambda \neq 0$, a classification of simple $C^{\lambda,\mu}$-modules is given in Theorem 4.26. A classification of simple $C^{0,\mu}$-modules is given in Theorem 4.29.

Using the classification of simple $C_A(H)$-modules (Section 4.4), we give a classification of simple weight A-modules in Section 4.5. A typical simple weight A-module depends on an arbitrarily large number of independent parameters. The set of simple A-modules is partitioned into 5 classes each of them is dealt separately with different techniques. This is too technical to explain in the introduction.

In Section 4.6, a central extension of the Lie algebra $\mathfrak{a} = \mathfrak{sl}_2 \ltimes V_2$ is studied, which is called in the literature the *Schrödinger algebra*. The *Schrödinger algebra* $\mathfrak{s}$ is a 6-dimensional Lie algebra that admits a $\mathbb{K}$-basis $\{F, H, E, Y, X, Z\}$ elements of which satisfy the defining relations:

$$\begin{array}{cccc} [H,E]=2E, & [H,F]=-2F, & [E,F]=H, & [H,X]=X, \\ [H,Y]=-Y, & [E,Y]=X, & [E,X]=0, & [F,X]=Y, \\ [F,Y]=0, & [X,Y]=Z, & [Z,\mathfrak{s}]=0. & \end{array}$$

Let $\mathcal{S} := U(\mathfrak{s})$ be the universal enveloping algebra of the Schrödinger algebra $\mathfrak{s}$, then, clearly, $\mathcal{S}/(Z) \simeq A$. The localization $\mathcal{S}_Z$ of the algebra $\mathcal{S}$ at the powers of the central element Z is isomorphic to the tensor product of algebras $\mathbb{K}[Z^{\pm 1}] \otimes U(\mathfrak{sl}_2) \otimes A_1$, see (4.53). The tensor component $U(\mathfrak{sl}_2)$ is called the *hidden* $U(\mathfrak{sl}_2)$. Its explicit canonical generators are described in Lemma 4.37:

$$E' := E - \frac{1}{2}Z^{-1}X^2, \quad F' := F + \frac{1}{2}Z^{-1}Y^2, \quad H' := H + Z^{-1}XY - \frac{1}{2},$$

and $[H', E'] = 2E'$, $[H', F'] = -2F'$ and $[E', F'] = H'$. Using this fact, a short proof has been given of the fact that the centre of the algebra $\mathcal{S}$ is a polynomial algebra in two explicit generators (Proposition 4.39). The fact that the centre $Z(\mathcal{S})$ of $\mathcal{S}$ is a polynomial algebra $\mathbb{K}[Z, \mathsf{c}]$ was proved in [24] by using the Harish-Chandra homomorphism where

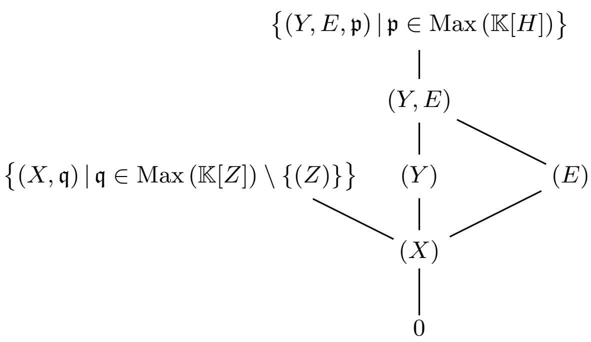
$$\mathsf{c} = Z\Big(4FE + H^2 + H\Big) + 2\Big(EY^2 + HXY - FX^2\Big).$$

In the above paper, it was not clear how this element was found. We clarify the 'origin' of c which is the (classical) Casimir element of the 'hidden' $U(\mathfrak{sl}_2)$ in the decomposition (4.53). It is conjectured that there is no simple *singular* Whittaker module for the algebra $\mathcal{S}$ [44, Conjecture 4.2]. We construct a family of such $\mathcal{S}$-modules (Proposition 4.44). We also prove that the conjecture holds 'generically' (Proposition 4.43). A classification of the simple singular Whittaker $\mathcal{S}$-module is given in [13].

1.2 The spatial ageing algebra $U(\mathfrak{b} \ltimes V_2)$

Let $\mathfrak{b} = \mathbb{K}H \oplus \mathbb{K}E$ be the Borel subalgebra of the Lie algebra $\mathfrak{sl}_2$. Then $\mathfrak{b} \ltimes V_2$ is a *solvable* Lie algebra which is a subalgebra of $\mathfrak{a} = \mathfrak{sl}_2 \ltimes V_2$. Let $\mathcal{A}$ be the universal enveloping algebra $U(\mathfrak{b} \ltimes V_2)$ of the Lie algebra $\mathfrak{b} \ltimes V_2$. It is called the *spatial ageing algebra*. Then $\mathcal{A}$ is a subalgebra of the universal enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$ generated by the elements H, E, X and Y. We study the algebra $\mathcal{A}$ in Chapter 3. The main result is Theorem 3.4 where we give classifications of prime, primitive and maximal ideals of $\mathcal{A}$, the generators and inclusions of prime ideals are given explicitly, we also give an explicit description of prime factor algebras.

Theorem 1.4. *(Theorem 3.4) The prime spectrum* Spec $(\mathcal{A})$ *of the algebra* $\mathcal{A}$ *is given below and all the inclusions of prime ideals are given (lines represent inclusions of primes):*



where $Z := EY^2$.

Generators and defining relations of the centralizers of the elements X, Y and E in the algebra $\mathcal{A}$ are given in Section 3.3. These results are used in classifications of $\mathbb{K}[X]$-, $\mathbb{K}[Y]$- and $\mathbb{K}[E]$-torsion $\mathcal{A}$-modules [16].

1.3 The smash product algebra $\mathbb{K}_q[X, Y] \rtimes U_q(\mathfrak{sl}_2)$

Fix an element $q \in \mathbb{K}^*$ such that q is not a root of unity. Recall that the *quantized enveloping algebra* of $\mathfrak{sl}_2$ is the $\mathbb{K}$-algebra $U_q(\mathfrak{sl}_2)$ with generators E, F, K and K^{-1} subject to the defining relations (see [29]):

$$KK^{-1} = K^{-1}K = 1, \ KEK^{-1} = q^2E, \ KFK^{-1} = q^{-2}F, \ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

There is a Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ defined by

$$\begin{aligned} &\Delta(K) = K \otimes K, && \varepsilon(K) = 1, && S(K) = K^{-1}, \\ &\Delta(E) = E \otimes 1 + K \otimes E, && \varepsilon(E) = 0, && S(E) = -K^{-1}E, \\ &\Delta(F) = F \otimes K^{-1} + 1 \otimes F, && \varepsilon(F) = 0, && S(F) = -FK, \end{aligned}$$

where Δ is the comultiplication on $U_q(\mathfrak{sl}_2)$, ε is the counit and S is the antipode of $U_q(\mathfrak{sl}_2)$.

We can make the *quantum plane* $\mathbb{K}_q[X, Y] := \mathbb{K}\langle X, Y \mid XY = qYX \rangle$ a $U_q(\mathfrak{sl}_2)$-module algebra by defining,

$$\begin{aligned} &K \cdot X = qX, && E \cdot X = 0, && F \cdot X = Y, \\ &K \cdot Y = q^{-1}Y, && E \cdot Y = X, && F \cdot Y = 0. \end{aligned}$$

Then one can form the smash product algebra $A := \mathbb{K}_q[X, Y] \rtimes U_q(\mathfrak{sl}_2)$, which is the main object of study in Chapter 6. As an abstract algebra, the generators and defining relations of the algebra A are given below.

Definition. The algebra A is an algebra generated over $\mathbb{K}$ by the elements E, F, K, K^{-1}, X and Y that satisfy the following defining relations (where K^{-1} is the inverse of K):

$$\begin{aligned}
&KEK^{-1} = q^2E, && KFK^{-1} = q^{-2}F, && [E,F] = \frac{K-K^{-1}}{q-q^{-1}},\\
&EX = qXE, && EY = X + q^{-1}YE, && \\
&FX = YK^{-1} + XF, && FY = YF, && \\
&KXK^{-1} = qX, && KYK^{-1} = q^{-1}Y, && qYX = XY.
\end{aligned}$$

Our aim is to study the prime spectrum of this algebra and to give a classification of simple weight A-modules. The smash product algebra A can be seen as a quantum analogue of the universal enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$ studied in Chapter 4. For example, the prime spectra of these two algebras have similar structure (compare Theorem 6.15 with Theorem 4.6); the representation theory of A has many parallels with that of $U(\mathfrak{sl}_2 \ltimes V_2)$; the centre of A is a polynomial algebra $\mathbb{K}[C]$ where

$$C = (FE - q^2EF)YX + q^2FX^2 - K^{-1}EY^2.$$

The study of quantum algebras usually requires more computations.

Recall that a *quantum Weyl field* over $\mathbb{K}$ is the skew field of fractions of a quantum affine space. We say that a $\mathbb{K}$-algebra A admitting a skew field of fractions $\mathrm{Frac}(A)$ satisfies the *quantum Gelfand-Kirillov conjecture* if $\mathrm{Frac}(A)$ is isomorphic to a quantum Weyl field over a purely transcendental field extension of $\mathbb{K}$; see [19, II.10, p. 230].

Theorem 1.5. *(Theorem 6.9) The quantum Gelfand-Kirillov conjecture holds for the algebra A.*

The next theorem gives generators and defining relations for the centralizer $C_A(K)$ of the element K in the algebra A.

Theorem 1.6. *(Theorem 6.29) Let $\varphi := (q^{-1}-q)YE + X$, $t := YX$, $u := K^{-1}Y\varphi$ and $\Theta := (1-q^2)FE + \frac{q^2(qK+q^{-1}K^{-1})}{1-q^2}$. Then the algebra $C_A(K)$ is generated by the elements $K^{\pm 1}$, C, Θ, t and u subject to the following defining relations:*

$$\begin{aligned}
t\cdot u &= q^2u\cdot t,\\
\Theta\cdot t &= q^2t\cdot\Theta + (q+q^{-1})u + (1-q^2)C,\\
\Theta\cdot u &= q^{-2}u\cdot\Theta - q(1+q^2)t + (1-q^2)K^{-1}C,\\
\Theta\cdot t\cdot u - \frac{1}{q(1-q^2)}u^2 - C\cdot u &= \frac{q^7}{1-q^2}t^2 - q^4K^{-1}C\cdot t,\\
[K^{\pm 1},\cdot] = 0, &\quad \text{and} \quad [C,\cdot] = 0.
\end{aligned}$$

Furthermore, $Z(C_A(K)) = \mathbb{K}[C, K^{\pm 1}]$.

The defining relations of the algebra $\mathscr{C} := C_A(K)$ are complex. From the outset, it is not obvious how to classify simple $\mathscr{C}$-modules. The key idea is based on the observation that this algebra has close connections with generalized Weyl algebras. Let $\mathscr{C}_t$ be the localization of the algebra $\mathscr{C}$ at the powers of the element t. We show that $\mathscr{C}_t$ is a generalized Weyl algebra (Proposition 6.32). For $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, we prove that the factor algebra

$$\mathscr{C}^{\lambda,\mu} := C_A(K)/(C-\lambda, K-\mu)$$

is a simple algebra if and only if $\lambda \neq 0$ (Theorem 6.34). Moreover, for any $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, the localization $\mathscr{C}_t^{\lambda,\mu}$ of the algebra $\mathscr{C}^{\lambda,\mu}$ at the powers of the element t is a central, simple generalized Weyl algebra (Proposition 6.32). Another key observation is that, for any $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, we can embed the algebra $\mathscr{C}^{\lambda,\mu}$ into a generalized Weyl algebra $\mathscr{A}$ (it is a central simple algebra, which plays the role of 'the quantum Weyl algebra'), see Proposition 6.38. Using these facts, a complete classification of simple $C_A(K)$-modules in given in Section 6.6. The problem of classifying simple $\mathscr{C}^{\lambda,\mu}$-modules splits into two distinct cases when $\lambda = 0$ and $\lambda \neq 0$. In the case $\lambda = 0$, we embed the algebra $\mathscr{C}^{0,\mu}$ into a skew polynomial algebra $\mathcal{R} = \mathbb{K}[h^{\pm 1}][t;\sigma]$ where $\sigma(h) = q^2 h$ (it is a subalgebra of the algebra $\mathscr{A}$) for which the classifications of simple modules are known. In the case $\lambda \neq 0$, we use the close relation of the algebra $\mathscr{C}^{\lambda,\mu}$ with its localization $\mathscr{C}_t^{\lambda,\mu}$, and the arguments are more complicated.

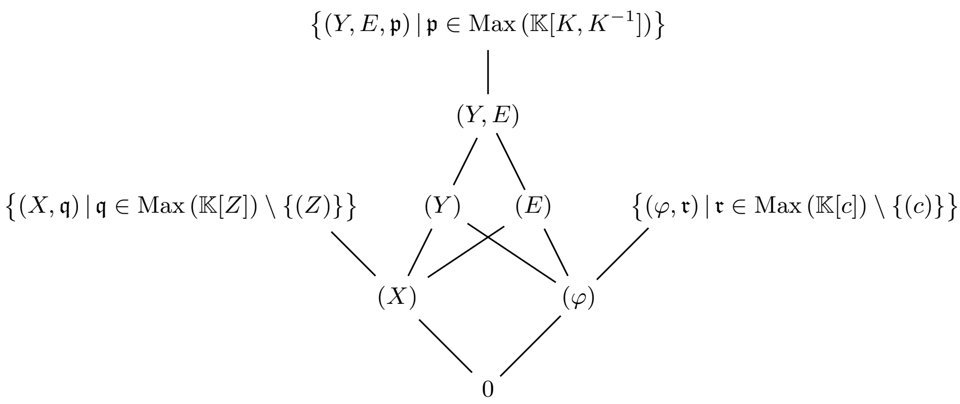
An A-module M is called a *weight module* if $M = \bigoplus_{\mu\in\mathbb{K}^*} M_\mu$ where $M_\mu = \{m \in M \mid Km = \mu m\}$. Using the classification of simple $C_A(K)$-modules (Section 6.6), we give a classification of simple weight A-modules in Section 6.7.

1.4 The quantum spatial ageing algebra

The subalgebra $\mathcal{A}$ of $\mathbb{K}_q[X,Y] \rtimes U_q(\mathfrak{sl}_2)$ generated by the elements E, K, K^{-1}, X and Y is called the *quantum spatial ageing algebra*, which is studied in Chapter 5. For the algebra $\mathcal{A}$,

(i) its prime, completely prime, primitive and maximal spectra are classified,
(ii) the generators of prime ideals and their inclusions are given explicitly,
(iii) generators and defining relations are given for all prime factor algebras $\mathcal{A}$,
(iv) the group of automorphisms of the algebra $\mathcal{A}$ is found (Theorem 5.14). In finding the group of automorphisms of the algebra $\mathcal{A}$, we use an explicit description of prime ideals of the algebra $\mathcal{A}$ and their inclusions.

Theorem 1.7. *(Theorem 5.8) The prime spectrum* Spec $(\mathcal{A})$ *of the algebra* $\mathcal{A}$ *is given below and all the inclusions of prime ideals are given (lines represent inclusions of primes):*



where $Z := \varphi Y K^{-1}$ *and* $c := XYK$.

The maximal and primitive ideals of the algebra $\mathcal{A}$ are classified.

Theorem 1.8. *(Corollary 5.9 and Proposition 5.11)*

1. $\mathrm{Max}\,(\mathcal{A}) = \{(Y,E,\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Max}\,(\mathbb{K}[K,K^{-1}])\} \sqcup \{(X,\mathfrak{q}) \mid \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}\}$ $\sqcup \{(\varphi,\mathfrak{r}) \mid \mathfrak{r} \in \mathrm{Max}\,(\mathbb{K}[c]) \setminus \{(c)\}\}$.
2. $\mathrm{Prim}(\mathcal{A}) = \mathrm{Max}(\mathcal{A}) \sqcup \{(Y),\ (E),\ 0\}$.

The group of automorphisms of $\mathcal{A}$ is determined.

Theorem 1.9. *(Theorem 5.14)* $\mathrm{Aut}_{\mathbb{K}}(\mathcal{A}) = \{\sigma_{\lambda,\mu,\gamma,i} \mid \lambda,\mu,\gamma \in \mathbb{K}^*,\ i \in \mathbb{Z}\} \simeq (\mathbb{K}^*)^3 \rtimes \mathbb{Z}$ *where* $\sigma_{\lambda,\mu,\gamma,i} : X \mapsto \lambda K^i X,\ Y \mapsto \mu K^{-i} Y,\ K \mapsto \gamma K,\ E \mapsto \lambda\mu^{-1} q^{-2i} K^{2i} E$ *(and* $\sigma_{\lambda,\mu,\gamma,i}(\varphi) = \lambda K^i \varphi$*). Furthermore,* $\sigma_{\lambda,\mu,\gamma,i}\sigma_{\lambda',\mu',\gamma',j} = \sigma_{\lambda\lambda'\gamma^j,\mu\mu'\gamma^{-j},\gamma\gamma',i+j}$ *and* $\sigma^{-1}_{\lambda,\mu,\gamma,i} = \sigma_{\lambda^{-1}\gamma^i,\mu^{-1}\gamma^{-i},\gamma^{-1},-i}$.

In general, to find centralizers is a challenging problem, especially to determine their defining relations as algebras. In section 5.4, we describe the centralizers of the elements K, X, φ, Y and E in the algebra $\mathcal{A}$. All the centralizers turned out to be generalized Weyl algebras. These facts are key ones for obtaining classifications of simple $\mathbb{K}[X]$-, $\mathbb{K}[\varphi]$-, $\mathbb{K}[Y]$- and $\mathbb{K}[E]$-torsion modules, see [11, 15] for details.

Chapter 2

Preliminaries

Ore extensions. Let α be an automorphism of a ring R. Recall that an *α-derivation* of R is any additive map $\delta : R \to R$ such that $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$ for all $r, s \in R$. If α is the identity map, then α-derivations are just the ordinary derivations.

Definition. Let R be a ring, α an automorphism of R, and δ an α-derivation of R. We define $S = R[x; \alpha, \delta]$ such that

1. S is a ring, containing R as a subring;
2. x is an element of S;
3. S is a free left R-module with basis $\{1, x, x^2, \dots\}$;
4. $xr = \alpha(r)x + \delta(r)$ for all $r \in R$.

Such a ring S is called an *Ore extension of* R. Any nonzero element $u \in S$ can be uniquely written in the form $u = r_n x^n + r_{n-1}x^{n-1} + \dots + r_1 x + r_0$ where $r_i \in R$ $(i = 0, 1, \dots, n)$. If $r_n \neq 0$ then the integer n is called the *degree* of u and r_n is called the *leading coefficient* of u.

Lemma 2.1. *Let $S = R[x; \alpha, \delta]$ be an Ore extension of R.*

1. *If R is a domain, then S is a domain.*
2. *If R is a prime ring, then S is a prime ring.*

Proof. See, [37, Theorem 1.2.9(i), (iii)]. □

The following theorem is a noncommutative version of the Hilbert Basis Theorem.

Theorem 2.2. *If R is a left (right) Noetherian ring, then so is the Ore extension $S = R[x; \alpha, \delta]$.*

Proof. See [37, Theorem 1.2.9]. □

Filtered and graded rings. A *filtered ring* is a ring R with a family $\{F_n \mid n \in \mathbb{Z}\}$ of additive subgroups of R such that

1. $F_i F_j \subseteq F_{i+j}$ for all i, j;

2. $1 \in F_0$;
3. $F_i \subseteq F_j$ for $i < j$;
4. $\bigcup_{n \in \mathbb{Z}} F_n = R$.

The family $\{F_n\}$ is called a *filtration* of R.

A $\mathbb{Z}$*-graded ring* is a ring R with a family $\{R_n, n \in \mathbb{Z}\}$ of additive subgroups of R such that

1. $R_i R_j \subseteq R_{i+j}$, and
2. $R = \bigoplus_n R_n$ as an abelian group.

The family $\{R_n\}$ is called a *grading* of R. A nonzero element of R_n is said to be *homogeneous* of degree n. For any filtered ring S one can construct a graded ring. We set

$$\mathrm{gr}_n S = F_n/F_{n-1} \quad \text{and} \quad \mathrm{gr}\, S = \bigoplus \mathrm{gr}_n S.$$

To define multiplication in $\mathrm{gr}\, S$ it suffices to consider multiplication of homogeneous elements. If $a \in F_n \setminus F_{n-1}$, then a is said to have *degree* n and $\bar{a} = a + F_{n-1} \in F_n/F_{n-1}$ is the *leading term* of a. Suppose c has degree m then we define

$$\bar{a}\bar{c} = ac + F_{m+n-1} \in \mathrm{gr}_{m+n} S.$$

This well-defined multiplication makes $\mathrm{gr}\, S$ into a ring. It is called the *associated graded ring* of S. In general, the associated graded ring of a filtered ring S has somewhat simpler structure than the ring S. In this case, one would like to transfer information from $\mathrm{gr}\, S$ back to S. Some connection between properties of a filtered ring S and its associated graded ring $\mathrm{gr}\, S$ is given in the following proposition.

Proposition 2.3. *1. If $\mathrm{gr}\, S$ is a domain, then S is a domain.*

2. If $\mathrm{gr}\, S$ is a prime ring, then S is a prime ring.

3. If $\mathrm{gr}\, S$ is right Noetherian, then S is right Noetherian.

Proof. See [37, Proposition 1.6.6] and [37, Theorem 1.6.9]. □

Here we note that the converses are not true, in general.

Prime ideals. A *prime ideal* in a ring R is any ideal P of R such that $P \neq R$ whenever I and J are ideals of R with $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$. The set of prime ideals of R is denoted by $\mathrm{Spec}\,(R)$. A *minimal prime ideal* of a ring R is any prime ideal of R which does not contain any other prime ideals. We have the following equivalent description of prime ideals.

Proposition 2.4. *For an ideal P of a ring R such that $P \neq R$, the following conditions are equivalent:*

1. *P is a prime ideal.*
2. *If $I, J \lhd R$ and $I, J \supsetneqq P$, then $IJ \nsubseteq P$.*
3. *R/P is a prime ring.*
4. *If I and J are right ideals of R such that $IJ \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$.*

5. *If $x, y \in R$ such that $xRy \subseteq P$, then either $x \in P$ or $y \in P$.*

Proof. See [28, Proposition 3.1]. □

Theorem 2.5. *Let R be a right or left Noetherian ring.*

1. *Any ideal in R contains a finite product of prime ideals.*
2. *R has only finitely many minimal prime ideals.*

Proof. See [28, Theorem 3.4] and its proof. □

A *semiprime ideal* in a ring R is any ideal of R which is an intersection of prime ideals.

Theorem 2.6. *An ideal in a ring R is semiprime if and only if whenever $x \in R$ with $xRx \subseteq I$ then $x \in I$.*

Proof. See [28, Theorem 3.7]. □

Let I be a two-sided ideal of a ring R. The ideal I is said to be *completely prime* if the factor ring R/I is a domain. The ideal I is said to be *primitive* if it is the annihilator of a simple left R-module. The set of primitive ideals of R is denoted by $\mathrm{Prim}\,(R)$. The ideal I is said to be a *maximal* ideal if it is maximal in the set of ideals of R distinct from R. We have the following implications (see [28, Proposition 2.15]):

$$I \text{ maximal } \Rightarrow I \text{ primitive } \Rightarrow I \text{ prime } \Rightarrow I \text{ semiprime.}$$

Localization. The technique of localization is a powerful tool in study of algebras. Let X be a multiplicative set in a ring R (i.e., X is multiplicative submonoid of $(R \setminus \{0\}, \cdot)$ and $1 \in X$). Then X is said to satisfy the *right Ore condition* if, for each $x \in X$ and $r \in R$, there exist $y \in X$ and $s \in R$ such that $ry = xs$, that is, $rX \cap xR \neq \emptyset$. X is said to be *right reversible* if

$$xr = 0 \text{ for some } x \in X, r \in R \text{ implies } rx' = 0 \text{ for some } x' \in X.$$

A *right denominator set* is any right reversible right Ore set. In a right Noetherian ring every right Ore set is right reversible; [28, Proposition 10.7].

Definition. Let X be a multiplicative set of a ring R. A *right quotient ring* (or *right Ore localization*) of R with respect to X is a ring Q together with a homomorphism $\phi : R \longrightarrow Q$ such that:

1. $\phi(x)$ is a unit of Q for all $x \in X$,
2. for all $q \in Q$, $q = \phi(r)\phi(x)^{-1}$ for some $r \in R$ and $x \in X$, and
3. $\ker \phi = \{r \in R \,|\, rx = 0 \text{ for some } x \in X\}$.

By abuse of notation, we will write elements of Q in the form rx^{-1} for $r \in R, x \in X$.

Theorem 2.7. *Let X be a multiplicative set in a ring R. Then there exists a right quotient ring of R with respect to X if and only if X is a right denominator set.*

Proof. See [28, Theorem 10.3]. □

Lemma 2.8. *Let X be a multiplicative set in a ring R.*

1. *If there exists a right quotient ring Q of R with respect to X, then it is unique up to isomorphism.*
2. *If R also has a left quotient ring Q' with respect to X then $Q \simeq Q'$.*

Proof. See [28, Corollary 10.5, Proposition 10.6]. □

Because of the uniqueness, we shall denote Q by RX^{-1} or R_X. The next result is useful in handling the passage between a ring R and its localization R_X.

Proposition 2.9. *Let X be a right denominator set in a ring R, and $Q = R_X$. If R is a Noetherian ring then there is a bijection*

$$\begin{aligned} \{P \in \operatorname{Spec}(R) \,|\, P \cap X = \emptyset\} &\longrightarrow \{P' \in \operatorname{Spec}(Q)\}, \\ P &\mapsto PQ, \end{aligned}$$

with the inverse $P' \mapsto P' \cap R$.

Proof. See [37, Proposition 2.1.16.(vii)]. □

Let X be a right Ore set in a ring R and M be a right R-module. The set

$$\operatorname{tor}_X(M) := \{m \in M \,|\, mx = 0 \text{ for some } x \in X\}$$

is a submodule of M [28, Lemma 4.21]. It is called the X-*torsion submodule* of M.

Definition. Let X be a right denominator set in a ring R and M be a right R-module. A *module of fraction* for M with respect to X consists of a right R_X-module N together with a R-module homomorphism $\psi : M \longrightarrow N$ such that:

1. for all $n \in N$, $n = \psi(m)x^{-1}$ for some $m \in M, x \in X$, and
2. $\ker \psi = \operatorname{tor}_X(M)$.

It can be shown that such a module of fraction exits and unique up to isomorphism; [28, Theorem 10.8, Corollary 10.10]. We denote this module by M_X or MX^{-1}.

Proposition 2.10. *Let X be a right denominator set in a ring R and M a right R-module.*

1. *$M \otimes_R R_X \simeq M_X$.*
2. *$\operatorname{tor}_X(M) = \ker(M \longrightarrow M \otimes_R R_X,\ m \mapsto m \otimes 1)$.*
3. *$\operatorname{tor}_X(M) = M \Leftrightarrow M \otimes_R R_X = 0$.*

Proof. See [37, Proposition 2.1.17]. □

Gelfand-Kirillov dimension of algebras and modules. Throughout this thesis $\mathbb{K}$ is a field. Let A be a finitely generated $\mathbb{K}$-algebra and let V be a $\mathbb{K}$-vector subspace of A spanned by $\{a_1, \dots, a_m\}$. If A is generated by $\{a_1, \dots, a_m\}$, or equivalently by the vector space V, then V is called a *generating subspace* of A. For $n \geqslant 1$, we will denote by V^n the subspace of A spanned by all monomials in $a_1, \dots, a_m$ of length n. We also define $V^0 = \mathbb{K}$. Then A has a standard finite dimensional filtration:

$$A = \bigcup_{n=0}^{\infty} V_n, \qquad \text{where } V_n := \mathbb{K} + V + V^2 + \cdots + V^n.$$

Definition. Let A be a finitely generated $\mathbb{K}$-algebra and let V be a generating subspace of A. The *Gelfand-Kirillov dimension*, or GK dimension for short, of A is defined by

$$\mathrm{GK}\,(A) = \overline{\lim_{n\to\infty}} \log_n(\dim V_n).$$

An equivalent definition of Gelfand-Kirillov dimension is

$$\mathrm{GK}\,(A) := \inf\{\gamma \in \mathbb{R} \,|\, \dim V_n \leqslant n^{\gamma},\, n \gg 0\}.$$

Remarks.

1. Let A be a finitely generated $\mathbb{K}$-algebra and suppose that V and W are two generating subspaces of A. Then $\overline{\lim_{n\to\infty}} \log_n(\dim V_n) = \overline{\lim_{n\to\infty}} \log_n(\dim W_n)$. Thus the GK dimension of A does not depend on the choice of generating subspaces.
2. If V contains 1, then $V_n = V^n$.

Let A be a finitely generated $\mathbb{K}$-algebra with a finite dimensional generating subspace V containing 1. If M is a finitely generated left A-module with a finite dimensional vector space F that generates M as an A-module, then

$$M = \bigcup_{n=0}^{\infty} V^n F.$$

The *Gelfand-Kirillov dimension of the module* M is defined by

$$\mathrm{GK}\,(M) := \overline{\lim_{n\to\infty}} \log_n(\dim V^n F).$$

We note that the Gelfand-Kirillov dimension $\mathrm{GK}\,(M)$ does not depend on the choice of the spaces V and F.

Gelfand-Kirillov dimension is a useful and important tool in the study of noncommutative algebras. We recall some basic properties of GK dimension in the following lemma. Further details concerning the GK dimension can be found in [35]. A non-zero-divisor of a ring is called a *regular element*. A derivation δ of a ring R is called a *locally nilpotent derivation* if $R = \bigcup_{n\geqslant 1} \ker(\delta^n)$.

Each element x of a ring R determines a derivation $\delta_x : R \to R,\ r \mapsto [x, r] := xr - rx$ which is called the *inner derivation* associated to x.

Lemma 2.11. *Let x be a regular element of the $\mathbb{K}$-algebra A such that the derivation $\delta_x : a \mapsto ax - xa$ is locally nilpotent. Then the set $X = \{1, x, x^2, \cdots\}$ is an Ore set in A and*

$$\mathrm{GK}\,(X^{-1}A) = \mathrm{GK}\,(AX^{-1}) = \mathrm{GK}\,(A).$$

Proof. See [35, Lemma 4.7]. □

An element r of ring R is called a *left regular element* (resp., a right regular element) if the map $\cdot r : R \to R,\ s \mapsto sr$ (resp., $r\cdot : R \mapsto R,\ s \mapsto rs$) is an injection.

Proposition 2.12. *Let I be an ideal of a $\mathbb{K}$-algebra A, and assume that I contains a right regular element or a left regular element of A. Then*

$$\mathrm{GK}\,(A/I) + 1 \leqslant \mathrm{GK}\,(A).$$

Proof. See [35, Proposition 3.15]. □

Proposition 2.13. *Let A be a right Noetherian $\mathbb{K}$-algebra and suppose that $\mathrm{GK}\,(A) < \infty$. If $P_0 \subset P_1 \subset \cdots \subset P_m$ is a chain of distinct prime ideals of A then*

$$\mathrm{GK}\,(A) \geqslant \mathrm{GK}\,(A/P_0) \geqslant \mathrm{GK}\,(A/P_m) + m.$$

Proof. See [37, Corollary 8.3.6(iv)]. □

Proposition 2.14. *Let A be a $\mathbb{K}$-algebra, and let M be a left A-module.*

1. *If $IM = 0$ for an ideal I of A, then $\mathrm{GK}\,({}_AM) = \mathrm{GK}\,({}_{A/I}M)$.*
2. $\mathrm{GK}\,({}_AM) \leqslant \mathrm{GK}\,(A)$.
3. *If M is finitely generated and $\alpha \in \mathrm{End}_A(M)$ is injective, then*

$$\mathrm{GK}\,(M/\alpha(M)) \leqslant \mathrm{GK}\,(M) - 1.$$

Proof. See [35, Proposition 5.1]. □

A $\mathbb{K}$-algebra A is *almost commutative* if there exists a filtration

$$\mathbb{K} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_i \subseteq \cdots \subseteq \bigcup_{i=0}^{\infty} A_i = A$$

such that (i) A_1 is finite dimensional and $A_i = A_1^i$ for all $i \geqslant 1$; (ii) the associated graded algebra $\mathrm{gr}\,A = \bigoplus_{i=0}^{\infty} A_i/A_{i-1}$ is commutative.

Proposition 2.15. *Let A be an almost commutative algebra, and let M be a finitely generated A-module with* $\mathrm{GK}\,(M) = d$ *and multiplicity $e(M)$. Let*

$$M = M_0 \supset M_1 \supset \cdots \supset M_i \supset M_{i+1} \supset \cdots \supset M_n$$

be a strictly descending chain of submodules with $\mathrm{GK}\,(M_i/M_{i+1}) = d$ *for all* $0 \leqslant i \leqslant n-1$*. Then*

1. $e(M/M_i) = \sum_{j=0}^{i-1} e(M_j/M_{j+1})$.
2. $n \leqslant e(M)$.

Proof. See [35, Corollary 7.8]. □

Generalized Weyl algebras. We only consider generalized Weyl algebra of degree 1.

Definition. Let D be a ring, $\sigma \in \mathrm{Aut}(D)$ and $a \in Z(D)$ where $Z(D)$ is the centre of D. The *generalized Weyl algebra* $A = D(\sigma, a) = D[X, Y; \sigma, a]$ is generated by D and two indeterminates X and Y subject to the defining relations

$$X\alpha = \sigma(\alpha)X \quad \text{and} \quad Y\alpha = \sigma^{-1}(\alpha)Y \quad \text{for all } \alpha \in D,$$
$$YX = a \quad \text{and} \quad XY = \sigma(a).$$

The algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is $\mathbb{Z}$-graded, where $A_n = Dv_n = v_nD$, $v_n = X^n\,(n > 0)$, $v_n = Y^{-n}\,(n < 0)$, $v_0 = 1$. It follows from the defining relations that

$$X^nY^m = \begin{cases} \sigma^n(a)\cdots\sigma^{n-m+1}(a)X^{n-m}, & \text{if } n \geqslant m, \\ \sigma^n(a)\cdots\sigma(a)Y^{m-n}, & \text{if } n \leqslant m. \end{cases}$$
$$Y^nX^m = \begin{cases} \sigma^{-n+1}(a)\cdots\sigma^{-n+m}(a)Y^{n-m}, & \text{if } n \geqslant m, \\ \sigma^{-n+1}(a)\cdots aX^{m-n}, & \text{if } n \leqslant m. \end{cases}$$

The following theorem gives a criterion of simplicity of generalized Weyl algebras of degree 1.

Theorem 2.16. *Let $A = D(\sigma, a)$ be a generalized Weyl algebra of degree 1. Then A is simple if and only if*

1. *D has no proper σ-invariant ideals;*
2. *no power of σ is an inner automorphism of D;*
3. *for each natural integer $n \in \mathbb{N}$, elements a and $\sigma^n(a)$ generate D as a left (or right) D-module;*
4. *a is not a zero divisor in D.*

Proof. See [6, Theorem 4.2]. □

Deleting derivations. The n-th *Weyl algebra* $A_n = A_n(\mathbb{K})$ is an associative algebra which is generated by elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ subject to the defining relations: $[x_i, x_j] = 0$, $[y_i, y_j] = 0$ and $[y_i, x_j] = \delta_{ij}$ where $[a, b] := ab - ba$ and δ_{ij} is the Kronecker delta function. The Weyl algebra A_n is a simple Noetherian domain of Gefand-Kirillov dimension $2n$ with $Z(A_n) = \mathbb{K}$. The next result is very useful.

Lemma 2.17. *[37, Lemma 14.6.5] Let B be a $\mathbb{K}$-algebra, δ be a $\mathbb{K}$-derivation on $S = B \otimes_{\mathbb{K}} A_n(\mathbb{K})$ and $T = S[t; \delta]$. There exists $s \in S$ such that the derivation $\delta' = \delta + \mathrm{ad}_s$ of S satisfies the following conditions,*

1. *$\delta'(B) \subseteq B$,*
2. *$\delta'\big(A_n(\mathbb{K})\big) = 0$, and*
3. *the algebra $T = B[t'; \delta'] \otimes_{\mathbb{K}} A_n(\mathbb{K})$ is a tensor product of algebras where $t' = t + s$.*

Proof. By writing $S = B \otimes A_{n-1}(\mathbb{K}) \otimes A_1(\mathbb{K})$ and using induction on n, it sufficient to prove the result for $n = 1$. Now, $\delta(x) = \sum_{i,j} b_{ij} x^i y^j$ for some elements $b_{ij} \in B$. Let

$$s_1 = \sum_{i,j} \frac{1}{j+1} b_{ij} x^i y^{j+1} \quad \text{and} \quad \delta_1 = \delta + \mathrm{ad}_{s_1}.$$

Then $[x, s_1] = \delta(x)$, and $\delta_1(x) = 0$. Similarly, $[x, y] = 1$ implies $[\delta_1(x), y] + [x, \delta_1(y)] = 0$ and therefore $[x, \delta_1(y)] = 0$. It follows that $\delta_1(y) \in B[x]$, with $\delta_1(y) = \sum b_j x^j$ say. Let

$$s_2 = -\sum \frac{1}{j+1} b_j x^{j+1}, \quad s = s_1 + s_2, \quad \text{and} \quad \delta' = \delta + \mathrm{ad}_s.$$

Then $\delta'(x) = \delta_1(x) = 0$, and $\delta'(y) = 0$. Thus $\delta'(A_1(\mathbb{K})) = 0$. Now let $b \in B$, $a \in A_1$, then $[b, a] = 0$ and so $0 = \delta'([b, a]) = [\delta'(b), a] + [b, \delta'(a)] = [\delta'(b), a]$. Hence $\delta'(b)$ centralizes $A_1(\mathbb{K})$, so $\delta'(b) \in B$. Then statement 3 is clear. □

The Diamond Lemma. For details and applications of the Diamond Lemma, see [19, I.11]. Suppose that A is a $\mathbb{K}$-algebra presented by generators and relations. Then A can be given as

$$A = \mathbb{K}\langle X \rangle / (w_\sigma - f_\sigma \,|\, \sigma \in \Sigma),$$

where $\mathbb{K}\langle X \rangle$ is the free algebra on a set X, $f_\sigma \in \mathbb{K}\langle X \rangle$ and the w_σ are words (products of elements from X). Let W be the free monoid on X, then $w_\sigma \in W$. Since W is a basis for $\mathbb{K}\langle X \rangle$ the cosets $\bar{w}$ for $w \in W$ span A. The set

$$S = \{(w_\sigma, f_\sigma) \,|\, \sigma \in \Sigma\} \subseteq W \times F$$

is called a *reduction system.*

For $\sigma \in \Sigma$ and $a, b \in W$, let $r_{a,\sigma,b} : \mathbb{K}\langle X \rangle \to \mathbb{K}\langle X \rangle$ be the linear map sending $a w_\sigma b \mapsto a f_\sigma b$ and fix all other words. We call $r_{a,\sigma,b}$ an *elementary reduction.* A *reduction* is a finite composition of elementary reductions. An element $f \in \mathbb{K}\langle X \rangle$ is *irreducible* if $r(f) = f$ for all reductions.

A semigroup ordering on W is a partial order $\leqslant$ such that

$$b < b' \Rightarrow abc < ab'c$$

for all $a, b, b', c \in W$. We say that the semigroup ordering $\leqslant$ is *compatible with the reduction system* S if for each $\sigma \in \Sigma$, the element f_σ is a linear combination of words $w < w_\sigma$.

We will require a semigroup ordering which satisfies the descending chain condition (the DCC). The typical example is the length-lexicographic ordering. The *length-lexicographic ordering* $\leqslant_{lex}$ on W is defined by

$$a \leqslant_{lex} b \Leftrightarrow a = b \text{ or } a <_{lex} b,$$

where $a <_{lex} b$ is given as

$$x_{i(1)}x_{i(2)} \cdots x_{i(s)} <_{lex} x_{j(1)}x_{j(2)} \cdots x_{j(t)}$$

if and only if either $s < t$, or $s = t$ and there is some $u \leqslant s$ such that $i(l) = j(l)$ for all $l < u$ and $i(u) < j(u)$. Therefore, to compare two different words, we first compare their lengths, if they have the same length, then we look at the leftmost place where they differ.

There are two kinds of ambiguities can arise in the reduction process. An *overlap ambiguity* is a 5-tuple $(a, b, c, \sigma, \tau) \in W^3 \times \Sigma^2$ such that $ab = w_\sigma$ and $bc = w_\tau$. The ambiguity lies in the fact that abc can be reduced in two ways:

$$r_{1,\sigma,c}(abc) = f_\sigma c \quad \text{and} \quad r_{a,\tau,1}(abc) = af_\tau.$$

An *inclusion ambiguity* is a 5-tuple $(a, b, c, \sigma, \tau) \in W^3 \times \Sigma^2$ such that $abc = w_\sigma$ and $b = w_\tau$. Again abc has two reductions:

$$r(abc) = f_\sigma \quad \text{and} \quad r_{a,\tau,c}(abc) = af_\tau c.$$

We say that the overlap (resp. inclusion) ambiguity (a, b, c, σ, τ) is *resolvable* if and only if there are reductions r, r' such that $r(f_\sigma c) = r'(af_\tau)$ (resp. $r(f_\tau) = r'(af_\tau c)$).

Theorem 2.18. (**Diamond Lemma**). Let $F = \mathbb{K}\langle X\rangle$ be a free algebra on a set X and W be the free monoid on X. Let $S = \{(w_\sigma, f_\sigma) \mid \sigma \in \Sigma\}$ be a reduction system and $\leqslant$ be a semigroup ordering on W which is compatible with S and satisfies the DCC. Assume that all overlap and inclusion ambiguities are resolvable. Then the cosets $\bar{w}$, for irreducible words $w \in W$, form a basis for the factor algebra $F/(w_\sigma - f_\sigma \mid \sigma \in \Sigma)$.

Smash product. Now, we recall the definition of smash product algebra, for details and examples see [38, 4.1].

Definition. If H is a Hopf algebra with comultiplication Δ, and A is an algebra which is an H-module such that

1. $h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b)$ for all $h \in H, a, b \in A$, where $\Delta(h) = h_1 \otimes h_2$ (Sweedler's notation), and
2. $h \cdot 1 = \varepsilon(h)1$ for all $h \in H$, where $1 \in A$ is the identity,

then A is called a *(left) H-module algebra*. Note that H is naturally a left H-module via the left multiplication. This yields a *left H-adjoint action on H* given by the rule

$$h \cdot l = h_1 l S(h_2) \quad \text{for } h, l \in H.$$

Definition. Let A be a left H-module algebra. The *smash product algebra* $A \rtimes H$ is defined as follows, for all a, $b \in A$ and h, $k \in H$:

1. as vector spaces, $A \rtimes H = A \otimes H$. To avoid confusion we write $a\#h$ for the element $a \otimes h$.
2. Multiplication is given by the rule

$$(a\#h)(b\#k) = a(h_1 \cdot b)\#h_2 k. \tag{2.1}$$

It is easy to see that $A \simeq A \otimes 1$ and $H \simeq 1 \otimes H$, so A and H can be naturally seen as subalgebras of $A \rtimes H$. For this reason we abbreviate the element $a\#h$ by ah. In this notation, we write $ha = (h_1 \cdot a)h_2$ using (2.1).

In this thesis, a $\mathbb{K}$-algebra A is called a *central simple algebra* if A is a simple algebra and $Z(A) = \mathbb{K}$.

Lemma 2.19. *Let A be a central simple algebra with unity, B an algebra with unity, $\mathscr{I}$ the set of two-sided ideals of B, and $\mathscr{J}$ the set of two-sided ideals of $A \otimes B$.*

1. *The map $\mathscr{I} \to \mathscr{J}$, $I \mapsto A \otimes I$, is a bijection.*
2. *Let $I \in \mathscr{I}$. Then I is a maximal (or prime) ideal of B if and only if $A \otimes I$ is a maximal (or prime) ideal of $A \otimes B$.*

Proof. See [21, Lemma 4.5.1]. □

Lemma 2.20. *Let A and B be $\mathbb{K}$-algebras. Then $Z(A \otimes_{\mathbb{K}} B) = Z(A) \otimes_{\mathbb{K}} Z(B)$.*

Proof. See [39, Corollary 1.7.24]. □

Chapter 3

The spatial ageing algebra $U(\mathfrak{b} \ltimes V_2)$

3.1 Introduction

In this thesis, module means a left module, $\mathbb{K}$ is a field of characteristic zero and $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$.

Recall that the Lie algebra $\mathfrak{sl}_2 = \mathbb{K}F \oplus \mathbb{K}H \oplus \mathbb{K}E$ is a simple Lie algebra over $\mathbb{K}$ where the Lie bracket is given by the rule: $[H, E] = 2E$, $[H, F] = -2F$ and $[E, F] = H$. Let $V_2 = \mathbb{K}X \oplus \mathbb{K}Y$ be the 2-dimensional simple $\mathfrak{sl}_2$-module with basis X and Y: $H \cdot X = X$, $H \cdot Y = -Y$, $E \cdot X = 0$, $E \cdot Y = X$, $F \cdot X = Y$ and $F \cdot Y = 0$. Let $\mathfrak{a} := \mathfrak{sl}_2 \ltimes V_2$ be the semi-direct product of Lie algebras, where V_2 is viewed as an abelian Lie algebra. In more detail, the Lie algebra $\mathfrak{a}$ admits the basis $\{H, E, F, X, Y\}$ and the Lie bracket is as follows

$$\begin{array}{lllll} [H,E]=2E, & [H,F]=-2F, & [E,F]=H, & [E,X]=0, & [E,Y]=X, \\ [F,X]=Y, & [F,Y]=0, & [H,X]=X, & [H,Y]=-Y, & [X,Y]=0. \end{array}$$

Let $\mathfrak{b} = \mathbb{K}H \oplus \mathbb{K}E$ be the Borel subalgebra of the Lie algebra $\mathfrak{sl}_2$. Then $\mathfrak{b} \ltimes V_2$ is a *solvable* Lie subalgebra of $\mathfrak{a}$. It admits a basis $\{H, E, X, Y\}$ and the Lie bracket on $\mathfrak{b} \ltimes V_2$ is given as follows

$$\begin{array}{lll} [H,E]=2E, & [H,X]=X, & [H,Y]=-Y, \\ [E,X]=0, & [E,Y]=X, & [X,Y]=0. \end{array}$$

The universal enveloping algebra $\mathcal{A} := U(\mathfrak{b} \ltimes V_2)$ of the Lie algebra $\mathfrak{b} \ltimes V_2$ is called the *spatial ageing algebra*. The algebra $\mathcal{A}$ is a subalgebra of the universal enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$. In this chapter, we study the prime spectrum and centralizers of some elements of the algebra $\mathcal{A}$, the algebra $U(\mathfrak{sl}_2 \ltimes V_2)$ will be studied in the next chapter. Let us describe the content of this chapter. In Section 3.2, an explicit description of the prime spectrum of the algebra $\mathcal{A}$ is given (Theorem 3.4). An explicit description of all the prime factor algebras of $\mathcal{A}$ is given in Theorem 3.4. All the possible inclusions of primes are given in (3.9). The sets of maximal, completely prime and primitive ideals of $\mathcal{A}$ are described (Corollary 3.5, Corollary 3.6 and Proposition 3.7, respectively). The centralizers of the elements X, Y and E are described in Section 3.3.

Much of this chapter is extracted from the joint paper with V. Bavula [16].

3.2 The prime ideals of $\mathcal{A}$

The aim of this section is to describe the prime ideals of the enveloping algebra $\mathcal{A}$ (Theorem 3.4). As a result, the sets of maximal, completely prime and primitive ideals are described (Corollary 3.5, Corollary 3.6 and Proposition 3.7). Theorem 3.4 also gives an explicit description of all prime factor algebras of $\mathcal{A}$.

For an algebra R, we denote by $Z(R)$ its centre. An element r of a ring R is called a *normal element* if $Rr = rR$.

The subalgebra $\mathbb{E}$ of $\mathcal{A}$. Let $\mathbb{E}$ be the subalgebra of $\mathcal{A}$ generated by the elements E, X and Y. The generators of the algebra $\mathbb{E}$ satisfy the defining relations

$$EY - YE = X, \quad EX = XE \quad \text{and} \quad YX = XY.$$

Clearly, X is a central element of the algebra $\mathbb{E}$. The algebra $\mathbb{E}$ is isomorphic to the universal enveloping algebra of the 3-dimensional Heisenberg Lie algebra. In particular, the algebra $\mathbb{E}$ is a Noetherian domain of Gelfand-Kirillov dimension 3. Let $\mathbb{E}_X$ be the localization of the algebra $\mathbb{E}$ at the powers of the element X. Then the algebra $\mathbb{E}_X$ is the tensor product of two algebras

$$\mathbb{E}_X = \mathbb{K}[X^{\pm 1}] \otimes A_1^+$$

where the algebra $A_1^+ := \mathbb{K}\langle EX^{-1}, Y\rangle$ is the (first) Weyl algebra since $[EX^{-1}, Y] = 1$. Since the algebra A_1^+ is a central algebra, i.e., $Z(A_1^+) = \mathbb{K}$, we have $Z(\mathbb{E}_X) = \mathbb{K}[X^{\pm 1}]$. Then $Z(\mathbb{E}) = Z(\mathbb{E}_X) \cap \mathbb{E} = \mathbb{K}[X]$.

The algebra $\mathcal{A}$. By the defining relations of the algebra $\mathcal{A}$,

$$\mathcal{A} = \mathbb{E}[H; \delta] \tag{3.1}$$

is an Ore extension where the $\mathbb{K}$-derivation δ of the algebra $\mathbb{E}$ is given by the rule: $\delta(E) = 2E$, $\delta(X) = X$ and $\delta(Y) = -Y$. Notice that X is a normal element of the algebra $\mathcal{A}$ since X is central in $\mathbb{E}$ and $XH = (H-1)X$. The localization $\mathcal{A}_X$ of the algebra $\mathcal{A}$ at the powers of the element X is an Ore extension

$$\mathcal{A}_X = \mathbb{E}_X[H; \delta] = (\mathbb{K}[X^{\pm 1}] \otimes A_1^+)[H; \delta] \tag{3.2}$$

where $\delta(E) = 2E$, $\delta(X) = X$ and $\delta(Y) = -Y$. The element $s = EX^{-1}Y \in \mathbb{E}_X$ satisfies the conditions of Lemma 2.17. In more detail, the element $H^+ := H + s = H + EX^{-1}Y$ commutes with the elements of A_1^+ and

$$\mathcal{A}_X = \mathbb{K}[X^{\pm 1}][H^+; \delta'] \otimes A_1^+ \quad \text{where } \delta'(X) = X. \tag{3.3}$$

Notice that the algebra $\mathbb{K}[X^{\pm 1}][H^+; \delta']$ can be presented as a skew Laurent polynomial algebra $\mathbb{K}[H^+][X^{\pm 1}; \sigma]$ where $\sigma(H^+) = H^+ - 1$. This is a central simple algebra of Gelfand-Kirillov dimension 2. Let $\partial := H^+X^{-1}$. Then $[\partial, X] = 1$ and so the subalgebra $A_1 = \mathbb{K}\langle \partial, X\rangle$ of $\mathcal{A}_X$ is the (first) Weyl algebra. Moreover, the algebra A_1 is a subalgebra of $\mathbb{K}[X^{\pm 1}][H^+; \delta']$ and the algebra $\mathbb{K}[X^{\pm 1}][H^+; \delta'] = A_{1,X}$ is the localization of the Weyl algebra A_1 at the powers of the element X. Now,

$$\mathcal{A}_X = A_{1,X} \otimes A_1^+. \tag{3.4}$$

So $\mathcal{A}_X$ is a localization of the second Weyl algebra.

Lemma 3.1. *1. The algebra $\mathcal{A}_X$ is a central simple algebra of Gelfand-Kirillov dimension 4.*
2. $Z(\mathcal{A}) = \mathbb{K}$.

Proof. 1. Since both the algebras $\mathbb{K}[X^{\pm 1}][H^+; \delta']$ and A_1^+ are central simple algebras of Gelfand-Kirillov dimension 2, statement 1 then follows from (3.3).

2. Since $\mathbb{K} \subseteq Z(\mathcal{A}) \subseteq Z(\mathcal{A}_X) = \mathbb{K}$, we have $Z(\mathcal{A}) = \mathbb{K}$. □

The factor algebra $\mathcal{B} := \mathcal{A}/(X)$. We still denote by H, E and Y the images of these elements in the factor algebra $\mathcal{B} := \mathcal{A}/(X)$. Then the algebra $\mathcal{B}$ is generated by the elements H, E and Y that satisfy the defining relations

$$[H, E] = 2E, \quad [H, Y] = -Y, \quad [E, Y] = 0.$$

Hence, the algebra $\mathcal{B}$ is an Ore extension,

$$\mathcal{B} = \mathbb{K}[E, Y][H; \delta] \quad \text{where } \delta(E) = 2E \text{ and } \delta(Y) = -Y. \tag{3.5}$$

It is clear that the element $Z := EY^2$ belongs to the centre of the algebra $\mathcal{B}$. The elements Y and E are normal elements in $\mathcal{B}$. Let $\mathcal{B}_Y$ be the localization of the algebra $\mathcal{B}$ at the powers of element Y. Then

$$\mathcal{B}_Y = \mathbb{K}[Z] \otimes \mathbb{K}[H][Y^{\pm 1}; \sigma] := \mathbb{K}[Z] \otimes \mathbb{Y} \tag{3.6}$$

where the skew polynomial algebra $\mathbb{Y} = \mathbb{K}[H][Y^{\pm 1}; \sigma]$ is a central simple algebra where the $\mathbb{K}$-automorphism σ of $\mathbb{K}[H]$ is defined as follows: $\sigma(H) = H + 1$. Hence, the centre of the algebra $\mathcal{B}_Y$ is $\mathbb{K}[Z]$. The algebras $\mathcal{B}$ and $\mathcal{B}_Y$ are Noetherian domains of Gelfand-Kirillov dimension 3.

Lemma 3.2. $Z(\mathcal{B}) = Z(\mathcal{B}_Y) = \mathbb{K}[Z]$ *where* $Z = EY^2$.

Proof. Since $\mathbb{K}[Z] \subseteq Z(\mathcal{B}) \subseteq Z(\mathcal{B}_Y) = \mathbb{K}[Z]$, we have $Z(\mathcal{B}) = \mathbb{K}[Z]$. □

The prime spectrum of the algebra $\mathcal{A}$. Recall that for an algebra R, we denote by $\mathrm{Spec}\,(R)$ the set of its prime ideals. The set $(\mathrm{Spec}\,(R), \subseteq)$ is a partially ordered set (poset) with respect to inclusion of prime ideals. Each element $r \in R$ determines two maps from R to R, $r\cdot : x \mapsto rx$ and $\cdot r : x \mapsto xr$ where $x \in R$. For an element $r \in R$, we denote by (r) the ideal of R generated by the element r.

Proposition 3.3. *Let R be a Noetherian ring and s be an element of R such that $\mathcal{S}_s := \{s^i \,|\, i \in \mathbb{N}\}$ is a left denominator set of the ring R and $(s^i) = (s)^i$ for all $i \geqslant 1$ (e.g., s is a normal element such that $\ker(\cdot s_R) \subseteq \ker(s_R \cdot)$). Then $\mathrm{Spec}\,(R) = \mathrm{Spec}(R, s) \sqcup \mathrm{Spec}_s(R)$ where $\mathrm{Spec}\,(R, s) := \{\mathfrak{p} \in \mathrm{Spec}\,(R) \,|\, s \in \mathfrak{p}\}$, $\mathrm{Spec}_s(R) := \{\mathfrak{q} \in \mathrm{Spec}\,(R) \,|\, s \notin \mathfrak{q}\}$ and*

- *(a) the map $\mathrm{Spec}\,(R, s) \to \mathrm{Spec}\,(R/(s))$, $\mathfrak{p} \mapsto \mathfrak{p}/(s)$, is a bijection with the inverse $\mathfrak{q} \mapsto \pi^{-1}(\mathfrak{q})$ where $\pi : R \to R/(s), r \mapsto r + (s)$,*
- *(b) the map $\mathrm{Spec}_s(R) \to \mathrm{Spec}\,(R_s)$, $\mathfrak{p} \mapsto \mathcal{S}_s^{-1}\mathfrak{p}$, is a bijection with the inverse $\mathfrak{q} \mapsto \sigma^{-1}(\mathfrak{q})$ where $\sigma : R \to R_s := \mathcal{S}_s^{-1}R$, $r \mapsto \frac{r}{1}$.*
- *(c) For all $\mathfrak{p} \in \mathrm{Spec}\,(R, s)$ and $\mathfrak{q} \in \mathrm{Spec}_s(R), \mathfrak{p} \nsubseteq \mathfrak{q}$.*

Proof. Clearly, $\mathrm{Spec}\,(R) = S_1 \sqcup S_0$ is a disjoint union where S_1 and S_0 are the subsets of $\mathrm{Spec}\,(R)$ that consist of prime ideals $\mathfrak{p}$ of R such that $\mathfrak{p} \cap \mathcal{S}_s \neq \emptyset$ and $\mathfrak{p} \cap \mathcal{S}_s = \emptyset$, respectively. If $\mathfrak{p} \in S_1$ then $s^i \in \mathfrak{p}$ for some $i \geqslant 1$, and so $\mathfrak{p} \supseteq (s^i) = (s)^i$, by the assumption. Therefore, $\mathfrak{p} \supseteq (s)$ (since $\mathfrak{p}$ is a prime ideal), i.e., $\mathfrak{p} \ni s$. This means that $S_1 = \mathrm{Spec}\,(R, s)$. We have shown that $s \in \mathfrak{p}$ iff $s^i \in \mathfrak{p}$ for some $i \geqslant 1$. By the very definition, $S_0 = \mathrm{Spec}\,(R) \setminus S_1 = \mathrm{Spec}\,(R) \setminus \{\mathfrak{p} \in \mathrm{Spec}\,(R) \,|\, s \in \mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Spec}\,(A) \,|\, s \notin \mathfrak{p}\} = \mathrm{Spec}_s(R)$.

The statement (a) is obvious since $s \in \mathfrak{p}$ iff $(s) \subseteq \mathfrak{p}$. The ring R is Noetherian, by [37, Proposition 2.1.16.(vii)], the map $\mathrm{Spec}_s(R) = \{\mathfrak{p} \in \mathrm{Spec}\,(R) \,|\, \mathfrak{p} \cap \mathcal{S}_s = \emptyset\} \to \mathrm{Spec}\,(R_s), \mathfrak{p} \mapsto \mathcal{S}_s^{-1}\mathfrak{p}$ is a bijection with the inverse $\mathfrak{q} \mapsto \sigma^{-1}(\mathfrak{q})$ and the statement (b) follows. The statement (c) is obvious. □

Remark. In the statements (a) and (b) of Proposition 3.3, we identify the sets via the bijections.

Let $U := U(\mathfrak{sl}_2)$ and U^+ be the 'positive part' of U, i.e., U^+ is the subalgebra of U generated by the elements H and E. Then $U^+ = \mathbb{K}[H][E; \sigma]$ is a skew polynomial algebra where $\sigma(H) = H - 2$. The localized algebra $U_E^+ = \mathbb{K}[H][E^{\pm 1}; \sigma]$ is a central simple domain. The following diagram explains the idea of finding the prime spectrum of the algebra $\mathcal{A}$ by repeated application of Proposition 3.3,

$$\begin{array}{ccc}
\mathcal{A} & \longrightarrow & \mathcal{A}_X \\
\downarrow & & \\
\mathcal{B} = \mathcal{A}/(X) & \longrightarrow & (\mathcal{A}/(X))_Y = \mathcal{B}_Y \\
\downarrow & & \\
U^+ = \mathcal{A}/(X, Y) & \longrightarrow & U_E^+ \\
\downarrow & & \\
\mathbb{K}[H] = U^+/(E). & &
\end{array} \tag{3.7}$$

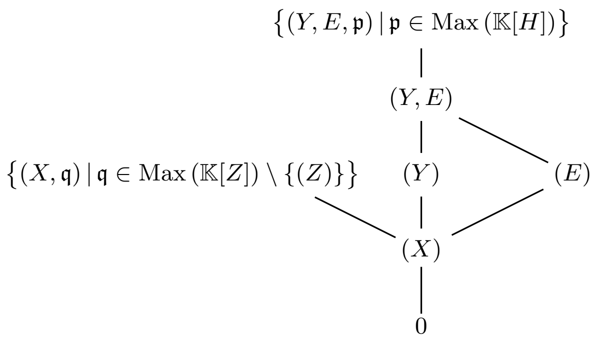
Using (3.7) and Proposition 3.3, we can represent the prime spectrum $\mathrm{Spec}\,(\mathcal{A})$ of the algebra $\mathcal{A}$ as the disjoint union of its subsets

$$\mathrm{Spec}\,(\mathcal{A}) = \mathrm{Spec}\,(\mathbb{K}[H]) \ \sqcup \ \mathrm{Spec}\,(U_E^+) \ \sqcup \ \mathrm{Spec}\,(\mathcal{B}_Y) \ \sqcup \ \mathrm{Spec}\,(\mathcal{A}_X) \tag{3.8}$$

where we identify the sets of prime ideals in (3.8) via the bijections given in the statements (a) and (b) of Proposition 3.3.

The next theorem gives an explicit description of the poset $(\mathrm{Spec}\,(\mathcal{A}), \subseteq)$ and of all the prime factor algebras of $\mathcal{A}$. It also shows that every prime ideal is a completely prime ideal.

Theorem 3.4. *The prime spectrum* $\mathrm{Spec}\,(\mathcal{A})$ *of the algebra* $\mathcal{A}$ *is the disjoint union of the sets in (3.8). More precisely,*



(3.9)

where

1. $\mathrm{Spec}\,(\mathbb{K}[H]) = \{(Y, E, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\,(\mathbb{K}[H])\} = \{(Y, E)\} \sqcup \{(Y, E, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Max}\,(\mathbb{K}[H])\}$ *and* $\mathcal{A}/(Y, E, \mathfrak{p}) \simeq \mathbb{K}[H]/\mathfrak{p}$.
2. $\mathrm{Spec}\,(U_E^+) = \{(Y)\}$, $(Y) = (X, Y)$ *and* $\mathcal{A}/(Y) \simeq U^+ = \mathbb{K}[H][E; \sigma]$ *is a skew polynomial algebra which is a domain where* $\sigma(H) = H - 2$.
3. $\mathrm{Spec}\,(\mathcal{B}_Y) = \{(X),\ (E),\ (X, \mathfrak{q}) \mid \mathfrak{q} \in \mathrm{Max}(\mathbb{K}[Z]) \setminus \{(Z)\}\}$ *and*
   (a) $\mathcal{A}/(X) = \mathcal{B} = \mathbb{K}[E, Y][H; \delta]$ *is an Ore domain (see 3.5) where* $\delta(E) = 2E$ *and* $\delta(Y) = -Y$,
   (b) $\mathcal{A}/(E) \simeq \mathbb{K}[H][Y; \sigma]$ *is a skew polynomial algebra which is a domain where* $\sigma(H) = H + 1$, *and*
   (c) $\mathcal{A}/(X, \mathfrak{q}) \simeq \mathcal{B}/(\mathfrak{q}) \simeq \mathcal{B}_Y/(\mathfrak{q})_Y \simeq L_{\mathfrak{q}} \otimes \mathbb{Y}$ *is a simple domain which is a tensor product of algebras where* $L_{\mathfrak{q}} := \mathbb{K}[Z]/\mathfrak{q}$ *is a finite field extension of* $\mathbb{K}$.
4. $\mathrm{Spec}\,(\mathcal{A}_X) = \{0\}$.

Proof. Recall that X is a normal element in the algebra $\mathcal{A}$. By Proposition 3.3,

$$\mathrm{Spec}\,(\mathcal{A}) = \mathrm{Spec}\,\big(\mathcal{A}/(X)\big) \ \sqcup \ \mathrm{Spec}\,(\mathcal{A}_X). \tag{3.10}$$

(i) *Statement 4 holds*: By Lemma 3.1.(1), the algebra $\mathcal{A}_X$ is a simple algebra. Hence, $\mathrm{Spec}\,(\mathcal{A}_X) = \{0\}$, as required.

Recall that Y is a normal element of the algebra $\mathcal{B} = \mathcal{A}/(X)$. By Proposition 3.3,

$$\mathrm{Spec}\,(\mathcal{A}/(X)) = \mathrm{Spec}\,(\mathcal{A}/(X,Y)) \,\sqcup\, \mathrm{Spec}\left((\mathcal{A}/(X))_Y\right) = \mathrm{Spec}\,(U^+) \,\sqcup\, \mathrm{Spec}\,(\mathcal{B}_Y). \tag{3.11}$$

(ii) $(Y) = (X,Y)$ *and* $(E) = (X,E)$: Both equalities follow from the relation $X = [E,Y]$.

(iii) *Statements 1 and 2 hold*: The element E is a normal element of the algebra U^+. By Proposition 3.3,

$$\mathrm{Spec}\,(U^+) = \mathrm{Spec}\,(U^+/(E)) \,\sqcup\, \mathrm{Spec}\,(U_E^+). \tag{3.12}$$

Since $\mathbb{K}[H] \simeq U^+/(E)$, statement 1 follows. Now, (3.8) holds by (3.10), (3.11) and (3.12). The algebra $U_E^+ \simeq \mathbb{K}[H][E^{\pm 1}; \sigma]$ is a central simple domain where $\sigma(H) = H - 2$. By the statement (ii), $\mathcal{A}/(Y) = \mathcal{A}/(X,Y) = U^+$ is a domain. The set $\mathrm{Spec}\,(U_E^+)$, as a subset of $\mathrm{Spec}\,(\mathcal{A})$, consists of the single ideal (Y), and statement 2 follows.

(iv) *Statement 3 holds*: By (3.6), $\mathcal{B}_Y = \mathbb{K}[Z] \otimes \mathbb{Y}$ where $\mathbb{Y}$ is a central simple algebra. Then, by Lemma 2.19, $\mathrm{Spec}\,(\mathcal{B}_Y) = \mathrm{Spec}\,(\mathbb{K}[Z])$. The set $\mathrm{Spec}\,(\mathcal{B}_Y)$, as a subset of $\mathrm{Spec}\,(\mathcal{A})$, is equal to $\{\mathcal{A} \cap (X)_Y, \mathcal{A} \cap (X,Z)_Y, \mathcal{A} \cap (X,\mathfrak{q})_Y \,|\, \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}\}$. We have to show that $\mathcal{A} \cap (X)_Y = (X)$, $\mathcal{A} \cap (X,Z)_Y = (E)$ and $\mathcal{A} \cap (X,\mathfrak{q})_Y = (X,\mathfrak{q})$.

$\mathcal{A} \cap (X)_Y = (X)$: Let $u \in \mathcal{A} \cap (X)_Y$, then $Y^i u \in (X)$ for some $i \in \mathbb{N}$. Since $\mathcal{A}/(X) = \mathcal{B}$ is domain and $Y \notin (X)$, we must have $u \in (X)$. Hence, $\mathcal{A} \cap (X)_Y = (X)$.

$\mathcal{A} \cap (X,Z)_Y = (E)$: By the statement (ii), $(E) = (X,E)$. So, $(E)_Y = (X,E)_Y = (X,Z)_Y$. Let $u \in \mathcal{A} \cap (X,Z)_Y = \mathcal{A} \cap (E)_Y$, then $Y^i u \in (E)$ for some $i \in \mathbb{N}$. Since $\mathcal{A}/(E) = \mathcal{A}/(X,E) \simeq \mathbb{K}[H][Y;\sigma]$ is a domain where $\sigma(H) = H + 1$ and $Y \notin (E)$, we have $u \in (E)$. Therefore, $\mathcal{A} \cap (X,Z)_Y = (E)$. So, statement (b) holds and (E) is a completely prime ideal of the algebra $\mathcal{A}$.

$\mathcal{A} \cap (X,\mathfrak{q})_Y = (X,\mathfrak{q})$ *for* $\mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}$: Let us first show that the statement (c) holds. It is clear that $\mathcal{A}/(X,\mathfrak{q}) \simeq \mathcal{B}/(\mathfrak{q})$. Since $\mathfrak{q} \neq (Z)$, the nonzero element $Z = EY^2$ of $L_\mathfrak{q}$ is invertible in the field $L_\mathfrak{q}$. Hence, the element Y is invertible in the algebra $\mathcal{B}/(\mathfrak{q})$. Now, $\mathcal{B}/(\mathfrak{q}) \simeq \mathcal{B}_Y/(\mathfrak{q})_Y \simeq L_\mathfrak{q} \otimes \mathbb{Y}$, see (3.6). This proves the statement (c). Since $\mathcal{A}/(X,\mathfrak{q})$ is a simple algebra (by the statement (c)), the ideal $(X,\mathfrak{q})$ of $\mathcal{A}$ is a maximal ideal and $(X,\mathfrak{q}) \subseteq \mathcal{A} \cap (X,\mathfrak{q})_Y \subsetneqq \mathcal{A}$, we must have $\mathcal{A} \cap (X,\mathfrak{q})_Y = (X,\mathfrak{q})$.

