Covariant Transforms on Locally Convex Spaces

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Abstract

In this thesis we are concerned with the study of covariant transform which generalizes the idea of wavelets transform on Hilbert and Banach spaces. The properties of covariant transform on Locally convex spaces like continuity and boundedness are studied. We illustrate the general theory by consideration the $ax + b$ group in details. This example has a close relation to various techniques in harmonic analysis. Moreover, some properties of the image space $\mathcal{W}_F V$ is discussed. The properties of the contravariant transform on locally convex spaces are investigated as well. Also, various examples of covariant and contravariant transforms are introduced.

Introduction

Wavelets transform is an active field of research. The concept of wavelets transform related to group representation has been extensively studied by A. Grossmann, I. Daubechies, H. Feichtinger, K. Grochenig, V. Kisil and others [7, 12, 11, 10, 29, 16, 17]. This concept has important applications in fields of signal processing and quantum mechanics.

A. Grossmann, J. Morlet and T. Paul published two papers [16, 17] which can be considered as the basic of the theory of wavelets transformations related to group representations. They considered the following wavelet transform:

$$[\mathcal{W}_{w_0} f](g) = \langle f, \rho(g) w_0 \rangle_H, \qquad (1),$$

which is a unitary transform from a Hilbert space H onto $L^2(G)$, for a suitable $w_0 \in H$, where ρ is a unitary, irreducible, square integrable representation of a locally compact group G in H.

H. Fitchinger and K. Grochenig in [10, 11, 12] defined wavelet transform in Banach spaces of functions using irreducible representations of groups in Hilbert spaces. These structures of Banach spaces are called coorbit spaces and defined as the following:

For a left invariant Banach space Y of functions on G, we have

$$co(Y) = \{v \in (H^1_w)^* : \mathcal{W}_u v \in Y\},$$

where $(H^1_w)^*$ is the conjugate dual of the Banach space H^1_w defined by

$$H^1_w = \{v \in H : \mathcal{W}_u v \in L^w_1(G)\}.$$

The vector u is a non-zero vector in H such that $\mathcal{W}_u u \in L^w_1(G)$. The Banach space $co(Y)$ is ρ-invariant which is isometrically isomorphic to the Banach space Y. However, Hilbert space techniques is considered for constructing and studying these Banach spaces.

In 1999, Vladimir V. Kisil described wavelets in Banach spaces without an explicit use of the techniques of Hilbert spaces [29]. This work freed the theory from unnecessary limitations, for example, the role of the Haar measure become not so crucial. The direction of research opened by Fitchinger and Grochenig was extended by Jens Christensen and Gestur Olafsson [5]. They gave a generalization of the theory of coorbit spaces by replacing the space H^1_w by a Frechet space S.

In 2009, Vladimir V. Kisil gave a generalization of the construction of wavelet transform related to group representations. This generalized transform is called covariant transform and defined as the following:

Definition 0.0.1 *Let ρ be a representation of a group G in a space V and F be an operator from V to a space U. We define a covariant transform $\mathcal{W}_F$ from V to the space $L(G,U)$ of U-valued function on G by the formula:*

$$\mathcal{W}_F : v \to \widetilde{v}(g) = F(\rho(g^{-1})v), \quad v \in V, g \in G.$$

consider a unitary representation ρ of a locally compact group G on H and fix $w_0 \in H$. Let F defined by a functional

$$F(v) = \langle v, w_0 \rangle, \quad v \in H.$$

The the covariant transform will be:

$$\mathcal{W}_F : v \to [\mathcal{W}_F v](g) = F(\rho(g^{-1})v) = \langle \rho(g^{-1})v, w_0 \rangle,$$

which is the wavelet transform (1).

Now, we consider example of the covariant transform which is not linear. Let G be the $ax+b$ group with its representation ρ_p on $L^p(\mathbb{R})$. Consider the functional $F_p : L^p(\mathbb{R}) \to \mathbb{R}^+$ defined by:

$$F_p(f) = \frac{1}{2}\int_{-1}^{1} |f(x)|^p \, dx,$$

then the covariant transform will be:

$$[\mathcal{W}_F^p f](a,b) = \frac{1}{2}\int_{b-a}^{b+a} |f(x)|^p \, dx.$$

It is clear that the supremum of the previous transform over $b \in \mathbb{R}$ is the Hardy-Littlewood maximal function.

In this thesis we are mainly concerned about covariant transform. This thesis is divided into four chapters. The first chapter gives basic definitions and results of the following topics:

1. Topological vector spaces.
2. Locally convex spaces.
3. Group representations on locally convex spaces.

In the second chapter, we are particularly interested in the non-linear covariant transform (2.4) which is related to the Hardy-Littlewood maximal function. This example has a close relation to various techniques in harmonic analysis. Our contributions in this chapter can be summarized in the following points:

1. The general properties of the covariant transform (2.4) such as continuity are studied.
2. Some estimations of this transform are given.
3. The relations between covariant transforms with different fiducial operators are studied.

Chapter 3 deals with covariant transform on locally convex spaces. In particular, the following points have been discussed:

1. The definitions of the covariant transforms on locally convex space V and its dual V^* are given.

2. We list some general properties about covariant transform such as continuity, boundedness and intertwining.

3. General definition of convolution through pairings is given and its properties are studied.

4. The reproducing kernel related to this generalized convolution are investigated.

5. Some examples of covariant transform from real and complex harmonic analysis are introduced.

Finally, in the forth chapter, we consider contravariant transform on locally convex spaces which is considered as a generalization of the inverse wavelet transform. The contributions in this chapter can be summarized in the following:

1. The general properties of the contravariant transform are studied and some examples are given.

2. The composition of covariant and contravariant transforms is considered.

3. Some well known transforms such as Hardy-Littlewood maximal function and singular integral operator are introduced as examples of this composition.

This work is dedicated with great respect and deep affection

to my parents, wife and wonderful children.

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Chapter 1

Basic Notations

In this chapter, we present some basic definitions and results on the topological vector spaces, locally convex spaces and group representation. The results and definitions below come mostly from the following references [3, 38, 39, 43, 45].

1.1 Topological Vector Spaces

Topological vector space is a set endowed with two structures: an algebraic structure and topological structure.

Definition 1.1.1 *Let V be a vector space over the field $\mathbb{K}$ of real or complex numbers and τ be a topology on V. The (V,τ) is called a topological vector space (TVS) if:*

1. *The addition map $(x,y) \to x+y$ of $V \times V$ onto V is continuous,*
2. *The scalar multiplication map $(\lambda, x) \to \lambda\, x$ of $\mathbb{K} \times V$ onto V is continuous.*

Here $V \times V$ and $\mathbb{K} \times V$ equipped with the product topology. The continuity of the addition map means that for any $(x,y) \in V \times V$ and any neighborhood W of $x+y$ in V, there exists neighborhoods U_x and U_y of x and y respectively, such that

$$U_x + U_y \subseteq W.$$

Similarly, The continuity of the scalar multiplication means that for any $(\lambda, x) \in \mathbb{K} \times V$ and any neighborhood W of $\lambda\, x$ in V, there exists neighborhoods K and U_x of λ and x respectively, such that

$$K\, U_x \subseteq W.$$

Let us define the following operators:

Definition 1.1.2 *Let V be a topological vector space. Let $a \in V$, then the translation map $T_a : V \to V$, defined by:*

$$T_a(x) = x + a. \tag{1.1}$$

For a non-zero $\alpha \in \mathbb{K}$, we define the multiplication map $S_\alpha : V \to V$ by

$$S_\alpha(x) = \alpha\, x. \tag{1.2}$$

The continuity of the addition map and scalar multiplication map imply the following.

Proposition 1.1.3 *Let the operators T_a and S_α be as defined above, then both T_a and S_α are homeomorphisms of V onto V.*

The following corollary is an immediate consequence of the previous proposition.

Corollary 1.1.4 *Let V be a TVS, then any $U \subseteq V$ is open if and only if $a + U$ is open for every $a \in V$.*

In particular, if U is a neighborhood of 0, then $a + U$ is a neighborhood of a and hence the topology of V is completely determined by a base of neighborhoods of the origin.

Now we define some geometric properties concerning subsets of space V.

Definition 1.1.5 *Let V be a vector space over $\mathbb{K}$ and $A \subseteq V$. Then*

1. *A is called balanced if*
$$\lambda A \subseteq A,$$
for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$.

2. *A is called convex if*
$$t A + (1-t) A \subseteq A,$$
for all $0 \leq t \leq 1$.

3. *A is called absorbing if given any $x \in V$, there exists $r > 0$ such that $x \in \lambda A$, for all $|\lambda| \geq r$.*

Now, we apply these properties to the neighborhoods of the origin.

Theorem 1.1.6 *Let (V, τ) be a TVS, then τ has a base $\mathcal{U}$ of neighborhoods of the origin such that each $U \in \mathcal{U}$ has the following:*

1. *Each $U \in \mathcal{U}$ is absorbing.*

2. *Given any $U \in \mathcal{U}$, there exists a balanced $W \in \mathcal{U}$ such that $W \subseteq U$.*

3. *Given any $U \in \mathcal{U}$, there exist some $E \in \mathcal{U}$ such that $E + E \subseteq U$.*

Next, we will introduce the definition of seminorms which are closely related to local convexity.

Definition 1.1.7 *Let V be a vector space over $\mathbb{K}$. A functional $p : V \to \mathbb{R}$ is called a seminorm on V if:*

1. *$p(x) \geq 0$, for all $x \in V$.*

2. $p(\lambda x) = |\lambda|\, p(x)$, *for all* $\lambda \in \mathbb{K}$ *and* $x \in V$.

3. $p(x+y) \leq p(x) + p(y)$, *for all* $x, y \in V$.

From condition (2) in the previous definition we have

$$p(0) = p(0 \cdot x) = 0 \cdot p(x) = 0.$$

A seminorm p is said to be norm if for any $x \neq 0$ we have $p(x) \neq 0$.

It follows from the definition of seminorms that

$$p(x) = p(x - y + y) \leq p(x-y) + p(y)$$

and

$$p(y) = p(y - x + x) \leq p(y-x) + p(x)$$

thus

$$|p(x) - p(y)| \leq p(x-y) \tag{1.3}$$

this inequality is useful in studying the continuity of seminorms.

Definition 1.1.8 *A family* $\{p_\alpha\}$ *of seminorms on* V *is said to be separating if to each* $x \neq 0$ *corresponds at least one* p_{α_0} *such that* $p_{\alpha_0}(x) \neq 0$.

Normed vector spaces possess a natural topology induced by norm.

Example 1.1.9 *Every normed vector space* V *is a TVS. The continuity of the addition map follows directly from the triangle inequality. Let* $(x_0, y_0) \in V \times V$, *and let* $(x_n) \subseteq V$ *and* $(y_n) \subseteq V$ *such that* $x_n \to x_0$ *and* $y_n \to y_0$, *then*

$$\begin{aligned} \|(x_n + y_n) - (x_0 + y_0)\| &= \|(x_n - x_0) + (y_n - y_0)\| \\ &\leq \|x_n - x_0\| + \|y_n - y_0\| \\ &\to 0. \end{aligned}$$

Also, the homogeneity of the norm leads to the continuity of the scalar multiplication map. Let $(\lambda_0, x_0) \in \mathbb{K} \times V$*, and let* $(x_n) \subseteq V$ *and* $(\lambda_n) \subseteq \mathbb{K}$ *such that* $x_n \to x_0$ *and* $\lambda_n \to \lambda_0$*, then*

$$\begin{aligned} \|\lambda_n x_n - \lambda_0 x_0\| &= \|\lambda_n x_n - \lambda_0 x_n + \lambda_0 x_n - \lambda_0 x_0\| \\ &= \|(\lambda_n - \lambda_0)x_n + \lambda_0(x_n - x_0)\| \\ &\leq |\lambda_n - \lambda_0| \, \|x_n\| + |\lambda_0| \, \|x_n - x_0\| \\ &\to 0. \end{aligned}$$

So, both addition and scalar multiplication maps are continuous and hence $(V, \|\cdot\|)$ *is a TVS.*

There are some important and useful topological vector spaces such that their topologies are not induced by norms.

Example 1.1.10 *Consider the Schwartz space* $S(\mathbb{R})$ *of rapidly decreasing functions. The space* $S(\mathbb{R})$ *consists of all infinitely differentiable functions such that*

$$p_{n,m}(v) = \sup_{x\in\mathbb{R}} \left| x^n \frac{d^m}{dx^m} v \right| < \infty, \quad n, m \in \mathbb{Z}^+. \tag{1.4}$$

Note that n *and* m *might be zeros.*

It is clear that $p_{n,m}(v) \geq 0$ *for all* $v \in S(\mathbb{R})$*. Also,*

$$p_{n,m}(\lambda v) = \sup_{x\in\mathbb{R}} \left| x^n \frac{d^m}{dx^m}(\lambda v) \right| = |\lambda| \sup_{x\in\mathbb{R}} \left| x^n \frac{d^m}{dx^m} v \right| = |\lambda| \, p_{n,m}(v).$$

For any $v_1, v_2 \in S(\mathbb{R})$*, we have:*

$$\begin{aligned} p_{n,m}(v_1 + v_2) &= \sup_{x\in\mathbb{R}} \left| x^n \frac{d^m}{dx^m}(v_1 + v_2) \right| \\ &= \sup_{x\in\mathbb{R}} \left| x^n \left(\frac{d^m}{dx^m} v_1 + \frac{d^m}{dx^m} v_2 \right) \right| \\ &\leq \sup_{x\in\mathbb{R}} \left(\left| x^n \frac{d^m}{dx^m} v_1 \right| + \left| x^n \frac{d^m}{dx^m} v_2 \right| \right) \\ &\leq \sup_{x\in\mathbb{R}} \left| x^n \frac{d^m}{dx^m} v_1 \right| + \sup_{x\in\mathbb{R}} \left| x^n \frac{d^m}{dx^m} v_2 \right| = p_{n,m}(v) + p_{n,m}(v), \end{aligned}$$

then $p_{n,m}$ are seminorms on $S(\mathbb{R})$.

The dual space of $S(\mathbb{R})$ is the space of all continuous linear functionals on $S(\mathbb{R})$ and is called the space of all tempered distributions $S'(\mathbb{R})$. Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$ and define functional F_f on $S(\mathbb{R})$ by:

$$F_f(v) = \int_{\mathbb{R}} f(x)\, v(x)\, dx.$$

Since $S(\mathbb{R}) \subset L^q(\mathbb{R})$, for any $1 \leq q \leq \infty$, then by the linearity of the functional F_f and by using Hölder inequality, we have:

$$|F_f(v)| = \left| \int_{\mathbb{R}} f(x)\, v(x)\, dx \right| \leq \|f\|_{L^p}\, \|v\|_{L^q} < \infty.$$

The continuity follows directly from this inequality. Thus every $f \in L^p(\mathbb{R})$ defines a tempered distribution.

1.2 Locally Convex Spaces

It follows from Theorem 1.1.6 that every TVS V has a base of neighborhood of 0 consisting of balanced absorbing sets. Some important topological spaces have bases of neighborhood of 0 consisting of convex sets.

Definition 1.2.1 *A TVS (V, τ) is called a locally convex space (LCS), if it has a base of neighborhoods of 0 consisting of convex sets.*

Next theorem shows that seminorms produce balanced convex absorbing sets.

Theorem 1.2.2 *Let p be a seminorm on a vector space V, then the set*

$$B_p(0, r) = \{x \in V : p(x) < r\}, \quad r > 0$$

is convex balanced absorbing.

Proof. let p be a seminorm on V. For any $r > 0$, define the following set

$$B_p(0, r) = \{x \in V : p(x) < r\}.$$

For any $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1$, and for any $x \in B_p(0, r)$, we have $p(\lambda\, x) = |\lambda|\, p(x) \leq p(x) < r$. So, $\lambda\, x \in B_p(0, r)$ and hence $B_p(0, r)$ is balanced.

Also, we can prove that $B_p(0, r)$ is convex: For any $x, y \in B_p(0, r)$ and $t \in [0, 1]$, we have

$$p(tx + (1-t)y) \leq t\, p(x) + (1-t)\, p(y) < t\, r + (1-t)\, r = r.$$

Now, we show that $B_p(0, r)$ is absorbing: Let $x \in V$, choose $t_x > \frac{p(x)}{r}$. If $|\lambda| \geq t_x > \frac{p(x)}{r}$, then $p(\frac{x}{\lambda}) = \frac{p(x)}{\lambda} < p(x) \cdot \frac{r}{p(x)} = r$, when $p(x) \neq 0$. If $p(x) = 0$, then $p(\frac{x}{\lambda}) = \frac{p(x)}{\lambda} = 0 < r$. Thus, $\frac{x}{\lambda} \in B_p(0, r)$ or $x \in \lambda\, B_p(0, r)$. $\square$

Example 1.2.3 *Let $S(\mathbb{R})$ be the Schwartz space with seminorms (1.4). Since $p_{n,m}$, where $n, m \in \mathbb{Z}^+$, are seminorms, then by Theorem 1.2.2 and for each $r > 0$ the following set:*

$$B_{n,m}(0, r) = \{v \in S(\mathbb{R}) : p_{n,m}(v) < r\}.$$

is convex balanced absorbing set.

Also, continuous seminorms produce open sets $B_p(0, r)$.

Lemma 1.2.4 *Let V be a TVS and p a seminorms on V. Then, the following are equivalent:*

1. *p is continuous at $0 \in V$.*
2. *p is continuous on V.*
3. *The set $B_p(0, 1) = \{x \in V : p(x) < 1\}$ is open in V.*

Seminorms and local convexity are related to each other. Next theorem shows one direction of this relation.

Theorem 1.2.5 *Suppose that $\mathcal{P}$ is a separating family of seminorms on a vector space V. For each $p \in \mathcal{P}$ and for each $r > 0$, we have the set*

$$B_p(0, r) = \{x \in V : p(x) < r\}.$$

Let $\mathcal{B}$ be the collection of all finite intersections of the sets $B_p(0, r)$. Then $\mathcal{B}$ is convex balanced absorbing base of neighborhoods of 0 generates the topology of V. Therefore, V is a locally convex space.

The previous theorem follows from Theorem 1.2.2.

In the converse direction, we can obtain a family of seminorms generating the LCS topology τ of V. Next definition is the first step to this goal.

Definition 1.2.6 *Let V be a vector space and A be an absorbing subset of V. Define a functional $p_A : V \to \mathbb{R}$ by*

$$p_A(x) = \inf\{t > 0 : x \in t\,A\}, \quad \textit{where} \;\; x \in V.$$

The function p_A is called the Minkowski functional of A.

Next lemma demonstrates the relation between the set A and the Minkowski functional.

Lemma 1.2.7 *Let A be an absorbing subset of a vector space V. If p_A is the Minkowski functional of A, then:*

1. $0 \leq p_A(x) < \infty$, *for all $x \in V$.*

2. $p_A(0) = 0$.

3. *If A is balanced, then $p_A(\lambda x) = |\lambda|\, p_A(x)$, for all $\lambda \in \mathbb{K}$ and $x \in V$.*

4. *If A is convex, then $p_A(x+y) \leq p_A(x) + p_A(y)$, for all $x, y \in V$.*

Hence, if A is absorbing, balanced and convex, then p_A is a seminorm on V.

Theorem 1.2.8 *Let A be an absorbing and balanced subset of a vector space V. Then*

$$\{x \in V : p_A(x) < 1\} \subseteq A \subseteq \{x \in V : p_A(x) \leq 1\}.$$

The continuity of the functional p_A is connected to A. Next theorem demonstrates this connection.

Theorem 1.2.9 *Let (V, τ) be a TVS and $A \subseteq V$, absorbing and balanced. Consider $p_A : V \to \mathbb{R}$. Then p_A is continuous on V if and only if A is a τ-neighborhood of 0 in V.*

The following theorem gives the converse relation between seminorms and local convexity.

Theorem 1.2.10 *Let (V, τ) be a locally convex space. Then there exists a family $\mathcal{P} = \{p_\alpha : \alpha \in I\}$ of seminorms which generates the topology τ of V.*

A special case of locally convex spaces is a Fréchet space.

Definition 1.2.11 *A Fréchet space is a metrizable, complete locally convex space.*

Example 1.2.12 *Let $(B, \|\cdot\|_B)$ be a Banach space. B is a Fréchet space because the norm $\|\cdot\|_B$ produces a translation invariant metric*

$$d(x, y) = \|x - y\|_B,$$

and the space B is complete with respect to this metric.

Since we will have to deal with a finite combination of seminorms, it is useful to consider collection of seminorms with special property.

Definition 1.2.13 *A collection of seminorms $\mathcal{P}$ on a vector space V is called directed if and only if for all $p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n} \in \mathcal{P}$ there is p_β and $C > 0$ such that*

$$p_{\alpha_1}(x) + p_{\alpha_2}(x) + \ldots + p_{\alpha_n}(x) \leq C\, p_\beta(x),$$

for all $x \in V$.

In normed spaces, the continuity of a linear map is equivalent to boundedness. Next theorem gives a similar result for maps on locally convex spaces.

Theorem 1.2.14 *Let V_1 and V_2 be two locally convex spaces with families of seminorms $\{p_\alpha\}$ and $\{q_\beta\}$. A linear map $f : V_1 \to V_2$ is continuous if and only if for every seminorm q_β on V_2 there are $p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}$ on V_1 and $C > 0$ such that, for all $x \in V_1$,*

$$q_\beta(f(x)) \leq C\,(p_{\alpha_1}(x) + p_{\alpha_2}(x) + \ldots + p_{\alpha_n}(x)).$$

If the $\{p_\alpha\}$ are directed family of seminorms, then f is continuous if and only if for every q_β, there is p_α and $D > 0$ such that

$$q_\beta(f(x)) \leq D\, p_\alpha(x).$$

The following theorem illustrates the continuity for a family of continuous maps.

Theorem 1.2.15 (Banach-Steinhause Theorem) *Let V_1 and V_2 be Fréchet spaces. Let $\mathcal{F}$ be a family of continuous linear maps from V_1 to V_2, so that for each seminorm q on V_2 and every $x \in V_1$, $\{q(f(x)) : f \in \mathcal{F}\}$ is bounded. Then for each q there is a seminorm p on V_1 and $C > 0$, so that*

$$q(f(x)) \leq C\, p(x)$$

for all $x \in V_1$ and $f \in \mathcal{F}$.

Note that the Banach-Steinhause theorem does not require the family of seminorms to be directed.

1.3 Representations of Topological Groups on Locally Convex Spaces

This section is devoted to study group representations which are important tools to understand wavelets on groups. Group representations represent elements of groups as linear continuous transformations of locally convex spaces.

If V is a locally convex space, write $GL(V)$ for the set of all invertible linear continuous transformations from V to itself.

Definition 1.3.1 *Let G be a topological group and V be a locally convex space, then a representation of G on V is a homomorphism $\rho : G \to GL(V)$ such that the action map $G \times V \to V$ given by*

$$(g, v) \to \rho(g)v \tag{1.5}$$

is continuous in both variables.

Thus each $\rho(g)$ is a continuous operator on V and by Theorem 1.2.15, for each seminorm q on V there exists a seminorm p and $C > 0$ on V such that

$$q(\rho(g)x) \leq C\, p(x), \quad \forall g \in G, x \in V.$$

If V is a Banach space then:

$$\|\rho(g)x\| \leq C\, \|x\|, \quad \forall g \in G, x \in V,$$

and thus ρ is bounded.

Note that, if V is finite dimensional, we say that ρ is finite dimensional. Otherwise, we say that ρ is infinite dimensional. For any group G we can define a trivial representation of G by $\rho(g) = I$, for all $g \in G$.

The following representation is a basic in most of this thesis and will be used frequently.

Example 1.3.2 *Consider the $ax+b$ group which is the group of all elements (a, b), where $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$, with the group law:*

$$(a, b) \cdot (a', b') = (a\,a', a\,b' + b).$$

The elements of the $ax + b$ group can be represented by matrix as the following:

$$(a, b) \approx \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

ince

$$(a, b) \cdot (1, 0) = (a, b) = (1, 0) \cdot (a, b)$$

then $(1, 0)$ is the identity of the $ax + b$ group. Also,

$$(a, b) \cdot (\frac{1}{a}, \frac{-b}{a}) = (1, 0) = (\frac{1}{a}, \frac{-b}{a}) \cdot (a, b),$$

so, $(a, b)^{-1} = (\frac{1}{a}, \frac{-b}{a})$.

