Infrared problem in the Faddeev–Popov sector in Yang–Mills Theory and Perturbative Gravity

Jos Gibbons

PhD

University of York

Mathematics

December 2015

Abstract

In a broad class of spacetimes including de Sitter space, the Faddeev–Popov ghost propagator is infrared-divergent in both BRST-quantised Yang–Mills theory and BRST-quantised perturbative gravity. Introducing a mass term for infrared regularisation, one may delete an infrared-divergent term from the propagator before taking the massless limit. This obtains an effective zero-mode sector Feynman propagator that is infrared-convergent and exhibits appropriate spacetime symmetries, such as de Sitter invariance in de Sitter space and time translation invariance on a flat static torus. This prescription, which dates to 2008, relies on free integration by parts (a term explained on page 11), so its generality to a broad class of spacetimes is limited. A further difficulty is that this prescription introduces a mass term in the action that breaks the theory's Becchi–Rouet–Stora–Tyutin invariance and anti-Becchi–Rouet–Stora–Tyutin invariance.

This thesis presents an alternative prescription in which it is shown that the modes responsible for the Faddeev–Popov ghost propagator's infrared divergence are cyclic in the Lagrangian formalism. These modes can then be obviated from the Lagrangian, Hamiltonian and Schrödinger wave functional formalisms. Neither of the aforementioned difficulties with the old prescription apply to the newer one discussed herein, which manifestly preserves both internal symmetries throughout. The prescriptions have equivalent perturbation theories in spacetimes in which free integration by parts is possible. The new prescription can then be regarded as a generalisation of the 2008 prescription to a broader class of spacetimes.

List of Contents

Acknowledgements

I would like to thank my supervisor A. Higuchi for all the useful discussions we have had during my time as his PhD student.

Author's declaration

I declare that this work on the obviation of zero modes of Faddeev–Popov (anti)ghosts is original. (The treatment of Yang–Mills theory has also been published in Ref. [1].) It has been done in close collaboration with my supervisor. It has not been submitted for any other degree or diploma before.

The three paragraphs below summarise which contributions to our joint research were mine. They are organised by placement in this thesis. The notation used below shall be defined later.

In Part I, I: generalised the treatment from metrics with factorisable γ_{ij} to metrics with factorisable $\det g_{\mu\nu}$; obtained the rules for rearranging products of the form $V \cdot (W \times X)$ with bosonic and/or fermionic fields[1]; noted $\left[\overline{Q},\, S\right] \propto 2k - 1$ for perturbative gravity; and proved Upadhyay's anti-BRST charge does not commute with S or anticommute with Q.

In Part II, I: provided the general definitions of zero modes, including in the momentum sector; proved $L_{\text{FP}}^{(00)} + L_{\text{FP}}^{(0+)} = 0$ for Yang–Mills theory and perturbative gravity; computed $\left\langle 0 \right| \left[A_{0(0)}\right]^2 \left| 0 \right\rangle$ for $n \in \{2,\, 3\}$, developed a succinct notation for perturbative gravity, especially for conservation laws, realised the importance of M_{AB}, which I also defined; defined the η_μ^A to facilitate the definition of the zero modes of canonical vector fields; computed perturbative gravity's L^{extra}; and computed perturbative gravity's Hamiltonian in the $\widetilde{B}^\mu$, c^μ, $\overline{c}^\mu$ formalism.

In Part III, I: explained, in Sec. 5.2, why the CMP readily induces extra terms; did the calculations for Appendices B, C.3, D and E alone; and made several corrections to earlier work on the material presented in Sec. C.2.

All sources acknowledged as references.

[1] These rules depend on exchange (anti)symmetries, i.e. the parity of Grassmann number grading. It is this, rather than spin, that defines the boson-fermion distinction herein, including for unphysical fields that violate the spin-statistics relation.

Part I Motivations

Chapter 1 A problem in scalar field theory

1.1 Overview of this thesis

In Sec. 1.1.1, I motivate the research described in this thesis. In Sec. 1.1.2, I summarise the content of the rest of this thesis. In Sec. 1.1.3, I conclude this section with a note on conventions I have adopted in this thesis for readability.

1.1.1 Infrared problems

This thesis is concerned with several related problems in quantum field theories in curved spacetimes, herein called *zero mode problems*. These are problems of *infrared divergences*, which afflict (for example) minimally coupled massless scalar fields in spacetimes specified below in Sec. 1.2. If the relevant fields are initially allowed an arbitrary mass, then associated quantities called *propagators* depend on those fields' masses. Further, depending on the field normalisation, the propagators either diverge in the massless limit or lose desirable spacetime symmetries. However, these fields are in fact massless. Therefore the infrared (IR) behaviour of the propagators causes the quantum field theory to lose one or more desirable symmetries. Addressing this is the motivation of the present thesis and a number of earlier works.

This thesis contrasts two different prescriptions for addressing these problems. These prescriptions are not equally recent. The less recent of the two prescriptions for addressing zero mode problems is attributable to Mir Faizal and Atsushi Higuchi in 2008 [2]. At this time, Atsushi Higuchi was supervising Mir Faizal's doctoral studies. I am Atsushi Higuchi's current PhD student and, like Mir Faizal, I have collaborated with Atsushi Higuchi on research considering the zero mode problems in BRST-quantised Yang–Mills theory and BRST-quantised perturbative gravity. The first published discussion of the younger of the two prescriptions for addressing zero mode problems is in a paper Atsushi Higuchi and I co-authored [1] in 2014. This paper presented the more recent prescription's treatment of the zero mode problem in BRST-quantised Yang–Mills theory. In this thesis, I present this prescription in more detail, and do not limit its scope to BRST-quantised Yang–Mills theory's zero mode problem.[2]

I discuss three zero mode problems. The first zero mode problem is not of historical interest, but is presented to illustrate several key concepts of my later arguments. It occurs in a toy model, namely

[2]Dr Higuchi and I are still drafting two further co-authored papers. One will provide a treatment of perturbative gravity analogous to Ref. [1]. The other will be a comment paper, covering the same material I discuss below in Sec. 2.6.4.

the theory of a minimally coupled scalar field. The field admits a decomposition into modes, and the IR divergence of the propagator is due to the field's spatially uniform mode, which is called its *zero mode* (hence the name "zero mode problem"). In this chapter, I will explain the toy model's zero mode problem and how each prescription addresses that problem. The less recent of the two prescriptions begins by giving the scalar field a mass. The prescription then deletes an IR-divergent term from the propagator, and takes the massless limit of the remaining terms to obtain an effective zero-mode sector propagator. I will therefore call this prescription the *fictitious mass prescription* or FMP. The more recent prescription verifies that, in the massless case, the zero mode is also *cyclic* (i.e. it does not appear undifferentiated in the theory's Lagrangian). This fact is integral to the more recent prescription's treatment of the zero mode problem that I discuss in this chapter. I therefore call this prescription the *cyclic modes prescription* or CMP.

The first zero mode problem and its treatment are presented in this chapter, serving as a preliminary for my later account of the other two zero mode problems. These problems occur in BRST-quantised Yang–Mills theory and BRST-quantised perturbative gravity, theories that are integral to modern physics. These theories and their zero mode problems will be discussed in Chapter 2 in the context of a literature review. As with the toy model's zero mode problem, the two prescriptions I present are an FMP and a CMP.

The reader may wonder why, if the FMP addressed these zero mode problems in 2008, it was nonetheless necessary to treat them with the CMP. The reason is that the FMP suffers from two difficulties that are absent in the CMP's treatment of zero mode problems. One difficulty is that, although the FMP's modification of propagators preserves spacetime symmetries, it adds a mass term to the Lagrangian that breaks internal symmetries. By contrast, the CMP manifestly preserves these internal symmetries throughout. The other is that, unlike the CMP, the FMP requires free integration by parts[3]; the CMP may be used in a broader class of spacetimes (see Sec. 1.2).[4] These difficulties are irrelevant to the zero mode problem considered in this chapter, but will be discussed in more detail later.

The main purposes of this thesis are to present the CMP, to clarify its advantages over the FMP (which were briefly summarised in the above paragraph), and show that the two prescriptions have equivalent perturbation-theoretic descriptions in spacetimes for which free integration by parts is possible. The CMP can therefore be seen as a generalisation of the FMP.

1.1.2 Structure of the rest of this thesis

Historically, maximally symmetric spacetimes have been of particular interest as toy spacetimes in both cosmology and quantum field theory. Examples include conformally Minkowski space, global de Sitter space and global anti de Sitter space. Global de Sitter space is notable for its accelerating expansion. The early cosmological inflation of spacetime is therefore an example of approximate

[3]Throughout this thesis, surface terms are assumed to vanish in integration by parts. The conditions for this depend on the integrands involved in the calculation, but also on the spacetime's geometry. The FMP's requirements are slightly stronger than those of the CMP, because its use of integration by parts is more extensive. In particular, multiple expressions for the perturbation theory's interaction Lagrangians must be equal despite surface terms. A review in Sec. 3.7 of the FMP's perturbation theory in Yang–Mills theory makes use of these requirements in Eqs. (3.7.18), (3.7.21) and (3.7.22). This definition of free integration by parts is repeated on page 164. By contrast, the CMP's only use of integration by parts is in non-perturbatively deriving equations of motion and conservation laws in the Lagrangian and/or Hamiltonian formalism.

[4]This concern is less pressing than the symmetries issue, however, as quantum field theory is more problematic, and of less interest, in spacetimes for which free integration by parts is unavailable.

de Sitter behaviour. Today, the universe's expansion is accelerating; a de Sitter approximation is applicable here too. While de Sitter space suffers from all three of the zero mode problems discussed in this thesis, it is but one example of a broad class of spacetimes to which all three zero mode problems are applicable. This class of spacetimes is specified in Sec. 1.2[5], and includes many *Friedmann–Lemaître–Robertson–Walker metrics* (see Sec. 1.2.1 below) in cosmology: specifically, those which are globally hyperbolic and have closed spatial sections. However, de Sitter space remains an example of especial interest, and at times this special case will warrant a discussion in detail. In Sec. 1.2.2, I motivate an especial interest in de Sitter space and discuss evidence for some of its applications, such as early cosmological inflation.

In Sec. 1.3, I review some of the theory on which the FMP and CMP rely. The infrared problems discussed in this thesis are a consequence of the behaviour of solutions of equations of motion. I must therefore give a brief overview of Lagrangian and Hamiltonian mechanics. Familiar results will be expressed in a non-standard notation that will be useful throughout this thesis.

The next step in explaining the first zero mode problem I discuss is reviewing a normalisation condition that is imposed on scalar fields, called *Klein–Gordon normalisation*. I explain the details of Klein–Gordon normalisation in Sec. 1.4. Klein–Gordon normalisation is integral to the IR behaviour of the scalar field's propagator. The IR behaviour of the propagator for a flat static torus is discussed in Sec. 1.5, while the analogous calculation for more general spacetimes is discussed in Sec. 1.6. Sec. 1.5 also contrasts two uses of the term "zero mode" to clarify the terminology of this thesis.

In Sec. 1.7, I discuss the FMP and CMP in more detail.

In Sec. 1.8, I discuss two spacetimes in more detail: the flat static torus and de Sitter space. Interest in the flat static torus originates from the fact that it simplifies statements of the zero mode problems and both prescriptions discussed herein for addressing them.

In Chapter 2, I explain the zero mode problems that concern the rest of this thesis. The primary purposes of Part I (Chapters 1 and 2) are to specify the problems of interest later in this thesis and to briefly introduce the motivation of the CMP.

Applying the CMP to BRST-quantised Yang–Mills theory and BRST-quantised perturbative gravity requires a lot of detail. These treatments are respectively provided in Chapters 3 and 4, which comprise Part II of this thesis. Each of these chapters show that the field modes responsible for a propagator's IR-divergence are cyclic and hence dynamically irrelevant. Next each chapter shows that the CMP adds new terms to the Lagrangian and Hamiltonian, and that these can be shown to imply that the CMP and FMP have equivalent perturbation-theoretic formulations in spacetimes for which free integration by parts is possible. My account of the gravity case closely parallels my account of the Yang–Mills case.

Part III contains my conclusions (in Chapter 5), appendices and other back matter. The appendices' functions are summarised in Chapter 5, although each appendix's function is also specified where appropriate in Chapters 1–4.

[5]The first number refers to the chapter; the second number refers to the section. I use the abbreviation Sec. whenever I refer to a section or subsection.

1.1.3 Writing conventions

I work throughout in units such that $c = \mu_0 = \hbar = 1$.[6] The "mass" of a field of rest mass M_0 may then be taken as $M := \frac{M_0 c}{\hbar}$, which has the units of inverse length.
I use Greek spacetime indices; for exclusively spatial indices, I use lower case Roman letters beginning at i. Another common convention is to use abstract indices, denoted with lower case Roman letters beginning at a. I use such letters for Lie algebra indices (except for the Lie algebra of Killing vector fields, for which I capitalise the Roman indices).

There are times when text that is inappropriate for footnotes or appendices must nonetheless be distinguished from the text body, to indicate that the text following it follows on from the text before it. Ref. [3] indicates this by using small text. I have instead chosen to use a different text colour. Heretofore I have used black text. By contrast, the text in this sentence is grey. It is my hope that this writing style will allow the reader to follow the structure[7], and crux, of my argument more closely. I have also taken one other measure to help this. Sections typically open with summaries of the sequence of their narratives. The stages of such narratives each occupy a subsection. The reader can satisfy themselves that this section of the chapter relies on this technique. A similar explanation of the roles of sections in a chapter will also be typical.

1.2 Spacetimes of interest

In Sec. 1.2.1, I specify the spacetimes that are considered in this thesis. One example is de Sitter space, but this case admits a number of generalisations that are straightforward throughout my analysis. This permits a rich class of spacetime metrics to be considered hereafter. I motivate an interest in de Sitter space in Sec. 1.2.2, which concludes with a one-sentence summary of the conditions demanded of spacetimes of interest in Sec. 1.2.1.
Throughout this section I make use of various results, definitions and notational conventions for general relativity, viz. Ref. [4].

1.2.1 Global hyperbolicity and compact spatial sections

Let n denote the dimension of the spacetime manifold. This thesis hereafter assumes one time dimension and at least one space dimension, so $n \geq 2$ and the dimension of spatial sections is $n - 1$. In a given coordinate system, the time coordinate may be written as x^0 or t, while space coordinates are denoted x^i and collected into a vector $\mathbf{x} \in \mathbb{R}^{n-1}$. An event in the manifold may be written as $x = (t, \mathbf{x})$ where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^{n-1}$ and $x^0 = t$.

[6] However, occasionally SI units will be used. I also normalise G so that the Einstein–Hilbert Lagrangian density is $\sqrt{|g|}\,(R - 2\Lambda)$; for example, in a spacetime of 1 time dimension and 3 space dimensions, the normalisation convention is $16\pi G = 1$. I discuss this in more detail in Sec. 2.6.1.

[7] The first example of my using grey text in this way will be a discussion of one of the assumptions of Friedmann–Lemaître–Robertson–Walker (FLRW) metrics.

One difference between grey text environments and footnotes is seen in their relation to grammar. If my grey text were rendered black, the only information that would be lost would be information regarding the relation between the information contents of black and grey regions of text. The grammatical structures of black and grey text are therefore identical and unrelated. By contrast, this footnote exemplifies a typical peculiarity of the relation between the grammar of a sentence and the positioning of a footnote within it. Indeed, a reader who read the sentence in which this footnote appeared, with the footnote interrupting it, would not correctly follow the grammar of said sentence.

Each spacetime considered is *globally hyperbolic*[8], i.e. each has a Cauchy surface that can be time-translated to generate the entire spacetime. I work hereafter in the timelike $(+-\cdots)$ convention. The line element of such spacetimes may then be written as

$$\mathrm{d}s^2 = N^2\mathrm{d}t^2 - \gamma_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j\mathrm{d}t\right) \tag{1.2.1}$$

where:

- $\gamma_{ij}(x)$ is a positive-definite invertible $(n-1)\times(n-1)$ real matrix-valued function;
- $N(x) > 0$ is the *lapse function*;
- and $N^i(x)$ is $(n-1)$–dimensional and is called the *shift vector*.

The spacetimes of interest are differentiable manifolds, so $\gamma_{ij},\ N,\ N^i$ are differentiable.
The choice of coordinates for a given globally hyperbolic metric fixes $N,\ N^i$, but a physically irrelevant change in the choice of coordinates can obtain arbitrary $N,\ N^i$. The simplest possible result for $N,\ N^i$ is the *synchronous gauge*

$$N = 1,\ N^i = 0. \tag{1.2.2}$$

However, the treatment of more general gauges is also of mathematical interest, and will be included herein.
The *Friedmann–Lemaître–Robertson–Walker* (FLRW) metrics of cosmology are an important special case. If these are considered in the synchronous gauge, they also impose two further conditions. The first such condition is

$$\gamma_{ij}(x) = a^2(t)\,\eta_{ij}(\mathbf{x}), \tag{1.2.3}$$

where $a > 0$ is called the *scale factor*. Note that γ_{ij} is invariant under the transformation

$$a \to \frac{a}{a_0},\ \eta_{ij} \to a_0^2\eta_{ij} \tag{1.2.4}$$

for any $a_0 \in \mathbb{R}_+$. We say a spacetime satisfying Eq. (1.2.3) is *expanding* if $\dot{a} > 0$ and *accelerating* (*decelerating*) if $\ddot{a}$ is positive (negative). Eq. (1.2.3) implies a more general condition, of the form

$$\sqrt{\gamma(x)} = a^{n-1}(t)\sqrt{\eta(\mathbf{x})} \tag{1.2.5}$$

where $a > 0$ as before, $\gamma := \det\gamma_{ij}$, and in the special case of Eq. (1.2.3) $\eta = \det\eta_{ij}$. Herein I will consider arbitrary spacetimes satisfying Eq. (1.2.5), thereby generalising Eq. (1.2.3).
The second condition which FLRW metrics impose is not assumed hereafter, since it is irrelevant to the statement and treatment of zero mode problems. For completeness, this condition is

$$\eta_{ij}(\mathbf{x})\mathrm{d}x^i\mathrm{d}x^j = \frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2, \tag{1.2.6}$$

where $\mathrm{d}\Omega^2$ is the line element of the $(n-2)$–dimensional unit sphere S^{n-2} and k is a constant for which kr^2 is dimensionless. For example, for $n = 4$ we may write

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 \tag{1.2.7}$$

[8]Globally hyperbolic spacetimes are of interest because they have a well-defined initial value formalism [4, 5]. Such spacetimes include conformally Minkowski space and de Sitter space, but not anti de Sitter space.

where $(r,\, \theta,\, \phi)$ are spherical polar coordinates. An analogous condition exists in some other important spacetimes, with a different value of $\eta_{ij}\mathrm{d}x^i\mathrm{d}x^j$. One example, which lacks the homogeneity in space of the FLRW metric, is the Schwarzschild metric. The Schwarzschild metric near a non-rotating, neutrally charged black hole of mass m satisfies

$$N = \sqrt{1 - \frac{2Gm}{rc^2}},\; N^i = 0,\; a = 1,\; \eta_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{\mathrm{d}r^2}{1 - \frac{2Gm}{rc^2}} + r^2\mathrm{d}\Omega^2. \tag{1.2.8}$$

However, $\gamma_{ij}\,(x)$ will for my purposes be any matrix for which $\det\gamma_{ij}$ is of the form given in Eq. (1.2.5). Even if γ_{ij} had the form in Eq. (1.2.3), the choice of η_{ij} would still be much more general than that used in the FLRW metric condition Eq. (1.2.6).

One last constraint on the spacetimes of interest in this thesis is that the spatial integral

$$V\,(t) := \int \mathrm{d}^{n-1}\mathbf{x}\sqrt{|g\,(t,\,\mathbf{x})|}g^{00}\,(t,\,\mathbf{x}) \tag{1.2.9}$$

(where $g := \det g_{\mu\nu}$), hereafter called the spacetime's *volume factor*, is finite in a suitable choice of coordinates. Throughout this thesis it will be convenient to use the succinct integral operator notations

$$\int_{\mathbf{x}} := \int \mathrm{d}^{n-1}\mathbf{x}\sqrt{|g|},\; \int_x := \int \mathrm{d}t \int_{\mathbf{x}} = \int \mathrm{d}^n x\sqrt{|g|} \tag{1.2.10}$$

so that $V = \int_{\mathbf{x}} g^{00}$. (Ref. [6] has a similar shorthand, defining $\mathrm{d}V_x := \sqrt{|g|}\mathrm{d}^{n-1}\mathbf{x}$ so $\int \mathrm{d}V_x = \int_{\mathbf{x}}$.) Defining $\gamma := \det\gamma_{ij}$, it is well-known that $\sqrt{|g|} = N\sqrt{\gamma}$ and $g^{00} = N^{-2}$, so $V = \int \mathrm{d}^{n-1}\mathbf{x}\frac{\sqrt{\gamma}}{N}$. All spacetimes with compact spatial sections[9] have finite volume factors[10]. An important example is de Sitter space, for which in *global coordinates* the line element may be written as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - H^{-2}\cosh^2 Ht\mathrm{d}\Omega^2, \tag{1.2.11}$$

where $\mathrm{d}\Omega^2$ is the dimensionless line element of the $(n-1)$–dimensional unit sphere S^{n-1} and the H^{-2} coefficient ensures that $\mathrm{d}s^2$ has the correct units.

1.2.2 Why de Sitter space?

The case of de Sitter space may be written as $a \propto \cosh Ht$,[11] where $H > 0$ is a constant for which Ht is dimensionless. This is the maximally symmetric accelerating choice[12], and we expect theories in de Sitter space and their vacua to exhibit a symmetry called *de Sitter invariance*. Empirically, the modern universe is expanding and accelerating [8, 9]. Further, a primordial period of exponential expansion known as cosmological inflation [10, 11, 12, 13, 14] explains the modern universe's flat-

[9]The spatial sections of a spacetime are topological spaces. A topological space is called *compact* if each of its open covers has a finite subcover. However, the only property of spacetimes with compact spatial sections that concerns us herein is that they have finite volume factors. The integration process in Eq. (1.2.9) requires coordinate patches, but a compact spatial section requires only finitely many of these. On each patch, the integral is finite because the integrand is continuous and hence bounded. A finite subcover of a compact spatial section is also obtainable, since a union of open balls covering the spatial section's points provides an open cover. Any finite subcover thereof imposes a finite upper bound on the volume factor.

[10]While this integral I have called the volume factor is convenient to name herein, the more usual definition of *volume* for a spacetime with compact spatial sections is $\int \mathrm{d}^{n-1}\mathbf{x}\sqrt{\gamma}$.

[11]I hereafter simply write $a = \cosh Ht$, using the invariance of γ_{ij} stated in Eq. (1.2.4). This removes the H^{-2} factor in Eq. (1.2.11); H, t have been nondimensionalised.

[12]More precisely, de Sitter space is the maximally symmetric vacuum solution of Einstein's field equations with positive cosmological constant [7]. I define *maximally symmetric spacetimes* in Sec. 2.6.2.

ness, the unobservably low modern concentration of magnetic monopoles, and the modern pattern of CMB anisotropy. (There is also recent tentative empirical evidence for such cosmological inflation. Primordial cosmological inflation's predicted spectrum of perturbations in the universe's primordial density is consistent with WMAP data [15, 16]. Another prediction of primordial cosmological inflation is primordial gravitational radiation. Unfortunately, apparent evidence [17] of this radiation has since been undermined [18].) The primordial and contemporary accelerating periods can both be treated with de Sitter space, if only as a toy model. A further motivation for its study is the proposed dS/CFT correspondence [19], which has much the same use for string theories in de Sitter space as the AdS/CFT correspondence does for string theories in anti de Sitter space.

A lot is known about quantum field theories in de Sitter space. Of critical importance in this thesis is the fact, which I discuss in Sec. 2.5.1, that minimally coupled massless free scalar fields in de Sitter space prevent the occurrence of a de Sitter-invariant Hadamard vacuum state [20]. However, I will usually be able to speak about zero mode problems in a much more general class of spacetimes. In summary, I hereafter assume a globally hyperbolic spacetime with one time dimension and at least one space dimension, which in appropriate coordinates satisfies Eq. (1.2.5) and has finite $V(t)$.

1.3 An overview of Lagrangian and Hamiltonian mechanics

A few facts about the Lagrangian and Hamiltonian formulations of classical field theory[13] need to be reviewed to explain the CMP in Sec. 1.7.2. In particular, I must introduce several non-standard notational conventions that ease the subsequent discussion of every zero mode problem considered in this thesis. (The aim is to avoid a need to calculate with a mixture of tensors and *tensor densities*.) The example of deriving the Klein–Gordon equation from a Lagrangian density will provide some familiarity. Other zero mode problems introduced in Chapter 2 include fermionic fields, which introduce a subtlety unaddressed in this section concerning the distinction between *left and right derivatives*. The partial derivatives in the Lagrangian (Hamiltonian) sector should in general be left (right) derivatives. I discuss this in more detail in Sec. (2.4.2) (Sec. (3.1)).

The notations $\int_{\mathbf{x}}$, $\int_x$ defined in Eq. (1.2.10) will be used throughout for succinctness. The action S, Lagrangian L, Lagrangian density $\mathcal{L}$ and scalar Lagrangian density[14] – which I will denote $\mathcal{L}_0$ – are related by

$$\mathcal{L} = \sqrt{|g|}\mathcal{L}_0,\ L = \int \mathrm{d}^{n-1}\mathbf{x}\mathcal{L} = \int_{\mathbf{x}} \mathcal{L}_0,\ S = \int \mathrm{d}tL = \int_x \mathcal{L}_0. \tag{1.3.1}$$

I will write the scalar $\mathcal{L}_0$ in terms of tensors. In particular, if $T(x)$ is an arbitrary tensor-valued field (or mode thereof), I write tensors (including $\mathcal{L}_0$) as a function of T and its covariant, *not partial*, derivatives. For example, $\nabla_\mu T$ will be considered T-independent for any T. Then

$$\frac{\partial \mathcal{L}}{\partial T} = \sqrt{|g|}\frac{\partial \mathcal{L}_0}{\partial T},\ \frac{\partial \mathcal{L}}{\partial \nabla_\mu T} = \sqrt{|g|}\frac{\partial \mathcal{L}_0}{\partial \nabla_\mu T},\ \cdots. \tag{1.3.2}$$

[13]Just as quantum mechanics is a quantisation of classical mechanics, quantum field theory is a quantisation of classical field theory. Throughout this thesis, I will call fields *classical* if they are not quantised.

[14]In the treatment of curved spacetime, there are multiple conventions regarding the term "Lagrangian density". One applies this term to a scalar quantity. Another convention calls this scalar the *scalar Lagrangian density* (see, e.g. Chapter 6 of Ref. [6]), and defines the Lagrangian density as $\sqrt{|g|}$ times the scalar Lagrangian density. (The $\sqrt{|g|}$ factor in the Lagrangian density is derived in Sec. 6.3 of Ref. [21].) I will use this second convention herein. Then the Lagrangian is the space integral of the Lagrangian density. However, the description of $\mathcal{L}_0$ as a "scalar Lagrangian density" is at risk of being confused with the fact that the Lagrangian density $\mathcal{L}$ is a *scalar density*, as noted in Chapter 2 of Ref. [6], whereas $\mathcal{L}_0$ is a scalar.

A transformation of fields may be written as $T \to T + \delta T$. Suppose such a transformation satisfies $\delta S = 0$. Then a stationary action principle is obtained by a process referred to as *varying* T. Working throughout to first order in δT and covariant derivatives thereof, this stationary action principle can be written as

$$0 = \delta S \approx \int_x \left(\delta T \frac{\partial \mathcal{L}_0}{\partial T} + \nabla_\alpha \delta T \frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha T} + \nabla_\beta \nabla_\alpha \delta T \frac{\partial \mathcal{L}_0}{\partial \nabla_\beta \nabla_\alpha T} + \cdots \right), \tag{1.3.3}$$

where even higher-order derivatives of T appear in the ellipsis $\cdots$. However, none of the Lagrangian densities considered in this thesis depend on third- or higher-order covariant derivatives of any tensor, so this ellipsis can be dispensed with hereafter. (I will only need to include second-order covariant derivatives in discussions in Secs. 4.4 and 4.6.)

Dropping boundary terms, integration by parts gives

$$0 = \int \mathrm{d}^n x \partial_\alpha \left(\sqrt{|g|} V^\alpha \right) = \int_x \nabla_\alpha V^\alpha \tag{1.3.4}$$

for any tensorial vector field V^α, since the identity $\sqrt{|g|}\Gamma^\alpha_{\alpha\beta} = \partial_\beta \sqrt{|g|}$ in *Christoffel symbols* implies $\sqrt{|g|}\nabla_\alpha V^\alpha = \partial_\alpha \left(\sqrt{|g|} V^\alpha \right)$ (see Sec. 3.4 of Ref. [4]). Thus

$$\begin{aligned} \delta S &= \int_x \left(\delta T \frac{\partial \mathcal{L}_0}{\partial T} + \nabla_\alpha \delta T \frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha T} + \nabla_\beta \nabla_\alpha \delta T \frac{\partial \mathcal{L}_0}{\partial \nabla_\beta \nabla_\alpha T} \right) \\ &= \int_x \delta T \left(\frac{\partial \mathcal{L}_0}{\partial T} - \nabla_\alpha \left(\frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha T} - \nabla_\beta \frac{\partial \mathcal{L}_0}{\partial \nabla_\beta \nabla_\alpha T} \right) \right). \end{aligned} \tag{1.3.5}$$

This is an example of a result of the form

$$\delta S = \int \mathrm{d}^n x \delta T \frac{\delta S}{\delta T}, \tag{1.3.6}$$

where $\frac{\delta S}{\delta T}$ is called a *functional derivative*.[15] By the fundamental lemma of the calculus of variations, the *stationary action principle* $\frac{\delta S}{\delta T} = 0$ may be equivalently stated as the *Euler–Lagrange equation* obtained by varying T. This equation may be rearranged as

$$\frac{\partial \mathcal{L}_0}{\partial T} = \nabla_\alpha J^\alpha, \; J^\alpha := \frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha T} - \nabla_\beta \frac{\partial \mathcal{L}_0}{\partial \nabla_\beta \nabla_\alpha T}. \tag{1.3.7}$$

A system is said to be on-shell if and only if all Euler–Lagrange equations are satisfied; results that do not require any Euler–Lagrange equations are said to hold off-shell.

As an example of Eq. (1.3.7), the choice

$$T = \phi^*, \; \mathcal{L}_0 = \nabla_\mu \phi^* \nabla^\mu \phi - \left(M^2 + \xi R \right) \phi^* \phi, \tag{1.3.8}$$

with R the Ricci scalar and ξ is a spacetime-constant dimensionless coupling parameter, gives the Klein–Gordon equation

$$\Box \phi = - \left(M^2 + \xi R \right) \phi \tag{1.3.9}$$

[15] If the calculation were done to all orders in T, the integrand would include additional terms, which by definition do not contribute to $\frac{\delta S}{\delta T}$. This is analogous to the fact that, in a Taylor series of a function, only one term is relevant to defining each of the function's derivatives at the point at which the Taylor series is expanded. Similarly, the aim of the above calculations is to compute $\frac{\delta S}{\delta T}$ in terms of specific choices for the tensor fields.

($\Box := \nabla_\mu \nabla^\mu$ is the *d'Alembert operator*), with $\xi = 0$ in the *minimally coupled* case[16], viz.

$$\Box\phi = -M^2\phi. \tag{1.3.10}$$

The momentum density π_T of T can now be defined as

$$\pi_T(x) := \frac{\delta L}{\delta \partial_0 T}, \tag{1.3.11}$$

which in the above example gives

$$\pi_\phi = \sqrt{|g|}\nabla^0\phi^*. \tag{1.3.12}$$

Next I introduce what I will call *reduced momentum densities*, viz.

$$\varpi_T := \frac{\pi_T}{\sqrt{|g|}} = \frac{1}{\sqrt{|g|}}\frac{\delta L}{\delta \partial_0 T}. \tag{1.3.13}$$

(The symbol ϖ is called by `\varpi` in LaTeX, and is a variant of π obtained by bending π's legs inward and bringing their feet into contact.) In many cases of interest[17]

$$\pi_T = \frac{\partial \mathcal{L}}{\partial \partial_0 T},\ \varpi_T = \frac{\partial \mathcal{L}_0}{\partial \partial_0 T}. \tag{1.3.14}$$

If this is true and, furthermore, second-order derivatives of T are absent from $\mathcal{L}_0$, then

$$\pi_T = \frac{\partial \mathcal{L}}{\partial \nabla_0 T} = \sqrt{|g|}\frac{\partial \mathcal{L}_0}{\partial \nabla_0 T} = \sqrt{|g|}J^0,\ \varpi_T = J^0. \tag{1.3.15}$$

In this case the reduced momentum densities are true tensors, unlike the usual momentum densities π_T. Further, the Euler–Lagrange equation may then be written as

$$\frac{\partial \mathcal{L}_0}{\partial T} = \nabla_0 \varpi_T + \nabla_i J^i. \tag{1.3.16}$$

Next I define

$$\Pi_T(t) := \int \mathrm{d}^{n-1}\mathbf{x}\,\pi_T(x) = \int_{\mathbf{x}} \varpi_T(x). \tag{1.3.17}$$

Since $S = \int \mathrm{d}tL$, the stationary action principle can be expressed another way, viz.

$$0 = \int \mathrm{d}t \left(\delta T\frac{\partial L}{\partial T} + \nabla_\mu \delta T \frac{\partial L}{\partial \nabla_\mu T}\right) = \int \mathrm{d}t \left(\delta T\left(\frac{\partial L}{\partial T} - \nabla_0 \Pi_T\right) + \nabla_i \delta T \frac{\partial L}{\partial \nabla_i T}\right) \tag{1.3.18}$$

(again, assuming the absence of second-order covariant derivatives). If a variation δT is spatially uniform and scalar-valued, $\nabla_i \delta T = 0$ so $\frac{\partial L}{\partial T} = \dot{\Pi}_T$ on-shell. For arbitrary T, the Euler–Lagrange equation Eq. (1.3.7) gives a conserved current if $\mathcal{L}_0$ is not explicitly T-dependent, in which case T is said to be *cyclic*. If δT is a spatially uniform scalar function (which can be taken as a variation only of spatially uniform scalar modes of T), δL is independent of the undifferentiated δT if and only if Π_T

[16]Some spacetimes have constant R (including two spacetimes I comment on in this thesis, the flat static torus and de Sitter space). For these, a mass–M solution of $\left(\Box + M^2 + \xi R\right)\phi = 0$ is a mass–$\sqrt{M^2 + \xi R}$ solution of $\left(\Box + M^2\right)\phi = 0$. Thus the results I will describe for $M = 0$ solutions of the minimally coupled Klein–Gordon equation, viz. $\Box\phi = 0$, apply in such spacetimes' coupled Klein–Gordon equation whenever $M = \pm\sqrt{-\xi R}$. Indeed, if R is a nonzero constant, as is the case for de Sitter space (but not the flat static torus), any choice of M yields such problems for $\xi = -\frac{M^2}{R}$.

[17]One exception is discussed in Sec. (3.4). It results from a change of field variables in a Lagrangian, which makes use of a nonlocal field defined in Eq. (3.4.1).

is conserved; and, in a slight abuse of terminology for brevity, I will call such δT *cyclic zero modes*. In the case of the Klein–Gordon field

$$\Pi_{\phi^*} = \int_{\mathbf{x}} \nabla^0 \phi, \tag{1.3.19}$$

$$\dot{\Pi}_{\phi^*} = \int_{\mathbf{x}} \nabla_\mu \nabla^\mu \phi = -M^2 \int_{\mathbf{x}} \phi, \tag{1.3.20}$$

so Π_{ϕ^*} is conserved when $M = 0$.
The Hamiltonian density is defined as

$$\mathcal{H} := -\mathcal{L} + \sum_T (\partial_0 T)(\pi_T), \tag{1.3.21}$$

where the sum runs over *all* canonical tensor fields which appear, differentiated and/or undifferentiated, in L. This construction of $\mathcal{H}$ in terms of $\mathcal{L}$ is a *Legendre transform*. A number of careful observations concerning its properties will be made at appropriate times throughout this thesis. One may write $\mathcal{H} = \sqrt{|g|}\mathcal{H}_0$ with

$$\mathcal{H}_0 := -\mathcal{L}_0 + \sum_T (\partial_0 T)(\varpi_T), \tag{1.3.22}$$

in analogy with the factorisation $\mathcal{L} = \sqrt{|g|}\mathcal{L}_0$. (However, unlike $\mathcal{L}_0$, $\mathcal{H}_0$ is not in general a scalar.) The Euler–Lagrange equation may instead be obtained by writing $\mathcal{H}$ in terms of Ts, $\partial_i T$s (if necessary) and π_Ts, which requires the definitions of π_T to be rearranged to make $\partial_0 T$s their subjects. This results in lower time derivatives disappearing from $\mathcal{H}_0$. The equations of motion are obtained as *Hamilton's equations*, viz.

$$\dot{T} = \frac{\delta H}{\delta \pi_T},\ \dot{\pi}_T = -\frac{\delta H}{\delta T}. \tag{1.3.23}$$

Thus the conservation of the canonical momentum of a cyclic T can also be proven in the Hamiltonian formalism.

1.4 Normalising and quantising Klein–Gordon fields

I do not present original material in this section, whose purpose is to review the results of scalar field theory that are key to my discussion of its zero mode problem later in this chapter. A good general reference is Chapter 2 of Parker and Toms's Ref. [6]. For example, their Eq. (2.48) introduces the same quantity as my Eq. (1.4.3) below, and their Eq. (1.47) is my Eq. (1.4.15).
In Sec. 1.4.1, I introduce a normalisation condition on classical Klein–Gordon fields that will be used throughout my analysis of the zero mode problem. In Sec. 1.4.2, I discuss enough of the quantum field theory of quantised Klein–Gordon fields to motivate this normalisation (one calculation is postponed to Appendix A). Sec. 1.4.2 also explains the concept of "modes" of scalar fields.

1.4.1 Classical fields

Minimally coupled scalar fields are of interest in this thesis. For the moment, I consider classical fields (later quantum fields will be indicated by circumflexes, viz. ϕ vs. $\hat{\phi}$). The Klein–Gordon equation may be rewritten, viz.

$$\Box\phi = \nabla_\mu\left(g^{\mu\nu}\nabla_\nu\phi\right) = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right), \tag{1.4.1}$$

$$\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right) = \sqrt{|g|}\Box\phi = -M^2\sqrt{|g|}\phi. \tag{1.4.2}$$

By homogeneity and linearity, the set of solutions of Eq. (1.3.10) for fixed M forms a vector space. One important example of a pseudo-inner product on this space is the *Klein–Gordon inner product*,

$$\langle\phi_1,\,\phi_2\rangle_{\mathrm{KG}} := \mathrm{i}\int_{\mathbf{x}}\left(\phi_1^*\nabla^0\phi_2 - \left(\nabla^0\phi_1^*\right)\phi_2\right). \tag{1.4.3}$$

(The factor of i ensures $\langle\phi_1,\,\phi_2\rangle^*_{\mathrm{KG}} = \langle\phi_2,\,\phi_1\rangle_{\mathrm{KG}}$.) This product is well-motivated; the Klein–Gordon inner product of two solutions of mass M is conserved. A *conserved current* is a vector V^μ with $\nabla_\mu V^\mu = 0$, so

$$0 = \sqrt{|g|}\nabla_\mu V^\mu = \partial_\mu\left(\sqrt{|g|}V^\mu\right), \tag{1.4.4}$$

$$0 = \int \mathrm{d}^{n-1}\mathbf{x}\,\partial_\mu\left(\sqrt{|g|}V^\mu\right) = \partial_0\left(\int_{\mathbf{x}} V^0\right) + \int \mathrm{d}^{n-1}\mathbf{x}\,\partial_i\left(\sqrt{|g|}V^i\right). \tag{1.4.5}$$

If $\sqrt{|g|}V^i$ contributes no boundary term, integration by parts implies the last integral vanishes, so $\int_{\mathbf{x}} V^0$ is a *conserved charge*. Indeed, $\mathrm{i}\int_{\mathbf{x}} V^0$ is simply the Klein–Gordon inner product if we choose

$$V^\mu := \phi_1^*\nabla^\mu\phi_2 - \left(\nabla^\mu\phi_1^*\right)\phi_2 \tag{1.4.6}$$

so that

$$\nabla_\mu V^\mu = \phi_1^*\Box\phi_2 - \left(\Box\phi_1^*\right)\phi_2 = \left(-M^2 + M^2\right)\phi_1^*\phi_2 = 0. \tag{1.4.7}$$

So this choice of V^μ is conserved, and so is the Klein–Gordon inner product. In particular, the *Klein–Gordon norm* $\langle\phi,\,\phi\rangle_{\mathrm{KG}}$ is conserved for any solution ϕ.

By inspection $\langle\phi,\,\phi^*\rangle_{\mathrm{KG}} = 0$ and $\langle\phi^*,\,\phi^*\rangle_{\mathrm{KG}} = -\langle\phi,\,\phi\rangle_{\mathrm{KG}}$. This last equation is important, because the solution set of the Klein–Gordon equation is closed under $\phi \mapsto \phi^*$, so $\langle\phi,\,\phi\rangle_{\mathrm{KG}}$ is indefinite. Quantisation obtains *quantised Klein–Gordon fields* expressible in terms of functions solving Eq. (1.3.10) called *modes*. One may (non-uniquely; see Sec. 1.4.2 for a fuller discussion) choose a basis of the space of solutions such that the basis elements are closed under $\phi \mapsto \phi^*$, and each basis element has non-zero Klein–Gordon norm. This allows us to draw a distinction between *positive-frequency* and *negative-frequency* modes of Klein–Gordon fields, which respectively have positive and negative Klein–Gordon norms. One popular normalisation of positive-frequency modes ϕ is to then set

$$\langle\phi,\,\phi\rangle_{\mathrm{KG}} = 1. \tag{1.4.8}$$

(I motivate this normalisation in Sec. 1.4.2 and Appendix A.) With such a convention, if ϕ is purely of positive (negative) frequency then ϕ^* is purely of negative (positive) frequency. (We say a solution ϕ is of purely positive (negative) frequency if it is a linear combination of modes of positive (negative) frequency only. A general solution is a superposition of both mode types.)

1.4.2 Quantisation

Let $\phi(x)$ denote a canonical bosonic scalar field solving a linear equation of motion such as the Klein–Gordon equation. Let $\hat{\phi}(x)$ denote a quantised canonical bosonic scalar field, and let $\pi(y)$ denote the conjugate momentum density of $\phi(x)$. Then $\pi(y)$ admits a quantisation $\hat{\pi}(y)$, which is the conjugate momentum density of $\hat{\phi}(x)$. Quantum theories are replete with *commutators*

$$\left[\hat{X},\, \hat{Y}\right] := \hat{X}\hat{Y} - \hat{Y}\hat{X}. \tag{1.4.9}$$

One axiom of the quantum field theory of $\hat{\phi}$ is the *equal-time canonical commutation relations* (CCRs). These are expressible in terms of the *Dirac delta function* (a measure) on spatial sections, viz.

$$\left[\hat{\pi}(t,\,\mathbf{x}),\, \hat{\phi}(t,\,\mathbf{y})\right] = -\mathrm{i}\delta(\mathbf{x},\,\mathbf{y}), \tag{1.4.10}$$

$$\left[\hat{\phi}(t,\,\mathbf{x}),\, \hat{\phi}(t,\,\mathbf{y})\right] = 0, \tag{1.4.11}$$

$$\left[\hat{\pi}(t,\,\mathbf{x}),\, \hat{\pi}(t,\,\mathbf{y})\right] = 0. \tag{1.4.12}$$

Note that the CCRs are written in a specific coordinate system. However, Eq. (1.4.10) has implications that can be stated in a coordinate-independent form. For example, $\left[\hat{\pi}(x),\, \hat{\phi}(y)\right] = 0$ when $x,\, y$ are causally disconnected, as occurs in spacetimes for which is $x - y$ is well-defined but spacelike (equivalently, when a coordinate system exists in which $x^0 = y^0$). This motivates the axiom; the fields commute in this case because of causality. (The other CCRs are even simpler; again spacelike separations require vanishing commutators, but an operator must also commute with itself at the same point in spacetime.) The CCR axioms can also be seen as generalisations of the Poisson brackets of classical mechanics.

The reason quantised fields do not in general commute is that they are operators on the Hilbert space other than multiples of the identity operator. To relate all this to classical field theory, we observe that the Klein–Gordon equation has *operator-valued solutions*. Indeed, given a basis of the vector space of ordinary function-valued solutions, the general operator-valued solution is a linear combination of said basis functions with spacetime-constant operator-valued coefficients. (The previous sentence was deliberately devoid of algebra that would have specified notation for these spacetime-constant operators. I present the details below, after some exposition, in Eq. (1.4.15).) It follows that the commutators of such coefficients should be consistent with the axiom Eq. (1.4.10).

Let $\hat{\phi}$ be a *Hermitian* quantised field associated with a particle species, and let $|0\rangle$ denote a zero-particle $\hat{\phi}$-sector vacuum state.[18] The most general one-particle state for that species is then a linear combination over labels σ of the kets $\hat{a}_\sigma^\dagger |0\rangle$, where $\hat{a}_\sigma^\dagger$ is the *creation operator* for a particle of state labelled by σ. (This is in fact a label of the ordinary function-valued solutions of the Klein–Gordon equation.) Then

$$\hat{a}_\sigma |0\rangle = 0,\ \langle 0| \hat{a}_\sigma^\dagger = 0. \tag{1.4.13}$$

[18] Such a vacuum need not exist; if it does, it is not in general unique; and such vacua, if they exist, may violate symmetries of interest. These are all common problems in quantum field theories in general spacetimes. I discuss such problems applicable to this thesis in more detail later as appropriate.

Another important equation is

$$[\hat{a}_\sigma,\, \hat{a}_{\sigma'}] = 0. \tag{1.4.14}$$

I assume hereafter that the σ labelling the states are discrete rather than continuous (this applies to the example of the flat static torus that will be discussed later), but the continuous case is analogous; sums over σ are then promoted to integrals. Explicitly

$$\hat{\phi}\,(t,\, \mathbf{x}) = \sum_\sigma \left(\hat{a}_\sigma \phi_\sigma\,(t,\, \mathbf{x}) + \hat{a}^\dagger_\sigma \phi^*_\sigma\,(t,\, \mathbf{x})\right), \tag{1.4.15}$$

where the solution subspace spanned by the ϕ_σ (the ϕ^*_σ) is called a *subspace of positive-frequency solutions* (*subspace of negative-frequency solutions*). The choice of these subspaces is not unique, but we can fix one choice to define positive- and negative-frequency modes. If $\langle \phi_\sigma,\, \phi_{\sigma'}\rangle_{\text{KG}} = \delta_{\sigma\sigma'}$ and

$$\varphi_\sigma := \sum_{\sigma'} \left(\alpha_{\sigma\sigma'} \phi_\sigma + \beta_{\sigma\sigma'} \phi^*_\sigma\right), \tag{1.4.16}$$

then $\langle \varphi_\sigma,\, \varphi_{\sigma'}\rangle_{\text{KG}} = \delta_{\sigma\sigma'}$ provided the complex matrices $\alpha,\, \beta$ satisfy

$$\left(\alpha^* \alpha^T - \beta^* \beta^T\right)_{\sigma\sigma'} = \delta_{\sigma\sigma'}. \tag{1.4.17}$$

(The transformation to the φ_σ subject to Eq. (1.4.17), partnered with the use of revised annihilation operators

$$\hat{b}_\sigma := \sum_{\sigma'} \left(\alpha^*_{\sigma\sigma'} \hat{a}_{\sigma'} - \beta^*_{\sigma\sigma'} \hat{a}^\dagger_{\sigma'}\right), \tag{1.4.18}$$

is called a *Bogoliubov transformation*. It satisfies

$$\hat{\phi}\,(t,\, \mathbf{x}) = \sum_\sigma \left(\hat{a}_\sigma \phi_\sigma\,(t,\, \mathbf{x}) + \hat{a}^\dagger_\sigma \phi^*_\sigma\,(t,\, \mathbf{x})\right) = \sum_\sigma \left(\hat{b}_\sigma \varphi_\sigma\,(t,\, \mathbf{x}) + \hat{b}^\dagger_\sigma \varphi^*_\sigma\,(t,\, \mathbf{x})\right); \tag{1.4.19}$$

and, if $[\hat{a}_\sigma,\, \hat{a}_{\sigma'}] = 0$ and $\left[\hat{a}_\sigma,\, \hat{a}^\dagger_{\sigma'}\right] = \delta_{\sigma\sigma'}$, then $\left[\hat{b}_\sigma,\, \hat{b}_{\sigma'}\right] = 0$ and $\left[\hat{b}_\sigma,\, \hat{b}^\dagger_{\sigma'}\right] = \delta_{\sigma\sigma'}$. A new vacuum state is also required, e.g. $\prod_\sigma \exp\left(\frac{\beta^*}{2\alpha^*} \hat{a}^{\dagger 2}_\sigma\right) |0\rangle \in \bigcap_\sigma \ker \hat{b}_\sigma$ if $\alpha_{\sigma\sigma'},\, \beta_{\sigma\sigma'} \propto \delta_{\sigma\sigma'}$.) However, in Minkowski space one convenient choice of the positive-frequency modes ϕ_σ are proportional to the familiar plane-wave functions $\exp \mathrm{i}\,(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t)$ with

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k} \cdot \mathbf{k} + M^2}. \tag{1.4.20}$$

Then complex conjugation yields negative-frequency modes proportional to $\exp \mathrm{i}\,(\mathbf{k} \cdot \mathbf{x} + \omega_{\mathbf{k}} t)$. (These superpositions integrate over $\mathbf{k}$-space.)

The ordinary functions $\phi_\sigma,\, \phi^*_\sigma$ span the space of solutions of Eq. (1.3.10) obtained with a fixed M. With the aforementioned procedure, $\phi_\sigma,\, \phi^*_\sigma$ also span the space of quantised Klein–Gordon fields. In the quantum theory we refer to these functions as *modes* of the quantised field $\hat{\phi}$. All vacuum-state p-point correlation functions of $\hat{\phi}$ with $p \in \mathbb{N}$, viz. $\left\langle 0 \right| \prod_{i=1}^p \hat{\phi}\,(x_i) \left| 0 \right\rangle$, can be computed from the x-space representations of these modes. In particular, the zero mode problem I describe later is due to specific ϕ_σs.

The momentum density is readily quantised, viz.

$$\hat{\pi}(t,\,\mathbf{y}) = \sqrt{|g|}\nabla^0\hat{\phi}(t,\,\mathbf{y})\,. \tag{1.4.21}$$

The trivial equal-time CCRs are then

$$\left[\hat{\phi}(t,\,\mathbf{x})\,,\,\hat{\phi}(t,\,\mathbf{y})\right] = \left[\nabla^0\hat{\phi}(t,\,\mathbf{x})\,,\,\nabla^0\hat{\phi}(t,\,\mathbf{y})\right] = 0. \tag{1.4.22}$$

The *Klein–Gordon orthonormality condition*

$$\delta_{\sigma\sigma'} = \langle\phi_\sigma,\,\phi_{\sigma'}\rangle_{\mathrm{KG}}\,,\,0 = \left\langle\phi_\sigma,\,\phi^*_{\sigma'}\right\rangle_{\mathrm{KG}} \tag{1.4.23}$$

(note the use of the *Kronecker delta*) manifestly generalises the requirement that

$$\langle\phi_\sigma,\,\phi_\sigma\rangle_{\mathrm{KG}} = 1,\,\langle\phi^*_\sigma,\,\phi^*_\sigma\rangle_{\mathrm{KG}} = -1,\,[\hat{a}_\sigma,\,\hat{a}_{\sigma'}] = 0, \tag{1.4.24}$$

It can be shown that imposing the Klein–Gordon orthonormality condition also implies the equivalence of two pairs of equations. The first pair comprises Eq. (1.4.14) and the equation[19]

$$\left[\hat{a}_\sigma,\,\hat{a}^\dagger_{\sigma'}\right] = \delta_{\sigma\sigma'}; \tag{1.4.25}$$

the second comprises Eq. (1.4.22) and the equation

$$\left[\hat{\phi}(t,\,\mathbf{x})\,,\,\nabla^0\hat{\phi}(t,\,\mathbf{y})\right] = \frac{\mathrm{i}\delta(\mathbf{x},\,\mathbf{y})}{\sqrt{|g(t,\,\mathbf{x})|}}. \tag{1.4.26}$$

Eqs. (1.4.25) and (1.4.26) are both crucial to theory. Eq. (1.4.25) is integral to the quantum description of particle numbers; for example, it can be used to show that $p!^{-1/2}\left(a^\dagger_\sigma\right)^p|0\rangle$ is a p-particle state. Appendix A presents a proof that Eq. (1.4.23) implies that Eqs. (1.4.14) and (1.4.25) are equivalent to Eqs. (1.4.22) and (1.4.26). I will hereafter impose the orthonormality condition in Eq. (1.4.23).

In the next section, the zero mode problem in minimally coupled scalar theory is explained in the context of a flat static torus.

[19]For continuous labels σ, all sums over σ become integrals, and the right-hand side of Eq. (1.4.25) becomes a σ-space Dirac delta, $\delta\left(\sigma,\,\sigma'\right)$.

The right-hand side of Eq. (1.4.25) may be rescaled as a matter of convention. This is achieved by rescaling the $\hat{a}_\sigma$ operators. However, in this thesis I will always use the scaling in Eq. (1.4.25).

In one important example in Minkowski space, the labels are wavevectors of a particle, i.e. a one-particle state of wavevector $\mathbf{k}$ is $\hat{a}^\dagger_{\mathbf{k}}|0\rangle$. It is then customary to take $\left[\hat{a}_{\mathbf{k}},\,\hat{a}^\dagger_{\mathbf{k}}\right] = (2\pi)^{n-1}2E(\mathbf{k})\,\delta\left(\mathbf{k},\,\mathbf{k}'\right)$, where $E = E(\mathbf{k})$ is the dispersion relation between $\mathbf{k}$ and the particle's energy E. The $(2\pi)^{n-1}2E(\mathbf{k})$ factor is included to obtain a Lorentz-invariant measure (see Sec. 3-1-2 of Ref. [3]). In this convention, the operator $\int \mathrm{d}^{n-1}\mathbf{k}$ should be replaced with $\int \frac{\mathrm{d}^{n-1}\mathbf{k}}{(2\pi)^{n-1}2E(\mathbf{k})}$.

1.5 An infrared problem in scalar field theory on a flat static torus

In Sec. 1.5.1, I derive a Klein–Gordon normalisation of massive solutions of Eq. (1.3.10) on a flat static torus. In Sec. 1.5.2, I contrast the behaviour of this normalisation, for spatially uniform solutions, in the massless and massive cases. In Sec. 1.5.3, I introduce the *propagator* of a Hermitian quantised Klein–Gordon field, and discuss its behaviour in the massless limit. It is this behaviour that constitutes the zero mode problem. Note that this section's discussion is limited to the case of the flat static torus, but in Sec. 1.6 I discuss the zero mode problem in an arbitrary spacetime of interest (viz. The summary concluding Sec. 1.2.2).

1.5.1 Klein–Gordon normalisation

Minkowski space may be considered in Cartesian coordinates satisfying

$$N = 1,\, N^i = 0,\, \gamma_{ij} = \delta_{ij}. \tag{1.5.1}$$

For a flat static torus, we also periodically identify each of the coordinates x^i, say with x^i having period $L^i > 0$. (We are concerned with spacetimes for which $\int_{\mathbf{x}} g^{00}$ is finite, so flat static tori are relevant whereas Minkowski space is not.) Let $L \in (0,\, \infty)^{(n-1)\times(n-1)}$ denote the diagonal matrix $\mathrm{diag}\left(L^i\right)$. A spacetime event that can be described with the coordinates $t = t_0,\, \mathbf{x} = \mathbf{x}_0$ may also be described with the coordinates $t = t_0,\, \mathbf{x} = \mathbf{x}_0 + L\mathbf{n}$ for any $\mathbf{n} \in \mathbb{Z}^{n-1}$.

Note that the spacetime satisfies $\partial_\alpha g_{\beta\gamma} = 0$. It is flat because of the case $\alpha = i$; it is static because of the case $\alpha = 0$.

Minimally coupled classical Klein–Gordon fields on this flat static torus satisfy

$$\left(\partial_0^2 - \boldsymbol{\nabla}^2 + M^2\right)\phi = 0. \tag{1.5.2}$$

The $M \neq 0$ plane-wave positive-frequency Klein–Gordon modes

$$\varphi_{\mathbf{k}} := \exp \mathrm{i}\left(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t\right) \tag{1.5.3}$$

satisfy Eq. (1.4.20) and

$$\partial^0 \varphi_{\mathbf{k}} = -\mathrm{i}\omega_{\mathbf{k}}\varphi_{\mathbf{k}}, \tag{1.5.4}$$

$$\partial^0 \varphi^*_{\mathbf{k}} = +\mathrm{i}\omega_{\mathbf{k}}\varphi^*_{\mathbf{k}}, \tag{1.5.5}$$

$$\langle\varphi_{\mathbf{k}},\, \varphi_{\mathbf{q}}\rangle_{\mathrm{KG}} = \mathrm{i}\int_{\mathbf{x}} \varphi^*_{\mathbf{k}}\varphi_{\mathbf{q}}\left(-2\mathrm{i}\omega_{\mathbf{k}}\right) = 2\omega_{\mathbf{k}}\int_{\mathbf{x}} \exp \mathrm{i}\left(\mathbf{q}-\mathbf{k}\right)\cdot\mathbf{x}. \tag{1.5.6}$$

The wavevector is discretised for the flat static torus, viz.

$$\mathbf{k} = 2\pi L^{-1}\mathbf{n},\, \mathbf{n} \in \mathbb{Z}^{n-1}. \tag{1.5.7}$$

Working in the coordinates chosen in Eq. (1.5.1),

$$\begin{aligned}\langle\varphi_{2\pi L^{-1}\mathbf{n}_1},\varphi_{2\pi L^{-1}\mathbf{n}_2}\rangle_{\mathrm{KG}}&=2\omega_{2\pi L^{-1}\mathbf{n}_1}\int\mathrm{d}^{n-1}\mathbf{x}\exp 2\pi\mathrm{i}L^{-1}\left(\mathbf{n}_2-\mathbf{n}_1\right)\cdot\mathbf{x}\\&=2\omega_{2\pi L^{-1}\mathbf{n}_1}L^{n-1}\delta_{\mathbf{n}_1\mathbf{n}_2},\end{aligned}\tag{1.5.8}$$

where I have introduced the geometric mean $L:=\left(\prod_{i=1}^{n-1}L^i\right)^{\frac{1}{n-1}}$ of the L^i so that

$$L^{n-1}=\prod_{i=1}^{n-1}L^i=\det L.\tag{1.5.9}$$

(It is common to use the symbol for a matrix to also denote its determinant, e.g. $g=\det g_{\mu\nu}$ is a determinant of the tensor $g_{\mu\nu}$ interpreted as a matrix. The determinant of the matrix L is not a spatial length but a product of $n-1$ spatial lengths, so it is more sensible to denote the determinant as L^{n-1}.) From Eq. (1.5.8), the Klein–Gordon normalisation for positive-frequency modes is

$$\phi_{\mathbf{n}}:=\frac{\varphi_{2\pi L^{-1}\mathbf{n}}}{\sqrt{2\omega_{2\pi L^{-1}\mathbf{n}}L^{n-1}}}=\frac{\exp\mathrm{i}\left(2\pi L^{-1}\mathbf{n}\cdot\mathbf{x}-\omega_{2\pi L^{-1}\mathbf{n}}t\right)}{\sqrt{2\omega_{2\pi L^{-1}\mathbf{n}}L^{n-1}}}.\tag{1.5.10}$$

1.5.2 The massless limit of spatially uniform solutions

The case $\mathbf{n}=\mathbf{0}$ is spatially uniform; and, by Eq, (1.4.20), for $M=0$ it is time-independent since it satisfies $\omega=0$ (i.e. this case has zero energy, which in the Schrödinger operator formalism $\omega=\mathrm{i}\partial_0$ is equivalent to time-independence). This zero-energy mode is called a *zero mode*, and this term is typically used in general for zero-energy states of time-independent wavefunction.
In this subsection, I show that the massless limit of $\mathbf{n}=\mathbf{0}$ creates a zero mode problem for scalar field theory in the flat static torus. This is a special case of an infrared problem that occurs in scalar field theory in curved spacetimes. It is convenient for the purposes of this thesis to define the zero mode of a scalar field in a non-standard way. In my terminology, *zero modes* of scalar fields are spatially uniform modes, so the variations $\delta T\in\bigcap_i\ker\nabla_i$ considered in Sec. 1.3 are zero-mode-sector variations. (A number of further details will need to be clarified in due course to define the zero mode of arbitrary scalar-valued functions, vector fields and conjugate momentum densities thereof.) Zero modes simplify the Klein–Gordon equation to an ordinary differential equation in time, viz.

$$\partial_0^2\phi=-M^2\phi.\tag{1.5.11}$$

For $M\neq 0$ there is one positive-frequency zero mode, viz.

$$\phi_{\mathbf{0}}=\frac{\exp\left(-\mathrm{i}Mt\right)}{\sqrt{2ML^{n-1}}}.\tag{1.5.12}$$

The conjugate $\phi_{\mathbf{0}}^*$ is a negative-frequency zero mode, so there are two zero modes overall. Notice that both zero modes diverge in the $M\to 0^+$ right-hand limit, which I will call the *IR limit* for brevity. Indeed, when $M=0$ the Klein–Gordon equation's solutions take a different form, because the eigenvalues of ∂_0 for which $\partial_0^2+M^2$ vanishes are no longer distinct. The most general massless solution is in fact $\phi=A+Bt$ for spacetime-constant coefficients $A,\ B$. It follows that, for small $M>0$, the approximate behaviour of $\phi_{\mathbf{0}}$ is of the form $A\left(M\right)+B\left(M\right)t$ for some functions $A\left(M\right),\ B\left(M\right)$.

An explicit calculation obtains the M-dependences of $A,\ B$, viz.

$$A+Bt=\frac{1-\mathrm{i}Mt}{\sqrt{2ML^{n-1}}}\approx\frac{\exp\left(-\mathrm{i}Mt\right)}{\sqrt{2ML^{n-1}}},\tag{1.5.13}$$

$$A=\frac{1}{\sqrt{2ML^{n-1}}}\propto\frac{1}{\sqrt{M}},\tag{1.5.14}$$

$$B=-\mathrm{i}MA\propto\sqrt{M}.\tag{1.5.15}$$

1.5.3 The Feynman propagator

The *Feynman propagator* of a Hermitian quantised Klein–Gordon field $\hat{\phi}$ is defined as

$$\mathrm{T}\left\langle 0|\hat{\phi}\left(x\right)\hat{\phi}\left(x^{'}\right)|0\right\rangle,\tag{1.5.16}$$

where T denotes time ordering.[20] Since $\left\langle 0|\hat{\phi}\left(x\right)\hat{\phi}\left(x^{'}\right)|0\right\rangle$ is a function of $x,\ x^{'}$, and hence in particular of the time coordinates $t:=x^0,\ t:=x^{'0}$, its *time ordering* is defined as thus: the $\hat{\phi}$ with the least time coordinate is placed rightmost in the product. More generally, for any function of the form[21]

$$K\left(t,\ t^{'}\right)=\left\langle 0|\widehat{A}\left(t\right)\widehat{A}\left(t^{'}\right)|0\right\rangle,\ \widehat{A}^{\dagger}=\widehat{A},\tag{1.5.17}$$

we have the time ordering

$$\mathrm{T}K\left(t,\ t^{'}\right):=K\left(\max\left\{t,\ t^{'}\right\},\ \min\left\{t,\ t^{'}\right\}\right),\tag{1.5.18}$$

which is symmetric by definition. An equivalent expression for this time ordering may be provided in terms of the *Heaviside step function*. We have

$$\begin{aligned}\mathrm{T}K\left(t,\ t^{'}\right)&=K\left(\max\left\{t,\ t^{'}\right\},\ \min\left\{t,\ t^{'}\right\}\right)\\&=\begin{cases}K\left(t,\ t^{'}\right)&t>t^{'}\\K\left(t,\ t\right)&t=t^{'}\\K\left(t^{'},\ t\right)&t<t^{'}\end{cases}\\&=\begin{cases}0&t>t^{'}\\\frac{1}{2}K\left(t,\ t\right)&t=t^{'}\\K\left(t^{'},\ t\right)&t<t^{'}\end{cases}+\begin{cases}K\left(t,\ t^{'}\right)&t>t^{'}\\\frac{1}{2}K\left(t,\ t\right)&t=t^{'}\\0&t<t^{'}\end{cases},\end{aligned}\tag{1.5.19}$$

so in terms of the Heaviside step function

[20] The interest in $\left\langle 0|\hat{\phi}\left(x\right)\hat{\phi}\left(x^{'}\right)|0\right\rangle$ is as a covariance of $\hat{\phi}$ at two spacetime events. Eqs. (1.4.15) and (1.4.25) imply $\left\langle 0|\hat{\phi}\left(x\right)|0\right\rangle=0$, i.e. $\hat{\phi}\left(x\right)$ has zero *vacuum expectation value* (VEV). The quantised Higgs field has nonzero VEV because it does not admit a decomposition analogous to Eq. (1.4.15). This is because the classical Higgs field solves a non-linear equation of motion. Indeed, an analogous treatment of a minimally coupled scalar Higgs field ρ takes $\mathcal{L}_0=\nabla_\mu\rho^*\nabla^\mu\rho-V\left(\rho\right)$, with *Mexican-hat potential* $V\left(\rho\right):=M^2\rho^*\rho+\frac{\lambda}{2}\left(\rho^*\rho\right)^2$. Here $M^2<0$ and $\lambda>0$. Varying ρ^* gives $\Box\rho=-\left(M^2+\lambda\rho^*\rho\right)\rho$. The VEV is a spacetime-constant global minimum of $V\left(\rho\right)$, so $\Box\rho=0$. Since $V\left(0\right)=0$ but $V\left(\sqrt{-\frac{M^2}{\lambda}}\right)=\frac{-M^4}{2\lambda}<0$, the solution $\rho^*\rho=-\frac{M^2}{\lambda}$ is the VEV. A brief discussion of this is provided in Sec. 12-6 of Ref. [22].

[21] In this grey text environment, dependences on variables other than time coordinates is possible but not displayed.

$$\theta(\tau) := \begin{cases} 0 & \tau < 0 \\ \frac{1}{2} & \tau = 0 \\ 1 & \tau > 0 \end{cases} = \int_{-\infty}^{\tau} \delta\left(\tau'\right) \mathrm{d}\tau' \tag{1.5.20}$$

we have

$$\begin{aligned} \mathrm{T}K\left(t, t'\right) &= K\left(\max\left\{t, t'\right\}, \min\left\{t, t'\right\}\right) \\ &= \begin{cases} K\left(t, t'\right) & t > t' \\ K(t, t) & t = t' \\ K\left(t', t\right) & t < t' \end{cases} \\ &= \begin{cases} 0 & t > t' \\ \frac{1}{2}K(t, t) & t = t' \\ K\left(t', t\right) & t < t' \end{cases} + \begin{cases} K\left(t, t'\right) & t > t' \\ \frac{1}{2}K(t, t) & t = t' \\ 0 & t < t' \end{cases} \\ &= K\left(t, t'\right)\theta\left(t - t'\right) + K\left(t', t\right)\theta\left(t' - t\right) \\ &= K\left(t, t'\right)\theta\left(t - t'\right) + K\left(t', t\right)\left(1 - \theta\left(t - t'\right)\right). \end{aligned} \tag{1.5.21}$$

In this thesis, important examples of Eq. (1.5.17) that are also expressible in the form

$$K\left(t, t'\right) = K_0(t) - K_0\left(t'\right) \tag{1.5.22}$$

for some K_0 satisfying $\dot{K}_0(\tau) > 0$ for all τ thereby additionally satisfy

$$\mathrm{T}K\left(t, t'\right) = \left|K_0(t) - K_0\left(t'\right)\right|. \tag{1.5.23}$$

But $\hat{\phi}$ has a mode decomposition, viz. Eq. (1.4.15). Thus

$$\hat{\phi}\left(x'\right)|0\rangle = \sum_{\sigma'} \phi^*_{\sigma'}\left(x'\right)\hat{a}^\dagger_{\sigma'}|0\rangle, \tag{1.5.24}$$

$$\begin{aligned} \left\langle 0|\hat{\phi}(x)\hat{\phi}\left(x'\right)|0\right\rangle &= \sum_{\sigma\sigma'} \phi_\sigma(x)\phi^*_{\sigma'}\left(x'\right)\left\langle 0|\hat{a}_\sigma\hat{a}^\dagger_{\sigma'}|0\right\rangle \\ &= \sum_{\sigma\sigma'} \phi_\sigma(x)\phi^*_{\sigma'}\left(x'\right)\delta_{\sigma\sigma'} = \sum_\sigma \phi_\sigma(x)\phi^*_\sigma\left(x'\right). \end{aligned} \tag{1.5.25}$$

In terms of the decompositions

$$x = (t, \mathbf{x}), \; x' = \left(t', \mathbf{x}'\right), \tag{1.5.26}$$

all contributions to $\sum_\sigma \phi_\sigma(x)\phi^*_\sigma\left(x'\right)$ due to $\mathbf{n} \neq \mathbf{0}$ solutions are of the form

$$\frac{\exp \mathrm{i}\left(2\pi L^{-1}\mathbf{n}\cdot\left(\mathbf{x} - \mathbf{x}'\right) - \omega_{2\pi L^{-1}\mathbf{n}}\left(t - t'\right)\right)}{2\omega_{2\pi L^{-1}\mathbf{n}} L^{n-1}}. \tag{1.5.27}$$

These terms are invariant under spacetime translations and, in particular, the case $\mathbf{x} = \mathbf{x}^{'}$ simplifies for $\mathbf{n} \neq \mathbf{0}$ to the spatially uniform, time-translation invariant result

$$\frac{\exp \mathrm{i}\omega_{2\pi L^{-1}\mathbf{n}}\left(t^{'} - t\right)}{2\omega_{2\pi L^{-1}\mathbf{n}} L^{n-1}}. \tag{1.5.28}$$

Note that these results are true for arbitrary M, including $M = 0$, and apply also in the $\mathbf{n} = \mathbf{0}$ case *provided that* $M \neq 0$. However, the $\mathbf{n} = \mathbf{0}$ case with a small mass instead has an approximate contribution

$$\left(A\left(M\right) + B\left(M\right)t\right)\left(A^{*}\left(M\right) + B^{*}\left(M\right)t^{'}\right) = \left(A + Bt\right)\left(A - Bt^{'}\right), \tag{1.5.29}$$

using the fact that Eqs. (1.5.14) and (1.5.15) imply $A \in \mathbb{R},\ B \in \mathrm{i}\mathbb{R}$.

These equations have a few further implications, resulting in an infrared problem for the theory. The spacetime-constant term AA^{*} is IR-divergent; the $BB^{*}tt^{'}$ term vanishes in the IR limit (which is fortunate given the effect it would otherwise have on spacetime symmetries); and the M-independent contribution is $AB\left(t - t^{'}\right)$, which is time-translation invariant. The IR limit of the massive Feynman propagator is therefore the sum of an IR-convergent time-translation invariant term and an IR-divergent spacetime-constant term. Indeed, if the term A^{2} were instead infrared-convergent and the $tt^{'}$ term vanished to preserve time-translation invariance, we would have $B = 0$ and the Klein–Gordon norm's massless limit would be 0. There is thus a conflict between time-translation invariance, infrared convergence and Klein–Gordon normalisation.

1.6 Generalisation to spacetimes with compact spatial sections

1.6.1 Low-mass behaviour of scalar "zero modes"

Working in the synchronous gauge for the background spacetime (viz. Eq. (1.2.2)), spatially uniform massless solutions of the Klein–Gordon equation satisfy

$$\partial_0\left(a^{n-1}\left(t\right)\sqrt{\eta\left(\mathbf{x}\right)}\dot{\phi}\left(t\right)\right) = 0. \tag{1.6.1}$$

(Note that I have assumed Eq. (1.2.5), not the less general Eq. (1.2.3). One of my contributions to our research was the realisation that this allowed all the results that follow to apply to a more general class of spacetimes.) This simplifies to

$$\dot{\phi} \propto a^{1-n}. \tag{1.6.2}$$

Defining

$$f\left(t\right) := \int_0^t a^{1-n}\left(\tau\right)\mathrm{d}\tau, \tag{1.6.3}$$

the general massless zero mode is of the form

$$\phi = A + Bf \tag{1.6.4}$$

for spacetime-constant coefficients $A,\ B$. (The approximate low-mass behaviour of a massive zero mode may behave differently; see, e.g. Eq. (1.8.3) for additional terms in the de Sitter case.) Two other important results are

$$\sqrt{|g|} = \sqrt{\gamma} = a^{n-1}\sqrt{\eta}, \tag{1.6.5}$$
$$V = a^{n-1}\int \mathrm{d}^{n-1}\mathbf{x}\sqrt{\eta} = \frac{\int \mathrm{d}^{n-1}\mathbf{x}\sqrt{\eta}}{\dot{f}}. \tag{1.6.6}$$

It is sufficient to choose a coordinate system for which the *coordinate volume* $V_c := \int \mathrm{d}^{n-1}\mathbf{x}\sqrt{\eta}$ is finite. One further important observation is that, since $\dot{f} = a^{1-n} > 0$, f is an example of a function K_0 of the form described in Eq. (1.5.22).
As in the case of the flat static torus, the IR limit of a massive solution satisfying $\langle \phi,\, \phi \rangle_{\mathrm{KG}} = 1$ is a massless solution satisfying $\langle \phi,\, \phi \rangle_{\mathrm{KG}} = 1$. Thus $A,\, B$ each have some appropriate M-dependence, and the normalisation condition is

$$\begin{aligned}
1 &= \mathrm{i}\int_{\mathbf{x}} \left(\phi^*\dot{\phi} - \dot{\phi}^*\phi\right) \\
&= \mathrm{i}\int_{\mathbf{x}} \left(\left(A^* + B^* f\right) B a^{1-n} - B^* a^{1-n}\left(A + Bf\right)\right) \\
&= \mathrm{i}\left(A^* B - A B^*\right) V_c,
\end{aligned} \tag{1.6.7}$$
$$A^* B - A B^* = \frac{-\mathrm{i}}{V_c}. \tag{1.6.8}$$

It suffices to impose $A > 0,\, B = \frac{-\mathrm{i}}{2AV_c}$. This constrains, but does not determine, the M-dependences of $A,\, B$. Curiously, the M-dependence is a-sensitive in a manner Dr Higuchi and I still do not know for general a. In particular, $a = 1$ (the flat static torus) gives $B \propto \sqrt{M}$, while $a = \cosh Ht$ (de Sitter space) gives $B \propto M$. This is one of the results I discuss in Sec. 1.8. In the IR limit, A diverges while B vanishes.

1.6.2 Analysis of Feynman propagator terms by spacetime symmetries and infrared behaviour

The general case modifies the M-dependence of $A,\, B$, and replaces the zero mode $A{+}Bt$ with $A{+}Bf$. However, much of what can be said about the Feynman propagator in the case of the flat static torus remains true because AB is M-independent and $A \in \mathbb{R},\, B \in \mathrm{i}\mathbb{R}$. The zero mode problem is entirely due to

$$\left(A + Bf\left(t\right)\right)\left(A^* + B^* f\left(t'\right)\right) = \left(A + Bf\left(t\right)\right)\left(A - Bf\left(t'\right)\right). \tag{1.6.9}$$

As in Sec. 1.5.3, there is a spacetime-constant IR-divergent term, a term that vanishes in the IR limit, and an M-independent term that is proportional to $f\left(t\right) - f\left(t'\right)$ and hence invariant under f-translation. The IR limit of the Feynman propagator therefore consists of a spacetime-constant IR-divergent term and an IR-convergent term that preserves a desirable spacetime symmetry.

1.7 Two prescriptions addressing the zero mode problem

Sec. 1.7.1 provides a brief summary of how the FMP addresses the zero mode problem discussed in Secs. 1.5 and 1.6. Sec. 1.7.2 discusses the CMP for the same purpose. However, discussion of the CMP is necessarily much more detailed.

1.7.1 The fictional mass prescription (FMP)

The only IR-divergent term in the massive Feynman propagator is a spacetime-constant contribution in the zero-mode sector, so its subtraction from the Feynman propagator does not modify which spacetime transformations preserve the result. The FMP subtracts the IR-divergent term and then takes the IR limit to obtain an effective zero-mode sector Feynman propagator, which is

$$\mathrm{T}\left\{AB\left(f\left(t\right)-f\left(t^{'}\right)\right)\right\}=\frac{-\mathrm{i}}{2V_c}\mathrm{T}\left\{f\left(t\right)-f\left(t^{'}\right)\right\}=\frac{-\mathrm{i}}{2V_c}\left|f\left(t\right)-f\left(t^{'}\right)\right|. \tag{1.7.1}$$

The theory has several spacetimes symmetries and a global internal symmetry, an invariance under $\phi\to\mathrm{e}^{\mathrm{i}\theta}\phi$ for spacetime-constant $\theta\in\mathbb{R}$. The other zero mode problems I discuss in this thesis occur in theories with local internal symmetries. The FMP's treatment of such theories adds a mass term that violates these symmetries. However, no such terms are introduced in the CMP.

1.7.2 The cyclic mode prescription (CMP)

The contents of this subsection are original.

In the Schrödinger picture of quantum field theory, the state has a formal wave functional $\Psi\left(T,\,\cdots\right)$ satisfying

$$\hat{\Pi}_T\Psi=-\mathrm{i}\frac{\delta\Psi}{\delta T}, \tag{1.7.2}$$

where $\frac{\delta}{\delta T}$ denotes a *functional derivative*. Suppose T is a spatially uniform scalar mode in a scalar Lagrangian density $\mathcal{L}_0$, no derivatives of T higher than $\dot{T}$ appear in $\mathcal{L}_0$, and the undifferentiated T also does not appear therein. Then Π_T is a conserved charge, and one may impose $\Pi_T=0$.

This fact takes some explaining. Consider first a 1-particle quantum-mechanical system of conserved linear kinetic momentum $\mathbf{p}\in\mathbb{R}^{n-1}$. In a state with $\mathbf{p}=\mathbf{p}_0$, the $\mathbf{x}$-space wave function's $\mathbf{x}$-dependence is realised as an $\exp\left(\mathrm{i}\mathbf{p}_0\cdot\mathbf{x}\right)$ factor (provided we set $\hbar=1$). Of interest are inner products on the theory's wave functions; in particular, norms have a probabilistic interpretation. But all unitary transformations of the wave functions preserve these inner products, and this includes multiplication by $\exp\left(-\mathrm{i}\mathbf{p}_1\cdot\mathbf{x}\right)$ for any constant with units of wavenumber (or equivalently, since we set $\hbar=1$, units of linear kinetic momentum). The $\mathbf{x}$-dependence of our wave function is now realised as an $\exp\left(\mathrm{i}\left(\mathbf{p}_0-\mathbf{p}_1\right)\cdot\mathbf{x}\right)$ factor. The effect of this unitary transformation is a $\mathbf{p}$-space translation of the conserved linear kinetic momentum, from $\mathbf{p}_0$ to $\mathbf{p}_0-\mathbf{p}_1$. Indeed, in terms of an interpretation in terms of classical physics, this is simply a frame shift. The choice of $\mathbf{p}_1$ is arbitrary. We can therefore choose $\mathbf{p}_1=\mathbf{p}_0$, which sets the vector-valued conserved linear kinetic momentum to $\mathbf{0}\in\mathbb{R}^{n-1}$. Again, an interpretation in terms of classical physics is readily available; the frame shift used here is to the *zero momentum frame* or *centre of mass frame*. The outcome is an $\mathbf{x}$-independent wavefunction.

This result can be extensively generalised. The original $\exp\left(\mathrm{i}\mathbf{p}_0\cdot\mathbf{x}\right)$ factor exists because of the operator identification $\hat{\mathbf{p}}=-\mathrm{i}\nabla$ (i.e. $\mathbf{p}=-\mathrm{i}\frac{\partial}{\partial\mathbf{x}}$), where again the choice of units is such that $\hbar=1$. Thus the multiples of $\exp\left(\mathrm{i}\mathbf{p}_0\cdot\mathbf{x}\right)$ are simply the mutual eigenfunctions of the $\hat{p}_i$ with respective eigenvalues $\left(\mathbf{p}_0\right)_i$. The crux of this is that the Schrödinger picture's operator formalism implies a canonical commutator $[\hat{p}_i, \hat{x}_j] = -\mathrm{i}\delta_{ij}$, in analogy with Eq. (1.4.10). We say $\hat{p}_i$, $\hat{x}_j$ are *conjugate variables*. And given any conserved quantity which has a conjugate variable, the method described in the previous paragraph obtains a wavefunction that is independent of this second conjugate variable, because the first conserved variable now vanishes.

The inner product of two quantum-mechanical wavefunctions ϕ, ψ is an ordinary integral, whose integrand is an ordinary function $\phi^*\psi$. The Schrödinger wave functional formalism promotes ϕ, ψ to Schrödinger wave functionals Φ, Ψ (say), and also promotes the inner product to a functional integral of $\Phi^*\Psi$. A quantum-mechanical $\hat{q}$ conjugate to some $\hat{p}$ satisfies

$$\hat{p}\Psi \propto \frac{\delta}{\delta q}\Psi, \tag{1.7.3}$$

a functional derivative. If $\hat{p}$ depends on neither space (e.g. due to being a definite integral over space) nor time (i.e. $\hat{p}$ is conserved), one can set $\hat{p} = 0$, in analogy with the classical case. This includes the Π_T discussed herein for which T is cyclic. Setting such a $\hat{p}$ to zero implies Ψ is $\hat{q}$-dependent, so q, p are obviated from the Schrödinger wave functional formalism.

Note that the existence of a variable conjugate to conserved charges is crucial to these inferences. We cannot, for example, use this line of reasoning to show that Klein–Gordon inner products may be set to 0.

In the above grey note, I explained the circumstances in which conditions of the form $\Pi_T = 0$ may be imposed. Indeed, the CMP simultaneously imposes all such conditions, one for each such T.[22] If one does this, several consequences result:

- Ψ is T-independent,
- $\mathcal{L}_0$ is $\dot{T}$-independent, and
- Hamilton's equations are applicable to an expression for $\mathcal{H}_0$ that contains neither T nor its derivatives.

The Lagrangian, Hamiltonian and Schrödinger wave functional formalisms therefore lose all explicit dependence on T and its derivatives. Thus T is dynamically irrelevant and has been obviated from all three formalisms, resulting in a theory whose fields and propagators no longer contain any contributions from T.

This is how the CMP resolves all zero mode problems discussed in this thesis. The crux of the CMP's usage is to argue that, when conserved charges are set to zero on all physical states, the effective Hamiltonian is obtainable as a Legendre transform of the effective Lagrangian. The difficult step is always verifying that zero modes are "cyclic", in the aforementioned sense that the Lagrangian does not explicitly depend on it undifferentiated. The Klein–Gordon example is instructive. For an arbitrary scalar field χ define

[22] It might be objected that the canonical (anti)commutation relation between a field ϕ and its conjugate momentum density π for physical states of non-zero norm is inconsistent with their having $\hat{\pi}$–eigenvalue 0. However, there is in fact no such inconsistency. The usual relation between ϕ and π holds for a specific pseudo-inner product on the Hilbert space. Deleting the $\int \mathrm{d}\phi_{(0)}$ integration operator from the pseudo-inner product's definition obtains a new pseudo-inner product with respect to which these physical states might have nonzero norm. (This $\int \mathrm{d}\phi_{(0)}$ deletion is legitimate because Ψ is $\phi_{(0)}$-independent for the subspace of physical states considered in the CMP.) This detail is discussed in more detail in Sec. 3.9 and Appendix B.

The canonical (anti)commutation relations are an attempt to quantise the *Poisson brackets* in classical field theory. Theories I consider in later chapters introduce a further complication requiring *Dirac brackets*. I mention this again in Sec. 3.9, and address it in Appendix C.

$$\chi_{(0)}(t) := \int_{\mathbf{x}} \frac{g^{00}\chi}{V},\ \chi_{(+)}(x) := \chi - \chi_{(0)}(t). \tag{1.7.4}$$

Thus $\chi_{(0)}$ is a spatially uniform contribution to χ, and all spatially uniform χ are fixed points of the map $\chi_{(0)}$. Thus

$$\left(\chi_{(0)}\right)_{(0)} = \chi_{(0)},\ \left(\chi_{(+)}\right)_{(0)} = \left(\chi - \chi_{(0)}\right)_{(0)} = \chi_{(0)} - \left(\chi_{(0)}\right)_{(0)} = 0. \tag{1.7.5}$$

An alternative notation for $\chi_{(0)}$ is

$$\chi_{(0)}(t) = \int \mathrm{d}^{n-1}\mathbf{x}\Upsilon(x)\chi(x),\ \Upsilon := \frac{\sqrt{|g|}g^{00}}{V} = \frac{a^{n-1}\sqrt{\eta}}{NV}. \tag{1.7.6}$$

Note that $\int \mathrm{d}^{n-1}\mathbf{x}\Upsilon = 1$. For any choice of N which is a product of a function of space and a function of time, so is Υ, and

$$\int \mathrm{d}^{n-1}\mathbf{x}\dot{\Upsilon}\chi_{(+)} = \frac{\dot{\Upsilon}}{\Upsilon}\left(\chi_{(+)}\right)_{(0)} = 0, \tag{1.7.7}$$

$$\begin{aligned}\partial_0\left(\chi_{(0)}\right) - (\dot{\chi})_{(0)} &= \int \mathrm{d}^{n-1}\mathbf{x}\dot{\Upsilon}\chi = \int \mathrm{d}^{n-1}\mathbf{x}\dot{\Upsilon}\chi_{(0)}\\ &= \chi_{(0)}\partial_0\int \mathrm{d}^{n-1}\mathbf{x}\Upsilon = 0.\end{aligned} \tag{1.7.8}$$

Thus for spacetimes of interest one can consider gauge choices for which $\dot{\chi}_{(0)}$ is unambiguous. For $M = 0$ the zero mode $\phi_{(0)}$ is cyclic. A massless Klein–Gordon field of manifestly real scalar Lagrangian density

$$\mathcal{L}_0 = g^{\mu\nu}\nabla_\mu\phi^*\nabla_\nu\phi \tag{1.7.9}$$

provides the conserved charge Π_{ϕ^*}, which for $N^i = 0$ is given by

$$\Pi_{\phi^*} = \int_{\mathbf{x}} g^{00}\dot{\phi} = V\dot{\phi}_{(0)}, \tag{1.7.10}$$

so that $\dot{\phi}_{(0)} \propto V^{-1}$. This corollary of the definition of $\chi_{(0)}$ in Eq. (1.7.4) is consistent with the spatially uniform Klein–Gordon modes found in Eq. (1.6.4), provided $\dot{N} = 0$ so

$$V = a^{n-1}\int \mathrm{d}^{n-1}\mathbf{x}\frac{\sqrt{\eta}}{N} \propto a^{n-1} = \dot{f}^{-1}. \tag{1.7.11}$$

Thus $\chi_{(0)}$ can be taken as the definition of *the zero mode of* χ, so that $\chi_{(+)}$ contains any "other" modes of χ.

