Cavity Quantum Electrodynamics of fibre-cavity networks

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The candidate confirms that the work submitted is his own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

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Abstract

Quantum mechanics, despite its abstract and unintuitive nature, is increasingly used in real-life applications. This thesis explores the current status of cavity quantum electrodynamics and its role in applications of quantum optics to quantum technologies. Various approaches to the treatment of optical cavities are discussed, with particular focus on a treatment in terms of cavities as linear optical devices, with a nonlinearity introduced by an atom, molecule, quantum dot etc in the cavity. Open quantum systems such as optical cavities coupled to a free external radiation field can be described by a quantum master equation. This thesis develops a description of such a system in which the Hamiltonian describes the coherent evolution of light inside the cavity, and the damping term describes the leaking of light out of the cavity mirrors. The goal of this approach is to describe couple networks of optical cavities in which information is transferred across the network coherently.

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Chapter 1

Introduction

This thesis explores the current status of cavity quantum electrodynamics (QED) and its role in applications of quantum optics to quantum technologies. Various approaches to the treatment of optical cavities are discussed, with particular focus on a treatment in terms of cavities as linear optical devices, with a nonlinearity introduced by an atom, molecule, quantum dot etc in the cavity.

The first chapter of the thesis is on background material, and covers the standard formulation of ideal cavities in terms of standing waves. It is possible to discuss these purely classically, since the classical equations of motion—i.e. Maxwell's equations—determine the mode structure of the quantum theory, which determines the transmission and reflection coefficients of the cavity. In chapter 4 the theory of leaky cavities is introduced. This is done using a master equation approach, in which a Hamiltonian describes the coherent evolution of light within the cavity and a damping term describes the decay of light out of the cavity through the mirrors. This approach is useful for describing multiple cavities connected by optical fibres.

In chapter 3 the quantization of the electromagnetic field in free space is discussed. In Section 3.1, we review Maxwell's equations and discuss their basic solutions in the absence of any charges and currents. In Section 3.2, we derive the corresponding observables of the quantised electromagnetic field for waves propagating along a particular axis from basic principles. In Section 3.3 we generalise these observables to the case of waves propagating in three dimensions. Finally, we summarise our findings in Section 7.

In order to address the issue of light leaking from a cavity with imperfect mirrors, some relevant analysis of electrodynamics in dielectric media is undertaken in chapter 5. This has been a subject of intense study in recent years.

Chapter 6 develops the master equation approach to leaky cavities further, and derives the appropriate Hamiltonian for such processes which match the classical trasmission of a cavity. The parameters of the model are then fixed in terms of the reflection coefficients of the mirrors of the cavity.

Chapter 2

Predictions of Maxwell's equations for light propagating through a cavity

2.1 Introduction

Consider the experimental setup in Fig. 6.1 of a dielectric slab of width d and refractive index n. If we assume normal incidence and consider the idealized case, where we can ignore diffraction, we can treat the system as one-dimensional. In this case Maxwell's equations are given by

$$\begin{aligned} \partial_x E(x,t) &= -\partial_t B(x,t) \\ \partial_x B(x,t) &= -\varepsilon(x)\,\partial_t E(x,t) \end{aligned} \tag{2.1}$$

with $E(x,t)$ and $B(x,t)$ being the electric and the magnetic field at position x and time t. Moreover $\varepsilon(x)$ is the permittivity of the dielectrics. For the setup which we consider here, we have

$$\varepsilon(x) = \begin{cases} n^2 & \text{for } x \in (0,L) \\ 1 & \text{elsewhere} \end{cases} \tag{2.2}$$

if the mirror surfaces are placed at $x = 0$ and $x = L$. In addition, there are two more equations which automatically apply, if we consider purely transverse modes. The solutions to these equations are traveling waves which change their wavelength and amplitude when being reflected and transmitted.

2.2 Boundary conditions

By integrating over a small interval crossing of the dielectric surface, it can be shown that the electric and magnetic fields are continuous at the boundary. The continuity equations at the mirror positions read

$$\begin{aligned} E_{\mathrm{I}}(x,t) + E_{\mathrm{R}}(x,t) &= E_{\mathrm{T}}(x,t)\,, \\ B_{\mathrm{I}}(x,t) + B_{\mathrm{R}}(x,t) &= B_{\mathrm{T}}(x,t)\,, \end{aligned} \tag{2.3}$$

for all times t and for $x = 0$ and $x = L$. The subscripts I, R and T denote incoming, reflected and transmitted components, respectively. The amplitudes of the electric and magnetic fields are related by

$$B(x,t) \;=\; \frac{\sqrt{\varepsilon(x)}}{c}\,\hat{k} \times E(x,t)\,, \tag{2.4}$$

where $\hat{k}$ is the unit wavevector. This equation allows us to eliminate B from Eq. (2.3) to get

$$\hat{k} \times (E_{\mathrm{I}} - E_{\mathrm{R}}) = n\,\hat{k} \times E_{\mathrm{T}}\,,$$

where the sign change is a result of the reflected field having the opposite sign of $\hat{k}$. From this we can obtain the Fresnel reflection coefficients for normal incidence. For example, when looking at waves traveling from free space into the dielectrics with refractive index n, these are defined as $r' \equiv E_{\mathrm{R}}/E_{\mathrm{I}}$ and $t' \equiv E_{\mathrm{T}}/E_{\mathrm{I}}$ and given by

$$r' = \frac{1-n}{1+n} \quad \text{and} \quad t' = \frac{2}{1+n}\,. \tag{2.5}$$

Here r' and t' are reflection and transmission coefficients which describe the relative change of amplitude when light passes through the surface. By looking at a wave which passes from the dielectric into free space, we moreover obtain the Fresnel coefficients

$$r = \frac{n-1}{n+1} \quad \text{and} \quad t = \frac{2n}{n+1} \tag{2.6}$$

with $r = -r'$. Here $r \equiv E_{\mathrm{R}}/E_{\mathrm{I}}$ and $t \equiv E_{\mathrm{T}}/E_{\mathrm{I}}$, where the indices I, R, and T denote the electric field component incoming from the slab, reflected back into the slab and transmitted into free space, respectively. Combining Eqs. (2.5) and (2.6), one can show that

$$r^2 + tt' \;=\; 1\,. \tag{2.7}$$

This relation is known as Stokes' relation.

2.3 Continuous laser driving

Suppose monochromatic light with frequency ω_0 and wave vector $k_{\text{laser}} = \omega_0/c$ enters the Fabry-Pérot cavity. Then the relative amplitude of the electric field which travels m times across the cavity equals

$$E_{\text{T}}(x,m) \;=\; t'\,r^{m-1}\,\mathrm{e}^{\mathrm{i}mnk_{\text{laser}}d}\,t \tag{2.8}$$

when it leaves the cavity. Here $x = 0$, when m is even and $x = d$ when m is odd. This takes into account that light accumulates a phase factor $\mathrm{e}^{\mathrm{i}nk_{\text{laser}}d}$ every time it propagates the length d of the cavity. Light that is ultimately reflected back has even m, light that is transmitted has odd m. The reflected light also has a contribution of r' from the component of the light that does not enter the cavity. The total reflection and transmission coefficients of the Fabry-Pérot cavity for normal incidence are therefore given by

$$\begin{aligned} r_{\text{cav}}(\omega_0) &= r' + \sum_{m\ \text{even}} E_{\text{T}}(0,m)\,, \\ t_{\text{cav}}(\omega_0) &= \sum_{m\ \text{odd}} E_{\text{T}}(d,m) \end{aligned} \tag{2.9}$$

which implies

$$\begin{aligned} r_{\text{cav}}(\omega_0) &= r' + t' \sum_{m\ \text{even}} r^{m-1}\,\mathrm{e}^{\mathrm{i}mnk_{\text{laser}}d}\,t\,, \\ t_{\text{cav}}(\omega_0) &= t' \sum_{m\ \text{odd}} r^{m-1}\,\mathrm{e}^{\mathrm{i}mnk_{\text{laser}}d}\,t\,. \end{aligned} \tag{2.10}$$

When calculating these geometric series, we finally obtain

$$\begin{aligned} r_{\text{cav}}(\omega_0) &= r\,\frac{\mathrm{e}^{2\mathrm{i}nk_{\text{laser}}d} - 1}{1 - r^2\,\mathrm{e}^{2\mathrm{i}nk_{\text{laser}}d}}\,, \\ t_{\text{cav}}(\omega_0) &= \frac{1 - r^2}{1 - r^2\,\mathrm{e}^{2\mathrm{i}nk_{\text{laser}}d}}\,\mathrm{e}^{\mathrm{i}nk_{\text{laser}}d}\,. \end{aligned} \tag{2.11}$$

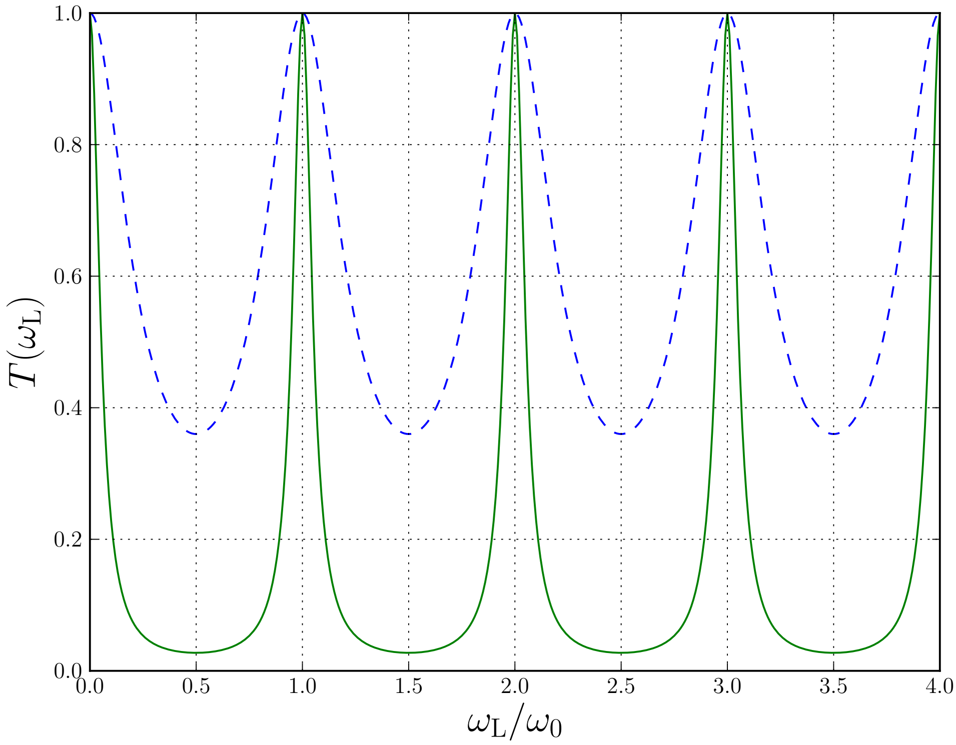

Figure 2.1: Transmission rate $T_{\rm cav}(\omega_0)$ in Eq. (2.12) of a Fabry-Pérot cavity which is driven by monochromatic light of frequency ω_0 for the refractive index $n = 3$ (dashed line) and $n = 20$ (solid line).

The overall cavity reflection and transmission rates $R_{\rm cav}(\omega_0)$ and $T_{\rm cav}(\omega_0)$ in Eq. (6.3) are given by the modulus squared of the corresponding relative amplitudes. Hence $R_{\rm cav}(\omega_0) = |r_{\rm cav}|^2$ and $T_{\rm cav}(\omega_0) = |t_{\rm cav}|^2$ which implies

$$R_{\rm cav}(\omega_0) = \frac{\frac{4r^2}{(1-r^2)^2}\sin^2(nk_{\rm laser}d)}{1+\frac{4r^2}{(1-r^2)^2}\sin^2(nk_{\rm laser}L)}\,,$$
$$T_{\rm cav}(\omega_0) = \frac{1}{1+\frac{4r^2}{(1-r^2)^2}\sin^2(nk_{\rm laser}d)} \tag{2.12}$$

with r as in Eq. (2.6). The factor $4r^2/(1-r^2)^2$ is known as the coefficient of Finesse.

Fig. 2.1 shows how the relative amplitude of the transmitted light depends on its frequency ω_0 and the refractive index n. For example, we see that laser light with a frequency equal to one of the cavity resonance frequencies ω_m in Eq. (4.2) does not get reflected by the cavity. This means, resonant light travels through the resonator, as if it were not there. In general, we find that the larger the refractive index n, the more the light is affected by the dielectric. For relatively large n, there is almost complete reflection for some frequencies ω_0. For n close to 1, traveling through the dielectrics is almost like traveling through the vacuum. In this paper, we seek a quantum master equation approach to optical cavities that reproduces these amplitudes.

2.4 Time evolution without laser driving

Suppose no external laser field is applied and a single wave packet bounces back and forth inside the two-sided cavity which is shown in Fig. 6.1. This wave packet is a superposition of plane waves. Again, we assume that all waves in the packet experience the same refractive index n, so that all parts of the wave packet travel with the same speed. After m bounces, the intensity of the wave at a fixed frequency ω_0 equals

$$I(t_m) \quad = \quad r^{2(m-1)}\, I(0)\,, \tag{2.13}$$

where $I(0)$ is the initial intensity of the wave and

$$t_m \quad \equiv \quad m\frac{nd}{c} \tag{2.14}$$

is the time it takes a wave packet to bounce m times through a medium of length d and with refractive index n. To simplify a later comparison with the predictions of a quantum model, we notice that the intensity

$$I(t) \quad = \quad r^{2ct/nd}\, I(0) \tag{2.15}$$

assumes exactly the same value as $I(t_m)$ for $t = t_m$.

