Logical Representations for Automated Reasoning about Spatial Relationships

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The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

Abstract

This thesis investigates logical representations for describing and reasoning about spatial situations. Previously proposed theories of spatial regions are investigated in some detail — especially the 1st-order theory of Randell, Cui and Cohn (1992). The difficulty of achieving effective automated reasoning with these systems is observed.

A new approach is presented, based on encoding spatial relations in formulae of 0-order ('propositional') logics. It is proved that entailment, which is valid according to the standard semantics for these logics, is also valid with respect to the spatial interpretation. Consequently, well-known mechanisms for propositional reasoning can be applied to spatial reasoning. Specific encodings of topological relations into both the modal logic S4 and the intuitionistic propositional calculus are given. The complexity of reasoning using the intuitionistic representation is examined and a procedure is presented which is shown to be of $O(n^3)$ complexity in the number of relations involved.

In order to make this kind of representation sufficiently expressive the concepts of *model constraint* and *entailment constraint* are introduced. By means of this distinction a 0-order formula may be used either to assert or to deny that a certain spatial constraint holds of some situation. It is shown how the proof theory of a 0-order logical language can be extended by a simple meta-level generalisation to accommodate a representation involving these two types of formula.

A number of other topics are dealt with: a decision procedure based on quantifier elimination is given for a large class of formulae within a 1st-order topological language; reasoning mechanisms based on the *composition* of spatial relations are studied; the non-topological property of *convexity* is examined both from the point of view of its 1st-order characterisation and its incorporation into a 0-order spatial logic. It is suggested that 0-order representations could be employed in a similar manner to encode other spatial concepts.

There is no branch of mathematics, however abstract, that will not eventually be applied to the phenomena of the real world.

—Lobachevsky, quoted in the American Mathematical Monthly, Feb 1984.

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Chapter 1

Introduction

Although spatial relationships pervade our comprehension of the world, we are almost completely unaware of how we manipulate spatial information. Our familiarity with spatial properties and arrangements of everyday objects makes the logical connections between different spatial relationships so transparent that it is extremely difficult to apprehend and make explicit the structure of this conceptual framework.

Spatial reasoning has a key role to play in a wide variety of computer applications. For example, it is of crucial importance in the following areas:

- geographical information systems (GIS)
- robot control
- computer aided design and manufacturing (CAD/CAM)
- virtual world modelling and animation
- medical analysis and diagnosis systems

In current computer systems representation of spatial information is based almost entirely on numerical coordinates and parameters. However, to specify the behaviour of the system a programmer will often need to evaluate high-level, *qualitative* relationships holding between data objects (e.g. to test whether one region *overlaps* another). Such information can be extracted when needed from numerical data-structures by special purpose algorithms. Writing such algorithms may often be quite straightforward for a competent programmer, but as large systems are developed problems are likely to emerge. There is potentially infinite variety in the form that spatial data can take, so a large number of similar algorithms operating on slightly different types of data must be written.[1] More seriously, the heterogeneity of data objects means that apparently equivalent properties of different data-types may diverge in extreme cases and this can lead to coding errors which are difficult to identify.

The primary cause of these problems is that current programming systems provide no general

[1] The *object-oriented* paradigm of computer programming can help overcome this problem but only if great care is taken in establishing a hierarchical organisation of data-types — even then it may be difficult to integrate new, unforeseen data-types neatly into an existing structure.

framework for manipulating high-level qualitative spatial information. In order to test whether a particular qualitative relationship holds (e.g. 'the sensor is in contact with the block') the programmer must first know about the details of how objects and their locations are represented and then formulate some test involving values contained in the relevant data-structures. This test will generally take the form of an equation or inequality (or perhaps some Boolean combination of equations and/or inequalities). Such tests determine qualitative (spatial) relationships according to the intended interpretation of data structures in the database. If a programmer could directly employ qualitative spatial vocabulary in place of complex test operations, many coding tasks would be greatly simplified; but providing such a facility is very far from straightforward. It requires the formulation of an adequate theory of spatial relations together with an effective means of computing inferences according to the theory.

In addition to its application in the context of well established kinds of computer system, spatial reasoning is of crucial importance to the field of Artificial Intelligence (AI). In attempting to construct computer programs that simulate 'intelligent' behaviour, many researchers have concluded that, as well as needing general purpose reasoning mechanisms, such systems must possess a large amount of background knowledge and, furthermore, in order to draw consequences from this information, detailed theories characterising the logical properties of the concepts and relations involved will be required. Spatial relations clearly form an extremely important conceptual domain — they are involved in a very high proportion of facts about the real world. Hence, in the development of this (logicist) approach to AI, theories of spatial relations will play a central role.

AI research into spatial reasoning is at the present largely dissociated from related branches of mathematics — geometry, topology and logic. This is partly because mathematical formalisms in these areas do not naturally lend themselves to effective automated computation of inferences. Another factor is the difficulty of assimilating these highly developed and complex disciplines into the relatively young and, as yet, rather fragmented field of AI. From the standpoint of AI, spatial reasoning is often seen as closely associated with the cognitive processing capabilities of humans and other animals. Mathematical theories on the other hand give a very abstract characterisation of reasoning, which is independent from biological or psychological processes. However, considerations of the *cognitive plausibility* of representations and algorithms employed in a computer program to provide reasoning capabilities are closely connected to considerations of the *computational complexity* of formal deductive systems.

In this thesis I shall adopt a mathematical view of the problem. However we shall see that certain conceptual frameworks which were in fact motivated by arguments of cognitive plausibility do lead to formal systems which are computationally manageable. Thus for example the idea of taking certain sets of relations as being of special significance in the classification of spatial situations and of taking the *composition*[2] of two relations as a primary mode of deduction appears to be both cognitively plausible and to lead to formal systems in which many useful inferences can

[2] Given two relations R_1 and R_2, their composition, '$R_1; R_2$' is the strongest relation such that for any three objects, a, b, c, if $R_1(a, b)$ and $R_2(b, c)$ hold, then $R_3(a, c)$ must hold. The nature and significance of relational composition will be studied in chapter 9.

be computed effectively (see for example (Freksa 1992b) and (Hernández 1994)).

The rest of this introductory chapter will be structured as follows: First I motivate the enterprise of automating spatial reasoning by exhibiting some of the more significant of the wide variety of spatial concepts and suggesting reasoning tasks and applications for which these concepts are significant. I then give a brief history of spatial reasoning in which I outline the major approaches to the subject that have been developed by mathematicians and philosophers. This is followed by a consideration of the relationships between different conceptual frameworks and formal systems for representing and reasoning about space. I then survey work on spatial reasoning in computer science, particularly from the perspective of AI. We shall see that (in addition to problems of adequate formal representation) automated reasoning about spatial information faces considerable problems of computational complexity. Finally I give a brief overview of each of the subsequent chapters of the thesis.

1.1 The Domain of Spatial Reasoning

1.1.1 Spatial Concepts and Information

Spatial information is presented to us by means of two very different modes: sensory perception and linguistic description. We acquire knowledge of spatial relationships either by some (more or less unconscious) processing or transformation of states produced in our sensory organs in response to bombardment by particles from the outside world, or by being told (or reading) about the spatial arrangement of parts of the world. The former, *sensory*, kind of information has been intensively studied by researchers in *computer vision* and *robotics* with some success; but it is the latter, *propositional* form of spatial information that will be the concern of this thesis. I shall pursue representations which can express information such as is contained in the following English sentences:

- Yorkshire is part of England.
- The hip bone is connected to the thigh bone.
- The fly is in the bottle.

I shall not, however, be concerned with the particular ways in which a natural language expresses spatial information but with precisely specified formal representations with definite rules of logical inference. Nevertheless, it will be seen that these formal expressions can be interpreted in terms of certain natural language expressions and, moreover, that logically valid deductions correspond to arguments which are intuitively sound under this interpretation.

1.1.2 Geometry of Points and Lines and its Primitive Concepts

The geometry of points and lines is the most ancient branch of spatial reasoning. Here the abstract dimensionless point is the basic element and all other spatial entities must be constructed out of points. One of the oldest theories of this mode of geometry is that of Euclid, whose axiomatic system is still used today.

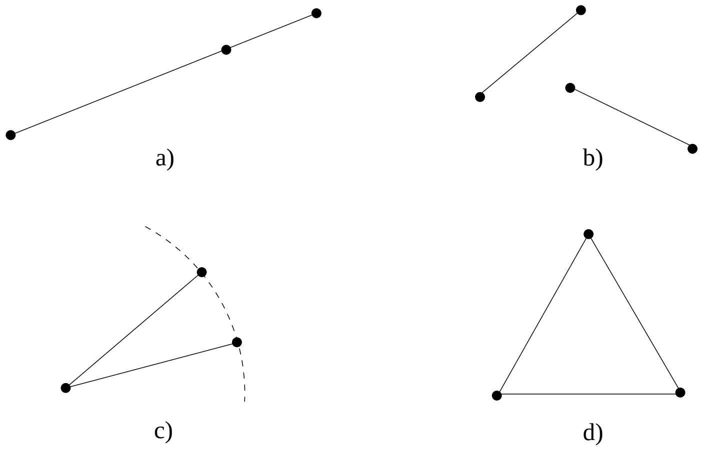

Figure 1.1: Some significant relations among points

Figure 1.1 presents four diagrams showing simple but very significant relationships which can hold between points. The figures are of course two dimensional but analogous relations can hold in 3 (or more) dimensions. Diagram a) shows the *betweenness* relation holding among three points (if order is disregarded one has the *collinearity* relation). b) depicts the equidistance of two pairs of points. Betweenness and equidistance are the two primitives in Tarski's (1959) formalisation of elementary geometry (see appendix A). c) shows the relation of equidistance of two points from a third (a situation easily constructed on paper using a compass). In fact, in Euclidean geometry both the relations a) and b) (and hence all relations of elementary Euclidean geometry) can be defined in terms of relation c). The ternary relation of equilaterality, d), can also serve as the sole primitive for Euclidean geometry of three or more dimensions (Tarski and Beth 1956).

If a coordinate frame and metric are specified for a Euclidean space algebraic methods can be applied to geometrical problems. Points, lines and surfaces are then represented by means of equations and inequalities relating the coordinates of points. This *analytic* geometry is the most widely used representation for spatial information; it forms the basis of almost all spatial representation and reasoning mechanisms employed in current computer systems.

1.1.3 Topology

Topology may be regarded as a sub-field of geometry but it is far more abstract than the geometry of points and lines. The topological properties of a spatial object are those that do not vary depending on scale or orientation. A good illustration of such invariance is provided by considering a drawing on a rubber sheet: the topological properties of the drawing are those which are preserved while the sheet is arbitrarily stretched and deformed.[3]

Figure 1.2 illustrates 8 particularly significant topological relations which can hold between two regions (although the diagram shows 2D regions, analogues of these relations apply to 1, 3 or higher

[3]By virtue of the very abstract way in which the theory of topology has been developed, 'topological' concepts have also been applied to areas of mathematics which are very far removed from this rubber-sheet interpretation.

dimensional regions). All of these relations are definable in the RCC (for Randell, Cohn and Cui — or alternatively Region Connection Calculus) theory of spatial regions (Randell, Cui and Cohn 1992) which will be investigated in detail in chapter 2. Essentially, the same set of relations has been independently identified as significant in the context of Geographical Information Systems (Egenhofer and Franzosa 1991, Egenhofer 1991, Clementini, Sharma and Egenhofer 1994).

The 8 relations form a *jointly exhaustive and pairwise disjoint* (JEPD) set, which means that any two regions stand to each other in exactly one of these relations. (JEPD sets are important in the *composition*-based approach to reasoning about binary relations, which will be explored in chapter 9.) This classification can be refined to introduce additional distinctions between relations. For instance amongst pairs of EC (externally connected) regions we could distinguish those connected at a boundary segment from those connected at a single point. Many such relations are also definable in the RCC theory.

Topological relationships are of a very general character and can be used to give a high-level description of all manner of spatial situations. For example, useful geographical information concerning countries, provinces and counties and the relationships between them can be expressed in terms of these relations. Non-spatial information can also often be represented metaphorically in terms of topological properties — e.g. the range of application of colour terms might be described in terms of regions in a 'colour space'.

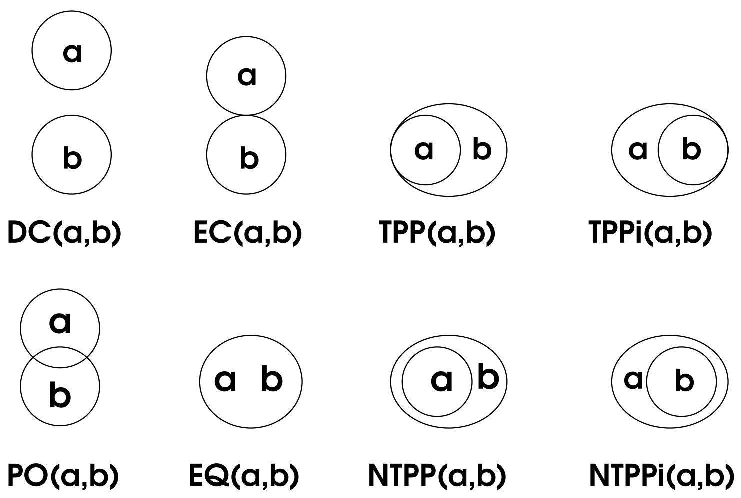

Figure 1.2: Basic relations in the RCC theory

Representation and effective automated reasoning about topological relations will be the main concern of this thesis. However, we shall see that representations and algorithms developed primarily for efficient topological reasoning can be extended to handle other aspects of spatial information. Formal characterisation of topological relationships has traditionally been carried out by axiomatising certain properties of sets of points. However, such an axiomatisation assumes a theory of sets. The resulting theory is extremely complex and consequently impractical as a basis for an automated reasoning system. An alternative approach to formalising topological notions is that of *algebraic* topology, in which the objects of the theory are n-dimensional polygons and polyhedra. This may

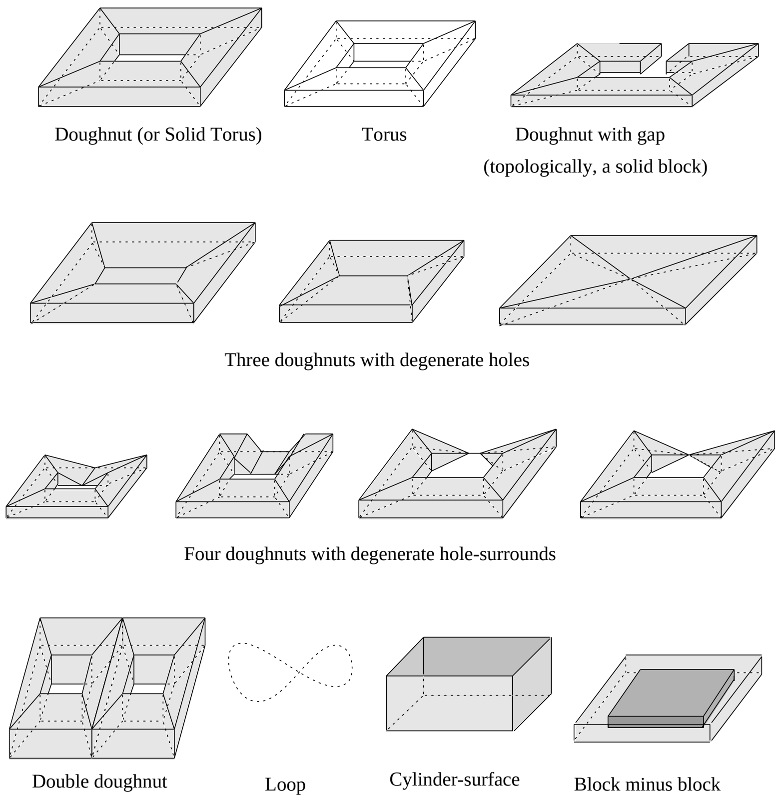

Figure 1.3: Shapes distinguished by Gotts using the RCC theory

prove to be more suitable for computational reasoning than the point-set representation but in its present form it is also far too complex. In view of the importance of topological concepts and the difficulty of carrying out automated reasoning using standard mathematical notations, much of this thesis will be concerned with the development of alternative representations for topological relationships.

1.1.4 Shape

Characterising the shape of objects or regions seems to involve a wide spectrum of spatial concepts. Although the shape of a region may be regarded as independent of its size and orientation, the relative proportions and positions of the parts of a region are essential to its shape, so size and orientation are in this way aspects of shape. In fact, if in describing any spatial situation we are only interested in distinguishing occupied regions from free space and are not concerned with the overall scale, then this can be accomplished by characterising the shape of the occupied (or free) space. Thus representing and reasoning about arbitrary shapes encompasses a very large part — if not the whole — of the domain of spatial reasoning.

Nevertheless a number of formalisms for describing shape have been developed. These can be divided into two broad classes: firstly there are the *constructive* representations in which complex shapes are described by structured combinations of primitive components; and secondly, there are approaches which might be called *constraining*, since shapes are characterised in terms of properties holding of a region and these properties are constrained to conform to some theory.

A well-known form of the constructive approach which is based on numerical/vector representations of objects is *Constructive Solid Geometry* (Requicha and Tilove 1978, Requicha 1980). More abstract examples of the approach include the many kinds of *shape grammar* that have been developed. A rather different method of shape construction is described by Leyton (1988). He specifies a *process grammar*, which generates shapes by means of a series of deformations starting from an initial disc shape. Constraining approaches to shape include those based on axiomatic theories such as the 1st-order RCC theory (Randell, Cui and Cohn 1992). Gotts (1994) has shown how many topologyically distinct 'shapes' can be distinguished in terms of this theory (see figure 1.3). Another approach to shape definition using RCC is described in (Cohn 1995).

1.1.5 Convexity and Containment

A limited but significant sub-domain of properties concerning shape comprises those concepts related to the notion of convexity: An object may be convex or may have a certain number of concavities. Even such a seemingly meagre range of distinctions can serve to discriminate between many different kinds of spatial region (Cohn 1995, Davis, Gotts and Cohn 1997).

In describing convexity-related properties it is useful not only to be able to say that a region is convex but also to be able to identify the smallest convex region which contains any given region. This is the convex-hull of the region. The (extended) RCC theory employs a convex-hull operator whose interpretation is the function from regions to their convex-hulls. In the present work I shall only be concerned with those notions of convexity and containment which are definable in terms of the convex-hull operator. Thus not only will many aspects of shape be overlooked but also the treatment of convexity will be limited.[4]

Several useful relationships concerning the 'containment' of one region within another may be defined in terms of convex-hulls. For instance, if a region a does not overlap b but is a part of the convex-hull of b, we may say that b *contains* a. This give a precise — although arguably unnatural — specification of a containment relation in terms of convex-hull together with some simple topological relations. Convexity and containment will be considered in detail in chapter 8.

1.1.6 Position and Orientation

Position and orientation are very important kinds of spatial information, which can be precisely represented by means of numerical coordinates. However, there are also a wide range of qualitative relationships involving these concepts. Figure 1.4 illustrates an analysis of qualitative orientation due to Freksa (1992b). 1.4a depicts a situation in which an observer, o, is heading towards a landmark, l, and sees a house, h, which is further away than and to the right of the landmark. Figure 1.4b is a qualitative representation of the relative position of the house with respect to the observer (at the lower intersection in the grid) and landmark (the upper intersection). 15 qualitat-

[4] A detailed examination of many subtle difficulties that arise when one tries to precisely characterise different kinds of cavity can be found in (Casati and Varzi 1994).

ively different relative locations can be distinguished by means of this representation, as indicated in 1.4c. Qualitative representations of orientation have also been investigated by Hernández (1994). Whilst position and orientation are clearly very important for many modes of spatial reasoning, further consideration of these aspects of spatial information is beyond the scope of this thesis.

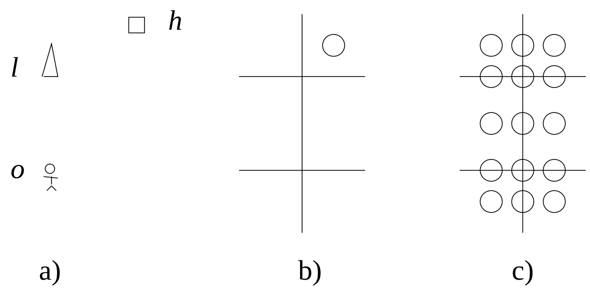

Figure 1.4: Qualitative orientation in a relative coordinate system

1.2 A Brief History of Spatial Reasoning

I shall now describe some of the more successful approaches to the characterisation of correct reasoning about spatial relationships. The ideas presented here predate or are independent from the use of electronic computers. More recent approaches to spatial reasoning taken by researchers in computer science (especially in the field of AI) will be reviewed later (in section 1.4).

1.2.1 Origins

I shall give only brief account of the early history of spatial reasoning: further details can be found in any good history of mathematics, such as that of Boyer (1968).

Geometry (literally *earth/land measurement*) dates back to the Egyptians. Egyptian mathematics was of a largely practical kind, concerned with simple calculations, very often of a spatial character (e.g. determining the area of a piece of land). The relations between lengths, areas and volumes were studied because of their value in commercial and architectural applications. The idea that all geometrical reasoning might be based on the application of a small number of fundamental principles appears to have originated in the ancient Greek civilisation. The almost mythical character Thales (who lived around 600 B.C.) is often credited with being the first person to demonstrate general principles of mathematical (and particularly geometrical) reasoning.

The idea of characterising valid reasoning in terms of *logical* modes of inference was taken up by many Greek thinkers and developed surprisingly rapidly, so that within a century Pythagoras (∼540 B.C.) and his followers had constructed very rigorous proofs of many theorems in number theory and geometry. Laws of valid argument were also studied independently of any particular subject matter. Early philosophers such as Plato (427–347 B.C.) realised that sequences of sentences that followed certain patterns always seemed to constitute a convincing argument. This is the basis of *formal* logic. Many principles of reason such as *modus ponens* and the law of the *excluded* *middle* were identified. Aristotle (384–322 B.C.) analysed *syllogisms*, which make up a significant fragment of quantificational logic.

At this time there was intense investigation of how principles of rigorous logical inference should be applied to reasoning about spatial relationships. Geometers strove to elicit the *elements* of geometry — that is, a set of fundamental definitions and postulates from which all geometrical truths could be logically derived. Many attempts were made to specify these elements until finally a system was discovered which seemed to yield all that was required. Euclid's *Elements* was written round about 300 B.C., while Euclid was a teacher in the *Museum* at Alexandria (an institution established by Ptolemy I). Despite a certain amount of quibbling about a postulate concerning parallel lines, Euclid's axiomatic geometry has been in use right up to the present day.

1.2.2 Development

For almost two millennia geometry was extensively developed; but it did not really go beyond the potentialities of its Euclidean foundation until it was investigated by Descartes (1596–1650). In *La Geometrie*, an appendix to his *Discours de la Méthode* (1637), Descartes introduced the idea of a *coordinate system*, in which points are identified with pairs (in 2D) or triples (in 3D) of real numbers. This interpretation provides the foundation for what is now known as *analytic* geometry in which lines, surfaces and volumes are represented by means of algebraic equations and inequalities involving the Cartesian coordinates of points. The uniformity of algebraic representation facilitates general and very effective methods for solving large classes of geometrical problems.

The 19th century saw a dramatic revolution in geometry. Euclid's fifth postulate (which states that for any point and any line there exists a unique line passing through the point and parallel to the first line) had long been the subject of investigation because it had long been hoped that it could be derived from the other (much simpler) postulates of the theory. The formal apparatus involved in representing and reasoning about Euclidean geometry had by this time become very precise and the general properties of formal systems had also become clearer. In particular the notions of logical equivalence, independence and consistency of axiom sets were now well understood. It was finally established that Euclid's fifth postulate (concerning the existence of unique parallels) was independent of the other simpler postulates so that consistent systems could be constructed in which it did not hold. Lobachevsky (1829) took the bold step of proposing a system of (hyperbolic) geometry, which explicitly contradicts the fifth postulate.[5]

The end of the 19th century also saw the birth of a radically new approach to the mathematical description of spatial relationships. The field of point-set topology was originated by Cantor (1845-1918) as an application of set theory to the study of Euclidean space. Investigations in topology (by Hausdorff (1914), Kuratowski (1933) and many others) lead to the clarification of many concepts in analysis (e.g. limits of infinite sequences).[6] The applications of modern topology are, for the

[5] The significance of non-Euclidean geometry is clearly explained in (Trudeau 1987), which also gives an illuminating view of the status of geometrical theories.

[6] A thorough introduction to basic topology can be found in (Kuratowski 1972).

most part, far-removed from spatial relationships in the physical world but are concerned with abstract mathematical structures. The complexity of point-set topology means that, although it is a powerful tool for the mathematician, it has not (as yet) yielded effective general purpose methods for reasoning about spatial relationships.

An alternative approach to characterising topological properties known as *algebraic topology* was created by Henri Poincaré in the last years of the 19th century. It was initially developed more or less independently of point-set topology although in the 1930s some unification of the approaches was attained (Alexandroff and Hopf 1935). The basis of the approach to topology is to use an algebraic object (often some kind of *group*) to describe the structure of a topological space. This formalism is in many respects more amenable to computational manipulation than the point-set approach and it is very likely that algebraic methods can provide a powerful tool for the development of spatial reasoning algorithms (see e.g. (Pigot 1992, Bertolotto, Floriani and Marzano 1995)). However, further consideration of the theory of algebraic topology is beyond the scope of this thesis.

1.2.3 New Foundations

During the early part of the 20th century the methods of logical analysis reached a state of extreme precision and were applied to many branches of mathematical and (to a lesser extent) physical science. Russell and Whitehead were both keen that the methods of logic should be applied not only to well established, objective physical theories but also to the development of phenomenological theories, describing the world as it is perceived through sense data. How such theories should be constructed was (and is still) far from clear. One idea, expounded by Whitehead in his book *The Concept of Nature* (Whitehead 1920), is that in a theory of the world of sensory experience, the basic entities of the logical representation should correspond directly to 'phenomena', these being objects of consciousness which are perceived *via* diverse sense-data but are conceived as integral objects or events — e.g. a cloud or the flight of a bird across the sky.

Treating such things as basic entities is at odds with the theoretical systems which have been developed to formalise classical science. In these systems the basic entities are typically points of space, instants of time and numerical quantities such as mass and velocity (in fact, points and instants are generally also identified with numerical coordinates). The spatial relationships between points are characterised by well-known geometrical theories and mathematical structures. Moreover, this analysis allows specification of physical laws in terms of differential equations, which form the axioms of nearly all physical theories. But the analysis also means that formal objects corresponding to physical bodies or events must be built up set-theoretically in terms of these basic entities. A complex and irregular region (e.g. that occupied by a cloud) then becomes an infinite set of points which may be extremely difficult or even impossible to characterise.

Under the alternative, phenomenological approach, objects and events become the basic entities of a theory. Geometry is now concerned with relationships between the regions occupied by bodies and dynamical laws must be formulated in terms of causal relationships between events: differential equations are replaced by qualitative relationships. Attempts to construct theories of this kind have been made by many philosophers and logicians as well as, more recently, by computer scientists; but this project has met with severe difficulties. In fact, I think it is fair to say that there is no widely accepted physical theory based on this type of ontology. This is perhaps not surprising, given firstly the relatively recent conception of the idea and secondly the difficulty in finding uses for such theories that would make their construction more than just a philosophical exercise.

An application which promises to motivate development of phenomenon-based theories is AI. Not only does this ontology appear to be closer to that employed in human reasoning (as evidenced by the structure of our ordinary language) but it also seems that it may be more appropriate as a vehicle for automated reasoning about real world situations, which if described in the terms of classical physics would be unmanageably complex. Nevertheless, despite considerable effort from AI researchers, the qualitative theories embedded in AI systems do not appear to have a power and generality comparable to classical theories of, for example, dynamics or electromagnetics.

One explanation for the lack of progress may be that researchers have assumed that, given the right formal framework, specifying theories of real world phenomena will be straightforward: much work has been directed towards providing general-purpose formal systems that are amenable to computation; but comparatively little has been concerned with providing theories of specific conceptual domains. However, in recent years, interest in such domain-specific theories has grown rapidly. By analogy with the role of point based geometry in classical physical theories, it is to be expected that characterisation of the geometrical relationships that may hold between extended objects will be of fundamental importance to many of these conceptual theories. Construction of general theories of these relationships is one of the primary goals of the sub-field of AI known as Qualitative Spatial Reasoning (henceforth QSR).

A detailed account of formal theories of spatial regions will be given in the next chapter.

1.3 Conceptual and Formal Frameworks

Let us now examine the plurality of possible frameworks for representing and reasoning about spatial information and the relationships between these frameworks.

The history of spatial reasoning shows that formalisation of its modes of inference can be carried out from a variety of different perspectives. Given our modern understanding of logical systems it is obvious that for any axiomatic theory there are infinitely many syntactically distinct but logically equivalent axiomatisations of the theory. In the context of geometry this is well illustrated by Euclid's fifth postulate which (when taken together with his other four postulates) has been proved logically equivalent to a host of other possible axioms (e.g. that the angles of a triangle add up to $180°$).

At a more fundamental level there are also many different concepts or sets of concepts that could be taken as primitives in a formal system. Given two sets of primitives, A and B, it may happen that each concept of B can be defined (by means of purely logical equivalences) in terms of the concepts of A; in this case the set A is at least as expressive as B. Moreover two sets of concepts may be equal in expressive power and so could serve as alternative sets of primitives for (essentially) the same theory.[7]

Analysis of Euclid's geometry led to several equivalent systems employing different primitive relations between points — equidistance of two pairs of points, equidistance of two points from a third, mutual equidistance of three points, the relation between five points which lie on the surface of the same sphere. However in all these formulations one primitive notion remains constant — that of *point*. Commitment to the notion of 'point' is easily overlooked in most axiomatic systems of geometry because it is often assumed that the domain of (1st-order) quantification coincides with the totality of points so there is no need to actually employ a predicate '$\mathsf{point}(x)$'. Nevertheless, as we shall see in the next chapter, a number of axiom systems have been proposed in which *regions* rather than points make up the domain of quantification. So, in formulating a theory of spatial relationships (or any other theory), we have a large degree of freedom, not only in how we state its axioms and which primitive predicates we employ, but also in choosing the type of objects that make up the domain of elementary individuals of the theory.

Nicod's doctoral thesis *Geometry in the Sensible World* (1924) opens with a penetrating analysis of the relationship between alternative systems of geometry based on different primitive notions. Here he introduces the ideas of *intrinsic* and *extrinsic* complexity of formal systems. The former resides in the structure of the system itself whereas the latter depends on how simply the elements and concepts of the formal theory can be matched to objects and properties in the domain of application of the theory. Thus, for specifying a theory of physical processes, a formal system in which points are the basic elements may be internally simple; but, because abstract points cannot be perceived directly or precisely located in the physical world, it would be deemed externally complex.

1.3.1 Basic Elements in a Spatial Theory

Five of the most promising candidates to serve as basic elements in a theory of spatial relationships are given in the following table:

Objects	*Existential Character*	*Proponents*
Points	abstract	Euclid, Descartes (1637)
Regions	spatial	Clarke (1981), Randell, Cui and Cohn (1992)
Bodies	physical	Sneed (1971)
Things	linguistic/metaphysical	Whitehead (1929), Simons (1987)
Sense-data	sensory	Whitehead (1929), Nicod (1924)

The most established ontological foundation for spatial reasoning is to construe *points* as the basic elements out of which more complex spatial objects are in some sense composed. Points are usually regarded as abstract theoretical entities because they have no physical extension nor mass.

The idea of developing a geometry based upon *sense-data* was pursued by Whitehead and Nicod

[7] A number of important theorems concerning the definability of concepts and the completeness of conceptual frameworks are given in (Tarski 1956b).

under the influence of Russell's epistemological theories, according to which the basic elements of reason must be correlated with simple sense data such as colour patches in the visual field (Russell 1912). Within such an ontology, points — if they are to exist at all — must be somehow constructed in terms of sense-data.

Taking *regions* as basic may be seen as a compromise between point-based and sense-data-based ontologies in that, although regions are strictly abstract partitions of a space, they seem to be much closer to sense-data than are points. (A given region may possess a certain property — 'greenness' say — and this will be perceived as a green patch.) Although Whitehead and Nicod saw sense-data as primary, they also gave axiomatisations whose objects are (abstract) regions; the correspondence between these regions and actual sense-data would then have to be given by an auxiliary definitional theory (c.f. the chapter 'The geometry of perspectives' in Nicod's thesis). Laguna and Tarski also developed theories of regions (which they, slightly misleadingly, called 'solids') but did not appear to be so concerned with the epistemological status of regions. Their theories are presented more as alternative abstract systems of geometry, in which the status of point and region is inverted with respect to set-theoretic construction.

Region-based formalisms have often been presented as relating to arbitrary 'solids' (de Laguna 1922, Tarski 1929). This might suggest that the objects of these theories are physical bodies — for 'solidity' is surely a physical property, which could not apply to an abstract region. However, a theory of physical bodies would have to take into account the material structure and properties of such objects rather than treating them as abstract volumes. Such formalisations of physical objects are at a relatively undeveloped stage, although a number of formal theories of Newtonian mechanics have been proposed (e.g. (Montague 1962)). A discussion of the problems involved in specifying physical theories in a fully formal framework can be found in (Sneed 1971).

A final existential perspective on the objects of spatial reason is given to us by our linguistic descriptions of objects in space. Such objects are generally individuated by means of *count nouns* (e.g. table, cup, saucer), each of which carries its own criteria for identification. These linguistic classifications and their associated criteria of recognition derive from the practical significance of certain types of physical entity conditioned to some extent by more or less arbitrary linguistic conventions. The utility of this classification is to a large extent determined by the physical nature of the world: the material properties of the world give rise to 'natural' ways of classifying it and breaking it into chunks. However, it may be argued these physical circumstances give rise to a framework of metaphysical categories which must underly any linguistic description of spatial entities.

1.3.2 Modes of Formalism

At a still more fundamental level, the very boundary between a logical representation language and a theory expressed in that language may be shifted. Three kinds of representation together with their apparatus for information manipulation are summarised in the following table:

axiomatic	theorem proving
algebraic (analytic)	coordinates and equations
purely logical	spatial logic and proof procedures

Applying the axiomatic method to spatial reasoning involves formulating a spatial theory in some general-purpose logical language (such as 1st-order logic) and then proving theorems in that system. It has been found that theorem-proving in all but the simplest logical languages is intractable. The algebraic approach is the one that is most commonly adopted. Information is coded in polynomial equations and/or inequalities. Disjunctive and quantificational constructs are avoided so that the expressive power of the system is limited. Effective methods for manipulating and extracting information from equations and inequalities are well-known. The possibility of a *purely logical* approach is not widely appreciated. It will be discussed in the next section.

1.3.3 Logical Theories of Space and Spatial Logics

The vocabulary of a formal language can be divided into two categories of atomic expression, which may be called logical and non-logical. In 1st-order logic the logical symbols are the truth-functional connectives and quantifiers, and the non-logical vocabulary consists of constants and predicates. (Variables may be regarded as notational devices associated with quantifiers as a means of indicating their scope.) We have seen how in representing a theory in a formal language there may be many possible sets of non-logical primitives in terms of which the theory could be specified. However, there is also a more radical kind of alternative formulation: concepts of the theory may be encoded directly into logical symbols (or complex logical structures) of the formal language. In doing this we arrive at a true spatial logic, rather than merely a theory of spatial relations specified in a general-purpose logical language.

In chapter 4 we shall investigate this possibility at some length. The most novel and substantial results of this thesis concern the representation of spatial relationships in terms of non-classical 0-order logics. One advantage of such encodings is that one often immediately obtains a decision procedure for the spatial theory.

1.4 Spatial Reasoning in Computer Science

In most existing computer programs, representation and manipulation of spatial data is very largely *numerical*. Objects and regions are represented by sets of coordinates and information is extracted from this data by means of arithmetic and trigonometric computations.

Numerical representation may be well suited for some purposes, in particular where the spatial information precisely describes some definite situation and where the output required from the system is itself primarily numerical. However, in many cases, useful spatial information does not describe a unique physical situation but qualitatively characterises a situation as being of a particular type. Extracting information from such data requires logical reasoning about the concepts involved in describing a situation; and hence requires a rigorous (formal) theory of qualitative spatial relationships.

From a computational point of view, qualitative theories of spatial relations are relatively undeveloped. Nevertheless some significant work has been done. Randell and Cohn (1989) and Randell, Cui and Cohn (1992) specify a 1st-order theory of spatial regions based on a primitive relation of connectedness, $\mathsf{C}(x, y)$, together with a number of (quasi-Boolean) functions. Despite containing very few non-logical primitives this theory has been found to be quite expressive: indeed a large number of significant spatial relations can be defined exclusively in terms of the relation, C (Gotts 1994). Egenhofer (1991) presents a much more limited framework in which a number of topological relations can be represented. He also shows how some simple inference rules can be used to generate the *composition* of any pair of these relations (see chapter 9 for a full discussion of composition-based reasoning).

1.4.1 Commonsense Knowledge

Many influential AI researchers have argued that representation of so-called 'commonsense' knowledge is of key importance in developing 'intelligent' computer systems; and a fair proportion of these researchers have employed formal representations and axiomatic theories as a means of encoding this knowledge (see e.g. (Hayes 1979, Hayes 1985b, Guha and Lenat 1990)). Qualitative spatial concepts are pervasive in everyday descriptions of the world so axiomatic theories of commonsense knowledge will have to incorporate many axioms governing the logical behaviour of spatial properties and relations.

A very large number of theories have been constructed, so detailed descriptions cannot be given here. Many of the papers which shaped this field of AI are contained in the collections (Hobbs, Blenko, Croft, Hager, Kautz, Kube and Shoham 1985) and (Hobbs and Moore 1985). A more recent reference on formal representations of commonsense knowledge is (Davis 1990).

1.4.2 Reasoning about Physical Systems

Another domain of knowledge representation that has received considerable attention is that of *physical systems*. Reasoning about physical systems may be treated as a type of commonsense reasoning or alternatively one may attempt to formalise the kind of reasoning employed by physicists, which involves manipulation of mathematical equations as well as the use of commonsense principles. Although spatial properties are of fundamental importance to the characterisation of physical systems, work in this area has tended to focus on their dynamical behaviour rather than their static properties. Key papers in this area can be found in (Weld and De Kleer 1990).

1.4.3 Spatial Reasoning in Robotics

Spatial reasoning is clearly of key importance in the field of robotics. But, because of the complexity of the domain, the use of formal representations has been limited. Most robot control systems rely on algorithms which are (from a logical point of view) rather *ad hoc*.

However, certain methods for classical robot path planning do make use of logical representations. A general representation for physical objects can be given in terms of *semi-algebraic sets*. These are sets of points defined by 1st-order formulae whose atoms are polynomial equalities or inequalities. The consistency of sets of such expressions can be determined by the decision procedure for algebra and geometry given by Tarski (1948), who showed how quantifiers could be eliminated from these formulae. The use of this decision procedure for computing collision free paths for a robot in an arbitrary workspace characterised in terms of semi-algebraic sets is described by Latombe (1991). Other uses of quantifier elimination methods in geometrical reasoning are discussed by Arnon (1988).

It is likely that spatial reasoning formalisms akin to those developed in this thesis will ultimately play an important role in robotic control systems. But before this can be done it will be necessary to develop representations with which one can express and reason about both spatial and dynamical aspects of physical systems. This is beyond the scope of the present research.

1.4.4 Spatial Reasoning and Computer Vision

The field of computer vision is an extremely active area of AI research and has produced systems which are actually used in applications. Vision is clearly very closely related to spatial reasoning. Nevertheless very little of the research done in this area is of direct relevance to the concerns of this thesis.

Computer vision is concerned primarily with *extracting* information from sensor data. The sensor data would typically take the form of two-dimensional pixel images. Various types of information may be extracted but the most common tasks would be to construct some kind of three-dimensional model of the scene or to locate types of object or region in the scene. Spatial reasoning on the other hand is concerned with *manipulating* spatial information and in particular in finding consequences holding among spatial propositions.

Although the concerns of vision and spatial reasoning are rather different, there is some interaction between the problems of the two fields. For instance, in extracting 3D information from a 2D scene, the ability to draw inferences from, and to test the consistency of, 3D spatial information may be very useful in narrowing down the range of possible interpretations of a scene. This technique would be akin to that used by Waltz (1975) for finding 3D descriptions of shaded 2D drawings.

1.4.5 Temporal Reasoning

Temporal reasoning is a distinct and very active area of research. Nevertheless space and time are often considered to be very closely related aspects of reality, so it is useful to consider similarities between spatial and temporal formalisms.

Temporal reasoning has been developed in a number of different ways.[8] Originating with the work of Prior (1955, 1967), *tense logics* have been developed in which temporal relationships

[8] A survey of temporal logics and their applications can be found in (Galton 1987).

between states of affairs are modelled in terms of propositional operators. This analysis of tense is much the same as that given by *modal logics* in respect of concepts such as necessity and belief, which are likewise represented in terms of propositional operators. Galton (1984) further analyses the structure of temporal operators by means of a language in which propositions and *events* are distinct types of expression.

Until recently languages such as tense logic, where temporality is modelled by special categories of logical operator, have not been widely employed by AI researchers. Theories of actions and change have rather been represented in more standard notation (1st-order logic or some variant), with semantic properties being specified by axioms or captured by special purpose inference rules. The best known work in this area is that of Allen. Allen identified a set of thirteen JEPD relations which can hold between two temporal intervals and studied reasoning procedures based on the composition of these relations (Allen 1981, Allen 1983). A 1st-order theory describing these temporal intervals and their relationship to actions and events was also developed (Allen 1984, Allen and Hayes 1985).

Whilst 1st-order theories may be very useful in establishing a sound theoretical framework for representing information in some domains, requirements of computational tractability mean that for most practical purposes it has been found that less expressive, more domain-specific languages must be used. These come in two basic varieties: on the one hand we have constraint languages capable of representing and reasoning with relational facts involving a fixed set of temporal relations (e.g. the 13 Allen relations — or perhaps some tractable subset of disjunctions of these relations); on the other hand we have languages containing temporal operators but less expressive than 1st-order logic (e.g. propositional or Horn clause languages). Formalisms of both these kinds are now (1997) extremely widespread and well-known in AI.

The content of this thesis reflects many parallels between the possible approaches which can be taken to representing spatial information and approaches which have been applied to temporal information. Construction of the RCC theory of spatial regions was greatly influenced by the works of Allen and Hayes (Allen and Hayes 1985, Hayes 1979, Hayes 1985a, Hayes 1985b) and consequently its development followed a similar pattern: a 1st-order theory was presented and investigated; then to provide a reasoning mechanism useful constraint languages were identified within which composition based reasoning could be conducted. The most original part of this thesis develops an alternative route to spatial reasoning *via* 0-order logical languages with spatial operators. Hence, spatial as well as temporal reasoning can be carried out within the broad framework of modal logic.[9]

I envisage that as the field of spatial reasoning is developed it will become increasingly linked to temporal reasoning. In order to represent and reason about changing situations a combined (spatio-temporal) formalism is clearly needed. Reasoning about action and change has very often

[9] In fact, it is perhaps more revealing to realise that what is common between all these modes of reasoning is that they are all representable in the very general framework of *Boolean algebras with additional (monadic) operators*. Logical languages whose semantics can be specified in terms of such algebras form a very natural class of formal systems whose expressive power is greater than that of propositional logic but which are still in many cases decidable. The use of such languages in spatial reasoning will be investigated in chapters 4,5 and 6.

been presented in formalisms in which there is a basic category of expression referring to *events*. In providing semantics for such formalisms events have very often been identified with temporal intervals. However, the temporal extent of an event is only one dimension of its existence. I believe that events (or at least most kinds of event) are spatial just as much as temporal entities and that an adequate semantics for events must take into account this spatial character.

The modal representation of spatial relations developed in chapter 4 is in many respects similar to propositional tense logics. If propositions in a tense logic are regarded as 1-dimensional regions on a 'time line' it is clear that temporal operators are closely related to spatial relationships. The main difference between tense logics and the spatial logics that I shall present is that a tense logical formula is evaluated to be true or false at a particular time.

1.5 Automating Spatial Reasoning

Automated reasoning has attracted a great deal of attention from computer scientists from the sixties onwards. Significant advances have been made in developing proof methods which are well-suited to computation.[10]

Despite this progress, fundamental problems remain. Most researchers in this area have focused on general-purpose 1st-order theorem proving. However, it is known that reasoning with this formalism is *undecidable*. This means that, although proof algorithms for 1st-order logic can be specified which are guaranteed to generate a proof of any *theorem* in finite time, there can be no algorithm that can determine whether any arbitrary 1st-order formula is a theorem in finite time. This is because, whatever proof procedure is used, there will always be a class of non-theorems for which the algorithm does not terminate. Unless this difficulty can somehow be circumvented it is unlikely that general-purpose 1st-order theorem provers will ever be used in practical applications.

There are essentially two ways of avoiding the undecidability problem: one is to use a general-purpose logical language which is less expressive than 1st-order logic; the other is to use some special purpose representation designed for reasoning in a particular conceptual domain. This thesis combines both these approaches: I focus on representing information in the restricted domain of spatial relationships but in order to reason about these relations I show (in chapter 4) that they can be encoded in a formalism which is normally regarded as a general purpose 0-order language.

1.5.1 Complexity of Mathematical Theories

As we have seen, spatial reasoning has long been a concern of mathematicians. Indeed the fields of geometry and topology are extremely well developed and are of direct relevance to automated reasoning about spatial situations. But the problem with nearly all mathematical theories is that they are too complex to reason with effectively. Topology is built upon a large amount of set theory, so any naive reasoning algorithm based on standard formulations of topology will have as its search space virtually all of mathematics. Whilst rather more succinct (1st-order) axiomatisations

[10] General texts on Automated Reasoning which describe these methods include (Bibel 1993) and (Duffy 1991).

of elementary geometry exist (e.g. (Tarski 1959)), these are still far too complex to be tackled by existing theorem proving techniques.

The need to employ such axiom systems can be avoided by employing the methods of analytic geometry. Lines and regions can then be represented in terms of formulae comprising polynomial equations and inequalities relating the Cartesian coordinates of points. If such an approach is to be effective the logical form of these formulae must be severely restricted: normally one simply has a set of equations/inequalities which is implicitly taken as a conjunction in which all variables are universally quantified. Under these restrictions disjunctive information cannot be represented, nor is it possible to specify relationships involving more subtle quantificational structure. Surprisingly, if one introduces Boolean operators and arbitrary quantification, the resulting language (known as the *Tarski language*) does actually remain decidable by means of a quantifier elimination method (Tarski 1948). Algorithms for quantifier elimination in the Tarski language have been the subject of considerable investigation (Collins 1975, Arnon 1988, Caviness and Johnson 1995, Mishra 1996) and, although the general problem is intractable, procedures have been found which are effective for large classes of formulae.

1.5.2 Tractability and Decidability

The major problem in developing a useful formalism for reasoning about spatial information (indeed for any domain) is the trade-off between expressive power and computational tractability. Whilst Egenhofer's representation does allow for certain inferences to be computed effectively, the scope of the theory is limited. On the other hand, although the formalism presented in Randell, Cui and Cohn (1992) is very expressive, since it is presented in 1st-order logic, reasoning in the calculus is extremely difficult (however the use of pre-calculated composition tables for relations definable in the theory does enable certain kinds of inference to be computed efficiently).

It is common in computer science to equate tractability with polynomial-time computability. But to a logician this will probably seem an overly harsh restriction, since proof procedures in nearly all interesting logics are at least exponentially hard. In this thesis I shall be primarily concerned with finding *decidable* representations for spatial information but we shall see in chapter 6 that by restricting the range of spatial relations which may be represented (to a class including all the RCC-8 relations illustrated in fig 1.2) a polynomial-time reasoning algorithm can be obtained.

1.6 The Content of this Thesis

The principal aim of this thesis is to investigate frameworks for representing spatial information that are both expressive enough to be useful for solving real problems and are in some sense tractable. I focus on topological relationships, which I consider to be the most fundamental of spatial concepts; but I also examine the non-topological property of convexity. The rest of the thesis is organised into the following chapters:

2: Axiomatic Theories of Spatial Regions

In the next chapter I survey previously proposed theories of spatial regions. I first give a brief description of classical point-set topology in which regions are treated as sets of points. I then consider theories in which regions are taken as basic entities. The earliest of these are the systems of Lesniewski and of Whitehead, put forward at the beginning of this (the 20th) century. Also covered are the theories of Tarski (1929) and Clarke (1981). I go on to describe in some detail the more recent theory of Randell, Cui and Cohn (1992) (the RCC theory), which is a modification of Clarke's calculus and was formulated with computational applications specifically in mind.

3: Analysis of the RCC Theory

The RCC theory is now investigated in some detail. I examine the axiom set and suggest certain modifications which seem to be required. Models of the theory in terms of classical point-set topology are given and the possibility of constructing a *complete* theory is considered. I observe that no adequate 1st-order theory can be either complete or decidable. I suggest a new theory constructed so as to avoid certain technical problems arising in the original RCC theory.

4: A 0-Order Representation

Since 1st-order theories such as RCC are undecidable they cannot be used as a basis for effective reasoning. Thus the representation language (or languages) used in a spatial reasoning system must be more restricted in their expressive power. 0-order logical calculi are normally regarded as *propositional* logics; but as we shall see, a spatial interpretation of expressions of these formalisms can be given, in which the non-logical constants refer to spatial *regions* rather than propositions. This idea is introduced using the classical propositional logic, which can be interpreted as a Boolean calculus of spatial regions. The formalism of classical logic is then augmented to provide a language $\mathcal{C}^+$, which is capable of expressing a considerably larger class of spatial facts. I give a decision procedure for this language obtained by adding simple meta-level reasoning to the basic proof theory of classical 0-order logic.

5: A Modal Representation

Further, extending the framework proposed in chapter **4**, I show how modal operators can be interpreted so as to correspond with further operations on spatial regions which are needed to capture more subtle differences between different spatial relationships. Specifically we shall see how the operator of the modal logic $S4$ can be interpreted as a topological interior operator. I then give an encoding for a large class of topological relations (also expressible in RCC) into an augmented form of the $S4$ language which I call $S4^+$. This provides a decision procedure for a quite expressive spatial language.

6: An Intuitionistic Representation and its Complexity

Whilst the modal logic representation of spatial reasoning exemplifies a general methodology for using 0-order languages in knowledge representation, its use for any practical application would require an efficient theorem prover for $S4$. In this chapter I describe the implementation of a spatial reasoning system using a representation in terms of 0-order intuitionistic formulae. The core of the system is a Gentzen-style sequent calculus, which is a restriction of a well known rules system for the full 0-order intuitionistic calculus. The intrinsic complexity of reasoning algorithms using this intuitionistic representation has been studied by Nebel (1995a). Nebel looked at reasoning using a tableau method and has shown that the inferences needed for reasoning with the fragment of the logic needed to represent a large class of spatial relations (including in particular the 8 basic relations considered in chapter 2) can be computed with a polynomial time algorithm.

7: Quantifier Elimination

In this short chapter I present a partial decision procedure for 1st-order theories of the connection relation. This is based on the method of *quantifier elimination.* This technique can be used as a preprocessing step applied to a restricted class of 1st-order spatial formulae prior to translation into the $S4$ or intuitionistic encodings.

8: Convexity and Containment

The main results of the thesis apply primarily to the significant but by no-means comprehensive range of spatial relations definable from the primitive relation of connectedness. However, similar methods can be applied to other aspects of spatial reasoning (and probably to other areas of knowledge representation). In this chapter I explain how the techniques of 0-order representation can be extended to handle non-topological information concerning the convexity of regions. This illustrates methods by which the techniques given for effective reasoning with topological relationships can be extended to handle non-topological information.

9: Composition-Based Reasoning

In this chapter I look at spatial reasoning based on the notion of relational composition. I examine the use of *composition tables* to compute inferences and their relation to 1st-order theories. I also present a *relation algebra* formalism for topological relations in which the role of the composition operation is much more prominent than in 1st-order representations.

10: Further Work and Conclusions

In the concluding chapter I evaluate the usefulness of the logical representations and reasoning systems presented in this thesis. I assess the prospects for development of more expressive representations for spatial reasoning which are computationally viable and look at how spatial reasoning might be incorporated into more general reasoning systems. Potential applications areas including Geographical Information Systems (GIS), Robot Motion Planning and Computer Vision are considered; and I describe a prototype GIS with a limited qualitative spatial reasoning capability.

1.6.1 Assumed Background and Notations Employed

In this thesis I assume a knowledge of classical logic and set theory. I also make use of established work in the areas of algebra, model theory, modal logic, and intuitionistic logic, so acquaintance with these fields will be useful. Standard formal notations of logic and set theory are employed. Other notations will be introduced and explained when required.

Chapter 2

Axiomatic Theories of Spatial Regions

This chapter surveys, in some detail, a number of formal theories of spatial regions. First I briefly explain classical point-set topology, in which regions are characterised as sets of points. The rest of the chapter is concerned with theories in which extended regions are treated as basic (0-order) entities. Although, some very eminent logicians have proposed and investigated region-based formalisms, they are still far less well understood than point-based theories. The following systems will be described in some detail: Leśniewski's Mereology, Tarski's Geometry of Solids, Clarke's theory of the Connection relation, and the Region Connection Calculus (RCC). Several other formalisms will also be considered.

2.1 Point-Set Topology

Classical point-set topology is based on set theory. The basic (0-order) elements of the theory are points. Regions are identified with sets of points. In developing the theory, the principle mathematical objects considered are *topological spaces*. These are sets of elements (points) associated with an auxiliary structure determining the topological properties of the space. A topological space can be formally defined in a number of ways. Perhaps the simplest is as a set of sets, which includes the empty set and is closed under arbitrary unions and finite intersections. This is the set of *open* sets of the space. The largest open set (which is the same as the union of all open sets) is called the universe of the topology. A topology can thus be represented by a structure $T = \langle U, O \rangle$, where U is the universe and O is the set of open sets.

In a topological space $T = \langle U, O \rangle$, given an arbitrary subset S of U, the *interior* of S is the largest member of O that is a subset of S. The interior function, i, on a topology $\langle U, O \rangle$ maps every subset of U to its interior (a member of O). Because of the conditions on the set of open sets, i must satisfy the axioms **PSTi1-4** given below. In **PSTi3**, $\mathcal{U}$ is a meta symbol referring to whatever is the universal set of the topological space under consideration — i.e. for topology

$T = \langle U, O \rangle$ we have $\mathcal{U} = U$.[1] X and Y are any subsets of the universe.

PSTi1) $i(X) \cup X = X$

PSTi2) $i(i(X)) = i(X)$

PSTi3) $i(\mathcal{U}) = \mathcal{U}$

PSTi4) $i(X \cap Y) = i(X) \cap i(Y)$

Given a set U, any function i that maps subsets of U to subsets of U and obeys the above axioms determines a unique topology $\langle U, O \rangle$: the elements of O are simply those subsets S of U such that $i(S) = S$. Hence, any topology $\langle U, O \rangle$ can be alternatively characterised by a structure $\langle U, i \rangle$, where i is an interior function.

A set is called *closed* iff it is the complement of some open set. The *closure* of a set is the smallest closed set of which it is a subset. The closure function, c, mapping arbitrary subsets of a space to their closures must satisfy the equations **PSTc1-4** given below. The set of closed sets of a space or the closure function, c, can each be used as further alternative ways of specifying the topology of a space. Interior and closure functions are inter-definable: $c(X) = \overline{i(\overline{X})}$ and $i(X) = \overline{c(\overline{X})}$. Here, and throughout the sequel, $\overline{X}$ is the complement of X w.r.t. the universe.

PSTc1) $X \cup c(X) = c(X)$

PSTc2) $c(c(X)) = c(X)$

PSTc3) $c(\emptyset) = \emptyset$

PSTc4) $c(X \cup Y) = c(X) \cup c(Y)$

As was mentioned in section 1.5, the language of set theory, in which point-set topology and many other mathematical theories are formulated, is highly intractable. Hence, this formalism is not well suited to computational applications. Nevertheless, it may be possible to find useful sub-languages of set theory for which effective reasoning procedures can be constructed. In section 5.3 I shall describe a purely algebraic sub-language of the formalism of point-set topology, which is both decidable and quite expressive.

Being built directly on set theory, point-set topology has an unambiguous set-theoretic semantics. This makes it a useful tool for studying the model theory of other spatial languages. In the rest of this chapter and the following chapter I shall consider several theories whose semantics are not so well defined. If it is possible to interpret such a language in point-set-theoretic terms, this immediately gives it a precise (though indirect) semantics. Hence such an interpretation can form the basis for soundness and completeness proofs. The methods of topological reasoning described in chapters 5 and 6 are both justified in this way.

[1] In considering a single topological space the symbol $\mathcal{U}$ is not really necessary since we can always refer directly to the universal set. However, in chapters 4, 5 and 6, I shall often use $\mathcal{U}$ to make statements about classes of algebras.

2.2 The Origins of Region-Based Theories

The early years of the 20th century saw intense activity in attempting to apply the methods of formal logic and set theory to mathematics and physics. Russell's epistemology and ideas about logical primitives were very influential at that time and Whitehead's book *Concept of Nature* (Whitehead 1920) proposed a view of physics and geometry which is a radical revision of traditional conceptions: he sought to found these disciplines on *sense data*, which according to Russell (1912) can be the only referents of truly primitive terms. To describe the spatial aspects of sense data, Whitehead proposed the construction of a geometry in which spatial regions rather than points would be the basic entities. Sense data could then be said to occupy spatial regions, whereas points would be abstract entities derived theoretically from regions.

In his book *Process and Reality* (1929) Whitehead suggested that a general theory of objects events and processes could be developed based on the primitive relation of *connectedness*; and he specified a large number of logical properties of this primitive. Since the only well-developed physical theories are formulated in terms of variables ranging over points in space (and time), Whitehead proposes the method of *extensive abstraction* (introduced in the earlier work (Whitehead 1920)) as a means of constructing points from regions of space (or space-time). The idea is to define a point in terms of certain infinitely nested sets of regions (a similar approach to characterising points in terms of regions has been followed by Clarke (1985) and is described below in section 2.5.3).

Nicod's doctoral thesis *Geometry in the Sensible World* (1924) developed Whitehead's approach in a number of directions.[2] Nicod adopted and modified Whitehead's method of 'extensive abstraction' for the construction of points from regions. He also proposed some highly original approaches to constructing geometrical systems from a phenomenological standpoint. One of these is a characterisation of geometry from the point of view of a being equipped only with a kinaesthetic sense of its own movement in space. Another takes into account the viewpoint and perspective of an observer in describing geometrical entities. It is also interesting to note that the chapter of the thesis on 'Temporal Relations and the Hypothesis of Durations' contains a discussion of temporal relationships between intervals and proposes a classification which is essentially the same as that adopted much later by Allen (1981). Another logician influenced by Whitehead was Theodore de Laguna who gave a theory of the 'geometry of solids'. This will be briefly described in section 2.7.1.

Contemporary with the investigations of Whitehead and his followers, the Polish logician and philosopher Stanislav Leśniewski was conducting an extensive enquiry into ontology and logical representations. He was particularly concerned with characterising the *part-whole* relation between objects and was critical of the set-theoretic treatment of this relationship. His theory was intended to describe entities of any kind; but in this chapter I shall only be concerned with its application to spatial regions.

[2] Russell regarded Nicod as potentially one of the greatest logicians of the 20th century and looked to him in particular to carry forward the project of founding logical theories of the physical world on the basis of sense-data. Tragically, Nicod died prematurely soon after the publication of his doctoral thesis.

2.3 Leśniewski's Mereology

Mereology, a formal theory of the part-whole relation was originally presented by Leśniewski (1927-1931) in his own logical calculus, which he called *Ontology*. This calculus is based on principles which are rather different from those of the standard predicate calculus. The principal distinctive features of Ontology are: firstly, that terms do not necessarily denote a single object (they may refer to nothing, a unique individual or any number of distinct individuals); and secondly, quantification is not associated with existential commitment (it has a more substitutional flavour). For certain purposes Leśniewski's Ontology has distinct advantages over standard logic. For example, in the spatial domain one may wish to employ a function 'the region of intersection of x and y' but this is a partial function, since if x and y are disjoint no such region exists. In standard logic terms always denote a unique individual, so partial functions are not legitimate; but in Ontology such functions present no problem.[3] A full description of Leśniewski's Ontology is beyond the scope of this thesis (see (Simons 1987) for a detailed account). However, the content of the theory of Mereology is not bound to the form in which it was initially stated. Hence I now present a formulation of Mereology, due to Tarski (1929), stated in standard classical logic.

Mereology is built on the single primitive relation $\mathsf{P}(x, y)$, whose interpretation is that x is a part of y. In terms of this, the relations of 'proper part' (PP) and 'disjointness' (DJ) are defined, as well as SUM, which is a relation between a set of individuals and an individual. I shall use small Roman letters for variables ranging over individuals and small Greek letters for variables ranging over sets of individuals. The definitions can then be given formally as:

Mdef1) $\mathsf{PP}(x, y) \equiv_{def} (\mathsf{P}(x, y) \wedge \neg(x = y))$

Mdef2) $\mathsf{DJ}(x, y) \equiv_{def} \neg\exists z[\mathsf{P}(z, x) \wedge \mathsf{P}(z, y)]$

Mdef3) $\mathsf{SUM}(\alpha, x) \equiv_{def} \forall y[y \in \alpha \rightarrow \mathsf{P}(y, x)]$
$\wedge \neg\exists z[\mathsf{P}(z, x) \wedge \forall y[y \in \alpha \rightarrow \mathsf{DJ}(y, z)]]$

In addition to the usual principles of classical logic and the theory of sets, the system is required to satisfy the following specifically mereological postulates:

Mpost1) $\forall x \forall y \forall z[\mathsf{P}(x, y) \wedge \mathsf{P}(y, z) \rightarrow \mathsf{P}(x, z)]$

Mpost2) $\forall \alpha[\exists x[x \in \alpha] \rightarrow \exists! x[\mathsf{SUM}(\alpha, x)]]$

These ensure firstly that the part relation is transitive and secondly (and rather controversially) that for any non-empty set of individuals there is a unique individual which is the sum of that set. Proofs of a number of theorems derivable from these axioms (e.g. that P is reflexive) can be found in (Woodger 1937, Appendix E). A short-coming of the theory of mereology, based as it is on the part relation, is that no distinction can be made between the relations of connectedness and overlapping: if two regions do not overlap they are simply discrete.

[3] Alternative formalisms in which partial functions can be handled are *sorted* classical logic (see e.g. (Cohn 1987)) and *free* logic (Bencivenga 1986).