The representation of the $ax + b$ group on $L^p(\mathbb{R})$ is given by the following [33]:

$$[\rho_p(a, b)f](x) = a^{-\frac{1}{p}} f\left(\frac{x - b}{a}\right). \tag{1.6}$$

These maps are isometries of $L^P(\mathbb{R})$ for any $(a, b) \in G$:

$$\begin{aligned} \|\rho_p(a, b)f\|^p_{L^P(\mathbb{R})} &= \int_{-\infty}^{\infty} \left| a^{-\frac{1}{p}} f\left(\frac{x - b}{a}\right)\right|^p dx \\ &= \int_{-\infty}^{\infty} \frac{1}{a} \left| f\left(\frac{x - b}{a}\right)\right|^p dx \\ &= \int_{-\infty}^{\infty} |f(x)|^p \, dx = \|f\|^p_{L^P(\mathbb{R})}. \end{aligned}$$

Let ρ be a representation of a group G on V and e be the identity of G, then as an immediate consequence of Definition 1.3.1 we have:

$$\rho(e)\rho(g) = \rho(eg) = \rho(g),$$

then $\rho(e) = I$. Also, for any $g \in G$ we have:

$$\rho(g)\rho(g^{-1}) = \rho(gg^{-1}) = \rho(e) = I,$$

and similarly we have

$$\rho(g^{-1})\rho(g) = \rho(g^{-1}g) = \rho(e) = I,$$

thus, $\rho(g)\rho(g^{-1}) = I = \rho(g^{-1})\rho(g)$ and hence $\rho(g^{-1}) = \rho(g)^{-1}$.

The continuity of the action map (1.5) implies the following.

Proposition 1.3.3 *Let ρ be defined as in the definition 1.3.1. The continuity condition of the action map (1.5) is equivalent to the following:*

1. *For every $v \in V$, the map $g \to \rho(g)v$ of G into V is continuous.*
2. *For every compact subset K of G, the set $\{\rho(g) : g \in K\}$ is equicontinuous (that is for any neighborhood W of 0 in V, there exists a neighborhood U of 0 in V such that $\rho(K)U \subset W$).*

If the topology of V generated by seminorms, then by Theorem 1.2.15, for each seminorm q on V there exists a seminorm p on V and $C > 0$ such that

$$q(\rho(g)x) \leq C\, p(x), \quad \forall g \in K, x \in V.$$

For Banach spaces, it means that

$$\|\rho(g)x\| \leq C\, \|x\|, \quad \forall g \in K, x \in V.$$

The next definition plays an important role in many results regarding covariant and contravariant transforms.

Definition 1.3.4 *The left regular representation $\Lambda(g)$ of a group G is the representation by the left shift in the space $L^p(G)$ of integrable functions on G with the left Haar measure $d\mu$:*

$$\Lambda(g) : f(h) \to f(g^{-1}h). \tag{1.7}$$

Example 1.3.5 *Let G be the $ax+b$ group with the left Haar measure $\frac{da\,db}{a^2}$ and consider the space $L^p(G)$ of integrable functions, then the map Λ (1.7) acts on $L^p(G)$ as the following:*

$$\begin{aligned}[\Lambda(c,d)f](a,b) &= f\left((\frac{1}{c}, \frac{-d}{c}) \cdot (a,b) \right) \\ &= f\left(\frac{a}{c}, \frac{b-d}{c} \right),\end{aligned}$$

where, $(a,b),(c,d) \in G$.

The representation Λ is isometric on $L^p(G)$, for any $(c,d) \in G$:

$$\begin{aligned}\|\Lambda(c,d)f\|^p_{L^p(G)} &= \int_{-\infty}^{\infty}\int_0^{\infty} |[\Lambda(c,d)f](a,b)|^p \,\frac{da\,db}{a^2} \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} \left| f\left(\frac{a}{c}, \frac{b-d}{c}\right)\right|^p \frac{da\,db}{a^2} \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} |f(a,b)|^p \,\frac{d(ac)\,d(cb-d)}{a^2\,c^2} \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} |f(a,b)|^p \,\frac{da\,db}{a^2} = \|f\|^p_{L^p(G)}.\end{aligned}$$

The intertwining operators are very important to classify all representations of a given group G.

Definition 1.3.6 *Let ρ_1 and ρ_2 be representations of a group G on V_1 and V_2 respectively. Let T be a linear continuous transformation from V_1 into V_2, then T is called intertwining operators if*

$$T\rho_1 = \rho_2 T.$$

If T is linear bijective from V_1 to V_2 and $T^{-1} : V_2 \to V_1$ is continuous, then we say the representations ρ_1 and ρ_2 are equivalent.

Definition 1.3.7 *Let V be a Banach space and V^* be its dual space. Let ρ be a representation of a group G on V. For a fixed $v_0 \in V^*$, the wavelet transform $\mathcal{W}_{v_0} : V \to C(G)$, where $C(G)$ is the space of continuous functions on G, is defined as the following[29]:*

$$\mathcal{W}_{v_0} : v \to \widetilde{v}(g) = \langle \rho(g^{-1})v, v_0 \rangle.$$

The wavelet transform $\mathcal{W}_{v_0}v$ intertwines the representation ρ and the left regular representation Λ:

$$\Lambda(g)[\mathcal{W}_{v_0}v](h) = [\mathcal{W}_{v_0}v](g^{-1}h) = \langle \rho(h^{-1}g)v, v_0 \rangle = \langle \rho(h^{-1})\rho(g)v, v_0 \rangle = [\mathcal{W}_{v_0}(\rho(g)v)](h).$$

Definition 1.3.8 *Let ρ be a representation of a group G in a linear space V. A closed subspace W of V is called an invariant subspace, if $\rho(g)w \in W$ for all $g \in G$ and $w \in W$.*

There are always two trivial invariant subspaces which are the null space and V. For a nontrivial invariant subspace W, the restriction of $\rho(g)$ to W is a representation of G, and we call it a subrepresentation of ρ.

Definition 1.3.9 *A representation ρ of a group G in V is irreducible if the null space and V are the only invariant subspaces of V. Otherwise, the representation ρ is called reducible.*

Example 1.3.10 *Consider the representation ρ (1.6) of the $ax + b$ group on $L^2(\mathbb{R})$. The space $L^2(\mathbb{R})$ can be written as a direct sum of two closed subspaces:*

$$\mathcal{H}^+ = \{f \in L^2(\mathbb{R}) : supp(\mathcal{F}f) \subset [0, \infty)\} \quad \text{and} \quad \mathcal{H}^- = \{f \in L^2(\mathbb{R}) : supp(\mathcal{F}f) \subset (-\infty, 0]\},$$

where $\mathcal{F}$ is the Fourier transform.

Then for any $f \in \mathcal{H}^+$ *and any* $(a, b) \in G$*, we have* $\rho(a, b)f \in L^2(\mathbb{R})$ *and*

$$\begin{aligned}
\mathcal{F}(\rho(a,b)f)(t) &= \int_{\mathbb{R}} e^{-2\pi ixt}\,[\rho(a,b)f](x)\,dx \\
&= \int_{\mathbb{R}} e^{-2\pi ixt}\,\frac{1}{\sqrt{a}}\,f\left(\frac{x-b}{a}\right)\,dx \\
&= \sqrt{a}\int_{\mathbb{R}} e^{-2\pi i(ax+b)t}\,f(x)\,dx \\
&= \sqrt{a}e^{-2\pi ibt}\,\int_{\mathbb{R}} e^{-2\pi iaxt}\,f(x)\,dx \\
&= \sqrt{a}e^{-2\pi ibt}\,\mathcal{F}f(at).
\end{aligned} \tag{1.8}$$

So, for any $t \in (-\infty, 0)$*, we have* $at \in (-\infty, 0)$ *and*

$$\mathcal{F}(\rho(a,b)f)(t) = \sqrt{a}e^{-2\pi ibt}\,\mathcal{F}f(at) = 0,$$

thus, $supp(\mathcal{F}(\rho(a,b)f)) \subset [0,\infty)$ *and* $\rho(a.b)f \in \mathcal{H}^+$*. Therefore,* $\mathcal{H}^+$ *is invariant subspace of* $L^2(\mathbb{R})$ *under the representation* ρ*.*

Similarly, we have $\mathcal{H}^-$ *is closed invariant subspace of* $L^2(\mathbb{R})$ *under the representation* ρ*. Then* ρ *is reducible representation on* $L^2(\mathbb{R})$ *to the closed subspaces* $\mathcal{H}^+$ *and* $\mathcal{H}^-$*.*

Chapter 2

Covariant Transform on the Real Line

This chapter is devoted to study covariant transform [33, 31, 32]. The covariant transform generalizes the idea of wavelet transform on Hilbert and Banach spaces. In particular, we are concerned about the properties of the covariant transform which maps functions on the real line into functions over the $ax + b$ group. Also, its relations to harmonic analysis are considered.

2.1 Covariant Transform

The covariant transform extends the construction of wavelets from group representations. In this section, the construction of covariant transform is defined and some basic theorems and examples concerning it are also given.

Definition 2.1.1 [*32*] *Let ρ be a representation of a group G in a space V and F be an operator from V to a space U. We define a covariant transform $\mathcal{W}_F$ from V to the space $L(G, U)$ of U-valued function on G by the formula:*

$$\mathcal{W}_F : v \to \widetilde{v}(g) = F(\rho(g^{-1})v), \quad v \in V, g \in G. \tag{2.1}$$

In this context, the operator F will be called fiducial operator. The operator F may not be linear.

The next theorem proves that the covariant transform $\mathcal{W}_F$ intertwines ρ and the left regular representation Λ on $L(G, U)$.

Theorem 2.1.2 [32] *Let $\mathcal{W}_F$ be the covariant transform (2.1), then we have*

$$\mathcal{W}_F \rho(g) = \Lambda(g) \mathcal{W}_F.$$

Proof. By using the properties of group representations and the definition of covariant transform , we have:

$$\begin{aligned}
[\mathcal{W}_F \rho(g) v](h) &= F(\rho(h^{-1}) \rho(g) v) \\
&= F(\rho(g^{-1} h)^{-1} v) \\
&= [\mathcal{W}_F v](g^{-1} h) \\
&= \Lambda(g)[\mathcal{W}_F v](h).
\end{aligned}$$

□

One immediate consequence of this result is the following corollary.

Corollary 2.1.3 [32] *The image space $\mathcal{W}_F(V)$ is invariant under the representation Λ.*

Proof. Let $u \in \mathcal{W}_F(V)$, then there exists $v \in V$ such that $u = \mathcal{W}_F(v)$. Now for any $h, g \in G$ and by using the previous theorem, we have

$$[\Lambda(g)u](h) = \Lambda(g)[\mathcal{W}_F(v)](h) = [\mathcal{W}_F(\rho(g)v)](h),$$

where $\rho(g)v \in V$. Therefore, $\Lambda(g)u \in \mathcal{W}_F(V)$ and hence the image space $\mathcal{W}_F(V)$ is Λ-invariant. □

In the following we introduce several examples of the covariant transform. We start with example of wavelet transform on Hilbert spaces [10, 42].

Example 2.1.4 *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let ρ be a unitary representation of a locally compact group G on H and fix $h_0 \in H$. Let $F : H \to \mathbb{C}$ be a functional defined by $F(v) = \langle v, h_0 \rangle$. Therefore, the covariant transform is*

$$\mathcal{W}_{h_0} : v \to \widetilde{v}(g) = F(\rho(g^{-1})v) = \langle \rho(g^{-1})v, h_0 \rangle.$$

which is the wavelet transform on Hilbert spaces.

Next is the wavelet transform on Banach spaces [29].

Example 2.1.5 *For a Banach space B and $F : B \to \mathbb{C}$ be a functional defined by*

$$F(b) = \langle b, l_0 \rangle, \quad \text{for } l_0 \in B^*.$$

Then the covariant transform is

$$\mathcal{W}_{l_0} : v \to \widetilde{v}(g) = F(\rho(g^{-1})v) = \langle \rho(g^{-1})v, l_0 \rangle.$$

Definition 2.1.6 *[36] The Hardy space $H^p(\mathbb{R}^2_+)$ is the space of holomorphic functions f on $\mathbb{R}^2_+$, with norm given by:*

$$\|f\|_{H^p} = \sup_{a>0} \left(\int_{-\infty}^{\infty} |f(a,b)|^p \, db \right)^{\frac{1}{p}}.$$

Some well known transforms like Cauchy and Poisson Integrals can be produced by covariant transform.

Example 2.1.7 *Let G be the $ax + b$ group. The representation of the $ax + b$ group on $L^p(\mathbb{R})$ is given by the following:*

$$[\rho_p(a,b)f](x) = a^{-\frac{1}{p}} f\left(\frac{x-b}{a}\right).$$

We consider the functionals $F_\pm : L^p(\mathbb{R}) \to \mathbb{C}$ defined by:

$$F_\pm(f) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(x)}{x \mp i}\, dx. \tag{2.2}$$

Since $(x \mp i)^{-1} \in L^q(\mathbb{R})$, for any $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then:

$$\begin{aligned} |F_\pm(f)| &= \frac{1}{2\pi i}\left|\int_{\mathbb{R}} \frac{f(x)}{x \mp i}\, dx\right| \\ &\le \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{|f(x)|}{|x \mp i|}\, dx \\ &\le \frac{1}{2\pi i}\|f\|_{L^p(\mathbb{R})} \cdot \|(x \mp i)^{-1}\|_{L^q(\mathbb{R})}, \quad \text{by Hölder inequality}, \end{aligned}$$

thus, the functional $F_\pm(f)$ is bounded.

Then the covariant transform (2.1) is the following:

$$\begin{aligned} \widetilde{f}(a,b) = F_\pm(\rho_p(a,b)^{-1} f) &= \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{a^{\frac{1}{p}} f(ax+b)}{x \mp i}\, dx \\ &= \frac{a^{\frac{1}{p}-1}}{2\pi i}\int_{\mathbb{R}} \frac{f(x)}{\left(\frac{x-b}{a}\right) \mp i}\, dx \\ &= \frac{a^{\frac{1}{p}}}{2\pi i}\int_{\mathbb{R}} \frac{f(x)}{(x-b) \mp ia}\, dx, \end{aligned}$$

which is the Cauchy integral that maps $L^p(\mathbb{R})$ to the space of functions $\widetilde{f}(a,b)$ such that $a^{-\frac{1}{p}}\widetilde{f}(a,b)$ is in the Hardy space in the upper/lower half-plane $H_p(\mathbb{R}^2_\pm)$.

Example 2.1.8 *Take again the $ax+b$ group and its representation (1.6). Let the fiducial operator $F : L^2(\mathbb{R}) \to \mathbb{C}$ defined by*

$$F : f \mapsto F_+ f - F_- f,$$

so,

$$\begin{aligned} F(f) = F_+(f) - F_-(f) &= \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(x)}{x-i}\, dx - \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(x)}{x+i}\, dx \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}} f(x)[\frac{1}{x-i} - \frac{1}{x+i}]\, dx \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}} f(x) \cdot \frac{2i}{x^2+1}\, dx = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x)}{x^2+1}\, dx \end{aligned}$$

then the covariant transform (2.1) reduces to the Poisson integral:

$$\begin{aligned}\widetilde{f}(a,b) = F(\rho_p(a,b)^{-1}f) &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{a^{\frac{1}{p}} f(ax+b)}{x^2+1}\,dx \\ &= \frac{a^{\frac{1}{p}-1}}{\pi}\int_{\mathbb{R}} \frac{f(x)}{\left(\frac{x-b}{a}\right)^2+1}\,dx \\ &= \frac{a^{\frac{1}{p}+1}}{\pi}\int_{\mathbb{R}} \frac{f(x)}{(x-b)^2+a^2}\,dx\end{aligned}$$

In the following example we will introduce a covariant transform such that F is non-linear.

Example 2.1.9 *Consider the* $ax+b$ *group and its representation (1.6). Define a functional* $F_p : L^p(\mathbb{R}) \to \mathbb{R}^+$ *by:*

$$F_p(f) = \frac{1}{2}\int_{-1}^{1} |f(x)|^p\,dx. \tag{2.3}$$

It is clear that the functional F_p *is bounded:*

$$F_p(f) = \frac{1}{2}\int_{-1}^{1} |f(x)|^p\,dx \le \frac{1}{2}\int_{-\infty}^{\infty} |f(x)|^p\,dx = \frac{1}{2}\|f\|^p_{L^p(\mathbb{R})}.$$

Then the covariant transform (2.1) will be:

$$[\mathcal{W}^p_F f](a,b) = F_p(\rho_p((a,b)^{-1})f) = \frac{1}{2}\int_{-1}^{1} |a^{\frac{1}{p}}\, f(ax+b)|^p\,dx = \frac{1}{2}\int_{b-a}^{b+a} |f(x)|^p\,dx. \tag{2.4}$$

In the forth chapter we will see the relation between the covariant transform (2.4) and the Hardy maximal function [40, 34].

2.2 Boundedness and Continuity of Covariant Transform

In this section we are concerned about the non-linear covariant transform (2.4). In what follows, the boundedness and continuity of the covariant transform (2.4) is discussed.

Definition 2.2.1 *Let G be the $ax+b$ group. A point $(a,b) \in G$ with angle $\alpha \in (0,\pi)$, construct a α-tent with the base $(b-m, b+m)$, where $m = a\tan(\frac{\alpha}{2})$. See Figure 2.1.*

The length of the base is $2a\tan(\frac{\alpha}{2})$.

Definition 2.2.2 *A point $(a,b) \in G$ is visible from a point (a_1,b_1) with respect to an angle α if and only if the α-tent of (a_1,b_1) is included inside the α-tent of (a,b), i.e. the point (a_1,b_1) lies inside the triangle $\Delta b-m,(a,b),b+m$. Also, (a,b) is visible from a real c, if and only if c lies in the base of the α-tent of (a,b).*

Any point (a,b) with angle α is visible from all reals $c \in (b-m, b+m)$, where $m = a\tan(\frac{\alpha}{2})$.

Definition 2.2.3 *Let X be a subset of the $ax+b$ group. Fix $\alpha \in (0,\pi)$ and Let $\mathcal{C}_X^\alpha$ be the set of all reals from which points of X are visible with respect to the angle α.*

If $X = \{(a,b)\}$, then $\mathcal{C}_X = (b-m, b+m)$. Let X be a subset of the $ax+b$ group then $\mathcal{C}_X^\alpha = \cup\{\mathcal{C}_{(a,b)}^\alpha : (a,b) \in X\}$. Also, if point $(a,b) \in G$ is visible from a point (a_1,b_1) with respect to an angle α then $\mathcal{C}_{(a_1,b_1)}^\alpha \subset \mathcal{C}_{(a,b)}^\alpha$.

Definition 2.2.4 *Let X and Y be two subsets of the $ax+b$ group. Fix $\alpha \in (0,\pi)$ and Let $\mathcal{C}_X^\alpha$ and $\mathcal{C}_Y^\alpha$ be defined as above. Then X and Y are said to be α-non-simultaneously visible (α-NSV) if and only if $\mathcal{C}_X^\alpha \cap \mathcal{C}_Y^\alpha = \emptyset$.*

Two points (a,b) and (a_1,b_1) are α-NSV if and only if $(b-m, b+m) \cap (b_1-m_1, b_1+m_1) = \emptyset$, where $m = a\tan(\frac{\alpha}{2})$ and $m_1 = a_1\tan(\frac{\alpha}{2})$.

Definition 2.2.5 *A subset X of the $ax+b$ group is called α-sparse if every two points on X are α-NSV, i.e. $\mathcal{C}_p^\alpha \cap \mathcal{C}_q^\alpha = \phi$ for any $p, q \in X$.*

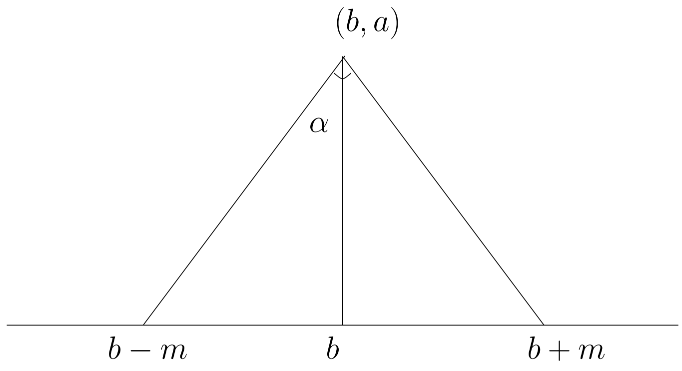

Figure 2.1: Tent (b, a) with angle α.

The above definitions depending on parameter α have the similar nature for all $0 < \alpha < \pi$. We do not need to consider separate subregions or exceptional values of this parameter in our proofs.

Although the following lemma seems to be obvious, we are going to need it several occasions.

Lemma 2.2.6 *Consider the covariant transform (2.4) and $f \in L_1(\mathbb{R})$. If X is an α-sparse, then*

$$\sum_{(a_i,b_i)\in X} [\mathcal{W}_F f](a_i, b_i) \leq \|f\|_{L_1(\mathbb{R})}. \tag{2.5}$$

Proof. Since no two points in X are visible from the same reals, then the bases of all tents (a_i, b_i), do not intersect with each other. Now, since $|f| > 0$, then the summation of all integrals of the function $|f|$ over all the bases of tents is less than the integral of $|f|$ over $\mathbb{R}$. □

The relation between the distance of two points on the $ax + b$ group and the length of the bases of their tents is determined by the following lemma.

Lemma 2.2.7 *Let (a, b) and (c, d) be two points on the $ax+b$ group. Let $A = [b-a, b+a]$ and $B = [d-c, d+c]$. Then for $L = \sqrt{(b-d)^2 + (a-c)^2}$, we have $|A \setminus B| < 2L$ and $|B \setminus A| < 2L$.*

Proof. We have three cases:

1. If A and B are disjoints, then $|A \setminus B| = |A| = 2a$. Also, since $A \cap B = \emptyset$, then $a < |b-d|$ and $|A \setminus B| = 2a < 2|b-d| < 2L$.

   Similarly,
   $$|B \setminus A| = |B| = 2c < 2|b-d| < 2L.$$
   Another way to prove this case: if $b < d$ then $b + a < d - c$ and this implies that $a < d - b - c < d - b$ and thus $|A \setminus B| = 2a < 2(d-b) < 2L$.

2. If $A \subseteq B$, then $|A \setminus B| = |\emptyset| = 0 < 2L$ and $|B \setminus A| = 2|a-c| < 2L$.

3. If $A \cap B \neq \emptyset$ but not contained in each other, then $|A \setminus B| \leq |b-d| + |a-c| < 2L$ and $|B \setminus A| \leq |b-d| + |a-c| < 2L$.

The proof is completed. □

The next theorem is a well known result called absolute continuity of Lebesgue integral theorem which is useful for us to investigate the continuity of the covariant transform.

Theorem 2.2.8 [28] *Let $f \in L^1(\mathbb{R})$. Then for any $\varepsilon > 0$, there is $\delta > 0$ such that*
$$\left| \int_A f \, d\mu \right| < \varepsilon \quad \textit{if} \quad |A| < \delta.$$

At this point we are ready to study the continuity of the covariant transform.

Proposition 2.2.9 *Let $f \in L^1(\mathbb{R})$, then the covariant transform (2.4) is a uniform continuous on the $ax+b$ group.*

Proof. For a given $f \in L^1(\mathbb{R})$, we have $|f| \in L^1(\mathbb{R})$. By theorem 2.2.8, for any $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that for any $C \subseteq \mathbb{R}$, $|C| < \delta_\varepsilon$ we have

$$\int_C |f(x)|\, dx < \varepsilon. \tag{2.6}$$

Now, let $(a, b) \in G$, then for any $(a_1, b_1) \in G$ such that $\sqrt{(b - b_1)^2 + (a - a_1)^2} < \frac{\delta_\varepsilon}{2}$, we have

$$\begin{aligned} \left|F(\rho(a,b)^{-1}f) - F(\rho(a_1,b_1)^{-1}f)\right| &= \left|\frac{1}{2}\int_{b-a}^{b+a} |f(x)|\, dx - \frac{1}{2}\int_{b_1-a_1}^{b_1+a_1} |f(x)|\, dx\right| \\ &= \left|\frac{1}{2}\int_{A\setminus B} |f(x)|\, dx - \frac{1}{2}\int_{B\setminus A} |f(x)|\, dx\right| \\ &\leq \frac{1}{2}\int_{A\setminus B} |f(x)|\, dx + \frac{1}{2}\int_{B\setminus A} |f(x)|\, dx, \end{aligned}$$

where $A = [b - a,\ b + a]$ and $B = [b_1 - a_1,\ b_1 + a_1]$. Then by Lemma 2.2.7 we have both of $|A \setminus B|$ and $|B \setminus A|$ are less than $2\sqrt{(b - b_1)^2 + (a - a_1)^2} < \delta_\varepsilon$. So,

$$\begin{aligned} \left|F(\rho(a,b)^{-1}f) - F(\rho(a_1,b_1)^{-1}f)\right| &\leq \frac{1}{2}\int_{A\setminus B} |f(x)|\, dx + \frac{1}{2}\int_{B\setminus A} |f(x)|\, dx, \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \quad \text{by (2.6)} \\ &= \varepsilon, \end{aligned}$$

thus the covariant transform is uniform continuous at (a, b). Since (a, b) is chosen arbitrary, then it is uniform continuous on G. □

Now we will discuss a stronger condition of the continuity which is the Lipschitz condition.