Chapter 3

Quantization in Free Space

As early as 1900, Planck introduced the idea of so-called basic *energy elements* into which the radiation of a blackbody could be divided. At first, this step was a mere mathematical trick which allowed him to derive a radiation law, which was consistent with experimental observations of the spectrum of a blackbody, from basic thermodynamical principles.[48] Later on, Planck's work became recognised as the origin of quantum physics and his basic energy elements became known as *photons*. But this only happened later on, after Einstein put the reality of the energy quanta of the electromagnetic field on a more firm footing by linking it to the photoelectric effect.[49] Although it can be shown that the essence of the photoelectric effect does not require the quantisation of the radiation field,[50] the photon concept becomes unavoidable when describing the key behaviour of quantum light fields that are beyond the domain of the classical.[51; 52]

Over the last several decades, a wide range of experiments have been performed that test the properties of light at the quantum level. Most importantly, recent decades have seen the introduction and rapid improvement of devices with the ability to register single photons in the optical regime. Single photon detectors typically work by sensing an electrical signal that results from the absorption of a photon.[53] Even single infrared photons can now be detected with an efficiency as high as 97%.[54] Moreover, single photon sources[55; 56] are now an essential tool in many quantum optics laboratories worldwide. In addition to enabling novel technologies, like quantum cryptography[57; 58; 59] and linear optics quantum computing,[60; 61] quantum optics experiments have helped us to answer a highly non-trivial but seemingly simple question,[62] namely “What is a photon?”.

When asked, most physicists now simply state that photons are the basic energy quanta of the electromagnetic field thus invoking the idea of the photon being a “particle” of light. This interpretation is not without its problems, a major one being that these energy quanta do not have a well-defined position, rather being infinite in extent. Others would avoid these problems by taking an instrumentalist approach where the photon is defined as simply whatever makes a detector click in an experiment sensitive enough to demonstrate the quantised nature of light. But this anti-realist interpretation is not particularly satisfying either. Field quantisation schemes in basic text books are often more mathematically than physically-motivated and therefore usually more detached from reality than is strictly necessary — it could be argued that this adds unnecessary difficulties.

The formal quantization of the electromagnetic field was first performed by Dirac in 1927.[63] Since then, most field quantisation schemes have relied on the mathematical fact that any function on a finite interval can be written as a Fourier series. More concretely, any real-valued function f with argument $x \in (0, d)$ can be expanded in a series of exponentials,[64]

$$f(x) \;=\; \sum_{m=-\infty}^{\infty} c_m \exp\left(\mathrm{i}m\,\frac{2\pi x}{d}\right) , \tag{3.1}$$

where the c_m are complex coefficients with $c_m = c^*_{-m}$. This is usually taken as the starting point when quantising the electromagnetic field inside a finite quantisation volume with certain electric field components vanishing at the boundaries.[52; 65; 66; 67; 68; 69; 70; 71; 72; 73; 74] Inspired by the above equation, the electromagnetic field observables are written as Fourier series of discrete sets of eigenfunctions which are the basic solutions of Maxwell's equations for the vector potential of the electromagnetic field in Coulomb gauge. The coefficients c_m and c^*_{-m} of these series are eventually replaced by photon annihilation and creation operators $\hat{c}_m$ and $\hat{c}^\dagger_m$, respectively. Subject to normalisation, the above-described canonical quantisation procedure automatically yields a harmonic oscillator Hamiltonian of the form

$$H_{\text{field}} \;=\; \sum_{m=1}^{\infty} \hbar\omega_m\, c_m^{\dagger} c_m + H_{\text{ZPE}} \tag{3.2}$$

which sums over a discrete set of cavity frequencies ω_m and where H_{ZPE} denotes the energy of the vacuum, the so-called zero point energy. Afterwards, the infinite-volume limit is taken to yield the field observables of the free radiation field, thereby introducing a continuum of eigenfrequencies.

The purpose of this chapter is to derive the field observables of the free radiation field from basic physical principles in a more direct way. The approach we present here is motivated by a recent experimental push towards integrated photonic devices for quantum information processing.[75; 76] Currently, a lot of effort is made worldwide to combine linear optics elements, optical cavities and single photon sources[55] to realise a so-called quantum internet[77] and quantum networks for quantum simulations.[78] When modelling such systems, it becomes important to use the same notion for the description of photons inside photonic devices as in linear optics scattering theory. One way of doing so is to extend the field quantisation scheme presented in this chapter to the scattering of light through mirrors and optical resonators.[79]

The physically-motivated field quantisation scheme which we present here has several advantages. For example, it does not invoke the solutions of Maxwell's equations in a specific gauge and there is no need to consider a finite quantisation volume with boundary conditions before being able to go to the infinite-volume limit. Instead, the starting point of our considerations is the experimental reality of what a photon is. We then notice that the basic principles of quantum physics for the construction of observables uniquely identify the relevant Hilbert space and the Hamiltonian H_{field} of the electromagnetic field inside a non-dispersive, non-absorbing, homogeneous medium. The usual expressions of the electric and magnetic field observables then follow from Heisenberg's equation of motion.

3.1 Classical Electrodynamics

We begin by considering Maxwell's equations in a non-dispersive, non-absorbing, homogeneous medium with (absolute) permittivity ε and permeability μ. In the absence of any currents and charges, these equations are given by [80]

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = 0, \qquad \nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t},$$
$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0, \qquad \nabla \times \mathbf{B}(\mathbf{r},t) = \varepsilon\mu \,\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}. \tag{3.3}$$

Here $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ are the electric and the magnetic field vector at time t and at position $\mathbf{r}$ within the medium, respectively. It is well known that the basic solutions of the above equations are travelling waves with wave vectors $\mathbf{k} = k\,\kappa$, where κ is a unit vector giving the relevant direction of the propagation.[80] The frequency ω of these waves can assume any positive value and relates to the magnitude of the wave vector via the dispersion relation

$$\omega \equiv \frac{k}{\sqrt{\varepsilon\mu}}. \tag{3.4}$$

In vacuum, this dispersion relation becomes $\omega = k/\sqrt{\varepsilon_0\mu_0} \equiv ck$ where ε_0 and μ_0 are the vacuum permittivity and permeability, respectively, and where c is the speed of light. The field vectors $\mathbf{E}$ and $\mathbf{B}$ of these travelling waves are perpendicular to $\mathbf{k}$ and each other in order that the divergence of the electric and magnetic fields vanish as required by Maxwell's equations.

3.1.1 Propagation along the x axis

Suppose a travelling wave propagates along the x axis, and the corresponding $\mathbf{E}$ and $\mathbf{B}$ fields are aligned along the y and z axes, respectively. For a right-travelling wave, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ can then be written as $\mathbf{E}(\mathbf{r},t) = (0, E(x,t), 0)$ and $\mathbf{B}(\mathbf{r},t) = (0,0,B(x,t))$. Analogously, one can write $\mathbf{E}(\mathbf{r},t) = (0, E(x,t), 0)$ and $\mathbf{B}(\mathbf{r},t) = (0,0,-B(x,t))$ for a left-travelling wave. Here $E(x,t)$ and $B(x,t)$ are defined as always having the same sign. Using this notation, the two Maxwell's equations in the right-hand column of Eq. (3.3) become

$$\begin{aligned} \partial_x E(x,t) &= \pm \partial_t B(x,t), \\ \partial_x B(x,t) &= \pm \varepsilon\mu\, \partial_t E(x,t). \end{aligned} \tag{3.5}$$

Which sign applies depends on whether the wave propagates in the positive or negative x direction. In the following, we use the indices L and R to distinguish between left and right travelling waves. As illustrated in Fig. 3.1, the plus sign in Eq. (6.6) applies to the first (L) and the minus sign applies to the second (R) case. In one dimension, the total energy of the electromagnetic field equals

$$H_{\text{field}} = A \int_{-\infty}^{\infty} dx \, \frac{1}{2} \left[\varepsilon E^2(x,t) + \frac{1}{\mu} B^2(x,t) \right] . \tag{3.6}$$

Here A is an area in the y-z plane in which the Hamiltonian H_{field} is defined. This above expression is quadratic in both field components and constant in time. The latter can be shown using Eq. (6.6).

3.1.2 Propagation in three dimensions

In three dimensions, the basic solutions of Maxwell's equations are analogous to the case of waves propagating along the x axis, but waves travelling in any direction of space and of any polarisation need to be considered. Each one of these travelling wave solutions is called a mode, and is characterised by a polarisation λ, which specifies the (positive) direction of its electric field, and a wave vector $\mathbf{k}$, which specifies its frequency and direction of propagation. The general solutions of Maxwell's equations are the superpositions of all possible modes $(\mathbf{k}, \lambda)$.

3.2 Field quantisation for propagation in one dimension

As in the previous section, we consider a non-dispersive, non-absorbing, homogeneous medium with permittivity ε and permeability μ. We then begin our quantisation by returning to the question in the introduction: "What is a photon?".[62] To answer this question, we point out that a detector that measures the energy of a very weak electromagnetic field produces discrete clicks. Single photon experiments have shown that these clicks are the signature of the fundamental property of the electromagnetic field, which is that the energy it carries is quantised.[53; 54; 55; 56; 57; 58; 59; 60; 61] These energy quanta are called photons.

3.2.1 The relevant Hilbert space

We know from observations that photons propagating in *one dimension* are characterised by their (positive) frequency, $\omega \in (0, \infty)$ and a direction of propagation, $\mathrm{X} = \mathrm{L}, \mathrm{R}$. In addition, they are characterised by their so-called polarisation, $\lambda = 1, 2$, which indicates the direction of their respective electric field vector. Combining these experimental facts with the rules of quantum physics, we then know that the Hilbert space for the description of the quantised electromagnetic field is spanned by tensor product states of the form

$$\prod_{\omega=0}^{\infty} \prod_{\mathrm{X}=\mathrm{L,R}} \prod_{\lambda=1,2} |n_{\mathrm{X}\lambda}(\omega)\rangle \ , \tag{3.7}$$

where $n_{\mathrm{X}\lambda}(\omega)$ is the number of excitations in the $(\mathrm{X}, \lambda, \omega)$ photon mode. By construction, photons in different modes are in pairwise orthogonal states. In the following, we denote the ground state of the electromagnetic field, the so-called vacuum state, by $|0\rangle$.

3.2.2 Field Hamiltonian

The Hamiltonian of a system in the Schrödinger picture is its energy observable. Experiments have shown that the energy of an electromagnetic field increases by $\hbar\omega$, whenever a photon of frequency ω is added. Moreover we know that the energy eigenstates of the field are the states in Eq. (3.7) with an integer number of photons in the field. Hence the Hamiltonian $\hat{H}_{\mathrm{field}}$ of the quantised electromagnetic field is such that

$$\hat{H}_{\mathrm{field}} \ |n_{\mathrm{X}\lambda}(\omega)\rangle \ = \ \left[\hbar\omega \, n_{\mathrm{X}\lambda}(\omega) + H_{\mathrm{ZPE}}\right] |n_{\mathrm{X}\lambda}(\omega)\rangle \ . \tag{3.8}$$

The constant H_{ZPE} in this equation denotes again the zero point energy, which we determine later on in this section. One way of obtaining an explicit expression for H_{field} is to sum over all the projectors onto the eigenstates of this operator multiplied by their respective eigenvalues, as it is usually done in quantum physics when constructing an observable.

Next we notice that the electromagnetic field has exactly the same energy level structure as a collection of independent harmonic oscillators with each of them characterised by a specific frequency ω, a polarisation λ, and a direction of propagation X. This analogy suggests that it is possible to write the above field Hamiltonian in a much more compact form. To do so, we define the harmonic oscillator annihilation and creation operators $\hat{a}_{\mathrm{X}\lambda}(\omega)$ and $\hat{a}^{\dagger}_{\mathrm{X}\lambda}(\omega)$ such that

$$\begin{aligned} \hat{a}_{\mathrm{X}\lambda}(\omega)\,|n_{\mathrm{X}\lambda}(\omega)\rangle &= \sqrt{n_{\mathrm{X}\lambda}(\omega)}\,|n_{\mathrm{X}\lambda}(\omega)-1\rangle\,, \\ \hat{a}^{\dagger}_{\mathrm{X}\lambda}(\omega)\,|n_{\mathrm{X}\lambda}(\omega)\rangle &= \sqrt{n_{\mathrm{X}\lambda}(\omega)+1}\,|n_{\mathrm{X}\lambda}(\omega)+1\rangle\,. \end{aligned} \tag{3.9}$$

These are photon annihilation and creation operators , i.e. harmonic oscillator operators for each mode (ω, λ, X), and can be shown to obey the commutation relation

$$\left[\hat{a}_{\mathrm{X}\lambda}(\omega), \hat{a}^{\dagger}_{\mathrm{X}'\lambda'}(\omega')\right] = \delta_{\mathrm{XX}'}\,\delta_{\lambda\lambda'}\,\delta(\omega-\omega')\,, \tag{3.10}$$

since the states $|n_{\mathrm{X}\lambda}(\omega)\rangle$ form an orthonormal basis. Using the above notation, $\hat{H}_{\mathrm{field}}$ simplifies to

$$\hat{H}_{\mathrm{field}} = \sum_{\mathrm{X=L,R}}\sum_{\lambda=1,2}\int_0^{\infty} \mathrm{d}\omega\,\hbar\omega\,\hat{a}^{\dagger}_{\mathrm{X}\lambda}(\omega)\hat{a}_{\mathrm{X}\lambda}(\omega) + H_{\mathrm{ZPE}}\,, \tag{3.11}$$

where we sum over all possible photon modes $(\mathrm{X}, \lambda, \omega)$. One can easily check that the eigenstates and eigenvalues of the operator in Eq. (3.11) are the same as the eigenstates and eigenvalues of $\hat{H}_{\mathrm{field}}$ in Eq. (3.8).