2.3.1 Other Mereological Systems

A number of theories have been developed which contain mereological primitives equivalent or similar to Leśneiwski's. Woodger's *The Axiomatic Method in Biology* (1937) uses the theory exactly as given above. Leonard and Goodman (Leonard and Goodman 1940) devised a formalism which they called the "calculus of individuals", based upon a predicate which holds when two individuals are discrete. This system is essentially the same as Leśneiwski's but uses different notation and contains many additional definitions. The theory is applied to a number of problems involving relations between individuals, groups and ensembles that cannot be handled by ordinary quantification. This formalism also appears in Goodman's book *The Structure of Appearance* (1951) which proposes an approach to formal description of the world based on principles of logical nominalism (a reluctance to admit the existence of abstract entities such as sets).

2.4 Tarski's Geometry of Solids

Building on Leśneiwski's *mereology* by introducing a new *sphere* primitive, Tarski (1929) gave a theory of the 'geometry of solids',[4] which is embedded, by means of definitions, into an axiomatisation of elementary Euclidean geometry (such as that given in (Tarski 1959)).

Tarski starts by postulating a domain of *spheres* (I use the predicate $\mathsf{SPH}(x)$ to mean x is a sphere), over which he defines the relations of *external tangency* (ET), *internal tangency* (IT), *external diametricity* (ED), *internal diametricity* (ID) and *concentricity* (CONC). $\mathsf{ED}(a,b,c)$ holds when a and b are externally tangent to c and touch diametrically opposite points on c's boundary. These relations are defined as follows:

SGdef1) $\mathsf{ET}(a,b) \equiv_{def} (\mathsf{SPH}(a) \wedge \mathsf{SPH}(b) \wedge \mathsf{DJ}(a,b) \wedge \forall x \forall y[(\mathsf{P}(a,x) \wedge \mathsf{P}(a,y) \wedge \mathsf{DJ}(b,x) \wedge \mathsf{DJ}(b,y)) \rightarrow (\mathsf{P}(x,y) \vee \mathsf{P}(y,x))])$

SGdef2) $\mathsf{IT}(a,b) \equiv_{def} (\mathsf{SPH}(a) \wedge \mathsf{SPH}(b) \wedge \mathsf{PP}(a,b) \wedge \forall x \forall y[(\mathsf{P}(a,x) \wedge \mathsf{P}(a,y) \wedge \mathsf{P}(x,b) \wedge \mathsf{P}(y,b)) \rightarrow (\mathsf{P}(x,y) \vee \mathsf{P}(y,x))])$

SGdef3) $\mathsf{ED}(a,b,c) \equiv_{def} (\mathsf{SPH}(a) \wedge \mathsf{SPH}(b) \wedge \mathsf{ET}(a,c) \wedge \mathsf{ET}(b,c) \wedge \forall x \forall y[(\mathsf{DJ}(x,c) \wedge \mathsf{DJ}(y,c) \wedge \mathsf{P}(a,x) \wedge \mathsf{P}(b,y)) \rightarrow \mathsf{DJ}(x,y)])$

SGdef4) $\mathsf{ID}(a,b,c) \equiv_{def} (\mathsf{SPH}(a) \wedge \mathsf{SPH}(b) \wedge \mathsf{SPH}(c) \wedge \mathsf{IT}(a,c) \wedge \mathsf{IT}(b,c) \wedge \forall x \forall y[(\mathsf{DJ}(x,c) \wedge \mathsf{DJ}(y,c) \wedge \mathsf{ET}(a,x) \wedge \mathsf{ET}(b,y)) \rightarrow \mathsf{DJ}(x,y)])$

SGdef5) $\mathsf{CONC}(a,b) \equiv_{def} (\mathsf{SPH}(a) \wedge \mathsf{SPH}(b) \wedge ((a=b) \vee (\mathsf{PP}(a,b) \wedge \forall x \forall y[(\mathsf{ED}(x,y,a) \wedge \mathsf{IT}(x,b) \wedge \mathsf{IT}(y,b)) \rightarrow \mathsf{ID}(x,y,b)]) \vee (\mathsf{PP}(b,a) \wedge \forall x \forall y[(\mathsf{ED}(x,y,b) \wedge \mathsf{IT}(x,a) \wedge \mathsf{IT}(y,a)) \rightarrow \mathsf{ID}(x,y,a)])))$

The next step in Tarski's formulation is to constrain the theory to be compatible with Euclidean geometry. To do this he defines the notions of *point* and *equidistance*, which can serve as the only

[4] As mentioned in section 1.3.1, this is perhaps better thought of as a theory of 'volumes', since the entities of the theory are allowed to inter-penetrate each other and the property of solidity is not considered. The same applies to de Laguna's theory which will be described in section 2.7.1.

primitives in such a theory of geometry.[5] Both these concepts can be defined in terms of the relations defined above.

SGdef6) A point is defined as the set of all spheres concentric with a given sphere:

$$\mathsf{POINT}(\pi) \ \equiv_{def} \ \exists x[x \in \pi \wedge \forall y[y \in \pi \leftrightarrow \mathsf{CONC}(x, y)]]$$

SGdef7) Equidistance of two points from a third, $ab = bc$, is defined as follows:

$$ab = bc \ \equiv_{def} \ \exists x[x \in b \wedge \neg\exists y[(y \in a \vee y \in c) \wedge (\mathsf{P}(y, x) \vee \mathsf{DJ}(y, x))]]$$

Identification with the corresponding notions in Euclidean geometry is then achieved by the following postulate:

SGpost1) The notions of point and of equidistance of two points from a third satisfy all the postulates of ordinary Euclidean geometry of three dimensions.

Having fixed the structure of the set of points we still need to specify how 'solids' are related to this structure.

SGdef8) A *solid* is an arbitrary sum of spheres:[6]

$$\mathsf{SOLID}(x) \ \equiv_{def} \ \exists X[\mathsf{SUM}(X, x) \wedge \forall y[y \in X \rightarrow \mathsf{SPH}(y)]]$$

SGdef9) The point π is interior to the solid a:

$$\mathsf{INTER}(\pi, a) \ \equiv_{def} \ \exists x[x \in \pi \wedge \mathsf{P}(x, a)]$$

We now correlate the set of interior points of a solid with the geometrically definable concept of a *regular open set* of points. To do this I define interior (int) and closure (cl) functions on sets of points (capital Greek letters). The definitions use the relation '$xy < yz$', which is definable from '$xy = yz$' (see appendix A). In the usual topology of Euclidean space the interior points of a set are those that can be surrounded by an 'open ball' all of whose points are within the set. This is the basis of the following definitions:

SGdef10) $\mathsf{int}(\Pi) = \Theta \ \equiv_{def} \ \forall x[x \in \Theta \leftrightarrow \exists y[y \neq x \wedge \forall z[zx < xy \rightarrow z \in \Pi]]$

SGdef11) $\mathsf{cl}(\Pi) = \Theta \ \equiv_{def} \ \forall x[x \in \Theta \leftrightarrow \forall y[y \neq x \rightarrow \exists z[zx < xy \wedge z \in \Pi]]$

SGdef12) $\mathsf{ROPEN}(\Pi) \ \equiv_{def} \ \mathsf{int}(\mathsf{cl}(\Pi)) = \Pi$

The next two postulates stipulate that the interior points of solids are to be identified with regular open sets of points.

[5] Tarski's own formulation of elementary geometry, which is given in appendix A, employs *equidistance* and *betweenness* as primitive relations; but betweenness can in fact be defined in terms of equidistance, so with the addition of such a definition, that axiom set could be used.

[6] In fact in Tarski's theory all 0-order entities are 'solids' so this predicate definition could be replaced with a universal axiom.

SGpost2) If x is a solid, then the class Π of all interior points of x is a non-empty regular open set:

$$\forall x \forall \Pi[(\mathsf{SOLID}(x) \wedge \Pi = \{\pi \mid \mathsf{INTER}(\pi, x)\}) \rightarrow (\mathsf{ROPEN}(\Pi) \wedge \Pi \neq \emptyset)]$$

SGpost3) If a class Π of points is a non-empty regular open set, there exists a solid x, such that Π is the class of all its interior points:

$$\forall \Pi[(\mathsf{ROPEN}(\Pi) \wedge \Pi \neq \emptyset) \rightarrow \exists x[\mathsf{SOLID}(x) \wedge \Pi = \{\pi \mid \mathsf{INTER}(\pi, x)\}]]$$

These two postulates ensure a one-to-one correspondence between solids and non-empty regular open sets of points. Thus the categorical axioms of elementary geometry which fix the structure of the domain of points are used to determine the structure of the domain of solids.

Finally the mereological part relation, P, must be fixed in terms of point geometry by identifying it with set inclusion among the sets of interior points associated with solids:

SGpost4) If a and b are solids, and all the interior points of a are at the same time interior to b, then a is part of b:

$$\forall \pi[\mathsf{INTER}(\pi, a) \rightarrow \mathsf{INTER}(\pi, b)] \leftrightarrow \mathsf{P}(a, b)$$

As a logical foundation for a conceptual scheme, Tarski's theory has the great merit of being *categorical*, which means that all its models are isomorphic. Hence the theory can be regarded as completely fixing the meanings of all the concepts covered by its vocabulary. However, the theory is only made categorical by indirect means: firstly the notions of *point*, *equidistance* and *betweenness* are introduced by a series of definitions; then it is stipulated that these defined concepts obey the axioms of Euclidean geometry (Tarski 1959). He admits that the resulting system is not ideal:

> The postulate system given above is far from simple and elegant; it seems very likely that this postulate system can be essentially simplified by using intrinsic properties of the geometry of solids. (Tarski 1929)

What makes Tarski's system so unwieldy as a tool for actually reasoning about spatial regions, is the hidden complexity involved in **SGdef6** and **SGpost1**. These bring in the whole of Euclidean geometry as a means of fixing the structure of the space of regions. Reasoning with the axioms of elementary geometry is in itself very hard (although it is known to be decidable, no effective general reasoning method is known for this system[7]) but in this context the complexity is far worse because the 'points' constrained by the Euclidean geometrical axioms correspond to *sets* of spheres in the solid geometry. Thus, if points were eliminated from the system by unpacking the Euclidean axioms in terms of the definition of point, the resulting formalism would be an enormously complex 2nd-order theory.

[7] Reasoning in elementary geometry can be carried out by translating geometric relations into algebraic polynomial equations and inequalities constraining the Cartesian coordinates of points. Consistency of such equations can be tested using a decision procedure also due to Tarski (1948).

Relation	*Interpretation*	*Definition of $R(x, y)$*
$\mathsf{DC}(x, y)$	x is disconnected from y	$\neg\mathsf{C}(x, y)$
$\mathsf{P}(x, y)$	x is a part of y	$\forall z[\mathsf{C}(z, x) \rightarrow \mathsf{C}(z, y)]$
$\mathsf{PP}(x, y)$	x is a proper part of y	$\mathsf{P}(x, y) \wedge \neg\mathsf{P}(y, x)$
$\mathsf{O}(x, y)$	x overlaps y	$\exists z[\mathsf{P}(z, x) \wedge \mathsf{P}(z, y)]$
$\mathsf{DR}(x, y)$	x is discrete from y	$\neg\mathsf{O}(x, y)$
$\mathsf{EC}(x, y)$	x is externally connected to y	$\mathsf{C}(x, y) \wedge \neg\mathsf{O}(x, y)$
$\mathsf{TP}(x, y)$	x is a tangential part of y	$\mathsf{P}(x, y) \wedge \exists z[\mathsf{EC}(z, x) \wedge \mathsf{EC}(z, y)]$
$\mathsf{NTP}(x, y)$	x is a nontangential part of y	$\mathsf{P}(x, y) \wedge \neg\exists z[\mathsf{EC}(z, x) \wedge \mathsf{EC}(z, y)]$

Table 2.1: Defined relations in Clarke's theory

2.5 Clarke's Theory

The formalism developed by Clarke (1981, 1985) is an attempt to construct a system more expressive than that of Leonard and Goodman (1940) and simpler than that of Tarski (1929), based on the primitive relation of connectedness used by Whitehead (1929). The domain of the theory is spatial or spatio-temporal regions and the C primitive is constrained to obey the following two axioms:

Cax1) $\forall x[\mathsf{C}(x, x) \wedge \forall y[\mathsf{C}(x, y) \rightarrow \mathsf{C}(y, x)]]$

Cax2) $\forall x \forall y[\forall z[\mathsf{C}(z, x) \leftrightarrow \mathsf{C}(z, y)] \rightarrow x = y]$

The first of these ensures the relation is reflexive and symmetric, whilst the second is an axiom of *extensionality*, which states that if two regions are connected to exactly the same other regions then they must be the same. From the C relation Clarke defines several other useful spatial relations. These are given in table 2.1

2.5.1 Fusions and Quasi-Boolean Operators

A fusion operator, f, is then defined as follows:

Cdef1) $x = f(X) \equiv_{def} \forall y[\mathsf{C}(y, x) \leftrightarrow \exists z[z \in X \wedge \mathsf{C}(y, z)]]$

This means that the fusion of a set of regions is that region which is connected to all and only those regions that are connected to at least one region in the set. (The intended interpretation of $f(\alpha) = x$ may be regarded as the same as Leśniewski's $\mathsf{SUM}(\alpha, x)$, although the latter is defined in terms of P rather than C.)

The theory also contains an axiom ensuring that for every non-empty set of regions a fusion region exists:

Cax3) $\forall X[\neg(X = \emptyset) \rightarrow \exists x[x = f(X)]]$

This axiom would be very odd in a completely standard 1st-order theory, since in such a theory it is normally assumed that all well formed terms denote an (existing) individual, all functions being unique and total. Clarke, however, introduces a slight modification into the logical interpretation of quantification in his theory. Specifically, the rule of universal instantiation, which normally allows one to replace a universally quantified variable by any ground term, is restricted so that one can only replace the variable by either an individual constant or a complex term τ for which it is provable that $\exists x[x = \tau]$.[8]

Clarke then defines functions similar to Boolean operators as follows:

Cdef2) $\mathsf{sum}(x, y) \; =_{def} \; f(\{z \mid (\mathsf{P}(z, x) \vee \mathsf{P}(z, y))\})$

Cdef3) $\mathsf{prod}(x, y) \; =_{def} \; f(\{z \mid (\mathsf{P}(z, x) \wedge \mathsf{P}(z, y))\})$

Cdef4) $\mathsf{compl}(x) \; =_{def} \; f(\{y \mid \neg\mathsf{C}(y, x)\})$

The definition of compl entails that every region is disconnected from its own complement:

$$\forall x[\neg\mathsf{C}(x, \mathsf{compl}(x))] \qquad (\neg\mathsf{Ccompl})$$

The principle $\neg$Ccompl is consistent with an interpretation of regions as arbitrary point-sets, compl as set complement and $\mathsf{C}(x, y)$ as true when x and y share a point. However, if one is interested in establishing a naturalistic theory of regions, one might prefer the complement function to be such that regions always connect with (but do not overlap) their complements.

2.5.2 Topological Functions

Clarke is now able to define the topological operators of interior, closure and exterior as functions from regions to regions:

Cdef5) $i(x) \; =_{def} \; f(\{y \mid \mathsf{NTP}(y, x)\})$

Cdef6) $c(x) \; =_{def} \; f(\{y \mid \neg\mathsf{C}(y, i(\mathsf{compl}(x)))\})$

Cdef7) $ex(x) \; =_{def} \; f(\{y \mid \mathsf{NTP}(y, \mathsf{compl}(x))\})$

An additional axiom is concerning these topological functions is given by Clarke as follows:

Cax4) $\forall x[\exists z[\mathsf{NTP}(z, x)] \wedge$
$\forall y \forall z[(\mathsf{C}(z, x) \rightarrow \mathsf{O}(z, x)) \wedge (\mathsf{C}(z, y) \rightarrow \mathsf{O}(z, y)) \rightarrow (\mathsf{C}(z, \mathsf{prod}(x, y)) \rightarrow \mathsf{O}(z, \mathsf{prod}(x, y)))]]$

It is provable that the condition $\forall z[\mathsf{C}(x, z) \rightarrow \mathsf{C}(y, z)$ is equivalent to $\mathsf{NTP}(x, y)$ and also that $\mathsf{NTP}(x, x) \leftrightarrow (x = i(x))$. Thus, this axiom asserts firstly that every region has a non-tangential part and secondly that the product of two open regions is itself open.

[8] This restriction may be regarded as enforcing a rudimentary sort theory: quantifiers range over a sort *region* and all individual constants refer to entities of this sort. However, functions (such as f) may have as their value either a region or an entity $\emptyset$ whose sort (which we may call *null*) is disjoint from *region*.

2.5.3 Points

Clarke (1985) subsequently extended his original theory of spatial regions by the introduction of *points*. These are not basic entities of the system but are identified with certain sets of regions. This is essentially the method of *extensive abstraction* first proposed by Whitehead and taken up by Nicod and de Laguna. Clarke stipulates that a set of regions π is a point, which we note as $\mathsf{PT}(\pi)$, iff it satisfies the following conditions:

Cpoint1) $\forall x \forall y[(x \in \pi \wedge y \in \pi) \rightarrow \mathsf{C}(x, y)]$

Cpoint2) $\forall x \forall y[(x \in \pi \wedge y \in \pi \wedge \mathsf{O}(x, y)) \rightarrow \mathsf{prod}(x, y) \in \pi]$

Cpoint3) $\forall x \forall y[(x \in \pi \wedge \mathsf{P}(x, y)) \rightarrow y \in \pi]$

Cpoint4) $\forall x \forall y[\mathsf{sum}(x, y) \in \pi \rightarrow (x \in \pi \vee y \in \pi)]$

He further requires that any pair of connected regions must share at least one point:

Cpoint5) $\forall x \forall y[\mathsf{C}(x, y) \rightarrow \exists \pi[\mathsf{PT}(\pi) \wedge x \in \pi \wedge y \in \pi]]$

The notion of a point's being incident in a region is defined simply as:

$$\mathsf{IN}(\pi, x) \ \equiv_{def} \ (\mathsf{PT}(\pi) \wedge x \in \pi)$$

so that point is identified with the set of regions in which it is incident.

A number of problems arise from Clarke's treatment of points. One is that **Cpoint2** is intuitively false: if a point is incident in two overlapping regions, this does not necessarily imply that it is incident in their product — the regions might be externally connected at one or more points that are not incident in the region of overlap. A further problem (noted by Biacino and Gerla (1991)), is that this treatment of points leads to a collapse of C to O because every pair of connected regions must also overlap. The proof (which does not depend on the discredited **Cpoint2**) is as follows:

> **proof:** Suppose $\mathsf{C}(a, b)$ then from **Cpoint5** we have $\exists \pi[a \in \pi \wedge b \in \pi]$. Now consider the region $r = \mathsf{sum}(\mathsf{compl}(a), \mathsf{compl}(b))$. Suppose r is equal to the universe. From **Cpoint3** we can derive that every point (incident in some region) is incident in the universe[9], so the point π must be incident in $\mathsf{sum}(\mathsf{compl}(a), \mathsf{compl}(a))$. By **Cpoint4** this means that either $\mathsf{compl}(a) \in \pi$ or $\mathsf{compl}(b) \in \pi$, so since $a \in \pi \wedge b \in \pi$ we have either $a \in \pi \wedge \mathsf{compl}(a) \in \pi$ or $b \in \pi \wedge \mathsf{compl}(b) \in \pi$. **Cpoint1** then requires that either $\mathsf{C}(a, \mathsf{compl}(a))$ or $\mathsf{C}(b, \mathsf{compl}(b))$ and both these alternatives contradict the $\neg\mathsf{Ccompl}$ principle. Thus z cannot equal the universe. This means that there exists a region w, such that $w = \mathsf{compl}(r) = \mathsf{compl}(\mathsf{sum}(\mathsf{compl}(a), \mathsf{compl}(b)))$. w must be part of both a and b. So we can conclude that $\mathsf{O}(a, b)$.

Thus, Clarke's introduction of points has the unintended consequence that connection is simply equivalent to overlap. The domain of the theory is then essentially a Boolean algebra with the null

[9] The possibility of an empty point not incident in any region does not appear to be ruled out by **Cpoint1-4**.

element removed and the topology of regions is discrete. It is apparent that the $\neg$Ccompl principle is instrumental in the collapse and this must cast further doubt on the definition of compl. One way to avoid these problems would be to use the following alternative definition of complement

$$\mathsf{compl}(x) =_{def} f(\{y \mid \neg\mathsf{O}(y, x)\}) .$$

However, this would render incorrect the definitions of the topological functions; and it is doubtful whether such functions could be reintroduced even by modified definitions. The theory would then become more like the RCC theory described in the next section. In the RCC theory distinctions between open and closed regions are not expressible.

2.6 The Region Connection Calulus (RCC)

With the intention of providing a logical framework for the incorporation of spatial reasoning into AI systems, Clarke's formalism was investigated and modified in the works (Randell and Cohn 1989) and (Randell 1991). A more radical re-working of the theory was presented in Randell, Cui and Cohn (1992) and it is this version which is described here. The new theory is known as the Region Connection Calculus (RCC). The research reported in this thesis has been very much influenced by this theory.

Like Clarke's theory, RCC is based on a primitive 'connectedness' relation, $\mathsf{C}(x, y)$ and the universe of quantification is intended to be a domain of spatial regions. The relation $\mathsf{C}(x, y)$ is reflexive and symmetric, which is ensured by the following two axioms:

RCC1) $\forall x \mathsf{C}(x, x)$ **(Cref)**

RCC2) $\forall xy[\mathsf{C}(x, y) \rightarrow \mathsf{C}(y, x)]$ **(Csym)**

Relation	*Interpretation*	*Definition of $R(x, y)$*
$\mathsf{DC}(x, y)$	x is disconnected from y	$\neg\mathsf{C}(x, y)$
$\mathsf{P}(x, y)$	x is a part of y	$\forall z[\mathsf{C}(z, x) \rightarrow \mathsf{C}(z, y)]$
$\mathsf{PP}(x, y)$	x is a proper part of y	$\mathsf{P}(x, y) \wedge \neg\mathsf{P}(y, x)$
$\mathsf{EQ}(x, y)$	x is identical with y	$\mathsf{P}(x, y) \wedge \mathsf{P}(y, x)$
$\mathsf{O}(x, y)$	x overlaps y	$\exists z[\mathsf{P}(z, x) \wedge \mathsf{P}(z, y)]$
$\mathsf{DR}(x, y)$	x is discrete from y	$\neg\mathsf{O}(x, y)$
$\mathsf{PO}(x, y)$	x partially overlaps y	$\mathsf{O}(x, y) \wedge \neg\mathsf{P}(x, y) \wedge \neg\mathsf{P}(y, x)$
$\mathsf{EC}(x, y)$	x is externally connected to y	$\mathsf{C}(x, y) \wedge \neg\mathsf{O}(x, y)$
$\mathsf{TPP}(x, y)$	x is a tangential proper part of y	$\mathsf{PP}(x, y) \wedge \exists z[\mathsf{EC}(z, x) \wedge \mathsf{EC}(z, y)]$
$\mathsf{NTPP}(x, y)$	x is a nontangential proper part of y	$\mathsf{PP}(x, y) \wedge \neg\exists z[\mathsf{EC}(z, x) \wedge \mathsf{EC}(z, y)]$

Table 2.2: Defined relations in the RCC theory

Using $\mathsf{C}(x,y)$, further dyadic relations are defined as shown in table 2.2. The relations: P, PP, TPPand NTPP, being non-symmetrical, support inverses. For the inverses the notation Φi is used, where $\Phi \in \{\mathsf{P},\mathsf{PP},\mathsf{TPP},\mathsf{NTPP}\}$. These relations are defined by definitions of the form $\Phi\mathsf{i}(x,y) \equiv_{def} \Phi(y,x)$. Of the defined relations, DC, EC, PO, EQ, TPP, NTPP, TPPi and NTPPi have been proven to form a JEPD[10] set (Randell, Cohn and Cui 1992a). This set is known as RCC-8. As the set is JEPD, any two regions stand in exactly one of these eight relations.

It can be seen that the RCC definitions are almost the same as those of Clarke. The new relations PO, TPP and NTPP have been introduced in order to partition all possible binary relations into a JEPD set. (The relation TP includes EQ as a special case and the universal region is both equal to and an NTP of itself.) Also, the defined relation EQ takes the place of the logical equality = used by Clarke. Consequences of this change will be examined in section 3.2 in the next chapter.

2.6.1 Functional Extension of the Basic Theory

RCC also incorporates a number of functions on regions as well as a constant denoting 'the universal region'. The functions are called quasi-Boolean, since they are intended to generate an algebra very similar to a standard Boolean algebra but with no least element (i.e. no 'null' region). The functions are specified as follows:

$$\begin{aligned}
\mathsf{u} &=_{def} \iota y[\forall z[\mathsf{C}(z,y)]] \\
\mathsf{sum}(x,y) &=_{def} \iota z[\forall w[\mathsf{C}(z,w) \leftrightarrow [\mathsf{C}(w,x) \vee \mathsf{C}(w,y)]]] \\
\mathsf{compl}(x) &=_{def} \iota y[\forall z[(\mathsf{C}(z,y) \leftrightarrow \neg\mathsf{NTPP}(z,x)) \wedge (\mathsf{O}(z,y) \leftrightarrow \neg\mathsf{P}(z,x))]] \\
\mathsf{prod}(x,y) &=_{def} \iota z[\forall u[\mathsf{C}(u,z) \leftrightarrow \exists v[\mathsf{P}(v,x) \wedge \mathsf{P}(v,y) \wedge \mathsf{C}(u,v)]]] \\
\mathsf{diff}(x,y) &=_{def} \iota w[\forall z[\mathsf{C}(z,w) \leftrightarrow \mathsf{C}(z,\mathsf{prod}(x,\mathsf{compl}(y)))]]
\end{aligned}$$

where $\alpha(\overline{x}) =_{def} \iota y[\Phi(y,\overline{x})]$ means $\forall\overline{x}[\Phi(\alpha(\overline{x}),\overline{x})]$. More will be said about these functions and this form of 'definition' in section 3.3.

2.6.2 The Sorted Logic LLAMA

It is important to note that all the quasi-Boolean functions except for sum are *partial* with respect to the domain of regions. This gives rise to a technical problem in that the standard proof-theory (and semantics) of 1st-order logic is based on an assumption that all function symbols correspond to *total* functions. To avoid this difficulty Randell, Cui and Cohn (1992) employ the *sorted* 1st-order logic, LLAMA,[11] as described by Cohn (1987).

The sorted logic allows the domain of discourse to be partitioned into a number of (base) *sorts*, each consisting of a (non-empty) set of entities of a particular kind. For each relation symbol in the vocabulary of a theory, certain combinations of argument sorts are specified. When the relation is combined with arguments whose sorts accord with one of these combinations, the resulting

[10] See section 1.1.3.

[11] Logic Lacking A Meaningful Acronym.

proposition is said to be *well-sorted*; if the argument sorts do not agree with the specification the proposition is *ill-sorted*. Likewise it is specified that application of function to a tuple of arguments will give a well-sorted term only for certain sort combinations of these arguments. Every function application will also have a *result sort* which is the sort of the entity denoted by the term formed by that application. In general the result sort will be an arbitrary (extensional) function of the sorts of the arguments given to a function. A Boolean combination of propositions is well sorted iff all its constituents are well sorted. This gives us a general notion of a well-sorted quantifier-free formula.[12]

In the LLAMA formalism, quantifiers and variables are not themselves associated with any sort restriction; rather, the range of any particular quantification is determined by the context of variables as arguments of sorted functions and relations. Suppose a predicate is formed by replacing one or more terms in a formula with a (new) variable symbol. If a quantifier is then applied to the predicate, the quantifier ranges over all entities in the domain which are such that, if a constant denoting that entity were substituted in the predicate in place of each occurrence of the quantified variable, the resulting formula would be well-sorted. If the domain of possible well-sorted values is empty then the entire formula is ill sorted and considered not to be a well-formed formula of the language.

In the case where multiple quantifiers occur in a formula, the situation is more complex. Here, the interpretation cannot be analysed in terms of successive applications of a single quantification operation; rather, multiple quantifiers serve to quantify over all sequences of individuals such that the formula, when instantiated with this sequence, is a well-sorted ground formula. Thus, in a formula $\forall x \forall y[\Phi(x, y)]$ quantification can be regarded as being over all pairs of entities $\langle a, b\rangle$ such that the formula $\Phi(a, b)$ is well-sorted. This treatment of quantification applies directly only to *prenex* formulae, with all quantifiers at the front, but any formula can be transformed into an equivalent prenex formula and the ranges of quantification determined from this.

A further feature of LLAMA, which makes it particularly expressive, is that for each sort there is a *sortal predicate*. These predicates can be used to specify explicit sortal restrictions on variables in a formula in addition to those determined from the sorts of the ordinary relations and functions.

2.6.3 Sorts in the RCC Theory

In considering the purely spatial aspects of the RCC theory, we may assume that there are just two disjoint (and non-empty) base sorts: REGION and NULL, plus the top sort '$\top$' (this is the join of REGION and NULL — all entities are of this sort) and the bottom sort '$\bot$' (no entity is of this sort).[13] We now declare that the arguments of all relations in the RCC theory are of sort REGION

[12] Note that this characterisation gives us a sorted logic which is *polymorphic*. This means that the permitted sorts of argument places are not individually restricted but may depend on other arguments (e.g. a predicate SPOUSE(x, y) might be allowed to have arguments of sorts $\langle male, female\rangle$ or $\langle female, male\rangle$ but not $\langle male, male\rangle$ or $\langle female, female\rangle$). Likewise, the result sort of a function can vary depending on its arguments.

[13] It is intended that the theory be embedded in a more comprehensive formalism incorporating temporal intervals and physical objects as well as spatial regions. This theory would make use of a much richer sort structure.

and the arguments and return values of all the quasi-Boolean functions are of sort $\top$.

2.6.4 Two Additional Axioms

Making use of the sorted framework a further axiom is given which links the quasi-Boolean functions to the relational part of the theory. The axiom states that the product of two regions is null, if and only if the two regions are discrete (i.e. non-overlapping):

$$\forall x \forall y[\mathsf{NULL}(\mathsf{prod}(x,y)) \leftrightarrow \mathsf{DR}(x,y)] \ .$$

Finally an existential axiom ensures that every region has a non-tangential proper part:

$$\forall x \exists y[\mathsf{NTPP}(y,x)] \qquad \textbf{(NTPP)}$$

The **NTPP** axiom rules out the possibility of atomic models of the theory, in which there is a class of regions (atoms) which have no proper parts. Several possibilities are considered for modifying the theory so as to allow the existence of atoms. These will be considered in section 3.4.

2.6.5 Further Development of RCC

As well as modifying certain axioms of Clarke's theory, Randell, Cui and Cohn (1992) develop their new theory so as to cover further non-topological information. They introduce a new *convex-hull* function, which enables properties involving convexity and containment to be represented. I shall examine this operator in chapter 8. The theory is also extended so as to describe possible modes of 'continuous' change which can occur in spatial configurations. This is done by identifying possible transitions which can occur amongst the topological relations holding between the regions occupied by bodies during some continuous process. I shall comment on this in section 10.3.2.

2.7 Other Relevant Work on Region-Based Theories

I conclude this chapter by briefly mentioning a number of other works which are relevant to the study of region-based theories of space.

2.7.1 de Laguna's Theory

In section 2.2 I referred to de Laguna's (1922) 'geometry of solids'. This theory is based on the primitive relation '*x can connect y and z*' (I shall write this as $\mathsf{CC}(x,y,z)$. This relation is true if it would be possible by displacement and/or rotation to bring x in to such a position that it connects (i.e. touches or overlaps) both y and z. The CC primitive is extremely expressive since it allows definitions of both connectedness: $\mathsf{C}(x,y) \equiv_{def} \forall z[\mathsf{CC}(z,x,y)]$; and relative length: $\mathsf{Longer}(x,y) \equiv_{def} \forall z \forall w[\mathsf{CC}(y,z,w) \rightarrow \mathsf{CC}(x,z,w)]$. Unfortunately this theory does not seem to have been explored or developed by any subsequent researcher in the field.

2.7.2 Grzegorczyk's Undecidability Results

Grzegorczyk's 1951 paper *Undecidability of some Topological Theories* (Grzegorczyk 1951) contains several important and very general results about the undecidability of certain kinds of spatial theory. This quite technical paper seems to be rarely cited by later researchers and came to my attention at a very late stage of my work on this thesis.[14] Although framed in terms of somewhat different formal apparatus from that found in the other spatial theories surveyed in this chapter, Grzegorczyk's undecidability results appear to apply (assuming appropriate notational modifications) to a very wide range of possible spatial theories. The nature and ramifications of these results will be considered in section 3.6.

2.7.3 Some Recent Research in the Field

The formalism of Bochman (1990) is a significant departure from all the others mentioned. A principal feature is that two types of basic mereological element are postulated — 'objects' and 'connections'. The *part* relation is primitive. Objects can have other objects and/or connections as parts, whereas connections are atomic, having only themselves as parts. A connection relation is then defined by saying that objects a and b are connected just in case there exists a connection κ such that every object of which κ is a part also shares a part with a and a part with b.

A survey by Gerla (1995) covers most of the formalisms described in this chapter but considers them from a rather different perspective, focusing on their correspondence to certain kinds of classical topological space.

A modification and development of Clarke's theory is proposed by Asher and Vieu (1995), who give an axiom set based on the C primitive, which is proved to be sound and complete with respect to a class of models based on point-set topology. A novel property of this theory is the definability of a relation of 'weak contact', which is supposed to hold when two bodies touch each other but are not physically joined.

Borgo, Guarino and Masolo (1996) give a theory of spatial regions based on three primitive concepts: the part relation, the property of being a (topologically) 'simple' region and the binary relation of congruence. This theory combines aspects of the connection-based theories derived from Clarke with the approach taken in Tarski's *Geometry of Solids*, whereby the logic of regions can be tied by means of definitions to the (classical, Euclidean) geometry of points.

Results of Pratt and Schoop (1997) concerning a complete axiomatic theory of the 2D Euclidean plane are of direct relevance to this thesis (particularly the next chapter) but their paper was published too recently to be fully considered in the present work. However, I shall make some comments in section 10.2.1.

Antother recent paper by Stell and Worboys (1979) considers the structure of sets of regions in terms of Heyting algebras. This work is closely related to the approach I shall describe in chapters 4, 5 and especially 6, where I use the intuitionistic logic, $\mathcal{I}$, to represent topological relations. $\mathcal{I}$ can also be interpreted in terms of Heyting algebras.

[14] Thanks to Nick Gotts.

Chapter 3

Analysis of the RCC Theory

In this chapter I examine in more detail the 1st-order RCC theory described in section 2.6. I start with a critique of its axioms. The consequences of 1st-order axioms are often far from obvious, so some new meta-level notations are developed to facilitate analysis of the theory. Using these tools I investigate the structures of possible models of the axioms. I go on to suggest an alternative axiom system which is in several respects easier to manipulate than the original theory. At the end of the chapter I specify a partial decision procedure for the revised theory based on the method of quantifier elimination.

3.1 RCC in Relation to this Thesis

Although the RCC theory was intended as a language for both representing and reasoning about spatial information, in its initial development representation was the primary focus. It was soon realised that, whilst the theory is very expressive, reasoning with RCC is extremely difficult. My research has been directed towards addressing this problem. Quite early in my investigation of the RCC reasoning problem I discovered a computationally feasible method for reasoning about certain spatial relationships. This does not make any use of the actual RCC axiom system but uses a radically different formalism to represent and reason about a large class of spatial relations, all of which are also definable in the RCC system. The representation, based on a topological interpretation of intuitionistic logic, is described in detail in chapter 6.

My intuitionistic encoding partly solved the RCC reasoning problem; but was only capable of handling a small (albeit significant) subset of the spatial relationships expressible in RCC. Thus the possibility of finding a much more comprehensive reasoning algorithm — possibly one that would cover everything expressible in RCC — still remained. Furthermore, many puzzles concerning the RCC formalism became apparent. It was clear that the axioms did not characterise a single unique model. The intended model was to accord with our intuitive (naïve) ideas about 'regions' (of fixed dimension) existing in a topologically simple space. However, the dimensionality and global topology of the space was not fixed by the axioms. Moreover the existential import of the theory appeared to be too weak to determine exactly which configurations of regions are possible.

The question arose as to whether RCC could be extended to yield an syntactically complete theory (see section 3.6) with a unique denumerable model (i.e. an $\aleph_0$-*categorical* theory). As well as remedying the representational shortcomings of RCC, such a theory would be a very significant step towards solving the reasoning problem. This is because any syntactically complete 1st-order theory must be decidable.

Recent discoveries by Nicholas Gotts and myself strongly suggest that this goal *cannot* be obtained. Specifically, there can be no complete 1st-order characterisation of the intended domain. This can be demonstrated by showing that if such a formalisation were given it would provide a complete theory and decision procedure for 1st-order arithmetic, which is known to be both undecidable and not characterisable by any axiomatic theory (Gödel 1931). The demonstration involves showing that the concepts of arithmetic can be defined in terms of spatial properties which are also definable in RCC. Details of the proof are beyond the scope of the present thesis.

Given that RCC is undecidable and a complete 1st-order characterisation of spatial regions is impossible, further enquiry into RCC can proceed in two directions. Firstly, it is almost certain that a complete characterisation of the intended domain can be given by adding one or more 2nd-order axioms (and perhaps also further 1st-order axioms) to the theory. Secondly, since a comprehensive reasoning algorithm for the domain of RCC is impossible, it will be important to identify more restricted languages for expressing spatial information, for which effective algorithms — or at least decision procedures — can be constructed.

The RCC theory provides a very expressive language for specifying spatial information. However, there are certain features that are problematic. In this chapter I attempt to clarify a number of aspects of the theory and suggest some modifications to its formalisation. Specifically, I consider: extensionality and identity conditions; the status of the quasi-Boolean functions; the *sort* theory and the 'null' region; the NTPP axiom; and models of the theory. I then present a revised axiom set constructed so as to avoid some of the main problems brought to light by the analysis. In chapter 7 I shall give a partial decision procedure for the new theory.

3.2 Identity and Extensionality

In contrast with the theory of Clarke, the RCC theory contains no 'axiom of extensionality'. In this section I consider whether or not such an axiom ought to be added to the theory.

Axiomatic theories (particularly those which seek to characterise a single primitive relation), often contain some kind of *axiom of extensionality*. This is an axiom which asserts that the identity of any two objects follows from their indiscernibility with respect to some property. Thus in set theory we have:

$$\forall x \forall y [\forall z [z \in x \leftrightarrow z \in y] \rightarrow (x = y)]$$

Such axioms can be regarded as strengthened forms of Leibniz' principle of the *identity of indiscernibles*. This principle is the left-to-right component of a second order axiom which can be regarded as defining identity:

$$\forall x \forall y[\ \forall \Psi[\Psi(x) \leftrightarrow \Psi(y)] \ \leftrightarrow \ x = y\ ]$$

Rather than requiring objects to be indiscernible with respect to all properties, we may require only that they cannot be distinguished in terms of a family of properties formed by (partially) instantiating some relation (over the universe of objects). The idea behind this specialisation of the axiom is that this family of properties is regarded as fixing all properties expressible in the theory. In the RCC calculus of regions the obvious axiom of extensionality would be:

$$\forall x \forall y[\forall z[\mathsf{C}(x,z) \leftrightarrow \mathsf{C}(y,z)] \rightarrow (x = y)] \qquad \textbf{(Cext)}$$

This states that if two regions x and y cannot be distinguished by some instance of $\mathsf{C}(\ldots, z)$ (i.e. we cannot find any region z such that $\mathsf{C}(x,z)$ does not have the same truth-value as $\mathsf{C}(y,z)$) they must be the same region. The force of this axiom is to claim that C is the defining relation for regions: regions can only be distinct if they differ with respect to their connectedness with other regions. Whether this is reasonable depends on what we take to be the domain of regions. If regions are made up of discrete atoms then configurations can easily arise where two distinct regions are indiscriminable in terms of the regions they are connected to. But, if every region has a non-tangential proper part and for every pair of non-identical regions there is some region which is part of one but not the other, then **Cext** must hold.

In the RCC theory we can derive something very similar to the axiom of extensionality. From the definitions of $\mathsf{EQ}(x,y)$ and $\mathsf{P}(x,y)$ given above we can very easily show that:

$$\forall x \forall y[\forall z[\mathsf{C}(x,z) \leftrightarrow \mathsf{C}(y,z)] \leftrightarrow \mathsf{EQ}(x,y)] \qquad \textbf{(CEQ)}$$

However, since the 'EQ' symbol is introduced by definition, this derived formulae does not have the force of the axiom of extensionality because 'EQ' need not necessarily have the properties of logical equality. Hence, the derivation does not show that an axiom of extensionality is redundant in the RCC calculus. What it shows rather is that if we take the equivalence

$$\forall x \forall y[(x = y) \ \leftrightarrow \ (\mathsf{P}(x,y) \wedge \mathsf{P}(y,x))] \qquad \textbf{(P =)}$$

as an axiom rather than a definition and assume that the symbol '=' is to have its usual logical properties, then this formula is equivalent to **Cext** and can thus serve as an axiom of extensionality for the RCC theory.

3.3 The Quasi-Boolean Functions

Most of the complexity of the RCC theory arises from the quasi-Boolean functions. In this section I examine the role of these functions in the theory and suggest how they could be handled in a more precise and economical way.

3.3.1 The Status of the Function Definitions

In Randell, Cui and Cohn (1992) the functions are introduced by means of a (non-standard) form of definite description operator. For example a 'sum' function is characterised as:

$$\mathsf{sum}(x,y) \ =_{def} \ \iota z[\forall w[\mathsf{C}(w,z) \leftrightarrow \ [\mathsf{C}(w,x) \vee \mathsf{C}(w,y)]]]$$

where the iota notation is to be interpreted as follows:

$$\alpha(\overline{x}) \ =_{def} \ \iota y[\Phi(y,\overline{x})] \qquad \text{means} \qquad \forall\overline{x}[\Phi(\alpha(\overline{x}),\overline{x})] \ .$$

Thus the sum 'definition' can be rewritten as:

$$\forall x\forall y\forall w[\mathsf{C}(w,\mathsf{sum}(x,y)) \leftrightarrow (\mathsf{C}(w,x) \vee \mathsf{C}(w,y))]$$

It should be noted that this formula is not purely definitional since, because all functions must have a value, the use of the sum function carries existential commitment. In general a formula which introduces a new function symbol into a theory cannot be regarded as a definition unless entities with appropriate properties to be values of the function are already guaranteed to exist as a consequence of the axioms of the theory.[1]

It is also important to note that the formula characterises the sum function only in the context of the C predicate. It can be contrasted with the following *explicit* characterisation:

$$\forall x\forall y\forall z[z = \mathsf{sum}(x,y) \leftrightarrow \forall w[\mathsf{C}(w,z) \leftrightarrow (\mathsf{C}(w,x) \vee \mathsf{C}(w,y))]]$$

This formula imposes a stronger condition on the domain of the C relation: namely that, given any two regions x and y, there is exactly one region that is connected to just the regions that are connected either to x or to y. It is quite easy to see that the contextual sum definition is logically equivalent to the left-to-right direction of the *explicit* sum definition. However the right-to-left direction does not follow. To get the right-to-left implication we also need the axiom of extensionality, **Cext**, given in the last section. Alternatively, one could replace the contextual sum axiom with the explicit one. If this is done and we also stipulate that $\mathsf{sum}(x,x) = x$, then the axiom of extensionality is immediately derivable.

3.3.2 RCC without Functions or Sorts

There are two reasons for the use of sort theory in formulating the RCC theory. Firstly, to accommodate functions which are partial with respect to the domain of regions; and secondly, because by casting a theory in sorted logic and using a proof procedure designed to treat sortal information in an efficient way, the effectiveness of automated theorem proving can often be greatly increased (Cohn 1987). However, the apparatus of functions and sorts does result in a formal language which is rather complex, both in its syntax and semantics. If we are primarily interested in investigating

[1] This applies whether or not we employ a sorted logic. However, if we use a sorted logic, we can allow that the values of functions need not be regions; so the existential commitment need not affect the theory with regard to the properties of regions.

the content and consequences of the RCC axioms, it is perhaps better to cast the theory in a simpler language. RCC can easily be modified so as to give a function-free unsorted version of the theory. The axioms introducing the quasi-Boolean functions are replaced by existential statements. Where the function is partial, the existential statement is nested within an implication. Thus the axioms introducing u, sum, compl and prod can respectively be replaced by the following:[2]

$$\exists x \forall y[\mathsf{C}(x,y)]$$

$$\forall x \forall y \exists! z\, \forall w[\mathsf{C}(w,z) \leftrightarrow (\mathsf{C}(w,x) \vee \mathsf{C}(w,y))]$$

$$\forall x[\exists y[\neg \mathsf{C}(x,y)] \leftrightarrow \exists! y[\forall z[(\mathsf{C}(z,y) \leftrightarrow \neg \mathsf{NTPP}(z,x)) \wedge (\mathsf{O}(z,y) \leftrightarrow \neg \mathsf{P}(z,x))]]]$$

$$\forall x \forall y[\mathsf{O}(x,y) \leftrightarrow \exists! z\, \forall w[\mathsf{C}(w,z) \leftrightarrow \exists v[\mathsf{P}(v,x) \wedge \mathsf{P}(v,y) \wedge \mathsf{C}(v,w)]]]$$

3.3.3 The Complement Function

Of all the axioms in the RCC theory, the one that introduces the complement function is the most complex and its consequences the hardest to fathom. In its original form the axiom is

$$\mathsf{compl}(x) \ =_{def} \ \iota y[\forall z[(\mathsf{C}(z,y) \leftrightarrow \neg \mathsf{NTPP}(z,x)) \wedge (\mathsf{O}(z,y) \leftrightarrow \neg \mathsf{P}(z,x))]] \qquad (\mathbf{complDef})$$

and if we assume **Cext** this is equivalent to:

$$\forall x \forall y[\ y = \mathsf{compl}(x) \leftrightarrow (\forall z[\mathsf{C}(z,y) \leftrightarrow \neg \mathsf{NTPP}(z,x)] \wedge \forall z[\mathsf{O}(z,y) \leftrightarrow \neg \mathsf{P}(z,x)])] \quad (\mathbf{complDef2})$$

From this it can readily be proved that

$$\forall x[\mathsf{EC}(x, \mathsf{compl}(x))] \ .$$

The definition of compl seems to be rather more complex than is desirable. The condition $y = \mathsf{compl}(x)$ is asserted to be equivalent to two separate universal constraints on x and y. Moreover, the first of these specifies exactly what is connected to y, the complement of x, in terms of the NTPPs of x. If the theory is extensional with respect to C then this specification alone should determine all the properties of any region's complement.

However, the second constraint specifying that the things that overlap the complement of x are exactly the things that are not part of x also appears to be true in the intended interpretation, and even seems to completely specify the complementation function. One might hope that the two conditions could be proved equivalent as a consequence of the definitions of the relations involved and the other functions. But despite considerable effort and extensive use of the OTTER theorem prover (McCune 1990), I have not been able to demonstrate this. Thus, the compl axiom seems to contain not only existential commitment but also to indirectly assert a universal equivalence between two ways of describing certain properties of regions.

In view of these observations I suggest that it is more perspicuous to replace the compl axiom by the following two axioms whose conjunction is equivalent to the original:

$$\forall x \exists! y \forall z[\mathsf{C}(z,y) \leftrightarrow \neg \mathsf{NTPP}(z,x)]$$

[2] Here, $\exists! x[\Phi(x)]$ means there is a unique entity satisfying $\Phi(\ldots)$ — i.e. $\exists x[\Phi(x) \wedge \forall y[\Phi(y) \rightarrow y = x]]$.

$$\forall z[(\mathsf{C}(z,x) \leftrightarrow \neg\mathsf{NTPP}(z,y))] \leftrightarrow \forall z[(\mathsf{O}(z,x) \leftrightarrow \neg\mathsf{P}(z,y))]$$

A further worry concerning the compl axiom is that I was unable (again despite considerable effort) to show that complementation is a symmetrical operation (i.e. that $x = \mathsf{compl}(y) \leftrightarrow y = \mathsf{compl}(x)$). This may mean that RCC is lacking the following clearly desirable theorem:

$$\forall x \forall y[(\forall z[\mathsf{C}(z,x) \leftrightarrow \neg\mathsf{NTPP}(z,y)] \rightarrow \forall z[\mathsf{C}(z,y) \leftrightarrow \neg\mathsf{NTPP}(z,x)])] \, .$$

This could also be derived if we adopted the simple formula $\forall x[\mathsf{compl}(\mathsf{compl}(x)) = x]$ as an axiom.

3.3.4 Relation to Orthodox Boolean Algebras

Boolean algebras are a very well understood class of mathematical structures. Since I will be making much use of these algebras (especially in the next chapter) it will be as well to give them a formal definition:

A Boolean algebra is a structure $\mathcal{A} = \langle S, +, \bot \rangle$, where S is a set of all the elements of the algebra, '$+$' is a function from $S \times S$ to S and '$\bot$' is a function from S to S.[3] These operations must satisfy the equations given in table 3.1, in which the '$x \cdot y$' operation is defined as equivalent to '$\bot(\bot x + \bot y)$' and the null and unit elements are defined by $0 \;=_{def}\; \bot(x + \bot x)$ and $1 \;=_{def}\; x + \bot x$. These equations are taken (with some modification of the presentation) from Kuratowski (1972) p.34.

$(x+y) = (y+x)$	$(x \cdot y) = (y \cdot x)$
$(x + (y+z)) = ((x+y)+z)$	$(x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)$
$x + (x \cdot y) = x$	$x \cdot (x+y) = x$
$x + 0 = x$	$x \cdot 1 = x$
$((x \cdot \bot y) + y) = (x+y)$	$((x \cdot \bot y) \cdot y) = 0$
$(x \cdot (y+z)) = ((x \cdot y) + (x \cdot z))$	

Table 3.1: An equational theory of Boolean algebras

It will be recalled that in the RCC theory there is no null region, which would correspond to the least element '$\emptyset$' in an orthodox Boolean algebra; and this is why the RCC functions are called 'quasi-Boolean'. But, there seems no reason why the functions in the RCC theory should not be regarded as genuine Boolean operators over the domain $\mathsf{REGION} \cup \mathsf{NULL}$. This would fix the properties of these operators by reference to a well understood structure. However, if we regard the RCC functions in this way we still have the problem of axiomatically linking the Boolean algebra to the relational part of the theory. This problem is complicated by the sort theory.

[3] I shall usually write the complementation operation as a prefix function '$-(\ldots)$'; but, where the algebra is a Boolean algebra of sets, I shall often write $\overline{X}$ to mean the complement of the set X.

3.3.5 A Single Generator for Boolean Functions

A standard Boolean algebra has the property that all operators are definable in terms of a single primitive function. In fact there are two possible primitives that can be used: in the terminology of electronic circuitry they are NAND and NOR. In a Boolean algebra of regions these operations correspond to 'complement of product' and 'complement of sum'. Thus, using the first alternative, starting with a function $\mathsf{cp}(x, y)$, the more familiar Boolean operations (together with null and universal constants) can be defined as follows:

- $\mathsf{compl}(x) \ =_{def} \ \mathsf{cp}(x, x)$
- $\mathsf{sum}(x) \ =_{def} \ \mathsf{cp}(\mathsf{cp}(x, x), \mathsf{cp}(y, y))$
- $\mathsf{prod}(x, y) \ =_{def} \ \mathsf{cp}(\mathsf{cp}(x, y), \mathsf{cp}(x, y))$
- $\emptyset \ =_{def} \ \mathsf{prod}(x, \mathsf{compl}(x))$
- $\mathsf{u} \ =_{def} \ \mathsf{sum}(x, \mathsf{compl}(x))$

This means of introducing the Boolean functions by pure definitions from a single function has the great advantage that in axiomatising the theory we need only be concerned with fixing the meaning of cp and its relationship with C — all properties of the other functions and constants will be consequences of their definitions.

3.3.6 Introduction of a Null Region

If we allow the null entity to be a *bona fide* region then the technical problems associated with the Boolean functions disappear. The functions become total rather than partial and hence there is no need to use a sorted logic in order to employ these functions in a 1st-order formalism.[4]

Introduction of a null region requires some revision of the fundamental RCC axioms. An intuitive consideration of the notion of connection suggests that the null-region should not be considered as connected to any other region. Thus we have the new axiom

$$\forall x[\neg \mathsf{C}(x, \emptyset)] \ .$$

Consequently, the reflexivity of the connection relation must be restricted so as only to hold for non-null regions. Thus the **Csym** axiom must be replaced with the weaker formula

$$\forall x \forall y[\mathsf{C}(x, y) \rightarrow \mathsf{C}(x, x)] \ .$$

3.4 Atoms and the NTPP Axiom

Randell, Cui and Cohn (1992) give an informal proof of the impossibility of having 'atomic' regions in a model of the axioms. These putative atoms would be regions having no proper parts. (Here we assume a theory without the **NTPP** axiom, which of course explicitly rules out such models.)

[4] Of course we may still wish to employ a sorted logic for the purpose of increasing the efficiency of automated deduction.

Suppose a region r has no proper parts. It therefore has no non-tangential proper parts; and thus, because of the compl axiom, it follows that every region is connected to the complement of r. Thus (assuming the **Cext** axiom) $\mathsf{compl}(r)$ must be the universal region, u. We can further conclude that $\mathsf{P}(r, \mathsf{compl}(r))$ and then from the definition of O we see that $\mathsf{O}(r, \mathsf{compl}(r))$. However, as mentioned above, from **complDef** and **Cext** one can derive $\forall x[\mathsf{EC}(x, \mathsf{compl}(x)]$. Thus $\mathsf{EC}(r, \mathsf{compl}(r))$. But, from the definition of EC, we must have $\neg\mathsf{O}(r, \mathsf{compl}(r))$ — a contradiction.

It has been suggested that there is a consistent atomic model of the RCC axioms in which only one (non-null) region exists.

Thus the NTPP axiom is derivable from the other axioms of the theory. Randell, Cui and Cohn suggest that the difficulty arises because the definition of the part relation is incompatible with the existence of atoms. Three possible solutions are given.

The first is to divide the domain of regions into three disjoint sorts: PROPER-REGIONs, ATOMs and PARTICLEs. All of these kinds of region must have NTPPs in accordance with the **NTPP** axiom. However, the proper parts of ATOMs are PARTICLEs and not PROPER-REGIONs. It is further required by additional axioms that: 1) if two ATOMs overlap they must be equal; and 2) every PROPER-REGION has a part which is an ATOM. Whilst this proposal may have some attractions as a conceptual scheme, it is far from clear whether it can really be made into a consistent theory and the added complexity of the sort structure would make the language far more unwieldy than the basic RCC theory.

The two further alternative treatments of atoms given by Randell, Cui and Cohn (1992) involve even more radical departures from the basic theory. One of them requires the function sum as well as the sort ATOM to be taken as primitives in addition to the original C. The other requires a new sort of POINTs to be added to the domain and is based on a new primitive relation, $\mathsf{IN}(p, r)$, of incidence, holding between points and regions — C is then introduced as a defined relation. These alternative theories are too far from the original to be considered in the present work.

In summary it must be said that the origin of the non-atomicity of regions in the RCC theory is not fully understood. Each of the alternatives proposed by Randell, Cui and Cohn (1992) seem more complex than is desirable and have not been worked out in detail. Another plausible suggestion made in that paper is that the problem lies with the definition of P; but a revised definition was not given.

3.5 Models of the RCC Theory

The RCC theory was initially developed through a methodology of specifying intuitively correct axioms rather than by considering mathematical models of space. However, in order to establish important meta-theoretic results such as completeness and categoricity (discussed further below) some kind of formal semantics is needed. Being formulated in 1st-order logic, the general purpose set-theoretic interpretation of that language may of course be employed; but consideration of the particular nature of the RCC theory suggests that other kinds of model may be more appropriate.

3.5.1 Graph Models of the C relation

Models of the C relation can be represented by symmetric, reflexive digraphs or more simply by non-directed graphs, in which each node is implicitly taken as being connected to itself. If we only require C to be reflexive and symmetric, then all such graphs will correspond to possible models of the theory. As we add further axioms we place constraints on admissible structures for the C relation. For examining these models it will be helpful to be able to refer to the set of all regions connected to some region, x. Thus $C(x)$, which may be called the C-*set*[5] of x, is defined as

$$C(x) \;=_{def}\; \{y \mid \mathsf{C}(x,y)\}$$

In terms of C-sets, symmetry and reflexivity correspond respectively to the facts

$$x \in C(x) \qquad \text{and} \qquad x \in C(y) \leftrightarrow y \in C(x) \;;$$

and the extensionality axiom **Cext** can be expressed by

$$C(x) = C(y) \;\leftrightarrow\; x = y \;.$$

Other logical properties of RCC, such as those stemming from the quasi-Boolean function axioms, would correspond to more subtle constraints on the domain of C-sets.

Models based on connection graphs and/or C-sets are very straightforwardly related to the relational vocabulary of the RCC theory and the ontological commitments embodied in such models do not go beyond what is implicit in the theory. However, they have a number of shortcomings. Graph models are very general and can be given for any theory based on a binary relation, so they do not characterise any properties which are particular to the spatial domain. Consequently they do not accord well with our perception of real situations. (In fact, as a means of building a mental picture of a situation described by some RCC formulae, graph models are worse than useless: if we visualise two connected regions as two blobs joined by an arc, we thereby picture the regions as disconnected!) A further problem for the researcher is that the graph models cannot readily be related to classical models of geometry and topology.

3.5.2 Models in Point-Set Topology

In contrast with graphs of the C relation, the topological spaces of classical point-set topology provide a well-understood class of mathematical structures, which — despite some subtleties — seem to accord much better with our perceptions of spatial situations. Whilst associating physical bodies with sets of points is an abstraction which requires a certain amount of imagination, spatial relationships between point-sets can be pictured in much the same way as relationships between physical bodies. One difference is that in the point-set model we can distinguish between *open* and *closed* sets, whereas physical bodies do not come in open and closed varieties. However, as we soon shall see, it is possible to give a point-set interpretation of 'region' under which no open/closed distinction arises.

[5] These sets are also employed in the analyses of Biacino and Gerla (1991) and Gerla (1995).

Advocates of the 'naïve' approach to knowledge representations may object to the use of topological models on the grounds that the mathematical content of these models goes far beyond the understanding of space enjoyed by the average person. Dogmatic adherents of the region-based approach may also object to the appearance of points in the models of a theory which is supposed to avoid commitment to the existence of points. Whilst I acknowledge the motivations for these objections, I take what I regard as a more pragmatic approach to the examination of region-based theories and am prepared to employ any mathematical apparatus that seems to be useful. I do think that one reason why region-based formalisms may be useful is that they are close to natural ways of describing space; but I do not think this means that in developing and investigating a formal theory of regions one should be restricted to employing only 'naïve' concepts.

3.5.3 Interpreting RCC in Point-Set Topology

To characterise the meaning of the non-logical vocabulary of RCC in terms of point-set topology we need to specify precisely how the individuals of the theory (i.e. regions) and the connection relation are to be interpreted by reference to a topological space. One possible specification is as follows:

- *Regions* are identified with non-empty open sets of points.
- Regions are *connected* if their closures share at least one point.

This interpretation is that suggested for the RCC theory in (Randell, Cui and Cohn 1992).

If we require that the theory should satisfy the extensionality principle, **Cext**, this immediately leads to a restriction on the class of open sets that can be considered regions: no two distinct regions can be identified with (open) sets that have the same closure. The most obvious way to ensure this is to specify that regions correspond only to *regular* open sets — i.e. those which are equal to interiors of their closures.

From the topological characterisation of C we ought to be able to derive interpretations in terms of point set-topology of all relations definable in RCC. Given the 1st-order definition of P ($\mathsf{P}(x,y) \equiv_{def} \forall z[\mathsf{C}(z,x) \rightarrow \mathsf{C}(z,y)]$) and the fact that for regular (open) sets $c(X) \subseteq c(Y)$ iff $X \subseteq Y$, it is clear that the parthood relation between regions corresponds to the subset relation in the point-set interpretation.

The intersection of two (regular) open sets is always a (regular) open set; so two open sets share a point just in case they share a non-empty (regular) open subset. If we assume that every non-empty regular open set of points corresponds to some region, we can say that two regions overlap if they share a point and this will accord with the 1st-order definition of overlap in terms of the C relation ($\mathsf{O}(x,y) \equiv_{def} \exists z[\mathsf{P}(z,x) \wedge \mathsf{P}(z,y)]$).

Formally the C, P and O relations can be defined as:

$$\mathsf{C}(x,y) \equiv_{def} \exists \pi[\pi \in c(X) \wedge \pi \in c(Y)]$$

$$\mathsf{P}(x,y) \equiv_{def} X \subseteq Y$$

$$\mathsf{O}(x, y) \equiv_{def} \exists\pi[\pi \in i(X) \wedge \pi \in i(Y)]$$

These definitions give us a rigorous formal specification of the RCC connection and overlap relations in terms of point-set topology. But they make use of a highly expressive set-theoretic language, including both quantification and the element relation, and hence are not very useful for automated reasoning. In the next chapter we shall see how essentially the same topological interpretation can be expressed algebraically, without the use of set-theory and quantification.

3.5.4 The Boolean Algebra of Regular Open Point-Sets

In the last section we saw that the RCC regions can be identified with non-empty *regular open* sets in a topological space. If this interpretation is to be adequate for the full theory equipped with quasi-Boolean functions, we need to be able to interpret these as functions operating on (non-empty) regular open sets. If we were simply to use the elementary Boolean set functions (complement, union, intersection) to model Boolean functions on regions we would immediately run into difficulties. The problem is that if we apply these operations to regular open sets, the resulting set is not necessarily regular open: the complement of a regular open set is regular closed; and the sum of two regular open sets is open but need not be regular.

This problem can be avoided by identifying Boolean functions on regions with the operators in the *regular open (Boolean) algebra* of a topological space. Given a topological space $\langle U, O\rangle$, the elements of this algebra are the regular open sets. The Boolean constants and functions are then defined as follows:

$$0 =_{def} \emptyset \qquad 1 =_{def} U$$

$$\perp(X) =_{def} i(\overline{X})$$

$$x \cdot y =_{def} X \cap Y \qquad x + y =_{def} i(c(X \cup Y))$$

Thus the regular complement is defined as the interior of the ordinary set complement and the regular sum is obtained by taking the interior of the closure of the set union. Product is simply defined as intersection. It can easily be verified that, given regular open sets as operands, the results of these operations are also regular open sets.

3.5.5 A Dual Topological Interpretation

There is also a dual interpretation under which regions are identified with *closed* sets — these are connected if they share a point and overlap if their *interiors* share a point. As before the requirements of the theory mean that the closed sets corresponding to regions must be non-empty and regular (a regular closed set is a set X such that $X = c(i(X))$). The regularity condition ensures that sets corresponding to regions must have a non-empty interior.

