PRIMITIVE NEAR-RINGS

BY

WILLIAM MICHAEL LLOYD HOLCOMBE

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CONTENTS

Introduction.

The theory of near-rings has arisen in a variety of ways. There is a natural desire to generalise the theory of rings and skew fields by relaxing some of their defining axioms. It has also been the hope of some mathematicians that certain problems in group theory, particularly involving permutation groups and group representations, may perhaps be clarified by developing a coherent algebraic theory of near-rings. Moreover, there is an increasing recognition by mathematicians in many branches of the subject, both pure and applied, of the ubiquity of near-ring like objects.

The first steps in the subject were taken by Dickson and Zassenhaus with their studies of 'near-fields', and by Wielandt with his classification of an important class of abstract near-rings. Papers by Frohlich, Blackett, Betsch and Laxton developed the theory considerably. Lately authors such as Beidleman, Ramakotaiah, Tharmanatram, Maxson, Malone and Clay have all added to our knowledge.

The history of the subject has been strongly influenced by our knowledge of ring theory, and although this has often been beneficial it must not be overlooked that a number of important problems in near-ring theory have no real parallel in the theory of rings. It is probably best to try to preserve a balance, and not to endeavour exclusively, either to generalise theorems from ring theory irrespective of their usefulness, or to ignore the theory of rings and attempt to formulate a completely independent theory. In many cases our results are generalisations of theorems from ring theory but at certain important junctures we will explicitly use the fact that we are dealing with a near-ring which is not a ring. This is a very interesting development in the subject.

We proceed, in the first chapter, with a review of the terms and notation that will be used in this thesis.

Where definitions and concepts are of a specialized or technical nature and only used in one section, it seems more sensible to postpone introducing them until a more natural point in the proceedings.

Chapter 2 gives a summary of the results on the various radicals corresponding to the Jacobson radical for associative rings. Most of these results are well known and readily available in the literature. We also consider near-rings with one, or more, of these radicals zero.

We defined, in Chapter 1, three different types of primitive near-ring, which are all genuine generalisations of the ring theoretic concept. Of these three, the two most important are 2-primitive and 0-primitive near-rings. In Chapter 3, we examine 2-primitive near-rings with certain natural conditions imposed on them. A theorem is obtained which could be considered to be the equivalent result for near-rings of the theorem classifying simple, artinian rings, due originally to Wedderburn and redeveloped by Jacobson.

Chapters 4 and 5 deal with 0-primitive near-rings satisfying certain conditions. Chapter 5 is a generalisation of Chapter 4, but we felt that the mathematical techniques involved would be clearer if the special case in Chapter 4 was expounded first. In these two chapters we classify a sizeable class of 0-primitive near-rings with identity and descending chain condition on right ideals.

Several types of prime near-rings have been developed in the literature. In Chapter 6 we examine these and related concepts.

In the theory of rings, Goldies' classification of prime and semi-prime ring with ascending chain conditions, has been of immense importance. Whether such a result could be obtained in the theory of near-rings is a matter for conjecture, at the moment. We have made a start on the problem with the construction of a class of near-rings which behave in a very similar way to Prime rings with the Goldie chain conditions. This is the content of Chapter 7. The inspiration for it, came mainly from the proof of Goldies' first theorem, due to C. Procesi, which is featured in Jacobson's book. (Jacobson [1]).

Chapter 8, is an attempt to initiate the development of a theory of vector groups and near-algebras which would play an important rôle in the future theory of near-rings, in a way, perhaps, similar to the rôle vector spaces and algebras play in ring theory. This may lead, in time, to results on 2-primitive near-rings with identity and a minimal right ideal, for example, or a Galois theory for certain 2-primitive near-rings. For the former problem, the experience of the semi-group theorists (Hoehnke [1] etc.) may prove useful.

Finally a note on the numbering of results and definitions etc.

If a reference is made, containing only two numbers, e.g. 1.12 then this means, "item 12 of section 1 of the present chapter". If a reference reads: 3.1.12, then this means "item 12 of section 1 of Chapter 3.

CHAPTER I

BASIC CONCEPTS OF NEAR-RINGS

Preliminary remarks

This chapter will include all the basic definitions and notation which will be required throughout the thesis.

We begin by defining what we mean by a 'near-ring'.

§1. Definitions of a near-ring. Examples.

1.1. A near-ring is an algebraic system consisting of a set, N, and two binary operations, addition (written +) and multiplication (written .), such that the following requirements are satisfied:

(a) The set N is a group under addition, (often written as N^+).

(b) The set N is a semigroup under multiplication.

(c) If $n_1, n_2, n_3 \in N$ then $n_1.(n_2 + n_3) = n_1.n_2 + n_1.n_3$

(d) If 0 is the additive identity of N, then

$$0.n = n.0 = 0 \quad \text{for all } n \in N.$$

We remark that the last condition (d) is not always insisted upon by some authors, but in the majority of work here it is required, and it seems sensible to insert it at the beginning to avoid undue confusion.

1.2. If G is an additive group, consider the set N of all mappings of G into itself which take the zero of G onto itself. We define addition on N by using the addition on G. Thus if $n, n_1 \in N$ we define a mapping $(n + n_1)\colon G \to G$ by $(g)(n + n_1) = (g)n + (g)n_1$ for all $g \in G$.

Multiplication on N is defined as the composition of mappings. This makes N into a near-ring and it is a fundamental one in the theory.

1.3. A division near-ring (or a near-field) is a near-ring N with the extra property that the set $N \setminus \{o\}$ (i.e. the non-zero elements) forms a group under multiplication.

1.4. If the additive group structure of a near-ring N is abelian, then we call N an abelian near-ring.

1.5. A commutative near-ring is a near-ring N which is commutative as a multiplicative simigroup.

1.6. A subnear-ring S of a near-ring N will simply mean a subset S of N, which, under the two binary operations induced on it by N, is a near-ring in its own right.

§2. The right modules with respect to a near-ring, homomorphisms and ideals.

2.1. If N is a near-ring and M any additively written group, then M will be said to be a right N-module if there exists a binary operation ξ: $M \times N \to M$, for short we will write

$$(m, n)\xi = m.n, \text{ for any } m \in M, n \in N,$$

satisfy the following properties.

(a) $m.(n_1 + n_2) = m.n_1 + m.n_2$

(b) $m.(n_1 . n_2) = (m.n_1).n_2$

for all $m \in M$, n_1, $n_2 \in N$.

(A right N-module will often be referred to simply as an N-module if no confusion arises).

2.2. An N-module M is unitary if the near-ring N has a multiplicative identity 1_N, such that

$$m \cdot 1_N = m \quad \text{for all } m \in M.$$

2.3. We remark, that some authors refer to these N-modules as 'N-groups' and reserve the title 'module' for a more specialised object. It should be noted that, under the operation of multiplication, the additive group of a near-ring N may be considered to be an N-module.

2.4. Let N and N_1 be near-rings. Then a mapping $f : N \to N_1$ is called a near-ring homomorphism if for all $n, n^1 \in N$,

(a) $(n + n^1)f = nf + n^1f$

(b) $(n.n^1)f = (nf).(n^1f)$.

2.5. If M and M_1 are N-modules for some near-ring N, then a mapping $\phi : M \to M_1$ is an N-homomorphism if for all $m, m^1 \in M$ and $n \in N$

(a) $(m + m^1)\phi = m\phi + m^1\phi$

(b) $(m.n)\phi = (m\phi).n$

2.6. In cases 2.4 and 2.5 we will use the term endomorphism if the domain and co-domain of the mapping are the same.

A monomorphism is a 1-1 homomorphism, an epimorphism is a homomorphism which is onto and an isomorphism is both an epimorphism and a monomorphism.

2.7. We introduce ideal-type concepts by studying the kernels of near-ring homomorphisms. Then a subset I of a near-ring N is an <u>ideal</u> if

(a) I is an additive subgroup of N which is normal in N.

(b) $N I \subset I$, where $NI = \{n.i \mid n \in N, i \in I\}$.

(c) $(n_1 + i)n_2 - n_1 n_2 \in I$ for all $i \in I$, $n_1, n_2 \in N$.

This is exactly a kernel of some near-ring homomorphism.

2.8. A <u>right ideal</u> R of N, is a set R such that,

(a) R is an additive normal subgroup of N.

(b) $(n_1 + r)n_2 - n_1 n_2 \in R$ for all $r \in R$, $n_1, n_2 \in N$.

2.9. A <u>left ideal</u> L of N is a set L such that

(a) L is an additive normal subgroup of N.

(b) $N L \subset L$.

2.10. The concept of a right ideal of a near-ring is rather more specialized than is desirable and we introduce a new object, which in our case, is more general. This gives us one of our important divergencies from ring theory, since in a ring both these concepts coincide.

We define, for a near-ring N, a <u>right N-subgroup of N^+</u>, or for short a<u>rt.N+-subgp</u>. to be a set K with

(a) K an additive subgroup of N^+ (<u>not</u> necessarily normal)

(b) $K N \subset K$.

Thus, for example, given any $n \in N$, we can look at the set $K = nN = \{nn_1;\ n_1 \in N\}$ and this set is easily seen to be a $rt.N^+$-subgp. using the elementary fact that for any

$n, n_1 \varepsilon N$

$-(n\, n_1) = n(-n_1)$ (This comes from the observation that

$0 = n.(-n_1) + n.(n_1)$ by 1.1 (c)

giving $n.(-n_1) = -[n(n_1)]$)

This set K may not be a right ideal. Some authors call these objects 'right modules'.

2.11. We return to our right N-modules and introduce subsets of them that will be required. If M is a rt.N-module, then a subset K is a right N-submodule of M if

(a) K is an additive normal subgroup of M

(b) $(m + k).n - m.n \varepsilon K$ for all $m \varepsilon M$, $k \varepsilon K$, $n \varepsilon N$.

Thus K is the kernel of a suitable N-homomorphism of M.

2.12. A subset P of a rt.N-module M is a rt.<u>N-subgroup of M</u> if

(a) P is an additive subgroup of M

(b) $PN \subset P$.

Thus a rt.N-submodule is a rt.N-subgroup.

2.13. We note that a right ideal of a near-ring N is simply a right N-submodule of N where N is considered as a rt. N-module. Also a rt.N^+-subgroup is a rt.N-subgroup of the rt.N-module N. So a right ideal is a right N^+-subgroup and also a right N-subgroup of N.

2.14. Naturally we may factor out ideals (rt. ideals and N-submodules) in the usual way, (these are like groups with operators) and we define direct sums (internal) with these objects only.)

§3. Special types of near-rings and modules

3.1. We will need, later, equivalent concepts to a minimal ring module and we clearly have two possibilities in the near-ring case. An N-module M (which is non-zero) is minimal if the only rt. N-subgroups of M are (0) and M itself

3.2 An N-module M ($\neq (0)$) is irreducible if the only rt. N-submodules of M are (0) and M itself.

We see that a minimal N-module is always irreducible but the converse is not true.

3.3. A near-ring N ($\neq (0)$) is simple if the only ideals of N are (0) and N itself.

3.4. We introduce some notation which will be invoked in many places. Let Γ be a non-zero additive group, E a multiplicative semigroup of endomorphisms of Γ^+, and define

$\mathfrak{M}_E(\Gamma)$ to be the set of all mappings $n : \Gamma \to \Gamma$ with the properties:

$$0_\Gamma . n = 0_\Gamma$$

$$(\gamma e) . n = (\gamma n) e \quad \text{for all } \gamma \in \Gamma, \ e \in E.$$

It is easily checked that $\mathfrak{M}_E(\Gamma)$ is a near-ring with a multiplicative identity namely the identity mapping.

In the special case of E consisting only of the identity endomorphism we shall often just write

$$\mathfrak{M}_{\{1\}}(\Gamma) = \mathfrak{M}(\Gamma).$$

3.5. An N-module M is of <u>type</u> 2 if

(a) $M.N \neq (0)$

(b) M is a minimal N-module.

3.6. An N-module M is of <u>type 1</u> if

(a) $M.N \neq (0)$

(b) M is an irreducible N-module

(c) $mN = (0)$ or $mN = M$ for all $m \in M$.

3.7. An N-module M is of <u>type 0</u> if

(a) M is an irreducible N-module (and $M \neq (0)$)

(b) There exists an $m \in M$, such that $M = m.N$.

Betsch introduced this notation (Betsch [2]) and it is becoming fairly standard.

3.8. A right ideal R of a near-ring is a <u>ν-modular right ideal</u> if (a) $\exists$ $e \in N$ such that $en-n \in R$ for all $n \in N$.

(b) $N^{+} \setminus R$ is a right N-module of type ν.

Here ν may take any of the values: 0,1,2.

3.9. Let M be a rt. N-module and suppose that S is an arbitrary subset of M. (non empty).

The set $(S)_r = \{n \in N \mid sn = 0; \ \forall s \in S\}$

is called the right annihilator of S in N.

It is easily checked that $(S)_r$ is a right ideal of N.

Also $(S)_r$ is an ideal of N if $S.N \subseteq S$.

3.10. A near ring N is a <u>ν-primitive near-ring</u>. (ν = 0,1.2) if there exists an N-module M of type ν such that $(M)_r = (0)$.

3.11. An ideal P of a near-ring N is a <u>ν-primitive ideal</u> ($\nu = 0,1,2$) if $N/_P$ is a ν-primitive near-ring, (where $N/_P$ is the near-ring obtained by factoring out the ideal P in the obvious way).