3.2.3 Electric and magnetic field observables

We now seek expressions for the quantised electric and magnetic field observables $\hat{\mathbf{E}}(x)$ and $\hat{\mathbf{B}}(x)$ for waves propagating along the x axis which correspond to the classical field amplitudes $E(x,t)$ and $B(x,t)$ in Section 3.1.1. To obtain these operators, we notice that the classical energy of the electromagnetic field is proportional to $E(x,t)^2$ and $B(x,t)^2$ (cf. Eq. (3.6)), while the field Hamiltonian $\hat{H}_{\mathrm{field}}$ is a quadratic function of the annihilation and creation operators $\hat{a}_{\mathrm{X}\lambda}(\omega)$ and $\hat{a}^{\dagger}_{\mathrm{X}\lambda}(\omega)$ (cf. Eq. (3.11)). This suggests the following ansatz for the respective polarisation-dependent amplitudes of $\hat{\mathbf{E}}(x)$ and $\hat{\mathbf{B}}(x)$,

$$\begin{aligned} \hat{E}_{\lambda}(x) &= \sum_{\mathrm{X=L,R}}\int_0^{\infty} \mathrm{d}\omega\, f_{\mathrm{X}\lambda}(x,\omega)\,\hat{a}_{\mathrm{X}\lambda}(\omega) + \mathrm{H.c.}\,, \\ \hat{B}_{\lambda}(x) &= \sum_{\mathrm{X=L,R}}\int_0^{\infty} \mathrm{d}\omega\, g_{\mathrm{X}\lambda}(x,\omega)\,\hat{a}_{\mathrm{X}\lambda}(\omega) + \mathrm{H.c.}\,, \end{aligned} \tag{3.12}$$

where $f_{X\lambda}(x,\omega)$ and $g_{X\lambda}(x,\omega)$ are complex coefficients. Splitting the field observables in this way is well-justified, since both fields are linear and additive with respect to all possible photon modes $(\mathrm{X},\lambda,\omega)$.

We demand in the following that the expectation values of the observables of the quantised electromagnetic field behave as predicted by Maxwell's equations. Taking into account that the time derivative of an observable $\hat{\mathcal{O}}$ is given by Heisenberg's equation of motion yields

$$\partial_t \langle \hat{\mathcal{O}}(x) \rangle \;=\; -\frac{\mathrm{i}}{\hbar} \left\langle \left[\hat{\mathcal{O}}(x), \hat{H}_{\text{field}} \right] \right\rangle \tag{3.13}$$

with $\hat{H}_{\text{field}}$ as in Eq. (3.11). Hence we find consistency with Maxwell's equations as long as

$$\begin{aligned} \partial_x f_{X\lambda}(x,\omega) &= \mp \mathrm{i}\omega \, g_{X\lambda}(x,\omega)\,, \\ \partial_x g_{X\lambda}(x,\omega) &= \mp \mathrm{i}\varepsilon\mu\omega \, f_{X\lambda}(x,\omega)\,. \end{aligned} \tag{3.14}$$

The minus sign in this equation corresponds to $\mathrm{X} = \mathrm{L}$ and the plus sign corresponds to $\mathrm{X} = \mathrm{R}$ (cf. Eq. (6.6)). The general solution of this equation can be written as

$$\begin{aligned} f_{X\lambda}(x,\omega) &= K_{\mathrm{X},1}(\omega)\,\mathrm{e}^{-\mathrm{i}kx} + K_{\mathrm{X},2}(\omega)\,\mathrm{e}^{\mathrm{i}kx}\,, \\ g_{X\lambda}(x,\omega) &= \pm\sqrt{\varepsilon\mu}\left[K_{\mathrm{X},1}(\omega)\,\mathrm{e}^{-\mathrm{i}kx} - K_{\mathrm{X},2}(\omega)\,\mathrm{e}^{\mathrm{i}kx}\right] \end{aligned} \tag{3.15}$$

with the always positive wave vector k given in Eq. (3.4) and with the K constants being complex functions of ω and X but independent of x, t and λ. They can assume any value without contradicting Maxwell's equations. However, if we want the index X to specify the direction L or R, we need to ensure that the corresponding time-dependent expectation values $\langle \hat{E}(x) \rangle$ and $\langle \hat{B}(x) \rangle$ are either functions of $kx+\omega t$ or of $kx-\omega t$. This implies $K_{\mathrm{L},2} = K_{\mathrm{R},1} = 0$, while $K_{\mathrm{L},1}$ and $K_{\mathrm{R},2}$ remain unspecified.

To determine $K_{\mathrm{L},1}$ and $K_{\mathrm{R},2}$ we now introduce a final constraint on the operators $\hat{E}(x)$ and $\hat{B}(x)$, which is that the expressions themselves must produce the quantum Hamiltonian (3.11) in the previous section when substituted in the classical electromagnetic Hamiltonian (3.6). In other words, we want that

$$\hat{H}_{\text{field}} = A \int_{-\infty}^{\infty} dx \, \frac{1}{2} \left[\varepsilon \hat{\mathbf{E}}^2(x) + \frac{1}{\mu} \hat{\mathbf{B}}^2(x) \right] \tag{3.16}$$

with A defined as in Eq. (3.6). Combining this Hamiltonian with the above equations and performing the x integration yields δ-functions. Subsequently performing another integration, this finally results in

$$\hat{H}_{\text{field}} = 2\pi\varepsilon A \sum_{\lambda=1,2} \int_0^\infty \mathrm{d}\omega \left[|K_{\mathrm{L},1}|^2 \left(2\hat{a}^\dagger_{\mathrm{L}\lambda}(\omega)\hat{a}_{\mathrm{L}\lambda}(\omega) + 1 \right) + |K_{\mathrm{R},2}|^2 \left(2\hat{a}^\dagger_{\mathrm{R}\lambda}(\omega)\hat{a}_{\mathrm{R}\lambda}(\omega) + 1 \right) \right] . \tag{3.17}$$

This operator becomes identical to the field Hamiltonian in Eq. (3.11) when we choose the K constants and the zero-point energy H_{ZPE} such that

$$|K_{\mathrm{L},1}(\omega)|^2 = |K_{\mathrm{R},2}(\omega)|^2 = \frac{\hbar\omega}{4\pi\varepsilon A} \quad \text{and} \quad H_{\mathrm{ZPE}} = A \int_0^\infty \mathrm{d}\omega \, \tfrac{1}{2}\hbar\omega \, . \tag{3.18}$$

After choosing phase factors (with no physical consequences) for the above constants $K_{\mathrm{L},1}(\omega)$ and $K_{\mathrm{R},2}(\omega)$ and substituting them into the above equations, we finally obtain the electric and magnetic field observables

$$\begin{aligned} \hat{\mathbf{E}}(x) &= \mathrm{i} \sum_{\lambda=1,2} \int_0^\infty \mathrm{d}\omega \sqrt{\frac{\hbar\omega}{4\pi\varepsilon A}} \, \mathrm{e}^{-\mathrm{i}kx} \left[\hat{a}_{\mathrm{L}\lambda}(\omega) - \hat{a}^\dagger_{\mathrm{R}\lambda}(\omega) \right] \mathbf{e}_\lambda + \mathrm{H.c.} \, , \\ \hat{\mathbf{B}}(x) &= -\mathrm{i}\sqrt{\varepsilon\mu} \sum_{\lambda=1,2} \int_0^\infty \mathrm{d}\omega \sqrt{\frac{\hbar\omega}{4\pi\varepsilon A}} \, \mathrm{e}^{-\mathrm{i}kx} \left[\hat{a}_{\mathrm{L}\lambda}(\omega) - \hat{a}^\dagger_{\mathrm{R}\lambda}(\omega) \right] \left(\hat{\mathbf{k}} \times \mathbf{e}_\lambda \right) + \mathrm{H.c.} \, , \end{aligned} \tag{3.19}$$

with the (positive) wave number k defined as in Eq. (3.4) and with $\hat{\mathbf{k}} \equiv \mathbf{k}/|\mathbf{k}|$ being a unit vector in the $\mathbf{k}$ direction. The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors orthogonal to x and orthogonal to each other. For example, $\mathbf{e}_1$ could be a vector oriented along the y axis, while $\mathbf{e}_2$ points in the direction of the z axis. The above operators $\hat{\mathbf{E}}(x)$ and $\hat{\mathbf{B}}(x)$ are consistent with the usual textbook expressions for the quantised electromagnetic field propagating in one dimension.[52; 73; 74]

3.3 Field quantisation for propagation in three dimensions

As pointed out in Section 3.1.2, the electromagnetic field for waves propagating in *three dimensions* has more degrees of freedom than in the case of waves propagating along a single axis. Otherwise, both have analogous properties. Taking this into account, we immediately see that the dimension of the relevant Hilbert space is significantly larger. Now photons traveling in all possible directions in a three-dimensional space have to be taken into account. To do so, we now introduce a set of annihilation and creation operators $\hat{a}_{\mathbf{k}\lambda}$ and $\hat{a}^\dagger_{\mathbf{k}\lambda}$ with the bosonic commutator relation

$$\left[\hat{a}_{\mathbf{k}\lambda}, \hat{a}^\dagger_{\mathbf{k}'\lambda'}\right] = \delta(\mathbf{k}-\mathbf{k}')\,\delta_{\lambda,\lambda'} \,. \tag{3.20}$$

These photon operators are analogous to the one-dimensional operators $\hat{a}_{\mathrm{X}\lambda}(\omega)$ and $\hat{a}^\dagger_{\mathrm{X}\lambda}(\omega)$ in Eq. (3.9) but their respective direction X and their respective frequency ω are now specified by the direction of the wave vector $\mathbf{k}$ and a frequency ω_k defined as

$$\omega_k \equiv \frac{|\mathbf{k}|}{\sqrt{\varepsilon\mu}} \,. \tag{3.21}$$

Using the above notation and proceeding as in the previous section, the Hamiltonian of the quantised electromagnetic field in three dimensions can be written as

$$\hat{H}_{\text{field}} = \sum_{\lambda=1,2} \int \mathrm{d}^3\mathbf{k}\, \hbar\omega_k\, \hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} + H_{\mathrm{ZPE}} \,, \tag{3.22}$$

where H_{ZPE} denotes again the (infinite) energy of the vacuum. The Hilbert space of the electromagnetic field is the states space obtained when applying the above annihilation and creation operators onto this vacuum state.

To obtain expressions for the electric and magnetic field observables $\hat{\mathbf{E}}(\mathbf{r})$ and $\hat{\mathbf{B}}(\mathbf{r})$ at a position $\mathbf{r}$ within the field, we demand again that the expectation values of these operators evolve as predicted by Maxwell's equations. Imposing this condition for the travelling waves of any wave vector $\mathbf{k}$, this yields

$$\begin{aligned} \hat{\mathbf{E}}(\mathbf{r}) &= \frac{\mathrm{i}}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \int \mathrm{d}^3\mathbf{k} \sqrt{\frac{\hbar\omega_k}{2\varepsilon}}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, \hat{a}_{\mathbf{k}\lambda}\, \mathbf{e}_{\mathbf{k}\lambda} + \mathrm{H.c.}\,, \\ \hat{\mathbf{B}}(\mathbf{r}) &= -\frac{\mathrm{i}}{(2\pi)^{3/2}} \sqrt{\varepsilon\mu} \sum_{\lambda=1,2} \int \mathrm{d}^3\mathbf{k} \sqrt{\frac{\hbar\omega_k}{2\varepsilon}}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\, \hat{a}_{\mathbf{k}\lambda} \left(\hat{\mathbf{k}} \times \mathbf{e}_{\mathbf{k}\lambda}\right) + \mathrm{H.c.} \end{aligned} \tag{3.23}$$

in analogy to Eq. (3.19). The vectors $\mathbf{e}_{\mathbf{k}\lambda}$ in these equations are polarisation vectors with $\mathbf{e}_{\mathbf{k}\lambda} \cdot \mathbf{e}_{\mathbf{k}\lambda'} = \delta_{\lambda,\lambda'}$ and $\mathbf{k} \cdot \mathbf{e}_{\mathbf{k}\lambda} = 0$. Allowing for negative wave numbers k, and not only positive ones, it is no longer necessary to distinguish left and right moving photons. The normalizing factors in the above equation are different from Eq. (3.19). However, one can easily check that a photon in the $(\mathbf{k}, \lambda)$ mode has the energy $\hbar\omega_k$, when substituting the above field operators into the three-dimensional analog of Eq. (3.16) with an infinite quantisation volume.

For simplicity, this chapter avoids a more rigorous derivation of Eq. (3.23) which would require a more detailed discussion of infinite-volume limits. Instead, the above derivation exploits the fact that the general form of Eq. (3.23), up to normalisation, is already well motivated by Eq. (3.19). Finally, we note that the above electric and magnetic field operators in Eq. (3.23) are again consistent with textbook expressions[52; 73; 74] but this time for the quantised electromagnetic field in three dimensions and in an infinite quantisation volume.

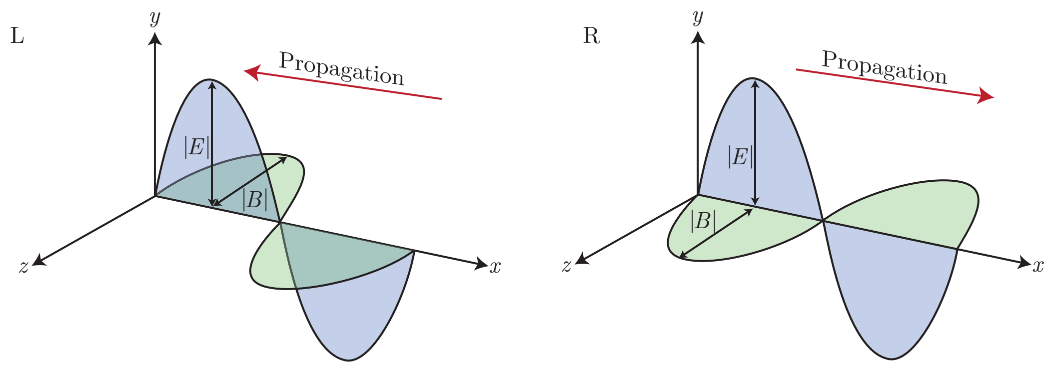

Figure 3.1: Schematic view of two travelling-wave solutions of Maxwell's equations propagating in one dimension at a fixed time t and with a fixed polarisation and a particular frequency. The directions of the $\mathbf{E}$ and $\mathbf{B}$ fields are chosen as usual in classical electrodynamics, according to a right-hand rule for waves propagating to the right (R) and to the left (L) of the x axis, respectively.