3.6 Completeness and Categoricity

In section 2.7.2, I mentioned that some results of Grzegorczyk (1951) have important consequences regarding the properties of spatial theories. This paper considers 1st (and higher) order theories of Boolean algebras supplemented with additional spatial functions and/or relations. The 1st-order theory of Boolean algebra is decidable but Grzegorczyk shows that the introduction of either a closure operation or an external connection relation, satisfying in each case a small set of algebraic conditions, results in a structure whose 1st-order theory is undecidable.

The assumed conditions on the closure operation are just those given in section 2.1 and the conditions on the external connection relation are as follows:

- $\mathsf{EC}(x, y) \rightarrow (\mathsf{prod}(x, y) = \emptyset)$
- $(\mathsf{prod}(x, y) = \emptyset) \wedge (\mathsf{sum}(x', x) = x) \wedge (\mathsf{sum}(y', y) = y) \wedge \mathsf{EC}(x', y') \rightarrow \mathsf{EC}(x, y)$
- $\mathsf{EC}(\mathsf{sum}(x, y), z) \wedge \neg\mathsf{EC}(x, z) \rightarrow \mathsf{EC}(y, z)$

These conditions are quite weak and one would expect them to be satisfied in any plausible theory of connection. This means that any 1st-order language containing Boolean (or quasi-Boolean) functions and a connection relation must be undecidable.

The question of what levels of expressiveness lead to undecidable languages is of crucial importance for automated reasoning. In the following chapters we shall see that it is possible to specify quite expressive representations for spatial information, which are decidable. The strategy is to find ways of expressing spatial relationships without the need for a full 1st-order language. One approach is to use a 1st-order language with limited forms of quantification. In chapter 7 I shall show that in a 1st-order theory based on the C relation it is in many cases possible to eliminate quantifiers by replacing quantified clauses with equivalent quantifier free formulae. Another approach is to use a 0-order representation language which is more expressive than classical propositional logic. Although augmenting a Boolean algebra with additional operators (such as a closure function) may lead to an undecidable 1st-order theory, it can also greatly extend the range of information which can be expressed in the form of algebraic equations without quantification. In chapter 4 we shall see how a 0-order (modal) logical language can be used to reason about such constraints.

An important corollary of the undecidability result is that no finitary[6] 1st-order theory of spatial regions (possessing a certain minimal expressivity) can be *complete*. A theory Θ is complete with respect to a language $\mathcal{L}$ iff for every formula ϕ expressed in the language $\mathcal{L}$, either $\Theta \rightarrow \phi$ or $\Theta \rightarrow \neg\phi$ is logically valid.[7] If a (finitary) 1st-order theory is complete, it is also decidable. This follows from the semi-decidability of (finitary) 1st-order logic: any logically valid 1st-order formula is provable in finite time; so to decide whether ϕ follows from Θ one can attempt to prove in parallel

[6] More will be said in section 10.2.1 about the restriction of this result to finitary systems.

[7] Note that, if this is the case, $\mathcal{L}$ can contain only a fixed finite vocabulary of non-logical expressions constrained by the theory. If it contained arbitrary relations, functions or constants it could not be complete.

(or by alternating from one proof to the other) both the sentences $\Theta \rightarrow \phi$ and $\Theta \rightarrow \neg\phi$. A proof of one of these formulae can always be obtained in finite time.

Any theory in which one can define a relation of external connection satisfying certain conditions must be undecidable. Moreover, since any complete 1st-order theory is decidable, any 1st-order theory in which this relation is definable must be incomplete. This means that there are purely theoretical RCC formulae (i.e. formulae not involving any arbitrary constants), which are contingent with respect to the RCC axioms — and indeed with respect to any sensible set of 1st-order axioms. Hence, RCC is not *categorical* — there must be multiple non-isomorphic models of the theory — and cannot be made categorical without adding some 2nd-order axiom. Because of this lack of categoricity, the entailments provable in the RCC theory are only those that hold in a very large class of possible models, many of which will have a very different structure to what is intended.

In fact, it is readily apparent that there is no single model of the axioms. For instance, the dimensionality of regions is not fixed: one can interpret them as being of two, three or even higher dimension. Moreover, spatial configurations which are impossible in (say) 2D may become possible in 3 or more dimensions. I have devoted considerable effort to the problem of finding a categorical version of the RCC theory and have shown how by adding extra axioms many unwanted models can be ruled out. I have concentrated specifically on characterising the dimensionality of RCC regions and on eliciting a complete set of existential axioms (this work is reported in (Bennett 1996a)). However, it was only towards the end of my PhD. research that I realised that categoricity could not be achieved by means of a (finite) 1st-order theory.

The undecidability of RCC and similar theories means that the problem of incorporating qualitative spatial information into AI systems divides into two parts: i) the foundational problem of providing a sufficiently rich theory of spatial concepts with a precise formal semantics; and ii) the problem of constructing inference algorithms for reasoning in terms of useful but less expressive representation languages.

3.7 A Revised Version of the RCC Theory

I now present an axiom set for an unsorted 1st-order theory of regions. The theory differs from Clarkes's and the RCC theory in that a null element is treated as a first-class region. This means that the Boolean component of the theory can be axiomatised much more straightforwardly than in the earlier theories. Apart from this the theory is intended to be much the same as RCC. Following RCC rather than Clarke, every non-null region is connected to its complement and no distinction can be made between open and closed regions.

It must be stressed that, although my revised axiom set avoids many of the problems with the RCC theory that were noted earlier in this chapter, a great deal of further work remains to be done on this theory. This is beyond the scope of the present work. In the remainder of the thesis I shall focus on alternatives to 1st-order theories, that are better suited to automated reasoning.

Preliminary Definitions

To make the axioms easier to state we need the following definitions:

D1) $\mathsf{P}(x,y) \equiv_{def} \forall z[\mathsf{C}(z,x) \rightarrow \mathsf{C}(z,y)]$

D2) $\mathsf{O}(x,y) \equiv_{def} \exists z[\mathsf{C}(z,z) \wedge \mathsf{P}(z,x) \wedge \mathsf{P}(z,y)]$

D3) $\mathsf{NTP}(x,y) \equiv_{def} \forall z[\mathsf{C}(z,x) \rightarrow \mathsf{O}(z,y)]$

Fundamental Axioms

A set of fundamental axioms can now be stated as follows:

A1) $\forall x \forall y[\mathsf{C}(x,y) \rightarrow \mathsf{C}(x,x)]$

A2) $\forall x \forall y[\mathsf{C}(x,y) \rightarrow \mathsf{C}(y,x)]$

A3) $\forall x \forall y[\forall z[\mathsf{C}(x,z) \leftrightarrow \mathsf{C}(y,z)] \rightarrow (x=y)]$

A4) $\forall x \forall y \exists z \forall u[\mathsf{C}(z,u) \leftrightarrow (\neg\mathsf{NTP}(u,x) \vee \neg\mathsf{NTP}(u,y))]$

A5) $\forall x[\mathsf{C}(x,x) \rightarrow \exists y[\mathsf{C}(y,y) \wedge \mathsf{NTP}(y,x)]]$

Axiom 1 is the new restricted reflexivity axiom, which allows only the null region to be disconnected from itself. 2 is the unchanged symmetry axiom and 3 is the extensionality axiom.

The fourth axiom guarantees that for any two regions, x and y, there is a region, z which is connected to every region which is not a non-tangential part of both x and y (and, because of the extensionality axiom, there can only be one such region). Under the intended interpretation, z is the complement of the product of x and y. A complement of product function, $\mathsf{cp}(x,y)$ can now be defined as:

D4) $(\mathsf{cp}(x,y) = z) \equiv_{def} \forall u[\mathsf{C}(u,z) \leftrightarrow (\neg\mathsf{NTP}(u,x) \vee \neg\mathsf{NTP}(u,y))]$

Unlike the function specifications in the original RCC theory, this is purely definitional because the existential import and uniqueness of the function are already entailed by the other axioms. As was explained in section 3.3.5 the Boolean functions and universal and null constants can all be easily defined in terms of the cp function. Moreover, because in the new theory the null entity is accepted as a true region, these will be proper rather than 'quasi' Boolean functions.

Finally, axiom 5 is a new version of the NTPP axiom modified to take account of null regions and using the simpler NTP in place of NTPP.

Additional Axioms

The system should also satisfy the theorems given below. At present I take these as additional axioms. However, it is likely that they are not all independent of each other and of the fundamental axioms, in which case they could be omitted from the axiom set. On the other hand, the observations made in section 3.6 mean that even with axioms **AA1-4** the system (being strictly 1st-order) cannot be complete, so one may wish to add still more axioms, to obtain a stronger theory with a more restricted set of models.

AA1) $\forall z[(\mathsf{C}(z,x) \leftrightarrow \neg\mathsf{NTP}(z,y))] \leftrightarrow \forall z[(\mathsf{O}(z,x) \leftrightarrow \neg\mathsf{P}(z,y))]$

AA2) $\forall x \forall y[\forall z[\mathsf{C}(z,x) \leftrightarrow \neg\mathsf{NTP}(z,y)] \rightarrow \forall z[\mathsf{C}(z,y) \leftrightarrow \neg\mathsf{NTP}(z,x)]]$

AA3) The structure $\langle \mathcal{R}, \mathsf{sum}, \mathsf{compl} \rangle$ is a Boolean algebra, where $\mathcal{R}$ is the domain of regions and sum and compl are defined from cp as specified in section 3.3.5.

AA4) $\forall x \forall y[\mathsf{P}(x,y) \leftrightarrow \mathsf{sum}(x,y) = y]$

AA1 and **AA2** were discussed in section 3.3.3 and relate to desired properties of the compl function. **AA3** is stated as a meta-level property but could be replaced by a set of 1st-order formulae characterising a Boolean algebra in terms of the Boolean functions of the object language. One could use equational formulae based on the theory given in table 3.1. It is clear that many (and perhaps all) of these formulae would be derivable from the other axioms of the theory. **AA4** ensures that the part relation coincides with the usual partial ordering on the elements of the Boolean algebra.

Models of the Revised Theory

Possible models of the revised theory include topological models which are much the same as those given above for the RCC theory (section 3.5), except that the domain of individuals contains the empty set. Thus, in an open set interpretation the regions will correspond to arbitrary regular open sets of a topological space T; and $\mathsf{C}(x,y)$ will hold just in case the closures of the sets corresponding to x and y share at least one point. The value of the function $\mathsf{cp}(x,y)$ would then be given by the interior of the complement of the product of the sets corresponding to x and y; and the Boolean algebra generated by cp would be the regular open Boolean algebra over T.

It is clear that axioms **A1**–**3** hold in such models. **A4** must also hold since it can be shown that

$$\forall x \forall y \forall u[\mathsf{C}(\mathsf{cp}(x,y),u) \leftrightarrow (\neg\mathsf{NTP}(u,x) \vee \neg\mathsf{NTP}(u,y))]$$

holds under the specified interpretation of cp.

Axiom **A5** imposes an additional density condition on the space T. Specifically, that every non-empty regular open set of T (i.e. every set corresponding to a non-null region) includes a non-empty regular closed subset. (The interior of this subset corresponds to a non-empty NTP of the region.)

Chapter 4

A 0-Order Representation

As in other areas of knowledge representation, constructing a formalism for representing spatial information involves a trade-off between expressive capability and the tractability of computing semantic relations (such as entailment) between expressions. In chapter 2 several very expressive theories of spatial regions were described. All of these addressed the problem of representing spatial information by employing logical languages of 1st (or higher) order — i.e. languages including *quantifiers*. But (as discussed in section 1.5.2) reasoning in 1st-order logic is not only intractable but undecidable; so, unless some special purpose reasoning algorithm is known, such a representation does not provide a practical mechanism for computing inferences. In this chapter I demonstrate how a 0-order (quantifier free) representation, which is an extension of the ordinary classical propositional calculus, can be used to represent a significant class of spatial relationships. This representation also yields a decision procedure for reasoning about this information.

4.1 Spatial Interpretation of 0-Order Calculi

The most familiar interpretations of 0-order logical calculi are as *propositional* logics: the non-logical constants are regarded as denoting *propositions* and the connectives as operating on their (propositional) arguments to form more complex propositions. Within such a conception, the classical connectives are interpreted as expressing truth-functional combinations of their arguments. However, taking non-logical constants as denoting propositions is not the only way that these calculi can be interpreted, which is why I describe them as '0-order' rather than 'propositional'. In this chapter I explain how the classical propositional logic (which I refer to as $\mathcal{C}$) can be employed as a language for spatial reasoning. Under this interpretation, the non-logical constants denote regions and the connectives correspond to operations forming new regions from their arguments.

This interpretation is compatible with well-known model-theoretic accounts of 0-order calculi, in which propositions are associated with sets rather than with truth-values. These sets are often thought of as sets of *possible worlds* in which a proposition is true but they can also be regarded as sets of *points* (or perhaps *atoms*) making up a spatial region. Such interpretations are generally employed as models of *modal* logics rather than the simple classical calculus (whose semantics is adequately captured by the simpler truth-functional semantics). In the next chapter we shall see that the non-truth functional operators of modal logics can also be given a spatial interpretation.

The possibility of representing spatial relations in classical propositional logic arises because the logic of spatial regions includes a Boolean algebra as a sub-structure. This has been known for a long time — it forms the basis of Venn diagrams (Venn 1881). By generalising the principles of Boolean reasoning, the rest of the chapter develops a rather more elaborate system in which it is possible to represent and reason about a much larger class of spatial relations. The generalisation involves a meta-level addition to the basic syntax and proof theory of a 0-order logic, which is needed to increase the expressive power of the representation: specifically it enables negative as well as positive constraints to be represented. This method of representing negative constraints in a 0-order logic is as far as I know completely original. It is also quite general and in subsequent chapters will be applied to modal and intuitionistic logical representations.

4.2 Set Semantics for the Classical Calculus

The 0-order classical calculus (henceforth $\mathcal{C}$) can be given a semantical interpretation in which the constants denote arbitrary subsets of some universe U and the logical connectives correspond to elementary set-theoretic operations. Specifically, a model for the logic $\mathcal{C}$ is a structure, $\langle U, K, \delta \rangle$, where U is a non-empty set, K is a denumerably infinite set of constants, and δ is a denotation function, which assigns to each constant, p, in K a subset, P, of U. The domain of δ is extended to all formulae formed from the constants by stipulating that:[1]

1. $\delta[\neg\phi] = \overline{\delta(\phi)}$
2. $\delta[\phi \wedge \psi] = \delta[\phi] \cap \delta[\psi]$
3. $\delta[\phi \vee \psi] = \delta[\phi] \cup \delta[\psi]$

where for any set S, $\overline{S}$ is the set of all elements of U that are not elements of S. (For example, if $\delta(a) = A$, $\delta(b) = B$ and $\delta(c) = C$, then $\delta(\neg(a \wedge (b \vee c))) = \overline{A \cap (B \cup C)}$.) Under this interpretation it can be shown that:

Classical Set-Semantics Theorem (CSST)
A formula, ϕ, is a theorem of $\mathcal{C}$ if and only if for every model $\langle U, K, \delta \rangle$, the equation $\delta(\phi) = U$ is satisfied.

The denotation function induces a correspondence between formulae and terms formed from constants denoting sets and elementary set operations (henceforth *set-terms*). It will be useful to define some notation to describe the relationship between these types of expression:

- For every propositional constant p_i there is a corresponding set constant P_i.

[1] With semantic and other meta-level functions such as δ I enclose the arguments in square rather than round brackets. The small Greek letters ϕ and ψ are employed as schematic variables standing for arbitrary propositional expressions.

- I write $\mathbf{ST}[\phi]$ as a means of expressing the set-term obtained from the formula ϕ by replacing 0-order constants, p_i, by set constants, P_i and the connectives $\neg$, $\wedge$ and $\vee$ respectively by $\perp$, $\cap$ and $\cup$. (Note that $\mathbf{ST}[\ldots]$ is a meta-level syntactic operation and not an ordinary (extensional) function.)

- The converse (meta-level) function from a set-term τ to a classical (propositional) formula will be written $\mathbf{CF}[\tau]$. If the empty set symbol, '$\emptyset$', occurs in τ it will be replaced by the falsity constant $\perp$ (its negation $\neg\perp$ will be written as $\top$).

- It will also be convenient to use the relational notation, $\phi \;{}_{\mathrm{CF}}\!\rightleftharpoons^{\mathrm{ST}}\; \tau$, to refer to the mapping between classical formulae and corresponding set-terms; thus we can write e.g. $(p \vee \neg q) \;{}_{\mathrm{CF}}\!\rightleftharpoons^{\mathrm{ST}}\; (P \cup \overline{Q})$. (Again this is a meta-level relation between expressions.)

The set semantics may be regarded as a generalisation of the usual truth-functional semantics for $\mathcal{C}$: if U is taken to be a singleton set, $\{1\}$, then all formulae are assigned one of two values: $\emptyset$ or $\{1\}$. Hence for any truth-value assignment $f : K \to \{\mathbf{t}, \mathbf{f}\}$ to the 0-order constants, there is a set assignment $\delta_f : K \to \{\{1\}, \emptyset\}$, such that $\delta_f[p] = \{1\}$ if $f[p] = \mathbf{t}$ and $\delta_f[p] = \emptyset$ if $f[p] = \mathbf{f}$. Moreover, the values of the truth-functions on truth-values are mirrored by corresponding values of the set operations on the two possible set values. So, if the domain of δ_f is extended to complex formulae according to the specification for the δ function given above, then $\delta_f[\phi] = \{1\}$ iff ϕ is given the value $\mathbf{t}$ under the truth-functional assignment f; and $\delta_f[\phi] = \emptyset$ iff ϕ is assigned $\mathbf{f}$ by f.

> **Proof of CSST:** If ϕ is converted to *conjunctive normal form* (CNF), then each conjunct will contain a pair of complementary literals (l and $\neg l$) if and only if ϕ is a tautology. The set term $\tau = \mathbf{ST}[\phi]$ can also be converted to an analogous normal form, *intersection normal form* (INF): by means of simple re-write rules any set-term can be expressed as an intersection of unions of set-constants and their complements. Thus τ can be expressed in the form
>
> $$(\tau_{11} \cup \ldots \cup \tau_{1i} \cup \overline{\upsilon_{11}} \cup \ldots \cup \overline{\upsilon_{1j}}) \cap \ldots \cap (\tau_{n1} \cup \ldots \cup \tau_{nk} \cup \overline{\upsilon_{n1}} \cup \ldots \cup \overline{\upsilon_{nl}}) \; .$$
>
> If a set-term corresponds to a tautological proposition then when expressed in INF each union in the expression must contain some pair, τ and $\overline{\tau}$, of a set constant and its complement. So, whatever the assignment to the set constants, each union, and hence the intersection of these unions, will denote the universal region.
>
> On the other hand suppose ϕ is not a tautology; then there is a truth-value assignment, $f(p_i)$, to the atomic propositions in ϕ such that ϕ is false according to truth-functional semantics. Hence, from the derived set assignment δ_f over the universe, $\{1\}$, (as described above) we can construct a model $\langle\{1\}, K, \delta_f\rangle$, in which ϕ does not denote the universe. ■

If we are only interested in the pure classical calculus, set-semantics may be considered a redundant generalisation of truth functional semantics, since **CSST** shows that a Boolean term has the value 1 in all assignments to its constants over any domain if and only if it takes the value 1 in all assignments over the domain $\{1\}$. Hence, consideration of a 2-element algebra is sufficient to determine validity of any entailment in $\mathcal{C}$. It is only when (as in the next chapter) we introduce additional operators corresponding to non-Boolean functions that we need to consider assignments over larger domains.

4.2.1 An Entailment Correspondence

The correlation between classical theorems and Boolean terms which are universal in any model is a special case of a more general correspondence between the entailment relation in classical logic and entailments among Boolean equations. These entailment relations will be represented with the following notation:

- $\phi_1, \ldots, \phi_n \models_C \phi_0$ means that in the calculus, $\mathcal{C}$, the formula ϕ_0 is entailed by the set of formulae, $\{\phi_1, \ldots, \phi_n\}$. (Thus $\models_C \phi$ means that ϕ is a theorem of $\mathcal{C}$.)
- $\xi_1, \ldots, \xi_n \models_S \xi_0$, where $\xi_0, \ldots, \xi_n$ are set-equations, means that in any model (i.e. assignment of sets to the constants occurring in the equations) for which the equations $\xi_1, \ldots, \xi_n$ hold, the equation ξ_0 also holds. ($\models_S \xi$ means that ξ holds in every model.)

The set-equations we shall be most often concerned with are *universal* — i.e of the form $\tau = U$, where U is the universe of whatever model is under question. This presents a slight notational difficulty if we want to say that a universal equation holds in all of some class of models, because the universal set will not generally be the same set in each model. For this purpose I employ the special symbol $\mathcal{U}$. We can regard this either as a special logical symbol equivalent to $\overline{\emptyset}$ or as a meta-variable standing for whatever set is the universe under consideration.[2]

Using these notations, the following theorem can now be stated:

Classical Entailment Correspondence Theorem (CECT)

$$\phi_1, \ldots, \phi_n \models_C \phi_0 \quad \text{if and only if} \quad \tau_1 = \mathcal{U}, \ldots, \tau_n = \mathcal{U} \models_S \tau_0 = \mathcal{U}$$

where $\phi_i \ {}_{\mathrm{CF}}{\Longleftarrow}^{\mathrm{ST}}\ \tau_i$ for each i.

Proof of CECT: If $\phi_1, \ldots, \phi_n \models_C \phi_0$ then the formula $(\phi_1 \wedge \ldots \wedge \phi_n) \rightarrow \phi_0$ must be a tautology; hence the equation $\overline{\tau_1 \cap \ldots \cap \tau_n} \cup \tau_0 = \mathcal{U}$ must hold in every model. But in any model satisfying $\tau_1 = \mathcal{U}, \ldots, \tau_n = \mathcal{U}$ one must have $\overline{\tau_1 \cap \ldots \cap \tau_n} = \emptyset$. Therefore $\tau_0 = \mathcal{U}$.

On the other hand suppose $\phi_1, \ldots, \phi_n \not\models_C \phi_0$; this means that there is some truth-functional assignment, f, under which $\phi_1, \ldots, \phi_n$ are all true whilst ϕ_0 is false. We then use the derived assignment δ_f over the domain $\{1\}$ as an assignment to corresponding set-constants occurring in the terms τ_i. Under this assignment we shall have $\tau_1 = \mathcal{U}, \ldots, \tau_n = \mathcal{U}$ and $\tau_0 = \emptyset$. So $\tau_1 = \mathcal{U}, \ldots, \tau_n = \mathcal{U} \not\models_S \tau_0 = \mathcal{U}$. ■

[2] Note that, if a term τ contains $\mathcal{U}$, then in $\mathbf{CF}[\tau]$ this will be replaced by $\top$.

4.2.2 Reasoning with Non-Universal Equations

The correspondence theorem, **CECT**, allows us to use classical propositional formulae to reason about universal set-equations and the formula $\mathbf{CF}[\tau]$ can be regarded as representing the equation $\tau = \mathcal{U}$. Moreover, because of the equivalence

$$X = Y \qquad \text{if and only if} \qquad (\overline{X} \cup Y) \cap (X \cup \overline{Y}) = \mathcal{U} \ ,$$

any set equation can be put into the universal form $\tau = \mathcal{U}$.

In order that we may use 0-order formulae to reason about arbitrary Boolean set equations it will be useful to define a transform $\mathbf{CFe}[\tau_1 = \tau_2]$, which gives a formula representative of any equation $\tau_1 = \tau_2$. Such a representative is provided by the formula $\mathbf{CF}[\tau]$, where $\tau = \mathcal{U}$ is equivalent to $\tau_1 = \tau_2$. For a universal equation $\mathbf{CFe}[\tau = \mathcal{U}]$ will just be equal to $\mathbf{CF}[\tau]$. For an arbitrary non-universal equation $\tau_1 = \tau_2$ we could use the definition

$$\mathbf{CFe}[\tau_1 = \tau_2] \ \equiv_{def} \ \mathbf{CF}[(\overline{\tau_1} \cup \tau_2) \cap (\tau_1 \cup \overline{\tau_2})] \equiv (\neg\mathbf{CF}[\tau_1] \vee \mathbf{CF}[\tau_2]) \wedge (\mathbf{CF}[\tau_1] \vee \neg\mathbf{CF}[\tau_2])$$

but since $(\neg\phi \vee \psi) \wedge (\phi \vee \neg\psi) \equiv (\phi \leftrightarrow \psi)$, it is more convenient to define **CFe** by

$$\mathbf{CFe}[\tau_1 = \tau_2] \ \equiv_{def} \ \mathbf{CF}[\tau_1] \leftrightarrow \mathbf{CF}[\tau_2] \ .$$

In terms of **CFe** we can state the following corollary of **CECT** which characterises entailment between arbitrary Boolean set-term equations:

$$\xi_1, \ldots, \xi_n \ \models_S \ \xi_0 \qquad \text{if and only if} \qquad \mathbf{CFe}[\xi_1], \ldots, \mathbf{CFe}[\xi_1] \ \models_C \ \mathbf{CFe}[\xi_0] \ .$$

In fact we shall almost always deal with equations which are in the universal form; but even in these cases the **CFe** operator is still a useful notation for translating from Boolean equations to their representative formulae.

4.3 Representing Topological Relationships in $\mathcal{C}$

Table 4.1 shows how four spatial relations can be characterised by constraints stated in terms of the classical propositional calculus, $\mathcal{C}$. The first column of the table specifies a spatial relation using the formal vocabulary of the RCC theory. The second column gives an informal description of the relation. The third column again describes the same relation in terms of an elementary set-term equation (all the equations are given in universal form). This characterisation is in accord with the interpretation of RCC regions as (non-empty regular open) subsets of a topological space given in section 3.5.2. The final column gives a formula of $\mathcal{C}$ that may be considered as representing the spatial relation. This formula is given by $\mathbf{CFe}[\xi]$, where ξ is the set equation of the third column.

The theorem **CECT** tells us that entailments among elementary set equations are faithfully mirrored by entailments among corresponding $\mathcal{C}$ formulae. Thus, in order to reason with spatial information expressible in terms of such set equations one can transform the equations into formulae of $\mathcal{C}$ and then test inferences using some method of propositional theorem proving.

Relation	*Description*	*Set Equation*	$\mathcal{C}$ *formula*
$\mathsf{DR}(x,y)$	x and y are discrete	$\overline{X \cap Y} = \mathcal{U}$	$\neg(x \wedge y)$
$\mathsf{P}(x,y)$	x is part of y	$\overline{X} \cup Y = \mathcal{U}$	$x \rightarrow y$
$\mathsf{Pi}(x,y)$	y is part of x	$X \cup \overline{Y} = \mathcal{U}$	$y \rightarrow x$
$\mathsf{EQ}(x,y)$	x and y are equal	$(\overline{X} \cup Y) \cap (X \cup \overline{Y}) = \mathcal{U}$	$x \leftrightarrow y$

Table 4.1: Definitions of four topological relations in $\mathcal{C}$

For example, the inference

$$\mathsf{DR}(a,b),\ \mathsf{P}(c,a)\ \vdash\ \mathsf{DR}(a,c)$$

depends on the following entailment between set equations:

$$\overline{A \cap B} = U,\ \overline{C} \cup A = U\ \models\ \overline{C \cap B} = U$$

and this can be shown to be valid because in $\mathcal{C}$ we have

$$\neg(a \wedge b),\ c \rightarrow a\ \models\ \neg(c \wedge b)\ .$$

Hence, even with this very simple encoding into $\mathcal{C}$, some significant spatial inferences can be determined.

Apart from the four relations given in table 4.1 a large class of other relations can also be represented including: x is the universe (x); x is null ($\neg x$); x is the complement of y ($\neg(x \leftrightarrow y)$); the sum of x and y is the universe ($x \vee y$); and x is the sum of y and z ($x \leftrightarrow (y \vee z)$).

The correspondence between binary topological relations among regions and the set equations or $\mathcal{C}$ formulae which can be used to represent them are illustrated in figure 4.1. The figure contains five sub-diagrams showing each of five JEPD relations that can hold between two regions. This classification does not distinguish between connection and overlapping or between tangential and non-tangential parts. Of the five relations only DR and EQ can be uniquely specified by a $\mathcal{C}$ formula.

It is not surprising that the distinction between connection and overlapping cannot be specified in terms of the purely Boolean formulae of $\mathcal{C}$. In the point-set interpretation of RCC this distinction depends on the topological closure operation; but in the simple language of Boolean set equations no such operation is available. To capture the distinction we shall need to use the more expressive representation described in the next chapter. However, it is more disappointing that the relation of partial overlap cannot be directly represented by any formula of $\mathcal{C}$; and even though the part relation corresponds directly to 'implication', the *proper* part relation cannot be uniquely specified: although we can easily say that one region is part of another, we cannot rule out the possibility that the two regions are equal.

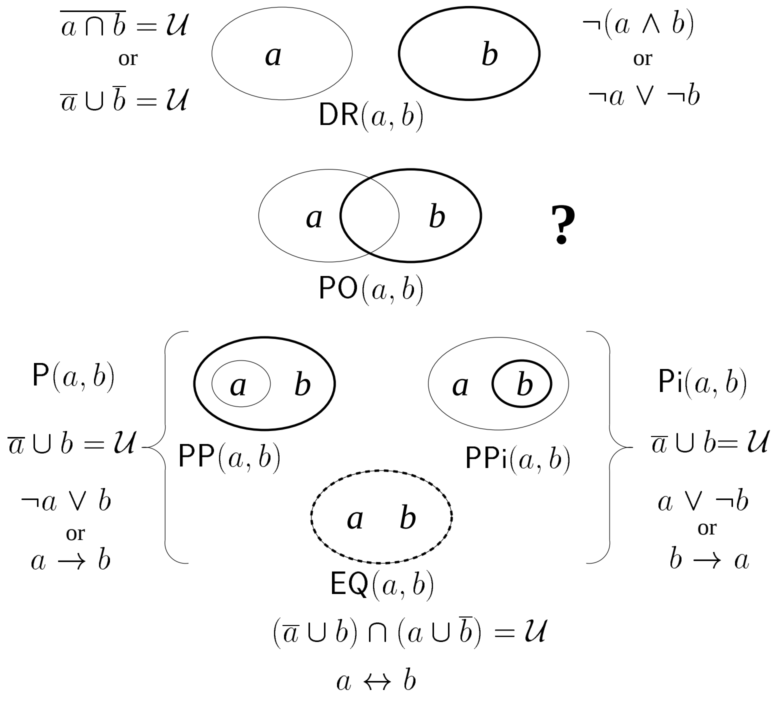

Figure 4.1: Topological relations representable in $\mathcal{C}$

4.4 Model and Entailment Constraints

As it stands, our representation is very limited: many simple spatial relations cannot be defined solely by means of universal set-equations. For example, we have observed that the relation $\mathsf{PP}(x, y)$, x is a *proper* part of y, cannot be so expressed. Nevertheless, informally this relation can be defined quite straightforwardly as that relation which holds whenever $\mathsf{P}(x, y)$ is true but not $\mathsf{EQ}(x, y)$. So it would seem that we can characterise the proper part relation if we can find a way to represent the absence of a relation which we can already define.

We must now ask how the negations of set-equation constraints should be represented. Take for example $\neg\mathsf{P}(x, y)$ (x is not part of y). Suppose we simply negate the classical formula representing $\mathsf{P}(x, y)$; we would then get $\neg(x \rightarrow y)$. But this formula corresponds to the set equation $\overline{\overline{X} \cup Y} = \mathcal{U}$ or equivalently $X \cap \overline{Y} = \mathcal{U}$; and this will only hold when both $X = \mathcal{U}$ and $Y = \emptyset$. So we see that the negation of a formula does not correspond to the absence of the relation enforced by that constraint. In terms of sets, what we really wanted to represent was $\overline{X} \cup Y \neq \mathcal{U}$ which is the direct negation of the set equation for $\mathsf{P}(x, y)$. But negating the formula in the propositional representation does not give us this because such a negation is interpreted as a complement operation on the set-term rather than a negation of the whole equation. This means that the absence of the relations defined so far cannot be represented directly as $\mathcal{C}$ formulae.

We need to increase the expressive capabilities of our representation language so we can represent situations in which we specify not only that a number of universal set-equations hold but also that certain such equations do not hold. Thus, we shall employ the more general constraint language of universal set-equations and their negations and use this to describe spatial situations. In order to use classical formulae to logically encode these constraints we need some way of indicating whether the formula is to be interpreted as an equality or an inequality. Thus a collection of constraints will be represented by a pair $\langle \mathcal{M}, \mathcal{E} \rangle$ where $\mathcal{M}$ is a set of formulae corresponding to equalities and $\mathcal{E}$ is a set of formulae corresponding to inequalities. The formulae in $\mathcal{M}$ are called *model constraints* because they correspond to equational constraints on possible models. The formulae in $\mathcal{E}$ are called *entailment constraints* for reasons which will be made clear in the next section. The language consisting of pairs of sets of $\mathcal{C}$ formulae will be called $\mathcal{C}^+$.

4.5 Consistency of $\mathcal{C}^+$ Situation Descriptions

What we now need is a method of determining from a pair of formula sets, $\langle \mathcal{M}, \mathcal{E} \rangle$, whether the corresponding spatial/algebraic constraints are consistent. $\langle \mathcal{M}, \mathcal{E} \rangle$ represents a set, Θ, of constraints of the form $\{m_1 = \mathcal{U}, \ldots, m_j = \mathcal{U},\ e_1 \neq \mathcal{U}, \ldots, e_k \neq \mathcal{U}\}$. Clearly, Θ is inconsistent if and only if the following entailment holds:

$$m_1 = \mathcal{U}, \ldots, m_j = \mathcal{U} \quad \models_S \quad e_1 = \mathcal{U} \ \vee \ \ldots \ \vee \ e_k = \mathcal{U} \qquad (DE)$$

Here, the r.h.s. is a disjunction of set-equations and as such cannot be translated into a union at the level of set-terms (just as negating a set equation is not equivalent to applying the complement operation to its set term). The correspondence theorem **CECT** does not tell us how to interpret disjunctions of set equations in $\mathcal{C}$. However, it can be established that in the domain of sets, entailments of this kind are *convex* in the sense of (Oppen 1980).[3] A class of entailments is convex in this sense iff

$$\text{whenever} \quad \Gamma \models \phi_1 \vee \ldots \vee \phi_n \quad \text{then} \quad \Gamma \models \phi_i, \text{ for some } i \in \{1 \ldots n\} .$$

The following theorem asserts the convexity of entailments of the form of DE:

Convexity of Disjunctive Boolean-Algebraic Entailments (BEconv)

$$\mu_1 = \mathcal{U}, \ldots, \mu_m = \mathcal{U} \ \models_S \ \varepsilon_1 = \mathcal{U} \ \vee \ \ldots \ \vee \ \varepsilon_n = \mathcal{U}$$

iff

$$\mu_1 = \mathcal{U}, \ldots, \mu_m = \mathcal{U} \ \models_S \ \varepsilon_i = \mathcal{U} \ \text{ for some } i \in \{1, \ldots n\}$$

Proof of BEconv: Consider a disjunctive entailment of the form of DE and let S be the set of set-constants which it contains. Suppose none of the disjuncts on the r.h.s. are entailed by the equations on the l.h.s.. This means that for each disjunct $e_i = \mathcal{U}$ there is an assignment, $\sigma_i : S \to 2^{U_i}$, of subsets of some universe, U_i, to the constants in S such that $e_i = \mathcal{U}$ is false, whilst the equations $m_i = \mathcal{U}$ are all true. We can assume, without loss of generality, that the universes in each of these assignments

[3] Note that later in this thesis I shall use the term *convex* with its ordinary sense, as a property of the surface of a region. Hopefully this will not cause too much confusion.

are disjoint. We now construct a new assignment, $\sigma_* : S \rightarrow 2^{U_*}$, such that $U_* = \bigcup_i U_i$ and $\sigma_*(X) = \bigcup_i \sigma_i(X)$. The U_i's thus form discrete subspaces of U_*. Clearly, this assignment must make all the l.h.s. equations true and each of the disjuncts on the r.h.s. false. Thus the r.h.s. will be entailed if and only if at least one of its disjuncts is individually entailed by the l.h.s.. This means that the class of entailments of the form of DE is convex. ■

In fact **BEconv** may be regarded as an immediate consequence of a general consistency property of equational literals, which I shall call **ELcons**. By an equational literal I mean a positive or negative equality relation, which may contain constants, function symbols and variables. Variables are assumed to be implicitly universally quantified. The property is as follows:

Consistency of Equational Literals (ELcons)

$$\mu_1 = \nu_1, \ldots, \mu_m = \nu_m,\ \neg(\sigma_1 = \tau_1), \ldots, \neg(\sigma_n = \tau_n)\ \models$$

iff

$$\mu_1 = \nu_1, \ldots, \mu_m = \nu_m\ \models\ \sigma_i = \tau_i\ \text{ for some } i \in \{1, \ldots n\}$$

ELcons can be established by considering possible proofs of inconsistency in some proof system for 1st-order logic with equality, which is known to be refutation complete. One such system, is that where the only proof rules are binary resolution, paramodulation and factoring (Duffy 1991). Since we are dealing with sets of literals (i.e. only unit clauses), factoring is not required and a simplified version of paramodulation can be employed. The details of the rules that are used do not matter, since **ELcons** can be demonstrated from quite general observations. The proof is as follows:

Proof of ELcons: Suppose we refute a set of equational literals by means of binary resolution and paramodulation. Once an application of binary resolution can be made, inconsistency is proved immediately; so any successful refutation must consist of a series of paramodulations followed by a single binary resolution. Note also that each paramodulation either involves two positive literals and generates a new positive literal or it involves a positive and a negative literal and generates a new negative literal. These observations enable us to show that any refutation makes essential use of exactly one negative literal. The key points are that the derivation of a positive literal cannot involve any negative literals and that no rule operates on more than one negative literal.

Consider the final step in the refutation; this is a resolution between a positive and a negative literal. The positive literal is either in the original set of literals or has been derived by a sequence of paramodulations involving only positive literals. The negative literal is either in the original set or has been generated from a positive and a negative literal. In the latter case, the positive literal must have been derived from only positive literals and the negative literal is either in the original set or is in turn derived from a positive and negative literal. However long this sequence continues, it is clear that exactly one negative literal from the original set is involved in the proof. ■

The negative literals in the left hand condition of **ELcons** may be moved over to the right to give an equivalent entailment,

$$\mu_1 = \nu_1, \ldots, \mu_m = \nu_m, \models \sigma_1 = \tau_1 \vee \ldots \vee \sigma_n = \tau_n \ . \qquad \textbf{(ELconv)}$$

From **ELcons** it immediately follows that entailments of the form of **ELconv** are convex. **ELconv** has the same syntactic form as DE; but, whereas **ELconv** specifies a purely logical entailment between equations, the entailment relation $\models_S$ occurring in DE specifies that the entailment holds if the terms are interpreted in accordance with elementary set theory. In general if the $\models$ in **ELconv** is replaced by a more specific entailment relation, $\models_\Theta$, the convexity property may no longer hold. However, if Θ can be expressed as a purely equational theory, entailments w.r.t. Θ can be expressed as purely logical equational entailments of the form of **ELconv**. Hence the convexity result will still hold for such theories. In particular, it holds for the relation $\models_S$, where S is elementary set theory (which is just an interpretation of Boolean algebra), since S can be specified purely in terms of equations. This gives us an alternative proof of **BEconv**.

If we combine **BEconv** with our interpretation of $\mathcal{C}^+$ expressions and then apply **CECT** we immediately get the following theorem characterising the consistency of $\mathcal{C}^+$ expressions.

> **$\mathcal{C}^+$ Consistency Theorem (C+CT)**
> A $\mathcal{C}^+$ expression $\langle \mathcal{M}, \mathcal{E} \rangle$ is consistent if and only if
> there is no formula $\phi \in \mathcal{E}$ such that $\mathcal{M} \models_C \phi$.

This should make it clear why the formulae in the set $\mathcal{E}$ are called entailment constraints.

4.6 Representing RCC Relations

We can now give $\mathcal{C}^+$ representations for a significant sub-class of the RCC relations. Let us first look at how the situation type "x is a proper part of y" is represented. We can say that $\mathsf{PP}(x, y)$ holds when x is part of y but the two regions are not equal. This gives us the equality $\overline{X} \cup Y = \mathcal{U}$ and the inequality $(\overline{X} \cup Y) \cap (X \cup \overline{Y}) \neq \mathcal{U}$. Equalities are encoded as model constraints and inequalities as entailment constraints so our propositional representation for the relation $\mathsf{PP}(x, y)$ is the pair

$$\langle \{x \rightarrow y\}, \{x \leftrightarrow y\} \rangle \ .$$

4.6.1 Non-Null Constraints

Recall that in discussing topological interpretations of RCC relations (section 3.5.2) I observed that point-sets corresponding to proper (non-null) RCC regions must be non-empty. An important use of entailment constraints is to ensure that regions involved in a situation description are non-null. If null regions are allowed they have properties which may seem counter-intuitive (for example the null region is both part of and disconnected from any other region) and many useful and apparently sound inferences may not hold if it is allowed that some of the regions involved may be null. The requirement that a region is non-null is expressed by the inequality $\overline{X} \neq \mathcal{U}$, which corresponds to the entailment constraint $\neg x$ in the $\mathcal{C}^+$ representation.

4.6.2 Representations of the RCC-5 Relations

The $\mathcal{C}^+$ representation allows us to represent each of the five topological relations shown in figure 4.1. These comprise a jointly exhaustive and pairwise disjoint (JEPD) set known as RCC-5. The model and entailment constraints (including non-null constraints) of the $\mathcal{C}^+$ representation for each of these relations are shown in table 4.2.

Relation	*Model Constraint*	*Entailment Constraints*
$\mathsf{DR}(x, y)$	$\neg(x \wedge y)$	$\neg x,\ \neg y$
$\mathsf{PO}(x, y)$	—	$\neg x \vee \neg y,\ x \rightarrow y,\ y \rightarrow x,\ \neg x,\ \neg y$
$\mathsf{PP}(x, y)$	$x \rightarrow y$	$y \rightarrow x,\ \neg x,\ \neg y$
$\mathsf{PPi}(x, y)$	$y \rightarrow x$	$x \rightarrow y,\ \neg x,\ \neg y$
$\mathsf{EQ}(x, y)$	$x \leftrightarrow y$	$\neg x,\ \neg y$

Table 4.2: The $\mathcal{C}^+$ encoding of some RCC relations

The model constraint associated with a relation is the strongest formula which holds in all models in which the relation holds. The entailment constraints serve to exclude models which, although consistent with the model constraint, are incompatible with the relation. Thus the entailment constraints associated with a relation in a JEPD set will normally correspond to model constraints of other relations in that set (plus the non-null constraints). The relation PO has no model constraint and is defined by excluding all of the other relations.

Certain entailment constraints which one might expect to be required can be eliminated or weakened because they are indirectly captured by other constraints. For example, in table 4.2 the entailment constraint $x \leftrightarrow y$, which occurred in the representation of PP worked out above, is replaced by the weaker formula $y \rightarrow x$, since in the presence of the model constraint $x \rightarrow y$, $y \rightarrow x$ would immediately entail $x \leftrightarrow y$.

4.7 Reasoning with $\mathcal{C}^+$

By making use of the results obtained so far one can use a classical propositional theorem prover as the basis of an effective automated spatial reasoning system. For clarity I concisely summarise the consistency checking algorithm for $\mathcal{C}^+$. Given a spatial description consisting of a set of relations of the form $R(\alpha, \beta)$, where R is one of the relations characterisable in $\mathcal{C}^+$ and α and β are constants denoting regions, the following simple algorithm will decide whether the description describes a possible situation:

- For each relation $R_i(\alpha_i, \beta_i)$ in the situation description find the corresponding propositional representation $\langle \mathcal{M}_i, \mathcal{E}_i \rangle$.
- Construct the overall $\mathcal{C}^+$ representation $\langle \bigcup_i \mathcal{M}_i,\ \bigcup_i \mathcal{E}_i \rangle$.
- For each formula $\phi \in \bigcup_i \mathcal{E}_i$ use a classical propositional theorem prover to determine whether the entailment $\bigcup_i \mathcal{M}_i \models_C \phi$ holds.
- If any of the entailments determined in the last step does hold then the situation is impossible.

For example we may want to know whether the following situation is possible: x is a proper part of y; y is disjoint with z; and x is a proper part of z. The $\mathcal{C}^+$ representations of the three spatial relations are respectively:

$$\langle \{x \rightarrow y\}, \{y \rightarrow x, \neg x, \neg y\} \rangle,\ \langle \{\neg(y \wedge z)\}, \{\neg y, \neg z\} \rangle \ \text{ and }\ \langle \{x \rightarrow z\}, \{z \rightarrow x, \neg x, \neg z\} \rangle\ .$$

So the overall $\mathcal{C}^+$ representation is

$$\langle \{x \rightarrow y,\ \neg(y \wedge z),\ x \rightarrow z\},\ \{y \rightarrow x,\ z \rightarrow x,\ \neg x,\ \neg y,\ \neg z\} \rangle\ .$$

We determine that this situation is impossible since

$$x \rightarrow y,\ \neg(y \wedge z),\ x \rightarrow z\ \models_C \neg x\ .$$

4.7.1 Determining Entailments

Computing inconsistency of $\mathcal{C}^+$ expressions is a special case of determining entailments between situation descriptions characterisable in $\mathcal{C}^+$. To refer to such an entailment, I shall use the notation $\langle \mathcal{M}, \mathcal{E} \rangle \models_{C^+} \langle \mathcal{M}', \mathcal{E}' \rangle$. We can express the meaning of this as an entailment between set-equations as follows:

$$\begin{array}{l} m_1 = \mathcal{U} \ \wedge \ \ldots \ \wedge m_h = \mathcal{U} \ \wedge e_1 \neq \mathcal{U} \ \wedge \ \ldots \ \wedge e_i \neq \mathcal{U} \\ \quad \models_S \\ m'_1 = \mathcal{U} \ \wedge \ \ldots \ \wedge m'_j = \mathcal{U} \ \wedge e'_1 \neq \mathcal{U} \ \wedge \ \ldots \ \wedge e'_k \neq \mathcal{U} \end{array}$$

If we then bring the r.h.s. over to the left and move the resulting negation inwards we get:

$$\begin{array}{l} m_1 = \mathcal{U} \ \wedge \ \ldots \ \wedge m_h = \mathcal{U} \ \wedge e_1 \neq \mathcal{U} \ \wedge \ \ldots \ \wedge e_i \neq \mathcal{U} \ \wedge \\ \qquad\qquad (m'_1 \neq \mathcal{U} \ \vee \ldots \vee m'_j \neq \mathcal{U} \ \vee e'_1 = \mathcal{U} \ \vee \ldots \vee e'_k = \mathcal{U}\,) \models_S \ . \end{array}$$

To show the validity of this we must show that whichever of the equations in the disjunction is chosen the resulting equation set is inconsistent. This is equivalent to showing that:

for all $p \in \mathcal{M}'$ we have $\langle \mathcal{M}, \mathcal{E} \cup \{p\} \rangle \models_{C^+}$ and for all $q \in \mathcal{E}'$ we have $\langle \mathcal{M} \cup \{q\}, \mathcal{E} \rangle \models_{C^+}$

Another equivalent way of expressing these which is more convenient from the point of view of actually calculating the entailments is the following:

$\mathcal{C}^+$ Entailment Theorem (C+ET)

$$\langle \mathcal{M}, \mathcal{E} \rangle \models_{C^+} \langle \mathcal{M}', \mathcal{E}' \rangle \quad \text{iff}$$

$$\text{either } \langle \mathcal{M}, \mathcal{E} \rangle \models_{C^+} \quad \text{or} \quad (\text{ for all } \phi \in \mathcal{M}' : \langle \mathcal{M}, \{\phi\} \rangle \models_{C^+}$$
$$\text{and for all } \psi \in \mathcal{E}' : \langle \mathcal{M} \cup \{\psi\}, \mathcal{E} \rangle \models_{C^+})$$

Informally, this means that a sequent is valid iff: either $\langle \mathcal{M}, \mathcal{E} \rangle$ is itself inconsistent; or, each of the model constraints in $\mathcal{M}'$ is entailed by the model constraints $\mathcal{M}$ and also each of the entailment constraints in $\mathcal{E}'$ in conjunction with the model constraints $\mathcal{M}$ entails one of the entailment constraints in $\mathcal{E}$. Determining the validity of a $\mathcal{C}^+$ entailment has thus been reduced to determining the inconsistency of certain $\mathcal{C}^+$ expressions and we already know that such an expression is inconsistent iff one of its entailment constraints is entailed by its model constraints.

4.7.2 Complexity of the Reasoning Algorithm

Consistency checking for sets of spatial relations representable in $\mathcal{C}^+$ is clearly NP-hard and essentially the same as the consistency checking problem for $\mathcal{C}$. The meta-level extension for handling the entailment constraints reduces each $\mathcal{C}^+$ consistency problem to n consistency problems of sets of $\mathcal{C}$ formulae, where n is the number of entailment constraints. Note that all these n problems could in principle be solved in parallel.

Another factor which can significantly limit the complexity of spatial reasoning using this encoding is that, in representing the five RCC relations given in table 4.2, only formulae containing at most two variables are employed. This means that the complexity of reasoning with these relations is that of '2-SAT', the satisfiability problem for binary clauses. This problem is computationally easy: it can be solved in time proportional to n^2, where n is the number of clauses involved. More specifically this problem is in the class NC of problems which can be solved in polylogarithmic time by using polynomially many parallel processors.

A detailed consideration of computational complexity is beyond the scope of this thesis. A survey of complexity classes can be found in (Johnson 1990).

Chapter 5

A Modal Representation

Using principles introduced in the last chapter, this chapter develops a more expressive representation for spatial relationships based on the 0-order modal logic $S4$. I explain how the Boolean set semantics for classical logics can be generalised to take account of additional non-truth functional operators. We shall see how the topological *interior* function can also be modelled in this way. In fact, considered in this way, the '$\Box$' operator of $S4$ obeys exactly the same constraints as an interior operator. This correspondence allows one to use deduction in $S4$ as a means for reasoning about equations between terms involving Boolean functions and an interior function. We shall see that these equations can express a large class of spatial relations. I go on to introduce the language $S4^+$ which extends the expressive power of $S4$ in exactly the same way as $\mathcal{C}^+$ extends the $\mathcal{C}$ representation. The $S4^+$ representation allows many RCC relations to be expressed including all the RCC-8 relations.

5.1 The Spatial Interpretation of Modal Logics

In this chapter I develop a 0-order representation for spatial information which is considerably more expressive than that given in the previous chapter. The principles upon which it is based are much the same as those employed in formulating $\mathcal{C}^+$ but, rather than using the simple classical logic to encode spatial information, I shall use modal logics whose language contains additional unary operators.

Modal operators are usually regarded as non-truth-functional operators on propositions. Many kinds of propositional modality have been studied: alethic modalities (necessity, possibility, contingency); propositional attitudes (knowledge, belief, certainty, etc.); deontic modalities (obligation, permission). However, in the context of a set-semantics — under which 0-order constants are interpreted as sets and Boolean operators as elementary set operation — modal operators can be regarded as mappings between subsets of some universe of elements. By thinking of these as sets of points within a space, we immediately get a spatial interpretation.

To specify the spatial interpretation of a modal operator in a more concrete way we can regard the universe of points as having the structure of a topological space. As we saw in section 2.1 the structure of a topological space determines (and is determined by) certain functions on subsets of the space, such as the *interior* and *closure* functions. We shall see that the modal '$\Box$' operator of the logic $S4$ can be interpreted as an interior operator on a topological space. This correspondence allows one to use deduction in $S4$ as a means for reasoning about equations between terms involving Boolean functions and an interior function. These can be regarded as topological constraints and can be used to express a large class of spatial relations. The connection between topological spaces and the logic $S4$ has been known since the work of Tarski and McKinsey (1948) but as far as I know has never been used as a vehicle for automated spatial reasoning.

5.1.1 Overview of the Approach Taken

In the next section I look at the semantics of modal logics and specifically at algebraic models based on modal algebras. I prove a correspondence between the deducibility relation of a modal logic and entailment among modal algebraic equations.

In section 5.3 I consider the algebraic interpretation of a topological space as a *closure algebra*, and show how many topological relationships can be expressed in terms of closure algebraic equations and the negations of such equations. I then observe (in section 5.4) that the modal algebras associated with the logic $S4$ are essentially the same as closure algebras. This means that $S4$ can be used to reason about equational closure algebra constraints.

Generalising the framework previously described in sections 4.4 and 4.5 of the last chapter, section 5.5 specifies the extension of a modal language L to a more expressive language, L^+. I prove a useful entailment convexity result for these languages. I then show (in section 5.6) how all the RCC-8 relations and many more topological relations can be encoded in the extended modal language $S4^+$. This provides a decision procedure for a significant class of topological relations.

Finally, in section 5.7, I explain how, in principle, modal representations allow us to replace the meta-level expressions of $\mathcal{C}^+$ and $S4^+$ by object level expressions in a modal logic incorporating an additional $S5$ operator.

5.2 Semantics for 0-Order (Modal) Logics

To generalise the spatial interpretation of $\mathcal{C}$ to 0-order languages with additional operators it is necessary to know some details of modal logics and their semantics. My presentation is very concise so the reader will need some prior knowledge of the subject. Two very good text books on modal logic are (Hughes and Cresswell 1968) and (Chellas 1980).

5.2.1 Modal Logics

A (propositional) modal language is obtained by adding to the language of classical propositional logic a monadic operator, '$\Box$'.[1] The inference rules of the modal logic consist of all the rules of classical propositional logic plus some additional rules concerning the modal operator. Many

[1] For some purposes one may wish to add several distinct modal operators to the language. The resulting system is called a *multi-modal* logic.

different sets of rules have been proposed capturing different intended meanings and properties of the operator. These give rise to a wide range of distinct modal logics. A rule common to most logics that have been called 'modal' is the rule of necessitation (**RN**): this states that if any formula, ϕ, is a logical theorem then so is the formula $\Box\phi$.

Further logical properties of the modal operator are usually presented in terms of axiom schemata. A schema specifies that all formulae exhibiting a certain logical form have the status of axioms. Thus if the proof system of the underlying classical propositional logic is presented in axiomatic style (i.e. as a set of axiom schema and the rule of *modus ponens*) then the proof system of a modal logic, L, is obtained by simply adding further axiom schemata and the rule **RN**. I write

$$\phi_1, \ldots, \phi_n \vdash_L \phi_0$$

to mean that the formula ϕ_0 is deducible from the set of formulae $\{\phi_1, \ldots, \phi_n\}$ in the logic L.

For every modal $\Box$ operator there is a dual operator, $\Diamond$, defined by $\Diamond\phi \leftrightarrow \neg\Box\neg\phi$. Consequently (since negation obeys the usual classical principles), it is easily proved that $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$; so one can equally well take $\Diamond$ as the primary modal operator and introduce $\Box$ by definition.

5.2.2 The Logic $S4$

$S4$ is one of the simpler and better known modal logics. It may also be called $KT4$ since it is obtained from classical propositional logic by adding the the rule of necessitation and the following axiom schemas:

K. $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

T. $\Box\phi \rightarrow \phi$

4. $\Box\phi \rightarrow \Box\Box\phi$

A modal logic which satisfies the schema **K**, as well as obeying the rule of necessitation, is known as *normal*.

5.2.3 Kripke Semantics

Currently the best known interpretations of modal logics are those in terms of *Kripke semantics*. In a Kripke semantics a *model* consists of a set of possible worlds together with an *accessibility relation* — a binary relation between worlds — associated with each modal operator. Propositions denote sets of possible worlds (the set of worlds in which they are true). A Kripke model, $\mathcal{M}$, is thus a structure $\langle W, R, P, d\rangle$, where W is a set of worlds, R is the accessibility relation, P is a set of constants, $\{p_i\}$, and d is a function mapping elements of P to subsets of W.

Such a model determines the truth of each modal formula at each possible world. Classical formulae are interpreted as follows:

- Atomic formulae, p_i are true in exactly the worlds in the set $d(p_i)$.

- Conjunctions, $\phi \wedge \psi$, are true in worlds where both ϕ and ψ are true.

- Disjunctions, $\phi \vee \psi$, are true in worlds where either ϕ or ψ (or both) is true.
- Negations, $\neg\phi$, are true in worlds where ϕ is not true.

We write $\models^{\mathcal{M}}_{\alpha} \phi$ to mean that formula ϕ is true at world α in model $\mathcal{M}$. A modal operator, $\Box$, is then interpreted as follows: in a model $\mathcal{M} = \langle W, R, P, d \rangle$

$$\models^{\mathcal{M}}_{\alpha} \Box\phi \quad \text{iff} \quad \models^{\mathcal{M}}_{\beta} \phi \quad \text{for all } \beta \in W \text{ s.t. } R(\alpha, \beta)$$

A *frame* is a set of all Kripke models satisfying some specification of the properties of the accessibility relation, R. For example, the set of all Kripke models in which R is reflexive and symmetric constitutes a frame. Finally we say that a formula is *valid* in some frame, F, if it is true at every world in every model in F.

The logic $S4$ is characterised by the frame, $\mathcal{F}_{S4}$, consisting of all Kripke models whose accessibility relations are *reflexive* and *transitive* (R is a *quasi-ordering* on W). Every theorem provable according to the proof system for $S4$ specified above is valid in $\mathcal{F}_{S4}$; and conversely every formula valid in $\mathcal{F}_{S4}$ is provable in the proof system.

A vast spectrum of different modal operators can be specified by placing more or less general restrictions on the corresponding accessibility relation.[2] Furthermore, Kripke semantics allows one to specify operators whose logic seems to correspond well with intuitive properties of modal concepts employed in natural language. Indeed, a number of logics proposed for natural language modalities, which were originally specified proof theoretically (by axiom schemata intended to capture intuitive properties of modal concepts) can be captured very easily within the Kripke paradigm by quite simple restrictions on the accessibility relation.

Whilst the Kripke approach certainly provides a very flexible approach to modal semantics, its generality is often overstated. Consequently, many researchers in both AI and philosophical logic tend to think of possible worlds semantics as essentially based upon accessibility relations. However, although Kripke models may be appropriate for certain types of modal operator, in other cases it may be more natural to suppose a quite different structuring of possible worlds or even a semantics that is not based on possible worlds at all.

5.2.4 Modal Algebras

A *modal algebra* is a mathematical structure that provides a semantics for modal logics which is more general than a Kripke model. Just as the formulae of classical propositional logic can be interpreted as referring to elements of a Boolean algebra, modal formulae can be interpreted as elements of a Boolean algebra supplemented with an additional unary operation obeying certain constraints. This is a modal algebra. Boolean algebras with additional operators were first studied in detail by Jónsson and Tarski (1951). Their connection to modal logics was investigated by Lemmon (1966a, 1966b). A clear account of the essential properties of modal algebras and their

[2] Often such restrictions are thought of as defining a logic rather than an operator but this is misleading since the possible worlds semantics allows any number of different operators to be encompassed in a single logical language.

relation to Kripke semantics is given by Hughes and Cresswell (1968, Chapter 17) and a much more detailed examination can be found in (Goldblatt 1976).

A modal algebra can be represented by a structure $\mathcal{M} = \langle S, +, \perp, * \rangle$, where $\langle S, +, \perp \rangle$ is a Boolean algebra and, for all elements x and y of the algebra, the operator '$*$' satisfies the equation

$$*(x + y) = *x + *y \tag{add}$$

Operators obeying this equation are known as *additive*.[3] A direct consequence of additivity is the following monotonicity property (which will be useful later):[4]

$$\text{if} \quad x \leq y \quad \text{then} \quad *x \leq *y \tag{mon}$$

Further equational restrictions may be placed on the '$*$' operator. Of particular importance are

$$x \leq *x \qquad (\text{i.e. } x + *x = *x) \tag{epis}$$

$$*0 = 0 \tag{norm}$$

$$*(*(x)) = *(x) \tag{idem}$$

5.2.5 Algebraic Models

We can now define an *algebraic model* for a modal language[5] as a structure $\langle S, +, \perp, *, P, \delta \rangle$, where $\langle S, +, \perp, * \rangle$ is a modal algebra, P is the set of constants of the language and δ is a function mapping modal formulae to elements of S. For each constant $p \in P$, $\delta[p]$ may be any element of S. This assignment to the constants determines the value $\delta[\phi]$ of all complex formulae according to the following recursive specification:[6]

- $\delta[\alpha \vee \beta] = \delta[\alpha] + \delta[\beta]$
- $\delta[\neg\alpha] = \perp\delta[\alpha]$
- $\delta[\Diamond \alpha] = *(\delta[\alpha])$

Note that under this interpretation the $*$ operation of the algebra is associated with the modal $\Diamond$ rather than $\Box$. This is because of the additivity of the algebraic $*$ operator: the algebraic equation characterising additivity corresponds to the modal schema $\Diamond(\phi \vee \psi) \leftrightarrow (\Diamond \phi \vee \Diamond \psi)$ which is true in every normal modal logic.

We say that a formula, ϕ, is *universal* in a model $\langle S, \cup, \perp, *, P, \delta \rangle$ iff $\delta[\phi] = 1$ — i.e. if the model assigns to the formula the unit (universal) element of the modal algebra $\langle S, \cup, \perp, * \rangle$. An

[3] It is additive operators which are the primary focus of the investigations of Jónsson and Tarski (1951).

[4] Proof: $(x \leq y) \rightarrow (x + y = y) \rightarrow (*y = *(x + y)) \rightarrow (*y = *x + *y) \rightarrow (*x \leq *y)$ QED.

[5] I am assuming here that the language has only one modal operator. For a multi-modal language the model would have several functions $*_i$ — one for each modality.

[6] Specifications for the connectives $\wedge$, $\rightarrow$, $\leftrightarrow$ and $\Box$ can easily be derived from their definitions in terms of $\neg$, $\vee$ and $\Diamond$.

algebraic frame, $\mathcal{F}_E$, is a set of all algebraic models whose algebras satisfy some set of equations, E, constraining the '$*$' operator. Finally we say that a formula is valid with respect to some algebraic frame, $\mathcal{F}_E$, if it is universal in every model in $\mathcal{F}_E$.

In order that algebraic models provide a semantics for some modal logic, L, we must find a set of characteristic equations, E_L such that a formula ϕ is valid in the frame $\mathcal{F}_{E_L}$ if and only if it is a theorem of L. For brevity I shall denote the frame associated with the logic L by $\mathcal{F}_L$, rather than $\mathcal{F}_{E_L}$. For instance, the frame $\mathcal{F}_{S4}$ is the set of all models satisfying the equations **add**, **epis**, **norm**, and **idem**. It is known that a formula is valid with respect to $\mathcal{F}_{S4}$ iff it is a theorem of the logic $S4$ (Hughes and Cresswell 1968, Chapter 17).