3.12. An element $a \varepsilon N$ is rt.quasi-regular if the smallest right ideal containing all elements of the form $n - an, \forall\, n \varepsilon N$, also contains a. We usually write quasi-regular to mean rt.quasi-regular, and abbreviate it to just 'q.r.'

3.13. A non-zero rt.N^+subgroup K of a near-ring N is <u>nilpotent</u> if there exists a positive integer q with the property that

$$k_1.k_2 \ldots k_q = 0 \text{ for any } k_1, k_2, \ldots, k_q \varepsilon K.$$

3.14. A non-zero rt.N^+-subgroup L of a near-ring N is <u>nil</u> if there exists, for each non-zero element $d \varepsilon L$, a positive integer s (depending possibly on d) such that the product of d with itself taken s times,

$$d^s = d \ldots d = 0$$

3.15. A non-zero rt.N^+-subgroup Q of a near-ring N is <u>quasi-regular</u> if every element is quasi-regular.

Chapter 2. Radicals and Semisimplicity

The radicals considered in this chapter will be generalisations of the Jacobson radical for rings. Because the concepts of irreducible and minimal N-modules do not coincide, several generalisations of the Jacobson radical exist and between them they more or less satisfy all the properties that the Jacobson radical of a ring possesses. Betsch and Laxton were the first to formulate these radicals and many of the following results are due to them, although in Laxton's case he considered only special near-rings, distributively generated near-rings.

§1. The Jacobson Radicals of a near-ring

1.1. N is a near-ring, write $\Sigma(\upsilon)$ for the collection of all N-modules of type υ. ($\upsilon = 0,1,2$).

1.2. We define three radicals by

$$J_\upsilon(N) = \bigcap_{\Gamma \in \Sigma(\upsilon)} (\Gamma)_r \quad , \quad \upsilon = 0,1,2$$

with the convention that if $\Sigma(\upsilon) = \phi$ then $J_\upsilon(N) = N$.

1.3. We see that $J_\upsilon(N)$ is an ideal of N. If N is in fact a ring then the $J_\upsilon(N)$ all coincide and equal the Jacobson radical of N.

1.4. We may factor $J_\upsilon(N)$ out of the near-ring N. This leads us to the question "what is $J_\upsilon(N/J_\upsilon(N))$?"

Betsch ([2], 2.13) has shown that

$$J_\upsilon(N/J_\upsilon(N)) = J_\upsilon(N)/J_\upsilon(N) = (0) \quad \text{for } \upsilon = 0,1,2.$$

1.5. If $P(\nu)$ is the collection of all ν-primitive ideals of N, then

$$J_\nu (N) = \bigcap_{P \varepsilon P(\nu)} P \quad ; \quad \nu = 0,1.2.$$

(Betsch [2], 2.12), (Laxton [2], 1.1).

1.6. If K is an ideal of N and $J_\nu (N/_K) = K/_K = \{0\}$,

then $J_\nu (N) \subseteq K$, for $\nu = 0,1,2.$

(Betsch [2], 2.14).

1.7. If N is a near-ring with a multiplicative identity then

$$J_1 (N) = J_2 (N) \qquad \text{(Betsch [2]. 2.8).}$$

1.8. In an arbitrary near-ring, the following inequalities hold;

$$J_0 (N) \subseteq J_1 (N) \subseteq J_2 (N).$$

Examples are known when equality does not occur. (Betsch [2]. §4).

§2. Basic Properties of the radicals. Another radical object.

Up to this point all the three radicals J_0, J_1 and J_2 have exhibited similar properties, and although whenever $1 \varepsilon N$ we have $J_1 = J_2$, there is a certain conformity in the results 1.4, 1.5, 1.6. We now try to determine the relationship between these radicals and the intersection of all the ν-modular right ideals. This gives trouble because the natural result depends heavily on the right ideals being maximal, or 'nearly maximal' as rt.N^+-subgroups. Anyway we have the following result.

2.1. $$J_\mu (N) = \bigcap_{R \varepsilon R(\mu)} R \quad , \text{ for } \quad \mu = 1.2$$

where $R(\mu)$ is the set of all μ-modular right ideals of N.

This result is due to Betsch [2]. 2.7., and for $\mu = 2$, also to Laxton [2]. 1.3.

2.2. It is not possible to prove an analogous result for $\mu = 0$ and we notice that the intersection of all the 0-modular right ideals is not necessarily an ideal. Even so, it is a very interesting set and we can use it as a radical-like object to obtain some useful results. We make a definition.

2.3. For any near-ring N, define

$$D(N) = \bigcap_{R \in \mathcal{R}(0)} R$$, where $\mathcal{R}(0)$ is the collection of 0-modular right ideals of N.

2.4. It has been shown by Betsch and Laxton that $J_0(N) \subseteq D(N) \subseteq J_1(N) \subseteq J_2(N)$, and here again examples are known where $J_0(N) \neq D(N)$ and $D(N) \neq J_1(N)$, e.g. (Betsch [2]. § 4).

2.5. $J_2(N)$ contains all the nilpotent N^+-subgroups, all the nil N^+-subgroups and all the quasi-regular N^+-subgroups. (These results appear in Ramakotoiah [1], Th. 2.1.; Cor. 2.3; Cor. 2.4).

2.6. D(N) contains all the nilpotent right ideals, all the nil right ideals and all the quasi-regular right ideals. (Ramakotaiah [1], Th. 2.2; Cor. 2.5; Cor. 2.6).

2.7. $J_0(N)$ contains all the quasi-regular ideals, all nil ideals and all nilpotent ideals.

(Ramakotaiah [1], Th. 2.3; Cor. 2.7; Cor. 2.8).

2.8. So far, we have seen that, the four radicals we have defined satisfy many of the properties that the Jacobson radical of a ring satisfied, but now we encounter one of the more difficult problems. This concerns the possibility of the nilpotency of any of the radicals under suitable chain conditions. We have introduced quasi-regular elements, but unlike ring theory, it has not, so far, been possible to express any of our radicals directly in terms of quasi-regular elements. It would have been nice to show that $J_2(N)$ consisted solely of quasi-regular elements, but then $J_2(N)$, being a quasi-regular ideal would be contained in $J_0(N)$, by 2.6, and this we know is not always the case.

However, we can show that $D(N)$ is quasi-regular, and hence so is $J_0(N)$.

2.9. $D(N)$ is a quasi-regular right ideal (Laxton [2], 3.2), (Ramakotaiah [1], Thm. 2.2.).

Finally we note the following results.

2.10. An ideal P is ν-primitive if and only if $P = (L:N) = \{n \mid Nn \subseteq L\}$, where L is a ν-modular right ideal of N. (Laxton [3], Prop. 2) (Ramakotaiah [1], Thm. 1.2.).

2.11. $J_0(N) = (D(N):N) = \{n \in N \mid Nn \subseteq D(N)\}$ (Laxton [3], 3.2). (Ramakotaiah. [1], Th. 2.2.).

From this last result we may deduce that $J_0(N)$ is the largest ideal contained in $D(N)$. (Ramakotaiah [1], Cor. 1.2.).

§3. Near-rings with descending chain conditions

We shall summarize the results concerning near-rings with descending chain conditions and with one of the radicals zero. In the majority of cases, these results are well known and widely available in the literature

3.1. There are two descending chain conditions of interest.

(i) The descending chain condition (d.c.c.) on right ideals

(ii) The descending chain condition (d.c.c.) on rt.N^+-subgroups

3.2. Theorem. Suppose that N has d.c.c. on rt. N^+-subgroups. Then

(i) $J_2(N) = (0) \Leftrightarrow$ N possesses no nilpotent, non-zero rt.N^+-subgroups.

(ii) $D(N) = (0) \Leftrightarrow$ N possesses no nilpotent, non-zero right ideals

(iii) $J_0(N) = (0) \Leftrightarrow$ N possesses no nilpotent, non-zero ideals.

Proof. In all the three cases $\Rightarrow$ follows from 2.5, 2.6, 2.7, respectively.

(i) $\Leftarrow$ (Betsch [1]. Th. 4.1.) if N possesses no non-zero nilpotent rt.N^+-subgroups, suppose $J_2(N) \neq (0)$.

Hence $J_2(N)$ contains an N^+-subgroup which is minimal, say M. Now $0 \neq M \subseteq J_2(N)$. If $M.N \neq (0)$, then M is of type 2 and so $M.J_2(N) = (0) \Rightarrow M.M = (0)$ If $M.N = (0)$, then $M.M = (0)$. Hence in either situation M is a non-zero nilpotent N^+-subgroup, a contradiction.

(ii) $\Leftarrow$ This follows if we can show that D(N) is a nilpotent rt. ideal.

(iii) $\Leftarrow$ If D(N) is a nilpotent right ideal then J (N) is a nilpotent ideal and the result is immediate.

We have thus reduced the problem to showing that D(N) is nilpotent under d.c.c. for rt.N^+-subgroups.

We already know that D(N) is quasi-regular, and we can apply the following theorem.

3.3. If N has d.c.c. on rt.N^+-subgroups, then any quasi-regular rt.N^+-subgroups is nilpotent. (Ramakotaiah [1], Th. 5.1.).

3.4. If N has d.c.c. on rt.N^+-subgroups then D(N) is nilpotent.

3.5. The natural questions to ask now are, whether we can decompose a near-ring into a direct sum of right ideals under suitable conditions on the radicals.

The first steps in this direction were made by Blackett, Laxton and Betsch have produced further results. We look at these now.

3.6. Theorem. (Betsch [2], 3.4) if N has d.c.c. on right ideals and J_2 (N) = (0), then N is a direct sum of right ideals which are N-modules of type 2.

Proof. Now $J_2 (N) = \bigcap R$, where the R's are taken over all the 2-modular right ideals. Because of d.c.c. on right ideals, we can find $R_1, \ldots, R_m$ amongst these R's, such that $J_2 (N) = \bigcap_{i=1}^{m} R_i$ and we cannot reduce the number of these R's further. Then $\bigcap_{i=1}^{m} R_i = (0)$.

Define $K_i = R_1 \cap R_2 \cap \dots \cap R_{i-1} \cap R_{i+1} \cap \dots \cap R_m$

for each $1 \leq i \leq m$ $(K_1 = R_2 \cap \dots \cap R_m,\ K_m = R_1 \cap \dots \cap R_{m-1})$

Then $K_i \neq (0)$ for $1 \leq i \leq m$.

We notice that $K_i \cap R_i = (0)$ for $1 \leq i \leq m$ and so

$N = K_i \oplus R_i$ for $1 \leq i \leq m$, since the R_i are 2-modular rt.ideals.

In particular $N = K_1 \oplus R_1$.

We show by induction that $N = K_1 \oplus K_2 \oplus \dots \oplus K_i \oplus$

$(R_1 \cap R_2 \cap \dots \cap R_i)$

for any $1 \leq i \leq m$. This is evident for $i = 1$.

Assume that it is true for $i = s$.

Now $$R_1 \cap \dots \cap R_s \Big/ R_1 \cap R_2 \cap \dots \cap R_s \cap R_{s+1}$$

$$\cong (R_1 \cap \dots \cap R_s + R_{s+1}) \Big/ R_{s+1}$$

from the isomorphism theorems.

Now $R_{s+1} \not\supseteq R_1 \cap \dots \cap R_s$, because of the irredundant nature of the R_i's. Hence

$$(R_1 \cap \dots \cap R_s) \Big/ (R_1 \cap R_2 \cap \dots \cap R_s \cap R_{s+1}$$

$$\cong N \Big/ R_{s+1}$$ as N-modules.

Now $(R_1 \cap R_2 \cap \ldots \cap R_s) \cap R_{s+1} \cap R_{s+2} \cap \ldots \cap R_m = (0)$,

thus we have the direct sum decomposition

$$R_1 \cap \ldots \cap R_s = (R_1 \cap \ldots \cap R_{s+1}) \oplus K_{s+1}$$

Hence, as $N = K_1 \oplus \ldots \oplus K_s \oplus (R_1 \cap R_2 \cap \ldots \cap R_s)$

$$N = K_1 \oplus \ldots \oplus K_s \oplus K_{s+1} \oplus (R_1 \cap R_2 \cap \ldots \cap R_s \cap R_{s+1})$$

This shows the induction process, so

$$N = K_1 \oplus K_2 \oplus \ldots \oplus K_m \oplus (R_1 \cap \ldots \cap R_m)$$

$$= K_1 \oplus K_2 \oplus \ldots \oplus K_m , \text{ as } R_1 \cap \ldots \cap R_m = (0).$$

Since $N = R_i \oplus K_i$ $(1 \leq i \leq m)$ and the R_i are 2-modular, then $K_i \cong N/R_i$ are N-modules of type 2. $(1 \leq i \leq m)$.

3.7. Theorem. If N has d.c.c. on right ideals and $D(N) = (0)$, then N is a direct sum of right ideals which are type 0 as N-modules.

Proof. This proof is essentially similar to the preceding one. This theorem may be found in Betsch [2], 3.4 and Laxton [3], Thm. 3.

3.8. Theorem. (Betsch [2]. 3.4) If N has d.c.c. on right ideals and $J_1(N) = (0)$, then N is a direct sum of right ideals which are of type 1 as N-modules.

Proof. See 3.6 also.

3.9. The questions concerning the decomposition of these near-rings as direct sums of ν-primitive ideals is only partially resolved and we must wait until we have dealt with the density theorems before looking at them.

§4. Identity elements in near-rings with zero radicals.