To obviate zero modes from the Hamiltonian, we first need to write

$$\mathcal{H}_0 = \dot{\phi}\varpi_\phi + \dot{\phi}^*\varpi_{\phi^*} - \nabla_\mu\phi^*\nabla^\mu\phi \tag{1.7.12}$$

in terms of $\varpi_\phi = \nabla^0\phi^*$, $\varpi_{\phi^*} = \nabla^0\phi$, $\nabla_i\phi$ and $\nabla_i\phi^*$ to use Hamilton's equations. We thus need to express ∇_0, ∇^i in terms of ∇^0, ∇_j. Since $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$, Eq. (1.2.1) gives $g_{\mu\nu}$ as

$$g_{00} = N^2 - \gamma_{ij}N^iN^j,\ g_{0i} = -\gamma_{ij}N^j,\ g_{ij} = -\gamma_{ij}. \tag{1.7.13}$$

It is well-known [4] that $g_{\mu\nu}$ has an inverse $g^{\mu\nu}$ given by

$$g^{00} = N^{-2},\, g^{0i} = -N^{-2}N^i,\, g^{ij} = N^{-2}N^iN^j - \gamma^{ij}, \tag{1.7.14}$$

where γ^{ij} be the inverse of γ_{ij} as a matrix, i.e. $\gamma^{ij}\gamma_{jk} = \delta^i_k$. However this result was first realised, it is obvious once stated because Eqs. (1.7.13) and (1.7.14) together imply $g_{\mu\rho}g^{\rho\nu} = \delta^\nu_\mu$. Hence

$$\begin{aligned}\nabla_0 &= N^2\left(N^{-2}\nabla_0 - N^{-2}N^i\nabla_i\right) + N^i\nabla_i\\ &= N^2\nabla^0 + N^i\nabla_i, &(1.7.15)\\ \nabla^i &= -N^{-2}N^i\nabla_0 + \left(N^{-2}N^iN^j - \gamma^{ij}\right)\nabla_j\\ &= N^i\left(N^{-2}N^j\nabla_j - N^{-2}\nabla_0\right) - \gamma^{ij}\nabla_j\\ &= -N^i\nabla^0 - \gamma^{ij}\nabla_j, &(1.7.16)\\ \dot{\phi} &= N^2\varpi_{\phi^*} + N^i\nabla_i\phi, &(1.7.17)\\ \nabla^i\phi &= -N^i\varpi_{\phi^*} + \gamma^{ij}\nabla_j\phi. &(1.7.18)\end{aligned}$$

Eqs. (1.7.15)-(1.7.18) imply

$$\begin{aligned}\mathcal{H}_0 &= \dot{\phi}\varpi_\phi + \dot{\phi}^*\varpi_{\phi^*} - \frac{1}{2}\left(\mathcal{L}_0^* + \mathcal{L}_0\right)\\ &= \dot{\phi}\varpi_\phi - \frac{1}{2}\nabla_0\phi\nabla^0\phi^* - \frac{1}{2}\nabla_i\phi\nabla^i\phi^*\\ &+ \dot{\phi}^*\varpi_{\phi^*} - \frac{1}{2}\nabla_0\phi^*\nabla^0\phi - \frac{1}{2}\nabla_i\phi^*\nabla^i\phi\\ &= \frac{1}{2}\left(N^2\varpi_{\phi^*} + N^i\nabla_i\phi\right)\varpi_\phi + \frac{1}{2}\nabla_i\phi\left(N^i\varpi_\phi + \gamma^{ij}\nabla_j\phi^*\right)\\ &+ \frac{1}{2}\left(N^2\varpi_\phi + N^i\nabla_i\phi^*\right)\varpi_{\phi^*} + \frac{1}{2}\nabla_i\phi^*\left(N^i\varpi_{\phi^*} + \gamma^{ij}\nabla_j\phi\right)\\ &= N^2\varpi_{\phi^*}\varpi_\phi + N^i\left(\nabla_i\phi\varpi_\phi + \nabla_i\phi^*\varpi_{\phi^*}\right) + \gamma^{ij}\nabla_i\phi\nabla_j\phi^*. &(1.7.19)\end{aligned}$$

Since $\nabla_i\phi_{(0)} = 0$, $\phi_{(0)}$ and its derivatives have been obviated, as have $\phi^*_{(0)}$ and its derivatives. Momentum densities can be split into "zero" and "other" modes too, viz. Sec. 3.2. When the CMP sets conserved momenta to 0, it reduces $\varpi_\phi,\, \varpi_{\phi^*}$ to $\varpi_{\phi_{(+)}},\, \varpi_{\phi^*_{(+)}}$. The end result is of the form

$$\mathcal{H}_0 = \mathcal{H}_0\left(\phi_{(+)},\, \nabla_i\phi_{(+)},\, \varpi_{\phi_{(+)}},\, \phi^*_{(+)},\, \nabla_i\phi^*_{(+)},\, \varpi_{\phi^*_{(+)}}\right). \tag{1.7.20}$$

Thus zero modes have been obviated, including from the momentum sector.

This strategy removes $\phi_{(0)}$ from $\mathcal{H}_0$. In Sec. 3.1, I will discuss removal of zero modes from Hamiltonians and Lagrangians, and show that the two are equivalent. Indeed, we can switch to the Lagrangian formalism now, which in this case gives a shorter treatment. Setting $\Pi_{\phi^*} = 0$ in the CMP gives $\dot{\phi}_{(0)} = 0$, so $\nabla_\mu\phi_{(0)} = 0$ and

$$\mathcal{L}_0 = \nabla_\mu\phi^*_{(+)}\nabla^\mu\phi_{(+)}. \tag{1.7.21}$$

Thus $\phi_{(0)}$ is obviated in this formalism.

1.8 Comments on the flat static torus and de Sitter space

Throughout this section I consider minimally coupled solutions of the Klein–Gordon equation. Both the flat static torus and de Sitter space can be considered in Cartesian coordinates satisfying

$$N = 1,\ N^i = 0,\ \gamma_{ij} = a^2(t)\,\delta_{ij},\ \dot{f} = a^{1-n}. \tag{1.8.1}$$

The Klein–Gordon equation for zero modes then simplifies to

$$\partial_0\left(\frac{\dot{\phi}}{\dot{f}}\right) = -\frac{M^2\phi}{\dot{f}}. \tag{1.8.2}$$

Thus $A + Bf$ is a solution if and only if $M = 0$. In particular, massless solutions are of this form and massive ones are not. However, the flat static torus obtains $f = t$, and approximation of a massive mode at sufficiently low order in M is of the form $A + Bf$. In particular, $\frac{1-\mathrm{i}Mt}{\sqrt{2ML^{n-1}}}$ is Klein–Gordon normalised.

In Sec. 1.8.1 I show how to compute $f(t)$ for de Sitter space. I write $f = I_{n-1}$, where the functions I_k are defined in Eq. (1.8.8); I obtain I_1, I_2, and an expression for I_{k+2} in terms of I_k. In Sec. 1.8.2 I discuss Ref. [1]'s computation of the Klein–Gordon normalised de Sitter-invariant massive zero modes in de Sitter space at order M. Ref. [1] discusses this in its Sec. II and Appendices A and F. The paper's Sec. II normalises massive spatially uniform Klein–Gordon fields in de Sitter space, although some details of this are postponed to the paper's Appendix A. The most important finding is that, in the de Sitter–invariant case, such modes are of the form

$$\frac{1}{\sqrt{2b_0}}\left\{\frac{1}{M} - M\left[f_2(t) + b_1 + \mathrm{i}b_0 f(t)\right]\right\} + o(M) \tag{1.8.3}$$

where

$$f_2 := \int_0^t \left[\dot{f}\left(t'\right)\int_0^{t'} \frac{\mathrm{d}\tau}{\dot{f}(\tau)}\right]\mathrm{d}t', \tag{1.8.4}$$

$$b_0 := \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)H^n}, \tag{1.8.5}$$

$$b_1 := \frac{\psi(n-1) - \psi\left(\frac{n}{2}\right) - \psi(1) + \psi\left(\frac{1}{2}\right)}{2(n-1)H^2} \tag{1.8.6}$$

(with $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$ the *digamma function*, viz. Chapter 6 of Ref. [23]). This approximation is not of the form $A(M) + B(M)f$, because the $\mathcal{O}(M)$ contribution is not proportional to f. I make further comments on this in Sec. 1.8.3.

Ref. [1] also discusses the case of the flat static torus in more detail in its Appendix E. I will discuss this material herein in Sec. 3.8. In particular, Ref. [1]'s Eq. (F8) will warrant a more detailed proof than in Ref. [1]. This proof must distinguish the cases $n = 2$, $n = 3$ and $n \geq 4$, but only the $n \geq 4$ case is considered in Ref. [1].

1.8.1 Massless zero modes in de Sitter space

For any n, an analytic expression exists for $f(t)$ in the case of de Sitter space. For the purposes of this calculation I nondimensionalise time by setting $H = 1$, viz.

$$a = \cosh t,\ f(t) = \int_0^t \operatorname{sech}^{n-1} u \mathrm{d}u. \tag{1.8.7}$$

(I will preserve this nondimensionalisation throughout Sec. 1.8. As a result, a ratio $\frac{M}{H}$ in a discussion of massive solutions of the Klein–Gordon equation is simplified to M. This has the effect of nondimensionalising mass too.) Thus $f(t) = I_{n-1}(t)$, where

$$I_k(t) := \int_0^t \operatorname{sech}^k u \mathrm{d}u. \tag{1.8.8}$$

The method for computing the I_k is standard. By inspection $I_0 = t$ and $I_2 = \tanh t$, and I_1 is the famous *Gudermannian function* [24], viz.

$$I_1 = \operatorname{gd}(t) := \int_0^t \operatorname{sech} u \mathrm{d}u = \int_0^t \frac{2\mathrm{e}^u}{1+\mathrm{e}^{2u}} \mathrm{d}u = \int_1^{\mathrm{e}^t} \frac{2\mathrm{d}y}{1+y^2} = [2\arctan y]_1^{\mathrm{e}^t} = 2\arctan \mathrm{e}^t - \frac{\pi}{2}. \tag{1.8.9}$$

For greater values of k a recursion is needed. Write

$$I_{k+2} = \int_0^t g(u) \frac{\mathrm{d}h}{\mathrm{d}u} \mathrm{d}u \tag{1.8.10}$$

where $g(u) := \operatorname{sech}^k t,\ h(u) := \tanh t$. Integrating by parts, and using $\tanh^2 t = 1 - \operatorname{sech}^2 t$, gives

$$\begin{aligned} I_{k+2}(t) &= \operatorname{sech}^k t \tanh t + \int_0^t k \operatorname{sech}^k t \left(1 - \operatorname{sech}^2 t\right) \mathrm{d}t \\ &= \operatorname{sech}^k t \tanh t + kI_k(t) - kI_{k+2}(t), \qquad (1.8.11) \\ I_{k+2}(t) &= \frac{kI_k(t) + \operatorname{sech}^k t \tanh t}{k+1}. \qquad (1.8.12) \end{aligned}$$

(Note that this recursion would break down if we attempted to use it to obtain I_1 from $I_{-1} = \sinh t$.) For example, for $n = 4$ de Sitter spacetime

$$f(t) = I_3(t) = \frac{I_1(t) + \operatorname{sech} t \tanh t}{2} = \frac{\operatorname{gd}(t) + \operatorname{sech} t \tanh t}{2}. \tag{1.8.13}$$

1.8.2 Massive zero modes

Eq. (1.8.2) is a second-order ordinary differential equation, so its solution space is 2-dimensional. For $i \in \{0, 1\}$ and fixed M define the functions

$$\begin{aligned} a_0^{(i,M)}(t) &:= f^i, \qquad (1.8.14) \\ a_{k+1}^{(i,M)}(t) &:= \int_0^t \left[\dot{f}\left(t'\right) \int_0^{t'} \frac{a_k^{(i,M)}(\tau)\,\mathrm{d}\tau}{\dot{f}(\tau)}\right] \mathrm{d}t', \qquad (1.8.15) \\ a^{(i,M)}(t) &:= \sum_{k=0}^{\infty} \left(-M^2\right)^k a_k^{(i,M)}(t). \qquad (1.8.16) \end{aligned}$$

Then $\partial_0\left(\dot{f}^{-1}\dot{a}_0^{(i,\,M)}\right)=0$, so

$$\partial_0\left(\frac{\dot{a}^{(i,\,M)}}{\dot{f}}\right)=\sum_{k=1}^{\infty}\left(-M^2\right)^k\frac{a_k^{(i,\,M)}(t)}{f(t)}=-\frac{M^2a^{(i,\,M)}}{f}, \tag{1.8.17}$$

i.e. $a^{(i,\,M)}$ solves Eq. (1.8.2). Thus $a^{(M)}:=a^{(0,\,M)}$, $b^{(M)}:=a^{(1,\,M)}$ form a basis of the solution space, with

$$\lim_{M^2\to 0}a^{(M)}=1,\ \lim_{M^2\to 0}b^{(M)}=f. \tag{1.8.18}$$

In practice, however, there is a problem with attempting to associate a massless solution $A+Bf$ with the mass–M solution $Aa^{(M)}+Bb^{(M)}$. Doing so would be inconsistent with the implications of commonly required spacetime symmetries for the M-dependence if A, B are assumed M-independent. For example, we previously saw that, for the flat static torus, we require $B=-\mathrm{i}MA$ if we seek time translation invariance, and normalisation requires $AB^*\neq 0$. More generally, a Klein–Gordon normalised solution $Aa^{(M)}+Bb^{(M)}$ satisfies

$$1=\frac{2V_c\,\mathrm{Im}\left(AB^*\right)}{\dot{f}}a^{(M)}\overleftrightarrow{\partial_0}b^{(M)},\ a^{(M)}\overleftrightarrow{\partial_0}b^{(M)}=\frac{\dot{f}}{2V_c\,\mathrm{Im}\left(AB^*\right)}. \tag{1.8.19}$$

If A, $\mathrm{i}B>0$ as in the massless case in Sec. (1.6), $AB^*=-AB\in\mathrm{i}\mathbb{R}$ and $\mathrm{Im}\left(AB^*\right)=\mathrm{i}AB>0$. For example, the solution in Eqs. (1.8.4)–(1.8.6) satisfies

$$A\approx\frac{M^{-1}-b_1M}{\sqrt{2b_0}},\ B\approx -\mathrm{i}M\sqrt{\frac{b_0}{2}}. \tag{1.8.20}$$

The $o\left(M\right)$ term in Eq. (1.8.3) explains why these results are only approximate. Due to higher-order terms, A, B are infinite power series in M (with $A=\mathcal{O}\left(M^{-1}\right)$, $B=\mathcal{O}\left(M\right)$) and $\mathrm{i}AB=\frac{1}{2}$, so $a^{(M)}\overleftrightarrow{\partial_0}b^{(M)}=\dot{f}$.

It remains to be verified that the conditions in Eq. (1.8.20) are applicable up to a phase to the de Sitter-invariant solution. Note that these conditions are equivalent to Eqs. (1.8.4)–(1.8.6). It is for this that we turn to some technical details previously published in Appendix A of Ref. [1]. I will assume the reader's familiarity with the *hypergeometric function*

$${}_2F_1\left(\alpha,\,\beta;\,\gamma;\,z\right):=\sum_{k=0}^{\infty}\frac{\left(\alpha\right)_k\left(\beta\right)_k}{\left(\gamma\right)_k}\frac{z^k}{k!}, \tag{1.8.21}$$

where $\left(\alpha\right)_k:=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ is the *Pochhammer symbol*. (A good discussion of this function is given in Chapter 15 of Ref. [23].) The mass-M zero modes are spanned by the functions [1, 25]

$$f_\ell\left(t\right)=\frac{N_\ell\left(\cosh t\right)^\ell}{2^{\ell+\frac{n-2}{2}}}{}_2F_1\left(b_{\ell+},\,b_{\ell-};\,\ell+\frac{n}{2},\,\frac{1-\mathrm{i}\sinh t}{2}\right), \tag{1.8.22}$$

with ℓ a non-negative integer and

$$\lambda:=\sqrt{\left(\frac{n-1}{2}\right)^2-M^2},\ b_{\ell\pm}:=\ell+\frac{n-1}{2}\pm\lambda,\ N_\ell:=\sqrt{\frac{\Gamma\left(b_{\ell+}\right)\Gamma\left(b_{\ell-}\right)}{2}}. \tag{1.8.23}$$

The de Sitter-invariant Bunch–Davies vacuum state is obtained from $\ell = 0$. Note that

$$b_{\ell+} + b_{\ell-} = 2\ell + n - 1,\ b_{\ell+} b_{\ell-} = \ell\left(\ell + n - 1\right) + M^2, \tag{1.8.24}$$

so for small M we have the approximations

$$b_{0+} \approx n - 1 - \frac{M^2}{n-1},\ b_{0-} \approx \frac{M^2}{n-1},\ N_0 \approx \sqrt{\frac{\Gamma\left(n\right)}{2}} M^{-1}. \tag{1.8.25}$$

Formula 9.136.1 in Ref. [26] is the identity

$$\begin{aligned} {}_2F_1\left(2\alpha,\, 2\beta;\, \alpha + \beta + \frac{1}{2};\, \frac{1 - \sqrt{z}}{2}\right) &= \frac{\Gamma\left(\alpha + \beta + \frac{1}{2}\right)\sqrt{\pi}}{\Gamma\left(\alpha + \frac{1}{2}\right)\Gamma\left(\beta + \frac{1}{2}\right)}\, {}_2F_1\left(\alpha,\, \beta;\, \frac{1}{2};\, z\right) \\ &\quad - \frac{\Gamma\left(\alpha + \beta + \frac{1}{2}\right) 2\sqrt{\pi}}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}\, {}_2F_1\left(\alpha + \frac{1}{2},\, \beta + \frac{1}{2};\, \frac{3}{2};\, z\right). \end{aligned} \tag{1.8.26}$$

For small ϵ,

$$\frac{\Gamma\left(z\right)}{\Gamma\left(z \pm \epsilon\right)} = \exp\left\{\ln\Gamma\left(z\right) - \ln\Gamma\left(z \pm \epsilon\right)\right\} \approx \mathrm{e}^{\mp\epsilon\psi(z)}. \tag{1.8.27}$$

Taking $\alpha = \frac{b_{0+}}{2}$, $\beta = \frac{b_{0-}}{2}$, $\sqrt{z} = \mathrm{i}\sinh t$ in Eq. (1.8.26) gives

$$\alpha + \beta + \frac{1}{2} = \frac{n}{2}, \tag{1.8.28}$$

$$\begin{aligned} \frac{\Gamma\left(\alpha + \beta + \frac{1}{2}\right)\sqrt{\pi}}{\Gamma\left(\alpha + \frac{1}{2}\right)\Gamma\left(\beta + \frac{1}{2}\right)} &\approx \frac{\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n}{2} - \frac{M^2}{2(n-1)}\right)\Gamma\left(\frac{1}{2} + \frac{M^2}{2(n-1)}\right)} \\ &= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2} - \frac{M^2}{2(n-1)}\right)} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{M^2}{2(n-1)}\right)} \\ &\approx \exp\left\{\frac{M^2}{2\left(n-1\right)}\left[\psi\left(\frac{n}{2}\right) - \psi\left(\frac{1}{2}\right)\right]\right\} \\ &\approx 1 + \frac{\left[\psi\left(\frac{n}{2}\right) - \psi\left(\frac{1}{2}\right)\right] M^2}{2\left(n-1\right)}, \end{aligned} \tag{1.8.29}$$

$$\begin{aligned} \frac{\Gamma\left(\alpha + \beta + \frac{1}{2}\right) 2\sqrt{\pi}}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)} &\approx \frac{\Gamma\left(\frac{n}{2}\right) 2\sqrt{\pi}}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{M^2}{2(n-1)}\right)} \\ &\approx \frac{\Gamma\left(\frac{n}{2}\right) 2\sqrt{\pi}}{\Gamma\left(\frac{n-1}{2}\right)\frac{2(n-1)}{M^2}} = \frac{\Gamma\left(\frac{n}{2}\right)\sqrt{\pi} M^2}{2\Gamma\left(\frac{n+1}{2}\right)}, \end{aligned} \tag{1.8.30}$$

and

$$
\begin{aligned}
{}_2F_1\left(b_{0+},\, b_{0-};\, \frac{n}{2},\, \frac{1-\mathrm{i}\sinh t}{2}\right) &\approx \left(1+\frac{\left[\psi\left(\frac{n}{2}\right)-\psi\left(\frac{1}{2}\right)\right]M^2}{2\left(n-1\right)}\right) \\
&\times {}_2F_1\left(\frac{n-1}{2},\, \frac{M^2}{2\left(n-1\right)};\, \frac{1}{2};\, -\sinh^2 t\right) \\
&-\frac{\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}M^2}{2\Gamma\left(\frac{n+1}{2}\right)}\,{}_2F_1\left(\frac{n}{2},\, 1+\frac{M^2}{2\left(n-1\right)};\, \frac{3}{2};\, -\sinh^2 t\right) \\
&\approx 1-M^2\left\{\frac{\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{n+1}{2}\right)}\,{}_2F_1\left(\frac{n}{2},\, 1;\, \frac{3}{2};\, -\sinh^2 t\right)\right. \\
&\left.-\frac{\psi\left(\frac{n}{2}\right)-\psi\left(\frac{1}{2}\right)}{2\left(n-1\right)}\,{}_2F_1\left(\frac{n-1}{2},\, 0;\, \frac{1}{2};\, -\sinh^2 t\right)\right\} \\
&\approx 1-M^2\left\{f_2\left(t\right)-\frac{\psi\left(\frac{n}{2}\right)-\psi\left(\frac{1}{2}\right)}{2\left(n-1\right)}-\mathrm{i}b_0 f\left(t\right)\right\}. \qquad (1.8.31)
\end{aligned}
$$

The right-hand side of Eq. (1.8.31) reduces $f_0\left(t\right)$ to Eq. (1.8.3) (since $N_0 \approx \sqrt{\frac{\Gamma(n)}{2}}M^{-1}$).

1.8.3 Discussion of de Sitter-invariant Klein–Gordon normalised massive zero modes in de Sitter space

In the low-mass limit $M^{\pm 1}$ terms appear. This differs from the $M^{\pm 1/2}$ factors obtained for the flat static torus.

When Klein–Gordon normalised massive zero modes are written as power series in M, they have infinitely many terms. With time translation-invariant zero modes on the flat static torus, the f term is the only term $\in \mathcal{O}\left(\sqrt{M}\right)$. In de Sitter space, the f and f_2 terms are both $\in \mathcal{O}\left(M\right)$ in a de Sitter-invariant zero mode.

However, the only relevant terms are those which are leading order in $\operatorname{Re} f$ or $\operatorname{Im} f$. For $\operatorname{Re} f$, only the $\mathcal{O}\left(M^{-1}\right)$ term is relevant; for $\operatorname{Im} f$, only the $\mathcal{O}\left(M\right)$ term is relevant. Since $f_2 \in \mathcal{O}\left(M\right)$ is real-valued, it is irrelevant to the comparison of Klein–Gordon normalised massless solutions of the Klein–Gordon equation with Klein–Gordon normalised massive solutions of the Klein–Gordon equation.

Chapter 2 A review of literature on two theories and their zero mode problems

This chapter has three main purposes. The first purpose is to present the background required to understand the zero mode problems I address in Part II and associated appendices. The second purpose is to explain these zero mode problems. The third purpose is to review the literature relevant to the aforementioned zero mode problems. Our research forms a small part of that literature review, and will not be discussed until the later stages of this chapter. I will also discuss flaws I have identified in some of the discussion of perturbative gravity's internal symmetries.

In Sec. 2.1, I explain the construction of Yang–Mills theory as a generalisation of classical electromagnetism. Yang–Mills theory provides a description of all known non-gravitational fundamental interactions. In Sec. 2.2, I discuss and motivate a modification of Yang–Mills theory, due to Faddeev and Popov [27, 28], that is required in the path integral formalism. Following Ref. [27], I call this the *Faddeev–Popov method*; the result is called *BRST quantisation*, since it motivates the *BRST and anti-BRST transformations* discussed in Sec. 2.3. The gauge invariance of Yang–Mills theory is broken when the theory is BRST-quantised. In Sec. 2.3, I discuss two symmetries of BRST-quantised Yang–Mills theory. The BRST quantisation of Yang–Mills theory introduces three new scalar fields. In Sec. 2.4, I describe a zero mode problem concerning these new fields. This zero mode problem is analogous to the one discussed in Chapter 1.

A number of researchers have considered the zero mode problem in BRST-quantised Yang–Mills theory. In Sec. 2.5, I review the history of such research. The FMP is one prescription for addressing this zero mode problem. However, the FMP is not manifestly consistent with the preservation of the symmetries described in Sec. 2.3. I discuss this difficulty in more detail in Sec. 2.5.2 to motivate the use of the CMP. This treatment will require the entirety of Chapter 3 to be properly developed.

Perturbative gravity has a number of important similarities with Yang–Mills theory. Both theories can be BRST-quantised; in both cases, BRST quantisation presents certain symmetries and a zero mode problem; and the benefits of the CMP compared with the FMP are similar in the two theories. In Sec. 2.6, I discuss the case of perturbative gravity as concisely as possible, making use of its various similarities with the case of Yang–Mills theory. My discussion of perturbative gravity will include a criticism of an occasional mistake by researchers regarding perturbative gravity's internal symmetries. My treatment of the zero mode problem in BRST-quantised perturbative gravity with the CMP is provided in Chapter 4.

2.1 From electromagnetism to Yang–Mills theory

In Sec. 2.1.1, I introduce the manifestly Lorentz-covariant formalism of classical electromagnetism. In Sec. 2.1.2, I discuss classical electromagnetism's conservation laws. In Sec. 2.1.3, I discuss a generalisation of classical electromagnetism called *Yang–Mills theory*, which was developed in Ref. [29]. Any use of Yang–Mills theory as a physical model requires the specification of a Lie group called the *gauge group* of the model. For example, electromagnetism is the special case of Yang–Mills theory obtained when the gauge group chosen is denoted $\mathrm{U}(1)$. I discuss the relevant theory and notation in more detail in Sec. 2.1.3. In Sec. 2.1.4, I discuss the gauge groups that are relevant to uses of Yang–Mills theory in the Standard Model.

2.1.1 An overview of classical electromagnetism

The scalar field theory considered in Chapter 1 admits a global internal symmetry, $\phi(x) \to \mathrm{e}^{\mathrm{i}\theta}\phi(x)$ for constant $\theta \in \mathbb{R}$. To promote this to a local internal symmetry $\phi(x) \to \mathrm{e}^{\mathrm{i}\theta(x)}\phi(x)$ requires an amendment to the choice of $\mathcal{L}_0$, viz.

$$\mathcal{L}_0 = D_\mu\phi^* D^\mu\phi - M^2\phi^*\phi,\ D^\mu\phi := \nabla^\mu\phi + \mathrm{i}qA^\mu\phi. \tag{2.1.1}$$

Here a vector field A^μ has been introduced, as has a coupling constant q called a *charge*. The internal transformation $\phi(x) \to \mathrm{e}^{\mathrm{i}\theta(x)}\phi(x)$ must also be accompanied with an appropriate transformation of A^μ, say $A^\mu \to A'^\mu$, so that $D^\mu\phi$ transforms to $\mathrm{e}^{\mathrm{i}\theta}\left(\nabla^\mu\phi + qA'^\mu\phi\right)$. (Explicitly, $A'_\mu = A_\mu + \frac{\mathrm{i}}{q}\partial_\mu\theta$.) Classical electromagnetism can be explained in terms of the behaviour of A^μ and its associated charges. The scalar Lagrangian density of classical electromagnetism with an external current j_μ may be written as[23]

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A^\mu j_\mu,\ F_{\mu\nu} := \nabla_\mu A_\nu - \nabla_\nu A_\mu. \tag{2.1.2}$$

Since $(\nabla_\mu - \partial_\mu)A_\nu = -\Gamma^\rho_{\mu\nu}A_\rho$ is $\mu \leftrightarrow \nu$–symmetric, we could alternatively write $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ even for curved spacetimes, and indeed this expression is used ubiquitously. The main advantage of the ∇-based definition in Eq. (2.1.2) is that all terms that appear are tensors ($\nabla_\mu A_\nu$ is a tensor in any spacetime, while $\partial_\mu A_\nu$ is not). In Sec. 1.3 I described a Lagrangian formalism that never missed an opportunity to use true tensors. This approach provides an easy derivation of the Euler–Lagrange equation due to varying A^μ; repeatedly using the fact that $F_{\mu\nu}$ is antisymmetric,

$$j^\nu = -\frac{\partial\mathcal{L}_0}{\partial A_\nu} = -\nabla_\mu\frac{\partial}{\partial\nabla_\mu A_\nu}\left(-\frac{1}{2}\nabla_\rho A_\sigma F^{\rho\sigma}\right) = \nabla_\mu F^{\mu\nu}. \tag{2.1.3}$$

Note that $\frac{\delta S}{\delta j_\mu} = -A^\mu$ is not set to zero as part of the "stationary action principle"; j_μ is not dynamical.

[23] Eq. (2.1.2) does not include contributions due to matter fields, for two key reasons. The first is that such terms are irrelevant to all subsequent calculations in this thesis, except for the use of Eq. (2.1.9) below to motivate Yang–Mills theory. In particular, the Euler–Lagrange equations of interest are not contingent on matter fields. The second is that such matter terms depend on the matter theorised. The Dirac spinor term $\overline{\psi}(\mathrm{i}\gamma^\mu D_\mu - m)\psi$ for a Dirac spinor ψ of mass m and gamma matrices γ^μ is *just one* form such a term could take. The scalar theory in Eq. (2.1.9), which generalises that of Chapter 1, is another possible form for a matter term.

2.1.2 Conservation laws in electromagnetism

Any antisymmetric tensor $X^{\mu\nu}$, such as $F^{\mu\nu}$, satisfies

$$\nabla_\mu X^{\mu\nu} = \partial_\mu X^{\mu\nu} + \Gamma^\mu_{\mu\rho} X^{\rho\nu} + \Gamma^\nu_{\mu\rho} X^{\mu\rho} = \partial_\rho X^{\rho\nu} + X^{\rho\nu}\partial_\rho \ln\sqrt{|g|} = \frac{1}{\sqrt{|g|}}\partial_\rho\left(\sqrt{|g|}X^{\rho\nu}\right) \tag{2.1.4}$$

(the term $\Gamma^\nu_{\mu\rho}X^{\mu\rho}$ vanishes because $\Gamma^\nu_{\mu\rho}$ is $\mu \leftrightarrow \rho$–antisymmetric). If V_ν is a vector field for which $\nabla_\mu V_\nu$ is symmetric (e.g. $V_\nu = \partial_\nu \chi$ for some scalar field χ),

$$\begin{aligned}\partial_0 \int_{\mathbf{x}} V_i X^{0i} &= \partial_0 \int_{\mathbf{x}} V_\nu X^{0\nu} = \int \mathrm{d}^{n-1}\mathbf{x}\partial_0\left(\sqrt{|g|}V_\nu X^{0\nu}\right) \\ &= \int \mathrm{d}^{n-1}\mathbf{x}\partial_\mu\left(\sqrt{|g|}V_\nu X^{\mu\nu}\right) = \int_{\mathbf{x}} \nabla_\mu\left(V_\nu X^{\mu\nu}\right) \\ &= \int_{\mathbf{x}}\left(V_\nu \nabla_\mu X^{\mu\nu}\right) = \int \mathrm{d}^{n-1}\mathbf{x}\left(V_\nu\partial_\mu\left(\sqrt{|g|}X^{\mu\nu}\right)\right).\end{aligned} \tag{2.1.5}$$

Like vector fields, antisymmetric rank-two tensors provide a natural source of conservation laws; $\int_{\mathbf{x}} V_i X^{0i}$ is conserved if $\nabla_\mu X^{\mu\nu} = 0$. This condition reduces on-shell to $j^\nu = 0$ for the special case $X^{\mu\nu} = F^{\mu\nu}$. (The condition $j^\nu = 0$, which may be restated as the absence of electric charges or motion thereof, is valid in a vacuum. In that case the field equation is homogeneous, viz. $\nabla_\mu F^{\mu\nu} = 0$. Thus the quantised field $\hat{A}^\mu$ has a mode decomposition analogous to Eq. (1.4.15) that can be used to show the VEV is 0.)

There is also an important example of a spatial integral that vanishes off-shell for any antisymmetric $X^{\mu\nu}$. Since $X^{00} = 0$,

$$\int_{\mathbf{x}} \nabla_i X^{i0} = \int_{\mathbf{x}} \nabla_\mu X^{\mu 0} = \int \mathrm{d}^{n-1}\mathbf{x}\partial_\mu\left(\sqrt{|g|}X^{\mu 0}\right) = \int \mathrm{d}^{n-1}\mathbf{x}\partial_i\left(\sqrt{|g|}X^{i0}\right) = 0. \tag{2.1.6}$$

(Conserved charges that "differ" by a quantity of this form are therefore equal; this is integral to a treatment of Noether charges that will be discussed in Sec. 4.3 and Appendix E.) In the special case $X^{\mu\nu} = F^{\mu\nu}$, this vanishing integral is proportional to the electric charge $\int_{\mathbf{x}} j^0$ by Gauss's law.[24] Related to this is the fact that

$$\nabla_\mu \nabla_\nu X^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\nabla_\nu X^{\mu\nu}\right) = \frac{1}{\sqrt{|g|}}\partial_\mu\partial_\nu\left(\sqrt{|g|}X^{\mu\nu}\right) = 0 \tag{2.1.7}$$

(since $\partial_\mu\partial_\nu$, $X^{\mu\nu}$ are respectively symmetric and antisymmetric), so on-shell

$$\nabla_\mu j^\mu = -\nabla_\mu \nabla_\nu F^{\mu\nu} = 0. \tag{2.1.8}$$

This provides an alternative proof that $\int_{\mathbf{x}} j^0$ is conserved on-shell, a considerably weaker result than the fact that $\int_{\mathbf{x}} j^0 = 0$. However, the on-shell result $\nabla_\mu j^\mu = 0$ is interesting in its own right. It is typical to call conserved spatial integrals *charges*, and to call a solution of $\nabla_\mu W^\mu = 0$ a *conserved current*.

Symmetric $X^{\mu\nu}$ satisfying $\nabla_\mu X^{\mu\nu} = 0$ provide an analogous source of conservation laws that become relevant in Sec. 2.6.2.

[24] The boundary terms of integration by parts are herein assumed to vanish throughout; for example, the universe's total electric charge $\int_{\mathbf{x}} j^0$ vanishes. Indeed, we would not expect the universe's electric charge to have a nonzero conserved value that has persisted since the Big Bang.

2.1.3 Yang–Mills theory

Classical electromagnetism for flat spacetime dates to the nineteenth century. When the strong and weak nuclear interactions were discovered, physicists considered whether a generalisation of the electromagnetic formalism could be used to describe nuclear interactions. It was eventually realised [29] that this was indeed the case, with a few key modifications. The result is *Yang–Mills theory*, a brief overview of which is given in Chapter IV.5 of Ref. [30].

Not only does the vector field A^μ of electromagnetism have counterparts for nuclear interactions; all of these vector fields could be explained in terms of the promotion of a scalar field theory's global internal symmetry to a local one. Explicitly, consider a theory with a scalar Lagrangian density of the form

$$\mathcal{L}_0 = \nabla_\mu \phi^\dagger \nabla^\mu \phi - M^2 \phi^\dagger \phi, \tag{2.1.9}$$

where ϕ is a multiplet of $\mathfrak{n}$ scalar fields[25] and † denotes a Hermitian adjoint. For any $\mathfrak{n}$, the scalar Lagrangian density is invariant under the global transformation $\phi \to U\phi$ for constant $U \in \mathbb{C}^{\mathfrak{n}\times\mathfrak{n}}$ with

$$U^\dagger U = \mathbb{I}_\mathfrak{n}, \tag{2.1.10}$$

the $\mathfrak{n} \times \mathfrak{n}$ identity matrix, and we seek a modification of the theory that is invariant under a local generalisation of the transformation, so that U can be spacetime-dependent. This requires $\mathfrak{n}$ fields analogous to the electromagnetic A^μ.

A matrix U solving Eq. (2.1.10), which may not be constant, is called a *unitary matrix*, and the group of such matrices is denoted $\mathrm{U}(\mathfrak{n})$.[26] Note that

$$\forall U \in \mathrm{U}(\mathfrak{n})\ |\det U| = 1. \tag{2.1.11}$$

Electromagnetism is an example of the case $\mathfrak{n} = 1$; we say that the theory has gauge group $\mathrm{U}(1)$. A general element of $\mathrm{U}(\mathfrak{n})$ may be written in the form $U = \mathrm{e}^{\mathrm{i}\vartheta} V$ where

$$V \in \mathrm{U}(\mathfrak{n}),\ \det V = 1,\ \vartheta \in \mathbb{R} \tag{2.1.12}$$

(i.e. $|\mathrm{e}^{\mathrm{i}\vartheta}| = 1$).[27] Such V are called *special unitary matrices*, and the group of these is denoted $\mathrm{SU}(\mathfrak{n})$. Thus there is a group factorisation, viz.

$$\mathrm{U}(\mathfrak{n}) = \mathrm{SU}(\mathfrak{n}) \otimes \mathrm{U}(1). \tag{2.1.13}$$

While $\mathrm{U}(1)$ is an *Abelian group* (i.e. its elements commute), the groups $\mathrm{U}(\mathfrak{n})$, $\mathrm{SU}(\mathfrak{n})$ are Abelian if and only if $\mathfrak{n} = 1$.

[25] The symbols n or N are more commonly used, but in this thesis these symbols each have another meaning. To avoid ambiguity, I have decided to use `mathfrak` for this variable. I will soon discuss *special unitary groups*, for which the symbols SU, $\mathfrak{su}$ are commonly used. This motivates my use of `mathfrak`.

[26] Whenever sets of square matrices are referred to herein as groups, the group operation of interest is matrix multiplication, so that the identity element of the group is the identity matrix conformable with the matrices of interest. The matrix elements used throughout are complex-valued, so the associativity required for a group is obtained.

[27] The symbol ϑ is a θ variant called in LaTeX by `\vartheta`.

These groups characterise continuous symmetries; they are examples of *Lie groups*. The *infinitesimal* elements of these groups are those matrices that differ infinitesimally from the identity matrix. These admit the first-order approximation

$$e^{iM} \approx \mathbb{I}_{\mathfrak{n}} + iM,\ M^{\dagger} = M. \tag{2.1.14}$$

When multiplying infinitesimal elements, second-order terms may be neglected so that

$$\left(\mathbb{I}_{\mathfrak{n}} + iM_1\right)\left(\mathbb{I}_{\mathfrak{n}} + iM_2\right) = \mathbb{I}_{\mathfrak{n}} + i\left(M_1 + M_2\right). \tag{2.1.15}$$

The Hermitian Ms form a vector space called a *Lie algebra* that is closed under commutators. While a Lie algebra admits a matrix representation for which the definition of commutators is immediate, it can be considered more abstractly, so that a "commutator" need only be an antisymmetric binary operator under which the Lie algebra is closed. Since a Lie algebra is a vector space, a basis $\{T^a\}$ of a matrix representation of the Lie algebra may be considered, so that real coefficients f^{abc} exist with

$$\left[T^a,\, T^b\right] = i\sum_c f^{abc}T^c,\ f^{abc} = -f^{bac}. \tag{2.1.16}$$

These are called *structure constants*, and they all vanish if and only if the Lie group is Abelian. Hereafter indices that appear twice will be subjected to implicit summation, so that the above result can be written as

$$\left[T^a,\, T^b\right] = if^{abc}T^c. \tag{2.1.17}$$

The T^as are Hermitian matrix representations of the so-called *generators* of the Lie algebra. For $\mathfrak{n} > 1$ the Lie algebra of SU$(\mathfrak{n})$ has $\mathfrak{n}^2 - 1$ generators. Since U(1) is Abelian, the additional generator of U$(\mathfrak{n})$ commutes with those of SU$(\mathfrak{n})$. Indeed, this generator is represented by the identity matrix, since $e^{i\theta} \approx 1 + i\theta$ for small real θ. So the structure constants of an Abelian group vanish, and U$(\mathfrak{n})$ admits a generator index a for which $T^a = \mathbb{I}_{\mathfrak{n}}$ and each f^{abc} vanishes.

Yang–Mills theory generalises classical electromagnetism by replacing A_μ with a multiplet A_μ^a.[28] The scalar Lagrangian density is [29]

$$\mathcal{L}_0 = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - A^{a\mu}j^a_\mu,\ F^a_{\mu\nu} := \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + qf^{abc}A^b_\mu A^c_\nu. \tag{2.1.18}$$

As before, $F^a_{\mu\nu}$ is $\mu \leftrightarrow \nu$–antisymmetric. It is now natural to introduce *dot and cross products*

$$V \cdot W := V^aW^a,\ (V \times W)^a := f^{abc}V^bW^c \tag{2.1.19}$$

on multiplets, and to define $V^2 := V \cdot V$. Thus

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu} - A^\mu \cdot j_\mu,\ F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + qA_\mu \times A_\nu. \tag{2.1.20}$$

The Euler–Lagrange equation is $D_\mu F^{\mu\nu} = j^\nu$ where $D_\mu := \nabla_\mu + qA_\mu\times$ [31], so that replacing the $\nabla_\mu\phi^\dagger\nabla^\mu\phi$ term in Eq. (2.1.9) with $D_\mu\phi^\dagger D^\mu\phi$ achieves an invariance under a local transformation of

[28]This multiplet must transform appropriately so that, under an $\mathcal{L}_0$-preserving transformation $\phi \to U\phi$, $D_\mu\phi$ transforms to $UD'_\mu\phi = D_\mu(U\phi)$ where $D'_\mu = \nabla_\mu + qA'_\mu$. Thus $D'_\mu\phi = U^\dagger D_\mu(U\phi)$ and $A'_\mu = q^{-1}U^\dagger D_\mu U$.

the multiplet ϕ. The special case of electromagnetism has only one value for each "multiplet index" and vanishing structure constants, so the cross products are lost and D_μ reduces to ∇_μ. This recovers the scalar Lagrangian density of Eq. (2.1.2). Furthermore, Yang–Mills theory is invariant under the gauge transformation $A_\mu \to A_\mu + D_\mu \chi$ for arbitrary scalar multiplet χ^a. The electromagnetic special case of this result is an invariance under the gauge transformation $A_\mu \to A_\mu + \nabla_\mu \chi$, with χ an arbitrary differentiable scalar field. Indeed, this is a familiar symmetry of classical electromagnetism.

2.1.4 The gauge groups of the Standard Model

The electroweak interaction was obtained as a unification of earlier $\mathrm{U}(1)$, $\mathrm{SU}(2)$ approximations of the electromagnetic and weak nuclear interactions. A brief overview of the electroweak unification is given in Chapter VII.2 of Ref. [30]. A $\mathrm{U}(\mathfrak{n})$ interaction would differ from an otherwise identical $\mathrm{SU}(\mathfrak{n})$ interaction in the number of gauge bosons. The $\mathrm{SU}(\mathfrak{n})$ interaction has $\mathfrak{n}^2 - 1$ gauge bosons that have a charge type associated with the interaction. In the $\mathrm{U}(\mathfrak{n})$ case there is an additional gauge boson, for a total of $\mathfrak{n}^2$. Such a gauge boson (e.g. a gluon colour singlet) would be analogous to a photon, and in fact the photon is an example of this. At low energies, electromagnetism and the weak nuclear interaction appear unrelated, and their respective gauge groups are approximately $\mathrm{U}(1)$, $\mathrm{SU}(2)$. These interactions are in fact different aspects of an *electroweak interaction* observable at higher energies. However, the gauge group is not simply obtained from the product

$$\mathrm{SU}(2) \otimes \mathrm{U}(1) = \mathrm{U}(2), \tag{2.1.21}$$

for two key reasons:

- The Standard Model's $\mathrm{SU}(2)$ doublets have different $\mathrm{U}(1)$ charges[29];
- The weak and electromagnetic parts of the electroweak interaction for one $\mathrm{SU}(2)$ doublet have respective gauge groups $\mathrm{SU}_\mathrm{L}(2)$, $\mathrm{U}_Y(1)$, giving an electroweak gauge group factorisation $\mathrm{SU}_\mathrm{L}(2) \otimes \mathrm{U}_Y(1)$.[30] Indeed, the electromagnetic $\mathrm{U}(1)$ is a mixture of the $\mathrm{U}(1)$ factor in $\mathrm{SU}(2) \otimes \mathrm{U}(1)$ and a $\mathrm{U}(1)$ subgroup of $\mathrm{SU}(2)$. This is due to $Q = T_3 + \frac{Y}{2}$, where Q is the electromagnetic charge, T_3 is an electroweak isospin component and Y is the weak hypercharge.

This interaction's gauge bosons are the photon (which has an electroweak charge but zero electric charge) and three other bosons ($W^\pm$, Z^0) that have electroweak charges due to the non-zero structure constants of the electroweak gauge group.

The strong nuclear interaction is a little different. The empirical interaction between hadrons is a residual interaction, an emergent consequence of the interactions between elementary quarks, antiquarks and gluons. The three "colour charges" of quarks and antiquarks that bind them in hadrons suggest that gluons carry a "colour interaction" of gauge group $\mathrm{U}(3)$ or $\mathrm{SU}(3)$. Each option results in 8 species of gluon, but in the $\mathrm{U}(3)$ case there would also be a colour-neutral "colour singlet" analogous to the photon. This colour singlet's would easily be detected if it existed, as it would not be subject to colour confinement. Since the singlet does not exist empirically, the gauge group of the colour interaction is $\mathrm{SU}(3)$ [35].

[29]For example, in convenient units the $\mathrm{U}(1)$ charges of the left-handed quark doublet u_L, d_L is $\frac{1}{3}$, while the $\mathrm{U}(1)$ charges of the left-handed electron neutrino-electron doublet ν_L, e_L is -1.

[30]The electroweak theory of Sheldon Glashow [32], Abdus Salam [33] and Steven Weinberg [34] provides the Standard Model's description of the gauge group.

It follows that Yang–Mills theory is sufficient to describe the strong and electroweak interactions of the Standard Model of particle physics, i.e. all fundamental interactions other than gravity (for which general relativity is currently our best description). A Yang–Mills description of the Standard Model requires only that appropriate gauge groups be used, as described above. The Lie groups of interest are semisimple [29], so without loss of generality the structure constants may be chosen to be fully antisymmetric [36], i.e.

$$f^{abc} = -f^{bac} = -f^{acb} = f^{cab}. \tag{2.1.22}$$

This choice will be used hereafter, and implies that

$$V \cdot (W \times X) = V^a f^{abc} W^b X^c = f^{cab} V^a W^b X^c = (V \times W) \cdot X. \tag{2.1.23}$$

I call this the *triple product on multiplets. The* scalar triple product on $\mathbb{C}^3$ has this form.[31] The usual dot and cross products on $\mathbb{C}^3$ are respectively symmetric and antisymmetric. However, fermion-valued multiplets anticommute in the dot product; their classical description is in terms of Grassmann numbers. Thus if the classical multiplets $b_1,\ b_2$ ($f_1,\ f_2$) are bosonic (fermionic) fields

$$b_1 \cdot b_2 = b_2 \cdot b_1, \tag{2.1.24}$$

$$b_1 \times b_2 = -b_2 \times b_1, \tag{2.1.25}$$

$$f_1 \cdot f_2 = -f_2 \cdot f_1, \tag{2.1.26}$$

$$f_1 \times f_2 = f_2 \times f_1, \tag{2.1.27}$$

$$b_1 \cdot f_1 = f_1 \cdot b_1, \tag{2.1.28}$$

$$b_1 \times f_1 = -f_1 \times b_1, \tag{2.1.29}$$

and by Eq. (2.1.23) an exchange of adjacent fields in a triple product may or may not cause a sign change, as can be deduced by placing a cross product between these fields. In particular:

- the triple product of three fields, of which at most one is a fermionic field, changes sign under the exchange of adjacent fields, so changes sign under an odd permutation of the three fields;
- the triple product of three fermionic fields is unchanged under the exchange of adjacent fields, and hence under any permutation of the three fields; and
- a triple product of a bosonic field and two fermionic fields changes sign under the exchange of adjacent fields if and only if one is a bosonic field, and so changes sign precisely when the boson moves into or out of the central position.

The last of these results allows the rearrangement of a number of triple products throughout this thesis, and will be used without comment hereafter.

2.2 The BRST quantisation of Yang–Mills theory

In Sec. 2.2.1, I discuss the path integral method that is used to BRST-quantise Yang–Mills theory. The derivation from the path integral method is not worth presenting herein in great detail because there are still questions surrounding the rigour of the path integral method. The motivation of this

[31] The special case of the scalar triple product on $\mathbb{C}^3$ takes $f^{abc} = \epsilon^{abc}$, the Levi–Civita symbol. The gauge group with these structure constants is SU (2).

section is to justify, in a level of detail sufficient for present purposes, a choice of scalar Lagrangian density [37, 38] summarised in Eq. (2.2.11). A good discussion of BRST quantisation is given in Secs. 9.4 and 16.2 of Ref. [27].

In Sec. 2.2.2, I discuss various choices of the Faddeev–Popov Lagrangian density, and specify the choice that will be used thereafter. The result introduces three new massless scalar-valued multiplets in Yang–Mills theory. These fields are denoted $B,\ c,\ \overline{c}$. The zero mode problem I discuss in Sec. 2.4 is due to the fact that $c,\ \overline{c}$ are massless. In Sec. 2.2.3, I discuss reasons why this cannot be resolved by giving $c,\ \overline{c}$ mass.

2.2.1 The motivation for the Faddeev–Popov method

Let $\widehat{\mathcal{O}}$ denote an operator-valued quantisation of an empirical quantity $\mathcal{O}$ in a theory whose action S is expressible as a functional of quantities hereafter written as A. A general overview of the *path integral formulation of quantum field theory*, and the behaviour of Grassmann variables, is given in Sec. 9-1 of Ref. [3].[32] In this formalism, the vacuum-state mean $\left\langle \widehat{\mathcal{O}} \right\rangle$ of $\mathcal{O}$ may be expressed as a ratio of *functional integrals*, viz.

$$\left\langle \widehat{\mathcal{O}} \right\rangle = \frac{\int \mathcal{D}A\, \mathcal{O}\left[A\right] \mathrm{e}^{\mathrm{i}S[A]}}{\int \mathcal{D}A\, \mathrm{e}^{\mathrm{i}S[A]}} \tag{2.2.1}$$

where A is a collection of functions (such as the vector field A^{μ}), $\int \mathcal{D}A$ denotes a functional integration over the function space of A, $\mathcal{O}\left[A\right]$ is a functional of A whose quantum-mechanical operator promotion is $\hat{\mathcal{O}}\left[A\right]$, and the action $S\left[A\right]$ is a scalar-valued functional of A.

If $S,\ \mathcal{O}$ are each invariant under some *gauge transformation* of A, then choices of A related by a gauge transformation form equivalence classes. Functional integration over such an equivalence class provides an infinite factor in each of the functional integrals in Eq. (2.2.1). This motivates the addition to $\mathcal{L}_0$ of appropriate terms $\Delta\mathcal{L}_0$ that are not gauge-invariant. These terms multiply $\mathrm{e}^{\mathrm{i}S}$ by some functional $\mathrm{e}^{\mathrm{i}\int_x \Delta\mathcal{L}_0}$ and, if new fields are also introduced, $\int \mathcal{D}A$ by some integration operator. For suitable new terms $\Delta\mathcal{L}_0$ in $\mathcal{L}_0$, this inserts an identity operator into $\int \mathcal{D}A\, \mathrm{e}^{\mathrm{i}S[A]}$, $\int \mathcal{D}A\, \widehat{\mathcal{O}}\left[A\right] \mathrm{e}^{\mathrm{i}S[A]}$. Identifying such terms makes use of functional-integral generalisations of the results

$$1 = \int \mathrm{d}u \delta\left(g\left(u\right)\right) \left|g^{'}\left(g^{-1}\left(0\right)\right)\right|, \tag{2.2.2}$$

$$\det R = \int \mathrm{e}^{\overline{\eta}^{\mathrm{T}} R \eta} \mathrm{d}\eta \mathrm{d}\overline{\eta} \tag{2.2.3}$$

for functions g with a unique root $g^{-1}\left(0\right)$ and a square matrix R conformable with Grassmann-valued vectors $\eta,\ \overline{\eta}$ satisfying the normalisation

$$\int \eta_i \mathrm{d}\eta_j = \int \overline{\eta}_i \mathrm{d}\overline{\eta}_j = \delta_{ij},\ \int \eta_i \mathrm{d}\overline{\eta}_j = \int \overline{\eta}_i \mathrm{d}\eta_j = 0. \tag{2.2.4}$$

2.2.2 The Nakanishi–Lautrup and Faddeev–Popov fields

One appropriate choice of $\Delta\mathcal{L}_0$ is [27]

$$\Delta\mathcal{L}_0 = \mathcal{L}_{\mathrm{GF}}^{\alpha_0} + \mathcal{L}_{\mathrm{FP}}^{(2)},\ \mathcal{L}_{\mathrm{GF}}^{\alpha_0} := -\frac{\left(\nabla_\mu A^\mu\right)^2}{2\alpha_0},\ \mathcal{L}_{\mathrm{FP}}^{(2)} = \mathrm{i}\overline{c} \cdot \nabla_\mu D^\mu c \tag{2.2.5}$$

[32] The action is therein denoted I, not S.

where α_0 is a real-valued gauge parameter[33] and c^a, $\overline{c}^a$ are Hermitian[34] multiplet-valued spin-0 fermionic fields which, by the spin-statistics theorem, cannot be physical. These fermionic fields are respectively called the *Faddeev–Popov ghost* and *Faddeev–Popov antighost*.[35] The prefix anti- is not a reference to antimatter; the ghost and antighost are each their own "antiparticle species", insofar as that concept is applicable to unphysical fields. The use of an overline in $\overline{c}$ also does not indicate either complex conjugation or an adjoint; the fields c, $\overline{c}$ are each Hermitian, and hence are each their own adjoint.

The choice of $\Delta\mathcal{L}_0$ in Eq. (2.2.5) can be modified in several ways. For example, a total derivative may be added. Indeed, since $\mathcal{L}_{\text{FP}}^{(2)}$ as defined above contains a second-order derivative, which complicates use of the Lagrangian formalism in Sec. 1.3, $\mathcal{L}_{\text{FP}}^{(2)}$ is typically replaced with

$$\mathcal{L}_{\text{FP}} := \mathcal{L}_{\text{FP}}^{(2)} - \mathrm{i}\nabla_\mu \left(\overline{c} \cdot D^\mu c\right) = -\mathrm{i}\nabla_\mu \overline{c} \cdot D^\mu c. \tag{2.2.6}$$

Alternatives to $\mathcal{L}_{\text{GF}}^{\alpha_0}$ may also be obtained; they are of interest because they allow an extension to the case $\alpha_0 = 0$, which would otherwise yield a divergence. We begin by introducing a scalar boson multiplet $B^{'}$ and adding a term $\frac{\alpha_0}{2} B^{'2}$ to the scalar Lagrangian density. This process is legitimate because, by inspection, this $B^{'2}$ term decouples from all others. Introducing the *Nakanishi–Lautrup auxiliary field* $B := B^{'} - \frac{\nabla_\mu A^\mu}{\alpha_0}$ implies

$$\mathcal{L}_{\text{GF}}^{\alpha_0} + \frac{\alpha_0}{2} B^{\prime 2} = \frac{\alpha_0}{2} B^2 + B \cdot \nabla_\mu A^\mu. \tag{2.2.7}$$

This allows the choice $\Delta\mathcal{L}_0 = \mathcal{L}_{\text{GF}}^{B} + \nabla_\mu \left(B \cdot A^\mu\right) + \mathcal{L}_{\text{FP}}$ with

$$\mathcal{L}_{\text{GF}}^{B} := \frac{\alpha_0}{2} B^2 - \nabla_\mu B \cdot A^\mu = \mathcal{L}_{\text{GF}}^{\alpha_0} - \nabla_\mu \left(B \cdot A^\mu\right). \tag{2.2.8}$$

Note that

$$\nabla_\mu A^\mu + \alpha_0 B = 0 \tag{2.2.9}$$

(equivalently, $B^{'} = 0$) is an Euler–Lagrange equation of both $\mathcal{L}_{\text{GF}}^{B} + \mathcal{L}_{\text{FP}}$ and $\mathcal{L}_{\text{GF}}^{\alpha_0} + \frac{\alpha_0}{2} B^2$, so that on-shell we may equivalently define B as $-\frac{\nabla_\mu A^\mu}{\alpha_0}$. Thus Eq. (2.2.9) is an Euler–Lagrange equation if $\mathcal{L}_{\text{GF}}^{B}$ replaces $\mathcal{L}_{\text{GF}}^{\alpha_0}$ in $\Delta\mathcal{L}_0$. Such a replacement can then be used, including in the gauge choice $\alpha_0 = 0$, which is called the *Landau gauge* [41]. This observation motivates both the introduction of the field B and the use of a B-dependent scalar Lagrangian density, viz.

$$\Delta\mathcal{L}_0 = \frac{\alpha_0}{2} B^2 + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}},\ \mathcal{L}_{\text{GF}} := -\nabla_\mu B \cdot A^\mu. \tag{2.2.10}$$

The quantity $\frac{\alpha_0}{2} B^2 + \mathcal{L}_{\text{GF}}$ is called the *gauge-fixing term*, and $\mathcal{L}_{\text{GF}}$ is this term's value in the Landau gauge. The choice of α_0 in the gauge-fixing term fixes a gauge. In the Landau gauge, varying B gives the Euler–Lagrange equation $\nabla_\mu A^\mu = 0$. Without BRST quantisation, this constraint on A^μ may

[33] I follow Ref. [39] in using the symbol α_0; the alternative ξ has also been used, but is avoided herein (despite our use of it in Ref. [1]) because ξ is typically used for Killing vectors, which I discuss extensively in Chapter 4. Another alternative [40] is α. I use α_0 to avoid confusion with any other uses of α.

[34] The first elucidation of a theory in which these fields were Hermitian was the landmark paper Ref. [39], which established the unitarity of BRST-quantised Yang–Mills theory. The i factor in $\mathcal{L}_{\text{FP}}$ is required for the Hermiticity of the action S.

[35] Hereafter Faddeev–Popov is abbreviated as FP, and any use of the latter in subsequently defined technical terms is implied to abbreviate the former.

instead be imposed as a gauge choice. This gauge choice is familiar in classical electromagnetism as the *Lorenz gauge*.[36] The quantity $\mathcal{L}_{\text{FP}}$ is called the *FP-ghost term*, and is analogous to the $\nabla_\mu\phi^*\nabla^\mu\phi$ term in the scalar field theory of Chapter 1. Its inclusion in $\Delta\mathcal{L}_0$ is actually optional for an Abelian interaction such as electromagnetism, but is otherwise necessary.

2.2.3 On the masslessness of the FP-sector fields

In summary, the scalar Lagrangian density of BRST-quantised Yang–Mills theory may be chosen as

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}\cdot F^{\mu\nu} - A^\mu\cdot(j_\mu + \nabla_\mu B) + \frac{\alpha_0}{2}B^2 - \mathrm{i}\nabla_\mu\overline{c}\cdot D^\mu c, \tag{2.2.11}$$

and α_0 may be taken to be any real number, including 0. The fields c, $\overline{c}$ could each be granted a mass M by including a *mass term* $\mathrm{i}M^2\overline{c}\cdot c$ in $\mathcal{L}_0$. Indeed, the FMP introduces such a term in its analysis, before taking the right-hand limit $M \to 0^+$. However, the above path integral method does not justify a mass term, because $\int \mathcal{D}c\mathcal{D}\overline{c}\exp\left\{\int_x\left(\nabla_\mu\overline{c}\cdot D^\mu c - M^2\overline{c}\cdot c\right)\right\}$ is non-trivially M-dependent. An additional problem with attempting to include a mass term is discussed in Sec. 2.3.3.

2.3 The BRST and anti-BRST transformations in BRST-quantised Yang–Mills theory

While BRST-quantising Yang–Mills theory breaks its invariance under the transformation

$$A_\mu \to A_\mu + D_\mu\chi, \tag{2.3.1}$$

the transformations $A_\mu \to A_\mu + \eta c$, $A_\mu \to A_\mu + \eta\overline{c}$ with η a constant Grassmann number are action-preserving gauge transformations provided c, $\overline{c}$ are also appropriately transformed. These gauge transformations are respectively called the *BRST and anti-BRST transformations*. The purpose of this section is to elucidate these transformations, which are of interest because they show that BRST-quantised Yang–Mills theory still exhibits some useful symmetries with associated conserved currents. Since each of these transformations preserves the classical terms in the scalar Lagrangian density of Yang–Mills theory, only the terms added in its BRST quantisation are of interest. In Sec. 2.3.1, I discuss several important general properties of the BRST and anti-BRST transformations. I completely specify these transformations in Sec. 2.3.2, wherein I use some of the results of Sec. 2.3.1 to show that the scalar Lagrangian density of Eq. (2.2.11) is invariant under both transformations if the external current j_μ vanishes. On the other hand, a mass term would violate these symmetries, viz. Sec. 2.3.3. My discussions of the BRST and anti-BRST transformations are respectively based on Refs. [39] and [40].[37]

[36] This gauge is named for Ludvig Lorenz, who should not be confused with Hendrik Lorentz, for whom Lorentz invariance is named. Such confusion is, however, occasionally observed. For example, the gauge may be restated as $\boldsymbol{\nabla}\cdot\mathbf{A} + c^{-1}\partial_t\Phi = 0$ in SI units, and this is often called the Lorentz condition. A historical overview may be found in Ref. [42].

[37] Ref. [39]'s treatment of Yang–Mills theory is limited to Minkowski space, but the generalisation herein to curved spacetimes is trivial.

2.3.1 An overview of two transformations

The BRST[38] transformation may be written as $X \to X + \delta X$, where X is an arbitrary bosonic or fermionic field and the dimensionless bosonic operator δ is called the *BRST operator*. In particular we may write

$$\delta b = \theta \left[Q,\, b\right],\, \delta f = \theta \left\{Q,\, f\right\}, \tag{2.3.2}$$

where Q is a fermionic operator called the *BRST charge*, θ is an arbitrary constant Grassmann number, and b (f) is an arbitrary bosonic (fermionic) field. Notice the use of commutators and anticommutators $\{X,\, Y\} = XY + YX$, which may be collectively denoted by

$$[X,\, Y]_{\pm} := XY \pm YX. \tag{2.3.3}$$

The BRST transformation satisfies the usual Leibniz rule

$$\delta\left(XY\right) = \left(\delta X\right) Y + X \delta Y \tag{2.3.4}$$

whenever $X,\, Y$ are each either a bosonic field or a fermionic field. For arbitrary bosonic fields $b_1,\, b_2$ and fermionic fields $f_1,\, f_2$ we obtain

$$\begin{aligned}
\delta\left(b_1 b_2\right) &= \theta\left[Q,\, b_1 b_2\right] = \theta\left[Q,\, b_1\right] b_2 + \theta b_1 \left[Q,\, b_2\right] \\
&= \theta\left[Q,\, b_1\right] b_2 + b_1 \theta\left[Q,\, b_2\right] = \left(\delta b_1\right) b_2 + b_1 \delta b_2, && (2.3.5) \\
\delta\left(b_1 f_1\right) &= \theta\left\{Q,\, b_1 f_1\right\} = \theta\left[Q,\, b_1\right] f_1 + \theta b_1 \left\{Q,\, f_1\right\} \\
&= \theta\left[Q,\, b_1\right] f_1 + b_1 \theta\left\{Q,\, f_1\right\} = \left(\delta b_1\right) f_1 + b_1 \delta f_1, && (2.3.6) \\
\delta\left(f_1 f_2\right) &= \theta\left[Q,\, f_1 f_2\right] = \theta\left\{Q,\, f_1\right\} f_2 - \theta f_1 \left\{Q,\, f_2\right\} \\
&= \theta\left\{Q,\, f_1\right\} f_2 + f_1 \theta\left\{Q,\, f_2\right\} = \left(\delta f_1\right) f_2 + f_1 \delta f_2. && (2.3.7)
\end{aligned}$$

Another important identity, which follows trivially from $\partial_\mu Q = 0$, is

$$\delta \partial_\mu X = \partial_\mu \delta X. \tag{2.3.8}$$

A solution of $\delta X = 0$ is called *BRST-invariant* (this is both an adjective and a noun), and BRST-invariants include $g_{\alpha\beta},\, g^{\alpha\beta}$ and hence Christoffel symbols. Thus

$$\delta \nabla_\mu X = \nabla_\mu \delta X, \tag{2.3.9}$$

and $[Q,\, A^\mu],\, [Q,\, B],\, \{Q,\, c\},\, \{Q,\, \bar{c}\}$ completely specify the BRST transformation of fields other than matter fields.[39]

[38] This abbreviation refers to Carlo Becchi, Alain Rouet, Raymond Stora [43] and Igor Tyutin [44], four physicists who co-discovered the BRST transformation. At one time the name *BRS transformation* was common in the literature [39, 45], as Tyutin's contributions were not fully appreciated. This was still true when what is now known as the *anti-BRST transformation* was discovered [40, 46, 47]. While the prefix anti- refers to the fact that the roles of the ghost and antighost are approximately exchanged in a comparison of the transformations, Ojima settled for the name "another BRS transformation", consistently placing this name in quotation marks. During the early 1980s the name "dual BRS transformation" was also used [48]. In modern terminology, the two transformations and related concepts are referred to with the terms BRST and anti-BRST. So at last Tyutin shares credit for the BRST transformation.

[39] The matter fields not discussed herein, such as Dirac spinors, contribute BRST-invariant terms to the scalar Lagrangian density, although the fields are not in general BRST-invariant themselves.

The above facts about the BRST transformation apply also to the anti-BRST transformation $\overline{\delta}$ and anti-BRST charge $\overline{Q}$ [46, 47], viz.

$$\overline{\delta} b = \overline{\theta}\left[\overline{Q},\, b\right],\ \overline{\delta} f = \overline{\theta}\left\{\overline{Q},\, f\right\}. \tag{2.3.10}$$

A further important result is the fact that Q, $\overline{Q}$ *anticommute* and are *nilpotent* [46, 47], viz.

$$Q^2 = \overline{Q}^2 = \left\{Q,\, \overline{Q}\right\} = 0. \tag{2.3.11}$$

Thus any BRST transformation is a BRST-invariant, and similarly for anti-BRST. Further important results are $\left[\overline{Q},\, [Q,\, X]_\pm\right]_\mp = -\left[Q,\, \left[\overline{Q},\, X\right]_\pm\right]_\mp$ and $\delta^2 = \overline{\delta}^2 = \left[\delta,\, \overline{\delta}\right] = 0$, i.e. $\delta,\, \overline{\delta}$ *commute* and are nilpotent. Indeed

$$\left[Q,\, \left[\overline{Q},\, X\right]_\pm\right]_\mp = Q\overline{Q}X \pm QX\overline{Q} \mp \overline{Q}XQ - X\overline{Q}Q, \tag{2.3.12}$$

$$\begin{aligned}\left[\overline{Q},\, [Q,\, X]_\pm\right]_\mp &= \overline{Q}QX \pm \overline{Q}XQ \mp QX\overline{Q} - XQ\overline{Q} = -Q\overline{Q}X \mp QX\overline{Q} \pm \overline{Q}XQ + X\overline{Q}Q\\ &= -\left[Q,\, \left[\overline{Q},\, X\right]_\pm\right]_\mp.\end{aligned} \tag{2.3.13}$$

Since any operator $\mathcal{Q}$, including Q or $\overline{Q}$, satisfies

$$\left[\mathcal{Q},\, [\mathcal{Q},\, X]_\pm\right]_\mp = \mathcal{Q}^2 X \pm \mathcal{Q}X\mathcal{Q} \mp \mathcal{Q}X\mathcal{Q} - X\mathcal{Q}^2 = \left[\mathcal{Q}^2,\, X\right], \tag{2.3.14}$$

it follows that $\delta^2 X = \overline{\delta}^2 X = 0$. Similarly, if X is a bosonic or fermionic field then

$$\delta\overline{\delta}X = \theta\left[Q,\, \overline{\theta}\left[\overline{Q},\, X\right]_\pm\right]_\pm, \tag{2.3.15}$$

where commutators (anticommutators) are used if X is a boson (fermion), so

$$\delta\overline{\delta}X = \theta Q\overline{\theta}\left[\overline{Q},\, X\right]_\pm \pm \theta\overline{\theta}\left[\overline{Q},\, X\right]_\pm Q = -\theta\overline{\theta}\left(Q\left[\overline{Q},\, X\right]_\pm \mp \left[\overline{Q},\, X\right]_\pm Q\right) = -\theta\overline{\theta}\left[Q,\, \left[\overline{Q},\, X\right]_\pm\right]_\mp \tag{2.3.16}$$

(note the use of $\left\{\overline{\theta},\, Q\right\} = 0$) and

$$\overline{\delta}\delta X = -\overline{\theta}\theta\left[\overline{Q},\, [Q,\, X]_\pm\right]_\mp = \theta\overline{\theta}\left[\overline{Q},\, [Q,\, X]_\pm\right]_\mp, \tag{2.3.17}$$

$$\left[\delta,\, \overline{\delta}\right] X = -\theta\overline{\theta}\left(\left[Q,\, \left[\overline{Q},\, X\right]_\pm\right]_\mp + \left[\overline{Q},\, [Q,\, X]_\pm\right]_\mp\right) = 0. \tag{2.3.18}$$

Applying one or more BRST or anti-BRST transformations obtains a nonzero result only if each transformation type is used at most once, providing an at most fourfold extension of the dimension of vector spaces closed under one or both transformations, viz.

$$X \to X,\, \delta X,\, \overline{\delta}X,\, \delta\overline{\delta}X = -\overline{\delta}\delta X. \tag{2.3.19}$$

2.3.2 Explicit BRST and anti-BRST transformations

It remains to specify that [43, 44]

$$[Q,\, A^\mu] = D^\mu c,\; [Q,\, B] = 0,\; \{Q,\, c\} = -\frac{q}{2} c \times c,\; \{Q,\, \overline{c}\} = \mathrm{i} B \tag{2.3.20}$$

and that [40, 46, 47]

$$\left[\overline{Q},\, A^\mu\right] = D^\mu \overline{c},\; \left[\overline{Q},\, \overline{B}\right] = 0,\; \left\{\overline{Q},\, \overline{c}\right\} = -\frac{q}{2} \overline{c} \times \overline{c},\; \left\{\overline{Q},\, c\right\} = \mathrm{i} \overline{B} \tag{2.3.21}$$

where

$$\overline{B} := \mathrm{i} q \overline{c} \times c - B. \tag{2.3.22}$$

Since $\left\{\overline{Q},\, f_1 \times f_2\right\} = \left\{\overline{Q},\, f_1\right\} \times f_2 - f_1 \times \left\{\overline{Q},\, f_2\right\}$ we have

$$\begin{aligned} \left[\overline{Q},\, B\right] &= \mathrm{i} q \left[\overline{Q},\, \overline{c} \times c\right] = -\frac{\mathrm{i} q^2}{2} \left(\overline{c} \times \overline{c}\right) \times c + q \overline{c} \times \overline{B} \\ &= q \overline{c} \times \left(\overline{B} - \mathrm{i} q \overline{c} \times c\right) = -q \overline{c} \times B, \end{aligned} \tag{2.3.23}$$

$$\left[\overline{Q},\, B^2\right] = 2B \cdot \left[\overline{Q},\, B\right] = 0. \tag{2.3.24}$$

The multiplet-valued external current j_μ is hereafter set to 0. Excluding classical terms, the scalar Lagrangian density is

$$\begin{aligned} \mathcal{L}_0 &= \frac{\alpha_0}{2} B^2 - \nabla_\mu B \cdot A^\mu - \mathrm{i} \nabla_\mu \overline{c} \cdot D^\mu c \\ &= \left\{Q,\, -\frac{\mathrm{i}\alpha_0}{2} B \cdot \overline{c} + \mathrm{i} \nabla_\mu \overline{c} \cdot A^\mu\right\} \end{aligned} \tag{2.3.25}$$

$$\begin{aligned} &= \frac{\alpha_0}{2} B^2 + \nabla_\mu \overline{B} \cdot A^\mu + \mathrm{i} \nabla_\mu c \cdot D^\mu \overline{c} \\ &= \frac{\alpha_0}{2} B^2 + \left\{\overline{Q},\, -\mathrm{i} \nabla_\mu c \cdot A^\mu\right\}, \end{aligned} \tag{2.3.26}$$

so is BRST-invariant by Eq. (2.3.25) and anti-BRST-invariant by Eq. (2.3.26). Note the use of the identity

$$\{\mathcal{Q},\, f_1 b_1\} = \{\mathcal{Q},\, f_1\}\, b_1 - f_1 \left[\mathcal{Q},\, b_1\right] \tag{2.3.27}$$

to establish results such as $\{Q,\, \mathrm{i} \nabla_\mu \overline{c} \cdot A^\mu\} = -\nabla_\mu B \cdot A^\mu - \mathrm{i} \nabla_\mu \overline{c} \cdot D^\mu c$. I have also used the fact that Eq. (2.3.22) implies

$$\begin{aligned} \nabla_\mu \left(B + \overline{B}\right) \cdot A^\mu + \mathrm{i} \left(\nabla_\mu \overline{c} \cdot D^\mu c + c \leftrightarrow \overline{c}\right) &= \nabla_\mu \left(\mathrm{i} q \overline{c} \times c\right) \cdot A^\mu \\ &\quad + \mathrm{i} \left(\nabla_\mu \overline{c} \cdot \nabla^\mu c + \nabla_\mu c \cdot \nabla^\mu \overline{c}\right) \\ &\quad + \mathrm{i} q \left(\nabla_\mu \overline{c} \cdot A^\mu \times c + \nabla_\mu c \cdot A^\mu \times \overline{c}\right) \\ &= \mathrm{i} q \left\{\nabla_\mu \left(\overline{c} \times c\right) - \nabla_\mu \overline{c} \times c - \overline{c} \times \nabla_\mu c\right\} \cdot A^\mu \\ &= 0. \end{aligned} \tag{2.3.28}$$

The set of Euler–Lagrange equations must then be closed under both BRST and anti-BRST transformations. So are the conserved currents; if $\nabla_\mu W^\mu = 0$ then

$$\nabla_\mu \left[Q,\, W^\mu\right]_\pm = \left[Q,\, \nabla_\mu W^\mu\right]_\pm = 0, \tag{2.3.29}$$

i.e. if W^μ is conserved so is $\left[Q,\, W^\mu\right]_\pm$ (and, similarly, so is $\left[\overline{Q},\, W^\mu\right]_\pm$). I discuss these closure rules in more detail in Sec. 3.3.

2.3.3 A mass term would violate both symmetries

One last observation is that $\overline{c}{\cdot}c$ is neither BRST-invariant nor anti-BRST invariant. Indeed, the identity $\left[\mathcal{Q},\, f_1 f_2\right] = \left\{\mathcal{Q},\, f_1\right\} f_2 - f_1 \left\{\mathcal{Q},\, f_2\right\}$ implies

$$\left[Q,\, \overline{c}\cdot c\right] = \left\{Q,\, \overline{c}\right\}\cdot c - \overline{c}\cdot\left\{Q,\, c\right\} = \mathrm{i}B\cdot c + \frac{q}{2}\overline{c}\cdot\left(c\times c\right), \tag{2.3.30}$$

$$\left[\overline{Q},\, \overline{c}\cdot c\right] = \left\{\overline{Q},\, \overline{c}\right\}\cdot c - \overline{c}\cdot\left\{\overline{Q},\, c\right\} = -\frac{q}{2}\overline{c}\times\overline{c}\cdot c - \mathrm{i}\overline{c}\cdot\overline{B}. \tag{2.3.31}$$

The fact that a $\overline{c}\cdot c$ term would break the BRST and anti-BRST invariances of BRST-quantised Yang–Mills theory has implications for the relative merits of the FMP and CMP, as will be discussed in Sec. 2.5.2.

2.4 The zero mode problem of the FP-ghost propagator in Yang–Mills theory

Since all cross products in Eq. (2.2.11) are q-weighted, the case $q = 0$ is *non-interacting*. In Sec. 2.4.1, I show that the non-interacting case has an infrared problem very similar to the one considered in Chapter 1. I use discrete mode labels throughout; a zero mode problem does not occur where there are continuous mode labels in Minkowski space. In Sec. 2.4.2, I discuss the outcome for more general q, which obtains a zero mode problem that is harder to explicitly describe.

2.4.1 The non-interacting case

In the non-interacting case the FP-ghost term is $-\mathrm{i}\nabla_\mu\overline{c}\cdot\nabla^\mu c$, and the fields c, $\overline{c}$ are each cyclic, so varying either fermionic field provides a conserved current. Varying c gives the Euler–Lagrange equation $\nabla_\mu\nabla^\mu\overline{c} = 0$; varying $\overline{c}$ gives the Euler–Lagrange equation $\nabla_\mu\nabla^\mu c = 0$. Note that the Euler–Lagrange equation obtained by varying either of these fields is a differential equation in the *other* field. This is analogous to the fact that Eq. (1.3.9) is a differential equation in ϕ and an Euler–Lagrange equation obtained by varying ϕ^*. Note also that c, $\overline{c}$ are massless Klein–Gordon fields, so for $q = 0$ their quantisation on the flat static torus is of the form

$$\widehat{c}\left(x\right) = \sum_\sigma \left\{\hat{c}_\sigma\phi_\sigma\left(x\right) + \hat{c}^\dagger_\sigma\phi^*_\sigma\left(x\right)\right\},\, \hat{\overline{c}}\left(x\right) = \sum_\sigma \left\{\hat{\overline{c}}_\sigma\phi_\sigma\left(x\right) + \hat{\overline{c}}^\dagger_\sigma\phi^*_\sigma\left(x\right)\right\}. \tag{2.4.1}$$

Sec. 3.7.1 will observe that, for $q = 0$, the fermionic operators are spacetime-constant and satisfy

$$\left\{\widehat{c}_\sigma,\, \widehat{\overline{c}}^\dagger_{\sigma'}\right\} = -\mathrm{i}\delta_{\sigma\sigma'},\, \left\{\widehat{c}_\sigma,\, \widehat{c}_{\sigma'}\right\} = \left\{\widehat{c}_\sigma,\, \widehat{c}^\dagger_{\sigma'}\right\} = \left\{\widehat{\overline{c}}_\sigma,\, \widehat{\overline{c}}_{\sigma'}\right\} = \left\{\widehat{\overline{c}}_\sigma,\, \widehat{\overline{c}}^\dagger_{\sigma'}\right\} = 0, \tag{2.4.2}$$

in analogy with Eqs. (1.4.14) and (1.4.25). The *FP-ghost propagator* is defined as $\mathrm{T}\left\langle 0|\widehat{c}\,(x)\,\widehat{\overline{c}}\,(y)\,|0\right\rangle$. Since

$$\begin{aligned}\left\langle 0|\widehat{c}\,(x)\,\widehat{\overline{c}}\,(y)\,|0\right\rangle &= \left(\sum_{\sigma}\langle 0|\,\widehat{c}_{\sigma}\phi_{\sigma}\,(x)\right)\left(\sum_{\sigma'}\phi_{\sigma'}\,(y)\,\widehat{\overline{c}}^{\dagger}_{\sigma'}\,|0\rangle\right)\\ &= \sum_{\sigma\sigma'}\phi_{\sigma}\,(x)\,\phi_{\sigma'}\,(y)\left\langle 0|\left\{\widehat{c}_{\sigma},\,\widehat{\overline{c}}^{\dagger}_{\sigma'}\right\}|0\right\rangle\\ &= -\mathrm{i}\sum_{\sigma}\phi_{\sigma}\,(x)\,\phi^{*}_{\sigma}\,(y)\,,\end{aligned} \tag{2.4.3}$$

it is natural to linearly extend Sec. 1.5.3's definition of time ordering so that

$$\mathrm{T}\left\langle 0|\widehat{c}\,(x)\,\widehat{\overline{c}}\,(y)\,|0\right\rangle = -\mathrm{i}\mathrm{T}\sum_{\sigma}\phi_{\sigma}\,(x)\,\phi^{*}_{\sigma}\,(y)\,. \tag{2.4.4}$$

In the non-interacting case, this propagator results in the same problematic infrared behaviour discussed in Sec. 1.6.

2.4.2 Comments on the interacting case

For any q, $\overline{c}$ is cyclic in $\mathcal{L}_{\mathrm{FP}}$. Varying $\overline{c}$ gives $\nabla_{\mu}D^{\mu}\overline{c}=0$. Varying c gives

$$0=\left(\frac{\partial}{\partial c}-\nabla_{\mu}\frac{\partial}{\partial\nabla_{\mu}c}\right)\left(\nabla_{\mu}\overline{c}\cdot\nabla^{\mu}c+q\nabla_{\mu}\overline{c}\cdot A^{\mu}\times c\right). \tag{2.4.5}$$

Because classical fermionic fields anticommute, Euler–Lagrange equations are obtained with a convention known as *left-differentiation*. Any fermionic field with respect to which differentiation is sought is placed to the left of any other factors in a term. Thus

$$0=\left(\frac{\partial}{\partial c}-\nabla_{\mu}\frac{\partial}{\partial\nabla_{\mu}c}\right)\left(-\nabla^{\mu}c\cdot\nabla_{\mu}\overline{c}+qc\cdot A^{\mu}\times\nabla_{\mu}\overline{c}\right)=qA^{\mu}\times\nabla_{\mu}\overline{c}+\nabla_{\mu}\nabla^{\mu}\overline{c}=\nabla_{\mu}D^{\mu}\overline{c}-q\nabla_{\mu}A^{\mu}\times\overline{c}. \tag{2.4.6}$$

On-shell the result $\nabla_{\mu}A^{\mu}=-\alpha_{0}B$ gives

$$\nabla_{\mu}D^{\mu}\overline{c}=-q\alpha_{0}B\times\overline{c}. \tag{2.4.7}$$

In the Landau gauge $\nabla_{\mu}D^{\mu}\overline{c}=0$, so c, $\overline{c}$ solve the same interacting generalisation of the massless Klein–Gordon equation. Indeed, the conserved charges are $\int_{\mathbf{x}}D^{0}c$, $\int_{\mathbf{x}}D^{0}\overline{c}$. The FP-(anti)ghost fields are in general q-dependent (the modes are modified and $\widehat{c}_{\sigma}$, $\widehat{\overline{c}}_{\sigma}$ are in general spacetime-dependent operators), and it is natural to ask:

- whether the FP-ghost propagator has an infrared problem for $q\neq 0$;
- and whether the infrared-divergent contributions to the massive propagator in the FMP respect the spacetime symmetries desired for the full propagator in the IR limit (e.g. time translation invariance on a flat static torus or de Sitter invariance in de Sitter space).

If both these things are true, the non-interacting case admits an infrared problem that the FMP can address while preserving the propagator's spacetime symmetries. In Chapter 3, I show that any such infrared problem may be solved in the CMP. I will not prove that an infrared problem results for arbitrary q; for the present purposes it matters only that the CMP works when such an infrared problem results. The $q = 0$ infrared problem justifies constructing a BRST-invariant perturbation theory that does not encounter an infrared problem.

2.5 The history of the zero mode problem

The zero mode problem in BRST-quantised Yang–Mills theory has been considered for several decades, as have its implications and possible prescriptions for addressing it. In this section, I review the history of this analysis. In Sec. 2.5.1, I relate the zero mode problem to Hadamard states, which are a well-motivated concept of a physically acceptable state [5].

In 2008, Atsushi Higuchi and Mir Faizal introduced a prescription for addressing the zero mode problem [2]. I have called this prescription the fictitious mass prescription (FMP). In Sec. 2.5.2, I discuss problems with the FMP that have motivated the development of the CMP in my collaboration with Atsushi Higuchi [1].