Proposition 2.2.10 *For any $f \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $1 \leq p < \infty$ the covariant transform (2.4) is Lipschitz continuous.*

Proof. Let $f \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then as in the Proposition 2.2.9

$$\begin{aligned} \left|F_p(\rho_p(a,b)^{-1}f) - F_p(\rho_p(a_1,b_1)^{-1}f)\right| &\leq \frac{1}{2}\int_{A\setminus B} |f(x)|^p\, dx + \frac{1}{2}\int_{B\setminus A} |f(x)|^p\, dx, \\ &\leq \frac{1}{2}\|f\|_\infty^p\, |A \setminus B| + \frac{1}{2}\|f\|_\infty^p\, |B \setminus A| \end{aligned}$$

where $A = [b-a,\, b+a]$ and $B = [b_1 - a_1,\, b_1 + a_1]$. Again by lemma 2.2.7 we have both of $|A \setminus B|$ and $|B \setminus A|$ are less than $2L$, where $L = \sqrt{(b-b_1)^2 + (a-a_1)^2}$. Thus,

$$\begin{aligned} \left|F_p(\rho_p(a,b)^{-1}f) - F_p(\rho_p(a_1,b_1)^{-1}f)\right| &\le \frac{1}{2}\|f\|_\infty^p\, |A \setminus B| + \frac{1}{2}\|f\|_\infty^p\, |B \setminus A| \\ &\le \|f\|_\infty^p\, L + \|f\|_\infty^p\, L \\ &= 2L\, \|f\|_\infty^p, \end{aligned}$$

where $L = |(a,b) - (a_1,b_1)|$ and $2\,\|f\|_\infty^p$ is the Lipschitz constant. Hence $\mathcal{W}_F^p f$ is Lipschitz continuous. □

Next example shows that an unbounded L^p function may not produce a Lipschitz covariant transform $\mathcal{W}_F^p f$.

Example 2.2.11 *Let*

$$f(x) = \begin{cases} x^{-\frac{1}{2p}} & : 0 \le x \le 1 \\ 0 & : \textit{otherwise} \end{cases}$$

It is clear that $f \in L^p(\mathbb{R})$, for $1 \le p < \infty$. Note that f is not bounded. The covariant transform $\mathcal{W}_F^p f$ is

$$[\mathcal{W}_F^p f](a,b) = \frac{1}{2}\int_{b-a}^{b+a} |f(x)|^p\, dx.$$

For the point $(\frac{1}{n}, 0) \in G$, for $n \ge 1$:

$$[\mathcal{W}_F^p f](\frac{1}{n}, 0) = \frac{1}{2}\int_0^{\frac{1}{n}} \frac{1}{\sqrt{x}}\, dx = \left[\sqrt{x}\,\right]_0^{\frac{1}{n}} = \frac{1}{\sqrt{n}}.$$

For any two points $(\frac{1}{n}, 0)$ and $(\frac{1}{m}, 0)$ where $n, m \ge 1$ and $n \ne m$:

$$\begin{aligned} \frac{|[\mathcal{W}_F^p f](\frac{1}{n},0) - [\mathcal{W}_F^p f](\frac{1}{m},0)|}{|\frac{1}{n} - \frac{1}{m}|} &= \frac{|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}|}{|\frac{1}{n} - \frac{1}{m}|} \\ &= \frac{|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}}|}{|(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}})(|\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}|)|} \\ &= \frac{1}{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}} = \frac{\sqrt{nm}}{\sqrt{n} + \sqrt{m}}, \end{aligned}$$

Since $\frac{\sqrt{nm}}{\sqrt{n}+\sqrt{m}}$ can not be bounded by any constant K. Then $\mathcal{W}_F^p f$ is not Lipschitz.

2.3 Some Estimations of Covariant Transform

Consider again the non-linear covariant transform (2.4). In this section the relation between this transform and harmonic analysis is studied and structure of the $ax + b$ group is used explicitly in proofs.

Definition 2.3.1 *Let G be the $ax + b$ group and let X be a subset of G. Let $\mathcal{C}_X^\alpha$ be a set of all real numbers from which the points of X are visible with respect to the angle α. Then the measure of $\mathcal{C}_X^\alpha$ is called the horizontal capacity of X and denoted by $h(X)$, i.e. $h(X) = \mu(\mathcal{C}_X^\alpha)$. Also for a countable set X, vertical capacity $v(X)$ is defined as the sum of all the coordinates a of the points of X, i.e. $v(x) = \sum\{a : (a, b) \in X\}$.*

The vertical capacity $v(X)$ admits $+\infty$ and in case that X is uncountable, then $v(X) = +\infty$.

The horizontal and vertical capacities have the following basic properties:

Proposition 2.3.2 *Let G be the $ax + b$ group. Let X and Y be two subsets of G. Then*

1. $h(X) \leq 2\tan\left(\frac{\alpha}{2}\right)\, v(X)$*, the identity is obtained when X is a α-sparse.*
2. $h(X \cup Y) \leq h(X) + h(Y)$*, the identity is obtained when X and Y are α-NSV.*
3. $v(X \cup Y) \leq v(X) + v(Y)$*, the identity is obtained when X and Y are disjoint.*

Proof. Let X and Y be as above, then:

1. Any point $(a, b) \in X$ is visible from a set of real numbers with measure $2a\tan\left(\frac{\alpha}{2}\right)$,

thus $h(\{(a,b)\}) = 2a\tan\left(\frac{\alpha}{2}\right)$. Therefore,

$$\begin{aligned} h(X) &\leq \sum_{(a_i,b_i)\in X} h(\{(a_i,b_i)\}) \\ &= \sum_{(a_i,b_i)\in X} 2a_i \tan\left(\frac{\alpha}{2}\right) \\ &= 2\tan\left(\frac{\alpha}{2}\right) \sum_{(a_i,b_i)\in X} a_i = 2\tan\left(\frac{\alpha}{2}\right) v(X). \end{aligned}$$

If X is a α-sparse then it is obvious that

$$h(X) = \sum_{(a_i,b_i)\in X} h(\{(a_i,b_i)\})$$

and thus, $h(X) = 2\tan\left(\frac{\alpha}{2}\right) v(X)$.

2. Let $\mathcal{C}_X^\alpha$ and $\mathcal{C}_Y^\alpha$ be the sets of all real numbers from which the points of X and Y are visible respectively. It is clear that $\mathcal{C}_{X\cup Y}^\alpha = \mathcal{C}_X^\alpha \cup \mathcal{C}_Y^\alpha$ and thus,

$$\begin{aligned} h(X\cup Y) = \mu(\mathcal{C}_{X\cup Y}^\alpha) &= \mu(\mathcal{C}_X^\alpha \cup \mathcal{C}_Y^\alpha) \\ &\leq \mu(\mathcal{C}_X^\alpha) + \mu(\mathcal{C}_Y^\alpha) = h(X) + h(Y). \end{aligned}$$

If X and Y are α-NSV then $\mathcal{C}_X^\alpha \cap \mathcal{C}_Y^\alpha = \phi$ and thus

$$h(X\cup Y) = \mu(\mathcal{C}_{X\cup Y}^\alpha) = \mu(\mathcal{C}_X^\alpha) + \mu(\mathcal{C}_Y^\alpha) = h(X) + h(Y).$$

3. Since $v(X\cup Y) = \sum_{(a_i,b_i)\in X\cup Y} a_i$, then it is clear that $v(X\cup Y) \leq v(X) + v(Y)$. Also, it is obvious that if X and Y are disjoints then $v(X\cup Y) = v(X) + v(Y)$.

□

The following lemma is a modified version of Vitali Lemma [10, 11]. A new proof is given by using structure of the $ax+b$ group. Let X be a collection of points of the $ax+b$ group. Let $p \in X$, then we use the notation $X\hat{\setminus}p$ to denote the maximal subset of $X \setminus p$ such that p and $X\hat{\setminus}p$ are α-NSV, i.e. $\mathcal{C}_p^\alpha \cap \mathcal{C}_{X\hat{\setminus}p}^\alpha = \phi$.

Lemma 2.3.3 *Let X be a finite collection of points of the $ax + b$ group. Then there is a α-sparse subset Y of X such that*

$$h(X) < 6 \tan\left(\frac{\alpha}{2}\right) v(Y).$$

Proof. Choose a point $p_1 = (a_1, b_1)$ in X such that p_1 has the biggest vertical capacity, i.e. a_1 is the biggest. The set $(X \setminus p_1) \setminus (X \hat{\setminus} p_1)$ has points which are not α-NSV with p_1 and have smaller vertical capacities than p_1. Again choose $p_2 \in X \hat{\setminus} p_1$ such that p_2 has the biggest vertical capacity on $X \hat{\setminus} p_1$. Assume that $p_1, p_2, \ldots, p_n$ have been chosen, then choose $p_{n+1} \in X \hat{\setminus} \{p_1, p_2, \ldots, p_n\}$ such that p_{n+1} has the biggest vertical capacity on $X \hat{\setminus} \{p_1, p_2, \ldots, p_n\}$. By continue this, we get an α-sparse sub-collection Y of X.

Now, for any point $p = (a, b) \in X \setminus Y$, there exists a point $p_n \in Y$ with a bigger vertical capacity such that they are not α-NSV. Since a_n is bigger than a, then the point $p'_n = (3a_n, b_n)$ is visible from all real numbers from which p or (p_n) are visible as in the Figure 2.2. Therefore, the horizontal capacity of p'_n is bigger than the horizontal capacities of p, p_n. Thus, the vertical capacity of Y multiplied by $6\tan(\frac{\alpha}{2})$ exceed the horizontal capacity of X. $\square$

In the case that α is right angle, the vertical capacity of Y multiplied by 6 exceed the the horizontal capacity of X, i.e.

$$6v(Y) > h(X).$$

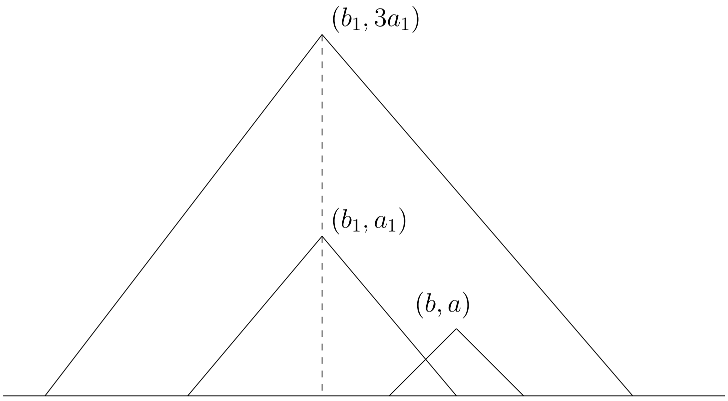

Figure 2.2: Tent $(b_1, 3a_1)$ contains tents (b, a) and (b_1, a_1).

The following is an infinite version of the Lemma 2.3.3.

Lemma 2.3.4 *Let X be any collection of points of the $ax + b$ group such that*

$$\sup_{(a,b)\in X} a < \infty.$$

Then there is a countable sparse subset Y of X such that

$$h(X) \leq 10 \tan\left(\frac{\alpha}{2}\right) v(Y).$$

Proof. Let $M_1 = \sup_{(a,b)\in X} a$, then choose a point $p_1 = (a_1, b_1) \in X$ such that $a_1 > \frac{1}{2} M_1$. Now consider $Y_1 = X \hat{\setminus} p_1$ and let $M_2 = \sup_{(a,b)\in Y_1} a$. Choose a point $p_2 = (a_2, b_2) \in Y_1$ such that $a_2 > \frac{1}{2} M_2$. Assuming that $p_1, \ldots, p_n$ have been chosen, let $Y_n = X \hat{\setminus} \{p_1, \ldots, p_n\}$ and choose $p_{n+1} = (a_{n+1}, b_{n+1}) \in Y_n$ such that $a_{n+1} < \frac{1}{2} M_{n+1}$, where $M_{n+1} = \sup_{(a,b)\in Y_n} a$. Now let $Y = \{p_n : n \in \mathbb{N}\}$, then Y is a countable sparse subset of X.

Let $(a, b) \in X$ be arbitrary. If $(a, b) \in Y$, then we are done. Otherwise, let $n \in \mathbb{N}$ be the first index such that $(a, b) \notin Y_n$, then the points (a, b) and $p_n = (a_n, b_n) \in Y_{n-1}$ are not NSV. So, we have

$$a_n \geq \frac{1}{2} M_n \geq \frac{1}{2} a.$$

Also

$$|b_n - b| \leq a_n + a < 3a_n,$$

and

$$a < a + |b_n - b| < a + a_n \leq 5a_n.$$

Thus, the tent $(5a_n, b_n)$ contains tents of (a_n, b_n) and (a, b) and hence the vertical capacity of Y multiplied by $10\ \tan\left(\frac{\alpha}{2}\right)$ exceed the horizontal capacity of X. □

Next proposition shows that the visibility between points of the $ax + b$ group does not affect by shift and scaling.

Proposition 2.3.5 *The visibility relation between two points of the $ax + b$ group is invariant under the left action of the $ax + b$ group:*

$$\Lambda_g : h \to gh.$$

Proof. Assume that (a, b) is a point on the upper half plane visible from another point (a_1, b_1), then by definition 2.2.2 we have

$$b - a < b_1 - a_1 < b_1 + a_1 < b + a. \tag{2.7}$$

Let $g_1 = (1, d)$, then $\Lambda_{g_1}(a, b) = (a, b + d)$ and $\Lambda_{g_1}(a_1, b_1) = (a_1, b_1 + d)$. Then by using (2.7) we have

$$(b+d) - a = d + (b - a) < d + (b_1 - a_1) = (d + b_1) - a_1 < (d + b_1) + a_1 < (d + b) + a.$$

Therefore, $\Lambda_{g_1}(a, b)$ is visible from $\Lambda_{g_1}(a_1, b_1)$ and the visibility is invariant under Λ_{g_1}. Now, let $g_2 = (c, 0)$, then $\Lambda_{g_2}(a, b) = (ac, bc)$ and $\Lambda_{g_2}(a_1, b_1) = (a_1c, b_1c)$. Since $c > 0$, then we have

$$bc - ac < b_1c - a_1c < b_1c + a_1c < bc + ac.$$

So, $\Lambda_{g_2}(a,b)$ is visible from $\Lambda_{g_2}(a_1,b_1)$, thus visibility is invariant under Λ_{g_2}. Since

$$\Lambda_g(a,b) = (\Lambda_{g_1} \circ \Lambda_{g_2})(a,b), \quad g = (c,d),$$

then, the visibility is invariant under Λ_g. □

The following is an immediate consequence of the previous proposition.

Corollary 2.3.6 *Let X be a subset of the $ax+b$ group and Y be a sparse subset of X. Then $\Lambda_g(Y)$ is a sparse subset of $\Lambda_g(X)$.*

Proof. Let (a,b) and (a_1,b_1) be two points in Y, then they are NSV and hence $|b-b_1| > a+a_1$. Let $g_1 = (1,d)$, then $\Lambda_{g_1}(a,b) = (a,b+d)$ and $\Lambda_{g_1}(a_1,b_1) = (a_1,b_1+d)$. So,

$$|(b_1+d)-(b+d)| = |b_1-b| > a+a_1$$

and hence $\Lambda_{g_1}(a,b)$ and $\Lambda_{g_1}(a_1,b_1)$ are NSV. Now, let $g_2 = (c,0)$, then $\Lambda_{g_2}(a,b) = (ac,bc)$ and $\Lambda_{g_2}(a_1,b_1) = (a_1c,b_1c)$. So,

$$|b_1c - bc| = c\,|b_1-b| > c\,(a+a_1) = ca + ca_1,$$

thus $\Lambda_{g_2}(a,b)$ and $\Lambda_{g_2}(a_1,b_1)$ are NSV. Therefore, $\Lambda_g(Y)$ is a sparse subset of $\Lambda_g(X)$, where $g=(c,d)$.

There is an easier proof as the following: Let $b < b_1$, and this implies that $b+a < b_1 - a_1$, then

$$b+d < b_1+d$$

and

$$(b+d)+a < (b_1+d)+a_1.$$

Therefore, $\Lambda_{g_1}(a,b)$ and $\Lambda_{g_1}(a_1,b_1)$ are NSV. Also, since $c > 0$ then $bc < b_1c$ and $c(b+a) < c(b_1-a_1)$. Thus, $\Lambda_{g_2}(a,b)$ and $\Lambda_{g_2}(a_1,b_1)$ are NSV and hence $\Lambda_g(Y)$ is a sparse subset of $\Lambda_g(X)$, where $g = (c, d)$. □

Next proposition shows that the horizontal and vertical capacities of a subset X of the $ax + b$ group are preserved under shift but not under scaling. Therefore, horizontal and vertical capacities are not invariant under the left action Λ_g, where $g = (c, d)$.

Proposition 2.3.7 *Let X be a collection of points of the $ax + b$ group. Then*

1. $h(\Lambda_{(1,d)}X) = h(X)$ *and* $v(\Lambda_{(1,d)}X) = v(X)$, *where* $(1, d) \in G$.
2. $h(\Lambda_{(c,0)}X) = c\, h(X)$ *and* $v(\Lambda_{(c,0)}X) = c\, v(x)$, *where* $(c, 0) \in G$.

Proof. Let X be a set of points of the $ax + b$ group, then:

1. Let $g_1 = (1, d) \in G$, then $\Lambda_{g_1}(X) = \{(a, b + d) : (a, b) \in X\}$. It is clear that
$$v(\Lambda_{g_1}(X)) = \sum_{(a_i,b_i)\in X} a_i = v(X).$$
Also, since the visibility relation is left invariant and Λ_{g_1} acts on X by shift, then $h(\Lambda_{g_1}(X)) = h(X)$.
2. Let $g_2 = (c, 0) \in G$, then $\Lambda_{g_2}(X) = \{(ac, bc) : (a, b) \in X\}$. Therefore,
$$v(\Lambda_{g_2}(X)) = \sum_{(a_i,b_i)\in X} ca_i = c \sum_{(a_i,b_i)\in X} a_i = c\, v(X).$$
Since Λ_{g_2} acts on points of X by scaling, then all the tents are scaled and thus $h(\Lambda_{g_2}(X)) = ch(X)$.

□

Although horizontal and vertical capacities are not preserved under scaling, the ratio $\frac{h(X)}{v(X)}$ are preserved.

Corollary 2.3.8 *Let X be a collection of points of the $ax+b$ group. Then $\frac{h(\Lambda_g(X))}{v(\Lambda_g(X))} = \frac{h(X)}{v(X)}$.*

Proof. Let $g = (c, d) \in G$, then $\Lambda_g = \Lambda_{g_1} \circ \Lambda_{g_2}$, where $g_1 = (1, d)$ and $g_2 = (c, 0)$ and

$$h(\Lambda_g(X)) = h((\Lambda_{g_1} \circ \Lambda_{g_2})(X)) = h(\Lambda_{g_1}(\Lambda_{g_2}(X)))$$

From the previous proposition, we have:

$$h(\Lambda_{g_1}(\Lambda_{g_2}(X))) = h(\Lambda_{g_2}(X)) = ch(X)$$

Also, $v(\Lambda_g(X)) = v(\Lambda_{g_1}(\Lambda_{g_2}(X)))$. Again, by using the previous proposition we have:

$$v(\Lambda_{g_1}(\Lambda_{g_2}(X))) = v(\Lambda_{g_2}(X)) = cv(X),$$

thus $\frac{h(\Lambda_g(X))}{v(\Lambda_g(X))} = \frac{c\,h(X)}{c\,v(X)} = \frac{h(X)}{v(X)}$. □

Lemma 2.3.9 *Let G be the $ax+b$ group, and let $\mathcal{W}_F$ be the covariant transform (2.4) and $f \in L_1(G)$. For $\lambda > 0$, define the following set $X_\lambda = \{(a, b) \in G : [\mathcal{W}_F f](a, b) > \lambda\, a\}$. Then, we have a sparse subset Y of X_λ such that*

$$v(Y) < \frac{\|f\|_{L_1}}{\lambda}. \tag{2.8}$$

Proof. By using lemma 2.3.4, we can extract a subset Y from X_λ such that no two points in Y are visible from the same real numbers. Furthermore, for any $(a, b) \in Y$, we have

$$\lambda\, a < [\mathcal{W}_F f](a, b) \quad \text{or} \quad a < \frac{1}{\lambda}\, [\mathcal{W}_F f](a, b),$$

so,

$$\begin{aligned} \sum_{(a_i,b_i)\in Y} a_i &< \frac{1}{\lambda} \sum_{(a_i,b_i)\in Y} [\mathcal{W}_F f](a_i, b_i) \\ &< \frac{\|f\|_{L_1}}{\lambda} \quad \text{by (2.5)}. \end{aligned}$$

Since $v(Y) = \sum_{(a_i,b_i)\in Y} a_i$, then $v(Y) < \frac{\|f\|_{L_1}}{\lambda}$. □

By combining Lemma 2.3.4 with Lemma 2.3.9, we obtain the following corollary.

Corollary 2.3.10 *Let G be the $ax + b$ group and let the covariant transform $\mathcal{W}_F$ and f be as in the Lemma 2.3.9. For $\lambda > 0$, define the following set $X_\lambda = \{(a, b) \in G : [\mathcal{W}_F f](a, b) > \lambda\, a\}$. Then,*

$$h(X_\lambda) < 10\,\tan\left(\frac{\alpha}{2}\right)\frac{\|f\|_{L_1}}{\lambda}. \tag{2.9}$$

Proof. By using Lemma 2.3.4 we can extract a sparse subset Y of X_λ such that

$$h(X_\lambda) < 10\,\tan\left(\frac{\alpha}{2}\right) v(Y),$$

and by using Lemma 2.3.9 we have $h(X_\lambda) < 10\,\tan\left(\frac{\alpha}{2}\right)\frac{\|f\|_{L_1}}{\lambda}$. □

Lemma 2.3.9 gives an upper bound for coordinates a of points of the set X_λ. The next example proves that we can not find lower bound for the coordinates a bigger than 0.

Example 2.3.11 *Let*

$$f(x) = \begin{cases} 1, & 0 < x \le 1; \\ 0, & otherwise. \end{cases}$$

It is clear that $f \in L^1(\mathbb{R})$:

$$\|f\|_{L_1} = \int_{-\infty}^{\infty} |f(x)|\,dx = \int_0^1 dx = 1 < \infty.$$

Let $\lambda = \frac{1}{2}$ and $b = \frac{1}{2}$, then

1. *For $0 < a \le \frac{1}{2}$, the covariant transform is*

$$[\mathcal{W}_F f](a, \frac{1}{2}) = \frac{1}{2}\int_{\frac{1}{2}-a}^{\frac{1}{2}+a} |f(x)|\,dx = a > \frac{a}{2} = \lambda\, a.$$

2. *For $\frac{1}{2} < a < 1$, then*

$$\begin{aligned}[\mathcal{W}_F f](a, \frac{1}{2}) &= \frac{1}{2}\int_{\frac{1}{2}-a}^{\frac{1}{2}+a} |f(x)|\,dx \\ &= \frac{1}{2}\int_0^1 dx = \frac{1}{2},\end{aligned}$$

since $a < 1$, then $\lambda\, a < \frac{1}{2} = [\mathcal{W}_F f](a, \frac{1}{2})$.

3. *For $a \geq 1$, we have*

$$\begin{aligned}[\mathcal{W}_F f](a, \frac{1}{2}) &= \frac{1}{2} \int_{\frac{1}{2}-a}^{\frac{1}{2}+a} |f(x)|\, dx \\ &= \frac{1}{2} \int_0^1 dx = \frac{1}{2},\end{aligned}$$

again, since $a \geq 1$, we have $\lambda\, a \geq \frac{1}{2} = [\mathcal{W}_F f](a, \frac{1}{2})$.

Therefore, $(a, \frac{1}{2}) \in X_{\frac{1}{2}}$, for $0 < a < 1$, but $(a, \frac{1}{2}) \notin X_{\frac{1}{2}}$ for $a \geq 1$.

Since we estimate the bounds of the coordinates a of points of the set X_λ and the upper bound of the horizontal capacity $h(X_\lambda)$, then we can estimate the size of the set X_λ.