We ask now, whether, under suitable chain conditions, any of the radicals being zero implies the existence in the near-ring of a multiplicative identity. We exhibit a simple example of a finite near-ring with all its radicals zero and with every non-zero element a left identity.

4.1. Example. Let G be a non-trivial, finite additive group. We write N for the set of mappings of G into itself, with the properties that: (i) given any $n \in N$, then $\exists\, g \in G$ such that

$$h.n = g \text{ for all } h \in G \text{ with } h \neq 0.$$

$$\text{and (ii) } 0.n = 0.$$

It is easy to verify that N is a near-ring (sometimes called the near-ring of constant mappings), and every non-zero element is a left identity. For if n_1, $n_2 \in N$ and $n_1 \neq 0$.

Let $xn_1 = g_1$, $xn_2 = g_2$ for all $0 \neq x \in G$.

Then $xn_1n_2 = g_1n_2 = g_2 = xn_2$.

Thus $n_1n_2 = n_2$

If $(0) \neq K \subseteq N$ was a nilpotent N^+-subgroups, then $\exists\, k \in K$ with $k \neq 0$. Let $n \in N$, then $k.n = n \in K$ and so $N = K$, and clearly N cannot be nilpotent. Thus $J_2(N) = (0)$

We have shown that there exists a finite near-ring N such that $J_2(N) = (0)$ and N does not possess an identity, We may ask now, whether all near-rings N with $J_2(N) = (0)$ and d.c.c. on rt. N^+-subgroups possess a left identity? In fact, better results are available. The following theorem is due to Betsch (Betsch [2], 3.4).

4.2. <u>Theorem.</u> If N has d.c.c. on right ideals and $D(N) = (0)$, then N possesses a left identity.

<u>Proof.</u> Let F be a 0-modular right ideal and K any ideal such that there exists an $h \in N$ for which $n - hn \in K$ for all $n \in N$. We show that there exists an $x \in N$ with $n - xn \in K \cap F$ for all $n \in N$. Suppose that $K \subseteq F$, then $K \cap F = K$ and we just need $x = h$.

However if $K \not\subseteq F$, then $N = F + K$ as F is 0-modular.

Let $e \in N$ such that $n - en \in F$, for all $n \in N$.

Put $e = u + e'$ where $u \in F$, $e' \in K$

$h = f' + v$ where $v \in K$, $f' \in F$.

Suppose $m \in N$, then $(e' + f')m - e'm \in F$

and $e'm - em = (e - u)m - em \in F$.

Thus $(e' + f')m - em \in F$, say $(e' + f')m - em = f_0$

Now $m - (e' + f')m = m - (f_0 + em) = m - em - f_0$.

So $m - (e' + f')m \in F$ as $m - em \in F$

Also $(e' + f')m - f'm \in K$

$f'm - hm = (h - v)m - hm \in K$

thus $(e' + f')m - hm \in K$, say $(e' + f')m - hm = k_0$.

Now $m - (e' + f')m = m - hm - k_0 \in K$ as $m - hm \in K$.

We have $m - (e' + f')m \in K \cap F$ for all $m \in M$ and thus $x = e' + f'$ will be suitable.

We can show by induction that if N has d.c.c. on rt. ideals, $D(N) = \bigcap_{i=1}^{q} F_i$ where the F_i are 0-modular right ideals, and that there is an $x' \varepsilon N$ such that

$$n' - x'n' \varepsilon \bigcap_{i=1}^{q} F_i \quad \text{for all} \quad n' \varepsilon N.$$

But in our case $D(N) = (0)$ and so we have found an $x' \varepsilon N$ such that $n' - x'n' = 0$ for all $n' \varepsilon N$.

That means that x' is a left identity for N.

4.3. Because of the lack of adaptability of the ring theory concept of quasi-regularity to our case, this proof cannot be extended to give us an identity in this situation, and clearly our hopes in this direction are dulled by the example 4.1.

We examine now, the possibility of replacing $D(N) = (0)$ by $J_0(N) = (0)$ in the hypothesis of 4.2. We construct the following example.

4.4. Example. Let Γ be a finite, additive, non-zero group, and K, Δ subgroups of Γ such that $K \neq (0)$, $\Delta \neq (0)$ and $K \cap \Delta = (0)$ and $\Gamma \neq \Delta \cup K$ and $|\Delta| \geq 3$.

Let N be set of all mappings of $\Gamma \to \Gamma$ such that $\Delta \to \Delta$ and $K \to \Delta$ and zero is preserved under each mapping. This is a near-ring. If $n, n_1 \varepsilon N$, then $\gamma(n + n_1) \varepsilon \Gamma$, for all $\gamma \varepsilon \Gamma$.

$\delta(n + n_1) = \delta n + \delta n_1 \varepsilon \Delta$ for all $\delta \varepsilon \Delta$.

$k(n + n_1) = kn + kn_1 \varepsilon \Delta$, for all $k \varepsilon K$.

$\gamma(n.n_1) = (\gamma n)n_1 \varepsilon \Gamma$. $\delta(n.n_1) = (\delta n)n_1 \varepsilon \Delta$.

$k(n.n_1) \varepsilon \Delta \quad n_1 \subseteq \Delta$.

We claim that $J_0(N) = (0)$; in fact that N is 0-primitive. Γ is certainly a rt. N-module and $(\Gamma)_r = (0)$.

Let $0 \neq \gamma_1 \in \Gamma$ and $\gamma_1 \notin K \cup \Delta$. We show that $\Gamma = \gamma_1 N$.

Pick any $\gamma' \in \Gamma$, define $m: \Gamma \to \Gamma$ by:

$m: \gamma_1 \to \gamma'$

$m: \gamma'' \to 0$ for all $\gamma'' \in \Gamma \setminus \{\gamma_1\}$.

Clearly $m \in N$ and $\gamma_1 m = \gamma'$, so $\gamma_1 N = \Gamma$. We now show that if L is any non-zero N-submodule of Γ, then $L = \Gamma$.

We have that for all $\ell \in L$, $\gamma_2 \in \Gamma$, $n_2 \in N$, $(\gamma_2 + \ell)n_2 - \gamma_2 n_2 \in L$.

Suppose $L \neq \Gamma$ such that $\exists \bar{\gamma} \in \Gamma$ with $\bar{\gamma} \notin L$. If $\bar{\gamma} \notin K$.

Then define $m' : \Gamma \to \Gamma$ by

$m' : \bar{\gamma} \to \bar{\gamma}$

$m' : \gamma_3 \to 0$ for all $\gamma_3 \in \Gamma \setminus \{\bar{\gamma}\}$.

Then $m' \in N$ and for $d_1 \neq 0$, $(\bar{\gamma} + d_1)m' - \bar{\gamma}m' = -\bar{\gamma} \notin L$, $(d_1 \in L)$ a contradiction to L being an N-submodule.

The only remaining possibility is if $\bar{\gamma} \in K$ whenever $\bar{\gamma} \notin L$.

Choose a $\beta' \notin \Delta \cup K$. For any $y \notin L$, we may define $\bar{m} : \Gamma \to \Gamma$ by

$\beta' . \bar{m} = y$

$\gamma_4 \bar{m} = 0$ for all $\gamma_4 \in \Gamma \setminus \{\beta'\}$.

Clearly $\bar{m} \in N$ and if $d_2 \neq 0$, $d_2 \in L$,

$(\beta' + d_2)\bar{m} - \beta' . \bar{m} = -y \notin L$, a contradiction.

Hence Γ is of type 0 and so N is a 0-primitive near-ring.

In particular $J_0(N) \subseteq (\Gamma)_r = (0)$.

Has N a left identity? Suppose there is $e \neq o$ such that $em_3 = n_3$ for all $n_3 \in N$.

Then for all $k' \varepsilon K$, $k'e \varepsilon \Delta$, since $e \varepsilon N$.

Suppose $k_1 \varepsilon K$ and $k_1 e = \delta_1 \neq 0$. $(\delta_1 \varepsilon \Delta)$.

Define $n_4 \varepsilon N$ by $n_4 : \Gamma \to \Gamma$,

$$\gamma_5 n_4 = 0 \quad \forall \gamma_5 \varepsilon \Gamma \smallsetminus (\Delta \cup K)$$

$$\delta_3 n_4 = 0 \quad \forall \delta_3 \varepsilon \Delta$$

$$k_1 n_4 = \delta_2 \quad \text{where } \delta_2 \neq \delta_1, \ \delta_2 \varepsilon \Delta \quad \text{and } \delta_2 \neq 0.$$

$$k_2 n_4 = 0 \quad \forall k_2 \varepsilon K \smallsetminus \{ k_1 \}.$$

Then $k_1 en_4 = \delta_1 n_4 = 0$

but $k_1 n_4 = \delta_2 \neq 0$

Thus $en_4 \neq n_4$. This leaves the possibility that

$k''e = 0$ for all $k'' \varepsilon K$.

Then $k_1 en_4 = 0$ and $k_1 n_4 = \delta_2 \neq 0$.

N has therefore no left identity, yet it is finite and 0-primitive.

§5. The radicals of related near-rings.

The final consideration of this chapter is the relationship of the radicals $J_0(N)$, $J_2(N)$, and $D(N)$ with near-rings which have a close connection with the original near-ring N.

For instance, if B is an ideal of a near-ring N, then B is itself a near-ring. How does $J_2(B)$ relate to $J_2(N)$?

We need some preliminary lemmas.

5.1. Lemma. Suppose that B is an ideal of a near-ring N. If a is a quasi-regular element of the near-ring B, then a is a quasi-regular element of the near-ring N.

Proof. Since a is q.r. in B, then $a \in R = \bigcap_{L \in \mathcal{L}} L$,

where $\mathcal{L}$ is the set of all right ideals of B containing all the elements of the form $b - ab$, $\forall\, b \in B$.

Let $T = \bigcap_{S \in \mathcal{S}} S$, where $\mathcal{S}$ is the set of all right ideals of N containing the elements $n - an$, $\forall\, n \in N$.

For any $S \in \mathcal{S}$, then S^+ is normal in N^+.

Put $F = B \cap S$, then F^+ is normal in B^+.

Also F is a right ideal, of B

Now $n - an \in S$ $\forall\, n \in N$ and so

$b - ab \in F$ $\forall\, b \in B$.

Thus $F \in \mathcal{L}$ and so $a \in F$. Thus $a \in T$ and a is q.r. in N.

5.2. Lemma. If $\psi : N \to N'$ is a epimorphism of near-rings, then a quasi-regular element x in N, is mapped onto a quasi-regular element $x\psi$ in N'

Proof. If x is q.r. in N, then $x \in T = \bigcap_{S \in \mathcal{S}} S$, where $\mathcal{S}$ is the set of all right ideals of N containing the elements $n - xn, \forall\, n \in N$. If $S \in \mathcal{S}$, let $S' = S\psi$. Since ψ is an epimorphism of near-rings, S' is a right ideal of N'.

Also $n - xn \in S$ so $n\psi - (x\psi)(n\psi) \in S'$ $\forall\, n \in N$ and $S' \in \mathcal{S}'$ the set of all right ideals of N' containing all elements of the form $n' - x'n'$, $\forall\, n' \in N'$.

Let $T' = \bigcap_{S_1' \in \mathcal{S}'} S_1'$. Then if $S_1' \in \mathcal{S}'$, then S_1' corresponds to a right ideal S_1 of N such that $S_1 \supseteq \ker\psi$ and $S_1 \in \mathcal{S}$. Hence if $x \in T = \bigcap_{S \in \mathcal{S}} S$, then $x' \in T'$.

That is, $x' = x\psi$ is q.r. in N'.

5.3. Lemma. If $1 \in N$ and $J_2(N) = (0)$, then for any ideal B of N, $D(B) = (0)$.

Proof. Let $y \in D(B)$ and assume that $y \neq 0$. $D(B)$ is a right ideal of B with all its elements quasi-regular, so y is q.r. in B and also in N by 5.1.

$D(B).B \subseteq D(B)$ and so $y.B$ is a rt.B^+-subgroup which is q.r. (i.e. all its elements are q.r.). Clearly yB is a q.r. rt. N^+-subgroup and so $yB \subseteq J_2(N) = (0)$ by 2.5.

This holds for any $0 \neq y \in D(B)$.

Now $yN.B \subseteq yB = 0$ for any $y \in D(B)$.

Thus yN is nilpotent and so by 2.5.

$yN \subseteq J_2(N) = (0)$

$1 \in N \Rightarrow y.1 = (0)$. Thus $D(B) = (0)$.

5.4. <u>Theorem</u>. If B is an ideal in a near-ring N with identity, then $D(B) \subseteq B \cap J_2(N)$.

<u>Proof</u>. Let $\bar{N} = N / J_2(N)$

then $\bar{B} = (B + J_2(N)) / J_2(N)$ is an ideal of $\bar{N}$.

Hence $D(\bar{B}) = (0)$ (by 1.4 and 5.3).

Now $(B + J_2(N)) / J_2(N) \cong B / B \cap J_2(N)$

Then, since $D(B)$ is quasi-regular in B,

$$D(B) / B \cap J_2(N) \text{ is quasi-regular in } B / B \cap J_2(N) \quad (5.2).$$

Now $J_2(\bar{N}) = (0)$ and so $D(B / B \cap J_2(N)) = (0)$

$$D(B) / B \cap J_2(N) \subseteq D(B / B \cap J_2(N)) = (0)$$

Hence $D(B) \subseteq B \cap J_2(N)$.