2.5.1 On minimally coupled massless scalar fields

A vacuum state is denoted $|0\rangle$, and should respect whatever symmetries are imposed on the classical field theory. For example, suppose a de Sitter-invariant classical field theory is sought in de Sitter space; then a de Sitter-invariant vacuum state would also be desired. Therefore, the non-existence of any de Sitter-invariant Hadamard state would be problematic. In fact, it is known [20] that the theory of the minimally coupled massless scalar field discussed in Chapter 1 does not have a de Sitter-invariant Hadamard state in de Sitter space.[40]

This is (not immediately obviously) a consequence of de Sitter space's special case of the zero mode problem I discussed in Chapter 1. In my discussion of the flat static torus in Sec. 1.5, I showed that in such a spacetime Klein–Gordon normalisation resulted in an infrared problem for the propagator of a minimally coupled massless scalar field. I also showed that, if this normalisation condition were relaxed to prevent this infrared problem, the propagator would no longer be time-translation invariant. One could equivalently say that, if time-translation invariance is required, then Klein–Gordon normalisation is imposed and the infrared problem is obtained. The analogous result for de Sitter space [49] is that the propagator must lose its de Sitter invariance (if the infrared problem is to be prevented, which is necessary to safeguard the quantum field the theory's consistency). This symmetry breaking can be restated as the non-existence of a de Sitter-invariant Hadamard state to describe the state of the scalar field. In Sec. 1.6, I discussed this problem in the language of the symmetries and infrared behaviour of the propagator instead. I showed that this problem survives in the spacetimes of interest in this thesis.

The scalar field discussed in Chapter 1 is bosonic. In Sec. 2.4, I explained that this problem has a fermionic analogue in the FP-ghost sector of BRST-quantised Yang–Mills theory. It is therefore natural to suspect that the FP-ghost sector would, for example, lack de Sitter invariance in de Sitter space. If this is so, the implication would be that the FP-(anti)ghosts lack a de Sitter-invariant perturbative vacuum state in the theory's Hilbert space. However, the FMP and CMP both aim to show that the theory can be constructed in spacetimes of interest so as to preserve appropriate

[40] A theoretical interest in de Sitter space is far from new, for reasons discussed in Sec. 1.2.2. Historical interest in de Sitter space has led to much being discovered about quantum field theories in de Sitter space. The aforementioned result concerning de Sitter-invariant Hadamard states has analogues for other spacetimes.

spacetime symmetries. In particular, de Sitter invariance is sought in de Sitter space. I discuss how the FMP and CMP achieve this in Sec. 2.5.2.

Incidentally, the situation for vector fields is quite different. A de Sitter-invariant Hadamard state exists for the theory of a massless vector field in maximally symmetric spacetimes, including de Sitter space [50]. For example, the multiplets $A^a_\mu(x)$ of Yang–Mills theory have a propagator

$$\mathrm{T}\left\langle 0\right| A^a_\mu(x)\, A^b_\nu(y) \left|0\right\rangle \tag{2.5.1}$$

that lacks the aforementioned symmetry and infrared problems of the FP-ghost propagator.

2.5.2 Motivating a shift from the FMP to the CMP

The formalism considered thus far contains a spatially uniform ϕ_σ, say ϕ_0, with q-dependent generalisation φ_0 . Any formalism containing such a ϕ_0 requires infrared regularisation. The FMP makes use of one method of infrared regularisation, namely that of deleting an infrared-divergent term from the massive FP-ghost propagator [2].

One issue with this approach is that an FP-sector mass term is added to the Lagrangian before the massless limit is taken. Since such a mass term is not BRST-invariant, it is not obvious that the BRST and anti-BRST symmetries of the theory are preserved. In particular we hope for an FP-sector perturbative vacuum state that respects the BRST and anti-BRST symmetries in addition to appropriate spacetime symmetries. The outcome of this would be that Q, $\overline{Q}$ each annihilate all physical states (including vacua). The physical states comprise a Fock space that the operator-valued charges Q, $\overline{Q}$ annihilate, just as Q^2, $\overline{Q}^2$ annihilate the full Hilbert space.

The CMP allows for a formalism in which φ_0 never appears [1], resulting in a different FP-ghost propagator. In Chapter 3, I discuss the CMP's treatment of the zero mode problem in BRST-quantised Yang–Mills theory. I show therein that manifest BRST- and anti-BRST-invariance are preserved throughout. The desirable spacetime symmetries and internal symmetries are then preserved, *and* the theory's propagator is infrared-convergent.

A further advantage of the CMP is that, unlike the FMP, it does not assume free integration by parts is possible. This fact broadens the class of spacetimes this thesis is able to consider. I show in Sec. 3.7 that, in spacetimes that allow free integration by parts, the FMP and CMP have equivalent perturbation theories. The CMP is in this sense a generalisation of the FMP.

2.6 The story for perturbative gravity

In Sec. 2.6.1, I discuss the formalism with which general relativity describes perturbations of a background metric. In Sec. 2.6.2, I list a few standard properties of Lie derivatives and Killing vector fields. This is a necessary preamble for my discussion of the BRST quantisation in Sec. 2.6.3.

Like Yang–Mills theory, BRST and anti-BRST transformations can be defined for general relativity[41],

[41] The formalism of BRST-quantised Yang–Mills theory and of its (anti-)BRST transformations, respectively summarised in Secs. 2.2 and 2.3, is known as the *BRST formalism*. All phase space constraints in the Hamiltonian formulation of Yang–Mills theory are related to the Lie algebra, even after BRST quantisation. The same is not true of general relativity. This motivated the development of a generalisation of the BRST formalism, the *Batalin–Vilkovisky formalism* or BV formalism. This formalism can incorporate all phase space constraints in the Hamiltonian formulation of general relativity. However, I have concluded that the BV formalism is unnecessary for my treatment herein of the zero mode problem of perturbative gravity. An explanation of this conclusion requires a review of the BV formalism, so I can highlight what motivates it in other contexts. I present the explanation in Appendix F.

and these transformations preserve the BRST-quantised action. Explicit expressions for these transformations, in analogy with Eqs. (2.3.20)–(2.3.22), are dependent on a parameter κ introduced below in Eq. (2.6.1). Unlike the case of Yang–Mills theory, there are two radically different popular versions of the BRST quantisation, which differ in their expressions for the (anti-)BRST transformations and whether they make use of the *vielbein formalism*.[42] In Sec. 2.6.4, I discuss perturbative gravity's BRST and anti-BRST transformations, and the history of the understanding and description of these transformations. I show in particular that some of the discussion of the anti-BRST transformation in the literature is mistaken on key issues.

In Sec. 2.6.5, I explain the zero mode problem for perturbative gravity, and briefly comment on the FMP's solution to it. I only explicitly verify infrared divergence in the case of de Sitter space, for which an analytic treatment is feasible. For the purposes of Sec. 2.6.5, the FP-ghost sector need not be explicitly considered. Indeed, just as Chapter 1 discussed a bosonic analogue of the Yang–Mills zero mode problem, the formalism of Sec. 2.6.5 is applicable to an FP-ghost sector problem in perturbative gravity.

In Sec. 2.6.6, I discuss qualitative similarities between an infrared problem in the FP-ghost sector of BRST-quantised Yang–Mills theory, an infrared problem in the FP-ghost sector of BRST-quantised perturbative gravity, and controversies regarding the graviton sector of BRST-quantised perturbative gravity (the last is discussed only in the context of de Sitter space).

2.6.1 Some conventions for notation and terminology

General relativity describes gravity. The analogue of the electromagnetic tensor A_μ might be assumed to be the metric tensor $g_{\mu\nu}$. In the case of perturbative gravity, the full metric tensor $g^f_{\mu\nu}$ differs from a background metric tensor $g_{\mu\nu}$, viz.

$$g^f_{\mu\nu}(x) = g_{\mu\nu}(x) + \kappa h_{\mu\nu}(x), \tag{2.6.1}$$

where $\kappa h_{\mu\nu}$ is a perturbation metric for some small constant κ. Perturbation-dependent results can be described as κ-dependent, and perturbative results can be expressed as power series in κ. It will sometimes be beneficial to compare qA_μ with $\kappa h_{\mu\nu}$, and to identify both of these as measures of an "interaction" in their respective theories. Indeed, gravity is referred to as *interacting* only when $\kappa h_{\mu\nu}$ does not vanish, so *interacting gravity* and *perturbative gravity* are taken herein as synonymous.[43]

A treatment of perturbative gravity should work with covariant derivatives that commute with (and hence annihilate) both the index-lowering metric and its inverse as a matrix, which is the index-raising metric. Which of $g_{\mu\nu}$, $g^f_{\mu\nu}$ should be the index-lowering metric? One can, in fact, use either, and hereafter I will use $g_{\mu\nu}$. This choice is known as *linearised gravity*, a fact that merits some explanation. The index-raising metric tensor is $g^{\mu\nu}$, so

$$g^{f\,\nu}_{\mu} = \delta^\nu_\mu + \kappa h^\nu_\mu. \tag{2.6.2}$$

[42]This is not a spelling error; *vierbein* (in German, "four legs") is the $n = 4$ special case of *vielbein* (in German, "many legs"). Alternative names include *tetrad* (again only advisable for $n = 4$, since this word is derived from the Greek for four) and *frame field*.

[43]There is one subtlety here I should briefly mention. *Perturbative* is usually taken to imply the use of perturbative techniques to approximately express results as perturbative corrections to the properties of a simpler system. However, a non-perturbative treatment of interacting gravity is also possible (although this remains speculative).

The *trace* of a tensor of the form $x_{\mu\nu}$ is then $x := x_{\mu\nu}g^{\mu\nu}$, so $g_{\mu\nu}$ has trace n and $g^f_{\mu\nu}$ has trace $n + \kappa h$. Throughout my analysis it will be important to consider the map

$$x_{\mu\nu} \to X_{\mu\nu} := x_{\mu\nu} - kxg_{\mu\nu}, \tag{2.6.3}$$

where $k \in \mathbb{R}$ is a spacetime-constant parameter. (I will discuss this map in more detail in Sec. 2.6.3. One especially important example is $h_{\mu\nu} \to H_{\mu\nu}$). Thus

$$X = (1 - kn)\, x, \tag{2.6.4}$$

and for $k \neq n^{-1}$ we have the inverse relation

$$x = \frac{X}{1 - kn}. \tag{2.6.5}$$

The relation between h and H is therefore linear. But the situation is quite different if $g^f_{\mu\nu}$ is chosen as the index-lowering metric and replaces $g_{\mu\nu}$ in Eq. (2.6.3), viz.

$$X_{\mu\nu} = x_{\mu\nu} - kx_{\rho\sigma}g^{f\rho\sigma}g^f_{\mu\nu}, \tag{2.6.6}$$

$$H_{\mu\nu} = h_{\mu\nu} - kh_{\rho\sigma}g^{f\rho\sigma}\left(g_{\mu\nu} + \kappa h_{\mu\nu}\right). \tag{2.6.7}$$

Eq. (2.6.4) obtains a linear relation between $h_{\mu\nu}$ and $H_{\mu\nu}$, whereas for nonzero k Eq. (2.6.7) instead obtains a non-linear equation. It is therefore natural to describe the use of $g_{\mu\nu}$ as an index-lowering metric that obtains this result as linearised, and to describe the alternative use of $g^f_{\mu\nu}$ as an index-lowering metric as *non-linearised*.

Prior to BRST quantisation, the Lagrangian density of general relativity may be taken as the *Einstein–Hilbert Lagrangian density*. For any n, the scalar Lagrangian density is proportional to $R - 2\Lambda$ (see, e.g. Ref. [51]), where R is the Ricci scalar and Λ is the cosmological constant. For example, in SI units the case $n = 4$ gives[44]

$$\mathcal{L}_0 = \frac{R - 2\Lambda}{16\pi Gc^{-4}}. \tag{2.6.8}$$

Hereafter I impose an n-dependent nondimensionalised choice of G so that in general we may write

$$\mathcal{L}_0 = R - 2\Lambda. \tag{2.6.9}$$

This result is very different from the $-\frac{1}{4}F_{\mu\nu} \cdot F^{\mu\nu}$ term in Yang–Mills theory, so the terms that the Faddeev–Popov method adds are also quite dissimilar, as I show explicitly in Sec. 2.6.3. However, the effect of the Faddeev–Popov method is still to add a gauge-fixing term (expressible in terms of a Nakanishi–Lautrup auxiliary field) and an FP-ghost term (expressible in terms of FP-ghosts and FP-antighosts). As with Yang–Mills theory, the fields that the Faddeev–Popov method introduces are all massless, and an FP-ghost propagator can be defined. The zero mode problem of perturbative gravity is, again, a matter of an infrared-divergent propagator.

[44] Feynman's definition in Ref. [21] of the $\Lambda = 0$ special case of the Einstein–Hilbert action includes a $-$ sign relative to mine (viz., my Eq. (2.6.8)). The $\frac{1}{16\pi}$ factor in Feynman's result follows from Ref. [21]'s Eqs. (4.1.1) and (10.1.2). He does not make the power of c explicit. However, the result will always be $\propto c^4G^{-1}$. The dimension of $\int \mathrm{d}^n x\,(R - 2\Lambda)\,c^4G^{-1}$ is $\mathrm{L}^{n-1}\mathrm{TL}^{-2}\mathrm{L}^4\mathrm{T}^{-4}\mathrm{L}^{1-n}\mathrm{MT}^2 = \mathrm{L}^2\mathrm{MT}^{-1}$, where L, M, T respectively denote dimensions of length, mass and time in SI units. This is the correct dimension for an action.

2.6.2 A preamble on Lie derivatives and Killing vectors

The *Lie derivative* of a tensor $T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}$ with respect to a vector V^γ may be defined as

$$\pounds_V T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q} := V^\gamma\nabla_\gamma T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q} - \sum_{j=1}^{p}\nabla_\gamma V^{\alpha_k} T^{\alpha_1\cdots\alpha_{j-1}\gamma\alpha_{j+1}\cdots\alpha_p}_{\beta_1\cdots\beta_q} + \sum_{i=1}^{q}\nabla_{\beta_i} V^\gamma T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_{i-1}\gamma\beta_{i+1}\cdots\beta_q}, \tag{2.6.10}$$

or equivalently as

$$\pounds_V T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q} := V^\gamma\partial_\gamma T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q} - \sum_{j=1}^{p}\partial_\gamma V^{\alpha_k} T^{\alpha_1\cdots\alpha_{j-1}\gamma\alpha_{j+1}\cdots\alpha_p}_{\beta_1\cdots\beta_q} + \sum_{i=1}^{q}\partial_{\beta_i} V^\gamma T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_{i-1}\gamma\beta_{i+1}\cdots\beta_q}, \tag{2.6.11}$$

viz. Ref. [52]. Important implications of Eq. (2.6.10) include the Leibniz law

$$\pounds_V\left(T_1T_2\right) = T_1\pounds_V T_2 + \left(\pounds_V T_1\right)T_2, \tag{2.6.12}$$

the results

$$\pounds_V\phi = V^\gamma\nabla_\gamma\phi = V^\gamma\partial_\gamma\phi, \tag{2.6.13}$$

$$\pounds_{\alpha V}W = \alpha\pounds_V W - V\pounds_W\alpha \tag{2.6.14}$$

for ϕ (W) a scalar (vector) field, and

$$\pounds_V g_{\alpha\beta} = \nabla_\alpha V_\beta + \nabla_\beta V_\alpha. \tag{2.6.15}$$

The statement that this vanishes is the *Killing equation* [52],

$$\pounds_V g_{\alpha\beta} = \nabla_\alpha V_\beta + \nabla_\beta V_\alpha = 0. \tag{2.6.16}$$

A vector field V^γ that solves this is called a *Killing vector field* or *Killing vector* [52]. It is customary to denote a Killing vector field as ξ^γ, and I will only ever use ξ^γ for this purpose.

Vector fields admit a commutator $[V, W]^\mu := \pounds_V W^\mu$, which is antisymmetric if components of V, W are c-number valued. Killing vectors are closed under this commutator, so they form a Lie algebra. Any basis $\{\xi^\mu_A\}$ of the space of Killing vectors therefore satisfies a result of the form

$$\pounds_{\xi_A}\xi^\mu_B = f_{AB}{}^C\xi^\mu_C. \tag{2.6.17}$$

For the flat static torus, one choice of basis of the Killing vectors is $\xi^\mu_A = \delta^\mu_A$, where A runs over spacetime indices. Thus the Lie algebra of Killing vectors is Abelian for flat static tori.

Since $\nabla_\alpha\xi_\beta$ is antisymmetric, ξ^α is conserved:

$$\nabla_\alpha\xi^\alpha = g^{\alpha\beta}\nabla_\alpha\xi_\beta = 0. \tag{2.6.18}$$

If $X^{\mu\nu}$ is symmetric and $\nabla_\mu X^{\mu\nu} = 0$, then $\nabla_\mu (\xi_\nu X^{\mu\nu}) = \nabla_\mu \xi_\nu X^{\mu\nu} = 0$ and $\xi_\nu X^{\mu\nu}$ ($\int_{\mathbf{x}} \xi_\nu X^{0\nu}$) is a conserved current (conserved charge). Since Killing vectors generate symmetries in this way, a spacetime with more linearly independent Killing vectors than another of the same manifold dimension may be regarded as "more symmetric". The Killing vector fields of an n–dimensional manifold form a vector space of dimension $\leq \frac{1}{2} n (n+1)$, and some spacetimes achieve this maximum. These are therefore called *maximally symmetric* spacetimes. Examples include Minkowski space, de Sitter space and anti de Sitter space
Since ξ is conserved, $\pounds_\xi \phi = \nabla_\alpha (\xi^\alpha \phi)$. If the behaviour of ϕ at infinity does not result in a boundary term when integration by parts is used, corollaries include

$$\int_{\mathbf{x}} \pounds_\xi \phi = \partial_0 \int_{\mathbf{x}} \xi^0 \phi, \tag{2.6.19}$$

$$\int_x \pounds_\xi \phi = 0. \tag{2.6.20}$$

Thus terms of the form $\pounds_\xi \phi$ in a scalar Lagrangian density make no contribution to the associated action.
One last relation between Killing vectors and Lie derivatives is that any tensor T satisfies

$$\nabla^\alpha \pounds_\xi T = \pounds_\xi \nabla^\alpha T \tag{2.6.21}$$

(see, e.g. Ref. [53]).

2.6.3 The BRST-quantised Einstein–Hilbert Lagrangian density

The Lagrangian density may be written as

$$\sqrt{|g|} (R - 2\Lambda) + \sqrt{|g|} (\mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}), \tag{2.6.22}$$

where $\mathcal{L}_{\text{GF}}$ is a gauge-fixing term dependent on a real-valued gauge parameter α_0 and $\mathcal{L}_{\text{FP}}$ is a FP-ghost term. As with Yang–Mills theory, the term "gauge-fixing term" is used because varying the Nakanishi–Lautrup auziliary field obtains a common gauge choice as an Euler–Lagrange equation. Introducing an auxiliary field as appropriate, two formalisms are possible.

- In the vielbein formalism [40] a mixture of Lie algebra indices and Greek indices are used, viz.

$$\mathcal{L}_{\text{GF}} = h g^{\mu\nu} \Gamma_\mu^{ab} \nabla_\nu s^{ab} + \alpha_0 h s^{ab} s^{ab}, \tag{2.6.23}$$

$$\mathcal{L}_{\text{FP}} = -\mathrm{i} h \nabla^\mu \bar{t}^{ab} \left(t^{ac} \Gamma_\nu^{cb} - t^{bc} \Gamma_\nu^{ca} + \nabla_\nu t^{ab} \right). \tag{2.6.24}$$

  As with Yang–Mills theory, I use lower-case Roman upper Lie algebra indices.
  In Eqs. (2.6.23) and (2.6.24):
  - $h_\mu{}^a$ is a vielbein component;
  - h denotes the trace of $h_\mu{}^a$, *not* of the term $h_{\mu\nu}$ discussed in Sec. 2.6.1;
  - $\Gamma_\mu^{ab} = \left(\nabla_\mu h_\nu{}^a - \Gamma_{\mu\nu}^\lambda h_\lambda{}^a \right) h^{\nu b}$ is called the *spin-affine connection*;
  - the Lie algebra indices are those of the spin affine connection's Lie algebra;

  – and the fields s^{ab}, t^{ab}, $\bar{t}^{ab}$ are respectively analogous to the fields B, c, $\bar{c}$ in BRST-quantised Yang–Mills theory. While s_{ab} is a spin-0 bosonic field, t^{ab}, $\bar{t}^{ab}$ are unphysical spin-0 fermionic fields. The FP-ghost propagator is then $\left\langle 0|t^{ab}(x)\,\bar{t}^{cd}(y)\,|0\right\rangle$.

- A second formalism[45] is used, for example, in Sec. 4.3 of Ref. [39].[46]

  The mapping in Eq. (2.6.3) can be rewritten as $X_{\mu\nu} := \gamma^{\rho\sigma}_{\mu\nu} x_{\rho\sigma}$ where, to simplify Ref. [39]'s notation, I have introduced the tensor

$$\gamma^{\rho\sigma}_{\mu\nu} := \delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - k g_{\mu\nu} g^{\rho\sigma}. \tag{2.6.25}$$

  The constant $k \in \mathbb{R}$ is a gauge choice. It is often written as $k = 1 + \beta^{-1}$ where $\beta := (k-1)^{-1}$. The *de Donder gauge* is the choice $k = \frac{1}{2}$ (equivalently, $\beta = -2$). Appendix H.3 provides a motivation for the de Donder gauge.

  The BRST quantisation of perturbative gravity may be written as

$$\mathcal{L}_{\text{GF}} = -\gamma^{\rho\sigma}_{\mu\nu}\nabla^{\mu}B^{\nu}\kappa h_{\rho\sigma} + \frac{\alpha_0}{2}B^{\mu}B_{\mu}, \tag{2.6.26}$$

$$\mathcal{L}_{\text{FP}} = -\gamma^{\rho\sigma}_{\mu\nu}\nabla^{\mu}\bar{c}^{\nu}\,\mathrm{i}\pounds_{c} g^{f}_{\rho\sigma}, \tag{2.6.27}$$

  where B^{μ}, c^{μ}, $\bar{c}^{\mu}$ are spin-1 promotions of the fields B, c, $\bar{c}$ of Yang–Mills theory. The FP-ghost propagator is then $\langle 0|c_{\mu}(x)\,\bar{c}_{\nu}(y)\,|0\rangle$. As in the Yang–Mills case, the Nakanishi–Lautrup auxiliary field is bosonic, while the FP-(anti)ghosts are unphysical fermionic fields. While Yang–Mills theory is a theory of an interaction carried by spin-1 bosonic fields $A^{\mu a}$, general relativity is a theory of an interaction carried by a spin-2 bosonic field $g_{\mu\nu}$. In both cases, the Nakanishi–Lautrup and FP-(anti)ghost fields have a spin that is one less than that of the interaction.

I will work hereafter with this second formalism instead of the vielbein formalism, for several reasons:

- The use of vielbeins and spin affine connections would unnecessarily complicate the use of general relativity in this thesis;
- The vielbein formalism places two multiplet indices on the fields introduced in the Faddeev–Popov method, while the formalism I favour requires only one index on each field (this is of particular convenience in Sec. 3.2);
- While the vielbein formalism requires both multiplet and Greek indices, the alternative I use requires only one index type, a virtue not even enjoyed by BRST-quantised Yang–Mills theory (viz. A^{a}_{μ}).

I will also work in the Landau gauge so that $\alpha_0 = 0$. This removes the $B^{\mu}B_{\mu}$ term from $\mathcal{L}_0$.

The modes of c^{μ} ($\bar{c}^{\mu}$) that cause the infrared problem, the so-called zero modes, are of the form $\theta^{A}(t)\,\xi^{\mu}_{A}$ ($\bar{\theta}^{A}(t)\,\xi^{\mu}_{A}$), where $\theta^{A}(t)$, $\bar{\theta}^{A}(t)$ are spatially uniform unphysical Grassmann-valued scalar fields. Off-shell (on-shell), the θ^{A}, $\bar{\theta}^{A}$ are arbitrary (constrained by equations of motion in the scalars θ^{A}, $\bar{\theta}^{A}$).

[45] By my order of discussion; this formalism is actually the less recent of the two, although its application to the question of anti-BRST transformation is more recent (see Sec. 2.6.4).

[46] Again Ref. [39] is formative, but does not consider general curved spacetimes. Thus my Eqs. (2.6.26) and (2.6.27) below are slight generalisations of equations in Ref. [39] that use partial rather than covariant derivatives.

2.6.4 History of the BRST and anti-BRST transformations, with and without the vielbein formalism

Eqs. (2.3.2) and (2.3.10) respectively relate the BRST operator to the BRST charge and the anti-BRST operator to the anti-BRST charge. Both of the formulations of BRST-quantised perturbative gravity described in Sec. 2.6.3 admit BRST and anti-BRST transformations of this form. The vielbein formalism was first formulated in 1979 in Ref. [54], which also specified the BRST transformation in this formalism. The first source that defined the anti-BRST transformation [40] did so for both Yang–Mills theory and general relativity, and relied for the latter on the vielbein formalism. I will only define the BRST and anti-BRST transformations in my preferred formalism. One source for these formulations is Ref. [48].[47] Explicitly

$$[Q,\, B^\mu] = 0,\; \left[Q,\, g^f_{\mu\nu}\right] = \pounds_c g^f_{\mu\nu},\; \{Q,\, c^\mu\} = c^\nu \nabla_\nu c^\mu,\; \{Q,\, \overline{c}^\mu\} = \mathrm{i}B^\mu, \tag{2.6.28}$$

and similarly

$$\left[\overline{Q},\, \overline{B}^\mu\right] = 0,\; \left[\overline{Q},\, g^f_{\mu\nu}\right] = \pounds_{\overline{c}} g^f_{\mu\nu},\; \left\{\overline{Q},\, \overline{c}^\mu\right\} = \overline{c}^\nu \nabla_\nu \overline{c}^\mu,\; \left\{\overline{Q},\, c^\mu\right\} = \mathrm{i}\overline{B}^\mu \tag{2.6.29}$$

where

$$\overline{B}^\mu := -\mathrm{i}\pounds_{\overline{c}} c^\mu - B^\mu. \tag{2.6.30}$$

Note that

$$\pounds_{\overline{c}} c^\mu = \overline{c}^\nu \nabla_\nu c^\mu - \nabla_\nu \overline{c}^\mu c^\nu = \overline{c}^\nu \nabla_\nu c^\mu + c^\nu \nabla_\nu \overline{c}^\mu \tag{2.6.31}$$

is symmetric, because each argument is fermionic. Note also that the covariant derivatives may be replaced throughout with partial derivatives. Since $g_{\mu\nu}$ is (anti-)BRST-invariant,

$$[Q,\, \kappa h_{\mu\nu}] = \left[Q,\, g^f_{\mu\nu}\right] = \pounds_c g^f_{\mu\nu} = \nabla_\mu c_\nu + \nabla_\nu c_\mu + \pounds_c \kappa h_{\mu\nu}, \tag{2.6.32}$$

$$\left[\overline{Q},\, \kappa h_{\mu\nu}\right] = \nabla_\mu \overline{c}_\nu + \nabla_\nu \overline{c}_\mu + \pounds_{\overline{c}} \kappa h_{\mu\nu}. \tag{2.6.33}$$

The terms $\sqrt{|g|}\,(R - 2\Lambda)$ that exist in the Lagrangian density before it is BRST-quantised vary by a total derivative under any transformation of the form

$$\delta g^f_{\mu\nu} = \pounds_V\, g^f_{\mu\nu}, \tag{2.6.34}$$

and the BRST and anti-BRST transformations are examples of this that take $V \in \left\{\theta c,\, \overline{\theta c}\right\}$. Other terms in the scalar Lagrangian density are BRST-invariant, since $\gamma^{\rho\sigma}_{\mu\nu}$ is BRST- and anti-BRST-invariant and so

$$\left\{Q,\, \mathrm{i}\nabla^\mu \overline{c}^\nu \gamma^{\rho\sigma}_{\mu\nu} \kappa h_{\rho\sigma}\right\} = -\nabla^\mu B^\nu \gamma^{\rho\sigma}_{\mu\nu} \kappa h_{\rho\sigma} - \mathrm{i}\nabla^\mu \overline{c}^\nu \gamma^{\rho\sigma}_{\mu\nu} \pounds_c g^f_{\mu\nu} = \mathcal{L}_{\mathrm{GF}}\big|_{\alpha_0=0} + \mathcal{L}_{\mathrm{FP}} =: \mathcal{L}^{Bc\overline{c}}_0. \tag{2.6.35}$$

It can be shown that the anti-BRST invariance of these terms is equivalent to the gauge choice $k = \frac{1}{2}$. This gauge choice is assumed throughout Ref. [48] (although it does not acknowledge that there is a

[47] This 1983 paper does not claim originality for the results I summarise in Eqs. (2.6.28)–(2.6.30). Kuusk traces the BRST case to two papers [55, 56] published in 1976 and 1977. The anti-BRST case is of course more recent, as no discussion of anti-BRST can predate Ref. [46]. Kuusk traces the anti-BRST case to two papers [57, 58] published in 1981 and 1982.

gauge choice), which correctly claims $\left[\overline{Q},\, \mathcal{L}_0^{Bc\overline{c}}\right] = 0$ in this case. The FP-ghost contribution to $\mathcal{L}_0^{Bc\overline{c}}$ is

$$\mathcal{L}_{\text{FP}} := -\mathrm{i}\nabla^\mu \overline{c}^\nu \left(\nabla_\mu c_\nu + \nabla_\nu c_\mu - 2k g_{\mu\nu} \nabla_\lambda c^\lambda\right) - \mathrm{i}\nabla^\mu \overline{c}^\nu \kappa £_c h_{\mu\nu} + \mathrm{i}k \nabla_\alpha \overline{c}^\alpha g^{\beta\gamma} \kappa £_c h_{\beta\gamma}. \tag{2.6.36}$$

Exchanging c with $\overline{c}$ and multiplying by -1 by exchanging fermionic ghost and antighost factors, define

$$\mathcal{L}_{\overline{\text{FP}}} := -\mathrm{i}\left(\nabla_\mu \overline{c}_\nu + \nabla_\nu \overline{c}_\mu - 2k g_{\mu\nu} \nabla_\lambda \overline{c}^\lambda\right) \nabla^\mu c^\nu - \mathrm{i}\kappa £_{\overline{c}} h_{\mu\nu} \nabla^\mu c^\nu + \mathrm{i}k g^{\beta\gamma} \kappa £_{\overline{c}} h_{\beta\gamma} \nabla_\alpha c^\alpha. \tag{2.6.37}$$

Anti-BRST invariance, with the anti-BRST transformation defined by Eqs. (2.6.29) and (2.6.30), is equivalent to the condition $\mathcal{L}_{\text{FP}} - \nabla^\mu B^\nu \kappa H_{\mu\nu} \approx \nabla^\mu \overline{B}^\nu \kappa H_{\mu\nu} - \mathcal{L}_{\overline{\text{FP}}}$ (where $\approx$ denotes equality up to a total derivative), i.e.

$$\mathcal{L}_{\text{FP}} + \mathcal{L}_{\overline{\text{FP}}} \approx \nabla^\mu \left(B^\nu + \overline{B}^\nu\right) \kappa H_{\mu\nu} = \mathrm{i}\nabla^\mu \left(£_c \overline{c}^\nu\right) \kappa H_{\mu\nu}. \tag{2.6.38}$$

The only terms in $\mathcal{L}_{\text{FP}} + \mathcal{L}_{\overline{\text{FP}}}$ are those which appear in $\mathcal{L}_{\text{FP}}$ or $\mathcal{L}_{\overline{\text{FP}}}$ and are *not* $c^\mu \leftrightarrow \overline{c}^\mu$–antisymmetric. Let $\sim$ indicate equality up to such antisymmetric terms and total derivatives, so a replacement $h_{\mu\nu} \to h g_{\mu\nu}$ in the second and third terms renders them $c^\mu \leftrightarrow \overline{c}^\mu$–antisymmetric, viz.

$$\mathrm{i}\nabla^\mu \left(£_c \overline{c}^\nu\right) \kappa h g_{\mu\nu} = \mathrm{i}\kappa h \nabla_\nu \left(£_c \overline{c}^\nu\right) = \mathrm{i}\kappa h \nabla_\nu \left(c^\rho \nabla_\rho \overline{c}^\nu + \overline{c}^\rho \nabla_\rho c^\nu\right). \tag{2.6.39}$$

Thus

$$\begin{aligned}
\mathcal{L}_{\text{FP}} &\sim -\mathrm{i}\nabla^\mu \overline{c}^\nu \kappa c^\alpha \nabla_\alpha H_{\mu\nu} - \mathrm{i}\nabla^\mu \overline{c}^\nu \nabla_\nu c^\alpha \kappa H_{\alpha\mu} + 2\mathrm{i}k \nabla^\lambda \overline{c}_\lambda \nabla^\mu \overline{c}^\nu \kappa H_{\mu\nu} \\
&\sim \mathrm{i}\nabla_\alpha \nabla_\mu \overline{c}_\nu \kappa c^\alpha H^{\mu\nu} + \mathrm{i}\nabla_\mu \overline{c}_\nu \nabla_\alpha c^\alpha \kappa H^{\mu\nu} \\
&\quad - \mathrm{i}\nabla^\mu \overline{c}^\alpha \nabla_\alpha c^\nu \kappa H_{\mu\nu} + 2\mathrm{i}k \nabla^\lambda \overline{c}_\lambda \nabla_\mu \overline{c}_\nu \kappa H^{\mu\nu}.
\end{aligned} \tag{2.6.40}$$

Since $\left[\nabla_\alpha,\, \nabla_\mu\right] \overline{c}_\nu = R_{\nu\beta\alpha\mu} \overline{c}^\beta$, which when multiplied by c^α is $c^\mu \leftrightarrow \overline{c}^\mu$–antisymmetric,

$$\mathcal{L}_{\text{FP}} \sim \mathrm{i}\nabla_\mu \nabla_\alpha \overline{c}_\nu \kappa c^\alpha H^{\mu\nu} + \mathrm{i}\nabla_\mu \overline{c}_\nu \nabla_\alpha c^\alpha \kappa H^{\mu\nu} - \mathrm{i}\nabla_\mu \overline{c}^\alpha \nabla_\alpha c_\nu \kappa H^{\mu\nu} + 2\mathrm{i}k \nabla_\alpha \overline{c}^\alpha \nabla_\mu \overline{c}_\nu \kappa H^{\mu\nu}. \tag{2.6.41}$$

Thus $\mathcal{L}_{\text{FP}} + \mathcal{L}_{\overline{\text{FP}}} \approx K_{\mu\nu} \kappa H^{\mu\nu}$, where

$$\begin{aligned}
K_{\mu\nu} &:= \mathrm{i}\left(\nabla_\mu \nabla_\alpha \overline{c}_\nu\right) c^\alpha - \mathrm{i}\nabla_\mu \overline{c}^\alpha \nabla_\alpha c_\nu + \mathrm{i}\nabla_\mu \overline{c}_\nu \nabla_\alpha c^\alpha + 2\mathrm{i}k \nabla_\alpha \overline{c}^\alpha \nabla_\mu c_\nu \\
&\quad - \mathrm{i}\overline{c}^\alpha \nabla_\mu \nabla_\alpha c_\nu + \mathrm{i}\nabla_\alpha \overline{c}_\nu \nabla_\mu c^\alpha - \mathrm{i}\nabla_\alpha \overline{c}^\alpha \nabla_\mu c_\nu - 2\mathrm{i}k \nabla_\mu \overline{c}_\nu \nabla_\alpha c^\alpha.
\end{aligned} \tag{2.6.42}$$

Thus anti-BRST invariance is equivalent to

$$K^{\mu\nu} \approx \nabla^\mu \left(£_c \overline{c}^\nu\right). \tag{2.6.43}$$

But

$$K_{\mu\nu}=\mathrm{i}\left[\left(\nabla_{\mu}\nabla_{\alpha}\bar{c}_{\nu}\right)c^{\alpha}+\nabla_{\alpha}\bar{c}_{\nu}\nabla_{\mu}c^{\alpha}-\bar{c}^{\alpha}\nabla_{\mu}\nabla_{\alpha}c_{\nu}-\nabla_{\mu}\bar{c}^{\alpha}\nabla_{\alpha}c_{\nu}\right]$$
$$+\mathrm{i}\nabla_{\mu}\left[\left(\nabla_{\alpha}\bar{c}_{\nu}\right)c^{\alpha}-\bar{c}^{\alpha}\nabla_{\alpha}c_{\nu}\right]+\mathrm{i}\left(1-2k\right)\left[\nabla_{\mu}\bar{c}_{\nu}\nabla_{\alpha}c^{\alpha}-\nabla_{\alpha}\bar{c}^{\alpha}\nabla_{\mu}c_{\nu}\right]. \tag{2.6.44}$$

The desired result therefore occurs if and only if $1-2k=0$, as claimed.
A number of authors [59, 60] mistakenly claim that the action $\int_x\mathcal{L}_0$ is anti-BRST invariant for all k, where $\mathcal{L}_0:=R-2\Lambda+\mathcal{L}_0^{Bc\bar{c}}+\frac{\alpha_0}{2}B^{\mu}B_{\mu}$. However, the above calculation shows that $[S,\,\mathcal{L}_0]\propto 2k-1$. Also, Upadhyay defines the anti-BRST transformation incorrectly [60]. He replaces my Eq. (2.6.29) with

$$\left[\overline{Q},\,B^{\mu}\right]=0,\;\left[\overline{Q},\,g^{f}_{\mu\nu}\right]=\pounds_{\bar{c}}g^{f}_{\mu\nu},\;\left\{\overline{Q},\,\bar{c}^{\mu}\right\}=\bar{c}^{\nu}\nabla_{\nu}\bar{c}^{\mu},\;\left\{\overline{Q},\,c^{\mu}\right\}=-\mathrm{i}B^{\mu}, \tag{2.6.45}$$

i.e. he modifies Eq. (2.6.30) by instead taking $\overline{B}^{\mu}=-B^{\mu}$. (Note that this alternative to Eq. (2.6.30) is implausible, given the form of Eq. (2.3.22).) Unfortunately, this definition of the anti-BRST transformation is invalid; I now show its $\overline{Q}$ does not anticommute with Q, and does not commute with S for any k. I will abbreviate "$[X,\,x]$ is a total derivative" as "x is X-invariant". For example, $\mathcal{L}_0^{Bc\bar{c}}$ is Q-invariant (i.e. BRST-invariant). The "anti-BRST transformation" of Refs. [59] and [60] is obtainable from the BRST transformation by the replacement

$$Q,\,B^{\mu},\,c^{\mu},\,\bar{c}^{\mu}\to\overline{Q},\,-B^{\mu},\,\bar{c}^{\mu},\,c^{\mu}, \tag{2.6.46}$$

so the well-known fact that $Q^2=0$ implies that $\overline{Q}^2=0$. Since

$$\begin{aligned}\left[\left\{Q,\,\overline{Q}\right\},\,c^{\mu}\right]&=\left[Q,\,\left\{\overline{Q},\,c^{\mu}\right\}\right]+\left[\overline{Q},\,\left\{Q,\,c^{\mu}\right\}\right]\\&=\left[\overline{Q},\,c^{\nu}\nabla_{\nu}c^{\mu}\right]\\&=-\mathrm{i}\left(B^{\nu}\nabla_{\nu}c^{\mu}-c^{\nu}\nabla_{\nu}B^{\mu}\right),\end{aligned} \tag{2.6.47}$$

which is nonzero because the derivative contracts with the B-field in one term but the c-field in the other, $\overline{Q}$ does not anticommute with Q. Also, since $\mathcal{L}_0^{Bc\bar{c}}$ is Q-invariant,

$$\mathcal{L}_0^{**}:=\nabla^{\mu}B^{\nu}\kappa H_{\mu\nu}-\mathrm{i}\nabla^{\mu}c^{\nu}\gamma^{\rho\sigma}_{\mu\nu}\pounds_{\bar{c}}g^{f}_{\rho\sigma} \tag{2.6.48}$$

is $\overline{Q}$-invariant, so for $\mathcal{L}_0^{Bc\bar{c}}$ to be $\overline{Q}$-invariant would require the following to also be $\overline{Q}$-invariant:

$$\mathcal{L}_0^{Bc\bar{c}}+\mathcal{L}_0^{**}\propto\gamma^{\rho\sigma}_{\mu\nu}\left(\nabla^{\mu}\bar{c}^{\nu}\pounds_{c}g^{f}_{\rho\sigma}+\nabla^{\mu}c^{\nu}\pounds_{\bar{c}}g^{f}_{\rho\sigma}\right)=A\left(c,\,\bar{c}\right)+\kappa\nabla_{\gamma}H_{\mu\nu}\left(\left(\nabla^{\mu}\bar{c}^{\nu}\right)c^{\gamma}+c\leftrightarrow\bar{c}\right) \tag{2.6.49}$$

with

$$A=\kappa\nabla^{\mu}\bar{c}^{\nu}\left(\gamma^{\rho\sigma}_{\mu\nu}\nabla_{\rho}c^{\gamma}h_{\gamma\sigma}+\rho\leftrightarrow\sigma\right)+c\leftrightarrow\bar{c}=2\kappa\left(\left(\nabla^{\rho}\bar{c}^{\sigma}-kg^{\rho\sigma}\nabla_{\nu}\bar{c}^{\nu}\right)\nabla_{\rho}c^{\gamma}h_{\gamma\sigma}\right)+c\leftrightarrow\bar{c} \tag{2.6.50}$$

so

$$\mathcal{L}_0^{Bc\bar{c}}+\mathcal{L}_0^{**}=\kappa\left(\nabla_{\gamma}H_{\mu\nu}\left(\nabla^{\mu}\bar{c}^{\nu}\right)c^{\gamma}-2kh^{\rho}_{\gamma}\nabla_{\nu}\bar{c}^{\nu}\nabla_{\rho}c^{\gamma}\right)+c\leftrightarrow\bar{c} \tag{2.6.51}$$

(the term $2\kappa h_{\gamma\sigma}\left(\nabla^{\rho}\overline{c}^{\sigma}\nabla_{\rho}c^{\gamma}+c\leftrightarrow\overline{c}\right)$ has been dropped because it vanishes). Thus

$$\left[Q,\,\mathcal{L}_0^{Bc\overline{c}}+\mathcal{L}_0^{**}\right]\propto B_1\left(c,\,\overline{c}\right)+B_2\left(h,\,B,\,c,\,\overline{c}\right)+B_3\left(h,\,B,\,c,\,\overline{c}\right)+B_4\left(c,\,\overline{c}\right) \tag{2.6.52}$$

with

$$B_1=\gamma^{\rho\sigma}_{\mu\nu}\nabla_{\gamma}\left(\pounds_{\overline{c}}g^{f}_{\rho\sigma}\right)\left(\left(\nabla^{\mu}\overline{c}^{\nu}\right)c^{\gamma}+c\leftrightarrow\overline{c}\right), \tag{2.6.53}$$

$$B_2=\kappa\nabla_{\gamma}H_{\mu\nu}\left(\nabla^{\mu}\left(\overline{c}^{\rho}\nabla_{\rho}\overline{c}^{\nu}\right)c^{\gamma}+\mathrm{i}\nabla^{\mu}\overline{c}^{\nu}B^{\gamma}-\mathrm{i}\left(\nabla^{\mu}B^{\nu}\right)\overline{c}^{\gamma}-\nabla^{\mu}c^{\nu}\left(\overline{c}^{\rho}\nabla_{\rho}\overline{c}^{\gamma}\right)\right), \tag{2.6.54}$$

$$B_3=-2k\kappa h^{\rho}_{\gamma}\left(\nabla_{\nu}\left(\overline{c}^{\sigma}\nabla_{\sigma}\overline{c}^{\nu}\right)\nabla_{\rho}c^{\gamma}+\mathrm{i}\nabla_{\nu}\overline{c}^{\nu}\nabla_{\rho}B^{\gamma}-\mathrm{i}\nabla_{\nu}B^{\nu}\nabla_{\rho}\overline{c}^{\gamma}-\nabla_{\nu}c^{\nu}\nabla_{\rho}\left(\overline{c}^{\sigma}\nabla_{\sigma}\overline{c}^{\gamma}\right)\right), \tag{2.6.55}$$

$$B_4=-2k\left(g^{\rho\sigma}\pounds_{\overline{c}}g^{f}_{\gamma\sigma}\right)\left(\nabla_{\nu}\overline{c}^{\nu}\nabla_{\rho}c^{\gamma}+c\leftrightarrow\overline{c}\right). \tag{2.6.56}$$

But $\sum_{p=1}^{4}B_p$ is not zero even up to a total derivative. For example, the terms dependent on a (possibly differentiated) Nakanishi–Lautrup auxiliary field are

$$\mathrm{i}\kappa\nabla_{\gamma}H_{\mu\nu}\left(B^{\gamma}\nabla^{\mu}\overline{c}^{\nu}-\nabla^{\mu}B^{\nu}\overline{c}^{\gamma}\right)-2\mathrm{i}k\kappa h^{\rho}_{\gamma}\left(\nabla_{\rho}B^{\gamma}\nabla_{\nu}\overline{c}^{\nu}-\nabla_{\nu}B^{\nu}\nabla_{\rho}\overline{c}^{\gamma}\right). \tag{2.6.57}$$

In light of this, I will hereafter always define the anti-BRST transformation by Eqs. (2.6.29) and (2.6.30) instead of Eq. (2.6.45).

2.6.5 The zero mode problem in de Sitter space

As with Yang–Mills theory, the non-interacting theory's zero mode problem is easier to demonstrate with explicit calculations. The FMP adds an effective mass, so that the Faddeev–Popov sector without a metric perturbation is

$$-\mathrm{i}\left(\nabla^{\mu}\overline{c}^{\nu}Z_{\mu\nu}-M^2\overline{c}^{\nu}c_{\nu}\right),\;Z_{\mu\nu}=\nabla_{\mu}c_{\nu}+\nabla_{\nu}c_{\mu}-2kg_{\mu\nu}\nabla_{\rho}c^{\rho}. \tag{2.6.58}$$

The field equation obtained by varying the antighost is

$$\begin{aligned}0&=\nabla^{\mu}\left(\nabla_{\mu}c_{\nu}+\nabla_{\nu}c_{\mu}-2kg_{\mu\nu}\nabla_{\rho}c^{\rho}\right)+M^2c_{\nu}\\&=\left(\delta^{\sigma}_{\nu}\Box+\nabla^{\sigma}\nabla_{\nu}-2k\nabla_{\nu}\nabla^{\sigma}+M^2\delta^{\sigma}_{\nu}\right)c_{\sigma}.\end{aligned} \tag{2.6.59}$$

Note the right-hand side of Eq. (2.6.59) includes $\left(\Box+M^2\right)c_{\nu}$, in analogy with the minimally coupled Klein–Gordon equation $\left(\Box+M^2\right)\phi=0$.

Next I do two things to rewrite Eq. (2.6.59). One is to write $k=1+\beta^{-1}$, where $\beta:=\frac{1}{k-1}$ was defined previously in the discussion of an alternative to the vielbein formalism. The other is to use the identity $\left[\nabla^{\sigma},\,\nabla_{\nu}\right]V_{\sigma}=R_{\nu\tau}V^{\tau}$, where $R_{\nu\tau}$ is the Riemann tensor. Thus

$$0=\left(\delta^{\sigma}_{\nu}\Box+2R_{\nu\tau}g^{\sigma\tau}-\nabla^{\sigma}\nabla_{\nu}-\frac{2}{\beta}\nabla_{\nu}\nabla^{\sigma}+M^2\delta^{\sigma}_{\nu}\right)c_{\sigma}. \tag{2.6.60}$$

For the rest of my discussion of the zero mode problem, I specialise to de Sitter space with the nondimensionalisation $H=1$, so that mass is also nondimensionalised and $R_{\nu\tau}=(n-1)\,g_{\nu\tau}$. The equation of motion is therefore

$$L_\nu^\sigma\left(-M^2-2\left(n-1\right)\right)c_\sigma=0,\; L_\nu^\sigma\left(-\mu^2\right):=\delta_\nu^\sigma\Box-\nabla^\sigma\nabla_\nu-\frac{2}{\beta}\nabla_\nu\nabla^\sigma+\mu^2\delta_\nu^\sigma. \tag{2.6.61}$$

Note that, if the nondimensionalisation $H=1$ had not been taken, instead of taking

$$\mu^2=M^2+2\left(n-1\right) \tag{2.6.62}$$

we would have

$$\mu^2=M^2+2\left(n-1\right)H^2. \tag{2.6.63}$$

From Eq. (1.2.11), the spatial part of $\mathrm{d}s^2$ in de Sitter space is the Euclidean $\mathrm{d}s^2$ of S^n multiplied by a function of time. It is therefore unsurprising that the FP-ghost propagator in de Sitter space can be expressed in terms of S^n-specific eigenfunctions of certain differential operators. Indeed, let $Y^{L\sigma}\left(x\right)$ denote the scalar spherical harmonic functions on S^n; these functions satisfy

$$\Box Y^{L\sigma}=L\left(L+n-1\right)Y^{L\sigma}. \tag{2.6.64}$$

Then any smooth vector field on S^n is a linear combination of the vectors

$$W_\mu^{L\sigma}:=\left(L\left(L+n-1\right)\right)^{-\frac{1}{2}}\nabla_\mu Y^{L\sigma} \tag{2.6.65}$$

and the vector spherical harmonics $A_\mu^{L\sigma}$ that satisfy

$$\nabla^\mu A_\mu^{L\sigma}=0, \tag{2.6.66}$$

$$\Box A_\mu^{L\sigma}=\left(L^2+\left(n-1\right)L-1\right)A_\mu^{L\sigma}, \tag{2.6.67}$$

$$\int_x g^{\mu\nu}A_\mu^{L\sigma}A_\nu^{L'\sigma'}=-\delta^{LL'}\delta^{\sigma\sigma'}. \tag{2.6.68}$$

Let $G_{\mu\nu'}\left(x,\,x'\right)$ denote the Green's function of $L_\mu^\nu\left(-M^2-2\left(n-1\right)\right)$ on S^n (note that this notation suppresses the Green's function's M-dependence). This Green's function admits a decomposition into $A_\mu^{L\sigma}\left(x\right)A_{\nu'}^{*L\sigma}\left(x'\right)$ terms and $W_\mu^{L\sigma}\left(x\right)W_{\nu'}^{*L\sigma}\left(x'\right)$ terms. These contributions to $G_{\mu\nu'}\left(x,\,x'\right)$ respectively comprise a "vector part" $G_{\mu\nu'}^V\left(x,\,x'\right)$ and "scalar part" $G_{\mu\nu'}^S\left(x,\,x'\right)$ (terminology borrowed from Sec. 5 of Ref. [61]), viz.

$$G_{\mu\nu'}\left(x,\,x'\right)=G_{\mu\nu'}^V\left(x,\,x'\right)+G_{\mu\nu'}^S\left(x,\,x'\right), \tag{2.6.69}$$

$$G_{\mu\nu'}^V\left(x,\,x'\right)=\sum_{L\sigma}k_1^{L\sigma}\left(M\right)A_\mu^{L\sigma}\left(x\right)A_{\nu'}^{*L\sigma}\left(x'\right), \tag{2.6.70}$$

$$G_{\mu\nu'}^S\left(x,\,x'\right)=\sum_{L\sigma}k_2\left(M\right)W_\mu^{L\sigma}\left(x\right)W_{\nu'}^{*L\sigma}\left(x'\right). \tag{2.6.71}$$

The terminology's motivation is immediate; the "scalar part" is expressed in terms of covariant derivatives of the scalar spherical harmonic functions, while the "vector part" is expressed in terms of the vector spherical harmonics instead. Solving

$$L_{\rho}^{\mu}\left(-M^{2}-2\left(n-1\right)\right)G_{\mu\nu'}\left(x,\,x^{'}\right)=\frac{g_{\rho\nu'}}{\sqrt{\left|g\left(x\right)\right|}}\delta\left(x,\,x^{'}\right) \tag{2.6.72}$$

for $k_{1}^{L\sigma}$ gives

$$k_{1}^{L\sigma}\left(M\right)=\frac{1}{\left(L+1\right)\left(L+n-2\right)-M^{2}-2\left(n-1\right)}. \tag{2.6.73}$$

Since $k_{1}^{L\sigma}\left(M\right)$ diverges at $L=1$ when $M^{2}=0$, the FP-ghost Feynman propagator is infrared-divergent. For general M, the $L=1$ term in $G_{\mu\nu'}^{V}\left(x,\,x^{'}\right)$ is

$$\frac{A_{\mu}^{L\sigma}\left(x\right)A_{\nu'}^{*L\sigma}\left(x^{'}\right)}{\left(1+1\right)\left(1+n-2\right)-M^{2}-2\left(n-1\right)}=-\frac{A_{\mu}^{1\sigma}\left(x\right)A_{\nu'}^{*1\sigma}\left(x^{'}\right)}{M^{2}}. \tag{2.6.74}$$

The FMP subtracts out this term to effect an infrared regularisation of $G_{\mu\nu'}^{V}\left(x,\,x^{'}\right)$, leaving the $L\geq2$ terms

$$\sum_{L\geq2,\,\sigma}\frac{A_{\mu}^{L\sigma}\left(x\right)A_{\nu'}^{*L\sigma}\left(x^{'}\right)}{\left(L+1\right)\left(L+n-2\right)-M^{2}-2\left(n-1\right)}. \tag{2.6.75}$$

Note that the infrared divergence in the $M\to0^{+}$ right-hand limit is entirely due to the $L=1$ mode, so the $M\to0^{+}$ right-hand limit of the $L\geq2$ series is an effective zero-mode sector propagator. It is customary to write the full propagator as

$$G_{\mu\nu'}^{V}\left(x,\,x^{'}\right)=Q_{\mu\nu'}^{\left(M^{2}\right)}\left(x,\,x^{'}\right)+\frac{1}{2\left(n-1\right)+M^{2}}\nabla_{\mu}\nabla_{\nu'}D_{0}^{\text{eff}}\left(x,\,x^{'}\right), \tag{2.6.76}$$

where $Q_{\mu\nu'}^{\left(M^{2}\right)}\left(x,\,x^{'}\right)$ is a solution of

$$L_{\rho}^{\mu}\left(-M^{2}-2\left(n-1\right)\right)Q_{\mu\nu'}^{\left(M^{2}\right)}\left(x,\,x^{'}\right)=\frac{g_{\rho\nu'}}{\sqrt{\left|g\left(x\right)\right|}}\delta\left(x,\,x^{'}\right) \tag{2.6.77}$$

(viz. Sec. 3 of Ref. [50]) and $D_{0}^{\text{eff}}\left(x,\,x^{'}\right)$ is the FMP's effective zero-mode sector propagator in scalar field theory.

2.6.6 A comparison of zero-mode problems

The discussion in this subsection is an expanded version of Sec. VI of Ref. [1]. Although that paper was primarily concerned with the Yang–Mills infrared problem I discussed in Sec. 2.4, and the CMP treatment thereof that I discuss in Chapter 3, Sec. VI of Ref. [1] discussed analogous concerns in perturbative gravity. The infrared limit of the FP-ghost propagator's behaviour in perturbative quantum gravity with massive FP-(anti)ghosts has not been discussed much in the literature. However, an analogous concern regarding the graviton's propagator has attracted much more interest. This has resulted in some controversy for the case of de Sitter space, as I will now discuss.

There are two approaches to gauge fixing the graviton two-point function. One approach includes a gauge-fixing term in the linearised theory, and obtains the propagator $\kappa^{2}\left\langle 0|h_{\mu\nu}\left(x\right)h_{\mu'\nu'}\left(x^{'}\right)|0\right\rangle$. The other obtains a graviton correlator after complete gauge fixing. The graviton two-point function obtained by the latter method, hereafter the *gauge-invariant graviton two point function*, is physical in the sense that its gauge degrees of freedom are completely fixed. This two-point function is infrared-divergent in the Poincaré patch of de Sitter space [62].

This discovery began the debate of gravity's infrared issues, and this controversy has some similarities with an issue in the FP-ghost sector. One subtlety was that the infrared divergences of the two-point function may be expressed in a pure-gauge form [63, 64, 65], and the infrared divergences may be removed entirely with a suitable choice of mode functions [66]. In this sense, these infrared divergences are gauge-artefact. Indeed, the two-point function has been given an infrared-finite construction in de Sitter space in some other coordinate patches [67, 68, 69] and covariant gauges [70, 71, 72].

After Faizal and Higuchi introduced the FMP in Ref. [2] to address the FP-ghost sector implications in 2008, they provided a treatment of the graviton sector in the global patch in 2012, which also relied on temporarily endowing a field (in this case the graviton) with a fictitious mass [73]. The resulting gauge-invariant graviton two-point function is known to be equivalent to the linearised Weyl tensor [74], which is both de Sitter invariant and infrared-convergent in a vacuum state of the theory that is like a Euclidean Bunch–Davies vacuum [75, 76, 77]. Higuchi and I have previously observed [1] that, since Ref. [2] obtains the Weyl tensor as its gauge-invariant graviton two-point function, *non-interacting* linearised gravity has no infrared problem in de Sitter space.

However, what is contentious is whether *interacting* linearised gravity retains an infrared problem for the graviton propagator. Some say it does [78], while others say it does not [79]. One could similarly ask whether interacting linearised gravity retains an infrared problem for the FP-ghost propagator. The debate between sources such as Refs. [78] and [79] regarding the graviton two-point function is analogous to the FP-ghost sector issues I consider in this thesis. However, I will consider the FP-ghost sector issues in all the spacetimes of interest identified in Sec. 1.2.

In Sec. 2.5.1, I observed that the behaviour of minimally coupled massless scalar fields raised the question of whether the FP-ghost sector's infrared problem breaks de Sitter invariance in de Sitter space. Indeed, one could ask this question of BRST-quantised Yang–Mills theory and BRST-quantised perturbative gravity. One similarity between the FP-ghost sector's infrared problems in Yang–Mills theory and perturbative gravity is that they are both amenable to the FMP [2]. In Ref. [1], Higuchi and I showed that the CMP can also address the Yang–Mills case, and we suggested it could address the gravity case too. I prove that this is so in Chapter 4.

Part II Applying the CMP to Yang–Mills theory and perturbative gravity

Chapter 3 Applying the CMP to Yang–Mills theory

The CMP obtains an effective theory in which conserved momenta are set to 0. Should these zero-momentum conditions be imposed first in the Lagrangian or Hamiltonian? Since the Lagrangian and Hamiltonian formalisms are equivalent, it would be concerning if it mattered. In Sec. 3.1, I show the two approaches are equivalent for any Hamiltonian with canonical coordinates and conjugate momenta thereof.

In Sec. 3.2, I define "zero" and "other" modes for any canonical scalar or vector field and for their conjugate momentum densities. This furthers the work of Sec. 1.7.2, and allows the explicit calculations that will obviate the zero mode problems introduced in Chapter 2[48].

The CMP's treatment of Yang–Mills theory will be considered entirely in the Landau gauge. This treatment begins in Sec. 3.3, where a problem is encountered. I show in Sec. 3.3 that, while the fields $B,\, \bar{c}$ are cyclic (and hence so are their zero modes), $c_{(0)}$ is not cyclic. This appears to prevent the CMP from successfully treating the infrared problem in BRST-quantised Yang–Mills theory. However, this is in fact not the case. In Sec. 3.4 I show how the issue of the non-cyclic $c_{(0)}$ can be addressed. This requires a change in the choice of fields in terms of which BRST-quantised Yang–Mills theory is expressed. In Sec. 3.5, I show that this choice results in all zero modes being cyclic. In Sec. 3.6, I show that the Lagrangian and Hamiltonian each gain an extra term as a result of this choice. I do not impose the synchronous gauge in Secs. 3.5 and 3.6.

In Sec. 3.7 I show that, in spacetimes for which free integration by parts is possible (so that the FMP may be used), the FMP is perturbatively equivalent to the CMP.

In Sec. 3.8, I complete the discussion of the flat static torus and de Sitter space that I began in Sec. 1.8. In Sec. 3.9, I discuss several issues that this chapter leaves unaddressed. My treatment of these issues will occur in appropriate appendices.

3.1 A note on the CMP

Suppose H is a Hamiltonian with conserved conjugate momenta π^{ζ}_{I} of cyclic canonical coordinates ζ^{I} and non-conserved conjugate momenta π^{ω}_{i} of non-cyclic canonical coordinates ω^{i}. We may write

$$H = H\left(\pi^{\omega}_{i},\, \pi^{\zeta}_{I},\, \omega^{i},\, \zeta^{I}\right), \tag{3.1.1}$$

[48]One exception is the infrared behaviour of the graviton two-point function, which was discussed in Sec. 2.6.6 but is not a concern hereafter.

and the CMP sets each π_I^ζ to 0, obtaining an *effective Hamiltonian*

$$H_{\text{eff}} := H\left(\pi_i^\omega,\, \mathbf{0},\, \omega^i,\, \zeta^I\right). \tag{3.1.2}$$

The Lagrangian and *effective Lagrangian* are respectively

$$L = \dot{\omega}^i \Pi_i^\omega\left(\dot{\boldsymbol{\omega}},\, \dot{\boldsymbol{\zeta}}\right) + \dot{\zeta}^I \Pi_I^\zeta\left(\dot{\boldsymbol{\omega}},\, \dot{\boldsymbol{\zeta}}\right) - H\left(\pi_i^\omega,\, \pi_I^\zeta,\, \omega^i,\, \zeta^I\right), \tag{3.1.3}$$

$$L_{\text{eff}} := \dot{\omega}^i P_i^\omega\left(\dot{\boldsymbol{\omega}},\, \dot{\boldsymbol{\zeta}}\right) - H\left(\pi_i^\omega,\, \mathbf{0},\, \omega^i,\, \zeta^I\right) \tag{3.1.4}$$

(with implicit summation over $i,\, I$), where $\Pi_i^\omega,\, \Pi_I^\zeta,\, P_i^\omega$ express conjugate momenta as functions of canonical coordinates' time derivatives (and, possibly, other variables not shown herein). These functions are obtainable implicitly from the Hamilton's equations

$$\dot{\omega}^i = \left.\frac{\partial_R H}{\partial_R \pi_i^\omega}\right|_{\pi_i^\omega = P_i^\omega,\, \pi_I^\zeta = 0}, \tag{3.1.5}$$

$$\dot{\omega}^i = \left.\frac{\partial_R H}{\partial_R \pi_i^\omega}\right|_{\pi_i^\omega = \Pi_i^\omega,\, \pi_I^\zeta = \Pi_I^\zeta}, \tag{3.1.6}$$

$$\dot{\zeta}^I = \left.\frac{\partial_R H}{\partial_R \pi_I^\zeta}\right|_{\pi_i^\omega = \Pi_i^\omega,\, \pi_I^\zeta = \Pi_I^\zeta}, \tag{3.1.7}$$

where ∂_R denotes *right-derivatives*. A consistent use of right-derivatives avoids a need for sign changes for fermionic fields. For example,

$$\frac{\partial_L}{\partial_L \pi_I^\zeta} = (-1)^{f(\zeta^I)}\, \frac{\partial_R}{\partial_R \pi_I^\zeta}, \tag{3.1.8}$$

where $f\left(\zeta^I\right)$ is the *fermion number* of ζ^I. Thus

$$\begin{aligned}
\frac{\partial_R L}{\partial_R \dot{\zeta}^I} &= (-1)^{f(\zeta^I)}\, \Pi_I^\zeta + \dot{\omega}^i \frac{\partial_R \Pi_i^\omega}{\partial_R \dot{\zeta}^I} + \dot{\zeta}_J \frac{\partial_R \Pi_J^\zeta}{\partial_R \dot{\zeta}^I} \\
&\quad - \left.\frac{\partial_R H}{\partial_R \pi_i^\omega}\right|_{\pi_i^\omega = \Pi_i^\omega,\, \pi_I^\zeta = \Pi_I^\zeta} \frac{\partial_R \Pi_i^\omega}{\partial_R \dot{\zeta}^I} - \left.\frac{\partial_R H}{\partial_R \pi_J^\zeta}\right|_{\pi_i^\omega = \Pi_i^\omega,\, \pi_I^\zeta = \Pi_I^\zeta} \frac{\partial_R \Pi_J^\zeta}{\partial_R \dot{\zeta}^I} \\
&= (-1)^{f(\zeta^I)}\, \Pi_I^\zeta \\
&\quad + \left[\dot{\omega}^i - \left.\frac{\partial_R H}{\partial_R \pi_i^\omega}\right|_{\pi_i^\omega = \Pi_i^\omega,\, \pi_I^\zeta = \Pi_I^\zeta}\right] \frac{\partial_R \Pi_i^\omega}{\partial_R \dot{\zeta}^I} + \left[\dot{\zeta}^J - \left.\frac{\partial_R H}{\partial_R \pi_J^\zeta}\right|_{\pi_i^\omega = \Pi_i^\omega,\, \pi_I^\zeta = \Pi_I^\zeta}\right] \frac{\partial_R \Pi_J^\zeta}{\partial_R \dot{\zeta}^I} \\
&= (-1)^{f(\zeta^I)}\, \Pi_I^\zeta,
\end{aligned} \tag{3.1.9}$$

$$\frac{\partial_L L}{\partial_L \dot{\zeta}^I} = \Pi_I^\zeta. \tag{3.1.10}$$

Hence L is $\dot{\zeta}^I$-independent if and only if $\Pi_I^\zeta = 0$, which is the condition the CMP imposes. However, such vanishing conjugate momenta are then also conserved, so the ζ^I are cyclic in L. The CMP's effect can therefore be equivalently stated as either the replacement of H with H_{eff} or as all canonical coordinates with conserved conjugate momenta in H's Hamilton's equations being obviated from L.

We may solve the equation $\Pi_I^\zeta = 0$, say as $\dot{\zeta}^I = \dot{Z}^I(\dot{\boldsymbol{\omega}})$ so that $\Pi_I^\zeta\left(\dot{\boldsymbol{\omega}},\, \dot{Z}(\dot{\boldsymbol{\omega}})\right) = 0$. Substituting $\dot{\zeta}^I = \dot{Z}^I(\dot{\boldsymbol{\omega}})$ in L gives

$$L = \dot{\omega}^i \Pi_i^\omega\left(\dot{\boldsymbol{\omega}},\, \dot{Z}(\dot{\boldsymbol{\omega}})\right) - H\left(\Pi_i^\omega\left(\dot{\boldsymbol{\omega}},\, \dot{Z}(\dot{\boldsymbol{\omega}})\right),\, \mathbf{0},\, \omega^i,\, \zeta^I\right). \tag{3.1.11}$$

Eq. (3.1.6) implies

$$\dot{\omega}^i = \left.\frac{\partial_R H}{\partial_R \pi_i^\omega}\right|_{\pi_i^\omega = \Pi_I^\zeta\left(\dot{\boldsymbol{\omega}},\, \dot{Z}(\dot{\boldsymbol{\omega}})\right),\, \pi_I^\zeta = 0}, \tag{3.1.12}$$

so $\Pi_i^\omega\left(\dot{\boldsymbol{\omega}},\, \dot{Z}(\dot{\boldsymbol{\omega}})\right) = P_i^\omega\left(\dot{\boldsymbol{\omega}},\, \dot{\boldsymbol{\zeta}}\right)$ and $L = L_{\text{eff}}$. Imposing the CMP's conditions $\Pi_I^\zeta = 0$ in the Lagrangian formalism is therefore equivalent to instead doing so in the Hamiltonian formalism. This is not trivial; it is in general invalid to use equations of motion to identify conserved charges and then set these to specific values in the Lagrangian formalism. For example, the classical simple harmonic oscillator of Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$, which has Euler–Lagrange equation $m\ddot{x} = -kx$, should not be rewritten using the conserved Hamiltonian $H = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$. This approach could achieve $L = H - kx^2$ (which would have a spurious Euler–Lagrange equation $kx = 0$) or $L = m\dot{x}^2 - H$ (which would have a spurious Euler–Lagrange equation $m\ddot{x} = 0$).
Eq. (3.1.4) may be restated as

$$H_{\text{eff}} = \dot{\omega}^i \pi_i^\omega\left(\dot{\boldsymbol{\omega}},\, \dot{\boldsymbol{\zeta}}\right) - L_{\text{eff}}. \tag{3.1.13}$$

Since we may begin by obviating zero modes from either the Lagrangian or Hamiltonian, we can think of L_{eff} as the Lagrangian that results from beginning in the Lagrangian formalism. Then Eq. (3.1.13) obtains a "Hamiltonian" from a Legendre transform that runs only over non-cyclic canonical coordinates. In other words, H_{eff} is a *Routhian* of L_{eff} that excludes cyclic coordinates' $\dot{q}p$ terms from the Legendre transform. (This is equivalent to imposing the conditions $\pi_I^\zeta = 0$.) The advantages in classical mechanics of using a Routhian formalism that separately treats cyclic and non-cyclic canonical coordinates are well-known. A brief overview thereof is provided in §41 of Chapter VII of Ref. [80].
The above calculations concern the Lagrangian and Hamiltonian of a theory with canonical coordinates and their conjugate momenta. For Yang–Mills theory and perturbative gravity, a respective promotion $\omega^i,\, \zeta^I$ to quantities of the form $\phi_{(+)},\, \phi_{(0)}$ for fields ϕ is necessary. Note that the momenta are still not momentum densities, since it is momenta, not momentum densities, that can be conserved. For example, if ϕ is a non-interacting massless Klein–Gordon field $\int_{\mathbf{x}} \nabla^0 \phi$ is its conserved conjugate momentum.
The promotion to classical fields is trivial. However, if the fields are quantised there is an additional subtlety. For each conserved momentum Π_I^ζ set to 0 in the CMP, we demand any physical state $|\psi\rangle$ satisfies $\hat{\pi}_I^\zeta |\psi\rangle = 0$, where the operator $\hat{\pi}_I^\zeta$ is the quantisation of π_I^ζ. The formal wave functional $\Psi\left(\zeta^I,\, \pi_I^\zeta,\, \cdots\right)$ then satisfies

$$\frac{\delta \Psi}{\delta \zeta^I} = 0. \tag{3.1.14}$$

Thus $\zeta^I,\, \pi_I^\zeta$ are also obviated from the Schrödinger wave functional formalism, viz.

$$\Psi\left(\zeta^I,\, \pi_I^\zeta\right) \to \Psi\left(\pi_I^\zeta = 0\right). \tag{3.1.15}$$

3.2 Defining zero modes of several types of field

In Sec. 1.7.2, I presented an explicit definition of the zero mode of a canonical scalar field, which I denoted χ. This was an improvement over the weaker statement that the zero mode should be a spatially uniform contribution to the field[49], viz.

$$\chi = \chi_{(0)} + \chi_{(+)},\, \partial_i \chi_{(0)} = 0. \tag{3.2.1}$$

In Sec. 2.6.3, I made an analogous statement about vector fields. If a decomposition of these fields into their "zero" and "other" modes is sought, it will be of the form

$$\chi^\mu = \chi^\mu_{(0)} + \chi^\mu_{(+)},\, \chi^\mu_{(0)} \in \left\{ \mathrm{X}^A \xi^\mu_A | \partial_i \mathrm{X}^A = 0 \right\}. \tag{3.2.2}$$

This raises the question of how to choose the X^A (such as the θ^A for $\phi^\mu = c^\mu$, or $\overline{\theta}^A$ for $\phi^\mu = \overline{c}^\mu$), which are named for the upper-case χ. While X^A constructions need not concern us until Chapter 4[50], I will mostly (see the discussion of Eq. (3.2.25) below) answer the question now for a reason that requires some explanation. Another issue I have left unaddressed heretofore is the conjugate momentum densities of quantities such as $\phi_{(0)},\, \phi_{(+)}$. In the construction I present below, it is shown that the conjugate momentum density of a tensor field T admits an analogous decomposition of its own. This decomposition is expressible in terms of the conjugate momentum densities of $T_{(0)},\, T_{(+)}$. Although I have no general zero mode decomposition of arbitrary tensor fields, I can provide a relation between the zero/other mode decompositions of canonical tensor fields and their conjugate momentum densities.

The (anti)commutators of quantised canonical fields and their quantised conjugate momentum densities are axiomatic in any quantum field theory, providing a generalisation of Eq. (1.4.10). Herein I do not place hats on these quantised fields, because this would unnecessarily clutter the notation, and would not emphasise that the mode decompositions I will present are intended for both quantised and classical fields. With that said, a general formalism for quantum field theories may be given. Consider fields $\phi^b(t,\, \mathbf{y})$[51], either all bosonic or all fermionic (these two special cases can be analysed separately.) Let $\pi_a(t,\, \mathbf{x})$ denote the conjugate momentum density of $\phi^b(t,\, \mathbf{y})$; then

$$\left[\pi_a(t,\, \mathbf{x}),\, \phi^b(t,\, \mathbf{y})\right]_\pm = -\mathrm{i}\delta^b_a \delta(\mathbf{x},\, \mathbf{y}) \tag{3.2.3}$$

(where the commutator or anticommutator's sign is determined by whether the fields are bosonic or fermionic). Next I need two sets of indices, which I denote by lower and upper case Roman letters

[49] In Sec. 1.5, I discuss a mode decomposition of a quantised scalar field on a flat static torus. This mode decomposition features one spatially uniform scalar field. However, before Sec. 1.7.2 it was not obvious what value this mode should have for a given field. Note in particular that a result of the form $\chi(x) = \chi_{(0)}(t) + \chi_{(+)}(x)$ is not unique, as the transformation $\chi_{(0)},\, \chi_{(+)} \to \chi_{(0)} + F(t),\, \chi_{(+)} - F(t)$ preserves this form of χ.

[50] The existence of a vector field multiplet A^a_μ in Yang–Mills theory will, it turns out, not require the zero modes of vector fields to be treated herein.

[51] My choice in Sec. 2.6.3 to avoid the vielbein formalism ensures that all tensor-valued canonical fields considered in this thesis have exactly one index. Had I chosen to work in the vielbein formalism, the "index" of ϕ^b would be interpreted as having multiple components in general.

beginning at a, A.[52] Throughout I implicitly sum over repeated indices, be they both lower case or both upper case. I now consider quantities of the form

$$\pi_{a(0)}\left(t,\,\mathbf{x}\right) := F_a^A\left(t,\,\mathbf{x}\right)\int \mathrm{d}^{n-1}\mathbf{w}G_A^c\left(t,\,\mathbf{w}\right)\pi_c\left(t,\,\mathbf{w}\right), \tag{3.2.4}$$

$$\phi^b_{(0)}\left(t,\,\mathbf{y}\right) := G_B^b\left(t,\,\mathbf{y}\right)\int \mathrm{d}^{n-1}\mathbf{z}F_d^B\left(t,\,\mathbf{z}\right)\phi^d\left(t,\,\mathbf{z}\right), \tag{3.2.5}$$

$$\pi_{a(+)}\left(t,\,\mathbf{x}\right) := \pi_a\left(t,\,\mathbf{x}\right) - \pi_{a(0)}\left(t,\,\mathbf{x}\right), \tag{3.2.6}$$

$$\phi^b_{(+)}\left(t,\,\mathbf{y}\right) := \phi^b\left(t,\,\mathbf{y}\right) - \phi^b_{(0)}\left(t,\,\mathbf{y}\right), \tag{3.2.7}$$

with functions F_a^A, G_B^a satisfying

$$\int \mathrm{d}^{n-1}\mathbf{z}F_a^A\left(t,\,\mathbf{z}\right)G_B^a\left(t,\,\mathbf{z}\right) = \delta_B^A. \tag{3.2.8}$$

I will call quantities with $_{(+)}$ subscripts *other modes*.

By inspection,

$$\begin{aligned}\left[\pi_{a(0)}\left(t,\,\mathbf{x}\right),\,\phi^b\left(t,\,\mathbf{y}\right)\right]_\pm &= -\mathrm{i}F_a^A\left(t,\,\mathbf{x}\right)\int \mathrm{d}^{n-1}\mathbf{w}G_A^c\left(t,\,\mathbf{w}\right)\delta_c^b\delta\left(\mathbf{w},\,\mathbf{y}\right)\\ &= -\mathrm{i}F_a^A\left(t,\,\mathbf{x}\right)G_A^b\left(t,\,\mathbf{y}\right),\end{aligned} \tag{3.2.9}$$

$$\begin{aligned}\left[\pi_a\left(t,\,\mathbf{x}\right),\,\phi^b_{(0)}\left(t,\,\mathbf{y}\right)\right]_\pm &= -\mathrm{i}G_A^b\left(t,\,\mathbf{y}\right)\int \mathrm{d}^{n-1}\mathbf{z}F_d^A\left(t,\,\mathbf{z}\right)\delta_a^d\delta\left(\mathbf{x},\,\mathbf{z}\right)\\ &= -\mathrm{i}F_a^A\left(t,\,\mathbf{x}\right)G_A^b\left(t,\,\mathbf{y}\right),\end{aligned} \tag{3.2.10}$$

$$\begin{aligned}\left[\pi_{a(0)}\left(t,\,\mathbf{x}\right),\,\phi^b_{(0)}\left(t,\,\mathbf{y}\right)\right]_\pm &= -\mathrm{i}F_a^B\left(t,\,\mathbf{x}\right)\int \mathrm{d}^{n-1}\mathbf{w}G_B^c\left(t,\,\mathbf{w}\right)F_c^A\left(t,\,\mathbf{w}\right)G_A^b\left(t,\,\mathbf{y}\right)\\ &= -\mathrm{i}\delta_A^BF_a^B\left(t,\,\mathbf{x}\right)G_A^b\left(t,\,\mathbf{y}\right) = -\mathrm{i}F_a^A\left(t,\,\mathbf{x}\right)G_A^b\left(t,\,\mathbf{y}\right).\end{aligned} \tag{3.2.11}$$

In summary

$$\begin{aligned}\left[\pi_{a(0)}\left(t,\,\mathbf{x}\right),\,\phi^b\left(t,\,\mathbf{y}\right)\right]_\pm = \left[\pi_a\left(t,\,\mathbf{x}\right),\,\phi^b_{(0)}\left(t,\,\mathbf{y}\right)\right]_\pm &= \left[\pi_{a(0)}\left(t,\,\mathbf{x}\right),\,\phi^b_{(0)}\left(t,\,\mathbf{y}\right)\right]_\pm\\ &= -\mathrm{i}F_a^A\left(t,\,\mathbf{x}\right)G_A^b\left(t,\,\mathbf{y}\right),\end{aligned} \tag{3.2.12}$$

so

$$\left[\pi_{a(0)}\left(t,\,\mathbf{x}\right),\,\phi^b_{(+)}\left(t,\,\mathbf{y}\right)\right]_\pm = \left[\pi_{a(+)}\left(t,\,\mathbf{x}\right),\,\phi^b_{(0)}\left(t,\,\mathbf{y}\right)\right]_\pm = 0. \tag{3.2.13}$$

The above construction obtains

$$\pi_a\left(t,\,\mathbf{x}\right) = \pi_{a(0)}\left(t,\,\mathbf{x}\right) + \pi_{a(+)}\left(t,\,\mathbf{x}\right), \tag{3.2.14}$$

$$\phi^b\left(t,\,\mathbf{y}\right) = \phi^b_{(0)}\left(t,\,\mathbf{y}\right) + \phi^b_{(+)}\left(t,\,\mathbf{y}\right), \tag{3.2.15}$$

[52] All previous associations of specific types of index with specific meanings, such as the use of a to denote Lie algebra indices, is hereafter dropped. In this context, indices may denote Lie algebra indices or spacetime indices. I will discuss specific possibilities towards the end of this section.

and the canonical momentum densities of $\phi^b_{(0)}$, $\phi^b_{(+)}$ are respectively proportional to $\pi_{b(0)}$, $\pi_{b(+)}$. The crucial implication for the CMP is this: if $\phi^b_{(0)}$ is cyclic, then its conjugate momentum is conserved and may be chosen to be 0, so that the proportional quantity $\pi_{b(0)}$ also vanishes.

The special cases relevant to this thesis can now be discussed; I recall the volume factor defined in Eq. (1.2.9).

- The previous result for $\phi_{(0)}$ in scalar field theory (viz. Eq. (1.7.4)) is a special case of Eq. (3.2.5). Consider a single field ϕ, so the indices $a, \cdots$ each have only one value, and such indices can be dropped. The indices $A, \cdots$ may also be dropped from F^A_a, G^a_A, by choosing

$$F\left(t,\,\mathbf{x}\right)=\frac{g^{00}\left(t,\,\mathbf{x}\right)\sqrt{\left|g\left(t,\,\mathbf{x}\right)\right|}}{V\left(t\right)},\,G\left(t,\,\mathbf{x}\right)=1. \tag{3.2.16}$$

  A few key results are worth noting. Zero modes of canonical scalar fields are spatially uniform, so may always be moved outside spatial integrals, including those which appear in the definitions of such zero modes. Hence

$$\left(X_{(0)}\right)_{(0)}=X_{(0)}, \tag{3.2.17}$$

$$\left(X_{(+)}\right)_{(0)}=\left(X-X_{(0)}\right)_{(0)}=0, \tag{3.2.18}$$

$$\left(X_{(0)}Y_{(+)}\right)_{(0)}=X_{(0)}\left(Y_{(+)}\right)_{(0)}=0. \tag{3.2.19}$$

  The dot and cross products for multiplet-valued canonical scalar fields therefore satisfy

$$\left(X_{(0)}\cdot Y_{(+)}\right)_{(0)}=0, \tag{3.2.20}$$

$$\left(X_{(0)}\times Y_{(+)}\right)_{(0)}=0, \tag{3.2.21}$$

$$\left(X\cdot Y\right)_{(0)}=X_{(0)}\cdot Y_{(0)}+\left(X_{(+)}\cdot Y_{(+)}\right)_{(0)}, \tag{3.2.22}$$

$$\left(X\times Y\right)_{(0)}=X_{(0)}\times Y_{(0)}+\left(X_{(+)}\times Y_{(+)}\right)_{(0)}. \tag{3.2.23}$$

- Another case of interest is a vector field ϕ^μ, so the indices $a, \cdots$ are simply spacetime indices. This time both indices of F^A_a, G^a_A are preserved, with upper case indices denoting the Lie algebra indices of Killing vector fields. Explicitly

$$F^A_\mu\left(t,\,\mathbf{x}\right)=\frac{g^{00}\left(t,\,\mathbf{x}\right)\sqrt{\left|g\left(t,\,\mathbf{x}\right)\right|}\eta^A_\mu\left(t,\,\mathbf{x}\right)}{V\left(t\right)},\,G^\nu_A\left(t,\,\mathbf{x}\right)=\xi^\nu_A\left(t,\,\mathbf{x}\right), \tag{3.2.24}$$

  where the ξ^ν_A are a basis of the Killing vectors and the η^A_μ are chosen so that

$$\int\mathrm{d}^{n-1}\mathbf{x}\left\{\frac{g^{00}\left(t,\,\mathbf{x}\right)\sqrt{\left|g\left(t,\,\mathbf{x}\right)\right|}}{V\left(t\right)}\xi^\mu_B\left(t,\,\mathbf{x}\right)\eta^A_\mu\left(t,\,\mathbf{x}\right)\right\}=\delta^A_B. \tag{3.2.25}$$

  Finding η^A_μ that satisfy Eq. (3.2.25) is a task I leave for Sec. 4.2.

3.3 Relation between conserved currents and the BRST and anti-BRST transformations

The Euler–Lagrange equation obtained by varying A^μ requires the inclusion in the Lagrangian density of those terms that predate BRST quantisation. However, this Euler–Lagrange equation is of no concern herein, and all other Euler–Lagrange equations are deducible entirely from the other terms. The scalar Lagrangian density may then be taken as[53]

$$\mathcal{L}_0 = -\nabla_\mu B \cdot A^\mu - \mathrm{i}\nabla_\mu \overline{c} \cdot D^\mu c \tag{3.3.1}$$

in the Landau gauge. Attempting to obtain an Euler–Lagrange equation by varying A^μ now would be illegitimate; A^μ no longer generates a stationary action principle.
One Euler–Lagrange equation is $\nabla_\mu A^\mu = 0$. This is a conservation law; the Noether charge is

$$Q_A := \int_{\mathbf{x}} A^0 = V(t)\left(N^2 A^0\right)_{(0)}. \tag{3.3.2}$$

Setting $Q_A = 0$ (the grey note in Sec. 1.7.2 explains why this is possible) gives

$$\left(N^2 A^0\right)_{(0)} = 0,\ A^0 = N^{-2}\left(N^2 A^0\right)_{(+)}. \tag{3.3.3}$$

Similarly, $\nabla_\mu D^\mu c = 0$ gives the Noether charge

$$Q_{Dc} := \int_{\mathbf{x}} D^0 c = \left(N^2 D^0 c\right)_{(0)}, \tag{3.3.4}$$

which I also set to 0. The fact that B ($\overline{c}$), and hence $B_{(0)}$ ($\overline{c}_{(0)}$), is cyclic implies their conjugate momenta are proportional to these vanishing Noether charges, and so $B_{(0)}$, $\overline{c}_{(0)}$ may be obviated. Unfortunately, the same is not true of $c_{(0)}$ unless $q = 0$. It appears undifferentiated in L because of the term

$$\begin{aligned}\int_{\mathbf{x}} \left(-\mathrm{i}q\nabla_\mu \overline{c} \cdot A^\mu \times c_{(0)}\right) &= -\mathrm{i}qc_{(0)} \cdot \int_{\mathbf{x}} A^\mu \times \nabla_\mu \overline{c} = -\mathrm{i}qc_{(0)} \cdot \left(N^2 A^\mu \times \nabla_\mu \overline{c}\right)_{(0)} \\ &= -\mathrm{i}qc_{(0)} \cdot \left\{\left(N^2 A^0 \times \nabla_0 \overline{c}\right)_{(0)} + \left(N^2 A^i \times \nabla_i \overline{c}\right)_{(0)}\right\} \\ &= -\mathrm{i}qc_{(0)} \cdot \left\{\left(N^2 A^0\right)_{(+)} \times \partial_0 \overline{c}_{(+)} + \left(N^2 A^i\right)_{(+)} \times \partial_i \overline{c}_{(+)}\right\}_{(0)}.\end{aligned} \tag{3.3.5}$$

Here the results $\left(N^2 A^0\right)_{(0)} = 0$, $\partial_i \overline{c}_{(0)} = 0$ have afforded some simplifications. However, the result is still clearly non-trivial for nonzero q. This is unsurprising, since

$$\frac{\partial \mathcal{L}_0}{\partial c^a_{(0)}} = \frac{\partial}{\partial c^a_{(0)}}\left(-\mathrm{i}q\nabla_\mu \overline{c} \cdot A^\mu \times c\right) = \frac{\partial}{\partial c^a_{(0)}}\left(-\mathrm{i}qc \cdot A^\mu \times \nabla_\mu \overline{c}\right) = -\mathrm{i}q\left(A^\mu \times \nabla_\mu \overline{c}\right)^a = -\mathrm{i}q\nabla_\mu\left(A^\mu \times \overline{c}\right)^a. \tag{3.3.6}$$

Although $\nabla_\mu\left(A^\mu \times \overline{c}\right)$ is a total derivative, $c \cdot \nabla_\mu\left(A^\mu \times \overline{c}\right)$ is not, so adding a total derivative to $\mathcal{L}_0$ to choose a different Lagrangian density that describes the same action will not suffice to obviate $c_{(0)}$.