Proposition 2.3.12 *Let G be the $ax + b$ group and let the covariant transform $\mathcal{W}_F$ and f be as in the Lemma 2.3.9. Let X_λ be a subset of G defined as above. Then*

$$\mu(X_\lambda) < \frac{c}{\lambda^2} \|f\|^2_{L_1}. \tag{2.10}$$

where, $c = 10 \tan(\frac{\alpha}{2})$

Proof. For any $(a, b) \in X_\lambda$, we have $0 < a < \frac{1}{\lambda}[\mathcal{W}_F f](a, b)$, and thus $0 < a < \frac{\|f\|_{L_1}}{\lambda}$. Moreover, the set $M = \{b : (a, b) \in X_\lambda\}$ is a subset of $\mathcal{C}_{X_\lambda}$. So, the set X_λ is contained inside the set $M \times \left[0, \frac{\|f\|_{L_1}}{\lambda}\right]$. Therefore

$$\begin{aligned}\mu(X_\lambda) &< \mu(M) \cdot \frac{\|f\|_{L_1}}{\lambda} \\ &< \mu(\mathcal{C}_{X_\lambda}) \cdot \frac{\|f\|_{L_1}}{\lambda} = h(X_\lambda)\, \frac{\|f\|_{L_1}}{\lambda}.\end{aligned}$$

By using the relation (2.9) we have

$$\mu(X_\lambda) < 10\, \tan\left(\frac{\alpha}{2}\right) \frac{\|f\|^2_{L_1}}{\lambda^2} = \frac{c}{\lambda^2} \|f\|^2_{L_1}$$

where $c = 10 \tan(\frac{\alpha}{2})$. $\square$

2.4 Covariant Transform with Different Fiducial Operators

In this section we are concerned about the connections between covariant transforms with different fiducial operators. Consider again the $ax+b$ group. Then any $w \in L_\infty[-1,1]$ such that $\|w\|_\infty \leq 1$ defines a linear functional F_w on $L_1(\mathbb{R})$:

$$F_w(f) = \frac{1}{2}\int_{-1}^{1} f(x)\, w(x)\, dx.$$

It is clear that the covariant transform $\mathcal{W}_{F_w} f$ is bounded:

$$\begin{aligned}[\mathcal{W}_{F_w} f](a,b) &= \frac{1}{2}\int_{-1}^{1} af(ax+b)\, w(x)\, dx \\ &\leq \frac{1}{2}\int_{-1}^{1} |af(ax+b)|\ dx \\ &= [\mathcal{W}_F f](a,b)\end{aligned}$$

Moreover, the non-linear covariant transform (2.4) can be expressed in terms of the linear covariant transform generated by F_w:

$$[\mathcal{W}_F f](a,b) = \sup\{[\mathcal{W}_{F_w} f](a,b) : w \in L_\infty[-1,1] \ \textit{and} \ \|w\|_\infty \leq 1\}.$$

The next lemma is a generalized version of Lemma 2.2.6.

Lemma 2.4.1 *Let $f \in L_1(\mathbb{R})$ and let X be a collection of points of the $ax+b$ group. If X is a sparse, then*

$$\sum_{(a_i,b_i)\in X} [\mathcal{W}_{F_w} f](a_i,b_i) \leq \|f\|_{L_1} \|w\|_\infty.$$

Proof. Let $f \in L_1(\mathbb{R})$, then

$$\begin{aligned}\sum_{(a_i,b_i)\in X} [\mathcal{W}_w f](a_i,b_i) &= \sum_{(a_i,b_i)\in X} \int_{-1}^{1} a_i f(a_i x + b_i)\, w(x)\, dx \\ &\leq \sum_{(a_i,b_i)\in X} \int_{-1}^{1} |a_i f(a_i x + b_i)|\; |w(x)|\, dx \\ &\leq \|w\|_\infty \sum_{(a_i,b_i)\in X} \int_{-1}^{1} |a_i f(a_i x + b_i)|\; dx \\ &= \|w\|_\infty \sum_{(a_i,b_i)\in X} \int_{b_i - a_i}^{b_i + a_i} |f(x)|\, dx \\ &\leq \|w\|_\infty \int_{-\infty}^{\infty} |f(x)|\, dx = \|w\|_\infty \, \|f\|_{L_1}\end{aligned}$$

□

Proposition 2.4.2 *Let w be as above and X_λ be a collection of points (a,b) of the $ax+b$ group such that $[\mathcal{W}_{F_w} f](a,b) > \lambda\, a$, for $\lambda > 0$. Let $\mathcal{C}^\alpha_{X_\lambda}$ be a collection of all reals from which the points of X_λ are visible. Then*

$$\int_{\mathcal{C}^\alpha_{X_\lambda}} w(x)\, dx \leq \frac{c}{\lambda} \|f\|_{L_1} \|w\|_\infty,$$

where $c = 10\tan(\frac{\alpha}{2})$

Proof. By using Lemma 2.3.4, we have a sparse subset Y of X such that

$$h(X_\lambda) = \mu(\mathcal{C}_{X_\lambda}) < 10 \tan\left(\frac{\alpha}{2}\right) \sum_{(a_i,b_i)\in Y} a_i .$$

Since, $a < \frac{1}{\lambda}[\mathcal{W}_{F_w} f](a,b)$ and $\|w\|_\infty \leq 1$, then

$$\begin{aligned}\int_{\mathcal{C}^\alpha_{X_\lambda}} w(x)\,dx \leq \mu(\mathcal{C}^\alpha_{X_\lambda}) &\leq 10\,\tan\left(\frac{\alpha}{2}\right) \sum_{(a_i,b_i)\in Y} a_i \\ &< \frac{10\,\tan(\frac{\alpha}{2})}{\lambda} \sum_{(a_i,b_i)\in Y} [\mathcal{W}_{F_w} f](a_i,b_i) \\ &\leq \frac{10\,\tan(\frac{\alpha}{2})}{\lambda} \int_{-\infty}^{\infty} |f(x)|\,|w(x)|\,dx \\ &\leq \frac{10\,\tan(\frac{\alpha}{2})}{\lambda} \|w\|_\infty \int_{-\infty}^{\infty} |f(x)|\,dx \\ &= \frac{c}{\lambda}\,\|w\|_\infty\,\|f\|_{L_1},\end{aligned}$$

where, $c = 10\tan(\frac{\alpha}{2})$. □

Let $K \in L^\infty(\mathbb{R})$, then we define the functional F_K on $L_1(\mathbb{R})$ as the following:

$$F_K(f) = \int_{-\infty}^{\infty} f(x)\,K(x)\,dx,$$

and the covariant transform will be

$$[\mathcal{W}_{F_K} f](a,b) = F_K(\rho(a,b)^{-1}f) = \int_{-\infty}^{\infty} af(ax+b)\,K(x)\,dx = \int_{-\infty}^{\infty} f(x)\,K\left(\frac{x-b}{a}\right)\,dx. \quad (2.11)$$

The covariant transform $\mathcal{W}_{F_K}$ can be approximated to $\mathcal{W}_F$ (2.4) by splitting integral over $\mathbb{R}$ into sum of integrals over small intervals. For a better control to the size of the split intervals, we need to put some conditions on the function K like uniform continuity.

Proposition 2.4.3 *Let G be the $ax+b$ group and let $\mathcal{W}_{F_K}$ be defined as in (2.11). Let $\mathcal{W}_F$ be defined as in (2.4) and let $f \in L_1(G)$. Assume that $K(x)$ is uniform continuous, bounded and summable. Then for any $\epsilon > 0$, there exists δ_ϵ such that*

$$\sum_{n=-\infty}^{\infty} c_n\Lambda(s_n)\,[\mathcal{W}_F f](a,b) \leq [\mathcal{W}_{F_K}|f|](a,b) \leq \sum_{n=-\infty}^{\infty} C_n\Lambda(s_n)\,[\mathcal{W}_F f](a,b), \quad (2.12)$$

where $s_n^{-1} = (\frac{\delta_\epsilon}{a}, 2n\delta_\epsilon) \in G$ and $|C_n - c_n| < \epsilon$.

Proof. Let $f \in L_1(\mathbb{R})$. Let $\epsilon > 0$, since $K(x)$ is a uniform continuous, then there is $\delta_\epsilon > 0$, such that $|K(x) - K(y)| < \frac{\epsilon}{2}$, when $|x - y| < \delta_\epsilon$.

Consider the covariant transform (2.4), then by Theorem 2.1.2 and for any $n \in \mathbb{Z}$, we have:

$$\Lambda\left(\frac{\delta_\epsilon}{a}, 2n\delta_\epsilon\right)^{-1}[\mathcal{W}_F f](a,b) = [\mathcal{W}_F \rho\left(\frac{\delta_\epsilon}{a}, 2n\delta_\epsilon\right)^{-1} f](a,b) = \int_{-1}^{1}\left|\left[\rho(a,b)^{-1}\rho\left(\frac{\delta_\epsilon}{a}, 2n\delta_\epsilon\right)^{-1} f\right](x)\right| dx$$
$$= \int_{-1}^{1}\left|\left[\rho(\frac{1}{\delta_\epsilon}, -2n - \frac{b}{a})f\right](x)\right| dx$$
$$= \int_{-1}^{1}\left|\delta_\epsilon f(\delta_\epsilon x + 2n\delta_\epsilon + \frac{\delta_\epsilon b}{a})\right| dx$$
$$= \int_{A_n^{a,b}} |f(x)|\, dx,$$

where the interval of integral is $A_n^{a,b} = [\frac{\delta_\epsilon b}{a} + (2n-1)\delta_\epsilon, \frac{\delta_\epsilon b}{a} + (2n+1)\delta_\epsilon]$ with centre $2n\delta_\epsilon + \frac{\delta_\epsilon b}{a}$ and $|A_n^{a,b}| = 2\delta_\epsilon$. it is clear that $A_n^{a,b} \cup A_m^{a,b} = \phi$ for any integers $n \neq m$ and $\cup_{n\in\mathbb{Z}} A_n^{a,b} = \mathbb{R}$.

Now consider the covariant transform $\mathcal{W}_{F_K}$:

$$[\mathcal{W}_{F_K}|f|](a,b) = \int_{-\infty}^{\infty} |f(x)|\, K\left(\frac{x-b}{a}\right) dx. \tag{2.13}$$

To compare between the covariant transforms (2.4) and (2.13), we need to shift the covariant transform (2.4) over $\mathbb{R}$ as the following:

$$\sum_{n=-\infty}^{\infty} \Lambda\left(\frac{\delta_\epsilon}{a}, 2n\delta_\epsilon\right)^{-1}[\mathcal{W}_F f](a,b) = \sum_{n=-\infty}^{\infty}\int_{A_n^{a,b}} |f(x)|\, dx = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

Now, we define

$$C_n = \max_{x\in A_n^{a,b}} K\left(\frac{x-b}{a}\right) \quad \text{and} \quad c_n = \min_{x\in A_n^{a,b}} K\left(\frac{x-b}{a}\right).$$

Therefor, for any $x \in A_n^{a,b}$ we have

$$c_n \le K\left(\frac{x-b}{a}\right) \le C_n,$$

and thus the relation 2.12 is satisfied.

Also, for any $n \in \mathbb{Z}$, we have

$$\begin{aligned}|C_n - c_n| &= |C_n - K\left(\frac{\delta_\epsilon b}{a} + 2n\delta_\epsilon\right) + K\left(\frac{\delta_\epsilon b}{a} + 2n\delta_\epsilon\right) - c_n| \\ &\leq \left|C_n - K\left(\frac{\delta_\epsilon b}{a} + 2n\delta_\epsilon\right)\right| + \left|K\left(\frac{\delta_\epsilon b}{a} + 2n\delta_\epsilon\right) - c_n\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

The last inequality follows from the uniform continuity of the function K. □

Chapter 3

Covariant Transforms on Locally Convex Spaces

This chapter is focused on the covariant transform on locally convex spaces. The definitions of covariant transforms on the locally convex space V and its dual V^* are given. We list some properties of the covariant transform such as intertwining and reproducing formula. Moreover, some properties of the image space $\mathcal{W}V$ are studied. Also, some examples of covariant transform are introduced in details.

3.1 Covariant Transform on Locally Convex Spaces

In this section we give the definition of covariant transform on locally convex spaces and study its properties.

Definition 3.1.1 *Let V be a locally convex space and V^* be its dual space. Let ρ be a continuous representation of a topological group G on V. Let $F_0 \in V^*$, then we define the covariant transform $\mathcal{W}_{F_0}$ from V to a space of functions on G by*

$$\mathcal{W}_{F_0} : v \to \widetilde{v}(g) = F_0(\rho(g^{-1})v), \quad g \in G. \tag{3.1}$$

It is clear that this definition is more general than Definition 1.3.7 because the Banach space V is replaced by a locally convex space and we consider a general functional F_0.

The covariant transform (3.1) satisfies the same intertwining property of the Theorem 2.1.2.

Proposition 3.1.2 *The covariant transform $\mathcal{W}_{F_0}$ (3.1) intertwines the representation ρ and the left regular representation Λ:*

$$\mathcal{W}_{F_0}\rho(g) = \Lambda(g)\mathcal{W}_{F_0}.$$

Proof. Let $v \in V$ and $g \in G$, then by using the properties of the representation ρ, we have:

$$\begin{aligned}[\mathcal{W}_{F_0}\rho(g)v](h) &= F_0(\rho(h^{-1})\rho(g)v)\\ &= F_0(\rho((g^{-1}h)^{-1})v)\\ &= [\mathcal{W}_{F_0}v](g^{-1}h)\\ &= \Lambda(g)[\mathcal{W}_{F_0}v](h).\end{aligned}$$

□

Again, the covariant transform (3.1) satisfies the following property as in the Corollary 2.1.3.

Corollary 3.1.3 *The image space $\mathcal{W}_{F_0}V$ is invariant under the left regular representation Λ.*

Proof. Let $u \in \mathcal{W}_{F_0}(V)$, then there exists $v \in V$ such that $u = \mathcal{W}_{F_0}(v)$. Now for any $h, g \in G$ and by using the previous theorem, we have

$$[\Lambda(g)u](h) = \Lambda(g)[\mathcal{W}_{F_0}(v)](h) = [\mathcal{W}_{F_0}(\rho(g)v)](h),$$

where $\rho(g)v \in V$. Therefore, $\Lambda(g)u \in \mathcal{W}_{F_0}(V)$ and hence the image space $\mathcal{W}_{F_0}(V)$ is Λ-invariant. □

The next corollary is an immediate consequence of the intertwining of the covariant transform $\mathcal{W}_{F_0}$.

Corollary 3.1.4 *The subspace $ker(\mathcal{W}_{F_0})$ of V is invariant under the representation ρ.*

Proof. Let $v \in ker(\mathcal{W}_{F_0})$, then $[\mathcal{W}_{F_0}v](h) = 0$, for all $h \in G$. Now for any $g, h \in G$, we have:

$$[\mathcal{W}_{F_0}(\rho(g)v)](h) = \Lambda(g)[\mathcal{W}_{F_0}v](h) = 0.$$

So, $\rho(g)v \in ker(\mathcal{W}_{F_0})$ for any $g \in G$ and hence $ker(\mathcal{W}_{F_0})$ is invariant under the representation ρ. □

Lemma 3.1.5 *Consider the covariant transform $\mathcal{W}_{F_0}$ (3.1). Let $F_0 \neq 0$ and ρ be irreducible, then $\mathcal{W}_{F_0}$ is injective.*

Proof. Since ρ is irreducible, then the only invariant subspaces under ρ is V and $\{0\}$. From Corollary 3.1.4, we have $ker(\mathcal{W}_{F_0})$ is invariant and since $F_0 \neq 0$, then $ker(\mathcal{W}_{F_0}) \neq V$. Therefore, $ker(\mathcal{W}_{F_0}) = \{0\}$ and hence $\mathcal{W}_{F_0}$ is injective. □

In the following, we transport seminorm p on V to a seminorm q on the image space $\mathcal{W}_{F_0}V$.

Proposition 3.1.6 *Consider the covariant transform $\mathcal{W}_{F_0}$ (3.1). Let $F_0 \neq 0$ and let ρ be irreducible, then for any seminorm p on V, we can define a map $q : \mathcal{W}_{F_0}V \to \mathbb{R}$ by*

$$q(\mathcal{W}_{F_0}v) = p(v), \quad v \in V, \tag{3.2}$$

and q is a seminorm.

Proof. We need first to prove that $\mathcal{W}_{F_0}$ is injective in order to ensure that the map q is well defined. Because if $v \in ker(\mathcal{W}_{F_0})$ is a non zero vector, then $\mathcal{W}_{F_0}v = 0$ and thus for any $u \notin ker(\mathcal{W}_{F_0})$, we have $\mathcal{W}_{F_0}(u+v) = \mathcal{W}_{F_0}u$ but $p(u+v)$ not necessary to be equal to $p(u)$. Now, since ρ is irreducible and $F_0 \neq 0$, then by Lemma 3.1.5, we have $\mathcal{W}_{F_0}$ is injective and hence for any seminorm p on V, we can define a map q on $\mathcal{W}_{F_0}V$:

$$q(\mathcal{W}_{F_0}v) = p(v), \quad v \in V.$$

Now, let λ be a scalar and $v \in V$, then:

1. $q(\lambda \mathcal{W}_{F_0}v) = q(\mathcal{W}_{F_0}(\lambda v)) = p(\lambda v) = |\lambda|\, p(v) = |\lambda|\, q(\mathcal{W}_{F_0}v)$,
2. Let $\mathcal{W}_{F_0}v_1$ and $\mathcal{W}_{F_0}v_2$ be two elements in $\mathcal{W}_{F_0}V$, then:

$$\begin{aligned} q(\mathcal{W}_{F_0}v_1 + \mathcal{W}_{F_0}v_2) = q(\mathcal{W}_{F_0}(v_1+v_2)) &= (v_1+v_2) \\ &\leq p(v_1) + p(v_2) \\ &= q(\mathcal{W}_{F_0}v_1) + q(\mathcal{W}_{F_0}v_2). \end{aligned}$$

Therefore, q is a seminorm on the image space $\mathcal{W}_{F_0}V$. □

So, let $\{p_\alpha\}$ be a collection of seminorms generating the topology on V, then $\{q_\alpha\}$ is a collection of seminorms generate a topology on $\mathcal{W}_{F_0}V$.

Lemma 3.1.7 *Let V be a locally convex space and let $v_n \to v$ in V. Then for each $g \in G$, we have $\mathcal{W}_{F_0}v_n(g) \to \mathcal{W}_{F_0}v(g)$.*

Proof. Let $v_n \to v$ in V. Then for each $g \in G$, we have:

$$|\mathcal{W}_{F_0}v_n(g) - \mathcal{W}_{F_0}v(g)| = |F_0(\rho(g^{-1})v_n) - F_0(\rho(g^{-1})v)| = |F_0(\rho(g^{-1})(v_n - v))|,$$

by the continuity of the functional F_0 and the representation $\rho(g^{-1})$ there exists $C > 0$ and a collection $p_{\alpha_1}, \ldots, p_{\alpha_n}$ of seminorms on V such that

$$\begin{aligned} |F_0(\rho(g^{-1})(v_n - v))| &\le C(p_{\alpha_1}(v_n - v) + \ldots + p_{\alpha_n}(v_n - v)) \\ &\to 0. \end{aligned}$$

So, $\mathcal{W}_{F_0} v_n(g) \to \mathcal{W}_{F_0} v(g)$, for each $g \in G$. □

Proposition 3.1.8 *The covariant transform $\mathcal{W}_{F_0}$ is continuous in the topology on V defined by transported norm (3.2).*

Proof. Let $v_n \to v$ on V. Then by the definition of the transported seminorm (3.2), we have

$$q_\alpha(\mathcal{W}_{F_0} v_n - \mathcal{W}_{F_0} v) = q_\alpha(\mathcal{W}_{F_0}(v_n - v)) = p_\alpha(v_n - v) \to 0.$$

So, $\mathcal{W}_{F_0}$ is continuous on V. □

The next proposition shows that $\mathcal{W}_{F_0}$ is a map from V to a space of bounded continuous functions on G.

Proposition 3.1.9 *Consider the covariant transform $\mathcal{W}_{F_0}$ (3.1). Then, the function $\mathcal{W}_{F_0} v = \widetilde{v}$ is bounded continuous function on G.*

Proof. First we show that $\widetilde{v}$ is bounded. Let $\{p_\alpha\}$ be a collection of seminorms generate the topology of V. Since F_0 is continuous functional on V, then by Theorem 1.2.14 there exists a collection of seminorms $p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}$ on V and $C > 0$ such that

$$|\widetilde{v}(g)| = |F_0(\rho(g^{-1})v)| \le C \left(p_{\alpha_1}(\rho(g^{-1})v) + p_{\alpha_2}(\rho(g^{-1})v) + \ldots + p_{\alpha_n}(\rho(g^{-1})v)\right), \ \forall g \in G.$$

We have $\{\rho(g^{-1}) : g \in G\}$ is a collection of continuous maps on V. Then by Theorem 1.2.15 we have for each p_{α_i},$1 \le i \le n$, the set $\{p_{\alpha_i}(\rho(g^{-1})v) : g \in G\}$

is bounded and there is a continuous seminorm p'_{α_i} on V and $D_i > 0$ such that $p_{\alpha_i}(\rho(g^{-1})v) \leq D_i\, p'_{\alpha_i}(v)$ for all $g \in G$ and $1 \leq i \leq n$. Thus,

$$|\widetilde{v}(g)| \leq C\, \left(D_1 p'_{\alpha_1}(v) + D_2 p'_{\alpha_2}(v) + \ldots + D_n p'_{\alpha_n}(v)\right), \quad \forall g \in G,$$

and hence $\widetilde{v}$ is bounded.

Now we prove the continuity of $\widetilde{v}$. Let $g \to h$ on G and by the continuity of F_0, there exist a collection of seminorms $p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}$ and $C > 0$ such that

$$\begin{aligned} |\widetilde{v}(g) - \widetilde{v}(h)| &= |F(\rho(g^{-1})v - \rho(h^{-1})v)| \\ &\leq C\, \left(p_{\alpha_1}(\rho(g^{-1})v - \rho(h^{-1})v) + \ldots + p_{\alpha_n}(\rho(g^{-1})v - \rho(h^{-1})v)\right), \ \forall g, h \in G. \end{aligned}$$

Since ρ is continuous on G then, $p_{\alpha_i}(\rho(g^{-1})v - \rho(h^{-1})v) \to 0$, $1 \leq i \leq n$, and thus $\widetilde{v}$ is continuous on G. □

3.2 Examples of covariant transform

In what follows, we give brief examples of covariant transform. More details of generalized versions of these examples are given later on in this chapter.

We start with examples of Cauchy and Poisson integrals. We are continuing the previous examples from the previous chapter.

Example 3.2.1 *Let G be the $ax + b$ group and $V = L^p(\mathbb{R})$. Let $v_0(x) = \frac{1}{2\pi i}\frac{1}{x+i}$ and let $F_0 \in V^*$ defined as the following:*

$$F_0(v) = \int_{\mathbb{R}} v_0(x)\, v(x)\, dx = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{v(x)}{x+i}\, dx.$$

Then the covariant transform (3.1) is the Cauchy integral:

$$\begin{aligned}\widetilde{v}(a,b) &= \frac{a^{\frac{1}{p}}}{2\pi i}\int_{\mathbb{R}} \frac{v(ax+b)}{x+i}\,dx \\ &= \frac{a^{\frac{1}{p}-1}}{2\pi i}\int_{\mathbb{R}} \frac{v(x)}{(\frac{x-b}{a})+i}\,dx \\ &= \frac{a^{\frac{1}{p}}}{2\pi i}\int_{\mathbb{R}} \frac{v(x)}{(x-b)+ai}\,dx\end{aligned}$$

Example 3.2.2 *Let G be the $ax+b$ group and $V = L^p(\mathbb{R})$. Let $v_0(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ and let $F_0 \in V^*$ defined as the following:*

$$F_0(v) = \int_{\mathbb{R}} v_0(x)\,v(x)\,dx = \frac{1}{\pi}\int_{\mathbb{R}} \frac{v(x)}{1+x^2}\,dx.$$

Then the covariant transform (3.1) is the Poisson kernel:

$$\begin{aligned}\widetilde{v}(a,b) &= \frac{a^{\frac{1}{p}}}{\pi}\int_{\mathbb{R}} \frac{v(ax+b)}{1+x^2}\,dx \\ &= \frac{a^{\frac{1}{p}-1}}{\pi}\int_{\mathbb{R}} \frac{v(x)}{1+(\frac{x-b}{a})^2}\,dx \\ &= \frac{a^{\frac{1}{p}+1}}{\pi}\int_{\mathbb{R}} \frac{v(x)}{a^2+(x-b)^2}\,dx\end{aligned}$$

Here, we give example of covariant transform on the Schwartz space $S(\mathbb{R})$.