Chapter 4

Predictions of the standard standing wave cavity Hamiltonian

In this chapter, we review the standard standing-wave description of the electromagnetic field between two mirrors and have a closer look at some of its predictions. As we shall see below, this model is only well suited for the description of the time evolution of the total number of photons inside an optical cavity with resonant or near-resonant laser driving. But, as pointed out in the Introduction, it cannot be used to calculate the photon emission rates through the different sides of a two-sided cavity.

4.1 The cavity-laser Hamiltonian

The Hamiltonian of the experimental setup in Fig. 6.1 is of the general form

$$H = H_{\rm cav} + H_{\rm laser} \, . \tag{4.1}$$

The first term describes the free energy of the electromagnetic field inside the resonator. The second term takes its external driving into account. When quantising the electromagnetic field in the way of most textbooks, one derives at the assumption that the field only contains standing-wave photon modes of frequency ω_m with

$$\omega_m = m\pi \frac{c}{nd} \, , \tag{4.2}$$

where m is a positive integer, c is the speed of light, n is the refractive index of the medium inside the cavity, and d is the distance of the resonator mirrors. If c_m is the corresponding photon annihilation operator, $H_{\rm cav}$ simply equals the Hamiltonian in Eq. (6.2). The laser field which drives the cavity is usually treated as a classical field. Denoting its Rabi frequencies by Ω_n and by frequency ω_0, it equals

$$H_{\rm laser} = \sum_{m=1}^{\infty} \frac{1}{2} \hbar \Omega_m \, {\rm e}^{-{\rm i}\omega_0 t} \, c_m + {\rm H.c.} \tag{4.3}$$

This Hamiltonian arises from a spatial overlap of the classical driving field and the quantised field in the vicinity of the cavity mirrors.

When changing into the interaction picture with respect to the free Hamiltonian $H_0 = \sum_{m=1}^{\infty} \hbar\omega_0 \, c_m^\dagger c_m$ and after applying the usual rotating-wave approximation, we obtain the time-independent interaction Hamiltonian

$$H_{\rm I} = \sum_{m=1}^{\infty} \frac{1}{2} \hbar \Omega_m \left(c_m + c_m^\dagger \right) + \hbar \Delta_m \, c_m^\dagger c_m \tag{4.4}$$

with the cavity-laser detuning Δ_m defined such that

$$\Delta_m \equiv \omega_m - \omega_0 \, . \tag{4.5}$$

For simplicity, we assume in the following that the frequency ω_0 is relatively close to only one of cavity resonance frequencies ω_m. Then only one of the cavity modes has to be taken into account and $H_{\rm I}$ becomes

$$H_{\rm I} = \hbar\Omega \left(c + c^\dagger \right) + \hbar\Delta \, c^\dagger c \, , \tag{4.6}$$

after neglecting the respective index m for operators and constants. This Hamiltonian is often used in the literature when describing a laser-driven optical cavity. However, notice that this model does not distinguish whether the laser drives the cavity from the left or from the right. The laser only excites standing-wave photon modes.

4.2 The corresponding master equation

The spontaneous leakage of photons through the cavity mirrors is in the following taken into account via the usual quantum optical master equation

$$\dot{\rho}_{\rm I} \;\;=\;\; -\frac{\rm i}{\hbar}\,[H_{\rm I},\rho_{\rm I}] + \frac{1}{2}\kappa\left(2c\rho_{\rm I}c^\dagger - c^\dagger c\rho_{\rm I} - \rho_{\rm I}c^\dagger c\right)\,, \tag{4.7}$$

where κ is the cavity decay rate and $\rho_{\rm I}$ denotes the density matrix of the quantised cavity field. This equation can be derived by coupling the c-mode to a continuum of free radiation field modes outside the cavity, letting the system evolve over a short time Δt, and tracing out the free radiation field to mimic the effects of a photon-absorbing environment. Using the above standing-wave description, it is not possible to assign different decay channels to cavity photons traveling in different directions.

4.3 Time evolution of expectation values

The most straightforward way of calculating the intensity of the emitted light is to adopt a rate equation approach. Taking into account that the expectation value of any observable $A_{\rm I}$ in the interaction picture equals $\langle A_{\rm I}\rangle = {\rm Tr}(A_{\rm I}\rho_{\rm I})$, we find that

$$\langle\dot{A}_{\rm I}\rangle \;\;=\;\; -\frac{\rm i}{\hbar}\,\langle[A_{\rm I},H_{\rm I}]\rangle - \frac{1}{2}\kappa\,\langle A_{\rm I}c^\dagger c + c^\dagger c A_{\rm I} - 2c^\dagger A_{\rm I} c\rangle\,. \tag{4.8}$$

Here we are especially interested in the time evolution of the mean photon number n,

$$n \;\;\equiv\;\; \langle c^\dagger c\rangle\,. \tag{4.9}$$

In order to obtain a closed set of rate equations, including one for n, we also need to consider the expectation values

$$k_1 \equiv \langle c + c^\dagger\rangle\,, \quad k_2 \equiv {\rm i}\langle c - c^\dagger\rangle\,. \tag{4.10}$$

Using Eq. (4.8), one can then show that n, k_1, and k_2 evolve according to the linear differential equations

$$\begin{aligned} \dot{n} &= \frac{1}{2}\Omega k_2 - \kappa n \, , \\ \dot{k}_1 &= -\Delta k_2 - \frac{1}{2}\kappa k_1 \, , \\ \dot{k}_2 &= \Omega + \Delta k_1 - \frac{1}{2}\kappa k_2 \, . \end{aligned} \tag{4.11}$$

To obtain expressions for the stationary state of the laser-driven resonator, we simply set these time derivatives equal to zero.

4.4 Stationary state photon emission rate

Doing so, we find for example that the stationary state cavity photon number $n^{\rm ss}$ equals

$$n^{\rm ss} = \frac{\Omega^2}{\kappa^2 + 4\Delta^2} \, . \tag{4.12}$$

The corresponding stationary state photon emission rates equals $I^{\rm ss} = \kappa n^{\rm ss}$ which implies

$$I^{\rm ss} = \frac{\Omega^2 \kappa}{\kappa^2 + 4\Delta^2} \, . \tag{4.13}$$

As we shall see in Section 6.4.2, this emission rate describes the leakage of photons through the left *and* the right cavity mirror.

Chapter 5

Normal mode approach to Quantum Electrodynamics in Dielectric Media

5.1 Normal Modes approach

The standard approach to QED in linear dielectrics is to expand the vector potential in a complete set of normal modes (eigenmodes),

$$A(x,t) = \sum_{\lambda} A_{\lambda}(x) q_{\lambda}(t). \tag{5.1}$$

So that in the Coulomb Gauge, in the absence of magnetization, Maxwell's equations reduce to

$$\nabla^2 A_{\lambda}(x) = \varepsilon(x)\omega_{\lambda}^2 A_{\lambda}(x) \tag{5.2}$$

$$\partial_t^2 q_{\lambda}(t) = \omega_{\lambda}^2 q_{\lambda}(t) \tag{5.3}$$

The modes are orthonormal with respect to the scalar product

$$(A_{\lambda}, A'_{\lambda}) = \int d^3x \varepsilon(x) A_{\lambda}^*(x) A_{\lambda'}(x) = \delta_{\lambda\lambda'} \tag{5.4}$$

The Hamiltonian is given by an infinite collection of harmonic oscillators,

$$H_{\text{normal modes}} = \int d^3x \left(\varepsilon(x)E^2(x) + B^2(x)\right) = \sum_{\lambda} \omega_\lambda a_\lambda^\dagger a_\lambda \tag{5.5}$$

Where $q(t) = \frac{1}{\sqrt{2\omega_\lambda}}(a(t) + a^\dagger(t))$.

The normal modes for a dielectric boundary are called triplet modes (Carniglia and Mandel) [47]

5.2 Scattering matrix approach

The scattering or transfer matrix approach begins by identifying incoming and outgoing modes, which are transformed into one another by a unitary scattering matrix. We can write the S-matrix in terms of an effective Hamiltonian H as $S = \lim_{T\to\infty} \exp\left(-i \int_{-T}^{T} H dt\right)$.

The effective Hamiltonian for a beamsplitter is given by

$$H_{\text{effective}} = \omega(a_1^\dagger a_1 + a_2^\dagger a_2) + J(a_1^\dagger a_2 - a_2^\dagger a_1), \tag{5.6}$$

where the a_1 and a_2 modes are traveling waves with frequency ω. J is a parameter which controls the strength of the exchange between the two modes, and is related to the transmissivity of the beamsplitter. The goal is to see how these traveling waves are related to the triplet modes, and to show that the effective Hamiltonian is the same as the normal mode Hamiltonian above.

5.3 Comparison

5.3.1 Dielectric Surface

First we will consider the simplest case of a single dielectric surface, that is a space with $\varepsilon(x) = \varepsilon_1$ for $x < 0$ and $\varepsilon(x) = \varepsilon_2$ for $x > 0$. We will also consider normal incidence and ignore diffraction and dispersion so that we can treat the problem as one dimensional. The problem is three dimensional, as lower dimensional electromagnetism is qualitatively different to three dimensional EM. The one-dimensional treatment is valid as long as the inputted light is very highly collimated, as for instance from a laser. In this situation we only need to consider propagation in one dimension, and the behaviour of the field in the orthogonal directions will play no role.

The normal modes in this case are the triplet modes of Carniglia and Mandel. However we will consider purely travelling waves and treat it as a scattering problem. The modes we consider are

$$A_{\rightarrow}(\omega) = A_0(\omega)e^{ik_1x}\Theta(-x) + A_0(\omega)e^{ik_2x}\Theta(x) \tag{5.7}$$

$$A_{\leftarrow}(\omega) = A_0(\omega)e^{-ik_1x}\Theta(-x) + A_0(\omega)e^{-ik_2x}\Theta(x) \tag{5.8}$$

where $k_i = \sqrt{\varepsilon_i}\omega$ and $A_0(\omega)$ is a normalization constant to be fixed later. The fact that each mode has a definite wavevector tells us that the modes are travelling waves. The vector potential is then given by

$$A(x,t) = \int_0^\infty d\omega\,(A_{\rightarrow}(\omega)a_{\rightarrow}(\omega) + A_{\leftarrow}(\omega)a_{\leftarrow}(\omega))e^{-i\omega t} + H.c$$

The Hamiltonian is

$$H = \frac{1}{2}\int_{-\infty}^{\infty} dx(\,\varepsilon(x)E(x)^2 + B(x)^2)$$

which can be re-written with the vector potential $A(x)$ as

$$H = \frac{1}{2}\int_{-\infty}^{\infty} dx\,\varepsilon(x)\,((\partial_t A)^2 + \partial_x A\partial_x A)$$

Where we have used the fact that A is transverse to the direction of propagation so that $B = \partial_x A$. After integrating by parts we use the wave equation $\varepsilon(x)\partial_t^2 A = \partial_x^2 A$ to express the Hamiltonian in terms of time derivatives as

$$H = \frac{1}{2}\int_{-\infty}^{\infty} dx\,\varepsilon(x)\,((\partial_t A)^2 - A\partial_t^2 A).$$

Plugging in the above expression for A and using the fact that $A_{\rightarrow} = A^*_{\leftarrow}$ we find each term involves an integral of the following form

$$F(\omega) := A_0^2(\omega) \left(\int_{-\infty}^{0} dx\, \varepsilon_1\, e^{in_1\omega x} + \int_{0}^{\infty} dx \varepsilon_2\, e^{in_2\omega x} \right) \tag{5.9}$$

$$= A_0^2(\omega) \left(\frac{1}{2}(n_1^2 + n_2^2)\delta(\omega) + 2i(n_1 - n_2)\mathcal{P}\frac{1}{\omega} \right) \tag{5.10}$$

Then in terms of F as defined above the Hamiltonian is

$$\begin{aligned} H = \int_0^\infty d\omega d\omega'\, \omega'(\omega + \omega') \Big[& \big(F(\omega - \omega') a_{\rightarrow}(\omega) a^\dagger_{\rightarrow}(\omega') + F(\omega - \omega')^* a_{\leftarrow}(\omega) a^\dagger_{\leftarrow}(\omega') \\ & + F(\omega + \omega') a_{\rightarrow}(\omega) a^\dagger_{\leftarrow}(\omega') + F(\omega + \omega')^* a_{\leftarrow}(\omega) a^\dagger_{\rightarrow}(\omega') \big) e^{-i(\omega - \omega')t} \\ & + \big(F(\omega - \omega')^* a^\dagger_{\rightarrow}(\omega) a_{\rightarrow}(\omega') + F(\omega - \omega') a^\dagger_{\leftarrow}(\omega) a_{\leftarrow}(\omega') \\ & + F(\omega + \omega')^* a^\dagger_{\rightarrow}(\omega) a_{\leftarrow}(\omega') + F(\omega + \omega') a^\dagger_{\leftarrow}(\omega) a_{\rightarrow}(\omega') \big) e^{i(\omega - \omega')t} \Big] \\ + \omega'(\omega - \omega') \Big[& \big(F(\omega + \omega') a_{\rightarrow}(\omega) a_{\rightarrow}(\omega') + F(\omega + \omega')^* a_{\leftarrow}(\omega) a_{\leftarrow}(\omega') \\ & + F(\omega - \omega') a_{\rightarrow}(\omega) a_{\leftarrow}(\omega') + F(\omega - \omega')^* a_{\leftarrow}(\omega) a_{\rightarrow}(\omega') \big) e^{-i(\omega + \omega')t} \\ & + \big(F(\omega + \omega')^* a^\dagger_{\rightarrow}(\omega) a^\dagger_{\rightarrow}(\omega') + F(\omega + \omega') a^\dagger_{\leftarrow}(\omega) a^\dagger_{\leftarrow}(\omega') \\ & + F(\omega - \omega')^* a^\dagger_{\rightarrow}(\omega) a^\dagger_{\leftarrow}(\omega') + F(\omega - \omega') a^\dagger_{\leftarrow}(\omega) a^\dagger_{\rightarrow}(\omega') \big) e^{i(\omega + \omega')t} \Big] \end{aligned}$$