[53] This dropping of "classical" terms (i.e. those which appear even without BRST quantisation) minimises clutter in my equations. An alternative approach is to use the subscript $_{\mathrm{cl}}$ for such "classical" terms, e.g. the right-hand side of Eq. (3.3.1) can instead be thought of as $\mathcal{L}_0 - \mathcal{L}_{0\mathrm{cl}}$, where $\mathcal{L}_0$ is the full BRST-quantised Yang–Mills Lagrangian density. Similarly, the Hamiltonian H that I use in Sec. 3.5 could instead be written as $H - H_{\mathrm{cl}}$. The Legendre transform provides an expression for $H + L$, which in a fuller treatment should also include a term $\partial_0 A_\mu \pi_{A_\mu}$.

An alternative way to think about this issue is by comparing conserved charges to momenta. The reduced momentum densities of B^a, $\overline{c}^a$ are respectively

$$\varpi_B = -A^0,\, \varpi_{\overline{c}} = -\mathrm{i}qD^0c. \tag{3.3.7}$$

Thus

$$Q_A = -\int_{\mathbf{x}} \varpi_B = -\int \mathrm{d}^{n-1}\mathbf{x}\pi_B,\, Q_{DC} = \frac{\mathrm{i}}{q}\int_{\mathbf{x}} \varpi_{\overline{c}} = \frac{\mathrm{i}}{q}\int \mathrm{d}^{n-1}\mathbf{x}\pi_{\overline{c}}. \tag{3.3.8}$$

In Sec. 2.4.2 I showed that in the Landau gauge $\nabla_\mu D^\mu \overline{c} = 0$, providing a further Noether charge I set to 0, viz.

$$Q_{D\overline{c}} := \int_{\mathbf{x}} D^0\overline{c} = \left(N^2 D^0 \overline{c}\right)_{(0)}. \tag{3.3.9}$$

Indeed, the three Noether charges set to 0 are

$$Q_A,\, Q_{Dc} = [Q,\, Q_A]\,,\, Q_{D\overline{c}} = \left[\overline{Q},\, Q_A\right], \tag{3.3.10}$$

so imposing the conditions $Q\left|\psi\right\rangle = 0,\, \overline{Q}\left|\psi\right\rangle = 0,\, Q_A\left|\psi\right\rangle = 0$ on all physical states $\left|\psi\right\rangle$ automatically obtains $Q_{Dc}\left|\psi\right\rangle = 0,\, Q_{D\overline{c}}\left|\psi\right\rangle = 0$ for all physical states $\left|\psi\right\rangle$. However, in the interacting case $Q_{D\overline{c}}$ is not proportional to $\int \mathrm{d}^{n-1}\mathbf{x}\pi_c$, since

$$\varpi_c = \frac{\partial}{\partial \nabla_0 c}\left(-\mathrm{i}\nabla_\mu \overline{c}\cdot\nabla^\mu c\right) = \mathrm{i}\nabla^0\overline{c}, \tag{3.3.11}$$

which differs from $\mathrm{i}D^0c$, i.e. $\int \mathrm{d}^{n-1}\mathbf{x}\pi_c$ is not conserved, unless $q = 0$. So not only is the zero mode problem harder to explicitly describe in the interacting case; it is also harder to treat. The entire strategy of the CMP hinges on identifying fields conjugate to conserved charges so the latter may be set to zero, as is possible (for example) in the relation between B and Q_A. Our problem is that $Q_{D\overline{c}}$ does not seem susceptible to this line of attack.

Yet another way to think about this is in terms of which transformations preserve $\mathcal{L}_0$. Spacetime-constant shifts in either B or $\overline{c}$ do this, because these fields are cyclic. But a spacetime-constant shift δc in c yields

$$\delta D^\mu c = qA^\mu \times \delta c, \tag{3.3.12}$$

$$\delta\left(-\mathrm{i}\nabla_\mu\overline{c}\cdot D^\mu c\right) = -\mathrm{i}q\nabla_\mu\overline{c}\cdot A^\mu\times\delta c = \mathrm{i}q\nabla_\mu\left(\overline{c}\times\delta c\right)\cdot A^\mu. \tag{3.3.13}$$

It follows that $\mathcal{L}_0$ is preserved if we also make the transformation

$$\delta B = \mathrm{i}q\overline{c}\times\delta c. \tag{3.3.14}$$

The simultaneous transformation of B, c in this way yields

$$\delta\left(c_{(0)}\right) = (\delta c)_{(0)} = \delta c,\, \delta\left(B - \mathrm{i}q\overline{c}\times c_{(0)}\right) = 0. \tag{3.3.15}$$

3.4 Redefinition of the Nakanishi–Lautrup auxiliary field

To say that $c_{(0)}$ is not cyclic is a comment on partial derivatives of $\mathcal{L}_0$. However, partial derivatives are defined, and computed, by adopting appropriate conventions concerning which other quantities are held fixed. If these conventions are changed, a new family of partial derivatives result. That is the motivation for the calculations in this section.

I now define a non-local field[54]

$$\widetilde{B} := B - \mathrm{i}q\overline{c} \times c_{(0)}, \tag{3.4.1}$$

so that a spacetime-constant shift in any one of $\widetilde{B}$, c, $\overline{c}$ preserves $\mathcal{L}_0$ if the other two fields are held fixed. This motivates the view that these fields are, in a sense, a more natural basis for the Yang–Mills formalism. To rewrite $\mathcal{L}_0$ in terms of $\widetilde{B}$, one need only note that

$$B^a = \widetilde{B}^a + \mathrm{i}qf^{abc}\overline{c}^b c^c_{(0)}. \tag{3.4.2}$$

Holding B, $\overline{c}$ fixed is *not* equivalent to holding $\widetilde{B}$, $\overline{c}$ fixed, since varying $c_{(0)}$ while holding $\overline{c}$ fixed varies $B - \widetilde{B}$. The partial derivatives with respect to fields in Yang–Mills theory are therefore not the same if the theory is written in terms of $\widetilde{B}$, c, $\overline{c}$ as they are if it is written in terms of B, c, $\overline{c}$. This change in the choice of fields changes one of the three fields, and two of the three momenta. One way to show this is by rewriting $\mathcal{L}_0$ in terms of $\widetilde{B}$, c, $\overline{c}$, viz.

$$\begin{aligned}
\mathcal{L}_0 &= -\nabla_\mu \left(\widetilde{B} + \mathrm{i}q\overline{c} \times c_{(0)}\right) \cdot A^\mu - \mathrm{i}\nabla_\mu \overline{c} \cdot (\partial^\mu c + qA^\mu \times c) \\
&= -\nabla_\mu \widetilde{B} \cdot A^\mu - \mathrm{i}q\nabla_\mu \left(\overline{c} \times c_{(0)}\right) \cdot A_\mu - \mathrm{i}\nabla_\mu \overline{c} \cdot \nabla^\mu c - \mathrm{i}q\nabla_\mu \overline{c} \cdot A^\mu \times c \\
&= -\nabla_\mu \widetilde{B} \cdot A^\mu - \mathrm{i}q \left(\nabla_\mu \left(\overline{c} \times c_{(0)}\right) - \nabla_\mu \overline{c} \times c\right) \cdot A_\mu - \mathrm{i}\nabla_\mu \overline{c} \cdot \nabla^\mu c \\
&= -\nabla_\mu \widetilde{B} \cdot A^\mu - \mathrm{i}q \left(\overline{c} \times \nabla_\mu c_{(0)} - \nabla_\mu \overline{c} \times c_{(+)}\right) \cdot A^\mu - \mathrm{i}\nabla_\mu \overline{c} \cdot \nabla^\mu c.
\end{aligned} \tag{3.4.3}$$

In Sec. 3.3, the original formalism obtained the reduced momentum densities

$$\varpi_B = -A^0,\ \varpi^{\mathrm{old}}_{\overline{c}} = -\mathrm{i}D^0 c,\ \varpi^{\mathrm{new}}_c = \mathrm{i}\nabla^0 \overline{c}. \tag{3.4.4}$$

However, the attempt to obtain reduced momentum densities from Eq. (3.4.3) yields

$$\varpi_{\widetilde{B}} = -A^0 = -N^{-2}\left(N^2 A^0\right)_{(+)}, \tag{3.4.5}$$

$$\begin{aligned}
\varpi^{\mathrm{new}}_{\overline{c}} &= -\mathrm{i}\nabla^0 c - \mathrm{i}qA^0 \times c_{(+)} \\
&= \varpi^{\mathrm{new}}_{\overline{c}} + \mathrm{i}qN^{-2}\left(N^2 A^0\right)_{(+)} \times c_{(0)},
\end{aligned} \tag{3.4.6}$$

$$\varpi^{\mathrm{new}}_c = \mathrm{i}\nabla^0 \overline{c} + \mathrm{i}q\frac{\delta}{\delta \nabla_0 c}\int_{\mathbf{x}} \nabla_0 c_{(0)} \cdot A^0 \times \overline{c}. \tag{3.4.7}$$

[54] Since $c_{(0)}$ is non-local by construction, so is the field defined in Eq. (3.4.1). An analogous construction of a non-local field occurs for perturbative gravity. I will provide this in Eq. (4.4.8).

The second and third results show momentum shifts, and in the third result's case a further calculation is required. Since

$$\begin{aligned}\frac{\delta}{\delta\dot{c}}\int_{\mathbf{x}}\partial_0 c_{(0)}\cdot A^0\times\bar{c} &= \frac{\delta}{\delta\dot{c}}\int_{\mathbf{x}}\partial_0\left(\int \mathrm{d}^{n-1}\mathbf{y}\Upsilon\left(t,\,\mathbf{y}\right)c\left(t,\,\mathbf{y}\right)\right)\cdot A^0\left(t,\,\mathbf{x}\right)\times\bar{c}\left(t,\,\mathbf{x}\right)\\ &= \frac{\delta}{\delta\dot{c}}\int \mathrm{d}^{n-1}\mathbf{y}\Upsilon\left(t,\,\mathbf{y}\right)c\left(t,\,\mathbf{y}\right)\cdot\int_{\mathbf{x}}A^0\left(t,\,\mathbf{x}\right)\times\bar{c}\left(t,\,\mathbf{x}\right)\\ &= \Upsilon\left(t,\,\mathbf{y}\right)\cdot\int_{\mathbf{x}}A^0\left(t,\,\mathbf{x}\right)\times\bar{c}\left(t,\,\mathbf{x}\right)\\ &= \frac{\sqrt{\left|g\left(t,\,\mathbf{y}\right)\right|}}{N^2\left(t,\,\mathbf{y}\right)}\left(N^2A^0\times\bar{c}\right)_{(0)}\left(t\right),\end{aligned}\tag{3.4.8}$$

the final reduced momentum density is

$$\varpi_c^{\mathrm{new}}\left(t,\,\mathbf{y}\right)=\mathrm{i}\nabla^0\bar{c}\left(t,\,\mathbf{y}\right)+\frac{\mathrm{i}q\left(N^2A^0\times\bar{c}\right)_{(0)}\left(t\right)}{N^2\left(t,\,\mathbf{y}\right)}=\varpi_c^{\mathrm{old}}\left(t,\,\mathbf{y}\right)+\frac{\mathrm{i}q\left(N^2A^0\times\bar{c}\right)_{(0)}\left(t\right)}{N^2\left(t,\,\mathbf{y}\right)}.\tag{3.4.9}$$

Since $\left(N^2A^0\right)_{(0)}=0$,

$$\varpi_c^{\mathrm{new}}\mathrm{i}\nabla^0\bar{c}+\mathrm{i}qN^{-2}\left(\left(N^2A^0\right)_{(+)}\times\bar{c}_{(+)}\right)_{(0)}=\varpi_c^{\mathrm{old}}+\mathrm{i}qN^{-2}\left(\left(N^2A^0\right)_{(+)}\times\bar{c}_{(+)}\right)_{(0)}.\tag{3.4.10}$$

I previously observed that a spacetime-constant shift in any of $\widetilde{B}$, c, $\bar{c}$ preserves $\mathcal{L}_0$. Such shifts are absorbed into zero modes, but a more general shift in a zero mode is possible; specifically, the shift may have any time-dependence. In Eq. (3.4.3), any undifferentiated $\widetilde{B}_{(0)}$, $c_{(0)}$, $\bar{c}_{(0)}$ need to be considered carefully. In fact $\widetilde{B}_{(0)}$ never appears undifferentiated, and neither does $c_{(0)}$ (although $c_{(+)}$ does). The term in $\mathcal{L}_0$ proportional to $\bar{c}_{(0)}$ is

$$-\mathrm{i}q\bar{c}_{(0)}\times\nabla_\mu c_{(0)}\cdot A^\mu.\tag{3.4.11}$$

In L, this becomes

$$\mathrm{i}q\bar{c}_{(0)}\cdot\left(\int_{\mathbf{x}}A^0\right)\times\partial_0 c_{(0)}=\mathrm{i}qV\bar{c}_{(0)}\cdot\left(N^2A^0\right)_{(0)}\times\partial_0 c_{(0)}=0.\tag{3.4.12}$$

So the zero modes $\left(N^2A^0\right)_{(0)}$, $\widetilde{B}_{(0)}$, $c_{(0)}$, $\bar{c}_{(0)}$ are completely lost from the $\widetilde{B}$, c, $\bar{c}$-based Lagrangian formalism in the Landau gauge when the conditions

$$Q\left|\psi\right\rangle=0,\,\overline{Q}\left|\psi\right\rangle=0,\,Q_A\left|\psi\right\rangle=0\tag{3.4.13}$$

are imposed for all physical states $\left|\psi\right\rangle$, which also obtains

$$Q_{Dc}\left|\psi\right\rangle=0,\,Q_{D\bar{c}}\left|\psi\right\rangle=0\tag{3.4.14}$$

for all such states.

A verification that the same is possible in the Hamiltonian formalism may seem unnecessary at this point. However, I do this in the following section, because it facilitates a treatment of the perturbation-theoretic comparison of the FMP and CMP for BRST-quantised Yang–Mills theory.

3.5 The Hamiltonian density without zero modes

In Sec. 3.5.1, I show that the field transformation $B,\, c,\, \bar{c} \to \widetilde{B},\, c,\, \bar{c}$ preserves the BRST-quantised Yang–Mills Hamiltonian. What is more, I show that a broad class of field transformations (including $B,\, c,\, \bar{c} \to \widetilde{B},\, c,\, \bar{c}$) of a broad class of Lagrangians (which includes the BRST-quantised Yang–Mills Lagrangian) preserves the numerical value of the Hamiltonian obtained from the resulting Legendre transform. This implies the $B,\, c,\, \bar{c}$-based BRST-quantised Yang–Mills Hamiltonian can be rewritten in terms of $B,\, c,\, \bar{c}$ by using Eq. (3.4.2).
However, the fact that the reduced momentum densities are different in the two formalisms introduces an important subtlety. One valid way to obtain $\mathcal{H}_0$ in terms of phase space fields is to write it explicitly in terms of elements of

$$\bigcup_{T\in\left\{\widetilde{B},\, c,\, \bar{c}\right\}} \left\{T_{(0)},\, \partial_0 T_{(0)},\, T_{(+)},\, \partial_\mu T_{(+)}\right\}, \tag{3.5.1}$$

and to then use expressions for "new" reduced momentum densities to remove all lower time derivatives. I use this method to obtain $\mathcal{H}_0$ in Sec. 3.5.2, where I show that setting conserved charges to 0 obviates zero modes from $\mathcal{H}_0$.

3.5.1 The response of the Legendre transform to a field transformation in the Lagrangian

Suppose a Lagrangian L depends only on fields and their time derivatives, with $\partial L/\partial t = 0$. The BRST-quantised Yang–Mills Lagrangian is of this form. Consider a field transformation satisfying

$$q_i = q_i\left(Q_j\right), \tag{3.5.2}$$

$$L = L_q\left(q_i,\, \dot{q}_i\right) \tag{3.5.3}$$

$$= L_Q\left(Q_j,\, \dot{Q}_j\right). \tag{3.5.4}$$

The definition of $\widetilde{B}$ in terms of B is of this form. Define a matrix

$$M_{ij} := \frac{\partial q_i}{\partial Q_j} = \frac{\partial \dot{q}_i}{\partial \dot{Q}_j} \tag{3.5.5}$$

so

$$\dot{q}_i = M_{ij}\dot{Q}_j, \tag{3.5.6}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{Q}_j}\left(M^{-1}\right)_{ji}, \tag{3.5.7}$$

$$\dot{\mathbf{q}}\cdot\frac{\partial L}{\partial \dot{\mathbf{q}}} = \dot{Q}_j M_{ij}\left(M^{-1}\right)_{ki}\frac{\partial L}{\partial \dot{Q}_k} = \dot{Q}_j\delta_{kj}\frac{\partial L}{\partial \dot{Q}_k} = \dot{\mathbf{Q}}\cdot\frac{\partial L}{\partial \dot{\mathbf{Q}}}. \tag{3.5.8}$$

Since $L_q = L_Q = L$, the expressions $L_q,\, L_Q$ for L have the same values of the associated Hamiltonian

$$H = \dot{\mathbf{q}} \cdot \frac{\partial L}{\partial \dot{\mathbf{q}}} - L = \dot{\mathbf{Q}} \cdot \frac{\partial L}{\partial \dot{\mathbf{Q}}} - L. \tag{3.5.9}$$

3.5.2 The BRST-quantised Yang–Mills Hamiltonian

Without the field transformation we get

$$\varpi_B = -A^0, \tag{3.5.10}$$

$$\varpi_c = \mathrm{i}\partial^0 \overline{c}, \tag{3.5.11}$$

$$\varpi_{\overline{c}} = -\mathrm{i}\partial^0 c - \mathrm{i}qA^0 \times c, \tag{3.5.12}$$

$$\begin{aligned} H = \int_{\mathbf{x}} &\left(\partial_\mu B \cdot A^\mu + \mathrm{i}\partial_\mu \overline{c} \cdot \nabla^\mu c + \mathrm{i}q\partial_\mu \overline{c} \cdot A^\mu \times c \right. \\ &\left. -\dot{B} \cdot A^0 + \dot{\overline{c}} \cdot \left(-\mathrm{i}\nabla^0 c - \mathrm{i}qA^0 \times c\right) - \mathrm{i}\nabla^0 \overline{c} \cdot \dot{c}\right). \end{aligned} \tag{3.5.13}$$

The first and second lines of the parentheses' contents in Eq. (3.5.13) are respectively the contributions from the Lagrangian and $\dot{B} \cdot \pi_B + \dot{\overline{c}} \cdot \pi_{\overline{c}} - \pi_c \cdot \dot{c}$. Eq. (3.5.13) simplifies to

$$H = \int_{\mathbf{x}} \left(\partial_i B_{(+)} \cdot A^i + \mathrm{i}\partial_i \overline{c}_{(+)} \cdot \nabla^i c + \mathrm{i}q\partial_i \overline{c}_{(+)} \cdot A^i \times c - \mathrm{i}\nabla^0 \overline{c} \cdot \dot{c}\right). \tag{3.5.14}$$

The result $B = \widetilde{B} + \mathrm{i}q\overline{c} \times c_{(0)}$ implies

$$B_{(0)} = \widetilde{B}_{(0)} + \mathrm{i}q\overline{c}_{(0)} \times c_{(0)}, \tag{3.5.15}$$

$$B_{(+)} = \widetilde{B}_{(+)} + \mathrm{i}q\overline{c}_{(+)} \times c_{(0)}, \tag{3.5.16}$$

$$\partial_i B_{(+)} = \partial_i \widetilde{B}_{(+)} + \mathrm{i}q\partial_i \overline{c}_{(+)} \times c_{(0)}. \tag{3.5.17}$$

Thus

$$H = \int_{\mathbf{x}} \left(\partial_i \widetilde{B}_{(+)} \cdot A^i + \mathrm{i}\partial_i \overline{c}_{(+)} \cdot \nabla^i c + \mathrm{i}q\partial_i \overline{c}_{(+)} \cdot A^i \times c_{(+)} - \mathrm{i}\nabla^0 \overline{c} \cdot \dot{c}\right). \tag{3.5.18}$$

Hereafter all reduced momentum densities that appear are "new", e.g. ϖ_c denotes ϖ_c^{new} instead of ϖ_c^{old}. These quantities must be used to remove quantities of the form $\partial_0 T$ from H, but Hamilton's equations may still be deployed if expressions of the form $\partial_j T$ survive. Since each $\varpi_{T'}$ is expressible in terms of some $\nabla^0 T$, the crux of the matter is rewriting ∇_0, ∇^i in terms of ∇^0, ∇_j. Indeed, one may derive

$$\dot{c} = N^2 \left(\mathrm{i}\varpi_{\overline{c}} + q\varpi_{\widetilde{B}} \times c_{(+)}\right) + N^i \partial_i c_{(+)}, \tag{3.5.19}$$

$$\nabla^i c = -N^i \left(\mathrm{i}\varpi_{\overline{c}} + q\varpi_{\widetilde{B}} \times c_{(+)}\right) - \gamma^{ij} \partial_j c_{(+)}, \tag{3.5.20}$$

$$\dot{\overline{c}} = N^2 \left(-\mathrm{i}\varpi_c + q\left(\varpi_{\widetilde{B}} \times \overline{c}\right)_{(0)}\right) + N^i \partial_i \overline{c}_{(+)}, \tag{3.5.21}$$

$$\nabla^i \overline{c} = -N^i \left(-\mathrm{i}\varpi_c + q\left(\varpi_{\widetilde{B}} \times \overline{c}\right)_{(0)}\right) - \gamma^{ij} \partial_j \overline{c}_{(+)}. \tag{3.5.22}$$

The reduced momentum densities are

$$\varpi_{\widetilde{B}} = -A^0, \tag{3.5.23}$$

$$\varpi_c = \mathrm{i}\left(\nabla^0 \overline{c} + qN^{-2}\left(N^2 A^0 \times \overline{c}\right)_{(0)}\right), \tag{3.5.24}$$

$$\varpi_{\overline{c}} = -\mathrm{i}\left(\nabla^0 c + qA^0 \times c_{(+)}\right). \tag{3.5.25}$$

Eqs. (3.5.24) and (3.5.25) may be rearranged to obtain

$$\nabla^0 c = \mathrm{i}\varpi_{\overline{c}} + q\varpi_{\widetilde{B}} \times c_{(+)}, \tag{3.5.26}$$

$$\nabla^0 \overline{c} = -\mathrm{i}\varpi_c + q\left(\varpi_{\widetilde{B}} \times \overline{c}\right)_{(0)}. \tag{3.5.27}$$

Eqs. (1.7.15), (1.7.16), (3.5.26) and (3.5.27) then imply Eqs. (3.5.19)–(3.5.22).

Eqs. (3.5.18)–(3.5.21) and (3.5.27) imply

$$H = \int_{\mathbf{x}} \Big(\mathrm{i}\partial_i \overline{c}_{(+)} \cdot \left(-N^i\left(\mathrm{i}\varpi_{\overline{c}} + q\varpi_{\widetilde{B}} \times c_{(+)}\right) - \gamma^{ij}\partial_j c_{(+)}\right) + \partial_i \widetilde{B}_{(+)} \cdot A^i + \mathrm{i}q\partial_i \overline{c}_{(+)} \cdot A^i \times c_{(+)}$$
$$-\mathrm{i}\left(-\mathrm{i}\varpi_c + q\left(\varpi_{\widetilde{B}} \times \overline{c}\right)_{(0)}\right) \cdot \left(N^2\left(\mathrm{i}\varpi_{\overline{c}} + q\varpi_{\widetilde{B}} \times c_{(+)}\right) + N^i \partial_i c_{(+)}\right)\Big). \tag{3.5.28}$$

This result has no explicit dependence on $\widetilde{B}_{(0)}$, $c_{(0)}$ and satisfies the Hamilton's equation

$$\begin{aligned}
\dot{\pi}_{\overline{c}_{(0)}} &= -\frac{\partial H}{\partial \overline{c}_{(0)}} \\
&= \mathrm{i}q\frac{\partial}{\partial \overline{c}_{(0)}} \int_{\mathbf{x}} \left(\varpi_{\widetilde{B}} \times \overline{c}\right)_{(0)} \cdot \left(N^2\left(\mathrm{i}\varpi_{\overline{c}} + q\varpi_{\widetilde{B}} \times c_{(+)}\right) + N^i \partial_i c_{(+)}\right) \\
&= \mathrm{i}q\frac{\partial}{\partial \overline{c}_{(0)}} \int_{\mathbf{x}} \left(\varpi_{\widetilde{B}_{(0)}} \times \overline{c}_{(0)} + \varpi_{\widetilde{B}_{(+)}} \times \overline{c}_{(+)}\right) \cdot \partial_0 c \\
&= -\mathrm{i}q\frac{\partial}{\partial \overline{c}_{(0)}} \left\{\overline{c}_{(0)} \cdot \varpi_{\widetilde{B}_{(0)}} \times \int_{\mathbf{x}} \partial_0 c\right\} \\
&= -\mathrm{i}qV\varpi_{\widetilde{B}_{(0)}} \times \left(N^2 \partial_0 c\right)_{(0)}.
\end{aligned} \tag{3.5.29}$$

Since $\dot{\pi}_{\overline{c}_{(0)}}$ is proportional to a conserved quantity that may be set to 0 on all physical states, $\pi_{\overline{c}_{(0)}}$ can be chosen as conserved, and hence also as 0, on all physical states.

3.6 Extra terms in the Hamiltonian and Lagrangian

The Hamiltonian and Lagrangian may be written in the form

$$H = H_0\left(B, c, \overline{c}\right), \; L = L_0\left(B, c, \overline{c}\right), \tag{3.6.1}$$

as linear operators that are not ordinary functions because the fields $B,\, c,\, \overline{c}$ are differentiated in some cases. The CMP's zero mode obviation obtains results of the form

$$H = H_0\left(\widetilde{B}_{(+)},\, c_{(+)},\, \overline{c}_{(+)}\right) + H^{\text{extra}}\left(\widetilde{B}_{(+)},\, c_{(+)},\, \overline{c}_{(+)}\right), \tag{3.6.2}$$

$$L = L_0\left(\widetilde{B}_{(+)},\, c_{(+)},\, \overline{c}_{(+)}\right) + L^{\text{extra}}\left(\widetilde{B}_{(+)},\, c_{(+)},\, \overline{c}_{(+)}\right). \tag{3.6.3}$$

In particular, none of the terms on the right-hand sides of Eqs. (3.6.2) and (3.6.3) depend on any zero modes. The purpose of this section is to compute $H^{\text{extra}},\, L^{\text{extra}}$.
By inspection of Eq. (3.5.28),

$$H^{\text{extra}} = \int_{\mathbf{x}} -\mathrm{i}q\left(\varpi_{\widetilde{B}} \times \overline{c}\right)_{(0)} \cdot \left(N^2 q \varpi_{\widetilde{B}} \times c_{(+)} + N^i \partial_i c_{(+)}\right). \tag{3.6.4}$$

Imposing the condition

$$\varpi_{\widetilde{B}_{(0)}}(t) := \int \mathrm{d}^{n-1}\mathbf{x}\Upsilon(x)\, \varpi_{\widetilde{B}}(x) = -\left(N^2 A^0\right)_{(0)}(t) = 0 \tag{3.6.5}$$

throughout affords the replacement

$$\varpi_{\widetilde{B}} \to \varpi_{\widetilde{B}_{(+)}} := \varpi_{\widetilde{B}} - \varpi_{\widetilde{B}_{(0)}} = -N^{-2}\left(N^2 A^0\right)_{(+)}, \tag{3.6.6}$$

so that

$$H^{\text{extra}} = -\mathrm{i}q\left(\varpi_{\widetilde{B}_{(+)}} \times \overline{c}_{(+)}\right)_{(0)} \cdot \int_{\mathbf{x}} \left(qN^2 \varpi_{\widetilde{B}_{(+)}} \times c_{(+)} + N^i \partial_i c_{(+)}\right), \tag{3.6.7}$$

a result that contains no zero modes of $B,\, \widetilde{B},\, c,\, \overline{c}$ or their derivatives.
A further result, which is unsurprising given the nature of the Legendre transform, is that

$$L^{\text{extra}} = -H^{\text{extra}}. \tag{3.6.8}$$

To verify this, I begin by defining $L_{\text{FP}}^{(0+)} := -\mathrm{i}\int_{\mathbf{x}} \nabla_\mu \overline{c}_{(0)} \cdot D^\mu c_{(+)}$ and similarly with $L_{\text{FP}}^{(00)},\, L_{\text{FP}}^{(+0)},\, L_{\text{FP}}^{(++)}$, so that

$$L_{\text{FP}} = L_{\text{FP}}^{(00)} + L_{\text{FP}}^{(0+)} + L_{\text{FP}}^{(+0)} + L_{\text{FP}}^{(++)}. \tag{3.6.9}$$

The condition $Q_{Dc} = 0$ implies

$$L_{\text{FP}}^{(00)} + L_{\text{FP}}^{(0+)} = -\mathrm{i}\partial_0 \overline{c}_{(0)} \cdot Q_{Dc} = 0, \tag{3.6.10}$$

$$L_{\text{FP}} = L_{\text{FP}}^{(+0)} + L_{\text{FP}}^{(++)}. \tag{3.6.11}$$

The condition $Q_{Dc} = Q_{D\bar{c}} = 0$ may be rearranged to obtain

$$\partial_0 c_{(0)} = (\dot{c})_{(0)} = -\frac{q}{V}\int_{\mathbf{x}} A^0 \times c = q\left(\varpi_{\widetilde{B}_{(+)}} \times c_{(+)}\right)_{(0)}, \tag{3.6.12}$$

$$\int_{\mathbf{x}} D^0 c_{(+)} = -qV\left(N^2 A^0 \times c_{(+)}\right)_{(0)}, \tag{3.6.13}$$

$$\partial_0 \bar{c}_{(0)} = q\left(\varpi_{\widetilde{B}_{(+)}} \times \bar{c}_{(+)}\right)_{(0)}, \tag{3.6.14}$$

$$\int_{\mathbf{x}} D^0 \bar{c}_{(+)} = -qV\left(N^2 A^0 \times \bar{c}_{(+)}\right)_{(0)}. \tag{3.6.15}$$

The term $L_{\text{FP}}^{(++)}$ is present whether $\mathcal{L}_0$ is written in terms of $B,\ c,\ \bar{c}$ or $\widetilde{B},\ c,\ \bar{c}$, so

$$\begin{aligned}
-L^{\text{extra}} &= L_{\text{FP}}^{(+0)} + L_{\text{GF}} + \int_{\mathbf{x}} \nabla_\mu \widetilde{B} \cdot A^\mu \\
&= -\mathrm{i}\int_{\mathbf{x}} \left(\nabla_\mu \bar{c}_{(+)} \cdot D^\mu c_{(0)} + q\nabla_\mu\left(\bar{c} \times c_{(0)}\right) \cdot A^\mu\right) \\
&= -\mathrm{i}\int_{\mathbf{x}} \left(\nabla^0 \bar{c}_{(+)} \cdot \partial_0 c_{(0)} + q\left\{\nabla_\mu\left(\bar{c} \times c_{(0)}\right) - \nabla_\mu \bar{c}_{(+)} \times c_{(0)}\right\} \cdot A^\mu\right) \\
&= -\mathrm{i}\int_{\mathbf{x}} \left(\nabla^0 \bar{c}_{(+)} \cdot \partial_0 c_{(0)} + q\left\{\partial_0\left(\bar{c}_{(0)} \times c_{(0)}\right) + \bar{c}_{(+)} \times \partial_0 c_{(0)}\right\} \cdot A^0\right) \\
&= -\mathrm{i}\int_{\mathbf{x}} \left(\nabla^0 \bar{c}_{(+)} \cdot \partial_0 c_{(0)} + q\bar{c}_{(+)} \times \partial_0 c_{(0)} \cdot A^0\right) \\
&= \mathrm{i}\partial_0 c_{(0)} \cdot \int_{\mathbf{x}} \left(\nabla^0 \bar{c}_{(+)} + qA^0 \times \bar{c}_{(+)}\right) \\
&= \mathrm{i}\partial_0 c_{(0)} \cdot \int_{\mathbf{x}} D^0 \bar{c}_{(+)} \\
&= -\mathrm{i}q\int_{\mathbf{x}} \left(qN^2 \varpi_{\widetilde{B}_{(+)}} \times c_{(+)} + N^i \partial_i c_{(+)}\right) \cdot \left(N^2 A^0 \times \bar{c}_{(+)}\right)_{(0)} \\
&= \mathrm{i}q\left(N^2 A^0 \times \bar{c}_{(+)}\right)_{(0)} \cdot \int_{\mathbf{x}} \left(qN^2 \varpi_{\widetilde{B}_{(+)}} \times c_{(+)} + N^i \partial_i c_{(+)}\right) \\
&= -\mathrm{i}q\left(\varpi_{\widetilde{B}_{(+)}} \times \bar{c}_{(+)}\right)_{(0)} \cdot \int_{\mathbf{x}} \left(qN^2 \varpi_{\widetilde{B}_{(+)}} \times c_{(+)} + N^i \partial_i c_{(+)}\right) \\
&= H^{\text{extra}}.
\end{aligned} \tag{3.6.16}$$

3.7 Comparison with the FMP in perturbation theory

The aim of this section is to obtain the effective action associated with the contribution of zero modes to the FMP's modified propagator, and then verify that this effective action is identical to one obtainable from the CMP. I obtain an effective action in the FMP (CMP) in Sec. 3.7.1 (3.7.2).

3.7.1 The perturbation theory of the FMP

The Feynman propagator is a perturbation-theoretic tool constructed from the non-interacting theory. In this case ($q = 0$), the FMP begins by granting $c,\ \bar{c}$ a common small mass, say $M > 0$, and then decompose $c,\ \bar{c}$ into field modes, viz.

$$\hat{c}(x) = \sum_\sigma \left[c_\sigma \varphi_\sigma(x) + c_\sigma^\dagger \varphi_\sigma^*(x)\right],\ \hat{\bar{c}}(x) = \sum_\sigma \left[\bar{c}_\sigma \varphi_\sigma(x) + \bar{c}_\sigma^\dagger \varphi_\sigma^*(x)\right] \tag{3.7.1}$$

where $c_\sigma,\ \bar{c}_\sigma$ are spacetime-constant annihilation operators, and the modes φ_σ are complex-valued functions of spacetime. The above decompositions sum over all modes, including zero modes, which in the $M \to 0^+$ right-hand limit become modes of zero frequency. The label $\sigma = 0$ is used for the zero mode, which is then $\varphi_0(t)$. (Note that the argument of φ_0 may be taken as t rather than x, by definition.) The $\sigma \neq 0$ modes are of positive frequency even for $M = 0$, and are hereafter called *positive-frequency modes*. These operators c_σ, $\bar{c}_\sigma$ are normalised by Eq. (2.4.2), provided that the canonical conjugate momentum density is defined using the left functional derivative, viz.

$$\pi_Y := \frac{\delta_L L}{\delta_L \dot{Y}},\ \dot{\pi}_Y = -\frac{\delta_L H}{\delta_L Y} \tag{3.7.2}$$

(the first equation is a definition while the second equation is a Hamilton's equation.)
The time-ordered FP-ghost propagator is a Green's function:

$$\begin{aligned} G_F\left(x,\, x^{'}\right) &:= \mathrm{T}\left\langle 0|c\left(x\right)\bar{c}\left(x^{'}\right)|0\right\rangle \\ &= -\mathrm{i}\theta\left(t-t^{'}\right)\sum_\sigma \varphi_\sigma\left(x\right)\varphi_\sigma^*\left(x^{'}\right) \\ &\quad -\mathrm{i}\theta\left(t^{'}-t\right)\sum_\sigma \varphi_\sigma\left(x^{'}\right)\varphi_\sigma^*\left(x\right). \end{aligned} \tag{3.7.3}$$

This can be decomposed into a "zero mode" part and an additional term, viz.

$$G_F\left(x,\, x^{'}\right) = G_{F(0)}\left(t,\, t^{'}\right) + G_{F(+)}\left(x,\, x^{'}\right), \tag{3.7.4}$$

$$\begin{aligned} G_{F(0)}\left(t,\, t^{'}\right) &:= -\mathrm{i}\theta\left(t-t^{'}\right)\varphi_0\left(t\right)\varphi_0^*\left(t^{'}\right) \\ &\quad -\mathrm{i}\theta\left(t^{'}-t\right)\varphi_0\left(t^{'}\right)\varphi_0^*\left(t\right), \end{aligned} \tag{3.7.5}$$

$$\begin{aligned} G_{F(+)}\left(x,\, x^{'}\right) &:= -\mathrm{i}\theta\left(t-t^{'}\right)\sum_{\sigma\neq 0} \varphi_\sigma\left(x\right)\varphi_\sigma^*\left(x^{'}\right) \\ &\quad -\mathrm{i}\theta\left(t^{'}-t\right)\sum_{\sigma\neq 0} \varphi_\sigma\left(x^{'}\right)\varphi_\sigma^*\left(x\right). \end{aligned} \tag{3.7.6}$$

Hereafter we impose the synchronous gauge, so the zero mode of a minimally coupled massless Klein–Gordon field is of the form $A + Bf$. One important result, if $M > 0$ and appropriate spacetime symmetries are imposed, is that $\varphi_0(t)$ takes approximately this form with

$$A = \frac{c_1}{M},\ B = -\mathrm{i}c_2 M,\ c_1,\, c_2 \in \mathbb{R}. \tag{3.7.7}$$

(For example, if de Sitter invariance is demanded in de Sitter space, the massive result is $A + B\left(f - \frac{\mathrm{i}}{b_0}\left(b_1 + f_2\right)\right) + o\left(M\right)$.) The Klein–Gordon inner product of a complex-valued spatially uniform $h(t)$ with itself is

$$\langle h,\, h\rangle_{\mathrm{KG}} = \mathrm{i}\int_{\mathbf{x}}\left(h^*\nabla^0 h - h\nabla^0 h^*\right) = \mathrm{i}V\left(t\right)\left(h^*\nabla^0 h - h\nabla^0 h^*\right). \tag{3.7.8}$$

The Klein–Gordon normalisation of $\varphi_0(t)$ is the condition

$$
\begin{aligned}
1 &= \left\langle \frac{c_1}{M} - \mathrm{i}c_2 M f,\, \frac{c_1}{M} - \mathrm{i}c_2 M f \right\rangle_{\mathrm{KG}} \\
&= \left(\left(\frac{c_1}{M} + \mathrm{i}c_2 M f \right) \left(-\mathrm{i}c_2 M \partial^0 f \right) - \left(\frac{c_1}{M} - \mathrm{i}c_2 M f \right) \left(\mathrm{i}c_2 M \partial^0 f \right) \right) \\
&\quad \times \mathrm{i}a^{n-1} \int \mathrm{d}^{n-1}\mathbf{x} \sqrt{\eta\left(\mathbf{x}\right)} \\
&= 2a^{n-1} c_1 c_2 \nabla^0 f \int \mathrm{d}^{n-1}\mathbf{x} \sqrt{\eta\left(\mathbf{x}\right)} = 2c_1 c_2 \int \mathrm{d}^{n-1}\mathbf{x} \sqrt{\eta\left(\mathbf{x}\right)},
\end{aligned}
\tag{3.7.9}
$$

since $\nabla^0 = \nabla_0$. Equivalently, $c_1 c_2 = \frac{1}{2V_c}$. Thus

$$
\begin{aligned}
\varphi_0\left(t\right) \varphi_0^*\left(t^{'}\right) &= \left(\frac{c_1}{M} - \mathrm{i}c_2 M f\left(t\right) \right) \left(\frac{c_1}{M} + \mathrm{i}c_2 M f\left(t^{'}\right) \right) \\
&= \frac{c_1^2}{M^2} + c_2^2 M^2 f\left(t\right) f\left(t^{'}\right) + \mathrm{i}c_1 c_2 \left(f\left(t^{'}\right) - f\left(t\right) \right) \\
&= \frac{c_1^2}{M^2} + \frac{M^2}{4c_1^2} f\left(t\right) f\left(t^{'}\right) + \frac{\mathrm{i}}{2V_c} \left(f\left(t^{'}\right) - f\left(t\right) \right),
\end{aligned}
\tag{3.7.10}
$$

$$
\begin{aligned}
-\mathrm{i}\theta\left(t - t^{'}\right) \varphi_0\left(t\right) \varphi_0^*\left(t^{'}\right) &= \theta\left(t - t^{'}\right) \\
&\quad \times \left\{ \frac{f\left(t^{'}\right) - f\left(t\right)}{2V_c} - \mathrm{i}\left(\frac{c_1^2}{M^2} + \frac{M^2}{4c_1^2} f\left(t\right) f\left(t^{'}\right) \right) \right\},
\end{aligned}
\tag{3.7.11}
$$

$$
\begin{aligned}
-\mathrm{i}\theta\left(t^{'} - t\right) \varphi_0\left(t^{'}\right) \varphi_0^*\left(t\right) &= \left\{ 1 - \theta\left(t - t^{'}\right) \right\} \\
&\quad \times \left\{ \frac{f\left(t\right) - f\left(t^{'}\right)}{2V_c} - \mathrm{i}\left(\frac{c_1^2}{M^2} + \frac{M^2}{4c_1^2} f\left(t\right) f\left(t^{'}\right) \right) \right\},
\end{aligned}
\tag{3.7.12}
$$

$$
\begin{aligned}
G_{F(0)}\left(t, t^{'}\right) &:= \theta\left(t - t^{'}\right) \left\{ \frac{f\left(t^{'}\right) - f\left(t\right)}{2V_c} - \mathrm{i}\left(\frac{c_1^2}{M^2} + \frac{M^2}{4c_1^2} f\left(t\right) f\left(t^{'}\right) \right) \right\} \\
&\quad + \left\{ 1 - \theta\left(t - t^{'}\right) \right\} \left\{ \frac{f\left(t\right) - f\left(t^{'}\right)}{2V_c} - \mathrm{i}\left(\frac{c_1^2}{M^2} + \frac{M^2}{4c_1^2} f\left(t\right) f\left(t^{'}\right) \right) \right\} \\
&= \frac{f\left(t^{'}\right) - f\left(t\right)}{2V_c} \left\{ 2\theta\left(t - t^{'}\right) - 1 \right\} - \mathrm{i}\left(\frac{c_1^2}{M^2} + \frac{M^2}{4c_1^2} f\left(t\right) f\left(t^{'}\right) \right) \\
&= \frac{f\left(t^{'}\right) - f\left(t\right)}{2V_c} \left\{ \theta\left(t - t^{'}\right) - \theta\left(t^{'} - t\right) \right\} \\
&\quad - \mathrm{i}\left(\frac{c_1^2}{M^2} + \frac{M^2}{4c_1^2} f\left(t\right) f\left(t^{'}\right) \right).
\end{aligned}
\tag{3.7.13}
$$

Note that $\frac{c_1^2}{M^2} + \frac{M^2}{4c_1^2} f\left(t\right) f\left(t^{'}\right)$ is the sum of two terms. One of these terms does not depend on t or $t^{'}$; the other term vanishes when $M = 0$. Therefore, $\frac{c_1^2}{M^2} + \frac{M^2}{4c_1^2} f\left(t\right) f\left(t^{'}\right)$ does not contribute to time derivatives of $G_{F(0)}\left(t, t^{'}\right)$ in the $M \to 0^+$ right-hand limit. Thus

$$
\begin{aligned}
\partial_t G_{F(0)}\left(t,\, t^{'}\right) &= \frac{-f^{'}\left(t\right)}{2V_c}\left\{\theta\left(t-t^{'}\right)-\theta\left(t^{'}-t\right)\right\} \\
&= \frac{\theta\left(t^{'}-t\right)-\theta\left(t-t^{'}\right)}{2V_c a^{n-1}\left(t\right)},
\end{aligned} \tag{3.7.14}
$$

$$
\partial_{t'}\partial_t G_{F(0)}\left(t,\, t^{'}\right) = \frac{-\delta\left(t-t^{'}\right)\mathbb{I}}{V_c a^{n-1}\left(t\right)}, \tag{3.7.15}
$$

where $\mathbb{I}$ is the unit matrix in the group's adjoint representation.
For $Y \in \{c,\, \overline{c}\}$ let $\mathcal{L}_{\text{int}}^{(Y)}$ denote the contribution to $\mathcal{L}_0$ involving $Y_{(0)}$ due to the term $-\mathrm{i}q\partial_\mu\overline{c}\cdot A^\mu \times c$. Suppose also that free integration by parts is legitimate in the spacetime considered. Using $\nabla_\mu A^\mu = 0$ and $\int_{\mathbf{x}} A^0 = 0$, and denoting equivalence up to a total derivative by $\sim$, we obtain

$$
\begin{aligned}
\int_{\mathbf{x}}\left(-\mathrm{i}q\nabla_\mu\overline{c}_{(0)}\cdot\left(A^\mu\times c_{(0)}\right)\right) &= -\mathrm{i}q\int_{\mathbf{x}}\left(\partial_0\overline{c}_{(0)}\cdot\left(A^0\times c_{(0)}\right)\right) \\
&= -\mathrm{i}q\partial_0\overline{c}_{(0)}\cdot\left(\int_{\mathbf{x}}A^0\right)\times c_{(0)} = 0,
\end{aligned} \tag{3.7.16}
$$

$$
\int_{\mathbf{x}}\left(-\mathrm{i}q\nabla_\mu\overline{c}_{(0)}\cdot\left(A^\mu\times c\right)\right) = \int_{\mathbf{x}}\left(-\mathrm{i}q\partial_0\overline{c}_{(0)}\cdot\left(A^0\times c_{(+)}\right)\right), \tag{3.7.17}
$$

$$
-\mathrm{i}q\nabla_\mu\overline{c}\cdot A^\mu\times c \sim \mathrm{i}q\overline{c}\cdot A^\mu\times\nabla_\mu c, \tag{3.7.18}
$$

$$
\int_{\mathbf{x}}\left(\mathrm{i}q\overline{c}_{(0)}\cdot A^\mu\times\nabla_\mu c_{(0)}\right) = \mathrm{i}q\overline{c}_{(0)}\cdot\left(\int_{\mathbf{x}}A^0\right)\times\partial_0 c_{(0)} = 0, \tag{3.7.19}
$$

$$
\int_{\mathbf{x}}\left(\mathrm{i}q\overline{c}\cdot A^\mu\times\nabla_\mu c_{(0)}\right) = \int_{\mathbf{x}}\left(-\mathrm{i}qA^0\times\overline{c}_{(0)}\cdot\partial_0 c_{(0)}\right). \tag{3.7.20}
$$

We may therefore take the interaction Lagrangians as

$$
\mathcal{L}_{\text{int}}^{(\overline{c})} = \mathrm{i}q\partial_0\overline{c}_{(0)}\cdot\left(A^0\times c_{(+)}\right) = -\mathrm{i}q\left(A^0\times c_{(+)}\right)\cdot\partial_0\overline{c}_{(0)},\, \mathcal{L}_{\text{int}}^{(c)} = -\mathrm{i}q\left(A^0\times\overline{c}_{(+)}\right)\cdot\partial_0 c_{(0)}. \tag{3.7.21}
$$

The next step is to integrate out the zero-mode contribution to the effective zero-mode sector propagator obtained in the FMP. The effective action thereby obtained has gained the extra term

$$
\begin{aligned}
S_{\text{extra}} &= -\mathrm{i}q^2\int\mathrm{d}t\mathrm{d}^{n-1}\mathbf{x}a^{n-1}\left(t\right)\sqrt{\eta\left(\mathbf{x}\right)}\int\mathrm{d}t^{'}\mathrm{d}^{n-1}\mathbf{x}^{'}a^{n-1}\left(t^{'}\right)\sqrt{\eta\left(\mathbf{x}^{'}\right)} \\
&\times\left(A^\mu\left(x\right)\times\overline{c}_{(+)}\left(x\right)\right)\cdot\partial_\mu\partial_{\nu^{'}}G_{F(0)}\left(t,\, t^{'}\right)\mathbb{I}\left(A^{\nu^{'}}\left(x^{'}\right)\times c_{(+)}\left(x^{'}\right)\right) \\
&= -\mathrm{i}q^2\int\mathrm{d}t\mathrm{d}^{n-1}\mathbf{x}a^{n-1}\left(t\right)\sqrt{\eta\left(\mathbf{x}\right)}\int\mathrm{d}t^{'}\mathrm{d}^{n-1}\mathbf{x}^{'}a^{n-1}\left(t^{'}\right)\sqrt{\eta\left(\mathbf{x}^{'}\right)} \\
&\times\left(A^0\left(x\right)\times\overline{c}_{(+)}\left(x\right)\right)\cdot\partial_t\partial_{t^{'}}G_{F(0)}\left(t,\, t^{'}\right)\mathbb{I}\left(A^0\left(x^{'}\right)\times c_{(+)}\left(x^{'}\right)\right) \\
&= \frac{\mathrm{i}q^2}{V_c}\int\mathrm{d}t\mathrm{d}^{n-1}\mathbf{x}a^{n-1}\left(t\right)\sqrt{\eta\left(\mathbf{x}\right)}\int\mathrm{d}t^{'}\mathrm{d}^{n-1}\mathbf{x}^{'}a^{n-1}\left(t^{'}\right)\sqrt{\eta\left(\mathbf{x}^{'}\right)} \\
&\times\left(A^0\left(x\right)\times\overline{c}_{(+)}\left(x\right)\right)\cdot\frac{\delta\left(t-t^{'}\right)\mathbb{I}}{T\left(t\right)}\left(A^0\left(x^{'}\right)\times c_{(+)}\left(x^{'}\right)\right) \\
&= \frac{\mathrm{i}q^2}{V_c}\int\mathrm{d}t\mathrm{d}^{n-1}\mathbf{x}a^{n-1}\left(t\right)\sqrt{\eta\left(\mathbf{x}\right)}\int\mathrm{d}^{n-1}\mathbf{x}^{'}\sqrt{\eta\left(\mathbf{x}^{'}\right)} \\
&\times\left(A^0\left(x\right)\times\overline{c}_{(+)}\left(x\right)\right)\cdot\left(A^0\left(x^{'}\right)\times c_{(+)}\left(x^{'}\right)\right)\mathbb{I}.
\end{aligned} \tag{3.7.22}
$$

This construction of an effective action term is encapsulated in the figure below, which shows a transformation of a Feynman diagram.

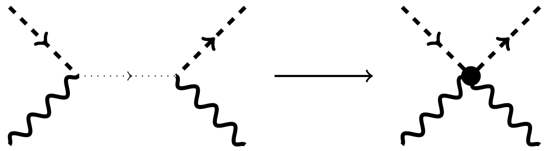

Figure 3.1 – This figure was previously published in Ref. [1]. The wavy, dashed and dotted lines respectively represent the gauge field, the nonzero-mode part of the FP-ghost propagator and its zero-mode part. The zero-mode contribution to the FP-ghost propagator is integrated out using $Q_A = 0$.

3.7.2 The perturbation theory of the CMP

I now obtain this effective action from the CMP, showing it is perturbatively equivalent to the FMP. The effective action is obtainable from the extra term which appears in the Lagrangian as a result of the coordinate transformation from B, c, $\bar{c}$ to $\widetilde{B}$, c, $\bar{c}$. This term is expressible as thus:

$$L^{\text{extra}} = \frac{\mathrm{i}q^2}{V_c} a^{n-1}(t)\, \overline{F} \cdot F, \tag{3.7.23}$$

$$\overline{F} := \int \mathrm{d}^{n-1}\mathbf{x}\sqrt{\eta(\mathbf{x})}A^0(x) \times \bar{c}_{(+)}(x), \tag{3.7.24}$$

$$F := \int \mathrm{d}^{n-1}\mathbf{x}\sqrt{\eta(\mathbf{x})}A^0(x) \times c_{(+)}(x). \tag{3.7.25}$$

The associated contribution to the action is therefore

$$\begin{aligned}\int \mathrm{d}t L^{\text{extra}}\mathbb{I} = \frac{\mathrm{i}q^2}{V_c}\int \mathrm{d}t a^{n-1}(t)\, \mathrm{d}^{n-1}\mathbf{x}\sqrt{\eta(\mathbf{x})}A^0(x) \times \bar{c}_{(+)}(x)\cdot\\ \mathrm{d}^{n-1}\mathbf{x}'\sqrt{\eta(\mathbf{x}')}A^0\left(x'\right) \times c_{(+)}\left(x'\right)\mathbb{I}\\ = S_{\text{extra}}.\end{aligned} \tag{3.7.26}$$

It is worth discussing another way to obtain the effective action in the CMP, because of the thereby "obvious" perturbation-theoretic interpretation. The Lagrangian's interaction term is

$$-\mathrm{i}q\int \mathrm{d}^{n-1}\mathbf{x}\sqrt{\gamma}\partial_\mu\bar{c}\cdot A^\mu \times c. \tag{3.7.27}$$

Imposing $Q_A = 0$, if $\partial_i\bar{c} = \partial_i c = 0$ then the left-hand integral is

$$-\mathrm{i}q\left(\int \mathrm{d}^{n-1}\mathbf{x}\sqrt{\gamma}\partial_0\bar{c} \times A^0_{(+)}\right)\cdot c = 0. \tag{3.7.28}$$

So the interaction term survives only if c and/or $\bar{c}$ is space-dependent, i.e. if and only if at least one of c, $\bar{c}$ is not identical with its zero mode. Thus L^{extra} is attributable to the perturbation the theory's FP-(anti)ghost sector. The interaction term in perturbation theory may be written as $-\mathrm{i}H_{\text{int}}$, so the factor in the ghost loop due to $G_{F(0)}\left(t, t'\right)$ is

$$q^2 \int \mathrm{d}t \mathrm{d}^{n-1}\mathbf{x} a^{n-1}(t) \sqrt{\eta(\mathbf{x})} \int \mathrm{d}t^{'} \mathrm{d}^{n-1}\mathbf{x}^{'} a^{n-1}\left(t^{'}\right) \sqrt{\eta(\mathbf{x}^{'})} \times \\ \left(A^{\mu}(x) \times \bar{c}_{(+)}(x)\right) \cdot \partial_{\mu}\partial_{\nu^{'}} G_{F(0)}\left(t, t^{'}\right) \mathbb{I}\left(A^{\nu^{'}}\left(x^{'}\right) \times c_{(+)}\left(x^{'}\right)\right). \tag{3.7.29}$$

Substituting Eq. (3.7.15) in Eq. (3.7.29) simplifies the latter to i times the right-hand side of Eq. (3.7.22). This expression is equal to both $-\mathrm{i}\int \mathrm{d}tH_{\text{int}} = \mathrm{i}\int \mathrm{d}tL_{\text{int}}$ (by the above argument) and $\mathrm{i}S_{\text{extra}}$ (by inspection of Eq. (3.7.22)), where S_{extra} is as in the FMP. Thus

$$S_{\text{extra}} = \int \mathrm{d}tL_{\text{int}}, \tag{3.7.30}$$

establishing perturbative equivalence.

3.8 The flat static torus and de Sitter space revisited

Ref. [1]'s Appendices E and F work in the synchronous gauge, and denote the perturbative vacuum state by $|0\rangle$. These appendices respectively consider the flat static torus and de Sitter space, and are concerned with the behaviours of $\left[A_{0(0)}(t), \dot{A}_{0(0)}\left(t^{'}\right)\right]$, $\left\langle 0 \right| \left[A_{0(0)}(t)\right]^2 \left|0\right\rangle$. (In particular, Appendix F uses the CMP to obtain $\left\langle 0 \right| \left[A_{0(0)}(t)\right]^2 \left|0\right\rangle$, and is concerned with verifying that known two-point functions compute the same value [50, 82, 83, 84].) Since A_{μ} has no analogue in scalar field theory, it was not possible to discuss this material in Chapter 1.

Time-translation invariance requires $A_{0(0)}(t)|0\rangle = 0$ for all t. In Sec. 3.8.1, I summarise a finding in Ref. [1] that this result is obtainable in the flat static torus, provided one works in the Landau gauge. In Sec. 3.8.2, I discuss the analogous case of de Sitter space. In Ref. [1], for brevity Dr Higuchi and I only considered the $n \geq 4$ case. The $n = 2$ and $n = 3$ cases require separate calculations. I present these, respectively, in Secs. 3.8.3 and 3.8.4. I also present the $n \geq 4$ case in Sec. 3.8.5.

3.8.1 Comments on the flat static torus

On the flat static torus, the equation of motion of A_0 forces the usual zero mode form of a scalar field. Quantising gives

$$A_{0(0)}(t) = \frac{\hat{Q} + t\hat{P}}{\sqrt{V}} \tag{3.8.1}$$

for spacetime-constant operators $\hat{P}$, $\hat{Q}$ (V is also constant, since $\dot{a} = 0$ on the flat static torus). Hence $|0\rangle$ is time-translation invariant if and only if $\hat{P} = 0$, in which case $\dot{A}_{0(0)}\left(t^{'}\right) = 0$. But it is shown that

$$\left[A_{0(0)}(t), \dot{A}_{0(0)}\left(t^{'}\right)\right] = -\frac{\mathrm{i}\alpha_0}{V}, \tag{3.8.2}$$

so outside the Landau gauge either $|0\rangle$ loses its time-translation invariance or the equal-time propagator $\left\langle 0 \right| \left[A_{0(0)}(t)\right]^2 \left|0\right\rangle$ diverges. However, in the Landau gauge

$$0 = \frac{1}{V}\left[\hat{Q} + t\hat{P}, \hat{P}\right] = \frac{1}{V}\left[\hat{Q}, \hat{P}\right], \tag{3.8.3}$$

which is consistent with $\hat{P} = 0$ and the requirement that $A_{0(0)}(t)|0\rangle = 0$ for all t.

3.8.2 Analogous comments on de Sitter space

The treatment in Appendix F of Ref. [1] observes that in de Sitter space

$$\left[A_{0(0)}\left(t\right),\,\dot{A}_{0(0)}\left(t\right)\right]=-\frac{\mathrm{i}\alpha_{0}}{U\left(t\right)},\tag{3.8.4}$$

where

$$U\left(t\right):=\frac{2\pi^{n/2}\cosh^{n-1}Ht}{\Gamma\left(\frac{n}{2}\right)H^{n-1}}\propto V\left(t\right)\tag{3.8.5}$$

(b_0 was defined in Eq. (1.8.5)). The equivalent of Eq. (3.8.1) is

$$A_{0(0)}\left(t\right)=\Phi\left(t\right)a+\Phi^{*}\left(t\right)a^{\dagger},\;\Phi\left(t\right):=\frac{\mathrm{i}b_{0}+\int_{0}^{t}U\left(\tau\right)\mathrm{d}\tau}{U\left(t\right)\sqrt{2b_{0}}},\tag{3.8.6}$$

with $a^{\dagger}$ a spacetime-constant operator satisfying $\left[a,\,a^{\dagger}\right]=-\alpha_0$ and $a\left|0\right\rangle=0$. This last condition is equivalent, for any $\alpha_0\neq0$, to the α_0-independent condition

$$\left\{\Phi^{*}\partial_{0}\left[UA_{0(0)}\right]-\frac{UA_{0(0)}}{\sqrt{2b_{0}}}\right\}\left|0\right\rangle=0.\tag{3.8.7}$$

It is therefore natural to demand Eq. (3.8.7) when $\alpha_0=0$. Appendix F of Ref. [1] proves that this requirement is equivalent to

$$\left\langle 0\right|\left[A_{0(0)}\left(t\right)\right]^{2}\left|0\right\rangle=-\frac{\alpha_{0}b_{0}}{2\left[U\left(0\right)\right]^{2}}\tag{3.8.8}$$

(which is proportional to H^{n-2}). Eq. (3.8.8) was verified in the Appendix for $n\geq4$ only. The cases $n\in\{2,\,3\}$ follow by a calculation that was not made explicit, but the necessary method was described. The rest of this section is concerned with presenting the details of this method for these cases. To be precise, I must show that

$$\left\langle 0\right|\left[A_{0(0)}\left(t\right)\right]^{2}\left|0\right\rangle=\begin{cases}-\frac{\alpha_{0}}{4\pi} & n=2\\ -\frac{H\alpha_{0}}{32} & n=3\end{cases}.\tag{3.8.9}$$

I check the cases $n=2$, $n=3$, $n\geq4$ separately in the next three subsections. Throughout I make use of numbered equations and previously unused notation that, unless stated otherwise, are taken from Ref. [84].

3.8.3 The case $n=2$

For $n=2$,

$$\left\langle 0\right|\left[A_{0(0)}\left(0\right)\right]^{2}\left|0\right\rangle=\frac{1}{8\pi^{2}}\int_{0}^{2\pi}\mathrm{d}\chi\left[1+\left(1+\alpha_{0}\right)\frac{\ln\left(1-\cos^{2}\frac{\chi}{2}\right)}{2\cos^{2}\frac{\chi}{2}}\right].\tag{3.8.10}$$

This result can be written as

$$\left\langle 0\right|\left[A_{0(0)}\left(0\right)\right]^{2}\left|0\right\rangle=A+B\left(1+\alpha_{0}\right)=A+B+B\alpha_{0},\tag{3.8.11}$$

with constants

$$A := \frac{1}{4\pi},\; B := \frac{1}{8\pi^2}\int_0^{2\pi} \mathrm{d}\chi \frac{\ln\left(1-\cos^2\frac{\chi}{2}\right)}{2\cos^2\frac{\chi}{2}}. \tag{3.8.12}$$

This result is proportional to α_0 if and only if $A+B=0$ i.e. if and only if $B=-\frac{1}{4\pi}$, in which case the result is simply $\frac{-\alpha_0}{4\pi}$ as required. So the entire problem reduces to proving that

$$\int_0^{2\pi} \mathrm{d}\chi \frac{\ln\left(1-\cos^2\frac{\chi}{2}\right)}{2\cos^2\frac{\chi}{2}} = -2\pi. \tag{3.8.13}$$

Substituting $\theta = \frac{\chi}{2}$, this becomes

$$\int_0^{\pi} \mathrm{d}\theta \frac{\ln\left(1-\cos^2\theta\right)}{\cos^2\theta} = -2\pi. \tag{3.8.14}$$

Let $u = \ln\sin^2\theta$, $v=\tan\theta$; integrating by parts, the desired integral is

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\left\{\left[\tan\theta\ln\sin^2\theta\right]_\varepsilon^{\pi-\varepsilon} - \int_0^{\pi-\varepsilon} 2\mathrm{d}\theta\right\} &= \lim_{\varepsilon\to 0^+}\left\{\tan\left(\pi-\varepsilon\right)\ln\sin^2\left(\pi-\varepsilon\right)\right.\\ &\quad\left. -\tan\varepsilon\ln\sin^2\varepsilon\right\} - 2\pi\\ &= -2\pi - 2\lim_{\varepsilon\to 0^+}\tan\varepsilon\ln\frac{\tan^2\varepsilon}{1+\tan^2\varepsilon}\\ &= -2\pi - 2\lim_{\delta\to 0^+}\delta\ln\frac{\delta^2}{1+\delta^2}\\ &= -2\pi - 4\lim_{\delta\to 0^+}\delta\ln\delta\\ &= -2\pi. \square\end{aligned} \tag{3.8.15}$$

3.8.4 The case $n = 3$

From Ref. [84], for $n=3$

$$\left\langle 0\right|\left[A_0\left(0\right)\right]^2\left|0\right\rangle = -\frac{H}{\left(4\pi\right)^{3/2}}A\left(Z\right) = -\frac{H}{8\pi^{3/2}}A\left(Z\right), \tag{3.8.16}$$

where from Eqs. (C4a) and (C5)

$$\sqrt{\pi}\sin\theta A\left(Z\right) = \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\left(\pi^2-\theta^2\right)\theta\cot\theta\right) + \left(\alpha_0+2\right)\theta, \tag{3.8.17}$$

with $\cos^2\frac{\chi}{2} = Z = -\cos\theta$. Thus

$$\begin{aligned}\left\langle 0\right|\left[A_{0(0)}\left(0\right)\right]^2\left|0\right\rangle &= \frac{1}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\sin\theta\left\langle 0\right|\left[A_0\left(0\right)\right]^2\left|0\right\rangle\\ &= -\frac{H}{16\pi^{3/2}}\int_0^{\pi}\mathrm{d}\theta\sin\theta A\left(Z\right)\\ &= -\frac{H}{16\pi^2}\int_0^{\pi}\mathrm{d}\theta\left\{\left(\alpha_0+2\right)\theta + +\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\left(\pi^2-\theta^2\right)\theta\cot\theta\right)\right\}.\end{aligned} \tag{3.8.18}$$

To evaluate $\left[\left(\pi^2-\theta^2\right)\theta\cot\theta\right]_0^\pi$, one may use the fact that $\lim_{\theta\to 0}\theta\cot\theta=1$ and

$$\lim_{\theta\to\pi}\left(\pi^2-\theta^2\right)\cot\theta=\lim_{\theta\to\pi}\left(\pi-\theta\right)\left(\pi+\theta\right)\cot\theta=2\pi\lim_{\theta\to\pi}\left(\pi-\theta\right)\cot\theta=-2\pi\lim_{\vartheta:=\pi-\theta\to 0}\vartheta\cot\vartheta=-2\pi, \tag{3.8.19}$$

so

$$\left[\left(\pi^2-\theta^2\right)\theta\cot\theta\right]_0^\pi=-3\pi^2, \tag{3.8.20}$$

$$\int_0^\pi \mathrm{d}\theta\left\{\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\left(\pi^2-\theta^2\right)\theta\cot\theta\right)+\left(\alpha_0+2\right)\theta\right\}=\frac{\pi^2\alpha_0}{2}, \tag{3.8.21}$$

$$\left\langle 0\right|\left[A_{0(0)}\left(0\right)\right]^2\left|0\right\rangle=-\frac{H\alpha_0}{32}, \tag{3.8.22}$$

as required.

3.8.5 The case $n\geq 4$

For $n\geq 4$ the two-point function is

$$\begin{aligned}\Delta_{\mu\nu'}\left(x,x'\right)=&\frac{n-2}{n-3}H^{-2}D_{\sqrt{n-2}H}\left(Z\right)\partial_\mu\partial_{\nu'}Z\\&+\frac{H^{-2}}{n-3}\frac{\partial D_{\sqrt{n-2}H}\left(Z\right)}{\partial Z}\partial_\mu Z\partial_{\nu'}Z\\&+\left(\alpha_0-\frac{n-1}{n-3}\right)\partial_\mu\partial_{\nu'}\lim_{M\to 0+}\partial_M^2\left[D_M\left(Z\right)-D_M\left(-1\right)\right].\end{aligned} \tag{3.8.23}$$

When $t=t'=0$ we have $\partial_t Z=0$, $\partial_t\partial_{t'}Z=-H^2$. The normalised zero mode is

$$G_0\left(t\right)=\sqrt{\frac{\Gamma\left(n-2\right)}{2HV\left(0\right)}}\operatorname{sech}^{\frac{n}{2}-1}Ht\,\mathrm{P}_{\frac{n-4}{2}}^{-\frac{n-2}{2}}\left(\mathrm{i}\sinh Ht\right) \tag{3.8.24}$$

(note the use of an associated Legendre function), and

$$\left\langle 0\right|\left[A_{0(0)}\left(0\right)\right]^2\left|0\right\rangle=-\frac{n-2}{n-3}\left|G_0\left(0\right)\right|^2-\left(\alpha_0-\frac{n-1}{n-3}\right)\lim_{M\to 0+}\partial_{M^2}\left|\dot{G}_0\left(0\right)\right|^2. \tag{3.8.25}$$

The identities

$$P_\nu^{-\mu}=\frac{2^{-\mu}\sqrt{\pi}}{\Gamma\left(\frac{\mu+\nu}{2}+1\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)},\ \Gamma\left(2z\right)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma\left(z\right)\Gamma\left(z+\frac{1}{2}\right) \tag{3.8.26}$$

imply

$$\left|G_0\left(0\right)\right|^2=\frac{\left(n-1\right)b_0}{2\left(n-2\right)\left[V\left(0\right)\right]^2},\ \lim_{M\to 0+}\partial_{M^2}\left|\dot{G}_0\left(0\right)\right|^2=\frac{b_0}{\left[V\left(0\right)\right]^2}. \tag{3.8.27}$$

This can be used to obtain Eq. (3.8.8).

3.9 Related appendices

The calculations in Sec. 3.5 show that the Hamiltonian formalism may be revised to contain no zero modes. New expressions for the Hamiltonian and Lagrangian are obtained. These are not simply due to a replacement $B,\,c,\,\overline{c}\to\widetilde{B}_{(0)},\,c_{(0)},\,\overline{c}_{(0)}$; the Lagrangian and Hamiltonian each also contain a new term, viz. Sec. 3.6. I discuss the perturbation-theoretic implications of this in Sec. 3.7. The FMP and CMP are equivalent in perturbation theory.

Let T be a Hermitian canonical field for which H does not depend on $T_{(0)}$. A Hamilton's equation implies $\pi_{T_{(0)}} \propto \frac{\delta}{\delta Y_{(0)}}$ is conserved, and setting this conserved charge to 0 on all physical states implies their Schrödinger wave functionals are $Y_{(0)}$-independent. These states then respect the symmetries which these conservation laws generate. Denote the wave functional Ψ; $\int \mathrm{d}T_{(0)}\Psi^\dagger\Psi$ is then infinite (unless $\Psi = 0$) for bosonic $T_{(0)}$ and zero for fermionic $T_{(0)}$, since Grassmann variables satisfy the normalisation condition $\int \mathrm{d}\theta = 0$. Physical states therefore do not have a finite positive norm. Indeed, $\left\langle \Psi \right| \left[\pi_{T_{(0)}},\, T_{(0)}\right]_\pm \left|\Psi\right\rangle$ is a linear combination of the vanishing or infinite quantity $\left\langle \Psi | T_{(0)}\pi_{T_{(0)}} | \Psi \right\rangle$ and its complex conjugate, and hence is zero or infinite. Thus $\langle\Psi|\Psi\rangle$ is proportional to this, and is zero or infinite because $\left[\pi_{T_{(0)}},\, T_{(0)}\right]_\pm$ is a nonzero multiple of the identity operator.

However, there is a resolution to this. Because Ψ is $Y_{(0)}$-independent, the $\int \mathrm{d}Y_{(0)}$ integration operator may be deleted from the pseudo-inner product. Such deletions revise the product, potentially allowing for a finitely positive-norm physical state. Matrix representations of pseudo-inner products can be used to find which pseudo-norms could allow physical states to have a finite positive norm (see Appendix B), but it remains to be shown that the revised pseudo-norms take the required forms.

The treatment in this Chapter does not use *Dirac brackets*, but it eventually must [81] because of two constraints on the phase space, viz.

$$\pi_{A_0} = 0,\ \pi_B + \sqrt{|g|}A^0 = 0 \tag{3.9.1}$$

(deriving the first of these requires inclusion of the $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\nabla_\mu A_\nu F^{\mu\nu}$ term in the scalar Lagrangian density). I address this in Appendix C.

If $Q_1,\, Q_2$ are Noether charges, $[Q_1,\, Q_2]$ should also be conserved. If the CMP sets $Q_i\,|\psi\rangle = 0$ for $i \in \{1,\, 2\}$ and any physical state $|\psi\rangle$, then $[Q_1,\, Q_2]_\pm\,|\psi\rangle = 0$ should also hold for all physical states $|\psi\rangle$ for consistency. This implies the anticommutators of pairs of conserved fermionic charges, and the commutators of conserved bosonic charges with conserved charges, should all be linear combinations of elements of a basis of the vector space of conserved charges. In other words, taking (anti)commutators should not generate "new" Noether charges linearly independent of pre-existing ones. In Appendix D, I verify this result for Yang–Mills theory and perturbative gravity.

Chapter 4 Applying the CMP to perturbative gravity

4.1 Comparison of Chapters 3 and 4

The procedure of this chapter is very similar to that of Chapter 3, because the zero mode problems these two chapters consider have many features in common:

- The infrared problem requires relevant modes of the FP-(anti)ghost fields to be obviated from the formalism;
- These modes and their conjugate momenta can be expressed in the forms discussed in Sec. 3.2;
- The zero modes of B^μ, c^μ, $\bar{c}^\mu$ are not all simultaneously cyclic in the usual Lagrangian formalism, but this can be remedied by appropriately redefining the Nakanishi–Lautrup auxiliary field;
- This redefinition of the Nakanishi–Lautrup auxiliary field causes new terms to appear in the Hamiltonian and Lagrangian;
- These terms can be used to prove that the FMP and CMP are perturbatively equivalent.

These facts motivate a structure for this chapter that is analogous to that of Chapter 3.

In Sec. 4.2, which is analogous to Sec. 3.2, I complete the construction of zero modes for perturbative gravity by obtaining η^A_μ that satisfy Eq. (3.2.25). In Sec. 4.3, which is analogous to Sec. 3.3, I review conservation laws to motivate a redefinition of the Nakanishi–Lautrup auxiliary field. In Sec. 4.4, which is analogous to Sec. 3.4, I obtain an appropriate redefinition of the Nakanishi–Lautrup auxiliary field. There is a complication compared with the Yang–Mills case; the shift in the Nakanishi–Lautrup auxiliary field is dependent on the first-order derivatives of the FP-(anti)ghost fields.

In Sec. 4.5, which is analogous to Secs. 3.5 , I compute the Lagrangian density in a revised formalism that uses $\widetilde{B}^\mu$, c^μ, $\bar{c}^\mu$ instead of B^μ, c^μ, $\bar{c}^\mu$. (Sec. 4.5 also computes an extra term in the Lagrangian, in analogy with a result in Sec. 3.6.) Unlike the B^μ, c^μ, $\bar{c}^\mu$-based formalism, the $\widetilde{B}^\mu$, c^μ, $\bar{c}^\mu$-based formalism obtains a Lagrangian density which features second-order derivatives. The Legendre transform can be modified so that it is still possible to define a Hamiltonian formalism, whose Hamilton's equations are equivalent to the Euler–Lagrange equations of a Lagrangian formalism containing second-order derivatives. This will be discussed in Sec. 4.6.

In Sec. 4.7, which is analogous to Sec. 3.7, I show that the FMP and CMP are perturbatively equivalent. To do this, I must review some standard facts about perturbation theory. Although this treatment is general, there is also much that is worth saying about the flat static torus as a special case. I postpone this treatment to Appendix H.

The treatment of the previous chapter requires an analysis of Dirac brackets to be truly rigorous. Such an analysis is offered in Appendix C. I have also analysed the Dirac brackets of perturbative gravity, in Appendix C.3.

4.2 The η_μ^A

This section considers the zero mode decomposition of a vector field ϕ^μ and its conjugate momentum density. In Sec. 3.2, I defined these as thus:

$$\pi_{\nu(0)}(t,\mathbf{x}) := F_\nu^A(t,\mathbf{x})\int \mathrm{d}^{n-1}\mathbf{w}G_A^\sigma(t,\mathbf{w})\,\pi_\sigma(t,\mathbf{w}), \tag{4.2.1}$$

$$\phi_{(0)}^\mu(t,\mathbf{y}) := G_B^\mu(t,\mathbf{y})\int \mathrm{d}^{n-1}\mathbf{z}F_\rho^B(t,\mathbf{z})\,\phi^\rho(t,\mathbf{z}), \tag{4.2.2}$$

$$\pi_{\nu(+)}(t,\mathbf{x}) := \pi_\nu(t,\mathbf{x}) - \pi_{\nu(0)}(t,\mathbf{x}), \tag{4.2.3}$$

$$\phi_{(+)}^\mu(t,\mathbf{y}) := \phi^\mu(t,\mathbf{y}) - \phi_{(0)}^\mu(t,\mathbf{y}) \tag{4.2.4}$$

with

$$F_\mu^A(t,\mathbf{x}) := \frac{g^{00}(t,\mathbf{x})\sqrt{|g(t,\mathbf{x})|}\eta_\mu^A(t,\mathbf{x})}{V(t)},\ G_A^\nu(t,\mathbf{x}) := \xi_A^\nu(t,\mathbf{x}) \tag{4.2.5}$$

so that

$$\int \mathrm{d}^{n-1}\mathbf{z}F_a^A(t,\mathbf{z})\,G_B^a(t,\mathbf{z}) = \delta_B^A \tag{4.2.6}$$

provided that the vectors η_μ^A satisfy Eq. (3.2.25). A simple calculation obtains an explicit example of such η_μ^A. In Chapter 3, I simplified the Yang–Mills Lagrangian density to only include terms attributable to the Faddeev–Popov method, and I do the same for perturbative gravity hereafter. The resulting scalar Lagrangian density may be written as

$$\mathcal{L}_0 := \frac{\alpha_0}{2}B^\nu B_\nu - \nabla^\mu B^\nu\kappa H_{\mu\nu} - \mathrm{i}\nabla^\mu\bar{c}^\nu Z_{\mu\nu} \tag{4.2.7}$$

where[55]

$$z_{\rho\sigma} := \pounds_c g_{\rho\sigma}^f = \left[Q,\, g_{\rho\sigma}^f\right] = \left[Q,\, \kappa h_{\rho\sigma}\right], \tag{4.2.8}$$

$$Z_{\mu\nu} := \gamma_{\mu\nu}^{\rho\sigma} z_{\rho\sigma} = \left[Q,\, \kappa H_{\mu\nu}\right]. \tag{4.2.9}$$

[55] I have introduced $Z_{\mu\nu}$ as a convenient abbreviation for subsequent formalism. Similar approaches have occurred before. Ref. [39] writes $\sqrt{|g|}Z_{\mu\nu}$ as $D^{\mu\nu}{}_\rho c^\rho$, where $D^{\mu\nu}{}_\rho$ is a differential operator. Since Ref. [39] asumes the unperturbed metric $g_{\mu\nu}$ is that of Minkowski space, it writes $D^{\mu\nu}{}_\rho$ in terms of partial derivatives instead of covariant ones. Also, Ref. [39] defines $D^{\mu\nu}{}_\rho$ for the de Donder gauge only; like Ref. [48], the existence of other gauges is not noted. The generalisation to curved spacetimes and other gauges is trivial.

Ref. [59] has a similar approach, which is applicable in arbitrary spacetimes and for any k. Since a total derivative may be added to $\mathcal{L}_0$, one may replace $-\mathrm{i}\nabla^\nu\bar{c}^\mu Z_{\mu\nu}$ with $\mathrm{i}\bar{c}^\mu\nabla^\nu Z_{\mu\nu}$. Ref. [59] then writes $\mathrm{i}\nabla^\nu Z_{\mu\nu}$ as $M_{\mu\nu}c^\nu$, with $M_{\mu\nu}$ a second-order differential operator expressed in terms of covariant derivatives. (If $D^{\mu\nu}{}_\rho$ is appropriately generalised with covariant derivatives, we have $M_{\mu\nu} = \mathrm{i}\nabla_\alpha\left(\sqrt{|g|}^{-1}D^\alpha{}_{\mu\nu}\right)$.) I have avoided such second-order derivatives herein, as they unnecessarily complicate the Legendre transform that obtains the Hamiltonian.

However, the redefinition of the Nakanishi–Lautrup auxiliary field will ultimately introduce such complications elsewhere. I show how to remedy this in Sec. 4.6.

Thus $Z_{\mu\nu}$ is symmetric, and the equation $\nabla^\mu Z_{\mu\nu} = 0$ is both the Euler–Lagrange equation obtained by varying $\bar{c}^\nu$ and the BRST-transform of the Euler–Lagrange equation $\kappa\nabla^\mu H_{\mu\nu} = 0$ obtained by varying B^ν. Thus $\xi_\nu Z^{\mu\nu}$ is a conserved current for any Killing vector field, and each $\int_{\mathbf{x}} \xi_{A\nu} Z^{0\nu}$ is a conserved charge. These are analogous to the conserved charges $\int_{\mathbf{x}} D^0 c^a$ for Yang–Mills theory; note that in both cases there is a vector space of charges bearing one Lie algebra index. By inspection

$$z_{\rho\sigma} = \kappa c^\alpha \nabla_\alpha h_{\rho\sigma} + \nabla_\beta c^\alpha \left(\delta_\rho^\beta g_{\alpha\sigma}^f + \delta_\sigma^\beta g_{\rho\alpha}^f\right) \tag{4.2.10}$$

so

$$Z_\nu^\mu = K_{\nu\alpha}^{\beta\mu} \nabla_\beta c^\alpha + \kappa c^\alpha \nabla_\alpha H_\nu^\mu,\ K_{\nu\alpha}^{\beta\mu} := g^{\lambda\mu} \gamma_{\lambda\nu}^{\rho\sigma} \left(\delta_\rho^\beta g_{\alpha\sigma}^f + \delta_\sigma^\beta g_{\rho\alpha}^f\right). \tag{4.2.11}$$

Define also

$$\hat{K}_{\nu\alpha}^{\beta\mu} := g^{\lambda\mu} \gamma_{\lambda\nu}^{\rho\sigma} \left(\delta_\rho^\beta g_{\alpha\sigma} + \delta_\sigma^\beta g_{\rho\alpha}\right), \tag{4.2.12}$$

the non-interacting value of $K_{\nu\alpha}^{\beta\mu}$. Note that

$$\hat{K}_{\nu\alpha}^{00} = g^{\lambda 0} \gamma_{\lambda\nu}^{0\sigma} g_{\alpha\sigma} + g^{\lambda 0} \gamma_{\lambda\nu}^{\rho 0} g_{\rho\alpha} = g^{00} g_{\nu\alpha} + (1 - 2k)\, \delta_\nu^0 \delta_\alpha^0 \tag{4.2.13}$$

is $\nu \leftrightarrow \alpha$–symmetric, and especially simple in the de Donder gauge. I now introduce two matrices

$$M_{BC}(t) := \int_{\mathbf{x}} \xi_B^\nu K_{\nu\alpha}^{00} \xi_C^\alpha,\ \hat{M}_{BC}(t) := \int_{\mathbf{x}} \xi_B^\nu \hat{K}_{\nu\alpha}^{00} \xi_C^\alpha. \tag{4.2.14}$$

The motivation for an interest in the matrix M_{BC} is given by its role in removing zero modes from the Lagrangian density of BRST-quantised perturbative gravity, viz. Sec. 4.5. The motivation for an interest in $\hat{M}_{BC}$, however, concerns the choice of the η_μ^A in this section. The existence of inverses is crucial, viz.

$$\left(\hat{M}^{-1}\right)^{AB}(t)\, \hat{M}_{BC}(t) = \left(M^{-1}\right)^{AB}(t)\, M_{BC}(t) = \delta_C^A. \tag{4.2.15}$$

Since each entry of M_{BC} is a polynomial in κ of degree ≤ 1, $\det M_{BC}$ is a polynomial in κ, which vanishes at $\kappa = 0$ if and only if $\hat{M}_{BC}$ is not invertible. However, in Appendix G I show that $\hat{M}_{BC}$ is in fact invertible, at least for the flat static torus and de Sitter space.[56] In these cases, $\det M_{BC}$ is a non-constant polynomial, so M_{BC} is invertible for all but finitely many κ.

If $\hat{M}_{BC}$ is invertible, Eq. (3.2.25) may be rewritten as

$$\begin{aligned}\int_{\mathbf{x}} \frac{\eta_\mu^A \xi_C^\mu}{N^2 V} &= \left(\hat{M}^{-1}\right)^{AB}(t) \int_{\mathbf{x}} \xi_B^\nu \hat{K}_{\nu\mu}^{00} \xi_C^\mu \\ &= \int_{\mathbf{x}} \left(\hat{M}^{-1}\right)^{AB}(t)\, \xi_B^\nu \hat{K}_{\nu\mu}^{00} \xi_C^\mu,\end{aligned} \tag{4.2.16}$$

and so it suffices to take

$$\eta_\mu^A := N^2 V \left(\hat{M}^{-1}\right)^{AB} \xi_B^\nu \hat{K}_{\nu\mu}^{00}. \tag{4.2.17}$$

Thus

$$F_\mu^A = \sqrt{|g|} \left(\hat{M}^{-1}\right)^{AB} \xi_B^\nu \hat{K}_{\nu\mu}^{00},\ \phi_{(0)}^\mu = \xi_A^\mu \left(\hat{M}^{-1}\right)^{AB} \int_{\mathbf{x}} \xi_B^\nu \hat{K}_{\nu\rho}^{00} \phi^\rho. \tag{4.2.18}$$

[56] Strictly speaking, I show this is so if appropriate choices are made. For example, for the flat static torus I set $\xi_A^\mu = \delta_A^\mu$ and $k < 1$.

The zero modes of B^μ, c^μ, $\overline{c}^\mu$ may then be written as

$$B^\mu_{(0)} = \beta^A \xi^\mu_A,\, \overline{c}^\mu_{(0)} = \theta^A \xi^\mu_A,\, \overline{c}^\mu_{(0)} = \overline{\theta}^A \xi^\mu_A \tag{4.2.19}$$

where

$$\beta^A := \left(\hat{M}^{-1}\right)^{AB} \int_{\mathbf{x}} \xi^\nu_B \hat{K}^{00}_{\nu\rho} B^\rho,\, \theta^A := \left(\hat{M}^{-1}\right)^{AB} \int_{\mathbf{x}} \xi^\nu_B \hat{K}^{00}_{\nu\rho} c^\rho,\, \overline{\theta}^A := \left(\hat{M}^{-1}\right)^{AB} \int_{\mathbf{x}} \xi^\nu_B \hat{K}^{00}_{\nu\rho} \overline{c}^\rho. \tag{4.2.20}$$

If the hats were dropped from these equations, the resulting definition of zero modes would be very different, involving a mixing of the zero modes and the perturbation metric. There are several reasons this would be an undesirable definition of zero modes:

- The decoupling of fields inherent in the treatment in Sec. 3.2 would not work, since we would have $\left[\hat{\theta}^A,\, \hat{\pi}_{\kappa h_{\mu\nu}}\right] \neq 0$ (these hats denote a promotion to operators);
- The mixing would complicate the perturbation theory.