Note that, if $\phi \leftrightarrow \psi$ is a theorem of some logic L, then ϕ and ψ must have the same denotation in every algebra in $\mathcal{F}_L$. Thus, since $\Diamond$ and $\Box$ are interpreted as extensional algebraic functions, $\Diamond\phi \leftrightarrow \Diamond\psi$ and $\Box\phi \leftrightarrow \Box\psi$ must also be theorems of L. Hence, any modal logic which can be given an algebraic semantics of this kind will be closed under the rule of equivalence: if $\vdash \phi \leftrightarrow \psi$ then $\vdash \Diamond\phi \leftrightarrow \Diamond\psi$, which I shall refer to as **RE**.

5.2.6 Power-Set Algebras

According to Stone's representation theorem (Stone 1936)[7] every Boolean algebra is isomorphic to a Boolean algebra whose elements are sets and whose operators are identified with the usual union, intersection and complementation operations of elementary set theory. Moreover, such an algebra can always be embedded in a Boolean algebra whose elements are all the subsets of some (universal) set W.

Jónsson and Tarski (1951) showed that a similar theorem holds for Boolean algebras with additional additive operators. This means that every modal algebra can be isomorphically embedded in a modal algebra whose elements are all members of the power set, 2^W, of some set, W. One may think of the elements of W as possible worlds; and since each proposition, p, of the modal language is interpreted as an element, α, in the modal algebra, α may be regarded as the set of worlds in which p is true.

Where an algebraic model is based on a power-set algebra, I shall represent it by a structure $\langle U, \cup, \perp, *, P, \delta \rangle$, where the sum operator is '$\cup$' to indicate that the Boolean operators correspond to the operators of elementary set theory. As in the previous chapter, I use the meta-symbol $\mathcal{U}$ to denote the universal set in whatever algebra is being considered. The power-set algebras are representative of the whole class of modal algebras in the sense that an equation which is true in all power-set algebras is true in every modal algebra (because every modal algebra can be embedded in a power-set algebra). This means that in characterising validity in terms of algebraic frames we can restrict the frames to contain only models based on power-set algebras. In the sequel I shall assume that we always consider only models based on power-sets and I shall refer to the resulting semantics as *algebraic set semantics*. A modal operator, $*$, in a power-set algebra, maps every subset, X, of the universe to another subset $*(X)$.

[7] A comprehensive study of this theorem can be found in (Johnstone 1982).

5.2.7 Mapping Between Algebraic and Logical Expressions

As with the classical set-semantics it will be useful to introduce meta-level notation for referring to the mapping between modal formulae and modal algebraic terms. I assume that these terms are interpreted as sets in a power-set algebra. Thus $\mathbf{MAT}[\phi]$ is the modal algebraic term obtained from the formulae ϕ by replacing the connectives $\neg$, $\vee$, $\wedge$ and $\Diamond$ by the operators $\perp$, $\cup$, $\cap$ and $*$ and the 0-order constants, p_i, by set constants, P_i. Since the $\Box$ operator is equivalent to $\neg\Diamond\neg$ this is replaced by the algebraic operator $\perp * \perp(\ldots)$. The function **MF** is the inverse of **MAT** so that $\mathbf{MF}[\tau]$ is the formulae ϕ such that $\mathbf{MAT}[\phi] = \tau$. I shall write $\phi \;_{\mathrm{MF}}\!\rightleftharpoons^{\mathrm{MAT}} \tau$ to refer to the mapping in the form of a relation.

I also define (by analogy with **CFe** introduced in section 4.2.2) the transform $\mathbf{MFe}[\xi]$, such that $\mathbf{MFe}[\tau = \mathcal{U}] = \mathbf{MF}[\tau]$ for universal equations and $\mathbf{MFe}[\tau_1 = \tau_2] = \mathbf{MF}[\tau_1] \leftrightarrow \mathbf{MF}[\tau_2]$, for non-universal equations. The expression $\mathbf{MFe}[\xi]$ refers to a modal formula which (because of the correspondence theorem, **Mcorr**, which will be given in section 5.2.9) may be regarded as representative of the modal algebraic equation ξ. However, because of the form of the entailment correspondence theorem, **S4ECT**, also proved in section 5.2.9, one might say that an equation ξ constraining an $S4$ modal algebra is better represented by $\Box\,\mathbf{MFe}[\xi]$ rather than $\mathbf{MFe}[\xi]$.

Equations characterising a class of algebraic structures (a frame) will in general contain free variables which are taken as implicitly universally quantified — the equations hold for all elements of the algebra. Thus, an equation with free variables will correspond to a class of modal formulae, which can be represented as a formula schema. Because of this, it is convenient to generalise **MF** so as to operate on terms with free variables. In such a case the resulting expression will be a modal schema rather than a formula and schematic logical variables will take the place of the free variables in the algebraic term. Accordingly, **MFe** can also be allowed to operate on equations containing free variables — again the result will be a schema rather than a formula.

By means of **MFe**, a set of algebraic equations defining a frame $\mathcal{F}_L$ can be translated directly into a set of modal schemas which specify the proof system of the corresponding logic L. To ensure the proof system is complete it will also be necessary to add the inference rule **RE** which is intrinsic to algebraic semantics (as explained at the end of section 5.2.5).

5.2.8 Entailment among Modal Algebraic Equations

If some entities of interest (in our case these will be spatial regions) are identified with elements in an algebra, then equations between algebraic terms can be used to specify relationships between these entities. One can then reason about these relations in terms of entailments among algebraic equations. Since set algebras are representative of the class of modal algebras the notion of entailment among modal algebraic equations can be defined in terms of possible set assignments to a language of modal algebraic terms:

- A set assignment to a language of algebraic terms is a structure $\Sigma = \langle S, U, \sigma, m\rangle$, where S is a set of constants, U is a universal set, $\sigma : S \to 2^U$ assigns a subset of U to each constant in S and $m : 2^U \to 2^U$ specifies the modal operator $*$ as a set function. If τ is a term built from the constants in S by means of Boolean and modal operators, then $\Sigma[\tau]$ is the set assigned to τ by Σ. This is determined by σ, m and the usual interpretation of Boolean operations on sets. If $\Sigma[\tau_1] = \Sigma[\tau_2]$ we say that Σ *satisfies* the equation $\tau_1 = \tau_2$.

Entailment relations among modal-algebraic equations can now be specified as follows:

- $\tau_1 = \upsilon_i, \ldots, \tau_n = \upsilon_n \models_{\mathrm{MA}_L} \tau_0 = \upsilon_0$ means that, for every assignment $\Sigma = \langle S, U, \sigma, m\rangle$ (where S includes all the constants occurring in the terms τ_i and υ_i) satisfying the equations associated with the frame $\mathcal{F}_L$, if Σ satisfies the equations $\tau_1 = \upsilon_1, \ldots, \tau_n = \upsilon_n$ it also satisfies the equation $\tau_0 = \upsilon_0$.

5.2.9 Relating $S4$ Modal-Algebraic Entailment to Deducibility

If a modal logic L is characterised by a modal algebraic frame $\mathcal{F}_L$ there is a correspondence between deduction in the logic and entailment between algebraic equations in the algebras in $\mathcal{F}_L$. Because of this we can use modal logics to reason about algebraic equations.

From the definition of an algebraic frame for the logic L we have the following correspondence between universal set equations and logical theorems:

$$\models_{\mathrm{MA}_L} \tau = \mathcal{U} \quad \text{iff} \quad \vdash_L \phi, \quad \text{where } \phi \ {}_{\mathrm{MF}}\!\!\rightleftharpoons^{\mathrm{MAT}} \tau \qquad \textbf{(Mcorr)}$$

In the last chapter we saw how classical propositional formulae can be used to reason about spatial properties that can be stated as equations of the form $\tau = \mathcal{U}$. The correctness of reasoning using this encoding was justified by the Classical Entailment Correspondence Theorem, **CECT**. Later in this chapter (sections 5.5.2 and 5.6) we shall see how, by using a similar correspondence theorem, modal formulae can be used to reason about a much wider range of spatial properties. To generalise the classical case to arbitrary modal logics we would need to establish the validity of a conjecture such as the following:

General Modal Entailment Correspondence Conjecture (GMECC)

$$\tau_1 = \mathcal{U}, \ldots, \tau_n = \mathcal{U} \models_{\mathrm{MA}_L} \tau_0 = \mathcal{U} \quad \text{iff} \quad \phi_1, \ldots, \phi_n \vdash_L \phi_0$$
$$\text{where } \phi_i \ {}_{\mathrm{MF}}\!\!\rightleftharpoons^{\mathrm{MAT}} \tau_i$$

Note that **GMECC** proposes a correspondence between an entailment relation and a deducibility relation, rather than between two entailment relations, as was the case for the theorem **CECT** of the last chapter. **CECT** relates entailments between Boolean set-term equations to 0-order entailments under the standard truth-functional semantics for $\mathcal{C}$. In using **CECT** to justify the use of classical theorem provers for spatial reasoning we took for granted the fact that any sound and complete proof system for classical 0-order logic is faithful to the truth-functional semantics. In attempting to establish **GMECC**, one is attempting to generalise **Mcorr**, which relates modal algebraic identities directly to modal theoremhood, and there is *prima facie* no need to to introduce another semantics.

An even more important thing to note about **GMECC** is that it is not true:[8] for many modal logics there are cases where an entailment between modal algebraic equations holds but the corresponding logical entailment between modal formulae is invalid. For example, in an $S4$ modal algebra $P = \mathcal{U}$ entails $\perp * \perp(P) = \mathcal{U}$; but $p \not\vdash_{S4} \Box p$. Nevertheless, as we would expect, $S4$ does respect **Mcorr**: applying the deduction theorem to the sequent $p \vdash \Box p$, yields $\vdash p \rightarrow \Box p$, which corresponds to a modal algebraic equation $(\overline{P} \cup \overline{*(\overline{P})}) = \mathcal{U}$; and this is not generally true for algebras in the frame $\mathcal{F}_{S4}$. The problem arises because all algebras in the frame $\mathcal{F}_{S4}$ must obey the identity $*0 = 0$, or equivalently $\perp * \perp\mathcal{U} = \mathcal{U}$. Indeed, this identity is satisfied by all algebras in any frame, $\mathcal{F}_L$, where the logic L obeys the rule of necessitation.

Although $S4$ does not obey the **GMECC** conjecture, the following correspondence between the entailment relation among universal set equations constraining algebras in $\mathcal{F}_{S4}$ and the deducibility relation of $S4$ can be proved:

> $S4$ **Entailment Correspondence Theorem (S4ECT)**
> $$\tau_1 = \mathcal{U}, \ldots, \tau_n = \mathcal{U} \models_{\mathrm{MA}_{S4}} \tau_0 = \mathcal{U} \quad \text{iff} \quad \Box\phi_1, \ldots, \Box\phi_n \vdash_{S4} \phi_0$$
> where $\phi_i \;{}_{\mathrm{MF}}\!\!\rightleftharpoons^{\mathrm{MAT}} \tau_i$

Proof of S4ECT: Since $S4$ is an extension of classical logic it obeys the deduction theorem: $\phi_1, \ldots, \phi_n \vdash_{S4} \phi_0$ iff $\vdash_{S4} (\phi_1 \wedge \ldots \wedge \phi_n) \rightarrow \phi_0$. By combining this with **Mcorr** we get the more general correspondence

$$\phi_1, \ldots, \phi_n \vdash_{S4} \phi_0 \quad \text{iff} \quad \models_{\mathrm{MA}_{S4}} \overline{(\tau_1 \cap \ldots \cap \tau_n)} \cup \tau_0 = \mathcal{U} \; .$$

Hence

$$\Box\phi_1, \ldots, \Box\phi_n \vdash_{S4} \phi_0 \quad \text{iff} \quad \models_{\mathrm{MA}_{S4}} \overline{(\perp * \perp\tau_1 \cap \ldots \cap \perp * \perp\tau_n)} \cup \tau_0 = \mathcal{U} \; .$$

From elementary set theory and the additivity of $*$ it can easily be shown that the equation on the r.h.s. is equivalent to $\perp * \perp(\tau_1 \cap \ldots \cap \tau_n) \subseteq \tau_0$, so we can establish **S4ECT** by showing that

$$\models_{\mathrm{MA}_{S4}} \perp * \perp(\tau_1 \cap \ldots \cap \tau_n) \subseteq \tau_0 \quad \text{iff} \quad \tau_1 = \mathcal{U}, \ldots, \tau_n = \mathcal{U} \models_{\mathrm{MA}_{S4}} \tau_0 = \mathcal{U} \; .$$

The r.h.s. can then be re-written to give

$$\models_{\mathrm{MA}_{S4}} \perp * \perp(\tau_1 \cap \ldots \cap \tau_n) \subseteq \tau_0 \quad \text{iff} \quad (\tau_1 \cap \ldots \cap \tau_n) = \mathcal{U} \models_{\mathrm{MA}_{S4}} \tau_0 = \mathcal{U}$$

and this equivalence can be more succinctly expressed as

$$\models_{\mathrm{MA}_{S4}} \perp * \perp(\upsilon) \subseteq \tau_0 \quad \text{iff} \quad \upsilon = \mathcal{U} \models_{\mathrm{MA}_{S4}} \tau_0 = \mathcal{U} \qquad (\dagger)\; .$$

[8] Despite the existence of simple counter-examples, for a long time I believed **GMECC** and I even published a faulty proof in (Bennett 1996b). Fortunately, the slightly weaker theorem **S4ECT**, which is provable, is sufficient to serve the purpose to which I originally put **GMECC**.

It is quite straightforward to show that the left to right direction of (†) holds for any *normal* modal algebra (and hence any algebra in $\mathcal{F}_{S4}$). Recall that a modal algebra is normal iff it obeys the equation **norm**, $*0 = 0$. This means that $\perp * \perp \mathcal{U} = \mathcal{U}$. Thus, if $\perp * \perp(\upsilon) \subseteq \tau_0$ in accordance with the l.h.s., then it is clear that any normal algebra satisfying $\upsilon = \mathcal{U}$ also satisfies $\tau_0 = \mathcal{U}$, which is what the r.h.s. says.

The right to left direction of (†) is considerably harder to show. I prove it by proving the contrapositive — i.e.:

$$\text{if} \qquad \not\models_{\mathrm{MA}_{S4}} \perp * \perp(\upsilon) \subseteq \tau_0 \qquad \text{then} \qquad \upsilon = \mathcal{U} \;\; \not\models_{\mathrm{MA}_{S4}} \;\tau_0 = \mathcal{U} \qquad (\dagger\dagger)$$

Let S be the set of all constants occurring in the terms υ and τ_0. If the antecedent of (††) is true, there must be some assignment $\Sigma = \langle S, U, \sigma, m\rangle$ satisfying the equational constraints of the frame $\mathcal{F}_{S4}$ and such that $\Sigma[\perp * \perp(\upsilon)] \not\subseteq \Sigma[\tau_0]$. From Σ we can construct an assignment, Σ', which verifies the consequent of (††) — i.e. $\Sigma'[\upsilon] = \mathcal{U}$ but $\Sigma'[\tau_0] \neq \mathcal{U}$:

Let $U' = \Sigma[\perp * \perp(\upsilon)]$. We define $\Sigma' = \langle S, U', \sigma', m'\rangle$ by stipulating that:

- $\sigma'[\kappa] = \sigma[\kappa] \cap U'$, for all constants, $\kappa \in S$,
- $m'(X) = U' \perp (U \perp m(U \perp (U' \perp X)))$, for all sets $X \subseteq U'$.

The specification of the modal function m' looks rather complicated; however, it is just the consequence of requiring that for any $X \subseteq U'$ the value of $\perp m' \perp (X)$ according to Σ' should be equal to $\perp m \perp (X)$ under Σ. To specify this precisely I define l to be the dual of m — i.e. $l(X) = \perp m \perp (X)$. From this definition it is easy to see that $m(X) = \perp l \perp (X)$. The interpretation of '$\perp$' as set complement is dependent on the specific value of the universal region, $\mathcal{U}$. To make this dependence explicit $m(X)$ can be expressed as $U \perp l(U \perp X)$; and conversely $l(X) = U \perp m(U \perp X)$. Similarly, $m'(X) = U' \perp l'(U' \perp X)$, where l' is the dual of m'. If we now stipulate that $l'(X) = l(X)$ we find that

$$m'(X) = U' \perp l'(U' \perp X) = U' \perp l(U' \perp X) = U' \perp (U \perp m(U \perp (U' \perp X))) \; .$$

By specifying m' in this way, I ensure that the operator $\perp * \perp(\ldots)$ is interpreted as the same function in Σ' as in Σ (except that the domain of m' is limited to subsets of U').

It can then be shown that for any term, τ (made up of constants in S), $\Sigma'[\tau] = \Sigma[\tau] \cap U'$. We know this identity holds for atomic terms because of the definition of σ', so to show it inductively for all terms we need to show that, if it holds for α and β, it must hold for $\overline{\alpha}$, $\alpha \cup \beta$, $\alpha \cap \beta$ and $*\alpha$. For the Boolean operators the required identities are demonstrated by the following sequences of equations:

$$\Sigma'[\overline{\alpha}] = U' \perp \Sigma'[\alpha] = U' \perp (\Sigma[\alpha] \cap U') = U' \perp \Sigma[\alpha] = (U \perp \Sigma[\alpha]) \cap U' = \Sigma[\overline{\alpha}] \cap U'$$

$$\Sigma'[\alpha \cup \beta] = \Sigma'[\alpha] \cup \Sigma'[\beta] = (\Sigma[\alpha] \cap U') \cup (\Sigma[\beta] \cap U') = (\Sigma[\alpha] \cup \Sigma[\beta]) \cap U' = \Sigma[\alpha \cup \beta] \cap U'$$

$$\Sigma'[\alpha \cap \beta] = \Sigma'[\alpha] \cap \Sigma'[\beta] = (\Sigma[\alpha] \cap U') \cap (\Sigma[\beta] \cap U') = (\Sigma[\alpha] \cap \Sigma[\beta]) \cap U' = \Sigma[\alpha \cap \beta] \cap U'$$

(In the first of these the identity $U' \perp \Sigma[\alpha] = (U \perp \Sigma[\alpha]) \cap U'$ depends on the fact that $U' \subseteq U$.)

For the case of $*\alpha$ we have $\Sigma'[*\alpha] = \Sigma'[\perp\perp * \perp\perp\alpha] = U' \perp \Sigma'[\perp * \perp(\perp\alpha)]$. We can now interpret the '$\perp * \perp$' operation as l', which has been defined so as to coincide with l: thus $U' \perp \Sigma'[\perp * \perp(\perp\alpha)] = U' \perp l'(\Sigma'[\perp\alpha]) = U' \perp l(\Sigma'[\perp\alpha])$. But I have already shown that $\Sigma'[\perp\alpha] = \Sigma[\perp\alpha] \cap U'$, so $U' \perp l(\Sigma'[\perp\alpha]) = U' \perp l(\Sigma[\perp\alpha] \cap U')$. Now, since m is additive, its dual l distributes over $\cap$ giving $U' \perp (l(\Sigma[\perp\alpha]) \cap l(U'))$. Because $U' = \Sigma[\perp * \perp(v)]$ and all algebras in $\mathcal{F}_{S4}$ satisfy **idem** it is easy to show that $l(U') = U'$.[9] Now, since $U' \subseteq U$, it immediately follows that $U' \perp (l(\Sigma[\perp\alpha]) \cap U') = (U \perp l(\Sigma[\perp\alpha])) \cap U'$. Finally this expression can be rewritten to give the desired result: $(U \perp l(\Sigma[\perp\alpha])) \cap U' = (U \perp \Sigma[\perp * \perp(\perp\alpha)]) \cap U' = \Sigma[\perp(\perp * \perp(\perp\alpha))] \cap U' = \Sigma[*\alpha] \cap U'$.

We must verify that the algebra specified by Σ' is a member of $\mathcal{F}_{S4}$. I have established that for every term (built from constants in S) $\Sigma'[\tau] = \Sigma[\tau] \cap U'$. This means that every equation, $\tau_1 = \tau_2$, satisfied by Σ will also be satisfied by Σ'. Since, by hypothesis, Σ must satisfy all the frame equations of $\mathcal{F}_{S4}$, Σ' must also satisfy these frame equations.

To complete the proof I must show that Σ' verifies the r.h.s. of (††). Since the algebra generated by Σ' is in $\mathcal{F}_{S4}$, it must satisfy **epis**, which means that for any term, τ, $\Sigma'[\tau] \subseteq \Sigma'[*\tau]$ and consequently $\Sigma'[\tau] \supseteq \Sigma'[\perp * \perp\tau]$. We know that $\Sigma'[\perp * \perp v] = U'$, so $\Sigma'[v] \supseteq U'$; but $\Sigma'[v] = \Sigma[v] \cap U'$, so $\Sigma'[v] = U'$. Recall that Σ was chosen to verify the antecedent of (††) because $\Sigma[\perp * \perp(v)] \not\subseteq \Sigma[\tau_0]$. Thus, $U' \not\subseteq \Sigma[\tau_0]$; and from this it follows that $\Sigma[\tau_0] \cap U' \subsetneq U'$. Hence we have $\Sigma'[\tau_0] \subsetneq U'$. ■

As with the classical case, an arbitrary modal set equation can be directly transformed into universal form and the formula $\mathbf{MFe}[\xi]$ can be regarded as representing the equational constraint ξ. The modal logic $S4$ can thus be used to reason about arbitrary equations constraining algebras in the frame $\mathcal{F}_{S4}$ according to the following generalisation of **S4ECT**:

$$\xi_1, \ldots, \xi_n \models_{\mathrm{MA}_{S4}} \xi_0 \quad \text{iff} \quad \Box\mathbf{MFe}[\xi_1], \ldots, \Box\mathbf{MFe}[\xi_1] \vdash_{S4} \mathbf{MFe}[\xi_0] \ .$$

The form of **S4ECT** is a bit awkward in that in the $S4$ deduction corresponding to an entailment between equations, we need to add an extra $\Box$ operator to the formulae on the left of $\vdash_{S4}$ but not to the formula on the right. This means that the question "What is the $S4$ representation of the equation $\xi\Gamma$" does not have a simple answer. However, it is easily shown that a sequent $\Box\phi_1, \ldots, \Box\phi_n \vdash_{S4} \phi_0$ is in fact valid if and only if $\Box\phi_1, \ldots, \Box\phi_n \vdash_{S4} \Box\phi_0$. Thus, for the purpose of testing entailments, it can be said that the representation of an equation ξ is $\Box\mathbf{MFe}[\xi]$.

[9] If the algebra specified by Σ satisfies **idem**, $*(*(x)) = *(x)$, then $m(m(X)) = m(X)$. Thus $l(U') = -m-(U') = -m-(\Sigma[-*-(v)]) = -m--m-(\Sigma[v]) = -mm-(\Sigma[v]) = -m-(\Sigma[v]) = \Sigma[-*-(v)] = U'$. The requirement that $l(U') = U'$ is of particular significance in that it is the reason why we need to have $\Box\phi_1, \ldots, \Box\phi_n \vdash_{S4} \phi_0$ on the r.h.s. of **S4ECT**, rather than the simpler (but stronger) condition $\phi_1, \ldots, \phi_n \vdash_{S4} \phi_0$.

5.3 Topological Closure Algebras

The purpose of my examining the algebraic semantics of modal logics, culminating in the demonstration of the theorem **S4ECT**, was that, just as **CECT** enabled us to use classical 0-order logics to reason about those spatial relationships which are essentially Boolean in character, **S4ECT** will enable us to reason about a wider range of relationships by means of deduction in the logic $S4$. This will enable us to reason in terms of certain topological properties which were not expressible in the classical representation. The key link is that modal algebras of the frame $\mathcal{F}_{S4}$ are essentially the same as *closure algebras*, which give an algebraic characterisation of topological spaces.

5.3.1 Closure and Interior Algebras

The theory of topological spaces is traditionally stated in the language of set theory. But, if we are concerned only with the structure of a topological space with respect to the Boolean operations and the interior and closure operations, we can do without the full language of set theory and give a purely algebraic account of the space, which does not involve any use of the elementhood relation, '$\in$'. This abstraction results in a Boolean algebra with an additional operator obeying appropriate conditions for either an interior or a closure function. In the first comprehensive treatment of these algebras (McKinsey and Tarski 1944) the closure operator was taken as primitive and the resulting algebra called a *closure algebra*. A closure algebra is a structure $\langle S, \cup, \perp, c \rangle$, where $\langle S, \cup, \perp \rangle$ is a Boolean Algebra and the operator 'c' satisfies the equations for a closure function given in section 2.1. These include in particular the equation $c(X \cup Y) = c(X) \cup c(Y)$, which means that c is an additive function. In other words $\langle S, \cup, \perp, c \rangle$ is a modal algebra.

An *interior algebra* is a structure $\langle S, \cup, \perp, i \rangle$, where $\langle S, \cup, \perp \rangle$ is a Boolean Algebra and i satisfies the equations characterising an interior operator. An interior can be interpreted in terms of a modal algebra but with i corresponding to the algebraic operation $\perp * \perp(\ldots)$.

Closure (or interior) algebraic equations provide a simple constraint language for describing topological relationships between arbitrary sets of points in a topological space. Some of the more significant constraints which can be expressed are given in table 5.1

Constraint	*Meaning*
$X = c(X)$	X is closed
$X = \perp c \perp (X)$	X is open
$X = \perp c \perp c(X)$	X is regular open
$X \cup Y = Y$	X is part of Y
$c(X) \cup Y = Y$	The closure of X is part of Y
$X \cap Y = \emptyset$	X and Y are disjoint
$c(X) \cap c(Y)$	The closures of X and Y are disjoint
$X = Y \cup Z$	X is the union of Y and Z

Table 5.1: Some constraints expressible as closure algebra equations.

5.3.2 RCC Relations Representable in Interior Algebra

I now consider how RCC relations can be represented in interior algebra. To do this I employ the topological interpretation of the RCC theory which was given in section 3.5.2. Recall that, under this interpretation, regions are identified with *non-empty regular open sets*. Two regions overlap if their corresponding sets share a point and are connected if the closures of these sets share a point. Thus the relations can be formally defined in terms of topology by:

$$\mathsf{O}(x,y) \equiv_{def} \exists\pi[\pi \in X \wedge \pi \in Y]$$

$$\mathsf{C}(x,y) \equiv_{def} \exists\pi[\pi \in c(X) \wedge \pi \in c(Y)]$$

These definitions give us a rigorous formal specification of the RCC connection and overlap relations in terms of point-set topology. But they make use of a highly expressive set-theoretic language, including both quantification and the element relation. Given that these relations are intuitively very simple, one may wonder whether it is possible to give an alternative characterisation of C and O in the much less expressive language of interior algebraic equations.

As it happens the negations of each of these relations can be quite easily defined as follows:

$$\mathsf{DC}(x,y) \equiv_{def} i(\overline{X}) \cup i(\overline{Y}) = \mathcal{U}$$

$$\mathsf{DR}(x,y) \equiv_{def} \overline{X \cap Y} = \mathcal{U}$$

But C and O cannot themselves be defined as interior algebraic equations. This follows from the general observation that purely equational constraints are always consistent with any purely equational theory (there must always be at least a trivial one-element model, in which all constants denote the same individual). Thus if the negation of some constraint can be expressed as an equation, then the constraint itself cannot be equationally expressible (otherwise that constraint would be consistent with its own negation).

To define C and O we would need a language containing both interior algebraic equations and the negations of such equations. This extended language will be considered later; but for now I shall consider only those topological relations definable with equations alone. Table 5.7 gives definitions of seven binary relations: DC, DR, P, Pi, NTP, NTPi and EQ. This set, which will be called RCC-7, is of particular significance because, as will be shown in the next section, each of the RCC-8 relations can be expressed as a conjunction of positive and negative RCC-7 relations. Note that RCC-7 is neither jointly exhaustive nor pairwise disjoint: if two regions partially overlap, they stand in none of the seven relations; and DR (being the disjunction of DC and EC) can hold of two regions which are also DC. A number of other binary RCC relations are expressible by means of interior/closure algebra equations.[10] For example, $\mathsf{EQ}(\mathsf{sum}(x,y),\mathsf{u})$ can be expressed by $X \cup Y = \mathcal{U}$.

[10] However, it appears that RCC-7 is the complete set of binary RCC relations expressible in interior/closure algebra, which are *essentially* binary in that they are not reducible to any monadic condition and specification of the relation in RCC does not involve reference to a third region such as u. Verifying this would require further examination of the class of interior algebraic equations.

RCC Relation	*Interior Algebra Equation*
$\mathsf{DC}(x, y)$	$i(\overline{X}) \cup i(\overline{Y}) = \mathcal{U}$
$\mathsf{DR}(x, y)$	$\overline{X \cap Y} = \mathcal{U}$
$\mathsf{P}(x, y)$	$\overline{X} \cup Y = \mathcal{U}$
$\mathsf{Pi}(x, y)$	$X \cup \overline{Y} = \mathcal{U}$
$\mathsf{NTP}(x, y)$	$i(\overline{X}) \cup Y = \mathcal{U}$
$\mathsf{NTPi}(x, y)$	$X \cup i(\overline{Y}) = \mathcal{U}$
$\mathsf{EQ}(x, y)$	$(\overline{X} \cup Y) \cap (X \cup \overline{Y}) = \mathcal{U}$

Table 5.2: Seven relations defined by interior algebra equations

Note that equations in table 5.2 assume that the regions are open. To make this explicit in an interior algebraic representation of RCC relations, one ought to include an equation of the form $X = i(X)$ for each region x occurring in the set of equations. In fact, in section 3.5.2 I argued that for a strictly correct point-set interpretation of RCC relations, one should require that regions should be *regular* open. This requirement is also easily enforced by equations of the form $X = i \perp i \perp (X)$.

One could equally well employ the interior algebra framework to specify RCC relations in terms of the dual set-theoretic interpretation of RCC mentioned in section 3.5.5. Under that interpretation, regions are taken as non-empty regular *closed* sets, which connect iff they share a point and overlap iff they share an interior point. The RCC-7 relations would then be specified as given in table 5.3. This encoding which is arguably simpler than that of table 5.2, was presented by me in (Bennett 1996b) and is the basis of subsequent analysis by Renz and Nebel (1997). However, in the next chapter, where I present an intuitionistic interpretation of interior algebraic constraints, we shall see that the open set interpretation is much more convenient.

RCC Relation	*Interior Algebra Equation*
$\mathsf{DC}(x, y)$	$\overline{X \cap Y} = \mathcal{U}$
$\mathsf{DR}(x, y)$	$\overline{i(X) \cap i(Y)} = \mathcal{U}$
$\mathsf{P}(x, y)$	$\overline{X} \cup Y = \mathcal{U}$
$\mathsf{Pi}(x, y)$	$X \cup \overline{Y} = \mathcal{U}$
$\mathsf{NTP}(x, y)$	$\overline{X} \cup i(Y) = \mathcal{U}$
$\mathsf{NTPi}(x, y)$	$i(X) \cup \overline{Y} = \mathcal{U}$
$\mathsf{EQ}(x, y)$	$(\overline{X} \cup Y) \cap (X \cup \overline{Y}) = \mathcal{U}$

Table 5.3: Alternative definitions for closed regions

5.3.3 Using Inequalities to Extend Expressive Power

I now consider a more expressive constraint language based on interior algebras, in which one can specify both interior algebraic equalities and negations of such equalities. Since each of the RCC-7 relations corresponds to an equation in interior algebra, the extended language allows straightforward representation of all those relations which can be expressed in the form

$$R_1(x,y) \wedge \ldots \wedge R_j(x,y) \wedge \neg R_{j+1}(x,y) \wedge \ldots \wedge \neg R_k(x,y) \;, \qquad \textbf{(RCC7conj)}$$

where each of the relations R_i is a member of RCC-7.

I have investigated the complete set of relations representable in this way by means of a simple Prolog program — the code (including inlined documentation) is given in appendix C.1. Since the RCC-7 relations are not logically independent many combinations of the form of **RCC7conj** are equivalent. It is easy to specify all the entailments and incompatibilities among pairs of RCC-7 relations and negated RCC-7 relations which are asserted of the same two objects. Any combination which contains an incompatibility is equivalent to the impossible relation and any combination which contains two relations, one of which is entailed by the other, is equivalent to the combination resulting from removing the entailed relation. Every combination, containing no incompatible pair and no relation that is entailed by another, specifies a distinct relation in its most simple form and can be regarded as its canonical representation. The Prolog program first generates every relation specification of the form of **RCC7conj** and identifies which of these are canonical.

We have seen that whether certain combinations of relations are regarded as possible depends upon whether we allow regions to be null (the null region is both part of and disconnected from every other region; but no two non-null regions can stand in both these two relations). If we allow that the regions involved may possibly be null we find that 171 distinct relations can be represented. The complete list of these relations is given in appendix C.1.1. If we require that both regions involved in a relation must be non-null then 115 of these relations are possible. These include each of the RCC-8 relations. Table 5.4 shows how each of the RCC-8 relations can be expressed

RCC Rel.	*Equivalent RCC-7 Conjunction*	*Algebraic Constraint(s)*
$\mathsf{DC}(x,y)$	$\mathsf{DC}(x,y)$	$(i(\overline{x}) \cup i(\overline{y}) = \mathcal{U})$
$\mathsf{EC}(x,y)$	$\mathsf{DR}(x,y) \wedge \neg\mathsf{DC}(x,y)$	$(\overline{x \cap y} = \mathcal{U}) \wedge (i(\overline{x}) \cup i(\overline{y}) \neq \mathcal{U})$
$\mathsf{PO}(x,y)$	$\neg\mathsf{DR}(x,y) \wedge \neg\mathsf{P}(x,y) \wedge \neg\mathsf{Pi}(x,y)$	$(\overline{x \cap y} \neq \mathcal{U}) \wedge (\overline{x} \cup y \neq \mathcal{U}) \wedge (x \cup \overline{y} \neq \mathcal{U})$
$\mathsf{TPP}(x,y)$	$\mathsf{P}(x,y) \wedge \neg\mathsf{EQ}(x,y) \wedge \neg\mathsf{NTPP}(x,y)$	$(\overline{x} \cup y = \mathcal{U}) \wedge (x \neq y) \wedge (i(\overline{x}) \cup y \neq \mathcal{U})$
$\mathsf{TPPi}(x,y)$	$\mathsf{Pi}(x,y) \wedge \neg\mathsf{EQ}(x,y) \wedge \neg\mathsf{NTPPi}(x,y)$	$(x \cup \overline{y} = \mathcal{U}) \wedge (x \neq y) \wedge (x \cup i(\overline{y}) \neq \mathcal{U})$
$\mathsf{NTPP}(x,y)$	$\mathsf{NTPP}(x,y)$	$(i(\overline{x}) \cup y = \mathcal{U})$
$\mathsf{NTPPi}(x,y)$	$\mathsf{NTPPi}(x,y)$	$(x \cup i(\overline{y}) = \mathcal{U})$
$\mathsf{EQ}(x,y)$	$\mathsf{EQ}(x,y)$	$(x = y)$

Table 5.4: The RCC-8 relations represented as interior algebra constraints

as a conjunction of RCC-7 relations and their negations and also gives the corresponding interior algebraic constraints.

The relations of the form **RCC7conj** form a *semi-lattice* with respect to the conjunction operation. This just means that conjunction is associative, symmetrical and idempotent. Clearly, the sub-structure comprising only those relations including the non-null constraints on both argument regions also forms a semi-lattice. It is fairly easy to show by inspection that the RCC-8 relations constitute a set of minimal elements (i.e. *atoms*) of this semi-lattice. One needs to check that the result of conjoining any RCC-8 relations with an additional RCC-7 constraint is either the impossible relation (corresponding to the $\perp$ element of the lattice) or is equivalent to the original RCC-8 relation.

Each RCC-7 relation is equivalent to some disjunction of RCC-8 relations; and, because RCC-8 is JEPD (jointly exhaustive and pairwise disjoint), the negations of RCC-7 relations also correspond to disjunctions of RCC-8 relations (provided that non-null constraints on the arguments are in force). This means that each of the 115 relations representable in this way is also a disjunction of RCC-8 relations. Hence, the language of interior algebraic equations and their negations provides a representation for almost half of the $2^8 = 256$ spatial relations which are disjunctions of the RCC-8 relations. In particular all of the RCC-8 relations can be expressed as well as the primitive C relation.

5.4 Encoding Closure Algebraic Constraints in $S4$

It was established by Tarski and McKinsey (1948) that the $S4$ box operator can be modelled algebraically by an interior operator. We have seen that, in the set algebra interpretation of a 0-order logical calculus, operators are identified with maps from subsets to subsets of some universe: the classical connectives are associated with Boolean functions and modal operators are associated with additive functions, which may be constrained by further equational constraints. A closure algebra is a Boolean algebra with an additive closure operator and is thus a modal algebra. c is the modal operator, which I have hitherto denoted by $*$. Hence c can be taken as the interpretation of a logical modal operator, '$\Box$'. I now show that the defining equations of the c operator mean that this is an $S4$ modal operator.

By making use of the meta-level notation relating modal algebraic equations and corresponding modal formulae it is easy to state precisely the relationship between closure/modal algebraic equations and modal formulae. The representation of a closure/modal algebraic equation ξ in modal logic is the formula $\mathbf{MFe}[\xi]$. Because the equations specifying properties of the closure operation contain free variables they will be mapped to modal schemata rather than formulae. The characteristic equations of a closure algebra and corresponding modal schemata are as follows:

Closure Axioms	*Modal Schemata*
$X \cup c(X) = c(X)$	$(\phi \vee \Diamond\phi) \leftrightarrow \Diamond\phi$
$c(c(X)) = c(X)$	$\Diamond\Diamond\phi \leftrightarrow \Diamond\phi$
$c(\emptyset) = \emptyset$	$\Diamond\bot = \bot$
$c(X \cup Y) = c(X) \cup c(Y)$	$\Diamond(\phi \vee \psi) \leftrightarrow (\Diamond\phi \vee \Diamond\psi)$

Table 5.5: Closure Axioms and Corresponding Modal Schemata

As modal formalisms are more often specified in terms of the $\Box$ operator, the transformation based on the dual correspondence between then interior operator and $\Box$ yields more familiar schemata:

Interior Axioms	*Modal Schemata*	
$i(X) \cup X = X$	$(\Box\phi \vee \phi) \leftrightarrow \phi$	(**T'**)
$i(i(X)) = i(X)$	$\Box\Box\phi \leftrightarrow \Box\phi$	(**4+**)
$i(U) = U$	$\Box\top$	(**N**)
$i(X \cap Y) = i(X) \cap i(Y)$	$\Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi)$	(**R**)

Table 5.6: Interior Axioms and Corresponding Modal Schemata

Clearly **T'** is equivalent to the schema **T**, $\Box\phi \rightarrow \phi$ (see section 5.2.2), and, given that **T** holds, **4+** can be weakened to $\Box\phi \rightarrow \Box\Box\phi$, which is the schema **4**. Furthermore it is well known that the schemata **N** and **R** in conjunction with the rule **RE** are equivalent to the combination of schema **K** and the rule of necessitation, **RN**. Thus specifying that **N**, **R** and **RE** hold is an alternative way of specifying that a modal logic is normal (see (Chellas 1980, chapter 4)). Recall that **RE** holds in any algebraic semantics for a modal operator. Hence, the modal logic derived from an interior or closure algebra by transforming equational algebraic constraints into modal schemata is exactly the logic $S4$. Consequently, in virtue of the correspondence theorem **S4ECT**, deduction in $S4$ can be used to reason about closure algebraic equations such as those given in tables 5.1, 5.2 and 5.3.

5.4.1 RCC Relations Representable in $S4$

Since the $S4$ modality can be interpreted as an interior function over a topological space, we can use this interpretation to encode topological relations as $S4$ formulae. The basis of this representation is exactly the same as for the $\mathcal{C}$ representation but by use of the additional modal operator it is possible to make a distinction between connection and overlapping which cannot be expressed in $\mathcal{C}$. Table 5.7 shows the $S4$ formula corresponding to each of the RCC-7 relations. The middle column shows the algebraic set-equation associated with the relation. We see that, if the interior operator i is identified with the corresponding modal algebra operator $\bot * \bot$, then the interior algebraic equation ξ, is represented by the $S4$ formula $\Box\,\mathbf{MFe}[\xi]$.

RCC Relation	*Interior Algebra Equation* (ξ)	*S4 formula* ($\Box\,\mathbf{MFe}[\xi]$)
$\mathsf{DC}(x,y)$	$i(\overline{X}) \cup i(\overline{Y}) = \mathcal{U}$	$\Box(\Box\neg x \vee \Box\neg y)$
$\mathsf{DR}(x,y)$	$\overline{X \cap Y} = \mathcal{U}$	$\Box\neg(x \wedge y)$
$\mathsf{P}(x,y)$	$\overline{X} \cup Y = \mathcal{U}$	$\Box(\neg x \vee y)$
$\mathsf{Pi}(x,y)$	$X \cup \overline{Y} = \mathcal{U}$	$\Box(x \vee \neg y)$
$\mathsf{NTP}(x,y)$	$i(\overline{X}) \cup Y = \mathcal{U}$	$\Box(\Box\neg x \vee y)$
$\mathsf{NTPi}(x,y)$	$X \cup i(\overline{Y}) = \mathcal{U}$	$\Box(x \vee \Box\neg y)$
$\mathsf{EQ}(x,y)$	$(\overline{X} \cup Y) \cap (X \cup \overline{X}) = \mathcal{U}$	$\Box((\neg x \vee y) \wedge (x \vee \neg y))$

Table 5.7: Seven relations defined by interior algebra equations and corresponding $S4$ formulae

I now illustrate how the correspondence theorem **S4ECT**, enables deduction in $S4$ to be used to reason about entailment among certain RCC relations. Consider the following argument:

$$\mathsf{NTP}(a,b) \wedge \mathsf{DR}(b,c) \models \mathsf{DC}(a,c)$$

This corresponds to the following entailment between interior algebraic equations:

$$i(\overline{A}) \cup B = \mathcal{U}\,,\ \ \overline{B \cap C} = \mathcal{U}\,,\ A = i(A),\ B = i(B),\ C = i(C) \ \models\ i(\overline{A}) \cup i(\overline{C}) = \mathcal{U}\ .$$

Here the equations of the form $\alpha = i(\alpha)$ constrain the regions to correspond to open sets.[11] By appealing to **S4ECT** this can be shown to be valid because we have

$$\Box(\Box\neg a \vee b),\ \ \Box\neg(b \wedge c),\ \Box(a \leftrightarrow \Box a),\ \Box(b \leftrightarrow \Box b),\ \Box(c \leftrightarrow \Box c)\ \ \vdash_{S4}\ \ \Box(\Box\neg a \vee \Box\neg c)\ .$$

The $S4$ representation is quite expressive but does have serious limitations. For instance, although both disconnection, $\mathsf{DC}(x,y)$, and discreteness, $\mathsf{DR}(x,y)$, can be represented it is still not possible to specify the relation of external connection, $\mathsf{EC}(x,y)$. We have also seen that (although their negations can be represented) the fundamental relations C and O cannot be represented. In order to overcome these deficiencies we need a language in which one can express closure-algebraic inequalities as well as equalities.

5.5 Extended Modal Logics, L^+

In order to increase the expressive power of $S4$, so that we can represent both positive and negative algebraic constraints, I use the same method that was applied to $\mathcal{C}$ in the last chapter. Given a 0-order modal logic, L, we can define an augmented representation language, L^+, whose expressions are pairs $\langle \mathcal{M}, \mathcal{E} \rangle$, where $\mathcal{M}$ and $\mathcal{E}$ are formulae of L and are called respectively *model* and *entailment* constraints. We stipulate that an L^+ expression $\langle \mathcal{M}, \mathcal{E} \rangle$ is consistent if and only if no formula in $\mathcal{E}$ is entailed by the set $\mathcal{M}$, according to the logic L.

[11] In general, to be faithful to RCC, one should ensure that regions are *regular* open by adding the stronger constraint $\alpha = i - i - (\alpha)$; but the inference in this example is valid for any open regions.

This kind of augmentation could in principle be applied to any logical language whatsoever. However, if it is to be useful, one must have some definite interpretation of the meanings of L^+ expressions (or at least some of these expressions). Just as $\mathcal{C}^+$ expressions may be interpreted as sets of positive and negative equational constraints on Boolean algebras, L^+ expressions can be interpreted as sets of positive and negative equational constraints on modal algebras. Defining such an interpretation requires a syntactic mapping between modal formulas and equations. A straightforward transform is given by $\mathbf{MFe}[\phi]$ and its inverse; but we shall see that for reasoning about most modal algebras slightly more complex mappings must be employed. To use L^+ to test consistency of arbitrary positive and negative equational constraints, every set of such constraints must be representable in L^+. However, it is not necessary that every L^+ expression be interpretable as a set of constraints: this interpretation may be applicable only to a sub-language of L^+. (For instance, we shall see that, in the case of $S4^+$, only those expressions where all model constraints have $\Box$ as the primary operator can be coherently interpreted as constraints on $S4$ algebras.)

To show that an interpretation of this kind is satisfactory one must show that (for all L^+ expressions that are interpretable as sets of algebraic constraints) the stipulated consistency checking method for L^+ expressions is sound and complete with respect to consistency of the corresponding constraints on modal algebras. As in the case of $\mathcal{C}$, this task can be divided into two parts: establishing a convexity result for entailment among modal algebraic constraints; and then exploiting an appropriate correspondence theorem relating entailments in the modal logic and entailments between modal algebraic equations. We shall see that because of the failure of **GMECC** for most modal logics, the second step does not seem to be achievable in a uniform way: a correspondence theorem — if one exists — must be established separately for each given modal logic.[12]

5.5.1 Convexity of Modal Algebras

In section 4.5 we saw that the theory of equational constraints on Boolean algebras is convex in the sense that a conjunction of equational constraints can only entail a disjunction of such constraints if it entails at least one disjunct of that disjunction. Consequently a set of positive and negative equational constraints is consistent if and only if the contrary of one of the negative constraints is entailed by the conjunction of the positive constraints. The same result can be proved for modal algebras — i.e. Boolean algebras supplemented with additional additive operators. Since all modal algebraic equations can be put in the form $\tau = \mathcal{U}$ this is guaranteed by the following theorem:

Convexity of Disjunctive Modal-Algebraic Entailments (MEconv)

$$\mu_1 = \mathcal{U}, \ldots, \mu_m = \mathcal{U} \models_{\mathrm{MA}_L} \varepsilon_1 = \mathcal{U} \lor \ldots \lor \varepsilon_n = \mathcal{U}$$

iff

$$\mu_1 = \mathcal{U}, \ldots, \mu_m = \mathcal{U} \models_{\mathrm{MA}_L} \varepsilon_i = \mathcal{U} \text{ for some } i \in \{1, \ldots n\}$$

Like **BEconv**, **MEconv** is closely related to **ELcons**. By appealing to **ELconv** and the

[12] However, in section 5.7 I shall give an alternative method of extending modal logics by adding an extra operator. This method does yield a general correspondence theorem for the extended languages.

fact that modal algebras can be specified by purely equational theories, we can deduce that any modal algebra is convex w.r.t. entailments of the form of DE. In appendix B I give an alternative, model-theoretic proof of **MEconv**, which may prove helpful to further study of the properties of entailments among modal-algebraic constraints. This proof relies only on the additivity of the modal operator and does not require that its algebraic properties be specifiable just in terms of equations. Nevertheless, because modal schemata correspond directly to universal equations in the algebraic semantics, any modal operator whose logical properties are specifiable in terms of schemata will correspond to an algebraic function which is equationally specifiable.

5.5.2 A Correspondence Theorem for $S4^+$

The convexity theorem **MEconv** means that checking consistency of sets of positive and negative modal algebraic constraints reduces to the problem of determining entailments among positive constraints. Thus if consistency of L^+ expressions is to be faithful to consistency of the associated algebraic constraints we only need to show that entailments between positive algebraic constraints hold just in case the corresponding entailments in L are valid. This requires a correspondence theorem such as **S4ECT**.

By combining **MEconv** with **S4ECT** a correspondence between the consistency of modal algebraic equations and inequalities and consistency of certain $S4^+$ expressions is immediately obtained. Also, because of the interpretation of interior algebras as $S4$ modal algebras, $S4^+$ can be used to test consistency of topological constraints. These results are encapsulated in the following theorem which ties together the main correspondence theorems of this chapter:

$S4^+$ Correspondence Theorem (S4+CT)

The following three conditions are equivalent:

1. The set $\{\mu_1 = \nu_1, \ldots, \mu_j = \nu_j,\ \sigma_1 \neq \tau_1, \ldots, \sigma_k \neq \tau_k\}$ of $S4$ modal algebraic equations and inequalities is consistent — i.e. is satisfied by some algebra in the frame $\mathcal{F}_{S4}$.

2. The corresponding set of interior algebraic equations and inequalities, resulting from replacing in the set of constraints given in 1. all occurrences of $\Box$ by i, is consistent — i.e. is satisfied by some topological space.

3. The $S4^+$ expression $\langle \mathcal{M}, \mathcal{E} \rangle$ given by
$\langle \{\Box\,\mathbf{MFe}[\mu_1 = \nu_1], \ldots, \Box\,\mathbf{MFe}[\mu_j = \nu_j]\},\ \{\mathbf{MFe}[\sigma_1 = \tau_1], \ldots, \mathbf{MFe}[\sigma_k = \tau_k]\}\rangle$
is consistent — i.e. there is no formula $\phi \in \mathcal{E}$ such that $\mathcal{M} \vdash_{S4} \phi$.

S4+CT enables one to test the consistency of sets of spatial relationships, representable in terms of interior algebra equations and inequalities, by carrying out a series of proof checks in the logic $S4$. The definition of consistency for $S4^+$ expressions also yields criteria for determining entailment between $S4^+$ expressions which is exactly analogous to that given for $\mathcal{C}^+$ in section 4.7.1:

> $S4^+$ **Entailment Theorem (S4+ET)**
>
> $\langle \mathcal{M}, \mathcal{E} \rangle \models_{S4^+} \langle \mathcal{M}', \mathcal{E}' \rangle$ iff
>
> either $\langle \mathcal{M}, \mathcal{E} \rangle \models_{S4^+}$ or (for all $\phi \in \mathcal{M}'$: $\langle \mathcal{M}, \{\phi\} \rangle \models_{S4^+}$
> and for all $\psi \in \mathcal{E}'$: $\langle \mathcal{M} \cup \{\psi\}, \mathcal{E} \rangle \models_{S4^+}$)

This enables a simple generalisation of **S4+CT** to give a correspondence between entailments between sets of modal or topological algebraic constraints and entailments between $S4^+$ expressions. One set of constraints entails another iff the entailment holds between the corresponding pair of $S4^+$ expressions.

5.6 Representing RCC Relations in $S4^+$

Since $S4^+$ can represent both equations and inequalities between terms made up of Boolean operations and an interior operator, it can express a very large class of spatial relationships. In particular, it can represent all those RCC relations which can be expressed in the form **RCC7conj** — i.e. as a conjunction of positive and negative RCC-7 relations (see section 5.3.3 above). In the $S4^+$ representation the positive relations $(R_1, \ldots, R_j)$ will correspond to model constraints and the negated relations $(R_{j+1}, \ldots, R_k)$ will correspond to entailment constraints.

Relation	*Model Constraint*	*Entailment Constraints*
$\mathsf{DC}(x, y)$	$\Box(\Box\neg x \vee \Box\neg y)$	$\neg x,\ \neg y$
$\mathsf{EC}(x, y)$	$\Box\neg(x \wedge y)$	$\Box\neg x \vee \Box\neg y,\ \neg x,\ \neg y$
$\mathsf{PO}(x, y)$	—	$\neg(x \wedge y),\ x \to y,\ y \to x,\ \neg x,\ \neg y$
$\mathsf{TPP}(x, y)$	$\Box(x \to y)$	$\Box\neg x \vee y,\ y \to x,\ \neg x,\ \neg y$
$\mathsf{TPPi}(x, y)$	$\Box(y \to x)$	$\Box\neg y \vee x,\ x \to y,\ \neg x,\ \neg y$
$\mathsf{NTPP}(x, y)$	$\Box(\Box\neg x \vee y)$	$y \to x,\ \neg x,\ \neg y$
$\mathsf{NTPPi}(x, y)$	$\Box(\Box\neg y \vee x)$	$x \to y,\ \neg x,\ \neg y$
$\mathsf{EQ}(x, y)$	$\Box(x \leftrightarrow y)$	$\neg x,\ \neg y$
$\mathsf{C}(x, y)$	—	$\Box\neg x \vee \Box\neg y\ \neg x,\ \neg y$
$\mathsf{EQ}(x, \mathsf{sum}(y, z))$	$\Box(x \leftrightarrow \Box\neg\Box\neg(y \vee z))$	$\neg x,\ \neg y$

Table 5.8: The $S4^+$ encoding of some RCC relations

The representations of the RCC-8 relations are given in table 5.8. The way they are obtained can be summarised as follows: express the RCC-8 relations in terms of RCC-7 relations and interpret these as equational constraints on interior algebras as given in table 5.4. Then translate these constraints into $S4$ according to table 5.7. The formulae corresponding to positive RCC-7 relations become model constraints in the $S4^+$ representation and those corresponding to negated RCC-7 relations become entailment constraints. (Note that the $S4^+$ correspondence theorem requires that model constraints have an extra initial $\Box$ added to the result of applying **MFe** to the modal algebraic equation but this is not required in the entailment constraints. This asymmetry stems from **S4ECT**.)

Let us now consider how the $S4^+$ representation can be used to test the consistency of a simple set of spatial relations. Take for example the following conjunction of RCC-8 relations:

$$\mathsf{TPP}(a,b) \wedge \mathsf{DC}(b,c) \wedge \mathsf{PO}(a,c) \ .$$

Translating into $S4^+$ according to table 5.8 we get the following representation:

$$\langle \{\Box(a \rightarrow b),\ \Box(\Box\neg b \vee \Box\neg c)\},\ \ \{\Box\neg a \vee b,\ b \rightarrow a,\ \neg(a \wedge c),\ a \rightarrow c,\ c \rightarrow a,\ \neg a,\ \neg b,\ \neg c\}\rangle$$

This is an ordered pair consisting of two sets of $S4$ formulae, the first set being model constraints and the second entailment constraints. Appealing to part 3 of **S4+CT** we determine that the relations are inconsistent because

$$\Box(a \rightarrow b),\ \Box(\Box\neg b \vee \Box\neg c) \vdash_{S4} \neg(a \wedge c)$$

i.e. one of the entailment constraints is entailed by the model constraints.[13]

As mentioned in section 5.3.2 one can also represent RCC relations in interior algebra in terms of the dual, closed set interpretation of RCC (see section 3.5.5). The result of encoding this in $S4^+$ is given in table 5.9.

Relation	*Model Constraint*	*Entailment Constraints*
$\mathsf{DC}(x,y)$	$\Box\neg(x \wedge y)$	$\neg x,\ \neg y$
$\mathsf{EC}(x,y)$	$\Box\neg(\Box x \wedge \Box y)$	$\neg(x \wedge y),\ \neg x,\ \neg y$
$\mathsf{PO}(x,y)$	—	$\neg(\Box x \wedge \Box y),\ x \rightarrow y,\ y \rightarrow x,\ \neg x,\ \neg y$
$\mathsf{TPP}(x,y)$	$\Box(x \rightarrow y)$	$x \rightarrow \Box y,\ y \rightarrow x,\ \neg x,\ \neg y$
$\mathsf{TPPi}(x,y)$	$\Box(y \rightarrow x)$	$y \rightarrow \Box x,\ x \rightarrow y,\ \neg x,\ \neg y$
$\mathsf{NTPP}(x,y)$	$\Box(x \rightarrow \Box y)$	$y \rightarrow x,\ \neg x,\ \neg y$
$\mathsf{NTPPi}(x,y)$	$\Box(y \rightarrow \Box x)$	$x \rightarrow y,\ \neg x,\ \neg y$
$\mathsf{EQ}(x,y)$	$\Box(x \leftrightarrow y)$	$\neg x,\ \neg y$
$\mathsf{C}(x,y)$	—	$\neg(x \wedge y),\ \neg x,\ \neg y$
$\mathsf{EQ}(x,\mathsf{sum}(y,z))$	$\Box(x \leftrightarrow (y \vee z))$	$\neg x,\ \neg y$

Table 5.9: $S4^+$ encoding based on the closed set interpretation of RCC

5.6.1 Regularity and Boolean Combination of Regions

In the topological interpretation of RCC given in section 3.5.2 it was argued that regions of the RCC theory should be identified only with (non-empty) *regular open* subsets of a topological space.

[13] Strictly speaking one should add extra model constraints of the form $x \leftrightarrow \Box\neg\Box\neg\alpha$ for each region α involved in the description (see the following section). However, these additional formulae are not relevant to the example.

(Recall that a region, x, is regular open iff $i(c(x)) = x$.) If our modal encoding is to be faithful to the intended meaning of RCC relations we need to enforce this regularity condition. Happily, regularity can easily be expressed in $S4$ as follows:

$$\Box\neg\Box\neg p \leftrightarrow p \qquad \text{or equivalently} \qquad \Box\Diamond p \leftrightarrow p$$

It must be noted that this condition is not a general schema such that every instance must be true. It is rather an additional model constraint that should be imposed on all the atomic constants used in describing a situation, because these are intended to be identified with regular sets.

The regularity of RCC regions is also relevant to the encoding of Boolean functions of regions. In section 3.5.4 I explained how, if the regions of the RCC theory are to be interpreted as regular open sets, then the Boolean operations (sum, prod and compl) of the theory correspond to operations within a regular open algebra rather than to the elementary Boolean set operators. In this algebra intersection corresponds to ordinary set intersection but the (regular open) complement of a set is the interior of its ordinary set complement and the (regular open) sum of two sets is the interior of the closure of their union. These operations can easily be represented in $S4$ and $S4^+$: $\mathsf{prod}(x, y)$ is translated as $x \wedge y$, $\mathsf{compl}(x)$ as $\Box\neg x$ and $\mathsf{sum}(x, y)$ as $\Box\neg\Box\neg(x \vee y)$.

5.7 Eliminating Entailment Constraints

The procedures for consistency checking and determining entailments for a logic L^+ of the kind described above rely on the use of simple meta-level reasoning. In this section I explain how, by introducing a further additional modal operator into the underlying logic L, reasoning can be conducted at the object level of this enriched language, which will be designated $L^{\boxtimes}$

In reasoning with an extended 0-order language L^+ the meanings of the two types of constraint are handled at the meta-level: determining entailments in these languages involves checking a number of different object-level entailments in the logic L. A set of algebraic constraints encoded in an L^+ expression $\langle \mathcal{M}, \mathcal{E} \rangle$ is consistent if and only if none of its entailment constraints in $\mathcal{E}$ is entailed by the set of all model constraints in $\mathcal{M}$. A natural question regarding these representations is whether it might be possible to extend the calculi involved so that the semantics of the two types of constraint was built directly into the object language. This would mean that computation of entailments could be carried out entirely at the object level.

In terms of algebraic semantics it is quite easy to introduce a new modal operator $\boxtimes$ by means of which the model/entailment constraint distinction can be made at the object level. If $\delta(\phi)$ is the algebraic denotation of a formula ϕ, we define $\boxtimes$ by:

- $\delta(\boxtimes\phi) = \mathcal{U}$ iff $\delta(\phi) = \mathcal{U}$.
- $\delta(\boxtimes\phi) = \emptyset$ iff $\delta(\phi) \neq \mathcal{U}$.

This operator is an $S5$ modal operator[14], since a formula $\boxtimes\phi$ is true in a model iff the formula

[14] $S5$ is the modal logic obtained by adding the schema $\neg\Box\neg\Box\phi \rightarrow \Box\phi$ to the schematic specification of $S4$. In terms of Kripke semantics $S5$ is characterised by the frame of all Kripke models whose accessibility relations are equivalence relations. See e.g. (Chellas 1980) for further information on $S5$.

ϕ is true at every point/world in the model. I shall call it a *strong-S5* operator because it does not allow the possibility, arising in the slightly weaker Kripke characterisation of $S5$, that there are worlds/points which are not relevant to evaluating the $\boxtimes$ at a particular world (because the set of worlds is partitioned into clusters which are not accessible to each other).[15] Given the definition of $\boxtimes$, we have $\neg\boxtimes\phi = \mathcal{U}$ iff $\phi \neq \mathcal{U}$. Thus, negations of universal set equations (and hence all equations) can be converted into positive equations. This obviates the need for entailment constraints, since a model constraint $\neg\boxtimes\phi$ has the same meaning as ϕ taken as an entailment constraint. More specifically, the translation of an L^+ expression

$$\langle\{\phi_1, \ldots, \phi_j\}, \{\psi_1 \ldots \psi_k\}\rangle$$

into $L^{\boxtimes}$ is the formula

$$\boxtimes\phi_1 \wedge \ldots \wedge \boxtimes\phi_j \wedge \neg\boxtimes\psi_1 \wedge \ldots \wedge \neg\boxtimes\psi_k \ .$$

Consequently any expression of L^+ can be represented by a simple object level formula in the multi-modal language $L^{\boxtimes}$.