5.5. Clearly $D(N) \cap B$ is a right ideal of B whose elements are q.r. in N. Are they q.r. in B?

In general it is not known but by introducing the descending Chain condition on rt.N^+-subgroups as an extra condition, we may use the result that tells us that a q.r. N^+-subgroup is nilpotent. (Ramakotaiah. [1], Thm. 5.1.).

5.6. <u>Theorem</u>. If B is an ideal of near-ring N with identity, and if N has descending chain condition on N^+-subgroups, then $D(N) \cap B \subseteq D(B) \subseteq B \cap J_2(N)$.

Proof. $D(N) \cap B$ is q.r. in N, hence $D(N) \cap B$ is a nilpotent right ideal of N, thus $D(N) \cap B$ is a nilpotent right ideal of B. By Ramakotaiah. [1], Cor. 2.2.

$D(N) \cap B$ is q.r. in B and thus $D(N) \cap B \subseteq D(B)$.

Chapter 3. 2-primitive near-rings with identity and descending chain condition on right ideals.

One of the central results in ring theory is the structure of simple artinian rings which then completes the classification of semi-simple artinian rings. In fact, simple artinian rings are equivalently primitive artinian rings and are characterized by being rings of homomorphisms of vector spaces over division rings. The same problem in near-ring theory, i.e. the structure of 2-primitive near-rings with identity and d.c.c. on right ideals has also been solved and gives us one of the finest results in the subject. The main result was announced by Wielandt ([1]) but no proof has so far appeared in the literature. The theory for distributively generated, finite, near-rings was discussed by Laxton ([1]) and it is his approach that we use here. We will first show that, for 2-primitive near-rings with identity and d.c.c. on right ideals we can obtain a 'density theorem'. We then restrict the case to finite near-rings and obtain a complete classification of these. The density theorem has been proved for 0-primitive near-rings with d.c.c. on right ideals and an identity by Wielandt and Betsch but is unpublished at the present time. These results will be stated at the appropriate places in this thesis.

§1. A Density theorem for 2-primitive near-rings with identity and d.c.c. on right ideals.

1.1. Throughout §1, N will denote a 2-primitive near-ring with an identity, 1_N, and d.c.c. on right ideals. Γ will be the faithful N-module of type 2. Hence $(\Gamma)_r = (0)$.

1.2. <u>Lemma</u>. $(\Gamma, +)$ is abelian if and only if $(N, +)$ is abelian.

<u>Proof</u>. If $(\Gamma, +)$ is abelian and $n_1, n_2 \varepsilon N$. Let $\gamma \varepsilon \Gamma$ be arbitrary.

Then $\gamma(n_1 + n_2 - n_1 - n_2) = \gamma n_1 + \gamma n_2 - (\gamma n_1) - (\gamma n_2) = 0$

So $n_1 + n_2 - n_1 - n_2 \varepsilon (\Gamma)_r = (0)$

So $n_1 + n_2 = n_2 + n_1$ for any $n_1, n_2 \varepsilon N$.

Now if $(N, +)$ is abelian, suppose $\gamma_1, \gamma_2 \varepsilon \Gamma$, then if $0 \neq \gamma \varepsilon \Gamma$ is arbitrary, $\gamma_1 = \gamma n_1$ for some $n_1 \varepsilon N$ and $\gamma_2 = \gamma n_2$ for some $n_2 \varepsilon N$

Then $\gamma_1 + \gamma_2 - \gamma_1 - \gamma_2 = \gamma n_1 + \gamma n_2 - \gamma n_1 - \gamma n_2$

$$= \gamma(n_1 + n_2 - n_1 - n_2) = \gamma.0 = 0$$

Thus $(\Gamma, +)$ is abelian.

1.3. <u>Lemma</u>. If N is finite and $(\Gamma, +)$ is nilpotent as a group, then $(N, +)$ is also nilpotent as a group.

<u>Proof</u>. Let $N = N_0 > N_1 > N_2 > \ldots > N_t = N_{t+1}$ be the lower central series for $(N, +)$, terminating at N_t. We assume that $N_t \neq (0)$. Let the lower central series for Γ be

$$\Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 > \quad \ldots > \Gamma_d = \Gamma_{d+1}$$

Now $\Gamma = \gamma N$ for each non-zero $\gamma \varepsilon \Gamma$. If $1 \leq j \leq d$

$\Gamma_j = \{[[\ldots[\Gamma, \Gamma], \ldots, \Gamma] \Gamma]\} =$

$\{[[\ldots [\gamma N, \gamma N], \ldots, \gamma N] \gamma N]\} = \gamma N_j$ for any $0 \neq \gamma \varepsilon \Gamma$.

If $d \geq t$ then $d = t$ as $\Gamma_d = \gamma N_d = \gamma N_t = \Gamma_t$

If $d \leq t$ then $\gamma N_d = \Gamma_d = (0) \Rightarrow N_d \subseteq (\Gamma)_r = (0)$

So $\Gamma_d \neq (0)$ and $N_t \neq (0) \Rightarrow \Gamma_d \neq (0)$.

This last lemma will be used in an application later. We return to the general case.

1.4. Definition. (i) The centralizer of Γ in N; $C_N(\Gamma)$ is the set of all endomorphisms ϕ of $(\Gamma, +)$ such that

$$(\gamma n)\phi = (\gamma\phi)n \quad \forall \gamma\varepsilon\Gamma, \forall n \varepsilon N.$$

(ii) We denote by $A_N(\Gamma)$ the set of all endomorphisms belonging to $C_N(\Gamma)$, that are in fact, automorphisms of $(\Gamma, +)$.

1.5. Proposition. $C_N(\Gamma) = (0) \cup A_N(\Gamma)$ where 0 denotes the zero endomorphism of Γ.

Proof. Let $x \varepsilon C_N(\Gamma)$ and $x \neq 0$. Clearly ker x is an N-submodule of Γ and so ker $x = (0)$.

Γx is an N-subgroup of Γ and so $\Gamma x = \Gamma$. Thus $x \varepsilon A_N(\Gamma)$.

1.6. Proposition. If $(0) \neq \gamma\varepsilon\Gamma$, then $(\gamma)_r$ is a right ideal,

$N/(\gamma)_r \cong \Gamma$ and hence is an N-module of type 2.

Proof. Let $\psi : N \to \Gamma$ be defined by

$$n\psi = \gamma n.$$

Then ψ is an N-homomorphism of N onto Γ.

Hence $N/\ker\psi \cong \Gamma$ as N-modules

i.e. $N/\ker\psi \cong N/(\gamma)_r$ is a type 2, N-module.

This means that $(\gamma)_r$ is maximal as a right ideal and as an N^+-subgroup of N.

1.7. Definition. We now define an equivalence relation $\mathcal{P}$ on the elements of Γ in the following way.

Let γ , $\gamma_1 \varepsilon \Gamma$, then we say

$$\gamma \mathcal{P} \gamma_1 \Leftrightarrow (\gamma)_r = (\gamma_1)_r$$

1.8. Proposition. If $\gamma , \gamma_1 \varepsilon \Gamma$ and $\gamma \neq 0$, $\gamma_1 \neq 0$, then $\gamma \mathcal{P} \gamma_1$ if and only if $\exists \ \phi \ \varepsilon \ C_N(\Gamma)$ such that $\gamma_1 = \gamma\phi$.

Proof. If $\gamma \mathcal{P} \gamma_1$ then $(\gamma)_r = (\gamma_1)_r$.

Let $\phi : \Gamma \to \Gamma$ be defined by

$(\gamma n) \phi = \gamma_1 n , \forall n \ \varepsilon N$. This is well-defined, for

$\gamma_1 n = 0 \Leftrightarrow n \ \varepsilon (\gamma_1)_r = (\gamma)_r \Leftrightarrow \gamma n = 0$

Thus $\phi \ \varepsilon \ C_N(\Gamma)$ and taking $n = 1_N$ shows that $\gamma \phi = \gamma_1$.

If $\phi \ \varepsilon \ C_N(\Gamma)$ then let $n' \varepsilon (\gamma)_r$,

$\gamma_1 n' = (\gamma\phi)n' = (\gamma n')\phi = 0$ so $(\gamma)_r \subseteq (\gamma_1)_r$.

$(\gamma)_r$ is maximal as a rt. N^+-group in N so $(\gamma)_r = (\gamma_1)_r$ by 1.6.

—.

We can therefore consider the equivalence classes of the group Γ to consist of the zero class $\{0\}$ and the classes $\gamma.\mathcal{A}_N(\Gamma)$ for suitable choices of γ . There exists a 1-1 relationship between the non-zero classes and the right ideals which are annihilators of elements (non-zero) of Γ. If we consider the group $\mathcal{A}_N(\Gamma)$ to be a permutation group acting on the non-zero elements of Γ , then we may regard the equivalent classes (different from zero) as being orbits on the non-zero elements of Γ. (An orbit is a minimal fixed block).

1.9. Theorem. Suppose that N is not a ring. Let Δ be a subset of $\Gamma\setminus\{0\}$ such that Δ is the union of m different orbits of Γ. Then $N/(\Delta)_r$ is a direct sum of m copies of Γ and $(\Delta)_r \not\subseteq (\gamma)_r$ for any non-zero $\gamma \in \Gamma\setminus\Delta$.

Proof. We use induction on the number m.

If $m = 1$, then $(\Delta)_r = (\delta)_r$ for any $\delta \in \Delta$. It follows from 1.6, that $N/(\delta)_r \cong \Gamma$ as N-modules. Clearly for $\gamma \notin \Delta, \gamma \neq 0$, we have $\gamma \not\sim \delta$, so $(\delta)_r \not\subseteq (\gamma)_r$.

It is assumed now, that the result is true for a value $m = k > 1$

Let Δ be a subset of $\Gamma\setminus\{0\}$ which is a union of $k+1$ different orbits, say $O(\gamma_1), \ldots, O(\gamma_{k+1})$ and no k orbits cover Δ.

Let Δ' be a subset of Δ, with

$$\Delta' = \Delta \cap \{\bigcup_{i=1}^{k} O(\gamma_i)\}.$$

(Clearly Δ' is covered by the k orbits $O(\gamma_1), \ldots, O(\gamma_k)$).

$(\Delta')_r$ is a rt. ideal and $(\Delta')_r \not\subseteq (\gamma_{k+1})_r$ by the inductive hypothesis. Thus $N = (\Delta')_r + (\gamma_{k+1})_r$.

Since $\Delta = \Delta' \cup O(\gamma_{k+1})$, $(\Delta)_r = (\Delta')_r \cap (\gamma_{k+1})_r$

$$N/(\Delta)_r = N/(\Delta')_r \cap (\gamma_{k+1})_r$$

$$= \left((\Delta')_r + (\gamma_{k+1})_r\right) / (\Delta')_r \cap (\gamma_{k+1})_r$$

$$= (\gamma_{k+1})_r / (\gamma_{k+1})_r \cap (\Delta')_r \quad + \quad (\Delta')_r / (\gamma_{k+1})_r \cap (\Delta')_r$$

now $(\Delta')_r / (\gamma_{k+1})_r \cap (\Delta')_r \cong N / (\gamma_{k+1})_r \cong \Gamma$;

$$(\gamma_{k+1})_r / (\Delta')_r \cap (\gamma_{k+1})_r \cong N / (\Delta')_r .$$

By the inductive hypothesis

$N / (\Delta')_r$ is a direct sum of K copies of Γ .

We have only to show now that

$N/(\Delta)_r \cong N/(\Delta')_r \oplus N/(\gamma_{k+1})_r$, and the result

follows by induction.

Let $x \in \left((\gamma_{k+1})_r / (\Delta')_r \cap (\gamma_{k+1})_r\right) \cap \left((\Delta')_r / (\Delta')_r \cap (\gamma_{k+1})_r\right)$

then $x = y + (\Delta')_r \cap (\gamma_{k+1})_r$ where $y \in (\gamma_{k+1})_r$

$x = z + (\Delta')_r \cap (\gamma_{k+1})_r$ where $z \in (\Delta')_r$

and $-z + y \in (\Delta')_r \cap (\gamma_{k+1})_r$, so $y \in (\Delta')_r$

$y \in (\Delta')_r \cap (\gamma_{k+1})_r$ so $x = \bar{0}$. The sum is direct.

We assume now that there is a $\bar{\gamma} \notin \Delta$ with $(\Delta)_r \subseteq (\bar{\gamma})_r$,

and obtain a contradiction.

Now by the inductive hypothesis, since $\bar{\gamma} \notin \Delta'$, $(\Delta')_r \not\subseteq (\bar{\gamma})_r$

Then $N = (\bar{\gamma})_r + (\Delta')_r$ and

$$\bar{\gamma}.(\Delta')_r = \gamma_{k+1}.(\Delta')_r = \Gamma .$$

Defining $\phi : \Gamma \to \Gamma$ by $(\gamma_{k+1}x)\phi = \bar{\gamma}x$ for all $x \in (\Delta')_r$.

(If $\gamma_{k+1}x = 0$ and $x \in (\Delta')_r$ then $x \in (\gamma_{k+1})_r \cap (\Delta')_r \subseteq (\bar{\gamma})_r$.)

It is easy to verify that $\phi \in C_N(\Gamma)$.