4.3 Conserved currents and symmetries

Each symmetric divergenceless tensor $X^{\mu\nu}$ provides a vector space of conserved currents $\xi_\nu X^{\mu\nu}$. Examples of such $X^{\mu\nu}$ include $\kappa H^{\mu\nu}$ and its BRST transform $Z^{\mu\nu}$. The gauge choice $k = \frac{1}{2}$ obtains anti-BRST invariance, so $\xi_\nu \left[\overline{Q},\, \kappa H^{\mu\nu}\right],\, \xi_\nu \left\{Q,\, \left[\overline{Q},\, \kappa H^{\mu\nu}\right]\right\}$ should also be conserved currents. Since other gauge choices violate anti-BRST invariance, conserved currents should be obtained by a method that does not rely on a consideration of anti-BRST transformations. There is more than one standard method of obtaining Noether currents; I will borrow the machinery of Eqs. (9.93)–(9.96) of Peskin and Schroeder's Ref. [27]. Let T_a denote an arbitrary canonical field (no assumptions are made about the meaning of the index a). Its spacetime indices are dropped herein, as in the discussion of Sec. 1.3. If a transformation of the form

$$\delta T_a\left(x\right) = \alpha \Delta T_a\left(x\right) \tag{4.3.1}$$

is action-preserving for a spacetime-constant scalar α, this transformation generates a global symmetry for which the scalar Lagrangian density $\mathcal{L}_0$ satisfies

$$\delta \mathcal{L}_0 = \alpha \nabla_\mu \mathcal{J}^\mu. \tag{4.3.2}$$

For local $\alpha\left(x\right)$ this result generalises to

$$\delta \mathcal{L}_0 = \alpha \nabla_\mu \mathcal{J}^\mu + \left(\left(\right) \nabla_\mu \alpha\right) \Delta T_a \frac{\partial \mathcal{L}_0}{\partial \nabla_\mu T_a}, \tag{4.3.3}$$

where summation over a indices is implicit throughout. Hence

$$\begin{aligned} \delta \mathcal{L}_0 = {} & \alpha \nabla_\mu \left(\mathcal{J}^\mu - \Delta T_a \frac{\partial \mathcal{L}_0}{\partial \nabla_\mu T_a}\right) \\ & + \nabla_\mu \left(\alpha \Delta T_a \frac{\partial \mathcal{L}_0}{\partial \nabla_\mu T_a}\right). \end{aligned} \tag{4.3.4}$$

Since the total derivative on the last line makes no contribution to the action, the stationary action principle gives

$$0 = \frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta\alpha} = \nabla_\mu j^\mu,\ j^\mu := \mathcal{J}^\mu - \Delta T_a \frac{\partial \mathcal{L}_0}{\partial \nabla_\mu T_a}. \tag{4.3.5}$$

Thus j^μ is a conserved current, and it can be obtained as the α coefficient in terms in $\delta\mathcal{L}_0$ in which α appears undifferentiated. This method can be used to obtain the *global gauge current* of Yang–Mills theory, a conserved current in the Landau gauge provided there is no $A_\mu \cdot j^\mu$ term. The global gauge transformation may be written as

$$\Delta A_\nu = -\frac{1}{q}D_\nu\alpha,\ \Delta B = \alpha\times B,\ \Delta c = \alpha\times c,\ \Delta\bar{c} = \alpha\times\bar{c}, \tag{4.3.6}$$

where α has become a multiplet-valued field. (The alternative choice $\Delta A_\nu = \alpha\times A_\nu$ may seem more natural, but is avoided here because $\Delta A_\nu = -\frac{1}{q}D_\nu\alpha$ preserves $\sqrt{|g^f|}\,(R-2\Lambda)$.) Thus

$$\begin{aligned}\Delta\varphi_a\frac{\partial\mathcal{L}_0}{\partial\nabla_\mu\varphi_a} &= -\frac{1}{q}\nabla_\nu\alpha\cdot F^{\nu\mu} - A_\nu\times\alpha\cdot F^{\nu\mu} - A^\mu\cdot\alpha\times B - \mathrm{i}\alpha\times\bar{c}\cdot D^\mu c + \mathrm{i}\alpha\times c\cdot\nabla^\mu\bar{c}\\ &= \alpha\cdot J^\mu_{gg} - \frac{1}{q}\nabla_\nu\alpha\cdot F^{\nu\mu}, \end{aligned} \tag{4.3.7}$$

$$J^\mu_{gg} := A_\nu\times F^{\nu\mu} + A^\mu\times B - \mathrm{i}\bar{c}\times D^\mu c + \mathrm{i}\nabla^\mu\bar{c}\times c. \tag{4.3.8}$$

Thus J^μ_{gg} is the conserved current associated with the global gauge transformation. Unsurprisingly, it is proportional to $\{\overline{Q},\ [Q,\ A^\mu]\}$ [40].

I return now to perturbative gravity. I show in Sec. 4.4 that one example of an action-preserving transformation is

$$\Delta c^\mu = \theta\xi^\mu,\ \Delta B^\mu = -\mathrm{i}\theta \pounds_\xi \bar{c}^\mu, \tag{4.3.9}$$

with

$$\Delta\mathcal{L}_0^{Bc\bar{c}} = \theta\nabla_\mu\left(\mathrm{i}\xi^\mu\nabla^\alpha\bar{c}^\beta\kappa H_{\alpha\beta}\right). \tag{4.3.10}$$

The resulting Noether current is

$$\begin{aligned}&\xi^\alpha\frac{\partial\mathcal{L}_0}{\partial\nabla_\mu c^\alpha} - \mathrm{i}\pounds_\xi\bar{c}^\alpha\frac{\partial\mathcal{L}_0}{\partial\nabla_\mu B^\alpha} - \mathrm{i}\xi^\mu\nabla^\alpha\bar{c}^\beta\kappa H_{\alpha\beta}\\ &= \xi^\alpha\mathrm{i}K^{\mu\beta}_{\nu\alpha}\nabla_\beta\bar{c}^\nu + \mathrm{i}\pounds_\xi\bar{c}^\alpha\kappa H^\mu_\alpha - \mathrm{i}\xi^\mu\nabla^\alpha\bar{c}^\beta\kappa H_{\alpha\beta}. \end{aligned} \tag{4.3.11}$$

(This is, in the anti-BRST invariant gauge choice $k=\frac{1}{2}$, analogous to $D^\mu\bar{c}$.) Another action-preserving transformation is the spacetime isometry transformation. It may be written as[57]

$$\Delta\kappa h_{\mu\nu} = \pounds_{\alpha\xi}g^f_{\mu\nu},\ \Delta B^\mu = \pounds_{\alpha\xi}B^\mu,\ \Delta c^\mu = \pounds_{\alpha\xi}c^\mu,\ \Delta\bar{c}^\mu = \pounds_{\alpha\xi}\bar{c}^\mu. \tag{4.3.12}$$

Thus

$$\Delta T_a\frac{\partial\mathcal{L}_0}{\partial\nabla_\mu T_a} = -\mathrm{i}\gamma^{\alpha\beta}_{\gamma\delta}\left(\nabla^\gamma\bar{c}^\delta\right)c^\mu\pounds_\Lambda g^f_{\alpha\beta} - \kappa H^\mu_\nu\pounds_\Lambda B^\nu + \mathrm{i}Z^\mu_\nu\pounds_\Lambda\bar{c}^\nu - \mathrm{i}\frac{\partial Z_{\alpha\beta}}{\partial\nabla_\mu c^\nu}\nabla^\alpha\bar{c}^\beta\pounds_\Lambda c^\nu. \tag{4.3.13}$$

(I have placed the ΔT_a factors on the right of terms on the right-hand side, which causes sign changes

[57]The reason $\Delta\kappa h_{\mu\nu} = \pounds_{\alpha\xi}g^f_{\mu\nu}$ is used instead of $\Delta\kappa h_{\mu\nu} = \kappa\pounds_{\alpha\xi}h_{\mu\nu}$ is to ensure that $R-2\Lambda$ is subject to a gauge transformation, and hence is invariant. In fact, this is true for any transformation of the form $\Delta\kappa h_{\mu\nu} = \pounds_V g^f_{\mu\nu}$ for some vector field V^ρ.

when T_a is a fermionic field.) The contribution proportional to undifferentiated α may be written as $\alpha C^\mu(\xi)$ where

$$C^\mu(\xi) := -\mathrm{i}\gamma^{\alpha\beta}_{\gamma\delta}\left(\nabla^\gamma\bar{c}^\delta\right)c^\mu \pounds_\xi \kappa h_{\alpha\beta} - \kappa H^\mu_\nu \pounds_\xi B^\nu + \mathrm{i}Z^\mu_\nu \pounds_\xi \bar{c}^\nu + \mathrm{i}\frac{\partial Z_{\alpha\beta}}{\partial\nabla_\mu c^\nu}\nabla^\alpha\bar{c}^\beta \pounds_\xi c^\nu$$
$$= -\mathrm{i}\left(\nabla^\gamma\bar{c}^\delta\right)c^\mu \pounds_\xi \kappa H_{\gamma\delta} - \kappa H^\mu_\nu \pounds_\xi B^\nu + \mathrm{i}Z^\mu_\nu \pounds_\xi \bar{c}^\nu - \mathrm{i}\frac{\partial Z_{\alpha\beta}}{\partial\nabla_\mu c^\nu}\nabla^\alpha\bar{c}^\beta \pounds_\xi c^\nu. \tag{4.3.14}$$

Thus the $C^\mu(\xi)$ are conserved currents. The conserved charges these generate ought to be obtainable by BRST-transforming the conserved charges generated by the Noether current obtained in Eq. (4.3.11). Indeed, this is verified in Appendix E. Further calculations show that anticommutators of pairs of fermionic conserved charges and commutators of bosonic conserved charges with conserved charges introduce no new conserved charges. These calculations are presented in Appendix D. Its placement reflects the fact that it is the only appendix which contains calculations for both Yang–Mills theory and perturbative gravity, which are respectively otherwise considered in earlier and later appendices.

4.4 Redefinition of the Nakanishi–Lautrup auxiliary field

In Sec. 4.4.1, I define a field $\widetilde{B}^\mu$ analogous to $\widetilde{B}$. As in the Yang–Mills case, the shift in the Nakanishi–Lautrup auxiliary field is linear in the FP-ghosts' zero modes rather than the full FP-ghost. In Yang–Mills theory, the zero modes are the non-local spatially uniform scalar functions $c^a_{(0)}(t)$. In perturbative gravity, the zero modes are the non-local spatially uniform scalar functions $\theta^A(t)$.

4.4.1 Three action-preserving transformations

The transformations $\delta B^\nu = \beta\xi^\nu$, $\delta\bar{c}^\nu = \bar{\theta}\xi^\nu$ (for spacetime-constant β, $\bar{\theta}$ with $\bar{\theta}$ Grassmann-valued) both preserve $\mathcal{L}_0$, since $\nabla^\mu\xi^\nu$ is antisymmetric so $\nabla^\mu\xi^\nu H_{\mu\nu} = 0$, $\nabla^\mu\xi^\nu Z_{\mu\nu} = 0$. These transformations generate the respective conserved currents $\xi_\nu\kappa H^{\mu\nu}$, $\xi_\nu Z^{\mu\nu}$. However, as with Yang–Mills theory, an analogous shift of the FP-ghost requires a more detailed discussion. The transformation $\delta c^\nu = \theta\xi^\nu$ for Grassmann-valued spacetime-constant θ effects

$$\delta c^\nu_{(0)} = \theta\xi^\nu, \tag{4.4.1}$$
$$\delta\pounds_c g^f_{\rho\sigma} = \pounds_{\theta\xi} g^f_{\rho\sigma} = \theta\kappa\pounds_\xi h_{\rho\sigma}, \tag{4.4.2}$$
$$\delta Z_{\mu\nu} = \theta\kappa\left(\pounds_\xi H_{\mu\nu} - \left(\pounds_\xi\gamma^{\rho\sigma}_{\mu\nu}\right)h_{\rho\sigma}\right) = \theta\kappa\pounds_\xi H_{\mu\nu}, \tag{4.4.3}$$
$$\delta\left(-\mathrm{i}\nabla^\mu\bar{c}^\nu Z_{\mu\nu}\right) = -\mathrm{i}\nabla^\mu\bar{c}^\nu\theta\kappa\pounds_\xi H_{\mu\nu}. \tag{4.4.4}$$

The next step is to simultaneously transform B^μ in an appropriate manner. The action will be preserved if this is done properly; $\mathcal{L}_0$ might not be, but it can vary by up to a total derivative. To preserve $\mathcal{L}_0$, which would be a more ambitious aim, would require $\delta\left(\nabla^\mu B^\nu H_{\mu\nu}\right) = -\mathrm{i}\nabla^\mu\bar{c}^\nu\theta\pounds_\xi H_{\mu\nu}$ (I have cancelled a $-\kappa$ factor in the gauge-fixing term from both sides). This result is not simply proportional to $H_{\mu\nu}$, so preserving $\mathcal{L}_0$ will not be possible. But $\pounds_\xi\left(\nabla^\mu\bar{c}^\nu H_{\mu\nu}\right) = \nabla_\gamma\left(\xi^\gamma\nabla^\mu\bar{c}^\nu H_{\mu\nu}\right)$ is a total derivative, so to preserve the action requires only that

$$\delta\left(\nabla^\mu B^\nu H_{\mu\nu}\right) = \mathrm{i}\left(\pounds_\xi\nabla^\mu\bar{c}^\nu\right)\theta H_{\mu\nu} = -\mathrm{i}\nabla^\mu\left(\pounds_{\theta\xi}\bar{c}^\nu\right)H_{\mu\nu}, \tag{4.4.5}$$

which is equivalent to $\delta\mathcal{L}_0^{Bc\overline{c}} = \mathrm{i}\theta\pounds_\xi\left(\nabla^\mu\overline{c}^\nu\kappa H_{\mu\nu}\right)$. It therefore suffices to choose

$$\delta B^\nu = -\mathrm{i}\pounds_{\theta\xi}\overline{c}^\nu = -\mathrm{i}\pounds_{\delta c_{(0)}}\overline{c}^\nu. \tag{4.4.6}$$

Indeed, since

$$\delta\nabla_\mu\left(-\mathrm{i}c^\mu\nabla^\alpha\overline{c}^\beta\kappa H_{\alpha\beta}\right) = \nabla_\mu\left(-\mathrm{i}\theta\xi^\mu\nabla^\alpha\overline{c}^\beta\kappa H_{\alpha\beta}\right) = -\mathrm{i}\theta\pounds_\xi\left(\nabla^\alpha\overline{c}^\beta\kappa H_{\alpha\beta}\right), \tag{4.4.7}$$

there is an invariant choice for the scalar Lagrangian density, namely $\mathcal{L}_1 := \mathcal{L}_0^{Bc\overline{c}} - \mathrm{i}\nabla_\mu\left(c^\mu\nabla^\alpha\overline{c}^\beta\kappa H_{\alpha\beta}\right)$. Note that $c^\mu\nabla^\alpha\overline{c}^\beta\kappa H_{\alpha\beta}$ is preserved by $\delta B^\mu = \beta\xi^\mu$ and $\delta\overline{c}^\mu = \overline{\theta}\xi^\mu$ if β, $\overline{\theta}$ are spacetime-constant, because $\nabla^\alpha\xi^\beta$ is antisymmetric. In terms of the $\theta^A(t)$ introduced in Eq. (4.2.20) I now define

$$\widetilde{B}^\nu := B^\nu + \mathrm{i}\theta^A(t)\pounds_{\xi_A}\overline{c}^\nu, \tag{4.4.8}$$

$$\widetilde{\beta}^A := \left(\hat{M}^{-1}\right)^{AB}\int_{\mathbf{x}}\xi_B^\nu\hat{K}_{\nu\rho}^{00}\widetilde{B}^\rho, \tag{4.4.9}$$

$$\widetilde{B}^\mu_{(0)} := \widetilde{\beta}^A\xi_A^\mu, \tag{4.4.10}$$

$$\widetilde{B}^\mu_{(+)} := \widetilde{B}^\mu - \widetilde{B}^\mu_{(+)}, \tag{4.4.11}$$

so $\delta\widetilde{B}^\nu = 0$. Thus a shift, by a spacetime-constant multiple of ξ^μ, in any of $\widetilde{B}^\mu$, c^μ, $\overline{c}^\mu$ is action-preserving.

The term $\mathrm{i}\theta^A\pounds_{\xi_A}\overline{c}^\nu$ is analogous to $-\mathrm{i}qc_{(0)}\times\overline{c} = \widetilde{B} - B$ in Yang–Mills theory[58], and in principle one can write $\pounds_{c_{(0)}}\overline{c}^\nu = c^\rho_{(0)}F^\nu{}_{\rho\sigma}\overline{c}^\sigma$ for appropriate operators $F^\nu{}_{\rho\sigma}$ analogous to the spacetime-constant structure constants f^{abc} introduced in Sec. 2.1.3. (Explicitly $F^\nu{}_{\rho\sigma} := \delta^\nu_\sigma\overrightarrow{\nabla}_\rho - \delta^\nu_\rho\overleftarrow{\nabla}_\sigma$.) The fact that these expressions are first-order differential operators results in second-order derivatives appearing in $\mathcal{L}_0$, since

$$\begin{aligned}\nabla^\mu B^\nu &= \nabla^\mu\left(\widetilde{B}^\nu - \mathrm{i}\theta^A\pounds_{\xi_A}\overline{c}^\nu\right) = \nabla^\mu\left(\widetilde{B}^\nu - \mathrm{i}\theta^A\xi_A^\rho\nabla_\rho\overline{c}^\nu + \mathrm{i}\theta^A\overline{c}^\rho\nabla_\rho\xi_A^\nu\right)\\ &= \nabla^\mu\widetilde{B}^\nu - \mathrm{i}\nabla^\mu\theta^A\xi_A^\rho\nabla_\rho\overline{c}^\nu + \mathrm{i}\nabla^\mu\theta^A\overline{c}^\rho\nabla_\rho\xi_A^\nu\\ &\quad - \mathrm{i}\theta^A\left(\nabla^\mu\xi_A^\rho\nabla_\rho\overline{c}^\nu - \nabla^\mu\overline{c}^\rho\nabla_\rho\xi_A^\nu + \xi_A^\rho\nabla^\mu\nabla_\rho\overline{c}^\nu - \overline{c}^\rho\nabla^\mu\nabla_\rho\xi_A^\nu\right).\end{aligned} \tag{4.4.12}$$

4.4.2 General zero modes

Consider the more general transformations

$$\delta\widetilde{B}^\mu = \widetilde{\beta}_0^A(t)\,\xi_A^\mu, \tag{4.4.13}$$

$$\delta c^\mu = \theta_0^A(t)\,\xi_A^\mu, \tag{4.4.14}$$

$$\delta\overline{c}^\mu = \overline{\theta}_0^A(t)\,\xi_A^\mu, \tag{4.4.15}$$

[58]The reason there is a factor of q in the Yang–Mills result but no factor of κ in the analogous result for perturbative gravity is a result of a subtle difference in the ways the Nakanishi–Lautrup auxiliary fields have been defined. The $-\nabla^\mu B^\nu\kappa H_{\mu\nu}$ term in the scalar Lagrangian density of perturbative gravity is analogous to the Yang–Mills term $-\nabla^\mu B\cdot A_\mu$. Note that, unlike the FP-ghost term $-\mathrm{i}\nabla^\mu\overline{c}\cdot(\nabla_\mu c + qA_\mu\times c)$, this term has no q-dependence in the term proportional to A_μ. This could have been remedied if the definition of B had been divided by q, viz. $B := -(\alpha_0 q)^{-1}\nabla_\mu A^\mu$. This would result in a $-q\nabla^\mu B\cdot A_\mu$ term, which would be more properly analogous to $-\nabla^\mu B^\nu\kappa H_{\mu\nu}$. However, because B would be divided by q in this approach, so would $\widetilde{B}$ and $\widetilde{B} - B$, and the latter would then be the q-independent quantity $-\mathrm{i}\overline{c}\times c_{(0)}$.

with spatially uniform scalar functions $\widetilde{\beta}_0^A$, θ_0^A, $\overline{\theta}_0^A$. The transformation in Eq. (4.4.13) may be restated as $\delta B^\mu = \widetilde{\beta}_0^A(t)\,\xi_A^\mu$, so that

$$\delta\left(\nabla^\mu B^\nu\right) = \widetilde{\beta}_0^A \nabla^\mu \xi_A^\nu + g^{0\mu}\dot{\widetilde{\beta}}_0^A \xi_A^\nu, \tag{4.4.16}$$

$$\delta\left(-\nabla^\mu B^\nu \kappa H_{\mu\nu}\right) = -\dot{\widetilde{\beta}}_0^A \xi_A^\nu \kappa H_\nu^0, \tag{4.4.17}$$

$$\delta \int_{\mathbf{x}} \left(-\nabla^\mu B^\nu \kappa H_{\mu\nu}\right) = -\dot{\widetilde{\beta}}_0^A \int_{\mathbf{x}} \xi_A^\nu \kappa H_\nu^0. \tag{4.4.18}$$

The CMP sets $\int_{\mathbf{x}} \xi_A^\nu \kappa H_\nu^0 = 0$, so the Lagrangian is unchanged. A similar argument establishes the same for the transformation in Eq. (4.4.15). The transformation in Eq. (4.4.14) may be restated in terms of B^μ, c^μ, $\overline{c}^\mu$ as

$$\delta B^\mu = -\mathrm{i}\theta_0^A(t)\,\pounds_{\xi_A}\overline{c}^\nu, \tag{4.4.19}$$

$$\delta c^\mu = \theta_0^A(t)\,\xi_A^\mu. \tag{4.4.20}$$

Hence

$$\begin{aligned}\delta\mathcal{L}_0^{Bc\overline{c}} &= \delta\left(-\nabla^\mu B^\nu \kappa H_{\mu\nu} - \mathrm{i}\nabla_\nu \overline{c}^\mu \left(K_{\mu\alpha}^{\beta\nu}\nabla_\beta c^\alpha + c^\alpha \kappa \nabla_\alpha H_\mu^\nu\right)\right) \\ &= \mathrm{i}\nabla^\mu\left(\theta_0^A(t)\,\pounds_{\xi_A}\overline{c}^\nu\right)\kappa H_{\mu\nu} - \mathrm{i}\nabla_\nu\overline{c}^\mu\left(K_{\mu\alpha}^{\beta\nu}\nabla_\beta\left(\theta_0^A(t)\,\xi_A^\alpha\right) + \theta_0^A(t)\,\xi_A^\mu \kappa\nabla_\alpha H_\mu^\nu\right).\end{aligned} \tag{4.4.21}$$

This expression was already established above to reduce to a total derivative if each θ^A is spacetime-constant, so the only important terms are multiples of $\dot{\theta}^A$.
Not including the Einstein–Hilbert terms, the scalar Lagrangian density may be taken as

$$-\gamma_{\mu\nu}^{\rho\sigma}\left[\nabla^\mu\left(\widetilde{B}^\nu - \mathrm{i}\theta^A \pounds_{\xi_A}\overline{c}^\nu\right)\kappa h_{\rho\sigma} + \mathrm{i}\nabla^\mu\overline{c}^\nu \pounds_c g_{\rho\sigma}^f\right], \tag{4.4.22}$$

although a total derivative may still be added. The aforementioned choice of $\mathcal{L}_0$ includes second-order derivatives of the FP-antighost, due to the term $\mathrm{i}\theta^A \pounds_{\xi_A}\left(\nabla^\mu \overline{c}^\nu\right)\kappa H_{\mu\nu}$. This presents a few complications:

- the proof in Sec. 3.1 that the CMP's analysis can be equivalently performed in either the Hamiltonian or Lagrangian formalism assumes the absence of second-order derivatives in $\mathcal{L}_0$;
- the absence of second-order derivatives is also assumed in the usual proof that fields and modes are cyclic if and only if their momenta are conserved;
- and the usual formulation of the Legendre transform and the Hamiltonian formalism requires a modification, which will be discussed in Sec. 4.6.

Adding a total derivative addresses all these concerns, viz.

$$\mathcal{L}_0 = -\nabla^\mu \widetilde{B}^\nu \kappa H_{\mu\nu} - \mathrm{i}\kappa\theta^A\left(\pounds_{\xi_A}\overline{c}^\nu\right)\nabla^\mu H_{\mu\nu} - \mathrm{i}\nabla^\mu\overline{c}^\nu Z_{\mu\nu}. \tag{4.4.23}$$

However, adding $\nabla_\mu K^\mu$ to $\mathcal{L}_0$ for some vector field K^μ adds $\int_{\mathbf{x}} K^0$ to L, so can be expected to shift the numerical value of the Hamiltonian. Therefore, herein I will not make use of such a modification of $\mathcal{L}_0$. Sec. 4.5 verifies the obviation of $\widetilde{\beta}^A$, θ^A, $\overline{\theta}^A$ from the action by using

$$\mathcal{L}_0 = -\nabla^\mu\left(\widetilde{B}^\nu - \mathrm{i}\theta^A \pounds_{\xi_A}\overline{c}^\nu\right)\kappa H_{\mu\nu} - \mathrm{i}\nabla^\mu\overline{c}^\nu Z_{\mu\nu}, \tag{4.4.24}$$

which implies the same obviation is also achievable with an alternative choice of $\mathcal{L}_0$ that has the same action $\int_x \mathcal{L}_0$, viz. Eq. (4.4.23).

4.5 The Lagrangian density without zero modes

The infrared problem for the FP-ghost propagator is due to a zero-mode contribution, viz.

$$\mathrm{T}\,\langle 0|\,\hat{\theta}^A \xi^\mu_A \hat{\overline{\theta}}^B\,\xi^\nu_B\,|0\rangle = \mathrm{T}\big(\langle 0|\,\hat{\theta}^A \hat{\overline{\theta}}^B\,|0\rangle\,\xi^\mu_A \xi^\nu_B\big) \tag{4.5.1}$$

(note that θ^A, $\overline{\theta}^B$ have been promoted to operators). The problem can therefore be thought of in terms of the scalar fields θ^A, $\overline{\theta}^B$, rather than vector-valued fields. These scalar fields are spatially uniform, so θ^A may only appear as θ^A, $\dot{\theta}^A$, $\ddot{\theta}^A$, and similarly for $\overline{\theta}^A$. The challenge is to verify that these scalar fields are cyclic, i.e. that they do not appear undifferentiated.

In Sec. 4.5.1, I verify zero mode obviation from the Lagrangian is possible, providing we work in terms of $\widetilde{B}^\mu$, c^μ, $\overline{c}^\mu$ instead of B^μ, c^μ, $\overline{c}^\mu$. Using the Hamiltonian instead would be a difficult first step. The very definition of the Hamiltonian requires care, because of the second-order derivatives introduced in L when it is written in terms of $\widetilde{B}^\mu$, c^μ, $\overline{c}^\mu$. I obtain the Hamiltonian in Sec. 4.6.

As with Yang–Mills theory, redefining the Nakanishi–Lautrup auxiliary field introduces an extra term in L. I compute this in Sec. 4.5.2. In fact, the method I use would appear at first to allow *two* expressions for this extra term, provided a matrix that I introduce in Eq. (4.5.23) below is invertible. In Sec. 4.5.3, I show that these expressions would not agree for the flat static torus, so this invertibility hypothesis is wrong in general.

4.5.1 Proof zero modes can be obviated

The Lagrangian is

$$\begin{aligned} L = \int_{\mathbf{x}} \Big\{ & -\nabla^\mu \left(\widetilde{B}^\nu_{(+)} + \widetilde{\beta}^A \xi^\nu_A - \mathrm{i}\theta^A \left(\pounds_{\xi_{|A}} \overline{c}^\nu_{(+)} + \overline{\theta}^B \pounds_{\xi_A} \xi^\nu_B + \xi^\nu_B \xi^0_A \dot{\overline{\theta}}^B \right)\right) \kappa H_{\mu\nu} \\ & -\mathrm{i}\nabla^\mu \left(\overline{c}^\nu_{(+)} + \overline{\theta}^A \xi^\nu_A \right) Z_{\mu\nu} \Big\} . \end{aligned} \tag{4.5.2}$$

Using

$$\int_{\mathbf{x}} \xi^\nu_A \kappa H^0_\nu = 0,\ \int_{\mathbf{x}} \xi^\nu_A Z^0_\nu = 0,\ \nabla_\mu \xi_{A\nu} = -\nabla_\nu \xi_{A\mu},\ H^{\mu\nu} = H^{\nu\mu},\ Z^{\mu\nu} = Z^{\nu\mu},\ [\pounds_{\xi_A}, \nabla_\beta] = 0, \tag{4.5.3}$$

and the factor that $\pounds_{\xi_A} \xi^\mu_B$ is a Killing vector, to simplify L gives

$$\begin{aligned} L = \int_{\mathbf{x}} \Big\{ & -\nabla^\mu \widetilde{B}^\nu_{(+)} \kappa H_{\mu\nu} - \mathrm{i}\dot{\theta}^A \left(\pounds_{\xi_{|A}} \overline{c}^\nu_{(+)} + \overline{\theta}^B \pounds_{\xi_A} \xi^\nu_B + \xi^\nu_B \xi^0_A \dot{\overline{\theta}}^B \right) \kappa H^0_\nu \\ & - \mathrm{i}\theta^A \left(\pounds_{\xi_{|A}} \nabla^\mu \overline{c}^\nu_{(+)} \kappa H_{\mu\nu} + \xi^\nu_B \left(\nabla^\mu \xi^0_A \dot{\overline{\theta}}^B \kappa H_{\mu\nu} + \xi^0_A \ddot{\overline{\theta}}^B \kappa H^0_\nu \right)\right) \\ & - \mathrm{i} \left(\nabla^\mu \overline{c}^\nu_{(+)} \right) \gamma^{\rho\sigma}_{\mu\nu} \times \\ & \left(\kappa \left(c^\alpha_{(+)} + \theta^A \xi^\alpha_A \right) \nabla_\alpha h_{\rho\sigma} + \left(\nabla_\beta c^\alpha_{(+)} + \nabla_\beta \theta^A \xi^\alpha_A + \theta^A \nabla_\beta \xi^\alpha_A \right) \left(\delta^\beta_\rho g^f_{\alpha\sigma} + \delta^\beta_\sigma g^f_{\rho\alpha} \right)\right) \Big\} . \end{aligned} \tag{4.5.4}$$

The $\overline{\theta}^B$ term is

$$\mathrm{i}\overline{\theta}^B\dot{\theta}^A\int_{\mathbf{x}}\left\{\left(\pounds_{\xi_A}\xi^\nu_B\right)\kappa H^0_\nu\right\}=0, \tag{4.5.5}$$

as required. For the θ^A term we return to the Lagrangian. Since $Z_{\mu\nu}$ is symmetric, we may rewrite the FP-ghost term as

$$-\mathrm{i}\int_{\mathbf{x}}\overline{C}^{\mu\nu}Z_{\mu\nu},\,\overline{C}^{\mu\nu}:=\frac{\nabla^\mu\overline{c}^\nu+\nabla^\nu\overline{c}^\mu}{2}. \tag{4.5.6}$$

Note that $\overline{C}^{\mu\nu}$ is symmetric. Thus

$$L=\int_{\mathbf{x}}\left\{-\nabla^\nu\left(\widetilde{B}^\mu+\mathrm{i}\theta^A\pounds_{\xi_A}\overline{c}^\mu\right)\kappa H_{\mu\nu}-\mathrm{i}\overline{C}^\mu_\nu\left(K^{\beta\nu}_{\mu\alpha}\nabla_\beta\left(c^\alpha_{(0)}+\theta^A\xi^\alpha_A\right)+\left(c^\alpha_{(0)}+\theta^A\xi^\alpha_A\right)\kappa\nabla_\alpha H^\nu_\mu\right)\right\}. \tag{4.5.7}$$

The θ^A term is then

$$\begin{aligned}&-\mathrm{i}\theta^A\int_{\mathbf{x}}\left\{\pounds_{\xi_A}\nabla^\nu\overline{c}^\mu\kappa H_{\mu\nu}-\left(K^{\beta\nu}_{\mu\alpha}\nabla_\beta\xi^\alpha_A+\xi^\alpha_A\kappa H^\nu_\mu\right)\overline{C}^\mu_\nu\right\}\\&=\mathrm{i}\theta^A\int_{\mathbf{x}}\left\{-\pounds_{\xi_A}\left(\kappa H^\nu_\mu\right)+\left(K^{\beta\nu}_{\mu\alpha}\nabla_\beta\xi^\alpha_A+\xi^\alpha_A\kappa H^\nu_\mu\right)\right\}\overline{C}^\mu_\nu\\&=\mathrm{i}\theta^A\int_{\mathbf{x}}\left\{\text{-}\pounds_{\xi_A}\left(\kappa H^\nu_\mu\right)+Z^\nu_\mu\left(c^\alpha=\xi^\alpha_A\right)\right\}\overline{C}^\mu_\nu,\end{aligned} \tag{4.5.8}$$

where in a slight abuse of notation the function $Z^\nu_\mu\left(c^\alpha\right)$ is extended to boson-valued c^α. Indeed

$$Z_{\mu\nu}\left(c^\alpha=\xi^\alpha_A\right)=\kappa\xi^\alpha_A\nabla_\alpha H_{\mu\nu}+\gamma^{\rho\sigma}_{\mu\nu}\nabla_\beta\xi^\alpha_A\left(\delta^\beta_\rho g^f_{\alpha\sigma}+\delta^\beta_\sigma g^f_{\rho\alpha}\right)=\kappa\xi^\alpha_A\nabla_\alpha H_{\mu\nu}, \tag{4.5.9}$$

so the θ^A term in L is

$$\begin{aligned}\mathrm{i}\kappa\theta^A\int_{\mathbf{x}}\left\{-\pounds_{\xi_A}H^\nu_\mu+\xi^\alpha_A\nabla_\alpha H^\nu_\mu\right\}\overline{C}^\mu_\nu&=\mathrm{i}\kappa\theta^A\int_{\mathbf{x}}\left\{H^\alpha_\mu\nabla_\alpha\xi^\nu_A-H^\nu_\alpha\nabla^\alpha\xi_{A\mu}\right\}\overline{C}^\mu_\nu\\&=\mathrm{i}\kappa\theta^A\int_{\mathbf{x}}\left\{\nabla_\alpha\left(H^\alpha_\mu\xi^\nu_A\right)-\nabla^\alpha\left(H^\nu_\alpha\xi_{A\mu}\right)\right\}\overline{C}^\mu_\nu\\&=\mathrm{i}\kappa\theta^A\int_{\mathbf{x}}\nabla_\alpha\left(H^\alpha_\mu\xi_{A\nu}-H^\alpha_\nu\xi_{A\mu}\right)\overline{C}^{\mu\nu}=0,\end{aligned} \tag{4.5.10}$$

as required. (The last line uses the fact that $H^\alpha_\mu\xi_{A\nu}-H^\alpha_\nu\xi_{A\mu}$ is antisymmetric.)
It is unsurprising that the above calculations explicitly obviate $\widetilde{\beta}^A$, θ^A, $\overline{\theta}^A$. If undifferentiated $\widetilde{\beta}^A$ could not be obviated, a spacetime-constant shift in $\widetilde{\beta}^A$ would preserve $\dot{\widetilde{\beta}}^A$ but not $S=\int_{\mathbf{x}}\mathcal{L}_0$. But such a transformation would add a Killing vector to $\widetilde{B}^\nu$, which preserves $\nabla^\mu\widetilde{B}^\nu\kappa H_{\mu\nu}$. Similarly, $\overline{\theta}^A$ can be trivially removed. For Eq. (4.4.23)'s choice of $\mathcal{L}_0$, $\delta\theta^A=\theta^A_0$ gives

$$\begin{aligned}\delta\mathcal{L}_0&=-\mathrm{i}\kappa\theta^A_0\left(\left(\pounds_{\xi_A}\overline{c}^\nu\right)\nabla^\mu H_{\mu\nu}-\nabla^\mu\overline{c}^\nu\pounds_{\xi_A}H_{\mu\nu}\right)\\&=-\mathrm{i}\kappa\theta^A_0\left(\pounds_{\xi_A}\left(\overline{c}^\nu\nabla^\mu H_{\mu\nu}\right)-\nabla^\mu\left(\overline{c}^\nu\pounds_{\xi_A}H_{\mu\nu}\right)\right),\end{aligned} \tag{4.5.11}$$

$$\delta S=0. \tag{4.5.12}$$

The treatment of differentiated zero modes is not much harder, since in the CMP $\dot{\widetilde{\beta}}^A_0\int_{\mathbf{x}}\xi^\mu_A\kappa H^0_\mu=0$ and $\dot{\overline{\theta}}^A_0\int_{\mathbf{x}}\xi^\mu_A Z^0_\mu=0$. The $\dot{\theta}^A$ dependence of L is due to the term

$$\dot{\theta}^A\Pi_A,\,\Pi_A:=\mathrm{i}\int_{\mathbf{x}}\xi^\alpha_A K^{0\nu}_{\mu\alpha}\nabla_\nu\overline{c}^\mu. \tag{4.5.13}$$

Setting the conserved momentum to zero to complete the obviation is equivalent, in the de Donder gauge, to setting $\int_{\mathbf{x}} \xi_A^{\alpha} \overline{Z}_{\alpha}^{0} = 0$.

4.5.2 An extra term in the Lagrangian due to the $\widetilde{B}^{\mu},\ c^{\mu},\ \overline{c}^{\mu}$ formalism

The change from $B^{\mu},\ c^{\mu},\ \overline{c}^{\mu}$ to $\widetilde{B}^{\mu},\ c^{\mu},\ \overline{c}^{\mu}$ introduces a new term in the Lagrangian of perturbative gravity. Let $L^{\text{extra}},\ L_{\text{FP}}$ respectively denote this extra term and the FP-ghost term. Define

$$L_{\text{FP}}^{(0+)} := -\mathrm{i} \int_{\mathbf{x}} \nabla^{\mu} \overline{c}_{(0)}^{\nu} \gamma_{\mu\nu}^{\rho\sigma} \pounds_{c_{(+)}} g_{\rho\sigma}^{f}, \tag{4.5.14}$$

and define $L_{\text{FP}}^{(00)},\ L_{\text{FP}}^{(+0)},\ L_{\text{FP}}^{(++)}$ similarly. Thus L_{FP} has been split into four parts, with

$$\begin{aligned} L_{\text{FP}}^{(00)} + L_{\text{FP}}^{(0+)} &= -\mathrm{i} \int_{\mathbf{x}} \nabla^{\mu} \overline{c}_{(0)}^{\nu} \gamma_{\mu\nu}^{\rho\sigma} \pounds_{c} g_{\rho\sigma}^{f} \\ &= -\mathrm{i} \int_{\mathbf{x}} \left(\overline{\theta}^{A} \nabla^{\mu} \xi_A^{\nu} + g^{0\mu} \dot{\overline{\theta}}^{A} \xi_A^{\nu} \right) \gamma_{\mu\nu}^{\rho\sigma} \pounds_{c} g_{\rho\sigma}^{f} \\ &= -\mathrm{i} \dot{\overline{\theta}}^{A} \int_{\mathbf{x}} \xi_A^{\nu} g^{0\mu} \gamma_{\mu\nu}^{\rho\sigma} \pounds_{c} g_{\rho\sigma}^{f} = -\mathrm{i} \dot{\overline{\theta}}^{A} \int_{\mathbf{x}} \xi_A^{\nu} Z_{\nu}^{0}, \end{aligned} \tag{4.5.15}$$

which vanishes when appropriate conserved charges are set to 0 in the CMP. Since both the $B^{\mu},\ c^{\mu},\ \overline{c}^{\mu}$ and $\widetilde{B}^{\mu},\ c^{\mu},\ \overline{c}^{\mu}$ formalisms retain $L_{\text{FP}}^{(++)}$ (which is independent of zero modes and their derivatives), only one of the four parts of L_{FP} contributes to L^{extra}, namely $L_{\text{FP}}^{(+0)}$. Next I compute an expression for L^{extra} that contains no zero modes or derivatives thereof. Since $c_{(0)}^{\mu} = \theta^{A} \xi_A^{\mu}$,

$$\nabla_0 c_{(0)}^{\mu} = \dot{\theta}^{A} \xi_A^{\mu} + \theta^{A} \nabla_0 \xi_A^{\mu},\ \nabla_i c_{(0)}^{\mu} = \theta^{A} \nabla_i \xi_A^{\mu}, \tag{4.5.16}$$

and

$$\mathrm{i} L_{\text{FP}}^{(+0)} = \int_{\mathbf{x}} \nabla_{\nu} \overline{c}_{(+)}^{\mu} \left(K_{\mu\alpha}^{\beta\nu} \nabla_{\beta} \left(\theta^{A} \xi_A^{\alpha} \right) + \kappa \theta^{A} \xi_A^{\alpha} \nabla_{\alpha} H_{\mu}^{\nu} \right) = A_A \dot{\theta}^{A} + B_A \theta^{A} \tag{4.5.17}$$

where

$$A_A := \int_{\mathbf{x}} \nabla_{\nu} \overline{c}_{(+)}^{\mu} K_{\mu\alpha}^{0\nu} \xi_A^{\alpha},,\ B_A := \int_{\mathbf{x}} \nabla_{\nu} \overline{c}_{(+)}^{\mu} \left(K_{\mu\alpha}^{\beta\nu} \nabla_{\beta} \xi_A^{\alpha} + \kappa \xi_A^{\alpha} \nabla_{\alpha} H_{\mu}^{\nu} \right) \tag{4.5.18}$$

are fermionic fields. We can obtain $\mathrm{i}L^{\text{extra}}$ from $\mathrm{i}L_{\text{FP}}^{(+0)}$ by taking account of $\widetilde{B}^{\mu} - B^{\mu}$, which effects the replacement

$$\begin{aligned} \kappa \theta^{A} \int_{\mathbf{x}} \nabla_{\nu} \overline{c}_{(+)}^{\mu} \xi_A^{\alpha} \nabla_{\alpha} H_{\mu}^{\nu} &\to \kappa \theta^{A} \int_{\mathbf{x}} \nabla_{\nu} \overline{c}_{(+)}^{\mu} \left(\xi_A^{\alpha} \nabla_{\alpha} H_{\mu}^{\nu} - \pounds_{\xi_A} H_{\mu}^{\nu} \right) \\ &= \kappa \theta^{A} \int_{\mathbf{x}} \nabla_{\nu} \overline{c}_{(+)}^{\mu} \left(\nabla_{\alpha} \xi_A^{\nu} H_{\mu}^{\alpha} - \nabla_{\mu} \xi_A^{\alpha} H_{\alpha}^{\nu} \right). \end{aligned} \tag{4.5.19}$$

so

$$\mathrm{i} L^{\text{extra}} = A_A \dot{\theta}^{A} + B_A^{'} \theta^{A} \tag{4.5.20}$$

with

$$B_A^{'} = \int_{\mathbf{x}} \nabla_{\nu} \overline{c}_{(+)}^{\mu} \left(K_{\mu\alpha}^{\beta\nu} \nabla_{\beta} \xi_A^{\alpha} + \kappa \left(H_{\mu}^{\alpha} \nabla_{\alpha} \xi_A^{\nu} - H_{\alpha}^{\nu} \nabla_{\mu} \xi_A^{\alpha} \right) \right) \tag{4.5.21}$$

also a fermionic field. Setting conserved charges to 0 as usual,

$$\begin{aligned}
0 &= \int_{\mathbf{y}} \xi_A^\mu Z_\mu^0 \\
&= \int_{\mathbf{y}} \xi_A^\mu \left(K_{\mu\alpha}^{00}\nabla_0 + K_{\mu\alpha}^{i0}\nabla_i + J_{\mu\alpha}^0\right)\left(\theta^B \xi_B^\alpha + c_{(+)}^\alpha\right) \\
&= \int_{\mathbf{y}} \left\{\dot{\theta}^B \left(K_{\mu\alpha}^{00}\xi_A^\mu \xi_B^\alpha\right) + \xi_A^\mu \left(\theta^B \left(K_{\mu\alpha}^{\beta 0}\nabla_\beta + J_{\mu\alpha}^0\right)\xi_B^\alpha + J_{\mu\alpha}^0 c_{(+)}^\alpha\right)\right\} \\
&= M_{AB}\dot{\theta}^B + N_{AB}\theta^B + P_A
\end{aligned} \tag{4.5.22}$$

where

$$N_{AB} := \int_{\mathbf{y}} \xi_A^\mu \left(K_{\mu\alpha}^{\beta 0}\nabla_\beta \xi_B^\alpha + \kappa \xi_B^\alpha \nabla_\alpha H_\mu^0\right), \tag{4.5.23}$$

$$P_A := \int_{\mathbf{y}} \xi_A^\mu \left(\kappa \nabla_\alpha H_\mu^0 + K_{\mu\alpha}^{\beta 0}\nabla_\beta\right) c_{(+)}^\alpha. \tag{4.5.24}$$

Note that M_{AB}, N_{AB} are c-number valued and independent of the FP-ghost fields, while P_A is ghost-dependent and fermionic. It can be shown that M_{AB} is invertible, at least for the flat static torus and de Sitter space; see Appendix G. Hence

$$\dot{\theta}^C = \left(M^{-1}\right)^{CA} M_{AB}\dot{\theta}^B = -\left(M^{-1}\right)^{CA}\left(N_{AB}\theta^B + P_A\right), \tag{4.5.25}$$

$$\begin{aligned}
\mathrm{i}L^{\text{extra}} &= -A_C \left(M^{-1}\right)^{CA}\left(N_{AB}\theta^B + P_A\right) + B_B^{'}\theta^B \\
&= \left(B_B^{'} - A_C\left(M^{-1}\right)^{CA} N_{AB}\right)\theta^B - A_C\left(M^{-1}\right)^{CA} P_A.
\end{aligned} \tag{4.5.26}$$

Since θ^B is cyclic, the θ^B term vanishes. (Indeed, varying θ^A in Eq. (4.5.26) obtains an Euler–Lagrange equation that is equivalent to the fact that theθ^B coefficient vanishes.) Thus

$$L^{\text{extra}} = \mathrm{i}\left(M^{-1}\right)^{CA} A_C P_A = \mathrm{i}\left(M^{-1}\right)^{CA}\int_{\mathbf{x}} K_{\mu\alpha}^{0\nu}\nabla_\nu \bar{c}_{(+)}^\mu \xi_C^\alpha \int_{\mathbf{y}} \xi_A^\rho \left(\kappa\nabla_\gamma H_\rho^0 + K_{\rho\gamma}^{\beta 0}\nabla_\beta\right) c_{(+)}^\gamma, \tag{4.5.27}$$

a result that is expressed entirely in terms of $_{(+)}$ modes.

4.5.3 A discussion of N_{AB}

That θ_B is cyclic guarantees it is removable from the Lagrangian formalism. The above calculation removed $\dot{\theta}_B$, thus entirely obviating θ_B from the effective field theory. The next thing to note is that there is *not* an analogous result in terms of N_{AB}. If N_{AB} were also invertible, θ^B could also be removed from L^{extra} using Eq. (4.5.22), viz.

$$\theta^C = \left(N^{-1}\right)^{CA} N_{AB}\theta^B = -\left(N^{-1}\right)^{CA}\left(M_{AB}\dot{\theta}^B + P_A\right), \tag{4.5.28}$$

$$\begin{aligned}
\mathrm{i}L^{\text{extra}} &= -B_C^{'}\left(N^{-1}\right)^{CA}\left(M_{AB}\dot{\theta}^B + P_A\right) + A_B\theta^B \\
&= \left(A_B - B_C^{'}\left(N^{-1}\right)^{CA} M_{AB}\right)\dot{\theta}^B - B_C^{'}\left(N^{-1}\right)^{CA} P_A.
\end{aligned} \tag{4.5.29}$$

By inspection, θ^B is cyclic in Eq. (4.5.29)'s expression for L^{extra} as required. Indeed, the above removal of θ^B implies

$$B^{'\mathrm{T}} = \theta^{\mathrm{T}} M^{-1} N,\ -B^{'\mathrm{T}} N^{-1} P = -\theta^{\mathrm{T}} M^{-1} P,\ B^{'\mathrm{T}} N^{-1} M = \theta^{\mathrm{T}}, \tag{4.5.30}$$

so either expression for L^{extra} is independent of both θ^B and $\dot{\theta}^B$, and the two expressions are then equal, viz. $\mathrm{i}A_C \left(M^{-1}\right)^{CA} P_A = \mathrm{i}B_C^{'} \left(N^{-1}\right)^{CA} P_A$. The fact that the CMP already removes θ^B from L should lead us to be suspicious of this alternative approach that makes use of Eq. (4.5.22) to remove θ^B. Indeed, the invertibility of N_{AB} would imply that there are two alternative, numerically equal expressions for L_{extra}: one in terms of M_{AB}, the other in terms of N_{AB}, viz.

$$L^{\text{extra}} = \mathrm{i}\left(N^{-1}\right)^{CA} \int_{\mathbf{x}} \nabla_\nu \bar{c}^\mu_{(+)} \left(K^{\beta\nu}_{\mu\alpha} \nabla_\beta \xi^\alpha_C + \kappa \left(\nabla_\alpha \xi^\nu_C H^\alpha_\mu - \nabla_\mu \xi^\alpha_C H^\nu_\alpha\right)\right) \int_{\mathbf{y}} \xi^\rho_A \left(\kappa \nabla_\beta H^0_\rho + K^{\gamma 0}_{\rho\alpha} \nabla_\gamma\right) c^\beta_{(+)}. \tag{4.5.31}$$

However, this result is not valid in general. In the flat static torus the basis of the Killing vector fields may be chosen as $\xi^\alpha_A = \delta^\alpha_A \in \ker \nabla_\beta$, so the $\int_{\mathbf{x}}$ integral on the right-hand side of Eq. (4.5.31) would vanish. By contrast, the right-hand side of Eq. (4.5.27) is instead equal to the non-vanishing quantity

$$\mathrm{i}\left(M^{-1}\right)^{\alpha\rho} \int_{\mathbf{x}} K^{0\nu}_{\mu\alpha} \nabla_\nu \bar{c}^\mu_{(+)} \int_{\mathbf{y}} \left(\kappa \nabla_\gamma H^0_\rho + K^{\beta 0}_{\rho\gamma} \nabla_\beta\right) \bar{c}^\gamma_{(+)}. \tag{4.5.32}$$

This contradiction establishes that N_{AB} is not, in fact, invertible for a flat static torus. Thus the correct general expression for L^{extra} is given by Eq. (4.5.27) instead of Eq. (4.5.31).

4.6 The Ostrogradski method's Hamiltonian

Theories in which second- or higher-order derivatives of canonical fields appear in the Lagrangian formalism still admit a Hamiltonian formalism. In classical mechanics, the conjugate momenta used in the appropriate Legendre formalism are defined by a method due to Mikhail Ostrogradski [85]. This method has been adapted to classical field theory; the notation used herein is adapted from Ref. [86]. I will only consider theories lacking third- and higher-order derivatives, since no theories in which such derivatives appear are relevant to this thesis.

In classical mechanics, outside of field theory, the Euler–Lagrange equation is of the form

$$0 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{q}}. \tag{4.6.1}$$

Define $q_1 := q$, $q_2 := \dot{q}$, and treat q_1, q_2 as independent variables, and define

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} - \dot{p}_2,\ p_2 := \frac{\partial L}{\partial \dot{q}_2}. \tag{4.6.2}$$

The Euler–Lagrange equation is then

$$\frac{\partial L}{\partial q_1} = \dot{p}_1. \tag{4.6.3}$$

Finally, define the Hamiltonian as

$$H := -L + \dot{q}_1 p_1 + \dot{q}_2 p_2 = -L + \dot{q}\left(\frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}}\right) + \ddot{q}\frac{\partial L}{\partial \ddot{q}_1}. \tag{4.6.4}$$

Hence

$$\frac{\partial H}{\partial q_1} = -\frac{\partial L}{\partial q} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = -\dot{p}_1, \tag{4.6.5}$$

$$\frac{\partial H}{\partial q_2} = -\frac{\partial L}{\partial \dot{q}} + p_1 = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}} = -\dot{p}_2, \tag{4.6.6}$$

$$\frac{\partial_R H}{\partial_R p_i} = \dot{q}_i \tag{4.6.7}$$

(note the use of a right-derivative in the final result; this disambiguation is necessary for fermionic q_i, p_i, as noted in Sec. 3.1). The rest of this section is concerned with how to adapt this approach to a field theory in curved spacetime, so that spatial derivatives also occur and covariant derivatives may be used instead of partial ones. Throughout a product of covariant derivatives will have the indices running along the left, viz. $\prod_{l=1}^{2}\nabla_{\alpha_1} = \nabla_{\alpha_1}\nabla_{\alpha_2}$. Consider a scalar Lagrangian density of the form

$$\mathcal{L}_0 = \mathcal{L}_0\left(\phi,\, \nabla_{\alpha_1}\phi,\, \nabla_{\alpha_1}\nabla_{\alpha_2}\phi\right) \tag{4.6.8}$$

for some tensor-valued canonical field ϕ (whose spacetime indices are suppressed herein, as is any possible dependence of $\mathcal{L}_0$ on other canonical fields and their derivatives). The stationary action principle is

$$\begin{aligned} 0 = \delta S &= \int \mathrm{d}t \int_{\mathbf{x}} \sum_{k=0}^{2}\left\{\delta\left(\prod_{l=1}^{k}\nabla_{\alpha_l}\right)\phi\frac{\partial \mathcal{L}_0}{\partial\left(\prod_{l=1}^{k}\nabla_{\alpha_l}\right)\phi}\right\} \\ &= \int \mathrm{d}t \int_{\mathbf{x}} \delta\phi \sum_{k=0}^{2}\left\{(-1)^k\left(\prod_{l=1}^{k}\nabla_{\alpha_{k+1-i}}\right)\frac{\partial \mathcal{L}_0}{\partial\left(\prod_{l=1}^{k}\nabla_{\alpha_l}\right)\phi}\right\}. \end{aligned} \tag{4.6.9}$$

The Euler–Lagrange equation is therefore

$$\sum_{k=0}^{m}\left\{(-1)^k\left(\prod_{l=1}^{k}\nabla_{\alpha_{k+1-i}}\right)\frac{\partial \mathcal{L}_0}{\partial\left(\prod_{l=1}^{k}\nabla_{\alpha_l}\right)\phi}\right\} = 0. \tag{4.6.10}$$

The Ostrogradski method's coordinates are $\phi_1 := \phi$, $\phi_{2\alpha} := \nabla_\alpha\phi$. Reduced momentum densities may be defined as

$$\varpi_1 := \frac{\partial \mathcal{L}_0}{\partial \nabla_0\phi} - \nabla_\alpha\frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha\nabla_0\phi},\; \varpi_2^\alpha := \frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha\nabla_0\phi}. \tag{4.6.11}$$

The Euler–Lagrange equation is

$$0 = \frac{\partial \mathcal{L}_0}{\partial \phi} - \nabla_\alpha\frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha\phi} + \nabla_\beta\nabla_\alpha\frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha\nabla_\beta\phi}, \tag{4.6.12}$$

while the Hamiltonian density is $\sqrt{|g|}\mathcal{H}_0$ with

$$\mathcal{H}_0 := -\mathcal{L}_0 + \nabla_0\phi_1\varpi_1 + \nabla_0\phi_{2\alpha}\varpi_2^\alpha = -\mathcal{L}_0 + \nabla_0\phi\left(\frac{\partial \mathcal{L}_0}{\partial \nabla_0\phi} - \nabla_\alpha\frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha\nabla_0\phi}\right) + \nabla_0\nabla_\alpha\phi\frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha\nabla_0\phi}, \tag{4.6.13}$$

and the Hamilton's equations are[59]

$$\frac{\partial_R \mathcal{H}_0}{\partial_R \varpi_k} = \nabla_0 \phi_k, \tag{4.6.14}$$
$$\frac{\partial \mathcal{H}_0}{\partial \phi_1} = -\frac{\partial \mathcal{L}_0}{\partial \phi} = -\nabla_i \left(\frac{\partial \mathcal{L}_0}{\partial \nabla_i \phi} - \nabla_\beta \frac{\partial \mathcal{L}_0}{\partial \nabla_\beta \nabla_i \phi} \right) - \nabla_0 \varpi_1, \tag{4.6.15}$$
$$\frac{\partial \mathcal{H}_0}{\partial \phi_{2\alpha}} = -\frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha \phi} + \delta_0^\alpha \left(\frac{\partial \mathcal{L}_0}{\partial \nabla_0 \phi} - \nabla_\beta \frac{\partial \mathcal{L}_0}{\partial \nabla_\beta \nabla_0 \phi} \right), \tag{4.6.16}$$
$$\frac{\partial \mathcal{H}_0}{\partial \phi_{20}} = -\nabla_\beta \frac{\partial \mathcal{L}_0}{\partial \nabla_\beta \nabla_0 \phi} = -\nabla_\beta \varpi_2^\beta, \tag{4.6.17}$$
$$\frac{\partial \mathcal{H}_0}{\partial \phi_{2i}} = -\frac{\partial \mathcal{L}_0}{\partial \nabla_i \phi}. \tag{4.6.18}$$

If ϕ is a spatially uniform scalar field, such as θ^A, then the coordinates are $\phi_1,\ \phi_{20}$. Hence

$$\varpi_1 = \frac{\partial \mathcal{L}_0}{\partial \dot{\phi}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}_0}{\partial \ddot{\phi}}, \tag{4.6.19}$$
$$\varpi_2 = \frac{\partial \mathcal{L}_0}{\partial \ddot{\phi}}, \tag{4.6.20}$$
$$\frac{\partial_R \mathcal{H}_0}{\partial_R \varpi_k} = \frac{\mathrm{d}^{k+1}\phi}{\mathrm{d}t^{k+1}}, \tag{4.6.21}$$
$$\frac{\partial \mathcal{H}_0}{\partial \phi_1} = -\dot{\varpi}_1, \tag{4.6.22}$$
$$\frac{\partial \mathcal{H}_0}{\partial \phi_{20}} = -\dot{\varpi}_2. \tag{4.6.23}$$

Therefore, verifying that a spatially uniform scalar field is cyclic, in the Lagrangian or Hamiltonian formalism, is the same calculation up to a sign change.
The scalar Lagrangian density $\mathcal{L}_0$ may be written as

$$\begin{aligned} \mathcal{L}_0 = &\left(-\nabla_\nu \widetilde{B}^\mu - \mathrm{i} \nabla_\nu \xi_A^\rho \nabla_\rho \bar{c}^\mu \theta^A - \mathrm{i} \xi_A^\rho \nabla_\nu \nabla_\rho \bar{c}^\mu \theta^A + \mathrm{i} \nabla_\nu \bar{c}^\rho \nabla_\rho \xi_A^\mu \theta^A + \mathrm{i} \bar{c}^\rho \nabla_\nu \nabla_\rho \xi_A^\mu \theta^A \right) \kappa H_\mu^\nu \\ &- \mathrm{i} \nabla_\nu \bar{c}^\mu \left(c_{(+)}^\alpha \nabla_\alpha \kappa H_\mu^\nu + K_{\mu\alpha}^{\beta\nu} \nabla_\beta c_{(+)}^\alpha + \theta^A \left(\xi_A^\alpha \nabla_\alpha \kappa H_\mu^\nu + K_{\mu\alpha}^{\beta\nu} \nabla_\beta \xi_A^\alpha \right) + \dot{\theta}^A K_{\mu\alpha}^{0\nu} \xi_A^\alpha \right), \end{aligned} \tag{4.6.24}$$

so the only field which is differentiated twice is the antighost, viz. $\frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha \nabla_\beta \bar{c}^\mu} = -\mathrm{i} \kappa \theta^A \xi_A^\beta H_\mu^\alpha$. Hence

$$\mathcal{H}_0 = -\mathcal{L}_0 + \nabla_0 \widetilde{B}^\mu \frac{\partial \mathcal{L}_0}{\partial \nabla_0 \widetilde{B}^\mu} + \nabla_0 c^\mu \frac{\partial \mathcal{L}_0}{\partial \nabla_0 c^\mu} + \nabla_0 \bar{c}^\mu \left(\frac{\partial \mathcal{L}_0}{\partial \nabla_0 \bar{c}^\mu} - \nabla_\alpha \frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha \nabla_0 \bar{c}^\mu} \right) + \nabla_0 \nabla_\alpha \bar{c}^\mu \frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha \nabla_0 \bar{c}^\mu}. \tag{4.6.25}$$

Since previous calculations have verified that $\widetilde{\beta}^A,\ \theta^A,\ \overline{\theta}^A$ are cyclic,

$$\begin{aligned} \mathcal{H}_0 = &-\mathcal{L}_0 + \nabla_0 \widetilde{B}_{(+)}^\mu \frac{\partial \mathcal{L}_0}{\partial \nabla_0 \widetilde{B}_{(+)}^\mu} + \nabla_0 c_{(+)}^\mu \frac{\partial \mathcal{L}_0}{\partial \nabla_0 c_{(+)}^\mu} \\ &\nabla_0 \bar{c}_{(+)}^\mu \left(\frac{\partial \mathcal{L}_0}{\partial \nabla_0 \bar{c}_{(+)}^\mu} - \nabla_\alpha \frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha \nabla_0 \bar{c}_{(+)}^\mu} \right) + \nabla_0 \nabla_\alpha \bar{c}_{(+)}^\mu \frac{\partial \mathcal{L}_0}{\partial \nabla_\alpha \nabla_0 \bar{c}_{(+)}^\mu}. \end{aligned} \tag{4.6.26}$$

But

$$\frac{\partial \mathcal{L}_0}{\partial \nabla_0 \widetilde{B}_{(+)}^\mu} = -\kappa H_\mu^0, \frac{\partial \mathcal{L}_0}{\partial \nabla_0 c_{(+)}^\mu} = \mathrm{i} K_{\rho\mu}^{0\nu} \nabla_\nu \bar{c}^\rho, \tag{4.6.27}$$

[59] If other fields appear in $\mathcal{L}_0$, the expression for $\mathcal{H}_0$ in Eq. (**??**) should be amended to include additional terms in the Legendre transform. However, these terms are in any case irrelevant to this calculation. Of course, we can make ϕ a multiplet to make the expression for $\mathcal{H}_0$ in Eq. (**??**) general, provided products are interpreted as contracting over all indices.

and

$$\frac{\partial\mathcal{L}_0}{\partial\nabla_\alpha\nabla_0\bar{c}^\mu_{(+)}}=-\mathrm{i}\kappa\theta^A\xi^0_A H^\alpha_\mu,\ \frac{\partial\mathcal{L}_0}{\partial\nabla_0\bar{c}^\mu_{(+)}}-\nabla_\alpha\frac{\partial\mathcal{L}_0}{\partial\nabla_\alpha\nabla_0\bar{c}^\mu_{(+)}}=-\mathrm{i}\kappa\dot{\theta}^A\xi^0_A H^0_\mu-\mathrm{i}Z^0_\mu, \tag{4.6.28}$$

so

$$\mathcal{H}_0=-\mathcal{L}_0-\nabla_0\widetilde{B}^\mu_{(+)}\kappa H^0_\mu+\mathrm{i}\nabla_0 c^\mu_{(+)}K^{0\nu}_{\rho\mu}\nabla_\nu\bar{c}^\rho+\nabla_0\bar{c}^\mu_{(+)}\left(-\mathrm{i}\kappa\dot{\theta}^A\xi^0_A H^0_\mu-\mathrm{i}Z^0_\mu\right)-\mathrm{i}\kappa\nabla_0\nabla_\alpha\bar{c}^\mu_{(+)}\theta^A\xi^0_A H^\alpha_\mu. \tag{4.6.29}$$

Note that the result of Sec. 3.5.1, together with the fact that variables $q_1=q$, $q_2=\dot{q}$ are treated as independent variables in Ostrogradski's method, implies that the $\mathcal{H}_0$ obtained in the $\widetilde{B}^\mu$, c^μ, $\bar{c}^\mu$ formalism has the same numerical value as the B^μ, c^μ, $\bar{c}^\mu$-formalism's $\mathcal{H}_0$, i.e. the usual $\mathcal{H}_0$.

4.7 Comparison with the FMP in perturbation theory

Tsung-Dao Lee and Chen-Ning Yang discovered [87, 88] that, in some contexts, a formulation of perturbation theory in terms of Feynman diagrams admits terms proportional to $\delta(0)$, where δ is the Dirac delta so that $\delta(0)$ is divergent. (The argument 0 of $\delta(0)$ is a difference between time coordinates.) However, they found also that such terms cancel in pairs. I will call such terms *Lee–Yang terms*, and will call the cancellation of them in pairs *Lee–Yang cancellation*.

For perturbative gravity, one difference between the perturbation theories of the CMP and the FMP is that the FMP exhibits the Lee–Yang cancellation of Lee–Yang terms that never arise in the CMP. In Sec. 4.7.1, I compute the effective zero-mode sector Lagrangian in the CMP of a broad class of Lagrangians that includes the zero mode sector of perturbative gravity.

To prove the FMP and CMP are equivalent in perturbation theory, I must verify the FMP's Lee–Yang cancellation, and then check the equivalence of other terms. The trick to verifying Lee–Yang cancellation is to compare what I will call *Lee–Yang determinants*. In Sec. 4.7.2, I explain the relevant theory and summarise the structure of the FMP calculations that will then conclude this section. These calculations will be divided into Secs. 4.7.3-4.7.5. For now, the following summary is appropriate. Sec. 4.7.3 computes a matrix determinant; Sec. 4.7.4 factorises that determinant; and Sec. 4.7.5 shows this factorisation implies that the CMP and FMP are perturbatively equivalent.

Throughout I impose the gauge choice in Eq. (1.2.2), and use N to denote non-negative integer dummy indices.

4.7.1 The perturbation-theoretic implications of the CMP

Let $X\sim Y$ abbreviate the condition that $X-Y$ is independent of the time derivatives of spatially uniform Hermitian canonical scalar modes b^A, $\bar{b}^A$, q^i. This section considers an arbitrary Lagrangian of the form

$$L\sim-\mathrm{i}\dot{\bar{b}}^A M_{AB}\dot{b}^B-\mathrm{i}\overline{K}_{iA}\dot{q}^i\dot{b}^A-\mathrm{i}\dot{\bar{b}}^A L_{Ai}\dot{q}^i+\frac{1}{2}\dot{q}^i N_{ij}\dot{q}^j \tag{4.7.1}$$

with matrices $\overline{K}$, L, $N \sim 0$.[60] The previously defined matrix M_{AB} and its inverse M^{AB} can be respectively denoted M, M^{-1}. The FP-(anti)ghost zero mode sector of perturbative gravity has a Lagrangian of the form in Eq. (4.7.1), provided that:

- the b^A are the spatially uniform scalar-valued mode coefficients of the field c^μ, such as the θ^A;
- the $\overline{b}^A$ are the spatially uniform scalar-valued mode coefficients of the field $\overline{c}^\mu$, such as the $\overline{\theta}^A$;
- the q^i are the spatially uniform scalar-valued mode coefficients of all fields outside the FP–ghost sector, such as the β^A.

Note that the lower case Roman indices no longer imply bijection with the $n-1$ space dimensions of the spacetime manifold.

The $-\mathrm{i}\dot{\overline{b}}^A M_{AB}\dot{b}^B$ term in L is worthy of special attention. In perturbative gravity's scalar Lagrangian density's FP-sector term $-\mathrm{i}\nabla_\nu\overline{c}^\mu K^{\beta\nu}_{\mu\alpha}\nabla_\beta c^\alpha$, the $\dot{\overline{\theta}}^A\dot{\theta}^B$ coefficient is $-\mathrm{i}\xi^\mu_A K^{00}_{\mu\alpha}\xi^\alpha_B$. The $\dot{\overline{\theta}}^A\dot{\theta}^B$ coefficient in the Lagrangian is therefore $-\mathrm{i}\int_{\mathbf{x}}\xi^\mu_A K^{00}_{\mu\alpha}\xi^\alpha_B = -\mathrm{i}M_{AB}$. (The term $-\mathrm{i}\int_{\mathbf{x}}\nabla_\nu\overline{c}^\mu K^{\beta\nu}_{\mu\alpha}\nabla_\beta c^\alpha$ is analogous to the Yang–Mills term $-\mathrm{i}\int_{\mathbf{x}}\partial_0\overline{c}_{(0)}\cdot g^{00}\partial_0 c_{(0)} = -\mathrm{i}V\dot{\overline{c}}_{(0)}\cdot\dot{c}_{(0)}$.) The full Lagrangian may then be written as $L_{(0)}+L_{(+)}$, where $L_{(0)}$ contains terms dependent on $\dot{\theta}^A$s and/or $\dot{\overline{\theta}}^A$s. Explicitly we have

$$L_{(0)} = \dot{\overline{\theta}}^A\left(\mathrm{i}Y_A - \mathrm{i}M_{AB}\dot{\theta}^B\right) + \mathrm{i}\overline{X}_A\dot{\theta}^A \tag{4.7.2}$$

for some $\overline{X}_A$, Y_A, and θ^A, $\overline{\theta}^A$, $\dot{\theta}^A$, $\dot{\overline{\theta}}^A$ are absent from $L_{(+)}$, $\overline{X}_A$, Y_A. Note that no terms dependent on θ^As and/or $\overline{\theta}^A$s appear in L because θ^A, $\overline{\theta}^A$ are cyclic.

This formalism affords a comparison of the zero mode problems of perturbative gravity and Yang–Mills theory, if only for non-interacting gravity on a flat static torus. If c^μ, $\overline{c}^\mu$ are each given an IR-regularising mass $M>0$ then $L_{(0)} = -\mathrm{i}V\left(\dot{\overline{\theta}}^A\dot{\theta}_A - M^2\overline{\theta}^A\theta_A\right)$, where the Kronecker delta lowers upper case indices. The non-interacting case's result $\hat{M}_{AB} = V(t)\,\delta_{AB}$ generalises in the interacting case to $M_{AB} = V(t)\left(\delta_{AB} - H_{AB}\right)$ for some matrix H_{AB}. For the non-interacting case,

$$0 = \left(\frac{\partial}{\partial\overline{\theta}^A} - \partial_0\frac{\partial}{\partial\dot{\overline{\theta}}^A}\right)\left(\mathrm{i}L_{(0)}\right) = -M^2V\theta^A - \partial_0\left(V\dot{\theta}^A\right), \tag{4.7.3}$$

and similarly with $\overline{\theta}^A$. This is the same field equation as for scalar zero modes in earlier chapters, so the resulting zero-mode sector of the propagator is the same as before, and so are its time derivatives.

With that grey note concluded, I return attention from the flat static torus to the more general case specified in Sec. 1.2.2. That θ^A, $\overline{\theta}^A$ are cyclic in L also implies their conjugate momenta are conserved and may be set to 0 in the CMP. Setting the conjugate momentum of $\overline{\theta}^A$ to 0 gives

$$0 = \frac{\partial L_{(0)}}{\partial\dot{\overline{\theta}}^A} = \mathrm{i}Y_A - \mathrm{i}M_{AB}\dot{\theta}^B, \tag{4.7.4}$$

so $\dot{\theta}^A = \left(M^{-1}\right)^{AB}Y_B$. Therefore $L_{(0)}$ has numerical value

$$L_{(0)} = \mathrm{i}\overline{X}_A\dot{\theta}^A = \mathrm{i}\overline{X}_A\left(M^{-1}\right)^{AB}Y_B. \tag{4.7.5}$$

[60] Note that b^A, $\overline{b}^A$, q^i need not be cyclic in L, since the equivalence relation $\sim$ is consistent with differences proportional to undifferentiated modes. However, Eq. (4.7.1) is sufficient to determine momenta.

Adding $L_{(+)}$ gives an effective zero-mode sector Lagrangian in the CMP,

$$L_{\text{eff}} = \mathrm{i}\overline{X}_A \left(M^{-1}\right)^{AB} Y_B + L_{(+)}. \tag{4.7.6}$$

4.7.2 An overview of Lee–Yang terms

The motivation of the entirety of Sec. 4.7 is to show that the FMP obtains the same result as Eq. (4.7.6). It can be shown [87] by the path integral method that, if the FMP's infrared regularisation method is used, Lagrangians of the form in Eq. (4.7.5) warrant an additional *Lee–Yang term* of value[61]

$$L_{\text{LY}} := \mathrm{i}\delta\left(0\right) \ln\det M_{AB} = \mathrm{i}\delta\left(0\right) \operatorname{tr}\ln M_{AB}. \tag{4.7.7}$$

The proof is as follows. Let b^A, $\overline{b}^A$ have respective momenta π_A, $\overline{\pi}_A$ in

$$L_{(0)} = \dot{\overline{b}}^A \left(\mathrm{i}Y_A - \mathrm{i}M_{AB}\dot{b}^B\right) + \mathrm{i}\overline{X}_A\dot{b}^A. \tag{4.7.8}$$

Define

$$\Pi_A := \pi_A + \mathrm{i}\overline{X}_A,\ \overline{\Pi}_A := \overline{\pi}_A - \mathrm{i}Y_A \tag{4.7.9}$$

so

$$\pi_A = \mathrm{i}M_{BA}\dot{\overline{b}}^B - \mathrm{i}\overline{X}_A, \tag{4.7.10}$$

$$\overline{\pi}_A = \mathrm{i}Y_A - \mathrm{i}M_{AB}\dot{b}^B, \tag{4.7.11}$$

$$\begin{aligned} L_{(0)} &= \delta^B_C\dot{\overline{b}}^C\overline{\pi}_B + \mathrm{i}\overline{X}_A\delta^A_C\dot{b}^C \\ &= \left(M^{-1}\right)^{AB} M_{CA}\dot{\overline{b}}^C\overline{\pi}_B + \mathrm{i}\overline{X}_A\left(M^{-1}\right)^{AB} M_{BC}\dot{b}^C && (4.7.12)\\ &= \left(M^{-1}\right)^{AB}\left(\left(\overline{X}_A - \mathrm{i}\pi_A\right)\overline{\pi}_B + \mathrm{i}\overline{X}_A\left(\mathrm{i}\overline{\pi}_B + Y_B\right)\right) \\ &= -\mathrm{i}\left(M^{-1}\right)^{AB}\left(\Pi_A\left(\overline{\Pi}_B - \mathrm{i}Y_B\right) - \mathrm{i}\overline{X}_A\overline{\Pi}_B\right) \\ &= -\mathrm{i}\left(M^{-1}\right)^{AB}\left(\Pi_A\overline{\Pi}_B + \mathrm{i}\left(Y_B\Pi_A - \overline{X}_A\overline{\Pi}_B\right)\right). && (4.7.13) \end{aligned}$$

The path integral formalism obtains the transition amplitude (say $\mathcal{A}$), up to a multiplicative factor that is important because it is independent of dynamical variables. First we discretise a finite time period. This gives the approximate proportionality relation

$$\begin{aligned} \mathcal{A} \propto \int \prod_{tAB}\prod_{k=0}^{m-1} \mathrm{d}b^A\left(t\right)\mathrm{d}\overline{b}^B\left(t\right)\mathrm{d}\pi_A\left(t\right)\mathrm{d}\overline{\pi}_B\left(t\right)\times \\ \exp\left[\Pi_C\left(t_k\right)\left(M^{-1}\left(t_k\right)\right)^{CD}\overline{\Pi}_D\left(t_k\right)\right]\exp\left(\mathrm{i}\int_{t_i}^{t_f} L_{(+)}\mathrm{d}t\right), \end{aligned} \tag{4.7.14}$$

where:

- t_i, t_f are initial and final times, so $t_i < t_f$;
- $t_k := t_i + k\Delta$, $k \in \{0, \cdots, m-1\}$ with $\Delta := \frac{t_f - t_i}{m} > 0$ for some large $m \in \mathbb{N}$.

(The $\left(M^{-1}\right)^{AB}\left(Y_B\Pi_A - \overline{X}_A\overline{\Pi}_B\right)$ term makes no contribution to the result, because of the identities $\int \mathrm{d}\theta\mathrm{d}\eta\theta = 0 \int \mathrm{d}\theta\mathrm{d}\eta\eta = 0$ for Grassmann variables θ, η.) The first exponential factor could in principle

[61] Replacing M with $-M$ or $|M|$ may add a multiple of $\delta\left(0\right)$ to L, but this addition is non-dynamical so can be neglected. We therefore need not insist, for example, on writing $\ln\left|\det M_{AB}\right|$ instead of $\ln\det M_{AB}$.

have instead been written with a $\pi_C \left(M^{-1}\right)^{CD} \overline{\pi}_D$ term, which can be removed in the CMP because π_A, $\overline{\pi}_A$ are the conserved momenta of cyclic modes. However, the FMP does not make use of this tactic, which is what ultimately yields the term L_{LY}.

Since Eq. (4.7.9) shifts momenta by non-momentum Grassmann numbers,

$$\int \mathrm{d}\Pi_A \overline{\Pi}_B = \int \mathrm{d}\pi_A \overline{\pi}_B \tag{4.7.15}$$

(this is an equation in integral operators). A discrete analogue is as follows: for a bosonic field B,

$$\theta_0^{'} := \theta_0 + a\theta_1 \implies \int \mathrm{d}\theta_0^{'} \mathrm{d}\theta_1 B\theta_1\theta_0 = \int \mathrm{d}\theta_0^{'} \mathrm{d}\theta_1 B\theta_1\theta_0^{'} = \int \mathrm{d}\theta_0 \mathrm{d}\theta_1 B\theta_1\theta_0. \tag{4.7.16}$$

Thus

$$\begin{aligned} \mathcal{A} \propto \int \prod_{tAB} \prod_{k=0}^{m-1} \mathrm{d}b^A(t)\, \mathrm{d}\overline{b}^B(t)\, \mathrm{d}\Pi_A(t)\, \mathrm{d}\overline{\Pi}_B(t) \times \\ \exp\left[\Pi_C(t_k) \left(M^{-1}(t_k)\right)^{CD} \overline{\Pi}_D(t_k)\right] \exp\left(\mathrm{i}\int_{t_i}^{t_f} L_{(+)} \mathrm{d}t^{'}\right). \end{aligned} \tag{4.7.17}$$

Eq. (2.2.3) simplifies this result in terms of $\det M^{-1} = (\det M)^{-1}$. Writing this determinant as the exponential of its logarithm, we have

$$\mathcal{A} \propto \int \prod_{tAB} \mathrm{d}b^A(t)\, \mathrm{d}\overline{b}^B(t) \exp\left[-\frac{1}{\Delta}\sum_{k=0}^{m-1} \ln\det M_{AB}(t_k)\,\Delta\right] \exp\left(\mathrm{i}\int_{t_i}^{t_f} L_{(+)} \mathrm{d}t^{'}\right). \tag{4.7.18}$$

The seemingly superfluous $\Delta^{\pm 1}$ factors now play an important role, as we make time continuous again. For any set S of times define

$$1_S(t) := \begin{cases} 1 & t \in S \\ 0 & t \notin S \end{cases}. \tag{4.7.19}$$

Since the measure $\delta(t - t_0)$ may be approximated by the nascent delta function $\frac{1}{\Delta} 1_{\left[t_0 - \frac{\Delta}{2},\, t_0 + \frac{\Delta}{2}\right]}(t)$ where we take the $\Delta \to 0^+$right-hand limit, setting $t = 0$ identifies $\frac{1}{\Delta}$ with $\delta(0)$. Similarly, summing over terms with a Δ factor is equivalent to a time integration, since Δ can be thought of as $\mathrm{d}t$. The continuous-time analogue of Eq. (4.7.18) is therefore

$$\begin{aligned} \mathcal{A} &\propto \int \mathrm{d}t \int \prod_{AB} \mathrm{d}b^A(t)\, \mathrm{d}\overline{b}^B(t) \exp\left[-\delta(0) \int \ln\det M_{AB}(t)\, \mathrm{d}t\right] \exp\left(\mathrm{i}\int L_{(+)} \mathrm{d}t^{'}\right) \\ &= \int \mathrm{d}t \int \prod_{AB} \mathrm{d}b^A(t)\, \mathrm{d}\overline{b}^B(t) \exp\left(\mathrm{i}\int \left(L_{(+)} + L_{\text{LY}}\right) \mathrm{d}t^{'}\right), \end{aligned} \tag{4.7.20}$$

where the time limits have been taken as $t_i = -\infty$, $t_f = \infty$. Note the addition of L_{LY} to L, verifying Eq. (4.7.7).

Although the appearance of this term $\propto \delta(0)$ seems troubling, it simply cancels with the loop Feynman diagrams' contribution to the Lagrangian. Once this cancellation is verified, the chain Feynman diagrams recover the effective zero-mode sector Lagrangian obtained from the CMP. The truth of the two previous sentences implies the CMP's perturbative equivalence to the FMP. They are proven in Sec. 4.7.5, the final subsection of Sec. 4.7's discussion of perturbation theory.

The determinant $\det M_{AB}$ is known as a *Lee–Yang determinant*. In Sec. 4.7.3, I compute the Lee–Yang determinant of the Lagrangian in Eq. (4.7.1) in terms of the matrices K, L, M, N. In Sec. 4.7.4, I show that this Lee–Yang determinant is expressible as the product of two Lee–Yang determinants. The first of these is the Lee–Yang determinant of Eq. (4.7.5). The second is the Lee–Yang determinant of an effective zero-mode sector Lagrangian obtained by setting the momenta of cyclic modes to zero. This effective zero-mode sector Lagrangian contains no zero modes, and appears in the Lagrangian of both the CMP and FMP. It is therefore irrelevant to the question of whether the prescriptions are perturbatively equivalent. The fact that Lee–Yang determinants multiply in the way shown in Sec. 4.7.4 implies that the determinants' logarithms add, as do the resulting Lee–Yang terms. Two Lee–Yang terms are thereby decoupled, and only one of these terms needs to be considered to verify that the CMP and FMP are perturbatively equivalent. In Sec. 4.7.5, I analyse the relevant Lee–Yang term to verify this perturbative equivalence.

4.7.3 A Lee–Yang determinant

In Eq. (4.7.1), the conjugate momenta are

$$P_A := \frac{\partial L}{\partial \dot{\bar{b}}^A} \sim -\mathrm{i}M_{AB}\dot{b}^B - \mathrm{i}L_{Ai}\dot{q}^i, \tag{4.7.21}$$

$$\overline{P}_A := \frac{\partial L}{\partial \dot{b}^A} \sim \mathrm{i}\dot{\bar{b}}^B M_{BA} + \mathrm{i}\overline{K}_{iA}\dot{q}^i, \tag{4.7.22}$$

$$p_i := \frac{\partial L}{\partial \dot{q}^i} \sim -\mathrm{i}\overline{K}_{iA}\dot{b}^A - \mathrm{i}\dot{\bar{b}}^A L_{Ai} + N_{ij}\dot{q}^j. \tag{4.7.23}$$

so the Hamiltonian H obtained by Legendre-transforming L satisfies

$$\begin{aligned} H + L &= \dot{\bar{b}}^A P_A - \overline{P}_A\dot{b}^A + \dot{q}^i p_i \\ &\sim -2\mathrm{i}\dot{\bar{b}}^A M_{AB}\dot{b}^B - 2\mathrm{i}\dot{\bar{b}}^A L_{Ai}\dot{q}^i - 2\mathrm{i}\overline{K}_{iA}\dot{q}^i\dot{b}^A + \dot{q}^i N_{ij}\dot{q}^j \\ &\sim 2L, \end{aligned} \tag{4.7.24}$$

$$H \sim L. \tag{4.7.25}$$

Inverting relations between canonical fields and momenta gives

$$\dot{b}^A \sim \left(M^{-1}\right)^{AB}\left(\mathrm{i}P_B - L_{Bi}\dot{q}^i\right), \tag{4.7.26}$$

$$\dot{\bar{b}}^A \sim \left(-\mathrm{i}\overline{P}_B - \overline{K}_{iB}\dot{q}^i\right)\left(M^{-1}\right)^{BA}, \tag{4.7.27}$$

$$\begin{aligned} p_i &\sim \overline{K}_{iA}\left(M^{-1}\right)^{AB}\left(P_B + \mathrm{i}L_{Bj}\dot{q}^j\right) \\ &\quad - \left(\overline{P}_A - \mathrm{i}\overline{K}_{jA}\dot{q}^j\right)\left(M^{-1}\right)^{AB}L_{Bi} + N_{ij}\dot{q}^j \\ &= \overline{K}_{iA}\left(M^{-1}\right)^{AB}P_B - \overline{P}_A\left(M^{-1}\right)^{AB}L_{Bi} + N_{ij}\dot{q}^j, \end{aligned} \tag{4.7.28}$$

$$\begin{aligned} N_{ij}\dot{q}^j &\sim p_i - \overline{K}_{iA}\left(M^{-1}\right)^{AB}P_B + \overline{P}_A\left(M^{-1}\right)^{AB}L_{Bi} \\ &=: \widetilde{p}_i, \end{aligned} \tag{4.7.29}$$

$$\dot{q}^i \sim \left(N^{-1}\right)^{ij}\widetilde{p}_j. \tag{4.7.30}$$

Thus

$$\begin{aligned}
\dot{\overline{b}}^A P_A - \overline{P}_A \dot{b}^A + \dot{q}^i p_i &\sim \left(-\mathrm{i}\overline{P}_B - \overline{K}_{iB}\left(N^{-1}\right)^{ij}\widetilde{p}_j\right)\left(M^{-1}\right)^{BA} P_A \\
&\quad - \overline{P}_A\left(M^{-1}\right)^{AB}\left(\mathrm{i}P_B - L_{Bi}\left(N^{-1}\right)^{ij}\widetilde{p}_j\right) \\
&\quad + \left(N^{-1}\right)^{ij}\widetilde{p}_j\left(\widetilde{p}_i + \overline{K}_{iA}\left(M^{-1}\right)^{AB}P_B - \overline{P}_A\left(M^{-1}\right)^{AB}L_{Bi}\right) \\
&\sim -2\mathrm{i}\overline{P}_A\left(M^{-1}\right)^{AB}P_B + \widetilde{p}_i\left(\hat{N}^{-1}_{\text{sym}}\right)^{ij}\widetilde{p}_j, && (4.7.31) \\
H,\, L &\sim \frac{\dot{\overline{b}}^A P_A - \overline{P}_A \dot{b}^A + \dot{q}^i p_i}{2} \\
&\sim -\mathrm{i}\overline{P}_A\left(M^{-1}\right)^{AB}P_B + \widetilde{p}_i\left(N^{-1}_{\text{sym}}\right)^{ij}\widetilde{p}_j, && (4.7.32)
\end{aligned}$$

where

$$\hat{N} := N + 2\mathrm{i}\overline{K}M^{-1}L,\; N^{-1}_{\text{sym}} := \frac{N^{-1} + \left(N^{-1}\right)^{\mathrm{T}}}{2}. \tag{4.7.33}$$

Eq. (4.7.32) implies the full theory's Lee–Yang determinant is $\frac{\det M^{-1}}{\sqrt{\det \hat{N}^{-1}_{\text{sym}}}}$, but in the subsection below I provide an alternative expression for this, viz. Eq. (4.7.37). The aim is to show that the full Lee–Yang determinant is the product of the Lee–Yang determinant of the zero-mode sector and the Lee–Yang determinant of the effective theory with its zero-mode contribution removed. (I obtain these two Lee–Yang determinants respectively in Eqs. (4.7.48) and (4.7.51).) If this is true – indeed, it is – we may separate out the Lee-Yang determinant for the zero-mode sector. However, proving this requires a block matrix decomposition of M_{AB}, because the required expressions for the Lee–Yang determinants are in terms of determinants of the blocks.