5.7.1 An Example of an Entailment Encoded in $\mathcal{C}^{\boxtimes}$

Let us look at a simple example of spatial reasoning carried out in $\mathcal{C}^{\boxtimes}$ — i.e. the classical 0-order calculus supplemented with a strong-$S5$ box operator. (Exactly the same principles apply to reasoning in $S4^{\boxtimes}$ but using $\mathcal{C}^{\boxtimes}$ makes for a simpler and clearer example.) We shall consider the transitivity of the proper part relation, PP:

$$\mathsf{PP}(a,b) \wedge \mathsf{PP}(b,c) \models \mathsf{PP}(a,c) \ .$$

$\mathsf{PP}(x,y)$ holds when $\overline{x} \cup y = \mathcal{U}$ but $\overline{y} \cup x \neq \mathcal{U}$. We also require that x and y are non-null. Non-null constraints on regions can now be expressed as $\neg\boxtimes\neg x$ for any region X. Thus the modal representation of $\mathsf{PP}(a,b)$ is:

$$\boxtimes(a \rightarrow b) \wedge \neg\boxtimes(b \rightarrow a) \wedge \neg\boxtimes\neg a \wedge \neg\boxtimes\neg b$$

Hence the transitivity of PP corresponds to the entailment:

$$\boxtimes(a \rightarrow b) \wedge \neg\boxtimes(b \rightarrow a),\ \boxtimes(b \rightarrow c) \wedge \neg\boxtimes(c \rightarrow b),\ \neg\boxtimes\neg a,\ \neg\boxtimes\neg b,\ \neg\boxtimes\neg c$$
$$\models \boxtimes(a \rightarrow c) \wedge \neg\boxtimes(c \rightarrow a) \wedge \neg\boxtimes\neg a \wedge \neg\boxtimes\neg c$$

In testing the validity of this entailment it is natural to proceed as follows. Since the r.h.s. is a conjunction, the sequent is valid iff each of the four sequents with the same l.h.s. but just one conjunct on the r.h.s. is valid. Of these four sequents, the two with $\neg\boxtimes\neg a$ and $\neg\boxtimes\neg c$ on the r.h.s. are trivially valid because these formulae also occur on the l.h.s.. To prove the validity of the other

[15] In most circumstances the strong and weak $S5$ operators cannot be distinguished at the object level. But the difference may sometimes be significant. For example a multi-modal logic may contain several distinct weak-$S5$ modalities but only one strong-$S5$ operator.

two, it is convenient to move all conjuncts on the l.h.s. which have an initial negation over to the right. We shall then have the following two sequents:

$$\boxtimes(a \to b) \wedge \boxtimes(b \to c) \models \boxtimes(a \to c) \vee \boxtimes(b \to a) \vee \boxtimes(c \to b) \vee \boxtimes\neg a \vee \boxtimes\neg b \vee \boxtimes\neg c$$

$$\boxtimes(a \to b) \wedge \boxtimes(b \to c) \wedge \boxtimes(c \to a) \models \boxtimes(b \to a) \vee \boxtimes(c \to b) \vee \boxtimes\neg a \vee \boxtimes\neg b \vee \boxtimes\neg c$$

We can verify these proof-theoretically by the application of just one modal rule (together with ordinary classical reasoning). This is the rule **RK** which holds in any normal modal logic:

$$\frac{(\phi_1 \wedge \ldots \wedge \phi_n) \to \phi}{(\boxtimes\phi_1 \wedge \ldots \wedge \boxtimes\phi_n) \to \boxtimes\phi}\ [\mathbf{RK}]$$

This rule together with the deduction theorem means that

$$\text{if} \quad \phi_1, \ldots, \phi_n \models \phi \quad \text{then} \quad \boxtimes\phi_1, \ldots, \boxtimes\phi_n \models \boxtimes\phi$$

Application of this principle validates both of our sequents, since

$$a \to b,\ b \to c \models a \to c \quad \text{and} \quad a \to b,\ b \to c,\ c \to a \models b \to a.$$

5.7.2 The Utility of $L^{\boxtimes}$ as Compared with L^{+}

Introduction of the new box operator to enable positive and negative constraints to be distinguished gives us a more uniform representation. Whereas previously the meaning of an expression was tied up essentially with the reasoning methods employed, in the new language expressions have a clear algebraic interpretation. We need no longer concern ourselves with the distinction between model and entailment constraints but can now describe spatial situations simply by a set of modal formulae; and can reason about consistency and entailment directly in this object language.

On the other hand it is not clear that this enriched language is more desirable from the computational point of view. Introduction of the new operator makes the language far more expressive and consequently much harder to reason with. However, we have seen that as long as the new modal operator is only used to express what was previously expressed by means of the model/entailment constraint distinction, then all $\boxtimes$ operators will only occur either up front or negated up front in the set of formulae describing a situation; and it seems likely that the optimal approach to reasoning with such formula sets is to mimic the $S4^{+}$ consistency checking algorithm described above. Specifically this means rewriting the sequents (according to simple classical principles) to obtain sets of sequents in which all formulae have a single $\boxtimes$ at the front: the l.h.s. is a conjunction and the r.h.s. a disjunction of such formulae. Once the sequents are in this form, it is easy to see that the sequents which correspond to entailments verifiable by the extended 0-order reasoning algorithm can all be proved using only the modal rule **RK** together with classical reasoning.

Since we know that the consistency checking method for $S4^{+}$ is correct we can conclude that only the rule **RK** is needed to prove all entailments in $L^{\boxtimes}$ involving formulae in which the $\boxtimes$ occurs either up-front or negated up-front. Since the logic of $S4$ obeys **RK** it follows that, if $S4^{\boxtimes}$ is used only to express the model and entailment constraints of $S4^+$, one can in fact treat $\boxtimes$ as if it were just another $S4$ $\Box$ operator. Nevertheless the more intuitive interpretation of the modal operator in this context is as the strong-$S5$ operator. In section 8.5 I shall use a modal representation in which strong-$S5$ operators are employed within complex formulae. In such contexts $\boxtimes$ cannot be treated as an $S4$ $\Box$ operator.

Chapter 6

An Intuitionistic Representation and its Complexity

In the last chapter I showed how spatial interpretation of the modal logic $S4$ enables a wide range of spatial relationships to be encoded. This means that entailments among these relations can be determined by means of an $S4$ theorem-prover. In this chapter I give an alternative encoding of spatial relations into the 0-order *intuitionistic* calculus. I also examine examine the complexity of reasoning using the intuitionistic representation. We shall see that the problem of determining entailments is in the polynomial complexity class known as NC.

6.1 The Topological Interpretation of $\mathcal{I}$

One of the most significant early applications of semantic methods to the investigation of logical systems is the topological interpretation of the intuitionistic calculus.[1] Tarski (1938) gave a semantics for 0-order intuitionistic logic (henceforth $\mathcal{I}$), which (like that just given for $S4$) makes use of an interior operator. Under Tarski's semantics, a model for $\mathcal{I}$ is a structure $\langle U, i, P, \delta \rangle$ where δ now assigns to each constant $p_i \in P$ an *open* subset of U (a set X such that $i(X) = X$). The domain of δ is then extended to all $\mathcal{I}$ formulae as follows:

1. $\delta(\sim\phi) = i(\ \overline{\delta(\phi)}\ )$
2. $\delta(\phi \wedge \psi) = \delta(\phi) \cap \delta(\psi)$
3. $\delta(\phi \vee \psi) = \delta(\phi) \cup \delta(\psi)$
4. $\delta(\phi \Rightarrow \psi) = i(\ \overline{\delta(\phi)} \cup \delta(\psi)\ )$

This denotation function is such that all intuitionistic theorems denote U under any assignment of *open* sets to non-logical constants.[2] Note that I use different symbols, '$\sim$' and '$\Rightarrow$', for negation

[1] Mostowski (1966, Lecture 1) gives an interesting account of the early work in this area.

[2] In fact, a more uniform presentation could be obtained by simply putting the set-definition of the classical connectives within the scope of an interior operator; but in the case of the conjunction and disjunction connectives the extra i operation would be redundant (since the unions or intersection of two open sets is always open).

and implication in $\mathcal{I}$ from those used in $\mathcal{C}$; but for conjunction and disjunction I use the same symbols, since their interpretations are the same in both systems. The topological interpretation of $\mathcal{I}$ means that each formula ϕ of $\mathcal{I}$ can be correlated with an interior algebraic term, which I shall refer to by the meta-notation **IAT**$[\phi]$. This term is obtained from ϕ by replacing propositional constants by set constants, '$\wedge$' by '$\cap$', '$\vee$' by '$\cup$', '$\sim$' by '$i(\overline{\ldots})$' and '$\ldots \Rightarrow \ldots$' by '$i(\overline{\ldots} \cup \ldots)$'.

The algebraic semantics for the intuitionistic calculus suggests a quite straightforward spatial interpretation which enables one to understand clearly why certain theorems that are classically valid do not hold intuitionistically. Consider the infamous law of the excluded middle, $p \vee \sim p$. The constant p will be identified with an open set, which we can think of as the set of interior points of some bounded region. Intuitionistic negation is associated with the operation of taking the interior of the complement of a region — in other words, where p is identified with the points within a boundary, $\sim p$ is identified with the points outside the boundary. The set associated with $p \vee \sim p$ is the union of the sets associated with p and with $\sim p$. Clearly this contains all points within our imagined boundary and all points outside the boundary, but does not contain any of the points lying on the boundary. Hence, the set associated with $p \vee \sim p$ does not necessarily contain all points in the universe, so formulae of this form are not in general theorems (in fact a formula of the form $p \vee \sim p$ is an intuitionistic theorem if and only if either p or $\sim p$ is a theorem). So, although it may be argued that such topological interpretations are not really in the spirit of intuitionism the spatial interpretation can serve to demystify and give a clearer understanding of the intuitionistic calculus.

One drawback of this representation is that no logical operator corresponding to the interior function appears explicitly in the language: the function occurs in the interpretations of intuitionistic negation and implication and is only referred to indirectly in logical formulae used to represent spatial constraints. Because of this, the $\mathcal{I}$ representations of spatial relations are less perspicuous than those of the $S4$ encoding, where the modal operator corresponds directly to the interior function.

6.1.1 Relation between $\mathcal{I}$ and $S4$

In order to understand the relationship between spatial representation in terms of $\mathcal{I}$ and the representation in terms of $S4$ developed in the last chapter, it will be useful to know something about how these two logical languages are themselves related. It has long been known (see Fitting (1969)) that formulae of the intuitionistic propositional calculus can be translated into modal formulae in such a way that an intuitionistic formula is a theorem if and only if the resulting modal formula is valid in the logic $S4$. The translation can be specified in terms of a recursive meta-level function,

trans[...], as follows:

$$\begin{aligned}
\mathbf{trans}[p_i] &= \Box\, p_i \quad \text{(where } p_i \text{ is a constant)}\\
\mathbf{trans}[\sim\phi] &= \Box\,\neg\mathbf{trans}[\phi]\\
\mathbf{trans}[\phi \vee \psi] &= \mathbf{trans}[\phi] \vee \mathbf{trans}[\psi]\\
\mathbf{trans}[\phi \wedge \psi] &= \mathbf{trans}[\phi] \wedge \mathbf{trans}[\psi]\\
\mathbf{trans}[\phi \Rightarrow \psi] &= \Box(\mathbf{trans}[\phi] \rightarrow \mathbf{trans}[\psi])
\end{aligned}$$

Algebraic set semantics brings out very clearly the affinity between $\mathcal{I}$ and $S4$. $\mathcal{I}$ can be regarded as a sub-language of $S4$ because the algebraic terms associated with $\mathcal{I}$ formulae form a subclass of the terms associated with $S4$ formulae. Actually this is not quite true because whereas atomic formulae in $S4$ may denote arbitrary subsets of a (topologically structured) universe, those of $\mathcal{I}$ denote only *open* subsets of the space. Thus in expressing an $\mathcal{I}$ formulae in $S4$, every atom, p, must be replaced by the formulae $\Box\, p$ — since $\Box$ corresponds to the interior function $\Box\, p$ will now denote an (arbitrary) open set. So intuitionistic formulae correspond only to (a subset of) *necessary* $S4$ formulae. An intuitionistic negation, $\sim(\ldots)$ is semantically equivalent to the $S4$ operation $\Box\,\neg(\ldots)$ and $(\ldots \Rightarrow \_\_)$ is equivalent to $\Box(\neg \ldots \vee \_\_)$ or $\Box(\ldots \rightarrow \_\_)$. Conjunction and disjunction have the same interpretation in the semantics of both logics and so are unchanged in the translation to $S4$.

6.1.2 Correspondence Theorem for $\mathcal{I}$

Tarski's "Second Principal Theorem" in the paper *Sentential Calculus and Topology* (Tarski 1938) establishes that a propositional formula is a theorem of $\mathcal{I}$ if and only if the corresponding set-term denotes the universe in any topological space under any assignment of open sets to the set constants occurring in the term. The proof of this is fairly involved and is not reconstructed here. I use the notation '$\vdash_{\mathcal{I}}$' to denote entailment in $\mathcal{I}$ and '$\models_T$' to denote topological entailment — i.e. entailment between set-equations which may contain the interior operator, i. Tarski's theorem can then be written formally as:[3]

Intuitionistic Correspondence Theorem (Icorr)

$\vdash_{\mathcal{I}} \ \phi$ if and only if $\models_T \ \mathbf{IAT}[\phi] = \mathcal{U}$

In using $\mathcal{I}$ to represent spatial relations we shall exploit very similar correspondence relations to those holding for $\mathcal{C}$ and $S4$. In order to secure the correspondence between entailment in $\mathcal{I}$ and entailment between set equations in the topological algebra of sets, we need to generalise Tarski's result to a correspondence between entailments:

[3] This theorem holds for any topology whatsoever. Adding conditions to the topology would mean the corresponding logic would be stronger. The limiting case is the *discrete* topology corresponding to classical logic.

Intuitionistic Entailment Correspondence Theorem (IECT)

$$\phi_1, \ldots, \phi_n \ \vdash_{\mathcal{I}} \ \phi_0$$

if and only if

$$\pi_1 = \mathcal{U}, \ldots, \pi_n = \mathcal{U} \ \models_T \ \pi_0 = \mathcal{U}$$

where $\pi_i = \mathbf{IAT}[\phi_i]$.

Proof of IECT: The positive half is simple: an $\mathcal{I}$ entailment $p_1, \ldots, p_n \ \vdash_{\mathcal{I}} \ p_0$ holds iff $\vdash_{\mathcal{I}} \ (p_1 \wedge \ldots \wedge p_n) \Rightarrow p_0$, so by **Icorr** we have $\models_T i(\overline{\pi_1 \cap \ldots \cap \pi_n} \cup \pi_0) = \mathcal{U}$. But if a set has $\mathcal{U}$ as its interior then it must be equal to $\mathcal{U}$. Consequently, the equation $(\overline{\pi_1 \cap \ldots \cap \pi_n} \cup \pi_0) = \mathcal{U}$ must hold in every model. Thus, whenever $\pi_i = \mathcal{U}$ for $i = 1 \ldots n$ we must also have $\pi_0 = \mathcal{U}$ — in other words $\pi_1 = \mathcal{U}, \ldots, \pi_n = \mathcal{U} \ \models_T \pi_0 = \mathcal{U}$.

Suppose on the other hand $p_1, \ldots, p_n \not\vdash_{\mathcal{I}} \ p_0$. Because of **Icorr** this means that $\not\models_T i(\overline{\pi_1 \cap \ldots \cap \pi_n} \cup \pi_0) = \mathcal{U}$, so there is some model, $\mathcal{M} = \langle U, i, P, \delta \rangle$, in which there is at least one element of $\pi_1 \cap \ldots \cap \pi_n$ which is not an element of π_0. On the basis of this model we now construct a model $\mathcal{M}' = \langle U', i', P, \delta' \rangle$ whose universe, U', is the set denoted by $\pi_1 \cap \ldots \cap \pi_n$ in $\mathcal{M}$. We set $i'(X) = i(X)$ for all $X \subseteq \mathcal{U}'$ and for all propositional constants p_i we set $\delta'(p_i) = \delta(p_i) \cap U'$. It is easy to see that if $\langle U, i \rangle$ is a topological space then so is $\langle U', i' \rangle$ (see section 2.1).

I now show that the new assignment is such that for any formula ϕ, $\delta'(\phi) = \delta(\phi) \cap U'$. This condition is clearly satisfied by atomic formulae so it can be proved by induction for all formulae if we can show that whenever formulae α and β satisfy the condition, it is also satisfied by $\alpha \wedge \beta$, $\alpha \vee \beta$, $\sim \alpha$ and $\alpha \Rightarrow \beta$. The first two cases are straightforward:

$$\delta'(\alpha \wedge \beta) = \delta'(\alpha) \cap \delta'(\beta) = (\delta(\alpha) \cap U') \cap (\delta(\beta \cap U')) = (\delta(\alpha) \cap \delta(\beta)) \cap U' = \delta(\alpha \wedge \beta) \cap U'$$

$$\delta'(\alpha \vee \beta) = \delta'(\alpha) \cup \delta'(\beta) = (\delta(\alpha) \cap U') \cup (\delta(\beta \cap U')) = (\delta(\alpha) \cup \delta(\beta)) \cap U' = \delta(\alpha \vee \beta) \cap U'$$

The case of $\sim \alpha$ is slightly harder to show:

$$\delta'(\sim \alpha) = i(U' \perp \delta'(\alpha)) = i(U' \perp (\delta(\alpha) \cap U')) = i(U' \perp \delta(\alpha))$$

Then, since $U' \subseteq U$, we have $i(U' \perp \delta(\alpha)) = i((U \perp \delta(\alpha)) \cap U')$ and i distributes over $\cap$ giving $i(U \perp \delta(\alpha)) \cap i(U')$. But U' is an intersection of the sets π_i, which are the denotations of formulae ϕ_i. So, since all formulae denote open sets, U' must also be open. Hence, $i(U') = U'$. So we have

$$i(U \perp \delta(\alpha)) \cap i(U') = i(U \perp \delta(\alpha)) \cap U' = \delta(\sim \alpha) \cap U' \ .$$

Now consider $\alpha \Rightarrow \beta$:

$$\delta'(\alpha \Rightarrow \beta) = i((U' \perp \delta'(\alpha)) \cup \delta'(\beta)\,) = i(\,(U' \perp (\delta(\alpha) \cap U')) \cup (\delta(\beta) \cap U')\,)$$

Since $U' \subseteq U$, it is easy to show that this last term is equivalent to

$$i(\,((U \perp \delta(\alpha)) \cup \delta(\beta)) \cap U'\,) \ ;$$

and because i distributes over intersections and $i(U') = U'$ this is equivalent to

$$i((U \perp \delta(\alpha)) \cup \delta(\beta)) \cap U' .$$

Finally we have

$$i((U \perp \delta(\alpha)) \cup \delta(\beta)) \cap U' = \delta(\alpha \Rightarrow \beta) \cap U'$$

Thus, $\delta'(\phi) = \delta(\phi) \cap U'$ for any formula, ϕ. So, in particular, for each $i = 1 \ldots n$, $\delta'(\phi_i) = \delta(\phi_i) \cap U' = \pi_i \cap U' = U'$; i.e. in the new model all antecedent formulae denote the universe. We also have $\delta'(\phi_0) = \delta(\phi_0) \cap U' = \pi_0 \cap U'$. Furthermore, we know that there is at least one element of U' which is not an element of π_0. This means that $\delta'(\phi_0) \neq U'$; so $\mathcal{M}'$ provides a counter-example to the entailment. This concludes the proof of **IECT**. ■

6.2 Intuitionistic Representation of RCC Relations

The topological interpretation of $\mathcal{I}$ enables one to use intuitionistic logic in much the same way as $S4$ to reason about spatial relationships. Paralleling the approach of the previous chapter, I characterise RCC relations as equational constraints in interior algebra and then rely on the correspondence theorem to reason about these constraints using a theorem prover for the intuitionistic logic. As noted above, the correspondence between terms in an interior algebra and formulae of $\mathcal{I}$ is more indirect than the correspondence with $S4$ formulae because in the interpretation of $\mathcal{I}$ (unlike that of $S4$) no logical connective corresponds either to the interior or to the complement operator of the algebra. However, the encoding of many topological relations is still straightforward. Table 6.1 shows encodings into $\mathcal{I}$ of each of the RCC-7 relations (introduced in section 5.4.1).

RCC	Set Equation	$\mathcal{I}$ formula
$\mathsf{DC}(x, y)$	$i(\overline{x}) \cup i(\overline{y}) = \mathcal{U}$	$\sim x \vee \sim y$
$\mathsf{DR}(x, y)$	$\overline{x \cap y} = \mathcal{U}$	$\sim(x \wedge y)$
$\mathsf{P}(x, y)$	$\overline{x} \cup y = \mathcal{U}$	$x \Rightarrow y$
$\mathsf{Pi}(x, y)$	$x \cup \overline{y} = \mathcal{U}$	$y \Rightarrow x$
$\mathsf{NTP}(x, y)$	$i(\overline{x}) \cup y = \mathcal{U}$	$\sim x \vee y$
$\mathsf{NTPi}(x, y)$	$x \cup i(\overline{y}) = \mathcal{U}$	$x \vee \sim y$
$\mathsf{EQ}(x, y)$	$(\overline{x} \cup y) \cap (x \cup \overline{y}) = \mathcal{U}$	$x \Leftrightarrow y$

Table 6.1: Representation of the RCC-7 relations in $\mathcal{I}$

In virtue of the theorem **IECT** an entailment among RCC-7 relations holds if and only if the corresponding intuitionistic entailment holds. Thus we can determine that the argument

$$\mathsf{NTP}(a, b) \wedge \mathsf{DR}(b, c) \models \mathsf{DC}(a, c)$$

is valid because it corresponds to the following intuitionistically valid sequent:

$$\sim a \vee b, \quad \sim(b \wedge c) \quad \vdash_{\mathcal{I}} \quad \sim a \vee \sim c$$

6.2.1 The $\mathcal{I}^+$ Encoding

The language $\mathcal{I}^+$ extends the expressive power of $\mathcal{I}$ in just the same way that $\mathcal{C}^+$ and $S4^+$ augment the languages $\mathcal{C}$ and $S4$. Thus it enables the specification of negative as well as positive equational constraints on interior algebras. Table 6.2 shows how each of the RCC-8 relations can be represented by sets of model and entailment constraints specified by means of $\mathcal{I}$ formulae. As with $S4^+$ the representations can be obtained by first analysing the RCC-8 relations into conjunctions of RCC-7 relations and their negations. The $\mathcal{I}$ formulae corresponding to the positive RCC-7 conjuncts then become model constraints and those corresponding to negative conjuncts become entailment constraints in the $S4^+$ representation. The table also shows how the fundamental relation, C, of the RCC theory can be represented as well as the quasi-Boolean function sum (see section 6.2.2 below).

Relation	*Model Constraint*	*Entailment Constraints*
$\mathsf{DC}(x, y)$	$\sim x \vee \sim y$	$\sim x, \ \sim y$
$\mathsf{EC}(x, y)$	$\sim(x \wedge y)$	$\sim x \vee \sim y, \sim x, \ \sim y$
$\mathsf{PO}(x, y)$	—	$\sim(x \wedge y), \ x \Rightarrow y, \ y \Rightarrow x, \sim x, \sim y$
$\mathsf{TPP}(x, y)$	$x \Rightarrow y$	$\sim x \vee y, \ y \Rightarrow x, \sim x, \sim y$
$\mathsf{TPPi}(x, y)$	$y \Rightarrow x$	$\sim y \vee x, \ x \Rightarrow y, \sim x, \sim y$
$\mathsf{NTPP}(x, y)$	$\sim x \vee y$	$y \Rightarrow x, \sim x, \sim y$
$\mathsf{NTPPi}(x, y)$	$\sim y \vee x$	$x \Rightarrow y, \sim x, \sim y$
$\mathsf{EQ}(x, y)$	$x \Leftrightarrow y$	$\sim x, \sim y$
$\mathsf{C}(x, y)$	—	$\sim x \vee \sim y, \sim x, \sim y$
$\mathsf{EQ}(x, \mathsf{sum}(y, z))$	$x \Leftrightarrow (y \vee z)$	$\sim x, \sim y, \sim z$

Table 6.2: Some RCC relations defined in $\mathcal{I}^+$ (including the RCC-8 relations)

Let us consider, for example, the representations of the relations $\mathsf{DC}(x, y)$ and $\mathsf{EC}(x, y)$. If two regions share no points they cannot overlap (although they may be connected). In such a case the equation $i(\overline{X \cap Y}) = \mathcal{U}$ must hold; this can be represented by the $\mathcal{I}$ formula $\sim(x \wedge y)$. In $\mathcal{I}$ (unlike $\mathcal{C}$) this formula is not equivalent to $\sim x \vee \sim y$. The latter corresponds to the set-equation $i(\overline{X}) \cup i(\overline{Y}) = \mathcal{U}$, which says that the union of the exteriors of two regions exhaust the space. If the regions touch at one or more points, then these points of contact will not be in the exterior of either region so this equation will not hold. Hence the second (stronger) formula can be employed as a model constraint to describe the relation $\mathsf{DC}(x, y)$. If the relation $\mathsf{EC}(x, y)$ holds then the weaker constraint $\sim(x \wedge y)$ holds but $\sim x \vee \sim y$ must not hold, so this stronger formula is an entailment constraint.

Consistency of $\mathcal{I}^+$ expressions is determined analogously to $\mathcal{C}^+$ and $S4^+$ expressions: an $\mathcal{I}^+$

expression $\langle \mathcal{M}, \mathcal{E} \rangle$ is inconsistent iff there is some $\phi \in \mathcal{E}$ such that $\mathcal{M} \vdash_{\mathcal{I}} \phi$. Again, the fact that each of the negative constraints can be considered separately is due to a convexity property of the class of constraints which are represented by this formalism. Since the theory of the topological interior operator is purely equational and the constraints corresponding to the model and entailment constraints are themselves also equations, this convexity property is a direct consequence of **ELcons** (which was proved in section 4.5).

In section 5.6 I explained how the inconsistency of the description $\mathsf{TPP}(a,b) \wedge \mathsf{DC}(b,c) \wedge \mathsf{PO}(a,c)$ could be demonstrated by means of the $S4^+$ representation. The corresponding $\mathcal{I}^+$ representation (according to table 6.2) is:[4]

$$\langle \{a \Rightarrow b,\ \sim b \vee \sim c\},\ \ \{\sim a \vee b,\ b \Rightarrow a,\ a \Rightarrow c,\ c \Rightarrow a,\ \sim(a \wedge c),\ \sim a,\ \sim b,\ \sim c\}\rangle\ .$$

This $\mathcal{I}^+$ expression is inconsistent because

$$a \Rightarrow b,\ \sim b \vee \sim c \quad \vdash_{\mathcal{I}} \quad \sim(a \wedge c)$$

i.e. one of the entailment constraints in entailed by the model constraints.

6.2.2 The Regularity Constraint and Boolean Functions Coded in $\mathcal{I}$

In section 5.6.1 I explained how regions could be constrained to be regular by means of an $S4$ model constraint. In $\mathcal{I}$ this constraint can be enforced by the model constraint formula

$$\sim\sim p \Rightarrow p\ .$$

In the topological semantics this corresponds to the condition $i \perp (i \perp (P)) \subseteq P$ or equivalently $i(c(P)) \subseteq P$. The condition $P \subseteq i(c(P))$ need not be explicitly added because $p \Rightarrow \sim\sim p$ is already a theorem of $\mathcal{I}$. It is interesting to note that the intuitionistic formulae assigned regular sets by the topological semantics are those for which the classical law of double negation holds.

As argued in section 3.5.2, the most coherent topological interpretation of the RCC theory is to identify the RCC regions with regular open sets (or alternatively regular closed sets). This means that in employing $\mathcal{I}^+$ to represent RCC relations, as well as adding model constraints ensuring regularity of the regions explicitly mentioned, one should also ensure that all Boolean combinations of these regions also correspond to regular sets. To ensure this, these operations can be represented in $\mathcal{I}$ as follows: $\mathsf{prod}(x,y)$ is translated as $x \wedge y$, $\mathsf{compl}(x)$ as $\sim x$ and $\mathsf{sum}(x,y)$ as $\sim\sim(x \vee y)$. Given the topological interpretations of the connectives involved, it is easy to see that, if its argument regions are regular, the regions denoted by any Boolean function will be regular.

[4] For full generality one ought to add extra model constraints constraining the regions to be regular, as explained in section 6.2.2. Note that (unlike $S4^+$) in $\mathcal{I}^+$ all regions are automatically constrained to be open.

6.3 Efficient Topological Reasoning Using $\mathcal{I}^+$

The implementation of the spatial reasoning algorithm described in (Bennett 1994b) used a sequent calculus proof system for intuitionistic logic, which contained certain optimisations making it more efficient in testing the sequents required by the topological reasoning algorithm (and rendering it incomplete for the full intuitionistic logic). Following the complexity analysis of Nebel (1995a) it became apparent that a far more effective special-purpose proof procedure could be constructed. This section examines the the proof-theory of the restricted class of sequents that need to be tested and shows how this analysis yields an efficient, clearly polynomial, proof method.

6.3.1 Sequent Calculus for $\mathcal{I}$

To formalise the proof theory I use a Gentzen-style (Gentzen 1955) sequent calculus for $\mathcal{I}$, which is essentially the same as that given by Dummett (1977). The proof rules of the calculus can be specified as follows:[5]

Axioms: $P, \Gamma \vdash P$ $\qquad$ $\mathbf{f}, \Gamma \vdash C$

Re-write $\quad {\sim}P \;\; \equiv_{def} \;\; P \Rightarrow \mathbf{f}$

Rules:

$$\frac{P,\ Q,\ \Gamma \vdash C}{P \wedge Q,\ \Gamma \vdash C}[\wedge\vdash] \qquad \frac{\Gamma \vdash P \quad and \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}[\vdash\wedge]$$

$$\frac{P,\ \Gamma \vdash C \quad and \quad Q,\ \Gamma \vdash C}{P \vee Q,\ \Gamma \vdash C}[\vee\vdash] \qquad \frac{\Gamma \vdash P \quad or \quad \Gamma \vdash Q}{\Gamma \vdash P \vee Q}[\vdash\vee]$$

$$\frac{P \Rightarrow Q,\ \Gamma \vdash P \quad and \quad Q,\ \Gamma \vdash C}{P \Rightarrow Q,\ \Gamma \vdash C}[\Rightarrow\vdash] \qquad \frac{\Gamma,\ P \vdash Q}{\Gamma \vdash P \Rightarrow Q}[\vdash\Rightarrow]$$

When applied in the top to bottom direction the rules preserve provability and generate all valid sequents. When used to prove a sequent, the rules are applied bottom to top in an attempt to show that the sequent is derivable from axioms. However, not all rules preserve provability when applied upwards, so the proof search is non-deterministic. Rules which preserve provability in both directions are called *invertible*. All the rules are invertible except $\vdash \vee$ and $\Rightarrow\vdash$.

From the computational point of view, the most serious defect of this rule set is that, in applying the $\Rightarrow\vdash$ rule, proving a sequent is reduced to proving two sequents, one of which may be more complex than the initial sequent. In a depth-first search for a proof, this may lead to infinite loops, whose detection is computationally expensive; on the other hand, a breadth first search is extremely expensive in terms of space.

[5] Roman capital letters denote arbitrary formulae, Roman small letters denote atomic formulae and Greek capitals denote arbitrary sets of formulae. The left hand side of a sequent is regarded as a set of formulae rather than a sequence, so the order of formulae on the left does not matter.

Theorem proving in the $\mathcal{I}$ sequent calculus is more complex than that of $\mathcal{C}$: in $\mathcal{C}$ all connectives can be eliminated deterministically because the rules produce Boolean combinations of sequents which are logically equivalent to the original sequent (so all rules are invertible). Thus, whereas theorem proving in $\mathcal{C}$ is NP-complete (so — assuming $P \neq NP$ — requires time which is exponential in the size of formula to be tested), checking $\mathcal{I}$ theorems is probably even more difficult: it is believed that it requires $O(n \log n)$ space as well as exponential time (Hudelmaier 1993).

6.3.2 Hudelmaier's $\Rightarrow\vdash$ Rules

The problem arising from the $\Rightarrow\vdash$ rule has recently been solved by Hudelmaier (1993). The idea is to replace the rule by four more specific rules, the applicability of which depends on the structure of the antecedent of the $\Rightarrow$ formula. Hudelmaier's rules are:

$$\frac{a,\ P,\ \Gamma\ \vdash\ C}{a,\ a \Rightarrow P,\ \Gamma\ \vdash\ C}\,[\mathrm{MP}{\Rightarrow}{\vdash}] \qquad \frac{P \Rightarrow (Q \Rightarrow R),\ \Gamma\ \vdash\ C}{(P \wedge Q) \Rightarrow R,\ \Gamma\ \vdash\ C}\,[\wedge \Rightarrow\vdash]$$

$$\frac{P \Rightarrow R,\ Q \Rightarrow R,\ \Gamma\ \vdash\ C}{(P \vee Q) \Rightarrow R,\ \Gamma\ \vdash\ C}\,[\vee \Rightarrow\vdash] \qquad \frac{Q \Rightarrow R,\ \Gamma\ \vdash\ P \Rightarrow Q \quad and \quad R,\ \Gamma\ \vdash\ C}{(P \Rightarrow Q) \Rightarrow R,\ \Gamma\ \vdash\ C}\,[\Rightarrow\Rightarrow\vdash]$$

Each of these except $\Rightarrow\Rightarrow\vdash$ is invertible. As indicated by the use of the small 'a', the *modus ponens* rule MP$\Rightarrow\vdash$ need only be applied when the antecedent of the implication is atomic. In upwards application of each of these rules, the resulting sequents can be shown to decrease in complexity according to a (specially constructed) measure of sequent complexity.

6.3.3 Spatial Reasoning Using Hudelmaier's Rules

We have seen that consistency of spatial relations which are instances of the RCC-8 set can be determined by testing the validity of certain $\mathcal{I}$ sequents. Moreover, if we are dealing only with the RCC-8 relations these sequents only contain formulae of the forms shown in table 6.2:

$$\sim a, \quad (a \Rightarrow b), \quad \sim(a \wedge b), \quad (\sim a \vee b), \quad (\sim a \vee \sim b)$$

In the remainder of this section I shall show how, given the limited range of formulae and the completeness of the Hudelmaier sequent rules, an effective consistency checking procedure for sets of RCC-8 relations can be constructed.

The sequent rules assume that negation is handled by replacing each negated formula $\sim \phi$ by the equivalent formula $\phi \Rightarrow \mathbf{f}$. This can be implemented as a simple deterministic re-write rule. After eliminating negations in this way another simplification can be made by applying Hudelmaier's $\wedge\Rightarrow\vdash$ rule. This means that formulae of the form $\sim(a \wedge b)$ are re-written first to $(a \wedge b) \Rightarrow \mathbf{f}$ and then to $(a \Rightarrow (b \Rightarrow \mathbf{f}))$. The resulting sequents will contain only formulae of the forms:

$$(a \Rightarrow \mathbf{f}), \quad (a \Rightarrow b), \quad (a \Rightarrow (b \Rightarrow \mathbf{f})), \quad ((a \Rightarrow \mathbf{f}) \vee b), \quad ((a \Rightarrow \mathbf{f}) \vee (b \Rightarrow \mathbf{f})) \qquad \textbf{(Iforms)}$$

Note that, amongst these formulae, the antecedents of all implications are atomic so (using the Hudelmaier rule set) the only rule applicable to implications is MP$\Rightarrow\vdash$. Apart from implications, the only other types of formulae are atomic propositions and two forms of disjunction. The disjunctions can be handled by the normal $\vee\vdash$ and $\vdash\vee$ rules. Both these rules give rise to a branch in the search space; and, because there may be any number of disjunctions amongst the premisses, the search space is exponential. Nevertheless, because the non-deterministic $\Rightarrow\vdash$ rule is not needed, this proof procedure can be used to test consistency of quite large sequents in reasonable time.

The Prolog program given in appendix C.2 is based on the method just described. A slight difference is that, rather than re-writing formulae of the form $(a \wedge b) \Rightarrow \mathbf{f})$ to $a \Rightarrow (b \Rightarrow \mathbf{f})$ and then applying the normal MP rule, I implemented the following variation of MP:

$$\frac{a,\ b,\ P,\ \Gamma\ \vdash\ C}{a,\ b,\ (a \wedge b) \Rightarrow P,\ \Gamma\ \vdash\ C}\ [\mathrm{MP2}{\Rightarrow}{\vdash}]$$

I also added a 'pruning' rule to delete redundant implications whose conclusion was already amongst the premisses. Small optimisations such as this, which are logically trivial, can often yield a marked improvement in the performance of an automated theorem prover. In the next section we shall see that pruning rules play a key part in the specification of a polynomial time proof procedure for these sequents.

6.3.4 Further Optimisation

In section 6.3.7 I shall present the model theoretic analysis given by Bernhard Nebel of the $\mathcal{I}$ sequents arising from the RCC-8 encoding. This analysis enabled Nebel to show that consistency checking of sets of RCC-8 relations can be performed in polynomial time. Inspired by this result I investigated how sequent calculus proofs of the relevant sequents could be optimised. As expected, proofs in the sequent calculus can also be carried out in polynomial time. In the rest of this section I present a series of sequent re-writing rules which achieves this end. I assume that all formulae in the sequents have been reduced to the forms **Iforms** as explained in the previous section.

Eliminating Disjunctions without Branching

Disjunctions would normally be eliminated by applying the rules $\vee\vdash$ and $\vdash\vee$. These create a branch in the proof: we attempt to verify each of the sequents obtained by replacing the disjunction by one of its disjuncts. Clearly this procedure leads to a search space which is exponential in the number of disjunctions (which is approximately proportional to the number of topological relations whose consistency is being tested). This situation is made worse because the $\vdash\vee$ rule must be applied non-deterministically. However, given the limited class of formulae appearing in the sequents, rather than carrying out this split we can work out the potential effects without actually applying a branching rule.

The plan will be first to take account of the disjunctive content of premisses and conclusion by applying certain 'pruning' rules, the simplest of which take the following forms[6]

$$\frac{Q,\ \Gamma\ \vdash\ P}{(P \vee Q),\ \Gamma\ \vdash\ P}\,[\mathrm{Pr}\vee] \qquad \frac{Q,\ \Gamma\ \vdash\ (P \vee R)}{(P \vee Q),\ \Gamma\ \vdash\ (P \vee R)}\,[\mathrm{Pr}\vee]$$

After carrying out all possible applications of these rules, we will have an equivalent sequent in which none of the disjunctive premisses have a disjunct which is the same as the conclusion or a disjunct of the conclusion. Such disjuncts will be called 'un-prunable'.

Another kind of pruning rule can be applied to implicative premisses:

$$\frac{(P \Rightarrow \mathbf{f}),\ \Gamma\ \vdash\ Q}{(P \Rightarrow Q),\ \Gamma\ \vdash\ Q}\,[\mathrm{Pr}\Rightarrow]$$

We notice that this rule, applicable where the consequent of an implication is the same as the conclusion, does not generalise to the case of a disjunctive conclusion: it is not sound to reduce a proof of $(p \Rightarrow q),\ \Gamma\ \vdash\ (q \vee r)$ to that of $(p \Rightarrow \mathbf{f}),\ \Gamma\ \vdash\ (q \vee r)$; and this is precisely the respect in which an intuitionistic implication $(P \Rightarrow Q)$ is logically weaker than the disjunction $(\sim P \vee Q)$.

Although, when the consequent of an implication is a disjunct of the conclusion, we cannot prune the implication itself, it may be that this circumstance justifies the pruning of some disjunctive premiss in accordance with the following rule (which has two variants[7]):

$$\frac{P,\ (q \Rightarrow r_1),\ (r_1 \Rightarrow r_2),\ \ldots,\ (r_{n-1} \Rightarrow r_n),\ (r_n \Rightarrow S),\ \Gamma\ \vdash\ (S \vee T)}{(P \vee q),\ (q \Rightarrow r_1),\ (r_1 \Rightarrow r_2),\ \ldots,\ (r_{n-1} \Rightarrow r_n),\ (r_n \Rightarrow S),\ \Gamma\ \vdash\ (S \vee T)}\,[\mathrm{Pr}\vee\Rightarrow]$$

This generalises Pr$\vee$ by taking account of chains of implication leading from a disjunct of a premiss to a disjunct of the conclusion.

In implementing this pruning rule it is more convenient first to compute the transitive closure of all formulae of the form $(p \Rightarrow Q)$ occurring in the sequent. Once this is done, chains of implication need not be considered so the pruning rule is simply applied to sequents of the form $(P \vee q),\ (q \Rightarrow S),\ \Gamma\ \vdash\ (S \vee T)$.

Reducing Disjunctions to Implications

I now show that in the sequents in question, the pruning rules fully take account of the extent to which the inferential power of disjunctions exceeds that of corresponding implications. Because of this, after applying the pruning rules, we can replace disjunctions with implications and deterministically apply the $\vdash \vee$ rule. Hence, testing validity is reduced to a 'Horn-like' problem.

Let us consider the inferential potential of the remaining un-prunable disjunctive premisses. The only rule that can directly be applied to these is the $\vee \vdash$ rule. However, this rule cannot directly yield the conclusion (or a disjunct of it) because otherwise one of the disjuncts would have

[6] Trivial variants of these rules must also be applied. These are obtained by replacing $(P \vee Q)$ by $(Q \vee P)$ and/or replacing $(P \vee R)$ by $(R \vee P)$ in the rules given above.

[7] $(S \vee T)$ may be replaced by $(T \vee S)$.

been prunable. This means that if the sequent is valid at least one of the disjuncts must be capable of taking part in a subsequent MP$\Rightarrow\vdash$ rule application. Because of the limited range of formulae in the sequents we can anticipate all the forms of potential modus ponens applications and give rules which yield the same consequences but bypass the $\vee\vdash$ rules. Moreover all these rules are invertible.

We have three rules where the implication of the MP is derived from a disjunction:

$$\frac{p,\ q,\ \Gamma\ \vdash\ C}{p,\ ((p\Rightarrow\mathbf{f})\vee q),\ \Gamma\ \vdash\ C}\,[\mathrm{MP}\vee 1]$$

$$\frac{p,\ (q\Rightarrow\mathbf{f}),\ \Gamma\ \vdash\ C}{p,\ ((p\Rightarrow\mathbf{f})\vee(q\Rightarrow\mathbf{f})),\ \Gamma\ \vdash\ C}\,[\mathrm{MP}\vee 2a] \qquad \frac{q,\ (p\Rightarrow\mathbf{f}),\ \Gamma\ \vdash\ C}{q,\ ((p\Rightarrow\mathbf{f})\vee(q\Rightarrow\mathbf{f})),\ \Gamma\ \vdash\ C}\,[\mathrm{MP}\vee 2b]$$

and three more rules where it is the antecedent that comes from the disjunction:

$$\frac{(p\Rightarrow\mathbf{f}),\ (q\Rightarrow\mathbf{f}),\ \Gamma\ \vdash\ C}{(p\Rightarrow\mathbf{f}),\ ((q\Rightarrow\mathbf{f})\vee p),\ \Gamma\ \vdash\ C}\,[\mathrm{MP}\vee 3]$$

$$\frac{((r\Rightarrow\mathbf{f})\vee p),\ (p\Rightarrow q),\ ((r\Rightarrow\mathbf{f})\vee q),\ \Gamma\ \vdash\ C}{((r\Rightarrow\mathbf{f})\vee p),\ (p\Rightarrow q),\ \Gamma\ \vdash\ C}\,[\mathrm{MP}\vee 4]$$

$$\frac{((p\Rightarrow\mathbf{f})\vee q),\ (q\Rightarrow(r\Rightarrow\mathbf{f})),\ ((p\Rightarrow\mathbf{f})\vee(r\Rightarrow\mathbf{f})),\ \Gamma\ \vdash\ C}{((p\Rightarrow\mathbf{f})\vee q),\ (q\Rightarrow(r\Rightarrow\mathbf{f})),\ \Gamma\ \vdash\ C}\,[\mathrm{MP}\vee 5]$$

Finally we have a number of rules such as the following, in which both the implication and its antecedent are derived from disjunctions.

$$\frac{((p\Rightarrow\mathbf{f})\vee q),\ ((q\Rightarrow\mathbf{f})\vee r),\ ((p\Rightarrow\mathbf{f})\vee r),\ \Gamma\ \vdash\ C}{((p\Rightarrow\mathbf{f})\vee q),\ ((q\Rightarrow\mathbf{f})\vee r),\ \Gamma\ \vdash\ C}\,[\mathrm{MP}\vee 6a]$$

It can now be seen that the proof possibilities afforded by these rules are retained when formulae of the form $((p\Rightarrow\mathbf{f})\vee q)$ are replaced by $(p\Rightarrow q)$ and formulae of the form $((p\Rightarrow\mathbf{f})\vee(q\Rightarrow\mathbf{f})$ by the two formulae $(p\Rightarrow(q\Rightarrow\mathbf{f}))$ and $(q\Rightarrow(p\Rightarrow\mathbf{f}))$:

The result of applying rules MP$\vee$1, MP$\vee$2 and MP$\vee$3 can equally be achieved by applying MP$\Rightarrow\vdash$ after this replacement.

Rules MP$\vee$4, MP$\vee$5 and MP$\vee$6 all produce new disjunctions; but prior application of the pruning rules ensures that these cannot contain as a disjunct either the conclusion or a disjunct of the conclusion. Hence these new disjuncts can only participate in a proof by means of further application of one of the MP$\vee$ rules. Moreover, if a chain of such applications is useful in constructing a proof it must eventually lead to an application of one of the rules MP$\vee$1, MP$\vee$2 or MP$\vee$3, which yield a new *non-disjunctive* formula. Examination of the MP$\vee$ rules will reveal that if disjunctions are replaced by implications (as specified above) the result of any such sequence of rules can be derived by a corresponding sequence of MP$\Rightarrow\vdash$ rules.

Given that this translation of disjunctions to implications preserves provability and noting that the formulae $(q \Rightarrow (p \Rightarrow \mathbf{f}))$ and $(p \Rightarrow (q \Rightarrow \mathbf{f}))$ are logically equivalent, it follows that provability is also preserved if formulae $((p \Rightarrow \mathbf{f}) \vee (q \Rightarrow \mathbf{f})$ are replaced by the single formula $(p \Rightarrow (q \Rightarrow \mathbf{f}))$.

Completion of the Proof Procedure

Having eliminated all disjunctive premisses, we are left with a sequent containing, on the left hand side, only atomic propositions and implications (with atomic antecedents), and on the right hand side a formula of one of the forms $p \Rightarrow \mathbf{f}$, $p \Rightarrow q$, $p \Rightarrow (q \Rightarrow \mathbf{f})$, $(p \Rightarrow \mathbf{f}) \vee q$ and $(p \Rightarrow \mathbf{f}) \vee (q \Rightarrow \mathbf{f})$. We proceed as follows:

Case a) For the non-disjunctive conclusions we can immediately apply the $\vdash\Rightarrow$ rule (twice in the case of a conclusion of the form $p \Rightarrow (q \Rightarrow \mathbf{f})$) so that the conclusion is reduced to a single atom. In the resulting sequent the only possible further rule applications are of Modus Ponens. This rule is applied until either the (atomic) conclusion is derived, in which case the sequent is valid, or else no possible applications remain, in which case the sequent is invalid.

Case b) If the conclusion is a disjunction we first make all possible applications of Modus Ponens and attempt to derive a disjunct of the conclusion. If this fails we then apply the $\vdash \vee$ rule splitting the proof into two branches. For each branch we proceed as for case a).

6.3.5 Complexity of the Improved Algorithm

The number of formulae of a given type occurring in a sequent generated by the RCC-8 reasoning algorithm is bounded by the size, n of the set, of topological relations to be tested. Checking for applications of the Pr$\vee$ and Pr$\Rightarrow$ rules is clearly linear in n. Determining applications of the Pr$\vee\Rightarrow$ rule involves determining the closure of the transitive relation of implication. This can be computed in order n^2 time. Once this closure has been computed application of all possible Pr$\vee\Rightarrow$ inferences becomes n^2 (since it involves checking pairs of formulae from the l.h.s. of the sequent).

The other non-trivial part of the proof algorithm is the application of Modus Ponens rules. Since the rule involves two formulae, one 'pass' of MP applications is order n^2. Because the transitive closure of implications has already been computed[8] and because the maximum number of antecedents in a formula is two, a maximum of two passes are required to exhaust all possible MP applications.

So, the proof method described provides an order n^2 (time) algorithm for checking consistency of those $\mathcal{I}$ sequents which arise in the topological consistency checking algorithm (as compared with Hudelmaier's $O(n \log n)$-*space* algorithm for arbitrary sequents). The number of such sequents which must be checked to determine the consistency of a set of RCC relations is equal to the number

[8] This will also have to be re-computed after disjunctions are replaced by implications; alternatively, as in the current implementation, all the implications derived from disjunctions can be added at the beginning of the decision procedure.

of entailment constraints in the $\mathcal{I}$ representation of the relations, which is itself approximately proportional to the number of relations. This means, that in terms of the number of topological relations whose consistency is to be checked, the new algorithm is of order n^3.

6.3.6 Implementation and Performance Results

The improved algorithm has been concisely prototyped in (SICStus) Prolog. The code is given in appendix C.3. Preliminary tests indicate that the algorithm can determine consistency of very large sets of topological relations in an acceptable time. The procedure performs particularly well if a database is accumulated incrementally so that at each stage computation of the closure of implications is linear in the number of implicative formulae already stored.

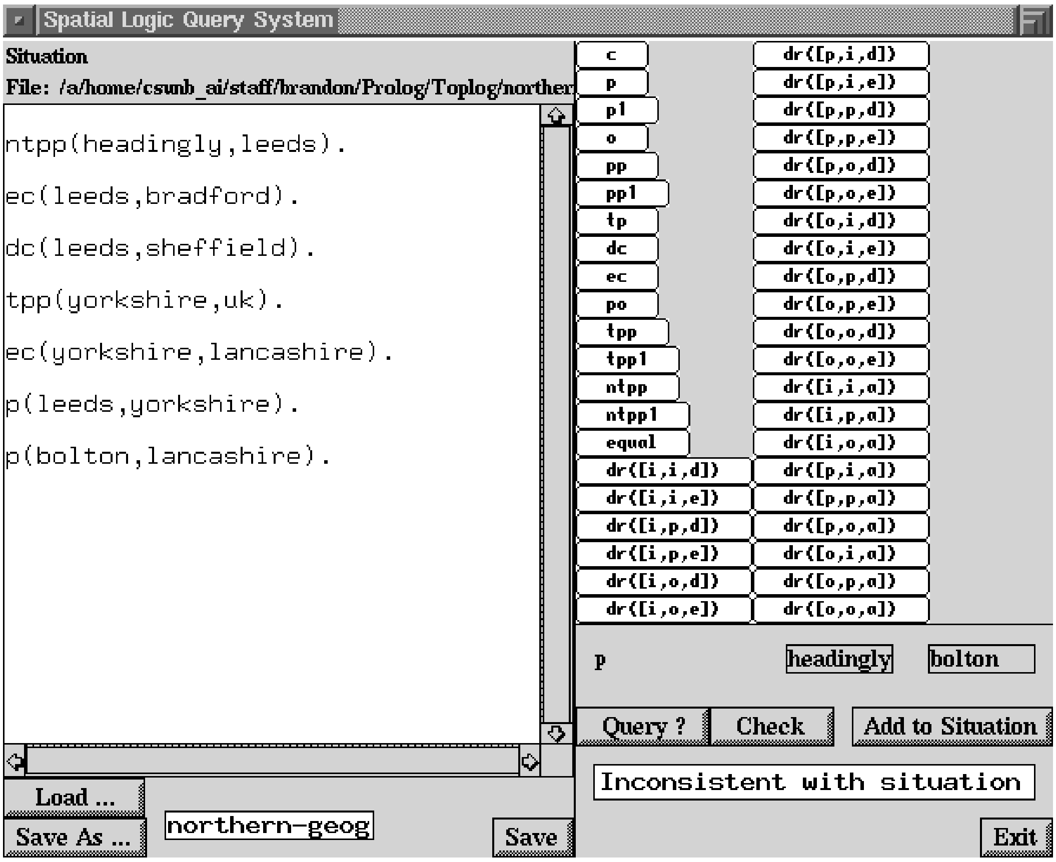

Figure 6.1: A spatial reasoner implemented in Prolog using $\mathcal{I}^+$

To test the effectiveness of the algorithm, a consistent database of n topological relations holding amongst r regions was randomly generated. The relations were generated by picking pairs of regions at random and a random relation from the RCC-8 relation set (with all regions required to be non-null). If the randomly generated relation was consistent with the database, it was added; otherwise it was rejected. This was repeated until n consistent relations had been added. The random database was then used to test query response time: random RCC-8 relations were generated and the $\mathcal{I}^+$ reasoner, exploiting the improved algorithm, was used to determine whether these relations were necessary, inconsistent or contingent with respect to the database.

The incremental addition of 300 consistent relations holding among 100 regions took on average 595 seconds and during this construction 53.7 (on average) inconsistent relations were rejected. The average query time for the resulting database was 2.6 seconds. Further analysis and revision of the program will be necessary in order to enhance its performance. This is beyond the scope of the present work but it seems very likely that an order of magnitude speed-up could be obtained quite easily.[9]

6.3.7 Nebel's Complexity Analysis

I conclude this chapter with a look at Bernhard Nebel's model theoretic analysis of the sequents arising from the $\mathcal{I}$ encoding of the RCC-8 relations. This analysis leads to an alternative proof that consistency of RCC-8 relations can be determined in polynomial time. It also reformulates the problem within the framework of classical constraints, which has received much attention from computer scientists (Mackworth 1977). To understand this section fully it will probably be necessary to refer to (Nebel 1995a) and to have some knowledge of intuitionistic model theory and proof theory (see e.g. (Kripke 1965) and (Nerode 1990)). By examining the intuitionistic sequents which are needed to reason with my $\mathcal{I}^+$ encoding of the RCC-8 relations, Nebel (1995a) has shown that the consistency of sets of RCC-8 relations can be computed in polynomial time.

Nebel's results are obtained by analysing a tableau-based proof procedure for intuitionistic logic — as described by Nerode (1990) — when it is applied to the restricted range of formula types used in encoding the RCC-8 relations. He showed that the consistency problem for these sequents can in fact be described in terms of a fairly simple set of classical constraints. This is because, for any invalid sequent involving only the formulae required to represent the RCC-8 relations, it is always possible to construct a Kripke model (Kripke 1965), containing exactly three worlds (which will be called v, w_1 and w_2), that provides a counter-example to the entailment. Nebel's encoding simply describes these models in classical predicate logic, by means of a binary relation $\mathsf{F}(w, a)$, which asserts that the (atomic) formula a is 'forced' (i.e. true) at the world w.

More, specifically, each world of the Kripke model is identified with a set of constants which are forced at that world. The worlds are (partially) ordered by the subset ordering on these sets. Whether a complex formulae is forced at a world w depends on whether its constituents are forced at w and also (in the case of negation and implication) whether they are forced at any 'larger' world:

- $\alpha \wedge \beta$ is forced at w iff both α and β are forced at w
- $\alpha \vee \beta$ is forced at w iff either α or β is forced at w

[9] One of the fundamental operations used in the Prolog program, the `assert` predicate, is known to be extremely slow and profile analysis of the program's run-time showed that over 80% of the execution time was spent carrying out this operation. By redesigning the data structures used by the algorithm, the use of `assert` could be avoided and the performance greatly enhanced. A lower level implementation — e.g. in C — would clearly be much faster still.

- $\sim\alpha$ is forced at w iff α is not forced at w nor at any world larger than w
- $\alpha \Rightarrow \beta$ is forced at w iff at w, and at any larger world, wherever α is forced so is β

The counter-models identified by Nebel always satisfy the ordering conditions $v \subseteq w_1$ and $v \subseteq w_2$ — so every formula forced by v is forced by w_1 and w_2. In these counter models the world v is constrained so as to demonstrate the invalidity of a sequent: v forces all the premiss formulae of the sequent but not its conclusion. The conditions under which a binary formula of $\mathcal{I}$ is forced at v in Nebel's counter-models can be specified classically as given in table 6.3. To test if a sequent is valid we consider a set of constraints consisting of the forcing constraint for each premiss formula, the negation of the forcing constraint of the conclusion formula and also all instances of $\mathsf{F}(v,x) \rightarrow (\mathsf{F}(w_1,x) \wedge \mathsf{F}(w_2,x))$ (where x is a any constant occurring in the sequent), which arise from the ordering conditions on the worlds. This set of classical formulae is consistent if and only if the original intuitionistic sequent is invalid.[10]

Formula	*Forcing constraint*
$\sim x$	$\neg\mathsf{F}(w_1,x) \wedge \neg\mathsf{F}(w_2,x)$
$\sim x \vee \sim y$	$(\neg\mathsf{F}(w_1,x) \wedge \neg\mathsf{F}(w_2,x)) \vee (\neg\mathsf{F}(w_1,y) \wedge \neg\mathsf{F}(w_2,y))$
$\sim(x \wedge y)$	$\neg(\mathsf{F}(w_1,x) \wedge \mathsf{F}(w_1,y)) \wedge \neg(\mathsf{F}(w_2,x) \wedge \mathsf{F}(w_2,y))$
$\sim x \vee y$	$(\neg\mathsf{F}(w_1,x) \wedge \neg\mathsf{F}(w_2,x)) \vee (\mathsf{F}(v,y)$
$x \Rightarrow y$	$(\mathsf{F}(v,x) \rightarrow \mathsf{F}(v,y)) \wedge (\mathsf{F}(w_1,x) \rightarrow \mathsf{F}(w_1,y)) \wedge (\mathsf{F}(w_2,x) \rightarrow \mathsf{F}(w_2,y))$

Table 6.3: Classical description of intuitionistic binary clause entailment

Remarkably, all the formulae in Nebel's classical encoding of the restricted $\mathcal{I}$ entailment problem are reducible to 2CNF form,[11] which means that the problem can be reduced to a 2-SAT problem. Thus the consistency sets of RCC-8 relations can be computed in polynomial time. More precisely, 2-SAT problems lie in the class NC, which means that they can be computed in polylogarithmic time on polynomially many processors, so parallel processing can be effectively exploited to speed up computation. This complexity result applies also to the larger class of relations expressible in terms of conjunctions of the RCC-7 relations and their negations; all such relations can be represented using the $\mathcal{I}$ formulae covered by Nebel's analysis. This includes almost half those relations that are disjunctions over the RCC-8 relations (the full set is given in appendix C.1.1). It is evident that applying parallelisation can improve the performance of almost any algorithm that exploits my $\mathcal{I}^+$ encoding. This is because each test of whether an entailment constraint is derivable from the model constraints can be carried out independently; so all these tests could be conducted simultaneously.

The forcing constraint analysis can also be used to identify classes of disjunctive relations over

[10] Note that Nebel's analysis does not cover the regularity condition on regions. Whether this can be represented within a tractable system is a matter for further research.

[11] A 2CNF formula is a conjunction, each of whose conjuncts is either a positive or negative literal or a disjunction of two positive or negative literals.

RCC-8 for which consistency checking of constraint networks is tractable. Clearly the completeness of compositional inference applies to any conjunction of RCC-7 relations and their negations and many disjunctions of RCC-8 relations are expressible in this way. In a network containing disjunctive relations it is possible to derive new information by composition without immediately getting a contradiction. So showing inconsistency may require repeated application of composition. Nevertheless, this procedure still leads to an algorithm which is polynomial in the number of nodes of the network (Nebel 1995a). In recent work by Renz and Nebel (1997) an analysis very similar to the forcing constraint interpretation of $\mathcal{I}$ is applied to the $S4$ encoding of RCC relations given in chapter 5. This enables a maximal tractable class of 148 disjunctive RCC-8 relations to be identified.

Chapter 7

Quantifier Elimination

This chapter explores the possibility of applying Quantifier Elimination transformations to RCC formulae. Such transformations provide a decision procedure for a large class of formulae in the 1st-order RCC language.

7.1 Quantifier Elimination Procedures

The undecidability of a logical system is very often associated with quantification. General 1st-order logic is only semi-decidable but, by restricting the forms of quantification permitted in formulae, a variety of decidable sub-languages can be found (Dreben and Goldfarb 1979, Börger, Grädel and Gurevich 1997). Most of the better known 0-order (i.e. quantifier free) logical formalisms are also decidable[1] These decidability results provide the basis for the method of constructing decision procedures by means of quantifier elimination. Suppose we have a 1st-order language which is in some way restricted — it may have restricted syntax or a limited vocabulary constrained to obey axioms of some theory. If we can show that every formula of this language can be converted *via* a series of transformations to a formula in a decidable language, which is consistent just in case the original formula were consistent, then we have a decision procedure for the original language. Typically the target language of such a conversion will be one with no (or limited) quantification, so the effect of transformation will be to eliminate quantifiers.

The method of quantifier elimination has been used to remarkable effect by Tarski (1948) to provide a decision procedure for 1st-order formulae composed by applying the Boolean connectives and quantification to propositions which are arbitrary polynomial equations and inequalities over the real numbers.[2]

[1] A notable exception is general Relation Algebra, which will be discussed in chapter 9.3.

[2] i.e. the non-logical vocabulary consists of the constants 0 and 1, the binary functions $+$, $-$, and $\times$, and the relations $=$ and $>$. The quantifiers range over the real numbers.

7.2 Quantifier Elimination in RCC

In this section I prove certain equivalences between RCC formulae which can be used to eliminate quantifiers in many contexts. In fact the proofs of theorems used in the quantifier theorem will be given in a way which is theory independent: I simply state every non-logical assumption which is used in the proofs. This means that the elimination is valid in any system in which these assumptions are theorems. All these assumptions are I believe provable from the RCC theory. However the treatment of Boolean functions assumes a complete Boolean algebra (with a null element) so their form might have to be altered to fit in with the sort structure of the RCC theory. All the assumptions are also theorems of my revised theory (given in section 3.7) and, since this theory incorporates a null region, assumptions involved in the treatment of the Boolean operations can be expressed directly in the theory.

Consider the definition of the part relation in terms of C:

$$\mathsf{P}(x,y) \ \equiv_{def}\ \forall z[\mathsf{C}(z,x) \rightarrow \mathsf{C}(z,y)] \qquad \textbf{(Pdef)}$$

If this definition is taken as a rewrite rule applied from right to left, it can be seen to achieve a quantifier elimination: a universally quantified expression involving Cis replaced by an unquantified expression in terms of P.

This elimination can be generalised to remove a universal quantifier operating on an arbitrary truth-functional combination of C relations. First the truth-functional matrix is converted to clausal normal form so that we have a conjunction of disjunctions of C literals. Since the universal quantifier distributes over conjunction it can be moved inwards to obtain a conjunction of universally quantified clauses. The quantifier can then be eliminated from each clause in virtue of the following equivalence:

C-clause Quantifier Elimination Theorem (CQE)

$$\forall x[(\mathsf{C}(x,a_1) \wedge \ldots \wedge \mathsf{C}(x,a_m)) \rightarrow (\mathsf{C}(x,b_1) \vee \ldots \vee \mathsf{C}(x,b_n))]$$
$$\leftrightarrow (\mathsf{P}(a_1,\mathsf{sum}\{b_1,\ldots,b_n\}) \vee \ldots \vee \mathsf{P}(a_m,\mathsf{sum}\{b_1,\ldots,b_n\}))$$

The left-hand (quantified) formula states that if any region x is connected to each of the regions $a_1,\ldots,a_m$, then x must also be connected to one of the regions $b_1,\ldots,b_n$. **CQE** states that this is equivalent to the condition that one of $a_1,\ldots,a_m$ is part of $\mathsf{sum}\{b_1,\ldots,b_n\}$.[3]

Proof of CQE: The equivalence of **CQE** is demonstrated by the following series of formula transformations:

1. $\forall x[(\mathsf{C}(x,a_1) \wedge \ldots \wedge \mathsf{C}(x,a_m)) \rightarrow (\mathsf{C}(x,b_1) \vee \ldots \vee \mathsf{C}(x,b_n))]$
2. $\forall x[(\mathsf{C}(x,a_1) \wedge \ldots \wedge \mathsf{C}(x,a_m)) \rightarrow \mathsf{C}(x,\mathsf{sum}\{b_1,\ldots,b_n\})]$
3. $\neg\exists x[\mathsf{C}(x,a_1) \wedge \ldots \wedge \mathsf{C}(x,a_m) \wedge \neg\mathsf{C}(x,\mathsf{sum}\{b_1,\ldots,b_n\})]$

[3] I write $\mathsf{sum}\{b_1,\ldots,b_n\}$ as an abbreviation for a term of the form $\mathsf{sum}(b_1,\mathsf{sum}(b_2,\ldots\mathsf{sum}(b_{n-1},b_n)))$.