From the definition of the function F the Hamiltonian involves terms including a δ-function, which can be integrated, and terms given by the principal part of $1/\omega$ which give interaction terms. The terms quadratic in creation or annihilation operators vanish since the frequencies are non-negative, and those terms vanish at $\omega' = 0$. That leaves the expected terms related to the free Hamiltonian,

$$H_0 = \int_0^\infty d\omega\, 2\omega^2 A_0^2(\omega)(n_1^2 + n_2^2) \left(a^\dagger_{\rightarrow}(\omega) a_{\rightarrow}(\omega) + a^\dagger_{\leftarrow}(\omega) a_{\leftarrow}(\omega) \right)$$

At this point we set $A_0 = 1/\sqrt{2\omega(n_1^2 + n_2^2)}$ to match the free-space theory. The rest of the Hamiltonian involves the principal value integral in F, in full

$$H_P = 2i\frac{n_1 - n_2}{n_1^2 + n_2^2}\mathcal{P}\int_0^\infty \frac{d\omega d\omega'}{\sqrt{\omega\omega'}} \times$$

$$\begin{aligned}
\omega'(\omega+\omega')\Big[&\Big(\frac{a_\rightarrow(\omega)a_\rightarrow^\dagger(\omega')}{\omega-\omega'} - \frac{a_\leftarrow(\omega)a_\leftarrow^\dagger(\omega')}{\omega-\omega'} + \frac{a_\rightarrow(\omega)a_\leftarrow^\dagger(\omega')}{\omega+\omega'} - \frac{a_\leftarrow(\omega)a_\rightarrow^\dagger(\omega')}{\omega+\omega'}\Big)e^{-i(\omega-\omega')t} \\
&+\Big(-\frac{a_\rightarrow^\dagger(\omega)a_\rightarrow(\omega')}{\omega-\omega'} + \frac{a_\leftarrow^\dagger(\omega)a_\leftarrow(\omega')}{\omega-\omega'} - \frac{a_\rightarrow^\dagger(\omega)a_\leftarrow(\omega')}{\omega+\omega'} + \frac{a_\leftarrow^\dagger(\omega)a_\rightarrow(\omega')}{\omega+\omega'}\Big)e^{i(\omega-\omega')t}\Big] \\
+\omega'(\omega-\omega')\Big[&\Big(\frac{a_\rightarrow(\omega)a_\rightarrow(\omega')}{\omega+\omega'} - \frac{a_\leftarrow(\omega)a_\leftarrow(\omega')}{\omega+\omega'} + \frac{a_\rightarrow(\omega)a_\leftarrow(\omega')}{\omega-\omega'} - \frac{a_\leftarrow(\omega)a_\rightarrow(\omega')}{\omega-\omega'}\Big)e^{-i(\omega+\omega')t} \\
&+\Big(-\frac{a_\rightarrow^\dagger(\omega)a_\rightarrow^\dagger(\omega')}{\omega+\omega'} + \frac{a_\leftarrow^\dagger(\omega)a_\leftarrow^\dagger(\omega')}{\omega+\omega'} + -\frac{a_\rightarrow^\dagger(\omega)a_\leftarrow^\dagger(\omega')}{\omega-\omega'} + \frac{a_\leftarrow^\dagger(\omega)a_\rightarrow^\dagger(\omega')}{\omega-\omega'}\Big)e^{i(\omega+\omega')t}\Big]
\end{aligned}$$

Let us consider the scattering matrix, $S = \exp\left(-i\int_{-\infty}^{\infty} H\,dt\right)$. The time integral turns the phase factors $e^{\pm i(\omega\pm\omega')t}$ into δ-functions. The terms quadractic in creation or annihilation operators are multiplied by $\delta(\omega+\omega')$ and consequently disappear. The terms multiplied by $1/(\omega-\omega')$ don't contribute to the principal part of the frequency integrals at $\omega = \omega'$ so we can get rid of them too. That leaves us with

$$\int_{-\infty}^{\infty} H_P\,dt = 2i\frac{n_1 + n_2}{n_1^2 + n_2^2}\int_0^\infty d\omega\,\left(r_{12}a_\leftarrow^\dagger(\omega)a_\rightarrow(\omega) + r_{21}a_\rightarrow^\dagger(\omega)a_\leftarrow(\omega)\right)$$

where $r_{ij} = (n_i - n_j)/(n_i + n_j)$.

5.4 Relativistic Kramers-Kronig relations

A full, first principles treatment of optical cavities would involve a description of the mirrors as dielectric media. Such a treatment is beyond the scope of this thesis, but the study of dielectic media in themselves has received interest in recent years with the development of metamaterials [82]. The general form for the polarization density in a linear, homogeneous dielectric medium is given by a convolution of the susceptibility with the electric field

$$P_i(t,\vec{x}) = \int \mathrm{d}t'\mathrm{d}^{n-1}x'\chi_{ij}(t-t',\vec{x}-\vec{x}')E_j(t',\vec{x}') \tag{5.11}$$

in units where $c = \varepsilon_0 = \mu_0 = 1$. Latin indices $i, j, ...$ run over spacelike coordinates $1, ..., n-1$, greek indices will run from $0, ..., n-1$. The Kramers-Kronig relations follow from a condition of causality: the polarization at time t is purely determined by the electric field at times t' in the past of t. Ignoring the position dependence this means that $\chi_{ij}(t-t')$ must vanish when t' is in the future of t. Without loss of generality we can set $t' = 0$ and write

$$\chi_{ij}(t) = \Theta(t)\chi_{ij}(t), \tag{5.12}$$

where Θ is the Heaviside step function. We can use the convolution theorem to re-write this in frequency space as

$$\chi_{ij}(\omega) = \frac{1}{2\pi}\mathcal{F}[\Theta(t)](\omega) * \chi_{ij}(\omega). \tag{5.13}$$

The Fourier transform of $\Theta(t)$ is a distribution (only defined under an integral)

$$\mathcal{F}[\Theta(t)](\omega) = \pi\delta(\omega) + i\mathcal{P}\frac{1}{\omega},$$

where $\mathcal{P}$ means we take the Cauchy principal value when integrating. Equation (5.13) becomes

$$\chi_{ij}(\omega) = \frac{1}{\pi i}\mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}\omega' \frac{\chi_{ij}(\omega')}{\omega' - \omega},$$

known as the Kramers-Kronig relations.

We have conveniently ignored the position dependence of χ to arrive at this. This is a problem if it is unconstrained, because then the polarization in (5.11) picks up contributions from the electric field anywhere in the past of time t. This includes contributions from outside the past light cone, which is unphysical. A popular choice is to assume that $\chi_{ij}(t-t', \vec{x}-\vec{x}') = \tilde{\chi}_{ij}(t-t', \vec{x})\delta^{n-1}(\vec{x}-\vec{x}')$ so that the polarization at $\vec{x}$ only responds to the electric field at the same place. This approximation becomes accurate in materials with no spatial dispersion, but will be inaccurate when this is present. A more general case consistent with special relativity would allow the susceptibility in (5.11) to take values whenever $(t', \vec{x}')$ is inside the past light cone of $(t, \vec{x})$, but it must vanish outside due to locality. Again without loss of generality set $(t', \vec{x}') = 0$ and analogously to (5.12) write

$$\chi_{ij}(t, \vec{x}) = \Theta(t - |\vec{x}|)\chi_{ij}(t, \vec{x}).$$

We use the convolution theorem again, but this time when we calculate the Fourier transform of the step function we need to use the shift theorem, and we get

$$\mathcal{F}[\Theta(t-|\vec{x}|)](\omega) = e^{i\omega|\vec{x}|}\mathcal{F}[\Theta(t)](\omega) = e^{i\omega|\vec{x}|}\left(\pi\delta(\omega) + i\mathcal{P}\frac{1}{\omega}\right)$$

This leads to the relativistic Kramers-Kronig relations

$$\chi_{ij}(\omega,\vec{x}) = \frac{1}{\pi i}\mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}\omega' \frac{e^{i(\omega-\omega')|\vec{x}|}\chi_{ij}(\omega',\vec{x})}{\omega'-\omega}.$$

Reinstating factors of c the phase factor becomes $e^{i(\omega-\omega')|\vec{x}|/c}$. It is easy to see that this approaches 1 as $c \to \infty$, so that the usual Kramers-Kronig relations can be viewed as the non-relativistic limit of the above relations. Now let us transform into $\vec{k}$-space. To do this we use the convolution theorem again, and we have to find the Fourier transform of $e^{i(\omega-\omega')|\vec{x}|}$. To do so go into spherical coordinates and write

$$\mathcal{F}[e^{i(\omega+i\epsilon)|\vec{x}|}](\vec{k}) = \int_0^\infty r^{n-2}\mathrm{d}r e^{i(\omega+i\epsilon)r}\int d\Omega_{n-2}e^{-i\vec{k}\cdot\vec{x}}$$

This integral involves Bessel functions and is only defined when $(\omega-\omega')$ has a non-zero imaginary part. The result can be found with Mathematica and in $n-1$ spatial dimensions is given by

$$\mathcal{F}[e^{i(\omega+i\epsilon)|\vec{x}|}](\vec{k}) = i2^{n-1}\pi^{(n/2)-1}\Gamma\left(\frac{n}{2}\right)\frac{(\omega+i\epsilon)}{(|\vec{k}|^2-(\omega+i\epsilon)^2)^{n/2}}.$$

This leads to the relativistic Kramers-Kronig relations in frequency space.

$$\chi_{ij}(k) = \frac{(n-2)!!}{(2\pi)^{n/2}}\mathcal{P}\int \mathrm{d}^n k' \frac{\chi_{ij}(k')}{(|\vec{k}-\vec{k}'|^2-(\omega-\omega'+i\epsilon)^2)^{n/2}}$$

Here k is a 4-vector with components $(\omega,\vec{k})$. The principal part of the frequency integral is taken.

Chapter 6

A Master Equation approach to dielectric media

In this chapter we will analyze the two-sided cavities relevant for quantum communication networks using a master equation approach. The theory we need is quantum electrodynamics (QED) in the presence of dielectric media, with a particular focus on non-dispersive media. We will study the relation between linear optics and standard cavity QED. Let us begin by considering the basic structure of classical electrodynamics and its role in the quantum theory.

6.1 Introduction

The starting point for any field quantization scheme is to consider a mode expansion of the field in a finite quantization volume. Mathematics teaches us that any function on a finite interval can be written as a Fourier series. For example, any real function $f(x)$ with $x \in (0, L)$ and $f(0) = f(d) = 0$ can be expanded in a series of sine functions,

$$f(x) \;\; = \;\; \sum_{m=0}^{\infty} c_m \, \sin\left(m\pi \frac{x}{d}\right) \, , \tag{6.1}$$

where the c_m are real coefficients. Most authors take this into account when quantising the electromagnetic field inside an optical resonator (cf. eg. Refs. [52; 73]). After considering a finite quantisation volume and imposing strict boundary conditions, a complete and discrete set of eigenfunctions is obtained, which are solutions of the electromagnetic wave equation in the Coulomb gauge. Associating these solutions with a discrete set of annihilation and creation operators, c_m and $c_m^\dagger$, eventually yields a standing-wave harmonic oscillator Hamiltonian of the form

$$H_{\rm cav} \;=\; \sum_{m=1}^{\infty} \hbar\omega_m \, c_m^\dagger c_m \tag{6.2}$$

with a discrete set of cavity frequencies ω_m (cf. App. 4). When modelling the electromagnetic field inside a dielectric slab or an open cavity, the normal modes of the system change but the electromagnetic field between its mirrors is still described by a harmonic oscillator Hamiltonian (cf. eg. Refs. [7; 8; 10; 11; 67; 69; 70; 72; 74]).

The cavity Hamiltonian in Eq. (6.2) has been probed successfully experimentally with the help of single atoms passing through the resonator (cf. eg. Refs. [12; 13; 14]). But there is a problem. The standard Hamiltonian $H_{\rm cav}$ cannot be used to analyse other rather simple quantum optics experiments in a straightforward way. Suppose a monochromatic laser field of frequency ω_0 drives a two-sided optical cavity from one side, thereby populating its eigen-modes. Moreover, suppose these modes are highly symmetric and couple equally well to the free radiation field on the left and on the right side of the resonator. Taking this point of view, one would expect equal stationary state photon emission rates through both sides of the cavity.

This is not the case. Analysing a laser-driven optical resonator, a so-called Fabry-Pérot cavity, with Maxwell's equations, shows for example that resonant laser light is entirely transmitted through the cavity, with no reflected component (cf. eg. Ref. [80] or App. 3.3). Off resonance, one part of the incoming laser beam is transmitted through the cavity, while the other part is reflected. The corresponding transmission and reflection rates $T_{\rm cav}(\omega_0)$ and $R_{\rm cav}(\omega_0)$ add up to one,

$$T_{\rm cav}(\omega_0) + R_{\rm cav}(\omega_0) \;=\; 1 \, , \tag{6.3}$$

but are in general different from each other. Quantum optical models of optical cavities should be able reproduce this behaviour easily.

Of course, the above problem has been noticed before by other authors. When having a closer look at the literature, we find many different descriptions of the electromagnetic field between two mirrors. For example, taking a phenomenological approach, Collett and Gardiner [16; 17] introduced the input-output formalism. They assumed a linear coupling between the field modes outside and the discrete set of photon modes inside the cavity and imposed boundary conditions for the electric field amplitudes on the mirrors. In this way, it becomes possible to model the coherent scattering of light through optical cavities in a way, which is consistent with Maxwell's equations (cf. eg. Refs. [18; 19]). The price for this consistency is a relatively large set of equations. These make it relatively hard to analyse more complex quantum optics experiments, like the scattering of light through cascaded cavities [20].

Moreover, there are different modes-of-the-universe descriptions of optical cavities [21; 22; 23; 24; 25]. These describe the electromagnetic field between two mirrors in terms of superpositions of the eigen-modes of a much larger surrounding cavity, the universe. For example, Refs. [24; 25] apply a macroscopical quantisation procedure to obtain a quasi-mode representation of the electromagnetic field. Quasi-modes are non-orthogonal photon modes and tunneling between photon modes associated with the inside and the outside of the resonator can occur. A relatively recent review of macroscopic QED can be found in Ref. [26]. For more recent developments in this are of research see eg. Ref. [27].