Example 3.2.3 *Let $V = S(\mathbb{R})$ be the Schwartz space. Let V^* be the dual space of V which is the space of tempered distributions. Let $v_0(x) = e^{-x^2} \in S(\mathbb{R})$ and let $F_0 \in V^*$ defined as the following:*

$$F_0(v) = \int_{\mathbb{R}} v_0(x)\,v(x)\,dx = \int_{\mathbb{R}} e^{-x^2}\,v(x)\,dx, \quad v \in S(\mathbb{R}).$$

Let G be the $ax+b$ group, then the covariant transform (3.1) is

$$\begin{aligned}[\mathcal{W}_{F_0}v](a,b) = F_0(\rho_p((a,b)^{-1})v) &= \int_{\mathbb{R}} v_0(x)\,[\rho_p((a,b)^{-1})v](x)\,dx \\ &= a^{\frac{1}{p}}\int_{\mathbb{R}} e^{-x^2}\,v(ax+b)\,dx = a^{\frac{1}{p}-1}\int_{\mathbb{R}} e^{-(\frac{x-b}{a})^2}\,v(x)\,dx.\end{aligned} \tag{3.3}$$

As an application of the Proposition 3.1.2 the transform $\mathcal{W}_{F_0}v$ intertwines the representation ρ of the $ax+b$ group and the left regular representation Λ:

$$\begin{aligned}[\widetilde{\rho(a,b)v}](c,d) &= \int_{\mathbb{R}} e^{-x^2}\,[\rho((c,d)^{-1})\rho(a,b)v](x)\,dx \\ &= \int_{\mathbb{R}} e^{-x^2}\,[\rho((a,b)^{-1}(c,d))^{-1}v](x)\,dx \\ &= \widetilde{v}((a,b)^{-1}(c,d)) = [\Lambda(a,b)\widetilde{v}](c,d).\end{aligned}$$

Cauchy and Poisson integrals in Examples 3.2.1 and 3.2.2 satisfy this intertwining property as well.

Let $V = S(\mathbb{R})$ be the Schwartz space and $\widetilde{v}$ be the covariant transform (3.3). Consider the following seminorms $r_{k,l}$ on the image space $\widetilde{V} = \mathcal{W}_{F_0}(V)$:

$$r_{k,l}(\widetilde{v}) = \sup_{b\in\mathbb{R}} \left| b^k\, D_b^l \widetilde{v}(a,b)\right|, \tag{3.4}$$

where $D_b^l = \frac{\partial^l}{\partial b^l}$.

In what follows, we will see that the seminorms $r_{k,l}$ (3.4) are related to the seminorms $p_{n,m}$ (1.4) on the Schwartz space $S(\mathbb{R})$. In order to investigate this relation, we need the following lemmas.

Lemma 3.2.4 *Let $v_0(x) = e^{\frac{-(x-b)^2}{a^2}}$, where $a, b \in \mathbb{R}$ and $a \neq 0$. Then,*

$$x^n\, e^{\frac{-(x-b)^2}{a^2}} = \sum_{j=0}^{n}\sum_{i=0}^{[\frac{j}{2}]} d_{j,i}\, b^{j-2i}\, D_b^{n-j} e^{\frac{-(x-b)^2}{a^2}}, \tag{3.5}$$

where $d_{j,i} = \frac{c'_{j,i}}{c_{0,0}}$, for some real constants $c'_{j,i}$ and $c_{0,0}$.

Proof. First we need to prove the following formula:

$$D_b^n e^{\frac{-(x-b)^2}{a^2}} = \sum_{j=0}^{n}\sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j-2i}\, x^{n-j} e^{\frac{-(x-b)^2}{a^2}},$$

where $c_{j,i}$ are real constants. The prove can be obtained by mathematical induction: For $n = 1$

$$\begin{aligned} D_b e^{\frac{-(x-b)^2}{a^2}} &= \left(\frac{2}{a^2}\right)(x-b)\, e^{\frac{-(x-b)^2}{a^2}} \\ &= \left(\frac{2}{a^2}\right) x\, e^{\frac{-(x-b)^2}{a^2}} - \left(\frac{2}{a^2}\right) b\, e^{\frac{-(x-b)^2}{a^2}} \\ &= \sum_{j=0}^{1} \sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j-2i} x^{n-j} e^{\frac{-(x-b)^2}{a^2}}. \end{aligned}$$

Now, assume that it is true for $n \in \mathbb{N}$. Then:

$$\begin{aligned} D_b^{n+1} e^{\frac{-(x-b)^2}{a^2}} &= \left(\frac{2}{a^2}\right)(x-b)\, e^{\frac{-(x-b)^2}{a^2}} \left[\sum_{j=0}^{n} \sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j-2i}\, x^{n-j}\right] \\ &\quad + e^{\frac{-(x-b)^2}{a^2}} \left[\sum_{j=1}^{n} \sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\,(j-2i)\, b^{j-1-2i} x^{n-j}\right] \\ &= \left(\frac{2}{a^2}\right) e^{\frac{-(x-b)^2}{a^2}} \left[\sum_{j=0}^{n} \sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j-2i}\, x^{n+1-j}\right] - \left(\frac{2}{a^2}\right) e^{\frac{-(x-b)^2}{a^2}} \\ &\quad \left[\sum_{j=0}^{n} \sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j+1-2i}\, x^{n-j}\right] + e^{\frac{-(x-b)^2}{a^2}} \left[\sum_{j=1}^{n} \sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\,(j-2i)\, b^{j-1-2i}\, x^{n-j}\right] \\ &= \sum_{j=0}^{n+1} \sum_{i=0}^{[\frac{j}{2}]} c'_{j,i}\, b^{j-2i} x^{n+1-j} e^{\frac{-(x-b)^2}{a^2}}. \end{aligned}$$

So, it is true for any $n \in \mathbb{N}$. Now,

$$\begin{aligned} D_b^n e^{\frac{-(x-b)^2}{a^2}} &= \sum_{j=0}^{n} \sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j-2i} x^{n-j} e^{\frac{-(x-b)^2}{a^2}} \\ &= c_{0,0}\, x^n\, e^{\frac{-(x-b)^2}{a^2}} + \sum_{j=1}^{n} \sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j-2i} x^{n-j} e^{\frac{-(x-b)^2}{a^2}}, \end{aligned}$$

so,

$$c_{0,0}\, x^n\, e^{\frac{-(x-b)^2}{a^2}} = D_b^n e^{\frac{-(x-b)^2}{a^2}} - \sum_{j=1}^{n} \sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j-2i} x^{n-j} e^{\frac{-(x-b)^2}{a^2}}.$$

By continuing this method, we obtain the following:

$$c_{0,0}\, x^n\, e^{\frac{-(x-b)^2}{a^2}} = \sum_{j=0}^{n} \sum_{i=0}^{[\frac{j}{2}]} c'_{j,i}\, b^{j-2i}\, D_b^{n-j} e^{\frac{-(x-b)^2}{a^2}}, \tag{3.6}$$

and thus,

$$x^n\, e^{\frac{-(x-b)^2}{a^2}} = \sum_{j=0}^{n} \sum_{i=0}^{[\frac{j}{2}]} d_{j,i}\, b^{j-2i}\, D_b^{n-j} e^{\frac{-(x-b)^2}{a^2}},$$

where $d_{j,i} = \frac{c'_{j,i}}{c_{0,0}}$. □

Since we need to differentiate $e^{\frac{-(x-b)^2}{a^2}}$ with respect to x and b, the relation between the two derivatives must be determined.

Lemma 3.2.5 *Let $v_0(x) = e^{\frac{-(x-b)^2}{a^2}}$, where $a, b \in \mathbb{R}$ and $a \neq 0$. Then,*

$$D_x^n e^{\frac{-(x-b)^2}{a^2}} = (-1)^n\, D_b^n e^{\frac{-(x-b)^2}{a^2}} \tag{3.7}$$

Proof. The proof is straight forward by using mathematical induction. For $n = 1$, we have

$$D_b e^{\frac{-(x-b)^2}{a^2}} = \frac{2}{a^2}\,(x-b)\, e^{\frac{-(x-b)^2}{a^2}}$$

and

$$D_x e^{\frac{-(x-b)^2}{a^2}} = \frac{-2}{a^2}\,(x-b)\, e^{\frac{-(x-b)^2}{a^2}} = -D_b e^{\frac{-(x-b)^2}{a^2}}.$$

Now, assume that it is true for $n \in \mathbb{N}$, then:

$$\begin{aligned} D_x^{n+1} e^{\frac{-(x-b)^2}{a^2}} = D_x\, D_x^n e^{\frac{-(x-b)^2}{a^2}} &= D_x\,(-1)^n\, D_b^n e^{\frac{-(x-b)^2}{a^2}} \\ &= (-1)^n\, D_b^n\, D_x e^{\frac{-(x-b)^2}{a^2}} = (-1)^{n+1}\, D_b^{n+1} e^{\frac{-(x-b)^2}{a^2}}. \end{aligned}$$

So, the relation (3.7) is true for any $n \in \mathbb{N}$. □

Definition 3.2.6 $[27, p.\ 79]$ *Dirac delta function δ is a distribution corresponding to the linear functional on the Schwartz space $S(\mathbb{R})$ defined as $\langle \delta, f \rangle = f(0)$, for any $f \in S(\mathbb{R})$.*

We can approximate the distribution δ by sequence of test functions. Consider the following sequence:

$$\delta_a(x) = \frac{1}{a\sqrt{\pi}}\, e^{\frac{-x^2}{a^2}},$$

then, for any $a > 0$ we have:

$$\frac{1}{a\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{a^2}}\, dx = \frac{1}{a\sqrt{\pi}} \cdot a\sqrt{\pi} = 1.$$

Then for any $f \in S(\mathbb{R})$, we have the following:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{a\sqrt{\pi}} e^{\frac{-x^2}{a^2}}\, f(x)\, dx &= \int_{-\infty}^{\infty} \frac{1}{a\sqrt{\pi}} e^{\frac{-x^2}{a^2}}\, f(0)\, dx + \int_{-\infty}^{\infty} \frac{1}{a\sqrt{\pi}} e^{\frac{-x^2}{a^2}}\, (f(x) - f(0))\, dx \\ &= f(0) + \int_{-\infty}^{\infty} \frac{1}{a\sqrt{\pi}} e^{\frac{-x^2}{a^2}}\, (f(x) - f(0))\, dx,\end{aligned}$$

the integral $\int_{-\infty}^{\infty} \frac{1}{a\sqrt{\pi}} e^{\frac{-x^2}{a^2}}\, (f(x) - f(0))\, dx \to 0$ as $a \to 0$ [25, p. 11]. Thus,

$$\int_{-\infty}^{\infty} \frac{1}{a\sqrt{\pi}} e^{\frac{-x^2}{a^2}}\, f(x)\, dx \to f(0) = \int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx, \quad \text{as} \quad a \to 0,$$

and hence, $\delta_a \to \delta$ as $a \to 0$ in S' in the weak-$*$ topology [27, p. 80].

Proposition 3.2.7 *Let $V = S(\mathbb{R})$ be the Schwartz space and consider the covariant transform (3.3). Then for any $v \in S(\mathbb{R})$, the seminorm $p_{n,m}(v)$ (1.4) is bounded by a collection of seminorms $r_{k,l}(\widetilde{v})$ (3.4):*

$$p_{n,m}(v) \le \lim_{a\to 0} \frac{1}{a\sqrt{\pi}} \sum_{j=0}^{n} \sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, r_{k,l}(\widetilde{v}),$$

where $k = j - 2i$ and $l = n - j + m$.

Proof. Let Dirac delta function δ be as above, then we have:

$$\begin{aligned}b^n\, v^{(m)}(b) &= \int_{-\infty}^{\infty} x^n\, v^{(m)}(x)\, \delta(x-b)\, dx \\ &= \lim_{a\to 0} \int_{-\infty}^{\infty} x^n\, v^{(m)}(x)\, \frac{1}{a\sqrt{\pi}}\, e^{\frac{-(x-b)^2}{a^2}}\, dx.\end{aligned}$$

By using lemma (3.2.4), we have:

$$\frac{1}{a\sqrt{\pi}}\int_{-\infty}^{\infty} x^n\, v^{(m)}(x)\, e^{\frac{-(x-b)^2}{a^2}}\, dx = \frac{1}{a\sqrt{\pi}}\int_{-\infty}^{\infty} v^{(m)}(x)\, [\sum_{j=0}^{n}\sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j-2i}\, D_b^{n-j} e^{\frac{-(x-b)^2}{a^2}}]\, dx$$

$$= \frac{1}{a\sqrt{\pi}}\sum_{j=0}^{n}\sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j-2i}\int_{-\infty}^{\infty} v^{(m)}(x)\, D_b^{n-j} e^{\frac{-(x-b)^2}{a^2}}\, dx$$

Now, by integrating by parts we have:

$$\int_{-\infty}^{\infty} v^{(m)}(x)\, D_b^{n-j} e^{\frac{-(x-b)^2}{a^2}}\, dx = (-1)^m\int_{-\infty}^{\infty} v(x)\, D_x^m\, D_b^{n-j} e^{\frac{-(x-b)^2}{a^2}}\, dx$$

$$= \int_{-\infty}^{\infty} v(x)\, D_b^{n-j+m} e^{\frac{-(x-b)^2}{a^2}}\, dx \quad \textit{by } (3.7).$$

Thus,

$$b^n\, v^{(m)}(b) = \lim_{a\to 0}\frac{1}{a\sqrt{\pi}}\sum_{j=0}^{n}\sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j-2i}\int_{-\infty}^{\infty} v(x)\, D_b^{n-j+m} e^{\frac{-(x-b)^2}{a^2}}\, dx$$

$$= \lim_{a\to 0}\frac{1}{a\sqrt{\pi}}\sum_{j=0}^{n}\sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, b^{j-2i}\, D_b^{n-j+m}\, \widetilde{v}(a,b),$$

and by taking the supremum over $\mathbb{R}$, we have:

$$\sup_{b\in\mathbb{R}} |b^n\, v^{(m)}(b)| \le \lim_{a\to 0}\frac{1}{a\sqrt{\pi}}\sum_{j=0}^{n}\sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, \sup_{b\in\mathbb{R}} |b^{j-2i}\, D_b^{n-j+m}\, \widetilde{v}(a,b)|,$$

thus,

$$p_{n,m}(v) \le \lim_{a\to 0}\frac{1}{a\sqrt{\pi}}\sum_{j=0}^{n}\sum_{i=0}^{[\frac{j}{2}]} c_{j,i}\, r_{k,l}(\widetilde{v}),$$

where $k = j - 2i$ and $l = n - j + m$. □

On the other hand, the seminorms $r_{i,j}$ can be bound by sum of $p_{n,m}$. For each $v \in S(\mathbb{R})$, we have:

$$
\begin{aligned}
|b^n\, D_b^m \widetilde{v}(a,b)| &= \left| b^n \int_{-\infty}^{\infty} e^{-x^2}\, D_b^m v(ax+b)\, dx \right| \\
&\leq |b|^n \int_{-\infty}^{\infty} e^{-x^2}\, |D_b^m v(ax+b)|\, dx \\
&\leq |b|^n \sup_{x\in\mathbb{R}} |D_b^m v(ax+b)| \int_{-\infty}^{\infty} e^{-x^2}\, dx \\
&= \sqrt{\pi}\, |b|^n \sup_{x\in\mathbb{R}} |D_b^m v(ax+b)|.
\end{aligned}
$$

Since we are taking the supremum over the entire $\mathbb{R}$, then

$$
\sup_{x\in\mathbb{R}} |D_b^m v(ax+b)| = \sup_{b\in\mathbb{R}} |D_b^m v(ax+b)|.
$$

and hence,

$$
\begin{aligned}
|b^n\, D_b^m \widetilde{v}(a,b)| &\leq \sqrt{\pi}\, |b|^n \sup_{b\in\mathbb{R}} |D_b^m v(ax+b)| \\
&\leq \sqrt{\pi} \sup_{b\in\mathbb{R}} |b^n\, D_b^m v(ax+b)|,
\end{aligned}
$$

thus,

$$
\begin{aligned}
r_{n,m}(\widetilde{v}) &= \sup_{b\in\mathbb{R}} |b^n\, D_b^m \widetilde{v}(a,b)| \\
&\leq \sqrt{\pi} \sup_{b\in\mathbb{R}} |b^n\, D_b^m v(ax+b)|.
\end{aligned}
$$

3.3 Covariant transform on dual spaces

Let V be a locally convex space and V^* be its dual. Let ρ be the representation of a topological group G on V, then the contragradient representation ρ^* of G on V^* is defined by the following:

$$
[\rho^*(g)F](v) = F(\rho(g^{-1})v), \quad F \in V^* \textit{ and } v \in V. \tag{3.8}
$$

Proposition 3.3.1 *Let ρ be a representation of a group G on a locally convex space V. For $g \in G$, if $\rho(g)$ is bounded on V, then the contragradient representation $\rho^*(g)$ (3.8) is bounded on V^*, i.e. there exists $C > 0$ and a collection of seminorms $p_{\alpha_1}, \ldots, p_{\alpha_n}$, such that:*

$$|[\rho^*(g)F](v)| \leq C(p_{\alpha_1}(v) + \ldots + p_{\alpha_n}(v)).$$

Proof. For any $g \in G$, let $\rho(g)$ is bounded on V, then for any $F \in V^*$ and $v \in V$ we have:

$$|[\rho^*(g)F](v)| = |F(\rho(g^{-1})v)|.$$

by using Proposition 3.1.9, there is $C > 0$ and a collection of seminorms $p_{\alpha_1}, \ldots, p_{\alpha_n}$ such that:

$$|F(\rho(g^{-1})v)| \leq C(p_{\alpha_1}(v) + \ldots + p_{\alpha_n}(v)),$$

thus $\rho^*(g)$ is bounded. □

Now, we give the definition of covariant transform on dual space V^*.

Definition 3.3.2 *Let V be a locally convex space and ρ be a continuous representation of a topological group G on V. Fix $v_0 \in V$. Define the covariant transform $\mathcal{W}^*_{v_0}$ from V^* to a space of functions on G by*

$$\mathcal{W}^*_{v_0} : F \to \widetilde{F}(g) = [\rho^*(g^{-1})F](v_0) = F(\rho(g)v_0), \quad g \in G. \tag{3.9}$$

To see the relation between the covariant transform $\mathcal{W}_{F_0}$ (3.1) and the covariant transform $\mathcal{W}^*_{v_0}$ (3.9), we need the following operator:

$$[\mathcal{S}f](h) = f(h^{-1}). \tag{3.10}$$

So, we have

$$[\mathcal{S}\mathcal{W}_{F_0}v_0](g) = [\mathcal{W}_{F_0}v_0](g^{-1}) = F_0(\rho(g)v_0) = [\mathcal{W}^*_{v_0}F_0](g).$$

Proposition 3.3.3 *The covariant transform $\mathcal{W}^*_{v_0}$ intertwines the contragradient representation ρ^* and the left regular representation Λ:*

$$\mathcal{W}^*_{v_0}\rho^*(g) = \Lambda(g)\mathcal{W}^*_{v_0}.$$

Proof. Let $F \in V^*$ and $g \in G$, then by using the properties of the representation ρ^*, we have:

$$\begin{aligned}[\mathcal{W}^*_{v_0}\rho^*(g)F](h) &= [\rho^*(h^{-1})\rho^*(g)F](v_0)\\ &= [\rho^*((g^{-1}h)^{-1})F](v_0)\\ &= [\mathcal{W}^*_{v_0}F](g^{-1}h)\\ &= \Lambda(g)[\mathcal{W}^*_{v_0}F](h).\end{aligned}$$

□

The following is an immediate consequence of the previous proposition.

Corollary 3.3.4 *The image space $\mathcal{W}^*_{v_0}V^*$ is invariant under the left regular representation Λ.*

Proof. Let $u \in \mathcal{W}^*_{v_0}V^*$, then there exists $F \in V^*$ such that $u = \mathcal{W}^*_{v_0}F$. Now for any $h, g \in G$ and by using the previous theorem, we have

$$[\Lambda(g)u](h) = \Lambda(g)[\mathcal{W}^*_{v_0}F](h) = [\mathcal{W}^*_{v_0}(\rho^*(g)F)](h),$$

where $\rho^*(g)F \in V^*$. Therefore, $\Lambda(g)u \in \mathcal{W}^*_{v_0}V^*$ and hence the image space $\mathcal{W}^*_{v_0}V^*$ is Λ-invariant. □

Example 3.3.5 *Let G be the $ax + b$ group and $V = S(\mathbb{R})$. Fix $v_0 = e^{-x^2} \in S(\mathbb{R})$, then the covariant transform (3.9) is:*

$$\widetilde{F_\alpha}(a,b) = [\rho^*_q((a,b)^{-1})F_\alpha](v_0) = F_\alpha(\rho_q(a,b)v_0) = F_\alpha\left(e^{-(\frac{(x-b)^2}{a^2})}\right). \qquad (3.11)$$

If F_α is represented by the following functional:

$$F_\alpha(v) = \int_{\mathbb{R}} v(x)\, v_\alpha(x)\, dx, \quad v \in S(\mathbb{R})$$

where v_α is a tempered distribution. Then

$$\widetilde{F_\alpha}(a,b) = F_\alpha(\rho_q(a,b)v_0) = a^{-\frac{1}{q}} \int_{\mathbb{R}} e^{-(\frac{x-b}{a})^2}\, v_\alpha(x)\, dx. \tag{3.12}$$

So, the covariant transform $\widetilde{F_\alpha}$ (3.12) coincide with the covariant transform $\widetilde{v}$ (3.3).

The image of the Dirac delta function δ is:

$$\widetilde{F_\delta}(a,b) = a^{-\frac{1}{q}} \int_{\mathbb{R}} e^{-(\frac{x-b}{a})^2}\, \delta(x)\, dx = a^{-\frac{1}{q}}\, e^{-(\frac{b}{a})^2}.$$

and the image of δ' is:

$$\begin{aligned}\widetilde{F_{\delta'}}(a,b) &= a^{-\frac{1}{q}} \int_{\mathbb{R}} e^{-(\frac{x-b}{a})^2}\, \delta'(x)\, dx \\ &= -a^{-\frac{1}{q}} \int_{\mathbb{R}} \left[-2\left(\frac{x-b}{a}\right) \cdot \frac{1}{a} \cdot e^{-(\frac{x-b}{a})^2}\right] \delta(x)\, dx \\ &= -a^{-\frac{1}{q}} \cdot -2(-\frac{b}{a}) \cdot \frac{1}{a} \cdot e^{-(\frac{b}{a})^2} = -2a^{-\frac{1}{q}-2} e^{-(\frac{b}{a})^2}.\end{aligned}$$

3.4 Convolutions and Pairings

This section presents the definition of convolution through an invariant pairing and its properties. Also, the reproducing formula on V is investigated.

Definition 3.4.1 [1] *Let V and U be two vector spaces. a pairing between V and U is a bilinear map:*

$$\langle \cdot, \cdot \rangle : V \times U \to \mathbb{C},$$

satisfying the following regularity conditions:

1. $\langle v, u \rangle = 0$, *for all $v \in V$ implies that $u = 0$.*

2. $\langle v, u\rangle = 0$, *for all* $u \in U$ *implies that* $v = 0$.

We assume that the pairing is linear in both variables.

Definition 3.4.2 *Let* L_1 *and* L_2 *be two left invariant spaces of functions on* G. *We say that a pairing* $\langle\cdot,\cdot\rangle_1 : L_1 \times L_2 \to \mathbb{C}$ *is left invariant if*

$$\langle \Lambda(g)f_1, \Lambda(g)f_2\rangle_1 = \langle f_1, f_2\rangle_1, \qquad \textit{for all} \quad f_1 \in L_1,\ f_2 \in L_2,\ g \in G. \tag{3.13}$$

Also we assume that the pairing $\langle\cdot,\cdot\rangle_1$ is continuous in both arguments separately, i.e. for any $f_n \to f$ in L_1 and any $h \in L_2$, we have:

$$\langle f_n, h\rangle_1 \to \langle f, h\rangle_1,$$

similarly, for any $h_n \to h$ in L_2 and any $f \in L_1$, we have:

$$\langle f, h_n\rangle_1 \to \langle f, h\rangle_1.$$

Example 3.4.3 *Let* G *be a group with left Haar measure* $d\mu$. *Consider the Haar pairing defined on* $L^p(G) \times L^q(G)$ *by:*

$$\langle f, h\rangle_1 = \int_G f(g)\, h(g)\, d\mu(g), \quad f \in L^p(G), h \in L^q(G).$$

Then if $f_n \to f$ *in* $L^p(G)$ *and for any* $h \in L^q(G)$, *we have:*

$$\begin{aligned} |\langle f_n, h\rangle - \langle f, h\rangle| &= |\langle f_n - f, h\rangle| \\ &= \left| \int_G (f_n - f)(g)\, h(g)\, d\mu(g) \right| \\ &\leq \|f_n - f\|_p \cdot \|h\|_q \quad \textit{by Hölder inequality} \\ &\to 0 \quad \textit{as} \quad n \to \infty. \end{aligned}$$

A similar calculation shows continuity in the second function.