(v) Clearly, we have the inclusions as in the diagram (3.9) (see the statement (ii)). It remains to show that there is no other inclusions. Recall that $Z = EY^2$. Hence, $(Z) \subseteq (E)$ and $(Z) \subseteq (Y)$. The ideals $\{(X,\mathfrak{q}) \,|\, \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}\}$ are maximal in $\mathcal{A}$ and $(\mathfrak{q}) + (Z) = (1)$. Therefore, none of the maximal ideals $(X,\mathfrak{q})$ contains (Y) or (E). Therefore, picture (3.9) represents the poset $(\mathrm{Spec}\,(\mathcal{A}), \subseteq)$. □

For an algebra R, let $\mathrm{Max}\,(R)$ be the set of its maximal ideals. The next corollary is an explicit description of the set $\mathrm{Max}(\mathcal{A})$.

Corollary 3.5. $\mathrm{Max}\,(\mathcal{A}) = \mathcal{P} \sqcup \mathcal{Q}$ *where* $\mathcal{P} := \big\{(Y,E,\mathfrak{p}) \,|\, \mathfrak{p} \in \mathrm{Max}\,(\mathbb{K}[H])\big\}$ and $\mathcal{Q} := \big\{(X,\mathfrak{q}) \,|\, \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}\big\}$.

Proof. The corollary follows from (3.9). $\square$

If $\mathfrak{g}$ is a solvable Lie algebra then every prime ideal of the universal enveloping algebra $U(\mathfrak{g})$ is completely prime, see [21, Theorem 3.7.2]. Since the spatial ageing algebra $\mathcal{A}$ is the enveloping algebra of a solvable Lie algebra, we have the following corollary (this corollary also follows from Theorem 3.4).

Corollary 3.6. *Every prime ideal of the algebra* $\mathcal{A}$ *is completely prime, i.e.,* $\mathrm{Spec}_c(\mathcal{A}) = \mathrm{Spec}\,(\mathcal{A})$.

Let R be an algebra and M be an R-module. For $a \in R$, let $a_M\cdot : M \to M$, $m \mapsto am$. The ideal of R, $\mathrm{ann}_R(M) := \{a \in R \,|\, aM = 0\}$, is called the *annihilator* of the R-module M. An R-module is called *faithful* if it has zero annihilator. The annihilator of each simple R-module is a prime ideal. Such prime ideals are called *primitive* and the set $\mathrm{Prim}\,(R)$ of all primitive ideals is called the *primitive spectrum* of R. The next proposition gives an explicit description of the set $\mathrm{Prim}\,(\mathcal{A})$.

Proposition 3.7. $\mathrm{Prim}(\mathcal{A}) = \mathrm{Max}(\mathcal{A}) \sqcup \{(Y),\ (E),\ 0\}$.

Proof. Clearly, $\mathrm{Prim}\,(\mathcal{A}) \supseteq \mathrm{Max}\,(\mathcal{A})$. The ideals (X) and (Y, E) are not primitive ideals as the corresponding factor algebras contain the central elements Z and H, respectively.

(i) $(Y) \in \mathrm{Prim}\,(\mathcal{A})$: For $\lambda \in \mathbb{K}^*$, let $I(\lambda) = (Y) + \mathcal{A}(E - \lambda)$. Since $\mathcal{A}/(Y) \simeq U^+$ (see Theorem 3.4.(2)), the left $\mathcal{A}$-module $M(\lambda) := \mathcal{A}/I(\lambda) \simeq U^+/U^+(E - \lambda) \simeq \mathbb{K}[H]\bar{1}$ is a simple $\mathcal{A}$-module/U^+-module where $\bar{1} = 1 + I(\lambda)$. By the definition of the module $M(\lambda)$, its annihilator $\mathfrak{p} := \mathrm{ann}_{\mathcal{A}}\big(M(\lambda)\big)$ contains the ideal (Y) but does not contain the ideal (Y, E), since otherwise we would have $0 = E\bar{1} = \lambda\bar{1} \neq 0$, a contradiction. By (3.9), we have $\mathfrak{p} = (Y)$.

(ii) $(E) \in \mathrm{Prim}\,(\mathcal{A})$: For $\lambda \in \mathbb{K}^*$, let $J_\lambda = (E) + \mathcal{A}(Y - \lambda)$. Since $\mathcal{A}/(E) \simeq \mathbb{K}[H][Y;\sigma]$ where $\sigma(H) = H + 1$ (see Theorem 3.4.(3b)), the left $\mathcal{A}$-module $T(\lambda) := \mathcal{A}/J_\lambda \simeq \mathbb{K}[H]\bar{1}$ is a simple module where $\bar{1} = 1 + J_\lambda$. Clearly, the prime ideal $\mathfrak{q} := \mathrm{ann}_{\mathcal{A}}\big(T(\lambda)\big)$ contains the ideal (E) but does not contain the ideal (Y, E) since otherwise we would have $0 = Y\bar{1} = \lambda\bar{1} \neq 0$, a contradiction. By (3.9), we have $\mathfrak{q} = (E)$.

(iii) *0 is a primitive ideal of* $\mathcal{A}$: For $\lambda \in \mathbb{K}^*$, we define the $\mathcal{A}$-module $S(\lambda) := \mathcal{A}/\mathcal{A}(X - \lambda, Y)$. Then $S(\lambda) = \bigoplus_{i\geqslant 0} \mathbb{K}[H]E^i\bar{1}$ where $\bar{1} = 1 + \mathcal{A}(X - \lambda, Y)$. Let $t = YX$ then $Ht = tH$ and $[t, E^i] = -iX^2E^{i-1}$. The fact that $S(\lambda)$ is a simple $\mathcal{A}$-module follows from the equality: $tE^i\bar{1} = (E^it - iX^2E^{i-1})\bar{1} = -i\lambda^2E^{i-1}\bar{1}$. Since $X \notin \mathrm{ann}_{\mathcal{A}}(S(\lambda))$, by (3.9), $\mathrm{ann}_{\mathcal{A}}(S(\lambda)) = 0$. Thus 0 is a primitive ideal of the algebra $\mathcal{A}$. $\square$

The next lemma is a faithfulness criterion for simple $\mathcal{A}$-modules.

Lemma 3.8. *Let* M *be a simple* $\mathcal{A}$*-module. Then* M *is a faithful* $\mathcal{A}$*-module iff* $\ker(X_M\cdot) = 0$.

Proof. The $\mathcal{A}$-module M is simple, so $\mathrm{ann}_{\mathcal{A}}(M) \in \mathrm{Prim}\,(\mathcal{A})$. Recall that the element X is a normal element of the algebra $\mathcal{A}$. So, $\ker(X_M\cdot)$ is a submodule of M. Then either $\ker(X_M\cdot) = 0$ or $\ker(X_M\cdot) = M$, and in the second case $\mathrm{ann}_{\mathcal{A}}(M) \supseteq (X)$. If $\ker(X_M\cdot) = 0$ then $\mathrm{ann}_{\mathcal{A}}(M) = 0$ since otherwise, by (3.9), $(X) \subseteq \mathrm{ann}_{\mathcal{A}}(M)$, a contradiction. $\square$

3.3 Centralizers of some elements of the algebra $\mathcal{A}$

Let R be an algebra and S be a non-empty subset of R. The algebra $C_R(S) := \{r \in R \,|\, rs = sr \text{ for all } s \in S\}$ is called the *centralizer* of S in R. The next lemma describes the centralizer of the element X in $\mathcal{A}$.

Lemma 3.9. $C_{\mathcal{A}}(X) = \mathbb{E}$.

Proof. Clearly, $\mathbb{E} \subseteq C_{\mathcal{A}}(X)$ and $XH^i = (H-1)^i X$ for all $i \geqslant 0$. So, the result follows from the equality $\mathcal{A} = \mathbb{E}[H;\delta]$, see (3.1). □

Let $h := H^+X = HX + EY$. Then the Ore extension $A_{1,X} = \mathbb{K}[X^{\pm 1}][H^+;\delta']$ where $\delta'(X) = X$ (see (3.4)) can be written as the Ore extension

$$A_{1,X} = \mathbb{K}[X^{\pm 1}][h;\delta] \quad \text{where} \quad \delta(X) = X^2 \ ([h,X] = X^2). \tag{3.13}$$

The next lemma describes the centralizers of the element Y in the algebras $\mathcal{A}_X$ and $\mathcal{A}$.

Lemma 3.10. *1.* $C_{\mathcal{A}_X}(Y) = A_{1,X} \otimes \mathbb{K}[Y]$.

2. The centralizer of the element Y in $\mathcal{A}$, $C_{\mathcal{A}}(Y) = \mathbb{K}[Y] \otimes R$, is a tensor product of algebras where $R := \mathbb{K}[X][h;\delta]$ is an Ore extension, $h = H^+X = HX + EY$ and $\delta(X) = X^2$.

3. The centre of the algebra $C_{\mathcal{A}}(Y)$ is $\mathbb{K}[Y]$.

Proof. 1. By (3.4), $\mathcal{A}_X = A_{1,X} \otimes A_1^+$ and $Y \in A_1^+$. Then $C_{\mathcal{A}_X}(Y) = A_{1,X} \otimes C_{A_1^+}(Y) = A_{1,X} \otimes \mathbb{K}[Y]$.

2. Now, $C_{\mathcal{A}}(Y) = \mathcal{A} \cap C_{\mathcal{A}_X}(Y) = \mathcal{A} \cap A_{1,X} \otimes \mathbb{K}[Y] \overset{(3.13)}{=} \mathcal{A} \cap \mathbb{K}[X^{\pm 1}][h;\delta] \otimes \mathbb{K}[Y] = \mathbb{K}[X][h;\delta] \otimes \mathbb{K}[Y]$ (since $h = HX + EY$ and X is a normal element of $\mathcal{A}$) and so the result.

3. By statement 2, $Z\big(C_{\mathcal{A}}(Y)\big) = \mathbb{K}[Y] \otimes Z(R) = \mathbb{K}[Y] \otimes \mathbb{K} = \mathbb{K}[Y]$. □

Using the equality $[E, YX^{-1}] = 1$, we see that the subalgebra $A_1' := \mathbb{K}\langle E, YX^{-1}\rangle$ of $\mathcal{A}_X$ is the (first) Weyl algebra. Then $\mathbb{E}_X = \mathbb{K}[X^{\pm 1}] \otimes A_1^+ = \mathbb{K}[X^{\pm 1}] \otimes A_1'$ is the tensor product of algebras. By (3.2), $\mathcal{A}_X = (\mathbb{K}[X^{\pm 1}] \otimes A_1')[H;\delta]$ where δ is as in (3.2). By Lemma 2.17, the algebra

$$\mathcal{A}_X = R' \otimes A_1' \tag{3.14}$$

is a tensor product of algebras where $R' := \mathbb{K}[X^{\pm 1}][H';\delta']$ is an Ore extension, $H' := H + 2YX^{-1}E$ and $\delta'(X) = X$. Then $h' := H'X = HX + 2YE \in \mathcal{A}$ and

$$R' = \mathbb{K}[X^{\pm 1}][h';\delta] \quad \text{where} \quad \delta(X) = X^2. \tag{3.15}$$

Notice that the elements $H'X^{-1} = h'X^{-2}$ and X of R' satisfy the commutation relation $[H'X^{-1}, X] = 1$. Therefore, the subalgebra $A_1 := \mathbb{K}\langle H'X^{-1}, X\rangle$ of R' is the (first) Weyl algebra and the algebra $R' = A_{1,X}$ is the localization of the Weyl algebra A_1 at the powers of the element X. In particular, the algebra R' is a central simple domain.

The next lemma describes the centralizers of the element E in the algebras $\mathcal{A}_X$ and $\mathcal{A}$.

Lemma 3.11. *1.* $C_{\mathcal{A}_X}(E) = R' \otimes \mathbb{K}[E]$.

2. The centralizer of the element E *in* $\mathcal{A}$, $C_{\mathcal{A}}(E) = \mathbb{K}[E] \otimes \mathcal{R}$, *is a tensor product of algebras where* $\mathcal{R} := \mathbb{K}[X][h'; \delta]$ *is an Ore extension,* $h' = H'X = HX + 2YE$ *and* $\delta(X) = X^2$.

3. The centre of the algebra $C_{\mathcal{A}}(E)$ *is* $\mathbb{K}[E]$.

Proof. 1. By (3.14), $\mathcal{A}_X = R' \otimes A_1'$ and $E \in A_1'$. Then $C_{\mathcal{A}_X}(E) = R' \otimes C_{A_1'}(E) = R' \otimes \mathbb{K}[E]$.

2. Now, $C_{\mathcal{A}}(E) = \mathcal{A} \cap C_{\mathcal{A}_X}(E) = \mathcal{A} \cap R' \otimes \mathbb{K}[E] \overset{(3.15)}{=} \mathcal{A} \cap \mathbb{K}[X^{\pm 1}][h'; \delta] \otimes \mathbb{K}[E] = \mathbb{K}[X][h'; \delta] \otimes \mathbb{K}[E]$ (since $h' = HX + 2EY$ and X is a normal element of $\mathcal{A}$) and so the result.

3. By statement 2, $Z\big(C_{\mathcal{A}}(E)\big) = \mathbb{K}[E] \otimes Z(R) = \mathbb{K}[E] \otimes \mathbb{K} = \mathbb{K}[E]$. □

Chapter 4

The universal enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$

4.1 Introduction

In this chapter we focus on the study of the universal enveloping algebra $A := U(\mathfrak{sl}_2 \ltimes V_2)$ of the Lie algebra $\mathfrak{sl}_2 \ltimes V_2$. We give explicit descriptions of the prime, maximal, primitive, completely prime and characteristic prime ideals of the algebra A. We investigate the centralizer $C_A(H)$ of the element H in the algebra A. In particular, the generators and defining relations of $C_A(H)$ are determined, a classification of simple $C_A(H)$-modules is given. We also give a classification of simple weight A-modules. The algebra A has a close relation with the *infinitesimal Hecke algebras of* $\mathfrak{sl}_2$, [41]. The first Hochschild cohomology of A was obtained in [41], which is a rank one free module over the center. The description of primitive ideals of the algebra A given in [41, Theorem 6.2] is not correct (for $z = 0$ in that paper). The Lie algebra $\mathfrak{sl}_2 \ltimes V_2$ admits a 1-dimensional central extension which is called the *Schrödinger algebra* $\mathfrak{s}$. Let $U(\mathfrak{s})$ be the universal enveloping algebra of the Schrödinger algebra $\mathfrak{s}$. We determine the primitive ideals of $U(\mathfrak{s})$. It is conjectured that there is no simple *singular* Whittaker module for the algebra A [44, Conjecture 4.2]. We construct a family of such A-modules (Proposition 4.44).

Spectra of the algebra A. In Section 4.2, an explicit description of the set of prime ideals of the algebra A together with their inclusions is given (Theorem 4.6). Using the classification of prime ideals of A, explicit descriptions of the sets of maximal, primitive and completely prime ideals are obtained (Corollary 4.7, Theorem 4.8 and Corollary 4.9, respectively). The group $\mathrm{Aut}_{\mathbb{K}}(A)$ of automorphisms of the algebra A is large as it contains plenty of locally nilpotent elements (an element $a \in A$ is called a locally nilpotent element if the inner derivation $\mathrm{ad}_a : A \to A$, $x \mapsto ax - xa$ is a locally nilpotent derivation, i.e., $A = \bigcup_{i \geqslant 1} \ker(\mathrm{ad}_a^i)$). An ideal of an algebra is called a *characteristic ideal* if it is invariant under all the automorphisms of the algebra. Corollary 4.10 is an explicit description of the characteristic prime ideals of A. It says that almost all prime ideals apart from an obvious set are characteristic ones.

The algebras $C_A(H), C^{\lambda,\mu}$ and their spectra. Let $\mathfrak{h} = \mathbb{K}H$ be the *Cartan* subalgebra of the Lie algebra $\mathfrak{sl}_2$ and $C_A(H)$ be the centralizer of the element H in A. The aim of Section 4.3 is to find explicit generators and defining relations for the algebra $C_A(H)$ (Theorem 4.14), to prove that the centre of the algebra $C_A(H)$ is a polynomial algebra $\mathbb{K}[C, H]$ (Theorem 4.14) and the algebra $C_A(H)$ is a free module over its centre (Proposition 4.16), to realize the algebra $C_A(H)$ as an algebra of differential operators, to prove various properties of the factor algebras $C^{\lambda,\mu}$ of $C_A(H)$. Results of this section are used in many proofs of this chapter. One of the important moments is a realization of the algebras $C_A(H)$ and $C^{\lambda,\mu}$ as algebras of differential operators (Proposition 4.17). The algebras $C^{\lambda,\mu}$ are simple iff $\lambda \neq 0$ (Theorem 4.23 and Proposition 4.27). For every $\lambda \neq 0$, the algebra $C^{\lambda,\mu}$ is a subalgebra of the first Weyl algebra A_1'. Theorem 4.26 classifies all simple $C^{\lambda,\mu}$-modules, it shows that the algebra $C^{\lambda,\mu}$ has *exactly one more* simple module than the Weyl algebra A_1'. A similar result holds for the algebras $C^{0,\mu}$ (Theorem 4.29) but the Weyl algebra A_1' is replaced by the skew polynomial algebra $R = \mathbb{K}[h][t;\sigma]$ where $\sigma(h) = h - 1$. In this case, *all* simple t-torsionfree R-modules are *also* simple t-torsionfree $C^{0,\mu}$-modules, and vice versa (Theorem 4.29.(2)).

Classification of simple weight A-modules. An A-module M is called a *weight* module if $M = \oplus_{\mu\in\mathbb{K}} M_\mu$ where $M_\mu = \{m \in M \mid Hm = \mu m\}$. Each nonzero component M_μ is a $C_A(H)$-module. If, in addition, the weight A-module M is simple then all nonzero components M_μ are *simple* $C_A(H)$-modules. So, the problem of classification of *simple weight* A-modules is closely related to the problem of classification of *all simple* $C_A(H)$-modules, which can be seen as the first, the more difficult, of two steps. The second one is about how 'to assemble' simple $C_A(H)$-modules in order to have a simple A-module. The difficulty of the first step stems from the fact that the algebra $C_A(H)$ is of comparable size to the algebra A itself ($\mathrm{GK}\,(C_A(H)) = 4$ and $\mathrm{GK}\,(A) = 5$ where GK stands for the Gelfand-Kirillov dimension) and the defining relations of the algebra $C_A(H)$ are much more complex than the defining relations of the algebra A (see, (4.12)–(4.15)). An advantage is that the algebra $C_A(H)$ has an additional central element H. Moreover, the centre of $C_A(H)$ is a polynomial algebra $\mathbb{K}[C, H]$ (Theorem 4.14) where $C = FX^2 - HXY - EY^2$ is a central element of the algebra A. The problem of classification of simple $C_A(H)$-modules is equivalent to the same problem but for all the factor algebras $C^{\lambda,\mu} := C_A(H)/(C - \lambda, H - \mu)$ where $\lambda, \mu \in \mathbb{K}$. We assume that the field $\mathbb{K}$ is algebraically closed. There are two distinct cases: $\lambda \neq 0$ and $\lambda = 0$. They require different approaches. The common feature is a discovery of the fact that in order to study simple modules over the algebras $C^{\lambda,\mu}$ we embed them into algebras for which classifications of simple modules are known. A surprise is that the sets of simple modules of the algebras $C^{\lambda,\mu}$ and their over-algebras are tightly connected. In the case $\lambda \neq 0$, such an algebra is the first Weyl algebra, but in the second case when $\lambda = 0$, it is a skew polynomial algebra $\mathbb{K}[h][t;\sigma]$ where $\sigma(h) = h - 1$. Classifications of simple $C^{\lambda,\mu}$-modules is given in Section 4.4 (Theorem 4.26 and Theorem 4.29). Using it a classification of simple weight A-modules is given in Section 4.5. A typical simple weight A-module depends on an arbitrarily large number of independent parameters. The set of simple A-modules is partitioned into 5 classes each of them is dealt separately with different techniques (Lemma 4.30, Proposition 4.33, Theorem 4.35 and Theorem 4.36).

Much of this chapter is extracted from the joint paper with V. Bavula [14].

4.2 The prime ideals of A

The aim of this section is to describe the prime ideals of the algebra A (Theorem 4.6). As a result, the sets of maximal, primitive, completely prime and prime characteristic ideals are described (Corollary 4.7, Theorem 4.8, Corollary 4.9 and Corollary 4.10, respectively). An explicit classification of prime ideals that are invariant under all automorphisms of the algebra A is given (Corollary 4.10).

Recall that the Lie algebra $\mathfrak{a} = \mathfrak{sl}_2 \ltimes V_2$ admits the basis $\{H, E, F, X, Y\}$ and the Lie bracket is defined as follows

$$[H,E]=2E, \quad [H,F]=-2F, \quad [E,F]=H, \quad [E,X]=0, \quad [E,Y]=X,$$
$$[F,X]=Y, \quad [F,Y]=0, \quad [H,X]=X, \quad [H,Y]=-Y, \quad [X,Y]=0.$$

Recall that $A = U(\mathfrak{a})$ is the enveloping algebra of the Lie algebra $\mathfrak{a}$.

An involution $*$ of A. Let Λ be an algebra. An anti-isomorphism τ of the algebra Λ (i.e., a linear map τ such that $\tau(ab) = \tau(b)\tau(a)$ for all $a, b \in \Lambda$) is called an *involution* if $\tau^2 = \mathrm{id}_\Lambda$. The algebra A admits the following involution $*$:

$$F^* = -E, \quad H^* = H, \quad E^* = -F, \quad Y^* = X, \quad X^* = Y. \tag{4.1}$$

There is an automorphism S of the algebra A such that

$$S(F) = E, \quad S(H) = -H, \quad S(E) = F, \quad S(Y) = -X, \quad S(X) = -Y, \tag{4.2}$$

and $S^2 = \mathrm{id}_A$.

The spatial ageing algebra $\mathcal{A}$. Let $\mathcal{A}$ be the subalgebra of A generated by the elements H, E, X and Y. The algebra $\mathcal{A}$ is the *spatial ageing algebra* which is studied in the previous chapter. Let $\mathcal{A}_X$ be the localization of $\mathcal{A}$ at the powers of X. Let $\partial := HX^{-1} + EYX^{-2} \in \mathcal{A}_X$. Then $[\partial, X] = 1$ and so the subalgebra $A_1 := \mathbb{K}\langle \partial, X\rangle$ of $\mathcal{A}_X$ is the first Weyl algebra. Recall that, the algebra $\mathcal{A}_X$ is a central simple algebra of Gelfand-Kirillov dimension 4 (see Lemma 3.1.(1)), and $\mathcal{A}_X$ is a tensor product of two central simple algebras

$$\mathcal{A}_X = A_{1,X} \otimes A_1^+ \tag{4.3}$$

where $A_{1,X}$ is the localization of A_1 at the powers of X and $A_1^+ := \mathbb{K}\langle EX^{-1}, Y\rangle$ is the first Weyl algebra since $[EX^{-1}, Y] = 1$ (see (3.4)).

The centre of the algebra A. Using the defining relations of the algebra A, the algebra A is a skew polynomial algebra

$$A = \mathcal{A}[F; \sigma, \delta] \tag{4.4}$$

where σ is an automorphism of $\mathcal{A}$ such that $\sigma(H) = H+2, \sigma(E) = E, \sigma(Y) = Y, \sigma(X) = X$; and δ is a σ-derivation of the algebra $\mathcal{A}$ such that $\delta(H) = 0, \delta(E) = -H, \delta(Y) = 0$ and $\delta(X) = Y$.

Then the localization A_X of A at the powers of X is a skew polynomial algebra

$$A_X = \mathcal{A}_X[F; \sigma, \delta] \tag{4.5}$$

where σ and δ are defined as in (4.4). The key idea of finding the centre of A is by 'deleting the automorphism' σ first and then using Lemma 2.17 'deleting the derivation'. In more detail, let $\Phi := FX^2$, then by (4.5) and (4.3),

$$A_X = \mathcal{A}_X[\Phi; \delta'] = (A_{1,X} \otimes A_1^+)[\Phi; \delta'] \tag{4.6}$$

is an Ore extension where δ' is a derivation of the algebra $\mathcal{A}_X$ given by the rule: $\delta'(\partial) = -2\partial YX$, $\delta'(X) = YX^2$, $\delta'(EX^{-1}) = -\partial X^2$ and $\delta'(Y) = 0$. The element $s = -\partial X^2 Y$ satisfies the conditions of Lemma 2.17. Specifically, the element $C := \Phi + s = FX^2 - HXY - EY^2$ commutes with the elements of A_1^+, moreover, the element C commutes with the elements of $A_{1,X}$ and hence,

$$A_X = \mathbb{K}[C] \otimes A_{1,X} \otimes A_1^+ = \mathbb{K}[C] \otimes \mathcal{A}_X \tag{4.7}$$

is a tensor product of algebras. By (4.7), the skew field $\text{Frac}\,(A)$ is isomorphic to the skew field of fractions of the second Weyl algebra $A_2\big(\mathbb{K}(C)\big)$ over the field $\mathbb{K}(C)$ of rational functions. Moreover, $Z\big(\text{Frac}(A)\big) = \mathbb{K}(C)$.

Lemma 4.1. *1. $Z(A) = Z(A_X) = \mathbb{K}[C]$ where $C = FX^2 - HXY - EY^2$.*

2. $S(C) = -C$ where S is the automorphism (4.2) of A.

Proof. 1. By (4.7), $Z(A_X) = Z(\mathbb{K}[C]) \otimes Z(A_{1,X}) \otimes Z(A_1^+) = \mathbb{K}[C]$. Since $\mathbb{K}[C] \subseteq Z(A) \subseteq A \cap Z(A_X) = \mathbb{K}[C]$, we have $Z(A) = \mathbb{K}[C]$.

2. Statement 2 is obvious. □

Lemma 4.2. *1. In the algebra A, $(X) = (Y) = AX + AY = XA + YA$.*

2. Let $U := U(\mathfrak{sl}_2)$. Then $A/(X) \simeq U$.

Proof. 1. The equality $(X) = (Y)$ follows from the equalities $FX - XF = Y$ and $EY - YE = X$. So, $(X) = (Y) = (X, Y)$. Let us show that $XA \subseteq AX + AY$ and $YA \subseteq AX + AY$. Recall that $A = \mathcal{A}[F; \sigma, \delta]$ (see (4.4)) and X is a normal element of $\mathcal{A}$, $XA = X\sum_{i\geqslant 0}\mathcal{A}F^i = \sum_{i\geqslant 0}\mathcal{A}XF^i = \mathcal{A}X + \sum_{i\geqslant 1}\mathcal{A}XF^i = \mathcal{A}X + \sum_{i\geqslant 1}\mathcal{A}(F^iX - iF^{i-1}Y) \subseteq AX + AY$. The second inclusion follows from the first one by applying the automorphism S (see (4.2)). So, $(X, Y) = AX + AY$. By applying the involution $*$ to this equality we obtain that $(X, Y) = XA + YA$.

2. By statement 1, $A/(X) = A/(X, Y) \simeq U$. □

Lemma 4.3. *For all $i \geqslant 1$, $(X^i) = (X)^i$.*

Proof. To prove the statement we use induction on i. The case $i = 1$ is obvious. Suppose that $i > 1$ and the equality $(X^j) = (X)^j$ holds for all $1 \leqslant j \leqslant i - 1$. By Lemma 4.2.(1), $AX \subseteq XA + YA$. It follows from the equality $FX^i = X^iF + iX^{i-1}Y$ that $X^{i-1}Y \in (X^i)$. Now,

$(X)^i = (X)^{i-1}(X) = (X^{i-1})(X) = AX^{i-1}AXA \subseteq AX^{i-1}(XA+YA) \subseteq (X^i) + AX^{i-1}YA \subseteq (X^i)$. Therefore, $(X)^i = (X^i)$. □

Proposition 4.4. *Let* $\mathfrak{q} \in \mathrm{Max}(\mathbb{K}[C]) \setminus \{(C)\}$. *Then*

1. *The ideal* $(\mathfrak{q}) := A\mathfrak{q}$ *of* A *is a maximal, completely prime ideal.*
2. *The factor algebra* $A/(\mathfrak{q})$ *is a simple algebra.*

Proof. Notice that $\mathfrak{q} = \mathbb{K}[C]q$ where $q = q(C) \in \mathbb{K}[C]$ is an irreducible polynomial such that $q(0) \in \mathbb{K}^*$.

(i) *The factor algebra* $A/(\mathfrak{q})$ *is a simple algebra, i.e.,* $(\mathfrak{q})$ *is a maximal ideal of* A: By (4.7), $A_X/(\mathfrak{q})_X \simeq L_\mathfrak{q} \otimes A_{1,X} \otimes A_1^+$ is a central simple algebra where $L_\mathfrak{q} := \mathbb{K}[C]/\mathfrak{q}$ is a finite field extension of $\mathbb{K}$. Hence, the algebra $A/(\mathfrak{q})$ is a simple algebra iff $(X^i, \mathfrak{q}) = A$ for all $i \geqslant 1$. By Lemma 4.3, $(X^i) = (X)^i$ for all $i \geqslant 1$. Therefore, $(X^i, \mathfrak{q}) = (X^i) + (\mathfrak{q}) = (X)^i + (\mathfrak{q})$ for all $i \geqslant 1$. It remains to show that $(X)^i + (\mathfrak{q}) = A$ for all $i \geqslant 1$. By Lemma 4.2.(1), $(X) = (X, Y)$. If $i = 1$ then $(X) + (\mathfrak{q}) = (X, Y, \mathfrak{q}) = \big(X, Y, q(0)\big) = A$, since $q(0) \in \mathbb{K}^*$. Now, $A = A^i = \big((X) + (\mathfrak{q})\big)^i \subseteq (X)^i + (\mathfrak{q}) \subseteq A$, i.e., $(X)^i + (\mathfrak{q}) = A$, as required.

(ii) $(\mathfrak{q})$ *is a completely prime ideal of* A: Since $A_X/(\mathfrak{q})_X \simeq L_\mathfrak{q} \otimes A_{1,X} \otimes A_1^+$ is a domain, the ideal $A \cap (\mathfrak{q})_X$ is a completely prime ideal of A. Now, it suffices to show that $(\mathfrak{q}) = A \cap (\mathfrak{q})_X$. But this is obvious since by statement (i), the ideal $(\mathfrak{q})$ is a maximal ideal of A.

(iii) $Z\big(A/(\mathfrak{q})\big) = L_\mathfrak{q}$: Since $L_\mathfrak{q} \subseteq Z\big(A/(\mathfrak{q})\big) \subseteq Z\big(A_X/(\mathfrak{q})_X\big) = L_\mathfrak{q}$, we have $Z\big(A/(\mathfrak{q})\big) = L_\mathfrak{q}$. □

Proposition 4.5. $A \cap (C)_X = (C)$ *and the ideal* (C) *of* A *is a completely prime ideal.*

Proof. Recall that $A = \mathcal{A}[F; \sigma, \delta]$ (see (4.4)), X is a normal element in $\mathcal{A}$ and the central element C can be written as $C = X^2F + s'$ where $s' = -X(H-1)Y - EY^2$.

(i) *If* $Xf \in (C)$ *for some* $f \in A$ *then* $f \in (C)$: Notice that $Xf = Cg$ for some $g \in A$. To prove the statement (i) we use induction on the degree $m = \deg_F(f)$ of the element $f \in A$. Since A is a domain, $\deg_F(fg) = \deg_F(f) + \deg_F(g)$ for all $f, g \in A$. The case when $m \leqslant 0$ (i.e., $f \in \mathcal{A}$) is obvious since the equality $Xf = Cg$ holds iff $f = g = 0$ (since $\deg_F(Xf) \leqslant 0$ and $\deg_F(Cg) \geqslant 1$ providing $g \neq 0$). So, we may assume that $m \geqslant 1$. We can write the element f as a sum $f = f_0 + f_1F + \cdots + f_mF^m$ where $f_i \in \mathcal{A}$ and $f_m \neq 0$. The equality $Xf = Cg$ implies that $\deg_F(g) = \deg_F(Xf) - \deg_F(C) = m - 1$. Therefore, $g = g_0 + g_1F + \cdots + g_{m-1}F^{m-1}$ for some $g_i \in \mathcal{A}$ and $g_{m-1} \neq 0$. Then

$$\begin{aligned} Xf_0 + Xf_1F + \cdots + Xf_mF^m &= (X^2F + s')\big(g_0 + g_1F + \cdots + g_{m-1}F^{m-1}\big) \\ &= X^2\Big(\sigma(g_0)F + \delta(g_0)\Big) + X^2\Big(\sigma(g_1)F + \delta(g_1)\Big)F + \cdots + X^2\Big(\sigma(g_{m-1})F + \delta(g_{m-1})\Big)F^{m-1} \\ &\quad + s'g_0 + s'g_1F + \cdots + s'g_{m-1}F^{m-1} \\ &= X^2\delta(g_0) + s'g_0 + \Big(X^2\sigma(g_0) + X^2\delta(g_1) + s'g_1\Big)F + \cdots + X^2\sigma(g_{m-1})F^m. \end{aligned}$$

Comparing the terms of degree zero we have the equality $Xf_0 = X^2\delta(g_0) + s'g_0 = X^2\delta(g_0) - \big(X(H-1)Y + EY^2\big)g_0$, i.e., $X\big(f_0 - X\delta(g_0) + (H-1)Yg_0\big) = -EY^2g_0$. All terms in the equality belong to the subalgebra $\mathcal{A}$. Since X is a normal element of $\mathcal{A}$ such that $\mathcal{A}/\mathcal{A}X$ is a domain and the element EY^2 does not belong to the ideal $\mathcal{A}X$ (see (3.5)), we have $g_0 \in \mathcal{A}X$, i.e., $g_0 = Xh_0$ for some $h_0 \in \mathcal{A}$. Now, the element g can be written as $g = Xh_0 + g'F$ where $g' = 0$ if $m = 1$, and $\deg_F(g') = m - 2$ if $m \geqslant 2$. Then $Xf = C(Xh_0 + g'F)$ and so $X(f - Ch_0) = Cg'F$. Notice that $Cg'F$ has zero constant term as a noncommutative polynomial in F (where the coefficients are written on the left). Therefore, the element $f - Ch_0$ has zero constant term, and hence can be written as $f - Ch_0 = f'F$ for some $f' \in A$ with $\deg_F(f') < \deg_F(f)$. Now, $Xf'F = Cg'F$, hence $Xf' = Cg' \in (C)$ (by deleting F). By induction, $f' \in (C)$, and then $f = Ch_0 + f'F \in (C)$, as required.

(ii) $A \cap (C)_X = (C)$: Let $u \in A \cap (C)_X$. Then $X^i u \in (C)$ for some $i \in \mathbb{N}$. By statement (i), $u \in (C)$.

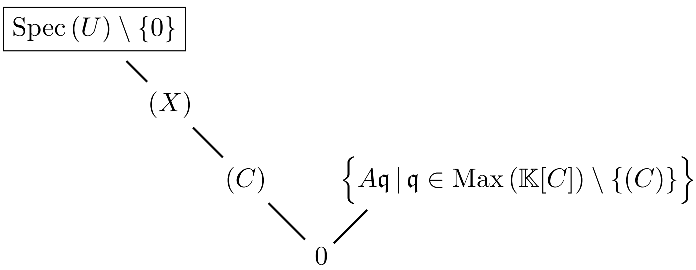
(iii) *The ideal* (C) *of* A *is a completely prime ideal*: By (4.7), $A_X/(C)_X \simeq A_{1,X} \otimes A_1^+$ is a domain. By statement (ii), the algebra $A/(C)$ is a subalgebra of $A_X/(C)_X$, so $A/(C)$ is a domain. This means that the ideal (C) is a completely prime ideal of A. □

The next theorem gives an explicit description of the poset $(\mathrm{Spec}\,(A), \subseteq)$.

Theorem 4.6. *Let* $U := U(\mathfrak{sl}_2)$. *The prime spectrum of the algebra* A *is a disjoint union*

$$\mathrm{Spec}(A) = \mathrm{Spec}(U) \sqcup \mathrm{Spec}(A_X) = \{(X, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(U)\} \sqcup \{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}(\mathbb{K}[C])\}. \tag{4.8}$$

Furthermore,



(4.9)

Proof. By Lemma 4.2.(2), $A/(X) \simeq U$. By Lemma 4.3 and Proposition 3.3,

$$\mathrm{Spec}\,(A) = \mathrm{Spec}\,(A, X) \ \sqcup \ \mathrm{Spec}\,(A_X).$$

Therefore, $\mathrm{Spec}\,(A) = \{(X, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\,(U)\} \sqcup \{A \cap A_X\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}\,(\mathbb{K}[C])\}$. By Proposition 4.4.(1), $A \cap A_X\mathfrak{q} = (\mathfrak{q})$ for all $\mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}$. By Proposition 4.5, $A \cap A_X C = (C)$. Therefore, (4.8) holds. For all $\mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}$, the ideals $A\mathfrak{q}$ of A are maximal. Notice that $(C) \subseteq (X)$. Therefore, (4.9) holds. □

For a list of prime ideals of U, see [3, Section 4] or [20, Theorem 4.5]. We note that any nonzero prime ideal of U is primitive, i.e., $\mathrm{Prim}\,(U) = \mathrm{Spec}\,(U) \setminus \{0\}$. For any ideal I of U and any automorphism $\sigma \in \mathrm{Aut}_{\mathbb{K}}(U)$, $\sigma(I) = I$, see [3] for details.

The next result is an explicit description of the set of maximal ideals of the algebra A.

Corollary 4.7. $\mathrm{Max}\,(A) = \mathrm{Max}\,(U) \ \sqcup \ \big\{A\mathfrak{q} \,|\, \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}\big\}$.

Proof. It is clear by (4.9). □

A ring R is called a *Jacobson ring* if every prime ideal of R is an intersection of primitive ideals. The enveloping algebras of finite dimensional Lie algebras are Jacobson rings, [37, Corollary 9.1.8]. The next theorem is a description of the set of primitive ideals of the algebra A.

Theorem 4.8. $\mathrm{Prim}\,(A) = \mathrm{Prim}\,(U) \ \sqcup \ \big\{A\mathfrak{q} \,|\, \mathfrak{q} \in \mathrm{Spec}\,\mathbb{K}[C] \setminus \{0\}\big\}$.

Proof. Clearly, $\mathrm{Prim}\,(U) \subseteq \mathrm{Prim}\,(A)$ and $\{A\mathfrak{q} \,|\, \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}\} \subseteq \mathrm{Prim}\,(A)$ since $A\mathfrak{q}$ is a maximal ideal (Corollary 4.7). The ideal (X) is not a primitive ideal, since the factor algebra $A/(X) \simeq U$ contains central elements. 0 is not a primitive ideal since the centre of A is non-trivial. In view of (4.9) it suffices to show that $(C) \in \mathrm{Prim}\,(A)$. The algebra A is a Jacobson algebra since it is a universal enveloping algebra of a finite dimensional Lie algebra [37, Corollary 9.1.8]. Therefore, any prime ideal of A is an intersection of primitive ideals lying over it. Clearly, $(X) = \bigcap_{(X) \subseteq P, P \in \mathrm{Spec}\,(U) \setminus \{0\}} P$. Since (C) is a prime ideal it must be primitive, by the diagram (4.9). □

The next corollary is a description of the set $\mathrm{Spec}_c(A)$ of completely prime ideals of the algebra A.

Corollary 4.9. *The set* $\mathrm{Spec}_c(A)$ *of completely prime ideals of* A *is equal to*

$$\begin{aligned}\mathrm{Spec}_c(A) &= \mathrm{Spec}_c(U) \ \sqcup \ \big\{A\mathfrak{q} \,|\, \mathfrak{q} \in \mathrm{Spec}\,(\mathbb{K}[C])\big\} \\ &= \big\{(X, \mathfrak{p}) \,|\, \mathfrak{p} \in \mathrm{Spec}\,(U),\ \mathfrak{p} \neq \mathrm{ann}_U(M) \text{ for some simple finite dimensional} \\ &\qquad U\text{-module } M \text{ of } \dim_{\mathbb{K}}(M) \geqslant 2\big\} \ \sqcup \ \big\{A\mathfrak{q} \,|\, \mathfrak{q} \in \mathrm{Spec}\,(\mathbb{K}[C])\big\}.\end{aligned}$$

Proof. The result follows from Proposition 4.4.(1) and Proposition 4.5. □

By Theorem 4.6 and Corollary 4.9, the set of prime ideals that are not completely prime is equal to $\{(X, \mathfrak{p}) \,|\, \mathfrak{p} = \mathrm{ann}_U(M)$ for some simple finite dimensional U-module M of $\dim_{\mathbb{K}}(M) \geqslant 2\big\}$.

Let A be a $\mathbb{K}$-algebra and $\mathrm{Aut}_{\mathbb{K}}(A)$ be its group of automorphisms. An ideal $\mathfrak{a}$ of the algebra A is called a *characteristic ideal* if $\sigma(\mathfrak{a}) = \mathfrak{a}$ for all $\sigma \in \mathrm{Aut}_{\mathbb{K}}(A)$. Let $\mathrm{Spec}_{ch}(A)$ be the set of prime characteristic ideals of A, the, so-called, *characteristic prime spectrum* of A.

For each element $(\lambda, \mu) \in (\mathbb{K}^*)^2$, there is an automorphism $t_{\lambda,\mu}$ of the algebra A given by the rule

$$t_{\lambda,\mu} : A \to A, \quad E \mapsto \lambda E, \ \ F \mapsto \lambda^{-1}F, \ \ H \mapsto H, \ \ X \mapsto \mu X, \ \ Y \mapsto \lambda^{-1}\mu Y. \tag{4.10}$$

Clearly, $t_{\lambda,\mu}t_{\lambda',\mu'} = t_{\lambda\lambda',\mu\mu'}$ and $t_{\lambda,\mu}^{-1} = t_{\lambda^{-1},\mu^{-1}}$. So, the 2-dimensional algebraic torus $\mathbb{T}^2 := \big\{t_{\lambda,\mu} \,|\, (\lambda, \mu) \in (\mathbb{K}^*)^2\big\} \simeq (\mathbb{K}^*)^2$ is a subgroup of $\mathrm{Aut}_{\mathbb{K}}(A)$. We note that $t_{\lambda,\mu}(C) = \lambda^{-1}\mu^2 C$.

Corollary 4.10. *Let* $G := \mathrm{Aut}_{\mathbb{K}}(A)$ *and* $\mathcal{Q} := \Big\{A\mathfrak{q} \,|\, \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}\Big\}$. *Then the set* $\mathcal{Q}$ *is* G*-invariant and* $\mathrm{Spec}_{ch}(A) = \mathrm{Spec}(A) \setminus \mathcal{Q}$.

Proof. By the diagram (4.9), the ideals $0, (C)$ and (X) are characteristic. Then, for each $\sigma \in G$, $\sigma(C) = \lambda_\sigma C$ for some $\lambda_\sigma \in \mathbb{K}^*$. Hence, the subset $\mathcal{Q}$ of Spec(A) is G-invariant. Since (X) is a characteristic ideal of A, there is a group homomorphism

$$\mathrm{Aut}_{\mathbb{K}}(A) \to \mathrm{Aut}_{\mathbb{K}}(U), \quad \sigma \mapsto \bar{\sigma} : a + (X) \mapsto \sigma(a) + (X).$$

All ideals of U are characteristic ideals, [3]. To finish the proof notice that none of the ideals in $\mathcal{Q}$ is $\mathbb{T}^2$-invariant (since $t_{\lambda,\mu}(C) = \lambda^{-1}\mu^2 C$). □

4.3 The centralizer $C_A(H)$ and its defining relations

The aim of this section is to find explicit generators and defining relations for the centralizer $C_A(H)$ of the element H in A (Theorem 4.14), to prove that the centre of the algebra $C_A(H)$ is a polynomial algebra $\mathbb{K}[C, H]$ (Theorem 4.14) and the algebra $C_A(H)$ is a free module over its centre (Proposition 4.16), to realize the algebra $C_A(H)$ as an algebra of differential operators and to prove various properties of the factor algebras $C^{\lambda,\mu}$ of $C_A(H)$. Results of this section is used in many proofs of this chapter.

The centralizer of the element H. The next lemma describes the structure of the algebras A_X and $C_{A_X}(H)$.

Lemma 4.11. *1. $C_{\mathcal{A}_X}(H) = \mathbb{K}[H] \otimes A_1'$ is a tensor product of algebras where $A_1' := \mathbb{K}\langle e, t\rangle$ is the (first) Weyl algebra with canonical generators $e := EX^{-2}$ and $t := XY$ (where $[e, t] = 1$).*

2. $C_{A_X}(H) = \mathbb{K}[C, H] \otimes A_1'$ and $Z\big(C_{A_X}(H)\big) = \mathbb{K}[C, H]$.

3. $A_X = C_{A_X}(H)[X^{\pm 1}; \sigma]$ is a skew polynomial algebra where $\sigma(C) = C, \sigma(H) = H-1, \sigma(e) = e$ and $\sigma(t) = t$. In particular, the algebra $A_X = \mathbb{K}[C] \otimes A_1' \otimes B_1$ is a tensor product of algebras where $B_1 = \mathbb{K}[H][X^{\pm 1}; \sigma]$ is a central simple algebra and $\sigma(H) = H - 1$.

Proof. 1. By (3.2), $\mathcal{A}_X = \mathbb{E}_X[H; \delta]$. So, $C_{\mathcal{A}_X}(H) = \mathbb{E}_X^\delta[H]$ where $\mathbb{E}_X^\delta = \{a \in \mathbb{E}_X \,|\, \delta(a) = 0\}$. Let us show that $\mathbb{E}_X^\delta = A_1'$. By the explicit nature of the derivation δ,

$$\mathbb{E}_X^\delta = \bigoplus_{i,k\in\mathbb{N}; j\in\mathbb{Z}} \{\mathbb{K}E^i X^j Y^k \,|\, \delta(E^i X^j Y^k) = 0\}.$$

Now, $\delta(E^i X^j Y^k) = (2i + j - k)E^i X^j Y^k = 0$, i.e., $j = k - 2i$. So, $E^i X^j Y^k = E^i X^{k-2i} Y^k = (EX^{-2})^i \cdot (XY)^k$. Therefore, $\mathbb{E}_X^\delta = A_1'$.

2. By (4.7), $A_X = \mathbb{K}[C] \otimes \mathcal{A}_X$. So, $C_{A_X}(H) = \mathbb{K}[C] \otimes C_{\mathcal{A}_X}(H) = \mathbb{K}[C, H] \otimes A_1'$, by statement 1. The Weyl algebra A_1' is a central algebra, hence $Z\big(C_{A_X}(H)\big) = \mathbb{K}[C, H]$.

3. Statement 3 follows from statement 2. □

Lemma 4.12. *Let $t := XY$. For $i \geqslant 1$, the following identities hold in the algebra A.*

1. $F^i X^{2i} = FX^2\Big(FX^2 + 2t\Big)\Big(FX^2 + 4t\Big)\cdots\Big(FX^2 + 2(i-1)t\Big)$.

2. $E^iY^{2i} = EY^2\Big(EY^2+2t\Big)\Big(EY^2+4t\Big)\cdots\Big(EY^2+2(i-1)t\Big).$

Proof. 1. We use induction on $i \geqslant 1$. The initial case when $i=1$ is obvious. So, let $i>1$ and suppose that the identity holds for all integers $<i$. Then

$$\begin{aligned} F^iX^{2i} &= F\cdot FX^2\Big(FX^2+2t\Big)\Big(FX^2+4t\Big)\cdots\Big(FX^2+2(i-2)t\Big)\cdot X^2 \\ &= FX^2\Big(FX^2+2t\Big)\Big(FX^2+4t\Big)\cdots\Big(FX^2+2(i-1)t\Big) \end{aligned}$$

since $FX^2\cdot X^2 = X^2\cdot\Big(FX^2+2t\Big)$.

2. Statement 2 follows from statement 1 by applying the automorphism S, see (4.2). □

The algebra U is a generalized Weyl algebra,

$$U \simeq \mathbb{K}[H,\Delta]\Big(\sigma, a = \frac{1}{4}(\Delta - H(H+2))\Big) \tag{4.11}$$

where $\Delta := 4FE + H(H+2)$ is the Casimir element of the enveloping algebra U and σ is the automorphism of the algebra $\mathbb{K}[H,\Delta]$ defined by $\sigma(H)=H-2$ and $\sigma(\Delta)=\Delta$, [1]. In particular, U is a $\mathbb{Z}$-graded algebra $U=\bigoplus_{i\in\mathbb{Z}} Dv_i$ where $D:=\mathbb{K}[H,\Delta]=\mathbb{K}[H,FE]$, $v_i=E^i$ if $i\geqslant 1$, $v_0=1$ and $v_i=F^{|i|}$ if $i\leqslant -1$. The polynomial algebra $\mathbb{K}[X,Y]\subset A$ is also a $\mathbb{Z}$-graded algebra $\mathbb{K}[X,Y]=\bigoplus_{j\in\mathbb{Z}}\mathbb{K}[t]w_j$ where $t=XY$, $w_j=X^j$ if $j\geqslant 1$, $w_0=1$ and $w_j=Y^{|j|}$ if $j\leqslant -1$. Note that the algebra A is a $\mathbb{Z}$-graded algebra $A=\bigoplus_{i\in\mathbb{Z}}A_i$ where $A_i:=\{a\in A\,|\,[H,a]=ia\}$. Clearly, $C_A(H)=A_0$. The following lemma gives the generators of the algebra $C_A(H)$.

Lemma 4.13. *The algebra* $C_A(H)=\mathbb{K}\langle H,FE,XY,FX^2,EY^2\rangle=\mathbb{K}\langle C,H,FE,XY,FX^2\rangle$ *is a Noetherian algebra.*

Proof. Since $A=\oplus_{i\in\mathbb{Z}}A_i$ is a $\mathbb{Z}$-graded Noetherian algebra, the algebra $A_0=C_A(H)$ is a Noetherian algebra. The algebra $A=U\otimes\mathbb{K}[X,Y]$ is a tensor product of vector spaces. Hence $A=\bigoplus_{i\in\mathbb{Z}}Dv_i\otimes\bigoplus_{j\in\mathbb{Z}}\mathbb{K}[t]w_j$ where D, v_i, t and w_j are as above. Using the relations $[E,t]=X^2$ and $[F,t]=Y^2$, we see that $A=\sum_{i,j\in\mathbb{Z}}D[t]v_iw_j$ where $D[t]=\bigoplus_{i\geqslant 0}Dt^i$ is a vector space. Notice that $A_k=\sum\{D[t]v_iw_j\,|\,i,j\in\mathbb{Z}; 2i+j=k\}$. In particular,

$$C_A(H)=A_0=\sum_{\substack{i,j\in\mathbb{Z};\\2i+j=0}}D[t]v_iw_j=\sum_{i\in\mathbb{Z}}D[t]v_iw_{-2i}=\sum_{i\geqslant 1}D[t]F^iX^{2i}+D[t]+\sum_{i\geqslant 1}D[t]E^iY^{2i}.$$

Now, using Lemma 4.12 and the equalities $[FX^2,t]=t^2$ and $[EY^2,t]=t^2$, we see that

$$C_A(H)=\sum_{i\geqslant 1}D[t]\big(FX^2\big)^i+D[t]+\sum_{i\geqslant 1}D[t]\big(EY^2\big)^i.$$

Hence, $C_A(H)=\mathbb{K}\langle H,FE,XY,FX^2,EY^2\rangle$. Since $C=FX^2-HXY-EY^2$, the second equality in the lemma follows. □

The next theorem describes defining relations of the algebra $C_A(H)$ and shows that its centre is a polynomial algebra $\mathbb{K}[H,C]$.

Theorem 4.14. *Let $\Phi := FX^2$ and $\Theta := FE$. Then the algebra $C_A(H)$ is of Gelfand-Kirillov dimension 4 and generated by the elements C, H, t, Φ and Θ subject to the following defining relations (where C and H are central in the algebra $C_A(H)$):*

$$[\Phi, t] = t^2, \tag{4.12}$$

$$[\Theta, t] = 2\Phi - (H+2)t - C, \tag{4.13}$$

$$[\Theta, \Phi] = 2\Theta t + H\Phi, \tag{4.14}$$

$$\Theta t^2 = \Phi(\Phi - Ht - C). \tag{4.15}$$

Furthermore, $Z\big(C_A(H)\big) = \mathbb{K}[C, H]$.

Proof. (i) *Generators of $C_A(H)$*: By Lemma 4.13, the algebra $C_A(H)$ is generated by the elements C, H, t, Φ and Θ. It is clear that C and H are central in $C_A(H)$ and the elements satisfy the relations (4.12)–(4.15). It remains to show that these relations are defining relations.

(ii) $\mathrm{GK}\,\big(C_A(H)\big) = 4$: Let $\mathcal{D}$ be the subalgebra of $C_A(H)$ generated by the elements C, H, t and Φ. Then $\mathcal{D} = \mathbb{K}[C, H] \otimes \mathbb{K}[t][\Phi; \delta]$ is a tensor product of algebras where δ is the $\mathbb{K}$-derivation of the algebra $\mathbb{K}[t]$ defined by $\delta(t) = t^2$. Clearly, $\mathcal{D}$ is a Noetherian domain of Gelfand-Kirillov dimension 4. Now, the inclusions $\mathcal{D} \subseteq C_A(H) \subseteq C_{A_X}(H)$ yield the inequalities $4 = \mathrm{GK}\,(\mathcal{D}) \leqslant \mathrm{GK}\,\big(C_A(H)\big) \leqslant \mathrm{GK}\,\big(C_{A_X}(H)\big) = 4$ (see Lemma 4.11.(2)). Hence, $\mathrm{GK}\,\big(C_A(H)\big) = 4$.

Let $\mathcal{C}$ be the $\mathbb{K}$-algebra generated by the symbols C, H, t, Φ and Θ subject to the defining relations (4.12)–(4.15) with C and H central in $\mathcal{C}$.

(iii) $\mathrm{GK}\,(\mathcal{C}) = 4$: There is a natural epimorphism of algebras $f : \mathcal{C} \twoheadrightarrow C_A(H)$. Our aim is to show that f is an algebra isomorphism. Let $\mathcal{C}_t$ be the localization of $\mathcal{C}$ at the powers of the element t. Then by (4.15), we see that $\mathcal{C}_t \simeq \mathcal{D}_t = \mathbb{K}[C, H] \otimes \mathbb{K}[t^{\pm 1}][\Phi; \delta]$ where $\mathcal{D} = \mathbb{K}[C, H] \otimes \mathbb{K}[t][\Phi; \delta]$ is a subalgebra of $\mathcal{C}$. Hence, $\mathrm{GK}\,(\mathcal{C}_t) = 4$. Now, the inclusions $\mathcal{D} \subseteq \mathcal{C} \subseteq \mathcal{C}_t$ yield that $4 = \mathrm{GK}\,(\mathcal{D}) \leqslant \mathrm{GK}\,(\mathcal{C}) \leqslant \mathrm{GK}\,(\mathcal{C}_t) = 4$. Hence, $\mathrm{GK}\,(\mathcal{C}) = 4$.

(iv) *The algebra $\mathcal{C}$ is a domain*: Let $\mathcal{E}$ be the algebra generated by the symbols C, H, t, Φ and Θ subject to the defining relations (4.12)–(4.14) with C and H central in $\mathcal{E}$. Then $\mathcal{E}$ is an Ore extension

$$\mathcal{E} = \mathbb{K}[C, H, t][\Phi; \delta][\Theta; \sigma, \delta'] = \mathcal{D}[\Theta; \sigma, \delta'] \tag{4.16}$$

where σ is the automorphism of the algebra $\mathcal{D}$ defined by $\sigma(C) = C, \sigma(H) = H, \sigma(t) = t$ and $\sigma(\Phi) = \Phi + 2t$; δ' is the σ-derivation of the algebra $\mathcal{D}$ given by the rule: $\delta'(C) = \delta'(H) = 0$, $\delta'(t) = 2\Phi - (H+2)t - C$ and $\delta'(\Phi) = (H+4)\Phi - 2(H+2)t - 2C$. In particular, $\mathcal{E}$ is a Noetherian domain. Let $Z := \Theta t^2 - \Phi(\Phi - Ht - C)$. Then Z is a central element of the algebra $\mathcal{E}$. Clearly, $\mathcal{C} \simeq \mathcal{E}/(Z)$. To prove that $\mathcal{C}$ is a domain, it suffices to show that the ideal (Z) of $\mathcal{E}$ is a completely prime ideal. Let $\mathcal{E}_t$ be the localization of the algebra $\mathcal{E}$ at the powers of the element t. Then $\mathcal{E}_t \simeq \mathbb{K}[C, H, Z, t^{\pm 1}][\Phi; \delta] = \mathbb{K}[C, H, Z] \otimes \mathbb{K}[t^{\pm 1}][\Phi; \delta]$ is a tensor product of algebras where δ is a derivation of the algebra $\mathbb{K}[t^{\pm 1}]$ such that $\delta(t) = t^2$. Hence, $\mathcal{E}_t/(Z)_t \simeq \mathbb{K}[C, H] \otimes \mathbb{K}[t^{\pm 1}][\Phi; \delta]$ is a domain.

Claim 1: If $tu \in (Z)$ *for some* $u \in \mathcal{E}$*, then* $u \in (Z)$.

Proof of Claim 1: Recall that $\mathcal{E} = \mathcal{D}[\Theta; \sigma, \delta']$ (see (4.16)), t is a normal element of the algebra $\mathcal{D}$, $Z = t^2\Theta + \xi$ is a central element of $\mathcal{E}$ where $\xi := (H+4)t\Phi - (H+2)t^2 - 2Ct + C\Phi - \Phi^2 \in \mathcal{D}$. Notice that $tu = Zv$ for some element $v \in \mathcal{E}$. To prove Claim 1, we use induction on the degree $m = \deg_\Theta(u)$ of the element $u \in \mathcal{E}$. Since $\mathcal{E}$ is a domain, $\deg_\Theta(fg) = \deg_\Theta(f) + \deg_\Theta(g)$ for all $f, g \in \mathcal{E}$. The case when $m \leqslant 0$, i.e., $u \in \mathcal{D}$ is obvious. So, we may assume that $m \geqslant 1$. The element u can be written as $u = u_0 + u_1\Theta + \cdots + u_m\Theta^m$ where $u_i \in \mathcal{D}$ and $u_m \neq 0$. The equality $tu = Zv$ implies that $\deg_\Theta(v) = m - 1$, since $\deg_\Theta(Z) = 1$. Therefore, $v = v_0 + v_1\Theta + \cdots + v_{m-1}\Theta^{m-1}$ for some $v_i \in \mathcal{D}$ and $v_{m-1} \neq 0$. Then

$$\begin{aligned} tu_0 + tu_1\Theta + \cdots + tu_m\Theta^m &= (t^2\Theta + \xi)(v_0 + v_1\Theta + \cdots + v_{m-1}\Theta^{m-1}) \\ &= t^2\Big(\sigma(v_0)\Theta + \delta'(v_0)\Big) + t^2\Big(\sigma(v_1)\Theta + \delta'(v_1)\Big)\Theta + \cdots + t^2\Big(\sigma(v_{m-1})\Theta + \delta'(v_{m-1})\Big)\Theta^{m-1} \\ &\quad + \xi v_0 + \xi v_1\Theta + \cdots + \xi v_{m-1}\Theta^{m-1} \\ &= t^2\delta'(v_0) + \xi v_0 + \Big(t^2\sigma(v_0) + t^2\delta'(v_1) + \xi v_1\Big)\Theta + \cdots + t^2\sigma(v_{m-1})\Theta^m. \end{aligned}$$

Comparing the terms of degree zero we have the equality $tu_0 = t^2\delta'(v_0) + \xi v_0$, i.e.,

$$t\Big(u_0 - t\delta'(v_0) - (H+4)\Phi v_0 + (H+2)tv_0 + 2Cv_0\Big) = \Phi(C - \Phi)v_0.$$

All terms in the equality belong to the algebra $\mathcal{D}$. Since t is a normal element of the algebra $\mathcal{D}$ such that $\mathcal{D}/\mathcal{D}t \simeq \mathbb{K}[C, H, \Phi]$ is a domain and the elements Φ and $C - \Phi$ do not belong to the ideal $\mathcal{D}t$, we have $v_0 \in \mathcal{D}t$, i.e., $v_0 = tw_0$ for some $w_0 \in \mathcal{D}$. Now, the element v can be written as $v = tw_0 + v'\Theta$ where $v' = 0$ if $m = 1$, and $\deg_\Theta(v') = m - 2$ if $m \geqslant 2$. Then $tu = Z(tw_0 + v'\Theta)$ and so $t(u - Zw_0) = Zv'\Theta$. Hence, $u - Zw_0 = u'\Theta$ for some $u' \in \mathcal{E}$ with $\deg_\Theta(u') < \deg_\Theta(u)$. Now, $tu'\Theta = Zv'\Theta$, hence $tu' = Zv' \in (Z)$ (by deleting Θ). By induction, $u' \in (Z)$, and then $u = Zw_0 + u'\Theta \in (Z)$. This completes the proof of the Claim 1.

Claim 2: $\mathcal{E} \cap (Z)_t = (Z)$.

Proof of Claim 2: Clearly, $(Z) \subseteq \mathcal{E} \cap (Z)_t$. It remains to establish the reverse inclusion. Let $u \in \mathcal{E} \cap (Z)_t$. Then $t^i u \in (Z)$ for some $i \in \mathbb{N}$. Then by the Claim 1, $u \in (Z)$. Hence, $\mathcal{E} \cap (Z)_t = (Z)$.

By Claim 2, the algebra $\mathcal{E}/(Z)$ is a subalgebra of $\mathcal{E}_t/(Z)_t$. So, $\mathcal{E}/(Z)$ is a domain. In particular, the algebra $\mathcal{C} \simeq \mathcal{E}/(Z)$ is a Noetherian domain.

(v) $\mathcal{C} \simeq C_A(H)$: Since $\mathrm{GK}\,(\mathcal{C}) = \mathrm{GK}\,\big(C_A(H)\big) = 4$ and the algebra $\mathcal{C}$ is a domain. The algebra epimorphism $f : \mathcal{C} \twoheadrightarrow C_A(H)$ must be an isomorphism, i.e., $\mathcal{C} \simeq C_A(H)$, by Proposition 2.12. This means that the relations (4.12)–(4.15) (together with the condition that C and H are central elements) are defining relations of the algebra $C_A(H)$. By Lemma 4.11.(2), $Z\big(C_A(H)\big) = \mathbb{K}[C, H]$. □

The Weyl algebra $A_1' = \mathbb{K}[h][t, e; \sigma, a = h]$ is a GWA where $\sigma(h) = h - 1$ and $h := et$. So, $A_1' = \bigoplus_{i \in \mathbb{Z}} A_{1,i}'$ is a $\mathbb{Z}$-graded algebra where $A_{1,0}' = \mathbb{K}[h]$ is a polynomial algebra in h and, for $i \geqslant 1$,

$A'_{1,\pm i} = \mathbb{K}[h]v_{\pm i}$ where $v_i = t^i$, $v_{-i} = e^i$ and $v_0 = 1$. The algebra $C_{A_X}(H) = \bigoplus_{i\in\mathbb{Z}} C_{A_X}(H)_i$ is a $\mathbb{Z}$-graded algebra where $C_{A_X}(H)_i = \mathbb{K}[C, H] \otimes A'_{1,i}$.

By Lemma 4.11, the algebra $C_A(H)$ is a subalgebra of $C_{A_X}(H) = \mathbb{K}[C, H] \otimes A'_1$ where

$$\Phi = C + Ht + et^2 = C + (h + H)t, \tag{4.17}$$

$$\Theta = FE = FX^2 \cdot EX^{-2} = \Phi e = Ce + (h + H)(h - 1), \tag{4.18}$$

since $et = h$ and $te = h - 1$.

In order to prove Proposition 4.16, we need to change the generators of the algebra (we replace Φ by $\phi = ht$).

Corollary 4.15. *Let* $\phi := EY^2$. *Then* $\phi = et^2 = ht$ *and the algebra* $C_A(H)$ *is generated by the elements* C, H, t, ϕ *and* Θ *subject to the defining relations*

$$[\phi, t] = t^2, \tag{4.19}$$

$$[\Theta, t] = 2\phi + (H - 2)t + C, \tag{4.20}$$

$$[\Theta, \phi] = 2\Theta t + (-\phi + 2t)H, \tag{4.21}$$

$$\Theta t^2 = (\phi + Ht + C)\phi. \tag{4.22}$$

Proof. Since $\phi = EX^{-2}X^2Y^2 = et^2 = ht = \Phi - C - Ht$, the algebra $C_A(H)$ is generated by the elements C, H, t, ϕ and Θ. It is routine to check that the defining relations (4.12)–(4.15) can be written as (4.19)–(4.22), respectively. □

By (4.18), for all $n \geqslant 1$, $\Theta^n = \sum_{i=0}^{n} \Theta_{n,i} e^i$ for some $\Theta_{n,i} \in \mathbb{K}[C, H, h]$ with $\deg_h \Theta_{n,i} = 2(n - i)$. Moreover, $\Theta_{n,n} = C^n$ and $\Theta_{n,0} = (h + H)^n(h - 1)^n$. For all $n \geqslant 1$,

$$\phi^n = \phi_n t^n \quad \text{where } \phi_n := h(h - 1)\cdots(h - n + 1). \tag{4.23}$$

For all $i \geqslant 1$ and $j \geqslant 0$,

$$\Theta^i\phi^j = \sum_{s=0}^{i} \Theta_{i,s}\sigma^{-s}(\phi_j)e^s t^j = \sum_{s=0}^{i} \Theta_{i,s}\sigma^{-s}(\phi_j)(-s, j)v_{-s+j} = \sum_{s=0}^{i} P_{i,j,s}v_{-s+j}$$

where $(-s, j) = h(h + 1)\cdots(h + s - 1)$ for $1 \leqslant s \leqslant j$; $(-s, j) = (h + s - 1)\cdots(h + s - j)$ for all $s \geqslant j$; and $(0, j) := 1$; $P_{i,j,s} \in \mathbb{K}[C, H, h]$ with

$$\deg_h P_{i,j,s} = 2(i - s) + j + \min(s, j) = (2i + j) - 2s + \min(s, j) \leqslant 2i + j$$

and $\deg_h P_{i,j,s} = 2i + j$ iff $s = 0$. For all $i \geqslant 1$ and $j \geqslant 0$,

$$\Theta^i\phi^j t = \sum_{s=0}^{i} \Theta_{i,s}\sigma^{-s}(\phi_j)e^s t^{j+1} = \sum_{s=0}^{i} \Theta_{i,s}\sigma^{-s}(\phi_j)(-s, j + 1)v_{-s+j+1} = \sum_{s=0}^{i} Q_{i,j,s}v_{-s+j+1}$$

where $Q_{i,j,s} \in \mathbb{K}[C, H, h]$ with

$$\deg_h Q_{i,j,s} = 2(i-s) + j + \min(s, j+1) = 2i + j - 2s + \min(s, j+1) \leqslant 2i + j$$

and $\deg_h Q_{i,j,s} = 2i + j$ iff $s = 0$.

Proposition 4.16. *The algebra $C_A(H)$ is a free module over its centre. Furthermore, the set $B(H) := \{\Theta^i \phi^j t^k, \phi^l t^m \,|\, i \geqslant 1, k = 0, 1$ and $j, l, m \in \mathbb{N}\}$ is a free basis of the $Z\big(C_A(H)\big)$-module $C_A(H)$.*

Proof. Let M be a free semigroup generated by the symbols Θ and ϕ, i.e., M is the set of all words in letters Θ and ϕ. Let a be an element of $C_A(H)$. By (4.19) and (4.20), the element a is a linear combination of the elements $m' t^k C^l H^m$ where $m' \in M$ and $k, l, m \in \mathbb{N}$. By (4.21), the element a is a linear combination of the elements $\phi^i \Theta^j t^k C^l H^m$ where $i, j, k, l, m \in \mathbb{N}$. Using the induction on the degree $\deg_\Theta$ with respect to the variable Θ (i.e., $\deg_\Theta(\Theta) = 1$ and $\deg_\Theta(\phi) = \deg_\Theta(t) = \deg_\Theta(C) = \deg_\Theta(H) = 0$) and the relation (4.22), i.e., $\Theta t^2 = (\phi + Ht + C)\phi$, and the relations (4.19)–(4.21), it follows that the element a is a linear combination of the elements $bC^l H^m$ where $b \in B(H)$.