4. $\neg(\exists x[\mathsf{C}(x, a_1) \wedge \neg\mathsf{C}(x, \mathsf{sum}\{b_1, ., b_n\})] \wedge ... \wedge \exists x[\mathsf{C}(x, a_m) \wedge \neg\mathsf{C}(x, \mathsf{sum}\{b_1, ., b_n\})])$

5. $\neg\exists x[\mathsf{C}(a_1, x) \wedge \neg\mathsf{C}(x, \mathsf{sum}\{b_1, ., b_n\})] \vee ... \vee \neg\exists x[\mathsf{C}(a_m, x) \wedge \neg\mathsf{C}(x, \mathsf{sum}\{b_1, ., b_n\})]$

6. $\mathsf{P}(a_1, \mathsf{sum}\{b_1, \ldots, b_n\})) \vee \ldots \vee \mathsf{P}(a_m, \mathsf{sum}\{b_1, \ldots, b_n\})$

The equivalence between 1 and 2 depends only on the definition of sum and that between 5 and 6 only on the definition of P. Steps 2–3 and 4–5 are purely logical equivalences and the entailment of 3 by 4 is also purely logical. That 3 entails 4 is shown by the following deduction sequence, by means of which (if we substitute $\mathsf{sum}\{b_1, \ldots, b_n\}$ for the arbitrary term τ), the negation of 3 can be derived from the negation of 4:

1. $\exists x[\mathsf{C}(x, a_1) \wedge \neg\mathsf{C}(x, \tau)] \wedge \ldots \wedge \exists x[\mathsf{C}(x, a_m) \wedge \neg\mathsf{C}(x, \tau)]$
2. $\mathsf{C}(k_1, a_1) \wedge \neg\mathsf{C}(k_1, \tau) \wedge \ldots \wedge \mathsf{C}(k_m, a_m) \wedge \neg\mathsf{C}(k_m, \tau)$
3. $\mathsf{C}(\mathsf{sum}\{k_1, .., k_m\}, a_1) \wedge \ldots \wedge \mathsf{C}(\mathsf{sum}\{k_1, .., k_m\}, a_m) \wedge \neg\mathsf{C}(\mathsf{sum}\{k_1, .., k_m\}, \tau)$
4. $\exists x[\mathsf{C}(x, a_1) \wedge \ldots \wedge \mathsf{C}(x, a_m) \wedge \neg\mathsf{C}(x, \tau)]$

■

Special cases of the reduction apply when either the left or right side of the quantified C-clause is empty. If the r.h.s. is empty then the clause is inconsistent since at least the universal region must connect with all of any set of regions. If the l.h.s. is empty, then the clause simply states that the sum of all the regions mentioned on the r.h.s. is equal to the universe. In terms of P, this can be written as $\mathsf{P}(\mathsf{u}, \mathsf{sum}(\{b_1, \ldots, b_n\})$. We can thus eliminate the innermost universal quantifier(s) of any pure C-formula[4] and in doing so end up with a formula containing only P and C relations (the remaining C-relations are those not originally within the scope of one of the innermost quantifiers) and the sum operator.

7.2.1 Extending the Procedure

To continue the procedure we would like to eliminate the innermost quantifiers of the resulting transformed formulae. Unfortunately, these formulae are no longer pure (they may contain other non-logical symbols apart from C) so the general case of further reduction is more complicated than **CQE**. The additional complexity takes the following forms.

1. The quantified variable may occur within the scope of a sum operator.
2. The P predicate is not symmetric so can act on a variable in two logically distinct ways.
3. Both P and C relations may be present.

[4]An RCC formula is a *pure C-formula* iff C is the only non-logical symbol occurring in it. Other symbols may always be eliminated by means of their definitions. The existential import of the functions must then be taken care of by suitable additional axioms for C.

Quantifier Elimination for P Clauses

My extension of the quantifier elimination procedure only addresses the first two of the problems just mentioned. I give a procedure which eliminates quantifiers from a clause containing only P-literals; the arguments of these literals may be arbitrary Boolean functions and the quantified variable can occur anywhere within these complex arguments. Such clauses will be called P-clauses. The decidability of P-clauses is an intuitive consequence of the decidability of the 1st-order theory of Boolean algebras (which is well known): the P relation can be correlated with the usual ordering on Boolean terms, so that $\mathsf{P}(\tau_1, \tau_2)$ can be identified with the Boolean equation $\tau_1 + \tau_2 = \tau_2$. However, it will be instructive to demonstrate the decidability of P-clauses by quantification elimination, within the language of RCC. This will expose exactly which mereological principles are essential to a decision procedure.

In the RCC language, the redundancy of quantification over truth-functions of P relations is in many cases obvious. For instance, $\forall x[\mathsf{P}(x, a) \rightarrow \mathsf{P}(x, b)] \leftrightarrow \mathsf{P}(a, b)$ follows immediately from reflexivity and transitivity of P. However, for a general quantifier elimination procedure we shall need to eliminate quantifiers from all forms of P-clause. To simplify this problem I first convert arbitrary P-clauses into a more restricted normal form. In virtue of axioms **AA3** and **AA4**, any P-literal, $\mathsf{P}(\phi(x), \psi(x))$, involving some variable x, where ϕ and ψ are any quasi-Boolean functions of constants and/or variables, can be regarded as a Boolean inequality of the form $\phi(x) \subseteq \psi(x)$. By applying appropriate and well known Boolean identities, such an inequality can always either be shown to be necessarily true or otherwise be transformed into a conjunction of inequalities of the forms $x \subseteq \tau$ and $\rho \subseteq x$, where x appears alone and only on one side of the '$\subseteq$' symbol. Thus, any P-literal involving x is either necessarily true or equivalent to a conjunction of P-literals of the form:

$$(\mathsf{P}(x, \tau_1) \wedge \ldots \wedge \mathsf{P}(x, \tau_i) \wedge \mathsf{P}(\rho_1, x) \wedge \ldots \wedge \mathsf{P}(\rho_j, x)) .$$

After applying this normalisation, quantifiers can be eliminated from an arbitrary clause made up of P literals in virtue of the following equivalence:

P-clause Quantifier Elimination Theorem (PQE)

$$\forall x[(\mathsf{P}(x, a_1) \wedge \ldots \wedge \mathsf{P}(x, a_i) \wedge \mathsf{P}(b_1, x) \wedge \ldots \wedge \mathsf{P}(b_j, x))$$
$$\rightarrow (\mathsf{P}(x, c_1) \vee \ldots \vee \mathsf{P}(x, c_k) \vee \mathsf{P}(d_1, x) \vee \ldots \vee \mathsf{P}(d_l, x))]$$
$$\leftrightarrow$$
$$(\ \neg\mathsf{P}(\mathsf{sum}\{b_1, \ldots, b_j\}, \mathsf{prod}\{a_1, \ldots, a_i\})$$
$$\vee\ \mathsf{P}(\mathsf{prod}\{a_1, \ldots, a_i\}, c_1) \vee \ldots \vee \mathsf{P}(\mathsf{prod}\{a_1, \ldots, a_i\}, c_k)$$
$$\vee\ \mathsf{P}(d_1, \mathsf{sum}\{b_1, \ldots, b_j\}) \vee \ldots \vee \mathsf{P}(d_l, \mathsf{sum}\{b_1, \ldots, b_j\})\ )$$

Proof of PQE: To see why this equivalence holds, first note that the left hand side is equivalent to

$$\forall x[\mathsf{P}(x,\ \mathsf{prod}\{a_1,\ldots,a_i\}) \wedge \mathsf{P}(\mathsf{sum}\{b_1,\ldots,b_j\},\ x)$$
$$\rightarrow (\mathsf{P}(x,c_1) \vee\ldots\vee \mathsf{P}(x,c_k) \vee \mathsf{P}(d_1,x) \vee\ldots\vee \mathsf{P}(d_l,x))]$$

To make the proof more concise I henceforth refer to $\mathsf{prod}\{a_1,\ldots,a_i\}$ by π and $\mathsf{sum}\{b_1,\ldots,b_j\}$ by σ. **PQE** then becomes:

$$\forall x[(\mathsf{P}(x,\pi) \wedge \mathsf{P}(\sigma,x)) \rightarrow (\mathsf{P}(x,c_1) \vee\ldots\vee \mathsf{P}(x,c_k) \vee \mathsf{P}(d_1,x) \vee\ldots\vee \mathsf{P}(d_l,x))]$$
$$\leftrightarrow (\ \neg\mathsf{P}(\sigma,\ \pi) \vee \mathsf{P}(\pi,c_1) \vee\ldots\vee \mathsf{P}(\pi,c_k) \vee \mathsf{P}(d_1,\sigma) \vee\ldots\vee \mathsf{P}(d_l,\sigma)\ )$$

Because the universal condition is hard to visualise I now transform it into an existential. If we negate both sides of this and then move the negations inwards we get

$$\exists x[\mathsf{P}(x,\pi) \wedge \mathsf{P}(\sigma,x) \wedge \neg\mathsf{P}(x,c_1) \wedge\ \ldots\ \wedge \neg\mathsf{P}(x,c_k) \wedge \neg\mathsf{P}(d_1,x) \wedge\ \ldots\ \wedge \neg\mathsf{P}(d_l,x)]$$
$$\leftrightarrow (\mathsf{P}(\sigma,\ \pi) \wedge \neg\mathsf{P}(\pi,c_1) \wedge\ \ldots\ \wedge\neg\mathsf{P}(\pi,c_k) \wedge \neg\mathsf{P}(d_1,\sigma) \wedge\ \ldots\ \wedge\neg\mathsf{P}(d_l,\sigma)) \quad (\mathbf{PQE2})$$

The left to right direction is relatively straightforward to demonstrate. It can easily be derived by making use of the following three principles describing properties of the P relation.

$$\exists x[\mathsf{P}(\alpha,x) \wedge \mathsf{P}(x,\beta)] \leftrightarrow \mathsf{P}(\alpha,\beta) \qquad (\mathbf{Pprin1})$$

$$\exists x[\mathsf{P}(x,\alpha) \wedge \neg\mathsf{P}(x,\beta)] \leftrightarrow \neg\mathsf{P}(\alpha,\beta) \qquad (\mathbf{Pprin2})$$

$$\exists x[\mathsf{P}(\alpha,x) \wedge \neg\mathsf{P}(\beta,x)] \leftrightarrow \neg\mathsf{P}(\alpha,\beta) \qquad (\mathbf{Pprin3})$$

The right to left direction is more difficult. We must show that, if the conditions on the right are satisfied, there must be some region satisfying all the conditions of the existentially quantified predicate on the left. It is clear that σ itself satisfies the conditions $\mathsf{P}(x,\pi)$ and $\mathsf{P}(\sigma,x)$ as well as all the conditions $\neg\mathsf{P}(d_n,x)$. However, it does not necessarily satisfy the conditions $\neg\mathsf{P}(x,c_n)$. To construct a region satisfying all these conditions we need to add extra bits to σ in such a way that the resulting region cannot be part of any of the c's and we must furthermore ensure that after this addition it still does not contain any of the d's as a part.

By applying the principle

$$\neg\mathsf{P}(\alpha,\beta) \leftrightarrow \exists x[\mathsf{P}(x,\alpha) \wedge \neg\mathsf{O}(x,\beta)] \qquad (\mathbf{POprin})$$

to the literal $\neg\mathsf{P}(\pi,c_1)$ we get $\exists x[\mathsf{P}(x,\pi) \wedge \neg\mathsf{O}(x,c_1)]$. We let e_1 be some region satisfying this condition. e_1 is disjoint from c_1 so clearly if we add it to σ then $\neg\mathsf{P}(\mathsf{sum}(\sigma,e_n),c_1)$. But we must construct a region that cannot violate any of the conditions $\neg\mathsf{P}(d_n,x)$. $\mathsf{sum}(\sigma,e_n)$ would violate one of these conditions if e_1 contained that part of d_n not contained in σ — i.e. if $\mathsf{P}(\mathsf{diff}(d_n, \sigma), e_1)$. Thus, rather than just adding e_1 to σ, we add a part of e_1 derived by means of the following principle:

$$\forall x \forall y \exists z[\mathsf{P}(z, x) \wedge \neg\mathsf{P}(y, z)] \ , \qquad \textbf{(Pprin4)}$$

which says that given any two regions there is always some region which is part of the first and does not contain the second as a part. If we instantiate this with e_1 and $\mathsf{diff}(d_1, \sigma)$ we get: $\exists z[\mathsf{P}(z, e_1) \wedge \neg\mathsf{P}(\mathsf{diff}(d_1, \sigma), z)]$.

Let e_1^1 denote a region which is an instance of this existential statement. e_1^1 is clearly disjoint from c_1 since it is part of e_1. Moreover, $\mathsf{sum}(\sigma, e_1^1)$ cannot contain d_1. However, it could still be the case that $\mathsf{sum}(\sigma, e_1^1)$ contains one of the other d's. Thus, we recursively apply **Pprin4** to e_1^1 to get a part of e_1^1 which does not contain $\mathsf{diff}(d_2, \sigma)$ as a part. This will be called e_1^2. Continuing this process we finally end up with e_1^l; and we can be sure that if this is added to σ the resulting region will not include any of the d's. Also, since e_1^l must be disjoint from c_1 the result will not be a part of c_1. We let $\sigma_1 = \mathsf{sum}(\sigma, e_1^l)$.

σ_1 is part of π, does not contain any of the d's and is not part of c_1. To complete the proof we need to successively extend σ_1 to derive a region which is definitely not part of any of the c's and also does not contain any of the d's. Thus to construct σ_2 we first identify a region e_2 which is part of π but not part of c_2; we then form the sequence of regions $c_2^1, \ldots, c_2^l$, where c_2^l is disjoint from c_2 and does not contain any of the regions $\mathsf{diff}(d_n, \sigma_1)$. σ_2 is then equal to $\mathsf{sum}(\sigma_1, e_2^l)$. After repeating this process for each of the c's we finally reach σ_k; and this region satisfies all the literals in the existential formula on the left of **PQE2**, so this formula is proved. Hence, the equivalent formula **PQE** is also a theorem. ■

7.3 Limitations and Uses of the Procedure

We would like to iterate quantifier elimination transforms to obtain a quantifier-free formula; but a problem arises when we encounter, in the course of the reduction, a matrix containing both C and P relations, since we have no way of eliminating a quantifier from a mixed clause of this kind. Indeed, the undecidability of RCC means that no general quantifier elimination procedure could exist. I have studied possible ways of eliminating quantifiers in various restricted forms of mixed clause. In some cases the elimination is straightforward but in other cases there seems to be no way to get an equivalent quantifier free formula, except by introducing additional relational vocabulary. This is not in itself a problem but it means that, for successive iterations of quantifier elimination, clauses containing an increasingly extended vocabulary of relations must be considered.

Despite its limitations, the partial quantifier elimination procedure described in this chapter can be used to extend the range of formulae that can be handled by a decision procedure which employs one of the 0-order encoding techniques described in chapters 5 and 6. Specifically, one can provide a decision procedure for a language which, as well as allowing one to specify the wide range of spatial relationships that can be encoded directly into $S4^+$ or $\mathcal{I}^+$, also allows the assertion of certain kinds of quantified clause, whose quantifiers can be eliminated by applying the equivalences **CQE** and **PQE** as re-write rules, prior to translating into the 0-order encoding.

Chapter 8

Convexity

In this chapter I investigate how the representations described so far may be extended to handle concepts related to convexity. I first present a 1st-order axiomatisation of a *convex-hull* operator. I then consider how the logical properties of the operator can be encoded into intuitionistic and modal representations.

8.1 Beyond Topology

Hitherto, I have considered only properties of regions that are purely topological in nature — i.e. properties that are invariant under continuous deformations. Whilst these properties are fundamental, they cannot provide the basis for a fully comprehensive spatial description language. A fully expressive spatial language would be capable of expressing metrical information, at least of a relative kind — if we introduce an absolute metric unit, we then have a language with the expressive power of arithmetic and which is not completely axiomatisable.[1] The language of elementary point geometry with a relative (but not an absolute) metric is completely axiomatisable (See e.g. (Tarski 1959) and appendix A);[2] but computing inferences within this language is highly intractable.

The value of a representation language for AI depends on its expressive power and its tractability. We saw in chapter 6 that it is possible to reason effectively with certain topological relations. An obvious question is whether one can find a more expressive language which is still tractable; and, more specifically, whether one can find a tractable language capable of expressing non-topological spatial concepts. Such a language would contain one or more primitive concepts that are not topological in character.

Intermediate in expressive power between topology and metrical geometries (such as Euclidean geometry) is *affine* geometry. An affine geometry articulates the concept of *betweenness* but cannot express orthogonality or say anything about angular relationships between objects. In this chapter I consider affine geometry from the point of view of reasoning in a region-based theory.

[1] Tarski (1956b) demonstrates that a formal geometrical language containing a congruence relation and a unit element as primitives is, in some sense, *maximally* expressive.

[2] In fact, several distinct complete geometries can be formulated (see e.g. (Trudeau 1987)).

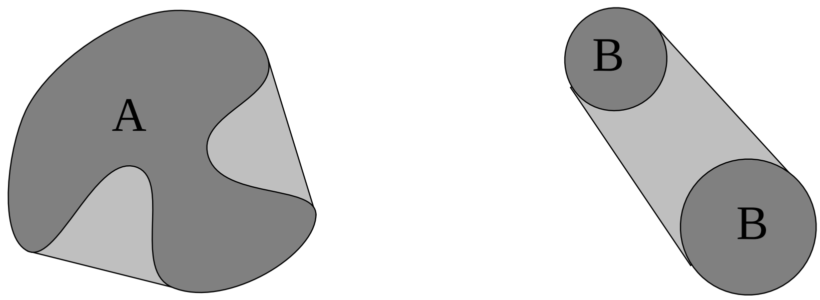

Figure 8.1: Illustration of convex-hulls in 2 dimensions

8.2 The Convex-Hull Operator, conv

The relation of betweenness is intimately connected with *convexity*: a region is convex if it is closed with respect to betweenness — i.e. if every point lying between two points in the region is itself in the region. This thesis is primarily concerned with expressing spatial properties of regions, rather than points[3] and in a region-based theory it can be argued that convexity is a more primitive notion than betweenness: to decide whether one region 'lies between' two others, one must choose between a variety of stronger or weaker notions of betweenness (can the regions overlapΓ must *all* points of the inner region be between the outer regionsΓ); but the property of a region's being convex is not so ambiguous. Of course a rigorous semantical definition of convexity requires regions to be considered as subspaces of some affine space, so the class of convex regions will be dependent on the properties of this space. In the purely region based analysis of convexity carried out in this chapter I assume that the axiomatised property of convexity is intended to be consistent with an interpretation in Euclidean space.

Following Randell, Cui and Cohn (1992) I take the *convex-hull* operator, conv, as a primitive function mapping regions to their convex-hulls. By the convex-hull of a region is meant the smallest convex region of which it is a part. If one were to stretch an elastic membrane round a region then the convex-hull would be the whole of the region enclosed.[4] Figure 8.1 shows convex-hulls of two regions in 2 dimensions (region B is a two piece region).

The conv function and a predicate, CONV, true of convex regions, are inter-definable:

$$\mathsf{CONV}(x) \equiv_{def} (x = \mathsf{conv}(x))$$

$$(\mathsf{conv}(x) = y) \equiv_{def} \mathsf{CONV}(y) \wedge \forall z[(\mathsf{CONV}(z) \wedge \mathsf{P}(x,z)) \rightarrow \mathsf{P}(y,z)]$$

There are many possible ways in which a ternary relation $\mathsf{Between}(x, y, z)$, read '$y$ is between x and z', could be defined in terms of conv. These capture different precise senses of the, somewhat ambiguous, natural concept of betweenness. Most of the ambiguity of 'betweenness' arises in connection with its application to extended bodies rather than points.[5] A very weak definition of

[3] For an axiomatic and algebraic analysis of computing convex-hulls of sets of points see (Knuth 1992).

[4] One might say the term is slightly inappropriate, since 'hull' normally refers to an outer shell, rather than a volume or area.

[5] The intuitive meaning of the betweenness relation on points leaves little scope for ambiguity, except that we

betweenness is the following:

$$\mathsf{W\_Between}(x, y, z) \equiv_{def} \mathsf{P}(y, \mathsf{conv}(\mathsf{sum}(x, z))) \ .$$

W_Between could itself be taken as primitive and CONV could then be defined by

$$\mathsf{CONV}(x) \equiv_{def} \forall y[\exists v \exists w[(\mathsf{P}(v, x) \wedge \mathsf{P}(w, x) \wedge \mathsf{W\_Between}(v, y, w))] \rightarrow \mathsf{P}(y, x)] \ .$$

W_Between does not really capture the intuitive notion of betweenness because it allows cases such as where y is in a cavity of x which is on the opposite side of x to that facing z. It also allows y to overlap or even be part of either x or z. Before giving a better definition we need to be clear about the main aspects of ambiguity in the concept. One source of ambiguity concerns whether the regions involved may overlap. Probably the most natural way to settle this is to require that y cannot overlap either x or z but allow that x and z may possibly overlap. A second source of ambiguity is whether y must be completely between x and z or may be only partly between them. Both senses are easy to define but it seems most straightforward to define partial betweenness first:

$$\begin{aligned}\mathsf{P\_Between}(x, y, z) \equiv_{def} & \neg\mathsf{O}(x, y) \wedge \neg\mathsf{O}(z, y) \wedge \\ & \exists x' \exists z' [\mathsf{P}(x', x) \wedge \mathsf{P}(z', z) \wedge \mathsf{CONV}(x') \wedge \mathsf{CONV}(z') \wedge \mathsf{O}(y, \mathsf{conv}(\mathsf{sum}(x', z')))] \ .\end{aligned}$$

We can then say that y is (completely) between x and z, if every part of y is partially between them:

$$\mathsf{Between}(x, y, z) \equiv_{def} \forall y' [\mathsf{P}(y', y) \rightarrow \mathsf{P\_Between}(x, y', z)] \ .$$

Interestingly, CONV can be defined from Between in exactly the same way that it is defined from W_Between.

8.2.1 Containment Relations Definable with conv

A large number of new binary relations can be defined in terms of the conv together with other RCC relations. For example Randell, Cui and Cohn (1992) give the following definitions of three possible *containment* relations which form a disjoint and exhaustive partition of the DR relation:[6]

- $\mathsf{INSIDE}(x, y) \equiv_{def} \mathsf{DR}(x, y) \wedge \mathsf{P}(x, \mathsf{conv}(y))$
- $\mathsf{P\text{-}INSIDE}(x, y) \equiv_{def} \mathsf{DR}(x, y) \wedge \mathsf{PO}(x, \mathsf{conv}(y))$
- $\mathsf{OUTSIDE}(x, y) \equiv_{def} \mathsf{DR}(x, \mathsf{conv}(y))$

may wish to distinguish between *strict* betweenness, where y may not be equal to x or z, and the weaker version (used in Tarski's Elementary Geometry — see appendix A), which does allow this possibility.

[6] It may be argued that, for many purposes, relations involving convex-hulls are most informative when we are considering non-overlapping regions. Such regions can correspond to discrete physical bodies, regarding which we will often be interested in spatial properties that are much more complex than simply whether or not the regions touch.

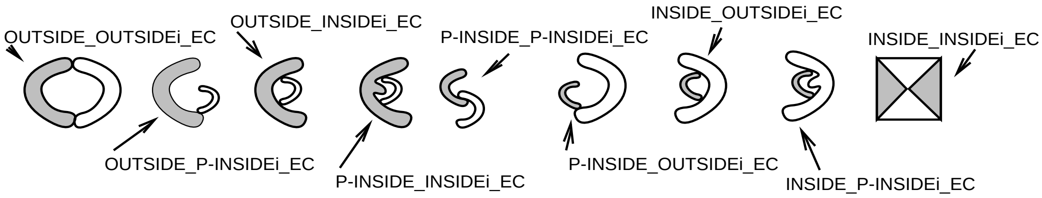

Figure 8.2: Nine refinements of EC

Randell, Cui and Cohn (1992) use these relations to differentiate a set of 24 relations which can hold among any two regions. The relationship between overlapping regions is simply described in terms of one of six possible RCC-8 relations. For EC and DC regions we additionally specify the two containment relations $R(x, y)$ and $R'(y, x)$, where $R, R' \in \{\mathsf{INSIDE}, \mathsf{P\text{-}INSIDE}, \mathsf{OUTSIDE}\}$. For DR regions each of the resulting nine combinations of R and R' is possible — the EC cases are illustrated in figure 8.2. This yields a set consisting of $6 + 9 + 9 = 24$ JEPD relations. However, if two regions are finite and mutually INSIDE each other, then because of axiom 8.3 they cannot be DC; so, in the case where the regions are required to be finite, only 23 of these relations are possible. This set is known as RCC-23.

Following (Randell, Cui and Cohn 1992) I represent the RCC-23 relations that are specialisations of EC and DC by expressions of the form $[\sigma_1, \sigma_2, \tau](x, y)$, where σ_1 is either 'I', 'P' or 'O' according as either $\mathsf{INSIDE}(x, y)$, $\mathsf{P\text{-}INSIDE}(x, y)$ or $\mathsf{OUTSIDE}(x, y)$; σ_2 refers to the corresponding inverse relation (i.e. one of these 3 relations but with the arguments reversed); and τ is either 'D' or 'E' according to whether the regions are completely disconnected or externally connected. Thus, for example, $[\mathsf{P}, \mathsf{I}, \mathsf{E}](x, y)$ means that $\mathsf{P\text{-}INSIDE}(x, y)$, $\mathsf{INSIDE}(y, x)$ and $\mathsf{EC}(x, y)$.

More generally by combining basic RCC-8 relations with the conv operator we can specify a large number of relations by means of expressions of the form

$$R_1(x, y) \wedge R_2(x, \mathsf{conv}(y)) \wedge R_3(\mathsf{conv}(x), y) \wedge R_4(\mathsf{conv}(x), \mathsf{conv}(y)) \ .$$

Although there are $8^4 = 4096$ different expressions of this form, the logical properties of convexity mean that many of these are equivalent — indeed, many are equivalent to the empty/impossible relation, $\perp(x, y)$. The number of distinct relations expressible in this way has not yet been determined; but, despite the equivalences, it is clearly quite large.

8.3 1st-Order axioms for conv

In order to construct a logical language in which the operation of forming the convex-hull of a region is incorporated into the vocabulary, it is necessary to understand and formalise the logical properties of the new operator. An obvious starting point is to specify fundamental properties of the convex-hull operator in 1st-order logic. I give seven axioms specifying important properties of conv.[7] For readability I make use of the CONV predicate defined above (section 8.2). I also introduce a

[7] Earlier versions of the axioms can be found in (Randell, Cui and Cohn 1992, Bennett 1994b, Cohn 1995).

predicate $\mathsf{Fin}(x)$ to assert that x is finite. This is needed to express a property of convexity that only holds for finite regions.[8] I shall not assume any specific set-theoretic interpretation of regions. My intention is that the axioms should be compatible with any of the possible interpretations of RCC described in section 3.5.

$$\forall x[\mathsf{TP}(x, \mathsf{conv}(x)) \vee (x = \mathsf{u})] \tag{8.1}$$

$$\forall x \forall y[\mathsf{P}(x, y) \;\rightarrow\; \mathsf{P}(\mathsf{conv}(x), \mathsf{conv}(y))] \tag{8.2}$$

$$\forall x \forall y[(\mathsf{Fin}(x) \wedge (\mathsf{conv}(x) = \mathsf{conv}(y))) \;\rightarrow\; \mathsf{C}(x, y)] \tag{8.3}$$

$$\forall x \forall y[\mathsf{conv}(x + \mathsf{conv}(y)) = \mathsf{conv}(x + y)] \tag{8.4}$$

$$\forall x \forall y[\mathsf{CONV}(\mathsf{conv}(x) * \mathsf{conv}(y))] \tag{8.5}$$

$$\forall x \forall y[\mathsf{DC}(x, y) \rightarrow \neg\mathsf{CONV}(x + y)] \tag{8.6}$$

$$\forall x \forall y[(\mathsf{NTPP}(x, y) \wedge \neg(\mathsf{conv}(x) = \mathsf{u})) \rightarrow \neg\mathsf{CONV}(y \perp x)] \tag{8.7}$$

Axiom 8.1 states the obvious fact that a region must be a tangential part of its convex hull. An exception to this requirement is the universal region, u: if u is convex then it will be equal to its own convex-hull; but $\mathsf{TP}(\mathsf{u}, \mathsf{u})$ is false (at least under the definition of TP given by Randell, Cui and Cohn (1992)). If u is not convex then conv cannot be a total function. Axiom 8.2 expresses a monotonicity property: taking convex-hulls preserves parthood relationships. Axiom 8.3 ensures that any two *finite* regions having the same convex hull must be connected.[9] The next three axioms connect the properties of convexity to the Boolean functions. Axiom 8.4 says that if we take the convex hull of a sum, then any convex-hull operators on the summands are redundant. Axiom 8.5 asserts that the intersection of any two convex regions must itself be convex. Axiom 8.6 expresses the obvious fact that the sum of two DC regions cannot be convex. Axiom 8.7 expresses a similar property: that shapes with interior holes cannot be convex. The condition $\neg(\mathsf{conv}(x) = \mathsf{u}))$ rules out anomalous counter-examples, where the complement of a convex region is subtracted from $\mathsf{u}$to yield a convex region.

This list is not guaranteed to be a complete axiomatisation of the conv operator. It is very difficult to be sure that a set of axioms fully captures a concept unless we have a formal model (or set of models) within which the concept is defined and show that the axioms are sound and complete with respect to that model (those models). Investigating such models is the subject of ongoing work. Short of proving completeness, we can gain confidence in our axiom set by showing that expected properties of convexity can be derived from our axiom set. For instance the following

[8] Introduction of the Fin predicate is methodologically dubious, since finitude is not 1st-order axiomatisable. Nevertheless, for present purposes it is convenient to assume Fin as primitive, in order to state one of the properties required of the convex-hull function under its intended interpretation.

[9] It might be imagined that certain finite, but infinitely complex, regions could have the same convex-hull and yet not be connected. However, I have not been able to find a reasonable set-theoretic interpretation in which this can occur.

theorems are quite easy to prove:[10]

$$\forall x[\mathsf{conv}(\mathsf{conv}(x)) = \mathsf{conv}(x)] \quad \text{(from 8.5)} \tag{8.8}$$

$$\forall x \forall y[\mathsf{P}((\mathsf{conv}(x) + \mathsf{conv}(y), \mathsf{conv}(x+y))] \quad \text{(from 8.2)}^{11} \tag{8.9}$$

The first of these expresses the simple fact that applying the convex hull operator a second time in succession is redundant and the second asserts the distributivity of conv and + with respect to P. Since these properties are very simple, we originally included them in our axiom set. It may still be the case that one or more of our current axioms is derivable from the rest.

Apart from the implicit existential import of the conv function itself, all the conv axioms given so far are universal in nature. However, one might expect there to be other existential axioms involving convexity. Indeed, since the domain of regions in the RCC theory is atomless, it seems reasonable to require that every region has both convex non-tangential proper parts and convex tangential proper parts:

$$\forall x \exists y[\mathsf{NTPP}(y,x) \wedge \mathsf{CONV}(y)] \qquad \text{and} \qquad \forall x \exists y[\mathsf{TPP}(y,x) \wedge \mathsf{CONV}(y)]$$

8.4 Encoding $\mathsf{conv}(x)$ in $\mathcal{I}^+$

In (Bennett 1994b) I described a method of reasoning about convexity by means of a meta-level extension of the intuitionistic encoding described in chapter 6. The language $\mathcal{I}^+$, is extended to a language $\mathcal{I}^+_{\mathsf{conv}}$, in which, as well as having ordinary constant symbols c_i denoting regions, one can also employ terms $\mathsf{conv}(c_i)$ to refer to the convex hull of the region c_i. Here, conv is to be regarded as a meta-level syntactic device rather than a real function symbol: the $\mathcal{I}^+$ reasoning algorithm simply treats $\mathsf{conv}(c_i)$ as an atomic constant. The meaning of conv is then characterised by an additional meta-level reasoning mechanism which enforces constraints associated with convexity.

The constraints enforced in my original system correspond to the following axiom set:

1. $\forall x[\mathsf{conv}(\mathsf{conv}(x)) = \mathsf{conv}(x)]$
2. $\forall x[\mathsf{TP}(x, \mathsf{conv}(x))]$
3. $\forall x \forall y[\mathsf{P}(x,y) \rightarrow \mathsf{P}(\mathsf{conv}(x), \mathsf{conv}(y))]$
4. $\forall x \forall y[(\mathsf{conv}(x) = \mathsf{conv}(y)) \rightarrow \mathsf{C}(x,y)]$

This set amended and slightly extended a previous axiom set that had been given in (Randell, Cui and Cohn 1992); however, as we saw in section 8.3, it is now clear that further axioms are needed to adequately characterise conv. It is also known that the last of these axioms only applies to finite regions. Nevertheless, it is worth describing how the limited axiom system can be enforced and considering how this approach could be extended to take account of additional properties of conv.

Observe that none of the axioms contains any Boolean operators and also that in our extended $\mathcal{I}^+$ the conv pseudo-operator can only be applied to an atomic constant. Consequently the

[10] Thanks to Stephano Borgo.

[11] Conversely, 8.2 can be derived from 8.9. I prefer to take the former as an axiom, since it does not involve any Boolean operators.

relationships possible between Boolean combinations of region constants and/or their convex hulls are not in any way constrained by the limited axiom set.[12] Moreover, since all the axioms are universal (apart from the implicit existential import of the conv function) they are equivalent to the sets of all their ground instances. In determining whether a set of spatial facts stated in $\mathcal{I}^+_{\mathsf{conv}}$ is consistent with the axioms, the only instances of the axioms which can be relevant are those where the variables are replaced by constants occurring in the facts. We thus treat the 1st-order axioms as schemas and instantiate them in all possible ways using the region constants occurring in the spatial facts under consideration. This will result in a finite number of ground constraints.

We must now consider how to test whether the facts are consistent with the additional convexity constraints. Axiom 1 can have no effect on consistency since expressions of the form $\mathsf{conv}(\mathsf{conv}(x))$ do not occur in $\mathcal{I}^+_{\mathsf{conv}}$ — indeed the axiom tells us that there is no reason why we should need to employ such expressions. The constraints arising from axiom 2 can immediately be translated into $\mathcal{I}^+$ formulae, just as any other TP relation. Instances of axioms 3 and 4 are of most interest and illustrate a general method by which $\mathcal{I}^+$ could be extended. We see that each of these is a simple Boolean combination of topological constraints (P, = and C) that can be directly represented in $\mathcal{I}^+$.

These Boolean combinations of $\mathcal{I}^+$ expressible constraints can be interpreted at the meta-level in terms of Boolean combinations of $\mathcal{I}^+$ consistency problems. For example if we have a set of facts Φ expressible in $\mathcal{I}^+$ and add to these a fact ψ, such that $\psi \equiv \phi_1 \vee \phi_2$, where both ϕ_1 and ϕ_2 are expressible in $\mathcal{I}^+$, then the set of facts $\{\Phi,\ \psi\}$ is consistent if and only if either $\{\Phi,\ \phi_1\}$ is consistent or $\{\Phi,\ \phi_2\}$ is consistent. However, it is clear that the number of $\mathcal{I}^+$ consistency checks required to test consistency of a spatial situation description, involving Boolean combinations of $\mathcal{I}^+$ expressible conditions, is exponential in the number of disjunctions occurring in these Boolean combinations. Moreover, since enforcing axioms such as the conv axioms requires one to consider all possible instantiations over the regions mentioned in the situation description, the number of disjunctive constraints may be quite large.

Treatment of axioms 3. and 4. is encompassed by a general procedure which enables enforcement of all axioms of the form:

$$\forall x_1, \ldots, x_n[\Phi(x_1, \ldots, x_n) \rightarrow \Psi(x_1, \ldots, x_n)],$$

where $\Phi(x_1, \ldots, x_n)$ and $\Psi(x_1, \ldots, x_n)$ specify situations which can be described by means of $\mathcal{I}^+$.

To test whether a given $\mathcal{I}^+$ situation description satisfies such an axiom an iterative fixed-point method can be used:

1. Test the $\mathcal{I}^+$ description for consistency. If it is inconsistent, stop.

2. Check whether any instance of the antecedent is entailed by the $\mathcal{I}^+$ description. This involves translating $\Phi(\ldots)$ into $\mathcal{I}^+$ and substituting all combinations of constants occurring in the description for the free variables. If any such instance is entailed, add the corresponding $\mathcal{I}^+$ representation of $\Psi(\ldots)$, under the same substitution, to the description.

[12] In a more complete set we would have axioms such as 8.4, which relates conv to the Boolean sum operator.

3. Check whether any instance of the consequent is inconsistent with the $\mathcal{I}^+$ description — i.e. translate $\Psi(\ldots)$ into $\mathcal{I}^+$ and substitute all combinations of constants occurring in the description for the free variables. If any such $\Psi(\ldots)$ is inconsistent, add the corresponding $\mathcal{I}^+$ representation of the *negation* of $\Phi(\ldots)$, under the same substitution, to the description.

4. If no new information was added by steps 2 and 3, stop: the situation is consistent with the axiom. Otherwise, go to 1 to test the new extended $\mathcal{I}^+$ description.

This process must terminate; and if the final situation description is still consistent then the axiom is satisfiable, since for all substitutions either the antecedent is not entailed by the description or the consequent has been explicitly added; and the consequent is either consistent with the description or the negation of the antecedent has been added. Clearly the convex-hull axioms 3. and 4. are of the form which can be captured in this manner. In fact, since their antecedents are simple, they can be enforced quite efficiently.

In section 9.2.3 I shall present a table of *compositions* of the RCC-23 relations, which was computed using the $\mathcal{I}^+$ reasoning algorithm given in chapter 6, augmented with the meta-level reasoning for conv which has just been described. A full discussion of relational composition can be found in the next chapter.

8.5 Modal Representation of Convexity

We have seen how the topological interior function corresponds to the $S4$ modal box operator. Such a correspondence may suggest that other useful functions of spatial regions can be captured by modal operators in a 0-order calculus. In the remainder of this chapter I specify a multi-modal language with a convex hull operator.[13] This language contains usual classical connectives (which will be interpreted algebraically in accordance with section 4.2) plus three modal operators:

I — an interior operator, constrained to behave exactly as the $S4$ modality,

⊠ — the strong-S5 operator,

○ — the convexity operator, whose properties are to be specified.

To fix the meaning of the new operator, we need to find 0-order axiom schemata (or rule schemata) to enforce the desired properties of ○. These schemata will correspond to the 1st-order axioms given above. I do not know of a general method for performing this kind of transformation and it seems unlikely that such a method exists. However, in each case we can see that under the algebraic interpretations of the logical operators the schemata are equivalent to the axioms.

The schema corresponding to axiom 1 is very simple:

$$\bigcirc \bigcirc X \leftrightarrow \bigcirc X \qquad (Sch1)$$

[13] This material is a slight revision of what I presented in (Bennett 1996b).

Axiom 2 is a little harder to represent as a modal schema. $\mathsf{TP}(X, Y)$ means that X is a *tangential part* of X. This holds if either X is a tangential *proper* part of Y or X is equal to Y. Thus to represent this we use the encoding for $\mathsf{TPP}(X, Y)$ given in table 5.8 but drop the second entailment constraint $Y \to X$ which would ensure that X and Y are non equal. Hence, using the strong-$S5$ $\boxtimes$ rather than the model/entailment-constraint distinction, axiom 2 can be represented by the schema

$$\boxtimes(X \to \bigcirc X) \;\wedge\; \neg\boxtimes(X \to \mathbf{I} \bigcirc X), \tag{$Sch2$}$$

which says that all regions are part of their convex-hull but not part of the interior of their convex-hull. We may note that the initial $\boxtimes$ in the first conjunct is redundant, since it is implicit in modal axiom schemata that they are true in all possible worlds — or, in the context of algebraic semantics, that their denotation is $\mathcal{U}$.

Axiom 3, which states that if X is part of Y then $\bigcirc X$ is part of $\bigcirc Y$ can be represented by

$$\boxtimes(X \to Y) \to (\bigcirc X \to \bigcirc Y). \tag{$Sch3$}$$

This requires some explanation. In general, where we have a 1st-order axiom of the form $p \to q$, this will be translated by $\boxtimes\tau(p) \to \tau(q)$ (where $\tau(\alpha)$ is the representation of α), which ensures that if $\tau(p) = \mathcal{U}$ then $\tau(q) = \mathcal{U}$. Note that we do not need $\boxtimes\tau(p) \to \boxtimes\tau(q)$ because the antecedent must either denote $\emptyset$, in which case the schema is trivially satisfied, or it denotes $\mathcal{U}$, in which the consequent must also denote $\mathcal{U}$. If we were to write simply $\tau(p) \to \tau(q)$ this would represent the stronger requirement that $\tau(p)$ is always a subset of $\tau(q)$ whether or not $\tau(p) = \mathcal{U}$.

Using a similar transformation axiom 4 can be straightforwardly represented by:

$$\boxtimes(\bigcirc X \leftrightarrow \bigcirc Y) \;\to\; \neg\boxtimes\neg(X \wedge Y) \tag{$Sch4$}$$

$\neg\boxtimes\neg(X \wedge Y)$ corresponds to the entailment constraint representing $\mathsf{C}(X, Y)$ and asserts that X and Y share at least one point.

Finally axiom 5 can be straightforwardly captured by:

$$\bigcirc(\bigcirc X \wedge \bigcirc Y) \leftrightarrow (\bigcirc X \wedge \bigcirc Y) \tag{$Sch5$}$$

It should be noted that the strong-$S5$ operator, $\boxtimes$, is not needed if we specify the logic by means of *rule* schemata rather than only *axiom* schemata. For example, *Sch3* becomes:

$$\frac{\vdash X \to Y}{\vdash \bigcirc X \to \bigcirc Y}\,[\bigcirc\ \mathbf{Mon}]$$

which tells us that $\bigcirc$ is monotonic with respect to the part relation (i.e. $\to$).

The second conjunct of *Sch2* would correspond to the rule:

$$\frac{\vdash X \to \mathbf{I} \bigcirc X}{\vdash \bot}\,[\bigcirc \mathsf{TP}]$$

and *Sch4* to the rule:

$$\frac{\vdash (\bigcirc X \leftrightarrow \bigcirc Y) \wedge \neg(X \wedge Y)}{\vdash \bot} [\leftrightarrow \bigcirc \mathsf{C}].$$

8.5.1 Practicality of the Modal Representation

The possibility of specifying convex-hull as a modal operator illustrates the potential expressive power of multi-modal formalisms as representations for spatial information. However, whether such logics could actually be used as vehicles for effective reasoning remains debatable. As in the case of the simpler $S4^+$ and $\mathcal{I}^+$ representations of purely topological relations, it is likely that, by limiting the range of formulae that can be employed to simple syntactic forms, one might be able to construct effective decision procedures for some sub-language of this multi-modal language of convexity. The crucial question is whether useful expressive power can be provided within a tractable representation.

Chapter 9

Composition Based Reasoning

Originating in Allen's analysis of temporal relations, the use of *Composition Tables* has become a key technique in providing an efficient inference mechanism for a wide class of theories. In this chapter I examine compositional reasoning in general and its use in spatial reasoning. I present composition tables for several important sets of RCC relations including the RCC-23 relations (introduced in section 8.2.1). This table was computed using the intuitionistic encoding described in chapter 1 together with the meta-level encoding of convexity axioms specified in section 8.4. Finally I look at the formalism of *Relation Algebra* and show how it allows algebraic definition of the RCC-8 relations in terms of the primitive connectedness relation.

9.1 Composition Tables

A *compositional inference* is a deduction, from two relational facts of the forms $R(a, b)$ and $S(b, c)$, of a relational fact of the form $T(a, c)$, involving only a and c. Such inferences may be useful in their own right or may be employed as part of a larger inference mechanism, such as a consistency checking procedure for sets of relational facts. In either case, one will normally want to deduce the strongest relation $T(a, c)$ that is entailed by $R(a, b) \wedge S(b, c)$ and which is expressible in whatever formalism is being employed.

In many cases the validity of a compositional inference does not depend on the particular constants involved but only on logical properties of the relations R, S and T. Where this is so it makes sense, from a computational point of view, to record the compositions of pairs of relations, so that the result of a compositional inference can simply be looked up when required. This technique is particularly appropriate where we are dealing with relational information involving a fixed set of relations. One can then store the result of composing any pair from a set of n relations in an $n \times n$ *composition table*. The simplicity of this idea makes it very attractive as a potential means of achieving effective capabilities for reasoning about any domain within which significant information can be represented by a limited set of binary relations. Since their introduction by

Allen (1983) composition tables[1] have received considerable attention from researchers in AI and related disciplines (Vilain and Kautz 1986, Egenhofer and Franzosa 1991, Freksa 1992a, Randell, Cohn and Cui 1992a, Röhrig 1994, Cohn, Gooday and Bennett 1994, Schlieder 1995).

Given a set Rels of binary relations a composition table can be identified with a mapping $CT : \mathsf{Rels} \times \mathsf{Rels} \rightarrow 2^{\mathsf{Rels}}$ — i.e. if R_1 and R_2 are elements of Rels, then the value of $CT(R_1, R_2)$ is a subset of Rels, which is the composition table entry for the pair $\langle R_1, R_2 \rangle$. The set Rels will be called the *basis* set of CT. Clearly, if there are n relations in Rels then the composition table for Rels can be represented by an $n \times n$ array or table. In fact, because of the nature of relational composition, such an array is a very inefficient way to store this information. I described the redundancy inherent in composition tables in (Bennett 1994a) and an abbreviated version of this material is included as appendix D of this thesis.

For many purposes a composition table entry is associated with a disjunctive relation. Because of this it is convenient to be able to write a set of relation names as if it were the name of a disjunctive relation. Thus

$$\{R_1, \ldots R_n\}(a, b) \qquad \text{means} \qquad \lambda xy[R_1(x, y) \vee \ldots \vee R_n(x, y)](a, b) \ .$$

It is usual to assume that the elements of Rels form a JEPD partition of the possible relations which can hold between pairs of objects in the domain under consideration (i.e. every pair of objects in the domain is related by exactly one of the members of Rels). Under these conditions any Boolean combination of relations is equivalent to a disjunction of members of Rels.

The precise meaning of a composition table depends to some extent on the context in which it is employed. Sometimes it is a record of certain kinds of consequence of some underlying theory which may already be fully or partially formalised. Alternatively, the specification of a composition table may precede the development of a formal theory of the relations involved and is an initial step in specifying the theory of some set of intuitively understood relations. In either case, the fundamental mode of reasoning encoded in a composition table is to test consistency of triads of relations of the forms $R(a, b)$, $S(b, c)$, $T(a, c)$, where $R, S, T \in \mathsf{Rels}$: such a triad is consistent if and only if $T \in CT(R, S)$.

Compositional reasoning can be generalised to the case where one composes relations which are themselves disjunctions. Here it is usually assumed that the composition of two disjunctive relations $R(a, b)$ and $S(b, c)$ is simply the disjunction of all possible compositions $R_i(a, b)$ and $S_j(b, c)$, where R_i and S_j are respectively disjuncts of R and S. Thus, the domain of the function CT can be extended to disjunctive relations as follows:

$$CT(R, S) \ =_{def} \ \bigcup_{ij} CT(R_i, S_j) \ .$$

If $R(a, b)$, $S(b, c)$ and $T(a, c)$ are disjunctive relations then by computing the generalised composition of R and S it may be found that some of the disjuncts of T are not possible. Eliminating

[1] In fact Allen called his table a 'transitivity table' but 'composition table' is arguably more appropriate and it seems that this is becoming the standard term.

such disjuncts can be regarded as a generalisation of the simple triad consistency checking procedure for non-disjunctive relations. The more general composition rule for disjunctive relations can be formally specified by the following inference rule schema:

$$\frac{R(a,b) \wedge S(b,c) \wedge T(a,c)}{(CT(R,S) \cap T)(a,c)} \text{[Comp]}$$

If $T \subseteq CT(R,S)$ no new information is generated, otherwise $T(a,c)$ is replaced by the stronger relation $(CT(R,S) \cap T)(a,c)$. If $CT(R,S)$ is disjoint from T then an inconsistency has been detected.

Repeated application of the inference rule Comp is known as 'compositional constraint propagation'. It is clear that, given any set of instances of the disjunctive relations over Rels, after repeated application of Comp, one will either generate an inconsistency or reach a state where no new information can be generated by Comp. If an inconsistency has been detected we can say that the relation set is inconsistent with respect to CT, otherwise it is consistent with respect to CT.

9.1.1 Soundness and Completeness of a Composition Table

The issue of the conditions under which a composition table can provide a complete consistency checking procedure for relational facts was raised and discussed by Bennett, Isli and Cohn (1997). The notions of soundness and completeness of a composition table appeal to some underlying theory or intuition of the meanings of the relations involved. To say that a composition table is sound is to say that, whenever a set of relations is determined by that composition table to be inconsistent, then that set of relations is indeed inconsistent with the underlying theory or intuition. Likewise, a composition table is complete (perhaps one should say 'refutation complete') if, whenever a set of relations is inconsistent with the background theory/intuitions, this can be detected by reference to the composition table.

These ideas need to be made more precise. I stipulate that:

- A composition table CT for a relation set Rels is sound w.r.t. some (possibly unformalised) theory Θ if, whenever we find among some set of instances of Rels a triad $R(a,b)$, $S(b,c)$, $T(a,c)$, such that $T \notin CT(R,S)$, then this set of instances is inconsistent with Θ.

To make the completeness property fully precise we first need the following definition: a set of relation instances is *total* if every pair of constants occurring in these instances occur together in exactly one instance — i.e. every pair of constants are uniquely related.[2] I then say that:

- A composition table CT for a relation set Rels is (refutation) complete w.r.t. some (possibly unformalised) theory Θ if, whenever some total set $\mathcal{S}$ of instances of Rels is inconsistent with Θ, we can find relations $R(a,b)$, $S(b,c)$, $T(a,c) \in \mathcal{S}$, s.t. $T \notin CT(R,S)$.

[2] If a set of relation instances is not total this means that some pair of constants are not constrained by any relation. Any pair of unconstrained constants are implicitly related by the universal relation ($\top(x,y)$). When we are dealing with a JEPD relation set Rels, the universal relation is just the disjunction of all relations in Rels. This means that a non-complete set of relation instances contains implicit disjunctive relations. The requirement that the relation set is total, can then be seen as part of the requirement that the relation set is non-disjunctive.

Suppose a composition table is sound and complete with respect to Θ for non-disjunctive relations; does this mean that by employing compositional constraint propagation (i.e. repeated application of Comp) we get a consistency checking procedure which is sound and complete (w.r.t. Θ) for disjunctive relationsΓ It is quite easy to show that compositional constraint propagation must be sound if triad consistency checking for non-disjunctive relations is sound. This is because, given any relations $R(a,b)$, $S(b,c)$ and $T(a,c)$, the rule Comp only eliminates those disjuncts of T that are inconsistent with any possible non-disjunctive strengthening of R and S. However, compositional constraint propagation is not in general complete. The problem is that although each triad of disjunctive relations between three constants may be consistent, there may be no single non-disjunctive specialisation of all the disjunctive relations such that every triad is consistent.

On the other hand if a composition table is complete for non-disjunctive relations, this does always yield a complete refutation procedure for disjunctive relations by use of a back-tracking search algorithm. Clearly a set of disjunctive relation instances is consistent just in case there is some non-disjunctive strengthening of these instances which is itself consistent. This can always be found by exhaustive search of all possible combinations of non-disjunctive specialisations of the disjunctive relations. Computationally, this method requires time which is exponential in the number of disjunctions, whereas the application of compositional constraint propagation requires only $O(n^3)$ time, where n is the number of constants occurring in the set of relations to be tested. Consequently there has been much interest in discovering specific sets of disjunctive relations for which the compositional constraint propagation method is indeed complete (Vilain and Kautz 1986, Nebel 1995a, Nebel 1995b, Renz and Nebel 1997). In the rest of this chapter I shall not be much concerned with the tractability of reasoning with disjunctive relation sets; so my attention will be largely confined to total sets of non-disjunctive relations.

9.1.2 Formal Theories and Composition Tables

In the previous section, the properties of soundness and completeness of a composition table were defined on the assumption that one has some method of testing consistency of sets of ground relations. If the basis relations are defined in some formal theory then this can be tested by means of some refutation proof procedure for the logical language in which the theory is formulated. I shall now look in more detail at how a composition table can be computed from a formal theory and what the table means in terms of the theory. We shall see that the possibility of specifying a sound and complete composition table for a set of relations, with respect to some theory, depends upon certain properties of that theory.

Although the definitions of composition table soundness and completeness in terms of consistency seem at first sight to be very straightforward, when we try to describe exactly how composition table entries should be logically deduced from a formal theory, certain difficulties arise. At the heart of these problems is the way in which the compositional properties of relations should be abstracted from properties of ground instances of these relations. Whilst compositional reasoning and its soundness and completeness are characterised in terms of ground instances, the table itself contains only relation names. Because of this, if a composition table is to be coherent, the logic of relational composition must be in some sense homogeneous with respect to the domain of individuals.

Let us assume that the essential characteristic of a composition table is its ability to discriminate between consistent and inconsistent triads of relations. This leads to the following stipulation for the composition function:

- **CTdef**: Given a theory Θ in which a set Rels of base relations is defined, the composition, $CT(R, S)$ (where $R, S \in \mathsf{Rels}$), is the set of all relations $T_i \in \mathsf{Rels}$, for which the formula $\exists x \exists y \exists z[R(x, y) \wedge S(y, z) \wedge T_i(x, z)]$ is consistent with Θ.

Here I have used existential quantification to indicate that, if the combination $R(x, y) \wedge S(y, z) \wedge T_i(x, z)$ is possible for any three individuals in the domain, then T_i must be included in the composition of R and S. This ensures soundness of the composition table since only triads that are impossible under any instantiation are ruled out by the composition table.

However, it is not at all clear that this definition gives rise to a complete composition table. One possible problem occurs if we consider a language containing constants denoting entities with special logical properties (e.g. the universal region, denoted by u in the RCC theory): if the facts $R(x, y)$ and $S(x, y)$ involve one of these constants, certain possibilities for the relation $T(x, z)$ might in this case be impossible; and, in such special cases, the compositional inference justified by the composition table would be too weak to ensure completeness. Even if our language does not contain special constants, it is still by no means obvious that compositional reasoning provides a complete refutation procedure. It may be that there are theories and relation sets for which one may have a total network of relation instances which is inconsistent even though every triad of these instances is consistent with the theory.

Nevertheless, **COMPdef** must surely be the correct definition of the CT function: any stronger definition would be unsound because it would tell us that some triad of relations is impossible when in fact there is at least one instantiation for which it is possible. Consequently we must identify conditions under which **COMPdef** yields a composition table which is complete with respect to Θ. To this end I introduce the concept of *k-compactness* applicable to a relation set relative to a theory, within which the relations are defined.[3]

> A relation set Rels is *k-compact* w.r.t. a theory Θ iff: for any total network of instances of Rels, the network is inconsistent with Θ iff it includes a sub-network of size k or less, which is inconsistent with Θ.

For some sets of relations we may find that there can be arbitrarily large inconsistent (total) networks all of whose sub-networks are consistent. We say that these are not finitely compact. If there can be an infinite inconsistent network with no finite inconsistent sub-network the relation set is not compact at all.[4]

[3] This concept was first introduced in (Bennett et al. 1997).

[4] My notion of compactness is directly analogous to that which is applied to logical languages: such a language is compact if every inconsistent set of formulae has a finite inconsistent subset.

From the definition of k-compactness it immediately follows that a composition table for a set of relations Rels can be complete only if Rels is 3-compact with respect to Θ. Furthermore, if we are concerned with a language in which all individual constants are arbitrary (i.e. we have no constants referring to particular individuals with special properties), then, if Rels is 3-compact with respect to Θ, the composition table for Rels constructed according to **COMPdef** must be complete with respect to Θ.

Not all relation sets are 3-compact: consider a theory in which individuals have the properties of equal sized discs in the plain and a set of relations including the relation of *external connection*. The theory requires that any given circle can be externally connected to a maximum of six other circles (this could be specified directly as an axiom of the theory or could be a consequence of the axiom set). Hence, a situation in which seven regions are all mutually externally connected is inconsistent; but this cannot be detected by checking any triad of relations between three regions.[5] Hence, no set of relations including a relation of external connection can be 3-compact with respect to this theory.

9.1.3 The Extensional Definition of Composition

The notion of 3-compactness yields a precise specification of what relationship is necessary between a set of relations and a theory in order that one might construct a complete composition table for that relation set. However, being stated in terms of the relationship between local and overall consistency, this specification is essentially meta-theoretic. Establishing 3-compactness will typically involve first showing that some class of models is canonical for the theory (i.e. every consistent set of relational constraints has a model in this class which is consistent with the theory); and then demonstrating (by reasoning about these models) that, if there is a model which is locally consistent with every triad of relational constraints, there must also be a model which is consistent with the whole set of constraints. Such proofs are often difficult and very much dependent on the specific relational theory under consideration. Hence, it would be very desirable to have some general criteria for 3-compactness that could be stated in terms of the theory in question. It seems plausible that one might be able to demonstrate that, given a set of relations and a theory, the relations are 3-compact with respect the theory just in case certain formulae are theorems of that theory.

A promising approach to this problem is to try to cast the requirements of 3-compactness (and hence composition table completeness) in terms of the operation of *extensional composition*, which is definable within any 1st-order theory. This operation is based on the following definition of the composition of two relations which is standard in set theory:

- **EXCOMPdef**: Let R_1 be a relation from A to B and R_2 be a relation from B to C (i.e. A, B and C are sets, $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$). Then the *composition* of R_1 with R_2, $(R_1; R_2)$ is the set of all ordered pairs, $\langle a, c\rangle \in A \times C$, such that, for some $b \in B$, $\langle a, b\rangle \in R_1$ and $\langle b, c\rangle \in R_2$.

[5] This example is described in (Cui, Cohn and Randell 1993).

In 1st-order logic the extensional composition operator can be defined by:

$$\forall x \forall y[(R;S)(x,y) \leftrightarrow \exists z[R(x,z) \wedge S(z,y)]] \qquad \textbf{(ExComp)}$$

This definition is strictly stronger than the consistency-based definition: not only does it ensure that whenever $R(a,b)$ and $S(b,c)$ hold $(R;S)(a,c)$ must also hold; it also requires that, whenever $(R;S)(a,c)$ holds, there must exist some region, say b, s.t. $R(a,b)$ and $S(b,c)$. In fact the inference from $R(a,b)$ and $S(b,c)$ to $(R;S)(a,c)$ must be the strongest compositional inference that is valid for any arbitrary constants a, b and c. Since b is arbitrary our premisses are equivalent to $\exists z[R(a,z) \wedge S(z,c)]$ and we can instantiate **ExComp** to get $(R;S)(a,c) \leftrightarrow \exists z[R(a,z) \wedge S(z,c)]$. Hence, the conclusion $(R;S)(a,c)$ is logically equivalent to the premisses and any inference to a stronger relation $T(a,c)$ would be unsound.

If a composition table CT satisfies the consistency-based definition of composition **CTdef**, it is easy to show that the extensional composition $(R;S)$ always denotes a relation whose extension is a subset of that of $CT(R,S)$. This means that for each composition table entry the following formula is provable:

$$\forall x \forall y[(R;S)(x,y) \rightarrow CT(R,S)(x,y)]$$

Unlike $CT(R,S)$, the relation $(R;S)$ need not necessarily be equivalent to some disjunction of a fixed set of base relations. If not then $CT(R,S)$ must be strictly weaker than $(R;S)$. Nevertheless, for a particular theory and set of relations, it may be that consistency-based composition coincides with the extensional definition — i.e.

$$\forall (R,S \in \mathsf{Rels})[\forall x \forall y[CT(R,S)(x,y) \leftrightarrow (R;S)(x,y)]\ ]\ .$$

Since $CT(R,S)$ is always simply a disjunction of relations taken from Rels, this formula can only be true if the set of disjunctive relations over Rels is closed under the extensional composition operator.

In (Bennett et al. 1997) it was suggested that if CT is not extensional (i.e. $CT(R,S)$ is weaker than $(R;S)$ for certain relations) then this must mean that information is lost when (consistency-based) compositions are computed *via* CT; and consequently that if consistency of a network is tested solely by propagation of constraints imposed by a non-extensional composition table we may find that it seems to be consistent when it is actually inconsistent. This conjecture is supported by the fact that $(R;S)$ gives the strongest possible compositional inference that is sound for arbitrary arguments. However, the conditions under which extensional composition provides a refutation-complete proof procedure have themselves not been established;[6] nor is it certain that there cannot be sets of relations for which a weaker form of compositional inference might be refutation-complete. Until these issues have been resolved, the connection between extensional composition and composition table completeness is not clear.

To clarify the preceding remarks it may be helpful to consider the case of the Allen relations. In his original presentation of a composition table for temporal relations Allen (1983) appears

[6] That the basic relations are JEPD and that they include equality are conditions that seem likely to be important.

to employ a consistency based interpretation of composition table entries. However, a 1st-order theory of temporal intervals was later given by Allen and Hayes (1985) and this theory justifies extensional interpretation of the Allen composition table. (Ladkin 1987) showed that these axioms are also faithful to the intended interpretation, in that their models are (isomorphic to) structures of intervals over an unbounded linear order. The 3-compactness can then be established by analysing these models in the light of Helly's theorem.[7] What is not clear is whether there is a connection between the fact that the Allen relations are 3-compact and the fact that the disjunctive Allen relations are closed under extensional composition.

9.1.4 Composition Tables and CSPs

A framework for problem solving that has received a great deal of attention from AI researchers is that of *Constraint Satisfaction Problems* (CSPs) (Mackworth 1977, Tsang 1993). A CSP consists of a set of variables and a set of constraints on possible values of these variables. These constraints can be regarded as a set of tuples of possible assignments (perhaps not explicitly given but checked on demand by some procedure) or as specified by some theory. The type of reasoning involved in solving a CSP has much in common with that employed in consistency checking by means of compositional reasoning. Although constraints of time and space permit only a very brief look at CSPs to be included in the current thesis, they may prove to be a powerful tool for spatial reasoning.

There are two ways in which the notion of a composition table can be assimilated into the framework of CSPs. One is to treat the composition table as a set of ternary constraints on variables ranging over relation names (see e.g. (Grigni, Papadias and Papadimitriou 1995)). Thus, for each (ordered) pair of objects, $\langle x, y \rangle$, the CSP has one variable, $v(x, y)$, whose domain is the set Rels. A composition table CT is then interpreted as a set of constraints which can be specified as all instances of formulae of the form

$$(v(x,y) = R \wedge v(y,z) = S) \rightarrow v(x,z) \in CT(R,S) .$$

This approach is applicable to any composition table and does not tell us anything about the relations involved.