4.7.4 Lee–Yang determinants of the zero-mode sector and effective theory

One may split the b^A into cyclic θ^A and non-cyclic c^I, and similarly with the $\overline{b}^A$. Note that there are now two types of upper case Roman indices, those beginning at A and those beginning at I. The matrix M_{AB} now admits a natural block matrix decomposition of the form

$$M = \begin{pmatrix} A_{IJ} & B_{IB} \\ C_{AJ} & D_{AB} \end{pmatrix}. \tag{4.7.34}$$

Hereafter $\mathbb{O}$s denote appropriate not-necessarily-square zero matrices, and I define

$$\widetilde{A}_{IJ} := A_{IJ} - A_{IA}D^{-1}_{AB}C_{BJ} \tag{4.7.35}$$

so that

$$M = \begin{pmatrix} \left(\widetilde{A} + BD^{-1}C\right)_{JK} & B_{JC} \\ C_{BK} & D_{BC} \end{pmatrix}. \tag{4.7.36}$$

The strategy is to first show $\det M = \det \widetilde{A} \det D$ so that

$$\frac{\det M^{-1}}{\sqrt{\det \hat{N}^{-1}_{\text{sym}}}} = \frac{1}{\det D \det \widetilde{A}\sqrt{\det \hat{N}^{-1}_{\text{sym}}}}, \tag{4.7.37}$$

and then show that $\frac{1}{\det D}$ is the Lee–Yang determinant of the zero mode sector while $\frac{1}{\det \widetilde{A}\sqrt{\det \hat{N}^{-1}_{\text{sym}}}}$ is the Lee–Yang determinant of the effective theory without the zero-mode sector.

Since adding a multiple of one row (column) to another row (column) preserves the determinant of a square matrix,

$$\det M = \det \begin{pmatrix} A_{IJ} - B_{IA} D^{-1}_{AB} C_{BJ} & \mathbb{O}_{IB} \\ \mathbb{O}_{AJ} & D_{AB} \end{pmatrix} = \det \begin{pmatrix} \widetilde{A}_{IJ} & \mathbb{O}_{IB} \\ \mathbb{O}_{AJ} & D_{AB} \end{pmatrix} = \det \widetilde{A} \det D. \tag{4.7.38}$$

By inspection the inverse of M is

$$M^{-1} = \begin{pmatrix} \widetilde{A}^{-1}_{IJ} & -\left(\widetilde{A}^{-1} B D^{-1}\right)_{IB} \\ -\left(D^{-1} C \widetilde{A}^{-1}\right)_{AJ} & \left(D^{-1}\left(\mathbb{I} + C\widetilde{A}^{-1} B D^{-1}\right)\right)_{AB} \end{pmatrix}, \tag{4.7.39}$$

since this claim implies

$$M^{-1} M = \begin{pmatrix} \left(\widetilde{A}^{-1}\widetilde{A} + E - E\right)_{IK} & \left(F - F D^{-1} D\right)_{IC} \\ \left(-G + G\right)_{AK} & \left(-J + D^{-1} D + J D^{-1} D\right)_{AC} \end{pmatrix} = \begin{pmatrix} \delta_{IK} & \mathbb{O}_{IC} \\ \mathbb{O}_{AK} & \delta_{AC} \end{pmatrix} \tag{4.7.40}$$

as required, where the matrices

$$E := \widetilde{A}^{-1} B D^{-1} C, \tag{4.7.41}$$

$$F := \widetilde{A}^{-1} B, \tag{4.7.42}$$

$$G := D^{-1} C \widetilde{A}^{-1}\left(\widetilde{A} + B D^{-1} C\right) = \left(D^{-1} + D^{-1} C \widetilde{A}^{-1} B D^{-1}\right) C, \tag{4.7.43}$$

$$J := D^{-1} C \widetilde{A}^{-1} B \tag{4.7.44}$$

have been defined for succinctness.

I have split the b^A into cyclic θ^A and non-cyclic c^I, and similarly with the $\overline{b}^A$. This also splits certain terms in the Lagrangian, viz.

$$\begin{aligned} L \sim & -\mathrm{i}\dot{\overline{\theta}}^A A_{AB}\dot{\theta}^B - \mathrm{i}\dot{\overline{\theta}}^A C_{AJ}\dot{c}^J - \mathrm{i}\dot{\overline{c}}^I B_{IB}\dot{\theta}^B - \mathrm{i}\dot{\overline{c}}^I D_{IJ}\dot{c}^J \\ & - \mathrm{i}\overline{K}_{iA}\dot{q}^i\dot{\theta}^A - \mathrm{i}\overline{K}_{iI}\dot{q}^i\dot{c}^I - \mathrm{i}\dot{\overline{\theta}}^A L_{Ai}\dot{q}^i - \mathrm{i}\dot{\overline{c}}^I L_{Ii}\dot{q}^i + \frac{1}{2}\dot{q}^i N_{ij}\dot{q}^j, \end{aligned} \tag{4.7.45}$$

and the $\dot{\theta}^A$ may be eliminated by setting

$$0 = \frac{\partial L}{\partial \dot{\theta}^B} = \mathrm{i}\left(\dot{\overline{\theta}}^A A_{AB} + \dot{\overline{c}}^I B_{IB} + \overline{K}_{iA}\dot{q}^i\right) \tag{4.7.46}$$

so

$$L \sim -\mathrm{i}\dot{\overline{\theta}}^A C_{AJ}\dot{c}^J - \mathrm{i}\dot{\overline{c}}^I D_{IJ}\dot{c}^J - \mathrm{i}\overline{K}_{iI}\dot{q}^i\dot{c}^I - \mathrm{i}\dot{\overline{\theta}}^A L_{Ai}\dot{q}^i - \mathrm{i}\dot{\overline{c}}^I L_{Ii}\dot{q}^i + \frac{1}{2}\dot{q}^i N_{ij}\dot{q}^j. \tag{4.7.47}$$

Due to the $-\mathrm{i}\dot{\overline{c}}^I D_{IJ}\dot{c}^J$ term, the associated Lee–Yang determinant is

$$\frac{1}{\det D}. \tag{4.7.48}$$

Other conserved quantities are also set to 0 in the CMP, viz.

$$0 = \frac{\partial L}{\partial \dot{\overline{\theta}}^A} \sim -\mathrm{i}\left(D_{AB}\dot{\theta}^B + \dot{\overline{c}}^I B_{IA} + \overline{K}_{IA}\dot{q}^i\right). \tag{4.7.49}$$

Eqs. (4.7.46) and (4.7.49) respectively give expressions for $\dot{\overline{\theta}}^A$ and $\dot{\theta}^A$, so that

$$L \sim \frac{1}{2}\dot{q}^i\hat{N}_{ij}\dot{q}^j - \mathrm{i}\left\{\dot{\overline{c}}^I\left(\widetilde{A}_{IJ}\dot{c}^J + \left(L - BD^{-1}L\right)_{Ij}\dot{q}^j\right) + \left(\overline{K} - \overline{K}D^{-1}C\right)_{iI}\dot{q}^i\dot{c}^I\right\}. \tag{4.7.50}$$

Due to the $\frac{1}{2}\dot{q}^i\hat{N}_{ij}\dot{q}^j$ and $-\mathrm{i}\dot{\overline{c}}^I\widetilde{A}_{IJ}\dot{c}^J$ terms, the associated Lee–Yang determinant is

$$\frac{1}{\det\widetilde{A}\sqrt{\det\widetilde{N}^{-1}_{\text{sym}}}}, \tag{4.7.51}$$

where

$$\widetilde{N} := N + 2\mathrm{i}\left(\overline{K}D^{-1}L + \left(\overline{K} - \overline{K}D^{-1}C\right)\widetilde{A}^{-1}\left(L - BD^{-1}L\right)\right), \tag{4.7.52}$$

$$\widetilde{N}^{-1}_{\text{sym}} := \frac{1}{2}\left(\widetilde{N}^{-1} + \left(\widetilde{N}^{-1}\right)^{\mathrm{T}}\right). \tag{4.7.53}$$

Lowering all indices,

$$\begin{aligned}
\frac{\widetilde{N}_{ij} - N_{ij}}{2\mathrm{i}} &= \overline{K}_{iA}D^{-1}_{AB}L_{Bj} + \left(\overline{K}_{iI} - \overline{K}_{iA}D^{-1}_{AB}C_{BI}\right)\widetilde{A}^{-1}_{IJ}\left(L_{Jj} - B_{JA}D^{-1}_{AB}L_{Bj}\right) \\
&= \overline{K}_{iI}\widetilde{A}^{-1}_{IJ}L_{Jj} - \overline{K}_{iA}D^{-1}_{AB}C_{BI}\widetilde{A}^{-1}_{IJ}L_{Jj} - \overline{K}_{iI}\widetilde{A}^{-1}_{IJ}L_{Jj} + D^{-1}_{AB} \\
&+ \overline{K}_{iA}\left(D^{-1}_{AC}C_{CI}\widetilde{A}_{IJ}B_{JD}D^{-1}_{DB}\right)L_{Bj} \\
&= \overline{K}_{iA}D^{-1}_{AB}L_{Bj} = \frac{\hat{N}_{ij} - N_{ij}}{2\mathrm{i}}, &(4.7.54) \\
\widetilde{N}_{ij} &= \hat{N}_{ij}, &(4.7.55)
\end{aligned}$$

in accordance with Eq. (4.7.33).
The product of the Lee–Yang determinants in Eqs. (4.7.48) and (4.7.51) is therefore the Lee–Yang determinant in Eq. (4.7.37), as required.

4.7.5 Perturbative equivalence

This subsection verifies that the FMP's chain Feynman diagrams reproduce Eq. (4.7.6), a contribution to the Lagrangian that is also present in the CMP. It verifies also that the FMP's loop Feynman diagrams cancel the FMP's Lee–Yang term, which has no CMP counterpart.
We may write $M_{AB} = \hat{M}_{AB} + H_{AB}$, so H_{AB} contains the contributions to M_{AB} attributable to the interaction $\kappa h_{\mu\nu}$. Similarly, we may write $\overline{X}_A = \hat{\overline{X}}_A + \widetilde{\overline{X}}_A$, $Y_A = \hat{Y}_A + \widetilde{Y}_A$, where the hats denote the non-interacting value and the tildes denote contributions due to the interaction.

The first task is to verify that the FMP's chain Feynman diagrams recover Eq. (4.7.6). Recall that

$$\dot{b}^A \sim \alpha^A - \left(M^{-1}\right)^{AB}L_{Bi}\dot{q}^i,\ \dot{\overline{b}}^A \sim \overline{\alpha}^A - \dot{q}^i\overline{K}_{iB}\left(M^{-1}\right)^{BA}, \tag{4.7.56}$$

where $\alpha^A := \mathrm{i}\left(M^{-1}\right)^{AB} P_B$, $\overline{\alpha}^A := -\mathrm{i}\overline{P}_B\left(M^{-1}\right)^{BA}$. We then have the non-perturbative $\alpha\overline{\alpha}$ propagator $\left\langle \alpha^A(t)\,\overline{\alpha}^B\left(t'\right)\right\rangle = -\delta\left(t-t'\right)\left(\hat{M}^{-1}\right)^{AB}(t)$. The right-hand side is a matrix, which perturbatively generalises to a power series

$$-\left(\hat{M}^{-1}(t)\sum_{k=0}^{\infty}\left\{\prod_{j=1}^{k}\int \mathrm{d}t_j\delta\left(t_{j-1}-t_j\right)\left(-H\hat{M}^{-1}\right)\left(t_j\right)\right\}\delta\left(t_k-t'\right)\right)^{AB} \tag{4.7.57}$$

where $t_0 := t$. Including ^{AB} indices, the full $\alpha\overline{\alpha}$ propagator is

$$\begin{aligned}-\left(\hat{M}^{-1}(t)\left\{\sum_{k=0}^{\infty}\left(-H\hat{M}^{-1}\right)^{k}(t)\right\}\right)^{AB}\left(\delta\left(t-t'\right)\right) &= \delta\left(t-t'\right)\left(-\hat{M}^{-1}\left(\mathbb{I}+H\hat{M}^{-1}\right)\right)^{AB}(t)\\ &= -\delta\left(t-t'\right)\left(\left(\hat{M}+H\right)^{-1}\right)^{AB}(t)\\ &= -\delta\left(t-t'\right)\left(M^{-1}\right)^{AB}(t).\end{aligned} \tag{4.7.58}$$

These expressions for $\dot{b}^A$, $\dot{\overline{b}}^A$ may be substituted into the Lagrangian, which may be decomposed as $L = L_0 + L_I$ where L_I is an interaction Lagrangian, i.e. L_0 is independent of $\kappa h_{\mu\nu}$ and derivatives thereof. Explicitly

$$\begin{aligned}L_I \sim{} & -\mathrm{i}\hat{\overline{X}}_A\left(\hat{M}^{-1}H\hat{M}^{-1}\right)^{AB}\hat{Y}_B + \mathrm{i}\overline{X}_A\left(\hat{M}^{-1}\right)^{AB}\widetilde{\hat{Y}}_B + \mathrm{i}\widetilde{\overline{X}}_A\left(\hat{M}^{-1}\right)^{AB}\hat{Y}_B\\ & + \mathrm{i}\overline{\alpha}^A\left(\left(H\hat{M}^{-1}\right)_A^{\ B}\hat{Y}_B - \widetilde{Y}_A\right) + \mathrm{i}\left(\hat{\overline{X}}_A\left(\hat{M}^{-1}H\right)^A_{\ B} - \widetilde{\overline{X}}_B\right)\alpha^B - \mathrm{i}\overline{\alpha}^A H_{AB}\alpha^B.\end{aligned} \tag{4.7.59}$$

Several of the terms in Eq. (4.7.59) have counterparts elsewhere in the FMP's perturbation theory. For example, the perturbation theory obtains a further $\hat{\overline{X}}_A\hat{Y}_B$ term, using an $\alpha\overline{\alpha}$ propagator to connect $-\hat{\overline{X}}_A\left(\hat{M}^{-1}H\right)^A_{\ B}\alpha^B$ to $-\overline{\alpha}^A\left(H\hat{M}^{-1}\right)_A^{\ B}\hat{Y}_B$. Since the term in the Lagrangian is obtained by including a $-\mathrm{i}$ factor, the $\hat{\overline{X}}_A\hat{Y}_B$ coefficient due to interaction terms is the ^{AB} component of

$$-\mathrm{i}\hat{M}^{-1}H\hat{M}^{-1} + \hat{M}^{-1}H\mathrm{i}M^{-1}H\hat{M}^{-1} = \mathrm{i}\left(M^{-1}-\hat{M}^{-1}\right), \tag{4.7.60}$$

since

$$\begin{aligned}M^{-1} = M^{-1}\left(M\hat{M}^{-1} - H\hat{M}^{-1}\right) &= \hat{M}^{-1} - \left(\hat{M}+H\right)^{-1}H\hat{M}^{-1}\\ &= \hat{M}^{-1}\left(\mathbb{I} - \left(MH^{-1}-\mathbb{I}\right)HM^{-1}H\hat{M}^{-1}\right)\\ &= \hat{M}^{-1} - \hat{M}^{-1}H\hat{M}^{-1} + \hat{M}^{-1}HM^{-1}H\hat{M}^{-1}.\end{aligned} \tag{4.7.61}$$

However, L_0 also provides an $\mathrm{i}\hat{\overline{X}}_A\left(\hat{M}^{-1}\right)^{AB}\hat{Y}_B$ term, giving a total of $\mathrm{i}\hat{\overline{X}}_A\left(M^{-1}\right)^{AB}\hat{Y}_B$. Further terms are as follows, and contain no contributions from L_0:

- Connecting $\widetilde{\overline{X}}_A\alpha^A$ to $\overline{\alpha}^A\widetilde{Y}_A$ gives $\mathrm{i}\widetilde{\overline{X}}_AM^{-1}\widetilde{Y}_B$;
- Connecting $-\hat{\overline{X}}_A\left(\hat{M}^{-1}H\right)^A_{\ B}\alpha^B$ to $\overline{\alpha}^A\widetilde{Y}_A$ and also including the $\mathrm{i}\hat{\overline{X}}_A\left(\hat{M}^{-1}\right)^{AB}\widetilde{Y}_B$ term gives

$$\mathrm{i}\hat{\overline{X}}_A\left(-\hat{M}^{-1}HM^{-1}+\hat{M}^{-1}\right)^{AB}\widetilde{Y}_B = \mathrm{i}\hat{\overline{X}}_A\left(M^{-1}\right)^{AB}\widetilde{Y}_B \tag{4.7.62}$$

since

$$\begin{aligned}-\hat{M}^{-1}HM^{-1}+\hat{M}^{-1}-M^{-1}&=\hat{M}^{-1}\left(\mathbb{I}-HM^{-1}\right)-M^{-1}\\&=\hat{M}^{-1}\left(M-H\right)M^{-1}-M^{-1}\\&=\hat{M}^{-1}\hat{M}M^{-1}-M^{-1}=0;\end{aligned}\tag{4.7.63}$$

- Similarly, an $\mathrm{i}\widetilde{\overline{X}}_A\left(M^{-1}\right)^{AB}\hat{Y}_B$ term is obtained.

Summing gives $\mathrm{i}\overline{X}_A\left(M^{-1}\right)^{AB}Y_B$, as required.

Next I show the loop Feynman diagrams cancel L_{LY}. The loop contribution is

$$\int \mathrm{d}t\mathrm{i}L_{\mathrm{loop}}\left(t\right)=-\operatorname{tr}\sum_{N\geq 1}\frac{1}{N}\int\left(\prod_{k=1}^{N}\mathrm{d}t_k\right)\left(\prod_{k=1}^{N}\left(\hat{M}^{-1}H\right)_{A_k}^{A_{k+1}}\left(t_k\right)\delta\left(t_k-t_{k+1}\right)\right),\tag{4.7.64}$$

where $t_1<\cdots<t_N$ are N successive values for a time coordinate, and $t_{N+1}:=t_1$. Note the N^{-1} symmetry factors and a -1 factor due to the loops being fermionic. Thus

$$\begin{aligned}L_{\mathrm{loop}}&=\mathrm{i}\delta\left(0\right)\sum_{N\geq 1}\frac{1}{N}\prod_{k=1}^{N}\left(\hat{M}^{-1}H\right)_{A_k}^{A_{k+1}}\left(t_k\right)=\mathrm{i}\delta\left(0\right)\sum_{N\geq 1}\frac{1}{N}\left(\left(\hat{M}^{-1}H\right)^{N}\right)_{A}^{A}\\&=-\mathrm{i}\delta\left(0\right)\ln\left(\mathbb{I}-\hat{M}^{-1}H\right)_{A}^{A}=-\mathrm{i}\delta\left(0\right)\operatorname{tr}\ln\left(\hat{M}^{-1}\left(\hat{M}-H\right)\right)_{AB}\\&=-\mathrm{i}\delta\left(0\right)\ln\det\left(\hat{M}^{-1}M\right)_{AB}=-\mathrm{i}\delta\left(0\right)\ln\det M_{AB}+\mathrm{i}\delta\left(0\right)\ln\det\hat{M}_{AB}\\&\equiv -L_{\mathrm{LY}}+\mathrm{i}\delta\left(0\right)\ln\det\hat{M}_{AB}.\end{aligned}\tag{4.7.65}$$

But $-\mathrm{i}\delta\left(0\right)\ln\det\hat{M}_{AB}$ contains no dynamical variables, so may be freely added to the Lagrangian. Doing so cancels $L_{\mathrm{loop}}+L_{\mathrm{LY}}$, as required.

Part III Conclusions and back matter

Chapter 5 Conclusions

5.1 Overview of Part III

In this chapter, I summarise the work of this thesis: its problem context, its findings and the outlook for future research. In Sec. 5.2, I discuss the problem context that motivated my research with Atsushi Higuchi. This requires me to discuss the findings of the CMP in some detail as well, in a grey note. The fictitious mass prescription (FMP) is an infrared regularisation of zero mode problems introduced in Ref. [2]. Difficulties with the FMP motivated an alternative treatment of zero mode problems. This is the cyclic modes prescription (CMP), described in this thesis and by myself and Dr Higuchi in Ref. [1].

In Sec. 5.3, I provide several examples of issues that are not addressed, or not fully addressed, in my and Dr. Higuchi's research. I recommend that such issues be investigated in future research.

This chapter begins Part III of this thesis. The rest of Part III is structured as follows. After this chapter, I present my Appendices. I have stated their purposes in earlier chapters, but here is a summary:

- Appendix A defends Klein–Gordon orthonormality in more detail;
- Appendix B addresses how the CMP can amend the pseudo-inner product of the theory;
- Appendix C verifies that the naïve Poisson brackets of Yang–Mills theory and perturbative gravity may be used;
- Appendix D shows that the (anti)commutators of Noether charges provide no new conserved charges for Yang–Mills theory or perturbative gravity;
- Appendix E shows that the spacetime-isometry Noether charges of perturbative gravity are obtained by BRST-transforming other Noether charges, and also by anti-BRST-transforming other Noether charges in the anti-BRST-invariant case (the de Donder gauge);
- Appendix F discusses the Batalin–Vilkovisky formalism in sufficient detail to defend my avoiding its use in Chapter 4;
- Appendix G verifies the matrix $\hat{M}_{AB}$ introduced in Chapter 4 is invertible on the flat static torus and in de Sitter space, if appropriate choices are made (e.g. $k < 1$ for the flat static torus);
- Appendix H provides further details of the perturbation theory of the CMP and FMP for perturbative gravity on a flat static torus.

Note that:

- Appendix A pertains to Chapter 1;
- no appendices pertain to Chapter 2;
- Appendix B pertains to Chapter 3;

- Appendices C and D pertain to Chapters 3 and 4;
- Appendices E-H pertain to Chapter 4.

I then provide *Definitions* and a *Glossary*, respectively beginning on pages 164 and 165. These resources differ in that, while *Definitions* defines terms I have introduced (or defined unusually) in this thesis, *Glossary* summarises the definitions of more standard terms. For example, I define the CMP in *Definitions*, whereas I define *scale factor* in the *Glossary*.

I conclude with my Bibliography, beginning on page 168.

5.2 Why this work was necessary

Two-point functions of interest in theoretical physics include the propagator of a scalar field, the FP-ghost propagator and the graviton two-point function. The infrared behaviours of these functions are determined by the normalisations of quantised fields' modes, which are classical solutions of field equations. No normalisation obtains two-point functions that simultaneously converge in the infrared limit and preserve desirable spacetime symmetries, such as time translation invariance on a flat static torus and de Sitter invariance in de Sitter space. One important normalisation convention is the Klein–Gordon normalisation of scalar modes, which is motivated by its rendering canonical (anti)commutation relations equivalent to the (anti)commutators of annihilation and creation operators. This convention, which is also applicable to the fields β^A, $\widetilde{\beta}^A$, θ^A, $\overline{\theta}^A$ introduced in Chapter 4, yields two-point functions that preserve appropriate spacetime symmetries but diverge in the infrared limit. These problematic behaviours of two-point functions are attributable to a limited subset of the modes of the quantised fields, and these have been called *zero modes*. The Faddeev–Popov method requires massless FP-(anti)ghost fields, so the FP-ghost propagator's problematic infrared behaviours in Yang–Mills theory and perturbative gravity cannot be remedied by postulating that FP-(anti)ghosts are massive.

In 2008, Mir Faizal and Atsushi Higuchi proposed a method for the infrared regularisation of these FP-ghost propagators [2]. This method assumes the FP-ghost and FP-antighost share a common positive mass, say M, and imposes Klein–Gordon normalisation to obtain M-dependent FP-ghost propagators in Yang–Mills theory and perturbative gravity. In each case, the result is expressed as the sum of three terms. Two of these terms respect the sought spacetime symmetries; the other term vanishes when $M = 0$. Of the two terms that respect spacetime symmetries, only one is infrared-convergent. The other term is spacetime-constant in Yang–Mills theory, and is a spacetime-constant multiple of the product of two Killing vector fields in perturbative gravity. In each case, the absence of this infrared-divergent term would not damage the FP-ghost propagator's spacetime symmetries. The method concludes by deleting the infrared-divergent term before taking the $M \to 0^+$ right-hand limit, which also results in the infrared-convergent term's deletion. Thus only one term survives, an M-independent (and hence infrared-finite) term that respects the intended spacetime symmetries. This is an effective zero-mode sector propagator. I have called this method the *fictitious mass prescription* (FMP), and explicitly discussed scalar field the theory's analogues of the three aforementioned terms in the FP-ghost propagator in Secs. 1.5.3, 1.6.2 and 1.7.1.

The fictitious masses temporarily granted to the FP-(anti)ghosts in the FMP requires a mass term in the Lagrangian. For $M \neq 0$, this term violates internal symmetries. I have discussed two important examples of such damaged symmetries, which are BRST invariance and (for Yang–Mills theory and the $k = \frac{1}{2}$ gauge choice in perturbative gravity) anti-BRST invariance. These symmetries are reinstated when $M = 0$, but they are not *manifestly* preserved before this condition is imposed to recover the usual massless BRST quantisation. If the FMP is used to obviate the effects of zero modes from two-point functions, are the (anti)-BRST internal symmetries preserved? Even if there is no doubt that they are, it would be desirable[62] to have a formalism that manifestly preserves these symmetries throughout. This fact motivated the development of such a formalism for the Lagrangian and Hamiltonian formulations of the field theory, and its formal Schrödinger wave functionals for physical states.

The result was the cyclic modes prescription (CMP), which aimed to obviate troublesome modes from the Lagrangian and Hamiltonian formalisms. It is always sufficient to obviate appropriate spatially uniform scalar modes; even in the gravity case, for which the FP-(anti)ghosts are vector fields, the "zero mode" of these fields is a linear combination of Killing vectors, and these Killing vectors' scalar coefficients are spatially uniform. The obviation of undifferentiated examples of these modes, i.e. their non-appearance in the formalism (so that these modes are *cyclic*), is equivalent to the conservation of the zero modes' conjugate momentum densities. In the Lagrangian (Hamiltonian) formalism, this follows from the Euler–Lagrange equations (Hamilton's equations). Before the CMP can be implemented, it must be shown that, in fact, the modes are cyclic.

Suppose these modes are indeed cyclic; they then do not appear undifferentiated. Similarly, the obviation of the modes' time derivatives is equivalent to the conserved momenta vanishing, and this may always be imposed from the aforementioned conservation laws. Space derivatives are in any case irrelevant, since these vanish for spatially uniform scalar fields. And the conjugate momenta have been set to 0, and therefore also no longer appear in an expression for the Hamiltonian in terms of phase space variables.

The Schrödinger picture implies that, when these conserved momenta vanish, physical states' formal Schrödinger wave functionals are independent of these scalar modes. Although this is also a desired outcome, an important technical issue now occurs. The classical field theory's naïve Poisson brackets can be promoted to the canonical (anti)commutation relations of the quantised field theory, provided the behaviour of the Dirac brackets is such as to allow this (which it is; see Appendix C). For Minkowski space, the resulting canonical (anti)commutation relations are true for the physical states of some Fock space, for an appropriate associated choice of pseudo-inner product on the Hilbert space.[63] All conserved charges set to 0 in the CMP then annihilate physical states, so it is fortunate that the set of them is closed under appropriate symmetries, and that any two of them commute or anticommute. However, the requirement in particular that some momenta be conserved implies that, in the aforementioned pseudo-inner product, physical states are zero-norm. This prevents a manifestly unitary formulation of the theory, *unless a different pseudo-inner product is used*.

An alternative choice of pseudo-inner product (see also Appendix B) must therefore be used, but it must also be motivated. Its motivation is found in the observation that formal Schrödinger wave

[62] There is at least one convenient analogy. The wave equations that follow from Maxwell's equations in free space are Lorentz-invariant, but it is worth asking whether Maxwell's equations are Lorentz-invariant for an arbitrary charge n-current. Even once it is known that this is the case, it is desirable to obtain a manifestly Lorentz-covariant formulation of classical electromagnetism. This makes use of tensors, viz. Sec. 2.1.1.

[63] The Fock space of one quantised field is the Hilbert space completion of the direct sum of the symmetric or antisymmetric tensors in the tensor powers of a single-particle Hilbert space, and the Fock space of multiple quantised fields is the tensor product of the fields' respective Fock spaces.

functionals are independent of the scalar modes that the CMP obviates from the Lagrangian and Hamiltonian formalisms. This implies that, in the case of such a scalar mode that is bosonic (fermionic), the integration over these modes in the functional integral representation of the usual pseudo-inner product on physical states causes the result to diverge (vanish). This observation can be thought of as an alternative characterisation of the aforementioned issue with the norm of physical states. However, the fact that the integrand is independent of these scalar modes motivates a modified definition of the functional integral, in which no integration over these scalar modes occurs. The usual canonical (anti)commutation relations for these scalar modes and their conjugate momentum densities are no longer true in this modified pseudo-inner product. Indeed, this modified pseudo-inner product has completely obviated the zero modes and their conjugate momentum densities from the Schrödinger wave functional formalism, just as they are obviated by conservation laws from the Hamiltonian and Lagrangian formalisms. Therefore, the potentially problematic canonical (anti)commutation relations cannot even be *stated*. Similarly, the mode decomposition of quantised fields, and the resulting computation of two-point functions, no longer features the scalar modes the CMP sought to obviate.

The CMP has one more advantage over the FMP. The FMP manually deletes a term from massive two-point functions before taking the infrared limit, thereby obtaining modified effective zero-mode sector propagators. For the FP-ghost sectors considered herein, the Hamiltonian and Lagrangian are then also modified in the FMP, by the addition to each of a new term (no such terms exist for the zero mode problem discussed in Chapter 1[64]). These terms in turn modify the formalism's perturbation theory. The FMP and CMP are each "true" in some sense. It is true that the terms the FMP deletes from two-point functions are irrelevant to spacetime symmetries. And it is true that, if the scalar modes responsible for an infrared problem which the CMP aims to address are indeed cyclic, conserved charges may be set to 0, thereby obviating these modes and their conjugate momenta from the theory. The prescriptions ought to be equivalent: specifically, in the sense of perturbation theory. When the FMP's perturbation theory is considered, free integration by parts is required, and not all of the spacetimes described in Sec. 1.2.2 allow this. The CMP, however, does not require this same use of free integration by parts. However, perturbative equivalence implies that the CMP also adds new terms to the Hamiltonian and Lagrangian, and this would not happen if all the scalar modes whose obviation is sought are cyclic in the usual formalism in terms of Nakanishi–Lautrup auxiliary field and FP-(anti)ghosts. Therefore, such obviation must require an alternative formalism. And indeed it does, which is why Dr Higuchi and I introduced the field $\widetilde{B}$ ($\widetilde{B}^\mu$) for Yang–Mills theory (perturbative gravity). Each time I treated an FP-ghost sector infrared problem in this thesis, I demonstrated the need for such a change in field variables before I provided details of the perturbation theories of the FMP and CMP. Indeed, Dr Higuchi and I realised the need for field variable change for Yang–Mills theory before comparing the perturbation theories of the prescriptions; and, when we subsequently considered the CMP for perturbative gravity, we defined

[64] The key difference is that, while the $\nabla_\mu\phi\nabla^\mu\phi$ term is analogous to $-\mathrm{i}\nabla_\mu\bar{c}\cdot\nabla^\mu c$, the theory discussed in Chapter 1 has no analogue of the interaction term $-\mathrm{i}q\nabla_\mu\bar{c}\cdot A^\mu\times c$. Indeed, if this term is deleted from Yang–Mills theory with the choice $q=0$, the new terms in $H,\ L$ vanish, as does $\widetilde{B}-B$. Note that all cross products contain a q factor, and a replacement of the form $q,\ \times\to\zeta q,\ \zeta^{-1}\times$ for $\zeta\in\mathbb{R}\backslash\{0\}$ would preserve such terms while redefining the structure constants. By contrast, setting $q=0$ deletes all terms containing cross products, which has the same effect as using an Abelian gauge group. Indeed, the Abelian case of electromagnetism does not obtain these "new terms" either. It is unsurprising, therefore, that general expressions for them in Yang–Mills theory are proportional to cross products, and therefore vanish in the Abelian case.

$\widetilde{B}^\mu$ before considering the perturbative implications. However, in hindsight the perturbation theory of the FMP, which has been understood since Ref. [2], can be seen by the above argument to imply the need for $\widetilde{B}$, $\widetilde{B}^\mu$.

The CMP has two consequences that, while not prohibitive, are such that researchers should bear them in mind. One is that $\widetilde{B}$, $\widetilde{B}^\mu$ are non-local. The other is specific to the case of perturbative gravity. It is a complication in zero-mode sector Feynman diagram analysis. I discussed this briefly at the end of Sec. 4.2.

5.3 Open questions

I will discuss two key respects in which our treatment with the CMP, of zero mode problems in globally hyperbolic compact spacetimes, is incomplete. I discuss each in a separate subsection below. There also remains the question of whether zero mode problems may be analogously addressed in other spacetimes of theoretical interest, such as anti de Sitter space.

5.3.1 Pseudo-inner product issues

I have noted that the calculation showing physical states are zero-norm in the CMP uses the usual pseudo-inner product that preserves the usual canonical (anti)commutation relations for quantised fields, and that the CMP uses a different pseudo-inner product. This does not prove the revised pseudo-norms of physical states are nonzero. In Appendix B, I introduce a notation for pseudo-inner products that distinguishes the usual pseudo-inner product from alternatives that would allow physical states that respect all desired spacetime and internal symmetries to have nonzero norms. This treatment only considers the case of Yang–Mills theory, and Dr Higuchi and I have not shown that the CMP's modified pseudo-inner product coincides with those which Appendix B shows are desirable. It is our expectation that it does, and that the case of perturbative gravity is analogous.

5.3.2 Theories beyond the Standard Model and General Relativity

The Standard Model and General Relativity are two theories which, between them, currently provide our best description of all known fundamental interactions in nature. The zero mode problems of scalar field theory, BRST-quantised Yang–Mills theory and BRST-quantised perturbative gravity are the zero mode problems that occur in these theories. However, theoretical physics has found reason to consider countless other theories that also have zero mode problems. Therefore, zero mode problems not considered in this thesis will surely be present in any comprehensive physics of the future.

What is more, the findings of this thesis do not immediately signal the details of the FMP or CMP for any such alternative theories. The details of FMP and CMP calculations in treatments 1, 3 and 4 are all idiosyncratically characteristic of the zero mode problem being considered therein. When these zero mode problems were considered case by case, many differences between them were identified. These include:

- the definition of zero modes;
- the terms that they introduce into infrared-divergent two-point functions;
- whether or not local internal symmetries worthy of preservation are also a consideration;

- how the Nakanishi–Lautrup auxiliary field should be shifted for the CMP;
- how the prescriptions' perturbation theories look; and
- which results of perturbation theory are required to demonstrate this.

The work on a zero mode problem discussed in Chapter 1 (Chapter 3) provides *insights* into the techniques that are useful when treating a zero mode problem in Chapter 3 (Chapter 4). However, the dissimilarities are sufficient to indicate that the work discussed in this thesis and Refs. [1] and [2] provides, at best, a roadmap for the treatment of other zero mode problems.

However, a number of things will clearly be true of such other zero mode problems. The FMP will modify any two-point function of concern. This *may* add new terms to the Hamiltonian and Lagrangian, and hence affect the perturbation theory. Suppose, for instance, that this is so. The CMP will, hopefully, be perturbatively equivalent to the FMP in a broad class of spacetimes. Suppose that this is also true. However, these assumptions together imply a change in the choice of field variables, analogous to $B,\, c,\, \overline{c} \to \widetilde{B},\, c,\, \overline{c}$, will be necessary to render all zero modes cyclic. This shows that a broad class of zero mode problems will require a result analogous to Eqs. (3.4.1) and (4.4.8). Indeed, it seems reasonable to suppose all the BRST quantisations of field theories will be examples of this, provided they feature FP-(anti)ghosts and an interaction term containing them. A further expectation is that the CMP's implications for the formulation of a manifestly unitary formalism will require analysis.

Each zero mode problem I have discussed in this thesis was derived from a specific choice of Lagrangian density of the form $\mathcal{L} = \sqrt{|g|}\mathcal{L}_0$, with $\mathcal{L}_0$ scalar-valued. I have spoken in this subsection of "other" zero mode problems, under the implicit assumption that these would occur in theories with different choices of $\mathcal{L}_0$. However, even a familiar choice of quantised fields, appearing in a familiar choice of $\mathcal{L}_0$, could have implications not explored in this thesis. I will close this chapter by discussing one subtle example. In Chapter 2, I showed that $k \neq \frac{1}{n}$ is necessary for $\gamma_{\mu\nu}^{\rho\sigma}$ to be invertible. In Chapter 4, I showed that $k = \frac{1}{2}$ is necessary for an anti-BRST invariant formulation of BRST-quantised perturbative gravity. These requirements for k are inconsistent if and only if $n = 2$. Empirically $n = 4$, and most discussion of the possible applicability of $n \neq 4$ spacetimes to our universe focus on $n > 4$ cases, such as superstring theories. However, $n = 2$ spacetimes have been of some interest [89, 90, 91], and it is dissatisfying if a proposed treatment of infrared problems cannot be made anti-BRST invariant in these cases. There are also efforts to describe Planck-scale spacetime in terms of $n = 2$ simplices (a good overview [92] is included in Ref. [93]). There are therefore at least some contexts in which the case $n = 2$ is of interest. For these, our treatment of BRST-quantised perturbative gravity's FP-ghost infrared problem does not provide anti-BRST invariance. Fortunately, no such invariance is required for the theory's consistency.

Appendices

A On Klein–Gordon normalisation

This appendix proves Klein–Gordon normalisation implies Eqs. (1.4.14) and (1.4.25) are equivalent to Eqs. (1.4.22) and (1.4.26). First I compute operator-valued Klein–Gordon inner products, viz.

$$\left\langle \phi_\sigma ,\, \hat{\phi} \right\rangle_{\text{KG}} = \mathrm{i} \int_{\mathbf{x}} \phi_\sigma^* \overleftrightarrow{\nabla^0} \sum_{\sigma'} \left(\phi_{\sigma'} \hat{a}_{\sigma'} + \phi_{\sigma'}^* \hat{a}_{\sigma'}^\dagger \right) = \sum_{\sigma'} \left\langle \phi_\sigma ,\, \phi_{\sigma'} \right\rangle_{\text{KG}} \hat{a}_{\sigma'} = \hat{a}_\sigma , \tag{A.1}$$

$$\left\langle \hat{\phi} ,\, \phi_\sigma \right\rangle_{\text{KG}} = \mathrm{i} \int_{\mathbf{x}} \sum_{\sigma'} \left(\phi_{\sigma'} \hat{a}_{\sigma'} + \phi_{\sigma'}^* \hat{a}_{\sigma'}^\dagger \right) \overleftrightarrow{\nabla^0} \phi_\sigma = \sum_{\sigma'} \left\langle \phi_{\sigma'} ,\, \phi_\sigma \right\rangle_{\text{KG}} \hat{a}_{\sigma'}^\dagger = \hat{a}_\sigma^\dagger , \tag{A.2}$$

where $X \overleftrightarrow{\nabla^0} Y := X \nabla^0 Y - \left(\nabla^0 X \right) Y$. Note that generalising the Klein–Gordon inner product to accept operator-valued arguments requires $\left\langle \hat{A} ,\, \hat{B} \right\rangle_{\text{KG}} := \mathrm{i} \int_{\mathbf{x}} \hat{A}^\dagger \overleftrightarrow{\nabla^0} B$. Hence

$$\begin{aligned} \left[\hat{a}_\sigma ,\, \hat{a}_{\sigma'}^\dagger \right] &= \left[\left\langle \phi_\sigma ,\, \hat{\phi} \right\rangle_{\text{KG}} ,\, \left\langle \hat{\phi} ,\, \phi_{\sigma'} \right\rangle_{\text{KG}} \right] \\ &= - \int_{\mathbf{x}} \int_{\mathbf{y}} \Big[\phi_\sigma^* \left(t ,\, \mathbf{x} \right) \nabla^0 \hat{\phi} \left(t ,\, \mathbf{x} \right) - \left(\nabla^0 \phi_\sigma^* \left(t ,\, \mathbf{x} \right) \right) \hat{\phi} \left(t ,\, \mathbf{x} \right) , \\ &\qquad \hat{\phi} \left(t ,\, \mathbf{y} \right) \nabla^0 \phi_{\sigma'} \left(t ,\, \mathbf{y} \right) - \left(\nabla^0 \hat{\phi} \left(t ,\, \mathbf{y} \right) \right) \phi_{\sigma'} \left(t ,\, \mathbf{y} \right) \Big] , \end{aligned} \tag{A.3}$$

$$\begin{aligned} \left[\hat{a}_\sigma ,\, \hat{a}_{\sigma'} \right] &= \left[\left\langle \phi_\sigma ,\, \hat{\phi} \right\rangle_{\text{KG}} ,\, \left\langle \phi_{\sigma'} ,\, \hat{\phi} \right\rangle_{\text{KG}} \right] \\ &= - \int_{\mathbf{x}} \int_{\mathbf{y}} \Big[\phi_\sigma^* \left(t ,\, \mathbf{x} \right) \nabla^0 \hat{\phi} \left(t ,\, \mathbf{x} \right) - \left(\nabla^0 \phi_\sigma^* \left(t ,\, \mathbf{x} \right) \right) \hat{\phi} \left(t ,\, \mathbf{x} \right) , \\ &\qquad \phi_{\sigma'}^* \left(t ,\, \mathbf{x} \right) \nabla^0 \hat{\phi} \left(t ,\, \mathbf{x} \right) - \left(\nabla^0 \phi_{\sigma'}^* \left(t ,\, \mathbf{x} \right) \right) \hat{\phi} \left(t ,\, \mathbf{x} \right) \Big] . \end{aligned} \tag{A.4}$$

One direction of the equivalence (deriving Eqs. (1.4.14) and (1.4.25) from Eqs. (1.4.22) and (1.4.26)) then follows easily:

$$\begin{aligned} \left[\hat{a}_\sigma ,\, \hat{a}_{\sigma'}^\dagger \right] &= \int_{\mathbf{x}} \int_{\mathbf{y}} \frac{\mathrm{i} \delta \left(\mathbf{x} ,\, \mathbf{y} \right)}{\sqrt{\left| g \left(t ,\, \mathbf{x} \right) \right|}} \left(\phi_\sigma^* \left(t ,\, \mathbf{x} \right) \nabla^0 \phi_{\sigma'} \left(t ,\, \mathbf{y} \right) - \nabla^0 \phi_\sigma^* \left(t ,\, \mathbf{x} \right) \phi_{\sigma'} \left(t ,\, \mathbf{y} \right) \right) \\ &= \mathrm{i} \int_{\mathbf{x}} \left(\phi_\sigma^* \left(t ,\, \mathbf{x} \right) \nabla^0 \phi_{\sigma'} \left(t ,\, \mathbf{x} \right) - \nabla^0 \phi_\sigma^* \left(t ,\, \mathbf{x} \right) \phi_{\sigma'} \left(t ,\, \mathbf{x} \right) \right) \\ &= \left\langle \phi_\sigma ,\, \phi_{\sigma'} \right\rangle_{\text{KG}} = \delta_{\sigma\sigma'} , \end{aligned} \tag{A.5}$$

$$\left[\hat{a}_\sigma ,\, \hat{a}_{\sigma'} \right] = 0. \tag{A.6}$$

For the other direction (deriving Eqs. (1.4.22) and (1.4.26) from Eqs. (1.4.14) and (1.4.25)), simply contrast Eq. (A.3) with the calculation

$$
\begin{aligned}
\left[\hat{a}_{\sigma},\, \hat{a}^{\dagger}_{\sigma'}\right] &= \delta_{\sigma\sigma'} = \langle \phi_{\sigma},\, \phi_{\sigma'} \rangle_{\mathrm{KG}} \\
&= \mathrm{i} \int_{\mathbf{x}} \left(\phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{x}\right) - \phi_{\sigma'}\left(t,\, \mathbf{x}\right) \nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \right) \\
&= - \int_{\mathbf{x}} \int_{\mathbf{y}} \frac{\mathrm{i}\delta\left(\mathbf{x},\, \mathbf{y}\right)}{\sqrt{\left|g\left(t,\, \mathbf{x}\right)\right|}} \left(\nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \phi_{\sigma'}\left(t,\, \mathbf{y}\right) - \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \right). \qquad \text{(A.7)}
\end{aligned}
$$

Thus

$$
\begin{aligned}
0 = \quad & \int_{\mathbf{x}} \int_{\mathbf{y}} \Big\{ \Big[\phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \nabla^{0} \hat{\phi}\left(t,\, \mathbf{x}\right) - \nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \hat{\phi}\left(t,\, \mathbf{x}\right), \\
& \hat{\phi}\left(t,\, \mathbf{y}\right) \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{y}\right) - \nabla^{0} \hat{\phi}\left(t,\, \mathbf{y}\right) \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \Big] \\
& - \frac{\mathrm{i}\delta\left(\mathbf{x},\, \mathbf{y}\right)}{\sqrt{\left|g\left(t,\, \mathbf{x}\right)\right|}} \left(\nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \phi_{\sigma'}\left(t,\, \mathbf{y}\right) - \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \right) \Big\} \\
= \quad & \int_{\mathbf{x}} \Big\{ \int_{\mathbf{y}} \Big[\phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \nabla^{0} \hat{\phi}\left(t,\, \mathbf{x}\right) - \left(\nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \right) \hat{\phi}\left(t,\, \mathbf{x}\right), \\
& \hat{\phi}\left(t,\, \mathbf{y}\right) \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{y}\right) - \nabla^{0} \hat{\phi}\left(t,\, \mathbf{y}\right) \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \Big] \\
& - \mathrm{i} \left(\nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \phi_{\sigma'}\left(t,\, \mathbf{x}\right) - \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{x}\right) \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \right) \Big\} \\
= \quad & \int_{\mathbf{x}} \Big\{ - \mathrm{i} \left(\nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \phi_{\sigma'}\left(t,\, \mathbf{x}\right) - \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{x}\right) \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \right) \\
& \quad + \int_{\mathbf{y}} \Big\{ \Big[\hat{\phi}\left(t,\, \mathbf{x}\right),\, \nabla^{0} \hat{\phi}\left(t,\, \mathbf{y}\right) \Big] \nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \\
& \quad - \Big[\hat{\phi}\left(t,\, \mathbf{y}\right),\, \nabla^{0} \hat{\phi}\left(t,\, \mathbf{x}\right) \Big] \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \\
& \quad - \Big[\hat{\phi}\left(t,\, \mathbf{x}\right),\, \hat{\phi}\left(t,\, \mathbf{y}\right) \Big] \nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \\
& \quad - \Big[\nabla^{0} \hat{\phi}\left(t,\, \mathbf{x}\right),\, \nabla^{0} \hat{\phi}\left(t,\, \mathbf{y}\right) \Big] \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \Big\} \Big\}.
\end{aligned}
\qquad \text{(A.8)}
$$

Hence

$$
\begin{aligned}
\mathrm{i} \nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \phi_{\sigma'}\left(t,\, \mathbf{x}\right) - \mathrm{i} \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{x}\right) \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) = & \int_{\mathbf{y}} \Big\{ \Big[\hat{\phi}\left(t,\, \mathbf{x}\right),\, \nabla^{0} \hat{\phi}\left(t,\, \mathbf{y}\right) \Big] \nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \\
& - \Big[\hat{\phi}\left(t,\, \mathbf{y}\right),\, \nabla^{0} \hat{\phi}\left(t,\, \mathbf{x}\right) \Big] \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \\
& - \Big[\hat{\phi}\left(t,\, \mathbf{x}\right),\, \hat{\phi}\left(t,\, \mathbf{y}\right) \Big] \nabla^{0} \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \\
& - \Big[\nabla^{0} \hat{\phi}\left(t,\, \mathbf{x}\right),\, \nabla^{0} \hat{\phi}\left(t,\, \mathbf{y}\right) \Big] \phi^{*}_{\sigma}\left(t,\, \mathbf{x}\right) \nabla^{0} \phi_{\sigma'}\left(t,\, \mathbf{y}\right) \Big\},
\end{aligned}
\qquad \text{(A.9)}
$$

$$
\Big[\hat{\phi}\left(t,\, \mathbf{x}\right),\, \nabla^{0} \hat{\phi}\left(t,\, \mathbf{y}\right) \Big] = \frac{\mathrm{i}\delta\left(\mathbf{x},\, \mathbf{y}\right)}{\sqrt{\left|g\left(t,\, \mathbf{x}\right)\right|}}, \qquad \text{(A.10)}
$$

$$
\Big[\hat{\phi}\left(t,\, \mathbf{x}\right),\, \hat{\phi}\left(t,\, \mathbf{y}\right) \Big] = 0, \qquad \text{(A.11)}
$$

$$
\Big[\nabla^{0} \hat{\phi}\left(t,\, \mathbf{x}\right),\, \nabla^{0} \hat{\phi}\left(t,\, \mathbf{y}\right) \Big] = 0. \qquad \text{(A.12)}
$$

B Pseudo-norm revision

The BRST and spacetime symmetries of the theory raise similar issues, which are compared and analysed herein. For this purpose, it is sufficient to identify each of the fields $N^2A^0,\, B,\, c,\, \bar{c}$ with its own zero mode. For brevity I define pseudo-norms as self-pseudo-inner products rather than square roots thereof. For example, a solution of $\langle\Phi|\Phi\rangle < 0$ is called *negative-norm*.

In Sec. B.1, I provide a normalisation convention for several FP-sector anticommutators. In Sec. B.2, I provide a matrix representation of solutions for these anticommutator conditions. In Secs. B.3 and B.4, I present explicit pseudo-inner products that obtain positive-norm states respectively preserving the internal and spacetime symmetries of BRST-quantised Yang–Mills theory. These sections use CMP methods for Yang–Mills theory. (An analogous treatment for perturbative gravity has not yet been produced.)

B.1 Anticommutators of the k_i, and required pseudo-inner product repairs by integration variable eliminations

For the non-interacting ($q = 0$) theory

$$c = k_0 + k_1 f,\,,\ \bar{c} = k_2 + k_3 f \tag{B.1}$$

for fermionic spacetime constants $k_0,\, k_1,\, k_2,\, k_3$. Zero-modes give $\mathcal{L}_{\text{FP}}^{(0)} = -\mathrm{i}\dot{\bar{c}}\cdot\dot{c}$ and

$$\varpi_{c_{(0)}} = \mathrm{i}\dot{\bar{c}}_{(0)} = \mathrm{i}k_3\dot{f},\ \varpi_{\bar{c}_{(0)}} = -\mathrm{i}\dot{c}_{(0)} = -\mathrm{i}k_1\dot{f}, \tag{B.2}$$

so canonical quantisation gives

$$\mathrm{i}\left\{k_0 + k_1 f,\, k_3\dot{f}\right\} = \left\{c_{(0)},\, \varpi_{c_{(0)}}\right\} = \frac{\mathrm{i}}{V\left(t\right)} = \left\{\bar{c}_{(0)},\, \varpi_{\bar{c}_{(0)}}\right\} = -\mathrm{i}\left\{k_2 + k_3 f,\, k_1\dot{f}\right\}. \tag{B.3}$$

Since $V = a^{n-1}V_c = \dot{f}^{-1}\int \mathrm{d}^{n-1}\mathbf{x}\sqrt{\eta\left(\mathbf{x}\right)}$ and $\sqrt{\eta\left(\mathbf{x}\right)} > 0$, some spacetime-constant $r > 0$ satisfies $\int \mathrm{d}^{n-1}\mathbf{x}\sqrt{\eta} = r^{-2}$. Then r is a spacetime constant, which can be set to 1 by rescaling $a\left(t\right)$. Thus $V^{-1} = r^2\dot{f}$ and $\left\{k_0 + k_1 f,\, k_3\dot{f}\right\} = r^2\dot{f} = -\left\{k_2 + k_3 f,\, k_1\dot{f}\right\}$. Equating coefficients of $\dot{f},\, f\dot{f}$ gives the anticommutation relations

$$\left\{k_i,\, k_j\right\} = r^2\left(\delta_{0i}\delta_{3j} - \delta_{1i}\delta_{2j}\right). \tag{B.4}$$

Obtaining such anticommutators is the motivation for considering the non-interacting special case, and Eq. (B.4) implies the $t = 0$ Schrödinger representation has $k_1 = -r^2\frac{\partial}{\partial k_2}$. If a k_1-annihilated wavefunction's normalization included integration over the Grassmann variable k_2, said wavefunction would be k_2-independent and hence of zero pseudo-norm. Thus k_2-integration must be excluded from our pseudo-inner product, and this trivially works at all orders, since $\delta c = \text{constant}$ is a symmetry of the full theory with Noether current $-\mathrm{i}D^\mu c$ and Noether charge $Q^{(0)} := -\mathrm{i}\int_{\mathbf{x}} D^0 c$.

It is important to prove that $\delta_{\text{BRST}}\left(D^\mu c\right) = 0$ implies $\left\{Q,\, Q^{(0)}\right\} = 0$, and to obtain $Q^{(0)}$ in the form $\mathcal{O}\left(k_1\right)$. These results, if obtained, establish that we can impose one $Q^{(0)}\left|\psi\right\rangle = 0$ condition for each Lie group generator, e.g. $N^2 - 1$ such conditions in $\mathrm{SU}\left(N\right)$. Doing this ensures a vacuum state $\left|\psi\right\rangle$ with $k_1\left|\psi\right\rangle = 0$ at zeroth-order satisfies the BRST invariance condition $Q\left|\psi\right\rangle = 0$. Indeed, the proof is simple. The BRST invariance of the fermionic field $D^\mu c$ is the statement $\{Q,\, D^\mu c\} = 0$. A $-\mathrm{i}\int \mathrm{d}^{n-1}\mathbf{x}$ integration then implies that $\{Q,\, Q^{(0)}\} = 0$. Finally, $D^0 c = k_1 \dot{f}$ implies $Q^{(0)} = -\mathrm{i}\int \mathrm{d}^{n-1}\mathbf{x}\, k_1 \dot{f}$.

B.2 Matrix representations of the ks

One matrix representation of the ks (placed together on lines in pairs which do not anticommute) is

$$k_1 = r\begin{pmatrix} A & \mathrm{O}_2 \\ \mathrm{O}_2 & A \end{pmatrix},\; k_2 = -r\begin{pmatrix} B & \mathrm{O}_2 \\ \mathrm{O}_2 & B \end{pmatrix}, \tag{B.5}$$

$$k_3 = \frac{-r}{2}\begin{pmatrix} C & C \\ -C & -C \end{pmatrix},\; k_0 = \frac{r}{2}\begin{pmatrix} C & -C \\ C & -C \end{pmatrix} \tag{B.6}$$

where

$$A := \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},\; B := \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},\; C := \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{B.7}$$

These satisfy

$$A^2 = B^2 = \mathrm{O}_2,\; C^2 = -\mathrm{I}_2,\; BA = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\; AB = \mathrm{I}_2 - BA,\; AC = \mathrm{i}A. \tag{B.8}$$

Note the factors of 2^{-1} in the matrix representations of $k_0,\, k_3$ are required for their anticommutator to be $-\mathrm{I}_4$ instead of $-4\mathrm{I}_4$. Define

$$\eta_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\; \eta = \begin{pmatrix} \eta_2 & \mathrm{O}_2 \\ \mathrm{O}_2 & -\eta_2 \end{pmatrix}. \tag{B.9}$$

One can show $A^\dagger \eta_2 = \eta_2 A$ (and similarly with $B,\, C$), and similarly $r^{-1}k_i^\dagger \eta = \eta r^{-1} k_i$ for each i. For example,

$$\begin{aligned} k_3^\dagger \eta &= \frac{-r}{2}\begin{pmatrix} C^\dagger & -C^\dagger \\ C^\dagger & -C^\dagger \end{pmatrix}\begin{pmatrix} \eta_2 & \mathrm{O}_2 \\ \mathrm{O}_2 & -\eta_2 \end{pmatrix} \\ &= \frac{-r}{2}\begin{pmatrix} C^\dagger\eta_2 & C^\dagger\eta_2 \\ C^\dagger\eta_2 & C^\dagger\eta_2 \end{pmatrix} = \frac{-r}{2}\begin{pmatrix} \eta_2 C & \eta_2 C \\ \eta_2 C & \eta_2 C \end{pmatrix}, \end{aligned} \tag{B.10}$$

$$\eta k_3 = \frac{-r}{2}\begin{pmatrix} \eta_2 & \mathrm{O}_2 \\ \mathrm{O}_2 & -\eta_2 \end{pmatrix}\begin{pmatrix} C & C \\ -C & -C \end{pmatrix} = \frac{-r}{2}\begin{pmatrix} \eta_2 C & \eta_2 C \\ \eta_2 C & \eta_2 C \end{pmatrix}. \tag{B.11}$$

One can similarly show that, had we attempted a metric $\eta = \begin{pmatrix} \eta_2 & \mathrm{O}_2 \\ \mathrm{O}_2 & \eta_2 \end{pmatrix}$, the result $k_3^\dagger \eta = \eta k_3$ would fail, so the $-$ sign in the definition of η in terms of η_2 is vital.

In the above matrix representation we have

$$\begin{aligned} k_0k_3 + k_1k_2 &= -r^2 \begin{pmatrix} \frac{1}{2}C^2 + AB & \frac{1}{2}C^2 \\ \frac{1}{2}C^2 & \frac{1}{2}C^2 + AB \end{pmatrix} \\ &= -r^2 \begin{pmatrix} \frac{1}{2}\mathrm{I}_2 - BA & -\frac{1}{2}\mathrm{I}_2 \\ -\frac{1}{2}\mathrm{I}_2 & \frac{1}{2}\mathrm{I}_2 - BA \end{pmatrix}, \qquad \text{(B.12)} \\ k_1k_3 &= \frac{-r^2}{2} \begin{pmatrix} AC & AC \\ -AC & -AC \end{pmatrix} = \frac{-r^2}{2} \begin{pmatrix} \mathrm{i}A & \mathrm{i}A \\ -\mathrm{i}A & -\mathrm{i}A \end{pmatrix}. \qquad \text{(B.13)} \end{aligned}$$

Let a ket $|\psi\rangle$ satisfying $\langle\psi|k_1k_3|\psi\rangle = \langle\psi|k_0k_3 + k_1k_2|\psi\rangle = 0$ have a column vector representation

$$\begin{pmatrix} \mathbf{U} \\ \mathbf{L} \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \; a,\, b,\, c,\, d \in \mathbb{C},\; \mathbf{U},\, \mathbf{L} \in \mathbb{C}^2 \qquad \text{(B.14)}$$

so $\langle\psi|k_1k_3|\psi\rangle = 0$ implies

$$\begin{aligned} 0 &= \begin{pmatrix} \mathbf{U}^\dagger & \mathbf{L}^\dagger \end{pmatrix} \begin{pmatrix} \eta_2 & \mathrm{O}_2 \\ \mathrm{O}_2 & -\eta_2 \end{pmatrix} \begin{pmatrix} A & A \\ -A & -A \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{L} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{U}^\dagger\eta_2 & -\mathbf{L}^\dagger\eta_2 \end{pmatrix} \begin{pmatrix} A(\mathbf{U}+\mathbf{L}) \\ -A(\mathbf{U}+\mathbf{L}) \end{pmatrix} \\ &= \left(\mathbf{U}^\dagger + \mathbf{L}^\dagger\right)\eta_2 A(\mathbf{U}+\mathbf{L}) \\ &= \frac{1}{2}\begin{pmatrix} a^* + c^* & b^* + d^* \end{pmatrix} \begin{pmatrix} a+b+c+d \\ a+b+c+d \end{pmatrix} \\ &= \frac{1}{2}\left(a^* + b^* + c^* + d^*\right)(a+b+c+d), \qquad \text{(B.15)} \\ 0 &= a+b+c+d. \qquad \text{(B.16)} \end{aligned}$$

Here I have used the result

$$\eta_2 A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \text{(B.17)}$$

from which I can obtain a further result:

$$\eta_2 BA = \eta_2\left(\mathrm{I}_2 - AB\right) = \eta_2 - \left(\eta_2 A\right)B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}. \qquad \text{(B.18)}$$

This result is of immediate use. Introduce complex numbers $w := a + d$, $z := b + c$ so $w = -z$ and $wz^* \in \mathbb{R}$. Then $\langle\psi|k_0k_3 + k_1k_2|\psi\rangle = 0$ implies

$$
\begin{aligned}
0 &= \begin{pmatrix} \mathbf{U}^\dagger & \mathbf{L}^\dagger \end{pmatrix} \begin{pmatrix} \eta_2 & \mathrm{O}_2 \\ \mathrm{O}_2 & -\eta_2 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\mathrm{I}_2 - BA & -\frac{1}{2}\mathrm{I}_2 \\ -\frac{1}{2}\mathrm{I}_2 & \frac{1}{2}\mathrm{I}_2 - BA \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{L} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{U}^\dagger \eta_2 & -\mathbf{L}^\dagger \eta_2 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\left(\mathbf{U}-\mathbf{L}\right) - BA\mathbf{U} \\ \frac{1}{2}\left(\mathbf{L}-\mathbf{U}\right) - BA\mathbf{L} \end{pmatrix} \\
&= \frac{1}{2}\left(\mathbf{U}+\mathbf{L}\right)^\dagger \eta_2 \left(\mathbf{U}-\mathbf{L}\right) - \left(\mathbf{U}^\dagger \eta_2 BA\mathbf{U} - \mathbf{L}^\dagger \eta_2 BA\mathbf{L}\right) \\
&= \frac{1}{2}\left\{\left(a^*+c^*\right)\left(a-c\right) - \left(b^*+d^*\right)\left(b-d\right) - \left(a^*-b^*\right)\left(a+b\right) + \left(c^*-d^*\right)\left(c+d\right)\right\} \\
&= \frac{1}{2}\left\{-a^*c + ac^* + b^*d - bd^* + ab^* - a^*b + c^*d - cd^*\right\} \\
&= \frac{1}{2}\left\{\left[\left(a+d\right)\left(b^*+c^*\right)\right] + \text{c.c.}\right\} = \frac{1}{2}\left\{wz^* + \text{c.c.}\right\} = wz^*.
\end{aligned} \tag{B.19}
$$

Thus $wz^* = 0,\ w = z = 0$, so $b = -a,\ d = -c$, and the most general representation of $|\psi\rangle$ is

$$
|\psi\rangle = \begin{pmatrix} a \\ -a \\ c \\ -c \end{pmatrix}. \tag{B.20}
$$

By inspection, this is precisely the solution to $k_1 |\psi\rangle = 0$, so is a null state in η's pseudo-inner product.

B.3 Repairing the pseudo-inner product

To specify a pseudo-inner product it is sufficient to obtain its pseudo-norms. Deleting two integration variables has the effect of removing two of the four terms in the pseudo-norm $|a|^2 - |b|^2 - |c|^2 + |d|^2$. Clearly, we need the result to be neither positive-definite (since then non-trivial nilpotent Hermitian operators no longer exist) nor negative-definite (as this has the same problem and also violates unitarity). In particular, the surviving terms do not both depend on either a or d, but neither do both of them depend on b or c. Taking one term from each pair gives either
(i) $|a|^2 - |b|^2$ or $-|c|^2 + |d|^2$ or
(ii) $|a|^2 - |c|^2$ or $-|b|^2 + |d|^2$.
A desirable pseudo-norm is of type (ii) rather than type (i), since type (i) choices imply $\langle\psi|\psi\rangle = 0$ whenever Eq. (B.20) holds. One thus anticipates obtaining either $|a|^2 - |c|^2$ or $-|b|^2 + |d|^2$. For solutions of Eq. (B.20), we simply have $-|b|^2 + |d|^2 = -\left(|a|^2 - |c|^2\right)$. With the pseudo-norm $|a|^2 - |c|^2$ the general solution to $\langle\psi|\psi\rangle = 1$ is

$$
|\psi\rangle = \begin{pmatrix} e^{i\alpha}\cosh\vartheta \\ -e^{i\alpha}\cosh\vartheta \\ e^{i\beta}\sinh\vartheta \\ -e^{i\beta}\sinh\vartheta \end{pmatrix},\ \alpha,\ \beta,\ \vartheta \in \mathbb{R}, \tag{B.21}
$$

whereas with the pseudo-norm $-|b|^2 + |d|^2$ the functions $\sinh\vartheta,\ \cosh\vartheta$ must be exchanged.

B.4 The spacetime symmetry

So far this appendix has considered internal symmetries, but a similar treatment is also possible for spacetime symmetries. With $k_1 \left|\psi\right\rangle = 0$ imposed, the zero-point modes' contribution to the FP-ghost propagator is

$$\left\langle \psi\right| \left(k_0 + k_1 f\left(t\right)\right) \left(k_2 + k_3 f\left(t^{'}\right)\right) \left|\psi\right\rangle = \left\langle \psi|k_0 k_2|\psi\right\rangle + \left\langle \psi|k_0 k_3|\psi\right\rangle f\left(t^{'}\right). \tag{B.22}$$

The spacetime symmetry condition is then $\left\langle \psi|k_0 k_3|\psi\right\rangle = 0$. If this inner product is in the full metric then only a zero-pseudo-norm state is consistent with this condition, because then

$$\left\langle \psi|\psi\right\rangle = r^{-2} \left\langle \psi\right| \left\{k_0,\, k_3\right\} \left|\psi\right\rangle = 2r^{-2}\,\mathrm{Re}\left\langle \psi|k_0 k_3|\psi\right\rangle = 0. \tag{B.23}$$

The elimination of integration variables is again necessary. Working throughout in the Landau gauge, one field equation may be rewritten as $\nabla_\mu D^\mu \overline{c}^a = 0$. (The operators are transposed in general, but in the Landau gauge $\nabla^\mu A^b_\mu = 0$ and the two versions of the field equation are equivalent.) The Noether charge is then $\overline{Q}^{(0)} = \mathrm{i}\int_{\mathbf{x}} D^0 \overline{c}$. We will identify an associated Noether symmetry. We now have $\overline{Q}^{(0)} \left|\psi\right\rangle = 0$, instead of $k_3 \left|\psi\right\rangle = 0$. To anticipate the new pseudo-inner product, observe $k_3 \left|\psi\right\rangle = 0$ has solution

$$\left|\psi\right\rangle = \begin{pmatrix} a \\ b \\ -a \\ -b \end{pmatrix}, \tag{B.24}$$

so $\left|a\right|^2 - \left|b\right|^2$ or $-\left|c\right|^2 + \left|d\right|^2$ will serve as an appropriate pseudo-norm. (These were previously called type (i) pseudo-norms.) Again, unitarity is easily satisfied.

C Dirac brackets for Yang–Mills theory and perturbative gravity

In Sec. C.1, I summarise the general theory of Poisson and Dirac brackets, viz. Ref. [81]. The notation and terminology I use when doing so will be used in my treatments of Yang–Mills theory and perturbative gravity. The former is the subject of Sec. C.2. The latter is the subject of Sec. C.3.

C.1 The theory of Dirac brackets

If $f = g$ holds on-shell, write $f \approx g$ and say $f,\, g$ are *weakly equal*. If $f = g$ holds off-shell, say $f,\, g$ are *strongly equal*. If the equation specifying the value of a canonical momentum is not invertible to obtain time derivatives, it gives a constraint on phase space expressible in the form $\varphi_j\left(\phi,\, \pi\right) \approx 0$. (The label j is an index over all such constraints, which are called *primary constraints*.) Here ϕ collectively denotes all canonical coordinates and π collectively denotes all canonical momenta. Then $\sum_j c_j\left(\phi,\, \pi\right)\varphi_j\left(\phi,\, \pi\right) \approx 0$ for any functions c_j. The usual Hamiltonian, defined by a Legendre transform, is hereafter called the *naïve Hamiltonian* and denoted $H^{\text{Naïve}}$. The Hamiltonian then admits the generalisation $H^{\text{Dirac}} := H^{\text{Naïve}} + \sum_j c_j\left(\phi,\, \pi\right)\varphi_j\left(\phi,\, \pi\right)$, where the functions c_j remain to be determined. The Poisson bracket $\left\{\cdot,\, \cdot\right\}_{\mathrm{P}}$ then satisfies

$$\dot{f} \approx \left\{f,\, H^{\text{Dirac}}\right\}_{\mathrm{P}} \approx \left\{f,\, H^{\text{Naïve}}\right\}_{\mathrm{P}} + \sum_k c_k \left\{f,\, \varphi_k\right\}_{\mathrm{P}} \tag{C.1}$$

for any function f. In particular $\dot{\varphi}_j \approx \left\{\varphi_j,\, H^{\text{Naïve}}\right\}_{\text{P}} + \sum_k c_k \left\{\varphi_j,\, \varphi_k\right\}_{\text{P}}$. But we require $\dot{\varphi}_j \approx 0$, so

$$\left\{\varphi_j,\, H^{\text{Naïve}}\right\}_{\text{P}} + \sum_k c_k \left\{\varphi_j,\, \varphi_k\right\}_{\text{P}} \approx 0. \tag{C.2}$$

This fact may introduce c_k-independent constraints other than primary constraints. Such further constraints are called *secondary constraints*. Let $\Phi\left(\phi, \pi\right) \approx 0$ be a secondary constraint. A term of the form $C\left(\phi, \pi\right)\Phi\left(\phi, \pi\right)$ can then be added to H^{Dirac}, thereby demoting $\Phi \approx 0$ to a primary constraint, and we can then attempt to find further secondary constraints. We continue until no new secondary constraints result, and thereafter do not distinguish between primary and secondary constraints.

The final form of Eq. (C.2) after such iterations may also allow us to solve for the c_k. Gauge degrees of freedom result in the c_k not being unique, which further complicates the form of H^{Dirac}.

Call a function $f\left(\phi,\, \pi\right)$ satisfying $\left\{f,\, \varphi_j\right\}_{\text{P}} \approx 0$ for all primary and secondary constraints φ_j a *first class function*. If $\left\{\varphi_j,\, \varphi_k\right\}_{\text{P}}$ is not weakly equal to zero call $\varphi_j,\, \varphi_k$ *second class constraints*. Label these constraints $\widetilde{\varphi}_a$ and define the matrix

$$M_{ab} := \left\{\widetilde{\varphi}_a,\, \widetilde{\varphi}_b\right\}_{\text{P}}. \tag{C.3}$$

This matrix is always invertible. The *Dirac bracket* $\left\{\cdot,\, \cdot\right\}_{\text{D}}$ is then defined as

$$\left\{f,\, g\right\}_{\text{D}} := \left\{f,\, g\right\}_{P} - \sum_{ab} \left\{f,\, \widetilde{\varphi}_a\right\}_{\text{P}} \left(M^{-1}\right)^{ab} \left\{\widetilde{\varphi}_b,\, g\right\}_{\text{P}}. \tag{C.4}$$

Note in particular that any weakly vanishing quantity, e.g. a primary constraint, has vanishing Poisson brackets and hence vanishing Dirac brackets.

C.2 Results for Yang–Mills theory

Let us consider an example relevant to us, namely that of Yang–Mills theory. We take all fields at equal time, say t. Our first class constraints are $\varphi_1\left(\mathbf{x}\right) = \pi_B\left(\mathbf{x}\right) + \sqrt{|g|}A^0\left(\mathbf{x}\right),\, \varphi_2\left(\mathbf{x}'\right) = \pi_{A_0}\left(\mathbf{x}'\right)$. Note the $\sqrt{|g|}$ factor and the distinction between A^0 and A_0. Thus

$$\left\{\varphi_1\left(\mathbf{x}\right),\, \varphi_2\left(\mathbf{x}'\right)\right\}_{\text{P}} = \sqrt{|g|}g^{00}\delta\left(\mathbf{x},\, \mathbf{x}'\right). \tag{C.5}$$

The primary constraints are thus both also second class constraints, and we define

$$\widetilde{\varphi}_i := \varphi_i, \quad i \in \left\{1,\, 2\right\}. \tag{C.6}$$

Thus

$$M_{ab} = \sqrt{|g|}g^{00}J_{ab}\delta\left(\mathbf{x},\, \mathbf{x}'\right), \quad J_{ab} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \delta_{a1}\delta_{b2} - \delta_{a2}\delta_{b1}. \tag{C.7}$$

Since $\delta\left(\mathbf{x},\, \mathbf{x}'\right)$ is the identity on the function space, it is self-inverse, whereas $J^{-1} = -J$. Thus

$$\left(M^{-1}\right)^{ab} = \frac{1}{\sqrt{|g|}g^{00}}\left(\delta_2^a\delta_1^b - \delta_1^a\delta_2^b\right)\delta\left(\mathbf{x},\, \mathbf{x}'\right). \tag{C.8}$$

Dirac brackets integrate over positions, viz.

$$\begin{aligned}\left\{f\left(\mathbf{x}\right),\,g\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{D}}&=\left\{f\left(\mathbf{x}\right),\,g\left(\mathbf{x}^{'}\right)\right\}_{P}-\int\mathrm{d}^{n-1}\mathbf{x}^{''}\mathrm{d}^{n-1}\mathbf{x}^{'''}\sum_{ab}\left(\delta_{a2}\delta_{b1}-\delta_{a1}\delta_{b2}\right)\\&\times\frac{\delta\left(\mathbf{x},\,\mathbf{x}^{'}\right)}{\sqrt{\left|g\left(x\right)\right|}g^{00}\left(x\right)}\left\{f\left(\mathbf{x}\right),\,\varphi_{a}\left(\mathbf{x}^{''}\right)\right\}_{\mathrm{P}}\left\{\varphi_{3-a}\left(\mathbf{x}^{'''}\right),\,g\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}\\&=\left\{f\left(\mathbf{x}\right),\,g\left(\mathbf{x}^{'}\right)\right\}_{P}+\frac{\delta\left(\mathbf{x},\,\mathbf{x}^{'}\right)}{\sqrt{\left|g\left(x\right)\right|}g^{00}\left(x\right)}\int\mathrm{d}^{n-1}\mathbf{x}^{''}\mathrm{d}^{n-1}\mathbf{x}^{'''}\times\\&\left(\left\{f\left(\mathbf{x}\right),\,\pi_{B}\left(\mathbf{x}^{''}\right)+\sqrt{|g|}A^{0}\left(\mathbf{x}^{''}\right)\right\}_{\mathrm{P}}\left\{\pi_{A_0}\left(\mathbf{x}^{'''}\right),\,g\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}\right.\\&\left.-\left\{f\left(\mathbf{x}\right),\,\pi_{A_0}\left(\mathbf{x}^{''}\right)\right\}_{\mathrm{P}}\left\{\pi_{B}\left(\mathbf{x}^{'''}\right)+\sqrt{|g|}A^{0}\left(\mathbf{x}^{'''}\right),\,g\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}\right).\end{aligned}\tag{C.9}$$

The Dirac brackets $\{f,\,g\}_{\mathrm{D}}$ with $f,\,g\in\left\{A^{0},\,B,\,\pi_{A_0},\,\pi_{B}\right\}$ must be calculated. Since π_{A_0} is a primary constraint, the result is always zero if $\pi_{A_0}\in\{f,\,g\}$, so only $f,\,g\in\left\{A^{0},\,B,\,\pi_{B}\right\}$ need be considered. By antisymmetry, it suffices to calculate three Dirac brackets, viz.

$$\begin{aligned}\left\{A^{0}\left(\mathbf{x}\right),\,B\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{D}}&=\left\{A^{0}\left(\mathbf{x}\right),\,B\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}+\frac{\delta\left(\mathbf{x},\,\mathbf{x}^{'}\right)}{\sqrt{\left|g\left(x\right)\right|}g^{00}\left(x\right)}\int\mathrm{d}^{n-1}\mathbf{x}^{''}\mathrm{d}^{n-1}\mathbf{x}^{'''}\times\\&\left(\left\{A^{0}\left(\mathbf{x}\right),\,\pi_{B}\left(\mathbf{x}^{''}\right)+\sqrt{|g|}A^{0}\left(\mathbf{x}^{''}\right)\right\}_{\mathrm{P}}\left\{\pi_{A_0}\left(\mathbf{x}^{'''}\right),\,B\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}\right.\\&\left.-\left\{A^{0}\left(\mathbf{x}\right),\,\pi_{A_0}\left(\mathbf{x}^{''}\right)\right\}_{\mathrm{P}}\left\{\pi_{B}\left(\mathbf{x}^{'''}\right)+\sqrt{|g|}A^{0}\left(\mathbf{x}^{'''}\right),\,B\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}\right)\\&=\left\{A^{0}\left(\mathbf{x}\right),\,B\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}+\frac{\delta\left(\mathbf{x},\,\mathbf{x}^{'}\right)}{\sqrt{\left|g\left(x\right)\right|}g^{00}\left(x\right)}g^{00}\left(x\right)\\&=\frac{\delta\left(\mathbf{x},\,\mathbf{x}^{'}\right)}{\sqrt{\left|g\left(x\right)\right|}},\end{aligned}\tag{C.10}$$

$$\begin{aligned}\left\{A^{0}\left(\mathbf{x}\right),\,\pi_{B}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{D}}&=\left\{A^{0}\left(\mathbf{x}\right),\,\pi_{B}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}+\frac{\delta\left(\mathbf{x},\,\mathbf{x}^{'}\right)}{\sqrt{\left|g\left(x\right)\right|}g^{00}\left(x\right)}\int\mathrm{d}^{n-1}\mathbf{x}^{''}\mathrm{d}^{n-1}\mathbf{x}^{'''}\times\\&\left(\left\{A^{0}\left(\mathbf{x}\right),\,\pi_{B}\left(\mathbf{x}^{''}\right)+\sqrt{|g|}A^{0}\left(\mathbf{x}^{''}\right)\right\}_{\mathrm{P}}\left\{\pi_{A_0}\left(\mathbf{x}^{'''}\right),\,\pi_{B}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}\right.\\&\left.-\left\{A^{0}\left(\mathbf{x}\right),\,\pi_{A_0}\left(\mathbf{x}^{''}\right)\right\}_{\mathrm{P}}\left\{\pi_{B}\left(\mathbf{x}^{'''}\right)+\sqrt{|g|}A^{0}\left(\mathbf{x}^{'''}\right),\,\pi_{B}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}\right)\\&=\left\{A^{0}\left(\mathbf{x}\right),\,\pi_{B}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}=0,\end{aligned}\tag{C.11}$$

$$\begin{aligned}\left\{B\left(\mathbf{x}\right),\,\pi_{B}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{D}}&=\left\{B\left(\mathbf{x}\right),\,\pi_{B}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}+\frac{\delta\left(\mathbf{x},\,\mathbf{x}^{'}\right)}{\sqrt{\left|g\left(x\right)\right|}g^{00}\left(x\right)}\int\mathrm{d}^{n-1}\mathbf{x}^{''}\mathrm{d}^{n-1}\mathbf{x}^{'''}\times\\&\left(\left\{B\left(\mathbf{x}\right),\,\pi_{B}\left(\mathbf{x}^{''}\right)+\sqrt{|g|}A^{0}\left(\mathbf{x}^{''}\right)\right\}_{\mathrm{P}}\left\{\pi_{A_0}\left(\mathbf{x}^{'''}\right),\,\pi_{B}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}\right.\\&\left.-\left\{B\left(\mathbf{x}\right),\,\pi_{A_0}\left(\mathbf{x}^{''}\right)\right\}_{\mathrm{P}}\left\{\pi_{B}\left(\mathbf{x}^{'''}\right)+\sqrt{|g|}A^{0}\left(\mathbf{x}^{'''}\right),\,\pi_{B}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}\right)\\&=\left\{B\left(\mathbf{x}\right),\,\pi_{B}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{P}}=\delta\left(\mathbf{x},\,\mathbf{x}^{'}\right).\end{aligned}\tag{C.12}$$

Therefore the naïve commutation relations used in this thesis are justified. Notice in particular that

$$0=\left\{B\left(\mathbf{x}\right),\,\pi_{B}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{D}}-\sqrt{|g|}\left\{A^{0}\left(\mathbf{x}^{'}\right),\,B\left(\mathbf{x}\right)\right\}_{\mathrm{D}}=\left\{B\left(\mathbf{x}\right),\,\widetilde{\varphi}_{1}\left(\mathbf{x}^{'}\right)\right\}_{\mathrm{D}}.\tag{C.13}$$

C.3 Results for perturbative gravity

We take all fields at equal time, say time t. I will use two references, Ref. [94] and Appendix E.2 of Ref. [4].

There are four primary constraints. They may be succinctly written by first defining [4, 94] the *Hamiltonian constraint*

$$H := \frac{1}{\sqrt{\gamma}} \left(\pi_{ij} \pi^{ij} - \frac{1}{2} \pi^2 \right) -^{(n-1)} R \sqrt{\gamma} \tag{C.14}$$

and the *momentum constraint*

$$H_i := -2 D_j \pi_i^j \tag{C.15}$$

where:

- $^{(n-1)}R$ is the Ricci scalar on a Cauchy surface;
- π^{ij} is the conjugate momentum densities of γ_{ij};
- γ_{ij} lowers the indices of π^{ij}; and
- D_j is the gauge covariant derivative.

The shift vector $N^i \in \mathbb{R}^{n-1}$ may also be denoted $\mathbf{N}$.

The primary constraints [4, 94] are shown below in Eqs. (C.16)–(C.19) (N^i is denoted $\mathbf{N}$ in Eq. (C.17)):

$$\varphi_1 = H_N := \int \mathrm{d}^{n-1}\mathbf{x} N(\mathbf{x}) H(\mathbf{x}), \tag{C.16}$$

$$\varphi_2 = H_{\mathbf{N}} := \int \mathrm{d}^{n-1}\mathbf{x} N^i(\mathbf{x}) H_i(\mathbf{x}), \tag{C.17}$$

$$\varphi_3^\mu = \pi_{H_{0\mu}}, \tag{C.18}$$

$$\varphi_{4\mu} = \pi_{B^\mu} + \sqrt{|g|} H_\mu^0. \tag{C.19}$$

(The constraint in Eq. (C.18) is due to the Ricci scalar in the Einstein–Hilbert action.) All of these constraints are also secondary.

The next task is to find the matrix M_{ab} defined in Eq. (C.3), and the inverse matrix thereof. Taking N, $\mathbf{N}$ (M, $\mathbf{M}$) in the first (second) argument of a Poisson bracket (hereafter PB) obtains M_{ab} when $a,\, b \in \{1,\, 2\}$, viz. Ref. [94]. This result is antisymmetric when $N = M$, $\mathbf{N} = \mathbf{M}$, but not more generally. The matrix M_{ab} may be written as a *block matrix*

$$M_{ab} = \begin{pmatrix} A & \mathbb{O}_2 \\ \mathbb{O}_2 & B \end{pmatrix}, \tag{C.20}$$

where A, B are 2×2 matrices and $\mathbb{O}_2$ is the 2×2 zero matrix. Explicitly

$$A = \begin{pmatrix} H_{\gamma^{ij}(N\nabla_j M - M \nabla_j N)} & -H_{\pounds_{\mathbf{N}} M} \\ H_{\pounds_{\mathbf{N}} M} & H_{[\mathbf{N},\, \mathbf{M}]} \end{pmatrix}, \tag{C.21}$$

$$B = \sqrt{|g|} g^{00} \delta^\mu_\nu \delta\left(\mathbf{x},\, \mathbf{x}'\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{C.22}$$

Note that, unlike A, the matrix B lacks any dependence on N, $\mathbf{N}$, M, $\mathbf{M}$, so is antisymmetric.