To finish the proof of the proposition it suffices to show that the elements of the set $B(H)$ are $\mathbb{K}(C, H)$-linearly independent in the algebra $\mathbb{K}(C, H) \otimes A_1'$ (since $C_A(H) \subseteq \mathbb{K}[C, H] \otimes A_1' \subseteq \mathbb{K}(C, H) \otimes A_1'$ where $\mathbb{K}(C, H)$ is the field of fractions of the polynomial algebra $\mathbb{K}[C, H]$). Let $\mathcal{F} := \mathbb{K}(C, H)$. Then the algebra $\mathcal{F} \otimes A_1'$ is the Weyl algebra $A_1'(\mathcal{F})$ over the field $\mathcal{F}$. By (4.19) and the equality $\phi = ht$ (Corollary 4.15), the $\mathcal{F}$-subalgebra of $A_1'(\mathcal{F})$ generated by the elements t and ϕ is equal to $\mathcal{F}[t][\phi; t^2 \frac{d}{dt}]$. Therefore, the elements $\{\phi^l t^m \,|\, l, m \in \mathbb{N}\}$ are $\mathcal{F}$-linearly independent.

Suppose that the elements of the set $B(H)$ are linearly dependent over the field $\mathcal{F}$. Fix a non-trivial linear combinations,

$$L := \sum_{i \geqslant 1, j \geqslant 0} \Theta^i \phi^j (\lambda_{ij} + \mu_{ij} t) + \sum_{k, l \geqslant 0} \gamma_{kl} \phi^k t^l$$

where $\lambda_{ij}, \mu_{ij}, \gamma_{kl} \in \mathcal{F}$. Then necessarily one of the elements $\lambda_{ij} + \mu_{ij} t$ is nonzero. We seek a contradiction. Let $N := \max\{2i + j \,|\, \lambda_{ij} + \mu_{ij} t \neq 0\}$. Then $N \geqslant 2$. Let $j_0 = \min\{j \,|\, 2i + j = N, \lambda_{ij} + \mu_{ij} t \neq 0\}$. Then either $\lambda_{i_0, j_0} \neq 0$ or $\mu_{i_0, j_0} \neq 0$ (or both) where $i_0 = \frac{1}{2}(N - j_0)$.

Notice that $L = \sum L_i v_i$ for some elements $L_i \in \mathcal{F}[h]$. Suppose that $\lambda_{i_0, j_0} \neq 0$. Then $L_{j_0} = \lambda_{i_0, j_0} P_{i_0, j_0, 0} + \alpha$ where $\alpha \in \mathcal{F}[h]$ with $\deg_h \alpha < N$ (since $\phi^k t^l = h(h-1) \cdots (h-k+1) t^{k+l}$ and $\deg_h h(h-1) \cdots (h-k+1) = k \leqslant k+l$, $\deg_h P_{i_0, j_0, 0} = 2i_0 + j_0 = N > j_0$ as $i_0 \geqslant 1$). Therefore, $\lambda_{i_0, j_0} = 0$, a contradiction. Similarly, if $\mu_{i_0, j_0} \neq 0$. Then $L_{j_0+1} = \mu_{i_0, j_0} Q_{i_0, j_0, 0} + \beta$ where $\beta \in \mathcal{F}[h]$ with $\deg_h \beta < N$ (since $\deg_h Q_{i_0, j_0, 0} = 2i_0 + j_0 = N > j_0 + 1$ as $i_0 \geqslant 1$). Therefore, $\mu_{i_0, j_0} = 0$, a contradiction. The proof of the proposition is complete. □

The algebras $C^{\lambda,\mu}$. For elements $\lambda, \mu \in \mathbb{K}$, let $C^{\lambda,\mu} := C_A^{\lambda,\mu}(H) := C_A(H)/(C - \lambda, H - \mu)$. By Theorem 4.14 and Corollary 4.15, the algebra $C^{\lambda,\mu}$ is generated by the images of the elements

$\{\Phi, \Theta, t\}$ or $\{\phi, \Theta, t\}$ in $C^{\lambda,\mu}$. For reasons of simplicity, we denote their images by the same letters. By Lemma 4.11.(2),

$$C^{\lambda,\mu}_{A_X} := C^{\lambda,\mu}_{A_X}(H) := C_{A_X}(H)/(C-\lambda, H-\mu) \simeq A'_1.$$

So, there is a natural algebra homomorphism $C^{\lambda,\mu} \to C^{\lambda,\mu}_{A_X} = A'_1$. The following proposition shows that the homomorphism is a monomorphism. We will identify the algebra $C^{\lambda,\mu}$ with its image in the Weyl algebra A'_1. This observation enable us to give a complete classification of simple $C^{\lambda,\mu}$-modules for all λ and $\mu \in \mathbb{K}$.

Proposition 4.17. *Let* $\lambda, \mu \in \mathbb{K}$. *Then*

1. *The algebra* $C^{\lambda,\mu}$ *is generated by the elements* ϕ, Θ *and* t *subject to the defining relations*

$$[\phi, t] = t^2, \tag{4.24}$$
$$[\Theta, t] = 2\phi + (\mu-2)t + \lambda, \tag{4.25}$$
$$[\Theta, \phi] = 2\Theta t + (-\phi + 2t)\mu, \tag{4.26}$$
$$\Theta t^2 = (\phi + \mu t + \lambda)\phi. \tag{4.27}$$

2. *The set* $B^{\lambda,\mu} = \{\Theta^i\phi^j t^k, \phi^l t^m \mid i \geqslant 1, k = 0,1 \text{ and } j,l,m \in \mathbb{N}\}$ *is a* $\mathbb{K}$*-basis for the algebra* $C^{\lambda,\mu}$.
3. *The algebra homomorphism*

$$C^{\lambda,\mu} \longrightarrow C^{\lambda,\mu}_{A_X} = A'_1, \quad t \mapsto t, \;\; \phi \mapsto ht, \;\; \Theta \mapsto \lambda e + (h+\mu)(h-1),$$

   is a monomorphism.
4. *The ideal* $(C-\lambda, H-\mu)$ *of the algebra* $C_A(H)$ *is equal to the intersection of* $C_A(H)$ *and the ideal* $(C-\lambda, H-\mu)$ *of the algebra* $C_{A_X}(H)$.
5. $\mathrm{GK}\,(C^{\lambda,\mu}) = 2$ *and* $Z(C^{\lambda,\mu}) = \mathbb{K}$.

Proof. 1. Statement 1 follows from Corollary 4.15.

2 and 3. By repeating the proof of Proposition 4.16 (where the elements C and H are replaced by λ and μ, respectively), we have that the elements of $B^{\lambda,\mu}$ span the vector space $C^{\lambda,\mu}$. Let $\overline{C}^{\lambda,\mu}$ be the image of the algebra $C^{\lambda,\mu}$ in A'_1 and $\overline{B}^{\lambda,\mu}$ be the image of the set $B^{\lambda,\mu}$ in A'_1. The set $\overline{B}^{\lambda,\mu}$ spans $\overline{C}^{\lambda,\mu}$. By repeating the proof of Proposition 4.16 (where the elements C and H are replaced by λ and μ, respectively), we have that the set $\overline{B}^{\lambda,\mu}$ is a $\mathbb{K}$-basis for the algebra $\overline{C}^{\lambda,\mu}$. Now, statements 2 and 3 follows.

4. Statement 4 follows from statement 3.

5. By statement 3, the subalgebra J of $C^{\lambda,\mu}$ generated by the elements t and ϕ is isomorphic to the algebra $\mathbb{K}[t][\phi; t^2\frac{d}{dt}]$. The inclusions $J \subseteq C^{\lambda,\mu} \subseteq A'_1$ yield the inequalities $2 = \mathrm{GK}\,(J) \leqslant \mathrm{GK}\,(C^{\lambda,\mu}) \leqslant \mathrm{GK}\,(A'_1) = 2$, i.e., $\mathrm{GK}\,(C^{\lambda,\mu}) = 2$. Notice that the centralizers of the elements t and $\phi = ht$ in the Weyl algebra A'_1 are $\mathbb{K}[t]$ and $\mathbb{K}[\phi]$, respectively. Therefore, $\mathbb{K} \subseteq Z(C^{\lambda,\mu}) = \mathbb{K}[t] \cap \mathbb{K}[\phi] = \mathbb{K}$, i.e., $Z(C^{\lambda,\mu}) = \mathbb{K}$. □

By Proposition 4.17.(1,3), we have the inclusions of algebras

$$C^{\lambda,\mu} \subset A_1' \subset A_{1,t}' = C_t^{\lambda,\mu} \tag{4.28}$$

where $A_{1,t}'$ and $C_t^{\lambda,\mu}$ are localizations of the algebras A_1' and $C^{\lambda,\mu}$ at the powers of the element t.

The Weyl algebra A_1' has a standard ascending filtration $\{A_{1,i}'\}_{i\in\mathbb{N}}$ by the total degree of the variables e and t ($\deg(e^i t^j) = i + j$ for all $i, j \geqslant 0$). The associated graded algebra $\mathrm{gr}A_1'$ is a polynomial algebra $\mathbb{K}[e,t]$, by abusing the notation. The subalgebra $C^{\lambda,\mu}$ of A_1' has the induced filtration $\{C^{\lambda,\mu} \cap A_{1,i}'\}_{i\in\mathbb{N}}$. Therefore, the associated graded algebra $\mathrm{gr}(C^{\lambda,\mu})$ is a subalgebra of the polynomial algebra $\mathrm{gr}(A_1')$. The elements t, ϕ and Θ have total degrees $1, 3$ and 4, respectively; and their images in $\mathrm{gr}(C^{\lambda,\mu})$ are t, et^2 and e^2t^2, respectively.

Now, let us consider $C^{\lambda,\mu}$ as an abstract algebra and equip it with the degree filtration $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ where $\deg(t) = 1, \deg(\phi) = 3$ and $\deg(\Theta) = 4$. By (4.24)–(4.27), the associated graded algebra $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$ is a commutative algebra which is an epimorphic image of the factor algebra $\mathbb{K}[t,\phi,\Theta]/(\Theta t^2 - \phi^2)$. So, by abusing the notation, the algebra $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$ is generated by (the images of) the elements t, ϕ and Θ that commute (see (4.24)–(4.26)) and satisfy the relation $\Theta t^2 = \phi^2$, see (4.27).

Lemma 4.18. *1. For all $i \in \mathbb{N}$, $\mathcal{F}_i = C^{\lambda,\mu} \cap A_{1,i}'$.*

2. $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu}) = \mathbb{K}[t,\phi,\Theta]/(\Theta t^2 - \phi^2)$, $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu}) = \mathrm{gr}(C^{\lambda,\mu}) \subset \mathrm{gr}(A_1') = \mathbb{K}[t,e]$ where $\phi = et^2$ and $\Theta = e^2t^2$ as elements of $\mathbb{K}[e,t]$.

3. The algebra $\mathrm{gr}(A_1')$ is not a finitely generated $\mathrm{gr}(C^{\lambda,\mu})$-module.

4. The algebra A_1' is not a finitely generated left/right $C^{\lambda,\mu}$-module.

Proof. 1. By Proposition 4.17.(2), the set $B^{\lambda,\mu}$ is a $\mathbb{K}$-basis of the algebra $C^{\lambda,\mu}$. We keep the notation as above. Since $\mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$ is a commutative algebra, each vector space $\mathcal{F}_n$ is a linear span of elements of $B^{\lambda,\mu}$ with degrees $\leqslant n$ ($\deg(\Theta^i\phi^j t^k) = 4i + 3j + k$ and $\deg(\phi^l t^m) = 3l + m$). Then, also each vector space $C^{\lambda,\mu} \cap A_{1,n}'$ is a linear space of elements of $B^{\lambda,\mu}$ with total degree $\leqslant n$ ($\deg(\Theta^i\phi^j t^k) = 4i + 3j + k$ and $\deg(\phi^l t^m) = 3l + m$). Therefore, $\mathcal{F}_n = C^{\lambda,\mu} \cap A_{1,n}'$ for all $n \geqslant 0$.

2. The set $B^{\lambda,\mu}$ is a $\mathbb{K}$-basis of the factor algebra $\Lambda = \mathbb{K}[t,\phi,\Theta]/(\Theta t^2 - \phi^2)$. Therefore, the algebra epimorphism $\Lambda \to \mathrm{gr}_{\mathcal{F}}(C^{\lambda,\mu})$ is an isomorphism. Now, statement 2 follows from statement 1.

3. By statement 2, $\mathrm{gr}(C^{\lambda,\mu}) \subseteq D = \mathbb{K}[t, et]$. Since the polynomial algebra $\mathbb{K}[t,e]$ is not a finitely generated D-module, it is not a finitely generated $\mathrm{gr}(C^{\lambda,\mu})$-module.

4. Statement 4 follows from statement 3. □

4.4 Classification of simple $C_A(H)$-modules

In this section, $\mathbb{K}$ is an algebraically closed field. A classification of simple $C_A(H)$-modules is given. This classification is used in a classification of simple weight A-modules which is obtained in Section 4.5. Two cases where the element C acts as zero or nonzero are very different cases, they are dealt with separately with different techniques. For an algebra $\mathcal{A}$, we denote by $\widehat{\mathcal{A}}$ the set of isomorphism classes of its simple modules. Clearly,

$$\widehat{C_A(H)} = \bigsqcup_{\lambda,\mu\in\mathbb{K}} \widehat{C^{\lambda,\mu}}. \tag{4.29}$$

The simple $C^{\lambda,\mu}$-module $M^{\lambda,\mu}$ (where $\lambda \neq 0$). Suppose that $\lambda \neq 0$ and μ is arbitrary. By Proposition 4.17.(3), $C^{\lambda,\mu}$ is a subalgebra of the Weyl algebra A_1' where $\phi = ht$ and $\Theta = \lambda e + (h+\mu)(h-1)$. The A_1'-module $M := A_1'/A_1't = \mathbb{K}[e]\bar{1}$ is a free $\mathbb{K}[e]$-module of rank 1 where $\bar{1} := 1 + A_1't$. The A_1'-module M is simple and can be identified with the algebra $\mathbb{K}[e]$ as a vector space. Then the element t acts on M as $-\frac{d}{de}$. The concept of $\deg_e$ is well-defined for $M \simeq \mathbb{K}[e]$. Since $\Theta \cdot e^i\bar{1} = \lambda e^{i+1}\bar{1} + \cdots$ for all $i \geqslant 0$ (where the three dots denote a polynomial of degree $< i+1$) and t acts on M as $-\frac{d}{de}$, the $C^{\lambda,\mu}$-module M is *simple*. We denote it by $M^{\lambda,\mu}$.

Lemma 4.19. *Let $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$. Then*

1. *The $C^{\lambda,\mu}$-module $M^{\lambda,\mu}$ is a simple module of* GK *dimension 1, $M^{\lambda,\mu} \simeq C^{\lambda,\mu}/C^{\lambda,\mu}(t,\phi)$ and the map $t : M^{\lambda,\mu} \to M^{\lambda,\mu}$, $m \mapsto tm$, is a surjection.*
2. *$C^{\lambda,\mu} = \mathbb{K}[\Theta] \oplus C^{\lambda,\mu}(t,\phi)$ and $C^{\lambda,\mu}(t,\phi) = \mathbb{K}[\Theta]\phi \oplus C^{\lambda,\mu}t$.*

Proof. We have shown already that the $C^{\lambda,\mu}$-module $M^{\lambda,\mu}$ is simple. It follows from Proposition 4.17.(3) that

$$1 = \mathrm{GK}_{\mathbb{K}[\Theta]}(M^{\lambda,\mu}) \leqslant \mathrm{GK}_{C^{\lambda,\mu}}(M^{\lambda,\mu}) \leqslant \mathrm{GK}_{A_1'}(M^{\lambda,\mu}) = 1,$$

i.e., $\mathrm{GK}_{C^{\lambda,\mu}}(M^{\lambda,\mu}) = 1$. The map $t\cdot = -\frac{d}{de} : M^{\lambda,\mu} \simeq \mathbb{K}[e] \to M^{\lambda,\mu} \simeq \mathbb{K}[e]$ is a surjection. Since $t\bar{1} = 0$ and $\phi\bar{1} = ht\cdot\bar{1} = 0$, there is a natural $C^{\lambda,\mu}$-module epimorphism $C^{\lambda,\mu}/C^{\lambda,\mu}(t,\phi) \twoheadrightarrow M^{\lambda,\mu}$ which is necessarily an isomorphism, by Proposition 4.16. In particular, $C^{\lambda,\mu} = \mathbb{K}[\Theta] \oplus C^{\lambda,\mu}(t,\phi)$. Then $C^{\lambda,\mu}(t,\phi) = \mathbb{K}[\Theta]\phi \oplus C^{\lambda,\mu}t$ (by Proposition 4.16, (4.24) and (4.27)). □

The simple $C^{\lambda,\mu}$-module $N^{\lambda,\mu}$ (where $\lambda \neq 0$). By Lemma 4.19, there is a short exact sequence of $C^{\lambda,\mu}$-modules

$$0 \longrightarrow N^{\lambda,\mu} \longrightarrow C^{\lambda,\mu}/C^{\lambda,\mu}t \longrightarrow M^{\lambda,\mu} \longrightarrow 0 \tag{4.30}$$

where $N^{\lambda,\mu} := C^{\lambda,\mu}(t,\phi)/C^{\lambda,\mu}t = \mathbb{K}[\Theta]\phi\tilde{1}$ and $\tilde{1} = 1 + C^{\lambda,\mu}t$. Clearly, ${}_{\mathbb{K}[\Theta]}N^{\lambda,\mu} \simeq \mathbb{K}[\Theta]$ (Lemma 4.19.(2)), $t\phi\tilde{1} = 0$ and $(\phi+\lambda)\phi\tilde{1} = 0$ (by (4.27)).

Lemma 4.20. *Let $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$. Then the $C^{\lambda,\mu}$-module $N^{\lambda,\mu}$ is a simple module of* GK *dimension 1, $N^{\lambda,\mu} \simeq C^{\lambda,\mu}/C^{\lambda,\mu}(t,\phi+\lambda)$ and the map $t\cdot : N^{\lambda,\mu} \to N^{\lambda,\mu}$, $n \mapsto tn$, is a surjection.*

Proof. Since ${}_{\mathbb{K}[\Theta]}N^{\lambda,\mu} \simeq \mathbb{K}[\Theta]$, the concept of $\deg_\Theta$ of the elements of $N^{\lambda,\mu}$ is well-defined ($\deg_\Theta(\Theta^i\phi\tilde{1}) := i$ for all $i \geqslant 0$). Let us show that, for all $n \geqslant 0$,

$$t \cdot \Theta^n\phi\tilde{1} = \lambda n\Theta^{n-1}\phi\tilde{1} + \cdots, \tag{4.31}$$

$$(\phi+\lambda) \cdot \Theta^n\phi\tilde{1} = -\lambda n(\mu+n-1)\Theta^{n-1}\phi\tilde{1} + \cdots, \tag{4.32}$$

where the three dots means a term of $\deg_\Theta < n-1$. We use induction on n. The case $n = 0$ was proved above ($t\phi\tilde{1} = 0$ and $(\phi+\lambda)\phi\tilde{1} = 0$). Suppose that $n > 0$ and the equalities are true for all $n' < n$. Then

$$\begin{aligned}
t \cdot \Theta^{n+1}\phi\tilde{1} &= \big([t,\Theta] + \Theta t\big)\Theta^n\phi\tilde{1} \\
&= -\Big(2\phi + (\mu-2)t + \lambda\Big)\Theta^n\phi\tilde{1} + \lambda n\Theta^n\phi\tilde{1} + \cdots \\
&= -\Big(-\lambda + 2(\phi+\lambda)\Big)\Theta^n\phi\tilde{1} + \lambda n\Theta^n\phi\tilde{1} + \cdots \\
&= \lambda(n+1)\Theta^n\phi\tilde{1} + \cdots, \\
(\phi+\lambda)\cdot\Theta^{n+1}\phi\tilde{1} &= \Big([\phi+\lambda,\Theta] + \Theta(\phi+\lambda)\Big)\Theta^n\phi\tilde{1} \\
&= -\Big(2\Theta t + (-\phi+2t)\mu\Big)\Theta^n\phi\tilde{1} - \lambda n(\mu+n-1)\Theta^n\phi\tilde{1} + \cdots \\
&= -\big(2\lambda n + \lambda\mu\big)\Theta^n\phi\tilde{1} - \lambda n(\mu+n-1)\Theta^n\phi\tilde{1} + \cdots \\
&= -\lambda(n+1)(\mu+n)\Theta^n\phi\tilde{1} + \cdots.
\end{aligned}$$

By (4.31), the $C^{\lambda,\mu}$-module $N^{\lambda,\mu}$ is simple. By (4.31) and (4.32), $\mathrm{GK}\,(N^{\lambda,\mu}) = 1$. By (4.31), the map $t\cdot : N^{\lambda,\mu} \to N^{\lambda,\mu}$ is a surjection. Finally, by Lemma 4.19.(2),

$$C^{\lambda,\mu} = \mathbb{K}[\Theta] \oplus \mathbb{K}[\Theta]\phi \oplus C^{\lambda,\mu}t = \mathbb{K}[\Theta] \oplus \mathbb{K}[\Theta](\phi+\lambda) \oplus C^{\lambda,\mu}t.$$

Therefore, the canonical $C^{\lambda,\mu}$-module epimorphism $C^{\lambda,\mu}/C^{\lambda,\mu}(t,\phi+\lambda) \to N^{\lambda,\mu}$ must be an isomorphism. □

Corollary 4.21. *The map* $t\cdot : C^{\lambda,\mu}/C^{\lambda,\mu}t \to C^{\lambda,\mu}/C^{\lambda,\mu}t$, $a + C^{\lambda,\mu}t \mapsto ta + C^{\lambda,\mu}t$, *is a surjection provided* $\lambda \neq 0$.

Proof. By Lemma 4.19.(1) and Lemma 4.20, the maps $t\cdot : N^{\lambda,\mu} \to N^{\lambda,\mu}$ and $t\cdot : M^{\lambda,\mu} \to M^{\lambda,\mu}$ are surjections, hence so is the map $t\cdot$ in the lemma, in view of the short exact sequence (4.30). □

Lemma 4.22. *Let R be a ring, $s, r \in R$ and $s_{m,n} : R/Rr^n \to R/Rr^n$, $a + Rr^n \mapsto s^m a + Rr^n$, for $m, n \geqslant 1$. If the map $s_{1,1}$ is a surjection then all the maps $s_{m,n}$ are surjections and $R = s^m R + Rr^n$ for all $m, n \geqslant 1$.*

Proof. If the map $s_{m,n}$ is a surjection then $R = s^m R + Rr^n$. For each $i \geqslant 1$, consider the map $s_i := s\cdot : Rr^i/Rr^{i+1} \to Rr^i/Rr^{i+1}$, $ar^i + Rr^{i+1} \mapsto sar^i + Rr^{i+1}$. In the commutative diagram

$$\begin{array}{ccc}
R/Rr & \xrightarrow{\;s_{1,1}\;} & R/Rr \\
{\scriptstyle \cdot r^i}\downarrow & & \downarrow{\scriptstyle \cdot r^i} \\
Rr^i/Rr^{i+1} & \xrightarrow[\;s_i\;]{} & Rr^i/Rr^{i+1}
\end{array}$$

the vertical maps are surjection. Since the map $s_{1,1}$ is surjective, the map s_i is also surjective. By considering the finite filtration of the abelian group R/Rr^n,

$$0 \subseteq Rr^{n-1}/Rr^n \subseteq Rr^{n-2}/Rr^n \subseteq \ldots \subseteq R/Rr^n,$$

we see that the map $s_{1,n}$ is a surjection. Then so is its powers $(s_{1,n})^m = s_{m,n}$. □

Theorem 4.23. *For all* $\lambda \in \mathbb{K}^*$ *and* $\mu \in \mathbb{K}$*, the algebra* $C^{\lambda,\mu}$ *is a central simple algebra of Gelfand-Kirillov dimension 2.*

Proof. In view of Proposition 4.17.(5), it remains to show that the algebra $C^{\lambda,\mu}$ is simple (where $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$). By Corollary 4.21 and Lemma 4.22 (where $s = r = t$), $C^{\lambda,\mu} = t^n C^{\lambda,\mu} + C^{\lambda,\mu} t^n$. In particular, $C^{\lambda,\mu} = (t^n)$ for all $n \geqslant 1$. Let $\mathfrak{a}$ be a nonzero ideal of the algebra $C^{\lambda,\mu}$. We have to show that $\mathfrak{a} = C^{\lambda,\mu}$. By (4.28), the algebra $C_t^{\lambda,\mu} = A'_{1,t}$ is a simple Noetherian algebra. Therefore, $t^n \in \mathfrak{a}$ for some $n \geqslant 1$, and so $\mathfrak{a} = C^{\lambda,\mu}$, as required. □

Proposition 4.24. *Let* $\lambda \in \mathbb{K}^*$ *and* $\mu \in \mathbb{K}$*. Then, for all nonzero elements* $a \in A'_1$*, the* $C^{\lambda,\mu}$*-module* $A'_1/A'_1 a$ *has finite length but the* $C^{\lambda,\mu}$*-module* A'_1 *has infinite length.*

Proof. The A'_1-module $M = A'_1/A'_1 a = A'_1 \bar{1}$ (where $\bar{1} = 1 + A'_1 a$) admits the standard filtration $\{M_i := A'_{1,i}\bar{1}\}$. Then $\dim(M_i) = e(M)i + s$ for all $i \gg 0$ where $e(M) \in \mathbb{N} \setminus \{0\}$ is the multiplicity of the A'_1-module M and $s \in \mathbb{Z}$. The algebra $C^{\lambda,\mu}$ is simple (since $\lambda \neq 0$). Hence, every simple $C^{\lambda,\mu}$-module has GK dimension 1. Then using a concept of multiplicity of a finitely generated $C^{\lambda,\mu}$-module (see Lemma 4.18.(2)), we must have that the $C^{\lambda,\mu}$-module M has finite length. By Lemma 4.18.(4), the $C^{\lambda,\mu}$-module A'_1 has infinite length. □

Classification of simple $C^{\lambda,\mu}$-modules where $\lambda \neq 0$. The Weyl algebra A'_1 is a subalgebra of the skew Laurent polynomial algebra $B = \mathbb{K}(h)[t, t^{-1}; \sigma]$ where $\sigma(h) = h - 1$. The algebra B is the localization $S^{-1}A'_1$ of the Weyl algebra A'_1 at $S := \mathbb{K}[h] \setminus \{0\}$. The algebra B is a Euclidean ring with left and right division algorithms. In particular, the algebra B is a principle left and right ideal domain. An element $b \in B$ is *irreducible* (or *indecomposable*) if $b = cd$ implies that c or d is invertible. Each simple B-module is isomorphic to B/Bb where b is an irreducible (indecomposable) element of B. B-modules B/Bb and B/Bc are isomorphic iff the elements b and c are *similar*, i.e., there exists an element $d \in B$ such that 1 is the greatest common right divisor of c and d, and bd is the least common left multiple.

Let $\alpha, \beta \in S = \mathbb{K}[h] \setminus \{0\}$. We write $\alpha < \beta$ if there are no roots λ and μ of the polynomials α and β, respectively, such that $\lambda - \mu \in \mathbb{N}$.

Definition. [5]. An element $b = e^m\beta_m + e^{m-1}\beta_{m-1} + \cdots + \beta_0$, where $m > 0$, $\beta_i \in \mathbb{K}[h]$ and $\beta_0, \beta_m \neq 0$, is called *normal* if $\beta_0 < \beta_m$ and $\beta_0 < h$.

The simple modules over the (first) Weyl algebra were classified by Block [18] and later using a different approach with a short proof by Bavula [2, 5]. For a simple A'_1-module M there are two options either $S^{-1}M = 0$ or $S^{-1}M \neq 0$. Accordingly, we say that the simple module is $\mathbb{K}[h]$-*torsion* or $\mathbb{K}[h]$-*torsionfree*, respectively.

Theorem 4.25. *[2, 5].* $\widehat{A'_1} = \widehat{A'_1}\,(\mathbb{K}[h]\text{-torsion}) \sqcup \widehat{A'_1}\,(\mathbb{K}[h]\text{-torsionfree})$ *where*

1. $\widehat{A'_1}\,(\mathbb{K}[h]\text{-torsion}) = \{A'_1/A'_1 t,\ A'_1/A'_1 e,\ A'_1/A'_1(h - \lambda_{\mathcal{O}}) \,|\, \mathcal{O} \in \mathbb{K}/\mathbb{Z} \setminus \{\mathbb{Z}\}\}$ *where* $\lambda_{\mathcal{O}}$ *is any fixed element of* $\mathcal{O} = \lambda_{\mathcal{O}} + \mathbb{Z}$*.*

2. *Each simple* $\mathbb{K}[h]$*-torsionfree* A_1'*-module is isomorphic to* $M_b := A_1'/A_1' \cap Bb$ *for a normal, irreducible element* b. *Simple* A_1'*-modules* M_b *and* $M_{b'}$ *are isomorphic iff the elements* b *and* b' *are similar.*

The following theorem gives a classification of simple $C^{\lambda,\mu}$-modules where $\lambda \neq 0$. It shows that there is a tight connection between the sets of simple $C^{\lambda,\mu}$-modules and A_1'-modules. The theorem gives an explicit construction for each simple $C^{\lambda,\mu}$-module as a factor module $C^{\lambda,\mu}/I$ where I is a left maximal ideal of $C^{\lambda,\mu}$. Let M be an A-module. The sum of all simple submodules of the A-module M is called the *socle* of M, denoted by $\mathrm{soc}_A(M)$. A submodule M' of M is called *essential* if its intersection with any nonzero submodule of M is nonzero. For $C^{\lambda,\mu}$-module M, we denote by $l_{C^{\lambda,\mu}}(M)$ its *length*.

Theorem 4.26. *Let* $\lambda \in \mathbb{K}^*$ *and* $\mu \in \mathbb{K}$. *Then*

1. *The map* $\mathrm{soc} = \mathrm{soc}_{C^{\lambda,\mu}} : \widehat{A_1'} \to \widehat{C^{\lambda,\mu}}$, $[M] \mapsto [\mathrm{soc}_{C^{\lambda,\mu}}(M)]$, *is not a bijection, and* $\widehat{C^{\lambda,\mu}} = \mathrm{soc}(\widehat{A_1'}) \sqcup \{N^{\lambda,\mu}\}$. *Furthermore,*
   (a) *the map* $\mathrm{soc}^{tf} : \widehat{A_1'}\,(t\text{-torsionfree}) \longrightarrow \widehat{C^{\lambda,\mu}}\,(t\text{-torsionfree})$, $[M] \mapsto [\mathrm{soc}_{C^{\lambda,\mu}}(M)]$, *is a bijection, but*
   (b) *the map* $\mathrm{soc}^{tt} : \widehat{A_1'}\,(t\text{-torsion}) = \{A_1'/A_1't\} \longrightarrow \widehat{C^{\lambda,\mu}}\,(t\text{-torsion}) = \{M^{\lambda,\mu}, N^{\lambda,\mu}\}$, $[A_1'/A_1't] \mapsto [M^{\lambda,\mu}]$, *is an injection which is not a bijection. In particular, the simple* $C^{\lambda,\mu}$*-module* $M^{\lambda,\mu}$ *and* $N^{\lambda,\mu}$ *are not isomorphic and the short exact sequence (4.30) splits.*
2. *For each* $[M] \in \widehat{A_1'}\,(\mathbb{K}[h]\text{-torsion})$, *the* $C^{\lambda,\mu}$*-module* M *is simple, i.e.,* $\mathrm{soc}_{C^{\lambda,\mu}}(M) = M$.
3. *For each* $[M] \in \widehat{A_1'}\,(\mathbb{K}[h]\text{-torsionfree})$, *i.e.,* $M = M_b = A_1'/A_1' \cap Bb$ *where* $b \in B$ *is as in Theorem 4.25.(2),* $N_b := C^{\lambda,\mu}/C^{\lambda,\mu} \cap Bb \subseteq M_b$ *and* $\mathrm{soc}_{C^{\lambda,\mu}}(M_b) = \mathrm{soc}_{C^{\lambda,\mu}}(N_b) \simeq N_{bt^{-n}}$ *for all* $n \gg 0$.

Proof. 1. Let M be a simple A_1'-module. By Proposition 4.24, the $C^{\lambda,\mu}$-module M has finite length. In particular, $\mathrm{soc}_{C^{\lambda,\mu}}(M) \neq 0$. *Let us show that* $\mathrm{soc}_{C^{\lambda,\mu}}(M)$ *is a simple* $C^{\lambda,\mu}$*-module.* Let M_t be the localization of the A_1'-module M at the powers of the element t. If $M_t = 0$, i.e., $M \simeq A_1'/A_1't$, then $\mathrm{soc}_{C^{\lambda,\mu}}(A_1'/A_1't) = A_1'/A_1't$ since the $C^{\lambda,\mu}$-module $A_1'/A_1't = M^{\lambda,\mu}$ is simple, as we have seen above. If $M_t \neq 0$, then the A_1'-module M is an essential submodule of M_t. By (4.28), the $C^{\lambda,\mu}$-module M is also an essential $C^{\lambda,\mu}$-submodule of M_t. Therefore, $\mathrm{soc}_{C^{\lambda,\mu}}(M)$ is a simple $C^{\lambda,\mu}$-module. This implies that the map soc is an injection (if simple A_1'-modules M and M' are isomorphic they are also isomorphic as $C^{\lambda,\mu}$-module, and so $\mathrm{soc}_{C^{\lambda,\mu}}(M) \simeq \mathrm{soc}_{C^{\lambda,\mu}}(M')$).

(a) *The map* soc^{tf} *is a bijection*: It remains to show that the map soc^{tf} is a surjection. Let N be a simple t-torsionfree $C^{\lambda,\mu}$-module. Then N_t is a simple $C_t^{\lambda,\mu}$-module which is automatically a simple $A_{1,t}'$-module, by (4.28). Then $N = \mathrm{soc}_{C^{\lambda,\mu}}(N_t) \subseteq M := \mathrm{soc}_{A_1'}(N_t) \subseteq N_t$ and $M \neq 0$ (by Proposition 4.24), and therefore M is a simple t-torsionfree A_1'-module (since M is an essential A_1'-submodule of $M_t = N_t$). Now, $N = \mathrm{soc}_{C^{\lambda,\mu}}(M)$, as we have seen above. So, the map soc^{tf} is a bijection.

(b) The A_1'-module $A_1'/A_1't$ is a simple t-torsion A_1'-module. Hence, $\widehat{A_1'}\,(t\text{-torsion}) = \{A_1'/A_1't\}$. Let us show that $\widehat{C^{\lambda,\mu}}\,(t\text{-torsion}) = \{M^{\lambda,\mu}, N^{\lambda,\mu}\}$. By Lemma 4.19.(1), the $C^{\lambda,\mu}$-module $M^{\lambda,\mu} = A_1'/A_1't$ is simple and t-torsion. By Lemma 4.20 and (4.31), the $C^{\lambda,\mu}$-module $N^{\lambda,\mu}$ is simple and

t-torsion. The $C^{\lambda,\mu}$-modules $M^{\lambda,\mu}$ and $N^{\lambda,\mu}$ are not isomorphic since $M^{\lambda,\mu} = \bigcup_{n\geqslant 1} \ker(\phi^n \cdot)$ and $N^{\lambda,\mu} = \bigcup_{n\geqslant 1} \ker((\phi+\lambda)^n \cdot)$ (by (4.32)). Let M be a simple t-torsion $C^{\lambda,\mu}$-module. It remains to show that either $M \simeq M^{\lambda,\mu}$ or $M \simeq N^{\lambda,\mu}$. The $C^{\lambda,\mu}$-module M is an epimorphic image of the $C^{\lambda,\mu}$-module $C^{\lambda,\mu}/C^{\lambda,\mu}t$. By (4.30), either $M \simeq M^{\lambda,\mu}$ or, otherwise, $M \simeq N^{\lambda,\mu}$. Since both cases do occur the short exact sequence (4.30) splits otherwise the only first case ($M \simeq M^{\lambda,\mu}$) would occur. Now, the statement (b) is obvious. Then, $\widehat{C^{\lambda,\mu}} = \mathrm{soc}(\widehat{A'_1}) \sqcup \{N^{\lambda,\mu}\}$.

2. By Theorem 4.25.(1), there are 3 cases to consider. The first case, i.e., $A'_1/A'_1t = M^{\lambda,\mu}$, is obvious. Let $M = A'_1/A'_1e$. Then $M = \mathbb{K}[t]\bar{1}$ where $\bar{1} := 1 + A'_1e$. If N is a nonzero $C^{\lambda,\mu}$-submodule of M then necessarily $N = f\mathbb{K}[t]\bar{1}$ for some nonzero polynomial $f \in \mathbb{K}[t]$. Since $\dim_{\mathbb{K}}(M/N) < \infty$ and the algebra $C^{\lambda,\mu}$ is a simple infinite dimensional algebra (Theorem 4.23) we must have $N = M$, i.e., the $C^{\lambda,\mu}$-module A'_1/A'_1e is simple.

Finally, let $M = A'_1/A'_1(h-\nu)$ where $\nu = \nu_{\mathcal{O}} \notin \mathbb{Z}$. Then $M = \bigoplus_{i\in\mathbb{Z}} \mathbb{K}v_i\bar{1}$ where $\bar{1} := 1+A'_1(h-\nu)$, $v_0 := 1$ and, for all $i \geqslant 1$, $v_i = t^i$ and $v_{-i} = e^i$. For all $i \geqslant 1$ and $j \in \mathbb{Z}$, $t^i v_j \bar{1} = \lambda_{ij} v_{i+j}$ for some $\lambda_{ij} \in \mathbb{K}^*$. Therefore, the element t acts as a bijection on the module M and $M = \mathbb{K}[t,t^{-1}]\bar{1} \simeq \mathbb{K}[t,t^{-1}]$, as $\mathbb{K}[t,t^{-1}]$-modules. If N is a nonzero submodule of M then $N = gM$ for some nonzero element g of $\mathbb{K}[t,t^{-1}]$. Since $\dim_{\mathbb{K}}(M/N) < \infty$ and the algebra $C^{\lambda,\mu}$ is a simple infinite dimensional algebra (Theorem 4.23) we must have $N = M$. This means that the $C^{\lambda,\mu}$-module M is simple.

3. Let $M = M_b = A'_1\bar{1}$ where $\bar{1} = 1 + A'_1 \cap Bb$. Recall that the $C^{\lambda,\mu}$-module M has finite length and let N be a simple $C^{\lambda,\mu}$-submodule of M (statement 1). Since $h = et$, the A'_1-module M is t-torsionfree, and so $0 \neq N_t \subseteq M_t$. The $C_t^{\lambda,\mu}$-module N_t is also an $A'_{1,t}$-module since $C_t^{\lambda,\mu} = A'_{1,t}$ (see (4.28)). Therefore, $N_t = M_t$, since the $A'_{1,t}$-module M_t is simple and N_t is its nonzero $A'_{1,t}$-submodule. Notice that $N_t = M_t = A'_{1,t}/A'_{1,t}\cap Bb = C_t^{\lambda,\mu}/C_t^{\lambda,\mu}\cap Bb \supseteq C^{\lambda,\mu}/C^{\lambda,\mu}\cap Bb = N_b \neq 0$ and $(N_b)_t = M_t$. Hence, $N = \mathrm{soc}_{C^{\lambda,\mu}}(N_b)$. Clearly, $N_b = C^{\lambda,\mu}\tilde{1}$ where $\tilde{1} := 1 + C^{\lambda,\mu} \cap Bb$. For each $n \in \mathbb{N}$, let $N_n = C^{\lambda,\mu}t^n\tilde{1}$. Then $N_n \neq 0$ since $(N_n)_t = (N_b)_t \neq 0$. Since the $C^{\lambda,\mu}$-module N_b has finite length the descending chain of $C^{\lambda,\mu}$-submodules of N_b, $N_b = N_0 \supseteq N_1 \supseteq \cdots$, stabilizes, say, at m'th step, i.e., $N_b = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_m = N_{m+1} = \ldots$ and $N_m \neq 0$. Since $(M/N)_t = M_t/N_t = M_t/M_t = 0$, we must have $t^n\bar{1} \in N$, i.e., $N = N_n$ for some n. Then necessarily $n \geqslant m$ and $N_m = N$. Now, for all $n \geqslant m$,

$$N = C^{\lambda,\mu}t^n\tilde{1} \simeq \frac{C^{\lambda,\mu}t^n + C^{\lambda,\mu}\cap Bb}{C^{\lambda,\mu}\cap Bb} \simeq \frac{C^{\lambda,\mu}t^n}{C^{\lambda,\mu}t^n \cap Bb} \simeq C^{\lambda,\mu}/C^{\lambda,\mu}\cap Bbt^{-n}. \qquad \square$$

The algebras $C^{0,\mu}$. The subalgebra R of the Weyl algebra A'_1 which is generated by the elements t and $h = et$ is a skew polynomial algebra $R = \mathbb{K}[h][t;\sigma]$ where $\sigma(h) = h - 1$. The algebra R is a homogeneous subalgebra of the $\mathbb{Z}$-graded algebra A'_1, it is the non-negative part of the $\mathbb{Z}$-grading of A'_1. By Proposition 4.17.(3), for all $\mu \in \mathbb{K}$, $C^{0,\mu} \subset R \subset A'_1$ and the subalgebra $C^{0,\mu}$ of R is generated by the elements $t, \phi = ht$ and $\Theta = (h+\mu)(h-1)$. Clearly, $\mathbb{K}[\Theta] \subseteq \mathbb{K}[h]$ and $\mathbb{K}[h] = \mathbb{K}[\Theta] \oplus \mathbb{K}[\Theta]h$. The element t is a normal element of the algebra R and $(t) = \bigoplus_{i\geqslant 1} \mathbb{K}[h]t^i$.

Proposition 4.27. *Let $\mu \in \mathbb{K}$.*

1. *$C^{0,\mu} = \mathbb{K}[\Theta] \oplus \bigoplus_{i\geqslant 1} \mathbb{K}[h]t^i$ and $C^{0,\mu} \cap Rt = Rt = \bigoplus_{i\geqslant 1} \mathbb{K}[h]t^i = (t,\phi) = (t)$ where (t,ϕ) is the ideal of $C^{0,\mu}$ generated by the elements t and ϕ. Furthermore, $(t,\phi) = C^{0,\mu}t + C^{0,\mu}\phi$.*

2. $C^{0,\mu} \subset R \subset R_t = C_t^{0,\mu} = A'_{1,t}$.
3. $\mathrm{Spec}(C^{0,\mu}) = \{0, (t), (t, \mathfrak{m}) \,|\, \mathfrak{m} \in \mathrm{Max}(\mathbb{K}[\Theta])\}$, $C^{0,\mu}/(t) \simeq \mathbb{K}[\Theta]$, *and* $C^{0,\mu}/(t, \mathfrak{m}) \simeq \mathbb{K}[\Theta]/\mathfrak{m}$. *In particular, all prime ideals of* $C^{0,\mu}$ *are completely prime.*
4. $\mathrm{Max}(C^{0,\mu}) = \{(t, \mathfrak{m}) \,|\, \mathfrak{m} \in \mathrm{Max}(\mathbb{K}[\Theta])\}$.

Proof. 1. The equality $(t, \phi) = (t)$ follows from (4.25). Multiplying the equality $\mathbb{K}[h] = \mathbb{K}[\Theta] \oplus \mathbb{K}[\Theta]h$ by the element t on the right yields $\mathbb{K}[h]t = \mathbb{K}[\Theta]t \oplus \mathbb{K}[\Theta]\phi \subseteq C^{0,\mu}$. For all $i \geqslant 1$, $C^{0,\mu} \supseteq (\mathbb{K}[h]t)^i = \mathbb{K}[h]t^i$, and so $C^{0,\mu} = \mathbb{K}[\Theta] \oplus \bigoplus_{i \geqslant 1} \mathbb{K}[h]t^i$ (since $C^{0,\mu} \subseteq R$). Then $C^{0,\mu} \cap Rt = Rt = \bigoplus_{i \geqslant 1} \mathbb{K}[h]t^i = (t, \phi)$. By Proposition 4.16, $(t, \phi) = C^{0,\mu}t + C^{0,\mu}\phi$.

2. Statement 2 follows from statement 1 and (4.28).

3 and 4. The ideal (t) of $C^{0,\mu}$ is a completely prime ideal since $C^{0,\mu}/(t) \simeq \mathbb{K}[\Theta]$, by statement 1. Therefore, the set of prime ideals that properly contain the ideal (t) is $\{(t, \mathfrak{m}) \,|\, \mathfrak{m} \in \mathrm{Max}(\mathbb{K}[\Theta])\}$. Each such an ideal is a completely prime, maximal ideal of $C^{0,\mu}$. The algebra $C^{0,\mu}$ is a domain, so 0 is the completely prime ideal of $C^{0,\mu}$. To finish the proof of statements 3 and 4 it suffices to show that if $\mathfrak{p}$ is a nonzero prime ideal of $C^{0,\mu}$ then $(t) \subseteq \mathfrak{p}$. Recall that $Rt = (t)$, by statement 1. By statement 2, $C_t^{0,\mu} = A'_{1,t}$ is a simple Noetherian domain. Therefore, $t^n \in \mathfrak{p}$. Hence, $\mathfrak{p} \supseteq Rt^n \cdot t = Rt^{n+1} = (Rt)^{n+1}$, and so $\mathfrak{p} \supseteq Rt$ since $\mathfrak{p}$ is a prime ideal and $Rt = (t)$. □

Classification of simple $C^{0,\mu}$-modules. The set $S = \mathbb{K}[h] \setminus \{0\}$ is a (left and right) Ore set of the domain $C^{0,\mu}$ and $B := S^{-1}C^{0,\mu} = \mathbb{K}(h)[t; \sigma]$ is a skew polynomial algebra where $\sigma(h) = h - 1$. The algebra B is a principle (left and right) ideal domain. Let $\mathrm{Irr}\,(B)$ be the set of irreducible elements of B.

In [10], simple modules for an arbitrary Ore extension $D[X; \sigma, \delta]$ are classified where D is a commutative Dedekind domain, σ is an automorphism of D and δ is a σ-derivation of D. The ring $R = \mathbb{K}[h][t; \sigma]$ is a very special case of such an Ore extension.

Theorem 4.28. *1.* $\widehat{R}\,(\mathbb{K}[h]\text{-torsion}) = \widehat{R}\,(t\text{-torsion}) = \widehat{R/(t)} = \big\{[R/R(h - \nu, t)] \,|\, \nu \in \mathbb{K}\big\}$.

2. $\widehat{R}\,(\mathbb{K}[h]\text{-torsionfree}) = \widehat{R}\,(t\text{-torsionfree}) = \{[M_b] \,|\, b \in \mathrm{Irr}(B),\ R = Rt + R \cap Bb\}$ *where* $M_b := R/R \cap Bb$*;* $M_b \simeq M_{b'}$ *iff the elements* b *and* b' *are similar (iff* $B/Bb \simeq B/Bb'$ *as* B*-modules).*

Proof. 1. The last two equalities in statement 1 follow from the fact that t is a normal element of R. Then, clearly, $\widehat{R}\,(\mathbb{K}[h]\text{-torsion}) \supseteq \widehat{R/(t)}$. It remains to show that the reverse inclusion holds. Let M be a simple $\mathbb{K}[h]$-torsion R-module. The field $\mathbb{K}$ is algebraically closed, so the R-module M is an epimorphic image of the R-module $R/R(h - \nu) = \mathbb{K}[t]\bar{1}$ for some $\nu \in \mathbb{K}$ where $\bar{1} = 1 + R(h - \nu)$. It follows from the equalities $ht^i\bar{1} = (\nu + i)t^i\bar{1}$ for all $i \geqslant 0$ that $t\mathbb{K}[t]\bar{1}$ is the only maximal R-submodule of $R/R(h - \nu)$. So, $M \simeq R/R(h - \nu, t) \in \widehat{R/(t)}$, as required.

2. The first equality in statement 1 implies (in fact, is equivalent to) the first equality in statement 2. By [10, Theorem 1.3] $\widehat{R}\,(\mathbb{K}[h]\text{-torsionfree}) = \{[M_b] \,|\, b \in \mathrm{Irr}(B),\ R = Rt + R \cap Bb\}$ (the condition (LO) of [10, Theorem 1.3] is equivalent to the condition $R = Rt + R \cap Bb$). □

Theorem 4.29 gives a classification of simple $C^{0,\mu}$-modules. It shows a close connection between the sets $\widehat{C^{0,\mu}}$ and $\widehat{R}$, they are almost identical, i.e., $\widehat{C^{0,\mu}}\,(t\text{-torsionfree}) = \widehat{R}\,(t\text{-torsionfree})$.

Theorem 4.29.

1. $\widehat{C^{0,\mu}}\,(t\text{-torsion}) = \{[M] \in \widehat{C^{0,\mu}} \,|\, (t)M = 0\} = \widehat{C^{0,\mu}/(t)} = \{[C^{0,\mu}/C^{0,\mu}(\Theta - \nu, t, \phi)] \,|\, \nu \in \mathbb{K}\}$.
2. $\widehat{C^{0,\mu}}\,(t\text{-torsionfree}) = \widehat{R}\,(t\text{-torsionfree}) = \widehat{R}\,(\mathbb{K}[h]\text{-torsionfree}) = \{[M_b = R/R \cap Bb] \,|\, b \in \mathrm{Irr}(B),\ R = Rt + R \cap Bb\}$ *(see Theorem 4.28).*

Proof. 1. The last two equalities are obvious. Clearly, $\widehat{C^{0,\mu}}\,(t\text{-torsion}) \supseteq \widehat{C^{0,\mu}/(t)}$. It remains to show that the reverse inclusion holds. If M is a simple t-torsion $C^{0,\mu}$-module that either $(t)M = 0$ or, otherwise, $(t)M = M$. The second case is impossible since otherwise, $M = (t)M = RtM \in \widehat{R}\,(t\text{-torsionfree})$, a contradiction ($t$ is a normal element of R). So, $(t)M = 0$, as required.

2. In view of Theorem 4.28.(2), it remains to show that the first equality holds. Let $[M] \in \widehat{C^{0,\mu}}\,(t\text{-torsionfree})$. By statement 1, $M = (t)M = RtM \in \widehat{R}\,(t\text{-torsionfree})$. Given $[N] \in \widehat{R}\,(t\text{-torsionfree})$. To finish the proof of statement 2, it suffices to show that N is a simple $C^{0,\mu}$-module. If L is a nonzero $C^{0,\mu}$-submodule of N then $N \supseteq L \supseteq (t)L \neq 0$, since N is t-torsionfree. Then $(t)L = RtL = N$, since N is a simple R-module. Hence, $L = N$, i.e., N is a simple $C^{0,\mu}$-module, as required. □

4.5 A classification of simple weight A-modules

The aim of this section is to give a classification of simple weight A-modules. They are partitioned into several classes of modules which are classified separately using different techniques. The key idea is to use the classification of simple $C_A(H)$-modules. In this section, we assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero. For each coset $\mathcal{O} \in \mathbb{K}/\mathbb{Z}$, we fix a representative $\mu_{\mathcal{O}} \in \mathcal{O} = \mu_{\mathcal{O}} + \mathbb{Z}$. An A-module M is called a *weight module* provided that $M = \bigoplus_{\mu\in\mathbb{K}} M_\mu$ where $M_\mu = \{m \in M \,|\, Hm = \mu m\}$. An element $\mu \in \mathbb{K}$ such that $M_\mu \neq 0$ is called a *weight* of M. Let $\mathrm{Wt}\,(M)$ be the set of all weights of the A-module M. For an algebra A, let $\widehat{A}$ be the set of isomorphism classes of simple A-modules and $\widehat{A}\,(\text{weight})$ be the set of isomorphism classes of simple weight A-modules. Let M be an A-module and $x \in A$. We say that M is *x-torsion* provided that for each element $m \in M$ there exists a natural number $i \in \mathbb{N}$ such that $x^i m = 0$, and that M is *x-torsionfree* if the only element of M annihilated by the element x is 0. Since the set $\{X^i \,|\, i \in \mathbb{N}\}$ is an Ore set in A,

$$\widehat{A}\,(\text{weight}) = \widehat{A}\,(\text{weight}, X\text{-torsion}) \ \sqcup\ \widehat{A}\,(\text{weight}, X\text{-torsionfree}). \tag{4.33}$$

Description of the set $\widehat{A}\,(\text{weight}, X\text{-torsion})$. An explicit description of the set $\widehat{A}\,(\text{weight}, X\text{-torsion})$ is given in Theorem 4.34. Clearly,

$$\widehat{A}\,(\text{weight}, X\text{-torsion}) = \widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsion}) \sqcup \widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsionfree}). \tag{4.34}$$

Lemma 4.30. *Let $M \in \widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsion})$. Then $XM = YM = 0$, i.e.,*

$$\widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsion}) = \widehat{U}\,(\text{weight}).$$

Proof. Let $M \in \widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsion})$. There exists a nonzero weight vector $m \in M$ such that $Xm = 0$ and $Ym = 0$, since $XY = YX$. Notice that $M = Am$ (since M is a simple A-module). So, the A-module M is an epimorphic image of the A-module $A/(AX + AY) = A/(X, Y) = U$, by Lemma 4.2. □

Lemma 4.31. *1. Let $M \in \widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsionfree})$. Then the central element C acts on M as a nonzero scalar C_M.*

2. Let $M \in \widehat{A}\,(\text{weight}, Y\text{-torsion}, X\text{-torsionfree})$. Then the central element C acts on M as a nonzero scalar C_M.

Proof. 1. Since M is a simple A-module, the central element C acts on M as a scalar C_M. It remains to show that $C_M \neq 0$. Suppose this is not the case, then there is a nonzero weight vector $m \in M$ such that $Xm = 0$ and $Cm = 0$. Since $C = FX^2 - (H+2)YX - Y^2E$, we have $Y^2Em = 0$ and so $Em = 0$, since M is Y-torsionfree. Let $m' = Ym$ then $m' \neq 0$ and $Xm' = Em' = 0$. So, the A-module $M' := Am' = \sum_{i,j\geqslant 0} \mathbb{K}F^iY^jm'$ is a proper submodule of the A-module M (since $m \notin M'$). This contradicts to the fact that M is a simple module.

2. Statement 2 follows from statement 1 by applying the automorphism S of A, see (4.2). □

For $\lambda, \mu \in \mathbb{K}$, we define the left A-modules $\mathscr{X}^\mu := A/A(H-\mu, X)$ and $\mathbb{X}^{\lambda,\mu} := A/A(C-\lambda, H-\mu, X)$. Clearly, $\mathbb{X}^{\lambda,\mu} \simeq \mathscr{X}^\mu/(C-\lambda)\mathscr{X}^\mu$. Since $XH = (H-2)X$, using the PBW Theorem we see that $\mathscr{X}^\mu = \bigoplus_{i,j,k\geqslant 0} \mathbb{K}F^iY^jE^k\tilde{1} = \mathbb{K}[F] \otimes \mathcal{V}\tilde{1}$ where $\tilde{1} := 1 + A(H-\mu, X)$ and $\mathcal{V} = \bigoplus_{j,k\geqslant 0} \mathbb{K}Y^jE^k$. It follows from the equalities $[E, Y] = X$ and $X\tilde{1} = 0$ and the fact that the element X commutes with E and Y that $Y^jE^k\tilde{1} = E^kY^j\tilde{1}$. Hence, abusing the notation we can write $\mathcal{V}\tilde{1} = \mathbb{K}[Y, E]\tilde{1}$ where $\mathbb{K}[Y, E]$ is a *polynomial* algebra in letters Y and E. Therefore, $\mathcal{V}\tilde{1} = \Sigma \otimes \mathbb{K}[EY^2]\tilde{1}$ where $\Sigma := \mathbb{K}[Y]Y^2 \oplus \mathbb{K}[E] \oplus Y\mathbb{K}[E]$ and $\mathbb{K}[EY^2]$ is a polynomial in EY^2. Now,

$$\mathscr{X}^\mu = \mathbb{K}[F] \otimes \Sigma \otimes \mathbb{K}[EY^2]\tilde{1} \simeq \mathbb{K}[F] \otimes \Sigma \otimes \mathbb{K}[EY^2]$$

is an isomorphism of vector spaces. Since $C = FX^2 - HYX - EY^2, (C-\lambda)\tilde{1} = -(EY^2+\lambda)\tilde{1}$,

$$(C-\lambda)\mathscr{X}^\mu = \mathbb{K}[F] \otimes \Sigma \otimes \mathbb{K}[EY^2](-EY^2 - \lambda)\tilde{1}.$$

Therefore,

$$\mathbb{X}^{\lambda,\mu} \simeq \mathscr{X}^\mu/(C-\lambda)\mathscr{X}^\mu \simeq \mathbb{K}[F] \otimes \Sigma\bar{1}$$

where $\bar{1} = 1 + A(C-\lambda, H-\mu, X)$, and the equality of statement 1 of the following proposition follows. Furthermore, the proposition shows that for all $\lambda \in \mathbb{K}^*$, the modules $\mathbb{X}^{\lambda,\mu}$ are simple, weight, X-torsion, Y-torsionfree A-modules. Later in Proposition 4.33.(1), we will see that the set $\widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsionfree})$ consists precisely of the modules $\mathbb{X}^{\lambda,\mu}$. Moreover, the $\mathbb{K}$-bases, weight space decompositions and annihilators of $\mathbb{X}^{\lambda,\mu}$ are given.

Proposition 4.32. *Let $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$. Then*

1. *The A-module $\mathbb{X}^{\lambda,\mu} = \bigoplus_{i\geqslant 0, j\geqslant 2} \mathbb{K}F^iY^j\bar{1} \oplus \bigoplus_{i,k\geqslant 0} \mathbb{K}F^iE^k\bar{1} \oplus \bigoplus_{i,k\geqslant 0} \mathbb{K}YF^iE^k\bar{1}$ is a simple module where $\bar{1} = 1 + A(C-\lambda, H-\mu, X)$.*

2. Recall that $\Theta = FE$. *Then*

$$\mathbb{X}^{\lambda,\mu} = \bigoplus_{i\geqslant 0, j\geqslant 2} \mathbb{K}F^iY^j\bar{1} \oplus \Big(\bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}F^i\Theta^k\bar{1} \oplus \bigoplus_{k\geqslant 0}\mathbb{K}\Theta^k\bar{1} \oplus \bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}E^i\Theta^k\bar{1}\Big) \oplus \Big(\bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}YF^i\Theta^k\bar{1} \oplus \bigoplus_{k\geqslant 0}\mathbb{K}Y\Theta^k\bar{1} \oplus \bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}YE^i\Theta^k\bar{1}\Big).$$

3. The weight space $\left(\mathbb{X}^{\lambda,\mu}\right)_{\mu+i}$ *of* $\mathbb{X}^{\lambda,\mu}$ *that corresponds to the weight* $\mu+i$ *(where* $i \in \mathbb{Z}$*) is*

$$\left(\mathbb{X}^{\lambda,\mu}\right)_{\mu+i} = \begin{cases} \mathbb{K}[\Theta]\bar{1}, & i=0,\\ E^r\mathbb{K}[\Theta]\bar{1}, & i=2r,\ r\geqslant 1,\\ YE^r\mathbb{K}[\Theta]\bar{1}, & i=2r-1,\ r\geqslant 1,\\ F^r\mathbb{K}[\Theta]\bar{1}\oplus \bigoplus_{j=0}^{r-1}\mathbb{K}F^jY^{2(r-j)}\bar{1}, & i=-2r,\ r\geqslant 1,\\ Y\mathbb{K}[\Theta]\bar{1}, & i=-1,\\ YF^{r-1}\mathbb{K}[\Theta]\bar{1}\oplus \bigoplus_{j=0}^{r-2}\mathbb{K}F^jY^{2(r-j)-1}\bar{1}, & i=-2(r-1)-1,\ r\geqslant 2. \end{cases}$$

In particular, $\mathrm{Wt}(\mathbb{X}^{\lambda,\mu}) = \{\mu+i \mid i\in\mathbb{Z}\}$ *and each weight space is infinite dimensional.*

4. $\mathrm{ann}_A\left(\mathbb{X}^{\lambda,\mu}\right) = (C-\lambda)$.

5. $\mathbb{X}^{\lambda,\mu}$ *is an* X*-torsion,* Y*-torsionfree weight* A*-module.*

Proof. 1. It remains to show that the A-module $\mathbb{X}^{\lambda,\mu}$ is simple. We use notation as above. Using the definition of C, we have the equality $EY^2\bar{1} = -\lambda\bar{1}$. Then, for all $k\geqslant 1$, $Y^{2k}E^k\bar{1} = (EY^2)^k\bar{1} = (-\lambda)^k\bar{1}$ (since $\mathcal{V}\bar{1} = \mathbb{K}[Y,E]\bar{1}$). Since $EY^2\bar{1} = -\lambda\bar{1}\neq 0$, the map $Y\cdot : \Sigma\bar{1}\to\Sigma\bar{1},\ s\bar{1}\mapsto Ys\bar{1}$, is an injection. Let u be a nonzero element of $\mathbb{X}^{\lambda,\mu}$. To prove that the A-module $\mathbb{X}^{\lambda,\mu}$ is simple it suffices to show that $au=\bar{1}$ for some $a\in A$. It follows from the equalities $XF^i = F^iX - iF^{i-1}Y$, $X\bar{1}=0$ and $\mathbb{X}^{\lambda,\mu} = \mathbb{K}[F]\otimes\Sigma\bar{1}$, that the map $X\cdot : \mathbb{X}^{\lambda,\mu}\to\mathbb{X}^{\lambda,\mu},\ u\mapsto Xu$, acts as $\frac{d}{dF}\otimes(-Y\cdot)_\Sigma$. So, we can assume that $u=s\bar{1}$ where $0\neq s\in\Sigma$.

Notice that $s = pY^2+\sum_{i=0}^m(\lambda_i+\mu_iY)E^i$ for some $p\in\mathbb{K}[Y]$ and $\lambda_i,\mu_i\in\mathbb{K}$. Then

$$\begin{aligned} Y^{2m}u = Y^{2m}s\bar{1} &= \Big(pY^{2m+2}+\sum_{i=0}^m(\lambda_i+\mu_iY)Y^{2(m-i)}Y^{2i}E^i\Big)\bar{1}\\ &= \Big(pY^{2m+2}+\sum_{i=0}^m(\lambda_i+\mu_iY)Y^{2(m-i)}(-\lambda)^i\Big)\bar{1} = f\bar{1} \end{aligned}$$

where $f\in\mathbb{K}[Y]\setminus\{0\}$ (since $s\neq 0$). So, we may assume that $u=f\bar{1}$ where $0\neq f\in\mathbb{K}[Y]$. Let $f=\sum_{i=0}^l\gamma_iY^i$ where $\gamma_i\in\mathbb{K}$ and $\gamma_l\neq 0$. Since $HY^i\bar{1} = (\mu-i)Y^i\bar{1}$ for all i and all the eigenvalues $\{\mu-i\mid i\geqslant 0\}$ are distinct, there is a polynomial $g\in\mathbb{K}[H]$ such that $gf\bar{1}=Y^l\bar{1}$. If $l=0$, we are done. We may assume that $l\geqslant 1$. Multiplying (if necessary) the equality above by Y we may assume that $l=2k$ for some natural number $k\in\mathbb{N}$. Then $(-\lambda)^{-k}E^kY^{2k}\bar{1}=\bar{1}$, as required.

2. Using the fact that the algebra U is a generalized Weyl algebra $U=\mathbb{K}[\Theta,H][E,F;\sigma,a=\Theta]$ where $\sigma(\Theta)=\Theta+H$, $\sigma(H)=H-2$ and the equality $F^iE^i = \Theta\sigma^{-1}(\Theta)\cdots\sigma^{-i+1}(\Theta)$, we see

that

$$\bigoplus_{i,k\geqslant 0} \mathbb{K}F^iE^k\bar{1} = \bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}F^i\Theta^k\bar{1} \oplus \bigoplus_{k\geqslant 0}\mathbb{K}\Theta^k\bar{1} \oplus \bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}E^i\Theta^k\bar{1}.$$

Then statement 2 follows from statement 1.

3. Statement 3 follows from statement 2.

4. It is clear that $(C-\lambda) \subseteq \mathrm{ann}_A(\mathbb{X}^{\lambda,\mu})$. By Proposition 4.4.(1), the ideal $(C-\lambda)$ of A is a maximal ideal, hence $(C-\lambda) = \mathrm{ann}_A(\mathbb{X}^{\lambda,\mu})$.

5. Clearly, $\mathbb{X}^{\lambda,\mu}$ is an X-torsion, weight A-module. By statement 1, $\mathbb{X}^{\lambda,\mu}$ is a simple module, it must be Y-torsionfree (since, otherwise, by Lemma 4.30, $C\mathbb{X}^{\lambda,\mu}=0$, a contradiction). □

The sets $\widehat{A}\,(\text{weight}, X\text{-torsion})$ and $\widehat{A}\,(\text{weight}, Y\text{-torsion})$. For $\lambda, \mu \in \mathbb{K}$, let us consider the A-module $\mathbb{Y}^{\lambda,\mu} := A/A(C-\lambda, H-\mu, Y)$. Then $\mathbb{Y}^{\lambda,\mu} \simeq {}^S\mathbb{X}^{-\lambda,-\mu}$ where ${}^S\mathbb{X}^{-\lambda,-\mu}$ is the A-module $\mathbb{X}^{-\lambda,-\mu}$ twisted by the automorphism S of the algebra A ($S(H)=-H$, $S(C)=-C$, $S(Y)=-X$). The subgroup $\mathbb{Z}$ of $(\mathbb{K},+)$ acts on $\mathbb{K}$ in the obvious way. For each $\lambda\in\mathbb{K}$, $\mathcal{O}(\lambda) := \lambda+\mathbb{Z}$ is the orbit of λ under the action of $\mathbb{Z}$. The set of all $\mathbb{Z}$-orbits can be identified with the elements of the factor group $\mathbb{K}/\mathbb{Z}$. For each orbit $\mathcal{O}\in\mathbb{K}/\mathbb{Z}$, we fix an element $\mu_{\mathcal{O}}\in\mathcal{O}$.

Proposition 4.33. *1. $\widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsionfree}) = \{[\mathbb{X}^{\lambda,\mu_{\mathcal{O}}}] \mid \lambda\in\mathbb{K}^*, \mathcal{O}\in\mathbb{K}/\mathbb{Z}\}$ and the A-modules $\mathbb{X}^{\lambda,\mu_{\mathcal{O}}}$ and $\mathbb{X}^{\lambda',\mu_{\mathcal{O}'}}$ are isomorphic iff $\lambda=\lambda'$ and $\mathcal{O}=\mathcal{O}'$.*

2. $\widehat{A}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsion}) = \{[\mathbb{Y}^{\lambda,\mu_{\mathcal{O}}}] \mid \lambda\in\mathbb{K}^*, \mathcal{O}\in\mathbb{K}/\mathbb{Z}\}$ and the A-modules $\mathbb{Y}^{\lambda,\mu_{\mathcal{O}}}$ and $\mathbb{Y}^{\lambda',\mu_{\mathcal{O}'}}$ are isomorphic iff $\lambda=\lambda'$ and $\mathcal{O}=\mathcal{O}'$.

Proof. 1. Let $M\in\widehat{A}\,(\text{weight}, X\text{-torsion}, Y\text{-torsionfree})$. By Lemma 4.31, $C_M=\lambda\neq 0$ for some $\lambda\in\mathbb{K}^*$. Then M is a factor module of $\mathbb{X}^{\lambda,\mu}$ for some $\mu\in\mathbb{K}$. By Proposition 4.32.(1), the module $\mathbb{X}^{\lambda,\mu}$ is a simple module, hence $M\simeq\mathbb{X}^{\lambda,\mu}$. Clearly, $\mathbb{X}^{\lambda,\mu}\simeq\mathbb{X}^{\lambda',\mu'}$ iff $\lambda=\lambda'$ and $\mu=\mu'+i$ for some $i\in\mathbb{Z}$.

2. Since $\mathbb{Y}^{\lambda,\mu}\simeq{}^S\mathbb{X}^{-\lambda,-\mu}$, statement 2 follows from statement 1. □

The next theorem gives a complete description of simple, weight, X-torsion A-modules and of simple, weight, Y-torsion A-modules.