A more illuminating approach is to regard the relations in a basis set Rels as themselves constituting the constraints of a CSP. This requires further analysis of the logical structure of the relations involved. In the case of the Allen relations, a natural interpretation is to identify the relations with order constraints on the end-points of temporal intervals and to take these end-points as elements of an ordered linear field such as the real or rational numbers (Vilain and Kautz 1986, Nebel 1995b). In sections 5.3.1 and 5.6 we saw how many topological RCC relations can be represented by equational (and disequational) constraints over interior algebras.

[7] Helly's theorem states that: the intersection of a set of convex subspaces of a space of dimension n has a non-empty intersection just in case every $n + 2$ members of that set have a non-empty intersection. Thus a set of linear intervals has a common intersection iff every subset of three intervals has a non-empty intersection. By characterising the Allen relations in terms of non-emptiness conditions one can then show that a total network of these relations is consistent iff every triad is consistent.

9.2 Composition Tables for RCC Relations

I now present composition tables for three of the most significant sets of RCC relations. These tables are constructed in accordance with the consistency-based specification of composition given by **CTdef**. Later, in section 9.2.5, I shall consider the possibility of an extensional interpretation of the RCC-8 table.

9.2.1 RCC-5

Recall that RCC-5 is the relation set {DR, PO, EQ, PP, PPi} resulting from ignoring the differences between connection and overlapping and between tangential and non-tangential parts, which are made by the RCC-8 relations. As we saw in chapter 4 each RCC-5 relation can be described by means of positive and negative Boolean equations and consequently RCC reasoning can be encoded in terms of classical model and entailment constraints within the 0-order language $\mathcal{C}^+$. In fact, given the limited expressive power of $\mathcal{C}^+$, I have not implemented a purely classical reasoner but have concentrated on reasoners for the more expressive language $\mathcal{I}^+$, which can express the more discriminating RCC-8 relation set. Hence, table 9.1 was actually obtained by merging entries in the RCC-8 composition table given in the next section. Note that the symbol $\top$ refers to the universal relation, which means that no base relation is excluded.

$R(a,b)$ \ $R(b,c)$	DR	PO	EQ	PP	PPi
DR	$\top$	DR, PO, PP	DR	DR, PO, PP	DR
PO	DR, PO, PPi	$\top$	PO	PO, PP	DR, PO, PPi
EQ	DR	PO	EQ	PP	PPi
PP	DR	DR, PO, PP	PP	PP	$\top$
PPi	DR, PO, PPi	PO, PPi	PPi	O	PPi

Table 9.1: Composition table for the RCC-5 Relations

9.2.2 RCC-8

The RCC-8 composition table was generated using the $\mathcal{I}^+$ encoding of the relations, by means of my first implementation of an optimised $\mathcal{I}$ theorem prover. The code is given in appendix C.2. The entry for relations R_1 and R_2 was computed by testing the consistency of the spatial configuration $R_1(a,b) \wedge R_2(b,c) \wedge R_i(a,c)$, where R_i is each of the RCC-8 relations. Running on a SPARC1 workstation the program generated the full *composition table* for RCC-8 in under 244 seconds.[8]

In section 9.2.4 I shall show that the RCC-8 relations are 3-compact with respect to their interpretation in the theory of interior algebras. This means that the composition table provides a refutation-complete proof procedure for sets of RCC-8 relational facts.

[8]By exploiting the results of appendix D concerning redundancy in composition tables, the table could have been computed in approximately one sixth of this time.

R2(b,c) / R1(a,b)	DC	EC	PO	TPP	NTPP	TPPi	NTPPi	EQ
DC	⊤	DR,PO,PP	DR,PO,PP	DR,PO,PP	DR,PO,PP	DC	DC	DC
EC	DR,PO,PPi	DR,PO TPP,TPi	DR,PO,PP	EC,PO,PP	PO,PP	DR	DC	EC
PO	DR,PO,PPi	DR,PO,PPi	⊤	PO,PP	PO,PP	DR,PO,PPi	DR,PO PPi	PO
TPP	DC	DR	DR,PO,PP	PP	NTPP	DR,PO TPP,TPi	DR,PO PPi	TPP
NTPP	DC	DC	DR,PO,PP	NTPP	NTPP	DR,PO,PP	⊤	NTPP
TPPi	DR,PO,PPi	EC,PO,PPi	PO,PPi	PO,TPP,TPi	PO,PP	PPi	NTPPi	TPPi
NTPPi	DR,PO,PPi	PO,PPi	PO,PPi	PO,PPi	O	NTPPi	NTPPi	NTPPi
EQ	DC	EC	PO	TPP	NTPP	TPPi	NTPPi	EQ

Table 9.2: Composition table for the RCC-8 relations

9.2.3 RCC-23

In section 8.2.1 we saw how various containment relations can be defined by means of the extended RCC theory with a convex-hull operator. In particular, the JEPD relation set RCC-23 was introduced in which the EC and DC relations of RCC-8 are further analysed in order to specify the relation holding between each region and the convex hull of the other. Table 9.3 gives the full composition table for the RCC-23 relations. If $R_1(a,b)$ and $R_2(b,c)$, where R_1 is the relation specified in the left hand column and R_2 is specified along the top, the corresponding table entry encodes the possible values of the relation $R_3(a,c)$.

Because each table entry is some subset of 23 possible base relations, there is not enough space to give the actual relation names. Hence, in order to present the table on a single page a specially concise notation was employed. Each of the 23 relations is represented by one of the two symbols '$\star$' and '$\circ$' at a certain position in a 3×4 matrix. These representations are shown in the second column. Table entries are constructed by superimposing the representations for each of the possible relations. Where '$\star$' and '$\circ$' should both be present in the same position, the symbol '$\bullet$' is used.

The table was generated using the meta-level enforcement of the conv axioms in the $\mathcal{I}^+$ representation, as described in section 8.4. Using an augmented version of the $\mathcal{I}^+$ reasoning program given in appendix C.2, the table was produced in 3h 31m on a SPARC10 workstation. It was subsequently published in (Bennett 1994b). The task of generating this table had been proposed two years earlier as a challenge for composition in (Randell, Cohn and Cui 1992a). (Cohn, Randell, Cui and Bennett 1993) contains a similar table constructed using a model building approach but it has subsequently been found that the table given there is too strict in that it rules out certain configurations, which are in fact possible for 3D spatial regions. My table has not been found to contain any false entries.

It is interesting to note that generation of this table was in fact one of the very first results on spatial reasoning that I obtained during my PhD research. The idea of the program was inspired by an account of Tarski's topological interpretation of $\mathcal{I}$ given by Mostowski (1966). After a period of intensive coding and experimentation, I found myself with a program that seemed to generate

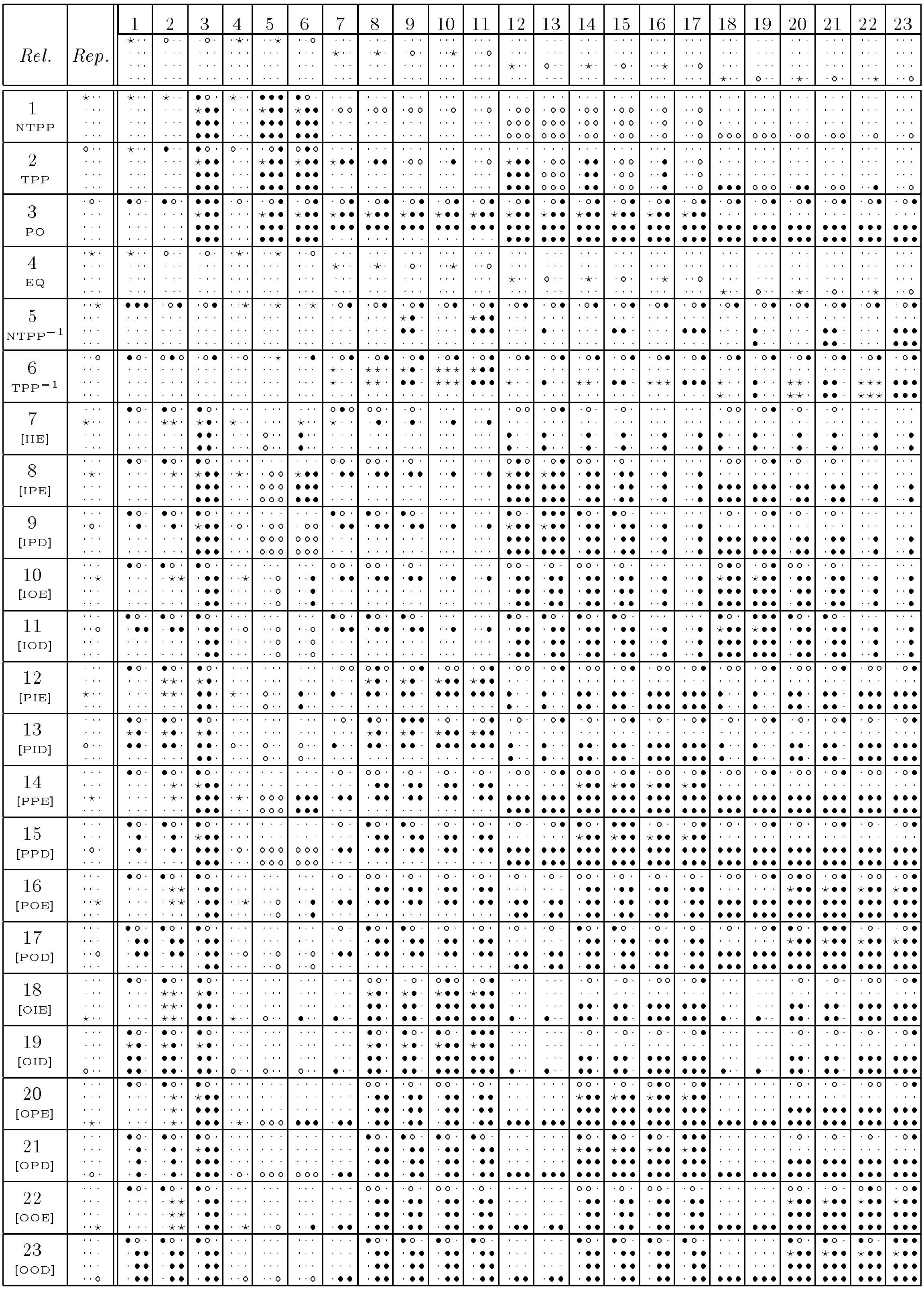

Table 9.3: Composition table for the RCC-23 relations

the correct composition table. Much of the rest of the work done during my PhD research was concerned with discovering exactly how this program worked.

9.2.4 3-Compactness of RCC-8

By analysing Nebel's classical encoding of the RCC-8 relations (described in section 6.3.7), I shall now show that the RCC-8 relations are 3-compact with respect to the consistency checking procedure provided by the $\mathcal{I}^+$ representation. Because of Tarski's topological interpretation of $\mathcal{I}$, it follows that the RCC-8 relations are 3-compact with respect to the general theory of topological spaces, within which these relations are characterised as specified in table 5.4.

Under the forcing constraint interpretation, each constant/region a is identified with three classical literals: $\mathsf{F}(v, a)$, $\mathsf{F}(w_1, a)$ and $\mathsf{F}(w_2, a)$; and each RCC-8 relation $R(a, b)$ is specified by a set of binary clauses involving the literals associated with a and b (Amongst these clauses I include those arising from the ordering condition on the worlds as well as directly from the model and entailment constraints.) For this representation it is clear that binary resolution provides a refutation complete proof procedure.

The forcing constraint clauses are consistent if and only if the $\mathcal{I}$ constraints from which they are derived are consistent; and these in turn are consistent if and only if the corresponding interior algebraic constraints are satisfiable in some topological space.[9] Thus inferences among RCC-8 relations are mirrored by logical derivations among the corresponding classical forcing constraints. Specifically, the forcing constraint clauses corresponding to the composition of two RCC-8 relations $R_1(a, b)$ and $R_2(b, c)$ are all those resolvents involving only a and c literals generated by applying binary resolution to the combined sets of forcing constraint clauses associated with the two relations. This set contains all derivable forcing constraints on a and c. Thus, an RCC-8 composition is associated with a set of binary resolutions among 2CNF clauses. Conversely, every binary resolution among forcing constraint clauses is correlated with the composition of a pair of RCC-8 relations. Because binary resolution is refutation-complete for classical clauses, it follows that an RCC-8 network can be shown to be inconsistent by means of compositional inference if and only if it is inconsistent with respect to the theory of interior algebras.

What I have just shown is not quite sufficient to conclude that the RCC-8 relation set is 3-compact with respect to the theory of interior algebras. It could be that showing inconsistency by compositional inference might require a chain of several such inferences, whereas if a relation set is 3-compact then any inconsistent network contains an inconsistent triad of relational facts, which can be detected by a single compositional inference. Happily, as I shall now show, it turns out that inconsistency of a total network of RCC-8 relations (as interpreted in interior algebra) can always be detected by a single compositional inference.

In the $\mathcal{I}$ encoding, the detection of an inconsistent triad corresponds to the discovery of two

[9] An intriguing question regarding this correspondence is whether there is an intuitive topological interpretation of the forcing constraints and the three 'worlds' associated with each region.

	$\sim y \vee \sim z$	$\sim(y \wedge z)$	$\sim y \vee z$	$\sim z \vee y$	$y \Rightarrow z$	$z \Rightarrow y$
$\sim x \vee \sim y$	$\top$	$\top$	$\top$	$\sim x \vee z$	$\top$	$\sim x \vee \sim z$
$\sim(x \wedge y)$	$\top$	$\top$	$\top$	$\sim x \vee \sim z$	$\top$	$\sim(x \wedge z)$
$\sim x \vee y$	$\sim x \vee \sim z$	$\sim x \vee z$	$\sim x \vee z$	$\top$	$\sim x \vee z$	$\top$
$\sim y \vee x$	$\top$	$\top$	$\top$	$\sim z \vee x$	$\top$	$\sim z \vee x$
$x \Rightarrow y$	$\sim x \vee \sim z$	$\sim(x \wedge z)$	$\sim x \vee z$	$\top$	$x \Rightarrow z$	$\top$
$y \Rightarrow x$	$\top$	$\top$	$\top$	$\sim z \vee x$	$\top$	$z \Rightarrow x$

Table 9.4: Compositional inferences among $\mathcal{I}$ formulae

model constraint formulae $\phi_1(x, y)$ and $\phi_2(y, z)$ that entail some entailment constraint $\psi(x, z)$.[10] $\phi_1(x, y)$ and $\phi_2(y, z)$ can each take one of seven possible forms. Table 9.4 gives, for each such combination, the strongest entailed formula involving only x and z. Where the entry is $\top$, this means that the only derivable formulae involving just x and z are theorems of $\mathcal{I}$. This can be verified by noting that for each of these combinations, either by supposing that y is a theorem of $\mathcal{I}$ or by supposing that y is inconsistent, one can derive both $\phi_1(x, y)$ and $\phi_2(y, z)$, for arbitrary instantiations of x and z. Thus, asserting $\phi_1(x, y)$ and $\phi_2(y, z)$ does not logically constrain the values of x and z. In all other table entries we see that the strongest derivable formula is itself one of the seven model constraints. So, binary composition of model constraint formulae either produces no new information or a new model constraint formula.

If one then considers the model and entailment constraints associated with each of the RCC-8 relations, one finds that each relation is 'saturated' with respect to to model constraint formulae, in the sense that each of the seven possible model constraints is either entailed by the model constraint associated with the relation or entails one of the entailment constraints associated with that relation. This means that if we add a new model constraint formula to the $\mathcal{I}$ representation of a total RCC-8 network it is either redundant or makes the network inconsistent. It follows that whenever a total RCC-8 network can be shown to be inconsistent by binary composition of $\mathcal{I}$ model constraints, this can be shown by a single application of this type of inference. Moreover, since compositional inference has been shown to be complete for testing inconsistency with respect to the interpretation in the theory of interior algebras, it must also follow that the RCC-8 relations are 3-compact with respect to this theory.

The 3-compactness of RCC-8 with respect to interior algebra can be contrasted with a result of (Grigni et al. 1995) concerning the *realisability* of a set of RCC-8 relational facts by a set of simply-connected planar regions. Drawing on results of Kratochvíl (1991) about the recognition of realisable string graphs Grigni et al. (1995) conclude that testing whether a set of such facts has a model, in which the constants refer to regions in the plane that are bounded by Jordan curves, is NP-hard. This means that the RCC-8 relations cannot be finitely compact with respect to a theory which constrains the regions in this way. Consequently, no composition table can be complete for testing consistency of RCC-8 relations in this restricted planar domain.

[10] A non-null entailment constraint $\sim x$ is equivalent to $\sim(x \wedge x)$ and can be treated as being of the form $\sim(x \wedge y)$.

9.2.5 Existential Import in RCC-8 Compositions

Examination of the composition table for RCC-8 (table 9.2.2) reveals that an extensional interpretation is not compatible with the 1st-order RCC theory. Consider the entry for $CT(\mathsf{DC}, \mathsf{DC})$, which is given as the universal relation $\top$. Interpreted extensionally this would mean that

$$\forall x \forall y [\exists z [\mathsf{DC}(x, z) \wedge \mathsf{DC}(z, y)] \leftrightarrow \top(x, y)] \ ,$$

which is equivalent to

$$\forall x \forall y \exists z [\mathsf{DC}(x, z) \wedge \mathsf{DC}(z, y)] \ .$$

This says that given any two regions, x and y, there is a region z disconnected from both of them. But this contradicts the RCC theory, which allows that the sum of x and y may be the universe, in which case no region would be disconnected from both these regions.

Another, slightly more complex, example is provided by the composition of EC and TPP, which is given as $\{\mathsf{EC}, \mathsf{PO}, \mathsf{TPP}, \mathsf{NTPP}\}$, corresponding to an extensional composition described by

$$\forall x \forall y [\ \exists z [\mathsf{EC}(x, z) \wedge \mathsf{TPP}(z, y)] \leftrightarrow$$
$$(\mathsf{EC}(x, y) \vee \mathsf{PO}(x, y) \vee \mathsf{TPP}(x, y) \vee \mathsf{NTPP}(x, y))] \ .$$

This says that whenever regions a, b are related by either of EC,PO,TPP or NTPP, there must be a third region c such that $\mathsf{EC}(a, c) \wedge \mathsf{TPP}(c, b)$. Situations satisfying these conditions are illustrated in Figure 9.1. As long as b is an ordinary bounded region, a region c satisfying the appropriate conditions can always be found. However, if a is an ordinary region and $b = \mathsf{u}$, then $\mathsf{NTPP}(a, b)$ but no region c can be found which is a TPP of b (the universe has no tangential proper parts).

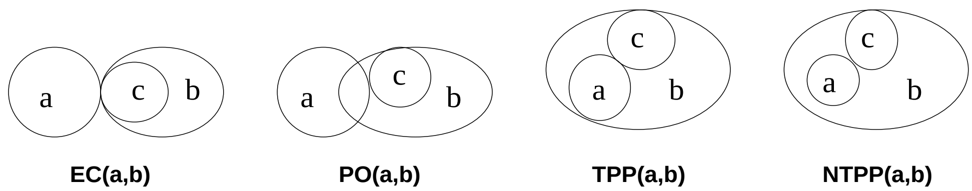

Figure 9.1: Composition of EC and TPP is not fully extensional

There are a number of ways that one might be able to avoid such problems and hence construct an extensional composition table. The most obvious is to remove the universal region u from the domain of possible referents of the region constants. All the exceptions to extensional composition that I am aware of involve u; so it seems that an extensional interpretation could be achieved with respect to a modified theory without a universal region. The domain of this new theory would then be more homogeneous and more similar to that of the Allen relations, where intervals are always bounded. Alternatively, it might be possible to retain u by refining the set of relations so as to differentiate relations involving u from those among ordinary regions. It seems plausible that by adding this additional expressive power to the base relations one could arrive at an extensional composition table. Of course the basis of the table would consist of considerably more than eight relations.

9.3 Relation Algebras

A formalism that has been valuable in the analysis of composition based reasoning algorithms for temporal relations (Ladkin and Maddux 1994) is *Relation Algebra*, in which relations are considered as elements of a Boolean algebra augmented with *composition* and *converse* operators obeying axioms first specified by Tarski (1941) and later investigated in great detail in (Tarski and Givant 1987). Although I have so far obtained only preliminary results concerning the characterisation of RCC relations within this formalism, I think it is appropriate to include these here. I believe that Relation Algebra may turn out to provide a very powerful language for automated reasoning.

A Relation Algebra is a Boolean algebra which has in addition to the usual *sum* (+), *product* (.) and *complement* ($\perp$), two additional operators: a binary *composition* operator, ';', and a unary *converse* operator, '$\smile$'. It also has constants $1'$, denoting the identity relation, and 1 the universal relation (this is not essential since it is definable by $1 = 1' + \perp 1'$). The objects of a Relation Algebra are intended to be binary relations conceived of as sets of pairs. (However, it turns out that this standard interpretation is not possible for every Relation Algebra.)

Under the intended interpretation ;, $\smile$ and $1'$ represent those operators which in a 1st-order theory of relations could be schematically defined as follows:

$$R;S(x,y) \quad \equiv_{def} \quad \exists z[R(x,z) \wedge S(z,y)]$$

$$R^{\smile}(x,y) \quad \equiv_{def} \quad R(y,x)$$

$$1'(x,y) \quad \equiv_{def} \quad (x = y)$$

But in a Relation Algebra relations are basic entities and the operators are given an algebraic characterisation so that they can be studied in a 0-order framework. Hence a Relation Algebra must obey (in addition to some axiom set characterising a Boolean algebra) the identities given in table 9.5 which fix the meanings of ';','$\smile$ and $1'$.

1.	$(x;y);z = x;(y;z)$	5.	$(x+y)^{\smile} = x^{\smile} + y^{\smile}$
2.	$(x+y);z = (x;z) + (y;z)$	6.	$(x;y)^{\smile} = y^{\smile};x^{\smile}$
3.	$x;1' = x$	7.	$x^{\smile};\perp(x;y) + \perp y = \perp y$
4.	$(x^{\smile})^{\smile} = x$		

Table 9.5: Equational axioms for a Relation Algebra

It is known that, in general, reasoning in a Relation Algebra is undecidable and this counts against the potential usefulness of these algebras in automated reasoning. However for many specific algebras, consistency checking is decidable and may even be polynomial.[11] Indeed if an extensionally interpreted composition table can be given for a vocabulary of basic relations in a Relation Algebra, this can be used to eliminate the composition operation from complex algebraic

[11] Some complexity results for reasoning with relation algebras are given by Ladkin and Maddux (1994).

terms and this will lead directly to a decision procedure. I believe that the viability of Relation Algebra as a formalism for automated reasoning deserves further exploration.

9.3.1 Defining Spatial Relations

The Relation Algebra formalism can be used to specify a spatial Relation Algebra which describes the same domain as the RCC theory. As in the 1st-order RCC theory, I start with a connectedness relation, which is axiomatised to be symmetric and reflexive. I now denote relations by the same letters as their RCC counterparts, but in lower case. Being symmetric and reflexive, c must obey the axioms

$$c^{\smile} = c \quad \text{(symmetry)} \qquad \text{and} \qquad c + 1' = c \quad \text{(reflexivity)} \ .$$

In terms of c one can easily define some of the more significant relations found in the RCC theory:

$$\begin{array}{ll} p =_{def} \perp(c; \perp c) & o =_{def} p^{\smile}; p \\ pp =_{def} p \cdot \perp 1' & tp =_{def} p \cdot (c; \perp o) \end{array}$$

In fact, making use of the relations just defined, we can go on to define all the RCC-8 relations as follows:

$$\begin{array}{ll} dc =_{def} \perp c & ntpp =_{def} pp \cdot \perp tp \\ ec =_{def} c \cdot \perp o & tppi =_{def} tpp^{\smile} \\ po =_{def} o \cdot \perp p \cdot \perp(p^{\smile}) & ntppi =_{def} ntpp^{\smile} \\ tpp =_{def} pp \cdot tp & eq =_{def} 1' \end{array}$$

It appears that many (if not all) relations definable in the RCC theory can be defined as relation algebraic expressions formed from the single primitive relation c. The resulting algebra can be obtained by factoring, with respect to the symmetry and reflexivity identities, the free relation algebra generated by a single relation. However, it is likely that additional axioms would be needed to capture adequately the existential properties of the domain of spatial regions. For example, if there is a universal region which connects with every region in the domain then the identity $c; c = 1$ must hold.

Chapter 10

Further Work and Conclusions

In this final chapter I summarise the main results of the thesis and point to areas that would benefit from further work. I also look at how logical spatial reasoning techniques fit into the wider context of AI and computer science in general.

10.1 What has been Achieved

In the course of this thesis a large number of possible spatial representations have been considered. The introductory chapter gave an overview of the origins and developments of various approaches to reasoning with spatial information. Chapter 2 surveyed some of the more important axiomatic theories of spatial regions, including point-set topology, Leśniewski's Mereology, Tarski's Geometry of solids, Clarke's theory of spatial regions and the RCC theory. In chapter 3 the RCC theory of spatial regions was examined in some detail and a number of modifications were suggested. The key meta-theoretic properties of completeness, categoricity, decidability were also considered. Chapters 4–6 developed a new approach to qualitative reasoning based on encodings of spatial concepts into 0-order logics. Because they are decidable, these representations are much better suited to computational applications than 1st-order formalisms. The next two chapters described different ways in which the expressive power of the 0-order representations might be extended: in chapter 7, I examined the use of quantifier elimination in RCC-like 1st-order spatial theories and showed that there are many classes of quantified expression whose quantifiers can be eliminated by syntactic transformation to logically equivalent quantifier-free forms; chapter 8 was concerned with extending the expressive power of 0-order representations beyond purely topological properties. Finally, in chapter 9, I examined the application of compositional reasoning to spatial relationships.

I hope that, from amongst the plethora of representational formalisms and the variety of reasoning methods that have been considered, certain general principles have emerged. Primary among these is the trade-off between expressive power and tractability, which confronts the attempt to turn theory into practice in all areas of AI. Whilst the intractability of reasoning within a given formal language is essentially indefeasible, I think that the findings of this thesis illustrate fruitful ways in which it can be circumvented. The key observation is that a language L within which a set of concepts C are easily expressed is not necessarily a good language for reasoning about those concepts. Indeed if L is highly expressive then it will be able to express logical connections in all manner of conceptual domains; but this generality means that L is over-expressive with respect to the problem of reasoning about the concepts in C. To achieve computational tractability, what we must look for is the minimally complex language capable of expressing the concepts of C — i.e. L should have just enough expressive power and no more. Consequently, the encoding of the logic of the concepts C into a tractable language L may be complex and indirect: capturing these concepts stretches the language to its limits.

In applying this principle of minimality, a wide variety of possible logical representations should be considered. In traditional logic and also in knowledge representation within the field of AI a fairly limited range of formalisms have been employed. Specifically, the range of available languages has often been seen as being restricted to 0-order propositional logic, 1st-order predicate logic (possibly with some limitations on the syntactic forms which can be employed) and higher-order logics. Since propositional logic is extremely limited in expressive power and logics of 2nd or higher order do not have complete proof procedures, some form of 1st-order logic has been the favourite language for representing factual information and expressing logical connections between concepts. The use of more expressive forms of 0-order logic has generally been confined to the characterisation of propositional modifiers (such as necessity and belief) by means of modal operators. Perhaps the most novel aspect of the work reported in this thesis is the use of these more expressive 0-order formalisms to capture the logic of purely extensional relational expressions. The use of modal and intuitionistic logic for representing spatial relations illustrates new potential uses in knowledge representation of logics whose expressive power is intermediate between the simple Boolean 0-order logic and quantificational logics. These augmented 0-order logics may prove to be applicable in many other conceptual domains.

The encoding of topological relations into $\mathcal{I}$ provides further support for the idea that, if effective reasoning is to be achieved, expressive power should be limited as much as possible, even at the expense of making the representation less natural. As I explained in section 6.1.1, the logic $\mathcal{I}$ can be regarded as an alternative syntax for a certain sub-language of $S4$. While the restricted expressivity of $\mathcal{I}$ means that more indirect encoding of topological constraints is required than with $S4$, this is compensated by $\mathcal{I}$ being better suited to automated reasoning.

10.2 Further Work

This study has sought to identify advantages and disadvantages of different possible representations of spatial information and to clarify the relationships between these different formalisms. However, many aspects of these theories still remain unclear. In this section I highlight a number of areas which I believe are particularly deserving of further research. Some of these are important because they concern the foundations of spatial reasoning, whilst others are areas which may lead to the further development of the theory in new directions.

10.2.1 Complete Spatial Theories

In section 3.6 I considered the possibility of constructing a complete and categorical theory having the same vocabulary as RCC. We saw that the undecidability of Grzegorczyk (1951) means that no complete finitary 1st-order theory of this kind can be specified. Nevertheless, for the sake of providing a theoretical foundation for spatial reasoning, a complete RCC-like theory is certainly desirable, even if formulated in a system such as 2nd-order or infinitary 1st-order logic for which a complete proof system cannot be specified.

As I was coming to the end of my work on this thesis a complete region-based spatial theory was indeed established by Pratt and Schoop (1997) using an infinitary extension of 1st-order logic. This theory is called '$' and is formulated for a language containing a monadic predicate of connectedness together with Boolean operations. $ consists of a set of 1st-order axioms and an infinitary inference rule[1] and is shown to be complete with respect to an interpretation in which the domain of regions consists of all those regular open regions of the Cartesian plane ($\Re^2$) that can be bounded by some finite number of linear edges. The restriction to linear bounded regions is inessential, since every configuration of regular open planar regions is topologically equivalent to some configuration in which all the edges are linear.

The vocabularies of RCC and $ are inter-definable, so the $ axiom set could be used to specify a version of RCC which is complete with respect to a natural interpretation in 2D space. However, the infinitary nature of $ means that it cannot be used as a practical tool for carrying out spatial inferences. The question also remains as to what axioms are needed to specify a theory which is complete with respect to a 3D interpretation.

10.2.2 Effective Modal and Intuitionistic Reasoning

Whilst modal representations of spatial relations can be shown to have a theoretical advantage over 1st-order representations (namely that decision procedures are known for the modal languages), nevertheless doubts may remain as to whether the modal representations could ever be of practical use. After all a decision procedure does not necessarily provide us with an *effective* means of computation. Ideally we would like to have polynomial algorithms for spatial reasoning. Recently, a lot of research has been directed towards the need for more efficient modal reasoning systems (Wallen 1990, Auffray, Enjalbert and Herbrard 1990, Catach 1991, Demri 1994, Giunchiglia and Sebastiani 1996, Nonnengart 1996, Balbiani and Demri 1997, Montanari and Policriti 1997, Hustadt and Schmidt 1997). If the modal approach to qualitative reasoning is to be of practical use it will be necessary to demonstrate that the modal representations can be effectively manipulated. One way to do this would be to identify tractable sub-languages of modal calculi which are capable of representing significant sets of spatial relations.

[1]This rule states that, if it can be shown for all n that every region which is a sum of n connected components has the property ϕ, then one can infer $\forall x[\phi(x)]$.

10.2.3 Extending Expressive Power

Another important direction for further work is to investigate how the expressive power of decidable spatial representations such as $\mathcal{I}^+$ can be extended. The main focus of my work has been on topological relations; but a fully expressive language for qualitative spatial information would be able to describe a wide range of non-topological properties and relations. For many purposes one would wish to characterise the relative position and orientation of regions or points (Freksa 1992b). It would also be very useful (e.g. for the task of object recognition) to have a richer vocabulary for distinguishing different shapes.

In chapter 8 I showed how the expressive power of the RCC language can be greatly increased by means of an additional conv function, giving the convex-hull of any region. This enables many useful relations concerning containment to be defined. Cohn (1995) has shown that, within this augmented RCC theory, a large class of shapes can be specified by defining shape concepts in a hierarchical manner; and the expressiveness and complexity of the language consisting of the RCC-8 relations and a convexity predicate is explored in detail in (Davis et al. 1997). However, properties involving orientation cannot be expressed within such a language. A logical treatment of orientation, convexity and related properties, in terms of points, has been given by Knuth (1992). This is based on a primitive ternary relation asserting that three points lie in an anti-clockwise orientation in the plane. There seems no reason why a similar predicate operating on regions could not be introduced into RCC. Appropriate axioms determining the logical properties of the new primitive would then have to be specified.

In section 8.5 I showed how properties of the convex-hull operator can be captured by means of modal schemata. It is possible that a similar technique could be applied to other spatial concepts. Indeed, Balbiani, Fariñas del Cerro, Tinchev and Vakarelov (1997) have shown that modal logics can be interpreted as specifying configurations in incidence geometry. My method of specifying properties of spatially interpreted modalities in terms of axiom schemata is somewhat *ad hoc* and does not provide a direct interpretation of the operator, in terms of model structures. To do this we would need richer mathematical structures as models. An obvious choice would be to use metrical Cartesian spaces. These are canonical models for Euclidean geometry and so provide an interpretation for any figure or property describable in this geometry. Having metrical spaces as models for qualitative languages also facilitates easy integration with quantitative information, as will be discussed in section 10.3.4.

Although the combination of topological concepts and convexity provides a very powerful spatial description language, the effective reasoning procedures that I have so far constructed only cover a small fragment of the properties and relations that can be expressed in terms of these concepts. Hence, the most useful further work will perhaps be directed towards expanding the range of information that can be handled by effective decision procedures, rather than the expressive power of spatial representations. (After all, highly expressive but intractable mathematical notations already exist.) Specifically, it is probable that there are effective algorithms for $\mathcal{I}$ reasoning that can deal with much larger classes of formulae than the restricted class needed to represent the RCC-8 relations. For instance, it would be useful to specify topological constraints involving Boolean combinations of regions, such as $\mathsf{DC}(x, \mathsf{prod}(y, z))$, corresponding to the $\mathcal{I}^+$ model constraint $\sim x \vee \sim(x \wedge z)$. An alternative means of increasing the expressive power of decidable systems is by means of quantifier elimination procedures such as the one given in chapter 7, which is by no means as general as it could be.

The next two sections focus on two aspects of extending expressive power that I consider to be of particular importance.

10.2.4 Reasoning with One-Piece and other Simplicity Constraints

An extremely important topological property of regions is that of being *self-connected* or *in one piece*. This property can be quite easily defined within the RCC theory as follows.

$$\mathsf{OP}(x) \equiv_{def} \forall y \forall z [x = \mathsf{sum}(y, z) \rightarrow \mathsf{C}(y, z)]$$

This property is particularly significant because the region occupied by what we think of as a 'physical body' is almost always in one piece; accordingly, in natural language descriptions of physical situations, implicit one-piece constraints are ubiquitous. In fact the objects of natural discourse are typically further constrained to be regular (i.e. of uniform dimension) and 'firmly' self-connected (a three dimensional physical object cannot be divided into two parts that are connected only at a point or along a line).

A serious deficiency with the reasoning systems described in this thesis is that I have not provided any means for reasoning under constraints specifying that certain regions are in one piece (let alone more subtle simplicity constraints). Handling such properties is an important goal for future research. It might be possible to apply a technique similar to that used to enforce the convexity of regions. In this case, if region a is supposed to be one-piece, we would check all pairs, b, c, of regions involved in the situation to see if $a = \mathsf{sum}(b, c)$ could be proved. If so the further condition $\mathsf{C}(b, c)$ must be added to the situation description. Whatever approach is taken, it is likely (contrary to what some might suppose) that the presence of simplicity constraints will make reasoning intrinsically more difficult (more will be said about this at the end of the next section).

10.2.5 Points and Dimensionality

In this thesis I have been primarily concerned with spatial relationships that can hold between *regions*. This restriction was motivated in the introduction by the observation that most 'natural' forms of description make reference to objects which occupy three-dimensional volumes (or, less commonly, two-dimensional areas). One can then argue that, although higher-dimensional objects can be constructed set-theoretically from points, it is much preferable from a computational point of view to formulate theories in which regions are basic entities (i.e. constitute a domain over which one can apply strictly 1st-order quantification) rather than to employ the highly complex language of set theory. Nevertheless, there is also strong evidence that many natural forms of expression do refer to point-like or linear entities; and it is clear that we also distinguish between two-dimensional regions on a surface and three dimensional volumes.

In chapter 2 we saw how Tarski and Clarke took regions as the basic entities of their theories but then introduced points set-theoretically as corresponding to certain sets of regions. For Tarski this was a means of rendering his theory categorical by constraining it to obey the axioms of elementary point geometry. Clarke's intention in introducing points is to show that his theory can encompass classical geometrical and topological concepts by the use of second order definitions. Neither of these treatments of points addresses the issue of how to construct a naturalistic logical formalism capable of expressing information about spatial entities of different dimensions. Preferably one would like to have a system which allowed this information to be represented without the use of second-order operators. This is particularly important for computational applications, since higher order formalisms are typically intractable and often not completely axiomatisable.

The RCC formalism does not place constraints on the dimensionality of regions except that (because of the existence of a non-tangential proper part of every region) all the regions in the domain must have the same dimensionality. For certain applications this will constitute a severe limitation in expressive power. A formalism having much in common with RCC but capable of expressing relations between entities of different dimensions has been given in (Gotts 1996). This 'INCH' calculus is based on the primitive $\mathsf{INCH}(x, y)$ read as 'x includes a chunk of y', meaning that the region x overlaps with some part of y which is of the maximum dimension of any part of y. In terms of just this primitive, predicates identifying regions of any finite dimensionality can be defined.

Handling properties involving dimensionality also presents major problems for automating spatial inferences. The reasoning algorithms described by me in chapters 4–6 only enforce entailments which hold in a very large class of topological spaces; and the same is true of reasoning using composition tables, as described in chapter 9. I mentioned at the end of section 9.2.4 that if we restrict the domain of regions involved in a set RCC-8 relational facts to planar regions bounded by Jordan curves, then testing consistency of these facts becomes NP-hard (Grigni et al. 1995). Solving this problem also involves enforcing the simplicity constraints mentioned in the previous section.

10.2.6 The Relation Between Logic and Algebra

The investigation carried out in this thesis was conducted primarily from the point of view of logical analysis. That is, my principal interest was in entailment relationships and inference rules involving formal expressions. However, in carrying out this analysis, algebraic structures have played a key role. In my encodings of spatial relationships into 0-order logics, equational algebraic theories acted as an intermediary between relational formalisms and 0-order formulae, and hence enabled me to show the correctness of these encodings. An alternative approach would be to start by adopting equational reasoning as a framework for computational inference and then look at what spatial theories could be expressed equationally.

Equational reasoning is a large research area in itself and its methods cannot be covered here. A collection of papers on many aspects of the area can be found in (Aït-Kaci and Nivat 1989), in which one chapter (Fearnley-Sander 1989) describes an interesting equational representation of spatial information based on vector spaces (this is very different to my closure algebraic treatment). The primary difference between equation-centred approaches and mine is in the formalism that is actually used for reasoning. I have suggested that 0-order logics should be used; but there may also be good reasons why it would be better to use equational reasoning.

Another issue concerning the relation between logic and algebra is that of notation. Although algebraic structures often occur as models for logical languages, there does not seem to be any standard way of stating correspondences between logical and algebraic properties; and I found considerable difficulty in arriving at a way of expressing the correlation theorems needed to justify my 0-order encodings. The general framework of *category theory* is well suited to describing relationships between different mathematical structures and may prove useful for this; but for further study of the connection between algebraic constraints and logical entailment, more specialised notation would be desirable.

10.2.7 Compositional Reasoning and Relation Algebra

In the last chapter I looked at the use of composition tables for consistency checking of sets of binary relations. We saw that compositional constraint propagation using such a table provides a consistency checking procedure that runs in $O(n^3)$ time, for a set of relational facts containing n constants. However, whether this procedure is complete depends on the particular relation set and the background theory with respect to which they are interpreted. Given the effectiveness of compositional reasoning, determining the conditions under which a composition table can be complete with respect to a theory (in the sense specified in section 9.1.1) is likely to be a fruitful area for further enquiry.

The formalism of Relation Algebra (briefly investigated in section 9.3) also deserves further study and may prove to be well suited to automated reasoning. Relation Algebra provides an extremely expressive alternative to 1st-order logic (it has almost the same expressive power (Tarski and Givant 1987)), particularly in formalising theories where binary relations play an important role.

10.3 Spatial Reasoning in a More General Framework

In this thesis I have treated spatial relationships as an isolated domain of information. However, if one wishes to develop a more general reasoning system, capable of processing the diverse kinds of information that humans routinely deal with, one must find some means by which purely spatial concepts and reasoning mechanisms can be interfaced or combined with representations and reasoning mechanisms for non-spatial concepts.

10.3.1 A General Theory of the Physical World

A tenet of AI research into Knowledge Representation is that, if an artificial agent is to act in an intelligent way to accomplish goals in the real world (or in some virtual simulation of the real world), it must have both factual knowledge about the actual state of the world and theoretical knowledge concerning possible states of the world and possible ways that one state can succeed another (Hayes 1979, Hayes 1985b, Guha and Lenat 1990). From the point of view of so-called 'symbolic' AI these laws of possibility and causality will constitute a formal theory of physical processes. This theory might be akin to those that have been established by physicists, except that, whereas the physicist is primarily concerned with the descriptive power and predictive accuracy of his theory, the computer scientist must also consider computational properties of the theory, such as the kinds of inference that can be effectively computed.

It has been suggested that, for the purposes of AI, what is needed is not a fully scientific theory of the physical world, but rather a *naïve* theory of those physical concepts which are relevant to 'commonsense' reasoning about the world (Hayes 1979, Hayes 1985a, Hayes 1985b, Randell, Cohn and Cui 1992b, Egenhofer and Mark 1995). But it is clear that, whatever style of theory is required, it must contain a sub-theory of spatial concepts. Just as a spatio-temporal geometry describes the underlying theory of coordinate systems upon which mathematical theories of physical processes are built, representations of spatial and temporal concepts must be fundamental to any formal description of these processes, which might be employed in AI. Formalisms for temporal reasoning have received a huge amount of attention from the AI community (see e.g. (Galton 1987)) and representations of spatial concepts are increasingly being studied. However, establishing a foundation for the specification of physical theories will require integrated representations and reasoning mechanisms capable of handling integrated spatial and temporal information and the construction of a suitable combined spatio-temporal theory poses formidable problems. Some of the more concrete proposals can be found in (Randell and Cohn 1989), (Randell, Cui and Cohn 1992), (Galton 1993) and (Galton 1997). I shall give some details of these proposals in the next section.

A suitable spatio-temporal theory ought to provide a framework within which theories of matter, kinematics and dynamics can be developed in such a way that these theories can be used to reason about descriptions of physical processes in a way which is amenable to effective automated reasoning. The theory of matter, whilst one of the principal focusses of physicists, has received comparatively little attention from logicians and AI researchers (a notable exception is Hayes' (1985a) analysis of the ontology of liquids). For instance, in formalising problems of robot motion planning it has generally been assumed that space can be neatly divided into two partitions: occupied space and empty space. This is clearly a very coarse approximation to the real nature and distribution of matter in the universe.

Quite a large body of work exists on qualitative kinematics and dynamics for AI (see e.g. (Weld and De Kleer 1990)). Nearly all this work is based upon some kind of abstraction of the spatio-temporal behaviour of a system into a sequence of transitions within a discrete space of possible states (state transitions among RCC relations will be considered in the next section). Given that a sufficiently expressive and computationally tractable representation for spatial-temporal information has not yet been discovered, this approach is certainly well justified. An alternative approach has been to develop formal theories in which actions, events or processes are proper entities constrained by temporal relationships (e.g. (Allen 1984)). In both these approaches the structure of space itself seems to all but disappear once phenomena are formally analysed. This lack of expressiveness in respect of spatial relationships seems to me to be an inherent weakness of most existing formalisms for describing physical processes.

10.3.2 Spatial Information and Change

One way of building a dynamical theory on top of a spatial theory is by specifying possible transitions among relations holding between the regions occupied by two bodies when the bodies undergo continuous displacement and/or deformation. Figure 10.1, taken from (Randell, Cui and Cohn 1992), shows a graph of possible transitions among the RCC-8 relations resulting from either continuous displacements or deformations of the regions involved. Transitions between qualitative spatial states have been studied in a number of papers by Antony Galton (1995, 1997).

Connected sub-graphs of a transition network are known as *conceptual neighbourhoods*, a term that was introduced in Freksa's (1992a) analysis of the Allen relations. Freksa noticed that all the entries in the composition table for the Allen relations correspond to conceptual neighbourhoods. The relationship between conceptual neighbourhoods and relational composition was also studied by me in (Bennett 1994a), where I showed that the correlation observed by Freksa does not apply to all sets of spatial relations.

An alternative method of accommodating change into a spatial representation is to introduce time as an extra dimension. 4-dimensional regions would then correspond to the space-time extensions of 3-dimensional objects throughout their history. This approach was adopted in (Randell and Cohn 1989) in which a theory of topological relations between spatio-temporal regions was augmented with a relation $\mathsf{B}(x, y)$ asserting that the spatio-temporal region x (wholly) temporally precedes region y.

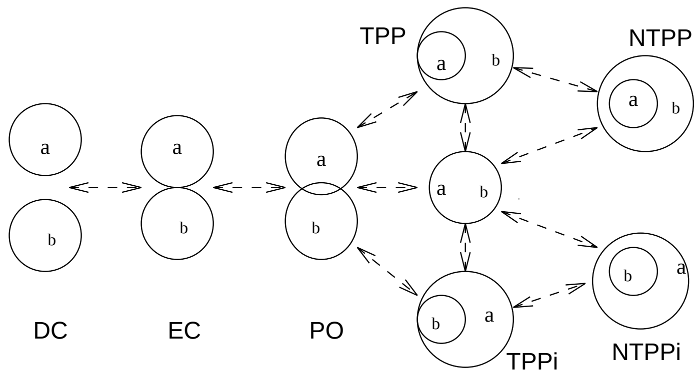

Figure 10.1: Transition network for eight topological relations

10.3.3 Vague and Uncertain Information

In real applications of reasoning systems, situations will often arise where information is vague or has some degree of uncertainty.[2] Ideally we would like a computer system to 'do its best' (whatever that means) even with vague or uncertain information. It is likely that an understanding of spatial vagueness will be very important in the development of many applications. Qualitative representations, such as the RCC language, have an intrinsic advantage over numerical representations when it comes to dealing with vague or uncertain facts: relevant qualitative distinctions can be made without any commitment to the precise details of a situation. For example, we may not know the exact geometry of a room, nor the exact size and position of a table situated somewhere in the room; however we can be certain that the table does not *overlap* the walls of the room. Using RCC we could simply assert something like '$\neg\mathsf{O}(\mathsf{table}, \mathsf{walls})$', whereas, in terms of numerical coordinates, stating this fact would require a complex and clumsy set of inequalities.

Although certain aspects of vagueness and uncertainty can be straightforwardly captured by the generality encapsulated in qualitative concepts, other aspects are not so easily represented. Certain types of region (e.g. a swamp or a cloud) have inherently vague boundaries and hence a sharp distinction between the topological relations holding among such regions cannot be made. An axiomatic theory which generalises RCC to take account of regions with vague boundaries has been developed in (Cohn and Gotts 1994a, Cohn and Gotts 1994b, Gotts and Cohn 1995, Cohn and Gotts 1996).

10.3.4 Relating Qualitative and Metric Representations

There has been a tendency among some researchers in the field of QSR to eschew metrical data, in the belief that significant AI tasks can be performed using only qualitative information. While in certain cases this may be possible, I believe that, in the majority of practical applications, one will want to combine both quantitative and qualitative information; and consequently, the interface between the two types of data will be increasingly studied.

Purely qualitative spatial reasoning systems provide an inference mechanism for determining whether a given qualitative fact follows from some set of such facts. Such systems can be used to answer queries relative to a qualitative database. However, a qualitative spatial reasoning system need not be employed in isolation from coordinate-based geometrical information and other kinds of numerical data. Indeed it is clear that for many useful functions, numerical information is essential. For instance, we may want to pose a query using qualitative concepts but requiring a quantitative answer (e.g. 'What is the area of the largest desert that lies entirely within the borders of one country?'). Moreover, the combination of qualitative and quantitative representations promises to be a powerful tool in system design and to enable novel program functionality. In the rest of this section I shall sketch a number of ways in which qualitative and metrical data could be combined.

In the introduction to this thesis I observed that current computer systems represent spatial

[2] Although vagueness and uncertainty have some logical properties in common, it is important to recognise them as very different phenomena. However, in the present brief discussion the differences are not important.

information almost entirely in terms of numerical coordinates. However, a high proportion of tests made on this data (e.g. in conditional statements of the form '`if` *test* `then` *command*'), although formulated in numerical terms, are actually designed to test qualitative relations between data objects. For example, we may wish to test whether two line segments cross. This is a qualitative relationship between the segments. To determine whether a qualitative relationship such as this holds between entities an algorithm is needed which will operate on numerical data-structures so as to extract the required information. In many cases — including the case of the crossing line segments – this can be achieved by formulating the relationship in terms of a Boolean combination of equalities and inequalities involving the coordinates of points; in other cases more complex iterative routines will be required.

Whilst it may be possible, on a case-by-case basis to devise an algorithm to extract specific qualitative information, when needed, from quantitative data-structures, it would be far preferable to have a general purpose method of testing all qualitative relationships which one may encounter. A qualitative representation, whose interpretation is linked in a precise way to the content of quantitative data-structures can go some way towards providing this capability. The idea is to associate the primitives of the qualitative representation with appropriate algorithmic operations on quantitative data. Given this interpretation of the primitives, any complex expression in the qualitative language would then be evaluated by combining these primitive operations in accordance with the semantics of logical operators in the representation. Constructing this evaluation mechanism may be very difficult (or even impossible) depending on the nature of the primitives and the logical operations involved; but once achieved it provides a general purpose procedure for evaluating a large (probably infinite) class of qualitative expressions. The qualitative representation can thus function directly as a query language as well as being used internally for program control.

A limited version of this approach is already found in nearly all computer programs. Whenever one defines some basic qualitative tests (as functions returning Boolean values) and then uses Boolean combinations of these tests in conditional statements, a simple qualitative language is in operation. To move from this limited capability to the use of a fully-fledged qualitative representation, one must identify a vocabulary of primitives and logical operators sufficient to represent any qualitative fact in some particular conceptual domain. The problem for the programmer then, rather than being 'how can I code an algorithm to test whether this relationship holdsΓ', becomes 'how can I express this relationship in terms of my qualitative languageΓ'. This architecture has the advantage that the evaluation of qualitative tests is independent of the particular data structures used to store quantitative data in the system, except in so far as operations corresponding to the primitives must be coded.

The main obstacle to achieving this kind of qualitative/quantitative interface is that, as we have seen, even modest logical vocabulary can give rise to a language which is highly intractable. In particular, a language which allows quantification over some potentially infinite domain of entities (i.e. a 1st-order language) will be undecidable unless, by taking account of the meanings of the specific vocabulary of the language, some special-purpose decision procedure can be devised. The 0-order representations of spatial relations developed in this thesis go some way towards solving this problem by providing quantifier representations capable of expressing a significant vocabulary of spatial relations.

Even where a decision procedure can be found it may be that the time taken to evaluate a qualitative test increases exponentially with the amount of information which has to be taken into account. This will make the language unsuitable for representing large amounts of data. However, in many cases it may be safe to assume that although, the database of quantitative spatial information may be very large, the qualitative tests/queries that the system will be required to evaluate will be comparatively concise. The time taken to evaluate a qualitative query will be a function of both the amount of quantitative information stored and the complexity of the query. Retrieving information from the quantitative database will typically take time which increases only polynomially in the size of the database (in most cases retrieval times will increase linearly or as some small power of the database size). Thus, even if the query-answer-time increases exponentially with query complexity this may be acceptable as long as all queries have complexity below a certain level. Also, on receiving a qualitative query it would be possible for the system to estimate the maximum time required to return an answer.

A useful generalisation of the capability of answering qualitative queries with respect to a metrical database, is the ability to generate a qualitative description from such a database. A simple example is that one may have a database consisting of a set of polygons, each corresponding to some geographical region, and from this one might wish to extract a complete description of the relationships between these regions in terms of the RCC-8 relation set — i.e. generate a set of facts in which each pair of regions is related by one of the RCC-8 relations. Having extracted a qualitative description from a quantitative database one could then combine this with additional purely qualitative information. Based on this idea a sophisticated and flexible architecture can be envisaged, in which quantitative data can be transparently combined when required with qualitative data in order to allow queries to be addressed to a hybrid information source containing both quantitative and qualitative data.

Yet another useful capability would be to generate numerical coordinate data satisfying a given set of spatial constraints. Thus, for example, one might wish to generate a possible quantitative specification for a mechanical component having certain prescribed qualitative properties. Perhaps, this could be done by means of some model-building automated theorem prover. An obvious difficulty is that there is usually no unique quantitative state satisfying given qualitative constraints; many solutions may be unnecessarily complex or deviate in subtle ways from what was really wanted, so it may be hard to pick a 'sensible' solution.

In section 10.2.3 I suggested that interpreting qualitative languages in terms of metrical models might be a way to develop more expressive languages. Clearly this would also be very useful for integrating qualitative and metrical information. The field of QSR has tended to eschew metrical models on the naïve assumption that such models are only appropriate for quantitative representations. But this is to misunderstand the relationship between a logical language and its models. Formal languages cannot ordinarily fully describe their own models: the fact that a model satisfies a given formal sentence is a matter of meta-logic. Nor does the ontological commitment of a formal language depend upon its models; but rather on its resources for asserting what exists and what does not (e.g. existential quantification) and the concomitant existential import of its theorems. Hence, there is no reason why qualitative languages should not have metrical models. Indeed, canonical metrical models arise naturally when the axioms of a theory enforce seemingly qualitative constraints which impose order on the domain of individuals (this is illustrated by the theory of Allen's interval relations (Allen and Hayes 1985, Ladkin 1987) — see the end of section 9.1.3 — and the spatial theory of Pratt and Schoop (1997) — see section 10.2.1).

10.4 Applications

Although this thesis has focused on devising spatial reasoning algorithms that can be effectively implemented, concrete applications have not been considered. In the introduction I observed that spatial information was of key importance to many areas of computer science, including such central fields of AI as computer vision and robotics. However, logical reasoning with formal languages has not become an established technique in any of these areas. It is therefore incumbent upon those developing QSR algorithms to indicate how these might be exploited to solve problems in the more pragmatic branches of computer science which are concerned with processing spatial data.

I shall first consider the possibility of applying QSR to robotics. The classical approach to robot control a robot is to compute precise movement instructions to achieve a desired goal (Schwartz and Sharir 1990, Latombe 1991). Whilst these instructions are predominantly metrical, the goal itself will typically correspond to a high-level qualitative prescription of an action (e.g. 'Put the box into the skip'). Computing the metrical instructions to achieve this goal can be seen as a generalisation of the problem of finding a spatial region satisfying given qualitative spatial constraints, which was mentioned at the end of the last section. But, in the case of a robotic goal, the constraints may not be purely spatial and one must generate a spatio-temporal movement path rather than simply a spatial region. One approach to this problem is to translate constraints into a numerical form and then use purely numerical constraint solving techniques (Schwartz and Sharir 1983, Arnon 1988). This can only be done effectively for fairly simple motions, so where more complex motions are required, planning techniques are often used to find a sequence of simpler subgoals which achieves the desired ultimate goal (Lozano-Pérez 1987, Schwartz, Sharir and Hopcroft 1987, del Pobil and Serna 1995).

Computing motion-plans is perhaps the aspect of robotics that is most likely to benefit from QSR techniques. Given a qualitative representation of initial and goal states, and a background theory of possible state changes, a qualitative movement plan can in principle be computed by abductive inference (Eshghi 1988, Shanahan 1991, Denecker, Missiaen and Bruynooghe 1992). Spatial concepts will play a very significant role in both the state descriptions and the background theory. However, adequate specification of robot states and goals will also require concepts for describing temporal relationships, material properties and perhaps abstract entities such as actions. Thus the reasoning problem is far from purely spatial. One might hope to be able to solve the problem in a much more general theory of physical situations and processes, as envisaged in section 10.3.1. However, it is doubtful whether a sufficiently general theory, which is also tractable, can be developed in the near future. In order to make use of purely spatial inference mechanisms one would have to factor out spatial aspects of the reasoning problem and show how these can be handled as a modular component of motion-planning computations. In my view, this is a much more realistic approach.

Applications to reasoning about physical systems face many of the same problems as arise in robotics. In fact, a robot can be seen as a rather simple example of a physical system, with a limited number of degrees of freedom. As noted above (section 10.3.1) adequate theories of physical processes will probably need to incorporate a very rich conceptual vocabulary. Hence, if qualitative spatial inferences are to be exploited the need for modularisation of reasoning problems is even more acute.

A task which is part of robot motion planning but is also useful for many other applications (e.g. route-finding aids for motor-vehicle drivers) is *navigation*. Here we are not concerned with the detailed mechanics of movement but with somewhat more abstract problems, such as finding a viable path between two spatial locations; for this purpose, the moving object can normally be considered to be a point rather than an extended body; and the required path can be represented by a line rather than a sequence of complex movements. Navigation problems are more purely spatial than robotic automation and consequently spatial reasoning techniques are easier to apply. A number of concrete proposals have been made for the use of qualitative representations in automated navigation systems (Kuipers and Levitt 1988, Schlieder 1993).

A very promising application for QSR is to GIS, which are increasingly in demand as a tool for business planning and land management. The need for qualitative spatial query languages to interact with these systems is clear (Egenhofer and Franzosa 1991, Egenhofer and Herring 1991, Egenhofer and Al-Taha 1992, Clementini et al. 1994, Egenhofer and Mark 1995). High-level queries of a naïve GIS user correspond to natural language questions and these typically involve qualitative concepts. In section 10.4.1 below I shall describe a prototype GIS that exploits topological reasoning.

Interpreting query languages is a special case of the more general problem of interpreting spatial expressions occurring in natural language, which tend to be predominantly qualitative rather than quantitative (consider prepositions such as 'in', 'on' and 'through') (Vieu 1991). But in applying QSR to natural language one faces the problem that spatial expressions are enmeshed in an unformalised and massively complex conceptual structure. By contrast, the limited spatial vocabulary employed in visual computer programming languages is much more amenable to formal description and a number of recent works have used qualitative representations to specify the syntax and semantics of visual programming languages[3] such as Pictorial Janus (Haarslev 1995, Gooday and Cohn 1995, Gooday and Cohn 1996).

Another branch of AI which may be well suited to exploit QSR techniques is computer vision.

[3] These are languages in which programs are created by editing pictures within a graphical environment. Program execution can also be visualised by means of animations of these graphical representations. This is intended to facilitate debugging and understanding of how a program works.

A computer vision system typically employs a fairly long series of transformation procedures culminating in a geometrical model of objects in the scene. Within this kind of architecture it is very easy to insert a procedure which exploits spatial reasoning. Indeed the use of semantic techniques, both for image segmentation and object recognition, has long been recognised (Winston 1975). Qualitative reasoning based on a set of orientation relations has been successfully applied to the analysis of traffic flow from video images (Fernyhough, Cohn and Hogg 1996, Fernyhough, Cohn and Hogg 1997).

In all areas involving spatial information it is easy to give hand-waving accounts of how QSR can be used to great advantage. However, the obstacles to attaining practical results cannot be overestimated. The results reported in this thesis indicate that achieving effective reasoning even with a very limited vocabulary of spatial concepts may require complex logical apparatus and reasoning algorithms, specifically tailored to handling that particular range of concepts. How these limited representations can be put to work on real problems is far from obvious.

It is tempting to suppose that once a sufficiently expressive representation has been devised, the manner in which it can be exploited will become obvious. But, without a hugely radical advance in computer hardware or software technology, it seems likely that the conflict between expressive power and tractability will always be a strong constraint on the use of AI techniques in computer systems. Thus, to find a practical application of QSR, one will have to show how some concrete task can be reduced to manipulating a small number of spatial concepts, or at least how the role of different types of spatial information in carrying out this task, can be isolated and handled in a modular fashion. This problem is especially acute if one attempts to work within an architecture in which all information and reasoning is handled by means of a purely qualitative representation; one cannot then rely on any of the well-understood mechanisms for quantitative data manipulation that have been developed over the years. In my opinion, the interface with quantitative information (discussed in the last section) is the key to opening up the path towards real applications. Embedding qualitative reasoning modules within a more conventional architecture enables one to explore the strengths of using qualitative representations without exposing all their weaknesses.[4]

As my main results are about reasoning with topological relations and (to a much lesser extent) convexity, I ought to suggest applications for this limited form of spatial reasoning. Topological relations are fundamental and pervasive in all spatial information, so one might expect the usefulness of topological reasoning to be equally general. But, what specific computational tasks can be reduced to topological reasoningΓ

I have observed that, in many potential application areas, adequate qualitative description of tasks requires not only non-topological concepts but also many non-spatial concepts. In such cases a modular analysis of relevant reasoning capabilities will be necessary in order to isolate useful topological inference procedures; and this is a research topic in itself. However, I believe that significant semantic constraints relevant to object recognition can be specified in terms of purely

[4](Fernyhough 1997) provides a good example of what can be achieved using this kind of architecture.

topological conditions and this may well lead to practical uses within the field of computer vision. The application for which purely topological reasoning has the most obvious uses is GIS. It easy to envisage situations in which a GIS user wants to pose a query that is essentially topological in nature. For example, in siting a factory one might wish to find an area of undeveloped land, which is adjacent (i.e. externally connected) to a water source, such as a lake, and is part of a particular urban district. What is not so obvious is how significant these topological queries are to the overall functionality of a GIS, which typically provides access to a vast amount of metrical information.

10.4.1 Topological Inference in a GIS Prototype

I shall conclude the discussion of applications with a description of a prototype 'spatial AI' system being developed as part of EPSRC project GR/K65041 on 'Logical Theories and Decision Procedures for Reasoning about Physical Systems'. This incorporates the ($O(n^3)$) topological reasoning algorithm based on my $\mathcal{I}^+$ encoding, which was described in section 6.3 (program code is given in appendix C.3). The system maintains a database of geographical information in the form of geometrical polygon data and also handles qualitative data in the form of topological relations between named regions. Some of these named regions are identified directly with polygons in the geometrical database, whereas for others the geometry is not precisely known but only constrained by the qualitative topological relations. The topological relationships determined by the the quantitative geometrical data can also be rapidly computed and accessed by the topological reasoning mechanism, allowing queries to be addressed to the combined qualitative and quantitative database. This capability is (as far as we know) not available in any other system. Work is also underway to demonstrate the use of topological reasoning in the control of artificial agents operating in a virtual world constituted by geographical data.