Consider the set

$$T = \{\bar{\gamma} n - \gamma_{k+1} \phi n \ ; \ n \in N\} \subseteq \Gamma .$$

$(\Delta')_r$ is a right ideal of N. so if $h \in (\Delta')_r$; $n, n' \in N$, then

$(h + n)n' - nn' \in (\Delta')_r$ so

$(\gamma_{k+1}.\phi)((h + n)n' - nn') = \bar{\gamma}((h + n)n' - nn')$

so $[(\gamma_{k+1}.\phi)h + (\gamma_{k+1}.\phi)n]n' - (\gamma_{k+1}.\phi)(nn') =$

$(\bar{\gamma}h + \bar{\gamma}n)n' - \bar{\gamma}nn'$

now as $h \in (\Delta')_r$, $(\gamma_{k+1}\phi)h = \bar{\gamma}h$,

thus $(\bar{\gamma}h + \gamma_{k+1}\phi n)n' - \gamma_{k+1}\phi nn' = (\bar{\gamma}h + \bar{\gamma}n)n' - \bar{\gamma}nn'$ (1)

Also $\gamma_{k+1}\,\phi(-n + h + n) = \bar{\gamma}(-n + h + n)$

so $-\gamma_{k+1}\,\phi n + \gamma_{k+1}\,\phi h + \gamma_{k+1}\,\phi n = -\bar{\gamma}n + \bar{\gamma}h + \bar{\gamma}n.$

Rearranging gives

$$(\bar{\gamma}n - \gamma_{k+1}\,\phi n) + \bar{\gamma}h = \bar{\gamma}h + (\bar{\gamma}n - \gamma_{k+1}\,\phi n) \qquad (2)$$

Hence $(\bar{\gamma}n - \gamma_{k+1}\,\phi n) \in$ centre of Γ^+, for all $n \in N$.

We will show that the subset T is in fact an N-subgroup of Γ.

$(\bar{\gamma}n - \gamma_{k+1}\,\phi n)n' = (\bar{\gamma}h + \bar{\gamma}z + \gamma_{k+1}\,\phi(-n))n'$ where

$n = h + z$, $h \in (\Delta')_r$ and $z \in (\bar{\gamma})_r$

so $(\bar{\gamma}n - \gamma_{k+1}\,\phi n)n' = (\bar{\gamma}h + \gamma_{k+1}\,\phi(-n))n'$

$= (\bar{\gamma}h + \bar{\gamma}(-n))n' - \bar{\gamma}(-n)n' + \gamma_{k+1}\,\phi(-n)n'$ by (1)

$= (\bar{\gamma}h - \bar{\gamma}n)n' + \gamma(-n)(-n') - \gamma_{k+1}\,\phi(-n)(-n')$

$= \bar{\gamma}(-n)(-n') - \gamma_{k+1}\,\phi(-n)(-n') + (\bar{\gamma}h - \bar{\gamma}n)n'$ by (2)

$= \bar{\gamma}(-n)(-n') - \gamma_{k+1}\,\phi(-n)(-n') + 0$ (since $n = h + z$) (3)

Hence $(\bar{\gamma}n - \gamma_{k+1}\,\phi n)n' \in T$ for all $n, n' \in N$.

Now if $n' = h' + z'$ with $h' \in (\Delta')_r$, $z' \in (\bar{\gamma})_r$,

$[\bar{\gamma}n + \gamma_{k+1}(-n)\,\phi] + [\bar{\gamma}n' + \gamma_{k+1}(-n')\,\phi]$

$= \bar{\gamma}h' + \bar{\gamma}n - \gamma_{k+1}n\phi - \gamma_{k+1}\,n'\phi$ by (2)

$= \bar{\gamma}(n' + n) - \gamma_{k+1}\,\phi(n' + n) \in T.$

Finally $-(\bar{\gamma}n - \gamma_{k+1}\,\phi n) = \gamma_{k+1}\,\phi n - \bar{\gamma}n$

$= -\gamma_{k+1}\,\phi(-n) + \bar{\gamma}(-n) - \gamma_{k+1}\,\phi(-n) + \gamma_{k+1}\,\phi(-n)$

$= \bar{\gamma}(-n) - \gamma_{k+1}\phi(-n) - \gamma_{k+1}\phi(-n) + \gamma_{k+1}\phi(-n)$

$= \bar{\gamma}(-n) - \gamma_{k+1}\phi(-n) \varepsilon T.$

Hence T is an N-subgroup of Γ and so $T = (0)$ or Γ.

We have to consider two distinct possibilities.

If N^+ is not abelian, then Γ^+ is not abelian and as the centre of Γ^+ is a proper subset of Γ, $T \subseteq$ centre of $\Gamma^+ \Rightarrow T = (0)$.

In the case when N^+ is abelian, we assume that $T = \Gamma$.

Since N is not a ring then there is $n_1, n_2, n^* \varepsilon N$ such that $(n_1 + n_2)n^* - (n_1 n^* + n_2 n^*) \neq 0$, and since Γ is faithful there is a $\gamma^* \varepsilon \Gamma$ such that $\gamma^*[(n_1 + n_2)n^* - (n_1 n^* + n_2 n^*)] \neq 0$.

Put $\gamma^* n_1 = \gamma_1^*$, $\gamma^* n_2 = \gamma_2^*$

Let $\gamma \varepsilon \Gamma$ so that $\gamma = \bar{\gamma}(x + y') - \gamma_{k+1}(x + y')\phi$ for some $x \varepsilon (\Delta')_r$, $y' \varepsilon (\gamma_{k+1})_r$.

$\gamma = \bar{\gamma}x + \bar{\gamma}y' - \gamma_{k+1}y'\phi - \gamma_{k+1}x\phi$

$= \bar{\gamma}x - \gamma_{k+1}y'\phi - \gamma_{k+1}x\phi$

$= \bar{\gamma}y' + \bar{\gamma}x - \bar{\gamma}x - \gamma_{k+1}y'\phi$ as Γ is abelian and

$\bar{\gamma}x = \gamma_{k+1}\phi x$ for $x \varepsilon (\Delta')_r$.

So $\gamma = \bar{\gamma}y'$. Hence $\Gamma = \bar{\gamma}.(\gamma_{k+1})_r$

Also $\Gamma = -(\gamma_{k+1}\phi(\bar{\gamma})_r)$.

Let $\gamma_1^* = \bar{\gamma}y$ where $y \varepsilon (\gamma_{k+1})_r$ and

$\gamma_2^* = -(\gamma_{k+1}\phi z)$ where $z \varepsilon (\bar{\gamma})_r$.

Then $(\gamma_1^* + \gamma_2^*)n^* = [\bar{\gamma}(y+z) - (\gamma_{k+1}\phi(y+z))]n^*$

$= \bar{\gamma}[-(y+z)(-n^*)] - [\gamma_{k+1}\phi(-(y+z)(-n^*)]$

by equation (3).

Thus $(\gamma_1^* + \gamma_2^*)n^* = -\{\bar{\gamma}(-(y+z).n^*\} + \gamma_{k+1}\phi(-(y+z))n^*.$

$= -\{-(\bar{\gamma}(y+z)).n^*\} + \{-(\gamma_{k+1}\phi(y+z)\}.n^*.$

$= -\{(-\gamma_1^*)n^*\} + \gamma_2^*.n^*.$

Now $(-\gamma_1^*)(-n^*) = [-(\bar{\gamma}\, y)](-n^*) = \bar{\gamma}(-y)(-n^*)$

$= \bar{\gamma}(-y)(-n^*) - \gamma_{k+1}\phi(-y)(-n^*)$

as $y \varepsilon (\gamma_{k+1})_r$.

So $(-\gamma_1^*)(-n^*) = (\bar{\gamma}y - \gamma_{k+1}\phi y)n^*$ by equation (3)

$= \bar{\gamma}yn^* = \gamma_1^* n^*.$

Thus $(\gamma_1^* + \gamma_2^*)n^* = \gamma_1^* n^* + \gamma_2^* n^*.$

This is a contradiction to the choice of γ_1^*, γ_2^* and n^*.

Hence $T \neq \Gamma$ and so $T = (0)$.

Therefore in all cases $T = (0)$.

Then $\bar{\gamma}n = \gamma_{k+1}\phi n$ for all $n \varepsilon N$.

Since $1 \varepsilon N$ we have

$$\bar{\gamma} = \gamma_{k+1}\phi$$

This shows that $\gamma_{k+1}\, \rho\, \bar{\gamma}$ that is, $\bar{\gamma}$ and γ_{k+1} lie in the same orbit. This is a contradiction to the assumption that

$$\bar{\gamma} \notin \Delta.$$

Hence we have shown that $(\bar{\gamma})_r \not\supseteq (\Delta)_r$.

1.10. We restate the theorem as follows.

Theorem. Let N be a 2-primitive near-ring with an identity and d.c.c. on right ideals. If Δ is any subset of $\Gamma \setminus \{0\}$, which is the union of m distinct orbits, then either

$\bar{\gamma} \in \Gamma$, $\bar{\gamma} \neq 0$ and $\bar{\gamma} \notin \Delta \Rightarrow (\bar{\gamma})_r \not\supseteq (\Delta)_r$ and $N = (\bar{\gamma})_r + (\Delta)_r$

or N is a ring.

1.11. Since N has d.c.c. on right ideals, an application of 1.10, in the case when N is not a ring, shows that $A_N(\Gamma)$ induces finitely many orbits on $\Gamma \setminus \{0\}$.

We proceed now to the density theorem.

1.12. Theorem.

(The Density Theorem for 2-primitive near-rings with identity and d.c.c. on right ideals.) If N is not a ring.

Suppose that $\gamma_1, \ldots, \gamma_p$ are non-zero elements of Γ with the property that $\gamma_i \rho \gamma_j \Rightarrow i = j$ for $1 \leq i, j \leq p$.

If $\gamma_1', \ldots, \gamma_p'$ are arbitrary elements of Γ, then there exists $n \in N$ such that $\gamma_i' = \gamma_i n$ for $1 \leq i \leq p$.

Proof. $\Gamma \setminus \{0\}$ is the union of a finite number of orbits, say $\Gamma = \{0\} \cup (\bigcup_{i=1}^{m} O(\gamma_i))$ where $p \leq m$, and $\gamma_{p+1}, \ldots, \gamma_m$ are representatives of the orbits of $\Gamma \setminus [\{0\} \cup (\bigcup_{i=1}^{p} O(\gamma_i))]$

We put $\Delta = \bigcup_{i=1}^{p} O(\gamma_i)$.

Then $\Gamma = \{0\} \cup \Delta \cup (\bigcup_{i=p+1}^{m} O(\gamma_i))$.

Let $\Delta'_i = \bigcup_{\substack{j=1 \\ j\neq i}}^{p} O(\gamma_j)$ where $1 \leq i \leq p$.

From Theorem 1.11, since $\gamma_i \notin \Delta'_i$, then $(\gamma_i)_r \not\supseteq (\Delta'_i)_r$ and

$$N = (\gamma_i)_r + (\Delta'_i)_r$$

Clearly $(\gamma_i)(\Delta'_i)_r = \Gamma$ from these statements, for each $1 \leq i \leq p$.

Thus $\gamma'_i = \gamma_i e_i$ where $e_i \varepsilon (\Delta'_i)_r$ and $1 \leq i \leq p$.

Put $n = \sum_{i=1}^{p} e_i$

Then $\gamma_i n = \gamma_i e_1 + .. + \gamma_i e_i + .. + \gamma_i e_p$.

But $e_j \varepsilon (\Delta'_j)_r = \bigcap_{\substack{k=1 \\ k\neq j}}^{p} (\gamma_k)_r$.

$j = i \Rightarrow e_j \varepsilon (\gamma_i)_r \Rightarrow \gamma_i e_j = 0$ for $i \neq j$

Thus $\gamma_i n = \gamma_i e_i = \gamma'_i$. $i = 1, \ldots, p$.

§2. The consequences of the density theorem.

2.1 We will be able to determine what a 2-primitive near-ring with identity and d.c.c. on right ideals looks like.

2.2. Theorem. If N is a 2-primitive near-ring with identity 1 and d.c.c. on right ideals, with Γ the faithful N-module of type 2, then either $N \cong M_{C_N(\Gamma)}(\Gamma)$ or N is an artinian primitive ring.

Proof. Assume N is not a ring. Let N_R be the set of right multiplications of elements of Γ by elements of N,

i.e., $x \varepsilon N_R \Longleftrightarrow \gamma x = \gamma n \ \forall \gamma \varepsilon \Gamma$ and some $n \varepsilon N$.

Clearly if $x \varepsilon N_R$ then $x \varepsilon \gamma\gamma\gamma_{C_N(\Gamma)}(\Gamma)$

Let $O(\gamma_1), . . , O(\gamma_p)$ be the orbits on $\Gamma \setminus \{0\}$ induced by the equivalence relation defined by the centralizer $C_N(\Gamma)$.

Then if $\gamma' \varepsilon \Gamma$ and $\gamma' \neq 0$, we can find $\phi \varepsilon C_N(\Gamma)$ such that $\gamma' = \gamma_i \phi$ for a particular $i \varepsilon \{1, . . , p\}$.

Let $x \varepsilon \gamma\gamma\gamma_{C_N(\Gamma)}(\Gamma)$, and pick any $\gamma \neq 0, \gamma \varepsilon \Gamma$. Then we can find $\phi \varepsilon C_N(\Gamma)$ and $i \varepsilon \{1, . . , p\}$ such that

$\gamma = \gamma_i \phi$ so $\gamma x = \gamma_i \phi x = \gamma_i x \phi$.

If the following are known,

$\gamma_1 x, \gamma_2 x, . . , \gamma_p x$, then the mapping x is completely determined on Γ.