Theorem 4.34. *1. $\widehat{A}\,(\text{weight}, X\text{-torsion}) = \widehat{U}\,(\text{weight}) \sqcup \{[\mathbb{X}^{\lambda,\mu_{\mathcal{O}}}] \mid \lambda\in\mathbb{K}^*, \mathcal{O}\in\mathbb{K}/\mathbb{Z}\}$.*

2. $\widehat{A}\,(\text{weight}, Y\text{-torsion}) = \widehat{U}\,(\text{weight}) \sqcup \{[\mathbb{Y}^{\lambda,\mu_{\mathcal{O}}}] \mid \lambda\in\mathbb{K}^*, \mathcal{O}\in\mathbb{K}/\mathbb{Z}\}$.

Proof. 1. The theorem follows from the equality (4.34), Lemma 4.30 and Proposition 4.33.

2. Statement 2 follows from statement 1. □

Description of the set $\widehat{A}\,(\text{weight}, X\text{-torsionfree})$. Since the element C belongs to the centre of the algebra A and the field $\mathbb{K}$ is algebraically closed,

$$\widehat{A}\,(\text{weight}) = \bigsqcup_{\lambda\in\mathbb{K}} \widehat{A(\lambda)}\,(\text{weight}) \tag{4.35}$$

where $A(\lambda) := A/(C-\lambda)$. Moreover,

$$\widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}) = \widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree}) \bigsqcup \widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsion}). \quad (4.36)$$

The simple modules in the set $\widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsion})$ are classified by Proposition 4.33.(2). So, in order to finish the classification of simple weight A-modules it remains to describe the set $\widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree})$.

The set $\widehat{A(0)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree})$**.** Let $C_t := C_A(H)_t$ be the localization of the algebra $C_A(H)$ at the powers of the element t. Then by Corollary 4.15, $C_t = \mathbb{K}[C,H] \otimes A'_{1,t}$. Clearly, $C_t = C_{A_t}(H)$. Let $[M] \in \widehat{C^{0,\mu}}\,(t\text{-torsionfree})$. By Theorem 4.29.(2), the element t acts *bijectively* on M (since t is a normal element of R and $(t) = Rt$). Therefore, the $C_A(H)$-module M is also a C_t-module. Using the equality $A_t = C_t[X^{\pm 1};\sigma]$, let us define an A_t-module

$$\widetilde{M} := A_t \otimes_{C_t} M = \bigoplus_{i\in\mathbb{Z}} X^i \otimes M = \bigoplus_{i\geqslant 1} Y^i \otimes M \oplus \bigoplus_{i\geqslant 0} X^i \otimes M. \quad (4.37)$$

Clearly, $\widetilde{M}$ is a weight A-module with $\text{Wt}(\widetilde{M}) = \mathcal{O}(\mu) = \mu + \mathbb{Z}$ and $\widetilde{M}_{\mu+i} = X^i \otimes M$ for all $i \in \mathbb{Z}$. The A-module $\widetilde{M}$ is X- and Y-torsionfree. Moreover, *the A-module $\widetilde{M}$ is simple* since if N is a nonzero submodule of $\widetilde{M}$ then it contains a nonzero element $X^i \otimes m$ for some $i \in \mathbb{Z}$ and $m \in M$. If $i = 0$ then $N = Am = \widetilde{M}$. If $i < 0$ then $N \ni X^{|i|}X^i \otimes m = 1 \otimes m$, and so $N = \widetilde{M}$. If $i > 0$ then $N \ni Y^i X^i \otimes m = 1 \otimes t^i m \neq 0$, and so $N = \widetilde{M}$. If $M' \in \widehat{C^{0,\mu'}}\,(t\text{-torsionfree})$ then the A-modules $\widetilde{M}$ and $\widetilde{M'}$ are isomorphic iff $\mathcal{O}(\mu) = \mathcal{O}(\mu')$ and the $C^{0,\mu}$-modules M and $X^i \otimes M'$ are isomorphic where $\mu = \mu' + i$ for a unique $i \in \mathbb{Z}$. Clearly, $\text{GK}\,(\widetilde{M}) = 2$. The following theorem is an explicit description of the set $\widehat{A(0)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree})$.

Theorem 4.35. $\widehat{A(0)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree}) = \{[\widetilde{M}] \mid [M] \in \widehat{C^{0,\mu_\mathcal{O}}}\,(t\text{-torsionfree}),\ \mathcal{O} \in \mathbb{K}/\mathbb{Z}\}$ *and* $\text{GK}\,(\widetilde{M}) = 2$ *for all* $\widetilde{M}$.

Proof. It suffices to show that if $\mathcal{M} \in \widehat{A(0)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree})$ then $\mathcal{M} \simeq \widetilde{M}$ for some $[M] \in \widehat{C^{0,\mu_\mathcal{O}}}\,(t\text{-torsionfree})$. Fix $\mu \in \text{Wt}(\mathcal{M})$. Since the elements X and Y act injectively on $\mathcal{M}$, we have that $\mathcal{O}(\mu) \subseteq \text{Wt}(\mathcal{M})$. So, we may assume that $\mu = \mu_\mathcal{O}$ where $\mathcal{O} = \mathcal{O}(\mu)$. Then $M := \mathcal{M}_\mu \in \widehat{C^{0,\mu_\mathcal{O}}}\,(t\text{-torsionfree})$, and so $\mathcal{M} \supseteq \bigoplus_{i\geqslant 1} Y^i M \oplus \bigoplus_{i\geqslant 0} X^i M = \widetilde{M}$, by (4.37) and simplicity of $\widetilde{M}$. So, $\mathcal{M} = \widetilde{M}$, as required. □

The set $\widehat{A(\lambda)}\,(\text{weight}, X\text{-torsionfree}, Y\text{-torsionfree})$ **where** $\lambda \neq 0$**.** Recall that $A_t = C_t[X^{\pm 1};\sigma]$. Let $[M] \in \widehat{C^{\lambda,\mu}}\,(t\text{-torsionfree})$. Then $[M_t] \in \widehat{C_t^{\lambda,\mu}}$. The A_t-module

$$M^\diamond := A_t \otimes_{C_t} M_t = \bigoplus_{i\in\mathbb{Z}} X^i \otimes M_t \quad (4.38)$$

is a simple, weight A_t-module with $\text{Wt}(M^\diamond) = \mathcal{O}(\mu) = \mu + \mathbb{Z}$ and $M^\diamond_{\mu+i} = X^i \otimes M_t$ for all $i \in \mathbb{Z}$ (if N is a nonzero submodule of $M^\diamond$ then it contains a nonzero element $X^i \otimes m$ for some $i \in \mathbb{Z}$

and $m \in M_t$. Then $N \ni X^{-i}X^i \otimes m = 1 \otimes m$, and so $N = M^\diamond$). For all $i \in \mathbb{Z}$,

$$M_i^\diamond = X^i \otimes M_t \simeq M_t^{\sigma^{-i}} \tag{4.39}$$

where $M_t^{\sigma^{-i}}$ is the C_t-module twisted by the automorphism σ^{-i} of the algebra C_t. (Recall that $A_t = C_t[X^{\pm 1};\sigma]$). By Theorem 4.26 and Theorem 4.25,

$$\widehat{C^{\lambda,\mu}}\,(t\text{-torsionfree}) = \{[\frac{A_1'}{A_1'e}],\, [\frac{A_1'}{A_1'(h-\nu_{\mathcal{O}})}]\,|\,\mathcal{O}\in\mathbb{K}/\mathbb{Z}\setminus\{\mathbb{Z}\}\}$$
$$\sqcup\ \{[\mathrm{soc}(N_b)]\,|\,b\in\mathrm{Irr}(B)/\sim,\ b \text{ is normal}\}$$

where $\mathrm{Irr}(B)/\sim$ is the set of equivalence classes of irreducible elements of the algebra $B = \mathbb{K}(h)[t^{\pm 1};\sigma]$ and $N_b = C^{\lambda,\mu}/C^{\lambda,\mu} \cap Bb$. Moreover, by Theorem 4.26.(3), $\mathrm{soc}(N_b) \simeq N_{bt^{-n}}$ for all $n \gg 0$.

For all $\lambda \in \mathbb{K}^*$ and $\mu \in \mathbb{K}$, the module $\mathfrak{m}^{\lambda,\mu} := C_t^{\lambda,\mu}/C_t^{\lambda,\mu}e$ is a simple $C_A(H)$-module. Hence, $\mathrm{soc}_{C_A(H)}(\mathfrak{m}^{\lambda,\mu}) = \mathfrak{m}^{\lambda,\mu}$. Notice that

$$\Big(\frac{A_1'}{A_1'e}\Big)^\diamond = \bigoplus_{i\in\mathbb{Z}} X^i \otimes \frac{A_{1,t}'}{A_{1,t}'e} \simeq \bigoplus_{i\in\mathbb{Z}} X^i \otimes \frac{C_t^{\lambda,\mu}}{C_t^{\lambda,\mu}e} = \bigoplus_{i\in\mathbb{Z}} X^i \otimes \mathfrak{m}^{\lambda,\mu}$$

and $X^i \otimes \mathfrak{m}^{\lambda,\mu} \simeq \mathfrak{m}^{\lambda,\mu+i}$ as $C_A(H)$-modules. Then there are equalities of A-modules

$$\mathrm{soc}_A\Big(\Big(\frac{A_1'}{A_1'e}\Big)^\diamond\Big) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{C_A(H)}(X^i\otimes\mathfrak{m}^{\lambda,\mu}) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{m}^{\lambda,\mu+i}. \tag{4.40}$$

For all $\lambda \in \mathbb{K}^*$, $\mu \in \mathbb{K}$ and $\mathcal{O} \in \mathbb{K}/\mathbb{Z}\setminus\{\mathbb{Z}\}$, the module $\mathsf{M}^{\lambda,\mu,\mathcal{O}} := C_t^{\lambda,\mu}/C_t^{\lambda,\mu}(h-\nu_{\mathcal{O}})$ is a simple $C_A(H)$-module. Hence, $\mathrm{soc}_{C_A(H)}(\mathsf{M}^{\lambda,\mu,\mathcal{O}}) = \mathsf{M}^{\lambda,\mu,\mathcal{O}}$. Since

$$\Big(\frac{A_1'}{A_1'(h-\nu_{\mathcal{O}})}\Big)^\diamond = \bigoplus_{i\in\mathbb{Z}} X^i \otimes \frac{A_{1,t}'}{A_{1,t}'(h-\nu_{\mathcal{O}})} \simeq \bigoplus_{i\in\mathbb{Z}} X^i \otimes \frac{C_t^{\lambda,\mu}}{C_t^{\lambda,\mu}(h-\nu_{\mathcal{O}})} = \bigoplus_{i\in\mathbb{Z}} X^i \otimes \mathsf{M}^{\lambda,\mu,\mathcal{O}}$$

and $X^i \otimes \mathsf{M}^{\lambda,\mu,\mathcal{O}} \simeq \mathsf{M}^{\lambda,\mu+i,\mathcal{O}}$ as $C_A(H)$-modules. Then there are equalities of A-modules

$$\mathrm{soc}_A\Big(\Big(\frac{A_1'}{A_1'(h-\nu_{\mathcal{O}})}\Big)^\diamond\Big) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{C_A(H)}(X^i\otimes\mathsf{M}^{\lambda,\mu,\mathcal{O}}) = \bigoplus_{i\in\mathbb{Z}} \mathsf{M}^{\lambda,\mu+i,\mathcal{O}}. \tag{4.41}$$

If $M \simeq N_b = C^{\lambda,\mu}/C^{\lambda,\mu} \cap Bb$ for an irreducible element b of $B = \mathbb{K}(h)[t^{\pm 1};\sigma]$. For all $i \in \mathbb{Z}$,

$$M_t^{\sigma^{-i}} \supseteq \frac{C_t^{\lambda,\mu+i}}{C_t^{\lambda,\mu+i}\cap B\sigma^i(b)} =: N_{\sigma^i(b)}.$$

By Theorem 4.26.(3),

$$\mathrm{soc}_{C_A(H)}(M_t^{\sigma^{-i}}) = \mathrm{soc}_{C_A(H)}(N_{\sigma^i(b)}) = N_{\sigma^i(b)t^{-n_i}}$$

for all $n_i \gg 0$. Then the A-module

$$\mathrm{soc}_A(M^\diamond) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{C_A(H)}(X^i \otimes M_t) \simeq \bigoplus_{i\in\mathbb{Z}} N_{\sigma^i(b)t^{-n_i}} \tag{4.42}$$

belongs to the set $\widehat{A(\lambda)}$ (weight, X-torsionfree, Y-torsionfree). The next theorem shows that all elements of the set $\widehat{A(\lambda)}$ (weight, X-torsionfree, Y-torsionfree) are precisely of this kind.

Theorem 4.36. *Let* $\lambda \in \mathbb{K}^*$ *and* $\mu \in \mathbb{K}$. *Then* $\widehat{A(\lambda)}$ (weight, X-torsionfree, Y-torsionfree) $= \{[\mathrm{soc}_A(M^\diamond)] \,|\, [M] \in \widehat{C^{\lambda,\mu_{\mathcal{O}}}}$ (t-torsionfree), $\mathcal{O} \in \mathbb{K}/\mathbb{Z}\}$ *and* $\mathrm{soc}_A(M^\diamond)$ *is explicitly described in (4.40), (4.41) and (4.42), the* A*-modules* $\mathrm{soc}_A(M^\diamond)$ *and* $\mathrm{soc}_A(M'^\diamond)$ *are isomorphic iff* $\lambda = \lambda'$, $\mathcal{O} = \mathcal{O}'$ *and* $M \simeq M'$; $\mathrm{GK}\,(\mathrm{soc}_A(M^\diamond)) = 2$.

Proof. Let $[\mathcal{M}] \in \widehat{A(\lambda)}$ (weight, X-torsionfree, Y-torsionfree). Then $\mathrm{Wt}(\mathcal{M}) = \mathcal{O} \in \mathbb{K}/\mathbb{Z}$. Let $\mu = \mu_{\mathcal{O}}$. Then $M := \mathcal{M}_\mu \in \widehat{C^{\lambda,\mu}}$ (t-torsionfree) and $M_t \in \widehat{C_t^{\lambda,\mu}}$. Clearly, $\mathcal{M} \subseteq \mathcal{M}_t = M^\diamond$, and so $\mathcal{M} = \mathrm{soc}_A(M^\diamond)$. Given $[M'] \in \widehat{C^{\lambda',\mu_{\mathcal{O}'}}}$ (t-torsionfree). If $\mathrm{soc}_A(M^\diamond) \simeq \mathrm{soc}_A(M'^\diamond)$ then $M^\diamond = \mathrm{soc}_A(M^\diamond)_t \simeq \mathrm{soc}_A(M'^\diamond)_t = M'^\diamond$ as A_t-modules, and so $\lambda = \lambda'$, $\mathcal{O} = \mathcal{O}'$ and $M_t \simeq M'_t$ as $C_t^{\lambda,\mu_{\mathcal{O}}}$-modules. Then $M = \mathrm{soc}_{C_A(H)}(M_t) \simeq \mathrm{soc}_{C_A(H)}(M'_t) = M'$ as $C_A(H)$-modules. Clearly, $\mathrm{GK}\,(\mathrm{soc}_A(M^\diamond)) = 2$. □

By (4.33), (4.35) and (4.36), Theorem 4.34, Theorem 4.35 and Theorem 4.36 give a complete classification of simple weight A-modules.

4.6 The Schrödinger algebra

The Schrödinger algebra is a non-semisimple Lie algebra, which plays an important role in mathematical physics. A classification of simple lowest weight modules for the Schrödinger algebra is given in [22]. The fact that all the weight spaces of a simple weight module have the same dimension is proved in [43]. By using Mathieu's twisting functor, a classification of simple weight modules with finite dimensional weight spaces over the Schrödinger algebra is given in [23]. In [42], the author studied the finite dimensional indecomposable modules for Schrödinger algebra. Quite recently, [24] studied the category $\mathcal{O}$ for the Schrödinger algebra and described primitive ideals with nonzero central charge.

The *Schrödinger algebra* $\mathfrak{s}$ is a 6-dimensional Lie algebra that admits a $\mathbb{K}$-basis $\{F, H, E, Y, X, Z\}$ elements of which satisfy the defining relations:

$$\begin{array}{llll} [H,E] = 2E, & [H,F] = -2F, & [E,F] = H, & [H,X] = X, \\ [H,Y] = -Y, & [E,Y] = X, & [E,X] = 0, & [F,X] = Y, \\ [F,Y] = 0, & [X,Y] = Z, & [Z,\mathfrak{s}] = 0. & \end{array}$$

The Lie algebra $\mathfrak{s}$ is not semisimple and can be viewed as the semidirect product $\mathfrak{s} = \mathfrak{sl}_2 \ltimes \mathcal{H}$ of Lie algebras where $\mathfrak{sl}_2 = \mathbb{K}F \oplus \mathbb{K}H \oplus \mathbb{K}E$ and $\mathcal{H} = \mathbb{K}X \oplus \mathbb{K}Y \oplus \mathbb{K}Z$ is the three dimensional *Heisenberg Lie algebra*. Let $\mathcal{S} := U(\mathfrak{s})$ be the universal enveloping algebra of the Schrödinger algebra $\mathfrak{s}$. The primitive ideals of $U(\mathfrak{s})$ with *non-zero* central charge were described by Dubsky, Lü, Mazorchuk and Zhao, [24]. In [24], they wrote that "the problem of classification of primitive ideals in $U(\mathfrak{s})$ for *zero* central charge might be very difficult". Using the classification of prime ideals of A (Theorem 4.6) we give a complete classification of the primitive ideals of $U(\mathfrak{s})$. It is conjectured that there is no simple *singular* Whittaker module for the algebra $\mathcal{S}$ [44, Conjecture 4.2]. We construct a family of such $\mathcal{S}$-modules (Proposition 4.44).

The centre of $\mathcal{S}$ and some related algebras. In this section, we show that the localization $\mathcal{S}_Z$ of the algebra $\mathcal{S}$ at the powers of the central element Z is isomorphic to the tensor product of algebras $\mathbb{K}[Z^{\pm 1}] \otimes U(\mathfrak{sl}_2) \otimes A_1$, see (4.53). Using this fact, a short proof is given of the fact that the centre of the algebra $\mathcal{S}$ is a polynomial algebra in two explicit generators (Proposition 4.39). The fact that the centre $Z(\mathcal{S})$ of $\mathcal{S}$ is a polynomial algebra was proved in [24] by using the Harish-Chandra homomorphism. In the above papers, it was not clear how the nontrivial central element c was found. In this paper, we clarify the 'origin' of c which is the Casimir element of the 'hidden' tensor component $U(\mathfrak{sl}_2)$ in the decomposition (4.53).

Let $U := U(\mathfrak{sl}_2)$ be the enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. Then the centre of the algebra U is a polynomial algebra, $Z(U) = \mathbb{K}[\Delta]$, where $\Delta := 4FE + H^2 + 2H$ is called the *Casimir* element of U.

An automorphism γ of $\mathcal{S}$. The algebra $\mathcal{S}$ admits an automorphism γ defined by

$$\gamma(F) = E,\ \gamma(H) = -H,\ \gamma(E) = F,\ \gamma(Y) = -X,\ \gamma(X) = -Y,\ \text{and}\ \gamma(Z) = -Z. \tag{4.43}$$

Clearly, $\gamma^2 = \mathrm{id}_{\mathcal{S}}$.

The subalgebra $\mathscr{H}$ of $\mathcal{S}$. Let $\mathscr{H}$ be the subalgebra of $\mathcal{S}$ generated by the elements X, Y and Z. Then the generators of the algebra $\mathscr{H}$ satisfy the defining relations

$$XY - YX = Z, \qquad ZX = XZ, \qquad \text{and}\ \ ZY = YZ.$$

So, $\mathscr{H} = U(\mathcal{H})$ is the universal enveloping algebra of the three dimensional Heisenberg algebra $\mathcal{H}$. In particular, $\mathscr{H}$ is a Noetherian domain of Gefand-Kirillov dimension 3. Let $\mathscr{H}_Z$ be the localization of $\mathscr{H}$ at the powers of the element Z and $\mathscr{X} := Z^{-1}X \in \mathscr{H}_Z$. Then the algebra $\mathscr{H}_Z$ is a tensor product of algebras

$$\mathscr{H}_Z = \mathbb{K}[Z^{\pm 1}] \otimes A_1 \tag{4.44}$$

where $A_1 := \mathbb{K}\langle \mathscr{X}, Y\rangle$ is the (first) Weyl algebra since $[\mathscr{X}, Y] = 1$.

The subalgebra $\mathscr{E}$ of $\mathcal{S}$. Let $\mathscr{E}$ be the subalgebra of $\mathcal{S}$ generated by the elements X, Y, Z and E. Then

$$\mathscr{E} = \mathscr{H}[E; \delta] \tag{4.45}$$

is an Ore extension where δ is the $\mathbb{K}$-derivation of the algebra $\mathscr{H}$ defined by $\delta(Y) = X, \delta(X) = 0$ and $\delta(Z) = 0$. Let $\mathscr{E}_Z$ be the localization of $\mathscr{E}$ at the powers of the element Z. Then

$$\mathscr{E}_Z = \mathscr{H}_Z[E;\delta] = \Big(\mathbb{K}[Z^{\pm 1}] \otimes A_1\Big)[E;\delta] \tag{4.46}$$

where δ is defined as in (4.45), in particular, $\delta(\mathscr{X}) = 0$. Now, the element $s = -\frac{1}{2}Z\mathscr{X}^2$ satisfies the conditions of Lemma 2.17. Specifically, the element $E' := E + s = E - \frac{1}{2}Z^{-1}X^2$ commutes with the elements of A_1. Hence, $\mathscr{E}_Z$ is a tensor product of algebras

$$\mathscr{E}_Z = \mathbb{K}[E', Z^{\pm 1}] \otimes A_1 = \mathbb{K}[E'] \otimes \mathscr{H}_Z. \tag{4.47}$$

In particular, $\mathscr{E}$ and $\mathscr{E}_Z$ are Noetherian domains of Gelfand-Kirillov dimension 4.

The subalgebra $\mathscr{F}$ of $\mathcal{S}$. Let $\mathscr{F} := \gamma(\mathscr{E})$. Then $\mathscr{F}$ is the subalgebra of $\mathcal{S}$ generated by the elements X, Y, Z and F. Notice that the automorphism γ (see (4.43)) can be naturally extended to an automorphism of $\mathcal{S}_Z$ by setting $\gamma(Z^{-1}) = -Z^{-1}$ where $\mathcal{S}_Z$ is the localization of the algebra $\mathcal{S}$ at the powers of the element Z. Let $\mathscr{F}_Z$ be the localization of $\mathscr{F}$ at the powers of the central element Z and $F' := \gamma(E') = F + \frac{1}{2}Z^{-1}Y^2 \in \mathscr{F}_Z$. Then $\mathscr{F}_Z$ is a tensor product of algebras

$$\mathscr{F}_Z = \mathbb{K}[F', Z^{\pm 1}] \otimes A_1 = \mathbb{K}[F'] \otimes \mathscr{H}_Z \tag{4.48}$$

where A_1 is as above, see (4.44).

The algebra $\mathscr{A}$. Let $\mathscr{A}$ be the subalgebra of $\mathcal{S}$ generated by the elements H, E, Y, X and Z. The algebra $\mathscr{A}$ is the enveloping algebra $U(\mathfrak{a})$ of the solvable Lie subalgebra $\mathfrak{a}$ of $\mathfrak{s}$ with basis elements H, E, Y, X and Z. The algebra $\mathscr{A}$ is an Ore extension

$$\mathscr{A} = \mathscr{E}[H;\delta] \tag{4.49}$$

where δ is a $\mathbb{K}$-derivation of the algebra $\mathscr{E}$ defined by $\delta(E) = 2E, \delta(Y) = -Y, \delta(X) = X$ and $\delta(Z) = 0$. Let $\mathscr{A}_Z$ be the localization of the algebra $\mathscr{A}$ at the powers of the central element Z. Then

$$\mathscr{A}_Z = \mathscr{E}_Z[H;\delta] = \Big(\mathbb{K}[E', Z^{\pm 1}] \otimes A_1\Big)[H;\delta] \tag{4.50}$$

where δ is defined as in (4.49), in particular, $\delta(\mathscr{X}) = \mathscr{X}$. The element $s = \mathscr{X}Y - \frac{1}{2} = Z^{-1}XY - \frac{1}{2}$ satisfies the conditions of Lemma 2.17. In particular, the element $H' := H + s = H + Z^{-1}XY - \frac{1}{2}$ commutes with the elements of A_1 and $[H', E'] = 2E'$. Hence, $\mathscr{A}_Z$ is a tensor product of algebras

$$\mathscr{A}_Z = \mathbb{K}[Z^{\pm 1}] \otimes \mathbb{K}[H'][E';\sigma] \otimes A_1 \tag{4.51}$$

where σ is the automorphism of the algebra $\mathbb{K}[H']$ such that $\sigma(H') = H' - 2$. In particular, $\mathscr{A}_Z$ is a Noetherian domain of Gelfand-Kirillov dimension 5.

The factor algebra $S/(Z)$. The set $\mathbb{K}Z$ is an ideal of the Lie algebra $\mathfrak{s}$ and $\mathfrak{s}/\mathbb{K}Z \simeq \mathfrak{sl}_2 \ltimes V_2$ is a semidirect product of Lie algebras where $V_2 = \mathbb{K}X \oplus \mathbb{K}Y$ is a 2-dimensional abelian Lie algebra. So,

$$\mathcal{S}/(Z) \simeq U(\mathfrak{s}/\mathbb{K}Z) \simeq U(\mathfrak{sl}_2 \ltimes V_2).$$

Recall that the centre of $A = U(\mathfrak{sl}_2 \ltimes V_2)$ is a polynomial algebra $Z(A) = \mathbb{K}[C]$ where $C = FX^2 - HXY - EY^2$, see Lemma 4.1.

Lemma 4.37. *1. Let* $E' := E - \frac{1}{2}Z^{-1}X^2$, $F' := F + \frac{1}{2}Z^{-1}Y^2$ *and* $H' := H + Z^{-1}XY - \frac{1}{2}$. *Then the following commutation relations hold in the algebra* $\mathcal{S}_Z$*:*

$$[H', E'] = 2E', \quad [H', F'] = -2F', \quad [E', F'] = H',$$

i.e., the Lie algebra $\mathbb{K}F' \oplus \mathbb{K}H' \oplus \mathbb{K}E'$ *is isomorphic to* $\mathfrak{sl}_2$. *Moreover, the subalgebra* U' *of* $\mathcal{S}_Z$ *generated by* H', E' *and* F' *is isomorphic to the enveloping algebra* $U(\mathfrak{sl}_2)$. *Furthermore, the elements* E', F' *and* H' *commute with* X *and* Y.

2. The localization $\mathcal{S}_Z$ *of the algebra* $\mathcal{S}$ *at the powers of* Z *is* $\mathcal{S}_Z = \mathbb{K}[Z^{\pm 1}] \otimes U' \otimes A_1$.

Proof. 1. It is straightforward to verify that the commutation relations in the lemma hold. The fact that the elements E', F' and H' commute with the elements X and Y follows from (4.47), (4.48) and (4.51), respectively. Let U be the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2 = \langle F', H', E' \rangle$. The algebra U' is an epimorphic image of the algebra U under a natural epimorphism $f : U \to U'$. The kernel of f, say $\mathfrak{p}$, is a (completely) prime ideal of U since U' is a domain. Suppose that $\mathfrak{p} \neq 0$, we seek a contradiction. Then $\mathfrak{p} \cap \mathbb{K}[\Delta] \neq 0$ (it is known fact) where Δ is the Casimir element of U. In particular, there is a non-scalar monic polynomial $P(t) = t^n + \lambda_{n-1}t^{n-1} + \cdots + \lambda_0 \in \mathbb{K}[t]$ such that $P(\Delta') = 0$ in $\mathcal{S}_Z$ where $\Delta' = 4F'E' + H'^2 + 2H'$. Then $Z^n P(\Delta') \in \mathcal{S}$ and necessarily $Z^n P(\Delta') \equiv 0 \mod \mathcal{S}Z$, i.e., $(EY^2 + HXY - FX^2)^n \equiv 0 \mod \mathcal{S}Z$, a contradiction since $\mathcal{S}/\mathcal{S}Z \simeq U(\mathfrak{sl}_2 \ltimes V_2)$.

2. Using the defining relations of the algebra $\mathcal{S}$, we see that the algebra $\mathcal{S}$ is a skew polynomial algebra

$$\mathcal{S} = \mathscr{A}[F; \sigma, \delta] \tag{4.52}$$

where σ is the automorphism of the algebra $\mathscr{A}$ defined by $\sigma(H) = H + 2, \sigma(E) = E, \sigma(Y) = Y, \sigma(X) = X$ and $\sigma(Z) = Z$; and δ is the σ-derivation of $\mathscr{A}$ given by the rule: $\delta(H) = \delta(Y) = \delta(Z) = 0$, $\delta(E) = -H$ and $\delta(X) = Y$. Then, by (4.51) and statement 1,

$$\mathcal{S}_Z = \mathscr{A}_Z[F'; \sigma', \delta'] = \Big(\mathbb{K}[Z^{\pm 1}] \otimes \mathbb{K}[H'][E'; \sigma] \otimes A_1\Big)[F'; \sigma', \delta'] = \mathbb{K}[Z^{\pm 1}] \otimes U' \otimes A_1 \tag{4.53}$$

is a tensor product of algebras where σ' is an automorphism of $\mathscr{A}_Z$ such that $\sigma'(Z) = Z, \sigma'(H') = H' + 2, \sigma'(E') = E', \sigma'(X) = X$ and $\sigma'(Y) = Y$; and δ' is a σ'-derivation of the algebra $\mathscr{A}_Z$ such that $\delta'(Z) = \delta'(H') = \delta'(X) = \delta'(Y) = 0$ and $\delta'(E') = -H'$. In particular, $\mathcal{S}_Z$ is a Noetherian domain of Gelfand-Kirillov dimension 6. □

The centre of the algebra $\mathcal{S}$**.** Let $\Delta' := 4F'E' + H'^2 + 2H'$ be the Casimir element of U', then the centre $Z(U') = \mathbb{K}[\Delta']$ is a polynomial algebra. Using the explicit expressions of the elements F', E' and H' (see Lemma 4.37.(1)), the element Δ' can be written as

$$\Delta' = \big(4FE + H^2 + H\big) + 2Z^{-1}\big(EY^2 + HXY - FX^2\big) - \frac{3}{4}. \tag{4.54}$$

Let

$$\mathsf{c} := Z\Delta' + \frac{3}{4}Z = Z\Big(4FE + H^2 + H\Big) + 2\Big(EY^2 + HXY - FX^2\Big). \tag{4.55}$$

Lemma 4.38. $Z\big(\mathcal{S}_Z\big) = \mathbb{K}[Z^{\pm 1}, \mathsf{c}]$.

Proof. By (4.53) and Lemma 2.20, $Z\big(\mathcal{S}_Z\big) = Z\big(\mathbb{K}[Z^{\pm 1}]\big) \otimes Z\big(U'\big) \otimes Z\big(A_1\big) = \mathbb{K}[Z^{\pm 1}] \otimes \mathbb{K}[\Delta'] = \mathbb{K}[Z^{\pm 1}, \Delta'] = \mathbb{K}[Z^{\pm 1}, \mathsf{c}]$. □

The next proposition shows that the centre of $\mathcal{S}$ is a polynomial algebra in two variables.

Proposition 4.39. $Z\big(\mathcal{S}\big) = \mathbb{K}[Z, \mathsf{c}]$.

Proof. By Lemma 4.38, $Z(\mathcal{S}) = \mathcal{S} \cap Z(\mathcal{S}_Z) = \mathcal{S} \cap \mathbb{K}[Z^{\pm 1}, \mathsf{c}] \supseteq \mathbb{K}[Z, \mathsf{c}]$. It remains to show that $Z(\mathcal{S}) = \mathbb{K}[Z, \mathsf{c}]$. Suppose that this is not the case, we seek a contradiction. Then $Z^{-1}f(\mathsf{c}) \in Z(\mathcal{S})$ for some non-scalar polynomial $f(\mathsf{c}) \in \mathbb{K}[\mathsf{c}]$ (since $Z^{-1} \notin \mathcal{S}$). Hence, by (4.55),

$$0 \equiv f(\mathsf{c}) \equiv f(-2C) \mod \mathcal{S}Z,$$

i.e., the element C is algebraic in $U(\mathfrak{sl}_2 \ltimes V_2)$, a contradiction. □

The primitive ideals and existence of singular Whittaker modules over the Schrödinger algebra. In this subsection, $\mathbb{K}$ is an algebraically closed field. Our aim is to give a classification of primitive ideals of the algebra $\mathcal{S}$ and to prove existence of simple singular Whittaker $\mathcal{S}$-modules.

For $\lambda \in \mathbb{K}$, let $\mathcal{S}(\lambda) := \mathcal{S}/\mathcal{S}(Z - \lambda)$. Then $\mathcal{S}(0) \simeq A$. If $\lambda \neq 0$ then, by (4.53),

$$\mathcal{S}(\lambda) = \mathcal{S}_Z/\mathcal{S}_Z(Z - \lambda) = U'_\lambda \otimes A_1 \tag{4.56}$$

is a tensor product of algebras. The algebra U'_λ, which is isomorphic to the enveloping algebra $U(\mathfrak{sl}_2)$, is generated by the elements

$$H_\lambda = H + \lambda^{-1}XY - \frac{1}{2}, \quad E_\lambda = E - \frac{1}{2}\lambda^{-1}X^2, \quad , F_\lambda = F + \frac{1}{2}\lambda^{-1}Y^2$$

and the elements H_λ, E_λ and F_λ are canonical generators of the Lie algebra $\mathfrak{sl}_2$ ($[H_\lambda, E_\lambda] = 2E_\lambda$, $[H_\lambda, F_\lambda] = -2F_\lambda$ and $[E_\lambda, F_\lambda] = H_\lambda$, see Lemma 4.37 for details). The algebra A_1 is the Weyl algebra generated by the elements $\lambda^{-1}X$ and Y. In particular, $\mathcal{S}(\lambda)$ is a Noetherian domain of Gelfand-Kirillov dimension 5, and the ideal of $\mathcal{S}$ generated by $Z - \lambda$ is completely prime. Furthermore, $Z(\mathcal{S}(\lambda)) = K[\mathsf{c}_\lambda]$ where $\mathsf{c}_\lambda = \lambda(4FE + H^2 + H) + 2(EY^2 + HXY - FX^2)$ is a non-standard Casimir element of the algebra U'_λ written via the new canonical generators H_λ, E_λ and F_λ, i.e., $4E_\lambda F_\lambda + H_\lambda^2 - 2H_\lambda = \lambda^{-1}\mathsf{c}_\lambda - \frac{3}{4}$.

For λ, $\mu \in \mathbb{K}$, let $\mathcal{S}(\lambda, \mu) := \mathcal{S}/(Z - \lambda, \mathsf{c} - \mu) \simeq \mathcal{S}(\lambda)/\mathcal{S}(\lambda)(\mathsf{c}_\lambda - \mu)$. The following lemma gives the condition for the factor algebra $\mathcal{S}(\lambda, \mu)$ to be a simple algebra.

Lemma 4.40. *Let* $\lambda \in \mathbb{K}^*$ *and* $\mu \in \mathbb{K}$.

1. $Z(\mathcal{S}(\lambda, \mu)) = \mathbb{K}$.
2. *The algebra* $\mathcal{S}(\lambda, \mu)$ *is a simple algebra iff* $\mu \neq \lambda(n^2 + 2n + \frac{3}{4})$ *for all* $n \in \mathbb{N}$.
3. *If* $\mu = \lambda(n^2 + 2n + \frac{3}{4})$ *for some* $n \in \mathbb{N}$ *then* $\mathcal{S}(\lambda, \mu)$ *has a unique proper two-sided ideal which is the tensor product of the annihilator of the unique simple* $(n + 1)$*-dimensional* $\mathfrak{sl}_2$*-module and the Weyl algebra* A_1.

Proof. Statement 1 follows from (4.56). Statements 2 and 3 follows from [21, 4.9.22] and the fact that $4E_\lambda F_\lambda + H_\lambda^2 - 2H_\lambda = \lambda^{-1}\mathsf{c}_\lambda - \frac{3}{4}$. □

Primitive ideals of the algebra $\mathcal{S}$. The next proposition gives a classification of prime, maximal and primitive ideals of the algebra $\mathcal{S}(\lambda)$ where $\lambda \neq 0$. By (4.56), the map $\mathrm{Spec}\,(U'_\lambda) \to \mathrm{Spec}\,(\mathcal{S}(\lambda))$, $\mathfrak{p} \mapsto \mathfrak{p} \otimes A_1$, is an injection, and we identify $\mathrm{Spec}\,(U'_\lambda)$ with its image in $\mathrm{Spec}\,(\mathcal{S}(\lambda))$.

Proposition 4.41. *Let* $\lambda \in \mathbb{K}^*$. *Then* $\mathrm{Spec}\,(\mathcal{S}(\lambda)) = \mathrm{Spec}\,(U'_\lambda)$, $\mathrm{Max}\,(\mathcal{S}(\lambda)) = \mathrm{Max}\,(U'_\lambda)$ *and* $\mathrm{Prim}\,(\mathcal{S}(\lambda)) = \mathrm{Prim}\,(U'_\lambda)$.

Proof. The first two equalities are obvious, by applying [21, Lemma 4.5.1]. Clearly, $\mathrm{Prim}\,(U'_\lambda) \subseteq \mathrm{Prim}\,(\mathcal{S}(\lambda))$ and $\mathrm{Prim}\,(U'_\lambda) = \mathrm{Spec}\,(U'_\lambda) \setminus \{0\} = \mathrm{Spec}\,(\mathcal{S}(\lambda)) \setminus \{0\}$. Since $\mathrm{Spec}\,(\mathcal{S}(\lambda)) = \mathrm{Spec}\,(U'_\lambda)$ and 0 is not a primitive ideal of $\mathcal{S}(\lambda)$ (since $Z(\mathcal{S}(\lambda)) = \mathbb{K}[\mathsf{c}_\lambda]$), we must have $\mathrm{Prim}\,(U'_\lambda) = \mathrm{Prim}\,(\mathcal{S}(\lambda))$. □

Remark. The primitive ideals of $\mathcal{S}(\lambda)$ for $\lambda \neq 0$ were described in [24, Corollary 30] as annihilators of Verma modules.

The next theorem together with Theorem 4.8 , gives an explicit description of the set of primitive ideals of $\mathcal{S}$.

Theorem 4.42. *Suppose that* $\mathbb{K}$ *is an algebraically closed field. Then*

$$\mathrm{Prim}(\mathcal{S}) = \big\{(Z - \lambda, \mathfrak{p}) \,|\, \lambda \in \mathbb{K}^*, \mathfrak{p} \in \mathrm{Spec}\,(U'_\lambda) \setminus \{0\}\big\} \sqcup \big\{(Z, \mathfrak{q}) \,|\, \mathfrak{q} \in \mathrm{Prim}(A)\big\}.$$

Proof. Since Z is a central element of $\mathcal{S}$ and $\mathbb{K}$ is algebraically closed, any primitive ideal of $\mathcal{S}$ contains $Z - \lambda$ for some $\lambda \in \mathbb{K}$. Hence, $\mathrm{Prim}(\mathcal{S}) = \sqcup_{\lambda \in \mathbb{K}^*} \mathrm{Prim}(\mathcal{S}(\lambda)) \sqcup \mathrm{Prim}(A) = \big\{(Z - \lambda, \mathfrak{p}) \,|\, \lambda \in \mathbb{K}^*, \mathfrak{p} \in \mathrm{Prim}(U'_\lambda)\big\} \sqcup \big\{(Z, \mathfrak{q}) \,|\, \mathfrak{q} \in \mathrm{Prim}(A)\big\}$, as required. Notice that $\mathrm{Prim}\,(U'_\lambda) = \mathrm{Spec}\,(U'_\lambda) \setminus \{0\}$. □

Singular Whittaker $\mathcal{S}$-modules. Simple, non-singular, Whittaker modules of the Schrödinger algebra were classified in [44]. They conjectured that there is no simple singular Whittaker module for the Schrödinger algebra [44, Conjecture 4.2]. Proposition 4.44 shows that there exists simple singular Whittaker A-modules (these are Whittaker Schrödinger modules of *zero* *level*), hence the conjecture is not true, in general. But we prove that the conjecture is true for Whittaker Schrödinger modules of *non-zero level*.

Let $R = \mathcal{S}$ or $\mathcal{S}(\lambda)$ for some $\lambda \in \mathbb{K}$ and let V be an R-module. A non-zero element $w \in V$ is called a *Whittaker vector* of type (μ, δ) if $Ew = \mu w$ and $Xw = \delta w$ where $\mu, \delta \in \mathbb{K}$. An R-module V is called a *Whittaker module* of type (μ, δ) if V is generated by a Whittaker vector of type (μ, δ). An R-module V is called a *singular Whittaker module* if V is generated by a Whittaker vector $w \in V$ of type $(0, 0)$ and $Hw \notin \mathbb{K}w$.

Using the decomposition (4.56), we can give a classification of simple Whittaker $\mathcal{S}$-modules of non-zero level easily (i.e., the simple Whittaker $\mathcal{S}(\lambda)$-modules where $\lambda \neq 0$).

Whittaker $\mathcal{S}(\lambda)$-modules where $\lambda \neq 0$. Let $\mu, \delta \in \mathbb{K}$. The *universal* Whittaker $\mathcal{S}(\lambda)$-module of type (μ, δ) is $W := W(\mu, \delta) := \mathcal{S}(\lambda)/\mathcal{S}(\lambda)\big(E - \mu, X - \delta\big)$. So, any Whittaker $\mathcal{S}(\lambda)$-module of type (μ, δ) is a homomorphic image of W. By (4.56),

$$\begin{aligned} W &= \frac{\mathcal{S}(\lambda)}{\mathcal{S}(\lambda)\big(E_\lambda + 1/2\lambda^{-1}X^2 - \mu,\ X - \delta\big)} = \frac{\mathcal{S}(\lambda)}{\mathcal{S}(\lambda)\big(E_\lambda + 1/2\lambda^{-1}\delta^2 - \mu,\ X - \delta\big)} \\ &= U'_\lambda/U'_\lambda\big(E_\lambda + 1/2\lambda^{-1}\delta^2 - \mu\big) \otimes A_1/A_1\big(X - \delta\big) \end{aligned} \tag{4.57}$$

The module $W_{U'_\lambda} := U'_\lambda/U'_\lambda\big(E_\lambda + 1/2\lambda^{-1}\delta^2 - \mu\big)$ is a Whittaker U'_λ-module of type $(-1/2\lambda^{-1}\delta^2 + \mu)$. The simple Whittaker U'_λ-modules are easily classified, see [18, Proposition 5.3]. Note that $A_1/A_1(X - \delta)$ is a simple A_1-module with $\mathrm{End}_{A_1}(A_1/A_1(X - \delta)) = \mathbb{K}$. Thus we have the following conclusion (which recovers the results of [44, Theorem 6.11]):

$$\begin{aligned} &\widehat{\mathcal{S}(\lambda)}\ \big(\text{Whittaker module of type}\,(\mu, \delta)\big) \\ &= \widehat{U'_\lambda}\ \big(\text{Whittaker module of type}\,(\mu - 1/2\lambda^{-1}\delta^2)\big) \otimes A_1/A_1(X - \delta). \end{aligned}$$

The next proposition shows that there is no simple singular Whittaker $\mathcal{S}$-module of nonzero level, i.e., all the simple Whittaker $\mathcal{S}(\lambda)$-modules of type $(0, 0)$ are weight modules where $\lambda \neq 0$.

Proposition 4.43. *If $\lambda \in \mathbb{K}^*$ then there is no simple singular Whittaker $\mathcal{S}(\lambda)$-module.*

Proof. By (4.57), the universal singular Whittaker $\mathcal{S}(\lambda)$-module $W = U'_\lambda/U'_\lambda E_\lambda \otimes A_1/A_1X$. Notice that A_1/A_1X is a simple A_1-module and $\mathrm{End}_{A_1}(A_1/A_1X) = \mathbb{K}$. Hence, each simple factor module L of W is equal to $M \otimes A_1/A_1X$ where M is a simple factor module of the U'_λ-module $U'_\lambda/U'_\lambda E_\lambda$. Then by [44, Theorem 6.10.(i)], M is a (highest) H_λ-weight U'_λ-module, i.e., M is a simple factor module of $U'_\lambda/U'_\lambda(H_\lambda - \mu, E_\lambda)$ for some $\mu \in \mathbb{K}$. Then L is a simple factor module of $U'_\lambda/U'_\lambda(H_\lambda - \mu, E_\lambda) \otimes A_1/A_1X \simeq \mathcal{S}(\lambda)/\mathcal{S}(\lambda)(H + \frac{1}{2} - \mu, E, X)$. Hence, L is a weight module. This completes the proof. □

Recall that $\mathcal{S}(0) = A$. Let $\mathcal{W} := A/A(X, E)$, a left A-module. Then any singular Whittaker A-module is an epimorphic image of $\mathcal{W}$. For any $\lambda \in \mathbb{K}^*$, we define the A-module

$$V(\lambda) := A/A(X, E, Y - \lambda) = \sum_{i,j \in \mathbb{N}} \mathbb{K}H^iF^j\bar{1} \quad \text{where } \bar{1} = 1 + A(X, E, Y - \lambda).$$

Clearly, $V(\lambda)$ is a singular Whittaker A-module. Then next proposition shows that $V(\lambda)$ is a simple A-module. Hence, the conjecture [44, Conjecture 4.2] does not hold in this case.

Proposition 4.44. *For any* $\lambda \in \mathbb{K}^*$*, the module* $V(\lambda)$ *is a simple* A*-module.*

Proof. We have to show that for any $0 \neq v = \sum_{i,j\in\mathbb{N}} \alpha_{i,j} H^i F^j \bar{1} \in V(\lambda)$ where $\alpha_{i,j} \in \mathbb{K}$, there exists an element $a \in A$ such that $av \in \mathbb{K}^*\bar{1}$. If $j > 0$ then $Xv = \sum_{i,j\in\mathbb{N}} \alpha_{i,j}(-\lambda)j(H-1)^i F^{j-1}\bar{1}$. By considering the leading term we see that $Xv \neq 0$. Therefore, $0 \neq X^n v \in \mathbb{K}[H]\bar{1}$ for some $n \in \mathbb{N}$. So, we may assume that $v = \sum_{i=0}^m \alpha_i H^i \bar{1}$ where $\alpha_i \in \mathbb{K}, m \in \mathbb{N}$ and $\alpha_m \neq 0$. Then $0 \neq (Y-\lambda)v = \sum_{i=0}^m \alpha_i \lambda\big((H+1)^i - H^i\big)\bar{1}$. By induction on m, we have $(Y-\lambda)^m v \in \mathbb{K}^*\bar{1}$, as required. □

Chapter 5

The quantum spatial ageing algebra

5.1 Introduction

Let $\mathbb{K}$ be a field and an element $q \in \mathbb{K}^* := \mathbb{K} \setminus \{0\}$ which is not a root of unity. The algebra $\mathbb{K}_q[X, Y] := \mathbb{K}\langle X, Y \,|\, XY = qYX\rangle$ is called the *quantum plane*. A classification of simple modules over the quantum plane is given in [9]. The *quantized enveloping algebra* $U_q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is generated over $\mathbb{K}$ by elements E, F, K and K^{-1} subject to the defining relations:

$$KEK^{-1} = q^2E, \; KFK^{-1} = q^{-2}F, \; EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

For basic properties and representation theory of the algebra $U_q(\mathfrak{sl}_2)$ the reader is referred to [29, 33]. The simple $U_q(\mathfrak{sl}_2)$-modules were classified in [2], see also [8], [9] and [17]. The quantum plane and the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$ are important examples of generalized Weyl algebras and ambiskew polynomial rings, see e.g., [5] and [32]. Let $U_q^{\geqslant 0}(\mathfrak{sl}_2)$ be the (positive) Borel part of $U_q(\mathfrak{sl}_2)$. It is the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by $K^{\pm 1}$ and E. There is a Hopf algebra structure on $U_q^{\geqslant 0}(\mathfrak{sl}_2)$ defined by

$$\Delta(K) = K \otimes K, \qquad \varepsilon(K) = 1, \qquad S(K) = K^{-1},$$
$$\Delta(E) = E \otimes 1 + K \otimes E, \qquad \varepsilon(E) = 0, \qquad S(E) = -K^{-1}E.$$

The notion of smash product has proved to be very useful in studying Hopf algebra actions [38]. For example, the enveloping algebra of a semi-direct product of Lie algebras can naturally be seen as a smash product algebra. The smash product is constructed from a module algebra, see [38, 4.1] for details and examples. We can make the quantum plane a $U_q^{\geqslant 0}(\mathfrak{sl}_2)$-*module algebra* by defining

$$K \cdot X = qX, \; E \cdot X = 0, \; K \cdot Y = q^{-1}Y, \; E \cdot Y = X,$$

and introduce the smash product algebra $\mathcal{A} := \mathbb{K}_q[X,Y] \rtimes U_q^{\geqslant 0}(\mathfrak{sl}_2)$. We call this algebra the *quantum spatial ageing algebra*. The defining relations for the algebra $\mathcal{A}$ are given explicitly in the following definition.

Definition. The *quantum spatial ageing algebra* $\mathcal{A} = \mathbb{K}_q[X,Y] \rtimes U_q^{\geqslant 0}(\mathfrak{sl}_2)$ is an algebra generated over $\mathbb{K}$ by the elements E, K, K^{-1}, X and Y subject to the defining relations:

$$EK = q^{-2}KE, \qquad XK = q^{-1}KX, \qquad YK = qKY,$$
$$EX = qXE, \qquad EY = X + q^{-1}YE, \qquad qYX = XY.$$

The algebra $\mathcal{A}$ can be seen as the quantum analogue of the spatial ageing algebra. This chapter is organized as follows. In Section 5.2, we describe the partially ordered sets of the prime, maximal and primitive ideals of the algebra $\mathcal{A}$. Using this description the prime factor algebras of $\mathcal{A}$ are given explicitly via generators and relations (Theorem 5.8). There are nine types of prime factor algebras of $\mathcal{A}$. For two of them, 'additional' non-obvious units appear under factorization at prime ideals. It is proved that every prime ideal of $\mathcal{A}$ is completely prime (Corollary 5.12). In Section 5.3, the automorphism group of $\mathcal{A}$ is determined, which turns out to be a 'small' non-commutative group that contains an infinite discrete subgroup (Theorem 5.14). The orbits of the prime spectrum under the action of the automorphism group are described. In Section 5.4, the centralizers of the elements K, X, φ, Y and E in the algebra $\mathcal{A}$ are determined.

Much of this chapter is extracted from the joint paper with V. Bavula [11].

5.2 Prime spectrum of the algebra $\mathcal{A}$

The aim of this section is to describe the prime, maximal and primitive spectra of the algebra $\mathcal{A}$ (Theorem 5.8, Corollary 5.9 and Proposition 5.11). Every prime ideal of $\mathcal{A}$ is completely prime (Corollary 5.12). For all prime ideals P of $\mathcal{A}$, the factor algebras $\mathcal{A}/P$ are given by generators and defining relations (Theorem 5.8).

Definition, [7]. Let D be an ring and σ be its automorphism. Suppose that elements b and ρ belong to the centre of the ring D, ρ is invertible and $\sigma(\rho) = \rho$. Then $E := D\langle \sigma; b, \rho\rangle := D[X, Y; \sigma, b, \rho]$ is a ring generated by D, X and Y subject to the defining relations:

$$X\alpha = \sigma(\alpha)X \text{ and } Y\alpha = \sigma^{-1}(\alpha)Y \text{ for all } \alpha \in D, \text{ and } XY - \rho YX = b.$$

If D is commutative domain, $\rho = 1$ and $b = u - \sigma(u)$ for some $u \in D$ (resp., if D is a commutative finitely generated domain over a field $\mathbb{K}$ and $\rho \in \mathbb{K}^*$) the algebras E were considered in [30] (resp., [31]).

The ring E is *the iterated skew polynomial ring* $E = D[Y; \sigma^{-1}][X; \sigma, \partial]$ where ∂ is the σ-derivation of $D[Y; \sigma^{-1}]$ such that $\partial D = 0$ and $\partial Y = b$ (here the automorphism σ is extended from D to $D[Y; \sigma^{-1}]$ by the rule $\sigma(Y) = \rho Y$).

Recall that an element d of a ring D is *normal* if $dD = Dd$. The next proposition shows that the rings E are GWAs and under a certain (mild) conditions they have a 'canonical' normal element.

Proposition 5.1. *Let $E = D[X, Y; \sigma, b, \rho]$. Then*

1. *[7, Lemma 1.2] The ring E is the GWA $D[H][X, Y; \sigma, H]$ where $\sigma(H) = \rho H + b$.*
2. *[7, Lemma 1.3] The following statements are equivalent:*
   (a) *[7, Corollary 1.4] $C = \rho(YX + \alpha) = XY + \sigma(\alpha)$ is a normal element in E for some central element $\alpha \in D$,*
   (b) *$\rho\alpha - \sigma(\alpha) = b$ for some central element $\alpha \in D$.*
3. *[7, Corollary 1.4] If one of the equivalent conditions of statement 2 holds then the ring $E = D[C][X, Y; \sigma, a = \rho^{-1}C - \alpha]$ is a GWA where $\sigma(C) = \rho C$.*

The algebra $\mathbb{E}$ is a GWA. Let $\mathbb{E}$ be the subalgebra of $\mathcal{A}$ generated by the elements X, E and Y. The generators of the algebra $\mathbb{E}$ satisfy the defining relations

$$EX = qXE, \quad YX = q^{-1}XY \quad \text{and } EY - q^{-1}YE = X.$$

So, $\mathbb{E} = \mathbb{K}[X][E, Y; \sigma, b = X, \rho = q^{-1}]$ where $\sigma(X) = qX$. The polynomial $\alpha = \frac{X}{q^{-1}-q}$ is a solution to the equation $q^{-1}\alpha - \sigma(\alpha) = X$. By Proposition 5.1, the algebra $\mathbb{E} = \mathbb{K}[X, C][E, Y; \sigma, a = qC - \alpha]$ is a GWA where $\sigma(X) = qX, \sigma(C) = q^{-1}C$ and $C = q^{-1}(YE + \frac{X}{q^{-1}-q}) = EY + \frac{qX}{q^{-1}-q}$ is a normal element of the algebra $\mathbb{E}$. Then the element $\varphi = q(q^{-1} - q)C$ is a normal element of the algebra $\mathbb{E}$. Clearly, $\varphi = (q^{-1} - q)YE + X = (1 - q^2)EY + q^2X$. Then

$$\mathbb{E} = \mathbb{K}[X, \varphi][E, Y; \sigma, a = \frac{\varphi - X}{q^{-1} - q}] \tag{5.1}$$

is a GWA where $\sigma(X) = qX$ and $\sigma(\varphi) = q^{-1}\varphi$. So, the algebra

$$\mathcal{A} = \mathbb{E}[K^{\pm1}; \tau] \tag{5.2}$$

is a skew Laurent polynomial algebra where $\tau(E) = q^2E, \tau(X) = qX, \tau(Y) = q^{-1}Y$ and $\tau(\varphi) = q\varphi$. The algebra $\mathcal{A}$ is a Noetherian domain of Gelfand-Kirillov dimension $\text{GK}\,(\mathcal{A}) = 4$.

Lemma 5.2. *The following identities hold in the algebra $\mathcal{A}$.*

1. $EY^i = \frac{q^{-2i}-1}{q^{-2}-1}XY^{i-1} + q^{-i}Y^iE$.
2. $YE^i = q^iE^iY - \frac{q(1-q^{2i})}{1-q^2}XE^{i-1}$.

Proof. Both equalities can be proved by induction on i and using the relation $EY = X + q^{-1}YE$. □

For a left denominator set $\mathcal{S}$ of a ring R, we denote by $\mathcal{S}^{-1}R = \{s^{-1}r \mid s \in \mathcal{S}, r \in R\}$ the left localization of the ring R at $\mathcal{S}$. If the left denominator set $\mathcal{S}$ is generated by elements $X_1, \ldots, X_n$, we also use the notation $R_{X_1,\ldots,X_n}$ to denote the ring $\mathcal{S}^{-1}R$. If M is a left R-module then the localization $\mathcal{S}^{-1}M$ is also denoted by $M_{X_1,\ldots,X_n}$.

By Lemma 5.2, the set $\mathcal{S}_Y := \{Y^i \,|\, i \geqslant 0\}$ is a left and right Ore set in the algebra $\mathcal{A}$. Let $\mathcal{A}_Y$ be the localization of $\mathcal{A}$ at the powers of Y. Recall that $\varphi = EY - qYE = X + (q^{-1} - q)YE$, we have

$$X\varphi = \varphi X, \qquad Y\varphi = q\varphi Y, \qquad E\varphi = q^{-1}\varphi E, \qquad K\varphi = q\varphi K.$$

So, the element φ is a normal element of the algebra $\mathcal{A}$. By (5.1), the localization $\mathbb{E}_Y$ of the algebra $\mathbb{E}$ at the powers of the element Y is the skew Laurent polynomial algebra $\mathbb{E}_Y = \mathbb{K}[X,\varphi][Y^{\pm 1};\sigma^{-1}]$ where $\sigma(X) = qX$ and $\sigma(\varphi) = q^{-1}\varphi$. Similarly, $\mathbb{E}_E = \mathbb{K}[X,\varphi][E^{\pm 1};\sigma] \simeq \mathbb{E}_Y$ where $\sigma(X) = qX$ and $\sigma(\varphi) = q^{-1}\varphi$. Then by (5.2),

$$\mathcal{A}_Y = \mathbb{E}_Y[K^{\pm 1};\tau] = \mathbb{K}[\varphi, X][Y^{\pm 1};\sigma][K^{\pm 1};\tau] \tag{5.3}$$

is an iterated skew polynomial ring where σ is the automorphism of $\mathbb{K}[\varphi, X]$ defined by $\sigma(\varphi) = q\varphi$, $\sigma(X) = q^{-1}X$; and τ is the automorphism of the algebra $\mathbb{K}[\varphi, X][Y^{\pm 1};\sigma]$ defined by $\tau(\varphi) = q\varphi, \tau(X) = qX, \tau(Y) = q^{-1}Y$. Let $\mathcal{A}_{Y,X,\varphi}$ be the localization of $\mathcal{A}_Y$ at the denominator set $\{X^i\varphi^j \,|\, i,j \in \mathbb{N}\}$, then $\mathcal{A}_{Y,X,\varphi} = \mathbb{K}[\varphi^{\pm 1}, X^{\pm 1}][Y^{\pm 1};\sigma][K^{\pm 1};\tau]$ is a quantum torus. For an algebra A we denote by $Z(A)$ its centre. The next result shows that the algebra $\mathcal{A}$ and some of its localizations have trivial centre.

Lemma 5.3. *1. $Z(\mathcal{A}_{Y,X,\varphi}) = \mathbb{K}$.*

2. $\mathcal{A}_{Y,X,\varphi}$ is a simple algebra.

3. $Z(\mathcal{A}_Y) = \mathbb{K}$.

4. $Z(\mathcal{A}) = \mathbb{K}$.

Proof. 1. Let $u = \sum \alpha_{i,j,k,l}\varphi^i X^j Y^k K^l \in Z(\mathcal{A}_{Y,X,\varphi})$, where $\alpha_{i,j,k,l} \in \mathbb{K}$ and $i,j,k,l \in \mathbb{Z}$. Since $Ku = uK$, we have $i + j - k = 0$. The equality $Xu = uX$ implies that $k - l = 0$. Similarly, the equality $Yu = uY$ implies that $i - j + l = 0$. Finally, using $\varphi u = u\varphi$ we get $-k - l = 0$. Therefore, we have $i = j = k = l = 0$, and so $u \in \mathbb{K}$. Thus $Z(\mathcal{A}_{Y,X,\varphi}) = \mathbb{K}$.

2. By [27, Corollary 1.5.(a)], contraction and extension provide mutually inverse isomorphisms between the lattices of ideals of a quantum torus and its centre. Then statement 2 follows from statement 1.

3. Since $\mathbb{K} \subseteq Z(\mathcal{A}_Y) \subseteq Z(\mathcal{A}_{Y,X,\varphi}) \cap \mathcal{A}_Y = \mathbb{K}$, we have $Z(\mathcal{A}_Y) = \mathbb{K}$.

4. Since $\mathbb{K} \subseteq Z(\mathcal{A}) \subseteq Z(\mathcal{A}_Y) \cap \mathcal{A} = \mathbb{K}$, we have $Z(\mathcal{A}) = \mathbb{K}$. □

Lemma 5.4. *The algebra $\mathcal{A}_{X,\varphi}$ is a central simple algebra.*

Proof. By Lemma 5.3.(1), the algebra $\mathcal{A}_{Y,X,\varphi}$ is central, hence so is the algebra $\mathcal{A}_{X,\varphi}$. By Lemma 5.3.(2), the algebra $(\mathcal{A}_{X,\varphi})_Y = \mathcal{A}_{Y,X,\varphi}$ is a simple Noetherian domain. So, if I is a nonzero ideal of the algebra $\mathcal{A}_{X,\varphi}$ then $Y^i \in I$ for some $i \geqslant 0$. To finish the proof it suffices to show that

$$(Y^i) = \mathcal{A}_{X,\varphi} \quad \text{for all } i \geqslant 1. \tag{5.4}$$

To prove the equality we use induction on i. Let $i = 1$. Then $X = EY - q^{-1}YE \in (Y)$. Since X is a unit of the algebra $\mathcal{A}_{X,\varphi}$, the equality (5.4) holds for $i = 1$. Suppose that $i \geqslant 1$

and equality (5.4) holds for all i' such that $i' < i$. By Lemma 5.2.(1), $Y^{i-1} \in (Y^i)$. Hence, $(Y^i) = (Y^{i-1}) = \mathcal{A}_{X,\varphi}$, by induction. □

The element φ is a normal element of the algebras $\mathbb{E}$ and $\mathcal{A}$. So, the localizations of the algebras $\mathbb{E}_Y$ and $\mathcal{A}_Y$ at the powers of φ are as follows

$$\mathbb{E}_{Y,\varphi} = \mathbb{K}[X, \varphi^{\pm 1}][Y^{\pm 1}; \sigma^{-1}] \text{ and } \mathcal{A}_{Y,\varphi} = \mathbb{E}_{Y,\varphi}[K^{\pm 1}; \tau]. \tag{5.5}$$

Now, we introduce several factor algebras and localizations of $\mathcal{A}$ that play a key role in finding the prime spectrum of the algebra $\mathcal{A}$ (Theorem 5.8) and all the prime factor algebras of $\mathcal{A}$ (Theorem 5.8). In fact, explicit sets of generators and defining relations are found for all prime factor algebras of $\mathcal{A}$ (Theorem 5.8). Furthermore, all these algebras are domains, i.e., all prime ideals of $\mathcal{A}$ are completely prime (Corollary 5.12).

The algebra $\mathcal{A}/(X)$. The element X is a normal element in the algebras $\mathbb{E}$ and $\mathcal{A}$. By (5.1), the factor algebra

$$\mathbb{E}/(X) = \mathbb{K}[\varphi][E, Y; \sigma, a = \frac{\varphi}{q^{-1} - q}], \quad \sigma(\varphi) = q^{-1}\varphi, \tag{5.6}$$

is a GWA. Since $YE = \frac{\varphi}{q^{-1}-q}$, $EY = q^{-1}\frac{\varphi}{q^{-1}-q} = q^{-1}YE$, the algebra

$$\mathbb{E}/(X) \simeq \mathbb{K}\langle E, Y \mid EY = q^{-1}YE\rangle \tag{5.7}$$

is isomorphic to the quantum plane. It is a Noetherian domain of Gelfand-Kirillov dimension 2. Now, the factor algebra

$$\mathcal{A}/(X) \simeq \mathbb{E}/(X)[K^{\pm 1}; \tau] \tag{5.8}$$

is a skew Laurent polynomial algebra where $\tau(E) = q^2E$ and $\tau(Y) = q^{-1}Y$. It is a Noetherian domain of Gelfand-Kirillov dimension 3. The element of the algebra $\mathcal{A}/(X)$, $Z := \varphi YK^{-1} = (1-q^2)EY^2K^{-1}$, belongs to the centre of the algebra $\mathcal{A}/(X)$. By (5.5), the localization of the algebra $\mathcal{A}/(X)$ at the powers of the central element Z,

$$\big(\mathcal{A}/(X)\big)_Z \simeq \frac{\mathcal{A}_{Y,\varphi}}{(X)_{Y,\varphi}} \simeq \mathbb{K}[Z^{\pm 1}] \otimes \mathbb{Y}, \tag{5.9}$$

is the tensor product of algebras where the algebra $\mathbb{Y} := \mathbb{K}[Y^{\pm 1}][K^{\pm 1}; \tau]$ is a central simple algebra since $\tau(Y) = q^{-1}Y$ and q is not a root of unity. Hence, the centre of the algebra $(\mathcal{A}/(X))_Z$ is $\mathbb{K}[Z^{\pm 1}]$. The algebra $(\mathcal{A}/(X))_Z$ is a Noetherian domain of Gelfand-Kirillov dimension 3.

Lemma 5.5. *1.* $Z\big(\mathcal{A}/(X)\big) = \mathbb{K}[Z]$.

2. $Z\Big(\big(\mathcal{A}/(X)\big)_Z\Big) = \mathbb{K}[Z^{\pm 1}]$.

Proof. $Z\big(\mathcal{A}/(X)\big) = \mathcal{A}/(X) \cap Z\Big(\big(\mathcal{A}/(X)\big)_Z\Big) = \mathcal{A}/(X) \cap \mathbb{K}[Z^{\pm 1}] = \mathbb{K}[Z]$, by (5.8). □

The algebra $\mathcal{A}/(\varphi)$. The element φ is a normal element in the algebras $\mathbb{E}$ and $\mathcal{A}$. By (5.1), the factor algebra

$$\mathbb{E}/(\varphi) = \mathbb{K}[X][E, Y; \sigma, a = -\frac{X}{q^{-1} - q}], \quad \sigma(X) = qX, \tag{5.10}$$

is a GWA. Since $YE = -\frac{X}{q^{-1}-q}$ and $EY = q(-\frac{X}{q^{-1}-q}) = qYE$, the algebra

$$\mathbb{E}/(\varphi) \simeq \mathbb{K}\langle E, Y \,|\, EY = qYE\rangle \tag{5.11}$$

is isomorphic to the quantum plane. It is a Noetherian domain of Gelfand-Kirillov dimension 2. Now, the factor algebra

$$\mathcal{A}/(\varphi) \simeq \mathbb{E}/(\varphi)[K^{\pm 1}; \tau] \tag{5.12}$$

is a skew Laurent polynomial algebra where $\tau(E) = q^2 E$ and $\tau(Y) = q^{-1}Y$. The algebra $\mathcal{A}/(\varphi)$ is a Noetherian domain of Gelfand-Kirillov dimension 3. The element $C := XYK \in \mathcal{A}/(\varphi)$ belongs to the centre of the algebra $\mathcal{A}/(\varphi)$.

The localization $\mathbb{E}_{X,Y}$ of the algebra $\mathbb{E}$ at the Ore set $\mathcal{S} = \{X^i Y^j \,|\, i, j \in \mathbb{N}\}$,

$$\mathbb{E}_{X,Y} = \mathbb{K}[X^{\pm 1}, \varphi][Y^{\pm 1}; \sigma^{-1}], \quad \sigma(X) = qX, \quad \sigma(\varphi) = q^{-1}\varphi, \tag{5.13}$$

is a skew Laurent polynomial algebra. Then the localization $\mathcal{A}_{X,Y}$ of the algebra $\mathcal{A}$ at the Ore set $\mathcal{S}$ is equal to $\mathcal{A}_{X,Y} = \mathbb{E}_{X,Y}[K^{\pm 1}; \tau]$. By (5.11) and (5.12), the localization of the algebra $\mathcal{A}/(\varphi)$ at the powers of the element C,

$$\left(\frac{\mathcal{A}}{(\varphi)}\right)_C \simeq \frac{\mathcal{A}_{X,Y}}{(\varphi)_{X,Y}} \simeq \left(\frac{\mathcal{A}_X}{(\varphi)_X}\right)_Y \simeq \mathbb{K}[C^{\pm 1}] \otimes \mathbb{Y} \tag{5.14}$$

is a tensor product of algebras where $\mathbb{Y}$ is the central simple algebra as in (5.9). Hence, the centre of the algebra $\big(\mathcal{A}/(\varphi)\big)_C$ is $\mathbb{K}[C^{\pm 1}]$.

Lemma 5.6. *1.* $Z\big(\mathcal{A}/(\varphi)\big) = \mathbb{K}[C]$.