Cavity models like the above treat optical cavities as closed quantum systems. What they ignore is that a detector placed some distance away from an optical resonator would register spontaneous photon emissions. These are analogous to the photon emissions from a laser-driven atomic system. Like atoms, optical cavities have a spontaneous decay rate, which we usually denote by κ. Photons from different optical cavities, can interfere, for example, when they simultaneously pass through a beamsplitter before hitting a detector (cf. eg. Refs. [29; 61]). The same applies to atomic systems. Interference of spontaneously emitted photons has been observed experimentally, for example, by Winelands group in a two-atoms double-slit experiment [30; 31; 32]. Atomic systems with spontaneous photon emission are routinely described by quantum optical master equations. The same should apply to optical cavities.

The purpose of this paper is to derive such a master equation. Before doing so, we notice that linear optics scattering theory and cavity QED both employ different notions of photons. In linear optics scattering theory, photons are the energy quanta of a free radiation field. No boundary conditions apply and a continuum of traveling-wave photon modes is considered. Scattering theory also suggests that cavity mirrors are half-transparent mirrors which either transmit or reflect any incoming photon. Since the mirrors affect the dynamics of traveling-wave photons, they cannot be the energy quanta of an optical cavity. As we shall see below, the energy expectation value of a cavity photon of frequency ω is in general different from $\hbar\omega$. Photons which are not in resonance with one of the cavity frequencies ω_m in Eq. (6.2) experience significant level shifts. As pointed out by Glauber and Lewenstein [70] in 1991, the photons and the energy quanta of an optical cavity seem to differ by some "virtual" excitation.

In this paper, we adopt the same notion of a photon as in linear optics scattering theory. This means, we no longer adopt a mathematical argument (cf. Eq. (6.1)) to define physical objects. In the following, there is no difference between the observables of the electromagnetic field in the presence and in the absence of cavity mirrors. Instead we model the electromagnetic field inside an optical cavity using the same Hilbert space as when modelling a free radiation field. A continuum of traveling-wave cavity photon modes with annihilation operators $a_{\rm A}(\omega)$ is considered. Here ω denotes the frequency of the respective photon mode. The index $\rm A = L, R$ helps to distinguish between photons travelling left and photons traveling right. Photons in different $(\omega, \rm A)$ modes are assumed to be in pairwise orthogonal states. Taking this approach makes it easy to guarantee that photons do not change their frequency when traveling through a resonator. Moreover, it allows us to assign different decay channels to photons traveling in different directions. It also enables us to assume that a laser which enters the cavity from the left excites only photons traveling right, as it should.

The effect of the cavity mirrors is to convert photons traveling left into photons traveling right and vice versa until they eventually leak out of the resonator. This is taken in the following into account by postulating a new cavity Hamiltonian $H_{\rm cav}$, which contains photon bouncing terms. These terms are already known to be the generators of the unitary operations associated with the scattering of photons through beamsplitters and other linear optics elements [26; 33; 34; 35]. As we shall see below, the corresponding photon bouncing rate $J(\omega)$ depends on the photon round trip time and on the amount of constructive and destructive interference within the cavity. Consequently, our theory contains two free parameters, namely $J(\omega)$ and the spontaneous cavity decay rate κ. These can be chosen such that our model yields the same predictions as Maxwell's equations (cf. eg. App. 3.3), when both theories apply.

In addition we find that our proposed traveling-wave cavity Hamiltonian predicts the same time evolution of the total number of photons inside the cavity as the usual standing-wave description (cf. eg. App. 4) for experiments with resonant and near-resonant laser driving. This means, the theory which we present in this paper is also consistent with current cavity QED experiments (cf. eg. Refs. [12; 13; 14]). Moreover, when the distance d between the cavity mirrors tends to infinity, the photon bouncing rate $J(\omega)$ tends to zero and our cavity Hamiltonian simplifies to the usual Hamiltonian of a free radiation field.

One advantage of the traveling-wave model which we propose here is that it makes it easy to analyse the spontaneous emission of photons through the different sides of an optical resonator. Moreover, it can be used to model the scattering of single photons through the fiber connections of coherent cavity networks. Already in 1997, Cirac *et al.* [36] proposed to build a quantum internet by connecting distant optical cavities via very long optical fibers. In the mean time, much effort has been made to realise such schemes in the laboratory [38; 39; 77]. Alternatively, cavities could be linked via fiber connections of intermediate length [40; 41; 42; 43; 44]. For example, Kyoseva *et al.* [44] proposed to create coherent cavity networks with very high or even complete connectivity by linking several cavities via linear optics elements and optical fibers, which are about 1m long. Using the approach which we propose here, it is relatively straightforward to analyse such networks analytically.

There are five sections in this paper. Section 6.2 postulates a traveling-wave cavity Hamiltonian and introduces the corresponding master equation of a laser-driven two-sided optical cavity. In Section 6.3, we use this equation to calculate the stationary state photon scattering rates through the left and through the right side of this experimental setup. Section 6.4 compares both rates with the stationary state scattering behaviour predicted by classical electrodynamics. As a result, we obtain expressions for the photon bouncing rate $J(\omega)$ and the spontaneous cavity decay rate κ which are consistent with Maxwell's equations. Afterwards, we summarise our findings in Section 7. A calculation of the reflection and transmission rates R and T in Eq. (6.3) as a function of the cavity parameters n and d and the respective laser frequency ω_0 with the help of Maxwell's equations can be found in App. 3.3. Moreover, App. 4 contains a detailed discussion of the experimental setup in Fig. 6.1 using the standard standing-wave description of laser-driven optical cavities.

6.2 A traveling-wave cavity Hamiltonian

In this section, we introduce a traveling-wave description of the electromagnetic field inside a two-sided laser-driven optical cavity. For simplicity we consider Fabry-Pérot cavity, that is, a planar, two-sided optical cavity, (cf. Fig. 6.1). It consists of a dielectric slab of fixed length d and with a refractive index $n > 1$, with infinite, parallel sides. An external monochromatic laser field with frequency ω_0 drives the resonator from the left at normal incidence. TO begin with we will assume that the absorption of photons in the dielectric medium is negligible, and furthermore the medium is. The generalisation of the proposed cavity description to arbitrary cavity designs is nevertheless straightforward, as long as the reflection and transmission coefficients of its mirrors are known [46]. The main reason for considering the experimental setup in Fig. 6.1 is that its stationary state behaviour can be modelled easily with the help of Maxwell's equations (cf. eg. Ref. [80] or App. 3.3), since the normal modes in the cavity are simply standing waves. But let us first have a closer look at the definition of photons in the absence of any cavity mirrors.

6.2.1 Photons in free space

When trying to explain the intensity-frequency dependence of the radiation emitted by a black body with the help of classical electrodynamics, Planck noticed that

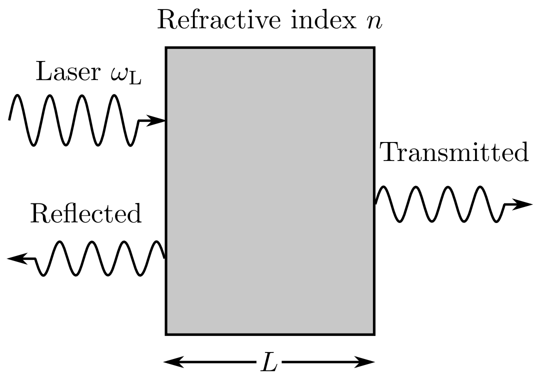

Figure 6.1: Schematic view of the experimental setup which we consider in this thesis. It consists of a laser-driven resonator (a dielectric slab) of length d. Detectors monitor the stationary state photon emission rate through both cavity mirrors.

electromagnetic energy could only be emitted in quantised form [48]. Nowadays, his discovery is regarded as the birth of Quantum Physics. In one dimension, photons are routinely modelled by bosonic annihilation operators $a_{\mathrm{A}}(\omega)$ with $\mathrm{A} = \mathrm{L}, \mathrm{R}$. Here ω denotes the respective photon frequency, which can assume any positive value. Moreover, the index $\mathrm{A} = \mathrm{L}, \mathrm{R}$ distinguish between photons traveling "left" and photons traveling "right." Photons in different (ω, A) modes are in general in pairwise orthogonal states. Annihilation and creation operators consequently obey the commutator relation

$$[a_{\mathrm{A}}(\omega), a_{\mathrm{B}}^{\dagger}(\omega')] \;=\; \delta_{\mathrm{A,B}}\, \delta(\omega - \omega') \tag{6.4}$$

with $\mathrm{A}, \mathrm{B} = \mathrm{L}, \mathrm{R}$. Using the above notation, the Hamiltonian H_{field} for the electromagnetic field energy equals the harmonic oscillator Hamiltonian

$$H_{\mathrm{field}} \;=\; \sum_{\mathrm{A=L,R}} \int_0^{\infty} \mathrm{d}\omega\, \hbar\omega\, a_{\mathrm{A}}^{\dagger}(\omega) a_{\mathrm{A}}(\omega) \tag{6.5}$$

up to a constant, the zero point energy. In the following, we set this constant to zero, since it plays no role when analysing scattering problems. The Hilbert space for the description of the electromagnetic field in free space contains all the states which are generated when applying the above photon creation operators to the vacuum state.

Let us now have a closer look at the electric and the magnetic field observables $E(x)$ and $B(x)$ at position x in a free space which is filled with a medium with permittivity ϵ and permeability μ. These need to be defined such that their expectation values evolve according to Maxwell's equations, whenever both theories apply. In one dimension, this requires in analogy to Eq. (2.1) in 3.3 that

$$\begin{aligned} \partial_x \langle E(x) \rangle &= -\partial_t \langle B(x) \rangle \, , \\ \frac{1}{\mu} \partial_x \langle B(x) \rangle &= -\varepsilon \, \partial_t \langle E(x) \rangle \end{aligned} \tag{6.6}$$

for any possible photon state. Since a local detector cannot distinguish between photons travelling left and photons traveling right, local field observables $E(x)$ and $B(x)$ have to be the sum of the respective contributions of left and right photons,

$$\begin{aligned} E(x) &= E_{\rm L}(x) + E_{\rm R}(x) \, , \\ B(x) &= B_{\rm L}(x) + B_{\rm R}(x) \, . \end{aligned} \tag{6.7}$$

In addition, we assume that respective components of the field observables are Hermitian as well as linear superpositions of their respective annihilation and creation operators, such that

$$\begin{aligned} E_{\rm A}(x) &= {\rm i} \int_0^\infty {\rm d}\omega \, \left(\xi_{\rm A}(x,\omega) \, a_{\rm A}(\omega) - {\rm H.c.} \right) \, , \\ B_{\rm A}(x) &= {\rm i} \int_0^\infty {\rm d}\omega \, \left(\zeta_{\rm A}(x,\omega) \, a_{\rm A}(\omega) - {\rm H.c.} \right) \, . \end{aligned} \tag{6.8}$$

Here $\mathrm{A}, \mathrm{B} = \mathrm{L}, \mathrm{R}$ and the $\xi_{\rm A}$'s and $\zeta_{\rm A}$'s are complex functions x and ω. Substituting these equations into Eq. (6.6), taking into account that the time derivative of the expectation value of any field observable $O(x)$ equals

$$\partial_t \langle O(x) \rangle = -\frac{\rm i}{\hbar} \langle [O(x), H_{\rm field}] \rangle \tag{6.9}$$

with $H_{\rm field}$ as in Eq. (6.5), and comparing coefficients, we obtain the differential equations

$$\begin{aligned} \partial_x \xi_{\rm A}(x,\omega) &= {\rm i}\omega \, \zeta_{\rm A}(x,\omega) \, , \\ \frac{1}{\mu} \partial_x \zeta_{\rm A}(x,\omega) &= {\rm i}\varepsilon\omega \, \xi_{\rm A}(x,\omega) \, . \end{aligned} \tag{6.10}$$

The general solution of these equations is of the form

$$\begin{aligned}\xi_{\rm A}(x,\omega) &= K_1\,{\rm e}^{-{\rm i}kx} + K_2\,{\rm e}^{{\rm i}kx}\,, \\ \zeta_{\rm A}(x,\omega) &= -\sqrt{\epsilon\mu}K_1\,{\rm e}^{-{\rm i}kx} + \sqrt{\epsilon\mu}K_2\,{\rm e}^{{\rm i}kx}\,,\end{aligned} \tag{6.11}$$

with the positive wavenumber k defined as

$$k \equiv \sqrt{\epsilon\mu}\,\omega \tag{6.12}$$

and where the constants K_1 and K_2 can be any complex function of ω. Since the index A can assume two different values, we now have four complex constants, which we can choose as we like, without contradicting Maxwell's equations.

Consistency with classical electrodynamics also requires that the expectation values of the field Hamiltonian

$$\tilde{H}_{\rm field} = \int_{-\infty}^{\infty} {\rm d}x \left(\epsilon|E(x)|^2 + \frac{1}{\mu}|B(x)|^2\right) \tag{6.13}$$

and the expectation values of $H_{\rm field}$ in Eq. (6.5) differ, for all possible photon states, at most by a constant.

If we want the annihilation operators $a_{\rm L}$ and $a_{\rm R}$ to represent photons with electric and magnetic field amplitudes, which travel left and right, respectively, then we need to choose

$$\begin{aligned}\xi_{\rm L}(x,\omega) &= K(\omega)\,{\rm e}^{-{\rm i}kx}\,, \\ \xi_{\rm R}(x,\omega) &= K(\omega)\,{\rm e}^{{\rm i}kx}\,, \\ \zeta_{\rm L}(x,\omega) &= -\sqrt{\epsilon\mu}\,K(\omega)\,{\rm e}^{-{\rm i}kx}\,, \\ \zeta_{\rm R}(x,\omega) &= \sqrt{\epsilon\mu}\,K(\omega)\,{\rm e}^{{\rm i}kx}\,.\end{aligned} \tag{6.14}$$

For symmetry reasons, we choose the same constant K for left and for right traveling photons. Moreover, we can safely assume that K is real by absorbing potential phase factors into the definition of the respective photon annihilation operators.