The inverse matrix of M_{ab} is obtainable from the identity

$$\begin{pmatrix} A & \mathbb{O}_2 \\ \mathbb{O}_2 & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & \mathbb{O}_2 \\ \mathbb{O}_2 & B^{-1} \end{pmatrix}. \tag{C.23}$$

Using also the fact that the Dirac delta and Kronecker delta are both self-inverse, we have

$$M^{-1} = \begin{pmatrix} \frac{H_{[\mathbf{N},\,\mathbf{M}]}}{\det} & \frac{H_{\pounds_{\mathbf{M}} N}}{\det} & 0 & 0 \\ \frac{-H_{\pounds_{\mathbf{N}} M}}{\det} & \frac{H_{\gamma^{ij}(N\nabla_j M - M\nabla_j N)}}{\det} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{|g|}g^{00}}\delta^{\nu}_{\rho}\delta\left(\mathbf{x},\,\mathbf{x}^{'}\right) \\ 0 & 0 & -\frac{1}{\sqrt{|g|}g^{00}}\delta^{\nu}_{\rho}\delta\left(\mathbf{x},\,\mathbf{x}^{'}\right) & 0 \end{pmatrix}, \tag{C.24}$$

where $\det := H_{\gamma^{ij}(N\nabla_j M - M\nabla_j N)} H_{[\mathbf{N},\,\mathbf{M}]} + H_{\pounds_{\mathbf{N}} M} H_{\pounds_{\mathbf{M}} N}$. The Dirac bracket (hereafter DB) is then

$$\{F,\,G\}_{\mathrm{D}} = \sum_{T=0}^{6} T_T, \tag{C.25}$$

where the terms T_T are

$$T_0 := \{F,\,G\}_{\mathrm{P}}\,, \tag{C.26}$$

$$T_1 := -\{F,\,H_N\}_{\mathrm{P}} \frac{H_{[\mathbf{N},\,\mathbf{M}]}}{\det} \{H_M,\,G\}_{\mathrm{P}}\,, \tag{C.27}$$

$$T_2 := -\{F,\,H_N\}_{\mathrm{P}} \frac{H_{\pounds_{\mathbf{M}} N}}{\det} \{H_{\mathbf{M}},\,G\}_{\mathrm{P}}\,, \tag{C.28}$$

$$T_3 := \{F,\,H_{\mathbf{N}}\}_{\mathrm{P}} \frac{H_{\pounds_{\mathbf{N}} M}}{\det} \{H_M,\,G\}_{\mathrm{P}}\,, \tag{C.29}$$

$$T_4 := -\{F,\,H_{\mathbf{N}}\}_{\mathrm{P}} \frac{H_{\gamma^{ij}(N\nabla_j M - M\nabla_j N)}}{\det} \{H_{\mathbf{M}},\,G\}_{\mathrm{P}}\,, \tag{C.30}$$

$$T_5 := \left\{F,\,\pi_{B^\nu} + \sqrt{|g|}H^0_\nu\right\}_{\mathrm{P}} \{\pi_{H_{0\nu}},\,G\}_{\mathrm{P}}\,, \tag{C.31}$$

$$T_6 := -\{F,\,\pi_{H_{0\nu}}\}_{\mathrm{P}} \left\{\pi_{B^\nu} + \sqrt{|g|}H^0_\nu,\,G\right\}_{\mathrm{P}}. \tag{C.32}$$

Computing DBs requires a few Poisson brackets. Some are straightforward:

$$\left\{B^\mu,\,\pi_{B^\nu} + \sqrt{|g|}H^0_\nu\right\}_{\mathrm{P}} = \delta^\mu_\nu \delta\left(\mathbf{x},\,\mathbf{x}^{'}\right), \tag{C.33}$$

$$\left\{H^0_\mu,\,\pi_{H_{0\nu}}\right\}_{\mathrm{P}} = g^{00}\delta^\nu_\mu \delta\left(\mathbf{x},\,\mathbf{x}^{'}\right), \tag{C.34}$$

$$\left\{\pi_{H_{0\mu}},\,\pi_{B^\nu} + \sqrt{|g|}H^0_\nu\right\}_{\mathrm{P}} = -\sqrt{|g|}g^{00}\delta^\mu_\nu \delta\left(\mathbf{x},\,\mathbf{x}^{'}\right). \tag{C.35}$$

Others require some calculation:

$$\begin{aligned}\{\gamma_{ij},\, H_N\}_{\mathrm{P}} &= \int \mathrm{d}^{n-1}\mathbf{x}\frac{N}{\sqrt{\gamma}}\left\{\gamma_{ij},\, \pi^{kl}\pi_{kl} - \frac{1}{2}\pi^2\right\}_{\mathrm{P}} \\ &= \int \mathrm{d}^{n-1}\mathbf{x}\frac{N}{\sqrt{\gamma}}\left(\gamma_{km}\gamma_{ln} - \frac{1}{2}\gamma_{kl}\gamma_{mn}\right)\left(\delta_i^k\delta_j^l\pi^{mn} + \delta_i^m\delta_j^n\pi^{kl}\right) \\ &= \int \mathrm{d}^{n-1}\mathbf{x}\frac{N}{\sqrt{\gamma}}\left(2\pi_{ij} - \pi\gamma_{ij}\right), \end{aligned} \tag{C.36}$$

$$\{\gamma_{ij},\, H_{\mathbf{N}}\}_{\mathrm{P}} = -2\int \mathrm{d}^{n-1}\mathbf{x}N^k\left\{\gamma_{ij},\, \Gamma^l_{lm}\pi_k^m\right\}_{\mathrm{P}} = -2\int \mathrm{d}^{n-1}\mathbf{x}N_j\Gamma^l_{li}. \tag{C.37}$$

It is now possible to obtain the DBs $\{F,\, G\}_{\mathrm{D}}$ with

$$F,\, G \in \mathcal{S} := \left\{\gamma_{ij},\, B^\mu,\, H^0_\mu,\, \pi^{ij},\, \pi_{B^\mu},\, \pi_{H_{0\mu}}\right\},\; F \neq G, \tag{C.38}$$

by beginning with the nonzero PBs of H_N, $H_{\mathbf{N}}$, $\pi_{H_{0\mu}}$, $\pi_{B^\mu} + \sqrt{|g|}H^0_\mu$ with the elements of $\mathcal{S}$.
For a DB to differ from a PB we need some T_T, $T \geq 1$ to be nonzero. The table below summarises those T_T which can be nonzero (provided the G-dependent PB is also nonzero) for a given choice of F.

F	γ_{ij}	B^μ	H^0_μ	π^{ij}	π_{B^μ}	$\pi_{H^0_\mu}$
T	1 to 4	5	6	None	None	5

A similar table for G is shown below.

G	γ_{ij}	B^μ	H^0_μ	π^{ij}	π_{B^μ}	$\pi_{H^0_\mu}$
T	1 to 4	6	5	None	None	6

Thus a choice of $F \neq G$ for which some $T_T \neq 0$ for some $T \geq 1$ requires

$$(F,\, G) \in \left\{\left(B^\mu,\, H^0_\nu\right),\, \left(\pi_{H_{0\mu}},\, H^0_\nu\right),\, \left(H^0_\mu,\, B^\nu\right),\, \left(H^0_\mu,\, \pi_{H_{0\nu}}\right)\right\} \tag{C.39}$$

(Eq. (C.39) uses round brackets to denote ordered pairs, not to denote antibrackets). In each of these cases, $T \in \{5,\, 6\}$. This simplifies the nonzero DBs distinct from PBs to

$$\{F,\, G\}_{\mathrm{D}} = \{F,\, G\}_{\mathrm{P}} + \left\{F,\, \pi_{B^\nu} + \sqrt{|g|}H^0_\nu\right\}_{\mathrm{P}}\{\pi_{H_{0\nu}},\, G\}_{\mathrm{P}} - \{F,\, \pi_{H_{0\nu}}\}_{\mathrm{P}}\left\{\pi_{B^\nu} + \sqrt{|g|}H^0_\nu,\, G\right\}_{\mathrm{P}}. \tag{C.40}$$

Such DBs are as follows:

$$\left\{B^\mu,\, H^0_\nu\right\}_{\mathrm{D}} = g^{00}\delta^\mu_\nu\delta\left(\mathbf{x},\, \mathbf{x}'\right), \tag{C.41}$$

$$\left\{H^0_\mu,\, \pi_{H^0_\nu}\right\}_{\mathrm{D}} = 0. \tag{C.42}$$

In particular, all DBs with the primary constraints vanish as required.
Further, the following Dirac bracket is identical with its usual Poisson bracket, which was also required:

$$\left\{\pi_{B^\mu} + \sqrt{|g|}H^0_\mu,\, \pi_{H_{0\nu}}\right\}_{\mathrm{D}} = \sqrt{|g|}g^{00}\delta^\nu_\mu\delta\left(\mathbf{x},\, \mathbf{x}'\right). \tag{C.43}$$

D There are no "new" Noether charges

Let ξ^μ denote an arbitrary Killing vector (if multiple Killing vectors need to be considered I write $\xi_1^\mu, \cdots$). Let Q_{00} denote a bosonic conserved charge that is neither BRST-invariant nor anti-BRST-invariant. In the Landau gauge, Yang–Mills theory and perturbative gravity in the de Donder gauge provide respective examples

$$Q_{00}^{\text{Example 1}} = \int_{\mathbf{x}} A^0, \qquad Q_{00}^{\text{Example 2}} = \int_{\mathbf{x}} \xi^\mu \kappa H_\mu^0. \tag{D.1}$$

Define

$$Q_{01} = [Q, Q_{00}], Q_{10} = [\overline{Q}, Q_{00}], Q_{11} = \{\overline{Q}, Q_{01}\} = \{\overline{Q}, [Q, Q_{00}]\}. \tag{D.2}$$

Thus Q_{ij} is fermionic if and only if $i+j$ is odd. The results $\nabla_\mu Q = \nabla_\mu \overline{Q} = 0$ imply that Q_{01}, Q_{10}, Q_{11} are also conserved.

Each of the four conserved charges has a 2-bit label that could instead be written as one base-4 digit. Of interest are the values of $Q_{ijkl} := [Q_{ij}, Q_{kl}]_\pm$ with $2i+j \le 2k+l$, where $\pm$ is a $+$ sign if and only if Q_{ij}, Q_{kl} are both fermionic. Since each Q_{ij} is conserved, so is each Q_{ijkl}. The aim is to show that, for the cases in Eq. (D.1), the Q_{ijkl} are linear combinations of terms of the form Q_{ij}, so that there are no "new" conservation laws. (Trivial conserved charges may also appear in expressions for Q_{ijkl}. Examples include $\int_{\mathbf{x}} V^0$ where $\nabla_\mu V^\mu = 0$ and integrals of the form $\int_x \xi^\mu A_\mu^0$ where $A^{\alpha\beta} = A^{\beta\alpha}$ and $\nabla_\alpha A^{\alpha\beta} = 0$ for some tensor $A^{\alpha\beta}$.)

For brevity we may use a single hexadecimal digit[65] to label the (anti)commutator, viz.

$$Q_{8i+4j+2k+l} := Q_{ijkl}. \tag{D.3}$$

(Since conserved charges are labelled with two bits, this creates no ambiguity.) Note that Q_{00} has a lower case Roman index in the Yang–Mills case, and for perturbative gravity there is one Q_{00} for each Killing vector, so that the vector spaces of charges Q_{00} is spanned by the Q_{00}^A (say) that ξ_A^μ generates. Hereafter upper Roman indices will be used for both cases, so (anti)commutators have two indices, viz. $Q_{ijkl}^{AB} = [Q_{ij}^A, Q_{kl}^B]_\pm$. The "diagonal terms" are $Q_0^{AB}, Q_5^{AB}, Q_{\mathrm{A}}^{AB}, Q_{\mathrm{F}}^{AB}$. They are called diagonal because they satisfy $2i+j = 2k+l$. They trivially vanish in the case $A = B$. The "non-diagonal" (anti)commutators of interest are $Q_1, Q_2, Q_3, Q_6, Q_7, Q_{\mathrm{B}}$. (If decimal labels were used instead of hexadecimal ones, the labels $10, 11$ would have been used for $Q_{\mathrm{A}}, Q_{\mathrm{B}}$, creating an ambiguity with the previously defined charges Q_{10}, Q_{11}.)

Some cases are trivial. For the cases in Eq. (D.1) $Q_1 = 0$, since Q_{00}, Q_{01} are proportional to momentum zero modes. Indeed $Q_{00} \propto \pi_B$ completely decouples from the FP-ghost sector, so $Q_2 = 0$. I discuss the other Q_{ijkl} below as follows:

- I discuss Q_3 in Sec. D.1;
- I discuss Q_6 in Sec. D.2;
- I discuss "diagonal" terms in Sec. D.3;
- I discuss Q_7, Q_{B} in Sec. D.4.

[65] The hexadecimal digits A-F will be deitalicised throughout.

D.1 The case of Q_3

A Nakanishi–Lautrup-sector term in the global gauge current is the only contribution to Q_3 in Yang–Mills theory that requires a detailed treatment. Dropping irrelevant factors and explicit t-dependence,

$$\begin{aligned}Q_3 &\propto \left[\pi_B(\mathbf{x}),\, \pi_B(\mathbf{y})\times B(\mathbf{y})\right]\\ &:= \int_{\mathbf{xy}} \left(\pi_B(\mathbf{x})\cdot\pi_B(\mathbf{y})\times B(\mathbf{y}) - \left(\pi_B(\mathbf{y})\times B(\mathbf{y})\right)\cdot\pi_B(\mathbf{x})\right)\\ &= \int_{\mathbf{xy}} f^{abc}\left(\pi_B^a(\mathbf{x})\,\pi_B^b(\mathbf{y})\,B^c(\mathbf{y}) - \pi_B^b(\mathbf{y})\,B^c(\mathbf{y})\,\pi_B^a(\mathbf{y})\right),\end{aligned} \tag{D.4}$$

where in the last line I use $f^{abc}=f^{bca}$. Using the identity $ABC-BCA=[A,\,B]\,C+B\,[A,\,C]$, the f^{abc} coefficient in the integrand of Eq. (D.4) is proportional to

$$\left[\pi_B^a(\mathbf{x}),\,\pi_B^b(\mathbf{y})\right]B^c(\mathbf{y})+\pi_B^b(\mathbf{y})\left[\pi_B^a(\mathbf{x}),\,B^c(\mathbf{y})\right]\propto\delta^{ac}. \tag{D.5}$$

But $f^{abc}\delta^{ac}=0$, so $Q_3=0$.
In perturbative gravity, $\pi_{B^\mu}=-\kappa H_\mu^0$. There are no terms in $J_\mu^{(B)}$ that contain both an $H_{\mu\nu}$ tensor and an undifferentiated B^ρ, so the only terms in $J_\mu^{(B)}$ that could contribute to Q_3 are those proportional to an undifferentiated B.
The only important term in Q_{11} is $\int_{\mathbf{y}} J^{(B)0}$, where $J_\mu^{(B)}$ is the B^μ-dependent vector field you defined. However, hereafter the Killing vector to which it is proportional will be denoted ξ_2^μ instead of ξ^μ. Thus $Q_3\propto\int_{\mathbf{xy}}\left[\xi_1^\mu(\mathbf{x})\,\pi_{B^\mu}(\mathbf{x}),\,\widetilde{J}^{(B)0}(\mathbf{y})\right]$, where $\widetilde{J}_\mu^{(B)}$ is obtained by deleting terms from $J_\mu^{(B)}$ that depend only on derivatives of B^μ. Define the integral operators

$$\int_{\mathbf{xy}} := \int \mathrm{d}^{n-1}\mathbf{x}\int_{\mathbf{y}},\ \int_{\mathbf{yx}} := \int \mathrm{d}^{n-1}\mathbf{y}\int_{\mathbf{x}}. \tag{D.6}$$

Explicitly $\widetilde{J}_\mu^{(B)}=-\xi_2^\nu\gamma_{\mu\nu}^{\rho\sigma}\kappa B^\gamma\nabla_\gamma h_{\rho\sigma}$ so

$$\begin{aligned}Q_3 &\propto \int_{\mathbf{xy}} \kappa\xi_1^\mu(\mathbf{x})\,\xi_2^\nu(\mathbf{y})\left[\pi_{B^\mu}(\mathbf{x}),\,B^\gamma(\mathbf{y})\,\nabla_\gamma H_\nu^0(\mathbf{y})\right]\\ &\propto \int_{\mathbf{xy}} \kappa\xi_1^\mu(\mathbf{x})\,\xi_2^\nu(\mathbf{y})\,\delta(\mathbf{x},\,\mathbf{y})\,\nabla_\mu H_\nu^0(\mathbf{y}) = \int_{\mathbf{y}} \kappa\xi_2^\nu\nabla_\mu\left(\xi_1^\mu H_\nu^0\right).\end{aligned} \tag{D.7}$$

Since the δ factor ultimately allows all arguments to be set to $\mathbf{y}$, Q_3 is $\xi_1\leftrightarrow\xi_2$–symmetric, so

$$Q_3\propto\int_{\mathbf{y}}\kappa\left[\xi_2^\nu\nabla_\mu\left(\xi_1^\mu H_\nu^0\right)+\xi_1^\nu\nabla_\mu\left(\xi_2^\mu H_\nu^0\right)\right]. \tag{D.8}$$

Define $\Phi := H^{0\nu}\xi_{1\nu}$, so $\int_{\mathbf{y}}\Phi$ is a conserved charge. Up to total derivatives

$$
\begin{aligned}
\xi_2^\nu \nabla_\mu \left(\xi_1^\mu H_\nu^0\right) &\approx -\left(\nabla_\mu \xi_{2\nu}\right)\left(\xi_1^\mu H^{0\nu}\right) \\
&= \left(\nabla_\nu \xi_{2\mu}\right)\left(\xi_1^\mu H^{0\nu}\right) = \left(\nabla_\mu \xi_{2\nu}\right)\left(\xi_1^\nu H^{0\mu}\right) \\
&\approx -\xi_{2\nu}\nabla_\mu\left(\xi_1^\nu H^{0\mu}\right) = -\left(\xi_{2\nu}\nabla_\mu \xi_1^\nu\right) H^{0\mu} \\
&= \left(\xi_{2\nu}\nabla^\nu \xi_{1\mu}\right) H^{0\mu} \\
&= \left(\xi_{1\nu}\nabla^\nu \xi_2^\mu + \pounds_{\xi_2}\xi_1^\mu\right) H_\mu^0 \\
&= \left(\pounds_{\xi_2}\xi_1^\mu - \xi_{1\nu}\nabla^\mu \xi_2^\nu\right) H_\mu^0 \\
&= H_\mu^0 \pounds_{\xi_2}\xi_1^\mu - \xi_{1\nu}\nabla^\mu\left(\xi_2^\nu H_\mu^0\right), \qquad \text{(D.9)} \\
\xi_2^\nu \nabla_\mu \left(\xi_1^\mu H_\nu^0\right) + \xi_1^\nu \nabla_\mu \left(\xi_2^\mu H_\nu^0\right) &\approx H_\mu^0 \pounds_{\xi_2}\xi_1^\mu - \xi_{1\nu}\nabla^\mu\left(\xi_2^\nu H_\mu^0 - \xi_{2\mu}H^{0\nu}\right) \\
&= H_\mu^0\left(\pounds_{\xi_2}\xi_1^\mu - \xi_{1\nu}\nabla^\mu \xi_2^\nu\right) + \xi_{1\nu}\xi_{2\mu}\nabla^\mu H^{0\nu} \\
&= \xi_{2\mu}\nabla^\mu \Phi = \nabla_\mu\left(\xi_2^\mu \Phi\right) \approx 0. \qquad \text{(D.10)}
\end{aligned}
$$

There is therefore no independent Noether charge.

D.2 The case of Q_6

For Yang–Mills theory, $\pi_{\overline{c}} = -\mathrm{i}\sqrt{|g|}D^0 c$ and $\pi_c = \mathrm{i}\sqrt{|g|}\nabla^0 \overline{c}$ so $Q_6 \propto \int_{\mathbf{yx}} \left[\pi_{\overline{c}}(\mathbf{x}), A^0(\mathbf{y}) \times c(\mathbf{y})\right]$, which is a linear combination of the $\int_{\mathbf{y}} A^{0a}(\mathbf{y})$. For gravity define

$$
\begin{aligned}
Q_c(\xi_1) &= \int \mathrm{d}^{n-1}\mathbf{x}\left(\xi_1^\alpha \pi_{c^\alpha} + \mathrm{i}\sqrt{|g|}\kappa H_\alpha^0 \pounds_{\xi_1}\overline{c}^\alpha - \mathrm{i}\sqrt{|g|}\xi_1^0 \nabla^\alpha \overline{c}^\beta \kappa H_{\alpha\beta}\right), \qquad \text{(D.11)} \\
Q_{\overline{c}}(\xi_2) &= -\mathrm{i}\int \mathrm{d}^{n-1}\mathbf{y}\,\xi_2^\gamma \pi_{\overline{c}^\gamma} = -\int \mathrm{d}^{n-1}\mathbf{y}\,\xi_2^\gamma Z_\gamma^0. \qquad \text{(D.12)}
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\{\xi_1^\alpha \pi_{c^\alpha}, \xi_2^\gamma \pi_{\overline{c}^\gamma}\right\} &= 0, \qquad \text{(D.13)} \\
\left\{H_\alpha^0 \pounds_{\xi_1}\overline{c}^\alpha, \xi_2^\gamma \pi_{\overline{c}^\gamma}\right\} &= \left\{\pounds_{\xi_1}\overline{c}^\alpha, \pi_{\overline{c}^\gamma}\right\}\xi_2^\gamma H_\alpha^0 \\
&= -\mathrm{i}\delta(\mathbf{x}, \mathbf{y})\,\delta_\gamma^\alpha \pounds_{\xi_1}\xi_2^\gamma H_\alpha^0 \\
&= -\mathrm{i}\delta(\mathbf{x}, \mathbf{y})\,\pounds_{\xi_1}\xi_2^\alpha H_\alpha^0, \qquad \text{(D.14)} \\
\left\{\xi_1^0 \nabla^\alpha \overline{c}^\beta H_{\alpha\beta}, \xi_2^\gamma \pi_{\overline{c}^\gamma}\right\} &= 0, \qquad \text{(D.15)} \\
\left\{Q_c(\xi_1), Q_{\overline{c}}(\xi_2)\right\} &= -\mathrm{i}\int \mathrm{d}^{n-1}\mathbf{x}\,\mathrm{d}^{n-1}\mathbf{y}\left(\mathrm{i}\sqrt{|g|}\right)\left(-\mathrm{i}\delta(\mathbf{x}, \mathbf{y})\,\kappa H_\alpha^0 \pounds_{\xi_1}\xi_2^\alpha\right) \\
&= -\mathrm{i}\kappa \int_{\mathbf{x}} \pounds_{\xi_1}\xi_2^\alpha H_\alpha^0. \qquad \text{(D.16)}
\end{aligned}
$$

D.3 The "diagonal" cases

Since Q_{00}^A, Q_{01}^B are proportional to momenta, $Q_0^{AB} = 0$, $Q_5^{AB} = 0$.

We may write Q_{10}^A in the form $\pi_{10}^A + q_{10}^A$, where π_{10}^A is proportional to a momentum and q_{10}^A has no dependence on the field to which said momentum is conjugate. For example, for Yang–Mills theory

$$
Q_{10} = \int_{\mathbf{x}} D^0 \overline{c},\ \pi_{10} = \int_{\mathbf{x}} \nabla^0 \overline{c} \propto \int_{\mathbf{x}} \pi_c,\ q_{10} = q\int_{\mathbf{x}} A^0 \times \overline{c}. \qquad \text{(D.17)}
$$

Thus $Q_{\mathrm{A}}^{AB} = \left\{q_{10}^{A},\, q_{10}^{B}\right\}$. For Yang–Mills theory $Q_{\mathrm{A}}^{ab} = q^2 f^{abc} f^{a'b'c'} \int_{\mathbf{x}}\int_{\mathbf{y}} \left\{A^{0b}\overline{c}^{c},\, A^{0b'}\overline{c}^{c'}\right\}$, which simplifies to 0 by the usual equal-time CARs and CCRs and the identity

$$\left\{b_1 f_1,\, b_2 f_2\right\} = b_1\left[f_1,\, b_2\right] f_2 + \left[b_1,\, b_2\right] f_1 f_2 + b_2 b_1 \left\{f_1,\, f_2\right\} - b_2\left[b_1,\, f_2\right] f_1, \tag{D.18}$$

where $b_1,\, b_2$ $(f_1,\, f_2)$ are bosonic (fermionic) fields. (Taking $b_1 = A_0^b,\, f_1 = \overline{c}^c$ etc. gives $[f_1,\, b_2] = 0$ etc.) The analogous calculation for perturbative gravity obtains

$$\begin{aligned} Q_{\mathrm{A}}^{AB} &\propto \left\{Q_c\left(\xi_A^{\mu}\right),\, Q_c\left(\xi_B^{\mu}\right)\right\} \\ &= \int \mathrm{d}^{n-1}\mathbf{x} \int \mathrm{d}^{n-1}\mathbf{y} \left\{\xi_A^{\alpha}\pi_{c^{\alpha}} + \mathrm{i}\sqrt{|g|}\kappa H_{\alpha}^{0}\pounds_{\xi_A}\overline{c}^{\alpha} - \mathrm{i}\sqrt{|g|}\xi_A^{0}\nabla^{\alpha}\overline{c}^{\beta}\kappa H_{\alpha\beta}\,,\right. \\ &\left.\underline{\xi_B^{\gamma}\pi_{c^{\gamma}} + \mathrm{i}\sqrt{|g|}\kappa H_{\gamma}^{0}\pounds_{\xi_B}\overline{c}^{\gamma} - \mathrm{i}\sqrt{|g|}\xi_B^{0}\nabla^{\gamma}\overline{c}^{\delta}\kappa H_{\gamma\delta}}\right\}, \end{aligned} \tag{D.19}$$

where underlined terms are $\mathbf{y}$-dependent. By inspection $Q_{\mathrm{A}}^{AB} = 0$.
To address Q_{F}^{AB} is trickier. For Yang–Mills theory

$$\left[A_0^a\left(\mathbf{x}\right),\, B^b\left(\mathbf{y}\right)\right] = \left[-\varpi_B^a\left(\mathbf{x}\right),\, B^b\left(\mathbf{y}\right)\right] = \frac{\mathrm{i}}{\sqrt{|g|}}\delta^{ab}\delta\left(\mathbf{x},\,\mathbf{y}\right), \tag{D.20}$$

$$\begin{aligned} Q_{11}^{b} &= f^{bcd}\int_{\mathbf{y}}\left(\underline{A^{\nu c}F_{\nu}^{\;0d} + A^{0c}B^{d} - \mathrm{i}\overline{c}^{c}D^{0}c^{d} + \mathrm{i}\left(\nabla^{0}\overline{c}^{c}\right)c^{d}}\right) \\ &= f^{bcd}\int_{\mathbf{y}}\left(\underline{A^{\nu c}F_{\nu}^{\;0d} + A^{0c}B^{d} + \overline{c}^{c}\varpi_{\overline{c}}^{d} + \varpi_{c}^{c}c^{d}}\right), \end{aligned} \tag{D.21}$$

$$\begin{aligned} Q_{\mathrm{F}}^{ab} &= f^{aef}f^{bcd}\int_{\mathbf{x}}\int_{\mathbf{y}} \\ &\times\left[A^{0\mathrm{e}}B^{f} + \overline{c}^{\mathrm{e}}\varpi_{\overline{c}}^{f} + \varpi_{c}^{\mathrm{e}}c^{f},\, \underline{A^{0c}B^{d} + \overline{c}^{c}\varpi_{\overline{c}}^{d} + \varpi_{c}^{c}c^{d}}\right]. \end{aligned} \tag{D.22}$$

From the identity $[b_1 b_2,\, b_3 b_4] = b_1 b_3\left[b_2,\, b_4\right] + b_1\left[b_2,\, b_3\right] b_4 + b_3\left[b_1,\, b_4\right] b_2 + \left[b_1,\, b_3\right] b_4 b_2$, the Nakanishi–Lautrup contribution is proportional to

$$\left(f^{ace}f^{bcd} - a\leftrightarrow b\right)\int_{\mathbf{x}}\pi_B^{\mathrm{e}}B^{d} = \left(f^{ace}f^{bcd} - d\leftrightarrow \mathrm{e}\right)\int_{\mathbf{x}}\pi_B^{\mathrm{e}}B^{d} = f^{ace}f^{bcd}\int_{\mathbf{x}}\left(\pi_B^{\mathrm{e}}B^{d} - \pi_B^{d}B^{\mathrm{e}}\right). \tag{D.23}$$

The first of these three expressions verifies an a$\leftrightarrow$ b–antisymmetry. The third shows that a $d\leftrightarrow \mathrm{e}$ exchange combines an $a\leftrightarrow b$ exchange with a further sign change, i.e. there is a $d\leftrightarrow \mathrm{e}$–symmetry. In the second expression, the factor $f^{ace}f^{bcd} - d\leftrightarrow \mathrm{e}$ is $d\leftrightarrow \mathrm{e}$–antisymmetric, but $\int_{\mathbf{x}}\pi_B^{\mathrm{e}}B^{d}$ is neither $d\leftrightarrow \mathrm{e}$–symmetric nor $d\leftrightarrow \mathrm{e}$–antisymmetric. To prevent a contradiction, the Nakanishi–Lautrup contribution is therefore zero. The same argument also vanishes the contributions from the ghost and antighost sectors, using the identity

$$\left[f_1 f_2,\, f_3 f_4\right] = -f_1 f_3\left\{f_2,\, f_4\right\} + f_1\left\{f_2,\, f_3\right\} f_4 - f_3\left\{f_1,\, f_4\right\} f_2 + \left\{f_1,\, f_3\right\} f_4 f_2. \tag{D.24}$$

Since all nonzero commutators in CCRs and nonzero anticommutators in CARs are proportional to $\delta\left(\mathbf{x},\,\mathbf{y}\right)$, for perturbative gravity a tensor $\mathrm{F}_{\mu\nu}$[66] exists for which $Q_{\mathrm{F}}^{AB} = \int_{\mathbf{x}}\xi_A^{\mu}\xi_B^{\nu}\mathrm{F}_{\mu\nu}$, which is an $A\leftrightarrow B$–antisymmetric pseudo-inner product on the Killing vector fields. We require this product

[66] The F is deitalicised because it is a hexadecimal digit, in reference to Q_{F}^{AB}. This notational convention will be used again in Sec. D.4 with the digits 7, B.

to be identically zero. It is completely specified by its pseudo-norms, so we need only verify these vanish. Since the pseudo-norms Q_{F}^{AA} vanish, and the pseudo-norm of a Killing vector field ξ^{μ} may be expressed in this form by choosing a basis of the Killing vector fields for which some A satisfies $\xi^{\mu}=\xi_{A}^{\mu}$, the pseudo-inner product's pseudo-norms all vanish as required.

D.4 The cases of $Q_7,\, Q_{\mathbf{B}}$

Since $Q_7^{AB},\, Q_{\mathrm{B}}^{AB}$ are fermionic, they should be expressible in terms of the $Q_{01}^{C},\, Q_{10}^{D}$ charges. Indeed, this is verified below.

The next few calculations make repeated use of the identity $[f_1,\, f_2 f_3]=\{f_1,\, f_2\}\, f_3-f_2\,\{f_1,\, f_3\}$ for classical fermionic fields $f_1,\, f_2,\, f_3$. For Yang–Mills theory

$$Q_7^{ab}\propto f^{bcd}\int \mathrm{d}^{n-1}\mathbf{x}\mathrm{d}^{n-1}\mathbf{y}\left[\pi_{\overline{c}}^{a},\, \underline{\overline{c}^{c}\pi_{\overline{c}}^{d}}\right]\propto f^{abd}\int \mathrm{d}^{n-1}\mathbf{x}\pi_{\overline{c}}^{d}, \tag{D.25}$$

$$\begin{aligned}
Q_{\mathrm{B}}^{ab}&=f^{bcd}\int_{\mathbf{x}}\int_{\mathbf{y}}\left[-\mathrm{i}\varpi_{c}^{a}+qf^{aef}A^{0\mathrm{e}}\overline{c}^{f},\, \underline{A^{0c}B^{d}+\overline{c}^{c}\varpi_{\overline{c}}^{d}+\varpi_{c}^{c}c^{d}}\right]\\
&=f^{bcd}\int_{\mathbf{x}}\int_{\mathbf{y}}\left(-\mathrm{i}\left[\varpi_{c}^{a},\, \underline{\varpi_{c}^{c}c^{d}}\right]+qf^{aef}\left[A^{0\mathrm{e}},\, \underline{B^{d}}\right]\underline{A^{0c}}\overline{c}^{f}+qf^{aef}A^{0\mathrm{e}}\left[\overline{c}^{f},\, \underline{\overline{c}^{c}\varpi_{\overline{c}}^{d}}\right]\right)\\
&=f^{bcd}\int_{\mathbf{x}}\int_{\mathbf{y}}\left(\delta^{ad}\underline{\varpi_{c}^{c}}+\mathrm{i}qf^{adf}\underline{A^{0c}}\overline{c}^{f}+\mathrm{i}qf^{aef}A^{0\mathrm{e}}\delta^{df}\underline{\overline{c}^{c}}\right)\frac{\delta\left(\mathbf{x},\, \mathbf{y}\right)}{\sqrt{\left|g\left(t,\, \mathbf{x}\right)\right|}}\\
&=f^{bcd}\int_{\mathbf{y}}\left(\delta^{ad}\underline{\varpi_{c}^{c}}+\mathrm{i}qf^{adf}\underline{A^{0c}\overline{c}^{f}}+\mathrm{i}qf^{aed}\underline{A^{0\mathrm{e}}\overline{c}^{c}}\right)\\
&=f^{bcd}\int_{\mathbf{x}}\left(\delta^{ad}\varpi_{c}^{c}+\mathrm{i}qf^{adf}\left(A^{0c}\overline{c}^{f}-A^{0f}\overline{c}^{c}\right)\right)\\
&=\int_{\mathbf{x}}\left(f^{abc}\varpi_{c}^{c}+\mathrm{i}qf^{adf}f^{bcd}\left(A^{0c}\overline{c}^{f}-A^{0f}\overline{c}^{c}\right)\right).
\end{aligned} \tag{D.26}$$

The result for Q_7^{ab} is expressible in terms of the Q_{01}^{d}. The result for Q_7^{ab} is expressible in terms of Q_{10}^{d} terms, since

$$f^{daf}f^{dbc}\int_{\mathbf{x}}\left(A^{0c}\overline{c}^{f}-A^{0f}\overline{c}^{c}\right)=\left(f^{daf}f^{dbc}-c\leftrightarrow f\right)\int_{\mathbf{x}}\left(A^{0c}\overline{c}^{f}\right)=\left(f^{daf}f^{dbc}-a\leftrightarrow b\right)\int_{\mathbf{x}}\left(A^{0c}\overline{c}^{f}\right) \tag{D.27}$$

vanishes by the same argument used in the above analysis of Eq. (D.23).

For perturbative gravity there exist tensors $7_{\mu\nu},\, \mathrm{B}_{\mu\nu}$ analogous to $\mathrm{F}_{\mu\nu}$, viz.

$$Q_7^{AB}=\int_{\mathbf{x}}\xi_{A}^{\mu}\xi_{B}^{\nu}7_{\mu\nu},\, Q_{\mathrm{B}}^{AB}=\int_{\mathbf{x}}\xi_{A}^{\mu}\xi_{B}^{\nu}\mathrm{B}_{\mu\nu}. \tag{D.28}$$

Since a fermionic field f satisfies $[f,\, \{Q,\, f\}]=[f,\, Qf+fQ]=fQf+f^{2}Q-Qf^{2}-fQf=0$ and similarly with $\overline{Q}$, the results $Q_7^{AA}=Q_{\mathrm{B}}^{AA}=0$ are trivial. The properties of pseudo-inner products used to show that $Q_{\mathrm{F}}^{AB}=0$ imply $Q_7^{AB}=0,\, Q_{\mathrm{B}}^{AB}=0$.

E The equivalence of “two Noether charges” in perturbative gravity

In Sec. E.1, I present a proof two conserved expressions are the same Noether charge. The rest of this appendix provides details that Sec. E.1 leaves unaddressed. There are two types of term that must be considered, and these are collected in Eq. (E.3) below. The terms differ in whether they depend on the Nakanishi–Lautrup auxiliary field. Those which do are considered in Sec. E.2; those which do not are considered in Sec. E.3.

E.1 Overview

The field equation obtained by varying c^μ is $\nabla_\nu \mathcal{T}^{\mu\nu} = 0$ where

$$\mathcal{T}_{\mu\nu} := T_{\mu\nu} + (2k-1)\,\kappa\nabla^\beta \bar{c}^\gamma \left(g_{\mu\nu}h_{\beta\gamma} - g_{\beta\gamma}h_{\mu\nu}\right), \quad T_{\mu\nu} := \gamma^{\rho\sigma}_{\mu\nu}\left(\nabla_\rho \bar{c}_\sigma + \kappa \pounds_{\bar{c}} h_{\rho\sigma}\right) = \left[\overline{Q},\, H_{\mu\nu}\right]. \quad \text{(E.1)}$$

Similarly, varying $\bar{c}^\mu$ gives $\nabla_\nu Z^{\mu\nu} = 0$. Since $Z_{\mu\nu}$, $\mathcal{T}_{\mu\nu}$ are symmetric elements of $\ker \nabla^\nu$, for any Killing vector field ξ^α the currents[67] $J_{(c)\mu} := \xi^\alpha Z_{\mu\alpha}$, $J_{(\bar{c})\mu} := \xi^\alpha \mathcal{T}_{\mu\alpha}$ are conserved.
Of interest is a comparison of $\left\{Q,\, J^\mu_{(\bar{c})}\right\}$ and $\left\{\overline{Q},\, J^\mu_{(c)}\right\}$ to the spacetime-isometry current. In fact

$$\begin{aligned}
\left\{\overline{Q},\, J^\mu_{(c)}\right\} + \left\{Q,\, J^\mu_{(\bar{c})}\right\} &= \xi^\alpha \left(\{\overline{Q},\, S_{\mu\alpha}\} + \{Q,\, \mathcal{T}_{\mu\alpha}\}\right) \\
&= \xi^\alpha \left(\{\overline{Q},\, S_{\mu\alpha}\} + \{Q,\, T_{\mu\alpha}\}\right) + \xi^\alpha \{Q,\, \mathcal{T}_{\mu\alpha} - T_{\mu\alpha}\} \\
&= \xi^\alpha \left(\{\overline{Q},\, [Q,\, H_{\mu\alpha}]\} + \{Q,\, [\overline{Q},\, H_{\mu\alpha}]\}\right) \\
&\quad + (2k-1)\,\kappa\xi^\alpha \left\{Q,\, \nabla^\beta \bar{c}^\gamma \left(g_{\mu\nu}h_{\beta\gamma} - g_{\beta\gamma}h_{\mu\nu}\right)\right\} \\
&= (2k-1)\,\kappa\xi^\alpha \left\{Q,\, \nabla^\beta \bar{c}^\gamma \left(g_{\mu\nu}h_{\beta\gamma} - g_{\beta\gamma}h_{\mu\nu}\right)\right\}, \qquad \text{(E.2)}
\end{aligned}$$

which vanishes if and only if $k = \frac{1}{2}$. What is shown hereafter is that $\int_{\mathbf{x}} \left\{Q,\, \mathrm{i}J^0_{(\bar{c})}\right\}$ is the spacetime-isometry charge, so that in the anti-BRST-invariant case $k = \frac{1}{2}$ this charge is $\int_{\mathbf{x}} \left\{\overline{Q},\, \mathrm{i}J^0_{(c)}\right\}$. The spacetime-isometry charge may therefore be represented as the transformation of some Noether charge under whichever of the BRST and anti-BRST transformations is action-preserving in the chosen gauge.

We first write

$$\left\{Q,\, \mathrm{i}J^\mu_{(\bar{c})}\right\} = J^{(B)}_\mu + \mathrm{i}J^{(\bar{c}c)}_\mu, \qquad \text{(E.3)}$$

where $J^{(B)}_\mu$ contains terms that depend on a (possibly differentiated) Nakanishi–Lautrup auxiliary field because of the BRST transformation of the FP-ghost, while $J^{(\bar{c}c)}_\mu$ contains other terms that result from the BRST transformation of the metric perturbation $\kappa h_{\mu\nu}$. The expressions

$$\begin{aligned}
J^{(B)}_\mu &= -\xi^\nu \left[\gamma^{\rho\sigma}_{\mu\nu}\left(\nabla_\rho B_\sigma + \kappa \pounds_B h_{\rho\sigma}\right) + (2k-1)\,\kappa\nabla^\beta B^\gamma \left(g_{\mu\nu}h_{\beta\gamma} - g_{\beta\gamma}h_{\mu\nu}\right)\right], \qquad \text{(E.4)} \\
J^{(\bar{c}c)}_\mu &= \xi^\alpha \left[\bar{c}^\beta \nabla_\beta Z_{\mu\alpha} + \nabla_\mu \bar{c}^\beta Z_{\beta\alpha} + \nabla_\alpha \bar{c}^\beta Z_{\beta\mu} - g_{\alpha\mu}\nabla^\beta \bar{c}^\gamma S_{\beta\gamma}\right. \\
&\quad \left. + (1-2k)\,\nabla_\beta \bar{c}^\beta Z_{\mu\alpha} + kT_{\mu\alpha}\left(2\nabla_\lambda c^\lambda + \kappa c^\beta \nabla_\beta h + 2\kappa \nabla^\beta c^\gamma h_{\beta\gamma}\right)\right] \qquad \text{(E.5)}
\end{aligned}$$

[67]Each Noether current has a subscript that refers to the fermionic field appearing in that current's definition. This is not the same fermionic field which is varied to obtain the Euler–Lagrange equation that implies a conservation law.

may be compared to the spacetime-isometry transformations

$$\delta_{\rm st}\kappa h_{\mu\nu} = \nabla_\mu\left(\alpha\xi_\nu\right) + \nabla_\nu\left(\alpha\xi_\mu\right) + \kappa\left(\alpha\xi^\lambda\nabla_\lambda h_{\mu\nu} + \nabla_\mu\left(\alpha\xi^\lambda\right)h_{\lambda\nu} + \nabla_\nu\left(\alpha\xi^\lambda\right)h_{\mu\lambda}\right), \tag{E.6}$$

$$\delta_{\rm st}B^\mu = \alpha\pounds_\xi B^\mu = \alpha\left(\xi^\lambda\nabla_\lambda B^\mu - \left(\nabla_\lambda\xi^\mu\right)B^\lambda\right), \tag{E.7}$$

$$\delta_{\rm st}c^\mu = \alpha\pounds_\xi c^\mu = \alpha\left(\xi^\lambda\nabla_\lambda c^\mu - \left(\nabla_\lambda\xi^\mu\right)c^\lambda\right). \tag{E.8}$$

The fact that $\nabla^\mu B^\nu\delta_{\rm st}H_{\mu\nu} - \nabla^\mu\delta_{\rm st}B^\nu H_{\mu\nu}$ is of the form $\alpha X + \left(\nabla_\mu\alpha\right)Y^\mu$ accounts for a contribution $Y^\mu + \kappa\xi^\mu\nabla^\beta B^\gamma H_{\beta\gamma}$ to $J^{(B)}_\mu$ (since $\xi^\mu\nabla^\beta B^\gamma H_{\beta\gamma}$ is the Nakanishi–Lautrup part of $-\xi^\mu\mathcal{L}_0$). When this contribution and $\delta_{\rm st}B^\mu$ are subtracted, what remains is

$$J^{(B)\rm rest}_\mu := -\kappa\xi^\alpha\left(B^\beta\nabla_\beta H_{\mu\alpha} + \nabla_\alpha B^\beta H_{\beta\mu} - H_\alpha{}^\nu\nabla_\nu B_\mu + H_{\mu\alpha}\nabla_\beta B^\beta\right). \tag{E.9}$$

The spacetime-isometry current obtained by varying B^μ is

$$J^{(B,\,\rm st,\,B)}_\mu := \kappa\left(B^\alpha\left(\nabla_\alpha\xi^\nu\right)H_{\mu\nu} - \left(\xi^\alpha\nabla_\alpha B^\beta\right)H_{\beta\mu}\right). \tag{E.10}$$

The difference between Eqs. (E.9) and (E.10) is

$$J^{(1)}_\mu := -\kappa\xi^\alpha\left(B^\beta\nabla_\beta H_{\mu\alpha} - H_\alpha{}^\nu\nabla_\nu B_\mu + H_{\mu\alpha}\nabla_\beta B^\beta - B^\alpha\left(\nabla_\alpha\xi^\nu\right)H_{\mu\nu}\right). \tag{E.11}$$

Since $H_{\mu\alpha} = 0$ and $\nabla_\alpha H^{\alpha\beta} = 0$, $J^{(1)}_\mu$ is a total derivative, viz. $J^{(1)}_\mu = -\kappa\nabla_\alpha\left(B^\alpha\xi_\beta H^{\mu\beta} - B^\mu\xi_\beta H^{\alpha\beta}\right)$. An analogous quantity in the treatment of $J^{(\bar{c}c)}_\mu$ is

$$J^{(2)}_\mu := \xi^\alpha\left(\bar{c}^\beta\nabla_\beta Z_{\mu\alpha} - Z_\alpha{}^\nu\nabla_\nu\bar{c}_\mu + Z_{\mu\alpha}\nabla_\beta\bar{c}^\beta - \bar{c}^\alpha\left(\nabla_\alpha\xi^\nu\right)Z_{\mu\nu}\right) = \nabla_\alpha\left(\xi^\beta\bar{c}^\alpha Z^\mu{}_\beta - \xi^\beta\bar{c}^\mu Z^\alpha{}_\beta\right). \tag{E.12}$$

Next we consider $J^{(X)}_\mu + J^{(Y)}_\mu + J^{(Z)}_\mu$, where:

- $J^{(X)}_\mu$ is obtained by varying c^μ;
- $J^{(Y)}_\mu$ is obtained by varying $h_{\mu\nu}$ in $h_{\mu\nu}$-independent terms;
- $J^{(Z)}_\mu$ is obtained by varying $h_{\mu\nu}$ in $\nabla_\alpha h_{\mu\nu}$-dependent terms.

Explicitly

$$J^{(X)}_\mu := T_{\mu\nu}\left(\xi^\alpha\nabla_\alpha c^\nu - \nabla_\alpha\xi^\nu c^\alpha\right) + \kappa\xi^\alpha T^{\mu\beta}h_{\beta\gamma}\nabla_\alpha c^\gamma - \kappa\nabla_\beta\xi^\alpha T_\mu{}^\nu h_{\alpha\nu}c^\beta, \tag{E.13}$$

$$J^{(Y)}_\mu := \xi^\alpha\left(T^\beta{}_\alpha\nabla_\beta c^\mu + T^{\beta\mu}\nabla_\beta c_\alpha + \kappa T^{\beta\gamma}\nabla_\beta c^\mu h_{\alpha\gamma} + \kappa T^{\gamma\mu}\nabla_\gamma c^\beta h_{\alpha\beta}\right), \tag{E.14}$$

$$J^{(Z)}_\mu := -\nabla_\alpha\left(T^{\mu\nu}c^\alpha\right)\left(\xi_\nu + \kappa\xi^\beta h_{\beta\nu}\right) + \frac{1}{2}\kappa T^{\beta\gamma}c^\mu\xi^\alpha\nabla_\alpha h_{\beta\gamma}. \tag{E.15}$$

Equivalence up to total derivatives may be denoted $\approx$ so that

$$-T^\mu{}_\nu\left(\nabla_\alpha\xi^\nu\right)c^\alpha \approx \xi^\nu\nabla_\alpha\left(T^\mu{}_\nu c^\alpha - T^\alpha{}_\nu c^\mu\right), \tag{E.16}$$

$$\begin{aligned}\nabla_\beta\xi^\alpha\left(T^{\beta\nu}c^\mu - T^{\mu\nu}c^\beta\right)h_{\alpha\nu} \approx \xi^\alpha\{&-\nabla_\beta\left(T^{\beta\nu}h_{\alpha\nu}\right)c^\mu - T^{\beta\nu}h_{\alpha\nu}\nabla_\beta c^\mu \\ &+\nabla_\beta\left(T^{\mu\nu}c^\beta\right)h_{\alpha\nu} + T^{\mu\nu}c^\beta\nabla_\beta h_{\alpha\nu}\}.\end{aligned} \tag{E.17}$$

Using Eqs. (E.16) and (E.17) in Eqs. (E.13)–(E.15), the only non-vanishing terms containing derivatives of $T_{\alpha\beta}$ in $J^{(X)}_\mu + J^{(Y)}_\mu + J^{(Z)}_\mu$ are $-\nabla_\beta\left(T^{\beta\alpha} + T^{\beta\nu}h^\alpha_{\ \nu}\right)c^\mu\xi^\alpha$.

The next step is to verify the equations

$$\nabla_\beta \left(T^{\beta\alpha} + T^{\beta\nu} h^\alpha_{\ \nu}\right) = \nabla^\nu \overline{c}^\beta \nabla^\alpha H_{\beta\nu}, \tag{E.18}$$

$$\frac{1}{2} T^{\beta\gamma} \nabla_\alpha h_{\beta\gamma} = \nabla^\nu \overline{c}^\beta \nabla_\alpha H_{\beta\nu}. \tag{E.19}$$

These imply that eliminating the contributions of spacetime-isometry and $J^{(2)}_\mu$ from $J^{(\overline{c}c)}_\mu$ leaves exactly $J^{(X)}_\mu + J^{(Y)}_\mu + J^{(Z)}_\mu$. Then $\int_{\mathbf{x}} \left\{Q, \, \mathrm{i}J^0_{(\overline{c})}\right\}$ is the spacetime-isometry Noether charge.
To prove Eq. (E.18), observe that $\nabla_\beta \mathcal{T}^{\beta\alpha} = 0$ implies

$$\begin{aligned}
\nabla_\beta T^{\beta\alpha} &= (1-2k)\,\kappa \nabla_\beta \left(\nabla^\kappa \overline{c}^\lambda \left(g^{\alpha\beta} h_{\kappa\lambda} - g_{\kappa\lambda} h^{\alpha\beta}\right)\right) \\
&= (1-2k)\,\kappa \left(\nabla^\alpha \left(h_{\kappa\lambda} \nabla^\kappa \overline{c}^\lambda\right) - \nabla_\beta \left(h^{\alpha\beta} \nabla_\lambda \overline{c}^\lambda\right)\right), \qquad \text{(E.20)}\\
\nabla_\beta \left(T^{\beta\nu} h^\alpha_{\ \nu}\right) &= h^\alpha_{\ \nu} \nabla_\beta T^{\beta\nu} + T^{\beta\nu} \nabla_\beta h^\alpha_{\ \nu} \\
&= (1-2k)\,\kappa h^\alpha_{\ \nu} \left(\nabla^\nu \left(h_{\kappa\lambda} \nabla^\kappa \overline{c}^\lambda\right) - \nabla_\beta \left(h^{\nu\beta} \nabla_\lambda \overline{c}^\lambda\right)\right) \\
&\quad + \left(\nabla^\beta \overline{c}^\nu + \nabla^\nu \overline{c}^\beta + \kappa \left(g^{\beta\rho} g^{\nu\sigma} - k g^{\beta\nu} g^{\rho\sigma}\right) \pounds_c h_{\rho\sigma}\right) \nabla_\beta h^\alpha_{\ \nu} \\
&= \nabla^\nu \overline{c}^\beta \left(\nabla^\alpha h_{\beta\nu} - k g_{\beta\nu} \nabla^\alpha h^\gamma_\gamma\right) = \nabla^\nu \overline{c}^\beta \nabla^\alpha H_{\beta\nu}. \qquad \text{(E.21)}
\end{aligned}$$

To prove Eq. (E.19), observe that the definition of $T_{\mu\nu}$ implies

$$\begin{aligned}
\frac{1}{2} T^{\beta\gamma} \nabla_\alpha h_{\beta\gamma} &= \frac{1}{2} \gamma^{\rho\sigma}_{\mu\nu} \left(\nabla_\rho \overline{c}_\sigma + \kappa \pounds_{\overline{c}} h_{\rho\sigma}\right) \nabla_\alpha h^{\mu\nu} \\
&= \frac{1}{2} \left(\nabla_\rho \overline{c}_\sigma + \kappa \pounds_{\overline{c}} h_{\rho\sigma}\right) \nabla_\alpha H^{\rho\sigma} = \nabla^\nu \overline{c}^\beta \nabla_\alpha H_{\beta\nu}. \qquad \text{(E.22)}
\end{aligned}$$

E.2 On $J^{(B)}_\mu$

We start from the conserved current

$$J_{(\overline{c})\mu} = \xi^\alpha \left[\gamma^{\beta\gamma}_{\alpha\mu} \left(\nabla_\beta \overline{c}_\gamma + \kappa \pounds_{\overline{c}} h_{\beta\gamma}\right) + (2k-1)\,\kappa \nabla^\beta \overline{c}^\gamma \left(g_{\mu\alpha} h_{\beta\gamma} - h_{\mu\alpha} g_{\beta\gamma}\right)\right]. \tag{E.23}$$

The contribution coming from $\overline{c} \to B$, which is $-\mathrm{i}$ times the effect of the BRST transformation, is

$$\xi^\alpha \left[\gamma^{\beta\gamma}_{\alpha\mu} \left(\nabla_\beta B_\gamma + \kappa \pounds_B h_{\beta\gamma}\right) + (2k-1)\,\kappa \nabla^\beta B^\gamma \left(g_{\mu\alpha} h_{\beta\gamma} - h_{\mu\alpha} g_{\beta\gamma}\right)\right], \tag{E.24}$$

which has κ-dependent ("non-linear") part

$$\kappa \xi^\alpha \left[B^\beta \nabla_\beta h_{\mu\alpha} + \nabla_\mu B^\beta h_{\beta\alpha} + \nabla_\alpha B^\beta h_{\beta\mu} - k g_{\mu\alpha} B^\beta \nabla_\beta h - g_{\mu\alpha} \nabla^\beta B^\gamma h_{\beta\gamma} + (1-2k)\, h_{\mu\alpha} \nabla_\beta B^\beta\right]. \tag{E.25}$$

The "spacetime current" from $-\kappa h_{\kappa\lambda} \gamma^{\kappa\lambda}_{\mu\nu} \nabla^\mu B^\nu$ has B-dependence

$$\xi^\alpha \gamma^{\beta\gamma}_{\alpha\mu} \left(\nabla_\beta B_\gamma + \kappa \pounds_B h_{\beta\gamma}\right). \tag{E.26}$$

Multiplying the perturbation-independent term of Eq. (E.26) by $\nabla_\mu \left(\alpha \xi_\nu\right)$ gives an expression in which the $\nabla^\mu \alpha$ coefficient is $\xi^\alpha \gamma^{\beta\gamma}_{\alpha\mu} \nabla_\beta B_\gamma$. Similarly, the $\nabla^\mu \alpha$ coefficient in $\nabla_\mu \left(\alpha \xi_\nu\right) \kappa h_{\beta\nu} \gamma^{\mu\nu}_{\kappa\lambda} \nabla^\kappa B^\lambda$ is

$\kappa\xi^{\alpha}h_{\alpha}{}^{\nu}\gamma^{\kappa\lambda}_{\mu\nu}\nabla_{\kappa}B_{\lambda}$. Subtracting this from Eq. (E.25) gives

$$\begin{aligned}&\kappa\xi^{\alpha}\left[B^{\beta}\nabla_{\beta}H_{\mu\alpha}+\nabla_{\alpha}B^{\beta}h_{\beta\mu}-h_{\alpha}{}^{\nu}\nabla_{\nu}B_{\mu}+h_{\mu\alpha}\nabla_{\beta}B^{\beta}-g_{\mu\alpha}\nabla^{\beta}B^{\gamma}h_{\beta\gamma}\right]\\&=\kappa\xi^{\alpha}\left[B^{\beta}\nabla_{\beta}H_{\mu\alpha}+\nabla_{\alpha}B^{\beta}H_{\beta\mu}-H_{\alpha}{}^{\nu}\nabla_{\nu}B_{\mu}+H_{\mu\alpha}\nabla_{\beta}B^{\beta}-g_{\mu\alpha}\nabla^{\beta}B^{\gamma}H_{\beta\gamma}\right]\end{aligned}\tag{E.27}$$

(note the addition of the trivially zero quantity $-kh\kappa\xi^{\alpha}\left(\nabla_{\alpha}B_{\mu}-\nabla_{\alpha}B_{\mu}+g_{\mu\alpha}\nabla_{\beta}B^{\beta}-g_{\mu\alpha}\nabla_{\beta}B^{\beta}\right)$). Including the conserved current's $-\xi^{\mu}\mathcal{L}_0$ term deletes the term $-\kappa\xi_{\mu}\nabla^{\beta}B^{\gamma}H_{\beta\gamma}$. The remaining terms, which by inspection equal $-J_{\mu}^{(B)\text{rest}}$, are simply the terms obtained by varying B^{μ}. Indeed, the $\nabla^{\mu}\alpha$ coefficient in $\kappa\nabla_{\mu}\left(\alpha\pounds_{\xi}B_{\beta}\right)H^{\mu\beta}$ is $\kappa\left(\xi^{\alpha}\nabla_{\alpha}B^{\beta}-B^{\alpha}\nabla_{\alpha}\xi^{\beta}\right)H_{\mu\beta}$, and

$$\begin{aligned}-\kappa B^{\alpha}\nabla_{\alpha}\xi_{\beta}H^{\mu\beta}&=-\kappa\nabla_{\alpha}\left(B^{\alpha}\xi_{\beta}H^{\mu\beta}\right)+\kappa\xi_{\beta}\left(H^{\mu\beta}\nabla_{\alpha}B^{\alpha}+B^{\alpha}\nabla_{\alpha}H^{\mu\beta}\right)\\&\approx-\kappa\nabla_{\alpha}\left(B^{\mu}\xi_{\beta}H^{\alpha\beta}\right)+\kappa\xi_{\beta}\left(H^{\mu\beta}\nabla_{\alpha}B^{\alpha}+B^{\alpha}\nabla_{\alpha}H^{\mu\beta}\right)\\&=-\kappa\xi_{\beta}H^{\alpha\beta}\nabla_{\alpha}B^{\mu}+\kappa\xi_{\beta}\left(H^{\mu\beta}\nabla_{\alpha}B^{\alpha}+B^{\alpha}\nabla_{\alpha}H^{\mu\beta}\right),\end{aligned}\tag{E.28}$$

as required. (Note that a total derivative has been added, by replacing one term of the form $\nabla_{\alpha}K^{\alpha}$ with another.) Similarly, the Lie derivative term in Eq. (E.26) is

$$\begin{aligned}\kappa\xi^{\alpha}\gamma^{\beta\gamma}_{\alpha\mu}\pounds_{B}h_{\beta\gamma}&=\kappa\xi^{\alpha}\left[B^{\beta}\nabla_{\beta}h_{\mu\alpha}+h_{\beta\alpha}\nabla_{\mu}B^{\beta}+h_{\beta\mu}\nabla_{\alpha}B^{\beta}-kg_{\mu\alpha}\left(B^{\beta}\nabla_{\beta}h+2h_{\beta\gamma}\nabla^{\beta}B^{\gamma}\right)\right]\\&=\kappa\xi^{\alpha}\left[B^{\beta}\nabla_{\beta}H_{\mu\alpha}+h_{\beta\alpha}\nabla_{\mu}B^{\beta}+h_{\beta\mu}\nabla_{\alpha}B^{\beta}-2kg_{\mu\alpha}h_{\beta\gamma}\nabla^{\beta}B^{\gamma}\right].\end{aligned}\tag{E.29}$$

E.3 On $J_{\mu}^{(\bar{c}c)}$

Next we consider the BRST transformation of the perturbation instead of the antighost, viz.

$$\begin{aligned}J_{\mu}^{(h)}&:=\kappa\xi^{\alpha}\left[\gamma^{\beta\gamma}_{\mu\alpha}\pounds_{\bar{c}}h_{\beta\gamma}+(2k-1)\nabla^{\beta}\bar{c}^{\gamma}\left(g_{\mu\alpha}h_{\beta\gamma}-h_{\mu\alpha}g_{\beta\gamma}\right)\right]\\&=\kappa\xi^{\alpha}\left[\bar{c}^{\beta}\nabla_{\beta}H_{\mu\alpha}+h_{\beta\alpha}\nabla_{\mu}\bar{c}^{\beta}+h_{\beta\mu}\nabla_{\alpha}\bar{c}^{\beta}\right.\\&\left.-2kh_{\beta\gamma}g_{\mu\alpha}\nabla^{\beta}\bar{c}^{\gamma}+(2k-1)\nabla^{\beta}\bar{c}^{\gamma}\left(g_{\mu\alpha}h_{\beta\gamma}-h_{\mu\alpha}g_{\beta\gamma}\right)\right]\\&=\kappa\xi^{\alpha}\left[\bar{c}^{\beta}\nabla_{\beta}H_{\mu\alpha}+H_{\beta\alpha}\nabla_{\mu}\bar{c}^{\beta}+H_{\beta\mu}\nabla_{\alpha}\bar{c}^{\beta}+kh\left(\nabla_{\mu}\bar{c}_{\alpha}+\nabla_{\alpha}\bar{c}_{\mu}\right)\right.\\&\left.-\nabla^{\beta}\bar{c}^{\gamma}\left(2kh_{\beta\gamma}g_{\mu\alpha}+(1-2k)\,g_{\mu\alpha}h_{\beta\gamma}+(2k-1)\,h_{\mu\alpha}g_{\beta\gamma}\right)\right]\\&=\kappa\xi^{\alpha}\left[\bar{c}^{\beta}\nabla_{\beta}H_{\mu\alpha}+H_{\beta\alpha}\nabla_{\mu}\bar{c}^{\beta}+H_{\beta\mu}\nabla_{\alpha}\bar{c}^{\beta}+kh\left(\nabla_{\mu}\bar{c}_{\alpha}+\nabla_{\alpha}\bar{c}_{\mu}\right)\right.\\&\left.-\nabla^{\beta}\bar{c}^{\gamma}\left(h_{\beta\gamma}g_{\mu\alpha}+(2k-1)\,h_{\mu\alpha}g_{\beta\gamma}\right)\right]\\&=\kappa\xi^{\alpha}\left[\bar{c}^{\beta}\nabla_{\beta}H_{\mu\alpha}+H_{\beta\alpha}\nabla_{\mu}\bar{c}^{\beta}+H_{\beta\mu}\nabla_{\alpha}\bar{c}^{\beta}+kh\left(\nabla_{\mu}\bar{c}_{\alpha}+\nabla_{\alpha}\bar{c}_{\mu}\right)\right.\\&\left.-\nabla^{\beta}\bar{c}^{\gamma}\left(H_{\beta\gamma}g_{\mu\alpha}+(2k-1)\,H_{\mu\alpha}g_{\beta\gamma}+khg_{\beta\gamma}g_{\mu\alpha}+(2k-1)\,khg_{\mu\alpha}g_{\beta\gamma}\right)\right]\\&=\kappa\xi^{\alpha}\left[\bar{c}^{\beta}\nabla_{\beta}H_{\mu\alpha}+H_{\beta\alpha}\nabla_{\mu}\bar{c}^{\beta}+H_{\beta\mu}\nabla_{\alpha}\bar{c}^{\beta}+kh\left(\nabla_{\mu}\bar{c}_{\alpha}+\nabla_{\alpha}\bar{c}_{\mu}\right)\right.\\&\left.-\nabla^{\beta}\bar{c}^{\gamma}\left(H_{\beta\gamma}g_{\mu\alpha}+(2k-1)\,H_{\mu\alpha}g_{\beta\gamma}+2k^{2}hg_{\beta\gamma}g_{\mu\alpha}\right)\right].\end{aligned}\tag{E.30}$$

Since $g^{\mu\nu}\pounds_{\bar{c}}g^{f}_{\mu\nu}=\nabla_{\beta}\bar{c}^{\beta}+\kappa\left(\bar{c}^{\beta}\nabla_{\beta}h+2h_{\beta\gamma}\nabla^{\beta}\bar{c}^{\gamma}\right)$,

$$\begin{aligned}\left\{Q,\,J_{\mu}^{(h)}\right\}&=\xi^{\alpha}\left(\bar{c}^{\beta}\nabla_{\beta}Z_{\mu\alpha}+\nabla_{\mu}\bar{c}^{\beta}Z_{\beta\alpha}+\nabla_{\alpha}\bar{c}^{\beta}Z_{\beta\mu}-g_{\mu\alpha}\nabla^{\beta}\bar{c}^{\gamma}Z_{\beta\gamma}+(1-2k)\,\nabla_{\beta}\bar{c}^{\beta}Z_{\mu\alpha}\right.\\&\left.+k\overline{T}_{\mu\alpha}\left(\nabla_{\beta}\bar{c}^{\beta}+\kappa\left(\bar{c}^{\beta}\nabla_{\beta}h+2h_{\beta\gamma}\nabla^{\beta}\bar{c}^{\gamma}\right)\right)\right)\end{aligned}\tag{E.31}$$

where $\overline{T}_{\mu\nu} := \gamma^{\kappa\lambda}_{\mu\nu}\nabla_\kappa \bar{c}_\lambda$. The $\bar{c} \to B$ and$-\xi^\mu \mathcal{L}_0$ contributions to the spacetime-isometry Noether current total

$$\begin{aligned}
\left(\pounds_\xi \bar{c}^\nu\right) Z^\mu{}_\nu - \xi^\mu \nabla^\beta \bar{c}^\gamma Z_{\beta\gamma} &= \xi^\alpha \nabla_\alpha \bar{c}^\beta Z^\mu{}_\beta - \left(\nabla_\alpha \xi^\beta\right) \bar{c}^\alpha Z^\mu{}_\beta - \xi^\mu \nabla^\beta \bar{c}^\gamma Z_{\beta\gamma} \\
&= \xi^\alpha \nabla_\alpha \bar{c}^\beta Z^\mu{}_\beta + \xi^\beta \nabla_\alpha \left(\bar{c}^\alpha Z^\mu{}_\beta\right) - \nabla_\alpha \left(\xi^\beta \bar{c}^\alpha Z^\mu{}_\beta\right) - \xi^\mu \nabla^\beta \bar{c}^\gamma Z_{\beta\gamma} \\
&\approx \xi^\alpha \nabla_\alpha \bar{c}^\beta Z^\mu{}_\beta + \xi^\beta \nabla_\alpha \left(\bar{c}^\alpha Z^\mu{}_\beta\right) - \nabla_\alpha \left(\xi^\beta \bar{c}^\mu Z^\alpha{}_\beta\right) - \xi^\mu \nabla^\beta \bar{c}^\gamma Z_{\beta\gamma} \\
&= \xi^\alpha \nabla_\alpha \bar{c}^\beta Z^\mu{}_\beta + \xi^\beta \nabla_\alpha \left(\bar{c}^\alpha Z^\mu{}_\beta\right) - \xi^\beta \left(\nabla_\alpha \bar{c}^\mu\right) Z^\alpha{}_\beta - \xi^\mu \nabla^\beta \bar{c}^\gamma Z_{\beta\gamma}, \qquad \text{(E.32)}
\end{aligned}$$

since $\left(\nabla_\alpha \xi^\beta\right) \bar{c}^\mu S^\alpha{}_\beta$ vanishes by $S_{\alpha\beta} = S_{\beta\alpha}$. A comparison with Eq. (E.31) benefits from changing the order of terms, viz.

$$\begin{aligned}
\left\{Q,\, J^{(h)}_\mu\right\} &= \xi^\alpha \left(\nabla_\beta \left(\bar{c}^\beta S_{\mu\alpha}\right) + \nabla_\alpha \bar{c}^\beta Z_{\beta\mu} - g_{\mu\alpha} \nabla^\beta \bar{c}^\gamma Z_{\beta\gamma}\right. \\
&\quad + \nabla_\mu \bar{c}^\beta Z_{\beta\alpha} - 2k \nabla_\beta \bar{c}^\beta Z_{\mu\alpha} \\
&\quad \left. + k\overline{T}_{\mu\alpha} \left(\nabla_\beta \bar{c}^\beta + \kappa \left(\bar{c}^\beta \nabla_\beta h + 2h_{\beta\gamma} \nabla^\beta \bar{c}^\gamma\right)\right)\right), \qquad \text{(E.33)} \\
\left(\pounds_\xi \bar{c}^\nu\right) Z^\mu{}_\nu - \xi^\mu \nabla^\beta \bar{c}^\gamma Z_{\beta\gamma} &\approx \xi^\beta \nabla_\alpha \left(\bar{c}^\alpha Z^\mu{}_\beta\right) + \xi^\alpha \nabla_\alpha \bar{c}^\beta Z^\mu{}_\beta - \xi^\mu \nabla^\beta \bar{c}^\gamma Z_{\beta\gamma} \\
&\quad - \xi^\beta \left(\nabla_\alpha \bar{c}^\mu\right) Z^\alpha{}_\beta. \qquad \text{(E.34)}
\end{aligned}$$

In particular, the first lines of the right-hand sides of Eqs. (E.33) and (E.34) are equal (save for the height of the spacetime index μ). The difference is

$$\begin{aligned}
\delta_\mu &:= \left\{Q,\, J^{(h)}_\mu\right\} - \left(\pounds_\xi \bar{c}^\nu\right) S^\mu{}_\nu + \xi^\mu \nabla^\beta \bar{c}^\gamma Z_{\beta\gamma} \\
&= \xi^\alpha \left[\nabla_\mu \bar{c}^\beta Z_{\beta\alpha} - 2k \nabla_\beta \bar{c}^\beta Z_{\mu\alpha} + \left(\nabla_\beta \bar{c}^\mu\right) Z_\alpha{}^\beta\right. \\
&\quad \left. + k\overline{T}_{\mu\alpha} \left(\nabla_\beta \bar{c}^\beta + \kappa \left(\bar{c}^\beta \nabla_\beta h + 2h_{\beta\gamma} \nabla^\beta \bar{c}^\gamma\right)\right)\right]. \qquad \text{(E.35)}
\end{aligned}$$

The aim is to show that this is obtained from the transformation of

$$\kappa \nabla^\mu \bar{c}^\nu \gamma^{\beta\gamma}_{\mu\nu} \pounds_c h_{\beta\gamma} = \kappa \overline{T}^{\mu\nu} \left(\frac{1}{2} c^\alpha \nabla_\alpha h_{\mu\nu} + h_{\alpha\nu} \nabla_\mu c^\alpha\right). \qquad \text{(E.36)}$$

Some variations are irrelevant, such as that of c^α in $\nabla_\alpha h_{\mu\nu}$ (because $\nabla_\mu \alpha$ is absent from $\alpha \pounds_\xi c^\nu$). The variation of $\nabla_\mu c^\alpha$ gives $\overline{T}^{\mu\nu} h_{\alpha\nu} \pounds_\xi c^\alpha = \overline{T}^{\mu\beta} h_{\beta\gamma} \xi^\alpha \nabla_\alpha c^\gamma - \overline{T}^{\mu\nu} h_{\alpha\nu} c^\beta \nabla_\beta \xi^\alpha$. The first of these terms provides a contribution equal to the $\nabla_\mu \alpha$ coefficient in

$$\overline{T}^{\delta\nu} \nabla_\delta c^\alpha \left[\nabla_\alpha \left(\alpha \xi_\nu\right) + \nabla_\nu \left(\alpha \xi_\alpha\right) + \kappa h_{\beta\nu} \nabla_\alpha \left(\alpha \xi^\beta\right) + \kappa h_{\alpha\beta} \nabla_\nu \left(\alpha \xi^\beta\right)\right], \qquad \text{(E.37)}$$

which is $\xi^\alpha \left(\overline{T}^\beta{}_\alpha \nabla_\beta c^\mu + \overline{T}^{\beta\mu} \nabla_\beta c_\alpha + \kappa h_{\alpha\gamma} \overline{T}^{\beta\gamma} \nabla_\beta c^\mu + \kappa h_{\alpha\beta} \overline{T}^{\gamma\mu} \nabla_\gamma c^\beta\right)$. The contribution from $\nabla_\alpha h_{\mu\nu}$ is the $\nabla_\mu \alpha$ coefficient in

$$\nabla_\alpha \left[\nabla_\mu \left(\alpha \xi_\nu\right) + \nabla_\nu \left(\alpha \xi_\mu\right) + \alpha \kappa \xi^\beta \nabla_\beta h_{\mu\nu} + \kappa \nabla^{(\alpha}\, \xi^{\beta)} h_{\beta\nu} + \kappa \left(\nabla_\nu \xi^\beta\right) h_{\beta\mu}\right]. \qquad \text{(E.38)}$$

Using $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$, Eq. (E.38) simplifies to

$$2\nabla_\alpha \left(\nabla_\mu \alpha \left(\xi_\nu + \kappa \xi^\beta h_{\beta\nu}\right)\right) + \left(\kappa \nabla_\alpha \alpha\right) \left(\xi^\beta \nabla_\beta h_{\mu\nu} + h_{\beta\nu} \nabla_\mu \xi^\beta\right). \tag{E.39}$$

The first of these terms is a total derivative that may be dropped by integration by parts. The contribution of interest is now

$$-\nabla_\alpha \left(\overline{T}^{\mu\nu} c^\alpha\right) \left(\xi_\nu + \kappa \xi^\beta h_{\beta\nu}\right) + \frac{1}{2}\kappa \overline{T}^{\beta\gamma} c^\mu \xi^\alpha \nabla_\alpha h_{\beta\gamma} + \kappa \overline{T}^{\beta\gamma} c^\mu h_{\alpha\gamma} \nabla_\beta \xi^\alpha, \tag{E.40}$$

but each of the terms $-\nabla_\alpha \left(\overline{T}^{\mu\nu} c^\alpha\right) \xi_\nu$, $-\kappa \nabla_\alpha \left(\overline{T}^{\mu\nu} c^\alpha\right) \xi^\beta h_{\beta\nu}$ is cancelled elsewhere. Indeed, the contribution from varying c^μ in the perturbation-independent ("linear") term is

$$\overline{T}_{\mu\nu} \left(\xi^\alpha \nabla_\alpha c^\nu - c^\alpha \nabla_\alpha \xi^\nu\right), \tag{E.41}$$

and the second term therein is

$$\begin{aligned} -\overline{T}^\mu{}_\nu c^\alpha \nabla_\alpha \xi^\nu &= \xi^\nu \nabla_\alpha \left(\overline{T}^\mu{}_\nu c^\alpha\right) - \nabla_\alpha \left(\overline{T}^\mu{}_\nu c^\alpha \xi^\nu\right) \\ &\approx \xi^\nu \nabla_\alpha \left(\overline{T}^\mu{}_\nu c^\alpha\right) - \nabla_\alpha \left(\overline{T}^\alpha{}_\nu c^\mu \xi^\nu\right) \\ &= \xi^\nu \nabla_\alpha \left(\overline{T}^\mu{}_\nu c^\alpha\right) - \nabla_\alpha \left(\overline{T}^\alpha{}_\nu c^\mu\right) \xi^\nu, \end{aligned} \tag{E.42}$$

giving a term $\xi^\nu \nabla_\alpha \left(T^\mu_\nu c^\alpha\right)$ that cancels $-\nabla_\alpha \left(\overline{T}^{\mu\nu} c^\alpha\right) \xi_\nu$. Next observe that

$$\nabla_\beta \left[\xi^\alpha \left(\overline{T}^{\beta\gamma} c^\mu h_{\alpha\gamma} - \overline{T}^{\mu\gamma} c^\beta h_{\alpha\gamma}\right)\right] \approx 0, \tag{E.43}$$

$$\begin{aligned} \nabla_\beta \xi^\alpha \left(\overline{T}^{\beta\gamma} c^\mu h_{\alpha\gamma} - \overline{T}^{\mu\gamma} c^\beta h_{\alpha\gamma}\right) &= -\xi^\alpha \nabla_\beta \left(\overline{T}^{\beta\gamma} c^\mu h_{\alpha\gamma} - \overline{T}^{\mu\gamma} c^\beta h_{\alpha\gamma}\right) \\ &= \xi^\alpha \left(-\nabla_\beta \left(\overline{T}^{\beta\gamma} h_{\alpha\gamma}\right) c^\mu - \overline{T}^{\beta\gamma} h_{\alpha\gamma} \nabla_\beta c^\mu \right. \\ &\quad \left. + \nabla_\beta \left(\overline{T}^{\mu\gamma} c^\beta\right) h_{\alpha\gamma} + \overline{T}^{\mu\gamma} c^\beta \nabla_\beta h_{\alpha\gamma}\right), \end{aligned} \tag{E.44}$$

and the penultimate term cancels $-\kappa \nabla_\alpha \left(\overline{T}^{\mu\nu} c^\alpha\right) \xi^\beta h_{\beta\nu}$.

The contribution still includes $\overline{T}_{\mu\nu} \xi^\alpha \nabla_\alpha c^\nu - \left(\nabla_\beta \overline{T}^\beta{}_\alpha\right) c^\mu \xi^\alpha - \overline{T}^\alpha{}_\nu \xi^\nu \nabla_\alpha c^\mu$. Indeed, Eq. (E.18) allows all surviving terms in the contribution to be combined. The $\kappa \xi^\alpha$ coefficient becomes

$$\begin{aligned} &\overline{T}^{\mu\beta} h_{\beta\gamma} \nabla_\alpha c^\gamma \cancel{+ \overline{T}^\beta{}_\alpha \nabla_\beta c^\mu} + \overline{T}^{\beta\mu} \nabla_\beta c_\alpha \cancel{+ \overline{T}^{\beta\gamma} h_{\alpha\gamma} \nabla_\beta c^\mu} + \overline{T}^{\gamma\mu} h_{\alpha\beta} \nabla_\gamma c^\beta + \frac{1}{2} \overline{T}^{\beta\gamma} c^\mu \xi^\alpha \nabla_\alpha h_{\beta\gamma} \\ &\quad + \overline{T}^\mu{}_\nu \nabla_\alpha c^\nu \cancel{- \overline{T}^\nu{}_\alpha \nabla_\nu c^\mu} - \left(\nabla^\nu \overline{c}^\beta\right) c^\mu \nabla_\alpha H_{\beta\nu} \cancel{- \overline{T}^{\beta\nu} h_{\alpha\nu} \nabla_\beta c^\mu} + \overline{T}^{\mu\nu} c^\beta \nabla_\beta h_{\alpha\nu}, \end{aligned} \tag{E.45}$$

where cancelling pairs have been indicated. Of what survives, the linear terms are

$$\overline{T}^{\beta\mu} \nabla_\beta c_\alpha + \overline{T}^\mu{}_\nu \nabla_\alpha c^\nu = \overline{T}^{\mu\beta} \left(\nabla_\alpha c_\beta + \nabla_\beta c_\alpha\right), \tag{E.46}$$

and we require these to match[68]

$$\left(\nabla^{\beta}\overline{c}_{\mu}+\nabla_{\mu}\overline{c}^{\beta}\right)\gamma_{\alpha\beta}^{\gamma\delta}\nabla_{\gamma}c_{\delta}-2k\left(\nabla_{\lambda}\overline{c}^{\lambda}\right)\gamma_{\alpha\mu}^{\gamma\delta}\nabla_{\gamma}c_{\delta}+2k\overline{T}_{\mu\alpha}\nabla_{\lambda}\overline{c}^{\lambda}=\overline{T}^{\beta}_{\ \mu}\left(\gamma_{\alpha\beta}^{\gamma\delta}\nabla_{\gamma}c_{\delta}+2kg_{\mu\alpha}\nabla_{\lambda}\overline{c}^{\lambda}\right)$$
$$=\overline{T}^{\beta}_{\ \mu}\left(\nabla_{\alpha}c_{\beta}+\nabla_{\beta}c_{\alpha}\right), \qquad \text{(E.47)}$$

as indeed they do. Now for the non-linear terms: the spacetime-isometry result is

$$\kappa\xi^{\alpha}T_{\mu}^{\ \beta}\left(c^{\gamma}\nabla_{\gamma}h_{\alpha\beta}+h_{\gamma\beta}\nabla_{\alpha}c^{\gamma}+h_{\gamma\alpha}\nabla_{\beta}c^{\gamma}\right). \qquad \text{(E.48)}$$

A comparison of this with Eq. (E.45) shows that the only remaining terms are

$$\frac{1}{2}T^{\beta\gamma}c^{\mu}\nabla_{\alpha}h_{\beta\gamma}-\left(\nabla^{\nu}\overline{c}^{\beta}\right)\left(\nabla_{\alpha}H_{\beta\nu}\right)c^{\mu}, \qquad \text{(E.49)}$$

which cancel by Eq. (E.19).

F Comments on the Batalin–Vilkovisky (BV) formalism

This discussion is primarily based on Sec. 15.9 of Weinberg's Ref. [95]. Weinberg denotes the BRST transformation s[69]; I will maintain this notation because the BV formalism affords a generalisation of the BRST transformations I have hitherto denoted δ. The BV formalism add terms to the action. Before these terms are added, he denotes the action as I. After new terms are added, he denotes the action S. I will also use this notation.

The BV formalism is used to address the following issues:

- There is a need to address Hamiltonian constraints whose origins are not in the Lie algebra. (This is a concern for BRST-quantised perturbative gravity.)
- If a theory's algebra is open, s^2 is a linear combination of functional derivatives $\frac{\delta I}{\delta\chi^A}$ (here χ^A is a field) rather than 0.
- A very early version of the formalism [96] was used to renormalise gauge theories. The sum of all 1-particle-irreducible diagrams in a background field does not obey the original action's BRST symmetries, but it does obey the *master equation* (see Sec. F.1).
- The BV formalism conveniently analyses possible violations of an action's symmetries by quantum effects.

The aim of this appendix is to show that the BV formalism need not be used in my treatment of perturbative gravity's FP–sector zero mode problem.

My treatment is classical in Secs. F.1–F.4. I introduce the formalism's classical machinery in Sec. F.1. Initially this serves only to provide an unusually complicated description of the ordinary BRST formalism. However, doing so also allows a subsequent explanation of how to generalise BRST. Next I summarise Ref. [95]'s proofs of a number of important consequences of the so-called *master equation* in Secs. F.2–F.4. These include a constructive definition in Sec. F.3 of a generalization of the BRST transformation.

[68] Note that it has been convenient to lower μ, due to the $\gamma_{\alpha\beta}^{\gamma\delta}$ formalism.

[69] Although the discussion of s in Weinberg's Secs. 15.8 and 15.9 makes clear it denotes the BRST transformation, he also calls it the *Slavnov operator* between his discussion of his Eqs. (15.8.8) and (15.8.9).

I quantise the theory in Sec. F.5 to discuss S–matrix computation. This results in a *quantum master equation*, which in at least some theories reduces to the ordinary master equation together with an additional condition abbreviated as $\Delta S = 0$ (the operator Δ will be defined in Eq. (F.21) in Sec. F.5). This will complete the account of how the BV formalism extends the BRST formalism. The extension can be understood in terms of which terms are added to the action. These terms depend only on new fields that the BV formalism introduces, so have no implications for the zero mode problems considered in this thesis. This cause of the BV formalism's irrelevance is similar to there being no need to include terms such as $-\frac{1}{4}F^2,\ R-2\Lambda$ in the scalar Lagrangian density.

F.1 Introduction to the formalism

Let $\mathrm{gh}\,T$ denote the *ghost number* of a field T. The first step is to introduce field partners for some fields χ^A, say $\chi^\ddagger_A$, satisfying $\mathrm{gh}\,\chi^A + \mathrm{gh}\,\chi^\ddagger_A = \mathrm{gh}\,\chi_A + \mathrm{gh}\,\chi^{\ddagger A} = -1$. The partner field $\chi^\ddagger$ is called the *antifield* of χ[70], which should not be confused with other concepts whose names include the prefix anti-. Indeed, since Eq. (F.1) implies $\chi,\ \chi^\ddagger$ are a bosonic field and a fermionic field in some order, a matter field's antifield is not associated with antimatter, nor is the ghost's antifield simply the antighost.

Until the antifields' numerical values are specified, they are external and the S–matrix is incalculable. (Until S–matrix calculation is considered, all discussion herein is classical, not quantised.) One way to provide such numerical values is as follows: for an arbitrary fermionic functional $\Psi\left[x\right]$ satisfying $\mathrm{gh}\,\Psi = -1$ so $\mathrm{gh}\,s\Psi = 0$, we define

$$\chi^\ddagger_A := \frac{\delta\Psi}{\delta\chi^A},\ \chi^{\ddagger A} := \frac{\delta\Psi}{\delta\chi_A} \tag{F.1}$$

The action gains a term (which of course must be bosonic and of ghost number 0), which for quantum gravity and Yang–Mills theories (i.e. theories of interest to us) may be chosen as

$$\sum_A s\chi^A\chi^\ddagger_A = \sum_A s\chi^A\frac{\delta\Psi}{\delta\chi^A} = s\Psi. \tag{F.2}$$

(Sums such as these are over fields with lower as well as upper labels A, and labels will be assumed to be upper except where explicit lower labels are necessary.) This choice obtains an action which is a linear functional of the antifields. One of the Euler–Lagrange equations is the *master equation*, viz.

$$\frac{\delta_R S}{\delta\chi^\ddagger_A}\frac{\delta_L S}{\delta\chi^A} = 0,\ \frac{\delta_R S}{\delta\chi^{\ddagger A}}\frac{\delta_L S}{\delta\chi_A} = 0, \tag{F.3}$$

where δ_L (δ_R) denotes left-differentiation (right-differentiation). The resulting action is numerically identical with the *gauge–fixed action*, so physical matrix elements are unaffected by small variations in Ψ.

The BV formalism begins to accomplish something new if Ψ is chosen as a *non-linear* functional of the antifields. (For reducible theories, ghosts among the χ^A gain not only antifields but also ghosts of their own. Weinberg's Sec. 15.8 discusses in detail the reasons ghosts of ghosts, ghosts of ghosts of

[70]The field χ usually has its antifield denoted χ^* but, since in this context * does not denote complex conjugation, Weinberg prefers ‡. I will have to follow his minority convention since some symbols require both a complex conjugation label and an antifield label.

ghosts etc. can be necessary.) Since $\omega^{*A},\ h_A$ have linear BRST transformations, their antifields appear linearly in S, which therefore is of the form $S = S_{\text{min}}\left[\phi,\,\omega,\,\phi^{\ddagger},\,\omega^{\ddagger}\right] - h^A\omega_A^{*\ddagger}$, where S_{min} is bosonic and $\text{gh}\,S_{\text{min}} = 0$, $\text{gh}\,\phi^{\ddagger} = -1$, $\text{gh}\,\omega^{\ddagger} = -2$, $\text{gh}\,\omega^{*\ddagger} = 0$. By inspection S_{min} also satisfies the master equation. Its arguments are called *minimal variables*, and the fields ω^{*A} and h_A and their antifields are called *trivial pairs*. (These antifields appear bilinearly in S.)