2. $Z\Big(\big(\mathcal{A}/(\varphi)\big)_C\Big) = \mathbb{K}[C^{\pm 1}]$.

Proof. $Z\big(\mathcal{A}/(\varphi)\big) = \mathcal{A}/(\varphi) \cap Z\Big(\big(\mathcal{A}/(\varphi)\big)_C\Big) = \mathcal{A}/(\varphi) \cap \mathbb{K}[C^{\pm 1}] = \mathbb{K}[C]$, by (5.12). □

Let $f : A \to B$ be an algebra epimorphism. Then Spec (B) can be seen as a subset of Spec (A) via the injection Spec $(B) \to$ Spec (A), $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$. So, Spec $(B) = \{\mathfrak{q} \in \text{Spec}\,(A) \,|\, \ker(f) \subseteq \mathfrak{q}\}$. Given a left denominator set $\mathcal{S}$ of the algebra A. Then $\sigma : A \to \mathcal{S}^{-1}A$, $a \mapsto 1^{-1}a$, is an algebra homomorphism. If the algebra A is a Noetherian algebra then Spec $(\mathcal{S}^{-1}A)$ can be seen as a subset of Spec (A) via the injection Spec $(\mathcal{S}^{-1}A) \to$ Spec (A), $\mathfrak{q} \mapsto \sigma^{-1}(\mathfrak{q})$.

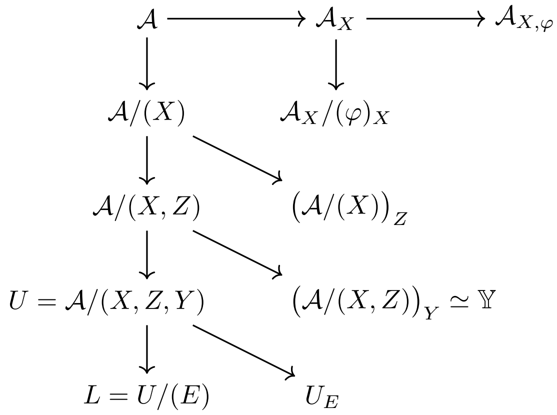
In the proof of Theorem 5.8 the following very useful lemma is used repeatedly.

Lemma 5.7. *Let A be a ring, $\mathcal{S}$ be a left denominator set of A and $\sigma : A \to \mathcal{S}^{-1}A$, $a \mapsto \frac{a}{1}$. Let $\mathfrak{q}$ be a completely prime ideal of $\mathcal{S}^{-1}A$, $\mathfrak{p}$ be an ideal of A such that $\mathfrak{p} \subseteq \sigma^{-1}(\mathfrak{q})$ and $\mathcal{S}^{-1}\mathfrak{p} = \mathfrak{q}$. Then $\mathfrak{p} = \sigma^{-1}(\mathfrak{q})$ iff $A/\mathfrak{p}$ is a domain.*

Proof. (⇒) Since $A/\sigma^{-1}(\mathfrak{q}) \subseteq \mathcal{S}^{-1}A/\mathfrak{q}$ and $\mathcal{S}^{-1}A/\mathfrak{q}$ is a domain (since $\mathfrak{q}$ is a completely prime ideal), the algebra $A/\sigma^{-1}(\mathfrak{q})$ is a domain.

(⇐) The left $\mathcal{S}^{-1}A$-module $\mathcal{S}^{-1}(A/\mathfrak{p}) \simeq \mathcal{S}^{-1}A/\mathcal{S}^{-1}\mathfrak{p} \simeq \mathcal{S}^{-1}A/\mathfrak{q}$ is not equal to zero. In particular, $\mathcal{S} \cap \mathfrak{p} = \emptyset$. So, for all $s \in \mathcal{S}$, the elements $s + \mathfrak{p}$ are nonzero in $A/\mathfrak{p}$. Since $A/\mathfrak{p}$ is a domain, $\mathrm{tor}_{\mathcal{S}}(A/\mathfrak{p}) = 0$. Clearly, $\sigma^{-1}(\mathfrak{q})/\mathfrak{p} \subseteq \mathrm{tor}_{\mathcal{S}}(A/\mathfrak{p})$. Hence, $\mathfrak{p} = \sigma^{-1}(\mathfrak{q})$. □

The prime spectrum of the algebra $\mathcal{A}$. The key idea in finding the prime spectrum of the algebra $\mathcal{A}$ is to use Proposition 3.3 repeatedly and the following diagram of algebra homomorphisms



(5.15)

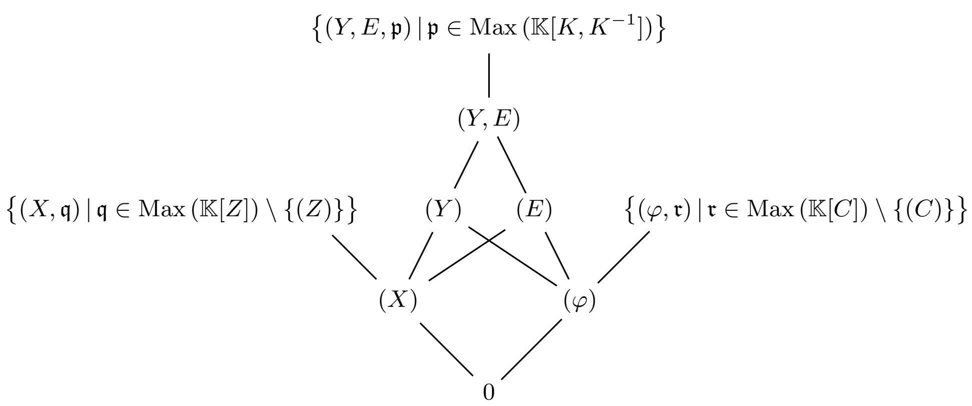
$\big($where $L = \mathbb{K}[K^{\pm 1}]$ and $U := U_q^{\geqslant 0}(\mathfrak{sl}_2)\big)$

that explains the choice of elements at which we localize. Using (5.15) and Proposition 3.3, we represent the spectrum Spec $(\mathcal{A})$ as the disjoint union of the following subsets where we identify the sets of prime ideals via the bijections given in the statements (a) and (b) of Proposition 3.3:

$$\begin{aligned}\mathrm{Spec}\,(\mathcal{A}) = \mathrm{Spec}\,(L) \,\sqcup\, \mathrm{Spec}\,(U_E) \,\sqcup\, \mathrm{Spec}\,(\mathbb{Y}) \\ \sqcup\, \mathrm{Spec}\,\big((\mathcal{A}/(X))_Z\big) \,\sqcup\, \mathrm{Spec}\,\big(\mathcal{A}_X/(\varphi)_X\big) \,\sqcup\, \mathrm{Spec}\,(\mathcal{A}_{X,\varphi}).\end{aligned} \tag{5.16}$$

The theorem below gives an explicit description of the prime ideals of the algebra $\mathcal{A}$ together with inclusions of prime ideals.

Theorem 5.8. *The prime spectrum* Spec $(\mathcal{A})$ *of the algebra* $\mathcal{A}$ *is the disjoint union of sets (5.16). In the diagram (5.17), all the inclusions of prime ideals are given (lines represent inclusions of primes). More precisely,*



(5.17)

where

1. $\mathrm{Spec}\,(L) = \{(Y, E, \mathfrak{p}) \,|\, \mathfrak{p} \in \mathrm{Spec}\,(\mathbb{K}[K^{\pm 1}]) \setminus \{0\}\}$ *and* $\mathcal{A}/(Y, E, \mathfrak{p}) \simeq L/\mathfrak{p}$ *where* $L = \mathbb{K}[K^{\pm 1}]$.
2. $\mathrm{Spec}\,(U_E) = \{(Y)\}$, $(Y) = (X, Y) = (X, Z, Y)$ *and* $\mathcal{A}/(Y) \simeq \mathbb{K}[E][K^{\pm 1}; \tau]$ *where* $\tau(E) = q^2 E$.
3. $\mathrm{Spec}\,(\mathbb{Y}) = \{(E)\}$, $(E) = (E, X) = (E, \varphi)$ *and* $\mathcal{A}/(E) \simeq \mathbb{K}[Y][K^{\pm 1}; \tau]$ *where* $\tau(Y) = q^{-1} Y$.
4. $\mathrm{Spec}\,\big((\mathcal{A}/(X))_Z\big) = \{(X), (X, \mathfrak{q}) \,|\, \mathfrak{q} \in \mathrm{Max}(\mathbb{K}[Z]) \setminus \{(Z)\}\}, Z = \varphi Y K^{-1}$,
   (a) $\mathcal{A}/(X) \simeq \mathbb{E}/(X)[K^{\pm 1}; \tau]$ *is a domain (see (5.8)), and*
   (b) $\mathcal{A}/(X, \mathfrak{q}) \simeq \frac{\mathcal{A}_{\varphi, Y}}{(X, \mathfrak{q})_{\varphi, Y}} \simeq L_{\mathfrak{q}} \otimes \mathbb{Y}$ *is a simple domain which is a tensor product of algebras where* $L_{\mathfrak{q}} := \mathbb{K}[Z]/\mathfrak{q}$ *is a finite field extension of* $\mathbb{K}$.
5. $\mathrm{Spec}\,(\frac{\mathcal{A}_X}{(\varphi)_X}) = \{(\varphi), (\varphi, \mathfrak{r}) \,|\, \mathfrak{r} \in \mathrm{Max}(\mathbb{K}[C]) \setminus \{(C)\}\}, C = XYK$, $\frac{\mathcal{A}_X}{(\varphi)_X} \simeq \frac{\mathcal{A}_{X,Y}}{(\varphi)_{X,Y}} \simeq \mathbb{K}[C^{\pm 1}] \otimes \mathbb{Y}$,
   (a) $\mathcal{A}/(\varphi) = \mathbb{E}/(\varphi)[K^{\pm 1}; \tau]$ *is a domain (see (5.12)).*
   (b) $\mathcal{A}/(\varphi, \mathfrak{r}) \simeq \frac{\mathcal{A}_{X,Y}}{(\varphi, \mathfrak{r})_{X,Y}} \simeq L_{\mathfrak{r}} \otimes \mathbb{Y}$ *is a simple domain which is a tensor product of algebras where* $L_{\mathfrak{r}} := \mathbb{K}[C]/\mathfrak{r}$ *is a finite field extension of* $\mathbb{K}$.
6. $\mathrm{Spec}\,(\mathcal{A}_{X,\varphi}) = \{0\}$.

Proof. As it was already mentioned above, we identify the sets of prime ideals via the bijection given in the statements (a) and (b) of Proposition 3.3. Recall that the set $\mathcal{S}_X = \{X^i \,|\, i \in \mathbb{N}\}$ is a left and right denominator set of $\mathcal{A}$ and $\mathcal{A}_X := \mathcal{S}_X^{-1}\mathcal{A} \simeq \mathcal{A}\mathcal{S}_X^{-1}$ is a Noetherian domain (since $\mathcal{A}$ is so). The element X is a normal element of $\mathcal{A}$. By Proposition 3.3,

$$\mathrm{Spec}\,(A) = \mathrm{Spec}\,(\mathcal{A}, X) \;\sqcup\; \mathrm{Spec}\,(\mathcal{A}_X) \tag{5.18}$$

and none of the ideals of the set $\mathrm{Spec}\,(\mathcal{A}, X)$ is contained in an ideal of the set $\mathrm{Spec}\,(\mathcal{A}_X)$. Similarly, the element φ is a normal element of $\mathcal{A}_X$ and, by Proposition 3.3

$$\mathrm{Spec}\,(\mathcal{A}_X) = \mathrm{Spec}\,\Big(\big(\mathcal{A}/(\varphi)\big)_X\Big) \;\sqcup\; \mathrm{Spec}\,(\mathcal{A}_{X,\varphi}). \tag{5.19}$$

By Lemma 5.4, the algebra $\mathcal{A}_{X,\varphi}$ is a simple domain. Hence, $\mathrm{Spec}\,(\mathcal{A}_{X,\varphi}) = \{0\}$, and statement 6 is proved.

(i) $\frac{\mathcal{A}_X}{(\varphi)_X} \simeq \frac{\mathcal{A}_{X,Y}}{(\varphi)_{X,Y}} \simeq \mathbb{K}[C^{\pm 1}] \otimes \mathbb{Y}$: The second isomorphism holds, by (5.14). Using the equalities $\varphi = (q^{-1} - q)YE + X = (1 - q^2)EY + q^2 X$ we see that the elements Y and E are invertible in the algebra $\frac{\mathcal{A}_X}{(\varphi)_X}$, and so the first isomorphism holds.

(ii) $\mathcal{A}/(\varphi, \mathfrak{r}) \simeq \frac{\mathcal{A}_{X,Y}}{(\varphi, \mathfrak{r})_{X,Y}} \simeq L_{\mathfrak{r}} \otimes \mathbb{Y}$ *for all prime ideals* $\mathfrak{r} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}$: Since $\mathfrak{r} \neq (C)$, the non-zero element $C = XYK \in L_{\mathfrak{r}}$ is invertible in the field $L_{\mathfrak{r}}$. Hence, the elements X and Y are invertible in the algebra $\mathcal{A}/(\varphi, \mathfrak{r})$. Hence,

$$\mathcal{A}/(\varphi, \mathfrak{r}) = \frac{\mathcal{A}_{X,Y}}{(\varphi, \mathfrak{r})_{X,Y}}. \tag{5.20}$$

Now, the statement (ii) follows from (5.20) and the statement (i).

(iii) *Statement 5 holds*: Recall that the algebra $\mathbb{Y}$ is a central simple algebra. By the statement (i), the set $\mathrm{Spec}\,(\mathcal{A}_X/(\varphi)_X)$, as a subset of $\mathrm{Spec}\,(\mathcal{A})$, is equal to $\{\mathcal{A} \cap (\varphi)_X,\, \mathcal{A} \cap (\varphi, \mathfrak{r})_X \,|\, \mathfrak{r} \in$

$\mathrm{Max}(\mathbb{K}[C]) \setminus \{(C)\}\}$. It remains to show that $\mathcal{A} \cap (\varphi)_X = (\varphi)$ and $\mathcal{A} \cap (\varphi, \mathfrak{r})_X = (\varphi, \mathfrak{r})$. The second equality follows from the statement (ii) since $(\varphi, \mathfrak{r}) \subseteq \mathcal{A} \cap (\varphi, \mathfrak{r})_X \subseteq \mathcal{A} \cap (\varphi, \mathfrak{r})_{X,Y} \overset{\text{(ii)}}{=} (\varphi, \mathfrak{r})$. Since $\mathbb{Y}$ is a central simple algebra, the statement (b) of statement 5 follows. The statement (a) of statement 5 is obvious (see (5.12)). Hence, $(\varphi) = \mathcal{A} \cap (\varphi)_X$, by Lemma 5.7. So, statement 5 holds.

By Proposition 3.3,

$$\mathrm{Spec}\left(\mathcal{A}/(X)\right) = \mathrm{Spec}\left(\mathcal{A}/(X,Z)\right) \sqcup \mathrm{Spec}\left(\left(\mathcal{A}/(X)\right)_Z\right) \tag{5.21}$$

and none of the ideals of the set $\mathrm{Spec}\,(\mathcal{A}/(X,Z))$ is contained in an ideal of the set $\mathrm{Spec}\,((\mathcal{A}/(X))_Z)$.

(iv) $\mathcal{A}/(X,\mathfrak{q}) \simeq \frac{\mathcal{A}_{Y,\varphi}}{(X,\mathfrak{q})_{Y,\varphi}} \simeq L_{\mathfrak{q}} \otimes \mathbb{Y}$ *is a simple domain for all prime ideals* $\mathfrak{q} \in \mathrm{Max}(\mathbb{K}[Z]) \setminus \{(Z)\}$: Since $\mathfrak{q} \neq (Z)$, the non-zero element $Z = \varphi Y K^{-1} \in L_{\mathfrak{q}}$ is invertible in the field $L_{\mathfrak{q}}$. Hence, the elements φ and Y are invertible in the algebra $\mathcal{A}/(X,\mathfrak{q})$. Therefore,

$$\mathcal{A}/(X,\mathfrak{q}) \simeq \frac{\mathcal{A}_{Y,\varphi}}{(X,\mathfrak{q})_{Y,\varphi}}. \tag{5.22}$$

Now, by (5.9), the statement (iv) holds.

(v) *Statement 4 holds*: The algebra $\mathbb{Y}$ is a central simple algebra. By (5.9), the set $\mathrm{Spec}\,((\mathcal{A}/(X))_Z)$, as a subset of $\mathrm{Spec}\,(\mathcal{A})$, is equal to $\{\mathcal{A} \cap (X)_{Y,\varphi}, \mathcal{A} \cap (X,\mathfrak{q})_{Y,\varphi} \,|\, \mathfrak{q} \in \mathrm{Max}(\mathbb{K}[Z]) \setminus \{(Z)\}\}$. We have to show that $\mathcal{A} \cap (X)_{Y,\varphi} = (X)$ and $\mathcal{A} \cap (X,\mathfrak{q})_{Y,\varphi} = (X,\mathfrak{q})$. The last equality follows from the statement (iv) (the algebra $\mathcal{A}/(X,\mathfrak{q})$ is simple and $(X,\mathfrak{q}) \subseteq \mathcal{A} \cap (X,\mathfrak{q})_{Y,\varphi} \subsetneqq \mathcal{A}$, hence $(X,\mathfrak{q}) = \mathcal{A} \cap (X,\mathfrak{q})_{Y,\varphi})$. Now, the statement (b) of statement 4 holds. The statement (a) is obvious, see (5.8). Hence, $(X) = \mathcal{A} \cap (X)_{Y,\varphi}$, by Lemma 5.7. So, the proof of the statement (v) is complete.

In the algebra $\mathcal{A}$, using the equality $\varphi = X + (q^{-1} - q)YE$ we see that

$$Z \equiv \varphi Y K^{-1} \equiv (q^{-1} - q)YEYK^{-1} \equiv (1 - q^2)EY^2K^{-1} \mod (X). \tag{5.23}$$

(vi) $(Y) = (X,Y) = (X,Z,Y)$: The first equality follows from the relation $X = EY - q^{-1}YE$. Then the second equality follows from (5.23).

(vii) $(E) = (E,X) = (E,\varphi)$: The first equality follows from the relation $X = EY - q^{-1}YE$. Then the second equality follows from the definition of the element $\varphi = X + (q^{-1} - q)YE$.

(viii) *The elements* Y *and* E *are normal in* $\mathcal{A}/(X)$: The statement follows from (5.7) and (5.8).

(ix) $\left(\frac{\mathcal{A}}{(X,Z)}\right)_Y \simeq \left(\frac{\mathcal{A}}{(E)}\right)_Y \simeq \mathbb{Y}$ *is a simple domain*: By (5.23), $(X,Z) = (X,EY^2) \subseteq (X,E) \overset{\text{(vii)}}{=} (E)$, hence $(X,EY^2)_Y = (X,E)_Y = (E)_Y$, by the statement (vii). Now, by (5.8), $\left(\frac{\mathcal{A}}{(X,Z)}\right)_Y \simeq \frac{\mathcal{A}_Y}{(X,Z)_Y} \simeq \frac{\mathcal{A}_Y}{(E)_Y} \simeq \left(\frac{\mathcal{A}}{(X,E)}\right)_Y \simeq \mathbb{Y}$.

By the statement (viii) and Proposition 3.3,

$$\mathrm{Spec}\left(\mathcal{A}/(X,Z)\right) = \mathrm{Spec}\left(\mathcal{A}/(X,Z,Y)\right) \sqcup \mathrm{Spec}\left(\left(\mathcal{A}/(X,Z)\right)_Y\right). \tag{5.24}$$

By the statement (vi), $\mathcal{A}/(X,Z,Y) \simeq \mathcal{A}/(X,Y) \simeq U := U_q^{\geqslant 0}(\mathfrak{sl}_2)$. By the statement (ix), $\big(\mathcal{A}/(X,Z)\big)_Y \simeq \mathcal{A}_Y/(E)_Y \simeq \mathbb{Y}$ is a simple domain. So, the set $\mathrm{Spec}\,\big(\big(\mathcal{A}/(X,Z)\big)_Y\big)$, as a subset of $\mathrm{Spec}\,(\mathcal{A})$, consists of the ideal (E). In more details, since $(X,Z) \subseteq (E), (X,Z)_Y = (E)_Y$ (see the proof of the statement (ix)) and $\mathcal{A}/(E) = \mathcal{A}/(E,X) = \mathbb{K}[Y][K^{\pm 1};\tau]$ is a domain, the result follows from Lemma 5.7. So, statement 3 holds.

The element E is a normal element of the algebra U. By Proposition 3.3,

$$\mathrm{Spec}\,(U) = \mathrm{Spec}\,(U/(E)) \sqcup \mathrm{Spec}\,(U_E). \tag{5.25}$$

Since $L = U/(E)$, statement 1 follows. The algebra $U_E \simeq \mathbb{K}[E^{\pm 1}][K^{\pm 1};\tau]$ is a central simple domain. Since $U = \mathcal{A}/(Y) = \mathcal{A}/(X,Z,Y)$ (the statement (vi)) is a domain, the set $\mathrm{Spec}\,(U_E)$, as a subset of $\mathrm{Spec}\,(\mathcal{A})$, consists of a single ideal (Y), and statement 2 follows.

We proved that (5.16) holds. Clearly, we have the inclusions as on the diagram (5.17). It remains to show that there are no other inclusions. The ideals $(Y,E,\mathfrak{p}), (X,\mathfrak{q})$ and $(\varphi,\mathfrak{r})$ are the maximal ideals of the algebra $\mathcal{A}$ (see statement 1, 4, and 5). By (5.22) and the relations given in (5.17), there are no additional lines leading to the maximal ideals $(X,\mathfrak{q})$. Similarly, by (5.20) and the relations given in (5.17), there are no additional lines leading to the maximal ideals $(\varphi,\mathfrak{r})$. The elements X and φ are normal elements of the algebra $\mathcal{A}$ such that $(X) \nsubseteq (\varphi)$ and $(X) \nsupseteq (\varphi)$, by (5.2). The proof of the theorem is complete. □

The next corollary is an explicit description of the set $\mathrm{Max}(\mathcal{A})$.

Corollary 5.9. $\mathrm{Max}\,(\mathcal{A}) = \mathcal{P} \sqcup \mathcal{Q} \sqcup \mathcal{R}$ *where* $\mathcal{P} := \big\{(Y,E,\mathfrak{p}) \,|\, \mathfrak{p} \in \mathrm{Max}\,(\mathbb{K}[K,K^{-1}])\big\}$, $\mathcal{Q} := \big\{(X,\mathfrak{q}) \,|\, \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}\big\}$ *and* $\mathcal{R} := \big\{(\varphi,\mathfrak{r}) \,|\, \mathfrak{r} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}\big\}$.

Proof. The corollary follows from (5.17). □

The $\mathcal{A}$-module is called *faithful* if it has zero annihilator. The next corollary is a faithfulness criterion for simple $\mathcal{A}$-modules.

Corollary 5.10. *Let M be a simple $\mathcal{A}$-module. Then M is a faithful $\mathcal{A}$-module iff* $\ker(X_M\cdot) = \ker(\varphi_M\cdot) = 0$ *iff $M_X \neq 0$ and $M_\varphi \neq 0$ (where M_X and M_φ are the localizations of the $\mathcal{A}$-module M at the powers of the elements X and φ, respectively).*

Proof. The $\mathcal{A}$-module M is simple, so $\mathrm{ann}_{\mathcal{A}}(M) \in \mathrm{Spec}\,(\mathcal{A})$. The elements X and φ are normal elements of the algebra $\mathcal{A}$. So, $\ker(X_M\cdot)$ and $\ker(\varphi_M\cdot)$ are submodules of M. Either $\ker(X_M\cdot) = 0$ or $\ker(X_M\cdot) = M$, and in the second case $\mathrm{ann}_{\mathcal{A}}(M) \supseteq (X)$. Similarly, either $\ker(\varphi_M\cdot) = 0$ or $\ker(\varphi_M\cdot) = M$, and in the second case $\mathrm{ann}_{\mathcal{A}}(M) \supseteq (\varphi)$. Conversely, if $\mathrm{ann}_{\mathcal{A}}(M) = 0$ then $\ker(X_M\cdot) = \ker(\varphi_M\cdot) = 0$. If $\ker(X_M\cdot) = \ker(\varphi_M\cdot) = 0$ then $\mathrm{ann}_{\mathcal{A}}(M) = 0$, by (5.17). So, the first 'iff' holds.

For a normal element $u = X, \varphi$, $\ker(u_M\cdot) = 0$ iff $M_u \neq 0$. Hence, the second 'iff' follows. □

The next proposition gives an explicit description of primitive ideals of the algebra $\mathcal{A}$.

Proposition 5.11. $\text{Prim}(\mathcal{A}) = \text{Max}(\mathcal{A}) \sqcup \{(Y),\ (E),\ 0\}$.

Proof. Clearly, $\text{Prim}(\mathcal{A}) \supseteq \text{Max}(\mathcal{A})$. The ideals (X), (φ) and (Y, E) are not primitive ideals as the corresponding factor algebras contain the central elements Z, C and K, respectively.

(i) Let us show that $(Y) \in \text{Prim}(\mathcal{A})$. For $\lambda \in \mathbb{K}^*$, let $I_\lambda = (Y) + \mathcal{A}(E - \lambda)$. Since $\mathcal{A}/(Y) \simeq U$, the left $\mathcal{A}$-module $M(\lambda) := \mathcal{A}/I(\lambda) \simeq U/U(E - \lambda) \simeq \mathbb{K}[K^{\pm 1}]\bar{1}$ is a simple $\mathcal{A}$-module/U-module where $\bar{1} = 1 + I(\lambda)$. By the very definition, the prime ideal $\mathfrak{a} := \text{ann}_{\mathcal{A}}(M(\lambda))$ contains the ideal (Y) but does not contain the ideal (Y, E) since otherwise we would have $0 = E\bar{1} = \lambda\bar{1} \neq 0$, a contradiction. By (5.17), $\mathfrak{a} = (Y)$.

(ii) Let us show that $(E) \in \text{Prim}(\mathcal{A})$. By Theorem 5.8, $(E) = (E, X)$ and $\bar{\mathcal{A}} := \mathcal{A}/(E) \simeq \mathbb{K}[Y][K^{\pm 1}; \sigma]$ where $\sigma(Y) = q^{-1}Y$. For $\lambda \in \mathbb{K}^*$, the $\mathcal{A}$-module $T(\lambda) := \bar{\mathcal{A}}/\bar{\mathcal{A}}(Y - \lambda) \simeq \mathbb{K}[K^{\pm 1}]\bar{1}$ is a simple module (since q is not a root of 1), where $\bar{1} = 1 + \bar{\mathcal{A}}(Y - \lambda)$. Clearly, the prime ideal $\mathfrak{b} := \text{ann}_{\mathcal{A}}(T(\lambda))$ contains the ideal (E) but does not contain the ideal (Y, E) since otherwise we would have $0 = Y\bar{1} = \lambda\bar{1} \neq 0$, a contradiction. By (5.17), $\mathfrak{b} = (E)$.

(iii) *0 is a primitive ideal of* $\mathcal{A}$: For $\lambda \in \mathbb{K}^*$, we define the $\mathcal{A}$-module $S(\lambda) := \mathcal{A}/\mathcal{A}(KX - \lambda, Y)$. Then $S(\lambda) = \bigoplus_{i \geqslant 0} \mathbb{K}[K^{\pm 1}]E^i\bar{1}$ where $\bar{1} = 1 + \mathcal{A}(KX - \lambda, Y)$. Let $t = YX$ then $Kt = tK$ and $tE^i = E^i t - \frac{1-q^{2i}}{1-q^2} X^2 E^{i-1}$. The fact that $S(\lambda)$ is a simple $\mathcal{A}$-module follows from the equality: $tE^i\bar{1} = -q^{2i-1}\frac{1-q^{2i}}{1-q^2}\lambda^2 K^{-2} E^{i-1}\bar{1}$. Since $X \notin \text{ann}_{\mathcal{A}}(S(\lambda))$ and $\varphi \notin \text{ann}_{\mathcal{A}}(S(\lambda))$, by (5.17), $\text{ann}_{\mathcal{A}}(S(\lambda)) = 0$. Thus 0 is a primitive ideal of the algebra $\mathcal{A}$. □

Corollary 5.12. *Every prime ideal of the algebra* $\mathcal{A}$ *is completely prime, i.e.,* $\text{Spec}_c(\mathcal{A}) = \text{Spec}\,(\mathcal{A})$.

Proof. See Theorem 5.8. □

Stratification and the Dixmier-Moeglin equivalence. The stratification theory of Goodearl and Letzter can be applied to the study of prime and primitive ideals of the quantum spatial ageing algebra $\mathcal{A}$. We will show that $\mathcal{A}$ satisfies the Dixmier-Moeglin equivalence (Theorem 5.13).

Let us recall the general strategy of Goodearl and Letzter briefly, for details see [19]. Let R be a Noetherian $\mathbb{K}$-algebra and $H = (\mathbb{K}^*)^r$ be an algebraic torus acting rationally on R by $\mathbb{K}$-algebra automorphisms. We denote by H-$\text{Spec}\,(R)$ the set of H-prime ideals of R (these coincide with the H-invariant prime ideals of R by [19, Proposition II.2.9]). Given an ideal I in R, $(I : H) := \bigcap_{h \in H} h \cdot I$ is the largest H-invariant ideal of R contained in I. It is well-known that $(P : H)$ is an H-prime ideal if P is a prime ideal of R. For an H-prime ideal J of R, the *H-stratum of* $\text{Spec}\,(R)$ *corresponding to* J is defined by

$$\text{Spec}_J(R) := \{P \in \text{Spec}\,(R) \mid (P : H) = J\}.$$

These H-strata give a partition of $\text{Spec}\,(R)$, namely

$$\text{Spec}\,(R) = \bigsqcup_{J \in H\text{-Spec}\,(R)} \text{Spec}_J(R),$$

called the H-*stratification of* Spec (R). The stratum $\mathrm{Spec}_J(R)$ can be described by using the Stratification Theorem [19, Thorem II.2.13].

Let $H = (\mathbb{K}^*)^3$ and let H act on the quantum spatial ageing algebra $\mathcal{A}$ by the $\mathbb{K}$-algebra automorphisms such that

$$(\lambda, \mu, \gamma) \cdot X = \lambda X, \quad (\lambda, \mu, \gamma) \cdot Y = \mu Y, \quad (\lambda, \mu, \gamma) \cdot K^{\pm 1} = \gamma^{\pm 1} K^{\pm 1}, \quad (\lambda, \mu, \gamma) \cdot E = \lambda \mu^{-1} E.$$

In particular, $(\lambda, \mu, \gamma) \cdot \varphi = \lambda \varphi$, $(\lambda, \mu, \gamma) \cdot Z = \lambda \mu \gamma^{-1} Z$ and $(\lambda, \mu, \gamma) \cdot C = \lambda \mu \gamma C$. Then the algebra $\mathcal{A}$ has 6 H-prime ideals,

$$H\text{-Spec}\,(\mathcal{A}) = \{0,\ (X),\ (\varphi),\ (Y),\ (E),\ (Y, E)\}.$$

Consequently, there are 6 H-strata in Spec $(\mathcal{A})$:

$$\begin{aligned}
\mathrm{Spec}_0(\mathcal{A}) &= \{0\}, \\
\mathrm{Spec}_{(X)}(\mathcal{A}) &= \{(X)\} \ \sqcup\ \{(X, \mathfrak{q}) \mid \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[Z]) \setminus \{(Z)\}\}, \\
\mathrm{Spec}_{(\varphi)}(\mathcal{A}) &= \{(\varphi)\} \ \sqcup\ \{(\varphi, \mathfrak{r}) \mid \mathfrak{r} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}\}, \\
\mathrm{Spec}_{(Y)}(\mathcal{A}) &= \{(Y)\}, \\
\mathrm{Spec}_{(E)}(\mathcal{A}) &= \{(E)\}, \\
\mathrm{Spec}_{(Y,E)}(\mathcal{A}) &= \{(Y, E)\} \ \sqcup\ \{(Y, E, \mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Max}\,(\mathbb{K}[K^{\pm 1}])\}.
\end{aligned} \tag{5.26}$$

A prime ideal P of a ring R is said to be *locally closed* if $\{P\}$ is locally closed in Spec (R) where Spec (R) is equipped with the Zariski topology. By [19, Lemma II.7.7], a prime ideal P in a ring R is locally closed iff the intersection of all prime ideals properly containing P is an ideal properly containing P. A prime ideal P of a Noetherian $\mathbb{K}$-algebra R is said to be *rational* if the field $Z(\mathrm{Frac} R/P)$ is algebraic over $\mathbb{K}$. The *Dixmier-Moeglin equivalence* states that if P is a prime ideal of a Noetherian $\mathbb{K}$-algebra then the following properties are equivalent:

$$P \text{ is locally closed} \iff P \text{ is primitive} \iff P \text{ is rational.}$$

Theorem 5.13. *The algebra $\mathcal{A}$ satisfies the Dixmier-Moeglin equivalence, and the primitive ideals of $\mathcal{A}$ are precisely the prime ideals that are maximal in their H-strata.*

Proof. The algebra $\mathcal{A}$ contains a sequence of subalgebras $\mathbb{K} \subseteq \mathbb{K}[X] \subseteq \mathbb{K}_q[X, Y] \subseteq \mathbb{E} \subseteq \mathcal{A}$ satisfying the hypotheses of [19, Proposition II.7.17], thus $\mathcal{A}$ satisfies the noncommutative Nullstellensatz over $\mathbb{K}$. Since $\mathcal{A}$ has finitely many H-prime ideals, the theorem follows from [19, Theorem II.8.4] □

Remark. Proposition 5.11 follows immediately from Theorem 5.13 and (5.26). This gives an alternative proof of Proposition 5.11 without considering simple $\mathcal{A}$-modules.

5.3 The automorphism group of $\mathcal{A}$

In this section, the group $G := \mathrm{Aut}_{\mathbb{K}}(\mathcal{A})$ of automorphisms of the algebra $\mathcal{A}$ is found (Theorem 5.14). Corollary 5.15 describes the orbits of the action of the group G on $\mathrm{Spec}\,(\mathcal{A})$ and the set of fixed points.

We introduce a degree filtration on the algebra $\mathcal{A}$ by setting $\deg(K) = \deg(K^{-1}) = 0$ and $\deg(E) = \deg(X) = \deg(Y) = 1$. So, $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}[n]$ where $\mathcal{A}[n] = \sum \mathbb{K}X^iY^jE^kK^l$ with $\deg(X^iY^jE^kK^l) := i+j+k \leqslant n$. Let $\mathrm{gr}\,\mathcal{A} := \bigoplus_{i\in\mathbb{N}} \mathcal{A}[i]/\mathcal{A}[i-1]$ (where $\mathcal{A}[-1] := 0$) be the associated graded algebra of $\mathcal{A}$ with respect to the filtration $\{\mathcal{A}[n]\}_{n\geqslant 0}$. For an element $a \in \mathcal{A}$, we denote by $\mathrm{gr}\, a \in \mathrm{gr}\,\mathcal{A}$ the image of a in $\mathrm{gr}\,\mathcal{A}$. It is clear that $\mathrm{gr}\,\mathcal{A}$ is an iterated Ore extension, $\mathrm{gr}\,\mathcal{A} \simeq \mathbb{K}[X][Y;\alpha][E;\beta][K^{\pm 1};\gamma]$ where $\alpha(X) = q^{-1}X$, $\beta(X) = qX$, $\beta(Y) = q^{-1}Y$, $\gamma(X) = qX$, $\gamma(Y) = q^{-1}Y$ and $\gamma(E) = q^2E$. In particular, $\mathrm{gr}\,\mathcal{A}$ is a Noetherian domain of Gelfand-Kirillov dimension $\mathrm{GK}\,(\mathrm{gr}\,\mathcal{A}) = 4$ and the elements X, Y and E are normal in $\mathrm{gr}\,\mathcal{A}$.

The group of units $\mathcal{A}^*$ of the algebra $\mathcal{A}$ is equal to $\{\mathbb{K}^*K^i \,|\, i \in \mathbb{Z}\} = \mathbb{K}^* \times \langle K \rangle$ where $\langle K \rangle = \{K^i \,|\, i \in \mathbb{Z}\}$. The next theorem is an explicit description of the group G.

Theorem 5.14. $\mathrm{Aut}_{\mathbb{K}}(\mathcal{A}) = \{\sigma_{\lambda,\mu,\gamma,i} \,|\, \lambda,\mu,\gamma \in \mathbb{K}^*,\, i \in \mathbb{Z}\} \simeq (\mathbb{K}^*)^3 \rtimes \mathbb{Z}$ *where* $\sigma_{\lambda,\mu,\gamma,i} : X \mapsto \lambda K^iX$, $Y \mapsto \mu K^{-i}Y$, $K \mapsto \gamma K$, $E \mapsto \lambda\mu^{-1}q^{-2i}K^{2i}E$ *(and* $\sigma_{\lambda,\mu,\gamma,i}(\varphi) = \lambda K^i\varphi$*). Furthermore,* $\sigma_{\lambda,\mu,\gamma,i}\sigma_{\lambda',\mu',\gamma',j} = \sigma_{\lambda\lambda'\gamma^j,\mu\mu'\gamma^{-j},\gamma\gamma',i+j}$ *and* $\sigma^{-1}_{\lambda,\mu,\gamma,i} = \sigma_{\lambda^{-1}\gamma^i,\mu^{-1}\gamma^{-i},\gamma^{-1},-i}$.

Proof. Using the defining relations of the algebra $\mathcal{A}$, one can verify that $\sigma_{\lambda,\mu,\gamma,i} \in G$ for all $\lambda,\mu,\gamma \in \mathbb{K}^*$ and $i \in \mathbb{Z}$. The subgroup G' generated by these automorphisms is isomorphic to the semi-direct product $(\mathbb{K}^*)^3 \rtimes \mathbb{Z}$. It remains to show that $G = G'$. Recall that the elements X and φ are normal in the algebra $\mathcal{A}$. Let $\sigma \in G$, we have to show that $\sigma \in G'$.

By (5.17), there are two options either the ideals (X) and (φ) are σ-invariant or, otherwise, they are interchanged. In more details, either, for some elements $\lambda, \lambda' \in \mathbb{K}^*$ and $i, j \in \mathbb{Z}$,

(a) $\sigma(X) = \lambda K^iX$ and $\sigma(\varphi) = \lambda' K^j\varphi$, or, otherwise,
(b) $\sigma(X) = \lambda K^i\varphi$ and $\sigma(\varphi) = \lambda' K^jX$.

(i) $\sigma(K) = \gamma K$ *for some* $\gamma \in \mathbb{K}^*$: The group of units $\mathcal{A}^*$ of the algebra $\mathcal{A}$ is equal to $\{\gamma K^s \,|\, \gamma \in \mathbb{K}^*,\, s \in \mathbb{Z}\}$. So, either $\sigma(K) = \gamma K$ or, otherwise, $\sigma(K) = \gamma K^{-1}$ for some $\gamma \in \mathbb{K}^*$. Let us show that the second case is not possible. Notice that $KX = qXK$ and $K\varphi = q\varphi K$, i.e., the elements X and φ have the same commutation relation with the element K. Because of that it suffices to consider one of the cases (a) or (b) since then the other case can be treated similarly. Suppose that the case (a) holds and that $\sigma(K) = \gamma K^{-1}$. Then the equality $\sigma(K)\sigma(X) = q\sigma(X)\sigma(K)$ yields the equality $\gamma K^{-1}\cdot\lambda K^iX = q\lambda K^iX\cdot\gamma K^{-1} = q\lambda\gamma qK^{i-1}X$. Hence, $q^2 = 1$, a contradiction.

(ii) $\deg\sigma(Y) = \deg\sigma(E) = 1$: Recall that $YE = q'^{-1}(\varphi - X)$ where $q' := q^{-1} - q$. By applying σ to this equality we obtain the equality $\sigma(Y)\sigma(E) = q'^{-1}\sigma(\varphi - X)$. Hence, $\deg\big(\sigma(Y)\sigma(E)\big) = \deg\big(q'^{-1}\sigma(\varphi - X)\big) = 2$, in both cases (a) and (b). Thus, there are three options for the pair $\big(\deg\sigma(Y), \deg\sigma(E)\big)$: $(1,1)$, $(0,2)$ or $(2,0)$. The last two options are not possible since otherwise we would have $\sigma(Y) \in \mathbb{K}[K^{\pm 1}]$ or $\sigma(E) \in \mathbb{K}[K^{\pm 1}]$, respectively. Hence, $\sigma(Y)\sigma(K) = \sigma(K)\sigma(Y)$

or $\sigma(E)\sigma(K) = \sigma(K)\sigma(E)$, respectively. But this is impossible since $YK \neq KY$ and $EK \neq KE$. Therefore, $\deg \sigma(Y) = \deg \sigma(E) = 1$.

(iii) *$\sigma(Y) = aY$ and $\sigma(E) = bE$ for some nonzero elements $a, b \in \mathbb{K}[K^{\pm 1}]$*: Applying σ to the relation $KEK^{-1} = q^2E$ we obtain the equality $K\sigma(E)K^{-1} = q^2\sigma(E)$. Since $\deg \sigma(E) = 1$, $\sigma(E) = bE + uX + vY + w$ for some elements $b, u, v, w \in \mathbb{K}[K^{\pm 1}]$. Using the relations $KXK^{-1} = qX$ and $KYK^{-1} = q^{-1}Y$, we see that $u = v = w = 0$, i.e., $\sigma(E) = bE$. Similarly, applying σ to the relation $KYK^{-1} = q^{-1}Y$, we obtain the equality $K\sigma(Y)K^{-1} = q^{-1}\sigma(Y)$. Since $\deg \sigma(Y) = 1$, $\sigma(Y) = aY + u'E + v'X + w'$ for some elements $a, u', v', w' \in \mathbb{K}[K^{\pm 1}]$. Using the relations $KXK^{-1} = qX$, $KYK^{-1} = q^{-1}Y$ and $KEK^{-1} = q^2E$, we see that $u' = v' = w' = 0$, i.e., $\sigma(Y) = aY$.

(iv) *$i = j$ (see the cases (a) and (b))*: The elements X and φ commute, hence $\sigma(X)\sigma(\varphi) = \sigma(\varphi)\sigma(X)$. Substituting the values of $\sigma(X)$ and $\sigma(\varphi)$ into this equality yields $q^{-i} = q^{-j}$ in both cases (a) and (b), i.e., $i = j$ (since q is not a root of unity).

(v) *The case (b) is not possible*: Suppose that the case (b) holds, i.e., $\sigma(X) = \lambda K^i\varphi$ and $\sigma(\varphi) = \lambda' K^i X$ (see the statement (iv)), we seek a contradiction. To find the contradiction we use the relations $qYX = XY$ and $Y\varphi = q\varphi Y$. Applying the automorphism σ to the first equality gives $\sigma(qYX) = qaY \cdot \lambda K^i\varphi = qa\lambda q^i K^i Y\varphi$ and $\sigma(XY) = \lambda K^i\varphi \cdot aY = \lambda K^i\tau(a)\varphi Y = \lambda K^i\tau(a)q^{-1}Y\varphi$ where τ is the automorphism of the algebra $\mathbb{K}[K^{\pm 1}]$ given by the rule $\tau(K) = q^{-1}K$. Hence, $\tau(a) = q^{i+2}a$, i.e., $a = \xi K^{-i-2}$ for some $\xi \in \mathbb{K}^*$. So, $\sigma(Y) = \xi K^{-i-2}Y$. Now applying σ to the second equality, $Y\varphi = q\varphi Y$, we have the equalities $\sigma(Y\varphi) = \xi K^{-i-2}Y \cdot \lambda' K^i X = \xi\lambda' q^i K^{-2}YX$ and $\sigma(q\varphi Y) = q\lambda' K^i X \cdot \xi K^{-i-2}Y = \xi\lambda' q^{i+3}K^{-2}XY = \xi\lambda' q^{i+4}K^{-2}YX$. Therefore, $q^4 = 1$, a contradiction (since q is not a root of unity). This means that the only case (a) holds. Summarizing, we have $\sigma(X) = \lambda K^i X$, $\sigma(K) = \gamma K$, $\sigma(\varphi) = \lambda' K^i\varphi$, $\sigma(Y) = aY$, and $\sigma(E) = bE$.

(vi) *$a = \mu K^{-i}$ for some $\mu \in \mathbb{K}^*$ (i.e., $\sigma(Y) = \mu K^{-i}Y$)*: Applying the automorphism σ to the relation $qYX = XY$ yields: $\sigma(qYX) = qaY \cdot \lambda K^i X = \lambda a q^i K^i qYX = \lambda a q^i K^i XY$ and $\sigma(XY) = \lambda K^i X \cdot aY = \lambda K^i\tau(a)XY$. Therefore, $\tau(a) = q^i a$, i.e., $a = \mu K^{-i}$ for some element $\mu \in \mathbb{K}^*$.

(vii) *$b = \delta K^{2i}$ for some $\delta \in \mathbb{K}^*$ (i.e., $\sigma(E) = \delta K^{2i}E$)*: Applying the automorphism σ to the relation $qXE = EX$ yields: $\sigma(qXE) = q\lambda K^i X \cdot bE = \lambda K^i\tau(b)qXE = \lambda K^i\tau(b)EX$ and $\sigma(EX) = bE \cdot \lambda K^i X = \lambda K^i b q^{-2i}EX$. Therefore, $\tau(b) = q^{-2i}b$, i.e., $b = \delta K^{2i}$ for some $\delta \in \mathbb{K}^*$.

(viii) *$\delta = \lambda\mu^{-1}q^{-2i}$*: Applying the automorphism σ to the relation $EY = X + q^{-1}YE$ gives: $\sigma(EY) = \delta K^{2i}E \cdot \mu K^{-i}Y = \delta\mu q^{2i}K^i EY = \delta\mu q^{2i}K^i(X + q^{-1}YE)$ and $\sigma(X + q^{-1}YE) = \lambda K^i X + q^{-1}\mu K^{-i}Y \cdot \delta K^{2i}E = K^i(\lambda X + \delta\mu q^{2i}q^{-1}YE)$. Therefore, $\delta\mu q^{2i} = \lambda$, and the statement (viii) follows. The proof of the theorem is complete. □

Corollary 5.15.

1. *The prime ideals $\mathcal{I} := \{0, (X), (\varphi), (Y), (E), (Y, E)\}$ are the only prime ideals of $\mathcal{A}$ that are invariant under action of the group G of automorphisms of $\mathcal{A}$ (i.e., $G\mathfrak{p} = \{\mathfrak{p}\}$ for all $\mathfrak{p} \in \mathcal{I}$).*

2. *If, in addition, $\mathbb{K}$ is an algebraically closed field, then each of the three series of prime ideals in* $\mathrm{Spec}(\mathcal{A})$ *is a single G-orbit. In particular, there are* 9 *G-orbits in* $\mathrm{Spec}(\mathcal{A})$.

Proof. By Theorem 5.14, $G\mathfrak{p} = \{\mathfrak{p}\}$ for all $\mathfrak{p} \in \mathcal{I}$. Let $\sigma = \sigma_{\lambda,\mu,\gamma,i}$, then $\sigma(Z) = \lambda\mu\gamma^{-1}q^iZ$, $\sigma(C) = \lambda\mu\gamma q^iC$ and $\sigma(K) = \gamma K$. Now the corollary is obvious since $\mathfrak{p} = (K-\alpha)$, $\mathfrak{q} = (Z-\beta)$ and $\mathfrak{r} = (C-\gamma)$ for some $\alpha, \beta, \gamma \in \mathbb{K}^*$ where the ideals $\mathfrak{p}, \mathfrak{q}$ and $\mathfrak{r}$ are as in Theorem 5.8. □

5.4 Centralizers of some elements of the algebra $\mathcal{A}$

The next proposition describes the centralizers of the elements K, X and φ in $\mathcal{A}$. The centralizers are large subalgebras of $\mathcal{A}$. Furthermore, they are GWAs and the algebra $\mathcal{A}$ is a skew polynomial algebra with coefficient ring $C_{\mathcal{A}}(X)$ or $C_{\mathcal{A}}(\varphi)$.

Proposition 5.16. 1. *$C_{\mathcal{A}}(K) = \mathbb{K}[K^{\pm 1}] \otimes \Lambda$ is a tensor product of algebras where $\Lambda = \mathbb{K}\langle t, u \,|\, tu = q^2ut\rangle$ is the quantum plane where $t := YX$ and $u := Y\varphi$. Moreover, $\mathcal{A}_Y = C_{\mathcal{A}}(K)[Y^{\pm 1}; \sigma']$ where $\sigma'(t) = q^{-1}t, \sigma'(u) = qu$ and $\sigma'(K) = qK$.*

2. *$C_{\mathcal{A}}(X) = \mathbb{K}[X,\varphi][\partial, y; \gamma, q\frac{\varphi - X}{q^{-1}-q}]$ is a GWA where $\gamma(X) = X, \gamma(\varphi) = q^{-2}\varphi$, $\partial := EK^{-1}$ and $y := KY$.*
   (a) *$Z\big(C_{\mathcal{A}}(X)\big) = \mathbb{K}[X]$.*
   (b) *The algebra $\mathcal{A} = C_{\mathcal{A}}(X)[K^{\pm 1}; \theta]$ is a skew polynomial ring where $\theta(X) = qX, \theta(\varphi) = q\varphi, \theta(\partial) = q^2\partial$ and $\theta(y) = q^{-1}y$.*
3. *$C_{\mathcal{A}}(\varphi) = \mathbb{K}[X,\varphi][\partial', y'; \nu, q^{-1}\frac{\varphi - X}{q^{-1}-q}]$ is a GWA where $\nu(X) = q^2X, \nu(\varphi) = \varphi$, $\partial' := EK$ and $y' := K^{-1}Y$.*
   (a) *$Z\big(C_{\mathcal{A}}(\varphi)\big) = \mathbb{K}[\varphi]$.*
   (b) *The algebra $\mathcal{A} = C_{\mathcal{A}}(\varphi)[K^{\pm 1}; \theta]$ is a skew polynomial ring where $\theta(X) = qX, \theta(\varphi) = q\varphi$, $\theta(\partial') = q^2\partial'$ and $\theta(y') = q^{-1}y'$.*

Proof. 1. By (5.2), $C_{\mathcal{A}}(K) = \mathbb{E}^{\omega_K}[K^{\pm 1}; \tau] = \mathbb{K}[K^{\pm 1}] \otimes \mathbb{E}^{\omega_K}$, where $\omega_K : a \mapsto KaK^{-1}$ is the inner automorphism of the algebra $\mathcal{A}$ determined by the element K. Notice that $\omega_K = \tau$ where τ is defined in (5.2). Using the explicit action of the automorphism τ on the elements X, φ, E and Y we see that $\mathbb{E}^{\tau} = \mathbb{K}\langle YX, Y\varphi\rangle = \Lambda$. Clearly, $\mathcal{A}_Y = C_{\mathcal{A}}(K)[Y^{\pm 1}; \sigma']$, by (5.3).

2. Clearly, $\partial, y \in \mathcal{C} := C_{\mathcal{A}}(X)$ and $\mathcal{D} := \mathbb{K}[X,\varphi] \subseteq \mathcal{C}$. By (5.1) and (5.2), the subalgebra, say $\mathcal{C}'$, of $\mathcal{A}$ generated by the elements $\mathcal{D}, \partial$ and y is the GWA $\mathcal{D}[\partial, y; \gamma, q\frac{\varphi - X}{q^{-1}-q}]$ since

$$\partial X = X\partial,\ yX = Xy,\ \partial\varphi = q^{-2}\varphi\partial,\ y\varphi = q^2\varphi y;\ y\partial = KYEK^{-1} = qYE = qa$$

where $a = \frac{\varphi - X}{q^{-1}-q}$ and $\partial y = EY = \frac{q^{-1}\varphi - qX}{q^{-1}-q} = q\frac{q^{-2}\varphi - X}{q^{-1}-q} = \gamma(qa)$. Since $\mathcal{C}' \subseteq \mathcal{C}$, it remains to show that $\mathcal{C}' = \mathcal{C}$. By (5.1) and (5.2),

$$\mathcal{A} = \bigoplus_{i\geqslant 0, j\in\mathbb{Z}} \mathcal{D}E^iK^j \oplus \bigoplus_{i\geqslant 1, j\in\mathbb{Z}} \mathcal{D}Y^iK^j = \bigoplus_{i\geqslant 0, j\in\mathbb{Z}} \mathcal{D}\partial^iK^j \oplus \bigoplus_{i\geqslant 1, j\in\mathbb{Z}} \mathcal{D}y^iK^j = \bigoplus_{j\in\mathbb{Z}} \mathcal{C}'K^j,$$

hence, $\mathcal{C} = \mathcal{C}'$, as required. Therefore, $\mathcal{A} = \mathcal{C}[K^{\pm 1}; \theta]$. It is easy to show that $Z(\mathcal{C}) = \mathbb{K}[X]$.

3. Clearly, $\partial', y' \in C := C_{\mathcal{A}}(\varphi)$ and $\mathcal{D} \subseteq C$. By (5.1) and (5.2), the subalgebra, say C', of $\mathcal{A}$ generated by the elements $\mathcal{D}, \partial'$ and y' is the GWA $\mathcal{D}[\partial', y'; \gamma', q^{-1}\frac{\varphi - X}{q^{-1}-q}]$ since

$$\partial' X = q^2 X\partial',\ y'X = q^{-2}Xy',\ \partial'\varphi = \varphi\partial',\ y'\varphi = \varphi y';\ y'\partial' = K^{-1}YEK = q^{-1}a$$

where $a = \frac{\varphi - X}{q^{-1}-q}$ and $\partial' y' = EY = \frac{q^{-1}\varphi - qX}{q^{-1}-q} = q^{-1}\frac{\varphi - q^2X}{q^{-1}-q} = \gamma'(q^{-1}a)$. Since $C' \subseteq C$, it remains to show that $C' = C$. By (5.1) and (5.2),

$$\mathcal{A} = \bigoplus_{i\geqslant 0, j\in\mathbb{Z}} \mathcal{D}E^iK^j \oplus \bigoplus_{i\geqslant 1, j\in\mathbb{Z}} \mathcal{D}Y^iK^j = \bigoplus_{i\geqslant 0, j\in\mathbb{Z}} \mathcal{D}\partial'^iK^j \oplus \bigoplus_{i\geqslant 1, j\in\mathbb{Z}} \mathcal{D}y'^iK^j = \bigoplus_{j\in\mathbb{Z}} C'K^j,$$

hence, $C = C'$, as required. Therefore, $\mathcal{A} = C[K^{\pm 1}; \theta]$. It is easy to show that $Z(C) = \mathbb{K}[\varphi]$. □

The next lemma describes the centralizers of the element Y in $\mathcal{A}$ and $\mathcal{A}_Y$.

Lemma 5.17. *1. $C_{\mathcal{A}_Y}(Y) = \mathbb{K}[Y^{\pm 1}] \otimes R$ where $R := \mathbb{K}\langle a, b \,|\, ab = q^2ba\rangle$ is the quantum plane, $a := KX$ and $b := K^{-1}\varphi$.*

2. $C_{\mathcal{A}}(Y) = \mathbb{K}[Y] \otimes R$.

3. $\mathcal{A}_Y = C_{\mathcal{A}_Y}(Y)[K^{\pm 1}; \tau]$ is a skew polynomial algebra where $\tau(Y) = q^{-1}Y$, $\tau(a) = qa$ and $\tau(b) = qb$.

Proof. 1. Statement 1 follows from (5.3).

2. $C_{\mathcal{A}}(Y) = \mathcal{A} \cap C_{\mathcal{A}_Y}(Y) = \mathcal{A} \cap \left(\mathbb{K}[Y^{\pm 1}] \otimes R\right) = \mathbb{K}[Y] \otimes R$.

3. Statement 3 follows from statement 1 and (5.3). □

The next lemma describes the centralizers of the element E in $\mathcal{A}$ and $\mathcal{A}_E$.

Lemma 5.18. *1. The centralizer of E in $\mathcal{A}_E$, $C_{\mathcal{A}_E}(E) = \mathbb{K}[E^{\pm 1}] \otimes P$, is the tensor product of algebras where $P := \mathbb{K}\langle \mathscr{X}_1, \mathscr{X}_2 \,|\, \mathscr{X}_2\mathscr{X}_1 = q^2\mathscr{X}_1\mathscr{X}_2\rangle$ is the quantum plane, $\mathscr{X}_1 := X\varphi$ and $\mathscr{X}_2 := KX^2$.*

2. $C_{\mathcal{A}}(E) = \mathbb{K}[E] \otimes P$.

3. $C_{\mathcal{A}_{E,X,\varphi}}(E) = \mathbb{K}[E^{\pm 1}] \otimes P_{\mathscr{X}_1, \mathscr{X}_2}$.

4. Let $\mathcal{C} := C_{\mathcal{A}_{E,X,\varphi}}(E)$. Then $\mathcal{A}_{E,X,\varphi} = \bigoplus_{i\in\mathbb{Z}}(K^i\mathcal{C} \oplus K^iX\mathcal{C})$ and for all $i \in \mathbb{Z}$ and $c \in \mathcal{C}$ $E \cdot K^ic = q^{-2i}K^ic \cdot E$ and $E \cdot K^iXc = q^{-2i+1}K^iXc \cdot E$.

Proof. 1. Let $\mathscr{R} = \mathbb{K}[\varphi, X][K^{\pm 1}; \theta]$ be the subalgebra of $\mathcal{A}$ where $\theta(X) = qX$ and $\theta(\varphi) = q\varphi$. By (5.2), we see that the localization $\mathcal{A}_E$ of $\mathcal{A}$ at the powers of the element E is a skew Laurent polynomial algebra

$$\mathcal{A}_E = \mathscr{R}[E^{\pm 1}; \sigma] \tag{5.27}$$

where $\sigma(\varphi) = q^{-1}\varphi, \sigma(X) = qX$ and $\sigma(K) = q^{-2}K$. Now, $C_{\mathcal{A}_E}(E) = \mathscr{R}^{\sigma}[E^{\pm 1}]$ where $\mathscr{R}^{\sigma} = \{r \in \mathscr{R} \,|\, \sigma(r) = r\}$. Let us show that $\mathscr{R}^{\sigma} = P$. In view of the explicit nature of the automorphism

σ of $\mathscr{R}$, $\mathscr{R}^{\sigma} = \bigoplus_{i\in\mathbb{Z},j,k\in\mathbb{N}}\{\mathbb{K}K^i\varphi^jX^k \mid \sigma(K^i\varphi^jX^k) = K^i\varphi^jX^k\}$. Notice that $\sigma(K^i\varphi^jX^k) = q^{-2i-j+k}K^i\varphi^jX^k$, we have $k = 2i+j$ (since q is not a root of unity), and so,

$$K^i\varphi^jX^{2i+j} = q^{i(2j+i-1)}(X\varphi)^j(KX^2)^i = q^{i(2j+i-1)}\mathscr{X}_1^j\mathscr{X}_2^i.$$

Therefore, $\mathscr{R}^{\sigma} = P$.

2. $C_{\mathcal{A}}(E) = \mathcal{A}\cap C_{\mathcal{A}_E}(E) = \mathcal{A}\cap\mathbb{K}[E^{\pm1}]\otimes P = \mathbb{K}[E]\otimes P$.

3 and 4. The elements X and φ are normal elements of $\mathscr{R} = \mathbb{K}[\varphi, X][K^{\pm1};\theta]$ where $\theta(X) = qX$ and $\theta(\varphi) = q\varphi$. The set $\mathcal{S}_{X,\varphi} = \{X^i\varphi^j \mid i,j\in\mathbb{N}\}$ is an Ore set of the domain $\mathscr{R}$. The localization $\mathscr{R}_{X,\varphi} := \mathcal{S}_{X,\varphi}^{-1}\mathscr{R}$ is equal to

$$\mathscr{R}_{X,\varphi} = \mathbb{K}[X^{\pm1},\varphi^{\pm1}][K^{\pm1};\theta] = \mathbb{K}[X^{\pm1},\mathscr{X}_1^{\pm1}][K^{\pm1};\theta] = \Big(P_{\mathscr{X}_1,\mathscr{X}_2}\oplus XP_{\mathscr{X}_1,\mathscr{X}_2}\Big)[K^{\pm1};\theta] \quad (5.28)$$

where $P_{\mathscr{X}_1,\mathscr{X}_2} = \mathcal{S}_{\mathscr{X}_1,\mathscr{X}_2}^{-1}P$ and $\mathcal{S}_{\mathscr{X}_1,\mathscr{X}_2} := \{\mathscr{X}_1^i\mathscr{X}_2^j \mid i,j\in\mathbb{N}\}$. The elements X and φ are normal elements of the algebras $\mathcal{A}$ and $\mathcal{A}_E$. Hence, $\mathcal{S}_{X,\varphi} = \{X^i\varphi^j \mid i,j\in\mathbb{N}\}$ is an Ore set of $\mathcal{A}$ and $\mathcal{A}_E$. By (5.27), the localization $\mathcal{A}_{E,X,\varphi} := \mathcal{S}_{X,\varphi}^{-1}\mathcal{A}_E$ of the algebra $\mathcal{A}_E$ at $\mathcal{S}_{X,\varphi}$ is equal to

$$\mathcal{A}_{E,X,\varphi} = \mathscr{R}_{X,\varphi}[E^{\pm1};\sigma] \overset{(5.28)}{=} \Big(P_{\mathscr{X}_1,\mathscr{X}_2}\oplus XP_{\mathscr{X}_1,\mathscr{X}_2}\Big)[K^{\pm1};\theta][E^{\pm1};\sigma] = \bigoplus_{i\in\mathbb{Z}}K^i(\mathcal{C}\oplus X\mathcal{C}) \quad (5.29)$$

where, for a moment, $\mathcal{C} := \mathbb{K}[E^{\pm1}]\otimes P_{\mathscr{X}_1,\mathscr{X}_2} = \Big(C_{\mathcal{A}_E}(E)\Big)_{\mathscr{X}_1,\mathscr{X}_2} \subseteq C_{\mathcal{A}_{E,X,\varphi}}(E)$. By (5.29), $C_{\mathcal{A}_{E,X,\varphi}}(E) = \mathcal{C}$, and so statement 3 holds. Now, by (5.29), statement 4 holds. □

Chapter 6

The smash product algebra $\mathbb{K}_q[X,Y] \rtimes U_q(\mathfrak{sl}_2)$

6.1 Introduction

The quantum plane $\mathbb{K}_q[X,Y]$ admits a well-known structure of $U_q(\mathfrak{sl}_2)$-module algebra (see, e.g., [33, 40]). In fact, there exists an uncountable family of non-isomorphic $U_q(\mathfrak{sl}_2)$-module algebra structures on the quantum plane and a complete description of those structures was presented in [25]. Given a module algebra over a Hopf algebra, one can form the smash product algebra [38, 4.1.3], which is a useful method to construct new algebras. In this chapter, our main object of study is the smash product algebra $A := \mathbb{K}_q[X,Y] \rtimes U_q(\mathfrak{sl}_2)$, where the quantum plane is endowed with the well-known $U_q(\mathfrak{sl}_2)$-module algebra structure, the precise definition is given below.

Fix a field $\mathbb{K}$ of characteristic zero, and an element $q \in \mathbb{K}^*$ such that q is not a root of unity. Recall that the *quantized enveloping algebra* of $\mathfrak{sl}_2$ is the $\mathbb{K}$-algebra $U_q(\mathfrak{sl}_2)$ with generators E, F, K, K^{-1} subject to the defining relations:

$$KEK^{-1} = q^2E,\ KFK^{-1} = q^{-2}F,\ EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

The centre of $U_q(\mathfrak{sl}_2)$ is a polynomial algebra $Z(U_q(\mathfrak{sl}_2)) = \mathbb{K}[\Omega]$ where $\Omega := FE + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2}$. There is a Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ defined by

$$\begin{aligned}
&\Delta(K) = K \otimes K, && \varepsilon(K) = 1, && S(K) = K^{-1},\\
&\Delta(E) = E \otimes 1 + K \otimes E, && \varepsilon(E) = 0, && S(E) = -K^{-1}E,\\
&\Delta(F) = F \otimes K^{-1} + 1 \otimes F, && \varepsilon(F) = 0, && S(F) = -FK,
\end{aligned}$$

where Δ is the comultiplication on $U_q(\mathfrak{sl}_2)$, ε is the counit and S is the antipode of $U_q(\mathfrak{sl}_2)$. Note that the Hopf algebra $U_q(\mathfrak{sl}_2)$ is neither cocommutative nor commutative. We can make the *quantum plane* $\mathbb{K}_q[X,Y] := \mathbb{K}\langle X, Y \mid XY = qYX\rangle$ a $U_q(\mathfrak{sl}_2)$-module algebra by defining,

$$K\cdot X = qX, \qquad E\cdot X = 0, \qquad F\cdot X = Y,$$
$$K\cdot Y = q^{-1}Y, \qquad E\cdot Y = X, \qquad F\cdot Y = 0.$$

Then one can form the smash product algebra $A := \mathbb{K}_q[X,Y] \rtimes U_q(\mathfrak{sl}_2)$. The generators and defining relations for this algebra are given below. Our aim is to study the prime spectrum of this algebra and give a classification of simple weight A-modules.

Definition. The algebra A is the algebra generated over $\mathbb{K}$ by the elements E, F, K, K^{-1}, X, Y with defining relations

$$KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F, \qquad [E,F] = \frac{K-K^{-1}}{q-q^{-1}},$$
$$EX = qXE, \qquad EY = X + q^{-1}YE,$$
$$FX = YK^{-1} + XF, \qquad FY = YF,$$
$$KXK^{-1} = qX, \qquad KYK^{-1} = q^{-1}Y, \qquad qYX = XY.$$

A PBW deformation of this algebra, the *quantized symplectic oscillator algebra of rank one*, was studied by Gan and Khare [26], where the PBW theorem was given and some basic representation theory of this algebra was considered. They also determined the centre of the deformed algebra (the centre is trivial), but for the algebra A, they did not give the central element. In this chapter we show that the centre of A is a polynomial algebra $\mathbb{K}[C]$ (Theorem 6.7), and the generator C is given explicitly, see (6.11)–(6.14). The method we use in finding the central element of A can be summarized as follows. The algebra A is 'covered' by a chain of subalgebras. These subalgebras are generalized Weyl algebras and the central elements can be determined by applying Proposition 6.1. At each step elements are getting more complicated but the relations are getting simpler. Finally, we find a central element in a large subalgebra $\mathbb{A}$ of A which turns out to be a central element of the algebra A.

We are interested in the algebra A because it can reasonably be seen as the quantum analogue of the enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$ of the semidirect product Lie algebra $\mathfrak{sl}_2 \ltimes V_2$. These two algebras are similar in many ways. For example, the prime spectra of these two algebras are the same, the representation theory of A has many parallels with that of $U(\mathfrak{sl}_2 \ltimes V_2)$, the Casimir element of A degenerates to the Casimir element of $U(\mathfrak{sl}_2 \ltimes V_2)$ as $q \to 1$. The study of quantum algebras usually requires more computations. Much work has been done on quantized enveloping algebras of semisimple Lie algebras (see, e.g., [29, 33]). In the contrast, few examples can be found on the quantized algebras of enveloping algebras of non-semisimple Lie algebras.

Let us briefly describe the contents of this chapter. In Section 6.2, we find the centre of the algebra A, it is a polynomial algebra and the generator is given explicitly. We also show that A satisfies the quantum Gelfand-Kirillov conjecture. An explicit description of the prime and primitive ideals of A together with the inclusions are given in Section 6.3. As the Weyl algebras play an important role in the study of enveloping algebras, in Section 6.4, we consider a quantum analogue of the Weyl algebra. It plays a similar role the Weyl algebra does but in the study of quantum algebras. The Weyl algebras and their quantum analogues are special examples of generalized Weyl algebras. An A-module M is called a weight module if $M = \bigoplus_{\mu \in \mathbb{K}^*} M_\mu$ where $M_\mu = \{m \in M \mid Km = \mu m\}$. A classification of simple weight A-modules is given in Section 6.7. In order to give a classification of simple weight A-modules, we need first study the centralizer $C_A(K)$ of the element K in the algebra A, which is also interesting in its own right. In Section 6.5, we give the generators and defining relations of the algebra $C_A(K)$ (Theorem 6.29). We have to choose the generators carefully to make the defining relations simpler. The centre of $C_A(K)$ is $\mathbb{K}[C, K^{\pm 1}]$ (Theorem 6.29). For $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, we prove that the factor algebra $\mathscr{C}^{\lambda,\mu} := C_A(K)/(C-\lambda, K-\mu)$ is a simple algebra if and only if $\lambda \neq 0$ (Theorem 6.34). One of the key observations is that the localization $\mathscr{C}_t^{\lambda,\mu}$ of the algebra $\mathscr{C}^{\lambda,\mu}$ at the powers of the element $t = YX$ is a central, simple, generalized Weyl algebra (Proposition 6.32). The other one is that, for any $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, we can embed the algebra $\mathscr{C}^{\lambda,\mu}$ into a generalized Weyl algebra $\mathscr{A}$ (which is also a central simple algebra), see Proposition 6.38. These two facts enable us to give a complete classification of simple $C_A(K)$-modules. The problem of classifying simple $\mathscr{C}^{\lambda,\mu}$-modules splits into two distinct cases, namely the case when $\lambda = 0$ and the case when $\lambda \neq 0$. In the case $\lambda = 0$, we embed the algebra $\mathscr{C}^{0,\mu}$ into a skew polynomial algebra $\mathcal{R} = \mathbb{K}[h^{\pm 1}][t;\sigma]$ where $\sigma(h) = q^2 h$ (it is a subalgebra of the algebra $\mathscr{A}$) for which the classifications of simple modules are known. In the case $\lambda \neq 0$, we use the close relation of $\mathscr{C}^{\lambda,\mu}$ with the localization $\mathscr{C}_t^{\lambda,\mu}$, and the argument is more complicated. A classification of simple $C_A(K)$-modules is given in Section 6.6.