6.2.2 Photons in optical cavities

Let us now have a closer look at the laser-driven dielectric slab illustrated in Fig. 6.1. Its Hamiltonian is of the general form

$$H = H_{\rm cav} + H_{\rm laser} \tag{6.15}$$

with the first term describing the electromagnetic field inside the cavity and the second term taking the external laser driving into account. To model this experimental setup, we adopt the same notion of photons as linear optics scattering theory. This means, we consider the same Hilbert space as in the previous subsection and consider again a continuum of photon modes $(\omega, {\rm A})$ with bosonic annihilation operators $a_{\rm A}(\omega)$. As in App. 4, we treat the laser field, which drives the cavity, as classical. As pointed out already in the Introduction, a laser which drives the cavity with frequency ω_0 from the left only excites photons which are of the same frequency and travel to the right. The laser Hamiltonian hence equals

$$H_{\rm laser} = \frac{1}{2}\hbar\Omega\, {\rm e}^{-{\rm i}\omega_0 t}\, a_{\rm R}(\omega_0) + {\rm H.c.} \tag{6.16}$$

in the Schrödinger picture with Ω being the respective laser Rabi frequency.

Considering free space photons and taking the same perspective as linear optics scattering theory, the cavity mirrors become beamsplitters, ie. semi-transparent mirrors. Their effect is to transmit and to reflect any incoming photon without changing its frequency. This frequency conservation suggests that a photon in the $a_{\rm R}(\omega)$-mode can only either remain in this mode or change into the $a_{\rm L}(\omega)$-mode. Taking this into account, we postulate the following cavity Hamiltonian

$$H_{\rm cav} = H_{\rm field} + H_{\rm bounce} \tag{6.17}$$

with $H_{\rm field}$ representing the free energy of the photons analogous to the free field Hamiltonian in Eq. (6.5) and with $H_{\rm bounce}$ being given by

$$H_{\rm bounce} = \frac{1}{2}\int_0^\infty {\rm d}\omega\, \hbar J(\omega)\, a_{\rm L}^\dagger(\omega) a_{\rm R}(\omega) + {\rm H.c.} \tag{6.18}$$

This second term describes the continuous conversion of photons traveling left into photons traveling right and vice versa with a conversion rate $J(\omega)$.

Using the same notion of photons as in free space also suggests that the observables for the electric and the magnetic field $E(x)$ and $B(x)$ are the same inside a finite dielectric slab and in a dielectric medium without boundaries. The reason for this is that, for any given photon state, a local detector at a position x with $x \in (0, L)$ should detect the same field amplitudes, independent of the presence or absence of any cavity mirrors. Moreover, inside the resonator, the expectation values of $E(x)$ and $B(x)$ should evolve again according to Maxwell's equations, ie. as stated in Eq. (6.6) in the previous subsection. To see, if this is indeed the case, we proceed again as above. Using Eqs. (6.6)–(6.9) but with $H_{\rm field}$ now replaced by the cavity Hamiltonian $H_{\rm cav}$ in Eq. (6.17) we derive a set of two differential equations for the $\xi_{\rm A}$ and $\zeta_{\rm A}$ coefficients in Eq. (6.8).

The presence of the photon bouncing term in the above Hamiltonian might seem surprising, since it is usually assumed that the photons of frequency ω are the energy quanta of light with their respective energy given by $\hbar\omega$. Taking this into account, one might conclude that the observable of the energy, ie. the cavity Hamiltonian, of a photon mode at frequency ω is $H_{\rm cav}(\omega) = \hbar\omega \, a^\dagger(\omega) a(\omega)$ with $a(\omega)$ being the respective photon annihilation operator. However, as we shall see below in Section 6.4.2, only the Hamiltonian $H_{\rm cav}$ of the free radiation field is of this relatively simple form. When diagonalising $H_{\rm cav}$ in Eq. (6.17), we find that

$$
\begin{aligned}
H_{\rm cav} \;&=\; \int_0^\infty {\rm d}\omega \, \left(\hbar\omega + \frac{1}{2}\hbar J(\omega) \right) \, a_+^\dagger(\omega) a_+(\omega) \\
&\quad + \left(\hbar\omega - \frac{1}{2}\hbar J(\omega) \right) \, a_-^\dagger(\omega) a_-(\omega) \, ,
\end{aligned}
\tag{6.19}
$$

where the $a_\pm$,

$$
a_\pm \;\equiv\; \frac{1}{\sqrt{2}} \left(a_{\rm L} \pm a_{\rm R} \right) \, ,
\tag{6.20}
$$

denote standing-wave photon annihilation and creation operators. This means, the energy quants of the cavity field are standing-wave photon modes. However, the energy of the ω mode is in general different from $\hbar\omega$. The additional energy of the standing wave photons which equals $+\frac{1}{2}\hbar J(\omega)$ or $-\frac{1}{2}\hbar J(\omega)$, respectively, takes into account that the cavity mirrors continuously convert "left" into "right" photons and vice versa. As we shall see below, the intensity of this scattering process depends on the relative size of ω with respect to the cavity parameters n and d.

6.2.3 A single-frequency model

A closer look at the above Hamiltonian shows that the frequencies of photons remain constant in time. If a single laser field with frequency ω_0 is the only source for photons in the resonator, then only the $a_\mathrm{L}(\omega_0)$ and the $a_\mathrm{R}(\omega_0)$ mode have to be taken into account. For simplicity, we consider only these two photon modes and ignore all other modes in the following. We also no longer state the ω dependence of constants of operators, if it is obvious. Using this notation and introducing the interaction picture with respect to

$$H_0 = \sum_{\mathrm{A}=\mathrm{L,R}} \hbar\omega_0 \, a_\mathrm{A}^\dagger a_\mathrm{A} \, , \tag{6.21}$$

the Hamiltonian H in Eq. (6.15) simplifies to the interaction Hamiltonian

$$H_\mathrm{I} = \frac{1}{2}\hbar\Omega \left(a_\mathrm{R} + a_\mathrm{R}^\dagger \right) + \frac{1}{2}\hbar J \left(a_\mathrm{L}^\dagger a_\mathrm{R} + \mathrm{H.c.} \right) . \tag{6.22}$$

We now have a time-independent Hamiltonian to describe the experimental setup in Fig. 6.1.

6.2.4 Cavity leakage

In order to take the possible leakage of photons through the resonator mirrors into account, we add a system-bath interaction term to the above Hamiltonian and then trace out the bath-degrees of freedom on a coarse grained time scale Δt [52]. Since we distinguish between “left” and “right” photons, it is straightforward to assign different decay channels to photons traveling in different directions. Cavity photons in the a_R-mode leave the cavity through the right mirror. Analogously, photons in the a_L-mode only leak out through the left mirror. In the following, we denote the corresponding spontaneous decay rate by κ. This decay rate is the same for “left” and “right” photons due to the symmetry of the considered experimental setup.

If we describe the system in Fig. 6.1 by a density matrix $\rho_{\rm I}$, then the corresponding photon emission rates $I_{\rm A}$ are given by

$$I_{\rm A} = \kappa \, {\rm Tr} \left(a_{\rm A}^\dagger a_{\rm A} \rho_{\rm I} \right) \tag{6.23}$$

with ${\rm A} = {\rm R}, {\rm L}$. In other words, the emission probability density is the mean number of photons in the $a_{\rm A}$-mode multiplied with κ. The quantum optical master equation of Lindblad form which reflects this emission behaviour is given by

$$\begin{aligned} \dot{\rho}_{\rm I} &= -\frac{\rm i}{\hbar} \left[H_{\rm I}, \rho_{\rm I} \right] + \sum_{\rm X=L,R} \frac{1}{2} \kappa \left(2 a_{\rm A} \rho_{\rm I} a_{\rm A}^\dagger - a_{\rm A}^\dagger a_{\rm A} \rho_{\rm I} \right. \\ & \quad \left. - \rho_{\rm I} a_{\rm A}^\dagger a_{\rm A} \right) . \end{aligned} \tag{6.24}$$

In the following, we use this equation to analyse the dynamics of the laser-driven optical cavity.

6.3 The corresponding time evolution

In this section, we calculate the stationary state photon emission rates $I_{\rm L}^{\rm ss}$ and $I_{\rm R}^{\rm ss}$ through the left and the right cavity mirror, respectively. Both sum up to the total photon emission rate

$$I_{\rm Tot}^{\rm ss} \equiv I_{\rm L}^{\rm ss} + I_{\rm R}^{\rm ss} . \tag{6.25}$$

To calculate these rates we use rate equations, ie. linear differential equation which describe the time evolution of expectation values.

6.3.1 Time evolution of expectation values

To obtain the relevant rate equations, we notice that the above master equation can be used to show that the expectation value $\langle A_{\rm I} \rangle$ of an observable $A_{\rm I}$ evolve according to the differential equation

$$\begin{aligned} \langle \dot{A}_{\rm I} \rangle &= -\frac{\rm i}{\hbar} \left\langle [A_{\rm I}, H_{\rm I}] \right\rangle + \sum_{\rm X=L,R} \frac{1}{2} \kappa \left\langle 2 a_{\rm A}^\dagger A_{\rm I} a_{\rm A} - A_{\rm I} a_{\rm A}^\dagger a_{\rm A} \right. \\ & \quad \left. - a_{\rm A}^\dagger a_{\rm A} A_{\rm I} \right\rangle . \end{aligned} \tag{6.26}$$

To find a closed set of rate equations, including equations for the time evolution of the mean photon number in the $a_{\rm L}$ and in the $a_{\rm R}$ photon mode, respectively, we need to consider the expectation values

$$\begin{aligned} n_{\rm L} &\equiv \langle a_{\rm L}^\dagger a_{\rm L} \rangle \, , \quad n_{\rm R} \equiv \langle a_{\rm R}^\dagger a_{\rm R} \rangle \, , \\ k_1 &\equiv \langle a_{\rm L} + a_{\rm L}^\dagger \rangle \, , \quad k_2 \equiv {\rm i} \langle a_{\rm R} - a_{\rm R}^\dagger \rangle \, , \\ k_3 &\equiv {\rm i} \langle a_{\rm L} a_{\rm R}^\dagger - a_{\rm L}^\dagger a_{\rm R} \rangle \, . \end{aligned} \tag{6.27}$$

These five variables evolve according to

$$\begin{aligned} \dot{n}_{\rm L} &= \frac{1}{2} J k_3 - \kappa n_{\rm L} \, , \\ \dot{n}_{\rm R} &= \frac{1}{2} \Omega k_2 - \frac{1}{2} J k_3 - \kappa n_{\rm R} \, , \\ \dot{k}_1 &= -\frac{1}{2} J k_2 - \frac{1}{2} \kappa k_1 \, , \\ \dot{k}_2 &= \Omega + \frac{1}{2} J k_1 - \frac{1}{2} \kappa k_2 \, , \\ \dot{k}_3 &= -\Omega k_1 - J (n_{\rm L} - n_{\rm R}) - \kappa k_3 \end{aligned} \tag{6.28}$$

which form a closed set of linear differential equations.

6.3.2 Photon scattering rates

Using Eq. (6.23), one can now show that the photon emission rate $I_{\rm A}$ with ${\rm A} = {\rm L}, {\rm R}$ is simply given by

$$I_{\rm A} = \kappa n_{\rm A} \, . \tag{6.29}$$

Different from the standing-wave description in App. 4, the emission rates $I_{\rm L}$ and $I_{\rm R}$ no longer depend on the same mean photon number. Proceeding as in App. 4 and setting all time derivatives equal to zero, we obtain the stationary state photon numbers

$$n_{\rm L}^{\rm ss} = \frac{\Omega^2 J^2}{\left(J^2 + \kappa^2\right)^2} \, , \quad n_{\rm R}^{\rm ss} = \frac{\Omega^2 \kappa^2}{\left(J^2 + \kappa^2\right)^2} \, . \tag{6.30}$$

Substituting these into Eq. (6.29) yields different stationary state photon emission rates for the different sides of a laser-driven resonator. More concretely, we find that

$$I_{\rm L}^{\rm ss} = \frac{\Omega^2 J^2 \kappa}{\left(J^2+\kappa^2\right)^2} \, , \quad I_{\rm R}^{\rm ss} = \frac{\Omega^2 \kappa^3}{\left(J^2+\kappa^2\right)^2} \, . \tag{6.31}$$

The total stationary state photon emission rate $I_{\rm Tot}^{\rm ss}$ in Eq. (6.25) hence equals

$$I_{\rm Tot}^{\rm ss} \;=\; \frac{\Omega^2 \kappa}{J^2+\kappa^2} \, . \tag{6.32}$$

This emission rate depends only on the total cavity photon number $n_{\rm Tot} \equiv n_{\rm L}+n_{\rm R}$. One can easily check that $I_{\rm Tot} = \kappa n_{\rm Tot}$, as it should.

6.3.3 Time evolution without laser driving

Before we compare the above photon emission rates with the predictions of Maxwell's equations, let us have a closer look at the case when there is no external laser driving. When $\Omega = 0$, then one can show that the time derivative of the total number of cavity photons $n_{\rm Tot}$ equals

$$\dot{n}_{\rm Tot} \;=\; -\kappa n_{\rm Tot} \tag{6.33}$$

without any approximations.

6.4 Consistency of different quantum and classical models

In principle, the predictions of classical physics should emerge from quantum physics, when certain approximations apply. In the following, we therefore require that Maxwell's equations and the above master equation approach yield the same predictions, when both models apply. This is the case, when the light inside the cavity behaves like a wave and its particle characteristics can be ignored. Such a situation is illustrated in Fig. 6.1, which is one of the reasons why we are interested in this experimental setup.