Definition 3.4.4 *Let $\widetilde{H}^p(\mathbb{R}^2_+)$, $1 < p < \infty$, be the space of all holomorphic functions f which satisfy the following norm:*

$$\|f\|_{\widetilde{H}^p} = \lim_{a\to 0} \frac{1}{a}\left(\int_{-\infty}^{\infty} |f(a,b)|^p \, db\right)^{\frac{1}{p}}.$$

In the following, we give some examples of different types of invariant pairings.

Example 3.4.5 1. *Consider the following invariant pairing on $L^2(G) \times L^2(G)$ which is the integration over the Haar measure $d\mu$:*

$$\langle f_1, f_2\rangle_1 = \int_G f_1(g)\, f_2(g)\, d\mu(g), \quad f_1, f_2 \in L^2(G). \tag{3.14}$$

The pairing (3.14) is left invariant: For $h \in G$,

$$\begin{aligned}\langle \Lambda(h)f_1, \Lambda(h)f_2\rangle &= \int_G [\Lambda(h)f_1](g)\, [\Lambda(h)f_2](g)\, d\mu(g)\\ &= \int_G f_1(h^{-1}g)\, f_2(h^{-1}g)\, d\mu(g)\\ &= \int_G f_1(g)\, f_2(g)\, d\mu(hg)\\ &= \int_G f_1(g)\, f_2(g)\, d\mu(g) = \langle f_1, f_2\rangle.\end{aligned}$$

2. *Let G be the $ax+b$ group. The following invariant pairing is called Hardy pairing:*

$$\langle f_1, f_2\rangle_1 = \lim_{a\to 0}\int_{-\infty}^{\infty} f_1(a,b)\, f_2(a,b)\, \frac{db}{a}, \tag{3.15}$$

where, $f_1 \in \widetilde{H}^p(\mathbb{R}^2_+)$ and $f_2 \in \widetilde{H}^q(\mathbb{R}^2_+)$ such that $\frac{1}{p}+\frac{1}{q}=1$, see Definition 3.4.4.

The pairing (3.15) is left invariant. For $(c,d) \in G$,

$$\begin{aligned}\langle \Lambda(c,d)f_1, \Lambda(c,d)f_2\rangle &= \lim_{a\to 0}\int_{-\infty}^{\infty} [\Lambda(c,d)f_1](a,b)\, [\Lambda(c,d)f_2](a,b)\, \frac{db}{a}\\ &= \lim_{a\to 0}\int_{-\infty}^{\infty} f_1\left(\frac{a}{c}, \frac{b-d}{c}\right)\, f_2\left(\frac{a}{c}, \frac{b-d}{c}\right)\, \frac{db}{a}\\ &= \lim_{a\to 0}\int_{-\infty}^{\infty} f_1(a,b)\, f_2(a,b)\, \frac{d(cb+d)}{ac}\\ &= \lim_{a\to 0}\int_{-\infty}^{\infty} f_1(a,b)\, f_2(a,b)\, \frac{db}{a} = \langle f_1, f_2\rangle.\end{aligned}$$

3. *Let G be the $ax+b$ group. Then the following invariant pairing $\langle\cdot,\cdot\rangle : L_\infty(G)\times L_\infty(G)\to\mathbb{C}$ is the $L_\infty-$version of the Haar pairing:*

$$\langle f_1, f_2\rangle_1 = \sup_{(a,b)\in G} |f_1(a,b)\cdot f_2(a,b)|\,, \quad f_1, f_2 \in L_\infty(G). \tag{3.16}$$

4. *Let L be the space of all functions on G, for which the limit (3.17) exists. Consider again the $ax+b$ group equipped with the following pairing which is the $L_\infty-$version of the Hardy pairing:*

$$\langle f_1, f_2\rangle_1 = \overline{\lim_{a\to 0}} \sup_{b\in\mathbb{R}} (f_1(a,b)\, f_2(a,b)), \quad f_1, f_2 \in L. \tag{3.17}$$

Proposition 3.4.6 *Let V be a non-trivial locally convex space and V^* be its dual space. Fix a non-zero $v_0\in V$ and a non-zero $F_0\in V^*$. Let $\langle\cdot,\cdot\rangle_1$ be a non-zero pairing defined on $\mathcal{W}_{F_0}V\times\mathcal{W}^*_{v_0}V^*$. Then there exist $v\in V$ and $F\in V^*$ such that $\langle v,F\rangle$ and $\langle\widetilde{v},\widetilde{F}\rangle_1$ are non-zero.*

Proof. Assume that $\langle\widetilde{v},\widetilde{F}\rangle_1 = 0$, for any v and F such that $\langle v,F\rangle\neq 0$. So, there are v_1 and F_1 with $\langle v_1,F_1\rangle = 0$ such that $\langle\widetilde{v}_1,\widetilde{F}_1\rangle_1\neq 0$. Let $u\in V$ be such that $\langle u,F_1\rangle\neq 0$, then by Definition 3.4.2:

$$0 = \langle v_1, F_1\rangle = \langle v_1+u-u, F_1\rangle = \langle v_1+u, F_1\rangle - \langle u, F_1\rangle,$$

so, $0\neq\langle u,F_1\rangle = \langle v_1+u,F_1\rangle$ and by our assumption we have $\langle\widetilde{u},\widetilde{F}_1\rangle_1 = \langle\widetilde{v_1+u},\widetilde{F}_1\rangle_1 = 0$. Now,

$$\begin{aligned} 0\neq\langle\widetilde{v}_1,\widetilde{F}_1\rangle_1 &= \langle(\widetilde{v_1+u-u}),\widetilde{F}_1\rangle_1 \\ &= \langle\widetilde{v_1+u},\widetilde{F}_1\rangle_1 - \langle\widetilde{u},\widetilde{F}_1\rangle_1 = 0. \end{aligned}$$

This is a contradiction. So, $\langle\widetilde{v},\widetilde{F}\rangle_1\neq 0$ for some v and F such that $\langle v,F\rangle\neq 0$. □

Now, fix $0\neq v_0\in V$ and $0\neq F_0\in V^*$. Let $\langle\cdot,\cdot\rangle_1$ be a left invariant pairing on $\mathcal{W}_{F_0}V\times\mathcal{W}^*_{v_0}V^*$ as in the previous proposition. Then by Proposition 3.4.6, there exist

F and v such that, the values $\langle \mathcal{W}_{F_0} v, \mathcal{W}^*_{v_0} F\rangle_1$ and $\langle v, F\rangle$ are non-zero. Then there exists $c \neq 0$ such that

$$\langle \mathcal{W}_{F_0} v, \mathcal{W}^*_{v_0} F\rangle_1 = c\,\langle v, F\rangle. \tag{3.18}$$

We can eliminate the constant c from the relation 3.18 by letting $v' = \frac{1}{c} v$, then

$$[\mathcal{W}_{F_0} v'](g) = F_0(\rho(g^{-1})v') = \frac{1}{c} F_0(\rho(g^{-1})v) = \frac{1}{c}[\mathcal{W}_{F_0} v](g).$$

By substituting v' in (3.18), we get

$$\begin{aligned} \langle \mathcal{W}_{F_0} v', \mathcal{W}^*_{v_0} F\rangle_1 &= \langle \frac{1}{c}\,\mathcal{W}_{F_0} v, \mathcal{W}^*_{v_0} F\rangle_1 \\ &= \frac{1}{c}\,c\,\langle v, F\rangle \\ &= \langle v, F\rangle. \end{aligned} \tag{3.19}$$

Now, we give a general definition of convolution on functions on G through invariant pairings.

Definition 3.4.7 *Let L_1 and L_2 be two spaces of functions on a group G. Let $\langle \cdot, \cdot\rangle_1$ be a left invariant pairing defined on $L_1 \times L_2$. Then the convolution of functions $f_1 \in L_1$ and $f_2 \in L_2$ is defined by the formula:*

$$(f_1 * f_2)(g) = \langle f_1, \Lambda(g)[\mathcal{S} f_2]\rangle_1, \quad g \in G, \tag{3.20}$$

where $\mathcal{S}$ is the operator (3.10).

It is clear that:

$$\Lambda(g)[\mathcal{S} f](h) = [\mathcal{S} f](g^{-1}h) = f(h^{-1}g).$$

Now, we list some examples of convolution (3.20) with different pairings.

Example 3.4.8 *1. Let f_1 and f_2 be two functions on $G = (\mathbb{R}, +)$. Consider the Haar Pairing (3.14), so*

$$\begin{aligned}(f_1 * f_2)(t) = \langle f_1, \Lambda(t)[\mathcal{S}f_2]\rangle_1 &= \int_{-\infty}^{\infty} f_1(x)\,\Lambda(t)[\mathcal{S}f_2](x)\,dx \\ &= \int_{-\infty}^{\infty} f_1(x)\,[\mathcal{S}f_2](x-t)\,dx \\ &= \int_{-\infty}^{\infty} f_1(x)\,f_2(t-x)\,dx.\end{aligned}$$

which is the usual convolution of functions on $\mathbb{R}$.

2. *Let G be a group with invariant measure $d\mu$. Let f_1 and f_2 be two functions on G, then the convolution through Haar pairing is:*

$$\begin{aligned}(f_1 * f_2)(g) = \langle f_1, \Lambda(g)[\mathcal{S}f_2]\rangle_1 &= \int_G f_1(h)\,\Lambda(g)[\mathcal{S}f_2](h)\,d\mu(h) \\ &= \int_G f_1(h)\,[\mathcal{S}f_2](g^{-1}h)\,d\mu(h) \\ &= \int_G f_1(h)\,f_2(h^{-1}g)\,d\mu(h).\end{aligned}$$

This is the standard convolution on G.

3. *Let G be the $ax+b$ group. Let f_1 and f_2 be two functions on G. So, the convolution through Hardy pairing (3.15) is:*

$$\begin{aligned}(f_1 * f_2)(a', b') &= \lim_{a\to 0}\int_{-\infty}^{\infty} f_1(a,b)\,\Lambda(a',b')[\mathcal{S}f_2](a,b)\,\frac{db}{a} \\ &= \lim_{a\to 0}\int_{-\infty}^{\infty} f_1(a,b)\,f_2((a,b)^{-1}(a',b'))\,\frac{db}{a} \\ &= \lim_{a\to 0}\int_{-\infty}^{\infty} f_1(a,b)\,f_2\left(\frac{a'}{a}, \frac{b-b'}{a}\right)\,\frac{db}{a}.\end{aligned}$$

4. *Let G be the $ax+b$ group. Let f_1 and f_2 be two functions on G. So, the convolution*

related to the pairing (3.16) is:

$$\begin{aligned}(f_1 * f_2)(a', b') &= \langle f_1, \Lambda(a', b')[\mathcal{S}f_2]\rangle_1\\ &= \sup_{(a,b)\in G} |f_1(a,b)\cdot \Lambda(a',b')[\mathcal{S}f_2](a,b)|\\ &= \sup_{(a,b)\in G} \left|f_1(a,b)\cdot [\mathcal{S}f_2]((a',b')^{-1}(a,b))\right|\\ &= \sup_{(a,b)\in G} \left|f_1(a,b)\cdot f_2((a,b)^{-1}(a',b'))\right| = \sup_{(a,b)\in G}\left|f_1(a,b)\cdot f_2(\frac{a'}{a},\frac{b-b'}{a})\right|\end{aligned}$$

5. *Let G be the $ax+b$ group. Let f_1 and f_2 be two functions on G. So, the convolution related to the pairing (3.17) is:*

$$\begin{aligned}(f_1 * f_2)(a', b') &= \langle f_1, \Lambda(a', b')[\mathcal{S}f_2]\rangle_1\\ &= \overline{\lim_{a\to 0}}\sup_{b\in\mathbb{R}} f_1(a,b)\cdot \Lambda(a',b')[\mathcal{S}f_2](a,b)\\ &= \overline{\lim_{a\to 0}}\sup_{b\in\mathbb{R}} f_1(a,b)\cdot f_2((a,b)^{-1}(a',b'))\\ &= \overline{\lim_{a\to 0}}\sup_{b\in\mathbb{R}} f_1(a,b)\cdot f_2(\frac{a'}{a},\frac{b-b'}{a})\end{aligned}$$

In the following we are going to show that the convolution (3.20) commutes with left regular and right regular representations.

Proposition 3.4.9 *Consider the convolution (3.20). Let f_1 and f_2 be two functions on G. Then:*

1. $(\Lambda(g)f_1) * f_2 = \Lambda(g)(f_1 * f_2), \qquad g\in G.$
2. $f_1 * (\mathcal{R}(g)f_2) = \mathcal{R}(g)(f_1 * f_2), \qquad g\in G.$

Proof. Let f_1 and f_2 be two functions on G and $g, h \in G$, then

1. By using the definition of convolution (3.20) and the definition of the left invariant pairing, we have:

$$\begin{aligned}[\Lambda(g)f_1 * f_2](h) &= \langle \Lambda(g)f_1, \Lambda(h)[\mathcal{S}f_2]\rangle \\ &= \langle f_1, \Lambda(g^{-1}h)[\mathcal{S}f_2]\rangle \\ &= [f_1 * f_2](g^{-1}h) = \Lambda(g)[f_1 * f_2](h).\end{aligned}$$

2. Let $h, g \in G$, then from the definition of convolution we have

$$\mathcal{R}(g)(f_1 * f_2)(h) = (f_1 * f_2)(hg) = \langle f_1, \Lambda(hg)[\mathcal{S}f_2]\rangle_1 = \langle f_1, \Lambda(h)\Lambda(g)[\mathcal{S}f_2]\rangle_1,$$

but we have

$$\Lambda(g)[\mathcal{S}f_2](h) = [\mathcal{S}f_2](g^{-1}h) = f_2(h^{-1}g) = [\mathcal{R}(g)f_2](h^{-1}) = \mathcal{S}[\mathcal{R}(g)f_2](h).$$

So,

$$\langle f_1, \Lambda(h)\Lambda(g)[\mathcal{S}f_2]\rangle_1 = \langle f_1, \Lambda(h)\mathcal{S}[\mathcal{R}(g)f_2]\rangle_1 = (f_1 * \mathcal{R}(g)f_2)(h).$$

Thus, we have $\mathcal{R}(g)(f_1 * f_2)(h) = (f_1 * \mathcal{R}(g)f_2)(h)$. □

Next proposition is a generalization of the relation (3.19).

Proposition 3.4.10 *Let V be a locally convex space and V^* be its dual. Fix non zeros $v_0 \in V$ and $F_0 \in V^*$ and let $\langle \cdot, \cdot \rangle_1$ be a left invariant pairing on $\mathcal{W}_{F_0}V \times \mathcal{W}^*_{v_0}V^*$. Then there exist $F \in V^*$ and $v \in V$ such that:*

$$(\mathcal{W}_{F_0}v * \mathcal{W}_F v_0)(g) = [\mathcal{W}_F v](g), \quad \forall\ g \in G. \tag{3.21}$$

Proof. Let v_0 and F_0 be as described above, then by the relation (3.19), there exist $v \in V$ and $F \in V^*$ such that

$$\langle \mathcal{W}_{F_0}v_0, \mathcal{W}^*_{v_0}F\rangle_1 = \langle v, F\rangle. \tag{3.22}$$

It is clear that:

$$\begin{aligned}\langle \mathcal{W}_{F_0} v_0, \mathcal{W}^*_{v_0} F\rangle_1 &= \langle \mathcal{W}_{F_0} v_0, \mathcal{S}\mathcal{W}_F v_0\rangle_1 \\ &= \langle \mathcal{W}_{F_0} v_0, \Lambda(e)\mathcal{S}\mathcal{W}_F v_0\rangle_1 \\ &= [\mathcal{W}_{F_0} v_0 * \mathcal{W}_F v_0](e).\end{aligned}$$

Also, we have

$$[\mathcal{W}_F v](e) = F(\rho(e^{-1})v) = F(v) = \langle v, F\rangle.$$

So, the relation (3.22) can be represented by convolution as the following:

$$(\mathcal{W}_{F_0} v * \mathcal{W}_F v_0)(e) = [\mathcal{W}_F v](e). \tag{3.23}$$

So, for any $g \in G$, we have:

$$\begin{aligned}(\mathcal{W}_{F_0} v * \mathcal{W}_F v_0)(g) &= \Lambda(g^{-1})(\mathcal{W}_{F_0} v * \mathcal{W}_F v_0)(e) \\ &= \Lambda(g^{-1})[\mathcal{W}_F v](e) = [\mathcal{W}_F v](g),\end{aligned}$$

the proof is completed. □

Moreover, the reproducing formula (3.21) can be expanded to the set:

$$E_v = Span(\{\rho(g)v : g \in G\}). \tag{3.24}$$

Proposition 3.4.11 *Let $v, v_0 \in V$ and $F, F_0 \in V^*$ be as above. Consider the set E_v (3.24), then*

$$\mathcal{W}_{F_0} u * \mathcal{W}_F v_0 = \mathcal{W}_F u, \quad \forall \quad u \in E_v. \tag{3.25}$$

Proof. Let $u \in E_v$, then u has the form $\sum_{k\in I} \lambda_k \, \rho(g_k)v$, for $I \subseteq \mathbb{N}$. So, by the linearity of the covariant transform

$$\mathcal{W}_{F_0} u = \sum_{k\in I} \lambda_k \, \mathcal{W}_{F_0}(\rho(g_k)v).$$

Now,

$$\begin{aligned}(\mathcal{W}_{F_0}u * \mathcal{W}_F v_0)(h) &= \langle \mathcal{W}_{F_0}u, \Lambda(h)\mathcal{S}\mathcal{W}_F v_0\rangle_1 \\ &= \sum_{k\in I} \lambda_k \left\langle \mathcal{W}_{F_0}\left(\rho(g_k)v\right), \Lambda(h)\mathcal{S}\mathcal{W}_F v_0\right\rangle_1.\end{aligned}$$

Now for any $k \in I$, and by using the intertwining property of $\mathcal{W}_{F_0}$ (Proposition 3.1.2), we have:

$$\left\langle \mathcal{W}_{F_0}\left(\rho(g_k)v\right), \Lambda(h)\mathcal{S}\mathcal{W}_F v_0\right\rangle_1 = \langle \Lambda(g_k)\mathcal{W}_{F_0}v, \Lambda(h)\mathcal{S}\mathcal{W}_F v_0\rangle_1,$$

and by the left invariance of the pairing, we have:

$$\begin{aligned}\langle \Lambda(g_k)\mathcal{W}_{F_0}v, \Lambda(h)\mathcal{S}\mathcal{W}_F v_0\rangle_1 &= \langle \mathcal{W}_{F_0}v, \Lambda(g_k^{-1}h)\mathcal{S}\mathcal{W}_F v_0\rangle_1 \\ &= [\mathcal{W}_F v](g_k^{-1}h),\end{aligned}$$

thus,

$$\begin{aligned}(\mathcal{W}_{F_0}u * \mathcal{W}_F v_0)(h) &= \sum_{k\in I} \lambda_k \, [\mathcal{W}_F v](g_k^{-1}h) \\ &= \sum_{k\in I} \lambda_k \, \Lambda(g_k)[\mathcal{W}_F v](h) \\ &= \sum_{k\in I} \lambda_k \, [\mathcal{W}_F\left(\rho(g_k)v\right)](h) \\ &= \left[\mathcal{W}_F\left(\sum_{k\in I} \lambda_k \, \rho(g_k)v\right)\right](h) = [\mathcal{W}_F u](h).\end{aligned}$$

Therefore, the reproducing formula (3.25) holds for any $u \in E_v$. □

Remark 3.4.12 *Since $(\rho^*(g)F)(v) = F(\rho(g^{-1})v)$, for any $g \in G$, then the set*

$$E_F = span(\{\rho^*(g)F : g \in G\}),$$

gives the same result of Proposition 3.4.11.

The following lemma is needed to expand the relation (3.25) to include any $u \in \overline{E_v}$.

Lemma 3.4.13 *Let ϕ be a map on E_v defined by*

$$\phi(u) = \mathcal{W}_{F_0} u * \mathcal{W}_F v_0,$$

then ϕ is continuous on E_v, i.e. if $u_n \to u$ in E_v then $\phi(u_n)(g) \to \phi(u)(g)$, for each $g \in G$.

Proof. Let $u_n \to u \in E_v$, then

$$\begin{aligned} \lim_{n\to\infty} [\phi(u_n)](g) &= \lim_{n\to\infty} (\mathcal{W}_{F_0} u_n * \mathcal{W}_F v_0)(g) \\ &= \lim_{n\to\infty} [\mathcal{W}_F u_n](g) \quad \textit{by } (3.25) \\ &= [\mathcal{W}_F u](g) \quad \textit{by Lemma } 3.1.7 \\ &= (\mathcal{W}_{F_0} u * \mathcal{W}_F v_0)(g) = [\phi(u)](g) \end{aligned}$$

So, ϕ is continuous on E_v. □

Proposition 3.4.14 *The reproducing formula (3.25) holds for any $u \in \overline{E_v}$.*

Proof. Let $u \in \overline{E_v}$, then there is a sequence $(u_n) \subseteq E_v$ converges to u. So, by the continuity of the covariant transform $[\mathcal{W}_F u](g) = \lim_{n\to\infty}[\mathcal{W}_F u_n](g)$, for each $g \in G$. Since $[\mathcal{W}_F u_n](h) = \langle \mathcal{W}_{F_0} u_n, \Lambda(h)[\mathcal{S}\mathcal{W}_F v_0]\rangle_1$, then $[\mathcal{W}_F u](h) = \lim_{n\to\infty}\langle \mathcal{W}_{F_0} u_n, \Lambda(h)[\mathcal{S}\mathcal{W}_F v_0]\rangle_1$. By the the previous lemma and the continuity of the pairing $\langle \cdot, \cdot \rangle_1$, we have $[\mathcal{W}_F u](h) = \langle \mathcal{W}_{F_0}(\lim_{n\to\infty} u_n), \Lambda(h)[\mathcal{S}\mathcal{W}_F v_0]\rangle_1 = (\mathcal{W}_{F_0} u * \mathcal{W}_F v_0)(h)$. □

If v is a cyclic vector on ρ, then $\overline{E_v} = V$ and hence the reproducing formula (3.25) holds for any $u \in V$.

3.5 Examples from Harmonic Analysis

In this section we list various examples of covariant transform from classical harmonic analysis.

The following proposition gives a characterization of the image space $\mathcal{W}_F(V)$ by left invariant vector fields on G.

Proposition 3.5.1 [33] *Let G be a Lie group with a Lie algebra $\mathfrak{g}$ and ρ be a smooth representation of G with derived representation $d\rho$. Let a fiducial operator F be a null solution for the operator $A = \sum_j a_j \, d\rho^{X_j}$, with $X_j \in \mathfrak{g}$ and a_j are constants. Then the covariant transform $[\mathcal{W}_F f](g) = F(\rho(g^{-1})f)$ for any f satisfies*

$$D(\mathcal{W}_F f) = 0, \qquad \textit{with} \quad D = \sum_j \overline{a_j} \, \mathcal{L}^{X_j},$$

where $\mathcal{L}^{X_j}$ are the left invariant fields on G corresponding to X_j.

Example 3.5.2 *Consider the representation ρ_p of the $ax+b$ group. Let $A = (1,0)$ and $N = (0,1)$, be the basis of the Lie algebra $\mathfrak{g}$ generating one-parameters subgroups of the $ax+b$ group:*

$$\mathcal{A} = \{(a,0) \in G : a \in \mathbb{R}_+\} \quad \textit{and} \quad \mathcal{N} = \{(1,b) \in G : b \in \mathbb{R}\}.$$

Then the derived representations are:

$$[d\rho_p^A f](x) = \frac{d}{dt}\left[\rho_p(e^{tA})f\right](x)|_{t=0} = \frac{d}{dt}\left[\rho_p(e^t,0)f\right](x)|_{t=0} = -\frac{1}{p}f(x) - xf'(x),$$

and

$$[d\rho_p^N f](x) = \frac{d}{dt}\left[\rho_p(e^{tN})f\right](x)|_{t=0} = \frac{d}{dt}\left[\rho_p(1,t)f\right](x)|_{t=0} = -f'(x).$$

The function $\frac{1}{(x+r)^{\frac{1}{p}}}$, where r is real number, is the null solution of the following differential equation:

$$d\rho_p^A + r\, d\rho_p^N = -\frac{1}{p}I - x\frac{d}{dx} - r\frac{d}{dx} = -\frac{1}{p}I - (x+r)\frac{d}{dx}.$$

In general, the function $\frac{1}{(x+r)^{\frac{1}{p}-s}}$, *where* r *and* s *are real numbers, is the null solution of the most general first order differential equation*

$$d\rho_p^A + r\, d\rho_p^N + sI = -\frac{1}{p}I - x\,\frac{d}{dx} - r\,\frac{d}{dx} + sI$$
$$= (-\frac{1}{p} + s)I - (x+r)\,\frac{d}{dx}.$$

Since $\frac{1}{p} - s$ *is real number then it can be replaced by* t. *Also, since* r *is a real number then it can be mapped to* 0 *by the action of the* $ax+b$ *group and then* $(x+r)^{-t}$ *can be mapped to* x^{-t}. *It is clear that* x^{-t} *is not integrable over* $\mathbb{R}$, *but for any real number* $a > 0$, *we have*

$$\int_0^a \frac{1}{x^t}\, dx = \lim_{\epsilon\to 0}\left[\frac{x^{-t+1}}{-t+1}\right]_\epsilon^a = \lim_{\epsilon\to 0}\left[\frac{a^{-t+1}}{-t+1} - \frac{\epsilon^{-t+1}}{-t+1}\right],$$

the last limit exists for any $0 < t < 1$. *So,* $x^{-t} \in L^q_{loc}(\mathbb{R})$, *for any* t *such that* $qt < 1$ *or* $t < \frac{1}{q}$.