Much of this chapter is extracted from the joint paper with V. Bavula [12].

6.2 The centre of the algebra $\mathbb{K}_q[X,Y] \rtimes U_q(\mathfrak{sl}_2)$

The aim of this section is to determine the centre of the algebra A (Theorem 6.7). The next proposition is a corollary of Proposition 5.1 when $\rho = 1$. The rings E with $\rho = 1$ admit a 'canonical' central element (under a mild condition). This proposition is a key one for this section and is used on many occasions to produce central elements. In the present section a full generality of the construction is needed, i.e. when the base ring D is *noncommutative*.

Proposition 6.1. *Let $E = D[X,Y;\sigma,b,\rho=1]$. Then*

1. *[7, Lemma 1.5] The following statements are equivalent:*
   (a) $C = YX + \alpha = XY + \sigma(\alpha)$ is a central element in E for some central element $\alpha \in D$,
   (b) $\alpha - \sigma(\alpha) = b$ for some central element $\alpha \in D$.
2. *[7, Corollary 1.6] If one of the equivalent conditions of statement 2 holds then the ring $E = D[C][X,Y;\sigma, a = C-\alpha]$ is a GWA where $\sigma(C) = C$.*

If D is commutative the implication $(b) \Rightarrow (a)$ also appeared in [31].

An involution τ of A. The algebra A admits the following involution τ (see [26], p. 693):

$$\tau(E) = -FK,\ \tau(F) = -K^{-1}E,\ \tau(K) = K,\ \tau(K^{-1}) = K^{-1},\ \tau(X) = Y,\ \tau(Y) = X. \quad (6.1)$$

The algebra $\mathbb{E}$ is a GWA. Let $\mathbb{E}$ be the subalgebra of A which is generated by the elements E, X and Y. The elements E, X and Y satisfy the defining relations

$$EX = qXE, \quad YX = q^{-1}XY, \quad \text{and} \quad EY - q^{-1}YE = X.$$

Recall that the algebra $\mathbb{E}$ is a GWA $\mathbb{E} = \mathbb{K}[\varphi, X][E, Y; \sigma, a = \frac{\varphi - X}{q^{-1}-q}]$ (see (5.1)) where $\varphi = (q^{-1} - q)YE + X = (1 - q^2)EY + q^2X$, $\sigma(\varphi) = q^{-1}\varphi$ and $\sigma(X) = qX$. Using the defining relations of the GWA $\mathbb{E}$, we see that the set $\{Y^i \,|\, i \in \mathbb{N}\}$ is a left and right Ore set in $\mathbb{E}$. The localization of the algebra $\mathbb{E}$ at this set, $\mathbb{E}_Y := \mathbb{K}[\varphi, X][Y^{\pm 1}; \sigma]$, is the skew Laurent polynomial ring. Similarly, the set $\{X^i \,|\, i \in \mathbb{N}\}$ is a left and right Ore set in $\mathbb{E}_Y$ and the algebra

$$\mathbb{E}_{Y,X} = \mathbb{K}[\varphi, X^{\pm 1}][Y^{\pm 1}; \sigma] = \mathbb{K}[\Phi] \otimes \mathbb{K}[X^{\pm 1}][Y^{\pm 1}; \sigma] \tag{6.2}$$

is the tensor product of the polynomial algebra $\mathbb{K}[\Phi]$ where $\Phi = X\varphi$ and the skew Laurent polynomial algebra $\mathbb{K}[X^{\pm 1}][Y; \sigma]$ which is a central simple algebra. In particular, $Z(\mathbb{E}_{Y,X}) = \mathbb{K}[\Phi]$. So, we have the inclusion of algebras $\mathbb{E} \subseteq \mathbb{E}_Y \subseteq \mathbb{E}_{Y,X}$. Recall that for any algebra A, we denote by $Z(A)$ its centre. The next lemma describes the centre of the algebras $\mathbb{E}$, $\mathbb{E}_Y$ and $\mathbb{E}_{Y,X}$.

Lemma 6.2. $Z(\mathbb{E}) = Z(\mathbb{E}_Y) = Z(\mathbb{E}_{Y,X}) = \mathbb{K}[\Phi]$ *is a polynomial algebra where* $\Phi := X\varphi$.

Proof. By (6.2), $\mathbb{K}[\Phi] \subseteq Z(\mathbb{E}) \subseteq Z(\mathbb{E}_Y) \subseteq Z(\mathbb{E}_{Y,X}) = \mathbb{K}[\Phi]$, and the result follows. □

We have the following commutation relations

$$X\varphi = \varphi X, \qquad Y\varphi = q\varphi Y, \qquad E\varphi = q^{-1}\varphi E, \qquad K\varphi = q\varphi K. \tag{6.3}$$

$$X\Phi = \Phi X, \qquad Y\Phi = \Phi Y, \qquad E\Phi = \Phi E, \qquad K\Phi = q^2\Phi K. \tag{6.4}$$

Lemma 6.3. *1.* $[F, \varphi] = YK$.

2. The powers of φ *form a left and right Ore set in* A.

3. The powers of X *form a left and right Ore set in* A.

4. The powers of Y *form a left and right Ore set in* A.

Proof. 1. $[F, \varphi] = [F, X + (q^{-1} - q)YE] = YK^{-1} + (q^{-1} - q)Y(-\frac{K-K^{-1}}{q-q^{-1}}) = YK$.

2. Statement 2 follows at once from the equalities (6.3) and statement 1.

3. The statement follows at once from the defining relations of the algebra A where X is involved.

4. The statement follows at once from the defining relations of the algebra A where Y is involved. □

The algebra $\mathbb{F}$ is a GWA. Let $\mathbb{F}$ be the subalgebra of A which is generated by the elements F, X and $Y' := YK^{-1}$. The elements F, X and Y' satisfy the defining relations

$$FY' = q^{-2}Y'F, \quad XY' = q^2Y'X \quad \text{and} \quad FX - XF = Y'.$$

Therefore, the algebra $\mathbb{F} = \mathbb{K}[Y'][F,X;\sigma,b=Y',\rho=1]$ where $\sigma(Y') = q^{-2}Y'$. The polynomial $\alpha = \frac{1}{1-q^{-2}}Y' \in \mathbb{K}[Y']$ is a solution to the equation $\alpha - \sigma(\alpha) = Y'$. By Proposition 6.1, the element $C' := XF + \frac{1}{1-q^{-2}}Y' = FX + \frac{1}{q^2-1}Y'$ belongs to the centre of the GWA

$$\mathbb{F} = \mathbb{K}[C',Y'][F,X;\sigma,a=C' - \frac{1}{1-q^{-2}}Y'].$$

Let $\psi := (1-q^2)C'$. Then $\psi = (1-q^2)FX - Y' = (1-q^2)XF - q^2Y' \in Z(\mathbb{F})$ and

$$\mathbb{F} = \mathbb{K}[\psi,Y'][F,X;\sigma,a=\frac{\psi+q^2Y'}{1-q^2}] \tag{6.5}$$

where $\sigma(\psi) = \psi$ and $\sigma(Y') = q^{-2}Y'$. Similar to the algebra $\mathbb{E}$, the localization of the algebra $\mathbb{F}$ at the powers of the element X is equal to

$$\mathbb{F}_X := \mathbb{K}[\psi,Y'][X^{\pm1};\sigma^{-1}] = \mathbb{K}[\psi] \otimes \mathbb{K}[Y'][X^{\pm1};\sigma^{-1}]$$

where σ is defined in (6.5). The centre of the algebra $\mathbb{K}[Y'][X^{\pm1};\sigma^{-1}]$ is $\mathbb{K}$. Hence, $Z(\mathbb{F}_X) = \mathbb{K}[\psi]$.

Lemma 6.4. $Z(\mathbb{F}) = Z(\mathbb{F}_X) = \mathbb{K}[\psi]$.

Proof. The result follows from the inclusions $\mathbb{K}[\psi] \subseteq Z(\mathbb{F}) \subseteq Z(\mathbb{F}_X) = \mathbb{K}[\psi]$. □

The GWA $\mathbb{A}$. Let T be the subalgebra of A generated by the elements $K^{\pm1}, X$ and Y. Clearly,

$$T := \Lambda[K^{\pm1};\tau] \quad \text{where } \Lambda := \mathbb{K}\langle X,Y \mid XY = qYX\rangle \text{ and } \tau(X) = qX \text{ and } \tau(Y) = q^{-1}Y. \tag{6.6}$$

It is easy to determine the centre of the algebra T.

Lemma 6.5. $Z(T) = \mathbb{K}[z]$ *where* $z := KYX$.

Proof. Clearly, the element $z = KYX$ belongs to the centre of the algebra T. The centralizer $C_T(K)$ is equal to $\mathbb{K}[K^{\pm1}, YX]$. Then the centralizer $C_T(K,X)$ is equal to $\mathbb{K}[z]$, hence $Z(T) = \mathbb{K}[z]$. □

Let $\mathbb{A}$ be the subalgebra of A generated by the algebra T and the elements φ and ψ. The generators $K^{\pm1}, X, Y, \varphi$ and ψ satisfy the following relations:

$$\begin{aligned} &\varphi X = X\varphi, && \varphi Y = q^{-1}Y\varphi, && \varphi K = q^{-1}K\varphi, && \\ &\psi X = X\psi, && \psi Y = qY\psi, && \psi K = qK\psi, && \varphi\psi - \psi\varphi = -q(1-q^2)z. \end{aligned}$$

These relations together with the defining relations of the algebra T are defining relations of the algebra $\mathbb{A}$. In more detail, let, for a moment, $\mathbb{A}'$ be the algebra generated by the defining relations as above. We will see $\mathbb{A}' = \mathbb{A}$. Then

$$\mathbb{A}' = T[\varphi,\psi;\sigma,b=-q(1-q^2)z,\rho=1].$$

Hence, the set of elements $\{K^iX^jY^k\varphi^l\psi^m \mid i \in \mathbb{Z}, j,k,l,m \in \mathbb{N}\}$ is a basis of the algebra $\mathbb{A}'$. This set is also a basis for the algebra $\mathbb{A}$. This follows from the explicit expressions for the elements $\varphi = (q^{-1}-q)YE + X$ and $\psi = (1-q^2)XF - q^2YK^{-1}$. In particular, that the leading terms of φ and ψ are equal to $(q^{-1}-q)YE$ and $(1-q^2)XF$, respectively ($\deg(K^{\pm 1}) = 0$). So, $\mathbb{A} = \mathbb{A}'$, i.e.,

$$\mathbb{A} = T[\varphi, \psi; \sigma, b = -q(1-q^2)z, \rho = 1]$$

where $\sigma(X) = X, \sigma(Y) = q^{-1}Y$ and $\sigma(K) = q^{-1}K$. Recall that the element b belongs to the centre of the algebra T (Lemma 6.5). The element $\alpha = q^3Z$ is a solution to the equation $\alpha - \sigma(\alpha) = b$. Then, by Proposition 6.1, the algebra $\mathbb{A}$ is the GWA

$$\mathbb{A} = T[C''][\varphi, \psi; \sigma, a = C'' - q^3z]$$

where $\sigma(C'') = C'', \sigma(X) = X, \sigma(Y) = q^{-1}Y, \sigma(K) = q^{-1}K$ and $C'' = \psi\varphi + q^3z = \varphi\psi + qz$ (since $\sigma(z) = q^{-2}z$). Let $C := \frac{C''}{1-q^2}$. Then

$$C = (1-q^2)^{-1}(\psi\varphi + q^3z) = (1-q^2)^{-1}(\varphi\psi + qz), \tag{6.7}$$

is a central element of the GWA

$$\mathbb{A} = T[C][\varphi, \psi; \sigma, a = (1-q^2)C - q^3z] \tag{6.8}$$ ATC

where $\sigma(C) = C, \sigma(X) = X, \sigma(Y) = q^{-1}Y$ and $\sigma(K) = q^{-1}K$. Since $\varphi = (q^{-1}-1)YE + X$ and $\psi = (1-q^2)XF - q^2YK^{-1}$, we see that

$$\mathbb{A}_{X,Y} = A_{X,Y}. \tag{6.9}$$

Hence, $C \in Z(A)$. In fact, $Z(A) = \mathbb{K}[C]$ (Theorem 6.7). In order to show this fact we need to consider the localization $A_{X,Y,\varphi}$.

Let $\mathbb{T} := T_{X,Y} = \Lambda_{X,Y}[K^{\pm 1}; \tau]$ where τ is defined in (6.6) and $\Lambda_{X,Y}$ is the localization of the algebra Λ at the powers of the elements X and Y. By (6.9) and (6.8),

$$A_{X,Y,\varphi} = \mathbb{A}_{X,Y,\varphi} = T_{X,Y}[C][\varphi^{\pm 1}; \sigma] = \mathbb{K}[C] \otimes \mathbb{T}[\varphi^{\pm 1}; \sigma] = \mathbb{K}[C] \otimes \Lambda' \tag{6.10}$$

where $\Lambda' = \mathbb{T}[\varphi^{\pm 1}; \sigma]$ and σ is as in (6.8). Notice that Λ' is a quantum torus, it is easy to compute its centre.

Lemma 6.6. *1. $Z(\Lambda') = \mathbb{K}$.*

2. The algebra Λ' is a simple algebra.

Proof. 1. Let $u = \sum \lambda_{i,j,k,l}K^iX^jY^k\varphi^l \in Z(\Lambda)$, where $\lambda_{i,j,k,l} \in \mathbb{K}$. Since $[K,u] = 0$, we have $j - k + l = 0$. Similarly, since $[X,u] = [Y,u] = [\varphi, u] = 0$, we have the following equations: $-i+k = 0$, $i-j+l = 0$, $-i-k = 0$, respectively. These equations imply that $i = j = k = l = 0$. Thus $Z(\Lambda) = \mathbb{K}$.

2. Since the algebra Λ' is central, it is a simple algebra, by [27, Corollary 1.5.(a)] □

Theorem 6.7. *The centre* $Z(A)$ *of the algebra* A *is the polynomial algebra in one variable* $\mathbb{K}[C]$.

Proof. By (6.10) and Lemma 6.6.(1), $Z(A_{X,Y,\varphi}) = \mathbb{K}[C]$. Hence, $Z(A) = \mathbb{K}[C]$. □

Using the defining relations of the algebra A, we can rewrite the central element C as follows

$$C = (1-q^2)FYXE + FX^2 - Y^2K^{-1}E - \frac{1}{1-q^2}YK^{-1}X + \frac{q^2}{1-q^2}YKX. \tag{6.11}$$

$$C = (FE - q^2EF)YX + q^2FX^2 - K^{-1}EY^2. \tag{6.12}$$

$$C = FX(EY - qYE) - K^{-1}EY^2 + \frac{q^3}{1-q^2}(K-K^{-1})YX. \tag{6.13}$$

$$C = (1-q^2)FEYX + \frac{q^3}{1-q^2}(K-K^{-1})YX + q^2FX^2 - K^{-1}EY^2. \tag{6.14}$$

The subalgebra $\mathcal{A}$ of A. Let $\mathcal{A}$ be the subalgebra of A generated by the elements $K^{\pm 1}, E, X$ and Y. Then $\mathcal{A}$ is the quantum spatial ageing algebra. The properties of this algebra was studied in the previous chapter. Recall that the algebra

$$\mathcal{A} = \mathbb{E}[K^{\pm 1}; \tau] \tag{6.15}$$

is a skew Laurent polynomial algebra where $\tau(E) = q^2E, \tau(X) = qX$ and $\tau(Y) = q^{-1}Y$. The elements $X, \varphi \in \mathcal{A}$ are normal elements of the algebra $\mathcal{A}$. The set $\mathcal{S}_{X,\varphi} := \{X^i\varphi^j \mid i, j \in \mathbb{N}\}$ is a left and right denominator set of the algebras A and $\mathcal{A}$. Clearly $\mathcal{A}_{X,\varphi} := \mathcal{S}_{X,\varphi}^{-1} \subseteq A_{X,\varphi} := \mathcal{S}_{X,\varphi}^{-1}A$. By Lemma 5.4, the algebra $\mathcal{A}_{X,\varphi}$ is a central simple algebra.

For an element $a \in A$, let $\deg_F(a)$ be its F-degree. Since the algebra A is a domain, $\deg_F(ab) = \deg_F(a) + \deg_F(b)$ for all elements $a, b \in A$. Using the defining relations of the algebra A, the algebra A is a skew polynomial algebra

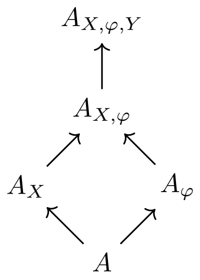
$$A = \mathcal{A}[F; \sigma, \delta] \tag{6.16}$$

where σ is an automorphism of $\mathcal{A}$ such that $\sigma(K) = q^2K, \sigma(E) = E, \sigma(X) = X, \sigma(Y) = Y$; and δ is a σ-derivation of the algebra $\mathcal{A}$ such that $\delta(K) = 0, \delta(E) = \frac{K-K^{-1}}{q-q^{-1}}, \delta(X) = YK^{-1}$ and $\delta(Y) = 0$.

Lemma 6.8. *The algebra* $A_{X,\varphi} = \mathbb{K}[C] \otimes \mathcal{A}_{X,\varphi}$ *is a tensor product of algebras.*

Proof. Recall that $\varphi = EY - qYE$. Then the equality (6.13) can be written as $C = FX\varphi - K^{-1}EY^2 + \frac{q^3}{1-q^2}(K-K^{-1})YX$. The element $X\varphi$ is invertible in $A_{X,\varphi}$. Now, using (6.16), we see that $A_{X,\varphi} = \mathcal{A}_{X,\varphi}[F; \sigma, \delta] = \mathcal{A}_{X,\varphi}[C] = \mathbb{K}[C] \otimes \mathcal{A}_{X,\varphi}$. □

Summarizing, we have the following inclusions of algebras



(6.17)

Quantum Gelfand-Kirillov conjecture for A. If we view A as the quantum analogue of the enveloping algebra $U(\mathfrak{sl}_2 \ltimes V_2)$, a natural question is whether A satisfies the quantum Gelfand-Kirillov conjecture. Recall that a *quantum Weyl skew field* over $\mathbb{K}$ is the skew field of fractions of a quantum affine space. We say that a $\mathbb{K}$-algebra A admitting a skew field of fractions $\mathrm{Frac}\,(A)$ satisfies the *quantum Gelfand-Kirillov conjecture* if $\mathrm{Frac}(A)$ is isomorphic to a quantum Weyl skew field over a purely transcendental field extension of $\mathbb{K}$; see [19, II.10, p. 230].

Theorem 6.9. *The quantum Gelfand-Kirillov conjecture holds for the algebra* A.

Proof. This follows immediately from (6.10). □

6.3 Prime, primitive and maximal spectra of A

The aim of this section is to give classifications of prime, primitive and maximal ideals of the algebra A (Theorem 6.15, Theorem 6.19 and Corollary 6.17). It is proved that every nonzero ideal of the algebra A has nonzero intersection with the centre of A (Corollary 6.16). The set of completely prime ideals of the algebra A is described in Corollary 6.20. Our goal is a description of the prime spectrum of the algebra A together with their inclusions. Next several results are steps in this direction, they are interesting in their own right.

Lemma 6.10. *The following identities hold in the algebra* A.

1. $FX^i = X^iF + \frac{1-q^{2i}}{1-q^2}YK^{-1}X^{i-1}$.
2. $XF^i = F^iX - \frac{1-q^{2i}}{1-q^2}YF^{i-1}K^{-1}$.

Proof. The equalities are proved by induction on i and using the defining relations of A. □

Lemma 6.11. *1. In the algebra* A, $(X) = (Y) = (\varphi) = AX + AY$.

2. $A/(X) \simeq U$.

Proof. 1. The equality $(X) = (Y)$ follows from the equalities $FX = YK^{-1} + XF$ and $EY = X + q^{-1}YE$. The inclusion $(\varphi) \subseteq (Y)$ follows from the equality $\varphi = EY - qYE$. The reverse inclusion $(\varphi) \supseteq (Y)$ follows from $Y = [F, \varphi]K^{-1}$ (Lemma 6.3). Let us show that $XA \subseteq AX + AY$. Recall that X is a normal element of $\mathcal{A}$. Then by (6.16), $XA = \sum_{k\geqslant 0} \mathcal{A}XF^k = \mathcal{A}X + \sum_{k\geqslant 1} \mathcal{A}XF^k \subseteq$

$AX + AY$ (the inclusion follows from Lemma 6.10.(2)). Then $(X) = AXA \subseteq AX + AY \subseteq (X,Y) = (X)$, i.e., $(X) = AX + AY$.

2. By statement 1, $A/(X) = A/(X,Y) \simeq U$. □

The next result shows that the elements X and φ are rather special.

Lemma 6.12. *1. For all $i \geqslant 1$, $(X^i) = (X)^i$.*

2. For all $i \geqslant 1$, $(\varphi^i)_X = (\varphi)^i_X = A_X$.

Proof. 1. To prove the statement we use induction on i. The case $i = 1$ is obvious. Suppose that $i > 1$ and the equality $(X^j) = (X)^j$ holds for all $1 \leqslant j \leqslant i-1$. By Lemma 6.10.(1), the element $YX^{i-1} \in (X^i)$. Now, $(X)^i = (X)(X)^{i-1} = (X)(X^{i-1}) = AXAX^{i-1}A \subseteq (X^i) + AYX^{i-1}A \subseteq (X^i)$. Therefore, $(X)^i = (X^i)$.

2. It suffices to show that $(\varphi^i)_X = A_X$ for all $i \geqslant 1$. The case $i = 1$ follows from the equality of ideals $(\varphi) = (X)$ in the algebra A (Lemma 6.11). We use induction on i. Suppose that the equality is true for all $i' < i$. By Lemma 6.3.(1), $[F, \varphi^i] = \frac{1-q^{-2i}}{1-q^{-2}} YK\varphi^{i-1}$, hence $Y\varphi^{i-1} \in (\varphi^i)$. Using the equalities $EY - q^{-1}YE = X$ and $E\varphi = q^{-1}\varphi E$, we see that $EY\varphi^{i-1} - q^{-i}Y\varphi^{i-1}E = (EY - q^{-1}YE)\varphi^{i-1} = X\varphi^{i-1}$. Now, $(\varphi^i)_X \supseteq (\varphi^{i-1})_X = A_X$, by induction. Therefore, $(\varphi^i)_X = A_X$ for all i. □

One of the most difficult steps in classification of prime ideals of the algebra A is to show that each maximal ideal $\mathfrak{q}$ of the centre $Z(A) = \mathbb{K}[C]$ generates the prime ideal $A\mathfrak{q}$ of the algebra A. There are two distinct cases: $\mathfrak{q} \neq (C)$ and $\mathfrak{q} = (C)$. The next theorem deals with the first case.

Theorem 6.13. *Let $\mathfrak{q} \in \mathrm{Max}(\mathbb{K}[C]) \setminus \{(C)\}$. Then*

1. *The ideal $(\mathfrak{q}) := A\mathfrak{q}$ of A is a maximal, completely prime ideal.*
2. *The factor algebra $A/(\mathfrak{q})$ is a simple algebra.*

Proof. Notice that $\mathfrak{q} = \mathbb{K}[C]q$ where $q = q(C)$ is an irreducible monic polynomial such that $q(0) \in \mathbb{K}^*$.

(i) *The factor algebra $A/(\mathfrak{q})$ is a simple algebra, i.e., $(\mathfrak{q})$ is a maximal ideal of A*: Consider the chain of localizations

$$A/(\mathfrak{q}) \longrightarrow \frac{A_X}{(\mathfrak{q})_X} \longrightarrow \frac{A_{X,\varphi}}{(\mathfrak{q})_{X,\varphi}}.$$

By Lemma 6.8, $\frac{A_{X,\varphi}}{(\mathfrak{q})_{X,\varphi}} \simeq L_{\mathfrak{q}} \otimes \mathcal{A}_{X,\varphi}$ where $L_{\mathfrak{q}} := \mathbb{K}[C]/\mathfrak{q}$ is a finite field extension of $\mathbb{K}$. By Lemma 5.4, the algebra $\mathcal{A}_{X,\varphi}$ is a central simple algebra. Hence, the algebra $\frac{A_X}{(\mathfrak{q})_X}$ is simple iff $(\varphi^i, \mathfrak{q})_X = A_X$ for all $i \geqslant 1$. By Lemma 6.12.(2), $(\varphi^i)_X = A_X$ for all $i \geqslant 1$. Therefore, the algebra $\frac{A_X}{(\mathfrak{q})_X}$ is simple. Hence, the algebra $A/(\mathfrak{q})$ is simple iff $(X^i, \mathfrak{q}) = A$ for all $i \geqslant 1$.

By Lemma 6.12.(1), $(X^i) = (X)^i$ for all $i \geqslant 1$. Therefore, $(X^i, \mathfrak{q}) = (X)^i + (\mathfrak{q})$ for all $i \geqslant 1$. It remains to show that $(X)^i + (\mathfrak{q}) = A$ for all $i \geqslant 1$. By Lemma 6.11.(1), $(X) = (X, Y)$.

If $i = 1$ then $(X) + (\mathfrak{q}) = (X, Y, \mathfrak{q}) = \big(X, Y, q(0)\big) = A$, by (6.11) and $q(0) \in \mathbb{K}^*$. Now, $A = A^i = \big((X) + (\mathfrak{q})\big)^i \subseteq (X)^i + (\mathfrak{q}) \subseteq A$, i.e., $(X)^i + (\mathfrak{q}) = A$, as required.

(ii) $(\mathfrak{q})$ *is a completely prime ideal of* A: The set $\mathcal{S} = \{X^i\varphi^j \mid i, j \in \mathbb{N}\}$ is a denominator set of the algebra A. Since $\frac{A_{X,\varphi}}{(\mathfrak{q})_{X,\varphi}} \simeq \mathcal{S}^{-1}(A/(\mathfrak{q}))$ is a (nonzero) algebra and $(\mathfrak{q})$ is a maximal ideal of the algebra A, we have that $\mathrm{tor}_{\mathcal{S}}(A/(\mathfrak{q}))$ is an ideal of the algebra $A/(\mathfrak{q})$ distinct from $A/(\mathfrak{q})$, hence $\mathrm{tor}_{\mathcal{S}}(A/(\mathfrak{q})) = 0$. This means that the algebra $A/(\mathfrak{q})$ is a subalgebra of the algebra $\frac{A_{X,\varphi}}{(\mathfrak{q})_{X,\varphi}} \simeq L_{\mathfrak{q}} \otimes \mathcal{A}_{X,\varphi}$ which is a domain. Therefore, the ideal $(\mathfrak{q})$ of A is a completely prime ideal.

(iii) $Z(A/(\mathfrak{q})) = L_{\mathfrak{q}}$: By Lemma 5.4, $Z(\mathcal{A}_{X,\varphi}) = \mathbb{K}$, and $A/(\mathfrak{q}) \subseteq \frac{A_{X,\varphi}}{(\mathfrak{q})_{X,\varphi}} \simeq L_{\mathfrak{q}} \otimes \mathcal{A}_{X,\varphi}$, hence $Z(A/(\mathfrak{q})) = L_{\mathfrak{q}}$. □

The case where $\mathfrak{q} = (C)$ is dealt with in the next proposition.

Proposition 6.14. *$A \cap (C)_{X,\varphi} = (C)$ and the ideal (C) of A is a completely prime ideal.*

Proof. Recall that $A = \mathcal{A}[F; \sigma, \delta]$ (see (6.16)), $\Phi = X\varphi \in \mathcal{A}$ is a product of normal elements X and φ in $\mathcal{A}$ and, by (6.13), the central element C can be written as $C = \Phi F + s$ where $s = -q^2K^{-1}EY^2 - X\tilde{y}$ and $\tilde{y} := \frac{q^4}{1-q^2}YK^{-1} - \frac{1}{1-q^2}YK$.

(i) *If $Xf \in (C)$ for some $f \in A$ then $f \in (C)$*: Notice that $Xf = Cg$ for some $g \in A$. To prove the statement (i), we use induction on the degree $m = \deg_F(f)$ of the element $f \in A$. Notice that A is a domain and $\deg_F(fg) = \deg_F(f) + \deg_F(g)$ for all $f, g \in A$. The case when $m \leqslant 0$ i.e., $f \in \mathcal{A}$, is obvious since the equality $Xf = Cg$ holds iff $f = g = 0$ (since $\deg_F(Xf) \leqslant 0$ and $\deg_F(Cg) \geqslant 1$ providing $g \neq 0$). So, we may assume that $m \geqslant 1$. We can write the element f as a sum $f = f_0 + f_1F + \cdots + f_mF^m$ where $f_i \in \mathcal{A}$ and $f_m \neq 0$. The equality $Xf = Cg$ implies that $\deg_F(g) = \deg_F(Xf) - \deg_F(C) = m - 1$. Therefore, $g = g_0 + g_1F + \cdots + g_{m-1}F^{m-1}$ for some $g_i \in \mathcal{A}$ and $g_{m-1} \neq 0$. Then

$$\begin{aligned} Xf_0 + Xf_1F + \cdots + Xf_mF^m &= (\Phi F + s)(g_0 + g_1F + \cdots + g_{m-1}F^{m-1}) \\ &= \Phi\big(\sigma(g_0)F + \delta(g_0)\big) + \Phi\big(\sigma(g_1)F + \delta(g_1)\big)F + \cdots + \Phi\big(\sigma(g_{m-1})F + \delta(g_{m-1})\big)F^{m-1} \\ &\quad + sg_0 + sg_1F + \cdots + sg_{m-1}F^{m-1} \\ &= \Phi\delta(g_0) + sg_0 + \big(\Phi\sigma(g_0) + \Phi\delta(g_1) + sg_1\big)F + \cdots + \Phi\sigma(g_{m-1})F^m. \end{aligned} \tag{6.18}$$

Comparing the terms of degree zero we have the equality $Xf_0 = \Phi\delta(g_0) + sg_0 = X\varphi\delta(g_0) + (-q^2K^{-1}EY^2 - X\tilde{y})g_0$, i.e., $X(f_0 - \varphi\delta(g_0) + \tilde{y}g_0) = -q^2K^{-1}EY^2g_0$. All the terms in this equality belong to the algebra $\mathcal{A}$. Recall that X is a normal element in $\mathcal{A}$ such that $\mathcal{A}/\mathcal{A}X$ is domain (see (5.8)) and the element $K^{-1}EY^2$ does not belong to the ideal $\mathcal{A}X$. Hence we have $g_0 \in \mathcal{A}X$, i.e., $g_0 = Xh_0$ for some $h_0 \in \mathcal{A}$. Now the element g can be written as $g = Xh_0 + g'F$ where $g' = 0$ if $m = 1$, and $\deg_F(g') = m - 2 = \deg_F(g) - 1$ if $m \geqslant 2$. Now, $Xf = C(Xh_0 + g'F)$ and so $X(f - Ch_0) = Cg'F$. Notice that $Cg'F$ has zero constant term as a noncommutative polynomial in F (where the coefficients are written on the left). Therefore, the element $f - Ch_0$

has zero constant term, and hence can be written as $f - Ch_0 = f'F$ for some $f' \in A$ with

$$\begin{aligned}\deg_F(f') + \deg_F(F) &= \deg_F(f'F) = \deg_F(f') + 1\\ &= \deg_F(f - Ch_0) \leqslant \max\left(\deg_F(f), \deg_F(Ch_0)\right) = \begin{cases} 1, & \text{if } m = 1,\\ m, & \text{if } m \geqslant 1.\end{cases}\end{aligned}$$

In both cases, $\deg_F(f') < \deg_F(f)$. Now, $Cg'F = X(f - Ch_0) = Xf'F$, hence $Xf' = Cg' \in (C)$ (by deleting F). By induction, $f' \in (C)$, and then $f = Ch_0 + f'F \in (C)$, as required.

(ii) *If* $\varphi f \in (C)$ *for some* $f \in A$ *then* $f \in (C)$: Notice that $\varphi f = Cg$ for some $g \in A$. To prove the statement (ii) we use similar arguments to the ones given in the proof of the statement (i). We use induction on $m = \deg_F(f)$. The case where $m \leqslant 0$, i.e., $f \in \mathcal{A}$ is obvious since the equality $\varphi f = Cg$ holds iff $f = g = 0$ (since $\deg_F(\varphi f) \leqslant 0$ and $\deg_F(Cg) \geqslant 1$ providing $g \neq 0$). So we may assume that $m \geqslant 1$. We can write the element f as a sum $f = f_0 + f_1F + \cdots + f_mF^m$ where $f_i \in \mathcal{A}$ and $f_m \neq 0$. Then the equality $\varphi f = Cg$ implies that $\deg_F(g) = \deg_F(\varphi f) - \deg_F(C) = m - 1$. Therefore, $g = g_0 + g_1F + \cdots + g_{m-1}F^{m-1}$ where $g_i \in \mathcal{A}$ and $g_{m-1} \neq 0$. Then replacing X by φ in (6.18), we have the equality

$$\varphi f_0 + \varphi f_1 F + \cdots + \varphi f_m F^m = \Phi\delta(g_0) + sg_0 + \cdots + \Phi\sigma(g_{m-1})F^m. \tag{6.19}$$

The element s can be written as a sum $s = (-\frac{q}{1-q^2}\varphi K^{-1} + \frac{1}{1-q^2}KX)Y$. Then equating the constant terms of the equality (6.19) and then collecting terms multiple of φ we obtain the equality in the algebra $\mathcal{A}$: $\varphi(f_0 - X\delta(g_0) + \frac{q}{1-q^2}K^{-1}Yg_0) = \frac{1}{1-q^2}KXYg_0$. The element $\varphi \in \mathcal{A}$ is a normal element such that the factor algebra $\mathcal{A}/\mathcal{A}\varphi$ is a domain (see (5.12)) and the element KXY does not belong to the ideal $\mathcal{A}\varphi$. Therefore, $g_0 \in \mathcal{A}\varphi$, i.e., $g_0 = \varphi h_0$ for some element $h_0 \in \mathcal{A}$. Recall that $\deg_F(g) = m - 1$. Now, $g = \varphi h_0 + g'F$ where $g' \in A$ and $g' = 0$ if $m = 1$, and $\deg_F(g') = m - 2 = \deg_F(g) - 1$ if $m \geqslant 2$. Now, $\varphi f = Cg = C(\varphi h_0 + g'F)$. Hence, $\varphi(f - Ch_0) = Cg'F$, and so $f - Ch_0 = f'F$ for some $f' \in A$ with

$$\begin{aligned}\deg_F(f') + \deg_F(F) &= \deg_F(f'F) = \deg_F(f') + 1\\ &= \deg_F(f - Ch_0) \leqslant \max\left(\deg_F(f), \deg_F(Ch_0)\right) = \begin{cases} 1, & \text{if } m = 1,\\ m, & \text{if } m \geqslant 1.\end{cases}\end{aligned}$$

In both cases, $\deg_F(f') < \deg_F(f)$. Now, $Cg'F = \varphi(f - Ch_0) = \varphi f'F$, hence $\varphi f' = Cg' \in (C)$ (by deleting F). Now, by induction, $f' \in (C)$, and then $f = Ch_0 + f'F \in (C)$, as required.

(iii) $A \cap (C)_{X,\varphi} = (C)$: Let $u \in A \cap (C)_{X,\varphi}$. Then $X^i\varphi^j u \in (C)$ for some $i, j \in \mathbb{N}$. It remains to show that $u \in (C)$. By the statement (i), $\varphi^j u \in (C)$, and then by the statement (ii), $u \in (C)$.

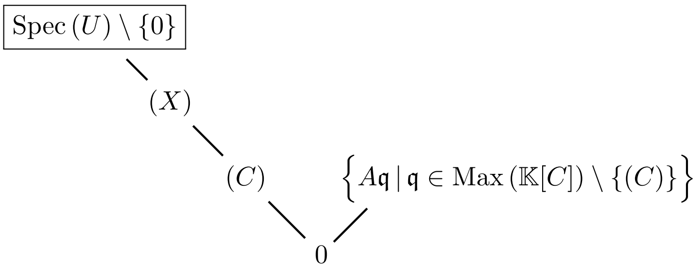
(iv) *The ideal* (C) *of* A *is a completely prime ideal*: By Lemma 6.8, $A_{X,\varphi}/(C)_{X,\varphi} \simeq \mathcal{A}_{X,\varphi}$, in particular, $A_{X,\varphi}/(C)_{X,\varphi}$ is a domain. By the statement (iii), the algebra $A/(C)$ is a subalgebra of $A_{X,\varphi}/(C)_{X,\varphi}$, so $A/(C)$ is a domain. This means that the ideal (C) is a completely prime ideal of A. □

The next theorem gives an explicit description of the poset $(\mathrm{Spec}\,(A), \subseteq)$.

Theorem 6.15. *Let* $U := U_q(\mathfrak{sl}_2)$*. The prime spectrum of the algebra* A *is a disjoint union*

$$\mathrm{Spec}(A) = \mathrm{Spec}(U) \sqcup \mathrm{Spec}(A_{X,\varphi}) = \{(X,\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(U)\} \sqcup \{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}(\mathbb{K}[C])\}. \tag{6.20}$$

Furthermore,



(6.21)

Proof. By Lemma 6.11.(2), $A/(X) \simeq U$. By Lemma 6.12.(1) and Proposition 3.3,

$$\mathrm{Spec}\,(A) = \mathrm{Spec}\,(A,X) \;\sqcup\; \mathrm{Spec}\,(A_X). \tag{6.22}$$

By Lemma 6.12.(2) and Proposition 3.3,

$$\mathrm{Spec}\,(A_X) = \mathrm{Spec}\,(A_{X,\varphi}). \tag{6.23}$$

Therefore, $\mathrm{Spec}\,(A) = \{(X,\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\,(U)\} \sqcup \{A \cap A_{X,\varphi}\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}\,(\mathbb{K}[C])\}$. Finally, by Theorem 6.13.(1), $A \cap A_{X,\varphi}\mathfrak{q} = (\mathfrak{q})$ for all $\mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}$. By Proposition 6.14, $A \cap A_{X,\varphi}C = (C)$. Therefore, (6.20) holds. For all $\mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}$, the ideals $A\mathfrak{q}$ of A are maximal. By (6.11), $AC \subseteq (X)$. Therefore, (6.21) holds. □

For a list of prime ideals of the algebra $U_q(\mathfrak{sl}_2)$ see [36, Theorem 4.6]. We note that every nonzero prime ideal of $U_q(\mathfrak{sl}_2)$ is a primitive ideal.

The next corollary shows that every nonzero ideal of the algebra A meets the centre of A.

Corollary 6.16. *If* I *is a nonzero ideal of the algebra* A *then* $I \cap \mathbb{K}[C] \neq 0$.

Proof. Suppose that the result is not true, let us choose an ideal $J \neq 0$ maximal such that $J \cap \mathbb{K}[C] = 0$. We claim that J is a prime ideal. Otherwise, suppose that J is not prime, then there exist ideals $\mathfrak{p}$ and $\mathfrak{q}$ such that $J \subsetneqq \mathfrak{p}$, $J \subsetneqq \mathfrak{q}$ and $\mathfrak{p}\mathfrak{q} \subseteq J$. By the maximality of J, $\mathfrak{p} \cap \mathbb{K}[C] \neq 0$ and $\mathfrak{q} \cap \mathbb{K}[C] \neq 0$. Then $J \cap \mathbb{K}[C] \supseteq \mathfrak{p}\mathfrak{q} \cap \mathbb{K}[C] \neq 0$, a contradiction. So, J is a prime ideal, but by Theorem 6.15 for all nonzero primes P, $P \cap \mathbb{K}[C] \neq 0$, a contradiction. Therefore, for any nonzero ideal I, $I \cap \mathbb{K}[C] \neq 0$. □

The next result is an explicit description of the set of maximal ideals of the algebra A.

Corollary 6.17. $\mathrm{Max}\,(A) = \mathrm{Max}\,(U) \;\sqcup\; \big\{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{(C)\}\big\}$.

Proof. The equality follows from (6.21). □

In the following lemma, we define a family of left A-modules. We will show that these modules are simple A-modules and their annihilators are equal to (C).

Lemma 6.18. *For $\lambda \in \mathbb{K}^*$, we define the left A-module $W(\lambda) := A/A(X-\lambda, Y, F)$. Then*

1. *The module $W(\lambda)$ is a simple A-module.*
2. $\mathrm{ann}_A(W(\lambda)) = (C)$.

Proof. 1. Let $\bar{1} = 1 + A(X-\lambda, Y, F)$ be the canonical generator of the A-module $W(\lambda)$. Then $W(\lambda) = \sum_{i\in\mathbb{N}} E^i \mathbb{K}[K^{\pm 1}]\,\bar{1}$. Suppose that V is a nonzero submodule of $W(\lambda)$, we have to show that $V = W(\lambda)$. Let $v = \sum_{i=0}^{n} E^i f_i \bar{1}$ be a nonzero element of the module V where $f_i \in \mathbb{K}[K^{\pm 1}]$ and $f_n \neq 0$. By Lemma 5.2.(2), $Yv = \sum_{i=1}^{n}(q^i E^i Y - \frac{q(1-q^{2i})}{1-q^2} X E^{i-1}) f_i \bar{1} = \sum_{i=1}^{n} -\frac{q(1-q^{2i})}{1-q^2} X E^{i-1} f_i \bar{1}$. By induction, we see that $Y^n v = P\bar{1} \in V$ where P is a nonzero Laurent polynomial in $\mathbb{K}[K^{\pm 1}]$. Then it follows that $\bar{1} \in V$, and so $V = W(\lambda)$.

2. It is clear that $\mathrm{ann}_A(W(\lambda)) \supseteq (C)$ and $X \notin \mathrm{ann}_A(W(\lambda))$. By (6.21), we must have $\mathrm{ann}_A(W(\lambda)) = (C)$. □

The next theorem is a description of the set of primitive ideals of the algebra A.

Theorem 6.19. $\mathrm{Prim}\,(A) = \mathrm{Prim}\,(U) \sqcup \{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}\,\mathbb{K}[C] \setminus \{0\}\}$.

Proof. Clearly, $\mathrm{Prim}\,(U) \subseteq \mathrm{Prim}\,(A)$ and $\{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Max}\,(\mathbb{K}[C]) \setminus \{C\mathbb{K}[C]\}\} \subseteq \mathrm{Prim}\,(A)$ since $A\mathfrak{q}$ is a maximal ideal (Corollary 6.17). By Corollary 6.16, 0 is not a primitive ideal. In view of (6.21), it suffices to show that $(C) \in \mathrm{Prim}\,(A)$. But this follows from Lemma 6.18. □

The next corollary is a description of the set $\mathrm{Spec}_c(A)$ of completely prime ideals of the algebra A.

Corollary 6.20. *The set $\mathrm{Spec}_c(A)$ of completely prime ideals of A is equal to*

$$\begin{aligned}\mathrm{Spec}_c(A) = \mathrm{Spec}_c(U) \sqcup \{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}\,(\mathbb{K}[C])\} \\ = \{(X,\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\,(U),\ \mathfrak{p} \neq \mathrm{ann}_U(M) \text{ for some simple finite dimensional} \\ U\text{-module } M \text{ of } \dim_{\mathbb{K}}(M) \geqslant 2\} \sqcup \{A\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}\,(\mathbb{K}[C])\}.\end{aligned}$$

Proof. The result follows from Theorem 6.13.(1) and Proposition 6.14. □

6.4 Action of A on the polynomial algebra $\mathbb{K}[x,y]$

In the study of universal enveloping algebras, the Weyl algebras play an important role as we have seen in the previous chapters. In this section, we consider a q-analogue of the (first) Weyl algebra, which is a central simple algebra of Gelfand-Kirillov dimension 2. It plays a role similar to the one the Weyl algebra does but in the study of quantum algebras.

Let σ_x and σ_y be the automorphisms of the polynomial algebra $\mathbb{K}[x, y]$ that are defined by the rule

$$\sigma_x(x) = qx, \quad \sigma_x(y) = y, \quad \sigma_y(x) = x, \quad \sigma_y(y) = qy.$$

The q-partial derivatives ∂_x^q and ∂_y^q on $\mathbb{K}[x, y]$ are defined by the rule

$$\partial_x^q(x^iy^j) = [i]\, x^{i-1}y^j \quad \text{and} \quad \partial_y^q(x^iy^j) = [j]\, x^iy^{j-1}$$

where $[n] := \frac{q^n - q^{-n}}{q - q^{-1}}$ is the 'quantum integer'. We denote the operator of 'multiplication by x (resp. y)' simply by x (resp. y). Nontrivial commutation relations between the elements x, σ_x, ∂_x^q, y, σ_y and ∂_y^q are

$$\sigma_x x = qx\sigma_x, \qquad \partial_x^q\sigma_x = q\sigma_x\partial_x^q, \qquad \partial_x^q x = q^{-1}x\partial_x^q + \sigma_x = qx\partial_x^q + \sigma_x^{-1}, \tag{6.24}$$

$$\sigma_y y = qy\sigma_y, \qquad \partial_y^q\sigma_y = q\sigma_y\partial_y^q, \qquad \partial_y^q y = q^{-1}y\partial_y^q + \sigma_y = qy\partial_y^q + \sigma_y^{-1}. \tag{6.25}$$

Let $\mathcal{W}_2$ be the subalgebra of $\mathrm{End}\,(\mathbb{K}[x, y])$ generated by the elements x, $\sigma_x^{\pm 1}$, ∂_x^q, y, $\sigma_y^{\pm 1}$ and ∂_y^q.

Lemma 6.21. *$\mathbb{K}[x, y]$ is a simple $\mathcal{W}_2$-module.*

Proof. Let N be a non-zero submodule of $\mathbb{K}[x, y]$. Let $0 \neq f \in N$. Notice that $\deg_x(\partial_x^q(f)) = \deg_x(f) - 1$ and $\deg_y(\partial_y^q(f)) = \deg_y(f) - 1$. Thus there exist $m, n \in \mathbb{N}$ such that $0 \neq (\partial_x^q)^m(\partial_y)^n(f) \in \mathbb{K}$. Therefore, $N = \mathbb{K}[x, y]$, and $\mathbb{K}[x, y]$ is a simple $\mathcal{W}_2$-module. □

Proposition 6.22. *There is an algebra homomorphism $\varrho : A \to \mathcal{W}_2$ defined by the rule*

$$K \mapsto (\sigma_x^{-1})^2\sigma_y^{-1}, \qquad K^{-1} \mapsto \sigma_y\sigma_x^2, \qquad E \mapsto -\sigma_y^{-1}x(\partial_x^q)^2 - \sigma_x\partial_x^q y\partial_y^q,$$
$$F \mapsto x, \qquad Y \mapsto y, \qquad X \mapsto -\sigma_x\partial_x^q y\sigma_y.$$

Proof. Let $M := A/A(K - 1, X, E)$, it is a left A-module. Then $M \simeq \mathbb{K}[F, Y]\tilde{1}$, where $\tilde{1} = 1 + A(K - 1, K^{-1} - 1, X, E)$. The linear map $\mathbb{K}[x, y] \to M$, $x^iy^j \mapsto F^iY^j\tilde{1}$ is a bijection. Let ϱ be the representation of A in $\mathbb{K}[x, y]$ obtained via this bijection from representation of A in M. Then we obtain the above correspondence. Let us show that $\varrho(E) = -\sigma_y^{-1}x(\partial_x^q)^2 - \sigma_x\partial_x^q y\partial_y^q$ for example. Notice that the following equality holds in the algebra A

$$EF^i = F^iE + [i]\, F^{i-1}\frac{Kq^{-i+1} - K^{-1}q^{i-1}}{q - q^{-1}}. \tag{6.26}$$

Notice further that for all integers a and b, $[a + b] = q^{-b}[a] + q^a[b]$. Then

$$EF^iY^j\tilde{1} = \Big(F^iE + [i]\, F^{i-1}\frac{Kq^{-i+1} - K^{-1}q^{i-1}}{q - q^{-1}}\Big)Y^j\tilde{1} = [i]\, F^{i-1}\frac{Kq^{-i+1} - K^{-1}q^{i-1}}{q - q^{-1}}Y^j\tilde{1}$$
$$= -[i]\,[i + j - 1]\, F^{i-1}Y^j\tilde{1} = \Big(-q^{-j}[i]\,[i - 1] - q^{i-1}[i]\,[j]\Big)\, F^{i-1}Y^j\tilde{1}.$$

We can see from this that $\varrho(E) = -\sigma_y^{-1}x(\partial_x^q)^2 - \sigma_x\partial_x^q y\partial_y^q$. □

q-analogue of the Weyl algebra. Recall that the (first) Weyl algebra A_1 is the subalgebra of $\mathrm{End}\,(\mathbb{K}[x])$ generated by the operators x and ∂_x where ∂_x is the derivative with respect to x. More precisely, $A_1 = \mathbb{K}\langle x, \partial_x \,|\, \partial_x x - x\partial_x = 1\rangle$.

Let $\mathcal{W}_1$ be the subalgebra of $\mathrm{End}\,(\mathbb{K}[x])$ generated by the operators σ_x, σ_x^{-1}, x and ∂_x^q. In particular, these generators of $\mathcal{W}_1$ satisfy the relations (6.24). The next proposition shows that $\mathcal{W}_1$ is a central simple GWA and the relations (6.24) (together with $\sigma_x\sigma_x^{-1} = \sigma_x^{-1}\sigma_x = 1$) are the defining relations of $\mathcal{W}_1$. Notice that $\mathcal{W}_2 \simeq \mathcal{W}_1 \otimes \mathcal{W}_1$ is a tensor product of algebras, then $\mathcal{W}_2$ is also a central simple algebra. The algebra $\mathcal{W}_1$ can be viewed as a q-analogue of the Weyl algebra A_1. Some ring theoretic properties of the algebra $\mathcal{W}_1$ were studied in [34].

Proposition 6.23. *The algebra $\mathcal{W}_1$ is a central simple GWA.*

Proof. Let $\mathcal{U}$ be the algebra generated by the symbols σ_x, σ_x^{-1}, x and ∂_x^q subject to the defining relations (6.24) (together with the relation $\sigma_x\sigma_x^{-1} = \sigma_x^{-1}\sigma_x = 1$). Notice that

$$x\partial_x^q = \frac{\sigma_x - \sigma_x^{-1}}{q - q^{-1}} \quad \text{and} \quad \partial_x^q x = \frac{q\sigma_x - q^{-1}\sigma_x^{-1}}{q - q^{-1}}.$$

Then $\mathcal{U} = \mathbb{K}[\sigma_x^{\pm 1}][x, \partial_x^q; \tau, a = \frac{q\sigma_x - q^{-1}\sigma_x^{-1}}{q-q^{-1}}]$ is a GWA where τ is an automorphism of the algebra $\mathbb{K}[\sigma_x^{\pm 1}]$ defined by $\tau(\sigma_x) = q^{-1}\sigma_x$. Moreover, there is a natural epimorphism of algebras $f : \mathcal{U} \twoheadrightarrow \mathcal{W}_1$. Let $\mathcal{U}_x$ be the localization of $\mathcal{U}$ at the powers of the element x. Then $\mathcal{U}_x = \mathbb{K}_q[\sigma_x^{\pm 1}, x^{\pm 1}]$ where $\mathbb{K}_q[\sigma_x^{\pm 1}, x^{\pm 1}] := \mathbb{K}\langle \sigma_x^{\pm 1}, x^{\pm 1} \,|\, \sigma_x x = qx\sigma_x\rangle$ is the central simple quantum torus. The inclusions $\mathbb{K} \subseteq Z(\mathcal{U}) \subseteq \mathcal{U} \cap Z(\mathcal{U}_x) = \mathbb{K}$ yield that $Z(\mathcal{U}) = \mathbb{K}$. The simplicity of $\mathcal{U}$ follows immediately from Theorem 2.16 ([6, Theorem 4.2]). Now, the epimorphism of algebras $f : \mathcal{U} \twoheadrightarrow \mathcal{W}_1$ must be an isomorphism. Hence, $\mathcal{W}_1$ is a central simple GWA. □

6.5 The centralizer of K in the algebra A

In this section, we find the explicit generators and defining relations of the centralizer $C_A(K)$ of the element K in the algebra A.

Proposition 6.24. *The algebra $C_A(K) = \mathbb{K}\langle K^{\pm 1},\ FE,\ YX,\ EY^2,\ FX^2\rangle$ is a Noetherian domain.*

Proof. Since A is a domain, then so is its subalgebra $C_A(K)$. Notice that the algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is a $\mathbb{Z}$-graded Noetherian algebra where $A_i = \{a \in A \,|\, KaK^{-1} = q^i a\}$. Then the algebra $A_0 = C_A(K)$ is a Noetherian algebra.

The algebra $U_q(\mathfrak{sl}_2)$ is a GWA: $U_q(\mathfrak{sl}_2) \simeq \mathbb{K}[K^{\pm 1}, \Omega][E, F; \sigma, a := \Omega - \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}]$ where $\Omega = FE + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2}$, $\sigma(K) = q^{-2}K$ and $\sigma(\Omega) = \Omega$. In particular, $U_q(\mathfrak{sl}_2)$ is a $\mathbb{Z}$-graded algebra $U_q(\mathfrak{sl}_2) = \bigoplus_{i\in\mathbb{Z}} Dv_i$ where $D := \mathbb{K}[K^{\pm 1}, \Omega] = \mathbb{K}[K^{\pm 1}, FE]$, $v_i = E^i$ if $i \geqslant 1$, $v_i = F^{|i|}$ if $i \leqslant -1$ and $v_0 = 1$. The quantum plane $\mathbb{K}_q[X,Y]$ is also a GWA: $\mathbb{K}_q[X,Y] \simeq \mathbb{K}[t][X, Y; \sigma, t]$ where $t := YX$ and $\sigma(t) = qt$. Therefore, the quantum plane is a $\mathbb{Z}$-graded algebra $\mathbb{K}_q[X,Y] = \bigoplus_{j\in\mathbb{Z}} \mathbb{K}[t]w_j$

where $w_j = X^j$ if $j \geqslant 1$, $w_j = Y^{|j|}$ if $j \leqslant -1$ and $w_0 = 1$. Since $A = U_q(\mathfrak{sl}_2)\otimes \mathbb{K}_q[X,Y]$ (tensor product of vector spaces), and notice that $Et = tE + X^2$, $Ft = tF + q^{-2}K^{-1}Y^2$, we have

$$A = U_q(\mathfrak{sl}_2)\otimes \mathbb{K}_q[X,Y] = \bigoplus_{i\in\mathbb{Z}} Dv_i \otimes \bigoplus_{j\in\mathbb{Z}} \mathbb{K}[t]w_j = \bigoplus_{i,j\in\mathbb{Z}} D[t]v_iw_j. \qquad (6.27)$$

By (6.27), for each $k\in\mathbb{Z}$, $A_k = \bigoplus_{i,j\in\mathbb{Z},\,2i+j=k} D[t]v_iw_j = \bigoplus_{i\in\mathbb{Z}} D[t]v_iw_{k-2i}$. Then $C_A(K) = A_0 = \bigoplus_{i\geqslant 0} D[t]E^iY^{2i}\oplus\bigoplus_{j\geqslant 1} D[t]F^jX^{2j}$. Notice that $EY^2\cdot t = q^{-2}t\cdot EY^2 + qt^2$ and $FX^2\cdot t = q^2t\cdot FX^2 + q^{-1}K^{-1}t^2$. By induction, one sees that for all $i,j\geqslant 0$, $E^iY^{2i}\in\bigoplus_{n\in\mathbb{N}}\mathbb{K}[t](EY^2)^n$ and $F^jX^{2j}\in\bigoplus_{n\in\mathbb{N}}\mathbb{K}[K^{\pm1},t](FX^2)^n$. Hence, $C_A(K) = A_0 = \bigoplus_{i\geqslant 0} D[t](EY^2)^i\oplus\bigoplus_{j\geqslant 1} D[t](FX^2)^j$. In particular, $C_A(K) = \mathbb{K}\langle K^{\pm1},\ FE,\ YX,\ EY^2,\ FX^2\rangle$. □

Lemma 6.25. *1. $C_{A_{X,Y,\varphi}}(K) = \mathbb{K}[C,K^{\pm1}]\otimes\mathbb{K}_{q^2}[(YX)^{\pm1},(Y\varphi)^{\pm1}]$ is a tensor product of algebras where $\mathbb{K}_{q^2}[(YX)^{\pm1},(Y\varphi)^{\pm1}]$ is a central, simple, quantum torus with $YX\cdot Y\varphi = q^2Y\varphi\cdot YX$.*

2. $\mathrm{GK}\left(C_{A_{X,Y,\varphi}}(K)\right) = 4$.

3. $\mathrm{GK}\,(C_A(K)) = 4$.

4. $A_{X,Y,\varphi} = \bigoplus_{i\in\mathbb{Z}} C_{A_{X,\varphi,Y}}(K)Y^i$.

Proof. 1. By (6.10), $A_{X,Y,\varphi} = \mathbb{K}[C]\otimes\Lambda'$ where Λ' is a quantum torus. Then $C_{A_{X,Y,\varphi}}(K) = \mathbb{K}[C]\otimes C_{\Lambda'}(K)$. Since Λ' is a quantum torus, it is easy to see that $C_{\Lambda'}(K) = \bigoplus_{i,j,k\in\mathbb{Z}} K^i(YX)^j(Y\varphi)^k$, i.e., $C_{\Lambda'}(K) = \mathbb{K}[K^{\pm1}]\otimes\mathbb{K}_{q^2}[(YX)^{\pm1},(Y\varphi)^{\pm1}]$. Then statement 1 follows.

2. Statement 2 follows from statement 1.

3. Let R be the subalgebra of $C_A(K)$ generated by the elements C, $K^{\pm1}$, YX and $Y\varphi$. Then $R = \mathbb{K}[C,K^{\pm1}]\otimes\mathbb{K}_{q^2}[YX,Y\varphi]$ is a tensor product of algebras. Clearly, R is a Noetherian algebra of Gelfand-Kirillov dimension 4. So $\mathrm{GK}\,(C_A(K))\geqslant\mathrm{GK}\,(R) = 4$. By statement 2, $\mathrm{GK}\,(C_A(K))\leqslant\mathrm{GK}\,(C_{A_{X,Y,\varphi}}(K)) = 4$. Hence, $\mathrm{GK}\,(C_A(K)) = 4$.

4. Statement 4 follows from statement 1 and (6.10). □

Proposition 6.26. *Let $h := \varphi X^{-1}$, $e := EX^{-2}$ and $t := YX$. Then*

1. $C_{A_{X,\varphi}}(K) = \mathbb{K}[C,K^{\pm1}]\otimes\mathscr{A}$ is a tensor product of algebras where $\mathscr{A} := \mathbb{K}[h^{\pm1}][t,e;\sigma,a = \frac{q^{-2}h-1}{1-q^2}]$ is a central simple GWA (where $\sigma(h) = q^2h$).

2. $\mathrm{GK}\,(C_{A_{X,\varphi}}(K)) = 4$.

3. $A_{X,\varphi} = \bigoplus_{i\in\mathbb{Z}} C_{A_{X,\varphi}}(K)X^i$.

Proof. 1. Let $\mathscr{A}$ be the subalgebra of $C_{A_{X,\varphi}}(K)$ generated by the elements $h^{\pm1}$, e and t.

(i) *$\mathscr{A}$ is a central simple GWA*: The elements $h^{\pm1}$, e and t satisfy the following relations

$$hh^{-1} = h^{-1}h = 1,\quad th = q^2ht,\quad eh = q^{-2}he,\quad et = \frac{q^{-2}h-1}{1-q^2},\quad te = \frac{h-1}{1-q^2}. \qquad (6.28)$$

Hence, $\mathscr{A}$ is an epimorphic image of the GWA $\mathscr{A}' = \mathbb{K}[h^{\pm1}][t,e;\sigma,a = \frac{q^{-2}h-1}{1-q^2}]$ where $\sigma(h) = q^2h$. Now, we prove that $\mathscr{A}'$ is a central simple algebra. Let $\mathscr{A}'_e$ be the localization of $\mathscr{A}'$ at the powers of the element e. Then $\mathscr{A}'_e = \mathbb{K}[h^{\pm1}][e^{\pm1};\sigma']$ where $\sigma'(h) = q^{-2}h$. Clearly, $Z(\mathscr{A}'_e) = \mathbb{K}$ and $\mathscr{A}'_e$

is a simple algebra. So, $Z(\mathscr{A}') = Z(\mathscr{A}'_e) \cap \mathscr{A}' = \mathbb{K}$. To show that $\mathscr{A}'$ is simple, it suffices to prove that $\mathscr{A}'e^i\mathscr{A}' = \mathscr{A}'$ for any $i \in \mathbb{N}$. The case $i = 1$ is obvious, since $1 = q^2et - te \in \mathscr{A}'e\mathscr{A}'$. By induction, for $i > 1$, it suffices to show that $e^{i-1} \in \mathscr{A}'e^i\mathscr{A}'$. This follows from the equality $te^i = q^{2i}e^it - \frac{1-q^{2i}}{1-q^2}e^{i-1}$. So, $\mathscr{A}'$ is a simple algebra. Now, the epimorphism of algebras $\mathscr{A}' \longrightarrow\!\!\!\!\!\rightarrow \mathscr{A}$ is an isomorphism. Hence, $\mathscr{A} \simeq \mathscr{A}'$ is a central simple GWA.

(ii) $C_{A_{X,\varphi}}(K) = \mathbb{K}[C, K^{\pm 1}] \otimes \mathscr{A}$: By Lemma 6.8, $A_{X,\varphi} = \mathbb{K}[C] \otimes \mathcal{A}_{X,\varphi}$. So, $C_{A_{X,\varphi}}(K) = \mathbb{K}[C] \otimes C_{\mathcal{A}_{X,\varphi}}(K)$. By (5.2), $\mathcal{A}_{X,\varphi} = \mathbb{E}_{X,\varphi}[K^{\pm 1}; \tau]$ where $\tau(E) = q^2E, \tau(X) = qX, \tau(Y) = q^{-1}Y$ and $\tau(\varphi) = q\varphi$. Then $C_{\mathcal{A}_{X,\varphi}}(K) = \mathbb{K}[K^{\pm 1}] \otimes \mathbb{E}^{\tau}_{X,\varphi}$. To finish the proof of statement (ii), it suffices to show that $\mathbb{E}^{\tau}_{X,\varphi} = \mathscr{A}$. By (5.1), $\mathbb{E}_{X,\varphi} = \mathbb{K}[X^{\pm 1}, \varphi^{\pm 1}][E, Y; \sigma, a = \frac{\varphi - X}{q^{-1}-q}]$ is a GWA. Then $\mathbb{E}_{X,\varphi} = \bigoplus_{i\geqslant 0} \mathbb{K}[X^{\pm 1}, \varphi^{\pm 1}]E^i \oplus \bigoplus_{j\geqslant 1} \mathbb{K}[X^{\pm 1}, \varphi^{\pm 1}]Y^j = \bigoplus_{i\geqslant 0, k\in\mathbb{Z}} \mathbb{K}[h^{\pm 1}]E^iX^k \oplus \bigoplus_{j\geqslant 1, k\in\mathbb{Z}} \mathbb{K}[h^{\pm 1}]Y^jX^k$. Now, it is clear that $\mathbb{E}^{\tau}_{X,\varphi} = \bigoplus_{i\geqslant 0} \mathbb{K}[h^{\pm 1}]e^i \oplus \bigoplus_{j\geqslant 1} \mathbb{K}[h^{\pm 1}]t^j = \mathscr{A}$.

2. Notice that $\mathrm{GK}\,(\mathscr{A}) = 2$, statement 2 follows from statement 1.

3. Notice that $\mathcal{A}_{X,\varphi} = \bigoplus_{i\in\mathbb{Z}} C_{\mathcal{A}_{X,\varphi}}(K)X^i$, statement 3 then follows from Lemma 6.8. □

Defining relations of the algebra $C_A(K)$. We have to select carefully generators of the algebra $C_A(K)$ in order to make the corresponding defining relations simpler. The next lemma indicates how we choose the generators.

Lemma 6.27. *We have the following relations.*

1. $YX \cdot Y\varphi = q^2Y\varphi \cdot YX$.
2. $FE \cdot YX = q^2YX \cdot FE + \frac{q+q^{-1}}{1-q^2}K^{-1}Y\varphi - \frac{q^2(qK+q^{-1}K^{-1})}{1-q^2}YX + C$.
3. $FE \cdot Y\varphi = q^{-2}Y\varphi \cdot FE + \frac{qK+q^{-1}K^{-1}}{1-q^2}Y\varphi - \frac{q(1+q^2)}{1-q^2}KYX + C$.

Proof. 1. Obvious.

2. Using the defining relations of A, the expression (6.11) of C, and $Y\varphi = q^4YX + q(1-q^2)EY^2$,

$$\begin{aligned} FE \cdot YX &= F(X + q^{-1}YE)X = FX^2 + YFXE = FX^2 + Y(YK^{-1} + XF)E \\ &= FX^2 + q^{-2}K^{-1}Y^2E + YXFE \\ &= q^2(YX)(FE) + (1+q^2)K^{-1}EY^2 - \frac{q^3K + (q - q^3 - q^5)K^{-1}}{1-q^2}YX + C \\ &= q^2YX \cdot FE + \frac{q+q^{-1}}{1-q^2}K^{-1}Y\varphi - \frac{q^2(qK + q^{-1}K^{-1})}{1-q^2}YX + C. \end{aligned}$$

3. $$\begin{aligned} FE \cdot Y\varphi &= F(X + q^{-1}YE)\varphi = FX\varphi + q^{-2}YF\varphi E = FX\varphi + q^{-2}Y(\varphi F + YK)E \\ &= q^{-2}Y\varphi FE + (q^2K + K^{-1})EY^2 - (\frac{q^3(K - K^{-1})}{1-q^2} + q(1+q^2)K)YX + C \\ &= q^{-2}Y\varphi \cdot FE + \frac{qK + q^{-1}K^{-1}}{1-q^2}Y\varphi - \frac{q(1+q^2)}{1-q^2}KYX + C. \end{aligned}$$ □

Let $\Theta := (1-q^2)\Omega = (1-q^2)FE + \frac{q^2(qK+q^{-1}K^{-1})}{1-q^2} \in Z(U_q(\mathfrak{sl}_2))$. By (6.12), We have

$$C = (\Theta - \frac{qK^{-1}}{1-q^2})YX + q^2FX^2 - \frac{1}{q(1-q^2)}K^{-1}Y\varphi. \tag{6.29}$$

By Lemma 6.27.(2), (3), we have

$$\Theta \cdot YX = q^2 YX \cdot \Theta + (q+q^{-1})K^{-1}Y\varphi + (1-q^2)C. \tag{6.30}$$

$$\Theta \cdot Y\varphi = q^{-2} Y\varphi \cdot \Theta - q(1+q^2)KYX + (1-q^2)C. \tag{6.31}$$

Lemma 6.28. *In the algebra $C_A(K)$, the following relation holds*

$$\Theta \cdot YX \cdot Y\varphi - \frac{1}{q(1-q^2)}K^{-1}(Y\varphi)^2 - C \cdot Y\varphi = \frac{q^7}{1-q^2}K(YX)^2 - q^4 C \cdot YX.$$

Proof. By (6.29), $\Theta \cdot YX = C + \frac{q}{1-q^2}K^{-1}YX - q^2FX^2 + \frac{1}{q(1-q^2)}K^{-1}Y\varphi$. So,

$$\Theta \cdot YX \cdot Y\varphi = C \cdot Y\varphi + \frac{q}{1-q^2}K^{-1}YX \cdot Y\varphi - q^2FX^2 \cdot Y\varphi + \frac{1}{q(1-q^2)}K^{-1}(Y\varphi)^2.$$

Then $\Theta \cdot YX \cdot Y\varphi - \frac{1}{q(1-q^2)}K^{-1}(Y\varphi)^2 - C \cdot Y\varphi = \frac{q}{1-q^2}K^{-1}YX \cdot Y\varphi - q^2FX^2 \cdot Y\varphi$. Notice that $YX \cdot Y\varphi = q^4(YX)^2 + q(1-q^2)YX \cdot EY^2$, $FX^2 \cdot Y\varphi = q^2FX\varphi \cdot YX$ and $EY^2 \cdot YX = q(YX)^2 + q^{-2}YX \cdot EY^2$. Then by (6.13) we obtain the identity as desired. □

Theorem 6.29. *Let $u := K^{-1}Y\varphi$ and recall that $t = YX$, $\Theta = (1-q^2)FE + \frac{q^2(qK+q^{-1}K^{-1})}{1-q^2}$. Then the algebra $C_A(K)$ is generated by the elements $K^{\pm 1}$, C, Θ, t and u subject to the following defining relations:*

$$t \cdot u = q^2 u \cdot t, \tag{6.32}$$

$$\Theta \cdot t = q^2 t \cdot \Theta + (q+q^{-1})u + (1-q^2)C, \tag{6.33}$$

$$\Theta \cdot u = q^{-2} u \cdot \Theta - q(1+q^2)t + (1-q^2)K^{-1}C, \tag{6.34}$$

$$\Theta \cdot t \cdot u - \frac{1}{q(1-q^2)}u^2 - C \cdot u = \frac{q^7}{1-q^2}t^2 - q^4K^{-1}C \cdot t, \tag{6.35}$$

$$[K^{\pm 1}, \cdot] = 0, \quad \text{and} \quad [C, \cdot] = 0 \tag{6.36}$$

where (6.36) means that the elements $K^{\pm 1}$ and C are central in $C_A(K)$. Furthermore, $Z(C_A(K)) = \mathbb{K}[C, K^{\pm 1}]$.