6.4.1 Consistency with Maxwell's equations

Below we list several conditions which guarantee the consistency of the predictions of Maxwell's equations and the predictions of our traveling-wave master equation:

1. In the relatively simple case with no laser driving, the relative energy flux out of the cavity predicted by both models should be the same. Using the same notation as in Sections 2.4 and 6.3.3, one can show that this condition is fulfilled when

$$\frac{\dot{I}(t)}{I(t)} = \frac{\dot{n}(t)}{n(t)} \,. \tag{6.34}$$

2. In the case of laser driving, the stationary state photon emission rates $I_{\rm L}^{\rm ss}$ and $I_{\rm R}^{\rm ss}$ should have the same dependence on ω_0, d, and n as the cavity reflection and transmission rates $R_{\rm cav}(\omega_0)$ and $T_{\rm cav}(\omega_0)$. More concretely, we want that

$$\frac{I_L^{\rm ss}}{I_{\rm Tot}^{\rm ss}} = R_{\rm cav}(\omega_0) \, , \quad \frac{I_R^{\rm ss}}{I_{\rm Tot}^{\rm ss}} = T_{\rm cav}(\omega_0) \, . \tag{6.35}$$

   This means, the ratio on the right hand side of this equation should not depend on the laser Rabi frequency Ω, since there is no Ω in the classical model.

In the following, we use the above conditions, to determine the two constants κ and J which we introduced in Section 6.2.

For example, substituting Eqs. (2.15) and (6.33) into Eq. (6.34), we find that the energy flux equality condition, ie. condition 1, applies when

$$\kappa = -\frac{2c}{nd} \ln r \, . \tag{6.36}$$

In this equation r is the Fresnel coefficient in Eq. (2.6) for the reflection of photons from the dielectric back into the dielectric. The logarithm of r guarantees that $\kappa = 0$ for $r = 1$. This means, for perfectly reflecting mirrors, light stays forever inside the cavity. When $r \to 0$, then $\kappa \to \infty$ and there is effectively no cavity.

To obtain an explicit expression for the bouncing rate J, we demand that our traveling wave description predicts the same photon transmission rate as in Eq. (2.12). Combining Eqs. (6.31)–(6.32) one can show that

$$\frac{I_L^{\text{ss}}}{I_{\text{Tot}}^{\text{ss}}} = \frac{J^2}{J^2+\kappa^2} \, , \quad \frac{I_R^{\text{ss}}}{I_{\text{Tot}}^{\text{ss}}} = \frac{\kappa^2}{J^2+\kappa^2} \, . \tag{6.37}$$

Comparing these two equations with Eq. (2.12), as suggested by the second condition in Section 6.4.1, and using the above result for κ, we can now show that

$$J = \frac{4c}{nd} \cdot \frac{r \ln r}{1-r^2} \sin\left(\omega_0 \frac{nd}{c}\right) \tag{6.38}$$

up to an overall phase factor. The bouncing rate J contains an interference term, which becomes zero when ω_0 is in resonance with the cavity, ie. when it assumes one of the frequencies ω_m in Eq. (4.2). Consequently, our model indeed predicts that resonant light does not get reflected inside the cavity. If the cavity is about 1 m long and the laser frequency is relatively far away from any of the cavity resonances, then J is of the order of 1 GHz.

Finally, let us have a closer look at the case of highly reflecting cavity mirrors. In this case, the Fresnel coefficient r is very close to one. Hence $-2\ln r = 1 - r^2$ to a very good approximation and Eqs. (6.36) and (6.38) simplify to

$$\begin{aligned} \kappa &= \frac{c}{nd}\,(1-r^2)\,, \\ J &= -2r\,\frac{c}{nd}\,\sin\left(\omega_0\frac{nd}{c}\right) \, . \end{aligned} \tag{6.39}$$

This result shows that the spontaneous decay rate κ of a two sided optical cavity (with zero photon absorption in the cavity mirrors) and the photon bouncing rate J depend only on the relative resonator length d, its refractive index n, and the frequency ω_0 of the incoming light.

6.4.2 Consistency with the standard single-mode description under certain conditions

We now know that the constants J and κ of the traveling-wave description of two-sided cavities which we propose in this paper can be adjusted such that it becomes consistent with the predictions of Maxwell's equation. However, there is already a well-established standing-wave model for optical cavities with external laser driving. The details of this model can be found in App. 4. The purpose of this subsection is to show that our model yields the same predictions as the standard standing-wave model for near-resonant laser driving. This means, our traveling-wave cavity Hamiltonian does not contradict already existing quantum optics experiments with resonant and near resonant optical cavities (cf. eg. Ref. [14]).

Resonant cavities

When the laser is on resonance, ie. when ω_0 equals one of the frequencies ω_m in Eq. (4.2), then J in Eq. (6.38) becomes zero,

$$J \;=\; 0\,. \tag{6.40}$$

This means, photons inside the cavity are not reflected and $n_{\rm L}$ remains zero. Using Eq. (6.28), one can indeed show that

$$\dot{n}_{\rm L} \;=\; -\kappa n_{\rm L} \tag{6.41}$$

in this case. Moreover, there is now a relatively simple set of rate equations which describe the time evolution of $n_{\rm R}$. Eq. (6.28) shows that

$$\begin{aligned} \dot{n}_{\rm R} &= \frac{1}{2}\Omega k_2 - \kappa n_{\rm R}\,, \\ \dot{k}_2 &= \Omega - \frac{1}{2}\kappa k_2 \end{aligned} \tag{6.42}$$

without any approximations. Consequently, the stationary state photon emission rates $I_{\rm L}^{\rm ss}$, $I_{\rm R}^{\rm ss}$, and $I_{\rm Tot}^{\rm ss}$ are given by

$$I_{\rm L}^{\rm ss} = 0 \quad\text{and}\quad I_{\rm R}^{\rm ss} = I_{\rm Tot}^{\rm ss} = \frac{\Omega^2}{\kappa}\,. \tag{6.43}$$

This means, the total stationary state photon emission rate $I_{\rm Tot}^{\rm ss}$ is exactly the same as the one we obtain when using the quantum optical standard standing-wave description in App. 4. We only need to identify the single-mode photon number n with $n_{\rm R}$ and set the detuning Δ in Eq. (4.13) equal to zero.

Near-resonant cavities

As we shall see below, the standard single-mode description of optical cavities also holds to a very good approximation for near-resonant laser driving, if we are only interested in the time evolution of the total cavity photon number $n_{\rm Tot}$. To do so, we notice that the photon bouncing rate J in Eq. (6.39) for near-resonant laser driving is to a very good approximation given by

$$J = -2\Delta \, , \tag{6.44}$$

as long as the cavity mirrors are highly reflecting and the Fresnel coefficient r is close to one. Here Δ equals the detuning Δ_m in Eq. (4.5) of the applied laser field from the nearest cavity resonance ω_m.

Taking this and Eq. (6.28) into account, we moreover notice that a closed set of rate equations for the time evolution of $n_{\rm Tot}$ is given by

$$\begin{aligned} \dot{n}_{\rm Tot} &= \frac{1}{2}\Omega k_2 - \kappa n_{\rm Tot} \, , \\ \dot{k}_1 &= -\Delta k_2 - \frac{1}{2}\kappa k_1 \, , \\ \dot{k}_2 &= \Omega + \Delta k_1 - \frac{1}{2}\kappa k_2 \, . \end{aligned} \tag{6.45}$$

These equations are exactly the same as the rate equations in Eq. (4.11), if we replace the single-mode photon number n by the total photon number $n_{\rm Tot}$ of the model which we propose in this paper. In other words, the single mode description in App. 4 correctly predicts the total photon emission rate $I_{\rm Tot}^{\rm ss}$ of a laser-driven optical cavity. In agreement with Eq. (4.13), it equals

$$I_{\rm Tot}^{\rm ss} = \frac{\Omega^2 \kappa}{\kappa^2 + 4\Delta^2} \tag{6.46}$$

which is a Lorentzian function of Δ. However, the standard standing wave description of optical cavities cannot predict the stationary state photon emissions rate through the different sides of two-sided cavities. In contrast to this, our standing-wave description of optical optical cavities (cf. Eq. (6.31)) predicts that

$$I_{\rm L}^{\rm ss} = \frac{4\Omega^2\Delta^2\kappa}{(4\Delta^2+\kappa^2)^2} \, , \quad I_{\rm R}^{\rm ss} = \frac{\Omega^2\kappa^3}{(4\Delta^2+\kappa^2)^2} \tag{6.47}$$

for near-resonant laser driving. The model predicts different emission rates for the two sides of the cavity, consistent with the classical transmission and reflection coefficients. This discrepancy is due to the presence of a laser input from one direction, breaking the left-right reflection symmetry of the cavity. However the total output of the cavity is fixed, the sum of the left and right outputs is

$$I_{\rm L}^{\rm ss} + I_{\rm R}^{\rm ss} + \frac{4\Omega^2\Delta^2\kappa}{(4\Delta^2+\kappa^2)^2} + \frac{\Omega^2\kappa^3}{(4\Delta^2+\kappa^2)^2} = \frac{\Omega^2\kappa}{4\Delta^2+\kappa^2} = I_{\rm Tot}^{\rm ss}\,,$$

which is in agreement with the total output from the single sided cavity. This shows that while total energy is conserved, the energy is split between the two different outputs in this model. This is due to the interference between the two modes travelling in opposite directions.

The free radiation field

Finally, let us have a closer look at the case where the distance d of the mirrors tends to infinity. From Eqs. (6.36) and (6.38) we immediately see that

$$\kappa = J = 0 \tag{6.48}$$

in this case. This is exactly as one would expect. If the resonator is infinitely long, then its photons remain inside forever and never change their direction. Moreover, for $J=0$, the cavity Hamiltonian $H_{\rm cav}$ in Eq. (6.17) simplifies to

$$H_{\rm cav} \;=\; \int_0^\infty {\rm d}\omega\, \hbar\omega \left(a_{\rm L}^\dagger(\omega)a_{\rm L}(\omega) + a_{\rm R}^\dagger(\omega)a_{\rm R}(\omega)\right)\,. \tag{6.49}$$

The quantised electromagnetic field inside the resonator simply becomes a free radiation field with a continuum of traveling wave photon modes, in agreement with the standard approach to QED in free space.

Chapter 7

Conclusions

The main motivation for this thesis is the fact that the usually assumed standing-wave description of optical cavities can be used for example to calculate the total photon scattering rate of experiments with resonant and near-resonant laser driving. However, it cannot be used to calculate the photon emission rates through the different sides of an optical cavity in a straightforward way. To overcome this problem, we require a more detailed description of the quantised electromagnetic field between two mirrors. More concretely, we require a cavity Hamiltonian which allows us to assign different decay channels to photons travelling in different directions. This Hamiltonian also needs to be able to guarantee that a photon traveling through an optical resonator does not change its frequency. The purpose of this paper is the introduction and justification of such a Hamiltonian for the electromagnetic field inside a two-sided optical cavity. Such a cavity consists of a dielectric slab with refractive index n and can be of any length d.

The only tractable way of obtaining cavity Hamiltonians is to postulate them such that their predictions are consistent with the predictions of classical physics, whenever both theories apply. This paper follows this philosophy and proposes the travelling-wave cavity Hamiltonian $H_{\rm cav}$ in Eq. (6.17) as an alternative to the usually assumed standing-wave cavity Hamiltonian. To justify its validity, we apply it to a situation which can also be analysed by taking a fully classical approach. As illustrated in Fig. 6.1, we assume that a two-sided optical cavity is driven by a monochromatic laser field with frequency ω_0. We then calculate the intensity of the transmitted and of the reflected light using either Maxwell's equations (cf. App. 3.3) or a quantum optical master equation which derives from Eq. (6.17). Both models are shown to yield the same stationary state reflection and transmission rates, if we choose the cavity decay rate κ and the photon bouncing rate J as suggested in Eqs. (6.36) and (6.38).

The cavity Hamiltonian $H_{\rm cav}$ in Eq. (6.17) acts on the Hilbert space which is usually only used for the modeling of free radiation fields. It contains photon annihilation and creation operators for a continuum of photon frequencies ω. Moreover, we distinguish photons traveling left and photons traveling right. This allows us to assume that a laser field which enters the setup from the left excites only photons traveling right, as it should. The cavity decay rate κ for the leakage of photons through either side of the cavity mirrors depends, as one would expect, only on the refractive index n and the length d of the resonator (cf. Eq. (6.36)). The effect of the cavity mirrors is to change the direction of photons inside resonator. They transfer "left" into "right" photons and vice versa. The corresponding photon bouncing rate J in Eq. (6.38) depends, like κ, on n and d but also on the laser frequency ω_0. This dependence accounts for constructive and destructive interference effects inside the resonator.

As predicted by Maxwell's equations, there is no conversion of photons when the cavity is resonantly driven by an applied laser field. In this case, J in Eq. (6.38) becomes zero. For near resonant laser driving, J becomes identical to -2Δ with Δ being the respective laser detuning. In this case one can show that the total cavity photon number $n_{\rm Tot}$ evolves in the same way as the photon number n in the usually assumed standing-wave description of optical cavities (cf. App. 4). This means, the proposed cavity theory does not contradict current cavity QED experiments (cf. eg. Refs. [12; 13; 14]). But now that a new cavity Hamiltonian is established, it can be used to describe physical scenarios which are beyond the scope of classical electrodynamics and beyond the scope of current field quantisation schemes. For example, it can be used to describe the scattering of single photons through the fiber connections of coherent cavity networks [44].

Our approach might be criticised for being phenomenological instead of deriving its equations via a rigorous field quantisation method. The same criticism has previously been applied to the input-output formalism. A lot of work has been done to reconcile different theories (cf. eg. Refs. [10; 27; 45]). However, macroscopic QED still contains several ad-hoc assumptions. It is not as rigorous as it might appear, since quantum physics does not tell us, which Hilbert space to choose, how to define photons in a gauge-independent way, and how to implement boundary conditions. For example, in our model, we implement boundary conditions by choosing its constants such that its stationary state is consistent with Maxwell's equations. But we do not restrict the Hilbert space in which photons live. Using the phenomenological approach which we propose here instead of previous models should make it much easier to analyse feasible cavity-fiber network experiments (cf. eg. Refs. [38; 39]).

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