Now we consider the space $B_0(\mathbb{R})$ *of all bounded functions with compact support. Fixed* $t \in (0,1)$, *consider the following collection of seminorms:*

$$p_K(v) = \int_K \left|\frac{1}{x^t}\right| dx \cdot \sup_{x\in K}|v(x)|, \quad K \text{ is a compact subset of } \mathbb{R}.$$

We define the functional F_t *on* $B_0(\mathbb{R}) \subseteq L^1_{loc}(\mathbb{R})$ *by:*

$$F_t(v) = \int_{\mathbb{R}} \frac{v(x)}{x^t}\, dx = \int_K \frac{v(x)}{x^t}\, dx, \quad K \text{ is the compact support of } v.$$

The functional F_t *is bounded:*

$$|F_t(v)| = \left|\int_K \frac{v(x)}{x^t}\, dx\right| \le \sup_{x\in K}|v(x)| \int_K \left|\frac{1}{x^t}\right| dx = p_K(v).$$

So, the covariant transform (3.1) is:

$$\widetilde{v}(a,b) = F_t(\rho_p(a,b)^{-1}v) = \int_{\mathbb{R}} \frac{a^{\frac{1}{p}}\, v(ax+b)}{x^t}\, dx = a^{\frac{1}{p}-1+t}\int_K \frac{v(x)}{(x-b)^t}\, dx,$$

which is the weak singular integral operator with kernel $(x-b)^{-t}$, $0 < t < 1$, [*35*, p.213].

The left invariant vector fields on the $ax+b$ group are:

$$[\mathcal{L}^A f](a,b) = \frac{d}{dt} f((a,b)\cdot e^{tA})|_{t=0} = \frac{d}{dt} f(ae^t, b)|_{t=0} = a\frac{\partial f}{\partial a}$$

and

$$[\mathcal{L}^N f](a,b) = \frac{d}{dt} f((a,b)\cdot e^{tN})|_{t=0} = \frac{d}{dt} f(a, at+b)|_{t=0} = a\frac{\partial f}{\partial b}.$$

Therefore by Proposition 3.5.1, the image of the covariant transform with mother wavelet $(x+r)^{\frac{1}{p}-s}$ consists of all the null solutions to the most general first order differential operator

$$\begin{aligned}\mathcal{L}^A + \overline{r}\,\mathcal{L}^N + sI &= a\partial_a + r\,a\,\partial_b + sI \\ &= a\,r(\partial_b + \frac{1}{r}\,\partial_a) + sI.\end{aligned}$$

Example 3.5.3 *Let $d\rho_p^A$ and $d\rho_p^N$ be the derived representations of the representation ρ_p of the $ax+b$ group. Take the function*

$$v_r(x) = \frac{1}{(x+r)^{s-\frac{1}{p}}},$$

for a complex number r and a real number s such that $s-\frac{1}{p}\in\mathbb{N}$. Since any $r\in\mathbb{C}$ can be mapped to i by the action of the $ax+b$ group, then $(x+r)^{-t}$ can be mapped to $(x+i)^{-t}$. Also, since $(x+i)^{-t}$ has no singularity on $\mathbb{R}$, we just need to discuss its behavior at ∞. So, for a positive $a\in\mathbb{R}$, we have:

$$\int_a^\infty \frac{1}{|x+i|^t}\,dx \le \int_a^\infty \frac{1}{x^t}\,dx = \left[\frac{x^{-t+1}}{-t+1}\right]_a^\infty,$$

so, $\int_a^\infty \frac{1}{|x+i|^t}\,dx$ is finite for any $t>1$. Thus, $(x+i)^{-t}\in L^q(\mathbb{R})$, for any real t such that $qt>1$.

Now, choose $v_0(x)=(x+i)^{-1}$ and define F_0 on $L^p(\mathbb{R})$, $1\le p<\infty$, by:

$$F_0(v) = \int_{\mathbb{R}} v(x)\,\overline{v_0(x)}\,dx = \int_{\mathbb{R}} \frac{v(x)}{x-i}\,dx.$$

The functional F_0 is bounded:

$$\begin{aligned}|F_0(v)| &= \left|\int_{\mathbb{R}} \frac{v(x)}{x-i}\,dx\right| \\ &\leq \int_{\mathbb{R}} \frac{|v(x)|}{|x-i|}\,dx \\ &\leq \|v\|_{L^p(\mathbb{R})} \left\|\frac{1}{x-i}\right\|_{L^q(\mathbb{R})}.\end{aligned}$$

So, the covariant transform (3.1) is:

$$\widetilde{v}(a,b) = F_0(\rho_p(a,b)^{-1}v) = \int_{\mathbb{R}} \frac{a^{\frac{1}{p}}\, v(ax+b)}{x-i}\,dx = a^{\frac{1}{p}} \int_{\mathbb{R}} \frac{v(x)}{x-b-ia},$$

and thus, for $v_0(x) = (x+i)^{-1}$

$$\begin{aligned}\mathcal{S}\widetilde{v_0}(a,b) = \widetilde{v_0}(a,b)^{-1} = F_0(\rho_p(a,b)v_0) &= \int_{\mathbb{R}} \frac{a^{-\frac{1}{p}}\, v_0(\frac{x-b}{a})}{x-i}\,dx \\ &= a^{1-\frac{1}{p}} \int_{\mathbb{R}} \frac{1}{x-b+ia} \cdot \frac{1}{x-i}\,dx \\ &= a^{1-\frac{1}{p}} \int_{\mathbb{R}} \left[\frac{A}{x-b+ia} + \frac{B}{x-i}\right]\,dx\end{aligned}$$

where $A = \frac{-1}{i(a+1)-b}$ and $B = -A$. So,

$$\begin{aligned}\int_{\mathbb{R}} \left[\frac{A}{x-b+ia} + \frac{B}{x-i}\right]dx &= A\int_{\mathbb{R}} \frac{(x-b)-ia}{(x-b)^2+a^2}\,dx + B\int_{\mathbb{R}} \frac{x+i}{x^2+1}\,dx \\ &= A\int_{\mathbb{R}} \left[\frac{x-b}{(x-b)^2+a^2} - \frac{x}{x^2+1}\right]dx - iA\int_{\mathbb{R}} \left[\frac{a}{(x-b)^2+a^2} + \frac{1}{x^2+1}\right]dx \\ &= \frac{A}{2}\left[\ln\left(\frac{(x-b)^2+a^2}{x^2+1}\right)\right]_{-\infty}^{\infty} - iA\left[\tan^{-1}\left(\frac{x-b}{a}\right) + \tan^{-1}(x)\right]_{-\infty}^{\infty} \\ &= 0 - iA[2\pi] = \frac{2i\pi}{i(a+1)-b}.\end{aligned}$$

Therefore,

$$\mathcal{S}\widetilde{v_0}(a,b) = \frac{a^{\frac{1}{q}} \cdot 2i\pi}{i(a+1)-b}, \quad \frac{1}{q} = 1 - \frac{1}{p}.$$

So, for $w = (c,d)$ and $z = (a,b)$, we have:

$$\Lambda(w)\mathcal{S}\widetilde{v_0}(z) = \mathcal{S}\widetilde{v_0}\left(\frac{a}{c}, \frac{b-d}{c}\right) = \frac{(\frac{a}{c})^{\frac{1}{q}} \cdot 2i\pi}{((\frac{a}{c}+1)i - (\frac{b-d}{c}))} = \frac{2i\pi \cdot a^{\frac{1}{q}} \cdot c^{\frac{1}{p}}}{(a+c)i-(b-d)},$$

where,

$$\frac{1}{(a+c)i-(b-d)} = \frac{-i}{(a+c)+i(b-d)} = \frac{-i}{(a+ib)+(c-id)} = \frac{-i}{z+\overline{w}}.$$

$\frac{1}{z+\overline{w}}$ *is the Hardy reproducing kernel on the upper half plane [9].*

Now consider the Hardy pairing (3.15):

$$\begin{aligned}(\widetilde{v} * \widetilde{v_0})(w) &= \langle \widetilde{v}, \Lambda(w)\mathcal{S}\widetilde{v_0}\rangle \\ &= \lim_{a\to 0}\int_{-\infty}^{\infty} \widetilde{v}(a,b)\,\mathcal{S}\widetilde{v_0}\left(\frac{a}{c},\frac{b-d}{c}\right)\frac{db}{a} \\ &= -i\cdot c^{\frac{1}{p}}\lim_{a\to 0} a^{-\frac{1}{p}}\int_{-\infty}^{\infty}\widetilde{v}(a,b)\,\frac{1}{((a+c)+i(b-d))}\,db \\ &= -i\cdot c^{\frac{1}{p}}\lim_{a\to 0} a^{-\frac{1}{p}}\int_{-\infty}^{\infty}\frac{\widetilde{v}(z)}{z+\overline{w}}\,dz.\end{aligned}$$

Now, by proposition 3.5.1, the image of the covariant transform with mother wavelet $(x+r)^{\frac{1}{p}-s}$ *consists of all the null solutions to the operator*

$$\begin{aligned}\mathcal{L}^A + \overline{r}\,\mathcal{L}^N + sI &= a\partial_a + \overline{r}\,a\,\partial_b + sI \\ &= a\,\overline{r}(\partial_b + \frac{1}{\overline{r}}\,\partial_a) + sI.\end{aligned}$$

and by the action of the $ax+b$*, the relation* $\partial_b + \frac{1}{\overline{r}}\,\partial_a$ *can be mapped to the* $\partial_b + i\,\partial_a$ *which is the C-R equation.*

Example 3.5.4 *Let* $d\rho_p^A$ *and* $d\rho_p^N$ *be the derived representations of the representation* ρ_p *of* $ax+b$ *group. Consider the following function*

$$v_r(x) = \frac{1}{(x^2+r)^t},$$

for a real numbers $r > 0$ *and* $t > 0$*, is a null solution of the most general second order differential equation:*

$$(d\rho_p^A)^2 + r\,(d\rho_p^N)^2 - \left(2t+1-\frac{2}{p}\right)d\rho_p^A - \left[2(\frac{1}{p}-1)t + \frac{1}{p} - \frac{1}{p^2}\right]I = 2t\,I + 3(t+1)x\frac{d}{dx} + (x^2+r)\frac{d^2}{dx^2}$$

Since the function $v_r(x)$ has no singularity on $\mathbb{R}$, then we need to check its behavior at ∞. So, for $a > 0$, we have

$$\int_a^\infty \frac{1}{(x^2+r)^t}\,dx \le \int_a^\infty \frac{1}{x^{2t}}\,dx = \lim_{\epsilon\to\infty}\left[\frac{x^{-2t+1}}{-2t+1}\right]_a^\epsilon = \lim_{\epsilon\to\infty}\frac{\epsilon^{-2t+1}}{-2t+1} - \frac{a^{-2t+1}}{-2t+1},$$

so, the last limit tends to zero for any real number $t > \frac{1}{2}$. Thus, $(x^2+r)^{-t} \in L^q(\mathbb{R})$, for any t such that $tq > \frac{1}{2}$ or $t > \frac{1}{2q}$.

Now, define the functional F_r on $L^p(\mathbb{R})$, $1 \le p \le \infty$, by:

$$F_r(v) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{v(x)}{x^2+r}\,dx.$$

It is clear that $\frac{1}{x^2+r} \in L^q(\mathbb{R})$, where $\frac{1}{p}+\frac{1}{q}=1$. Then the functional F_r is bounded:

$$\begin{aligned}|F_r(v)| &= \left|\frac{1}{\pi}\int_{\mathbb{R}} \frac{v(x)}{x^2+r}\right| dx\\ &\le \frac{1}{\pi}\int_{\mathbb{R}} \frac{|v(x)|}{x^2+r}\,dx\\ &\le \frac{1}{\pi}\|v\|_{L^p(\mathbb{R})}\left\|\frac{1}{x^2+r}\right\|_{L^q(\mathbb{R})}, \quad \text{by Hölder inequality.}\end{aligned}$$

So, the covariant transform (3.1) is

$$\widetilde{v}(a,b) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{a^{\frac{1}{p}}\,v(ax+b)}{x^2+r}\,dx = \frac{a^{\frac{1}{p}+1}}{\pi}\int_{\mathbb{R}} \frac{v(x)}{(x-b)^2+a^2r}\,dx,$$

and for $r = 1$, we have:

$$\widetilde{v}(a,b) = \frac{a^{\frac{1}{p}+1}}{\pi}\int_{\mathbb{R}} \frac{v(x)}{(x-b)^2+a^2}\,dx,$$

so, $a^{-\frac{1}{p}}\widetilde{v}(a,b)$ is the Poisson integral.

Now, the image of the covariant transform with mother wavelet $(x^2+r)^{-t}$ consists of the null solutions of the operator

$$\begin{aligned}(\mathcal{L}^A)^2 + r(\mathcal{L}^N)^2 - m\,\mathcal{L}^A - n\,I &= a^2\partial_a^2 + r\,a^2\partial_b^2 - ma\partial_a - nI\\ &= ra^2(\partial_b^2 + \frac{1}{r}\partial_a^2) - ma\partial_a - nI,\end{aligned}$$

where, $m = 2t+1-\frac{2}{p}$ and $n = 2(\frac{1}{p}-1)t+\frac{1}{p}-\frac{1}{p^2}$. So, for $r = 1$, $m = 0$ and $n = 0$ we have a Laplace equation.

Example 3.5.5 *Let G be the $ax+b$ group and ρ_p be its representation on $L^p(\mathbb{R})$. We consider Banach spaces which were introduced in [22] as the following: Let $\mathcal{L}^p(\mathbb{R})$, $1 \leq p \leq \infty$, be the space of all measurable functions $f : \mathbb{R} \to \mathbb{C}$, for which*

$$\widehat{f}(x) = \sum_{n\in\mathbb{Z}} |f(x-n)|,$$

belongs to the space $L^p[0,1]$. The space $\mathcal{L}^p(\mathbb{R})$ with the following norm

$$|f|_p := \|\widehat{f}\|_{L^p[0,1]},$$

becomes a Banach space [22].

It is clear that $\mathcal{L}^1(\mathbb{R}) = L^1(\mathbb{R})$:

$$\begin{aligned}\|f\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |f(x)|\,dx &= \sum_{n\in\mathbb{Z}} \int_0^1 |f(x-n)|\,dx \\ &= \int_0^1 \sum_{n\in\mathbb{Z}} |f(x-n)|\,dx \\ &= \int_0^1 |\widehat{f}(x)|\,dx = \|\widehat{f}\|_{L^1[0,1]}.\end{aligned}$$

Also, we have $\|f\|_{L^p(\mathbb{R})} \leq |f|_p$, for $1 < p < \infty$, and hence $\mathcal{L}^p(\mathbb{R}) \subset L^p(\mathbb{R})$ [22]. On the other hand $L^p(\mathbb{R})$, $1 < p < \infty$, contains functions which are not in $\mathcal{L}^p(\mathbb{R})$. Let

$$f(x) = \begin{cases} \frac{1}{x}, & x \geq 1; \\ 0, & otherwise. \end{cases}$$

then, for $1 < p \leq \infty$, we have:

$$\begin{aligned}\|f\|^p_{L^p(\mathbb{R})} = \int_{-\infty}^{\infty} |f(x)|^p\,dx &= \int_1^{\infty} \frac{1}{x^p}\,dx \\ &= \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \frac{1}{p-1} < \infty,\end{aligned}$$

thus $f \in L^p(\mathbb{R})$, for $p > 1$. But $f \notin L^1(\mathbb{R}) = \mathcal{L}^1(\mathbb{R})$ and hence $\widehat{f} \notin L^1[0,1]$. Since $L^p[0,1] \subseteq L^1[0,1]$, for any $p > 1$, then $\widehat{f} \notin L^p[0,1]$ and hence $f \notin \mathcal{L}^p(\mathbb{R})$ for any $p > 1$.

Let $f \in \mathcal{L}^p(\mathbb{R})$*, we are going to prove that* $\rho_p(2,0)f \in \mathcal{L}^p(\mathbb{R})$*:*

$$\begin{aligned} \|\widehat{\rho_p(2,0)}f\|^p_{L^p[0,1]} &= \int_0^1 |\widehat{\rho_p(2,0)}f(x)|^p\,dx \\ &= \frac{1}{2}\int_0^1 \left(\sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-n}{2}\right)\right|\right)^p dx \end{aligned} \tag{3.26}$$

The last summation can be divided into two summations:

$$\sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-n}{2}\right)\right| = \sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-2n}{2}\right)\right| + \sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-(2n-1)}{2}\right)\right|.$$

So the relation (3.26) will be

$$\|\widehat{\rho_p(2,0)}f\|^p_{L^p[0,1]} = \frac{1}{2}\int_0^1 \left(\sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-2n}{2}\right)\right| + \sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-(2n-1)}{2}\right)\right|\right)^p dx,$$

and by using the relation $(x+y)^p \le 2^{p-1}(x^p+y^p)$ *we have:*

$$\begin{aligned} \|\widehat{\rho_p(2,0)}f\|^p_{L^p[0,1]} &\le \frac{2^{p-1}}{2}\int_0^1 \left[\left(\sum_{n\in\mathbb{Z}} \left| f\left(\frac{x}{2}-n\right)\right|\right)^p + \left(\sum_{n\in\mathbb{Z}} \left| f\left(\frac{x+1}{2}-n\right)\right|\right)^p\right] dx \\ &= 2^{p-1}\int_0^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} |f(x-n)|\right)^p dx + 2^{p-1}\int_{\frac{1}{2}}^1 \left(\sum_{n\in\mathbb{Z}} |f(x-n)|\right)^p dx \\ &= 2^{p-1}\int_0^1 \left(\sum_{n\in\mathbb{Z}} |f(x-n)|\right)^p dx = 2^{p-1}\|\widehat{f}\|^p_{L^p[0,1]} < \infty. \end{aligned}$$

Also,

$$\begin{aligned} \|\widehat{f}\|^p_{L^p[0,1]} &= \int_0^1 |\widehat{f}(x)|^p\,dx \\ &= \frac{1}{2}\int_0^2 |\widehat{f}\left(\frac{x}{2}\right)|^p\,dx \\ &= \frac{1}{2}\int_0^1 |\widehat{f}\left(\frac{x}{2}\right)|^p\,dx + \frac{1}{2}\int_1^2 |\widehat{f}\left(\frac{x}{2}\right)|^p\,dx \\ &= \frac{1}{2}\int_0^1 |\widehat{f}\left(\frac{x}{2}\right)|^p\,dx + \frac{1}{2}\int_0^1 |\widehat{f}\left(\frac{x+1}{2}\right)|^p\,dx. \end{aligned}$$

Now,

$$\widehat{f}\left(\frac{x}{2}\right) = \sum_{n\in\mathbb{Z}} \left| f\left(\frac{x}{2} - n\right)\right| = \sum_{n\in\mathbb{Z}} \left| f\left(\frac{x}{2} - \frac{2n}{2}\right)\right| = \sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-2n}{2}\right)\right|,$$

and

$$\widehat{f}\left(\frac{x+1}{2}\right) = \sum_{n\in\mathbb{Z}} \left| f\left(\frac{x+1}{2} - n\right)\right| = \sum_{n\in\mathbb{Z}} \left| f\left(\frac{x+1}{2} - \frac{2n}{2}\right)\right| = \sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-(2n-1)}{2}\right)\right|.$$

So,

$$\begin{aligned}
\|\widehat{f}\|^p_{L^p[0,1]} &= \frac{1}{2}\int_0^1 \left(\sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-2n}{2}\right)\right|\right)^p dx + \frac{1}{2}\int_0^1 \left(\sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-(2n-1)}{2}\right)\right|\right)^p dx \\
&= \frac{1}{2}\int_0^1 \left[\left(\sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-2n}{2}\right)\right|\right)^p + \left(\sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-(2n-1)}{2}\right)\right|\right)^p\right] dx \\
&\le \frac{1}{2}\int_0^1 \left(\sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-2n}{2}\right)\right| + \sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-(2n-1)}{2}\right)\right|\right)^p dx \\
&= \frac{1}{2}\int_0^1 \left(\sum_{n\in\mathbb{Z}} \left| f\left(\frac{x-n}{2}\right)\right|\right)^p dx = \int_0^1 \left|\widehat{\rho_p(2,0)} f(x)\right|^p dx = \|\widehat{\rho_p(2,0)} f\|^p_{L^p[0,1]}.
\end{aligned}$$

Thus,

$$\|\widehat{f}\|^p_{L^p[0,1]} \le \|\widehat{\rho_p(2,0)} f\|^p_{L^p[0,1]} \le 2^{p-1}\|\widehat{f}\|^p_{L^p[0,1]}.$$

Now, let F be defined on $\mathcal{L}^p(\mathbb{R})$ by:

$$F(f) = \int_{-\infty}^{\infty} f(x)\, e^{-x^2}\, dx$$

The functional F is bounded:

$$\begin{aligned}
|F(f)| &\le \int_{-\infty}^{\infty} |f(x)|\, e^{-x^2}\, dx \\
&\le \|f\|_{L^p(\mathbb{R})}\, \|e^{-x^2}\|_{L^q(\mathbb{R})} \\
&\le |f|_p\, \|e^{-x^2}\|_{L^q(\mathbb{R})}.
\end{aligned}$$

So, the covariant transform (3.1) is:

$$\widetilde{f}(a,b) = \int_{-\infty}^{\infty} a^{\frac{1}{p}}\, f(ax+b)\, e^{-x^2}\, dx = a^{\frac{1}{p}-1} \int_{-\infty}^{\infty} f(x)\, e^{\frac{-(x-b)^2}{a^2}}\, dx.$$

Let $f_0(x) = e^{-x^2}$, *then*

$$\begin{aligned}
\mathcal{S}\widetilde{f_0}(a,b) = F(\rho_p(a,b)f_0) &= \int_{-\infty}^{\infty} a^{-\frac{1}{p}}\, f_0\left(\frac{x-b}{a}\right)\, e^{-x^2}\, dx \\
&= a^{-\frac{1}{p}} \int_{-\infty}^{\infty} e^{\frac{-(x-b)^2}{a^2}}\, e^{-x^2}\, dx \\
&= a^{-\frac{1}{p}} \int_{-\infty}^{\infty} e^{-(\frac{1}{a^2}+1)x^2+\frac{2b}{a^2}x-\frac{b^2}{a^2}}\, dx \\
&= a^{-\frac{1}{p}} \sqrt{\frac{\pi}{\frac{1}{a^2}+1}}\, e^{-\frac{b^2}{1+a^2}} = a^{-\frac{1}{p}+1} \sqrt{\frac{\pi}{1+a^2}}\, e^{-\frac{b^2}{1+a^2}}.
\end{aligned}$$

Now, consider the Hardy pairing (3.15), then:

$$\begin{aligned}
\langle \widetilde{f}, \mathcal{S}\widetilde{f_0}\rangle_1 &= \lim_{a\to 0} \int_{-\infty}^{\infty} \widetilde{f}(a,b)\, \mathcal{S}\widetilde{f_0}(a,b)\, \frac{db}{a} \\
&= \lim_{a\to 0} \int_{-\infty}^{\infty} \left(a^{\frac{1}{p}-1} \int_{-\infty}^{\infty} f(x)\, e^{\frac{-(x-b)^2}{a^2}}\, dx \right) \cdot a^{-\frac{1}{p}+1} \sqrt{\frac{\pi}{1+a^2}}\, e^{-\frac{b^2}{1+a^2}}\, \frac{db}{a} \\
&= \int_{-\infty}^{\infty} f(x) \lim_{a\to 0} \frac{1}{a} \sqrt{\frac{\pi}{1+a^2}} \int_{-\infty}^{\infty} e^{\frac{-(x-b)^2}{a^2}}\, e^{-\frac{b^2}{1+a^2}}\, db\, dx \\
&= \int_{-\infty}^{\infty} f(x) \lim_{a\to 0} \frac{1}{a} \sqrt{\frac{\pi}{1+a^2}} \int_{-\infty}^{\infty} e^{-(\frac{1}{a^2}+\frac{1}{1+a^2})b^2+\frac{2x}{a^2}b-\frac{x^2}{a^2}}\, db\, dx \\
&= \int_{-\infty}^{\infty} f(x) \lim_{a\to 0} \frac{1}{a} \sqrt{\frac{\pi}{1+a^2}} \left[a \sqrt{\frac{\pi(a^2+1)}{2a^2+1}}\, e^{-\frac{x^2}{2a^2+1}} \right] dx \\
&= \int_{-\infty}^{\infty} f(x) \cdot \pi \cdot e^{-x^2}\, dx = \pi\, F(f).
\end{aligned}$$

Thus $\frac{1}{\pi}\mathcal{S}\widetilde{f_0}$ *is a reproducing kernel on the space* $\mathcal{W}_F(\mathcal{L}^p(\mathbb{R}))$.