Let $\gamma_i x = \gamma_i'$ $i = 1, ... , p$.

We can find, by the Density theorem, 1.15, an element $n \varepsilon N$ such that $\gamma_i n = \gamma_i'$ $i = 1, ... , p.$, as N is not a ring.

Consider the mapping $\theta : \Gamma \to \Gamma$ defined by

$\gamma \theta = \gamma n \ \forall \gamma \varepsilon \Gamma$. , $\theta \varepsilon N_R$.

Then clearly θ is equal to x, since $\gamma_i \phi n = \gamma_i n \phi = \gamma_i x \phi = \gamma_i \phi x$

Hence $\gamma\gamma\gamma_{C_N(\Gamma)}(\Gamma) \subseteq N_R$.

Hence

$N_R = \gamma\gamma\gamma_{C_N(\Gamma)}(\Gamma)$, and it is an elementary matter to verify that

$N \cong N_R$ as near-rings.

2.3 We have already seen that $C_N(\Gamma) = (0) \cup A_N(\Gamma)$ and it is easily seen that every automorphism in $A_N(\Gamma)$, besides the identity automorphism, is, in fact, regular (fixed-point-free). For suppose that $a \epsilon A_N(\Gamma)$, $a \neq 1$, and if $\exists \gamma \epsilon \Gamma, \gamma \neq 0$ such that $\gamma a = \gamma$, then since

$\Gamma = \gamma N$, let $\gamma' \epsilon \Gamma$, $\gamma' = \gamma n'$ some $n' \epsilon N$.

Then $\gamma' a = \gamma n' a = \gamma a n' = \gamma n' = \gamma'$ which contradicts the assumption that $a \neq 1$.

2.4 Summing up we have shown that a 2-primitive near-ring N with identity and d.c.c. on right ideals is either a ring or it is isomorphic to the set of mappings of an additive group into itself commuting with 0 and a group of regular automorphisms of the group, where the additive group has a finite number of orbits under the automorphism group. This contrasts with the ring case where a primitive artinian ring is isomorphic to the set of homomorphisms of a vector space commuting with a division ring, the vector space having finite dimension over the division ring.

Notice that instead of homomorphisms we have mappings, instead of vector space we have an additive group with a multiplicative group operating on it and instead of a finite dimensional vector space we have a group with a finite number of orbits.

Wielandt [1] first noted this result but his proof, although available, is unpublished. It has been noted above, that our proof is a generalization of Laxton's proof for finite, distributively generated near-rings. The next problem is to take an arbitrary additive group Γ, a group G of regular automorphisms of Γ, with the property that as a permutation group on Γ, G induces a finite number of orbits, and ask whether the near-ring $\mathcal{M}_{0UG}(\Gamma)$ is 2-primitive with d.c.c. on right ideals. The only real difficulty occurs in showing that we have d.c.c. on right ideals, This problem did not arise in Laxton's case because everything was of finite order.

2.5 Theorem.

If Γ is a finite (additive) group and G is a group of regular automorphisms of Γ, then the near-ring $N = \mathcal{M}_G(\Gamma)$ is 2-primitive and finite.

Proof. Clearly N will be finite.

We show that Γ is a faithful N-module of type 2. Γ becomes an N-module by constructing multiplication of $\gamma \varepsilon \Gamma$ by $n \varepsilon N$ in the natural way,

ie. $\gamma . n = (\gamma)n$.

Clearly the zero mapping, ie. 0_N is the only element of $(\Gamma)_r$.

Let $0 \neq \gamma \varepsilon \Gamma$, We require $\gamma N = \Gamma$. Let γ' be an arbitrary element of Γ. Define a mapping, $\theta : \Gamma \to \Gamma$ as follows.

$(\gamma g)\theta = \gamma' g \quad \forall\, g \in G$

$\gamma_1 \theta = 0 \quad \forall\, \gamma_1 \in \Gamma \setminus \{\gamma G\}$

Then $\theta \in N$ is easily checked.

Thus $\gamma' = \gamma\theta \in \gamma N$

Hence $\Gamma = \gamma N$

The identity map on Γ is the multiplicative identity of the near-ring N.

2.6 In order to complete the structure theory we now prove the following theorem which was communicated with its proof to us privately by Wielandt and Betsch.

Theorem.

If Γ is an additive group and G is a group of regular automorphisms such that G induces only a finite nymber of orbits on Γ, as a permutation group, then $N = \gamma\gamma_G(\Gamma)$ is a 2-primitive near-ring with identity and descending chain condition on right ideals.

Proof. By adapting the proof of Theorem 2.5 we can see that N is 2-primitive with identity. We just need to show that N has descending chain condition on right ideals.

Let $\gamma_1, \ldots, \gamma_p$ be representatives (ie. members) of the orbits on $\Gamma \setminus \{0\}$.

Clearly $(\gamma_i)_r$ is a right ideal of N, $1 \le i \le p$.

and $N / (\gamma_i)_r \cong \Gamma$ as N-modules, $1 \le i \le p$.

Let $A_i = \bigcap_{\substack{j=1 \\ j\neq i}}^{p} (\gamma_j)_r$; these are right ideals of N, $1\leq i\leq p$,

and $N = A_i \oplus (\gamma_i)_r \quad r = 1, \ldots, p.$

So $A_i \cong \Gamma$ as N-modules $1\leq i\leq p$.

Moreover $A_i \cap A_k = (0) \quad i \neq k$

$$N = A_i \oplus \left[(\gamma_i)_r \cap \{(A_k) \oplus (\gamma_k)_r\}\right] \qquad i \neq k$$

$$= A_i \oplus A_k \oplus \left[(\gamma_i)_r \cap (\gamma_i)_r\right]$$

Thus $N = A_1 \oplus A_2 \oplus \ldots \oplus A_p$ and we have found a composition series for N, namely

$$N = A_1 \oplus A_2 \oplus \ldots \oplus A_p \supset A_1 \oplus A_2 \oplus \ldots \oplus A_{p-1} \supset A_1 \oplus A_2 \oplus \ldots \oplus A_{p-2} \supset \ldots \supset A_1 \oplus A_2 \supset A_1 \supset (0)$$

The factor terms

$$A_1 \oplus A_2 \oplus \ldots \oplus A_k \Big/ A_1 \oplus A_2 \oplus \ldots \oplus A_{k-1} \cong \Gamma \text{ as } N\text{-modules},$$

i.e. they are N-modules of type 2 .

Hence N has d.c.c. on right ideals

2.7 We may now collect our results together in a similar form to the well known, corresponding theorem, in ring theory.

<u>Theorem.</u> (Wielandt, Laxton). The following two statements are equivalent

(i) N is a 2-primitive near-ring with identity and d.c.c. on right ideals.

(ii) N is a primitive artinian ring or N is isomorphic to $\gamma\gamma\gamma_G(\Gamma)$ where Γ is an additive group, G is a group of regular automorphisms of Γ, which induce a finite number of orbits on Γ.

Remark.

N^+ is abelian if and only if Γ^+ is abelian, and so we know the abelian 2-primitive near-rings with 1 and d.c.c. on right ideals.

2.8 A well-known theorem on finite groups gives us the following corollary.

Corollary. If N is a finite 2-primitive near-ring with identity and if (N,+) is not nilpotent then

$$N \cong \gamma\gamma\gamma(\Gamma).$$

Proof. The centralizer of Γ consists of the zero endomorphism and regular automorphisms. $(\Gamma,+)$ is not nilpotent and so we apply the result of Thompson [1] which tells us that only the identity automorphism & the zero endomorphism can be in the centralizer of Γ. Thus $N \cong \gamma\gamma\gamma_G(\Gamma) = \gamma\gamma\gamma(\Gamma)$ as $G = \{0,1\}$.

§3. The connection with simple near-rings.

In what way is the structure of 2-primitive near-rings with identity and d.c.c. on right ideals connected with simple near-rings?

3.1 Theorem. A near-ring with descending chain condition on right N^+-groups and an identity is simple iff it is 2-primitive.

<u>Proof</u>. (After Laxton [1]). If N is simple, by the chain condition there is K, a non-zero N^+-subgroup which has no proper N^+-subgroups contained in it except (0). Then K is an N-module of type 2.

$(K)_r$ is an ideal and so $(K)_r = (0)$ or N.

But $1 \varepsilon N$ and $K.1 = K$ so $(K)_r = (0)$. Thus N is 2-primitive.

If N is 2-primitive, let $0 \neq T$, be an ideal of N.

$$N = A_1 \oplus A_2 \oplus \ldots \oplus A_p \text{ where } A_i = \bigcap_{\substack{j=1 \\ j \neq i}}^{p} (\gamma_j)_r$$

and the $\gamma_1, \ldots, \gamma_p$ are representatives of different orbits of Γ.

Each $A_i \cong \Gamma$ as N-modules.

Let $0 \neq x \varepsilon T$, then $x = a_1 + a_2 + \ldots + a_p$

$(a_i \varepsilon A_i \;;\; 1 \leq i \leq p)$

Since A_i is of type 2, $A_i = a_i N$ $1 \leq i \leq p$. Assume $a_1 \neq 0$.

Now $1 \varepsilon N$ so $1 = e_1 + e_2 + \ldots + e_p$ and

$A_i = e_i N$ $1 \leq i \leq p$. Where $e_i \varepsilon A_i$ $(i = 1, \ldots, p)$

But $e_j = (e_1 + e_2 + \ldots + e_p)e_j = e_1 e_j + e_2 e_j + \ldots + e_p e_j$

(direct sum)

where $e_i e_j \varepsilon A_i$ $1 \leq i \leq p$.

Thus $e_i e_j = 0$ $i \neq j$ and $e_j e_j = e_j$.

Hence $e_1 . x = e_1 a_1 + e_1 a_2 + \ldots + e_1 a_p$

now $a_i \varepsilon\, e_i N$ so $e_1 a_i = 0$ if $i \neq 1$ and $e_1 a_1 = a_1$

Therefore $e_1x = a_1 + 0 + . . + 0 = a_1 \neq 0$. Thus

$a_1 \varepsilon A_1 \cap T$ as $e_1x \varepsilon T$. But $A_1 \cap T = (0)$ or A_1

so $A_1 \cap T = A_1$ ie. $A_1 \subseteq T$.

Suppose $(A_2)_r \neq (0)$. Then $A_2y = 0 \Rightarrow \Gamma y = 0$

since $A_2 \cong \Gamma$, and if $\rho: \Gamma \to A_2$ is the isomorphism then

$(\gamma y)\rho = (\gamma\rho)y = 0, \forall \gamma \varepsilon \Gamma$. This contradicts $(\Gamma)_r = (0)$.

Therefore $A_2A_1 \neq (0)$. and $\exists$ $a'_2 \varepsilon A_2$ such that

$a'_2A_1 \neq (0)$ now $a'_2A_1 \subseteq A_2$ so $a'_2A_1 = A_2$

We have $A_1 \subseteq T$

and so $A_2 = a'_2 A_1 \subseteq a'_2T \subseteq T$

This may be repeated for $A_3, ... , A_p$

Thus $A_1, A_2, ... , A_p$ are all contained in T

Then $N \subseteq T$ and so N is simple

-

3.2 We have to restrict ourselves to near-rings with descending chain condition on rt.N^+-groups so that we can show that a simple near-ring is primitive. The other way round does not need this restriction, only d.c.c. on right ideals.

3.3 The questions we could now ask are concerned with relaxing conditions needed for the main theorem 2.7. Could we, for example, relax the chain condition and just insist on the existence of some minimal right ideal? Could we look at 0-primitive instead of 2-primitive near-rings? Neither of these questions have been completely investigated, we will consider the second question later on.

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§4. The decomposition of a near-ring N, with $J_2(N) = (0)$ and d.c.c. on right ideals

We now use some of the results in this chapter in a more general setting.

4.1 We remarked in 2.4.3 that the radical $J_2(N)$ is the intersection of the 2-primitive ideals of N.

We notice the following result.

4.2 Theorem. If N has d.c.c. on right ideals and an identity, and P is a 2-primitive ideal, then P is maximal as an ideal of N.

Proof. $N/_P$ is 2-primitive and so simple, thus P is a maximal ideal in N.

4.3 Theorem. Let N have an identity and descending chain condition on right ideals. If $J_2(N) = (0)$, then N is a direct sum of 2-primitive ideals of N.

Proof. $(0) = J_2(N) = \bigcap_{P \in \mathcal{P}_2} P$ where $\mathcal{P}_2$ is the set of all 2-primitive ideals of N.

With d.c.c. on right ideals we can find 2-primitive ideals $P_1, \ldots, P_k$ such that

$\bigcap_{i=1}^{k} P_i = J_2(N) = (0)$, and no proper subset of the P_i has zero intersection.

For each $1 \leq i \leq k$, define $Q_i = P_1 \cap \ldots \cap P_{i-1} \cap P_{i+1} \cap \ldots \cap P_k$

Then $P_i \cap Q_i = (0)$ $1 \leq i \leq k$. and $Q_i \neq (0)$.

Each P_i is maximal as an ideal in N by 4.2.

$(0) \neq Q_i \subseteq P_i \Rightarrow J_2(N) \neq (0)$ so we see immediately that

$N = P_i \oplus Q_i$ $1 \leq i \leq k$.