Proof. (i) *Generators of $C_A(K)$*: Notice that $Y\varphi = q^4YX + q(1-q^2)EY^2$. Then by Proposition 6.24 and (6.29), the algebra $C_A(K)$ is generated by the elements C, $K^{\pm 1}$, Θ, t and u. By (6.30), (6.31) and Lemma 6.28, the elements C, $K^{\pm 1}$, Θ, t and u satisfy the relations (6.32)–(6.36). It remains to show that these relations are defining relations.

Let $\mathscr{C}$ be the $\mathbb{K}$-algebra generated by the symbols C, $K^{\pm 1}$, Θ, t and u subject to the defining relations (6.32)–(6.36). Then there is a natural epimorphism of algebras $f : \mathscr{C} \twoheadrightarrow C_A(K)$. Our aim is to prove that f is an algebra isomorphism.

(ii) $\mathrm{GK}\,(\mathscr{C}) = 4$ *and* $Z(\mathscr{C}) = \mathbb{K}[C, K^{\pm 1}]$: Let R be the subalgebra of $\mathscr{C}$ generated by the elements C, $K^{\pm 1}$, t and u. Then $R = \mathbb{K}[C, K^{\pm 1}] \otimes \mathbb{K}_{q^2}[t, u]$ is a tensor product of algebra where $\mathbb{K}_{q^2}[t, u] := \mathbb{K}\langle t, u \,|\, tu = q^2ut \rangle$ is a quantum plane. Clearly, R is a Noetherian algebra of Gelfand-Kirillov dimension 4. Let $\mathscr{C}_{t,u}$ be the localization of $\mathscr{C}$ at the powers of the elements t and u. Then $\mathscr{C}_{t,u} = \mathbb{K}[C, K^{\pm 1}] \otimes \mathbb{K}_{q^2}[t^{\pm 1}, u^{\pm 1}] = R_{t,u}$. So, $\mathrm{GK}\,(\mathscr{C}_{t,u}) = 4$. Now, the inclusions

$R \subseteq \mathscr{C} \subseteq \mathscr{C}_{t,u}$ yield that $4 = \mathrm{GK}\,(R) \leqslant \mathrm{GK}\,(\mathscr{C}) \leqslant \mathrm{GK}\,(\mathscr{C}_{t,u}) = 4$, i.e., $\mathrm{GK}\,(\mathscr{C}) = 4$. Moreover, since $\mathbb{K}_{q^2}[t^{\pm 1}, u^{\pm 1}]$ is a central simple algebra, $Z(\mathscr{C}_{t,u}) = \mathbb{K}[C, K^{\pm 1}]$. Hence, $Z(\mathscr{C}) = \mathbb{K}[C, K^{\pm 1}]$.

By Lemma 6.25.(3), $\mathrm{GK}\,(\mathscr{C}) = \mathrm{GK}\,(C_A(K)) = 4$. In view of Proposition 2.12, to show that the epimorphism $f : \mathscr{C} \twoheadrightarrow C_A(K)$ is an isomorphism it suffices to prove that $\mathscr{C}$ is a domain.

Let $\mathscr{D}$ be the algebra generated by the symbols C, $K^{\pm 1}$, Θ, t and u subject to the defining relations (6.32)–(6.34) and (6.36). Then $\mathscr{D}$ is an Ore extension

$$\mathscr{D} = R[\Theta; \sigma, \delta]$$

where $R = \mathbb{K}[C, K^{\pm 1}] \otimes \mathbb{K}_{q^2}[t, u]$ is a Noetherian domain; $\sigma(C) = C$, $\sigma(K^{\pm 1}) = K^{\pm 1}$, $\sigma(t) = q^2 t$, $\sigma(u) = q^{-2} u$; δ is a σ-derivation of R given by the rule $\delta(C) = \delta(K^{\pm 1}) = 0$, $\delta(t) = (q + q^{-1})u + (1 - q^2)C$ and $\delta(u) = -q(1 + q^2)t + (1 - q^2)K^{-1}C$. In particular, $\mathscr{D}$ is a Noetherian domain. Let $Z := \Theta t u - \frac{1}{q(1-q^2)} u^2 - C u - \frac{q^7}{1-q^2} t^2 + q^4 K^{-1} C t = t u \Theta - \widehat{q}(u^2 + t^2) - q^2 C(u - K^{-1} t) \in \mathscr{D}$ where $\widehat{q} = \frac{q^3}{1-q^2}$. Then Z is a central element of $\mathscr{D}$ and $\mathscr{C} \simeq \mathscr{D}/(Z)$. To prove that $\mathscr{C}$ is a domain, it suffices to show that (Z) is a completely prime ideal of $\mathscr{D}$. Notice that $\mathscr{D}_{t,u} = \mathbb{K}[C, K^{\pm 1}, Z] \otimes \mathbb{K}_{q^2}[t^{\pm 1}, u^{\pm 1}]$ is a tensor product of algebras. Then

$$\mathscr{C}_{t,u} \simeq \mathscr{D}_{t,u}/(Z)_{t,u} \simeq \mathbb{K}[C, K^{\pm 1}] \otimes \mathbb{K}_{q^2}[t^{\pm 1}, u^{\pm 1}] \simeq R_{t,u}.$$

In particular, $\mathscr{C}_{t,u}$ is a domain and $(Z)_{t,u}$ is a completely prime ideal of $\mathscr{D}_{t,u}$.

(iii) *If $tx \in (Z)$ for some element $x \in \mathscr{D}$ then $x \in (Z)$*: Since Z is central in $\mathscr{D}$, $tx = Zd$ for some element $d \in \mathscr{D}$. We prove the statement (iii) by induction on the degree $\deg_\Theta(x)$ of the element x. Since $\mathscr{D}$ is a domain, $\deg_\Theta(dd') = \deg_\Theta(d) + \deg_\Theta(d')$ for all elements d, $d' \in \mathscr{D}$. Notice that $\deg_\Theta(Z) = 1$, the case $x \in R$ is trivial. So, we may assume that $m = \deg_\Theta(x) \geqslant 1$ and then the element x can be written as $x = a_0 + a_1\Theta + \cdots + a_m\Theta^m$ where $a_i \in R$ and $a_m \neq 0$. The equality $tx = Zd$ yields $\deg_\Theta(d) = m - 1$ since $\deg_\Theta(Z) = 1$. Hence, $d = d_0 + d_1\Theta + \cdots + d_{m-1}\Theta^{m-1}$ for some $d_i \in R$ and $d_{m-1} \neq 0$. Now, the equality $tx = Zd$ can be written as follows $t(a_0 + a_1\Theta + \cdots + a_m\Theta^m) = \big(tu\Theta - \widehat{q}(u^2 + t^2) - q^2 C(u - K^{-1}t)\big)\big(d_0 + d_1\Theta + \cdots + d_{m-1}\Theta^{m-1}\big)$. Comparing the terms of degree zero in the equality we have $ta_0 = tu\delta(d_0) - \big(\widehat{q}(u^2 + t^2) + q^2 C(u - K^{-1}t)\big)d_0$, i.e., $t\big(a_0 - u\delta(d_0) + \widehat{q}td_0 - q^2 CK^{-1}d_0\big) = -u(\widehat{q}u + q^2 C)d_0$. All terms in this equality are in the algebra R. Notice that t is a normal element of R, the elements $u \notin tR$ and $\widehat{q}u + q^2 C \notin tR$, we have $d_0 \in tR$. So $d_0 = tr$ for some element $r \in R$. Then $d = tr + w\Theta$ where $w = d_1 + \cdots + d_{m-1}\Theta^{m-2}$ if $m \geqslant 2$ and $w = 0$ if $m = 1$. If $m = 1$ then $d = tr$ and the equality $tx = Zd$ yields that $tx = tZr$, i.e., $x = Zr \in (Z)$ (by deleting t), we are done. So we may assume that $m \geqslant 2$. Now, the equality $tx = Zd$ can be written as $tx = Z(tr + w\Theta)$, i.e., $t(x - Zr) = Zw\Theta$. This implies that $x - Zr = x'\Theta$ for some $x' \in \mathscr{D}$ where $\deg_\Theta(x') < \deg_\Theta(x)$. Now, $tx'\Theta = Zw\Theta$ and hence, $tx' = Zw$ (by deleting Θ). By induction $x' \in (Z)$. Then $x = x' + Zr \in (Z)$.

(iv) *If $ux \in (Z)$ for some element $x \in \mathscr{D}$ then $x \in (Z)$*: Notice that the elements u and t are 'symmetric' in the algebra $\mathscr{D}$, the statement (iv) can be proved similarly to the statement (iii).

(v) $\mathscr{D} \cap (Z)_{t,u} = (Z)$: The inclusion $(Z) \subseteq \mathscr{D} \cap (Z)_{t,u}$ is obvious. Let $x \in \mathscr{D} \cap (Z)_{t,u}$. Then $t^i u^j x \in (Z)$ for some $i, j \in \mathbb{N}$. By the statement (iii) and the statement (iv), $x \in (Z)$. Hence, $\mathscr{D} \cap (Z)_{t,u} = (Z)$.

By the statement (v), the algebra $\mathscr{D}/(Z)$ is a subalgebra of $\mathscr{D}_{t,u}/(Z)_{t,u}$. Hence, $\mathscr{D}/(Z)$ is domain. This completes the proof. □

The next proposition gives a $\mathbb{K}$-basis for the algebra $\mathscr{C} := C_A(K)$.

Proposition 6.30.

$$\mathscr{C} = \mathbb{K}[C, K^{\pm 1}] \otimes_{\mathbb{K}} \Big(\bigoplus_{i,j\geqslant 1} \mathbb{K}\Theta^i t^j \oplus \bigoplus_{k\geqslant 1} \mathbb{K}\Theta^k \oplus \bigoplus_{l,m\geqslant 1} \mathbb{K}\Theta^l u^m \oplus \bigoplus_{a,b\geqslant 0} \mathbb{K}u^a t^b \Big).$$

Proof. The relations (6.32)-(6.35) can be written in the following equivalent form, respectively,

$$\begin{aligned} u\cdot t &= q^{-2} t\cdot u, \\ t\cdot\Theta &= q^{-2}\Theta\cdot t - q^{-2}(q+q^{-1})u - q^{-2}(1-q^2)C, \\ u\cdot\Theta &= q^2\Theta\cdot u + q^3(1+q^2)t - q^2(1-q^2)K^{-1}C, \\ \Theta\cdot t\cdot u &= \frac{1}{q(1-q^2)}u^2 + C\cdot u + \frac{q^7}{1-q^2}t^2 - q^4K^{-1}C\cdot t. \end{aligned}$$

On the free semigroup W generated by $C,\ K,\ K',\ \Theta,\ t$ and u (where K' play the role of K^{-1}), we introduce the length-lexicographic ordering such that

$$K' < K < C < \Theta < t < u.$$

With respect to this ordering the Diamond Lemma can be applied to $\mathscr{C}$ as there is only one ambiguity which is the overlap ambiguity $ut\Theta$ and it is resolvable as the following computations show:

$$\begin{aligned} (ut)\Theta &\to q^{-2}tu\Theta \to q^{-2}t\Big(q^2\Theta u + q^3(1+q^2)t - q^2(1-q^2)K'C\Big) \to t\Theta u + q(1+q^2)t^2 - (1-q^2)K'Ct \\ &\to \Big(q^{-2}\Theta t - q^{-2}(q+q^{-1})u - q^{-2}(1-q^2)C\Big)u + q(1+q^2)t^2 - (1-q^2)K'Ct \\ &\to q^{-2}\Theta tu - q^{-2}(q+q^{-1})u^2 - q^{-2}(1-q^2)Cu + q(1+q^2)t^2 - (1-q^2)K'Ct \\ &\to \frac{q}{1-q^2}u^2 + Cu + \frac{q}{1-q^2}t^2 - K'Ct, \\ u(t\Theta) &\to u\Big(q^{-2}\Theta t - q^{-2}(q+q^{-1})u - q^{-2}(1-q^2)C\Big) \to q^{-2}u\Theta t - q^{-2}(q+q^{-1})u^2 - q^{-2}(1-q^2)Cu \\ &\to q^{-2}\Big(q^2\Theta u + q^3(1+q^2)t - q^2(1-q^2)K'C\Big)t - q^{-2}(q+q^{-1})u^2 - q^{-2}(1-q^2)Cu \\ &\to \Theta ut + q(1+q^2)t^2 - (1-q^2)K'Ct - q^{-2}(q+q^{-1})u^2 - q^{-2}(1-q^2)Cu \\ &\to q^{-2}\Theta tu + q(1+q^2)t^2 - (1-q^2)K'Ct - q^{-2}(q+q^{-1})u^2 - q^{-2}(1-q^2)Cu \\ &\to \frac{q}{1-q^2}u^2 + Cu + \frac{q}{1-q^2}t^2 - K'Ct. \end{aligned}$$

So, by the Diamond Lemma, $\mathscr{C} = \mathbb{K}[C, K^{\pm 1}] \otimes_{\mathbb{K}} \Big(\bigoplus_{i,j\geqslant 1} \mathbb{K}\Theta^i t^j \oplus \bigoplus_{k\geqslant 1} \mathbb{K}\Theta^k \oplus \bigoplus_{l,m\geqslant 1} \mathbb{K}\Theta^l u^m \oplus \bigoplus_{a,b\geqslant 0} \mathbb{K}u^a t^b \Big)$. □

The algebra $\mathscr{C}^{\lambda,\mu}$. For elements $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, let $\mathscr{C}^{\lambda,\mu} := \mathscr{C}/(C-\lambda, K-\mu)$. By Theorem 6.29, the algebra $\mathscr{C}^{\lambda,\mu}$ is generated by the images of the elements Θ, t and u in $\mathscr{C}^{\lambda,\mu}$. For simplicity, we denote by the same letters their images.

Corollary 6.31. *Let $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$. Then*

1. *The algebra $\mathscr{C}^{\lambda,\mu}$ is generated by the elements Θ, t and u subject to the following defining relations*

$$t \cdot u = q^2 u \cdot t, \tag{6.37}$$

$$\Theta \cdot t = q^2 t \cdot \Theta + (q+q^{-1})u + (1-q^2)\lambda, \tag{6.38}$$

$$\Theta \cdot u = q^{-2} u \cdot \Theta - q(1+q^2)t + (1-q^2)\mu^{-1}\lambda, \tag{6.39}$$

$$\Theta \cdot t \cdot u = \frac{1}{q(1-q^2)}u^2 + \lambda u + \frac{q^7}{1-q^2}t^2 - q^4\mu^{-1}\lambda t. \tag{6.40}$$

2. $\mathscr{C}^{\lambda,\mu} = \bigoplus_{i,j\geqslant 1} \mathbb{K}\Theta^i t^j \oplus \bigoplus_{k\geqslant 1} \mathbb{K}\Theta^k \oplus \bigoplus_{l,m\geqslant 1} \mathbb{K}\Theta^l u^m \oplus \bigoplus_{a,b\geqslant 0} \mathbb{K}u^a t^b$.

Proof. 1. Statement 1 follows from Theorem 6.29.

2. Statement 2 follows from Proposition 6.30. □

Let $\mathscr{C}_t$ (resp. $\mathscr{C}_t^{\lambda,\mu}$) be the localization of the algebra $\mathscr{C}$ (resp. $\mathscr{C}^{\lambda,\mu}$) at the powers of the element $t = YX$. The next proposition shows that $\mathscr{C}_t$ and $\mathscr{C}_t^{\lambda,\mu}$ are GWAs.

Proposition 6.32. 1. *Let $v := \Theta t - \frac{1}{q(1-q^2)}u - C$. The algebra $\mathscr{C}_t = \mathbb{K}[C, K^{\pm 1}, t^{\pm 1}][u, v; \sigma, a]$ is a GWA of Gelfand-Kirillov dimension 4 where $a = \frac{q^7}{1-q^2}t^2 - q^4K^{-1}Ct$ and σ is the automorphism of the algebra $\mathbb{K}[C, K^{\pm 1}, t^{\pm 1}]$ defined by the rule: $\sigma(C) = C$, $\sigma(K^{\pm 1}) = K^{\pm 1}$ and $\sigma(t) = q^{-2}t$.*

2. *Let $\lambda \in \mathbb{K}$, $\mu \in \mathbb{K}^*$ and $v := \Theta t - \frac{1}{q(1-q^2)}u - \lambda$. Then the algebra $\mathscr{C}_t^{\lambda,\mu} = \mathbb{K}[t^{\pm 1}][u, v; \sigma, a]$ is a GWA of Gelfand-Kirillov dimension 2 where $a = \frac{q^7}{1-q^2}t^2 - q^4\mu^{-1}\lambda t$ and σ is the automorphism of the algebra $\mathbb{K}[t^{\pm 1}]$ defined by $\sigma(t) = q^{-2}t$.*
3. *For any $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, the algebra $\mathscr{C}_t^{\lambda,\mu}$ is a central simple algebra.*
4. *$Z(\mathscr{C}^{\lambda,\mu}) = \mathbb{K}$ and $\mathrm{GK}\,(\mathscr{C}^{\lambda,\mu}) = 2$.*

Proof. 1. By Theorem 6.29, the algebra $\mathscr{C}_t$ is generated by the elements C, $K^{\pm 1}$, v, $t^{\pm 1}$ and u. Note that the element v can be written as $v = -\frac{q^2}{1-q^2}\psi X = \frac{q}{1-q^2}\tau(u)$ where τ is the involution (6.1). It is straightforward to verify that the following relations hold in the algebra $\mathscr{C}_t$

$$ut = q^{-2}tu, \quad vt = q^2tv, \quad vu = \frac{q^7}{1-q^2}t^2 - q^4K^{-1}Ct, \quad uv = \frac{q^3}{1-q^2}t^2 - q^2K^{-1}Ct.$$

Then $\mathscr{C}_t$ is an epimorphic image of the GWA $T := \mathbb{K}[C, K^{\pm 1}, t^{\pm 1}][u, v; \sigma, a]$. Notice that T is a Noetherian domain of Gelfand-Kirillov dimension 4. The inclusions $\mathscr{C} \subseteq \mathscr{C}_t \subseteq \mathscr{C}_{t,u}$ yield that $4 = \mathrm{GK}\,(\mathscr{C}) \leqslant \mathrm{GK}\,(\mathscr{C}_t) \leqslant \mathscr{C}_{t,u} = 4$ (see Lemma 6.25.(3)), i.e., $\mathrm{GK}\,(\mathscr{C}_t) = 4$. So, $\mathrm{GK}\,(T) = \mathrm{GK}\,(\mathscr{C}_t)$. By Proposition 2.12, the epimorphism of algebras $T \twoheadrightarrow \mathscr{C}_t$ is an isomorphism.

2. Statement 2 follows from statement 1.

3. Let $\mathscr{C}_{t,u}^{\lambda,\mu}$ be the localization of $\mathscr{C}_t^{\lambda,\mu}$ at the powers of the element u. Then, by statement 2, $\mathscr{C}_{t,u}^{\lambda,\mu} = \mathbb{K}_{q^2}[t^{\pm 1}, u^{\pm 1}]$ is a central simple quantum torus. So, $Z(\mathscr{C}_t^{\lambda,\mu}) = Z(\mathscr{C}_{t,u}^{\lambda,\mu}) \cap \mathscr{C}_t^{\lambda,\mu} = \mathbb{K}$. For any nonzero ideal $\mathfrak{a}$ of the algebra $\mathscr{C}_t^{\lambda,\mu}$, $u^i \in \mathfrak{a}$ for some $i \in \mathbb{N}$ since $\mathscr{C}_{t,u}^{\lambda,\mu}$ is a simple

Noetherian algebra. Therefore, to prove that $\mathscr{C}_t^{\lambda,\mu}$ is a simple algebra, it suffices to show that $\mathscr{C}_t^{\lambda,\mu} u^i \mathscr{C}_t^{\lambda,\mu} = \mathscr{C}_t^{\lambda,\mu}$ for any $i \in \mathbb{N}$. The case $i = 1$ follows from the equality $vu = q^2uv - q^5t^2$. By induction, for $i > 1$, it suffices to show that $u^{i-1} \in \mathscr{C}_t^{\lambda,\mu} u^i \mathscr{C}_t^{\lambda,\mu}$. This follows from the equality $vu^i = q^{2i}u^iv + \frac{q^7(1-q^{-2i})}{1-q^2} t^2u^{i-1}$. Hence, $\mathscr{C}_t^{\lambda,\mu}$ is a simple algebra.

4. Since $\mathbb{K} \subseteq Z(\mathscr{C}^{\lambda,\mu}) \subseteq Z(\mathscr{C}_t^{\lambda,\mu}) \cap \mathscr{C}^{\lambda,\mu} = \mathbb{K}$, we have $Z(\mathscr{C}^{\lambda,\mu}) = \mathbb{K}$. It is obvious that $\mathrm{GK}\,(\mathscr{C}^{\lambda,\mu}) = 2$. □

Lemma 6.33. *In the algebra $\mathscr{C}^{\lambda,\mu}$ where $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, the following equality holds*

$$\Theta t^i = q^{2i}t^i\Theta + \frac{q^{-2i+1} - q^{2i+1}}{1-q^2} t^{i-1}u + (1-q^{2i})\lambda t^{i-1}.$$

Proof. By induction on i and using the equality (6.38). □

Theorem 6.34. *Let $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$.*

1. *The algebra $\mathscr{C}^{\lambda,\mu}$ is a simple algebra iff $\lambda \neq 0$.*
2. *The algebra $\mathscr{C}^{\lambda,\mu}$ is a domain.*

Proof. 1. If $\lambda = 0$ then the ideal (t) is a proper ideal of the algebra $\mathscr{C}^{0,\mu}$. Hence, $\mathscr{C}^{0,\mu}$ is not a simple algebra. Now, suppose that $\lambda \neq 0$, we have to prove that $\mathscr{C}^{\lambda,\mu}$ is a simple algebra. By Proposition 6.32.(3), $\mathscr{C}_t^{\lambda,\mu}$ is a simple algebra. Hence, it suffices to show that $\mathscr{C}^{\lambda,\mu}t^i\mathscr{C}^{\lambda,\mu} = \mathscr{C}^{\lambda,\mu}$ for all $i \in \mathbb{N}$. We prove this by induction on i.

Firstly, we prove the case for $i = 1$, i.e., $\mathfrak{a} := \mathscr{C}^{\lambda,\mu}t\mathscr{C}^{\lambda,\mu} = \mathscr{C}^{\lambda,\mu}$. By (6.38), the element $(q+q^{-1})u + (1-q^2)\lambda \in \mathfrak{a}$, so, $u \equiv \frac{q^2-1}{q+q^{-1}}\lambda \mod \mathfrak{a}$. By (6.40), $\frac{1}{q(1-q^2)}u^2 + \lambda u \in \mathfrak{a}$. Hence, $\frac{1}{q(1-q^2)}(\frac{q^2-1}{q+q^{-1}}\lambda)^2 + \lambda(\frac{q^2-1}{q+q^{-1}}\lambda) \equiv 0 \mod \mathfrak{a}$, i.e., $\frac{q^2(q^2-1)\lambda^2}{q^2+1} \equiv 0 \mod \mathfrak{a}$. Since $\lambda \neq 0$, this implies that $1 \in \mathfrak{a}$, thus, $\mathfrak{a} = \mathscr{C}^{\lambda,\mu}$.

Let us now prove that $\mathfrak{b} := \mathscr{C}^{\lambda,\mu}t^i\mathscr{C}^{\lambda,\mu} = \mathscr{C}^{\lambda,\mu}$ for any $i \in \mathbb{N}$. By induction, for $i > 1$, it suffices to show that $t^{i-1} \in \mathfrak{b}$. By Lemma 6.33, the element $\mathbf{u} := \frac{q^{-2i+1}-q^{2i+1}}{1-q^2}t^{i-1}u + (1-q^{2i})\lambda t^{i-1} \in \mathfrak{b}$. Then $v\mathbf{u} \in \mathfrak{b}$ where $v = \Theta t - \frac{1}{q(1-q^2)}u - \lambda$, see Proposition 6.32.(2). This implies that $(1-q^{2i})\lambda vt^{i-1} \in \mathfrak{b}$ and so, $vt^{i-1} \in \mathfrak{b}$. But then the inclusion $vt^{i-1} = (\Theta t - \frac{1}{q(1-q^2)}u - \lambda)t^{i-1} \in \mathfrak{b}$ yields that the element $\mathbf{v} := \frac{q^{-2i+1}}{1-q^2}t^{i-1}u + \lambda t^{i-1} \in \mathfrak{b}$. By the expressions of the elements $\mathbf{u}$ and $\mathbf{v}$ we see that $t^{i-1} \in \mathfrak{b}$, as required.

2. By Proposition 6.32.(2), the GWA $\mathscr{C}_t^{\lambda,\mu} \simeq \mathscr{C}_t/\mathscr{C}_t(C-\lambda, K-\mu)$ is a domain. Let $\mathfrak{a} = \mathscr{C}(C-\lambda, K-\mu)$ and $\mathfrak{a}' = \mathscr{C} \cap \mathscr{C}_t(C-\lambda, K-\mu)$. To prove that $\mathscr{C}^{\lambda,\mu}$ is a domain, it suffices to show that $\mathfrak{a} = \mathfrak{a}'$. The inclusion $\mathfrak{a} \subseteq \mathfrak{a}'$ is obvious. If $\lambda \neq 0$ then, by statement 1, the algebra $\mathscr{C}^{\lambda,\mu}$ is a simple algebra, so the ideal $\mathfrak{a}$ is a maximal ideal of $\mathscr{C}$. Then we must have $\mathfrak{a} = \mathfrak{a}'$. Suppose that $\lambda = 0$ and $\mathfrak{a} \subsetneq \mathfrak{a}'$, we seek a contradiction. Notice that the ideal $\mathfrak{a}'$ is a prime ideal of $\mathscr{C}$. Hence, $\mathfrak{a}'/\mathfrak{a}$ is a nonzero prime ideal of the algebra $\mathscr{C}^{0,\mu}$. By Proposition 6.32.(3), the algebra $\mathscr{C}_t^{0,\mu}$ is a simple algebra, so, $t^i \in \mathfrak{a}'/\mathfrak{a}$ for some $i \in \mathbb{N}$. Then $(\mathfrak{a}'/\mathfrak{a})_t = \mathscr{C}_t^{\lambda,\mu}$. But $(\mathfrak{a}'/\mathfrak{a})_t = \mathfrak{a}'_t/\mathfrak{a}_t = 0$, a contradiction. □

Proposition 6.35. *1. In the algebra $\mathscr{C}^{0,\mu}$, $(t) = (u) = (t,u) = \mathscr{C}^{0,\mu}t + \mathscr{C}^{0,\mu}u$.*

2. $\mathscr{C}^{0,\mu}/(t) \simeq \mathbb{K}[\Theta]$.

3. *In the algebra* $\mathscr{C}^{0,\mu}$, $(t^i) = (t)^i$ *for all* $i \geqslant 1$.
4. $\mathrm{Spec}\,(\mathscr{C}^{0,\mu}) = \{0,\ (t),\ (t,\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Max}\,(\mathbb{K}[\Theta])\}$.

Proof. 1. The equality $(t) = (u)$ follows from (6.38) and (6.39). The second equality then is obvious. To prove the third equality let us first show that $t\mathscr{C}^{0,\mu} \subseteq \mathscr{C}^{0,\mu}t + \mathscr{C}^{0,\mu}u$: In view of Corollary 6.31.(2), it suffices to prove that $t\Theta^i \in \mathscr{C}^{0,\mu}t + \mathscr{C}^{0,\mu}u$ for all $i \geqslant 1$. This can be proved by induction on i. The case $i = 1$ follows from (6.38). Suppose that the inclusion holds for all $i' < i$. Then $t\Theta^i = t\Theta^{i-1}\Theta \in (\mathscr{C}^{0,\mu}t + \mathscr{C}^{0,\mu}u)\Theta = \mathscr{C}^{0,\mu}\big(q^{-2}\Theta t - q^{-2}(q+q^{-1})u\big) + \mathscr{C}^{0,\mu}\big(q^2\Theta u + q^3(1+q^2)t\big) \subseteq \mathscr{C}^{0,\mu}t + \mathscr{C}^{0,\mu}u$. Hence, we proved that $t\mathscr{C}^{0,\mu} \subseteq \mathscr{C}^{0,\mu}t + \mathscr{C}^{0,\mu}u$. Now, the inclusions $(t) \subseteq \mathscr{C}^{0,\mu}t + \mathscr{C}^{0,\mu}u \subseteq (t,u) = (t)$ yield that $(t) = \mathscr{C}^{0,\mu}t + \mathscr{C}^{0,\mu}u$.

2. By statement 1, $\mathscr{C}^{0,\mu}/(t) = \mathscr{C}^{0,\mu}/(t,u) \simeq \mathbb{K}[\Theta]$.

3. The inclusion $(t^i) \subseteq (t)^i$ is obvious. We prove the reverse inclusion $(t)^i \subseteq (t^i)$ by induction on i. The case $i = 1$ is trivial. Suppose that the inclusion holds for all $i' < i$. Then $(t)^i = (t)(t)^{i-1} = (t)(t^{i-1}) = \mathscr{C}^{0,\mu}t\mathscr{C}^{0,\mu}t^{i-1}\mathscr{C}^{0,\mu} \subseteq (t^i) + (t^{i-1}u)$ since $t\mathscr{C}^{0,\mu} \subseteq \mathscr{C}^{0,\mu}t + \mathscr{C}^{0,\mu}u$ (see statement 1). By Lemma 6.33, the element $t^{i-1}u$ belongs to the ideal (t^i) of $\mathscr{C}^{0,\mu}$. Hence, $(t)^i \subseteq (t^i)$, as required.

4. By Proposition 3.3 and statement 3, $\mathrm{Spec}\,(\mathscr{C}^{0,\mu}) = \mathrm{Spec}\,(\mathscr{C}^{0,\mu}, t) \sqcup \mathrm{Spec}_t(\mathscr{C}^{0,\mu})$. Notice that $\mathscr{C}^{0,\mu}_t$ is a simple algebra (see Proposition 6.32.(3)) and $\mathscr{C}^{0,\mu}/(t) \simeq \mathbb{K}[\Theta]$ (see statement 2). Then $\mathrm{Spec}\,(\mathscr{C}^{0,\mu}) = \{0\} \sqcup \mathrm{Spec}\,(\mathbb{K}[\Theta]) = \{0,\ (t),\ (t,\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Max}\,(\mathbb{K}[\Theta])\}$. □

6.6 Classification of simple $C_A(K)$-modules

In this section, $\mathbb{K}$ is an algebraically closed field of characteristic zero. A classification of simple $C_A(K)$-modules is given in Theorem 6.37, Theorem 6.41 and Theorem 6.45. The set $\widehat{C_A(K)}$ of isomorphism classes of simple $C_A(K)$-modules are partitioned (according to the central character) as follows

$$\widehat{C_A(K)} = \bigsqcup_{\lambda\in\mathbb{K},\mu\in\mathbb{K}^*} \widehat{\mathscr{C}^{\lambda,\mu}}. \tag{6.41}$$

Given $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, the set $\widehat{\mathscr{C}^{\lambda,\mu}}$ can be partitioned further into disjoint union of two subsets consisting of t-torsion modules and t-torsionfree modules, respectively,

$$\widehat{\mathscr{C}^{\lambda,\mu}} = \widehat{\mathscr{C}^{\lambda,\mu}}\,(t\text{-torsion}) \ \sqcup\ \widehat{\mathscr{C}^{\lambda,\mu}}\,(t\text{-torsionfree}). \tag{6.42}$$

The set $\widehat{\mathscr{C}^{\lambda,\mu}}$ (t-torsion). An explicit description of the set $\widehat{\mathscr{C}^{\lambda,\mu}}$ (t-torsion) is given in Theorem 6.37. For λ and $\mu \in \mathbb{K}^*$, we define the left $\mathscr{C}^{\lambda,\mu}$-modules

$$\mathsf{t}^{\lambda,\mu} := \mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu}(t,u) \quad \text{and} \quad \mathsf{T}^{\lambda,\mu} := \mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu}\big(t, u-\hat{\lambda}\big)$$

where $\hat{\lambda} := q(q^2-1)\lambda$. By Corollary 6.31.(2), $\mathsf{t}^{\lambda,\mu} = \mathbb{K}[\Theta]\,\bar{1} \simeq {}_{\mathbb{K}[\Theta]}\mathbb{K}[\Theta]$ is a free $\mathbb{K}[\Theta]$-module where $\bar{1} = 1 + \mathscr{C}^{\lambda,\mu}(t,u)$ and $\mathsf{T}^{\lambda,\mu} = \mathbb{K}[\Theta]\,\tilde{1} \simeq {}_{\mathbb{K}[\Theta]}\mathbb{K}[\Theta]$ is a free $\mathbb{K}[\Theta]$-module where $\tilde{1} = 1 + \mathscr{C}^{\lambda,\mu}\big(t, u-\hat{\lambda}\big)$. Clearly, the modules $\mathsf{t}^{\lambda,\mu}$ and $\mathsf{T}^{\lambda,\mu}$ are of Gelfand-Kirillov dimension 1. The concept of $\deg_\Theta$ of the elements of $\mathsf{t}^{\lambda,\mu}$ and $\mathsf{T}^{\lambda,\mu}$ is well-defined ($\deg_\Theta(\Theta^i\,\bar{1}) = i$ and $\deg_\Theta(\Theta^i\,\tilde{1}) = i$ for all $i \geqslant 0$).

Lemma 6.36. *Let λ and $\mu \in \mathbb{K}^*$. Then*

1. *The $\mathscr{C}^{\lambda,\mu}$-module $\mathsf{t}^{\lambda,\mu}$ is a simple module.*
2. *The $\mathscr{C}^{\lambda,\mu}$-module $\mathsf{T}^{\lambda,\mu}$ is a simple module.*
3. *The modules $\mathsf{t}^{\lambda,\mu}$ and $\mathsf{T}^{\lambda,\mu}$ are not isomorphic.*

Proof. 1. Let us show that for all $i \geqslant 1$,

$$t \cdot \Theta^i\,\bar{1} = (1-q^{-2i})\lambda \cdot \Theta^{i-1}\,\bar{1} + \cdots, \tag{6.43}$$
$$u \cdot \Theta^i\,\bar{1} = -q^2(1-q^{2i})\mu^{-1}\lambda \cdot \Theta^{i-1}\,\bar{1} + \cdots \tag{6.44}$$

where the three dots means terms of $\deg_\Theta < i-1$. We prove the equalities by induction on i. By (6.38), $t\Theta\,\bar{1} = (1-q^{-2})\lambda\,\bar{1}$, and by (6.39), $u\Theta\,\bar{1} = -q^2(1-q^2)\mu^{-1}\lambda\,\bar{1}$. So, the equalities (6.43) and (6.44) hold for $i=1$. Suppose that the equalities hold for all integers $i' < i$. Then

$$\begin{aligned}
t \cdot \Theta^i\,\bar{1} &= \Big(q^{-2}\Theta t - q^{-2}(q+q^{-1})u - q^{-2}(1-q^2)\lambda\Big)\Theta^{i-1}\,\bar{1}\\
&= q^{-2}(1-q^{-2(i-1)})\lambda\Theta^{i-1}\,\bar{1} - q^{-2}(1-q^2)\lambda\Theta^{i-1}\,\bar{1} + \cdots\\
&= (1-q^{-2i})\lambda \cdot \Theta^{i-1}\,\bar{1} + \cdots,\\
u \cdot \Theta^i\,\bar{1} &= \Big(q^2\Theta u + q^3(1+q^2)t - q^2(1-q^2)\mu^{-1}\lambda\Big)\Theta^{i-1}\,\bar{1}\\
&= -q^4(1-q^{2(i-1)})\mu^{-1}\lambda\Theta^{i-1}\,\bar{1} - q^2(1-q^2)\mu^{-1}\lambda\Theta^{i-1}\,\bar{1} + \cdots\\
&= -q^2(1-q^{2i})\mu^{-1}\lambda \cdot \Theta^{i-1}\,\bar{1} + \cdots.
\end{aligned}$$

The simplicity of the module $\mathsf{t}^{\lambda,\mu}$ follows from the equality (6.43) (or the equality (6.44)).

2. Let us show that for all $i \geqslant 1$,

$$t \cdot \Theta^i\,\tilde{1} = (1-q^{2i})\lambda \cdot \Theta^{i-1}\,\tilde{1} + \cdots, \tag{6.45}$$
$$u \cdot \Theta^i\,\tilde{1} = q^{2i}\hat{\lambda} \cdot \Theta^i\,\tilde{1} - q^2(1-q^{2i})\mu^{-1}\lambda \cdot \Theta^{i-1}\,\tilde{1} + \cdots \tag{6.46}$$

where the three dots means terms of smaller degrees. We prove the equalities by induction on i. The case $i=1$ follows from (6.38) and (6.39). Suppose that the equalities (6.45) and (6.46) holds for all integers $i' < i$. Then

$$\begin{aligned}
t \cdot \Theta^i\,\tilde{1} &= \Big(q^{-2}\Theta t - q^{-2}(q+q^{-1})u - q^{-2}(1-q^2)\lambda\Big)\Theta^{i-1}\,\tilde{1}\\
&= q^{-2}(1-q^{2(i-1)})\lambda\Theta^{i-1}\,\tilde{1} - q^{-2}(q+q^{-1})q^{2(i-1)}\hat{\lambda}\Theta^{i-1}\,\tilde{1} - q^{-2}(1-q^2)\lambda\Theta^{i-1}\tilde{1} + \cdots\\
&= (1-q^{2i})\lambda \cdot \Theta^{i-1}\,\tilde{1} + \cdots,\\
u \cdot \Theta^i\,\tilde{1} &= \Big(q^2\Theta u + q^3(1+q^2)t - q^2(1-q^2)\mu^{-1}\lambda\Big)\Theta^{i-1}\,\tilde{1}\\
&= q^2\Big(q^{2(i-1)}\hat{\lambda}\Theta^i\,\tilde{1} - q^2(1-q^{2(i-1)})\mu^{-1}\lambda\Theta^{i-1}\,\tilde{1}\Big) - q^2(1-q^2)\mu^{-1}\lambda\Theta^{i-1}\,\tilde{1} + \cdots\\
&= q^{2i}\hat{\lambda} \cdot \Theta^i\,\tilde{1} - q^2(1-q^{2i})\mu^{-1}\lambda \cdot \Theta^{i-1}\,\tilde{1} + \cdots.
\end{aligned}$$

The simplicity of the module $\mathsf{T}^{\lambda,\mu}$ follows from the equality (6.45).

3. By (6.44), the element u acts locally nilpotently on the module $\mathsf{t}^{\lambda,\mu}$. But, by (6.46), the action of the element u on the module $\mathsf{T}^{\lambda,\mu}$ is not locally nilpotent. Hence, the modules $\mathsf{t}^{\lambda,\mu}$ and $\mathsf{T}^{\lambda,\mu}$ are not isomorphic. □

Theorem 6.37. *1.* $\widehat{\mathscr{C}^{0,\mu}}\,(t\text{-torsion}) = \big\{[\mathscr{C}^{0,\mu}/\mathscr{C}^{0,\mu}(t,u,\Theta-\alpha) \simeq \mathbb{K}[\Theta]/(\Theta-\alpha)] \,|\, \alpha \in \mathbb{K}\big\}$.

2. Let λ *and* $\mu \in \mathbb{K}^*$*. Then* $\widehat{\mathscr{C}^{\lambda,\mu}}\,(t\text{-torsion}) = \big\{[\mathsf{t}^{\lambda,\mu}],\ [\mathsf{T}^{\lambda,\mu}]\big\}$.

Proof. 1. We claim that $\mathrm{ann}_{\mathscr{C}^{0,\mu}}(M) \supseteq (t)$ for all $M \in \widehat{\mathscr{C}^{0,\mu}}\,(t\text{-torsion})$: In view of Proposition 6.35.(1), it suffices to show that there exists a nonzero element $m \in M$ such that $tm = 0$ and $um = 0$. Since M is t-torsion, there exists a nonzero element $m' \in M$ such that $tm' = 0$. Then, by the equality (6.40) (where $\lambda = 0$), we have $u^2m' = 0$. If $um' = 0$, we are done. Otherwise, the element $m := um'$ is a nonzero element of M such that $tm = um = 0$ (since $tu = q^2ut$). Now, statement 1 follows from the claim immediately.

2. Let $M \in \widehat{\mathscr{C}^{\lambda,\mu}}\,(t\text{-torsion})$. Then there exists a nonzero element $m \in M$ such that $tm = 0$. By (6.40), we have $(u-\hat{\lambda})um = 0$. Therefore, either $um = 0$ or otherwise the element $m' := um \in M$ is nonzero and $(u-\hat{\lambda})m' = 0$.

If $um = 0$ then the module M is an epimorphic image of the module $\mathsf{t}^{\lambda,\mu}$. By Lemma 6.36.(1), $\mathsf{t}^{\lambda,\mu}$ is a simple $\mathscr{C}^{\lambda,\mu}$-module. Hence, $M \simeq \mathsf{t}^{\lambda,\mu}$. If $m' = um \neq 0$ then $tm' = 0$ and $(u-\hat{\lambda})m' = 0$. So, the $\mathscr{C}^{\lambda,\mu}$-module M is an epimorphic image of the module $\mathsf{T}^{\lambda,\mu}$. By Lemma 6.36.(2), $\mathsf{T}^{\lambda,\mu}$ is a simple $\mathscr{C}^{\lambda,\mu}$-module. Then $M \simeq \mathsf{T}^{\lambda,\mu}$. By Lemma 6.36.(3), the two modules $\mathsf{t}^{\lambda,\mu}$ and $\mathsf{T}^{\lambda,\mu}$ are not isomorphic, this completes the proof. □

Recall that the algebra $C_{A_{X,\varphi}}(K) = \mathbb{K}[C,K^{\pm 1}] \otimes \mathscr{A}$ where $\mathscr{A}$ is a central simple GWA, see Proposition 6.26. The algebra $C_A(K)$ is a subalgebra of the algebra $C_{A_{X,\varphi}}(K)$ where

$$u = K^{-1}Y\varphi = K^{-1}\cdot YX\cdot\varphi X^{-1} = K^{-1}th, \tag{6.47}$$

$$\Theta = (1-q^2)Ceh^{-1} + \frac{qK^{-1}}{1-q^2}h + \frac{q^3K}{1-q^2}h^{-1}. \tag{6.48}$$

In more detail: by (6.13), $F = \Big(C + K^{-1}EY^2 - \frac{q^3}{1-q^2}(K-K^{-1})YX\Big)X^{-1}\varphi^{-1}$. Then the element FE can be written as

$$\begin{aligned}
FE &= CEX^{-1}\varphi^{-1} + K^{-1}EY^2EX^{-1}\varphi^{-1} - \frac{q^2}{1-q^2}(K-K^{-1})YE\varphi^{-1}\\
&= C\cdot EX^{-2}\cdot X\varphi^{-1} + K^{-1}\cdot EX^{-2}\cdot q^3(YX)^2\cdot EX^{-2}\cdot X\varphi^{-1} - \frac{q^3(K-K^{-1})}{1-q^2}\cdot YX\cdot EX^{-2}\cdot X\varphi^{-1}\\
&= Ceh^{-1} + q^3K^{-1}et^2eh^{-1} - \frac{q^3(K-K^{-1})}{1-q^2}teh^{-1}\\
&= Ceh^{-1} + \frac{qK^{-1}}{(1-q^2)^2}h + \frac{q^3K}{(1-q^2)^2}h^{-1} - \frac{q^2(qK+q^{-1}K^{-1})}{(1-q^2)^2}
\end{aligned}$$

where the last equality follows from (6.28). Then the equality (6.48) follows immediately since $\Theta = (1-q^2)FE + \frac{q^2(qK+q^{-1}K^{-1})}{1-q^2}$.

For $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, let $\mathscr{C}^{\lambda,\mu}_{A_{X,\varphi}} := C_{A_{X,\varphi}}(K)/(C-\lambda, K-\mu)$. Then by Proposition 6.26.(1), $\mathscr{C}^{\lambda,\mu}_{A_{X,\varphi}} \simeq \mathscr{A}$ is a central simple GWA. So, there is a natural algebra homomorphism $\mathscr{C}^{\lambda,\mu} \to \mathscr{C}^{\lambda,\mu}_{A_{X,\varphi}} \simeq \mathscr{A}$. The next proposition shows that this homomorphism is a monomorphism.

Proposition 6.38. *Let $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$. The following map is an algebra homomorphism*

$$\begin{aligned} \rho : \mathscr{C}^{\lambda,\mu} &\longrightarrow \mathscr{C}^{\lambda,\mu}_{A_{X,\varphi}} \simeq \mathscr{A} \\ t &\mapsto t \\ u &\mapsto \mu^{-1}th \\ \Theta &\mapsto (1-q^2)\lambda e h^{-1} + \frac{q\mu^{-1}}{1-q^2}h + \frac{q^3\mu}{1-q^2}h^{-1} \end{aligned}$$

Moreover, the homomorphism ρ is a monomorphism.

Proof. The fact that the map ρ is an algebra homomorphism follows from (6.47) and (6.48). Now, we prove that ρ is an injection. If $\lambda \neq 0$ then by Theorem 6.34.(1), the algebra $\mathscr{C}^{\lambda,\mu}$ is a simple algebra. Hence, the kernel $\ker \rho$ of the homomorphism ρ must be zero, i.e., ρ is an injection. If $\lambda = 0$ and suppose that $\ker \rho$ is nonzero, we seek a contradiction. Then $t^i \in \ker \rho$ for some $i \in \mathbb{N}$. But $\rho(t^i) = t^i \neq 0$, a contradiction. $\square$

Let $\mathscr{A}_t$ be the localization of the algebra $\mathscr{A}$ at the powers of the element t. Then $\mathscr{A}_t = \mathbb{K}[h^{\pm 1}][t^{\pm 1};\sigma]$ is a central simple quantum torus where $\sigma(h) = q^2 h$. It is clear that $\mathscr{C}^{\lambda,\mu}_{t,u} \simeq \mathscr{A}_t$. Let $\mathscr{B}$ be the localization of $\mathscr{A}$ at the set $S = \mathbb{K}[h^{\pm 1}] \setminus \{0\}$. Then $\mathscr{B} = S^{-1}\mathscr{A} = \mathbb{K}(h)[t^{\pm 1};\sigma]$ is a skew Laurent polynomial algebra where $\mathbb{K}(h)$ is the field of rational functions in h and $\sigma(h) = q^2 h$. The algebra $\mathscr{B}$ is a Euclidean ring with left and right division algorithms. In particular, $\mathscr{B}$ is a principle left and right ideal domain. For all $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, we have the following inclusions of algebras

$$\begin{array}{ccccc} \mathscr{C}^{\lambda,\mu} & \xrightarrow{\ \rho\ } & \mathscr{A} & & \\ \downarrow & & \downarrow & & \\ \mathscr{C}^{\lambda,\mu}_t & \longrightarrow & \mathscr{C}^{\lambda,\mu}_{t,u} = \mathscr{A}_t & \longrightarrow & \mathscr{B}. \end{array}$$

The set $\widehat{\mathscr{C}^{0,\mu}}$ (t-torsionfree). An explicit description of the set $\widehat{\mathscr{C}^{0,\mu}}$ (t-torsionfree) is given in Theorem 6.41. The idea is to embed the algebra $\mathscr{C}^{0,\mu}$ in a skew polynomial algebra $\mathcal{R}$ for which the simple modules are classified. The simple modules over these two algebras are closely related. It will be shown that $\widehat{\mathscr{C}^{0,\mu}}$ (t-torsionfree) $= \widehat{\mathcal{R}}$ (t-torsionfree).

Let $\mathcal{R}$ be the subalgebra of $\mathscr{A}$ generated by the elements $h^{\pm 1}$ and t. Then $\mathcal{R} = \mathbb{K}[h^{\pm 1}][t;\sigma]$ is a skew polynomial algebra where $\sigma(h) = q^2 h$. By Proposition 6.38, the algebra $\mathscr{C}^{0,\mu}$ is a subalgebra of $\mathcal{R}$. Hence, we have the inclusions of algebras

$$\mathscr{C}^{0,\mu} \subset \mathcal{R} \subset \mathscr{A} \subset \mathcal{R}_t = \mathscr{A}_t \subset \mathscr{B}.$$

We identify the algebra $\mathscr{C}^{0,\mu}$ with its image in the algebra $\mathcal{R}$.

Lemma 6.39. *Let* $\mu \in \mathbb{K}^*$. *Then*

1. $\mathscr{C}^{0,\mu} = \bigoplus_{i\geqslant 1} \mathbb{K}[h^{\pm 1}]t^i \oplus \mathbb{K}[\Theta]$.
2. $\mathcal{R} = \mathscr{C}^{0,\mu} \oplus \mathbb{K}[\Theta]h$.
3. $(t) = \bigoplus_{i\geqslant 1} \mathbb{K}[h^{\pm 1}]t^i = \mathcal{R}t$ *where* (t) *is the ideal of* $\mathscr{C}^{0,\mu}$ *generated by the element* t.

Proof. 1 and 2. Notice that $\mathbb{K}[\Theta] \subset \mathbb{K}[h^{\pm 1}]$ and $\mathbb{K}[h^{\pm 1}] = \mathbb{K}[\Theta] \oplus \mathbb{K}[\Theta]h$. Multiplying this equality on the right by the element t yields that $\mathbb{K}[h^{\pm 1}]t = \mathbb{K}[\Theta]t \oplus \mathbb{K}[\Theta]u \subseteq \mathscr{C}^{0,\mu}$. Then for all $i \geqslant 1$, $\mathbb{K}[h^{\pm 1}]t^i = \mathbb{K}[h^{\pm 1}]t \cdot t^{i-1} \subseteq \mathscr{C}^{0,\mu}t^{i-1} \subseteq \mathscr{C}^{0,\mu}$. Notice that

$$\mathcal{R} = \bigoplus_{i\geqslant 0} \mathbb{K}[h^{\pm 1}]t^i = \bigoplus_{i\geqslant 1} \mathbb{K}[h^{\pm 1}]t^i \oplus \mathbb{K}[h^{\pm 1}] = \bigoplus_{i\geqslant 1} \mathbb{K}[h^{\pm 1}]t^i \oplus \mathbb{K}[\Theta] \oplus \mathbb{K}[\Theta]h. \tag{6.49}$$

Then $\mathscr{C}^{0,\mu} = \mathscr{C}^{0,\mu} \cap \mathcal{R} = \bigoplus_{i\geqslant 1} \mathbb{K}[h^{\pm 1}]t^i \oplus \mathbb{K}[\Theta]$ since $\mathscr{C}^{0,\mu} \cap \mathbb{K}[\Theta]h = 0$. The statement 2 then follows from (6.49).

3. By Proposition 6.35.(1), $(t) = \mathscr{C}^{0,\mu}t + \mathscr{C}^{0,\mu}u$. Then the first equality follows from statement 1. The second equality is obvious. □

Proposition 6.40. *Let* $\mathrm{Irr}(\mathscr{B})$ *be the set of irreducible elements of the algebra* $\mathscr{B}$.

1. $\widehat{\mathcal{R}}\,(\mathbb{K}[h^{\pm 1}]\text{-torsion}) = \widehat{\mathcal{R}}\,(t\text{-torsion}) = \widehat{\mathcal{R}/(t)} = \{[\mathcal{R}/\mathcal{R}(h-\alpha,\, t)] \,|\, \alpha \in \mathbb{K}^*\}$.
2. $\widehat{\mathcal{R}}\,(\mathbb{K}[h^{\pm 1}]\text{-torsionfree}) = \widehat{\mathcal{R}}\,(t\text{-torsionfree}) = \{[M_b] \,|\, b \in \mathrm{Irr}(\mathscr{B}),\ \mathcal{R} = \mathcal{R}t + \mathcal{R} \cap \mathscr{B}b\}$ *where* $M_b := \mathcal{R}/\mathcal{R} \cap \mathscr{B}b$; $M_b \simeq M_{b'}$ *iff the elements* b *and* b' *are similar (iff* $\mathscr{B}/\mathscr{B}b \simeq \mathscr{B}/\mathscr{B}b'$ *as* $\mathscr{B}$*-modules).*

Proof. 1. The last two equalities are obvious, since t is a normal element of the algebra $\mathcal{R}$. Then it is clear that $\widehat{\mathcal{R}}\,(\mathbb{K}[h^{\pm 1}]\text{-torsion}) \supseteq \widehat{\mathcal{R}}\,(t\text{-torsion})$. Now, we show the reverse inclusion holds. Let $M \in \widehat{\mathcal{R}}\,(\mathbb{K}[h^{\pm 1}]\text{-torsion})$. Then M is an epimorphic image of the $\mathcal{R}$-module $\mathcal{R}/\mathcal{R}(h-\alpha) = \mathbb{K}[t]\bar{1}$ for some $\alpha \in \mathbb{K}^*$ where $\bar{1} = 1 + \mathcal{R}(h-\alpha)$. Notice that $t\mathbb{K}[t]\bar{1}$ is the only maximal $\mathcal{R}$-submodule of $\mathcal{R}/\mathcal{R}(h-\alpha)$. Then $M \simeq \mathcal{R}/\mathcal{R}(h-\alpha,\, t) \in \widehat{\mathcal{R}}\,(t\text{-torsion})$, as required.

2. The first equality follows from the first equality in statement 1. By [10, Theorem 1.3] $\widehat{\mathcal{R}}\,(\mathbb{K}[h^{\pm 1}]\text{-torsionfree}) = \{[M_b] \,|\, b \in \mathrm{Irr}(\mathscr{B}),\ \mathcal{R} = \mathcal{R}t + \mathcal{R} \cap \mathscr{B}b\}$ (the condition (LO) of [10, Theorem 1.3] is equivalent to the condition $\mathcal{R} = \mathcal{R}t + \mathcal{R} \cap \mathscr{B}b$). □

Theorem 6.41. $\widehat{\mathscr{C}^{0,\mu}}\,(t\text{-torsionfree}) = \widehat{\mathcal{R}}\,(t\text{-torsionfree}) = \widehat{\mathcal{R}}\,(\mathbb{K}[h^{\pm 1}]\text{-torsionfree}) = \{[M_b = \mathcal{R}/\mathcal{R} \cap \mathscr{B}b] \,|\, b \in \mathrm{Irr}(\mathscr{B}),\ \mathcal{R} = \mathcal{R}t + \mathcal{R} \cap \mathscr{B}b\}$ *(see Proposition 6.40).*

Proof. In view of Proposition 6.40.(2), it remains to show that the first equality holds. Let $[M] \in \widehat{\mathscr{C}^{0,\mu}}\,(t\text{-torsionfree})$. Then $M = (t)M = \mathcal{R}tM \in \widehat{\mathcal{R}}\,(t\text{-torsionfree})$. Given $[N] \in \widehat{\mathcal{R}}\,(t\text{-torsionfree})$. To finish the proof of statement 2, it suffices to show that N is a simple $\mathscr{C}^{0,\mu}$-module. If L is a nonzero $\mathscr{C}^{0,\mu}$-submodule of N then $N \supseteq L \supseteq (t)L \neq 0$, since N is t-torsionfree. Then $(t)L = \mathcal{R}tL = N$, since N is a simple $\mathcal{R}$-module. Hence, $L = N$, i.e., N is a simple $\mathscr{C}^{0,\mu}$-module, as required. □

The set $\widehat{\mathscr{C}^{\lambda,\mu}}$ (t-torsionfree) where $\lambda \in \mathbb{K}^*$. An explicit description of the set $\widehat{\mathscr{C}^{\lambda,\mu}}$ (t-torsionfree) where $\lambda \in \mathbb{K}^*$ is given in Theorem 6.45.

Recall that the algebra $\mathscr{C}_t^{\lambda,\mu} = \mathbb{K}[t^{\pm 1}][u, v; \sigma, a]$ is a GWA where $a = \frac{q^7}{1-q^2}t^2 - q^4\mu^{-1}\lambda t$ and σ is the automorphism of the algebra $\mathbb{K}[t^{\pm 1}]$ defined by $\sigma(t) = q^{-2}t$ (Proposition 6.32.(2)). Clearly,

$$\widehat{\mathscr{C}^{\lambda,\mu}}\,(t\text{-torsionfree}) = \widehat{\mathscr{C}^{\lambda,\mu}}\,(t\text{-torsionfree}, \mathbb{K}[t]\text{-torsion}) \sqcup \widehat{\mathscr{C}^{\lambda,\mu}}\,(\mathbb{K}[t]\text{-torsionfree}). \tag{6.50}$$

Lemma 6.42. *Let $\lambda, \mu \in \mathbb{K}^*$ and $\nu := q^{-3}(1-q^2)\mu^{-1}\lambda$. Then*

1. *The module $\mathfrak{f}^{\lambda,\mu} := \mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu}(t-\nu, u)$ is a simple $\mathscr{C}^{\lambda,\mu}$-module.*
2. *The module $\mathsf{F}^{\lambda,\mu} := \mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu}(t-q^2\nu, v)$ is a simple $\mathscr{C}^{\lambda,\mu}$-module.*
3. *Let $\gamma \in \mathbb{K}^* \setminus \{q^{2i}\nu \,|\, i \in \mathbb{Z}\}$. The module $\mathcal{F}_\gamma^{\lambda,\mu} := \mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu}(t-\gamma)$ is a simple $\mathscr{C}^{\lambda,\mu}$-module. The simple modules $\mathcal{F}_\gamma^{\lambda,\mu} \simeq \mathcal{F}_{\gamma'}^{\lambda,\mu}$ iff $\gamma = q^{2i}\gamma'$ for some $i \in \mathbb{Z}$ where $\gamma' \in \mathbb{K}^* \setminus \{q^{2i}\nu \,|\, i \in \mathbb{Z}\}$.*

Proof. 1. Note that $a = \frac{q^7}{1-q^2}(t-\nu)t$ and $\sigma(a) = \frac{q^3}{1-q^2}(t-q^2\nu)t$. By Corollary 6.31.(2) and the expression of the element v, $\mathfrak{f}^{\lambda,\mu} = \mathbb{K}[\Theta]\bar{1} = \mathbb{K}[v]\bar{1}$ where $\bar{1} = 1 + \mathscr{C}^{\lambda,\mu}(t-\nu, u)$. The simplicity of the module $\mathfrak{f}^{\lambda,\mu}$ follows from the equality: $uv^i\bar{1} = v^{i-1}\sigma^i(a)\bar{1} \in \mathbb{K}^*v^{i-1}\bar{1}$ for all $i \geqslant 1$.

2. Notice that $\mathsf{F}^{\lambda,\mu} = \mathbb{K}[u]\bar{1}$ where $\bar{1} = 1 + \mathscr{C}^{\lambda,\mu}(t-q^2\nu, v)$. The simplicity of the module $\mathsf{F}^{\lambda,\mu}$ follows from the equality: $vu^i\bar{1} = u^{i-1}\sigma^{-i+1}(a)\bar{1} \in \mathbb{K}^*u^{i-1}\bar{1}$ for all $i \geqslant 1$.

3. Notice that $\mathcal{F}_\gamma^{\lambda,\mu} = \sum_{i,j\geqslant 0}\mathbb{K}u^i\Theta^j\tilde{1} = \sum_{i,j\geqslant 0}\mathbb{K}u^iv^j\tilde{1} = \mathbb{K}[u]\tilde{1} + \mathbb{K}[v]\tilde{1}$ where $\tilde{1} = 1 + \mathscr{C}^{\lambda,\mu}(t-\gamma)$. Since $\gamma \in \mathbb{K}^* \setminus \{q^{2i}\nu \,|\, i \in \mathbb{Z}\}$, $\sigma^i(a)\bar{1} \in \mathbb{K}^*\bar{1}$ for all $i \in \mathbb{Z}$. Then the simplicity of the module $\mathcal{F}_\gamma^{\lambda,\mu}$ follows from the equalities in the proof of statements 1 and 2. The set of eigenvalues of the element $t_{\mathcal{F}_\gamma^{\lambda,\mu}}$ is $\mathrm{Ev}_{\mathcal{F}_\gamma^{\lambda,\mu}}(t) = \{q^{2i}\gamma \,|\, i \in \mathbb{Z}\}$. If $\mathcal{F}_\gamma^{\lambda,\mu} \simeq \mathcal{F}_{\gamma'}^{\lambda,\mu}$ then $\mathrm{Ev}_{\mathcal{F}_\gamma^{\lambda,\mu}}(t) = \mathrm{Ev}_{\mathcal{F}_{\gamma'}^{\lambda,\mu}}(t)$, so $\gamma = q^{2i}\gamma'$ for some $i \in \mathbb{Z}$. Conversely, suppose that $\gamma = q^{2i}\gamma'$ for some $i \in \mathbb{Z}$. Let $\tilde{1}$ and $\tilde{1}'$ be the canonical generators of the modules $\mathcal{F}_\gamma^{\lambda,\mu}$ and $\mathcal{F}_{\gamma'}^{\lambda,\mu}$, respectively. The map $\mathcal{F}_\gamma^{\lambda,\mu} \to \mathcal{F}_{\gamma'}^{\lambda,\mu}$, $\tilde{1} \mapsto u^i\tilde{1}'$ defines an isomorphism of $\mathscr{C}^{\lambda,\mu}$-modules if $i \geqslant 0$, and the map $\mathcal{F}_\gamma^{\lambda,\mu} \to \mathcal{F}_{\gamma'}^{\lambda,\mu}$, $\tilde{1} \mapsto v^i\tilde{1}'$ defines an isomorphism of $\mathscr{C}^{\lambda,\mu}$-modules if $i < 0$. □

Definition. ([4], l-normal elements of the algebra $\mathscr{C}_t^{\lambda,\mu}$.)

1. Let α and β be nonzero elements of the Laurent polynomial algebra $\mathbb{K}[t^{\pm 1}]$. We say that $\alpha < \beta$ if there are no roots λ and μ of the polynomials α and β, respectively, such that, $\lambda = q^{2i}\mu$ for some $i \geqslant 0$.
2. An element $b = v^m\beta_m + v^{m-1}\beta_{m-1} + \cdots + \beta_0 \in \mathscr{C}_t^{\lambda,\mu}$ where $m > 0$, $\beta_i \in \mathbb{K}[t^{\pm 1}]$ and $\beta_0, \beta_m \neq 0$ is called *l-normal* if $\beta_0 < \beta_m$ and $\beta_0 < \frac{q^7}{1-q^2}t^2 - q^4\mu^{-1}\lambda t$.

Theorem 6.43. *[2, 5]. Let $\lambda, \mu \in \mathbb{K}^*$. Then*

$$\widehat{\mathscr{C}_t^{\lambda,\mu}}\,(\mathbb{K}[t]\text{-torsionfree}) = \{[N_b := \mathscr{C}_t^{\lambda,\mu}/\mathscr{C}_t^{\lambda,\mu} \cap \mathscr{B}b] \,|\, b \text{ is } l\text{-normal}, b \in \mathrm{Irr}(\mathscr{B})\}.$$

Simple $\mathscr{C}_t^{\lambda,\mu}$-modules N_b and $N_{b'}$ are isomorphic iff the elements b and b' are similar.

Recall that, the algebra $\mathscr{C}^{\lambda,\mu}$ is generated by the canonical generators t, u and Θ. Let $\mathcal{F} = \{\mathcal{F}_n\}_{n\geqslant 0}$ be the standard filtration associated with the canonical generators. By Corollary 6.31, for $n \geqslant 0$,

$$\mathcal{F}_n = \bigoplus_{\substack{i,j\geqslant 1,\\ i+j\leqslant n}} \mathbb{K}\Theta^i t^j \oplus \bigoplus_{1\leqslant k\leqslant n} \mathbb{K}\Theta^k \oplus \bigoplus_{\substack{l,m\geqslant 1,\\ l+m\leqslant n}} \mathbb{K}\Theta^l u^m \oplus \bigoplus_{\substack{a,b\geqslant 0,\\ a+b\leqslant n}} \mathbb{K}u^a t^b.$$

For all $n \geqslant 1$, $\dim \mathcal{F}_n = \frac{3}{2}n^2+\frac{3}{2}n+1 = f(n)$ (where $f(s) = \frac{3}{2}s^2+\frac{3}{2}s+1 \in \mathbb{K}[s]$). For each nonzero element $a \in \mathscr{C}^{\lambda,\mu}$, the unique natural number n such that $a \in \mathcal{F}_n \setminus \mathcal{F}_{n-1}$ is called the *total degree* of the element a, denoted by $\deg(a)$. Set $\deg(0) := -\infty$. Then $\deg(ab) \leqslant \deg(a) + \deg(b)$ for all elements a, $b \in \mathscr{C}^{\lambda,\mu}$.

For an R-module M, we denote by $l_R(M)$ the *length* of the R-module M. The next proposition shows that $l_{\mathscr{C}^{\lambda,\mu}}(\mathscr{C}^{\lambda,\mu}/I) < \infty$ for all left ideals I of the algebra $\mathscr{C}^{\lambda,\mu}$.

Proposition 6.44. *Let λ, $\mu \in \mathbb{K}^*$. For each element nonzero element $a \in \mathscr{C}^{\lambda,\mu}$, the length of the $\mathscr{C}^{\lambda,\mu}$-module $\mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu}a$ is finite, more precisely, $l_{\mathscr{C}^{\lambda,\mu}}(\mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu}a) \leqslant 3\deg(a)$.*

Proof. Let $M := \mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu}a = \mathscr{C}^{\lambda,\mu}\bar{1} = \bigcup_{i\geqslant 0}\mathcal{F}_i\bar{1}$ be the standard filtration on M where $\bar{1} = 1 + \mathscr{C}^{\lambda,\mu}a$. Then $\mathcal{F}_i\bar{1} \simeq \frac{\mathcal{F}_i+\mathscr{C}^{\lambda,\mu}a}{\mathscr{C}^{\lambda,\mu}a} \simeq \frac{\mathcal{F}_i}{\mathcal{F}_i\cap\mathscr{C}^{\lambda,\mu}a}$. Let $d := \deg(a)$. Since, for all $i \geqslant 0$, $\mathcal{F}_{i-d}a \subseteq \mathcal{F}_i \cap \mathscr{C}^{\lambda,\mu}a$, we see that $\dim(\mathcal{F}_i\bar{1}) \leqslant f(i) - f(i-d) = 3di + \frac{3}{2}d - \frac{3}{2}d^2$. Recall that the algebra $\mathscr{C}^{\lambda,\mu}$ is a simple, infinite dimensional algebra since $\lambda \neq 0$ (Theorem 6.34.(1)). So, if $N = \mathscr{C}^{\lambda,\mu}n$ is a nonzero cyclic $\mathscr{C}^{\lambda,\mu}$-module (where $0 \neq n \in N$) and $\{\mathcal{F}_in\}_{i\geqslant 0}$ is the standard filtration on N then $\dim(\mathcal{F}_in) \geqslant i+1$ for all $i \geqslant 0$. This implies that $l_{\mathscr{C}^{\lambda,\mu}}(M) \leqslant 3d$. □

The group $q^{2\mathbb{Z}} = \{q^{2i} \mid i \in \mathbb{Z}\}$ acts on $\mathbb{K}^*$ by multiplication. For each $\gamma \in \mathbb{K}^*$, let $\mathcal{O}(\gamma) = \{q^{2i}\gamma \mid i \in \mathbb{Z}\}$ be the orbit of the element $\gamma \in \mathbb{K}^*$ under the action of the group $q^{2\mathbb{Z}}$. For each orbit $\mathcal{O} \in \mathbb{K}^*/q^{2\mathbb{Z}}$, we fix an element $\gamma_{\mathcal{O}} \in \mathcal{O}(\gamma)$.