Figure 10.2 shows a screen-dump of the current prototype system. Most of the code is written in (SICStus) Prolog but a Tcl/Tk sub-process is used to create the GUI. The window at the top left shows a simple cartographical display, whose geometry is determined by a database giving the coordinates and terrain type of a number of triangular regions. This data is shown in the bottom left window. The top right window presents a database of qualitative relations between regions. In the middle on the right is the Prolog top-level query window. All functions of the system can be accessed by typing commands and queries at the Prolog prompt (although common operations are more conveniently accessed via the GUI). The figure shows the Prolog interpreter being used for querying the qualitative database. Such queries are answered by means of the spatial reasoning algorithm described in chapter 6, which determines whether a relation given as a query is consistent with, inconsistent with or a necessary consequence of the database. (The bottom right window is one of a number of information screens which can be displayed via the system's 'help' function.)

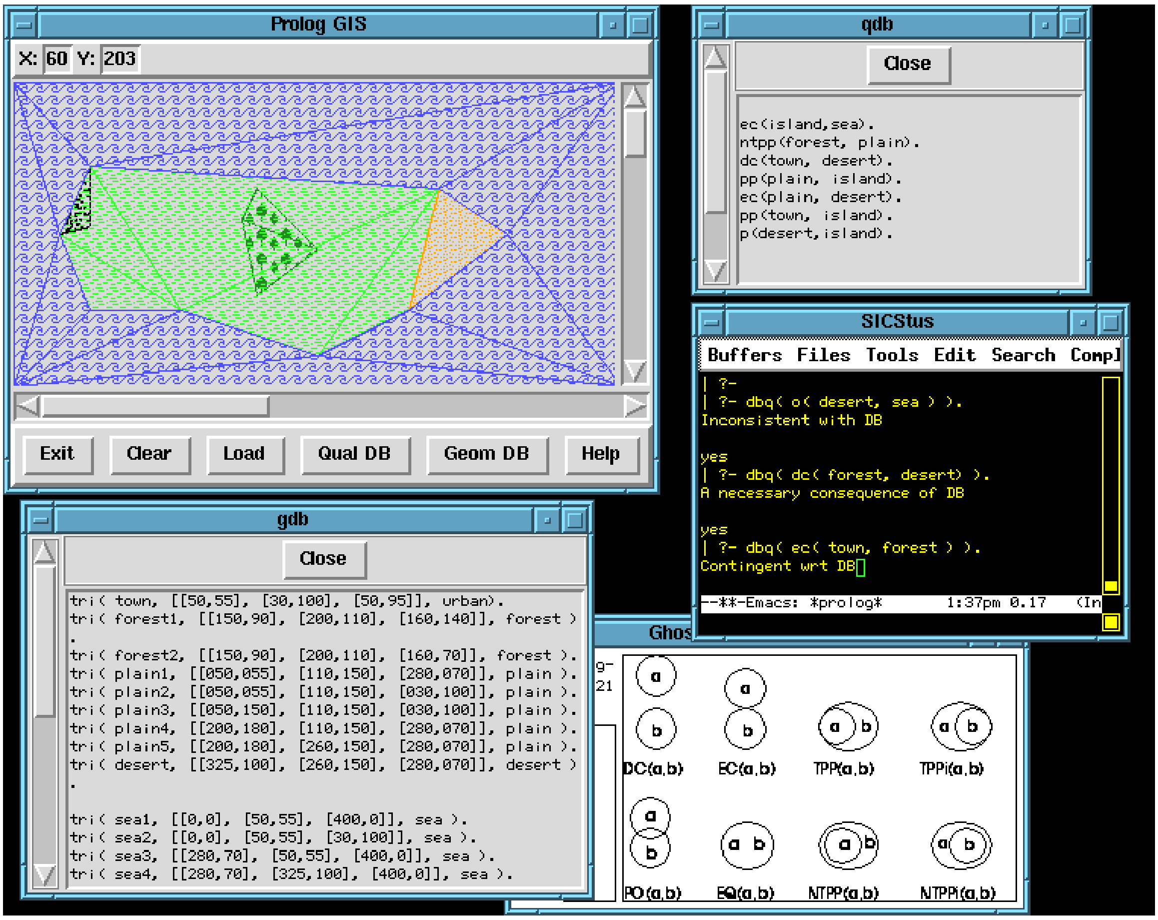

Figure 10.2: A prototype geographical information system

10.5 Conclusion

I shall conclude this thesis by making some general remarks about the prospects for automated reasoning based on insights I gained during my research.

When I started work on spatial reasoning, I was under the naïve impression that 1st-order logic, or something like it, could provide an ideal formalism for knowledge representation and reasoning in this and almost any conceptual domain. Although I was aware of the theoretical undecidability and intractability of 1st-order reasoning, I did not realise the seriousness of the difficulties that these properties pose for automated reasoning. I imagined that, with a powerful enough computer, it would be feasible to compute entailments between relations as determined by a simple axiomatic theory. However after attempting to compute RCC inferences using the OTTER theorem prover (McCune 1990) it soon became apparent that this is completely impractical. Even seemingly simple deductions would very often exhaust the available computational resources.

My experience of theorem proving probably has much in common with that of many others who have entered this field. It is now widely recognised that effective automated reasoning with logical representations cannot be achieved by general purpose proof systems but requires the construction of specialised reasoning algorithms. Even so, it seems to me surprising that a tractable proof procedure for spatial relations should be so far removed from one's intuitive picture of the problem. An interesting question is whether this is typical of effective solutions to reasoning problems. That this may be so was suggested by Alan Robinson (1979), who, having discovered the very powerful but extremely unnatural *hyper-resolution* inference rule, proposed that there may be a difference in kind between the style of reasoning intelligible to humans and the type of reasoning mechanisms which can be efficiently implemented in computer programs. This is also evidenced by the prodigious number-crunching abilities but poor conversational skills of computers. Though it does not give any reason for the divergence between styles of reasoning of humans and computers, Robinson's proposal does seem to concur with much of what has been discovered in the study of automated reasoning.

From another point of view, the use of modal and intuitionistic logics for spatial reasoning may not be so perverse as it first seems. It may just be that this use of these logics is unfamiliar. Although modal logics were originally intended to capture propositional modifiers and intuitionistic logic to specify an ontologically parsimonious form of mathematical reasoning, the structural manipulations embodied in the inference rules of these logics are of a very general nature. Hence, it is only to be expected that alternative interpretations can be given.

The success of the $\mathcal{I}$ and $S4$ encodings of spatial relations may also shed some light on why 1st-order reasoning is so intractable. In 1st-order predicate logic, the sub-structure of atomic propositions has no logical content. By this I mean that, although we may analyse an atomic proposition in terms of a relation between a number of functional terms, these components are arbitrary, having no special logical properties, except insofar as they may be constrained by axioms. Hence, the meanings of these symbols are not captured directly by rules of inference but only indirectly through axioms taking part in inference. Moreover, these axioms often take the form of quite complex quantificational formulae. It is these theoretical formulae that make 1st-order reasoning so computationally intensive, even when employed to compute seemly obvious consequences of simple factual information.

As an exception to this treatment of the meaning of predicates, the meaning of the equality relation is usually specified in terms of inference rules rather than axioms. One could treat equality as a non-logical symbol constrained by axioms[5] but it is easier to capture the logical properties of '=' by means of inference rules than in axioms. Axiomatic treatment of equality adds a large number of formulae to the specification of a 1st-order theory, which greatly increases the search space that an automated theorem prover has to deal with. Although adding inference rules for equality also increases the search space, it has been found that this method is in most cases much more conducive to automated reasoning (Wos 1988, Duffy 1991). When one reasons with a theory of equality in axiomatic form, a proof may involve a considerable amount of reasoning about the

[5]The equality relation can be characterised either by the 2nd-order axiom $(x = y) \leftrightarrow \forall\Phi[\Phi(x) \leftrightarrow \Phi(y)]$ or by a set 1st-order axioms ensuring that '=' is an equivalence relation and specifying all possible ways that an equality justifies substitution into the arguments of relations and function. The substitution axioms take the forms $\forall x \forall y \forall \overline{z} \forall \overline{w}[((x = y) \wedge \phi(\overline{z}, x, \overline{w})) \rightarrow \phi(\overline{z}, y, \overline{w})]$, where ϕ is a relation symbol of the theory, and $\forall x \forall y \forall \overline{z} \forall \overline{w}[((x = y) \rightarrow (\rho(\overline{z}, x, \overline{w}) = \rho(\overline{z}, y, \overline{w}))]$, where ρ is a function symbol. $\overline{z}$ and $\overline{w}$ represent (possibly empty) sequences of variables filling any additional argument places of ϕ and ρ.

concept of equality itself, as well as reasoning about other concepts; whereas, if equality is handled by an inference rule (such as paramodulation), then the theory of equality is encapsulated within this rule so the ramifying effect on the search space is greatly reduced.

Associating inferential meaning to other predicate and function symbols within a proposition can obviate the need for auxiliary axioms; and the findings concerning equality suggest that this may be extremely advantageous for automated reasoning. One example of this is the use of sorted logic (see e.g. (Cohn 1987) and section 2.6.2 of this thesis), where reasoning concerning the sorts of predicates and functions is built into inference rules.[6] Another example is the use of demodulation rules (see e.g. (Duffy 1991)) to rewrite and simplify terms in accordance with known identities. These rules must be tailored to the specific properties of a given theory but they have proved extremely effective in many domains (Wos 1988). A typical use of demodulation is to reduce Boolean and other algebraic terms to normal form, to avoid proliferation of equivalent but syntactically distinct terms. Algebraic terms are very common in mathematical theories but generally do not play a major role in theories of commonsense concepts. However, the analysis of RCC relations in terms of interior algebraic equations (see section 5.3) shows that an algebraic specification of such concepts may be possible, even where it is not immediately obvious. It is this analysis of the RCC relations that enables their meanings to be captured by means of inference rules rather than axioms.

Algebraic analysis may expose sub-structure in the meanings of relational concepts but in itself this is probably not helpful to automated reasoning. If we simply axiomatised the algebraic operators, the resulting theory might be even more complex than a direct axiomatisation of the concepts. To gain computational advantage we need a proof system that takes direct account of the inferential significance of the algebraic operators and hence encapsulates the meaning of the concepts within its inference rules. It is well-known that classical propositional logic can be interpreted as a Boolean algebra and that modal operators can also be identified with algebraic operators. Hence it should not be surprising that proof systems designed to compute inferences in these propositional languages can also be exploited to reason about algebraic equations. However, the detailed working-out of how this can be done is probably the most novel aspect of the work in this thesis.

Because they exceed the expressive power of simple Boolean algebra but avoid the intractability of 1st-order logic, I believe that decidable constraint languages based on Boolean algebras with additional operators are very well suited to computational manipulation. These encompass the modal algebras (which I explored in chapter 5) and also relation algebras (discussed in section 9.3). As well as providing a vehicle for effective automation of spatial reasoning, representations based on algebraic structures of this kind may be useful in many other areas of knowledge representation.

[6] Resolution-based inference rules are particularly well suited to incorporating sortal reasoning.

Appendix A

Elementary Geometry

A.1 Tarski's Axiom System

Tarski (1959) has given the following axiomatisation of *elementary geometry* in terms of the two primitives, *betweenness* and *equidistance*. Here $B(x, y, z)$ means that point y is between points x and z. This relation is taken as true if z is equal to either x or z. $xy = zw$ means that the distance between points x and y is equal to the distance between points y and z.

B1 [Identity Axiom for Betweenness]
$\forall xy[B(x,y,x) \rightarrow (x = y)$

B2 [Transitivity Axiom for Betweenness]
$\forall xyzu[(B(x,y,u) \wedge B(y,z,u)) \rightarrow B(x,y,z)]$

B3 [Connectivity Axiom for Betweenness]
$\forall xyzu[(B(x,y,z) \wedge B(x,y,u) \wedge (x \neq y)) \rightarrow (B(x,z,u) \vee B(x,u,z))]$

B4 [Reflexivity Axiom for Equidistance]
$\forall xy[xy = yx]$

B5 [Identity Axiom for Equidistance]
$\forall xyz[xy = zz \rightarrow (x = y)$

B6 [Transitivity Axiom for Equidistance]
$\forall xyzuvw[(xy = zu \wedge xy = vw) \rightarrow zu = vw]$

B7 [Pasch's Axiom]
$\forall txyzu\exists v[(B(x,t,u) \wedge B(y,u,z)) \rightarrow (B(x,v,y) \wedge B(z,t,v))]$

B8 [Euclid's Axiom]
$\forall txyzu\exists vw[(B(x,u,,t) \wedge B(y,u,z) \wedge (x \neq y))$
$\rightarrow (B(x,z,v) \wedge B(x,y,w) \wedge B(v,t,w))]$

B9 [Five-Segment Axiom]
$\forall xx'yy'zz'uu'[(xy = x'y' \wedge yz = y'z' \wedge xu = x'u' \wedge yu = y'u'$
$\wedge B(x,y,z) \wedge B(x',y',z') \wedge (x \neq y)) \rightarrow zu = z'u']$

B10 [Axiom of Segment Construction]
$\forall xyuv\exists z[B(x,y,z) \wedge yz = uv]$

B11 [Lower Dimension Axiom]
$\exists xyz[\neg B(x,y,z) \wedge \neg B(y,z,x) \wedge \neg B(z,x,y)]$

B12 [Upper Dimension Axiom]
$\forall xyzuv[(xy = xv \wedge yu = yv \wedge zu = zv \wedge (u \neq v))$
$\rightarrow (B(x,y,z) \vee B(y,z,x) \vee B(z,x,y))]$

B13 [Elementary Continuity Axioms]
All sentences of the form:

$$\forall vw \ldots [\exists z \forall xy[\phi \wedge \psi \rightarrow B(z,x,y)] \rightarrow \exists u \forall xy[\phi \wedge \psi \rightarrow B(x,u,y)]]$$

where ϕ stands for any formula in which the variables x, y, w, ..., but neither y nor z nor u, occur free, and similarly for ψ, with x and y interchanged.

B13' [Weak Continuity Axiom]
$\forall xyzx'z'u\exists y'[(ux = ux' \wedge uz = uz' \wedge B(u,x,z) \wedge B(x,y,z))$
$\rightarrow (uy = uy' \wedge B(x',y',z'))]$

A.2 Primitive Geometrical Concepts

The sequence of definitions given below shows how starting from the fundamental ternary relation $xy = yz$, which is true when two points, x and z, are equidistant from a third point, y, many other simple geometrical relations can be introduced. In these definitions, the juxtaposition xy of two variables x and y is intended to refer to the distance between these two points. Thus $xy \leq yz$ is a predicate which holds in case y is closer to x than to z. The other relations are: $B(x, y, z)$ — y is between x and z (including the case where y is identical with either x or z); $L(x, y, z)$ — x, y and z are *collinear*; and $M(x, y, z)$ — y is the mid-point between x and z.

The relation $xy = yz$ is of great geometrical significance as it relates the centre point (y) of a sphere to any pair of surface points (x and z). For a 2-dimensional figure, the truth of this relation for any three points can be determined by means of a compass. The relations $B(x, y, z)$ and $xy = zw$ are taken as primitives in Tarski's elementary geometry. A proof that the quaternary relation $xy = zw$ is definable in terms of the ternary $xy = yz$ is originally due to Pieri (1899). The following definitions showing how this can be done (together with further discussion of primitive notions in geometry) can be found in (Royden 1959).

$$\begin{aligned}
xy \leq yz &\equiv_{def} \forall w[yw = wz \rightarrow \exists u[xu = uy \wedge uy = yw]] \\
B(x, y, z) &\equiv_{def} \forall w[(wx \leq xy \wedge wz \leq zy) \rightarrow w = y] \\
L(x, y, z) &\equiv_{def} B(x, y, z) \vee B(y, x, z) \vee B(x, z, y) \\
M(x, y, z) &\equiv_{def} \forall w[(L(w, x, y) \wedge xy = yw) \leftrightarrow (w = x \vee w = z)] \\
wx = yz &\equiv_{def} \exists u \exists v[M(w, u, y) \wedge M(x, u, v) \wedge vy = yz]
\end{aligned}$$

Appendix B

An Alternative Proof of MEconv

In this appendix I give an alternative proof of the theorem **BEconv**, which was demonstrated in section 5.5.1. The statement of **BEconv** is as follows:

Convexity of Disjunctive Modal-Algebraic Entailments (MEconv)

$$\mu_1 = \mathcal{U}, \ldots, \mu_m = \mathcal{U} \models_{\mathrm{MA}_L} \varepsilon_1 = \mathcal{U} \vee \ldots \vee \varepsilon_n = \mathcal{U}$$

iff

$$\mu_1 = \mathcal{U}, \ldots, \mu_m = \mathcal{U} \models_{\mathrm{MA}_L} \varepsilon_i = \mathcal{U} \text{ for some } i \in \{1, \ldots n\}$$

The alternative proof relies only on the additivity of the modal operator and does not require that its algebraic properties be specifiable just in terms of equations. The basis of the proof is that given counter-models satisfying the premisses of the sequent and individually falsifying each disjunct of its conclusion, the additive nature of the operator allows one to construct a counter-model satisfying the premisses and falsifying the conclusion as a whole.

Proof of MEconv: Let S be the set of set-constants occurring in a disjunctive entailment, DE, of the form given in the theorem. Suppose none of the disjuncts on the r.h.s. is entailed by the equations on the l.h.s.. This means that for each disjunct $\varepsilon_i = \mathcal{U}$ there is an assignment, $\Sigma_i = \langle S, U_i, \sigma_i, m_i \rangle$ satisfying all the equations $\mu_j = \mathcal{U}$ but such that $\Sigma_i[\varepsilon_i] \neq \mathcal{U}$. We can assume, without loss of generality, that the universes, U_i in each of the assignments are disjoint. From these assignments we can construct a new assignment, Σ_0, again satisfying all the equations $\mu_j = \mathcal{U}$ and such that $\Sigma_0[\varepsilon_i] \neq \mathcal{U}$ for each ε_i:

Let $\Sigma_0 = \langle S, U_0, \sigma_0, m_0 \rangle$, where $U_0 = \bigcup_i U_i$, σ_0 is defined by $\sigma_0(\kappa) = \bigcup_i \sigma_i(\kappa)$ for each constant $\kappa \in S$ and m_0 is defined by $m_0(X) = \bigcup_i m_i(X \cap U_i)$ for every set $X \subseteq U_0$ (in each case the subscript i ranges from 1 to n).

We note that, because we are dealing with modal algebras, each of the functions m_i must be additive ($m_i(X \cup Y) = m_i(X) \cup m_i(Y)$). This means that m_0 is also additive:

$$m_0(X \cup Y) = \bigcup_i m_i((X \cup Y) \cap U_i) = \bigcup_i m_i((X \cap U_i) \cup (Y \cap U_i)) =$$

$$\bigcup_i (m_i(X \cap U_i) \cup m_i(Y \cap U_i)) = \bigcup_i (m_i(X \cap U_i)) \cup \bigcup_i (m_i(Y \cap U_i)) = m_0(X) \cup m_0(Y)$$

I now show that $\Sigma_0[\tau] = \bigcup_i \Sigma_i[\tau]$ for any term τ — i.e. the denotation of any term under Σ_0 is just the union of its denotations under the assignments Σ_i. If τ is a constant this is ensured directly by the specification of Σ', so we can prove it inductively for all terms by showing that if it holds for any terms α and β, it must also hold for the terms $\overline{\alpha}$, $\alpha \cup \beta$, $\alpha \cap \beta$ and $*\alpha$. For $\perp$ and $\cup$ we have:

$$\Sigma_0[\overline{\alpha}] = U_0 \perp \Sigma_0[\alpha] = \bigcup_i U_i \perp \bigcup_i \Sigma_i[\alpha] = \bigcup_i (U_i \perp \Sigma_i[\alpha]) = \bigcup_i \Sigma_i[\overline{\alpha}]$$

$$\Sigma_0[\alpha \cup \beta] = \Sigma_0[\alpha] \cup \Sigma_0[\beta] = \bigcup_i \Sigma_i[\alpha] \cup \bigcup_i \Sigma_i[\beta] = \bigcup_i \Sigma_i[\alpha \cup \beta]$$

Whence $\Sigma_0[\alpha \cap \beta] = \bigcup_i \Sigma_i[\alpha \cap \beta]$ must hold, since $\alpha \cap \beta = \perp(\overline{\alpha} \cup \overline{\beta})$.

The proof for the case of the modal $*$ operator is rather more involved. Since we are assuming $\Sigma_0[\alpha] = \bigcup_i \Sigma_i[\alpha]$ we have $\Sigma_0[*\alpha] = m_0(\Sigma_0[\alpha]) = m_0\left(\bigcup_i \Sigma_i[\alpha]\right)$ and since m_0 is additive this equals $\bigcup_i(m_0(\Sigma_i[\alpha]))$. If we now replace m_0 by its definition in terms of the functions m_i we get the expression $\bigcup_i \left(\bigcup_j m_j(\Sigma_i[\alpha] \cap U_j)\right)$ (where i and j both range from 1 to n). Notice that $\Sigma_i[\alpha]$ is always a subset of U_i; so, because the U_i's are disjoint, $\Sigma_i[\alpha] \cap U_j$ must equal $\Sigma_i[\alpha]$ if $i = j$ and $\emptyset$ otherwise. This means that the expression can be reduced to $\bigcup_i m_i(\Sigma_i[\alpha])$, which is equivalent to $\bigcup_i \Sigma_i[*\alpha]$.

Since $\Sigma_0[\tau] = \bigcup_i \Sigma_i[\tau]$ for any term τ and the ranges of the assignments Σ_i are disjoint, it follows that an equation is satisfied by Σ_0 if and only if it is satisfied by all of the Σ_i's. This ensures that Σ_0 satisfies all the frame equations of the logic L. It also means that Σ_0 must satisfy all the equations on the l.h.s. of the DE and none of the equations in the disjunction on the r.h.s. of DE.

Hence, the constructed assignment Σ_0 demonstrates that, if none of the disjuncts on the r.h.s. of DE is individually entailed by the equations on the l.h.s., their disjunction cannot be entailed. So the class of entailments of modal algebraic equations of the form of DE is convex. ■

Appendix C

Prolog Code

C.1 Generating all Conjunctions of RCC-7 Relations

The following program generates all logically distinct relations which can be specified as a conjunction of RCC-7 relations and their negations. The code includes documentation of how it works. Further explanation can be found in section 5.3.3 in the main thesis and also in the section following the program listing, where I present and explain the program's output.

```
%% rcc7cons.pl

% This program generates the complete set of logically distinct
% relations, which can be specified as  conjunctions of +ve and
% -ve literals taken from the RCC-7 relation set.

% The set can be generated with or without non-null constraints
% on the regions involved.
% For the sake of generality, non-null constraints are represented
% by adding the relations x0 and y0 to the set of RCC-7 relations.
% x0 is true just in case the 1st argument of the relation is null
% and y0 if the second argument is null.

%%  top-level calls

% Generate all combinations of RCC-7 relations and null relations
% There are 171 (including the impossible relation).
generate_rcc7_cons :-
                   setof( X, (rcc7con(X), complete(X)), Set),
                   showlist(Set),
                   length(Set,L), write(length(L)).
```

```
% Generate all combinations of RCC-7 relations for which the
% arguments are non-null.
% There are 115 (including the impossible relation).
generate_rcc7nn_cons :-
                   setof( X, (rcc7nncon(X), complete(X)), Set),
                   showlist(Set),
                   length(Set,L), write(length(L)).

% Find the most specific relations specifiable between non-null regions
% (excluding the impossible relation).
% This generates the RCC-8 relations.
generate_nnrcc7_base_rels :-
      setof( X, (rcc7nncon(X), complete(X)), Set),
      setof(B,(member(B,Set),\+( (member(C,Set), proper_subset(B,C) ))),Base),
      showlist(Base),
      length(Base,LB), write(base_length(LB)).

%% Subsidiary Predicates

rcc7con(CONJ) :- pick_conjunction([dc,dr,p,pi,ntpp,ntppi,eq,x0,y0], CONJ)
                 ; CONJ = impossible.

rcc7nncon( [not(x0), not(y0) | Rest] ) :-
                 pick_conjunction([dc,dr,p,pi,ntpp,ntppi,eq], Rest).
rcc7nncon( impossible ).

%% A set of relations is complete iff it is closed under implications.
complete(Set) :- \+( (member(R,Set), implies(R,S),
                     \+(member(S,Set)) )
                   ),
                 \+( (member(R1,Set),member(R2,Set),
                     implies(and(R1,R2),S2),
                     \+(member(S2,Set)) )
                   ).



%% Implications holding between rcc7 relations
implies(dc,dr).
```

```
implies(ntpp, p).
implies(ntppi, pi).
implies(eq, p).
implies(eq, pi).
implies(x0, ntpp).
implies(y0, ntppi).
implies(x0, dc).
implies(y0, dc).

implies( and(p,pi), eq ).
implies( and(dr,p), x0 ).
implies( and(dr,pi), y0 ).
implies( and(ntpp,ntppi), x0).
implies( and(ntpp,ntppi), y0).
implies( and(ntpp,eq), x0).
implies( and(ntpp,eq), y0).
implies( and(ntppi,eq), x0).
implies( and(ntppi,eq), y0).

implies( not(R), not(S) ):- implies(S,R), \+(S = and(_,_)).
implies( and(R,not(S)), not(T) ) :- implies(and(R,T), S).

%% Additional simple predicates

proper_subset(X, Y) :- \+(X=Y), \+( (member(E,X),\+(member(E,Y))) ).

%% write a list one element per line
showlist([]).
showlist([H|T]) :- write(H),nl,showlist(T).

% pick a conjunction of +ve or -ve literals from a list
pick_conjunction([],[]).
pick_conjunction([_|T], PT) :- pick_conjunction(T,PT).
pick_conjunction([H|T], [H | PT]) :- pick_conjunction(T,PT).
pick_conjunction([H|T], [not(H) | PT]) :- pick_conjunction(T,PT).


%% minimise/2 can be used to remove redundant rels from RCC7 conjunctions
%% (this is not actually used by the predicates defined above)
```

```
%% Minimising a set is removing all implied relations
minimise(Set,M) :- extract( R, Set, Rest ),
                   member( S, Rest ),
                   implies( S, R ), !,
                   minimise( Rest, M), !.

minimise(Set,M) :- extract( R, Set, Rest ),
                   extract( S, Rest, Rest2 ),
                   member( T, Rest2 ),
                   implies( and(S,T), R ), !,
                   minimise( Rest, M), !.
minimise(S,S).

%% extract an element from a list (non-deterministically)
extract(X,List,Rest) :- append( Front, [X | End], List ),
                        append( Front, End, Rest ).
```

C.1.1 171 Conjunctions of the RCC-7 Relations and their Negations

Here is the set of 171 logically distinct conjunctions of the RCC-7 relations and their negations generated by the program given in the last section. The relations are given in the form of a list of conjuncts, with negated conjuncts given as `not(`R`)`. Relations are denoted by their usual initials but in small letters (because of the syntax of Prolog). The empty list corresponds to the universal, holding between any two regions. Any conjunction containing a literal and its negation is equivalent to the `impossible` relation.

The fact that one or other of the regions involved in a relation is null is specified by the special pseudo-relations `x0` and `y0`, meaning respectively that the 1st or 2nd argument is null. In the RCC theory all regions are non-null. Thus, only those conjunctions including the conjuncts `not(x0)` and `not(y0)` correspond to legitimate RCC relations. There are 115 such relations (including the `impossible` relation which implicitly includes both non-null constraints).

The conjunction sets generated by the program are closed under implication and this ensures that they are all genuinely logically distinct. It also means that there is a lot of redundancy in the resulting specification of the relations. For instance, every conjunction which has `dc` as a conjunct also includes the weaker relation `dr` as a conjunct. This redundancy could be eliminated by post-processing the sets to remove implied conjuncts; however, there is not always a unique way to simplify a conjunction so I have not done this.

```
| ?- generate_rcc7_cons.
[]
impossible
[dc,dr]
```

```
[dc,dr,p,ntpp,x0]
[dc,dr,p,pi,ntpp,ntppi,eq,x0,y0]
[dc,dr,p,not(pi),ntpp,not(ntppi),not(eq),x0,not(y0)]
[dc,dr,pi,ntppi,y0]
[dc,dr,pi,ntppi,not(eq),y0]
[dc,dr,not(eq)]
[dc,dr,not(p),pi,not(ntpp),ntppi,not(eq),not(x0),y0]
[dc,dr,not(p),not(ntpp),not(eq),not(x0)]
[dc,dr,not(p),not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[dc,dr,not(pi),not(ntppi),not(eq),not(y0)]
[dr]
[dr,not(eq)]
[dr,not(p),not(ntpp),not(eq),not(x0)]
[dr,not(p),not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[dr,not(pi),not(ntppi),not(eq),not(y0)]
[p]
[p,ntpp]
[p,pi,eq]
[p,pi,eq,not(x0)]
[p,pi,eq,not(x0),not(y0)]
[p,pi,eq,not(y0)]
[p,pi,not(ntpp),eq,not(x0)]
[p,pi,not(ntpp),eq,not(x0),not(y0)]
[p,pi,not(ntpp),not(ntppi),eq,not(x0),not(y0)]
[p,pi,not(ntppi),eq,not(x0),not(y0)]
[p,pi,not(ntppi),eq,not(y0)]
[p,not(ntpp),not(ntppi),not(x0),not(y0)]
[p,not(ntpp),not(x0)]
[p,not(ntpp),not(x0),not(y0)]
[p,not(ntppi),not(x0),not(y0)]
[p,not(ntppi),not(y0)]
[p,not(pi),ntpp,not(ntppi),not(eq),not(x0),not(y0)]
[p,not(pi),ntpp,not(ntppi),not(eq),not(y0)]
[p,not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[p,not(pi),not(ntppi),not(eq),not(x0),not(y0)]
[p,not(pi),not(ntppi),not(eq),not(y0)]
[p,not(x0)]
[p,not(x0),not(y0)]
[p,not(y0)]
[pi]
```

```
[pi,ntppi]
[pi,ntppi,not(eq)]
[pi,ntppi,not(eq),not(x0)]
[pi,ntppi,not(eq),not(x0),not(y0)]
[pi,ntppi,not(eq),not(y0)]
[pi,not(eq)]
[pi,not(eq),not(x0)]
[pi,not(eq),not(x0),not(y0)]
[pi,not(eq),not(y0)]
[pi,not(ntpp),ntppi,not(eq),not(x0)]
[pi,not(ntpp),ntppi,not(eq),not(x0),not(y0)]
[pi,not(ntpp),not(eq),not(x0)]
[pi,not(ntpp),not(eq),not(x0),not(y0)]
[pi,not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[pi,not(ntpp),not(ntppi),not(x0),not(y0)]
[pi,not(ntpp),not(x0)]
[pi,not(ntpp),not(x0),not(y0)]
[pi,not(ntppi),not(eq),not(x0),not(y0)]
[pi,not(ntppi),not(eq),not(y0)]
[pi,not(ntppi),not(x0),not(y0)]
[pi,not(ntppi),not(y0)]
[pi,not(x0)]
[pi,not(x0),not(y0)]
[pi,not(y0)]
[not(dc),dr,not(p),not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),p,pi,eq,not(x0),not(y0)]
[not(dc),p,pi,not(ntpp),eq,not(x0),not(y0)]
[not(dc),p,pi,not(ntpp),not(ntppi),eq,not(x0),not(y0)]
[not(dc),p,pi,not(ntppi),eq,not(x0),not(y0)]
[not(dc),p,not(ntpp),not(ntppi),not(x0),not(y0)]
[not(dc),p,not(ntpp),not(x0),not(y0)]
[not(dc),p,not(ntppi),not(x0),not(y0)]
[not(dc),p,not(pi),ntpp,not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),p,not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),p,not(pi),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),p,not(x0),not(y0)]
[not(dc),pi,ntppi,not(eq),not(x0),not(y0)]
[not(dc),pi,not(eq),not(x0),not(y0)]
[not(dc),pi,not(ntpp),ntppi,not(eq),not(x0),not(y0)]
[not(dc),pi,not(ntpp),not(eq),not(x0),not(y0)]
```

```
[not(dc),pi,not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),pi,not(ntpp),not(ntppi),not(x0),not(y0)]
[not(dc),pi,not(ntpp),not(x0),not(y0)]
[not(dc),pi,not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),pi,not(ntppi),not(x0),not(y0)]
[not(dc),pi,not(x0),not(y0)]
[not(dc),not(dr),p,pi,eq,not(x0),not(y0)]
[not(dc),not(dr),p,pi,not(ntpp),eq,not(x0),not(y0)]
[not(dc),not(dr),p,pi,not(ntpp),not(ntppi),eq,not(x0),not(y0)]
[not(dc),not(dr),p,pi,not(ntppi),eq,not(x0),not(y0)]
[not(dc),not(dr),p,not(ntpp),not(ntppi),not(x0),not(y0)]
[not(dc),not(dr),p,not(ntpp),not(x0),not(y0)]
[not(dc),not(dr),p,not(ntppi),not(x0),not(y0)]
[not(dc),not(dr),p,not(pi),ntpp,not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(dr),p,not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(dr),p,not(pi),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(dr),p,not(x0),not(y0)]
[not(dc),not(dr),pi,ntppi,not(eq),not(x0),not(y0)]
[not(dc),not(dr),pi,not(eq),not(x0),not(y0)]
[not(dc),not(dr),pi,not(ntpp),ntppi,not(eq),not(x0),not(y0)]
[not(dc),not(dr),pi,not(ntpp),not(eq),not(x0),not(y0)]
[not(dc),not(dr),pi,not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(dr),pi,not(ntpp),not(ntppi),not(x0),not(y0)]
[not(dc),not(dr),pi,not(ntpp),not(x0),not(y0)]
[not(dc),not(dr),pi,not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(dr),pi,not(ntppi),not(x0),not(y0)]
[not(dc),not(dr),pi,not(x0),not(y0)]
[not(dc),not(dr),not(eq),not(x0),not(y0)]
[not(dc),not(dr),not(ntpp),not(eq),not(x0),not(y0)]
[not(dc),not(dr),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(dr),not(ntpp),not(ntppi),not(x0),not(y0)]
[not(dc),not(dr),not(ntpp),not(x0),not(y0)]
[not(dc),not(dr),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(dr),not(ntppi),not(x0),not(y0)]
[not(dc),not(dr),not(p),pi,not(ntpp),ntppi,not(eq),not(x0),not(y0)]
[not(dc),not(dr),not(p),pi,not(ntpp),not(eq),not(x0),not(y0)]
[not(dc),not(dr),not(p),pi,not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(dr),not(p),not(ntpp),not(eq),not(x0),not(y0)]
[not(dc),not(dr),not(p),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(dr),not(p),not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
```

```
[not(dc),not(dr),not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(dr),not(pi),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(dr),not(x0),not(y0)]
[not(dc),not(eq),not(x0),not(y0)]
[not(dc),not(ntpp),not(eq),not(x0),not(y0)]
[not(dc),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(ntpp),not(ntppi),not(x0),not(y0)]
[not(dc),not(ntpp),not(x0),not(y0)]
[not(dc),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(ntppi),not(x0),not(y0)]
[not(dc),not(p),pi,not(ntpp),ntppi,not(eq),not(x0),not(y0)]
[not(dc),not(p),pi,not(ntpp),not(eq),not(x0),not(y0)]
[not(dc),not(p),pi,not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(p),not(ntpp),not(eq),not(x0),not(y0)]
[not(dc),not(p),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(p),not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(pi),not(ntppi),not(eq),not(x0),not(y0)]
[not(dc),not(x0),not(y0)]
[not(eq)]
[not(eq),not(x0)]
[not(eq),not(x0),not(y0)]
[not(eq),not(y0)]
[not(ntpp),not(eq),not(x0)]
[not(ntpp),not(eq),not(x0),not(y0)]
[not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(ntpp),not(ntppi),not(x0),not(y0)]
[not(ntpp),not(x0)]
[not(ntpp),not(x0),not(y0)]
[not(ntppi),not(eq),not(x0),not(y0)]
[not(ntppi),not(eq),not(y0)]
[not(ntppi),not(x0),not(y0)]
[not(ntppi),not(y0)]
[not(p),pi,not(ntpp),ntppi,not(eq),not(x0)]
[not(p),pi,not(ntpp),ntppi,not(eq),not(x0),not(y0)]
[not(p),pi,not(ntpp),not(eq),not(x0)]
[not(p),pi,not(ntpp),not(eq),not(x0),not(y0)]
[not(p),pi,not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(p),not(ntpp),not(eq),not(x0)]
[not(p),not(ntpp),not(eq),not(x0),not(y0)]
```

```
[not(p),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(p),not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(pi),not(ntpp),not(ntppi),not(eq),not(x0),not(y0)]
[not(pi),not(ntppi),not(eq),not(x0),not(y0)]
[not(pi),not(ntppi),not(eq),not(y0)]
[not(x0)]
[not(x0),not(y0)]
[not(y0)]
length(171)
yes
| ?-
```

C.2 An $\mathcal{I}$ Theorem Prover for Spatial Sequents

The following code implements an intuitionistic theorem prover based on a Gentzen sequent calculus. The prover is optimised to perform better with the class of sequents generated by the encoding of RCC-8 reasoning in $\mathcal{I}$. This means that the prover is not complete for arbitrary $\mathcal{I}$ sequents. The main simplification of the calculus is that the rule for eliminating implications on the left of the sequent is replaced by modus ponens. Another variant of modus ponens in added to handle the case of an implication with a conjunction as its antecedent — see section 6.3.3.

```
% Gentzen system for propositional intuitionistic logic
% Restricted to give better performance on sets of spatial
% constraint formulae (as given in KR94).

%% set prooftrace to `on´ to see trace or use `prooftr´
%% command (below) to toggle mode.
:- dynamic prooftrace/1.
prooftrace(off).

% Output current goal if in tracing mode
entail(Prems,Conc) :- prooftrace(on),
                      format("trying to prove ~p |- ~p ~n", [Prems, Conc]),
                      fail.

%%------------------------------------------------
%% SEQUENT RULES FOR I
%%------------------------------------------------
%% Terminating conditions

entail(Prems, Conc)  :- member(Conc, Prems),   !,
                        entailtrace("Proven (conc is prem)~2n", []).

entail(Prems, _)     :- member(absurd, Prems),   !,
                        entailtrace("Proven (absurd prem)~2n", []).

%%--------------------------------------------------
%% Simple sequent re-writes

%%  |- equiv
entail(Prems, equiv(P,Q)) :- !,
                     setadd(P, Prems, P_Prems), !,
                     entail(P_Prems, Q), !,
```

```
                     setadd(Q, Prems, Q_Prems), !,
                     entail(Q_Prems, P), !.
%% equiv |-
entail(Prems, Conc) :-
                     extract(equiv(P,Q),Prems, Rest), !,
                     setadd2(if(P,Q), if(Q,P), Rest, NewPrems),
                     entail(NewPrems, Conc), !.

%% and |-
entail(Prems, Conc) :-
                     extract(and(P,Q),Prems,Rest), !,
                     setadd2(P, Q,  Rest, P_Q_Prems), !,
                     entail( P_Q_Prems, Conc), !.

%%  |- if
entail(Prems, if(P,Q)) :-  !,
                     setadd(P, Prems, P_Prems), !,
                     entail( P_Prems, Q), !.

%%  |- not
entail(Prems, not(P)) :-  !,
                     setadd(P, Prems, P_Prems), !,
                     entail(P_Prems, absurd), !.

%%-----------------------------------------------------
%% Conjunctive Splitting Rules (deterministic)

%% |- and
entail(Prems, and(P,Q)) :-  !,
                     entail(Prems, P), !,
                     entail(Prems, Q), !.

%%  or |-
entail(Prems, Conc) :-
                     extract(or(P,Q), Prems, Rest), !,
                     setadd(P, Rest, P_Prems),
                     entail(P_Prems, Conc), !,
                     setadd(Q, Rest, Q_Prems), !,
                     entail(Q_Prems, Conc), !.
```

```
%%-----------------------------------------------
%% Pruning Rules
%% Not necessary for completeness but save a lot of search.

%% Implications are redundant if you have the conclusion
entail(Prems, Conc) :-
                     extract(if(_,Q), Prems, Rest),
                     member(Q, Rest), !,
                     entail( Rest, Conc ), !.

%% More such rules could be added for greater efficiency

%%------------------------------------------------
%% Non-deterministic Rules
%% Application of these rules reduces sequent to a logically
%% stronger form: so must backtrack for completeness.
%% (rules xentail(..) are not used for the spatial reasoner
%% but would be needed for complete intuitionistic reasoning.)

%%  if |-
%%  The if rule is not used for the spatial constraints
%%  It is replaced by modus ponens and another similar rule.
%%  see below.
disabled_entail(Prems, Conc) :-
                     extract(if(P,Q), Prems, Rest),
                     entail(Rest, P), !,
                     setadd( Q, Rest, Q_Prems),
                     entail( Q_Prems, Conc ), !.

%%  not |-
%%  re-write not(X) to  if( X, absurd )
entail(Prems, Conc) :- extract(not(P),Prems, Rest), !,
                    setadd( if(P, absurd), Rest, NewPrems),
                    entail( NewPrems, Conc), !.


%% Using modus ponens for `if |-' is not complete for I in
%% general; but it is complete for the topological constraints
%% if used together with the similar rule following.
entail(Prems, Conc) :-
```

```
                     extract(if(P,Q), Prems, Rest),
                     member(P, Rest),
                     setadd(Q, Rest, Q_Prems),
                     entail( Q_Prems, Conc ).

%% Rule for constraint `not(and(X,Y))' on left
%% a Modus Ponens variant
entail(Prems, Conc) :-
                     extract(if(and(X,Y),Q), Prems, Rest),
                     member(X, Rest),
                     member(Y, Rest),
                     setadd(Q, Rest, Q_Prems),
                     entail( Q_Prems, Conc ).

%%  We still have non-determinism for disjunctive conclsions.
%%  This could also be eliminated by adding more prunig rules.
%%  |- or
entail(Prems, or(P,Q)) :-
                     ( entail(Prems, P)
                       ;
                       entail(Prems, Q)
                     ), !.

%%----------------------------------------------------

%% Conclusion may also be given as singleton list
%% (for compatibility with other sequent progs)
entail(Prems, [Conc]) :- !, entail(Prems, Conc).

%%% FAIL %%%
% If no rule applicable fail current entail goal
entail(_,_) :- entailtrace("Failed~2n", []),
               fail.


%%-----------------------------------------------------
% alternative top-level call for single premiss sequents
entails(P,Q):- entail([P], Q).

%---------------------------------------------------
```

```
% Simple predicates used above

% extract(X,List,Rest) -- X occurs in List, remainder is Rest
% the definition is now kept in ~/prolog/lib/mylib.pl
% extract(X, List, Rest) :- append(A, [X | B], List),
%                           append(A, B, Rest).

% add an element to a set
setadd( Elt, Set, Set) :- member(Elt, Set), !.
setadd( Elt, Set, [Elt | Set]).

% add two elts to a set
setadd2( Elt1, Elt2, SetIn, SetOut ) :-
               setadd( Elt1, SetIn, SetOut1 ),
               setadd( Elt2, SetOut1, SetOut).

%---------------------------------------------------
% Tracing the prover

% output with `format' if prooftrace is on
entailtrace(Str,Args) :-  ( prooftrace(on) ->
                            format(Str, Args)
                            ; true
                          ).

% Toggle proof tracing
prooftr :- (prooftrace(on) ->
            ( retractall(prooftrace(_)),
              assert(prooftrace(off)),
              write(prooftrace(off))
            )
            ;
            ( retractall(prooftrace(_)),
              assert(prooftrace(on)),
              write(prooftrace(on))
            )).

%------------------------------------------------------
% Some example test problems
```

```
check(emi) :- entail([], [or(p, not(p))]).
check(dn1) :- entails( p, not(not(p))).
check(dn2) :- entails( not(not(p)), p).

check(test):-
           entail([not(and(c,con(a))),not(and(b,con(c))),
                   if(c,con(b)),or(not(c),not(b)),
                   not(and(a,con(c))),if(c,con(a)),
                   or(not(c),not(a))],
                   not(and(c,con(b)))).

hard_test:- % Not so hard with restriced rules
             entail([not(and(a,b)), not(and(b,a)),
                     or(not(b),not(a)),not(and(b,c)),
                     if(c,con(b)),or(not(c),not(b)),
                     not(and(a,c)), not(and(c,a)),
                     or(not(c),not(a)),if(b,con(b)),
                     if(a,con(a)),if(c,con(c)),
                     not(and(con(b),con(a)))).
```

C.3 A Special Purpose $O(n^2)$ Algorithm for Spatial Sequents

The following code implements the $\mathcal{I}$ reasoning algorithm based on the optimised sequent calculus rules given in section 6.3.4. As with the program given in appendix C.2, this means that the proof system is only complete for sequents arising from the $\mathcal{I}$ encoding of RCC-8 consistency problems and not for arbitrary sequents of $\mathcal{I}$. For this restricted class of sequents it can be shown that the worst case run-time is of $O(n^2)$ in the number formulae on the l.h.s. of the sequent. This number is of the order of the number of RCC-8 relations which are to be tested for consistency; however, checking consistency of n relations requires $O(n)$ separate $\mathcal{I}$ sequents to be tested. Thus checking consistency of n RCC-8 relations requires $O(n^3)$ time. This result applies more generally to any set of relations which can be represented as a conjunction of RCC-7 relations and their negations.

```
%% n3top.pl

%% A decision procedure for spatial entailments encoded into sequents
%% of the binary fragment of intuitionistic propositional logic.

%% Declare dynamic predicates to store model and entailment constraints.
%% The last argument is a status flag used to keep track of formulae
%% which are asserted temporarily in the course of testing an entailment.

%% There are four kinds of model constraint:
:- dynamic mcon_or/3.
:- dynamic mcon_if/3.
:- dynamic mcon_nand/3.
%% Atoms are stored as mcon( AtomName, Status ).
:- dynamic mcon_atom/2.

%% Entailment constraints are stored as 'econ( Formula, Status )'.
:- dynamic econ/2.

%% Three status flags are used
%% db   -- formula is part of a consistent database encoding spatial facts
%% test -- formula is associated with a putative spatial fact whose
%%         consistency is to be tested.
%% pr   -- formula is asserted temporarily while testing particular sequent.

%% The flags ought to be further parameterised by some database id
%% (ie we would have: db(id), test(id) and pr(id).
%% Then we could use multiple databases.
```

```
%% ------------------------------------------------------------
%% Predicates for adding to the database

%% add_mcon( Formula, Status )
%% Add a model constraint formula to the database
%% also add extra implications entailed by disjunctions
%% and add closure of all implications
%% All added formulae have status S

add_mcon( if(X,Y), S ) :-
                    add_imp_and_close( if(X,Y), S ).
add_mcon( or(if(X,f),Y), S ) :-
                    % add the entailed implication if(X,Y)
                    add_imp_and_close( if(X,Y), S ),
                    assert_if_new( mcon_or( if(X,f), Y, S ) ).
add_mcon( or(if(X,f), if(Y,f)), S ) :-
                    add_imp_and_close( if(X,if(Y,f)), S ),
                    assert_if_new( mcon_or( if(X,f), if(Y,f), S ) ).

add_mcon(A, S) :- atom(A),
               assert_if_new( mcon_atom( A, S ) ).

add_mcon_list([], _).
add_mcon_list([H|T], S) :- add_mcon(H, S),
                        add_mcon_list(T, S).

add_econ_list([], _).
add_econ_list([H|T], S) :- assert( econ(H, S) ),
                        add_econ_list(T, S).

%% -----------------------------------------------------
%% add_imp_and_close( Implication, Status )
%% Add an implication to the database together with all its consequences
%% Status flag S also added, which allows temporary formulae to be removed.

% if already there do nothing
add_imp_and_close( if(X,Y), _ ) :- mcon_if(X,Y, _), !.

% if subsumed do nothing
add_imp_and_close( if(X,if(Y,f)), _ ) :-
```

```
                              ( mcon_nand(X,Y, _);
                                mcon_nand(Y,X, _);
                                mcon_if(X,f, _);
                                mcon_if(Y,f, _ )
                              ), !.

%% add if(X,if(Y,f)) and consequences
add_imp_and_close( if(X,if(Y,f)), S ) :- !,
             sweep( ( mcon_if(A,X, _),
                      assert_if_new( mcon_nand(A,Y, S) )
                    )
                  ),
             sweep( ( mcon_if(B,Y, _),
                      assert_if_new( mcon_nand(X,B, S) )
                    )
                  ),
             assert( mcon_nand(X,Y, S) ).

%% add simple if(X,Y) and consequences
add_imp_and_close( if(X,Y), S ) :-
             sweep( ( mcon_if(Y,Z, _),
                      assert_if_new( mcon_if(X,Z, S) )
                    )
                  ),
             sweep( ( mcon_if(Z,X, _),
                      assert_if_new( mcon_if(Z,Y, S) )
                    )
                  ),
             assert( mcon_if(X,Y, S) ).


%% ------------------------------------------------------------
%% prove( Formula )
%% This is true if Formula is a consequence of the model constraints
%% stored in the database.

prove( if(X,f) ) :- !,
                    prune_ors_wrt( if(X,f) ),
                    assert_if_new( mcon_atom(X, pr) ),
                    derive_by_modus_ponens(f).
```

```
prove( if(X,Y) ) :- assert( mcon_atom(X, pr) ),
                    derive_by_modus_ponens(Y).

prove( if(X,if(Y,f)) ) :- assert( mcon_atom(X, pr) ),
                          assert( mcon_atom(Y, pr) ),
                          derive_by_modus_ponens(f).

prove( or(if(X,f), if(Y,f)) ) :- !,
                    prune_ors_wrt( if(X,f) ),
                    prune_ors_wrt( if(Y,f) ),
                    assert( mcon_atom(X, pr) ),
                    ( derive_by_modus_ponens(f)
                      ;
                      ( clean(pr),
                        assert_if_new( mcon_atom(Y, pr)),
                        derive_by_modus_ponens(f)
                      )
                    ).

prove( or(if(X,f),Y) ) :- !,
                    prune_ors_wrt( if(X,f) ),
                    prune_ors_wrt( Y ),
                    assert( mcon_atom(X, pr) ),
                    ( derive_by_modus_ponens(f)
                      ;
                      ( clean(pr),
                        derive_by_modus_ponens(Y)
                      )
                    ).

%% --------------------------------------------------------
%% Specification of the PROOF RULES

%% Add all consequences of the pruning rule for disjunctions

prune_ors_wrt( F ) :-
               sweep( ( mcon_or(X,F, _), add_mcon( X, pr ) ) ),
               sweep( ( mcon_or(F,X, _), add_mcon( X, pr ) ) ).
```

```
derive_by_modus_ponens( Conc ) :-
          %% probably wont terminate as soon as Conc found
          % First sweep over all MP applications
          sweep( ( mcon_atom(A, _),
                   \+( mcon_atom(f, _) ),    %stop if inconsistent
                   \+( mcon_atom(Conc, _) ), %stop if proved
                   ( ( mcon_if( A, B, _ ),
                       assert_if_new( mcon_atom(B, pr) )
                     );
                     ( mcon_nand( A, B, _ ),
                       assert_if_new( mcon_if(B,f, pr) )
                     );
                     ( mcon_nand( C, A, _ ),
                       assert_if_new( mcon_if(C,f, pr) )
                     )
                 % Could also subsume if(X,A) clauses
                 % But must replace them if using an incremental DB
                   )
                 )
               ),
          % Then test whether Conc or f has been derived
          (mcon_atom(Conc, _) ; mcon_atom(f, _)).

%% -------------------------------------------------

%% ** Top-level predicate for testing intuitionistic sequents
test_sequent(Prems, Conc ) :-
               clean,
               add_mcon_list(Prems, test),
               prove(Conc),
               clean(test).

%% query database
%% Use check_new_cons_wrt_db to check consistency
%% Necessary if all query Mcons also entailed by db Mcons
%%             and all query Econs entailed by db Econs

%% This version only checks consistency
query_db( Rel, Ans ) :-
         rcc8i( Rel, Mcons, Econs ),
```

```
          (check_new_cons_wrt_db( Mcons, Econs )
             -> Ans = consistent
              ; Ans = inconsistent
          ),
          clean(test), clean(pr).

check_new_cons_wrt_db(Mcons, Econs) :-
                   add_mcon_list( Mcons, test ),
                   add_econ_list( Econs, test ),
                   all_econs_consistent.

all_econs_consistent :-
         \+( ( econ( F, _ ),
                clean(pr),
                prove( F )
              )
           ).

%%------------------------------------------------------------
%% Time random queries wrt a fixed database.

time_random_queries(No, Regs, T, AvT) :-
                  statistics(runtime,[_,_]),
                  do_n_random_queries(20, Regs),
                  statistics(runtime,[_,T]),
                  AvT is T/No.

do_n_random_queries(0, _) :- !.
do_n_random_queries(N, RegNo) :- !,
         random_rel( RegNo, RR ),
         gen_out( testing_rel(RR) ),
         query_db( RR, _ ),
         NextN is N-1,
         do_n_random_queries(NextN, RegNo), !.

%% ------------------------------------------------------------

generate_random_db( _, 0, 0).
generate_random_db( RegNo, RelNo, RelsTried ) :- !,
                    random_rel( RegNo, RR ),
```

```
                    gen_out( testing_rel(RR) ),
                    rcc8i( RR, Mcons, Econs),
                    ( add_if_consistent( Mcons, Econs )
                       -> (
                            MoreRels is RelNo -1,
                            gen_out('Consistent: more rels to add '(MoreRels)),
                            generate_random_db( RegNo, MoreRels, MoreTries),
                            RelsTried is MoreTries +1
                          )
                       ;  ( gen_out('Inconsistent wrt DB'),
                            generate_random_db( RegNo, RelNo, MoreTries),
                            RelsTried is MoreTries +1
                          )
                    ), !.

add_if_consistent( Mcons, Econs ) :-
           check_new_cons_wrt_db(Mcons, Econs),
           clean(pr),
           change_status(test,db), !.

add_if_consistent(_,_) :-
          clean(pr), clean(test), fail.


time_random_db(Regs, Rels, Tried, T) :-
                  clean,
                  statistics(runtime,[_,_]),
                  generate_random_db(Regs,Rels, Tried),
                  statistics(runtime,[_,T]),
                  nl, write(regions(Regs)),
                  nl, write(relations(Rels)),
                  nl, write(tried(Tried)),
                  nl, write(time(T)), nl, ttyflush.

%% ---------------------------------------------------
%% Predicates for adding removing and changing status of formulae
%% in the database

%% Add mcon unless already present
%% Note that the existing fact need not have same status
```

```
assert_if_new( mcon_if(X,Y,S) ) :-
                ( mcon_if(X,Y,_)   ; assert(mcon_if(X,Y,S)) ), !.
assert_if_new( mcon_nand(X,Y,S) ) :-
                ( mcon_nand(X,Y,_) ; assert(mcon_nand(X,Y,S)) ), !.
assert_if_new( mcon_or(X,Y,S) ) :-
                ( mcon_or(X,Y,_)   ; assert(mcon_or(X,Y,S)) ), !.
assert_if_new( mcon_atom(X,S) ) :-
                ( mcon_atom(X,_)   ; assert(mcon_atom(X,S)) ), !.

% clean(S) -- remove from the database all dynamic facts with status S
clean(S) :- retractall( mcon_or(_,_,S) ),
            retractall( mcon_if(_,_,S) ),
            retractall( mcon_nand(_,_,S) ),
            retractall( mcon_atom(_,S) ),
            retractall( econ(_,S) ).

% clean
% remove all dynamic facts from the database
clean :- clean(_).

% Change status of all mcons with status S1 to S2.
change_status(S1,S2) :-
           sweep( ( (retract( mcon_or(X,Y,S1) ),
                                assert(mcon_or(X,Y,S2)) );
                    (retract( mcon_if(X,Y,S1) ),
                                assert(mcon_if(X,Y,S2)) );
                    (retract( mcon_nand(X,Y,S1) ),
                                assert(mcon_nand(X,Y,S2)) );
                    (retract( mcon_atom(X,S1) ),
                                assert(mcon_atom(X,Y,S2)) )
                  )
                ).

%% ------------------------------------------------------------
%% rcc8i
%% This predicate specifies the mapping from RCC8 relations
%% to intuitionistic model and entailment constraint formulae

%%     RCC rel     Model Constraints       Entailment Constraints
```

```
rcc8i( dc(X,Y),    [or(if(X,f), if(Y,f))], [if(X,f), if(Y,f)] ).
rcc8i( ec(X,Y),    [if(X,if(Y,f))],        [or(if(X,f), if(Y,f)),
                                             if(X,f), if(Y,f)]).
rcc8i( po(X,Y),    [],                     [if(X,if(Y,f)), if(X,Y),
                                             if(Y,X), if(X,f), if(Y,f)]).
rcc8i( tpp(X,Y),   [if(X,Y)],              [or(if(X,f),Y),
                                             if(X,f), if(Y,f)]).
rcc8i( tppi(X,Y),  [if(Y,X)],              [or(if(Y,f),X),
                                             if(X,f), if(Y,f)]).
rcc8i( ntpp(X,Y),  [or(if(X,f),Y)],        [if(X,f), if(Y,f)] ).
rcc8i( ntppi(X,Y), [or(if(Y,f),X)],        [if(X,f), if(Y,f)] ).
rcc8i( eq(X,Y),    [if(X,Y), if(Y,X)],     [if(X,f), if(Y,f)] ).

%-----------------------------------------------------------
% Auxilliary Minor Predicates

gen_out_flag(on).
gen_out(_) :- gen_out_flag(off), !.
gen_out(O) :- write(O), nl, ttyflush, !.


:- use_module(library(random)).

random_rel( RegNo, Rel ) :-
            random_elt([dc,ec,po,tpp,tppi,ntpp,ntppi,eq], R),
            random(0,RegNo, R1),
            random(0,RegNo, R2),
            Rel =.. [R, r(R1), r(R2)].

random_elt(L,E) :- length(L,Len),
                   Lim is Len +1,
                   random(1,Lim,R),
                   nth(R,L,E).
```

Appendix D

Redundancy in Composition Tables

This appendix summarises the main results that were published in (Bennett 1994a) concerning the redundancy of information in composition tables.

If a basis set contains n relations, then there will be n^2 table entries and if computing each entry requires making n consistency checks then the total number of consistency checks required to construct the table will be n^3. However, a consideration of the structure of a composition table will reveal that it contains a large amount of redundant information. Hence much of the work done in consistency checking to compute such a table is also redundant. One sort of redundancy occurs because, if we compute each cell of a composition table separately, we end up checking the consistency of identical sets of relations several times. Further redundancy is introduced by the fact that any relation can be written in two ways: by inverting the relation and swapping the order of the arguments.

Clearly a composition table can be constructed very easily once we know the set of consistent triangular configurations of relations drawn from the basis set under consideration. Furthermore, once we have determined whether a triangle is consistent, we have already determined the consistency of the essentially equivalent triangles obtained by rotating the original or inverting each of its relations. The exact number of triangles equivalent to a given triangle depends upon the distribution of symmetric and asymmetric relations and whether it contains duplicate relations.

The question I now address is: how many essentially distinct triangles can be formed from s symmetric and a asymmetric relationsΓ Consider an arbitrary set of relations consisting of s symmetric relations, a asymmetric relations and a further asymmetric relations which are their converses. Figure D.1 shows all possible configurations of symmetric and asymmetric relations in a triangle, modulo rotation and flipping. The capital letters S and A stand for 'symmetric' and 'asymmetric' and indicate the numbers of each type of relation present in the triangle. The small letters 'c', 'd' and 'f', stand for 'converging', 'diverging' and 'following', which describe the different ways in which two asymmetric relations can be arranged. 'r' and 'n' denote rotating and not rotating configurations of three asymmetric relations.

To calculate the total number of essentially different triangles, the numbers of possible instantiations of each of these configurations were calculated case by case. After some manipulation, the

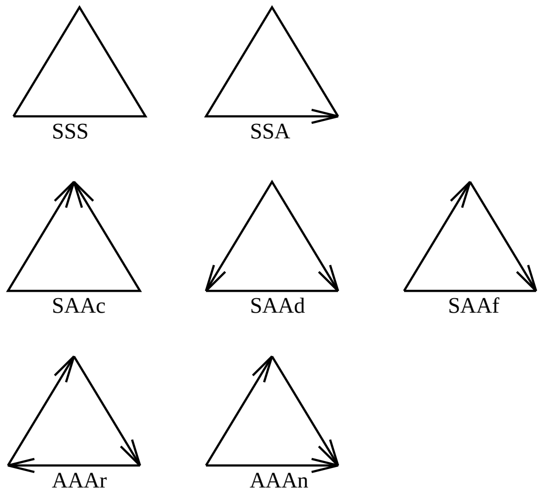

Figure D.1: Possible configurations of symmetric and asymmetric relations

following polynomial giving the total number, T of essentially distinct triangles in terms of s and a was arrived at:

$$T = \frac{1}{6}(s^3 + 3s^2 + 2s) + s^2a + s(2a^2 + a) + \frac{1}{3}(4a^3 + 2a)$$

We also know that the total number n of relations in a theory is equal to $s + 2a$, so $s = n \perp 2a$. By substituting $n \perp 2a$ for s in the polynomial we end up with a simpler equation primarily involving n:

$$T = \frac{1}{6}(n^3 + 3n^2 + 2n) \perp na$$

As the number of relations increases, the n^3 terms of the (second) equation will dominate. Thus for large n the number of distinct triangles will approach $n^3/6$.

The following table shows values of s, a, n^3, and T for a number of theories for which composition tables have been constructed. RCC-8 is the basis of eight topological relations defined in Randell, Cui and Cohn (1992). RCC-23 is a basis of spatial relations involving containment whose definition is discussed in Cohn et al. (1993) (the complete composition table is given in Bennett (1994b)). IC-13 is Allen's (1983) temporal interval calculus; and LOS-14 is Galton's (1994) Line of Sight calculus. The final column gives T as a percentage of n^3.

Basis Set	s	a	n^3	T	%
RCC-8	4	2	512	104	20.3
RCC-23	7	8	21167	2116	17.4
IC-13	1	6	2197	377	16.8
LOS-14	2	6	2744	476	17.3

Table D.1: Composition table redundancy figures for four relation sets

Hence, by looking at relational compositions as being charaterised by a set of consistent triangles rather than by a table and by taking advantage of rotational and mirror symmetry exhibited by these triangles, the computational work needed to determine the compositions of a set of relations can be reduced to approximately one sixth of what would be required using the naïve, table-based approach. Moreover, rather than storing a composition table, it is sufficient to store just the consistent triangles (or the inconsistent ones, if there are less of those). It is easy to see that from this information, composition table entries can be computed by a constant time bounded algorithm.

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