In particular $N = P_1 \oplus Q_1$

Also $= (N \cap P_1) \oplus Q_1 = [(P_2 \oplus Q_2) \cap P_1] \oplus Q_1$

$= (P_2 \cap P_1) \oplus (Q_2 \cap P_1) \oplus Q_1$

$= (P_2 \cap P_1) \oplus Q_2 \oplus Q_1$

Continuing $N = (P_3 \cap P_2 \cap P_1) \oplus Q_1 \oplus Q_2 \oplus Q_3$ etc.

until $N = (P_1 \cap P_2 \ldots \cap P_k) \oplus Q_1 \oplus Q_2 \oplus \ldots \oplus Q_k$

$= Q_1 \oplus Q_2 + \ldots + Q_k$ since $P_1 \cap P_2 \cap \ldots \cap P_k =$

$J_2(N) = (0)$.

Now $N = Q_i \oplus P_i$. Since P_i is 2-primitive,

$N/_{P_i}$ is a 2-primitive near-ring so $\exists$ an $N/_{P_i}$-module Γ_i of type 2

with $(\Gamma_i)_r = (\bar{0})$,

$\Gamma_i = \gamma_i(N/_{P_i})$ for all $0 \neq \gamma_i \epsilon \Gamma_i$

Let $\gamma_i \epsilon \Gamma_i$ and define $\gamma_i q_i = \gamma_i(q_i + P_i) \epsilon \Gamma_i$ for $q_i \epsilon Q_i$

Then Γ_i is a Q_i-module.

$\Gamma_i q_i = (0) \Rightarrow \Gamma_i(q_i + P_i) = (0) \Rightarrow q_i \epsilon P_i \Rightarrow q_i = 0$.

If $0 \neq \gamma_i \in \Gamma_i$ then $\gamma_i Q_i = \gamma_i (N/_{P_i}) = \Gamma_i$

Hence Q_i is 2-primitive as a near-ring.

-

4.4 Theorem. If N possesses d.c.c. on right ideals and an identity then $J_2(N) = (0)$ if and only if N is the direct sum of ideals Q_i $1<i<k$, which, as near-rings are of the form $Q_i \cong \Upsilon\Upsilon_{G_i}(\Gamma_i)$ $1\leq i\leq k$ where Γ_i is an additive group and G_i is a group of regular automorphisms of Γ_i, inducing finite number of orbits on $\Gamma_i (1\leq i\leq k)$, or are primitive rings.

-

This is due to Wielandt; and Laxton [1].

4.5 Blackett [1] has shown that in a near-ring N, with d.c.c. on rt.N^+-groups and $J_2(N) = 0$ we have the result that N is a (finite) direct sum of ideals A_i which are simple as near-rings. We thus get the result.

4.6 Theorem. If N has d.c.c. on right N^+-groups and $J_2(N) = (0)$ then N is a direct sum of ideals A_i of the form $A_i \cong \Upsilon\Upsilon_{G_i}(\Gamma_i)$ where Γ_i is an additive group and G_i is a group of regular automorphisms inducing finitely many orbits on Γ_i $(1\leq i\leq k)$; if N has a right identity & the A_i are not rings.

We notice that Blackett's theorem, (4.5), required a stronger chain condition but no right identity was needed. (It automatically had a left identity by (2.6.11).) However we need a right identity in 4.6.

§5. The centre of a near-ring with descending chain condition on rt. ideals.

Suppose N has d.c.c. on right ideals, $J_2(N) = (0)$ and $1 \varepsilon N$.

Let C be the centre of N, i.e.

$C = \{c \varepsilon N \mid nc = cn \ \forall n \varepsilon N\}$.

Clearly $1 \varepsilon C$ and $0 \varepsilon C$.

If $N = A_1 \oplus \ldots \oplus A_k$ where A_i are simple near-rings and ideals of N.

Let $c \varepsilon C$ then $c = c_1 + \ldots + c_k$ $(c_i \varepsilon A_i \ \ 1 \leq i \leq k.)$

Let C_i be the centre of the near-rings A_i $(1 \leq i \leq k.)$

If $n \varepsilon N$, $nc = nc_1 + \ldots + c_k n$ and $nc_i \varepsilon A_i \ 1 \leq i \leq k$.

$nc = cn = (c_1 + .. + c_k)n = c_1 n + .. + c_k n$ (direct sum) so $c_i n = nc_i$ $1 \leq i \leq k$. $\forall n \varepsilon N$

so $c_i a_i = a_i c_i \quad 1 \leq i \leq k. \ \forall a_i \varepsilon A_i$

If $x \varepsilon C \cap A_i$ then $x = 0 + \ldots + x + .. + 0$

and $xa_i = a_i x$ i.e. $x \varepsilon C_i$.

If $c_i \varepsilon C_i$ then $c_i a_j = 0 = a_j c_i$ if $i \neq j$

$nc_i = a_1 c_i + \ldots + a_k c_i = a_i c_i = c_i . n$ if $n = a_1 + \ldots + a_k$.

so $c_i \varepsilon A_i \cap C$, $\therefore$ $C_i = A_i \cap C$.

We note that C and the C_i are multiplicative semigroups.

Suppose $T = C_1 \times C_2 \times \ldots \times C_k =$

$\{(c_1, c_2, \ldots, c_k); \ c_i \varepsilon C_i, \ 1 \leq i \leq k\}$

T is a semigroup under the multiplication,

$(c_1, c_2, \ldots, c_k)\cdot(c'_1, c'_2, \ldots, c'_k) = (c_1c'_1, c_2c'_2, \ldots, c_kc_k')$

Define ψ: $C \to T$ by

fpr $c\varepsilon C$, $c\,\psi = (c_1, c_2, \ldots, c_k)$, where $c = c_1 + c_2 + \ldots + c_k$

and $c_i \varepsilon A_i \cap C = C_i$ $(1 \leq i \leq k)$.

ψ is a semigroup isomorphism of C onto T.

Now for $1 \leq i \leq k$, each A_i has an identity and so

$A_i \cong \gamma\gamma\gamma_{G_i}(\Gamma_i) = L_i$ as near-rings, where

Γ_i is an additive group and G_i a group of regular automorphisms

of Γ_i inducing a finite number of orbits on Γ_i.

Let $0 \neq \rho$ be a mapping of $\Gamma_i \to \Gamma_i$ which commutes with every

element of L_i. i.e. $\rho\ell_i = \ell_i\rho$ $\forall\, \ell_i \varepsilon L_i$

We pick $0 \neq \gamma_i \varepsilon \Gamma_i$, then γ_i and $\gamma_i\rho$ lie in the same orbit,

otherwise $\exists$ $\ell_i \varepsilon L_i$ such that

$\gamma_i\ell_i = 0$ and $\gamma_i\rho\ell_i \neq 0$ (by density theorem).

so $0 \neq \gamma_i\ell_i\rho = 0$ a contradiction

And so $\exists\, g_i \varepsilon G_i$ such that $\gamma_i\rho = \gamma_i g_i$.

For any $x_i \varepsilon \Gamma_i$, $\exists$ $b_i \varepsilon L_i$ such that $\gamma_i b_i = x_i$

so $x_i\rho = (\gamma_i b_i)\rho = (\gamma_i\rho)b_i = (\gamma_i g_i)b_i = (\gamma_i b_i)g_i = xg_i$

and hence ρ: $\Gamma_i \to \Gamma_i$ is simply a mapping of Γ obtained by

a right multiplication by an element of G_i. $x_i\rho = x_i g_i$ $\forall\, x_i \varepsilon \Gamma_i$.

Now let $\rho: \Gamma_i \to \Gamma_i$ be defined by $x_i\rho = x_i g_i \ \forall x_i \epsilon \Gamma_i$.

and a particular fixed $g_i \epsilon G_i$. Pick any $g_i' \epsilon G_i$, $x_i \epsilon \Gamma_i$,

Then $(x_i g_i')\rho = x_i g_i' g_i$ so for $\rho \epsilon \mathcal{M}_{G_i}(\Gamma_i)$

we must have

$(x_i g_i'\rho) = (x_i \rho g'_i) \quad \forall g_i' \epsilon G_i, \forall x_i \epsilon \Gamma_i$

i.e. $x_i g_i' g_i = x_i g_i' g_i \ \forall x_i \epsilon \Gamma, \ \forall \ g_i' \epsilon G_i$

i.e. $g_i' g_i = g_i' g_i$ i.e. g_i ϵ centre of G_i

Thus if $\rho \epsilon$ centre of L_i then $\rho: \Gamma_i \to \Gamma_i$ is defined by

$\forall x_i \epsilon \Gamma_i$, $x_i \rho = x_i g_i$ where $g_i \epsilon$ centre of G_i.

If $\rho: \Gamma_i \to \Gamma_i$ is defined by $\forall x_i \epsilon \Gamma_i$, $x_i \rho = x_i g_i$ where $g_i \epsilon$ Centre G_i

then $\rho \epsilon L_i$ and in fact $\rho \epsilon$ centre L_i

We can state the following result.

5.2 Theorem. If N has identity and d.c.c. on right ideals and $J_2(N) = (0)$ then if C = Centre of N and $N = A_1 \oplus \ldots \oplus A_k$ (A_i simple n.rings) then $C \cong (C_1 \times C_2 \times \ldots \times C_k)$ as mult.semigroups where C_i = centre A_i. $(1 \leq i \leq k)$ and if $A_i \cong \mathcal{M}_{G_i}(\Gamma_i)$ as near-rings $(1 \leq i \leq k)$,
then there is a group isomorphism between Δ_i = centre of G_i and D_i = centre of L_i

(These are commutative multiplicative groups).

5.3 Corollary. The centre of a near-ring N with identity, d.c.c. on right ideals, $J_2(N) = (0)$, is a multiplicative group.

§6. When are two N-modules of type 2, isomorphic in a 2-primitive near-ring?

6.1 Theorem. If N is a 2-primitive near-ring with identity and d.c.c. on right ideals then any two N-modules of type 2 are N-isomorphic. We need first a lemma.

6.2. Lemma If N is as in the statement of the theorem let Γ and Δ be N-modules of type 2, Then Γ and Δ are N-isomorphic, if they are faithful.

Proof. $N = A_1 \oplus \ldots \oplus A_k$ where A_i are of type 2, and rt.ideals so $A_i \cong N/(\gamma_i)_r$ for some $\gamma_i \varepsilon \Gamma$. $(1 \leq i \leq k.)$

$\Delta \cong N/(\delta)_r$ for some $\delta \varepsilon \Delta$. Then $(\delta)_r$ max.rt.ideal of N.

$N \neq (\delta)_r = N \cap (\delta)_r = A_1 \cap (\delta)_r \oplus \ldots \oplus A_k \cap (\delta)_r$,

where $A_i \cap (\delta)_r = (0)$ or A_i. Suppose $(\delta)_r = A_1 \oplus \ldots \oplus A_p$, $p<k$.

Then as $N/(\delta)_r \cong A_{p+1} \oplus \ldots \oplus A_k$ we must have $p+1 = k$.

Thus $N/(\delta)_r \cong A_k \cong N/(\gamma_k)_r$

If $(\gamma_k)_r \nsubseteq (\delta)_r$ then $N = (\delta)_r + (\gamma_k)_r$ so

$$((\delta)_r + (\gamma_k)_r) / (\delta)_r \cong ((\delta)_r + (\gamma_k)_r) / (\gamma_k)_r$$

i.e. $(\gamma_k)_r \Big/ (\delta)_r \cap (\gamma_k)_r \quad \cong \quad (\delta)_r \Big/ (\delta)_r \cap (\gamma_k)_r$

And so $(\gamma_k)_r = (\delta)_r$.

Define $\psi: \Gamma \rightarrow \Delta$ by

$(\gamma_k n)\psi = \delta n \ \forall n \varepsilon N.$

Then $\gamma_k n = 0 \Leftrightarrow \delta n = 0.$

Clearly ψ is an N-isomorphism of Δ and Γ.

6.3 <u>Proof of Theorem 6.1</u>

Let Δ be any N-module of type 2.

Then $(\Delta)_r$ is an ideal of N.

N is simple and so $(\Delta)_r = 0$ or N.

$1 \varepsilon N$ means that $(\Delta)_r = 0$

Thus $\Delta \cong \Gamma$ as N-modules where Γ is any other faithful N-module of type 2. by lemma 6.2.

Chapter 4. 0-primitive near-rings with identity and d.c.c. on right ideals (I)

There exists a large class of finite near-rings which are 0-primitive and yet not 2-primitive, in fact the radical J_2 is not only non-zero but contains an idempotent (non-zero) element. Naturally this situation is completely alien to the ring theoretic case, but it is possible to weaken the hypothesis of the Density theorem in order that it becomes valid for arbitrary 0-primitive near-rings with identity and d.c.c. on right ideals.

§1. A Density Theorem for 0-primitive near-rings.

We let N be a 0-primitive near-ring with identity 1 and d.c.c. on right ideals. Γ is a faithful N-module of type 0.

1.1 Definition. $C = \{\gamma \in \Gamma : \gamma N = \Gamma\}$ i.e. the set of cyclic generators of Γ.

$\Delta = \Gamma \setminus C$ or the set of 'non-generators' of Γ.

1.2 We will use the following fundamental theorem which, like the rest of this section, is due to Wielandt and Betsch, it has not, as yet, appeared in a published work.

Theorem (Wielandt-Betsch)

Let D and E be right ideals of N with the property that

$D \cap E \subseteq (\gamma)_r$, $D \not\subseteq (\gamma)_r$, $E \not\subseteq (\gamma)_r$ for some $\gamma \in C$

Then N is a ring.