F.2 Helpful corollaries of the master equation

By its ghost number, S_{min} has an antifield expansion of the form

$$\begin{aligned} S_{\text{min}} &= I\left[\phi\right] + \omega^A f_A^r\left[\phi\right]\phi_r^{\ddagger} + \frac{1}{2}\omega^A\omega^B f_{AB}^C\left[\phi\right]\omega_C^{\ddagger} + \frac{1}{2}\omega^A\omega^B f_{AB}^{rs}\left[\phi\right]\phi_r^{\ddagger}\phi_s^{\ddagger} \\ &\quad + \omega^A\omega^B\omega^C f_{ABC}^{rD}\left[\phi\right]\phi_r^{\ddagger}\omega_D^{\ddagger} + \frac{1}{2}\omega^A\omega^B\omega^C\omega^D f_{ABCD}^{EF}\left[\phi\right]\omega_E^{\ddagger}\omega_F^{\ddagger} + \cdots . \end{aligned} \tag{F.4}$$

From terms of zeroth order in antifields (and hence first order in ω),

$$f_A^r\frac{\delta I}{\delta\phi^r} = 0. \tag{F.5}$$

This is I-invariance under the transformation $\phi^r \to \phi^r + \epsilon^A f_A^r$ for arbitrary infinitesimals ϵ^A. The term using one $\phi^{\ddagger}$ on the right and two ωs on the left gives

$$f_A^r\frac{\delta f_B^s}{\delta\phi^r} - A \leftrightarrow B + f_{AB}^C f_C^s + \frac{\delta I}{\delta\phi^r}f_{AB}^{rs} = 0, \tag{F.6}$$

which given $\frac{\delta I}{\delta\phi} = 0$ becomes the commutation relation with structure constant f_{AB}^C for the above transformation. The master equation has one more term in the antifields, which has three ωs on the left and one $\omega^{\ddagger}$ on the right. This term gives

$$f_{[A}^r\frac{\delta f_{B\,C]}^D}{\delta\phi^r} - f_{[A\,B}^E f_{C]E}^D + f_{ABC}^{rD}\frac{\delta I}{\delta\phi^r} = 0. \tag{F.7}$$

This is a consistency condition for Eq. (F.5). Combining this with the field equations gives the Jacobi identity.

A further outcome of the master equation is that terms in the master equation of at least order p in antifields involve terms in S_{min} of at least order $p+1$ in antifields. This provides consistency conditions for Eqs. (F.6) and (F.7), as well as consistency conditions for those consistency conditions and so *ad infinitum*.

All this is achieved from the master equation alone, which in fact may be interpreted as an invariance of S under a generalised BRST transformation. To prove this requires a new concept discussed below.

F.3 A generalised BRST transformation

The *antibracket* of two functionals $F\left[\chi,\,\chi^{\ddagger}\right],\ G\left[\chi,\,\chi^{\ddagger}\right]$ is defined as

$$(F,\,G) := \frac{\delta_R F}{\delta\chi^A}\frac{\delta_L G}{\delta\chi_A^{\ddagger}} - \frac{\delta_R F}{\delta\chi_A^{\ddagger}}\frac{\delta_L G}{\delta\chi^A}. \tag{F.8}$$

Reversing the order of the derivatives in the second term multiplies each factor by either 1 or -1 if its functional differentiation is with respect to a fermionic or bosonic field, and each occurs exactly once so

$$(F,\,G)=\frac{\delta_R F}{\delta\chi^A}\frac{\delta_L G}{\delta\chi_A^\ddagger}+\frac{\delta_L F}{\delta\chi_A^\ddagger}\frac{\delta_R G}{\delta\chi^A}=\frac{\delta_R F}{\delta\chi^A}\frac{\delta_L G}{\delta\chi_A^\ddagger}\pm F\leftrightarrow G,$$

where the rules determining whether $\pm$ sign is $+$ or $-$ are trivial. Hence

$$(F,\,G)=\begin{cases}(G,\,F) & F,\,G\text{ both bosonic}\\ -(G,\,F) & \text{otherwise}\end{cases},\tag{F.9}$$

so $(F,\,F)=0$ for fermionic F. This condition does *not* suffice to prove

$$(S,\,S)=0,\tag{F.10}$$

but this non-trivial condition in fact follows from (indeed, is a restatement of) the master equation. A generalised BRST transformation can now be defined in terms of a fermionic infinitesimal constant θ:

$$\hat{\delta}_\theta\chi^A:=\theta\frac{\delta_R S}{\delta\chi_A^\ddagger}=-\theta\left(S,\,\chi^A\right),\,\hat{\delta}_\theta\chi_A^\ddagger:=-\theta\frac{\delta_R S}{\delta\chi^A}=-\theta\left(S,\,\chi_A^\ddagger\right).\tag{F.11}$$

When the term added to S is of the form given in Eq. (F.2), this reduces to the ordinary BRST transformation.

The antibracket is a derivation because it satisfies the Leibniz rule $(F,\,GH)=(F,\,G)\,H\pm G\,(F,\,H)$, where the $\pm$ sign is -1 if and only if F is bosonic and G is fermionic. Thus

$$\hat{\delta}_\theta G=-\theta\,(S,\,G)\,,\,\hat{\delta}_\theta H=-\theta\,(S,\,H)\implies\hat{\delta}_\theta GH=-\theta\,(S,\,GH)\,.\tag{F.12}$$

The solution set of $\hat{\delta}_\theta F=-\theta\,(S,\,F)$ therefore contains the closure of fields and antifields under multiplication, as required for a generalization of the BRST transformation. In particular, S is invariant under this transformation as required because of Eq. (F.10).
The antibracket also shares the ordinary BRST transformation's nilpotence, i.e. $(S,\,(S,\,H))=0$. This follows from its Jacobi identity, viz.

$$\pm\,(F,\,(G,\,H))+\text{cyclic permutations}=0,\tag{F.13}$$

where the $\pm$ sign of each of the three terms is $-$ if and only if the two outermost fields (e.g. $F,\,H$ in the leftmost of the three terms) are both bosonic. Taking $F=G=S$ gives

$$(S,\,(S,\,H))=-\tfrac{1}{2}\,(H,\,(S,\,S))=0.\tag{F.14}$$

F.4 Anticanonical transformations

However, this finding also implies the master equation's solution is not unique; rather, its solution set is closed under the transformation (of arbitrary functionals G, including S)

$$G \to G^{'} := G + (\delta F,\, G)\,, \tag{F.15}$$

where δF is an arbitrary infinitesimal fermionic functional of $\chi,\, \chi^{\ddagger}$ having ghost number -1. (This is one example of a *canonical transformation*. Since antifields are involved, Weinberg calls it an *anticanonical transformation*.) Such transformations may also require a shift in the definitions of antifields. For example, if $\delta F = \epsilon\Psi\,[\chi]$ with ϵ an infinitesimal bosonic constant,

$$S^{'}\left[\chi,\, \chi^{\ddagger}\right] = S\left[\chi,\, \chi^{\ddagger} + \epsilon\frac{\delta\Psi}{\delta\chi}\right]. \tag{F.16}$$

Following integration, satisfying the master equation also requires the shift

$$\chi^{\ddagger} \to \chi^{\ddagger'} := \chi^{\ddagger} - \frac{\delta\Psi}{\delta\chi}. \tag{F.17}$$

Note that these shifted antifields vanish if and only if Eq. (F.1) holds.

Anticanonical transformations are most generally definable as those which preserve the *fundamental antibracket relations*, viz. $\left(\chi^A,\, \chi^{\ddagger}_B\right) = \delta^A_B,\, \left(\chi^A,\, \chi^B\right) = \left(\chi^{\ddagger}_A,\, \chi^{\ddagger}_B\right) = 0$. Weinberg presents a proof, omitted here for brevity, that these follow from Eq. (F.15).

F.5 Comments on the S–matrix

The next task is the quantisation of the above classical treatment. We adopt Eq. (F.15) and use the gauge–fixed action I_Ψ, which is invariant under a BRST transformation acting only on fields, viz.

$$\delta_\theta\chi = \theta s\chi,\; s\chi := \left.\frac{\delta_R S\left[\chi,\, \chi^{\ddagger}\right]}{\delta\chi^{\ddagger}}\right|_{\chi^{\ddagger}=\frac{\delta\Psi}{\delta\chi}}. \tag{F.18}$$

(Indeed, sI_Ψ is then the sum of a term which the master equation sets to 0 and a term which vanishes by an antisymmetry argument.) While the transformation is the ordinary BRST transformation for closed algebras, more generally it is a transformation that is only nilpotent on-shell.

The vacuum–vacuum amplitude Z_Ψ and its shift δZ under a shift $\delta\Psi\,[\chi]$ in $\Psi\,[\chi]$ are respectively

$$Z_\Psi = \int\left[\prod \mathrm{d}\chi\right]\exp \mathrm{i}I_\Psi\,[\chi]\,, \tag{F.19}$$

$$\begin{aligned}\delta Z &= \mathrm{i}\int\left[\prod \mathrm{d}\chi\right]\exp \mathrm{i}I_\Psi \left.\frac{\delta_R S\left[\chi,\, \chi^{\ddagger}\right]}{\delta\chi^{\ddagger}_A}\right|_{\chi^{\ddagger}=\frac{\delta\Psi}{\delta\chi}} \frac{\delta\delta\Psi}{\delta\chi^A} \\ &= \int\left[\prod \mathrm{d}\chi\right]\exp \mathrm{i}I_\Psi\left\{\frac{\delta_R S}{\delta\chi^{\ddagger}_A}\frac{\delta_L I_\Psi}{\delta\chi^A} - \mathrm{i}\Delta S\left[\chi,\, \chi^{\ddagger}\right]\right\}\delta\Psi,\end{aligned} \tag{F.20}$$

where I have used integration by parts and introduced the second-order functional differential operator Δ defined as

$$\Delta := \frac{\delta_R}{\delta\chi^{\ddagger}_A}\frac{\delta_L}{\delta\chi^A}. \tag{F.21}$$

The condition for Z_Ψ to be Ψ-independent is then the *quantum master equation*, which is one of the few equations in this discussion in which an explicit power of $\hbar$ is worthwhile. The quantum master equation may be written as

$$(S,\, S) = 2\mathrm{i}\hbar\Delta S \,\text{at}\, \chi^\ddagger = \frac{\delta\Psi}{\delta\chi}. \tag{F.22}$$

The classical limit recovers the previous master equation because it takes the $\hbar \to 0^+$right-handed limit. Unless there are anomalies, an action may be constructed that satisfies the ordinary master equation, so that $(S,\, S) = 0,\, \Delta S = 0$. Assuming the quantum master equation, variations in operators' expectations are obtainable, viz.

$$\delta\left\langle \mathcal{O}\right\rangle = \frac{-\mathrm{i}}{Z_\Psi}\int\left[\prod \mathrm{d}\chi\right]\exp\mathrm{i}I_\Psi \frac{\delta_R\mathcal{O}}{\delta\chi^A}\left.\frac{\delta_R S\left[\chi,\,\chi^\ddagger\right]}{\delta\chi_A^\ddagger}\right|_{\chi^\ddagger=\frac{\delta\Psi}{\delta\chi}}. \tag{F.23}$$

In terms of the Slavnov operator of the generalised BRST transformation, this may be restated as

$$\delta\left\langle \mathcal{O}\right\rangle = \frac{-\mathrm{i}}{Z_\Psi}\int\left[\prod \mathrm{d}\chi\right]\exp\mathrm{i}I_\Psi s\mathcal{O}. \tag{F.24}$$

Thus expectation values of operators that are invariant under the generalised BRST transformation (i.e. satisfy $s\mathcal{O} = 0$) are unaffected by a change in the choice of Ψ.

G Proving $\hat{M}_{AB}$ is invertible for two important spacetimes

The aim is to show $\det\hat{M}_{AB} \neq 0$. This is equivalent to the existence of an inverse $\hat{M}^{AB}$ satisfying $\det\hat{M}^{AB} \neq 0$. The purpose of this appendix is to verify that this result is obtainable, with appropriate choices of the ξ_A^μ, for a flat static torus and de Sitter space. In the non-interacting case

$$\begin{aligned}\hat{M}_{AB} &= \int_{\mathbf{x}}\xi_A^\nu\xi_B^\beta\hat{K}_{\nu\beta}^{00} = \int_{\mathbf{x}}\xi_A^\nu\xi_B^\beta\left(\left(1-2k\right)\delta_\nu^0\delta_\beta^0 + g^{00}g_{\beta\nu}\right)\\ &= \int_{\mathbf{x}}\left(\left(1-2k\right)\xi_A^0\xi_B^0 + g^{00}\xi_A^\nu\xi_{B\nu}\right) = \int_{\mathbf{x}}\left(\left(1-2k\right)\xi_A^0 g^{0\nu} + g^{00}\xi_A^\nu\right)\xi_{B\nu}\\ &= \int_{\mathbf{x}}\left\{2g^{00}\left(1-k\right)\xi_A^0\xi_{B0} + \left(\left(1-2k\right)\xi_A^0 g^{0i} + g^{00}\xi_A^i\right)\xi_{Bi}\right\}.\end{aligned} \tag{G.1}$$

In the synchronous gauge, this simplifies to $\hat{M}_{AB} = \int_{\mathbf{x}}\left[2\left(1-k\right)\xi_A^0\xi_{B0} + \xi_A^i\xi_{Bi}\right]$. This result can be considered further for each of the spacetimes of interest.

G.1 The flat static torus

On a flat static torus, the choice $k > 1$ is undesirable because its FP-sector zero modes with $h_{\mu\nu} = 0$ include tachyons. I instead impose $k < 1$ so $2 - 2k > 0$, and choose the ξ_A^μ and η_ν^B as follows. Let η_{BC} be the n–dimensional Lorentzian metric of signature $1,\, -1,\, \cdots,\, -1$. A unique non-singular $\hat{M}^{AB}$ that serves my purposes is then easily obtained for $k < 1$:

$$\xi_A^\mu = \delta_A^\mu, \tag{G.2}$$

$$\begin{aligned}\eta_{BC}\delta_\mu^C = \eta_\mu^B &= N^2 V \hat{M}^{BC} \xi_C^\nu K_{\nu\mu}^{00} \\ &= N^2 V \hat{M}^{BC} \xi_C^\nu \left((1-2k)\, \delta_\nu^0 \delta_\mu^0 + g_{\mu\nu} \right) \\ &= N^2 V \hat{M}^{BC} \left((1-2k)\, \xi_C^0 \delta_\mu^0 + \xi_C^\nu g_{\mu\nu} \right) \\ &= N^2 V \left((1-2k)\, M^{B0} g^{0\nu} + \left(\hat{M}^{-1} \right)^{B\nu} \right) g_{\mu\nu},\end{aligned} \tag{G.3}$$

$$(1-2k)\, \hat{M}^{B0} g^{0\nu} + \hat{M}^{B\nu} = \frac{g^{B\nu}}{N^2 V}, \tag{G.4}$$

$$\hat{M}^{00} = \frac{1}{N^2 V\,(2-2k)}, \tag{G.5}$$

$$\hat{M}^{0i} = \hat{M}^{i0} = 0, \tag{G.6}$$

$$\hat{M}^{ij} = \frac{g^{ij}}{N^2 V}, \tag{G.7}$$

$$\det \left(\hat{M}^{-1} \right)^{AB} = \frac{\det g^{AB}}{2-2k} = \frac{1}{2g\,(1-k)} \neq 0. \tag{G.8}$$

G.2 de Sitter space

We show we may choose the ξ_μ^A so that there are no terms of the form $\hat{M}_{AB} \neq 0$ with $A \neq B$. We call such hypothetical terms off-diagonal terms. We split the Killing vectors in to two sets, those corresponding to space rotations and de Sitter boosts. (This decomposition is possible because de Sitter space has the topology $\mathbb{R} \times S^{n-1}$.) We show off-diagonal terms are found neither within one set nor between them (i.e. when A, B are chosen from different sets). Then $\hat{M}_{AB}$ is a diagonal matrix, and we can choose k so that the $\hat{M}_{AA}$, which are linear in k, are each nonzero. Indeed, from Eq. (4.2.13) we simply need to ensure

$$\nexists A:\ k = \frac{1}{2}\left(1 + \frac{\int_{\mathbf{x}} g^{00} \xi_A^\mu \xi_{A\mu}}{\int_{\mathbf{x}} \left(\xi_A^0 \right)^2} \right). \tag{G.9}$$

The inverse $\hat{M}^{-1}$ of $\hat{M}$ is then trivial to compute.

The usual basis of Killing vectors for SO$(n-1)$ space rotations comprises Killing vectors that have vanishing time components, and space components that are linear combinations of the vector spherical harmonics $A_\mu^{L\sigma}$. By the orthonormality condition Eq. (2.6.68), these Killing vectors provide no off-diagonal terms if we impose Eq. (1.2.2). Indeed

$$\hat{M}_{AB} = \int_{\mathbf{x}} \left(g^{00} \xi_A^\mu \xi_{B\mu} + (1-2k)\, \xi_A^0 \xi_B^0 \right) = \int_{\mathbf{x}} g^{00} \xi_A^i \xi_{Bi} = \delta_{AB} \int_{\mathbf{x}} g^{00} \xi_A^i \xi_{Ai}. \tag{G.10}$$

Not only have we found no off-diagonal terms; for rotational ξ_A^μ we have $\hat{M}_{AA} = \int_{\mathbf{x}} g^{00} \xi_A^i \xi_{Ai} \neq 0$, so the choice of k is so far irrelevant.

The usual basis of Killing vectors for de Sitter boosts have time components that are linear combinations of the $Y^{L\sigma}$ (say Y_A), and space components that are linear combinations of the $W_\mu^{L\sigma}$ (say $W_{A\mu}$).

Again we have an orthonormality condition. By parts, scalar harmonics ϕ_1, ϕ_2 satisfy

$$\int \mathrm{d}^{n-1}\mathbf{x}\nabla_i\phi_1\nabla^i\phi_2 = -\int \mathrm{d}^{n-1}\mathbf{x}\phi_1\nabla_i\nabla^i\phi_2 = (n-2)\int \mathrm{d}^{n-1}\mathbf{x}\phi_1\phi_2 \tag{G.11}$$

(these spherical harmonics have angular-momentum eigenvalue 1, so their $-\nabla_i\nabla^i$ eigenvalue is $1 \cdot (1+n-3) = n-2$). Thus

$$\hat{M}_{AB} = \int_{\mathbf{x}} \left(g^{00}W_A^\mu W_{B\mu} + (1-2k)\,Y_A Y_B\right) = \delta_{AB}\int_{\mathbf{x}} \left(g^{00}W_A^\mu W_{A\mu} + (1-2k)\,Y_A^2\right), \tag{G.12}$$

again giving no off-diagonal terms. However, for almost all k we have $\hat{M}_{AA} \neq 0$.

With the above conventions, there is also no off-diagonal term connecting a rotation Killing vector and a boost Killing vector because a rotational Killing vector ξ satisfies

$$\nabla_i\xi^i = -\nabla_0\xi^0 = 0 \rightarrow \int \mathrm{d}^{n-1}\mathbf{x}\xi_i\nabla^i\phi = -\int \mathrm{d}^{n-1}\mathbf{x}\nabla^i\xi_i\phi = 0 \tag{G.13}$$

for any scalar harmonic ϕ, so for rotational ξ_A^μ and boosting ξ_B^μ we have

$$\hat{M}_{AB} = \int_{\mathbf{x}} g^{00}\xi_A^i\xi_{Bi} = 0. \tag{G.14}$$

(We need not check the case where ξ_A^μ is boosting and ξ_B^μ rotational, since $\hat{M}_{AB} = \hat{M}_{BA}$.)

H L^{extra} and L_{eff} on the flat static torus

The purpose of this appendix is to compare the perturbation theories of the CMP and FMP for perturbative gravity on a flat static torus. Although a general proof of the prescriptions' perturbative equivalence has already been presented in Sec. 4.7, this discussion is formative for several reasons. In Sec. H.1, I compute L^{extra} in the CMP. In Sec. H.2, I deduce the Feynman diagrams that appear in the perturbation theory. These include chain diagrams and loop diagrams. In Sec. H.3, I discuss chain diagrams to compute important coefficients. In Sec. H.4, I use these coefficients to sum the amplitudes of chain diagrams. In Sec. H.5, I discuss loop diagrams.

H.1 The CMP

Consider a flat torus of spacetime dimension n and finite volume V with $g_{00} = 1$, $g_{0i} = 0$, $g_{ij} = -\delta_{ij}$ so that $\sqrt{|g|} = 1$ and $V = \int \mathrm{d}^{n-1}\mathbf{x}$ is a spacetime constant. The simplest choice for a basis of the Killing vectors is the δ_A^μ, with A any fixed spacetime index. The conserved charges then include

$$Q_\nu := \int \mathrm{d}^{n-1}\mathbf{x} Z_{0\nu}, \tag{H.1}$$

with $Z_{0\nu} = \partial_0 c_\nu + \partial_\nu c_0 - 2kg_{0\nu}\partial_\alpha c^\alpha + \kappa\left(c^\alpha\partial_\alpha H_{0\nu} + \partial_0 c^\alpha h_{\alpha\nu} + \partial_\nu c^\alpha h_{\alpha 0} - 2kg_{0\nu}\partial_\alpha c^\beta h_\beta^\alpha\right)$. The CMP sets $Q_\nu = 0$ to obtain $\dot{c}_{(0)}^\alpha$. The operator $\frac{1}{V}\int \mathrm{d}^{n-1}\mathbf{x}$ obtains all zero modes, which are spatially uniform and can be moved outside space integrals. Throughout we take $H_{0\gamma(0)} = H_{\gamma(0)}^0 = 0$; since $h_{\alpha\beta}$ and $H_{\alpha\beta} = h_{\alpha\beta} - khg_{\alpha\beta}$ are both symmetric, this choice implies $h_{\gamma(0)}^0 = k\delta_\gamma^0 h_{(0)}$. In the CMP,

$$
\begin{aligned}
0 = Q_0 = &\int \mathrm{d}^{n-1}\mathbf{x} \Big(2\left(1-k\right)\partial_0 c_0 - 2k\partial_i c^i_{(+)} + \kappa \times \\
&\Big(c^\alpha_{(+)}\partial_\alpha H_{00(+)} + 2\partial_0 c^\alpha_{(+)} h_{0\alpha(+)} + 2k\partial_0 c^0_{(0)} h_{(0)} - 2k\partial_\alpha c^\beta_{(+)} h^\alpha_{\beta(+)} - 2k^2\partial_0 c^0_{(0)} h_{(0)}\Big)\Big) \\
&= 2V\left(1-k\right)\left(1+k\kappa h_{(0)}\right)\partial_0 c^0_{(0)} \\
&+ \kappa \int \mathrm{d}^{n-1}\mathbf{x}\left(c^\alpha_{(+)}\partial_\alpha H_{00(+)} + 2\partial_0 c^\alpha_{(+)} h_{0\alpha(+)} - 2k\partial_\alpha c^\beta_{(+)} h^\alpha_{\beta(+)}\right), \qquad \text{(H.2)}
\end{aligned}
$$

$$
\dot{c}_{0(0)} = \frac{\kappa\left[2V\left(k-1\right)\right]^{-1}}{1+k\kappa h_{(0)}} \int \mathrm{d}^{n-1}\mathbf{x}\left(c^\alpha_{(+)}\partial_\alpha H_{00(+)} + 2\partial_0 c^\alpha_{(+)} h_{0\alpha(+)} - 2k\partial_\alpha c^\beta_{(+)} h^\alpha_{\beta(+)}\right). \qquad \text{(H.3)}
$$

Similarly

$$
0 = Q_i = \int \mathrm{d}^{n-1}\mathbf{x}\left\{ \partial_0 c_{i(0)} + \kappa c^\alpha \partial_\alpha H_{0i} + \partial_0 c_{\nu(+)} h^\nu_{i(+)} + \partial_0 c_{j(0)} h^j_{i(0)} + \partial_i c^\alpha_{(+)} h_{0\alpha(+)}\right\}. \qquad \text{(H.4)}
$$

This result may be rearranged as

$$
V G^j_{(0)i}\dot{c}_{j(0)} = -\kappa \int \mathrm{d}^{n-1}\mathbf{x}\left(c^\alpha\partial_\alpha H_{0i} + \partial_0 c_{\nu(+)} h^\nu_{i(+)} + \partial_i c^\alpha_{(+)} h_{0\alpha(+)}\right) \qquad \text{(H.5)}
$$

where $G^j_{(0)i} := \delta^j_i + \kappa h^j_{i(0)}$. In terms of an inverse matrix, viz. $\left(G^{-1}_{(0)}\right)^i_j G^k_{(0)i} = \delta^k_j$, Eq. (H.5) becomes

$$
\dot{c}_{j(0)} = -\frac{\kappa}{V}\left(G^{-1}_{(0)}\right)^i_j \int \mathrm{d}^{n-1}\mathbf{x}\left(c^\alpha\partial_\alpha H_{0i} + \partial_0 c_{\nu(+)} h^\nu_{i(+)} + \partial_i c^\alpha_{(+)} h_{0\alpha(+)}\right). \qquad \text{(H.6)}
$$

Define $V^\mu := \left(\partial^0 \overline{c}^\nu_{(+)} + \partial^\nu \overline{c}^0_{(+)} - 2k\delta^\nu_0 \partial_\sigma \overline{c}^\sigma_{(+)}\right)\left(\delta^\mu_\nu + \kappa h^\mu_\nu\right)$; then

$$
\begin{aligned}
Z_{\mu\nu}\big|_{c^\alpha = c^\alpha_{(0)}} &= \partial_\mu c_{\nu(0)} + \partial_\nu c_{\mu(0)} - 2k g_{\mu\nu}\partial_\alpha c^\alpha_{(0)} \\
&+ \kappa\left(c^\alpha_{(0)}\partial_\alpha H_{\mu\nu} + \partial_\mu c^\alpha_{(0)} h_{\alpha\nu} + \partial_\nu c^\alpha_{(0)} h_{\alpha\mu} - 2k g_{\mu\nu}\partial_\alpha c^\beta_{(0)} h^\alpha_\beta\right), \qquad \text{(H.7)} \\
\partial^\mu \overline{c}^\nu_{(+)}\; Z_{\mu\nu}\big|_{c^\alpha = c^\alpha_{(0)}} &= V^\mu \dot{c}_{\mu(0)} + \kappa\partial^\alpha \overline{c}^\nu_{(+)} c_{(0)\mu}\partial^\mu H_{\alpha\nu}. \qquad \text{(H.8)}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathrm{i}L^{(+0)}_{\mathrm{FP}} &= \int \mathrm{d}^{n-1}\mathbf{x}\left\{ V^0\dot{c}_{0(0)} + V^i\dot{c}_{i(0)} + \kappa\partial^\alpha\overline{c}^\nu_{(+)}\partial^\mu H_{\alpha\nu} c_{\mu(0)}\right\} \\
&= \int \mathrm{d}^{n-1}\mathbf{x}\mathrm{d}^{n-1}\mathbf{y}\left\{ V^0 \frac{\kappa\left[2V\left(k-1\right)\right]^{-1}}{1+k\kappa h_{(0)}} \times \right. \\
&\underline{\left(c^\alpha_{(+)}\partial_\alpha H_{00(+)} + 2\partial_0 c^\alpha_{(+)} h_{0\alpha(+)} - 2k\partial_\alpha c^\beta_{(+)} h^\alpha_{\beta(+)}\right)} \\
&- \frac{\kappa}{V} V^i \left(G^{-1}_{(0)}\right)^j_i \underline{\left(c^\alpha\partial_\alpha H_{0j} + \partial_0 c_{\nu(+)} h^\nu_{j(+)} + \partial_j c^\alpha_{(+)} h_{0\alpha(+)}\right)} \\
&\left. + \kappa\partial^\alpha\overline{c}^\nu_{(+)} c_{(0)\mu}\partial^\mu H_{\alpha\nu}\right\}. \qquad \text{(H.9)}
\end{aligned}
$$

The underlined quantities depend on $\mathbf{y}$; others depend on $\mathbf{x}$.

Denoting equality up to total derivatives by $\sim$,

$$
\begin{aligned}
-\kappa\partial^\mu B^\nu H_{\mu\nu} - \mathrm{i}\kappa\partial^\mu \overline{c}^\nu c^\alpha \partial_\alpha H_{\mu\nu} &= -\mathrm{i}\kappa\partial^\mu \overline{c}^\nu c^\alpha_{(+)}\partial_\alpha H_{\mu\nu} - \mathrm{i}\kappa\partial^\mu \overline{c}^\nu c^\alpha_{(0)}\partial_\alpha H_{\mu\nu} \\
&\quad - \kappa\partial^\mu \widetilde{B}^\nu H_{\mu\nu} - \mathrm{i}\kappa\partial^\mu\left(\partial_\alpha \overline{c}^\nu c^\alpha_{(0)}\right) H_{\mu\nu} \\
&\sim -\mathrm{i}\kappa\partial^\mu \overline{c}^\nu c^\alpha_{(+)}\partial_\alpha H_{\mu\nu} - \mathrm{i}\kappa\partial^\mu \overline{c}^\nu c^\alpha_{(0)}\partial_\alpha H_{\mu\nu} - \kappa\partial^\mu \widetilde{B}^\nu H_{\mu\nu} \\
&\quad - \mathrm{i}\kappa\partial^\mu\partial_\alpha \overline{c}^\nu c^\alpha_{(0)} H_{\mu\nu} + \mathrm{i}\kappa\partial^\mu \overline{c}^\nu \partial_\alpha c^\alpha_{(0)} H_{\mu\nu} \\
&\sim -\mathrm{i}\kappa\partial^\mu \overline{c}^\nu c^\alpha_{(+)}\partial_\alpha H_{\mu\nu} - \kappa\partial^\mu \widetilde{B}^\nu H_{\mu\nu} \\
&\quad + \mathrm{i}\kappa\partial^\mu \overline{c}^\nu \partial_\alpha c^\alpha_{(0)} H_{\mu\nu} - \mathrm{i}\kappa\partial_\alpha \overline{c}^\nu \partial^\mu c^\alpha_{(0)} H_{\mu\nu} \\
&\sim -\mathrm{i}\kappa\partial^\mu \overline{c}^\nu c^\alpha_{(+)}\partial_\alpha H_{\mu\nu} - \kappa\partial^\mu \widetilde{B}^\nu H_{\mu\nu} \\
&\quad + \mathrm{i}\kappa\partial^\mu \overline{c}^\nu \dot{c}^0_{(0)} H_{\mu\nu} - \mathrm{i}\kappa\partial_\alpha \overline{c}^\nu \dot{c}^\alpha_{(0)} H_{0\nu}.
\end{aligned} \tag{H.10}
$$

Therefore, the transformation

$$
\kappa\partial^\alpha \overline{c}^\nu_{(+)} c_{(0)\mu}\partial^\mu H_{\alpha\nu} \to \kappa\partial^\alpha \overline{c}^\nu_{(+)}\left(\dot{c}_{\alpha(0)}H_{0\nu} - \dot{c}_{0(0)}H_{\alpha\nu}\right) = \kappa\partial^i \overline{c}^\nu_{(+)}\left(\dot{c}_{i(0)}H_{0\nu} - \dot{c}_{0(0)}H_{i\nu}\right) \tag{H.11}
$$

effects $L^{(+0)}_{\mathrm{FP}} \to L^{\mathrm{extra}}$. Thus

$$
\begin{aligned}
\mathrm{i}L^{\mathrm{extra}} &= \int \mathrm{d}^{n-1}\mathbf{x}\left\{V^0 \dot{c}_{0(0)} + V^i \dot{c}_{i(0)} + \kappa\partial^\alpha \overline{c}^\nu_{(+)}\partial^\mu H_{\alpha\nu} c_{\mu(0)}\right\} \\
&= \int \mathrm{d}^{n-1}\mathbf{x}\mathrm{d}^{n-1}\mathbf{y}\left\{\frac{\kappa\left[2V(k-1)\right]^{-1}}{1+k\kappa h_{(0)}}\left(V^0 - \kappa\partial^i \overline{c}^\nu_{(+)}H_{i\nu}\right)\times\right. \\
&\quad \underline{\left(c^\alpha_{(+)}\partial_\alpha H_{00(+)} + 2\partial_0 c^\alpha_{(+)}h_{0\alpha(+)} - 2k\partial_\alpha c^\beta_{(+)}h^\alpha_{\beta(+)}\right)} \\
&\quad - \frac{\kappa}{V}\left(G^{-1}_{(0)}\right)^j_i\left(V^i + \kappa\partial^i \overline{c}^\nu_{(+)}H_{0\nu}\right)\times \\
&\quad \left.\underline{\left(c^\alpha\partial_\alpha H_{0j} + \partial_0 c_{\nu(+)}h^\nu_{j(+)} + \partial_j c^\alpha_{(+)}h_{0\alpha(+)}\right)}\right\}.
\end{aligned} \tag{H.12}
$$

Rearranging gives

$$
\begin{aligned}
L^{\mathrm{extra}} &= \mathrm{i}\frac{\left[2V(1-k)\right]^{-1}}{1+k\kappa h_{(0)}}\int \mathrm{d}^{n-1}\mathbf{x}A(t,\mathbf{x})\int \mathrm{d}^{n-1}\mathbf{y}B(t,y) \\
&\quad + \frac{\mathrm{i}}{V}\left(G^{-1}_{(0)}\right)^j_i\int \mathrm{d}^{n-1}\mathbf{x}A^i(t,\mathbf{x})\int \mathrm{d}^{n-1}\mathbf{y}B_j(t,y)
\end{aligned} \tag{H.13}
$$

where

$$
A := -\kappa\partial^i \overline{c}^\nu_{(+)}H_{i\nu(+)} + \left(\partial^0 \overline{c}^\nu_{(+)} + \partial^\nu \overline{c}^0_{(+)} - 2k\delta^\nu_0\partial_\sigma \overline{c}^\sigma_{(+)}\right)\kappa h^0_{\nu(+)}, \tag{H.14}
$$

$$
B := \kappa\left(c^\alpha_{(+)}\partial_\alpha H_{00(+)} + 2\dot{c}^\alpha_{(+)}h_{0\alpha(+)} - 2k\partial_\alpha c^\beta_{(+)}h^\alpha_{\beta(+)}\right), \tag{H.15}
$$

$$
A^i := \kappa\partial^i \overline{c}^\nu_{(+)}H_{0\nu(+)} + \left(\partial^0 \overline{c}^\nu_{(+)} + \partial^\nu \overline{c}^0_{(+)} - 2k\delta^\nu_0\partial_\sigma \overline{c}^\sigma_{(+)}\right)\kappa h^i_{\nu(+)}, \tag{H.16}
$$

$$
B_j := \kappa\left(c^\alpha_{(+)}\partial_\alpha H_{0j(+)} + \dot{c}_{\nu(+)}h^\nu_{j(+)} + \partial_j c^\alpha_{(+)}h_{0\alpha(+)}\right). \tag{H.17}
$$

(Quantities such as $A, \cdots, B_j$ can always be replaced with their own $_{(+)}$ mode, which effects such simplifications as $\delta^\alpha_{\beta(+)} = 0$.)

H.2 Feynman diagrams of interaction terms

The interaction terms in the Lagrangian density are

$$\mathcal{L} = -\mathrm{i}\kappa\partial^\mu \overline{c}^\nu \left(c^\alpha \partial_\alpha H_{\mu\nu} + \partial_\mu c^\alpha h_{\alpha\nu} + \partial_\nu c^\alpha h_{\alpha\mu} - 2kg_{\mu\nu}\partial_\alpha c_\beta h^{\alpha\beta}\right). \tag{H.18}$$

The next task is to obtain the zero-mode part $\mathcal{L}_{00}$ of $\mathcal{L}$. This involves setting both fields to their zero modes, and deleting any term which vanishes upon space integration (i.e. does not contribute to the Lagrangian) when setting $H^{0\nu}_{(0)} = 0$ so that $h^{0i}_{(0)} = 0$. Using $h_{00(0)} = h^0_{0(0)} = k\delta^0_0 h_{(0)} = kh_{(0)}$,

$$\begin{aligned}\mathcal{L}_{00} &= -\mathrm{i}\kappa\dot{\overline{c}}^\nu_{(0)} \left(\dot{c}^\alpha_{(0)} h_{\alpha\nu(0)} + \partial_\nu c^0_{(0)} h_{00(0)} - 2kg_{0\nu}\dot{c}_{0(0)} h^{00}_{(0)}\right) \\ &= -\mathrm{i}\kappa\left(\dot{\overline{c}}^i_{(0)}\dot{c}^j_{(0)} h_{ij(0)} + 2\left(1-k\right)\dot{\overline{c}}^0_{(0)}\dot{c}^0_{(0)} h_{00(0)}\right) \\ &= -\mathrm{i}\kappa\left(\dot{\overline{c}}^i_{(0)}\dot{c}^j_{(0)} h_{ij(0)} + 2k\left(1-k\right)\dot{\overline{c}}^0_{(0)}\dot{c}^0_{(0)} h_{(0)}\right).\end{aligned} \tag{H.19}$$

Perturbation theory attaches $\mathrm{i}\mathcal{L}$ factors to vertices. The factor $\mathrm{i}\mathcal{L}_{00}$ gives rise to a number of Feynman diagrams. These include loop diagrams (which vanish under Lee–Yang cancellation) and two series of chain diagrams (one scalar-valued and summing, say, to F; the other is matrix-valued and sums, say, to F_{ij}). From Eq. (H.18), with terms that spatial integration deletes dropped,

$$\begin{aligned}\mathcal{L}_{(0+)} &= -\mathrm{i}\kappa\left(\dot{\overline{c}}^\nu_{(0)}\left\{c^\alpha_{(+)}\partial_\alpha H_{0\nu(+)} + \dot{c}^\alpha_{(+)} h_{\alpha\nu(+)} + \partial_\nu c^\alpha_{(+)} h_{0\alpha(+)}\right\} - 2k\dot{\overline{c}}_{0(0)}\partial_\alpha c_{\beta(+)} h^{\alpha\beta}_{(+)}\right) \\ &= -\mathrm{i}\kappa\dot{\overline{c}}^\nu_{(0)} X_{\nu(+)} = \mathrm{i}\kappa X_{\nu(+)}\dot{\overline{c}}^\nu_{(0)},\end{aligned} \tag{H.20}$$

$$X_{\nu(+)} := c^\alpha_{(+)}\partial_\alpha H_{0\nu(+)} + \dot{c}^\alpha_{(+)} h_{\alpha\nu(+)} + \partial_\nu c^\alpha_{(+)} h_{0\alpha(+)} - 2kg_{0\nu}\partial_\alpha c_{\beta(+)} h^{\alpha\beta}_{(+)}, \tag{H.21}$$

$$\begin{aligned}\mathcal{L}_{(+0)} &= -\mathrm{i}\kappa\Big(\partial^\mu \overline{c}^\nu_{(+)} c^\alpha_{(0)}\partial_\alpha H_{\mu\nu(+)} - 2k\partial_\nu \overline{c}^\nu_{(+)}\dot{c}_{\beta(0)} h^{0\beta}_{(+)} \\ &\quad + \left(\dot{\overline{c}}^\nu_{(+)} h_{\alpha\nu(+)} + \partial^\mu \overline{c}^0_{(+)} h_{\alpha\mu(+)}\right)\dot{c}^\alpha_{(0)}\Big).\end{aligned} \tag{H.22}$$

Let $X \approx Y$ denote $\exists Z^\mu : X - Y = \partial_\mu Z^\mu$. Using $\partial^\mu H_{\mu\nu} = 0$ in Eq. (H.22), we can show that

$$\partial^\mu \overline{c}^\nu_{(+)} c^\alpha_{(0)}\partial_\alpha H_{\mu\nu(+)} \approx \left(\partial_\nu \overline{c}^\alpha_{(+)} H_{\mu\alpha(+)} - \partial^\alpha \overline{c}^\beta_{(+)} H_{\alpha\beta(+)} g_{\mu\nu}\right)\partial^\mu c^\nu_{(0)}, \tag{H.23}$$

since the difference between the two sides is

$$\begin{aligned}&\partial^\mu \overline{c}^\nu_{(+)} c^\alpha_{(0)}\partial_\alpha H_{\mu\nu(+)} + \partial^\alpha \overline{c}^\beta_{(+)} H_{\alpha\beta(+)}\partial_\nu c^\nu_{(0)} - \partial_\nu \overline{c}^\alpha_{(+)} H_{\mu\alpha(+)}\partial^\mu c^\nu_{(0)} \\ &= \partial^\mu \overline{c}^\nu_{(+)}\partial_\alpha\left(c^\alpha_{(0)} H_{\mu\nu(+)}\right) - \partial_\alpha \overline{c}^\nu_{(+)}\partial^\mu\left(H_{\mu\nu(+)} c^\alpha_{(0)}\right) \\ &\approx \partial^\mu \overline{c}^\nu_{(+)}\partial_\alpha\left(c^\alpha_{(0)} H_{\mu\nu(+)}\right) + \overline{c}^\nu_{(+)}\partial^\mu\partial_\alpha\left(H_{\mu\nu(+)} c^\alpha_{(0)}\right) \\ &= \partial^\mu\left(\overline{c}^\nu_{(+)}\partial_\alpha\left(c^\alpha_{(0)} H_{\mu\nu(+)}\right)\right) \approx 0.\end{aligned} \tag{H.24}$$

Hence

$$\begin{aligned}\mathcal{L}_{(+0)} &\approx -\mathrm{i}\kappa\left(\left(\partial_\nu\overline{c}^\alpha H_{\mu\alpha(+)}-\partial^\alpha\overline{c}^\beta_{(+)}H_{\alpha\beta(+)}g_{\mu\nu}\right)\partial^\mu c^\nu_{(0)}\right.\\&\left.+\left(\dot{\overline{c}}^\nu_{(+)}h_{\alpha\nu(+)}+\partial^\mu\overline{c}^0_{(+)}h_{\alpha\mu(+)}\right)\dot{c}^\alpha_{(0)}-2k\partial_\nu\overline{c}^\nu_{(+)}\dot{c}_{\beta(0)}h^{0\beta}_{(+)}\right)\\&=-\mathrm{i}\kappa Y_{\nu(+)}\dot{c}^\nu_{(0)},\end{aligned}\tag{H.25}$$

$$\begin{aligned}Y_{\nu(+)} &:= \partial_\nu\overline{c}^\alpha H_{0\alpha(+)}-\partial^\alpha\overline{c}^\beta_{(+)}H_{\alpha\beta(+)}g_{0\nu}\\&+\dot{\overline{c}}^\alpha_{(+)}h_{\alpha\nu(+)}+\partial^\mu\overline{c}^0_{(+)}h_{\mu\nu(+)}-2k\partial_\alpha\overline{c}^\alpha_{(+)}h^0_{\nu(+)}.\end{aligned}\tag{H.26}$$

(Note that, despite what the notation may suggest, $X_{\nu(+)}$, $Y_{\nu(+)}$ are not definitionally the plus modes of any quantities.) By inspection $X_{0(+)}=\frac{B}{\kappa}$, $X_{j(+)}=\frac{B_j}{\kappa}$, $Y_{0(+)}=\frac{A}{\kappa}$, $Y_{i(+)}=\frac{A_i}{\kappa}$. Extending the $(n-1)$-vectors A_i, B_i to n-vectors with $A_0:=A$, $B_0:=B$ at the vertices of Feynman diagrams we can put $\mathrm{i}\mathcal{L}_{(0+)}\approx\dot{\overline{c}}^\nu_{(0)}B_\nu$, $\mathrm{i}\mathcal{L}_{(+0)}\approx A_\nu\dot{c}^\nu_{(0)}$. We have

$$\mathrm{i}S_{\text{extra}}=\frac{1}{V}\int \mathrm{d}t\mathrm{d}^{n-1}\mathbf{x}\mathrm{d}t^{'}\mathrm{d}^{n-1}\mathbf{y}\delta\left(t-t^{'}\right)\left\{F_{ij}A^iB^j+FAB\right\},\tag{H.27}$$

$$\mathrm{i}L_{\text{eff}}=\frac{1}{V}\left\{F\int\mathrm{d}^{n-1}\mathbf{x}A\int\mathrm{d}^{n-1}\mathbf{y}B+F_{ij}\int\mathrm{d}^{n-1}\mathbf{x}A^i\int\mathrm{d}^{n-1}\mathbf{y}B^j\right\}.\tag{H.28}$$

Thus prescription equivalence ($L_{\text{eff}}=L^{\text{extra}}$) occurs if and only if

$$F=\frac{-\left[2\left(1-k\right)\right]^{-1}}{1+k\kappa h_{(0)}},\ F_{ij}=-\left(G^{-1}_{(0)}\right)_{ij}.\tag{H.29}$$

These equations are verified in Sec. (H.4).

H.3 The chain diagrams: a_s and a_t

For the flat static torus $\hat{K}^{00}_{\nu\beta}=\left(1-2k\right)\delta^0_\nu\delta^0_\beta+g^{00}g_{\beta\nu}$, so

$$\hat{K}^{00}_{00}=2-2k=-\frac{2}{\beta},\ \hat{K}^{00}_{0i}=\hat{K}^{00}_{i0}=0,\ \hat{K}^{00}_{ij}=-\delta_{ij},\tag{H.30}$$

where $\beta:=\frac{1}{k-1}$. (These results will be needed soon.) The FMP grants c^μ, $\overline{c}^\mu$ a common mass $M>0$, so that the FP-ghost term in the Lagrangian density is $-\mathrm{i}\left(\partial_\nu\overline{c}^\mu Z^\nu_\mu-M^2\overline{c}^\mu c_\mu\right)$. The Euler–Lagrange equation obtained by varying $\overline{c}^\mu$ is

$$-M^2c_\mu=\partial_\nu Z^\nu_\mu.\tag{H.31}$$

Replacing c^α with $\overline{c}^\alpha$ in the definition of Z^ν_μ, we may similarly write the other equation of motion as

$$-M^2\overline{c}_\mu=\partial_\nu\overline{Z}^\nu_\mu,\ \overline{Z}^\nu_\mu:=K^{\beta\nu}_{\mu\alpha}\nabla_\beta\overline{c}^\alpha+\kappa\overline{c}^\alpha\nabla_\alpha H^\mu_\nu.\tag{H.32}$$

In the non-interacting case, $h^\nu_\mu=0$ and $\mathcal{L}_0$ is $c\leftrightarrow\overline{c}$-antisymmetric (*antisymmetric* because classical fermionic fields anticommute). Next I obtain the ghost's zero mode by imposing $\partial_ic_\mu=0$, and the antighost case is analogous. Eq. (H.31) reduces to $-M^2c_\mu=\left(1-2k\right)\delta^0_\mu\partial^2_0c_0+\partial^2_0c_\mu$. This result concerning massive zero modes can be rewritten in terms of β, viz.

$$\partial^2_0c_{0(0)}=\frac{\beta M^2}{2}c_{0(0)},\ \partial^2_0c_{i(0)}=-M^2c_{i(0)}.\tag{H.33}$$

The general Hermitian solution is of the form

$$c_{0(0)} = \frac{A_0 \mathrm{e}^{-\mathrm{i}M_1 t} + A_0^\dagger \mathrm{e}^{\mathrm{i}M_1 t}}{\sqrt{M_1 V}},\ c_{i(0)} = \frac{A_i \mathrm{e}^{-\mathrm{i}Mt} + A_i^\dagger \mathrm{e}^{\mathrm{i}Mt}}{\sqrt{MV}}, \tag{H.34}$$

where $M_1 := M\sqrt{\frac{-\beta}{2}}$. The FP-ghost part of the Lagrangian density without a metric perturbation is $-\mathrm{i}K^{\beta\nu}_{\mu\alpha}\partial_\nu \overline{c}^\mu \partial_\beta c^\alpha$, so the general form of antighost zero modes is analogous, with constants $\overline{A}_\mu$. The constants A_μ, $\overline{A}_\mu$ are annihilation operators, whose anticommutators follow from CARs and determine the FP-ghost propagator. Explicitly

$$\pi_{c^\mu} = \mathrm{i}K^{0\nu}_{\gamma\mu}\partial_\nu \overline{c}^\gamma, \tag{H.35}$$

$$\pi_{\overline{c}^\mu} = -\mathrm{i}K^{\beta 0}_{\mu\alpha}\partial_\beta \overline{c}^\alpha, \tag{H.36}$$

$$-\mathrm{i}g_{\mu\nu} = \left\{\pi^\mu_{c_{(0)}},\ c_{\nu(0)}\right\} = \mathrm{i}\hat{K}^{00}_{\gamma\mu}\left\{\partial_0 \overline{c}^\gamma_{(0)},\ c_{\nu(0)}\right\}, \tag{H.37}$$

$$-1 = -\frac{2}{\beta}\left\{\partial_0 \overline{c}^0_{(0)},\ c_{0(0)}\right\}, \tag{H.38}$$

$$0 = \left\{\partial_0 \overline{c}^0_{(0)},\ c_{i(0)}\right\}, \tag{H.39}$$

$$0 = \left\{\partial_0 \overline{c}_{i(0)},\ c_{0(0)}\right\}, \tag{H.40}$$

$$-\delta_{ij} = \left\{\partial_0 \overline{c}_{i(0)},\ c_{j(0)}\right\}. \tag{H.41}$$

The only non-trivial anticommutators of ladder operators are $\left\{A_0,\ \overline{A}_0^\dagger\right\} = \frac{-\mathrm{i}\beta V}{4}$, $\left\{A_i,\ \overline{A}_j^\dagger\right\} = \frac{\mathrm{i}V}{2}\delta_{ij}$. Positive-frequency parts of a zero mode should annihilate physical kets, so the ghost-antighost vacuum $|0\rangle$ satisfies $A_\mu|0\rangle = 0$. Thus

$$\left\langle 0|c^0_{(0)}(t)\,\overline{c}^0_{(0)}\left(t'\right)|0\right\rangle = \frac{\mathrm{e}^{-\mathrm{i}M_1\left(t-t'\right)}}{M_1 V}\left\langle 0|A^0\overline{A}^{0\dagger}|0\right\rangle, \tag{H.42}$$

$$\left\langle 0|c^i_{(0)}(t)\,\overline{c}^j_{(0)}\left(t'\right)|0\right\rangle = \frac{\mathrm{e}^{-\mathrm{i}M\left(t-t'\right)}}{MV}\left\langle 0|A^i\overline{A}^{j\dagger}|0\right\rangle. \tag{H.43}$$

Replacing $t-t'$ throughout with $\left|t-t'\right|$ gives time-ordered results. Taking the massless limit (after an IR-divergent term is dropped) gives

$$G^{00} = \frac{-\mathrm{i}\left|t-t'\right|}{V}\left\langle 0|A^0\overline{A}^{0\dagger}|0\right\rangle,\ G^{ij} = \frac{-\mathrm{i}\left|t-t'\right|}{V}\left\langle 0|A^i\overline{A}^{j\dagger}|0\right\rangle. \tag{H.44}$$

Applying $\partial_t\partial_{t'}$ replaces each $-\mathrm{i}\left|t-t'\right|$ factor with $2\mathrm{i}\delta\left(t-t'\right)$. This implies

$$a_s\delta^{ij} = 2\mathrm{i}V^{-1}\left\langle 0|\left\{A^i,\ \overline{A}^{j\dagger}\right\}|0\right\rangle = -\delta^{ij}, \tag{H.45}$$

$$a_s = -1, \tag{H.46}$$

$$a_t = 2\mathrm{i}V^{-1}\left\langle 0|\left\{A^0,\ \overline{A}^{0\dagger}\right\}|0\right\rangle = \frac{-1}{2(1-k)}. \tag{H.47}$$

Note that the value of β for which $a_s = a_t$ ($\beta = -2$) is the same one for which the masses of c_0, c_i are equal. This condition is equivalent to the gauge choice $k = \frac{1}{2}$, i.e. the de Donder gauge.

H.4 The chain diagrams: F and F_{ij}

The matrix- (scalar-) valued series of Feynman diagrams is called the space- (time-) part. Before integration the Nth term ($N \geq 1$) of the space part is of the form

$$F_{sNij} := \partial_t \partial_1 G^{ij_1}(t,\, t_1) \prod_{k=1}^{N-1} \left(\kappa h_{j_k j_{k+1}(0)}(t_k)\, \partial_k \partial_{k+1} G^{j_{k+1} j_{k+2}}(t_k,\, t_{k+1})\right), \tag{H.48}$$

where $t_N := t'$ and $\partial_l := \partial_{t_l}$. Thus

$$\begin{aligned} F_{sNij} &= a_s \delta^{ij_1} \delta(t - t_1) \prod_{k=1}^{N-1} \left(\kappa h_{j_k j_{k+1}(0)}(t_k)\, a_s \delta^{j_{k+1} j_{k+2}} \delta(t_k - t_{k+1})\right) \\ &= -\delta(t - t_1) \prod_{k=1}^{N-1} \left(-\kappa h_{ij(0)}(t_k)\, \delta(t_k - t_{k+1})\right), \end{aligned} \tag{H.49}$$

$$\begin{aligned} \left(\prod_{k=1}^{N-1} \int \mathrm{d}t_k\right) F_{sNij} &= -\prod_{k=1}^{N-1} \int \mathrm{d}t_k \left\{\delta(t - t_1) \prod_{k=1}^{N-1} \left(-\kappa h_{ij(0)}(t_k)\, \delta(t_k - t_{k+1})\right)\right\} \\ &= -\prod_{k=1}^{N-1} \int \mathrm{d}t_k \left(-\kappa h_{ij(0)}(t_k)\right) = -\left(-\kappa h_{ij(0)}(t)\right)^{N-1}. \end{aligned} \tag{H.50}$$

Summing gives $F_{ij} := \sum_{N \geq 1} \left(\prod_{k=1}^{N-1} \int \mathrm{d}t_k\right) F_{sNij} = -\left(\mathbb{I} + \kappa h_{(0)}(t)\right)^{-1}_{ij} = -\left(G^{-1}_{(0)}\right)_{ij}$, where $\mathbb{I}$ is the identity matrix conformable with the matrix $h_{(0)ij}$.
Similarly, the time part is obtainable as follows:

$$\begin{aligned} F_{tN} &:= \partial_t \partial_1 G^{00}(t,\, t_1) \prod_{k=1}^{N-1} \left\{\kappa 2k(1-k)\, h_{(0)}(t_k)\, \partial_k \partial_{k+1} G^{00}(t_k,\, t_{k+1})\right\} \\ &= \frac{-1}{2(1-k)} \delta(t - t_1) \prod_{k=1}^{N-1} -\kappa k h_{(0)}(t_k)\, \delta(t_k - t_{k+1}), \end{aligned} \tag{H.51}$$

$$\left(\prod_{k=1}^{N-1} \int \mathrm{d}t_k\right) F_{tN} = \frac{-1}{2(1-k)} \left(-\kappa k h_{(0)}(t)\right)^{N-1}, \tag{H.52}$$

$$F := \sum_{N \geq 1} \left(\prod_{k=1}^{N-1} \int \mathrm{d}t_k\right) F_{tN} = \frac{-\left[2(1-k)\right]^{-1}}{1 + \kappa k h_{(0)}(t)}, \tag{H.53}$$

as required.

H.5 Loop diagrams

The loop diagrams of interest also split into "space part" and "time part" series. The space part is $\sum_{N \geq 1} \frac{L_{sN}}{N}$ where $L_{sN} := -\int \mathrm{d}t \prod_{k=1}^{N} \left\{\partial_k \partial_{k+1} \mathrm{i} G^{ij}(t_{k-1},\, t_k)\, h_{ij(0)}(t_{k-1})\right\}$, with $t_0 := t$. (The $\frac{1}{N}$ coefficient is a symmetry factor.) Thus

$$\begin{aligned} L_{sN} &= -\int \mathrm{d}t \prod_{k=1}^{N} \left\{\mathrm{i} a_s \delta(t_{k-1} - t_k)\, \kappa V h_{ii(0)}(t_{k-1})\right\} \\ &= -(-\kappa)^N \int \mathrm{d}t h^N_{ii(0)}(t) \prod_{k=1}^{N} \delta(t_{k-1} - t_k) = -(-\kappa)^N \delta(0) \int \mathrm{d}t h^N_{ii(0)}(t), \end{aligned} \tag{H.54}$$

$$\sum_{N \geq 1} \frac{L_{sN}}{N} = \delta(0) \int \mathrm{d}t \ln\left(\mathbb{I} + \kappa h_{(0)}(t)\right)_{ii}. \tag{H.55}$$

The time part is $\sum_{N\geq 1}\frac{L_{tN}}{N}$ where

$$\begin{aligned} L_{tN} &:= -\int \mathrm{d}t \prod_{k=1}^{N}\left\{\partial_k\partial_{k+1}\mathrm{i}G^{00}\left(t_{k-1},\, t_k\right)2\kappa k\left(1-k\right)h_{(0)}\left(t_{k-1}\right)\right\} \\ &= -\delta\left(0\right)\int \mathrm{d}t\left(-\kappa k h_{(0)}\left(t\right)\right)^{N}, \qquad \text{(H.56)} \\ \sum_{N\geq 1}\frac{L_{tN}}{N} &= \delta\left(0\right)\int \mathrm{d}t\ln\left(1+\kappa k h_{(0)}\left(t\right)\right). \qquad \text{(H.57)} \end{aligned}$$

These sums of amplitudes are contributions to $\mathrm{i}S_{\text{extra}}$; contributions to the action are obtained by multiplying by $-\mathrm{i}$. The contribution to the Lagrangian due to $\delta\left(0\right)$ terms, which is subject to a successful Lee–Yang cancellation, is

$$L_\delta := -\mathrm{i}\delta\left(0\right)\left(\ln\left(\mathbb{I}+\kappa h_{(0)}\left(t\right)\right)_{ii}+\ln\left(1+k\kappa h_{(0)}\left(t\right)\right)\right). \qquad \text{(H.58)}$$

These logarithms' arguments are coefficients of $-\mathrm{i}\partial_0\bar{c}^\mu\partial_0 c^\nu$ in the Lagrangian density. We may write $\mathcal{L}^{(00)}_{\text{FP}}=\mathcal{L}_t+\mathcal{L}_s$ where $\mathcal{L}_t,\,\mathcal{L}_s$ are "time" and "space" parts. We have a result of the form

$$\mathcal{L}_t=-\mathrm{i}K\left(t\right)\dot{\bar{c}}^0_{(0)}\dot{c}^0_{(0)}+\dot{\bar{c}}^0_{(0)}f\left(h_{(+)},\,c_{(+)}\right)+g\left(h_{(+)},\,\bar{c}_{(+)}\right)\dot{c}^0_{(0)} \qquad \text{(H.59)}$$

for some first-order differential operators $f,\,g$, and similarly with $\mathcal{L}_s$. Let $c^0_{(0)},\,\bar{c}^0_{(0)}$ have respective conjugate momenta $\pi^0_{(0)},\,\bar{\pi}^0_{(0)}$ and let $\mathcal{L}_t,\,\mathcal{L}_s$ have respective Legendre transforms $\mathcal{H}_t,\,\mathcal{H}_s$ so that

$$\pi^0_{(0)}=\mathrm{i}K\dot{\bar{c}}^0_{(0)}-g,\,\bar{\pi}^0_{(0)}=-\mathrm{i}K\dot{c}^0_{(0)}+f,\,\dot{\bar{c}}^0_{(0)}=\frac{-\mathrm{i}}{K}\left(\pi^0_{(0)}+g\right),\,\dot{c}^0_{(0)}=\frac{\mathrm{i}}{K}\left(\bar{\pi}^0_{(0)}-f\right). \qquad \text{(H.60)}$$

Hence

$$\mathcal{L}_t=\frac{-\mathrm{i}}{K}\left\{\pi^0_{(0)}\bar{\pi}^0_{(0)}+gf\right\},\,\mathcal{H}_t+\mathcal{L}_t=\frac{-\mathrm{i}}{K}\left\{g\bar{\pi}^0_{(0)}-\pi^0_{(0)}f\right\},\,\mathcal{H}_t=\frac{\mathrm{i}}{K}\left\{\left(\pi^0_{(0)}-g\right)\left(\bar{\pi}^0_{(0)}+f\right)+2gf\right\}. \qquad \text{(H.61)}$$

The standard choice of functional integral used in the path integral formalism integrates over $\pi^0_{(0)}$ and $\bar{\pi}^0_{(0)}$, but we can amend the measure to integrate instead over $\Pi:=\pi^0_{(0)}-g,\,\bar{\Pi}:=\bar{\pi}^0_{(0)}+f$. (This measure invariance is analogous to Eq. (4.7.15).) In terms of $\Pi,\,\bar{\Pi}$ we have

$$\mathcal{H}_t=\frac{\mathrm{i}}{K}\left\{\Pi\bar{\Pi}+2gf\right\},\,-\mathrm{i}\int \mathrm{d}^{n-1}\mathbf{x}\mathcal{H}_t=\frac{V}{K}\Pi\bar{\Pi}+2\left(\frac{V}{K}gf\right)_{(0)}, \qquad \text{(H.62)}$$

and the path integral is $I=\int \mathrm{d}\Pi\left(t\right)\mathrm{d}\bar{\Pi}\left(t\right)\mathrm{d}c^0_{(0)}\left(t\right)\mathrm{d}\bar{c}^0_{(0)}\left(t\right)\exp\left\{\frac{V}{K}\Pi\bar{\Pi}+2\left(\frac{V}{K}gf\right)_{(0)}\right\}$. We may discretise time with step period Δ, then take a continuous-time limit that identifies Δ^{-1} with $\delta\left(0\right)$:

$$\begin{aligned} I &\propto \prod_{k=1}^{N}\int \mathrm{d}\Pi\left(t_k\right)\mathrm{d}\bar{\Pi}\left(t_k\right)\frac{V}{K\left(t_k\right)}\Pi\left(t_k\right)\bar{\Pi}\left(t_k\right) \\ &= \prod_{k=1}^{N}\frac{-V}{K\left(t_k\right)}=\exp\sum_{k=1}^{N}\ln\frac{-V}{K\left(t_k\right)}\to\exp-\delta\left(0\right)\int \mathrm{d}t\ln K\left(t\right), \qquad \text{(H.63)} \\ L_{\mathrm{LY}t} &= \mathrm{i}\delta\left(0\right)\ln K\left(t\right). \qquad \text{(H.64)} \end{aligned}$$

The space part is similar. In terms of $K_{ij} := \left(\mathbb{I} - Va_s\kappa h_{(0)}(t)\right)^{-1}_{ij}$ the logarithmic term we wish to cancel is $-\mathrm{i}\delta(0)\,\mathrm{tr}\ln\left(\delta_{ij} - Va_s\kappa h_{ij(0)}(t)\right) = -\mathrm{i}\delta(0)\,\mathrm{tr}\ln K^{-1}$. Let space be $(n-1)$–dimensional. We have a result of the form

$$\mathcal{L}_s = -\mathrm{i}K_{ij}(t)\,\dot{\bar{c}}^i_{(0)}\dot{c}^j_{(0)} + \dot{\bar{c}}^i_{(0)} f_i\left(h_{(+)},\, c_{(+)}\right) + g_j\left(h_{(+)},\, \bar{c}_{(+)}\right)\dot{c}^j_{(0)}, \tag{H.65}$$

$$\mathcal{H}_s = \mathrm{i}\left(K^{-1}\right)^{ij}\left\{\left(\pi_{j(0)} - g_j\right)\left(\bar{\pi}_{i(0)} + f_i\right) + 2g_j f_i\right\}, \tag{H.66}$$

$$\Pi_j := \pi_{j(0)} - g_j, \tag{H.67}$$

$$\overline{\Pi}_i := \bar{\pi}_{i(0)} + f_i, \tag{H.68}$$

$$\begin{aligned} I &\propto \prod_{k=1}^{N}\prod_{h=1}^{n-1}\int \mathrm{d}\Pi_{k_h}(t_k)\,\mathrm{d}\overline{\Pi}_{l_h}(t_k)\exp\left(V\left(K^{-1}\right)^{ij}\Pi_j(t_k)\,\overline{\Pi}_i(t_k)\right) \\ &\propto \prod_{k=1}^{N}\prod_{h=1}^{n-1}\int \mathrm{d}\Pi_{k_h}(t_k)\,\mathrm{d}\overline{\Pi}_{l_h}(t_k)\left(K^{-1}\right)^{i_h j_h}\Pi_{j_h}(t_k)\,\overline{\Pi}_{i_h}(t_k), \end{aligned} \tag{H.69}$$

a product which gives factors of

$$I \propto \prod_{k=1}^{N}\det K^{-1}_{ij}(t) = \exp\sum_{K=1}^{N}\ln\det K^{-1}_{ij}(t) \to \exp\delta(0)\int \mathrm{d}t\,\mathrm{tr}\ln K^{-1}_{ij}(t), \tag{H.70}$$

$$L_{\mathrm{LY}s} = \mathrm{i}\delta(0)\,\mathrm{tr}\ln K_{ij}(t). \tag{H.71}$$

Eq. (H.71) is very similar to Eq. (H.64), and these results exactly cancel the $\delta(0)$ terms obtained above in Eq. (H.58).

Definitions

The **coordinate volume** is defined as $V_c := \int \mathrm{d}^{n-1}\mathbf{x}\sqrt{\eta}$, in the notation introduced in Sec. 1.2.1.

The **cyclic modes prescription** (CMP) is the prescription defended in this thesis for addressing the infrared problems I describe in Part I. The prescription sets conserved charges to zero (which, in some cases, begets further conserved charges that are then also set to zero). The result is a formalism which does not contain the field modes responsible for said infrared problems.

Dot and cross products of muliplet fields are defined in Eq. (2.1.19), and are analogous to their famous counterparts on $\mathbb{C}^3$.

The **fictitious mass prescription** (FMP) has the same motivation as the CMP. In the FMP, infrared-divergent terms are deleted from massive propagators, before a massless limit is taken to obtain an effective massless propagator. Several shortcomings with the FMP motivated the CMP's development.

The **flat static torus** is a finite-volume analogue of Minkowski space. In particular, the flat static torus admits Cartesian coordinates for which $\partial_\rho g_{\mu\nu} = 0$ and the spatial volume is finite.

Free integration by parts is a treatment of two putative interaction Lagrangians as equivalent if their difference is a surface term.

Reduced momentum densities are obtained by dividing conjugate momentum densities by $\sqrt{|g|}$, as explained in Sec. (1.3).

The **triple product on multiplets** is defined in Eq. (2.1.23). It is analogous to the scalar triple product on $\mathbb{C}^3$.

The **volume factor** is defined in Eq. (1.2.9).

For the purposes of this thesis, **zero modes** are defined as in Sec. 3.2.

Glossary

Annihilation and creation operators are, respectively, non-normal operators[71] that annihilate vacuum kets, and their adjoints (which therefore annihilate vacuum bras). If p creation operators act on a vacuum ket, the result is (up to a multiplicative constant) a p-particle state. Annihilation and creation operators therefore owe their names to their roles in shifting the particle numbers of physical states

Quantising a classical canonical field satisfying a linear equation of motion obtains a linear combination, with function-valued coefficients, of annihilation and creation operators (viz. Secs. 1.5 and 2.4). The (anti)commutators of annihilation and creation operators are equivalent, with an appropriate normalisation of field modes, to the canonical (anti)commutation relations on quantised canonical fields and their conjugate momentum densities.

The **Batalin–Vilkovisky formalism** is discussed in Appendix F. This formalism is an extension of the BRST formalism for physical theories whose Lie algebras do not explain all of the Hamiltonian formulation's phase space constraints.

The **BRST and anti-BRST transformations** are defined for the BRST-quantised theories described in Chapter 2. These transformations are action-preserving, except for one complication discussed in Sec. 2.6.3. They are nilpotent and commute (their associated fermionic charges anticommute).

Canonical (anti)commutation relations are (anti)commutators, equal to a multiple of the identity operator, between quantised conjugate momentum densities and quantised canonical fields.

A **Cauchy surface** of a spacetime is a hypersurface that intersects every inextendible past-directed causal curve exactly once. A spacetime which has Cauchy surfaces is **globally hyperbolic.** It is known that, if Σ is topologically equivalent to some Cauchy surface of a spacetime, said spacetime is topologically equivalent to $\Sigma \times \mathbb{R}$. Each event in the spacetime belongs to exactly one Cauchy surface. A coordinate system exists for which two events belong to the same Cauchy surface if and only if they are simultaneous. The $\mathbb{R}$-topology is due to that coordinate system's time coordinate. Globally hyperbolic spacetimes may then be equivalently defined as those which may be obtained from a Cauchy surface by time translation. Note the physical interpretation: the Cauchy surfaces are simply timeslices, and spatial sections, of the spacetime.

A topological space is **compact** if each of its open covers has a finite subcover. Physical space is described as **closed** in spacetimes whose spatial sections are compact topological spaces. The only property of such spacetimes of interest in this thesis is the fact that they have finite volume factor.

de Sitter invariance is invariance under de Sitter transformations. These are spacetime transformations in de Sitter space that are analogous to the Poincaré transformations on Minkowski space.

[71] An operator A is called *normal* if and only if $AA^\dagger = A^\dagger A$. Important examples include Hermitian and anti-Hermitian operators. It is a famous result that A is normal if and only if A is diagonalisable.

de Sitter space may be defined in any of several equivalent ways. For my purposes in this thesis, the definition of importance is that given in Eq. (1.2.11).

The **Dirac bracket** is a modified Poisson bracket required in a Hamiltonian formulation that admits phase space constraints. In particular, several important results on Poisson brackets in the absence of such constraints are applicable generally to Dirac brackets.

The **electroweak interaction** is a unified description of the empirical electromagnetic and weak interactions. The Higgs mechanism explains the fact that, unlike the photon, the weak interaction's gauge bosons are massive.

Faddeev–Popov ghosts and antighosts are integer-spin, and hence unphysical, fermionic fields introduced by the Faddeev–Popov method. Yang–Mills theory (perturbative gravity) introduces FP fields of spin 0 (1).

The **Faddeev–Popov method** is a modification of physical theories discussed in Chapter 2; it introduces a **BRST quantisation** of the Lagrangian. The resulting Lagrangian density loses its gauge invariance, and this fact facilitates the calculation of gauge-invariant quantities' means in the path integral formalism. A good account is found in Ref. [27].

A **Fock space** is most narrowly defined as the Hilbert space completion of the direct sum of the symmetric or antisymmetric tensors in the tensor powers of a single-particle Hilbert space. An extension to multiple-particle theories is trivial.

The **Friedmann–Lemaître–Robertson–Walker (FLRW) metric** is an approximation, in cosmology, of $\mathrm{d}s^2$ for the large-scale structure of the universe. It is of the form $\mathrm{d}s^2 = \mathrm{d}t^2 - a^2\left(t\right)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega^2\right)$.

Given a definite integral of the form $S = \int \mathrm{d}^p\mathbf{u}\mathcal{S}$, the **functional derivatives** $\frac{\delta S}{\delta\vartheta}$ of S are defined by $\delta S = \int \mathrm{d}^p\mathbf{u}\left\{\delta\vartheta\frac{\delta S}{\delta\vartheta} + o\left(\delta\vartheta\right)\right\}$, where δX is the change in an arbitrary field X due to the transformation $\vartheta \to \vartheta + \delta\vartheta$. Note that the dimensions of $\frac{\delta S}{\delta\vartheta}$ are those of $\frac{S}{u^p\vartheta}$.

A **gauge transformation** in fields is action-preserving. BRST quantisation breaks this symmetry of an action. However, BRST-quantised actions still exhibit some symmetries, such as BRST invariance.

The **Gudermannian function** [24] is defined in Sec. 1.8, and is required to calculate spatially uniform massless Klein–Gordon fields in de Sitter space. It is named for Christoph Gudermann.

Primary constraints in perturbative gravity's phase space are expressible in terms of the **Hamiltonian and momentum constraints** [94]. The constraints are defined by Eqs. (C.14) and (C.15).

An **infrared divergence** is a divergence in an $M \to 0^+$ right-hand limit, where the parameter M is a rest mass.

A **Killing vector field**, or more succinctly **Killing vector**, solves $\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = 0$, the **Killing equation**.

The **Klein–Gordon inner product** on solutions of the Klein–Gordon equation is defined in Eq. (1.4.3). Despite its name, it is not a true inner product, although it is a pseudo-inner product. **Klein–Gordon normalisation** is a normalisation convention definable in terms of this pseudo-inner product.

The **Landau gauge** is a gauge choice in the BRST quantisation. This gauge choice causes a divergent term, unless the formalism is rewritten with the Nakanishi–Lautrup auxiliary field. A term proportional to the square of this field appears in all other gauges.

The **lapse function** and **shift vector** are $N,\ N^i$, viz. Eq. (1.2.1).

Lee–Yang cancellation, **Lee–Yang determinants** and **Lee–Yang terms** are all defined in Sec. 4.7.2.

Left- and right-differentiation are conventions regarding the definition of derivatives with respect to a fermionic variable. In left-differentiation, any product containing the fermionic variable are written with the fermionic variable as the leftmost factor. Right-differentiation instead places the factor at the right.

The **Lie algebra** (defined in Sec. 2.1.3) is mostly unrelated to the **Lie derivative** (defined in Sec. 2.6.2). However, since Killing vectors are closed under the Lie derivative, they form an associated Lie algebra.

Linearised gravity is defined in Sec. 2.6.1.

The **master equation** and **quantum master equation** are respectively given in Eqs. (F.3) and (F.22).

The **naïve Hamiltonian** is a Legendre transform of the Lagrangian that admits phase space constraints. The use of Dirac brackets addresses the implications of these constraints for Poisson brackets. However, additional terms must also be added to the naïve Hamiltonian. This ensures that the Hamilton's equations are equivalent to the Euler–Lagrange equations.

The **Nakanishi–Lautrup auxiliary field** is a bosonic field, of the same spin as the FP-ghost, which may be used to rewrite a Faddeev–Popov Lagrangian density. The motivation is to ensure that the Landau gauge does not result in a divergent term.

The **Ostrogradski method** modifies the Legendre transform for theories in which the Lagrangian density contains second- and/or higher-order derivatives. It is discussed in Sec. 4.6.

Propagators are defined in Sec. 1.5.3.

A **pseudo-inner product** is a sesquilinear function $\langle\varphi,\,\psi\rangle$ on a Hilbert space $\mathscr{H}\ni\{\varphi,\,\psi\}$ satisfying $\langle\varphi,\,\psi\rangle=\langle\psi,\,\varphi\rangle^*$. A "true" inner product also has $\langle\varphi,\,\varphi\rangle\geq 0$, with equality if and only if $\varphi=0$. A **pseudo-norm** (often abbreviated to **norm**) may be defined as either $\langle\varphi,\,\varphi\rangle$ or $\sqrt{\langle\varphi,\,\varphi\rangle}$, depending on convenience. For example, if $\langle\varphi,\,\varphi\rangle<0$ it is customary to describe φ as **negative-norm**, using the first definition of pseudo-norms.

The Hamiltonian density and Lagrangian density may respectively be written as $\mathcal{H}=\sqrt{|g|}\mathcal{H}_0$ and $\mathcal{L}=\sqrt{|g|}\mathcal{L}_0$, where $\mathcal{L}_0$ is a scalar called the **scalar Lagrangian density**.

The **scale factor** $a\,(t)$ in an FLRW metric admits a generalisation for my purposes, as discussed in Sec. 1.2.1.

Structure constants are defined in Eq. (2.1.16).

The **synchronous gauge** is $N=1,\,N^i=0$.

The **vielbein formalism** is a formalism in general relativity. It provides one of two popular notations for BRST-quantised perturbative gravity. I decided not to use the vielbein formalism in my treatment of perturbative gravity (which is predominantly confined to Chapter 4). I defend this decision in Sec. 2.6.3.

Yang–Mills theory is an in general non-Abelian generalisation of electromagnetism, viz. Sec. 2.1.3. Yang–Mills theory is used in the Standard model to describe the electroweak unification of electromagnetism and the weak nuclear interaction, as well as the fundamental "colour" interaction. The strong nuclear interaction includes both this colour interaction and an emergent consequence thereof, the residual strong force. These interactions are respectively exchanged by virtual gluons and virtual mesons.

Bibliography

[1] J. Gibbons and A. Higuchi, Phys. Rev. D **91**, 024006 (2015); arXiv:hep-th/1410.7830 (2014).

[2] M. Faizal and A. Higuchi, Phys. Rev. D **78**, 067502 (2008).

[3] C. Itzykson and J. Zuber, *Quantum Field Theory*, Dover Publications, Courier Corporation (2006).

[4] R. Wald, *General Relativity*, University of Chicago Press, Chicago, IL (1984).

[5] R. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, Chicago Lectures in Physics, University of Chicago Press, Chicago, IL (1994).

[6] L. Parker and D. Toms, *Quantum field theory in curved spacetime: quantized fields and gravity*, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press (2009).

[7] L. Susskind and J. Lindesay, *An Introduction to Black Holes, Information and the String Theory Revolution: The Holographic Universe*, World Scientific, Singapore **10.5** 10 (2005).

[8] A. G. Riess, A. V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, P. M. Garnavich, R. L. Gilliland, C. J. Hogan, S. Jha, R. P. Kirshner, B. Leibundgut, M. M. Phillips, D. Reiss, B. P. Schmidt, R. A. Schommer, R. C. Smith, J. Spyromilio, C. Stubbs, N. B. Suntzeff and J. Tonry, Astron. J. **116**, 1009 (1998).

[9] S. Perlmutter et al., Astrophys. J. **518**, 565 (1999).

[10] D. Kazanas, Astrophys. J. **241**, L59 (1980).

[11] K. Sato, Mon. Not. R. Astron. Soc. **195**, 467 (1981).

[12] A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[13] A. D. Linde, Phys. Lett. **108B**, 389 (1982).

[14] A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[15] D. N. Spergel et al., The Astrophysical Journal Supplement Series **170.2** 377 (2007).

[16] M. Tegmark et al., Phys. Rev. D **74**, 123507, arXiv:astro-ph/0608632 (2006).

[17] P. A. R. Ade et al. (BICEP2 Collaboration), Phys. Rev. Lett. **112**, 241101 (2014).

[18] Planck Collaboration Team, arXiv:astro-ph/1409.5738 (2014).

[19] A. Strominger, J. High Energy Phys. **10** 034 (2001).

[20] B. Allen, Phys. Rev. D **32**, 3136 (1985).

[21] R. P. Feynman, *Feynman Lectures on Gravitation*, Reading: Addison-Wesley Pubi. Company (1995).

[22] C. Itzykson and J-B. Zuber, *Quantum Field Theory*, McGraw-Hill (1980).

[23] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York, Dover (1965).

[24] F. W. J Oliver, D. W. Lozier, R. F. Boisvert et al., eds., *NIST Handbook of Mathematical Functions*, Cambridge University Press, Section 4.23(viii) (2010).

[25] A. Higuchi, J. Math. Phys. (N.Y.) (N.Y.) **28**, 1553 (1987); **43**, 6385(E) (2002).

[26] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed., Elsevier/Academic Press, Amsterdam (2007).

[27] M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley Publishing Company, Westview Press Inc (1995).

[28] L. D. Faddeev and V. S. Popov, *Phys. Lett.* **25B**, 29 (1967).

[29] C. N. Yang and R. L. Mills, Phys. Rev. **96** 191 (1954).

[30] A. Zee, *Quantum field theory in a nutshell*, Princeton University Press, Princeton (2010).

[31] I. Ojima, Nucl. Phys. **B143**, 340 (1978).

[32] S. L. Glashow, *Nucl. Phys.* **22** 579 (1961).

[33] A. Salam, in *Elementary Particle Theory*, ed. N. Svartholm (Almquist and Wiksells, Stockholm), 367 (1969).

[34] S. Weinberg, *Phys. Rev. Lett.* **19** 1264 (1967).

[35] D. Griffiths (1987), *Introduction to Elementary Particles*, John Wiley & Sons.

[36] A. P. Stone, Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 57 Issue 03, 460, DOI: 10.1017/S0305004100035507 (1961).

[37] N. Nakanishi, Prog. Theor. Phys. **35**, 1111 (1966).

[38] B. Lautrup, Mat.Fys. Medd. - K. Dan. Vidensk. Selsk. **35**, 1 (1967).

[39] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. **66**, 1 (1979).

[40] I. Ojima, Prog. Theor. Phys. **64**, 625 (1980).

[41] L. Landau and E. Lifshitz, *The classical theory of fields*, Amsterdam: Elsevier Butterworth Heinemann (2007).

[42] J. D. Jackson and L. B. Okun, Rev. Mod. Phys. **73**, 663 (2001), arXiv:hep-ph/0012061 (2000).

[43] C. Becchi, A. Rouet and R. Stora, Ann. Phys. (N.Y.) **98**, 287 (1976).

[44] M. Z. Iofa and I. V. Tyutin, Theor. Math. Phys. **27**, 316 (1976).

[45] T. Kugo and I. Ojima, Phys. Lett. **73B**, 459 (1978).

[46] G. Curci and R. Ferrari, Phys. Lett. **63.1B**, 91 (1976).

[47] L. Baulieu and J. Thierry-Mieg, Nucl. Phys. **B197**, 477 (1982).

[48] P. Kuusk, Phys. Lett. B **122**, 202400 (1983).

[49] L. H. Ford and L. Parker, Phys. Rev. D **16**, 245 (1977).

[50] B. Allen and T. Jacobson, Commun. Math. Phys. **103**, 669 (1986).

[51] S. M. Carroll, *Lecture Notes on General Relativity*, Eq. (4.73), URL http://ned.ipac.caltech.edu/level5/March01/Carroll3/Carroll4.html (1997).

[52] K. Yano, *The Theory of Lie Derivatives and its Applications*, Nabu Press, North-Holland (1957).

[53] Various authors, edited by D. Pützfeld, C. Lämmerzahl and B. Schutz, *Equations of Motion in Relativistic Gravity*, *Fundamental Theories of Physics* series, Springer (2015).

[54] N. Nakanishi, Prog. Theor. Phys. **62** 779 (1979).

[55] R. Delbourgo and M. R. Medrano. Nucl Phys B **110** 467 (1976).

[56] K. Stelle, Phys Rev D **16** 953 (1977).

[57] R. Delbourgo and G. Thompson, J Phys. **G7** L133 (1981).

[58] M. Hirayama and N Hirano, Phys LelI **115B** 107 (1982).

[59] M. Faizal, Foundations of Physics February, Volume 41, Issue 2, 270 (2011).

[60] S. Upadhyay, Phys. Lett. **723B**, 470 (2013).

[61] B. Allen and M. Turyn, Nucl. Phys. B **292**, 813 (1987).

[62] L. H. Ford and L. Parker, Phys. Rev. D **16**, 1601 (1977).

[63] A. Higuchi, Nucl. Phys. **B282**, 397 (1987).

[64] B. Allen, Nucl. Phys. **B287**, 743 (1987).

[65] A. Higuchi and S. S. Kouris, Classical Quantum Gravity **17**, 3077 (2000).

[66] A. Higuchi, D. Marolf and I. A. Morrison, Classical Quantum Gravity **28**, 245012 (2011).

[67] S. W. Hawking, T. Hertog and N. Turok, Phys. Rev. D **62**, 063502 (2000).

[68] A. Higuchi and R. H. Weeks, Classical Quantum Gravity **20**, 3005 (2003).

[69] R. P. Bernar, L. C. B. Crispino and A. Higuchi, Phys. Rev. D **90**, 024045 (2014).

[70] B. Allen and M. Turyn, Nucl. Phys. **B292**, 813 (1987).

[71] A. Higuchi and S. S. Kouris, Classical Quantum Gravity **18**, 4317 (2001).

[72] I. A. Morrison, arXiv:gr-qc/1302.1860 (2013).

[73] M. Faizal and A. Higuchi, Phys. Rev. D **85**, 124021 (2012).

[74] A. Higuchi, arXiv:gr-qc/1204.1684 (2012).

[75] S. S. Kouris, Classical Quantum Gravity 18, 4961 (2001); **29**, 169501(E) (2012).

[76] P. J. Mora, N. C. Tsamis and R. P. Woodard, Phys. Rev. D **86**, 084016 (2012).

[77] M. B. Fröb, ArXiv:hep-th/1409.7964.

[78] N. C. Tsamis and R. P. Woodard, Class. Quant. Grav. **26**, 105006 (2009).

[79] T. Tanaka and Y. Urakawa, Prog. Theor. Exp. Phys. 073E01 (2014).

[80] L. D. Landau and E.M. Lifshitz, *Mechanics* (3rd ed.), Butterworth Heinemann (1976).

[81] P. A. M. Dirac, Canadian Journal of Mathematics **2** 129, doi:10.4153/CJM-1950-012-1 (1950).

[82] N. C. Tsamis and R. P. Woodard, J. Math. Phys. (N.Y.) **48**, 052306 (2007).

[83] A. Youssef, Phys. Rev. Lett. **107**, 021101 (2011).

[84] M. B. Fröb and A. Higuchi, J. Math. Phys. **55**, 062301 (2014).

[85] E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies, (*Cambridge Mathematical Library), Cambridge University Press (1904).

[86] F. J. de Urries and J. Julve, J. Phys. A: Math. Gen. **31** 6949, doi:10.1088/03054470/31/33/006 (1998).

[87] T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962).

[88] E. S. Abers and B. W. Lee, *Physics Reports* **9** 60 (1973).

[89] G. S. Birman and K. Nomizu, Michigan Math. J. **31**, no. 1, 77, doi:10.1307/mmj/1029002964 (1984).

[90] G. Mandal, A. M. Sengupta and S. R. Wadia, Mod. Phys. Lett. A **06**, 1685, DOI: 10.1142/S0217732391001822 (1991).

[91] R. da Rocha and W. A. Rodrigues, Jr., arXiv:hep-th/0512168 (2005).

[92] S. Carlip, arXiv:gr-qc/1009.1136 (2010).

[93] Various authors, edited by J. Murugan, A. Weltman and G. F. R. Ellis, *Foundations of Space and Time: Reflections on Quantum Gravity*, Cambridge University Press, Cambridge (2012).

[94] E. King, *Quantisation of a dynamical system with constraints forming an* $\mathrm{sl}(2,\mathbb{R})$ *Algebra*, M.Phil Thesis, University of York (2005).

[95] S. Weinberg, *The Quantum Theory of Fields, Volume 2*: *Modern Applications*, Nature **385.6614** 310 (1997), Cambridge University Press (2005).

[96] J. Zinn-Justin, in *Trends in Elementary Particle Theory – International Summer Institute on Theoretical Physics in Bonn 1974*, 2, Springer-Verlag, Berlin (1975).