Theorem 6.45. *Let $\lambda, \mu \in \mathbb{K}^*$. Then*

1. $\widehat{\mathscr{C}^{\lambda,\mu}}$ (t-torsionfree, $\mathbb{K}[t]$-torsion) $= \{[\mathfrak{f}^{\lambda,\mu}],\ [\mathsf{F}^{\lambda,\mu}],\ [\mathcal{F}^{\lambda,\mu}_{\gamma_{\mathcal{O}}}] \mid \mathcal{O} \in \mathbb{K}^*/q^{2\mathbb{Z}} \setminus \{\mathcal{O}(\nu)\}\}$.
2. *The map* $\widehat{\mathscr{C}^{\lambda,\mu}}$ ($\mathbb{K}[t]$-torsionfree) $\to \widehat{\mathscr{C}^{\lambda,\mu}_t}$ ($\mathbb{K}[t]$-torsionfree), $[M] \mapsto [M_t]$ *is a bijection with the inverse* $[N] \mapsto \mathrm{soc}_{\mathscr{C}^{\lambda,\mu}}(N)$.
3. $\widehat{\mathscr{C}^{\lambda,\mu}}$ ($\mathbb{K}[t]$-torsionfree) $= \{[M_b := \mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu} \cap \mathscr{B}bt^{-i}] \mid b$ is l-normal, $b \in \mathrm{Irr}(\mathscr{B})$, $i \geqslant 3\deg(b)\}$.

Proof. 1. Let $M \in \widehat{\mathscr{C}^{\lambda,\mu}}$ (t-torsionfree, $\mathbb{K}[t]$-torsion). There exists a nonzero element $m \in M$ such that $tm = \gamma m$ for some $\gamma \in \mathbb{K}^*$. Then M is an epimorphic image of the module $\mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu}(t-\gamma)$. If $\gamma \notin \mathcal{O}(\nu)$ then $M \simeq \mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu}(t-\gamma) = \mathcal{F}^{\lambda,\mu}_\gamma$ by Lemma 6.42.(3). It remains to consider the case when $\gamma \in \mathcal{O}(\nu)$, i.e., $\gamma = q^{2i}\nu$ for some $i \in \mathbb{Z}$.

(i) If $\gamma = q^{2i}\nu$ where $i \geqslant 1$ then $\sigma^i(a)m = 0$. Notice that $u^{i-1}v^{i-1}m = \sigma^{i-1}(a)\cdots\sigma(a)m \neq 0$, the element $m' := v^{i-1}m$ is a nonzero element of M. If $vm' = 0$, notice that $tm' = tv^{i-1}m = q^2\nu m'$, then M is an epimorphic image of the simple module $\mathsf{F}^{\lambda,\mu}$. Hence, $M \simeq \mathsf{F}^{\lambda,\mu}$. If $m'' := vm' \neq 0$, notice that $tm'' = tv^im = \nu m''$ and $um'' = uv^im = v^{i-1}\sigma^i(a)m = 0$, then M is an epimorphic image of the simple module $\mathfrak{f}^{\lambda,\mu}$. Hence, $M \simeq \mathfrak{f}^{\lambda,\mu}$.

(ii) If $\gamma = q^{-2i}\nu$ where $i \geqslant 0$ then $\sigma^{-i}(a)m = 0$. The element $e := u^im$ is a nonzero element of M. (The case $i = 0$ is trivial, for $i \geqslant 1$, it follows from the equality $v^iu^im = \sigma^{-i+1}(a)\cdots\sigma^{-1}(a)am \neq$

0). If $ue = 0$, notice that $te = tu^i m = \nu e$, then M is an epimorphic image of the simple module $\mathfrak{f}^{\lambda,\mu}$. Hence, $M \simeq \mathfrak{f}^{\lambda,\mu}$. If $e' := ue \neq 0$, notice that $te' = tu^{i+1}m = q^2\nu e'$ and $ve' = vu^{i+1}m = u^i\sigma^{-i}(a)m = 0$, then M is an epimorphic image of the simple module $\mathsf{F}^{\lambda,\mu}$. Hence, $M \simeq \mathsf{F}^{\lambda,\mu}$. This proves statement 1.

2. The result follows from Proposition 6.44.

3. Let $[M] \in \widehat{\mathscr{C}^{\lambda,\mu}}(\mathbb{K}[t]\text{-torsionfree})$. Then $[M_t] \in \widehat{\mathscr{C}_t^{\lambda,\mu}}(\mathbb{K}[t]\text{-torsionfree})$, and so $M_t \simeq \mathscr{C}_t^{\lambda,\mu}/\mathscr{C}_t^{\lambda,\mu} \cap \mathscr{B}b$ where $b = v^m\beta_m + v^{m-1}\beta_{m-1} + \cdots + \beta_0 \in \mathscr{C}^{\lambda,\mu}$ ($\beta_i \in \mathbb{K}[t]$, $m > 0$ and $\beta_m,\ \beta_0 \neq 0$) is an l-normal and irreducible in $\mathscr{B}$. Clearly, $0 \neq M_b := \mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu} \cap \mathscr{B}b \subseteq M_t$ and $M = \mathrm{soc}_{\mathscr{C}^{\lambda,\mu}}(M_t) = \mathrm{soc}_{\mathscr{C}^{\lambda,\mu}}(M_b)$, by statement 2. Let $I_b := \mathscr{C}^{\lambda,\mu} \cap \mathscr{B}b$, $J_n = \mathscr{C}^{\lambda,\mu}t^n + I_b$ for all $n \geqslant 0$ and $d = \deg(a)$. By Proposition 6.44, the following descending chain of left ideals of the algebra $\mathscr{C}^{\lambda,\mu}$ stabilizes:

$$\mathscr{C}^{\lambda,\mu} = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_n = J_{n+1} = \cdots, \quad n \geqslant 3d.$$

Hence, $\mathrm{soc}_{\mathscr{C}^{\lambda,\mu}}(M_b) = J_n/I_b \simeq \mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu} \cap \mathscr{B}bt^{-n}$. □

6.7 Simple weight A-modules

The aim of this section is to give a classification of simple weight A-module. The set $\widehat{A}\,(\text{weight})$ of isomorphism classes of simple weight A-modules is partitioned into disjoint union of four subsets, see (6.51). We will describe each of them separately.

An A-module M is called a *weight module* provided that $M = \bigoplus_{\mu \in \mathbb{K}^*} M_\mu$ where $M_\mu = \{m \in M \mid Km = \mu m\}$. We denote by $\mathrm{Wt}\,(M)$ the set of all weights of M, i.e., the set $\{\mu \in \mathbb{K}^* \mid M_\mu \neq 0\}$.

Verma modules and simple highest weight A-modules. For each $\lambda \in \mathbb{K}^*$, we define the Verma module $M(\lambda) := A/A(K-\lambda, E, X)$. Then $M(\lambda) = \mathbb{K}[Y,F]\tilde{1}$ where $\tilde{1} = 1 + A(K-\lambda, E, X)$. If M is an A-module, a *highest weight vector* is any $0 \neq m \in M$ such that m is an eigenvector of K and K^{-1} and $Em = Xm = 0$.

Lemma 6.46. *The set of highest weight vectors of the Verma module* $M(\lambda)$ *is* $\mathsf{H} := \{\, kY^n\tilde{1} \mid k \in \mathbb{K}^*,\ n \in \mathbb{N}\,\}$.

Proof. It is clearly that any element of H is a highest weight vector. Suppose that $m = \sum \alpha_{ij}Y^iF^j\tilde{1} \in M(\lambda)$ is a highest weight vector of weight μ where $\alpha_{ij} \in \mathbb{K}$. Then $Km = \sum \alpha_{ij}\lambda q^{-i-2j}Y^iF^j\tilde{1} = \mu m$. This implies that $i+2j$ is a constant, say $i+2j = n$. Then m can be written as $m = \sum \alpha_j Y^{n-2j}F^j\tilde{1}$ for some $\alpha_j \in \mathbb{K}$. By Lemma 6.10.(2), $Xm = \sum -q^{n-2j}\frac{1-q^{2j}}{1-q^2}\alpha_j\lambda^{-1}Y^{n-2j+1}F^{j-1}\tilde{1} = 0$. Thus, $\alpha_j = 0$ for all $j \geqslant 1$ and hence, $m \in \mathsf{H}$. □

By Lemma 6.46, there are infinite number of linear independent highest weight vectors. Let $N_n := \mathbb{K}[Y,F]Y^n\tilde{1}$ where $n \in \mathbb{N}$. Then N_n is a Verma A-module with highest weight $q^{-n}\lambda$, i.e., $N_n \simeq M(q^{-n}\lambda)$. Furthermore, $M(\lambda)$ is a submodule of $M(q^n\lambda)$ for all $n \in \mathbb{N}$. Thus, for any

$\lambda \in \mathbb{K}^*$, there exists an infinite sequence of Verma modules

$$\cdots \supset M(q^2\lambda) \supset M(q\lambda) \supset M(\lambda) \supset M(q^{-1}\lambda) \supset M(q^{-2}\lambda) \supset \cdots .$$

The following result of Verma $U_q(\mathfrak{sl}_2)$-modules is well-known; see [29, p. 20].

Lemma 6.47. *[29] Suppose that q is not a root of unity. Let $V(\lambda)$ be a Verma $U_q(\mathfrak{sl}_2)$-module. Then $V(\lambda)$ is simple if and only if $\lambda \neq \pm q^n$ for all integer $n \geqslant 0$. When $\lambda = q^n$ (resp. $-q^n$) there is a unique simple quotient $L(n,+)$ (resp. $L(n,-)$) of $V(\lambda)$. Each simple $U_q(\mathfrak{sl}_2)$-module of dimension $n+1$ is isomorphic to $L(n,+)$ or $L(n,-)$.*

Let $V(\lambda) := M(\lambda)/N_1$. Then $V(\lambda) \simeq \mathbb{K}[F]\bar{1}$, where $\bar{1} := 1 + A(K-\lambda, E, X, Y)$.

Theorem 6.48. *Up to isomorphism, the simple highest weight A-module are as follows*

(i) $V(\lambda)$, *when* $\lambda \neq \pm q^n$ *for any* $n \in \mathbb{N}$.
(ii) $L(n,+)$, *when* $\lambda = q^n$ *for some* $n \in \mathbb{N}$.
(iii) $L(n,-)$, *when* $\lambda = -q^n$ *for some* $n \in \mathbb{N}$.

In each case, the elements X and Y act trivially on the modules, and these modules are in fact simple highest weight $U_q(\mathfrak{sl}_2)$-modules.

Proof. In view of Lemma 6.11.(1), $\mathrm{ann}_A(V(\lambda)) \supseteq (X)$. So, $V(\lambda) \simeq U/U(K-\lambda, E)$ where $U = U_q(\mathfrak{sl}_2)$. Then the theorem follows immediately from Lemma 6.47. □

Simple weight modules that are neither highest nor lowest weight A-modules. Let A be an algebra, we denote by $\widehat{A}$ the set of isomorphism classes of simple left A-modules. Let $\mathcal{N}$ be *the set of simple weight A-modules M such that $XM \neq 0$ or $YM \neq 0$.* Then $\widehat{A}\,(\text{weight}) = \widehat{U_q(\mathfrak{sl}_2)}\,(\text{weight}) \sqcup \mathcal{N}$.

Lemma 6.49. *Let M be a simple A-module. If $x \in \{X, Y, E, F\}$ annihilates a non-zero element $m \in M$, then x acts locally nilpotently on M.*

Proof. For each element $x \in \{X, Y, E, F\}$, the set $S = \{x^i \,|\, i \in \mathbb{N}\}$ is an Ore set in the algebra A. Then $\mathrm{tor}_S(M)$ is a nonzero submodule of M. Since M is a simple module, $M = \mathrm{tor}_S(M)$, i.e., the element x acts locally nilpotently on M. □

Theorem 6.50. *Let $M \in \mathcal{N}$, then*

1. $\dim M_\lambda = \dim M_\mu$ *for any* $\lambda, \mu \in \mathrm{Wt}\,(M)$.
2. $\mathrm{Wt}\,(M) = \{q^n\lambda \,|\, n \in \mathbb{Z}\}$ *for any* $\lambda \in \mathrm{Wt}\,(M)$.

Proof. 1. Suppose that there exists $\lambda \in \mathrm{Wt}\,(M)$ such that $\dim M_\lambda > \dim M_{q\lambda}$. Then the map $X : M_\lambda \to M_{q\lambda}$ is not injective. Hence $Xm = 0$ for some non-zero element $m \in M_\lambda$. By Lemma 6.49, X acts locally nilpotently on M.

If $\dim M_{q^{-1}\lambda} > \dim M_{q\lambda}$, then the linear map $E : M_{q^{-1}\lambda} \to M_{q\lambda}$ is not injective. So $Em' = 0$ for some non-zero element $m' \in M_{q^{-1}\lambda}$. By Lemma 6.49, E acts on M locally nilpotently. Since

$EX = qXE$, there exists a non-zero weight vector m'' such that $Xm'' = Em'' = 0$. Therefore, M is a highest weight module. By Theorem 6.48, $XM = YM = 0$, this contradicts our assumption that $M \in \mathcal{N}$.

If $\dim M_{q^{-1}\lambda} \leqslant \dim M_{q\lambda}$, then $\dim M_{q^{-1}\lambda} < \dim M_\lambda$. Hence the map $Y : M_\lambda \to M_{q^{-1}\lambda}$ is not injective. It follows that $Ym_1 = 0$ for some non-zero element $m_1 \in M_\lambda$. By Lemma 6.49, Y acts on M locally nilpotently. Since $XY = qYX$, there exists some non-zero weight vector $m_2 \in M$ such that $Xm_2 = Ym_2 = 0$. By Lemma 6.11.(1), $\mathrm{ann}_A(M) \supseteq (X, Y)$, a contradiction. Similarly, one can show that there does not exist $\lambda \in \mathrm{Wt}(M)$ such that $\dim M_\lambda < \dim M_{q\lambda}$.

2. Clearly, $\mathrm{Wt}(M) \subseteq \{q^n\lambda \,|\, n \in \mathbb{Z}\}$. By the above argument we see that $\mathrm{Wt}(M) \supseteq \{q^n\lambda \,|\, n \in \mathbb{Z}\}$. Hence $\mathrm{Wt}(M) = \{q^n\lambda \,|\, n \in \mathbb{Z}\}$. □

Let M be an A-module and $x \in A$. We say that M is *x-torsion* provided that for each element $m \in M$ there exists some $i \in \mathbb{N}$ such that $x^i m = 0$.

Lemma 6.51. *Let $M \in \mathcal{N}$.*

1. *If M is X-torsion, then M is (φ, Y)-torsionfree.*
2. *If M is Y-torsion, then M is (X, φ)-torsionfree.*
3. *If M is φ-torsion, then M is (X, Y)-torsionfree.*

Proof. 1. Since $M \in \mathcal{N}$ is an X-torsion module, by the proof of Theorem 6.50, Y_M and E_M are injections. Let us show that φ_M is injective. Otherwise, there exists a nonzereo element $m \in M$ such that $\varphi m = 0$, i.e., $Xm = (q - q^{-1})YEm$. Since $X^i m = 0$ for some $i \in \mathbb{N}$ and $X(YE) = (YE)X$, we have $X^i m = (q-q^{-1})^i (YE)^i m = 0$. This contradicts to the fact that Y and E are injective maps on M.

2. Clearly, X_M is an injection. Let us show that φ_M is an injective map. Otherwise, there exists a nonzero element $m \in M$ such that $\varphi m = Ym = 0$ (since $Y\varphi = q\varphi Y$). Then $Xm = 0$ (since $\varphi = (1 - q^2)EY + q^2 X$), a contradiction.

3. Statement 3 follows from statments 1 and 2. □

By Lemma 6.51,

$$\begin{aligned} \widehat{A}\,(\text{weight}) &= \widehat{U_q(\mathfrak{sl}_2)}\,(\text{weight}) \sqcup \mathcal{N} \\ &= \widehat{U_q(\mathfrak{sl}_2)}\,(\text{weight}) \sqcup \mathcal{N}\,(X\text{-torsion}) \sqcup \mathcal{N}\,(Y\text{-torsion}) \sqcup \mathcal{N}\,((X,Y)\text{-torsionfree}). \end{aligned} \tag{6.51}$$

It is clear that $\mathcal{N}\,((X,Y)\text{-torsionfree}) = \widehat{A}\,(\text{weight}, (X,Y)\text{-torsionfree})$.

Lemma 6.52. *If $M \in \mathcal{N}\,(X\text{-torsion}) \sqcup \mathcal{N}\,(\varphi\text{-torsion}) \sqcup \mathcal{N}\,(Y\text{-torsion})$ then $C_M \neq 0$.*

Proof. Suppose that $M \in \mathcal{N}\,(X\text{-torsion})$, and let m be a weight vector such that $Xm = 0$. If $C_M = 0$, then, by (6.12), $Cm = -K^{-1}EY^2 m = 0$, i.e., $EY^2 m = 0$. This implies that either E_M or Y_M is not injective. By the proof of Theorem 6.50, we have a contradiction. Similarly, one can prove that for $M \in \mathcal{N}(Y\text{-torsion})$, $C_M \neq 0$. Now, suppose that $M \in \mathcal{N}(\varphi\text{-torsion})$, and let $m \in M_\mu$ be a weight vector such that $\varphi m = 0$. Since $Y\varphi = q(1-q^2)EY^2 + q^4YX$, we have

$$Y\varphi m = q(1-q^2)EY^2m + q^4YXm = 0. \tag{6.52}$$

If $C_M = 0$, then, by (6.13),

$$Cm = -\mu^{-1}EY^2m + \frac{q^3}{1-q^2}(\mu-\mu^{-1})YXm = 0. \tag{6.53}$$

The equalities (6.52) and (6.53) yield that $EY^2m = 0$ and $YXm = 0$, a contradiction. □

Theorem 6.53. *Let* $M \in \mathcal{N}$. *Then* $\dim M_\mu = \infty$ *for all* $\mu \in \mathrm{Wt}(M)$.

Proof. Since M is a simple A-module, the weight space M_μ of M is a simple $\mathscr{C}^{\lambda,\mu}$-module for some $\lambda \in \mathbb{K}$. If $M \in \mathcal{N}(X\text{-torsion}) \sqcup \mathcal{N}(Y\text{-torsion})$ then by Lemma 6.52, $\lambda = C_M \neq 0$. By Proposition 6.32.(4) and Theorem 6.34.(1), $\mathscr{C}^{\lambda,\mu}$ is an infinite dimensional central simple algebra. Hence, $\dim M_\mu = \infty$. It remains to consider the case where $M \in \mathcal{N}((X,Y)\text{-torsionfree})$. Suppose that there exists a weight space M_ν of M such that $\dim M_\nu = n < \infty$, we seek a contradiction. Then by Theorem 6.50, $\dim M_\mu = n$ for all $\mu \in \mathrm{Wt}(M)$ and $\mathrm{Wt}(M) = \{q^i\nu \mid i \in \mathbb{Z}\}$. Notice that the elements X and Y act injectively on M, then they act bijectively on M (since all the weight spaces are finite dimensional and of the same dimension). In particular, the element $t = YX$ acts bijectively on each weight space M_μ, and so, M_μ is a simple $\mathscr{C}_t^{\lambda,\mu}$-module. By Proposition 6.32.(2,3), the algebra $\mathscr{C}_t^{\lambda,\mu}$ is an infinite dimensional central simple algebra for any $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$. Then, $\dim M_\mu = \infty$, a contradiction. □

Description of the set $\mathcal{N}(X\text{-torsion})$. An explicit description of the set $\mathcal{N}(X\text{-torsion})$ is given in Theorem 6.55. It consists of a family of simple modules constructed below (see Proposition 6.54). For each $\mu \in \mathbb{K}^*$, we define the left A-module $\mathbb{X}^\mu := A/A(K-\mu,\, X)$. Then $\mathbb{X}^\mu = \bigoplus_{i,j,k\geqslant 0} \mathbb{K}F^iE^jY^k\,\bar{1}$ where $\bar{1} = 1 + A(K-\mu,\, X)$. Let $\lambda \in \mathbb{K}$. By (6.12), we see that the submodule of $\mathbb{X}^\mu$,

$$(C-\lambda)\,\mathbb{X}^\mu = \bigoplus_{i,j,k\geqslant 0} \mathbb{K}F^iE^jY^k\big(\mu^{-1}EY^2+\lambda\big)\,\bar{1} \overset{L.5.2}{=} \bigoplus_{i,j,k\geqslant 0} \mathbb{K}F^i\Big(\mu^{-1}q^kE^{j+1}Y^{k+2} + \lambda E^jY^k\Big)\,\bar{1}, \tag{6.54}$$

is a proper submodule and the map $(C-\lambda)\cdot : \mathbb{X}^\mu \longrightarrow \mathbb{X}^\mu$, $v \mapsto (C-\lambda)v$, is an injection, which is not a bijection. It is obvious that $\mathrm{GK}\,(\mathbb{X}^\mu) = 3$.

For $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, we define the left A-module $\mathbb{X}^{\lambda,\mu} := A/A(C-\lambda,\, K-\mu,\, X)$. Then,

$$\mathbb{X}^{\lambda,\mu} \simeq \mathbb{X}^\mu/(C-\lambda)\mathbb{X}^\mu \neq 0. \tag{6.55}$$

We have a short exact sequence of A-modules: $0 \longrightarrow \mathbb{X}^\mu \xrightarrow{(C-\lambda)\cdot} \mathbb{X}^\mu \longrightarrow \mathbb{X}^{\lambda,\mu} \longrightarrow 0$. The next proposition shows that the module $\mathbb{X}^{\lambda,\mu}$ is a simple module if λ is nonzero. Moreover, the $\mathbb{K}$-basis, the weight space decomposition and the annihilator of the module $\mathbb{X}^{\lambda,\mu}$ are given.

Proposition 6.54. *For* λ *and* $\mu \in \mathbb{K}^*$, *consider the left* A-*module* $\mathbb{X}^{\lambda,\mu} = A/A(C-\lambda,\, K-\mu,\, X)$.

1. *The A-module* $\mathbb{X}^{\lambda,\mu}=\bigoplus_{i\geqslant 0,j\geqslant 2}\mathbb{K}F^iY^j\,\bar{1}\oplus\bigoplus_{i,k\geqslant 0}\mathbb{K}F^iE^k\,\bar{1}\oplus\bigoplus_{i,k\geqslant 0}\mathbb{K}YF^iE^k\,\bar{1}$ *is a simple A-module where* $\bar{1}=1+A(C-\lambda,\,K-\mu,\,X)$.
2. $$\mathbb{X}^{\lambda,\mu}=\bigoplus_{i\geqslant 0,j\geqslant 2}\mathbb{K}F^iY^j\,\bar{1}\oplus\Big(\bigoplus_{i\geqslant 1,k\geqslant 0}\mathbb{K}F^i\Theta^k\,\bar{1}\oplus\bigoplus_{k\geqslant 0}\mathbb{K}\Theta^k\,\bar{1}\oplus\bigoplus_{i\geqslant 1,k\geqslant 0}\mathbb{K}E^i\Theta^k\,\bar{1}\Big)$$ $$\oplus\Big(\bigoplus_{i\geqslant 1,k\geqslant 0}\mathbb{K}YF^i\Theta^k\,\bar{1}\oplus\bigoplus_{k\geqslant 0}\mathbb{K}Y\Theta^k\,\bar{1}\oplus\bigoplus_{i\geqslant 1,k\geqslant 0}\mathbb{K}YE^i\Theta^k\,\bar{1}\Big).$$
3. *The weight subspace* $\mathbb{X}^{\lambda,\mu}_{q^s\mu}$ *of* $\mathbb{X}^{\lambda,\mu}$ *that corresponds to the weight* $q^s\mu$ *is*
$$\mathbb{X}^{\lambda,\mu}_{q^s\mu}=\begin{cases}\mathbb{K}[\Theta]\,\bar{1}, & s=0,\\ E^r\mathbb{K}[\Theta]\,\bar{1}, & s=2r,\ r\geqslant 1,\\ YE^r\mathbb{K}[\Theta]\,\bar{1}, & s=2r-1,\ r\geqslant 1,\\ F^r\mathbb{K}[\Theta]\,\bar{1}\oplus\bigoplus_{\substack{i+j=r,\\ j\geqslant 1}}\mathbb{K}F^iY^{2j}\,\bar{1}, & s=-2r,\ r\geqslant 1,\\ Y\mathbb{K}[\Theta]\,\bar{1}, & s=-1,\\ YF^{r-1}\mathbb{K}[\Theta]\,\bar{1}\oplus\bigoplus_{\substack{2i+j=2r-1,\\ j\geqslant 2}}\mathbb{K}F^iY^j\,\bar{1}, & s=-2(r-1)-1,\ r\geqslant 2.\end{cases}$$
4. $\mathrm{ann}_A(\mathbb{X}^{\lambda,\mu})=(C-\lambda)$.
5. $\mathbb{X}^{\lambda,\mu}$ *is an X-torsion and Y-torsionfree A-module.*
6. *Let* $(\lambda,\mu),\,(\lambda',\mu')\in\mathbb{K}\times\mathbb{K}^*$. *Then* $\mathbb{X}^{\lambda,\mu}\simeq\mathbb{X}^{\lambda',\mu'}$ *iff* $\lambda=\lambda'$ *and* $\mu=q^i\mu'$ *for some* $i\in\mathbb{Z}$.

Proof. 1. By (6.55), $\mathbb{X}^{\lambda,\mu}\neq 0$ and $\bar{1}\neq 0$. Using the PBW basis for the algebra A, we have $\mathbb{X}^{\lambda,\mu}=\sum_{i,j,k\geqslant 0}\mathbb{K}F^iY^jE^k\,\bar{1}$. Using (6.12), we have $\lambda\,\bar{1}=C\,\bar{1}=-\mu^{-1}EY^2\,\bar{1}$. Hence $EY^2\,\bar{1}=-\mu\lambda\,\bar{1}$, and then $Y^2E\,\bar{1}=-q^2\mu\lambda\,\bar{1}$. By induction on k and using Lemma 5.2, we deduce that

$$E^kY^{2k}\,\bar{1}=(-\mu\lambda)^kq^{-k(k-1)}\,\bar{1}\quad\text{and}\quad Y^{2k}E^k\,\bar{1}=(-q^2\mu\lambda)^kq^{k(k-1)}\,\bar{1}.\tag{6.56}$$

Therefore, $\sum_{j,k\geqslant 0}\mathbb{K}Y^jE^k\,\bar{1}=Y^2\mathbb{K}[Y]\,\bar{1}+\mathbb{K}[E]\,\bar{1}+Y\mathbb{K}[E]\,\bar{1}$, and then

$$\mathbb{X}^{\lambda,\mu}=\sum_{i\geqslant 0,j\geqslant 2}\mathbb{K}F^iY^j\,\bar{1}+\sum_{i,k\geqslant 0}\mathbb{K}F^iE^k\,\bar{1}+\sum_{i,k\geqslant 0}\mathbb{K}YF^iE^k\,\bar{1}=\mathbb{K}[F]\Big(\mathbb{K}[Y]Y^2+\mathbb{K}[E]+Y\mathbb{K}[E]\Big)\,\bar{1}.$$

So, any element u of $\mathbb{X}^{\lambda,\mu}$ can be written as $u=(\sum_{i=0}^nF^ia_i)\bar{1}$ where $a_i\in\Sigma:=\mathbb{K}[Y]Y^2+\mathbb{K}[E]+Y\mathbb{K}[E]$. Statement 1 follows from the following claim: if $a_n\neq 0$, then there is an element $a\in A$ such that $au=\bar{1}$.

(i) $X^nu=a'\,\bar{1}$ for some nonzero element $a'\in\Sigma$: Using Lemma 6.10, we have $Xu=\sum_{i=0}^{n-1}F^ib_i\,\bar{1}$ for some $b_i\in\Sigma$ and $b_{n-1}\neq 0$. Repeating this step $n-1$ times (or using induction on n), we obtain the result as required. So, we may assume that $u=a_0\bar{1}$ where $0\neq a_0\in\Sigma$.

(ii) Notice that the element $a_0\in\Sigma$ can be written as $a_0=pY^2+\sum_{i=0}^m(\lambda_i+\mu_iY)E^i$ where $p\in\mathbb{K}[Y]$, λ_i and $\mu_i\in\mathbb{K}$. Then, by (6.56), $Y^{2m}u=Y^{2m}a_0\,\bar{1}=\Big(pY^{2m+2}+\sum_{i=0}^m(\lambda_i+\mu_iY)Y^{2(m-i)}Y^{2i}E^i\Big)\,\bar{1}=\Big(pY^{2m+2}+\sum_{i=0}^m(\lambda_i+\mu_iY)Y^{2(m-i)}\gamma_i\Big)\,\bar{1}=f\bar{1}$ for some $\gamma_i\in\mathbb{K}^*$ where f is a nonzero polynomial in $\mathbb{K}[Y]$ (since $a_0\neq 0$). Hence, we may assume that $u=f\bar{1}$ where $0\neq f\in\mathbb{K}[Y]$.

(iii) Let $f = \sum_{i=0}^{l} \gamma_i Y^i$ where $\gamma_i \in \mathbb{K}$ and $\gamma_l \neq 0$. Since $KY^i\,\bar{1} = \mu q^{-i} Y^i\,\bar{1}$ and all eigenvalues $\{\mu q^{-i} \,|\, i \geqslant 0\}$ are distinct, there is a polynomial $g \in \mathbb{K}[K]$ such that $gf\bar{1} = Y^l\,\bar{1}$. If $l = 0$, we are done. We may assume that $l \geqslant 1$. By multiplying by Y (if necessary) on the equality above we may assume that $l = 2k$ for some natural number k. Then, by (6.56), $\omega_k^{-1} E^k Y^{2k}\,\bar{1} = \bar{1}$ where $\omega_k = (-\mu\lambda)^k q^{-k(k-1)}$, as required.

2. Recall that that the algebra $U_q(\mathfrak{sl}_2) = \mathbb{K}[\Theta, K^{\pm 1}][E, F; \sigma, a = (1-q^2)^{-1}\Theta - \frac{q^2(qK+q^{-1}K^{-1})}{(1-q^2)^2}]$ is a GWA where $\sigma(\Theta) = \Theta$ and $\sigma(K) = q^{-2}K$. Then for all $i \geqslant 1$, $F^iE^i = a\sigma^{-1}(a)\cdots\sigma^{-i+1}(a)$. Therefore,

$$\bigoplus_{i,k\geqslant 0} \mathbb{K}F^iE^k\,\bar{1} = \bigoplus_{i\geqslant 1, k\geqslant 0} \mathbb{K}F^i\Theta^k\,\bar{1} \oplus \bigoplus_{k\geqslant 0} \mathbb{K}\Theta^k\,\bar{1} \oplus \bigoplus_{i\geqslant 1, k\geqslant 0} \mathbb{K}E^i\Theta^k\,\bar{1}.$$

Then statement 2 follows from statement 1.

3. Statement 3 follows from statement 2.

4. Clearly, $(C-\lambda) \subseteq \mathrm{ann}_A(\mathbb{X}^{\lambda,\mu})$. Since $\lambda \in \mathbb{K}^*$, by Corollary 6.17, the ideal $(C-\lambda)$ is a maximal ideal of A. Then we must have $(C-\lambda) = \mathrm{ann}_A(\mathbb{X}^{\lambda,\mu})$.

5. Clearly, $\mathbb{X}^{\lambda,\mu}$ is an X-torsion weight module. Since $\mathbb{X}^{\lambda,\mu}$ is a simple module, then by Lemma 6.51, $\mathbb{X}^{\lambda,\mu}$ is Y-torsionfree.

6. $(\Rightarrow)$ Suppose that $\mathbb{X}^{\lambda,\mu} \simeq \mathbb{X}^{\lambda',\mu'}$. By statement 4, $(C-\lambda) = \mathrm{ann}_A(\mathbb{X}^{\lambda,\mu}) = \mathrm{ann}_A(\mathbb{X}^{\lambda',\mu'}) = (C-\lambda')$. Hence, $\lambda = \lambda'$. By Theorem 6.50 (or by statement 3), $\{q^i\mu \,|\, i \in \mathbb{Z}\} = \mathrm{Wt}(\mathbb{X}^{\lambda,\mu}) = \mathrm{Wt}(\mathbb{X}^{\lambda',\mu'}) = \{q^i\mu' \,|\, i \in \mathbb{Z}\}$. Hence, $\mu = q^i\mu'$ for some $i \in \mathbb{Z}$.

$(\Leftarrow)$ Suppose that $\lambda = \lambda'$ and $\mu = q^i\mu'$ for some $i \in \mathbb{Z}$. Let $\bar{1}$ and $\bar{1}'$ be the canonical generators of the modules $\mathbb{X}^{\lambda,\mu}$ and $\mathbb{X}^{\lambda',\mu'}$, respectively. If $i \leqslant 0$ then the map $\mathbb{X}^{\lambda,\mu} \to \mathbb{X}^{\lambda',\mu'}$ $\bar{1} \mapsto Y^{|i|}\,\bar{1}'$ defines an isomorphism of A-modules. If $i \geqslant 1$ then the map $\mathbb{X}^{\lambda,\mu} \to \mathbb{X}^{\lambda',\mu'}$, $\bar{1} \mapsto (YE)^i\,\bar{1}'$ defines an isomorphism of A-modules. □

We define an equivalent relation $\sim$ on the set $\mathbb{K}^*$ as follows: for μ and $\nu \in \mathbb{K}^*$, $\mu \sim \nu$ iff $\mu = q^i\nu$ for some $i \in \mathbb{Z}$. Then the set $\mathbb{K}^*$ is a disjoint union of equivalent classes $\mathcal{O}(\mu) = \{q^i\mu \,|\, i \in \mathbb{Z}\}$. Let $\mathbb{K}^*/\sim$ be the set of equivalent classes. Clearly, $\mathbb{K}^*/\sim$ can be identified with the factor group $\mathbb{K}^*/\langle q\rangle$ where $\langle q\rangle = \{q^i \,|\, i \in \mathbb{Z}\}$. For each orbit $\mathcal{O} \in \mathbb{K}^*/\langle q\rangle$, we fix an element $\mu_{\mathcal{O}} \in \mathcal{O}$.

Theorem 6.55. $\mathcal{N}(X\text{-torsion}) = \left\{[\mathbb{X}^{\lambda,\mu_{\mathcal{O}}}] \,|\, \lambda \in \mathbb{K}^*,\ \mathcal{O} \in \mathbb{K}^*/\langle q\rangle\right\}$.

Proof. Let $M \in \mathcal{N}(X\text{-torsion})$. By Lemma 6.52, the central element C acts on M as a nonzero scalar, say λ. Then M is an epimorphic image of the module $\mathbb{X}^{\lambda,\mu}$ for some $\mu \in \mathbb{K}^*$. By Proposition 6.54.(1), $\mathbb{X}^{\lambda,\mu}$ is a simple A-module, hence $M \simeq \mathbb{X}^{\lambda,\mu}$. Then the theorem follows from Proposition 6.54.(6). □

Lemma 6.56. *1. For all $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$,* $\mathrm{GK}\,(\mathbb{X}^{\lambda,\mu}) = 2$.

2. $A(C, K-\mu, X) \subsetneq A(K-\mu, X, Y, E) \subsetneq A$.

3. For all $\mu \in \mathbb{K}^*$, the module $\mathbb{X}^{0,\mu}$ is not a simple A-module.

Proof. 1. By Proposition 2.14.(3), $\mathrm{GK}\,(\mathbb{X}^{\lambda,\mu}) \leqslant \mathrm{GK}\,(\mathbb{X}^{\mu}) - 1 = 2$. If $\lambda \neq 0$ then it follows from Proposition 6.54.(1) that $\mathrm{GK}\,(\mathbb{X}^{\lambda,\mu}) = 2$. If $\lambda = 0$ then consider the subspace $V =$

$\bigoplus_{i,j\geqslant 0} \mathbb{K}F^iE^j\,\bar{1}$ of the A-module $\mathbb{X}^\mu$. By (6.54), we see that $V \cap C\mathbb{X}^\mu = 0$. Hence, the vector space V can be seen as a subspace of the A-module $\mathbb{X}^{0,\mu}$. In particular, $\text{GK}\,(\mathbb{X}^{0,\mu}) \geqslant 2$. Therefore, $\text{GK}\,(\mathbb{X}^{0,\mu}) = 2$.

2. Let $\mathfrak{a} = A(C, K-\mu, X)$ and $\mathfrak{b} = A(K-\mu, X, Y, E)$. Since $C \in \mathfrak{b}$ we have the equality $\mathfrak{b} = A(C, K-\mu, X, Y, E)$. Clearly, $\mathfrak{a} \subseteq \mathfrak{b}$. Notice that $A/\mathfrak{b} \simeq U/U(K-\mu, E)$ where $U = U_q(\mathfrak{sl}_2)$. Then $\text{GK}\,(A/\mathfrak{b}) = 1$, in particular, $\mathfrak{b} \subsetneq A$ is a proper left ideal of A. It follows from statement 1 that, $2 = \text{GK}\,(A/\mathfrak{a}) > \text{GK}\,(A/\mathfrak{b})$, hence the inclusion $\mathfrak{a} \subseteq \mathfrak{b}$ is strict.

3. By statement 2, the left ideal $A(C, K-\mu, X)$ is not a maximal left ideal. Thus, the A-module $\mathbb{X}^{0,\mu}$ is not a simple module. □

Corollary 6.57. *Let $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$. The A-module $\mathbb{X}^{\lambda,\mu}$ is a simple module iff $\lambda \neq 0$.*

Proof. The result follows from Proposition 6.54.(1) and Lemma 6.56.(3). □

Description of the set $\mathcal{N}\,(Y$-torsion). An explicit description of the set $\mathcal{N}\,(Y$-torsion) is given in Theorem 6.59. It consists of a family of simple modules constructed below (see Proposition 6.58). The results and arguments are similar to that of the case for X-torsion modules. But for completeness, we present the results and their proof in detail. Let $\mu \in \mathbb{K}^*$, we define the left A-module $\mathbb{Y}^\mu := A/A(K-\mu,\, Y)$. Then $\mathbb{Y}^\mu = \bigoplus_{i,j,k\geqslant 0} \mathbb{K}E^iF^jX^k\bar{1}$ where $\bar{1} = 1 + A(K-\mu,\, Y)$. It is obvious that $\text{GK}\,(\mathbb{Y}^\mu) = 3$. Let $\lambda \in \mathbb{K}$. By (6.12), we have $(C-\lambda)\,\bar{1} = (q^2FX^2 - \lambda)\,\bar{1}$. Then using Lemma 6.10, we see that the submodule of $\mathbb{Y}^\mu$,

$$\begin{aligned}(C-\lambda)\,\mathbb{Y}^\mu = \bigoplus_{i,j,k\geqslant 0} \mathbb{K}E^iF^jX^k(C-\lambda)\,\bar{1} &= \bigoplus_{i,j,k\geqslant 0} \mathbb{K}E^iF^jX^k\big(q^2FX^2-\lambda\big)\,\bar{1}\\ &= \bigoplus_{i,j,k\geqslant 0} \mathbb{K}E^iF^j\big(q^2FX^{k+2}-\lambda X^k\big)\,\bar{1}.\end{aligned} \tag{6.57}$$

Therefore, the submodule $(C-\lambda)\mathbb{Y}^\mu$ of $\mathbb{Y}^\mu$ is a proper submodule, and the map $(C-\lambda)\cdot : \mathbb{Y}^\mu \to \mathbb{Y}^\mu,\ v \mapsto (C-\lambda)v$, is an injection, which is not a bijection.

For $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, we define the left A-module $\mathbb{Y}^{\lambda,\mu} := A/A(C-\lambda,\, K-\mu,\, Y)$. Then

$$\mathbb{Y}^{\lambda,\mu} \simeq \mathbb{Y}^\mu/(C-\lambda)\mathbb{Y}^\mu \neq 0. \tag{6.58}$$

We have a short exact sequence of A-modules: $0 \longrightarrow \mathbb{Y}^\mu \xrightarrow{(C-\lambda)\cdot} \mathbb{Y}^\mu \longrightarrow \mathbb{Y}^{\lambda,\mu} \longrightarrow 0$. The next proposition shows that the module $\mathbb{Y}^{\lambda,\mu}$ is a simple module if λ is nonzero. Moreover, the $\mathbb{K}$-basis, the weight space decomposition and the annihilator of the module $\mathbb{Y}^{\lambda,\mu}$ are given.

Proposition 6.58. *For λ and $\mu \in \mathbb{K}^*$, consider the left A-module $\mathbb{Y}^{\lambda,\mu} = A/A(C-\lambda,\, K-\mu,\, Y)$.*

1. *The A-module $\mathbb{Y}^{\lambda,\mu} = \bigoplus_{i\geqslant 0, j\geqslant 2} \mathbb{K}E^iX^j\,\bar{1} \oplus \bigoplus_{i,k\geqslant 0} \mathbb{K}E^iF^k\,\bar{1} \oplus \bigoplus_{i,k\geqslant 0} \mathbb{K}E^iF^kX\,\bar{1}$ is a simple A-module where $\bar{1} = 1 + A(C-\lambda,\, K-\mu,\, Y)$.*
2. $$\begin{aligned}\mathbb{Y}^{\lambda,\mu} = \bigoplus_{i\geqslant 0, j\geqslant 2} \mathbb{K}E^iX^j\,\bar{1} &\oplus \Big(\bigoplus_{i\geqslant 1, k\geqslant 0} \mathbb{K}\Theta^kE^i\bar{1} \oplus \bigoplus_{k\geqslant 0}\mathbb{K}\Theta^k\,\bar{1} \oplus \bigoplus_{i\geqslant 1, k\geqslant 0} \mathbb{K}\Theta^kF^i\,\bar{1}\Big)\\ &\oplus \Big(\bigoplus_{i\geqslant 1, k\geqslant 0} \mathbb{K}\Theta^kE^iX\bar{1} \oplus \bigoplus_{k\geqslant 0}\mathbb{K}\Theta^kX\,\bar{1} \oplus \bigoplus_{i\geqslant 1, k\geqslant 0} \mathbb{K}\Theta^kF^iX\,\bar{1}\Big).\end{aligned}$$

3. *The weight subspace* $\mathbb{Y}^{\lambda,\mu}_{q^s\mu}$ *of* $\mathbb{Y}^{\lambda,\mu}$ *that corresponds to the weight* $q^s\mu$ *is*

$$\mathbb{Y}^{\lambda,\mu}_{q^s\mu} = \begin{cases} \mathbb{K}[\Theta]\,\bar{1}, & s=0,\\ \mathbb{K}[\Theta]E^r\,\bar{1} \oplus \bigoplus_{\substack{i+j=r,\\ j\geqslant 1}} \mathbb{K}E^iX^{2j}\,\bar{1}, & s=2r,\ r\geqslant 1,\\ \mathbb{K}[\Theta]X\,\bar{1}, & s=1,\\ \mathbb{K}[\Theta]E^{2r}X\,\bar{1} \oplus \bigoplus_{\substack{2i+j=2r+1,\\ j\geqslant 2}} \mathbb{K}E^iX^j\,\bar{1}, & s=2r+1,\ r\geqslant 1,\\ \mathbb{K}[\Theta]F^r\,\bar{1}, & s=-2r,\ r\geqslant 1,\\ \mathbb{K}[\Theta]F^rX\,\bar{1}, & s=-2r+1,\ r\geqslant 1. \end{cases}$$

4. $\mathrm{ann}_A(\mathbb{Y}^{\lambda,\mu}) = (C-\lambda)$.
5. $\mathbb{Y}^{\lambda,\mu}$ *is a* Y*-torsion and* X*-torsionfree* A*-module.*
6. *Let* $(\lambda,\mu), (\lambda',\mu') \in \mathbb{K}\times\mathbb{K}^*$. *Then* $\mathbb{Y}^{\lambda,\mu} \simeq \mathbb{Y}^{\lambda',\mu'}$ *iff* $\lambda=\lambda'$ *and* $\mu = q^i\mu'$ *for some* $i\in\mathbb{Z}$.

Proof. 1. Notice that $\mathbb{Y}^{\lambda,\mu} = \sum_{i,j,k\geqslant 0}\mathbb{K}E^iF^jX^k\,\bar{1}$. By (6.12), we have $\lambda\,\bar{1} = C\,\bar{1} = q^2FX^2\,\bar{1}$, i.e., $FX^2\,\bar{1} = q^{-2}\lambda\,\bar{1}$. By induction on k and using Lemma 6.10.(1), we deduce that

$$F^kX^{2k}\,\bar{1} = (FX^2)^k\,\bar{1} = q^{-2k}\lambda^k\,\bar{1}. \tag{6.59}$$

Therefore, $\sum_{j,k\geqslant 0}\mathbb{K}F^jX^k\,\bar{1} = \mathbb{K}[X]X^2\,\bar{1} + \mathbb{K}[F]\,\bar{1} + \mathbb{K}[F]X\,\bar{1}$, and so

$$\mathbb{Y}^{\lambda,\mu} = \sum_{i\geqslant 0, j\geqslant 2}\mathbb{K}E^iX^j\bar{1} + \sum_{i,k\geqslant 0}\mathbb{K}E^iF^k\,\bar{1} + \sum_{i,k\geqslant 0}\mathbb{K}E^iF^kX\,\bar{1}.$$

So, any element u of $\mathbb{Y}^{\lambda,\mu}$ can be written as $u = \sum_{i=0}^n E^ia_i\bar{1}$ where $a_i \in \Gamma := \mathbb{K}[X]X^2 + \mathbb{K}[F] + \mathbb{K}[F]X$. Statement 1 follows from the following claim: if $a_n \neq 0$, then there exists an element $a\in A$ such that $au = \bar{1}$.

(i) $Y^nu = a'\,\bar{1}$ for some nonzero element $a'\in\Gamma$: By Lemma 5.2, we have $Yu = \sum_{i=0}^{n-1}E^ib_i$ for some $b_i\in\Gamma$ and $b_{n-1}\neq 0$. Repeating this step $n-1$ times, we obtain the result as desired. So, we may assume that $u = a'\,\bar{1}$ for some nonzero $a'\in\Gamma$.

(ii) Notice that the element a' can be written as $a' = pX^2 + \sum_{i=0}^m F^i(\lambda_i + \mu_iX)$ where $p\in\mathbb{K}[X]$, λ_i and $\mu_i\in\mathbb{K}$. By Lemma 6.10, we see that $F^iX\bar{1} = XF^i\bar{1}$. Then

$$\begin{aligned} X^{2m}u &= (pX^{2m+2} + \sum_{i=0}^m(\lambda_i+\mu_iX)X^{2m}F^i)\,\bar{1}\\ &= (pX^{2m+2} + \sum_{i=0}^m(\lambda_i+\mu_iX)X^{2(m-i)}X^{2i}F^i)\,\bar{1}\\ &= (pX^{2m+2} + \sum_{i=0}^m(\lambda_i+\mu_iX)X^{2(m-i)}\gamma_i)\,\bar{1} = f\,\bar{1} \end{aligned}$$

for some $\gamma_i\in\mathbb{K}^*$ (by (6.59)) and f is a nonzero element in $\mathbb{K}[Y]$. Hence, we may assume that $u = f\,\bar{1}$ where $f\in\mathbb{K}[X]\setminus\{0\}$.

(iii) Let $f = \sum_{i=0}^l \alpha_iX^i$ where $\alpha_i\in\mathbb{K}$ and $\alpha_l\neq 0$. Since $KX^i\bar{1} = q^i\mu X^i\bar{1}$ and all eigenvalues $\{q^i\mu \mid i\in\mathbb{N}\}$ are distinct, there is a polynomial $g\in\mathbb{K}[K]$ such that $gf\,\bar{1} = X^l\,\bar{1}$. If $l=0$, we are done. We may assume that $l \geqslant 1$. By multiplying by X (if necessary) on the equality we may assume that $l = 2k$ for some natural number k. Then, by (6.59), we have $q^{2k}\lambda^{-k}F^kX^{2k}\,\bar{1} = \bar{1}$, as required.

2. Recall that $U_q(\mathfrak{sl}_2)$ is a generalized Weyl algebra, then $E^iF^i = \sigma^i(a)\sigma^{i-1}(a)\cdots\sigma(a)$ holds for all $i \geqslant 1$. Hence,

$$\bigoplus_{i,k\geqslant 0} \mathbb{K}F^iE^k\,\bar{1} = \bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}\Theta^kE^i\,\bar{1} \oplus \bigoplus_{k\geqslant 0}\mathbb{K}\Theta^k\,\bar{1} \oplus \bigoplus_{i\geqslant 1,k\geqslant 0} \mathbb{K}\Theta^kF^i\,\bar{1}.$$

Then statement 2 follows from statement 1.

3. Statement 3 follows from statement 2.

4. Clearly, $(C-\lambda) \subseteq \mathrm{ann}_A(\mathbb{Y}^{\lambda,\mu})$. Then we must have $(C-\lambda) = \mathrm{ann}_A(\mathbb{X}^{\lambda,\mu})$ since $(C-\lambda)$ is a maximal ideal of A.

5. Clearly, $\mathbb{Y}^{\lambda,\mu}$ is Y-torsion. Since $\mathbb{Y}^{\lambda,\mu}$ is a simple module, then by Lemma 6.51, $\mathbb{Y}^{\lambda,\mu}$ is X-torsionfree.

6. ($\Rightarrow$) Suppose that $\mathbb{Y}^{\lambda,\mu} \simeq \mathbb{Y}^{\lambda',\mu'}$. By statement 4, $(C-\lambda) = \mathrm{ann}_A(\mathbb{Y}^{\lambda,\mu}) = \mathrm{ann}_A(\mathbb{Y}^{\lambda',\mu'}) = (C-\lambda')$. Hence, $\lambda = \lambda'$. By Theorem 6.50 (or by statement 3), $\{q^i\mu \,|\, i \in \mathbb{Z}\} = \mathrm{Wt}(\mathbb{Y}^{\lambda,\mu}) = \mathrm{Wt}(\mathbb{Y}^{\lambda',\mu'}) = \{q^i\mu' \,|\, i \in \mathbb{Z}\}$. Hence, $\mu = q^i\mu'$ for some $i \in \mathbb{Z}$.

($\Leftarrow$) Suppose that $\lambda = \lambda'$ and $\mu = q^i\mu'$ for some $i \in \mathbb{Z}$. Let $\bar{1}$ and $\bar{1}'$ be the canonical generators of the modules $\mathbb{Y}^{\lambda,\mu}$ and $\mathbb{Y}^{\lambda',\mu'}$, respectively. If $i \geqslant 0$ then the map $\mathbb{Y}^{\lambda,\mu} \to \mathbb{Y}^{\lambda',\mu'}$ $\bar{1} \mapsto X^i\,\bar{1}'$ defines an isomorphism of A-modules. If $i \leqslant -1$ then the map $\mathbb{Y}^{\lambda,\mu} \to \mathbb{Y}^{\lambda',\mu'}$, $\bar{1} \mapsto (FX)^i\,\bar{1}'$ defines an isomorphism of A-modules. □

Theorem 6.59. $\mathcal{N}(Y\text{-torsion}) = \{[\mathbb{Y}^{\lambda,\mu_{\mathcal{O}}}] \,|\, \lambda \in \mathbb{K}^*,\ \mathcal{O} \in \mathbb{K}^*/\langle q\rangle\}$.

Proof. Let $M \in \mathcal{N}(Y\text{-torsion})$. By Lemma 6.52, the central element C acts on M as a nonzero scalar, say λ. Then M is an epimorphic image of the module $\mathbb{Y}^{\lambda,\mu}$ for some $\mu \in \mathbb{K}^*$. By Proposition 6.58.(1), $\mathbb{Y}^{\lambda,\mu}$ is a simple A-module, hence $M \simeq \mathbb{Y}^{\lambda,\mu}$. Then the theorem follows from Proposition 6.58.(6). □

Lemma 6.60. *1. For all $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$, $\mathrm{GK}\,(\mathbb{Y}^{\lambda,\mu}) = 2$.*

2. $A(C, K-\mu, Y) \subsetneq A(K-\mu, X, Y, E) \subsetneq A$.

3. For all $\mu \in \mathbb{K}^*$, the module $\mathbb{Y}^{0,\mu}$ is not a simple A-module.

Proof. 1. By Proposition 2.14.(3), $\mathrm{GK}\,(\mathbb{Y}^{\lambda,\mu}) \leqslant \mathrm{GK}\,(\mathbb{Y}^{\mu}) - 1 = 2$. If $\lambda \neq 0$ then it follows from Proposition 6.58.(1) that $\mathrm{GK}\,(\mathbb{Y}^{\lambda,\mu}) = 2$. If $\lambda = 0$ then consider the subspace $V = \bigoplus_{i,j\geqslant 0}\mathbb{K}E^iF^j\,\bar{1}$ of the A-module $\mathbb{Y}^{\mu}$. By (6.57), we see that $V \cap C\mathbb{Y}^{\mu} = 0$. Hence, the vector space V can be seen as a subspace of the A-module $\mathbb{Y}^{0,\mu}$. In particular, $\mathrm{GK}\,(\mathbb{Y}^{0,\mu}) \geqslant 2$. Therefore, $\mathrm{GK}\,(\mathbb{Y}^{0,\mu}) = 2$.

2. Let $\mathfrak{a}' = A(C, K-\mu, Y)$ and $\mathfrak{b} = A(K-\mu, X, Y, E)$. Since $C \in \mathfrak{b}$ we have the equality $\mathfrak{b} = A(C, K-\mu, X, Y, E)$. Clearly, $\mathfrak{a}' \subseteq \mathfrak{b}$. By Lemma 6.56.(2) and its proof, $\mathfrak{b}$ is a proper left

ideal of A and $\mathrm{GK}\,(A/\mathfrak{b}) = 1$. Then it follows from statement 1 that, $2 = \mathrm{GK}\,(A/\mathfrak{a}') > \mathrm{GK}\,(A/\mathfrak{b})$, hence the inclusion $\mathfrak{a}' \subseteq \mathfrak{b}$ is strict.

3. By statement 2, the left ideal $A(C, K-\mu, Y)$ is not a maximal left ideal. Thus, the A-module $\mathbb{Y}^{0,\mu}$ is not a simple module. □

Corollary 6.61. *Let $\lambda \in \mathbb{K}$ and $\mu \in \mathbb{K}^*$. The A-module $\mathbb{Y}^{\lambda,\mu}$ is a simple module iff $\lambda \neq 0$.*

Proof. The result follows from Proposition 6.58.(1) and Lemma 6.60.(3). □

The set $\mathcal{N}\,((X,Y)$-torsionfree). Theorem 6.63 and Theorem 6.64 give explicit description of the set $\mathcal{N}\,((X,Y)$-torsionfree). Recall that $\mathcal{N}\,((X,Y)\text{-torsionfree}) = \widehat{A}\,(\text{weight}, (X,Y)\text{-torsionfree})$. Then clearly,

$$\mathcal{N}\,((X,Y)\text{-torsionfree}) = \widehat{A(0)}\,\big(\text{weight},\ (X,Y)\text{-torsionfree}\big) \sqcup \bigsqcup_{\lambda\in\mathbb{K}^*} \widehat{A(\lambda)}\,\big(\text{weight},\ (X,Y)\text{-torsionfree}\big). \tag{6.60}$$

Let A_t be the localization of the algebra at the powers of the element $t = YX$. Recall that the algebra $\mathscr{C}_t$ is a GWA, see Proposition 6.32.(1).

Lemma 6.62. *$A_t = \mathscr{C}_t[X^{\pm 1};\iota]$ is a skew polynomial algebra where ι is the automorphism of the algebra $\mathscr{C}_t$ defined by $\iota(C) = C$, $\iota(K^{\pm 1}) = q^{\mp 1}K^{\pm 1}$, $\iota(t) = qt$, $\iota(u) = q^2u$ and $\iota(v) = v$.*

Proof. Clearly, the algebra $\mathscr{C}_t[X^{\pm 1};\iota]$ is a subalgebra of A_t. Notice that all the generators of the algebra A_t are contained in the algebra $\mathscr{C}_t[X^{\pm 1};\iota]$, then $A_t \subseteq \mathscr{C}_t[X^{\pm 1};\iota]$. Hence, $A_t = \mathscr{C}_t[X^{\pm 1};\iota]$, as required. □

The set $\widehat{A(0)}\,\big(\text{weight},\ (X,Y)\text{-torsionfree}\big)$. Let $[M] \in \widehat{\mathscr{C}^{0,\mu}}\,(t\text{-torsionfree})$. By Theorem 6.41, the element t acts *bijectively* on the module M (since t is a normal element of $\mathcal{R}$). Therefore, the $\mathscr{C}$-module M is also a $\mathscr{C}_t$-module. Then by Lemma 6.62, we have the induced A_t-module

$$\widetilde{M} := A_t \otimes_{\mathscr{C}_t} M = \bigoplus_{i\in\mathbb{Z}} X^i \otimes M = \bigoplus_{i\geqslant 1} Y^i \otimes M \oplus \bigoplus_{i\geqslant 0} X^i \otimes M.$$

Clearly, $\widetilde{M}$ is an (X,Y)-torsionfree, weight A-module and $\mathrm{Wt}\,(\widetilde{M}) = \{q^i\mu \,|\, i \in \mathbb{Z}\} = \mathcal{O}(\mu)$. We claim that $\widetilde{M}$ *is a simple A-module*. Suppose that N is a nonzero A-submodule of $\widetilde{M}$ then $X^i \otimes m \in N$ for some $i \in \mathbb{Z}$ and $m \in M$. If $i = 0$ then $N = Am = \widetilde{M}$. If $i \geqslant 1$, since $Y^i(X^i \otimes m) \in \mathbb{K}^*(1 \otimes t^i m)$, then $1 \otimes tm \in N$ and so $N = \widetilde{M}$. If $i \leqslant -1$ then $X^{|i|}X^i \otimes m = 1 \otimes m \in N$, so $N = \widetilde{M}$. If $M' \in \widehat{\mathscr{C}^{0,\mu'}}\,(t\text{-torsionfree})$ then the A-modules $\widetilde{M}$ and $\widetilde{M}'$ are isomorphic iff the $\mathscr{C}^{0,\mu}$-modules M and $X^i \otimes M'$ are isomorphic where $\mu = q^i\mu'$ for a unique $i \in \mathbb{Z}$.

Theorem 6.63. $\widehat{A(0)}\,\big(\text{weight},\ (X,Y)\text{-torsionfree}\big) = \big\{[\widetilde{M}] \,|\, [M] \in \widehat{\mathscr{C}^{0,\mu_{\mathcal{O}}}}\,(t\text{-torsionfree}),\ \mathcal{O} \in \mathbb{K}^*/q^{\mathbb{Z}}\big\}.$

Proof. Let $V \in \widehat{A(0)}\,\big(\text{weight},\ (X,Y)\text{-torsionfree}\big)$. Then the elements X and Y act injectively on the module V. For any $\mu \in \mathrm{Wt}\,(V)$, the weight space V_μ is a simple t-torsionfree $\mathscr{C}^{0,\mu}$-module. Then $V \supseteq \bigoplus_{i\geqslant 1} Y^i \otimes V_\mu \oplus \bigoplus_{i\geqslant 0} X^i \otimes V_\mu = \widetilde{V_\mu}$. Hence, $V = \widetilde{V_\mu}$ since V is a simple module. □

The set $\widehat{A(\lambda)}$ (weight, (X,Y)-torsionfree) **where** $\lambda \in \mathbb{K}^*$**.** Let $M \in \widehat{\mathscr{C}^{\lambda,\mu}}$ (t-torsionfree). Then $M_t \in \widehat{\mathscr{C}_t^{\lambda,\mu}}$. By Lemma 6.62, we have the induced A_t-module

$$M^{\blacklozenge} := A_t \otimes_{\mathscr{C}_t} M_t = \bigoplus_{i\in\mathbb{Z}} X^i \otimes M_t.$$

Clearly, $M^{\blacklozenge}$ is a simple weight A_t-module and $\mathrm{Wt}\,(M^{\blacklozenge}) = \{q^i\mu \,|\, i \in \mathbb{Z}\} = \mathcal{O}(\mu)$. For all $i \in \mathbb{Z}$, the weight space $M_i^{\blacklozenge} := X^i \otimes M_t \simeq M_t^{\iota^{-i}}$ as $\mathscr{C}_t$-modules where $M_t^{\iota^{-i}}$ is the $\mathscr{C}_t$-module twisted by the automorphism ι^{-i} of the algebra $\mathscr{C}_t$ (the automorphism ι is defined in Lemma 6.62). The set $\widehat{\mathscr{C}^{\lambda,\mu}}$ (t-torsionfree) is described explicitly in Theorem 6.45.(1,3). If $M = \mathfrak{f}^{\lambda,\mu}$ then $X^i \otimes \mathfrak{f}_t^{\lambda,\mu} \simeq (\mathfrak{f}_t^{\lambda,\mu})^{\iota^{-i}} \simeq \mathfrak{f}_t^{\lambda,q^i\mu}$ as $\mathscr{C}_t$-modules. It is clear that $\mathrm{soc}_{\mathscr{C}}(\mathfrak{f}_t^{\lambda,\mu}) = \mathfrak{f}^{\lambda,\mu}$. Hence, $\mathrm{soc}_{\mathscr{C}}(X^i \otimes \mathfrak{f}_t^{\lambda,\mu}) = \mathrm{soc}_{\mathscr{C}}(\mathfrak{f}_t^{\lambda,q^i\mu}) = \mathfrak{f}^{\lambda,q^i\mu}$. Then the A-module

$$\mathrm{soc}_A\Big(\big(\mathfrak{f}^{\lambda,\mu}\big)^{\blacklozenge}\Big) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{\mathscr{C}}(X^i \otimes \mathfrak{f}_t^{\lambda,\mu}) \simeq \bigoplus_{i\in\mathbb{Z}} \mathfrak{f}^{\lambda,q^i\mu}. \tag{6.61}$$

Similarly, if $M = \mathsf{F}^{\lambda,\mu}$ then $X^i \otimes \mathsf{F}_t^{\lambda,\mu} \simeq (\mathsf{F}_t^{\lambda,\mu})^{\iota^{-i}} \simeq \mathsf{F}_t^{\lambda,q^i\mu}$ as $\mathscr{C}_t$-modules. It is clear that $\mathrm{soc}_{\mathscr{C}}(\mathsf{F}_t^{\lambda,\mu}) = \mathsf{F}^{\lambda,\mu}$. Hence, $\mathrm{soc}_{\mathscr{C}}(X^i \otimes \mathsf{F}_t^{\lambda,\mu}) = \mathrm{soc}_{\mathscr{C}}(\mathsf{F}_t^{\lambda,q^i\mu}) = \mathsf{F}^{\lambda,q^i\mu}$. Then the A-module

$$\mathrm{soc}_A\Big(\big(\mathsf{F}^{\lambda,\mu}\big)^{\blacklozenge}\Big) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{\mathscr{C}}(X^i \otimes \mathsf{F}_t^{\lambda,\mu}) \simeq \bigoplus_{i\in\mathbb{Z}} \mathsf{F}^{\lambda,q^i\mu}. \tag{6.62}$$

If $M = \mathcal{F}_\gamma^{\lambda,\mu}$ where $\gamma \in \mathbb{K}^* \setminus \{q^{2i}\nu \,|\, i \in \mathbb{Z}\}$. Then $X^i \otimes \mathcal{F}_{\gamma,t}^{\lambda,\mu} \simeq (\mathcal{F}_{\gamma,t}^{\lambda,\mu})^{\iota^{-i}} \simeq \mathcal{F}_{q^{-i}\gamma,t}^{\lambda,q^i\mu}$ as $\mathscr{C}_t$-modules. It is cleat that $\mathrm{soc}_{\mathscr{C}}(\mathcal{F}_{\gamma,t}^{\lambda,\mu}) = \mathcal{F}_\gamma^{\lambda,\mu}$. Hence, $\mathrm{soc}_{\mathscr{C}}(X^i \otimes \mathcal{F}_{\gamma,t}^{\lambda,\mu}) = \mathcal{F}_{q^{-i}\gamma}^{\lambda,q^i\mu}$ is a simple $\mathscr{C}$-module. Then the A-module

$$\mathrm{soc}_A\Big(\big(\mathcal{F}_\gamma^{\lambda,\mu}\big)^{\blacklozenge}\Big) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{\mathscr{C}}(X^i \otimes \mathcal{F}_{\gamma,t}^{\lambda,\mu}) \simeq \bigoplus_{i\in\mathbb{Z}} \mathcal{F}_{q^{-i}\gamma}^{\lambda,q^i\mu}. \tag{6.63}$$

If $M \in \widehat{\mathscr{C}^{\lambda,\mu}}$ ($\mathbb{K}[t]$-torsionfree) then , by Theorem 6.45.(3), $M \simeq \mathscr{C}^{\lambda,\mu}/\mathscr{C}^{\lambda,\mu} \cap \mathscr{B}bt^{-n}$ for some l-normal element $b \in \mathrm{Irr}\,(\mathscr{B})$ and for all $n \gg 0$. For all $i \in \mathbb{Z}$,

$$M_t^{\iota^{-i}} \supseteq \frac{\mathscr{C}_t^{\lambda,q^i\mu}}{\mathscr{C}_t^{\lambda,q^i\mu} \cap \mathscr{B}\iota^i(b)t^{-n}} := \mathcal{M}_{\iota^i(b)t^{-n}}.$$

Then $\mathrm{soc}_{\mathscr{C}}(M_t^{\iota^{-i}}) = \mathrm{soc}_{\mathscr{C}}(\mathcal{M}_{\iota^i(b)t^{-n}}) = \mathcal{M}_{\iota^i(b)t^{-n_i}}$ for all $n_i \gg 0$. Then the A-module

$$\mathrm{soc}_A(M^{\blacklozenge}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{soc}_{\mathscr{C}}(X^i \otimes M_t) \simeq \bigoplus_{i\in\mathbb{Z}} \mathcal{M}_{\iota^i(b)t^{-n_i}}. \tag{6.64}$$

The next theorem describes the set $\widehat{A(\lambda)}$ (weight, (X,Y)-torsionfree) where $\lambda \in \mathbb{K}^*$.

Theorem 6.64. *Let* $\lambda, \mu \in \mathbb{K}^*$. *Then* $\widehat{A(\lambda)}$ (weight, (X,Y)-torsionfree) $= \{[\mathrm{soc}_A(M^{\blacklozenge})] \,|\, [M] \in \widehat{\mathscr{C}^{\lambda,\mu_{\mathcal{O}}}}$ (t-torsionfree), $\mathcal{O} \in \mathbb{K}^*/q^{\mathbb{Z}}\}$ *and* $\mathrm{soc}_A(M^{\blacklozenge})$ *is explicitly described in (6.61), (6.62), (6.63) and (6.64).*

Proof. Let $\mathcal{M} \in \widehat{A(\lambda)}$ (weight, (X,Y)-torsionfree). Then $\mathrm{Wt}\,(\mathcal{M}) = \mathcal{O}(\mu) \in \mathbb{K}^*/q^{\mathbb{Z}}$ for any $\mu \in \mathrm{Wt}\,(\mathcal{M})$. Then $M := \mathcal{M}_\mu \in \widehat{\mathscr{C}^{\lambda,\mu_{\mathcal{O}}}}$ (t-torsionfree) and $M_t \in \widehat{\mathscr{C}_t^{\lambda,\mu_{\mathcal{O}}}}$. Clearly, $M^{\blacklozenge} = \mathcal{M}_t \supseteq \mathcal{M}$. So, $\mathcal{M} = \mathrm{soc}_A(M^{\blacklozenge})$. □

By (6.51) and (6.60), Theorem 6.55, Theorem 6.59, Theorem 6.63 and Theorem 6.64 give a complete classification of simple weight A-modules.

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Notations

We use the standard notations $\mathbb{Z}$ for the ring of integers, $\mathbb{N}$ for the natural numbers. Throughout the thesis $\mathbb{K}$ is a field of characteristic zero.