Also,

$$\begin{aligned}
(\widetilde{f} * \widetilde{f_0})(g) = \Lambda(g^{-1})(\widetilde{f} * \widetilde{f_0})(e) &= \Lambda(g^{-1})\langle \widetilde{f}, \mathcal{S}\widetilde{f_0}\rangle_1 \\
&= \Lambda(g^{-1})\, [\pi F(f)] \\
&= \Lambda(g^{-1})\, [\pi\, \widetilde{f}(e)] = \pi\, \widetilde{f}(g).
\end{aligned}$$

Chapter 4

The Contravariant Transform

The purpose of this chapter is to investigate the general properties of the contravariant transform. Some examples of the contravariant transform related to the $ax + b$ group are considered. Also, the general properties of composition of covariant and contravariant transforms are studied.

4.1 Introduction

Let $\langle \cdot, \cdot \rangle$ be a left invariant pairing on $L \times L'$ and assume that the pairing can be extended in its second component to V-valued functions, i.e. $\langle \cdot, \cdot \rangle : L \times L' \to V$. Let ρ be a representation of a group G on a locally convex space V.

Definition 4.1.1 [33] *Let ρ be a representation of G in a locally convex space V, we define the function $w(g) = \rho(g)w_0$ for $w_0 \in V$ such that $w(g) \in L'$ in a suitable sense. The contravariant transform $\mathcal{M}^{\rho}_{w_0}$ is a map $L \to V$ defined by the pairing:*

$$\mathcal{M}^{\rho}_{w_0} : f \to \langle f, w \rangle, \qquad \textit{where } f \in L. \tag{4.1}$$

We assume that the pairing $\langle \cdot, \cdot \rangle$ is linear, then we have the following:

Lemma 4.1.2 *Let $\mathcal{M}^{\rho}_{w_0}$ be the contravariant transform (4.1), where the pairing $\langle \cdot, \cdot \rangle$ is linear. Then $\mathcal{M}^{\rho}_{w_0}$ is linear.*

Proof. Let $f_1, f_2 \in L$ and $\lambda \in \mathbb{C}$, then by the linearity of the pairing we have:

$$\mathcal{M}^{\rho}_{w_0}(\lambda f_1) = \langle \lambda f_1, w \rangle = \lambda \langle f_1, w \rangle = \lambda\, \mathcal{M}^{\rho}_{w_0}(f_1),$$

and

$$\mathcal{M}^{\rho}_{w_0}(f_1 + f_2) = \langle f_1 + f_2, w \rangle = \langle f_1, w \rangle + \langle f_2, w \rangle = \mathcal{M}^{\rho}_{w_0} f_1 + \mathcal{M}^{\rho}_{w_0} f_2.$$

Therefore $\mathcal{M}^{\rho}_{w_0}$ is linear. □

In the following, we transport seminorm p on V to a seminorm q on L.

Proposition 4.1.3 *Let p be a seminorm on V. Define a map q on L by*

$$q(f) = p(\mathcal{M}^{\rho}_{w_0} f), \qquad f \in L.$$

Then q is a seminorm on L.

Proof.

Let p be a seminorm on V and q be defined as above. Then for any $f_1, f_2 \in L$ and $\lambda \in \mathbb{C}$, we have:

1. It is clear that $q(f_1) = p(\mathcal{M}^{\rho}_{w_0} f_1) \geq 0$ and for $f_1 = 0$ the contravariant transform $\mathcal{M}^{\rho}_{w_0} f_1 = 0$. Therefore, $q(0) = p(0) = 0$.

2. Since the contravariant transform $\mathcal{M}^{\rho}_{w_0}$ is linear, then we obtain the following:

   $$q(\lambda\, f_1) = p(\mathcal{M}^{\rho}_{w_0}(\lambda\, f_1)) = |\lambda|\, p(\mathcal{M}^{\rho}_{w_0} f_1) = |\lambda|\, q(f_1),$$

   so, q is homogeneous.

3. Again by the linearity of the contravariant transform $\mathcal{M}^{\rho}_{w_0}$ we have:

$$\begin{aligned} q(f_1+f_2) &= p(\mathcal{M}^{\rho}_{w_0}(f_1+f_2)) \\ &= p(\mathcal{M}^{\rho}_{w_0}f_1+\mathcal{M}^{\rho}_{w_0}f_2) \\ &\leq p(\mathcal{M}^{\rho}_{w_0}f_1)+p(\mathcal{M}^{\rho}_{w_0}f_2) = q(f_1)+q(f_2), \end{aligned}$$

thus q satisfies the triangle inequality and hence q is a seminorm on L. □

So, for any collection p_α of seminorms on V, the collection q_α is a collection of seminorms on L generating a topology.

The following proposition follows directly from the above definition of seminorm.

Proposition 4.1.4 *The contravariant transform $\mathcal{M}^{\rho}_{w_0} : L \to V$ is a continuous map in the topology on L defined by transported seminorms.*

Proof. Let $f_n \to f$ on L. Then by the above definition of seminorm we have:

$$p_\alpha(\mathcal{M}^{\rho}_{w_0}f_n - \mathcal{M}^{\rho}_{w_0}f) = p_\alpha(\mathcal{M}^{\rho}_{w_0}(f_n-f)) = q_\alpha(f_n-f) \to 0.$$

So, $\mathcal{M}^{\rho}_{w_0}$ is continuous on L. □

By Theorem 1.2.14, for any seminorm p on V, there exist a collection $q_1, q_2, \ldots, q_n$ of seminorms on L and $C > 0$ such that for all $f \in L$,

$$p(\mathcal{M}^{\rho}_{w_0}f) \leq C(q_1(f)+q_2(f)+\ldots+q_n(f)),$$

thus $\mathcal{M}^{\rho}_{w_0}$ is bounded.

Proposition 4.1.5 *The contravariant transform intertwines the representation ρ and the left regular representation Λ:*

$$\mathcal{M}^{\rho}_{w_0}\Lambda(g) = \rho(g)\mathcal{M}^{\rho}_{w_0}.$$

Proof. Let $g \in G$ and $f \in L$, then by the left invariance of the pairing we have

$$\mathcal{M}^{\rho}_{w_0}\Lambda(g)f = \langle \Lambda(g)f, w\rangle = \langle f, \Lambda(g^{-1})w\rangle,$$

since,

$$\Lambda(g^{-1})w(h) = w(gh) = \rho(g)\rho(h)w_0 = \rho(g)w(h),$$

then, by the linearity of the pairing we have:

$$\mathcal{M}^{\rho}_{w_0}\Lambda(g)f = \langle f, \rho(g)w\rangle = \rho(g)\langle f, w\rangle = \rho(g)\mathcal{M}^{\rho}_{w_0}f.$$

□

The following is an immediate consequence of the previous proposition.

Corollary 4.1.6 *The image space $\mathcal{M}^{\rho}_{w_0}(L)$ is invariant under the representation ρ.*

Proof. Let $u \in \mathcal{M}^{\rho}_{w_0}(L)$, then there exist $f \in L$ such that $u = \mathcal{M}^{\rho}_{w_0}f$. Let $g, h \in G$, then

$$[\rho(g)u](h) = [\rho(g)\mathcal{M}^{\rho}_{w_0}f](h) = [\mathcal{M}^{\rho}_{w_0}(\Lambda(g)f)](h).$$

Since L is left invariant space, then $\Lambda(g)f \in L$ and hence $\mathcal{M}^{\rho}_{w_0}(\Lambda(g)f) \in \mathcal{M}^{\rho}_{w_0}(L)$. Therefore, $\rho(g)$ maps $u \in \mathcal{M}^{\rho}_{w_0}(L)$ into $\rho(g)u \in \mathcal{M}^{\rho}_{w_0}(L)$ and hence $\mathcal{M}^{\rho}_{w_0}(L)$ is invariant under the representation ρ. □

The following is an another consequence of the Proposition 4.1.5.

Corollary 4.1.7 *The subspace $ker(\mathcal{M}^{\rho}_{w_0})$ of L is invariant under the left regular representation Λ.*

Proof. Let $f \in ker(\mathcal{M}^{\rho}_{w_0})$, then $\mathcal{M}^{\rho}_{w_0}f = \langle f, w\rangle = 0$. Now, for any $g \in G$, we have:

$$\mathcal{M}^{\rho}_{w_0}\Lambda(g)f = \rho(g)\mathcal{M}^{\rho}_{w_0}f = 0.$$

So, $\Lambda(g)f \in ker(\mathcal{M}^{\rho}_{w_0})$, for all $g \in G$ and hence $ker(\mathcal{M}^{\rho}_{w_0})$ is Λ-invariant. □

Now, we list some examples of contravariant transform.

Example 4.1.8 *Let G be a group with a Haar measure $d\mu$ and ρ be the representation of G on Hilbert space V. Then the usual invariant pairing defined on $L^2(G) \times L^2(G)$ is Haar pairing (3.14)*

$$\langle f_1, f_2\rangle = \int_G f_1(g)\, f_2(g)\, d\mu(g), \quad f_1, f_2 \in L^2(G).$$

To extend the pairing in its second component to the function $w(g) = \rho(g)w_0$, we need to consider a square integrable representation ρ and a fixed vector $w_0 \in V$ such that

$$\int_G |(w_0, \rho(g)w_0)|^2\, d\mu(g) < \infty,$$

which is the condition of admissibility. Then the contravariant transform is:

$$\mathcal{M}_{w_0} f = \int_G f(g)\, w(g)\, d\mu(g) = \int_G f(g)\, \rho(g)w_0\, d\mu(g). \tag{4.2}$$

The integral (4.2) is understood in the weak sense and its value belongs to the algebraic dual of V^*. If this element is continuous on V^* then it is an element of the second dual V^{**}.

In the general case we can estimate:

$$\begin{aligned} |\langle \mathcal{M}_{w_0} f, F\rangle| &= \left|\langle \int_G f(g)\, w(g)\, d\mu(g), F\rangle\right| \\ &= \left|\int_G f(g)\langle w(g), F\rangle\, d\mu(g)\right| \\ &\le \int_G |f(g)|\, |\langle w(g), F\rangle|\, d\mu(g) \\ &\le \|f\|_2\, \|W_F\|_2, \quad \textit{where} \quad W_F(g) = \langle w(g), F\rangle. \end{aligned}$$

We do not put general conditions which guarantees $\|W_F\|_2$ is bounded and will verify it on the case-by-case basis. Furthermore, for a reflexive space V, in particular if V is a Hilbert space, we have $V = V^{**}$ and thus $\mathcal{M}_{w_0} f$ belongs to V.

The following example express the boundary value of Cauchy integral.

Example 4.1.9 *Let G be the $ax+b$ group with measure $d\mu(a,b) = \frac{db}{a}$ and ρ_p be its representation (1.6) on $L^p(\mathbb{R})$, $1<p<\infty$. Then Hardy pairing is defined in (3.15) by*

$$\langle f_1, f_2\rangle_H = \lim_{a\to 0}\int_{-\infty}^{\infty} f_1(a,b)\, f_2(a,b)\,\frac{db}{a}, \quad f_1 \in \widetilde{H}^p(\mathbb{R}^2_+), f_2 \in \widetilde{H}^q(\mathbb{R}^2_+),$$

where $\widetilde{H}^p(\mathbb{R}^2_+)$ is given in Definition 3.4.4. In this case we can choose function which is not admissible like $w_0 = \frac{1}{\pi i}\frac{1}{x+i} \in L^p(\mathbb{R})$. So, the contravariant transform is:

$$[\mathcal{M}_{w_0} f](x) = \frac{1}{\pi i}\lim_{a\to 0} a^{-\frac{1}{p}} \int_{-\infty}^{\infty} \frac{f(a,b)}{b-(x+ia)}\, db. \tag{4.3}$$

The contravariant transform (4.3) is the boundary value of the Cauchy integral as $a \to 0$.

Now, we give an example of non-linear contravariant transform.

Example 4.1.10 *Let G be the $ax+b$ group with representation ρ_∞. Consider the following non-linear invariant pairing which is the L^∞-version of the Haar pairing:*

$$\langle f_1, f_2\rangle_\infty = \sup_{g\in G} |f_1(g)\cdot f_2(g)|.$$

Consider the following two functions on $\mathbb{R}$:

$$w_0^+(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0, \end{cases} \quad \textit{and} \quad w_0^*(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1. \end{cases} \tag{4.4}$$

The contravariant transforms related to the previous mother wavelets are:

$$[\mathcal{M}_{w_0^+} f](x) = \langle f, w\rangle_\infty = \sup_{(a,b)\in G} |f(a,b)\,\rho_\infty(a,b) w_0^+(x)| = \sup_{a\in\mathbb{R}_+} |f(a,x)|, \tag{4.5}$$

$$[\mathcal{M}_{w_0^*} f](x) = \langle f, w\rangle_\infty = \sup_{(a,b)\in G} |f(a,b)\,\rho_\infty(a,b) w_0^*(x)| = \sup_{a>|b-x|} |f(a,x)|. \tag{4.6}$$

The last two transforms are the vertical and non-tangential maximal functions respectively [15].

4.2 Composing the covariant and the contravariant transforms

In this section we gather the common properties of the covariant transform $\mathcal{W}_F$ (3.1) and the contravariant transform $\mathcal{M}_w$ (4.1) to deduce some properties for the composition $\mathcal{M}_w \circ \mathcal{W}_F$.

Proposition 4.2.1 *The composition $\mathcal{M}_w \circ \mathcal{W}_F : V \to V$ of a contravariant $\mathcal{M}_w$ and covariant $\mathcal{W}_F$ transforms intertwines the representation ρ with itself.*

Proof. The proof follows immediately from Proposition 3.1.2 and Proposition 4.1.5. Let $v \in V$ and $g \in G$, then

$$\begin{aligned}[]
[\mathcal{M}_w \circ \mathcal{W}_F](\rho(g)v) &= \mathcal{M}_w(\mathcal{W}_F \rho(g)v) \\
&= \mathcal{M}_w(\Lambda(g)\mathcal{W}_F v) \\
&= \rho(g)\mathcal{M}_w(\mathcal{W}_F v) = \rho(g)[[\mathcal{M}_w \circ \mathcal{W}_F]v,
\end{aligned}$$

the proof is completed. □

The ρ-invariance of the space $[\mathcal{M}_w \circ \mathcal{W}_F](V)$ is an immediate consequence of the previous proposition.

Corollary 4.2.2 *The image space $[\mathcal{M}_w \circ \mathcal{W}_F](V)$ is invariant under the representation ρ.*

Proof. Let $u \in [\mathcal{M}_w \circ \mathcal{W}_F](V)$, then there is $v \in V$ such that $u = [\mathcal{M}_w \circ \mathcal{W}_F](v)$. Then for any $g \in G$,

$$\rho(g)u = \rho(g)[\mathcal{M}_w \circ \mathcal{W}_F](v) = [\mathcal{M}_w \circ \mathcal{W}_F](\rho(g)v).$$

Since $\rho(g)v \in V$, then $\rho(g)u \in [\mathcal{M}_w \circ \mathcal{W}_F](V)$ and hence $[\mathcal{M}_w \circ \mathcal{W}_F](V)$ is invariant under ρ. $\square$

Similarly, let $\mathcal{M}_w$ be the contravariant transform (4.1) and $\mathcal{W}_F$ be the covariant transform (3.1) where F is an operator from V to L. Then the composition $\mathcal{W}_F \circ \mathcal{M}_w : L \to L$ commutes with the left regular representation Λ.

Proposition 4.2.3 *The composition $\mathcal{W}_F \circ \mathcal{M}_w : L \to L$ intertwines the left regular representation Λ with itself.*

Proof. Again the proof follows immediately from the intertwining property of covariant and contravariant transforms. Let $f \in L$ and $g \in G$, then

$$\begin{aligned}\Lambda(g)[\mathcal{W}_F \circ \mathcal{M}_w](f) &= \Lambda(g)\mathcal{W}_F(\mathcal{M}_w f)\\ &= \mathcal{W}_F(\rho(g)\mathcal{M}_w f)\\ &= \mathcal{W}_F(\mathcal{M}_w(\Lambda(g)f)) = [\mathcal{W}_F \circ \mathcal{M}_w]\Lambda(g)f,\end{aligned}$$

so, $\mathcal{W}_F \circ \mathcal{M}_w$ commutes with the representation Λ. $\square$

Again, we obtain the following as an immediate consequence of the previous proposition.

Corollary 4.2.4 *The image space $[\mathcal{W}_F \circ \mathcal{M}_w](L)$ is invariant under the left regular representation Λ.*

Proof. Let $u \in [\mathcal{W}_F \circ \mathcal{M}_w](L)$, then there is $f \in L$ such that $u = [\mathcal{W}_F \circ \mathcal{M}_w](f)$. Then for any $g \in G$, we have

$$\Lambda(g)u = \Lambda(g)[\mathcal{W}_F \circ \mathcal{M}_w](f) = [\mathcal{W}_F \circ \mathcal{M}_w](\Lambda(g)f).$$

Since the space L is left invariant, then $\Lambda(g)f \in L$ and hence $\Lambda(g)u \in [\mathcal{W}_F \circ \mathcal{M}_w](L)$, for any $g \in G$. Therefore, $[\mathcal{W}_F \circ \mathcal{M}_w](L)$ is Λ-invariant. □

In what follows, some well known transforms are introduced as examples of composition of covariant and contravariant transforms.

Example 4.2.5 *The composition of the contravariant $\mathcal{M}_{w_0^*}$ (4.6) with the covariant transform (2.4):*

$$[\mathcal{W}_F^\infty f](a,b) = \frac{1}{2a}\int_{b-a}^{b+a} |f(x)|\,dx,$$

is

$$\begin{aligned}[\mathcal{M}_{w_0^*} \circ \mathcal{W}_F^\infty f](x) &= \sup_{a>|b-x|} \frac{1}{2a}\int_{b-a}^{b+a} |f(x)|\,dx,\\ &= \sup_{b_1<x<b_2} \frac{1}{b_2-b_1}\int_{b_1}^{b_2} |f(x)|\,dx.\end{aligned}$$

The supremum is over all the points $(a,b) \in G$ such that (a,b) is visible from x with angle $\alpha = 90°$. In this case $\mathcal{M}_{w_0^*} \circ \mathcal{W}_F^\infty f$ is the Hardy-Littlewood maximal function M_f [19]. By Proposition 4.2.1, we deduce that the maximal function commutes with ρ_∞.

Now, let $X_\lambda = \{x : [\mathcal{M}_{w_0^*} \circ \mathcal{W}_F^\infty f](x) > \lambda\}$, so for any $x \in X_\lambda$ we have

$$\sup_{a>|b-x|} \frac{1}{2a}\int_{b-a}^{b+a} |f(x)|\,dx > \lambda.$$

Therefore, there is $(a_x, b_x) \in G$ such that

$$\int_{b_x-a_x}^{b_x+a_x} |f(x)|\,dx > 2a_x\,\lambda. \tag{4.7}$$

Now, consider the set $X'_\lambda = \{(a_x,b_x) : x \in X_\lambda\}$, then it is clear that

$$X_\lambda \subseteq \mathcal{C}_{X'_\lambda}$$

and hence

$$\mu(X_\lambda) \le \mu(\mathcal{C}_{X'_\lambda}) = h(X'_\lambda).$$

By the Lemma 2.3.4 there exists a sparse subset Y *of* X'_λ *such that* $h(X'_\lambda) \leq 10\, v(Y)$. *So,*

$$\begin{aligned}\mu(X_\lambda) \leq h(X'_\lambda) &\leq 10\, v(Y)\\ &< \frac{5}{\lambda}\int_Y |f(x)|\,dx \quad \textit{by } (4.7)\\ &\leq \frac{5}{\lambda}\|f\|_{L^1}.\end{aligned}$$

which is the weak type inequality [26].

Example 4.2.6 *Consider the representation* ρ *of the* $ax+b$ *group on* $L^2(\mathbb{R})$. *Let* $v_0 \in L^2(\mathbb{R})$ *be an admissible mother wavelet. Define the functional* F *on* $L^2(\mathbb{R})$ *by:*

$$F(v) = \langle v, v_0\rangle,$$

then the covariant transform (3.1) is:

$$[\mathcal{W}_F v](a,b) = \langle \rho(a,b)^{-1} v, v_0\rangle = \langle v, \rho(a,b) v_0\rangle. \tag{4.8}$$

Consider the function w_0^+ *(4.4) and the left invariant pairing (3.17), then the contravariant transform (4.1) is:*

$$\begin{aligned}[\mathcal{M}_{w_0^+} f](t) = \langle f, w\rangle &= \overline{\lim_{a\to 0}} \sup_{b\in\mathbb{R}} f(a,b)\cdot[\rho_\infty(a,b) w_0^+](t)\\ &= \overline{\lim_{a\to 0}} f(a,t).\end{aligned} \tag{4.9}$$

Then the composition of the covariant transform (4.8) and the contravariant transform (4.9) is:

$$\begin{aligned}[\mathcal{M}_{w_0^+} \circ \mathcal{W}_F v](t) &= \overline{\lim_{a\to 0}} [\mathcal{W}_F v](a,t)\\ &= \overline{\lim_{a\to 0}} \sqrt{a}\int_{-\infty}^{\infty} v(ax+t)\,\overline{v_0(x)}\,dx\end{aligned}$$

Since $\int_{-\infty}^{\infty} v(ax+t)\,\overline{v_0(x)}\,dx < \infty$, *then the limit tends to* 0 *and hence*

$$[\mathcal{M}_{w_0^+} \circ \mathcal{W}_F v](t) = 0.$$

Example 4.2.7 *Consider the representation ρ_∞ (1.6) of the $ax+b$ group on $C(\mathbb{R})$. Consider the Dirac delta function $\delta(x)$ and define the following fiducial operator F_δ on $C(\mathbb{R})$:*

$$F_\delta(f) = \int_{-\infty}^{\infty} f(x)\,\delta(x) = f(0).$$

The covariant transform (3.1) will be:

$$\begin{aligned}[\mathcal{W}_\delta f](a,b) &= \int_{-\infty}^{\infty} [\rho_\infty(a,b)^{-1} f](x)\,\delta(x)\,dx \\ &= \int_{-\infty}^{\infty} f(ax+b)\,\delta(x)\,dx \\ &= f(b)\end{aligned}$$

Now, consider the following function:

$$w_0(x) = \frac{1}{\pi}\,\frac{1-\mathcal{X}_{[-1,1]}(x)}{x}.$$

So,

$$[w(a,b)](x) = [\rho_\infty(a,b)w_0](x) = w_0\left(\frac{x-b}{a}\right) = \frac{a}{\pi}\,\frac{1-\mathcal{X}_{[-a,a]}(x-b)}{x-b}.$$

Now, consider the contravariant transform $\mathcal{M}_{w_0}$ defined by the Hardy pairing (3.15):

$$[\mathcal{M}_{w_0} f](x) = \lim_{a\to 0}\int_{-\infty}^{\infty} f(a,b)\,[\rho_\infty(a,b)w_0](x)\,\frac{db}{a}.$$

So, the composition of covariant and contravariant transforms is:

$$\begin{aligned}[\mathcal{M}_{w_0}\circ\mathcal{W}_\delta f](x) &= \lim_{a\to 0}\int_{-\infty}^{\infty} [\mathcal{W}_\delta f](a,b)\,[\rho_\infty(a,b)w_0](x)\,\frac{db}{a} \\ &= \frac{1}{\pi}\lim_{a\to 0}\int_{-\infty}^{\infty} f(b)\cdot\frac{1-\mathcal{X}_{[-a,a]}(x-b)}{x-b}\,db \\ &= \frac{1}{\pi}\lim_{a\to 0}\int_{|x-b|>a}\frac{f(b)}{x-b},\end{aligned}$$

which is the Hilbert transform [34].

4.3 Further Plans

The main group considered in this thesis is the $ax + b$ group. A possible extension can be achieved by considering more general group. For example, the relation between covariant transform and the semidirect product of Heisenberg group and its representation can be studied.

The properties of covariant transform on locally convex spaces was discussed in this thesis. However, there are some interesting examples of Topological vector spaces which are not locally convex spaces. So, the properties of covariant transform on non-locally convex spaces like boundedness and continuity can be discussed as well.

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