1.3 Definition. G = Group of all N-automorphisms of Γ.

1.4 Lemma. (i) G acts as a fixed-point-free automorphism group on C

(ii) if $g \in G$ and $g \neq$ identity auotmorphism, then the fixed points of g form an N-subgroup contained in Δ.

Proof. (i) if $\gamma \in C$ and $\gamma g = \gamma$ then $\gamma N = \Gamma$ so $\gamma n = (\gamma g)n = \gamma n g$ i.e. g is identity auotmorphism on Γ.

(ii) Let $g \in G$. $g \neq$ identity automorphism.

Put $F_g = \{\gamma \in \Gamma \mid \gamma g = \gamma\}$.

Then $\gamma_1, \gamma_2 \in F_g, \Rightarrow (\gamma_1 - \gamma_2)g = \gamma_1 g - \gamma_2 g = \gamma_1 - \gamma_2$

so $\gamma_1 - \gamma_2 \in F_g$. F_g is an additive group

$(\gamma_1 n)g = (\gamma_1 g)n = \gamma_1 n$ so F_g is an N-group of Γ.

Clearly $F_g \cap C = \phi$ by (i).

1.5. We can define an equivalence relation $\sim$ on the elements of C. If $\gamma, \gamma_1 \in C$ then

$\gamma \sim \gamma_1 \Leftrightarrow \gamma_1 = \gamma g$ for some $g \in G$.

1.6 Proposition. $\gamma \sim \gamma_1 \Leftrightarrow (\gamma)_r = (\gamma_1)_r$ (for $\gamma, \gamma_1 \in C$).

Proof If $\gamma_1 = \gamma g$ then

$(\gamma_1 n) = 0 \Leftrightarrow (\gamma g)n = 0 \Leftrightarrow \gamma n g = 0 \Leftrightarrow \gamma n = 0$

If $(\gamma)_r = (\gamma_1)_r$ define $g : \Gamma \to \Gamma$ by

$\forall n \in N$, $(\gamma n)g = \gamma_1 n$; $g \in G$ and so $1 \in N$ gives $\gamma g = \gamma_1$.

1.7. Proposition. If $\gamma \in C$ then $(\gamma)_r$ is a right ideal of N, maximal as a right ideal.

Proof $\Gamma \cong N/(\gamma)_r$ as N-modules, and Γ is of type 0

and so possesses no proper N-submodules, except (0).

1.8 Theorem Density theorem for 0-primitive near-rings with identity (Wielandt-Betsch). Suppose N is not a ring.

Let $\gamma_1, \ldots, \gamma_k \varepsilon C$ such that $\gamma_i \sim \gamma_j \Rightarrow i = j$

If $\gamma_1', \ldots, \gamma_k'$ are arbitrary elements of Γ, then there exists an $n \varepsilon N$ such that $\gamma_i' = \gamma_i n$; $1 \leq i \leq k$

Proof. By Theorem I.2.

§2. The theory for the generators of Γ.

From this last theorem it is clear that we are in a position to find out what happens to elements of C, but since the theorem tells us nothing about the behaviour of elements of Δ we are very severely restricted. Thus it is in Δ, the set of non-generators, that we have to make added assumptions. In general it is not obvious that Δ has any algebraic structure; it might not always be closed under addition for example. We make the following assumptions.

2.1 Assumptions In this section N will denote a near-ring with identity and d.c.c. on right ideals such that

a) N is not a ring

b) N is 0-primitive and not 2-primitive

c) If Γ is the faithful N-module of type 0, and $\Delta = \Gamma \setminus C$, the set of non-generators of Γ, then Δ is an N-module of type 2.

2.2 Proposition If $\gamma \varepsilon C$ then $(\Delta)_r \not\subseteq (\gamma)_r$

Proof If $(\Delta)_r \subseteq (\gamma)_r$ then

$\Gamma(\Delta)_r = \gamma N.(\Delta)_r \subseteq \gamma.(\Delta)_r$ since $(\Delta)_r$ is an ideal as Δ is an N-module.

Thus $\Gamma.(\Delta)_r = (0)$ contradicting $(\Delta)_r \neq (0)$ since if $(\Delta)_r = (0)$ then N would be 2-primitive.

2.3 <u>Proposition.</u> If $\gamma\varepsilon C$ then $N = (\gamma)_r + (\Delta)_r$.

<u>Proof</u> By 2.2 and 1.7

2.4 <u>Theorem.</u> If $\gamma_1, \ldots, \gamma_n \varepsilon C$ and $\gamma_i \sim \gamma_j \Rightarrow i = j$, let $\gamma_1', \ldots, \gamma_n' \varepsilon \Gamma$. Then $\exists\, x\varepsilon(\Delta)_r$ such that

$\gamma_i' = \gamma_i x\,;\ 1 \leq i \leq n.$

<u>Proof.</u> We proceed by induction on n.

If $n = 1$. $\gamma_1 \varepsilon C$ and $(\Delta)_r \not\subseteq (\gamma_1)_r$ so

$\gamma_1.(\Delta)_r \neq (0)$, but $\gamma_1.(\Delta)_r$ is an N-submodule of Γ and so

$\gamma_1.(\Delta)_r = \Gamma$ i.e. $\gamma_1' = \gamma_1 x$ for some $x\varepsilon(\Delta)_r$.

Assume the result is true for $n = k-1$ ($k>1$).

Let $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}, \gamma_k \varepsilon C$, $\gamma_i \sim \gamma_j \Rightarrow i = j$

By the inductive hypothesis, if $\bar{\gamma}_k \neq 0$,

$\exists\, \bar{x}\varepsilon(\Delta)_r$ such that

$0 = \gamma_1\bar{x} = \gamma_2\bar{x} = \ldots = \gamma_{k-2}\bar{x}$ and $\gamma_k\bar{x} = \bar{\gamma}_k \neq 0.$

Then $\left[(\Delta)_r \cap (\gamma_1)_r \cap (\gamma_2)_r \cap \ldots \cap (\gamma_{k-2})_r\right] \not\subseteq (\gamma_k)_r$

Also $(\gamma_{k-1})_r \not\subseteq (\gamma_k)_r$ as $\gamma_{k-1} \not\sim \gamma_k$ and both

$(\gamma_{k-1})_r$ and $(\gamma_k)_r$ are maximal as right ideals of N.

We now apply Theorem 1.2, with the result that, as N is not a ring,

$\left[(\Delta)_r \cap (\gamma_1)_r \cap \ldots \cap (\gamma_{k-2})_r \cap (\gamma_{k-1})_r\right] \not\subseteq (\gamma_k)_r.$

Hence $\exists\ x_1 \varepsilon\left[(\Delta)_r \cap (\gamma_1)_r \cap \cdots \cap (\gamma_{k-2}) \cap (\gamma_{k-1})_r\right]$ and $\gamma_k x_1 \neq 0$.

Now $\gamma_k\left[(\Delta)_r \cap (\gamma_1)_r \cap \cdots \cap (\gamma_{k-2})_r \cap (\gamma_{k-1})_r\right] = \Gamma$

and so $\gamma_k' = \gamma_k t$ for some $t\ \varepsilon\left[(\Delta)_r \cap (\gamma_1)_r \cap \cdots \cap (\gamma_{k-1})_r\right]$.

Also by the inductive hypothesis $\exists x'$ such that

$\gamma_i x' = \gamma_i'$ for $i = 1, \ldots, k-1$ and $x' \varepsilon (\Delta)_r$

Also $\gamma_k x' = {}_k t'$ for some $t' \varepsilon\left[(\Delta)_r \cap (\gamma_1)_r \cap \cdots \cap (\gamma_{k-2})_r \cap (\gamma_{k-1})_r\right]$

Put $x = x' - t' + t$. Then $x \varepsilon (\Delta)_r$.

$$\gamma_i x = \gamma_i x' - \gamma_i t' + \gamma_i t = \gamma_i' - 0 + 0 = \gamma_i'. \quad 1 \leq i \leq k-1$$

$\gamma_k x = \gamma_k x' - \gamma_k t' + \gamma_k t = \gamma_k t' - \gamma_k t' + \gamma_k'$.

Hence $\gamma_j x = \gamma j'$ for $1 \leq j \leq k$.

The theorem follows by the principle of induction.

2.5 We may turn Δ into a $N/_{(\Delta)_r}$-module in the following way.

Let $n + (\Delta)_r \varepsilon N/_{(\Delta)_r}$ and $\delta \varepsilon \Delta$

If we define $\delta(n + (\Delta)_r) = \delta n$, then Δ is a $N/_{(\Delta)_r}$-module of type 2.

Clearly Δ is faithful with respect to the near-ring $N/_{(\Delta)_r}$

Thus $N/_{(\Delta)_r}$ is a 2-primitive near-ring. It is quite possible

that $N/_{(\Delta)_r}$ is in fact a ring and we will have to consider this possibility separately.

Let $g \varepsilon G = Aut_N(\Gamma)$. Then $\Delta g \subseteq \Delta$ and if we denote by g^{Δ} the map g restricted to the group Δ, we see that the elements of Δ which remain fixed under the action of g^{Δ} must form an N-subgroup of Δ. (Prop. 1.4)

Thus N-subgroup must be (0) or Δ. Thus for each $g \varepsilon G$, g^{Δ} acts identically on Δ or is regular on Δ. Thus g acts regularly on Γ, or regularly on C and identically on Δ.

2.6. We now introduce a compatibility criterion connecting the groups Γ, Δ and G which is necessarily satisfied in our present situation. Since Γ has no N-submodules except (0) and Γ itself, in particular Δ is not an N-submodule of Γ. Thus either

$(\Delta,+)$ is not normal in $(\Gamma,+)$ or

$\exists\, \delta \varepsilon \Delta, \gamma \varepsilon \Gamma, n \varepsilon N$ st. $(\gamma + \delta)n - \gamma n \notin \Delta$.

Suppose that $\forall \gamma \varepsilon C$, and $\forall \delta \varepsilon \Delta$ we can find a $g \varepsilon G$ such that $\gamma + \delta = \gamma g$ then if given a $g \varepsilon G$, such that $\gamma + \delta = \gamma g$ for some $\gamma \varepsilon C$ and $\delta \varepsilon \Delta$, $\bar{\gamma} g - \bar{\gamma} \varepsilon \Delta$ for all $\bar{\gamma} \varepsilon C$, we see immediately that

$(\gamma + \delta)n - \gamma n = \gamma g n - \gamma n = (\gamma n)g - \gamma n \varepsilon \Delta$.

This gives rise to the following definition.

2,7 <u>Definition</u>. Let A be an additive group, B a non-empty subset of A, and H a group of automorphisms of A, such that for each $h \varepsilon H$, $b \varepsilon B$

$$bh \varepsilon B.$$

Then $\{A,B,H\}$ satisfy the <u>compatibility criterion</u> (or <u>are compatible</u>) if and only if:-

not all the following conditions are satisfied.

(i) B is a normal additive subgroup of $(A,+)$

(ii) Given any $x \in A \setminus B$, any $y \in B$, there is an $h \in H$ (where h depends on x and y), such that $x + y = xh$.

(iii) If $h' \in H' = \{h \in H \mid x + y = xh$ for some $x \in A \setminus B$ and $y \in B\}$ then $x'.h' - x' \in B$ for all $x' \in A \setminus B$.

2.8 Proposition. In the situation of this section, $\{\Gamma, \Lambda, G\}$ satisfies the compatability criterion.

S3. The case when $N/_{(\Delta)_r}$ is not a ring.

$N/_{(\Delta)_r}$ has an identity and d.c.c. on right ideals and is 2-primitive. We can use the theory of Chapter 3.

Keeping the same notation and assumptions as section 2 we consider now the case when $N/_{(\Delta)_r}$ is not a ring

We introduce an equivalence relation on Δ.

3.1 Definitions If $\delta_1, \delta_2 \in \Delta$ then we define

$$\delta_1 \sim^* \delta_2 \Leftrightarrow \delta_2 = \delta_1 \bar{g}$$

for some $\bar{g} \in \mathrm{Aut}_{N(\Delta)_r}(\Delta) = \bar{G}$

If $\delta \in \Delta$ then we define

$$(\delta)_r^* = \{\bar{x} \in N/_{(\Delta)_r} \mid \delta\bar{x} = 0\}.$$

Clearly $(\delta)^*_r$ is a right ideal of $N/_{(\Delta)_r}$ and $\sim^*$ is an equivalence relation on Δ since $\bar{G}$ is a group of regular automorphisms of Δ.

In fact if $g\varepsilon G$ then $g^{\Delta}\varepsilon\bar{G}$ for g^{Δ} is an automorphism of Δ and if $\delta\varepsilon\Delta$, $n + (\Delta)_r \varepsilon N/_{(\Delta)_r}$ then

$\delta(n + (\Delta)_r)g^{\Delta} = (\delta n)g^{\Delta} = \delta g^{\Delta} n = (\delta g^{\Delta})(n + (\Delta)_r)$

3.2 <u>Proposition</u> If $\delta_1, \delta_2 \varepsilon\Delta$ then $\delta_1 \sim^* \delta_2 \quad (\delta_1)_r{}^* = (\delta_2)_r{}^*$

3.3 <u>Theorem</u> If $\delta_1, \ldots, \delta_m \varepsilon\Delta$ such that $\delta_i \sim^* \delta_j \Longrightarrow i = j$ and $\delta_1', \ldots, \delta'_m \varepsilon\Delta$, then $\exists\ \bar{y}\varepsilon N/_{(\Delta)_r}$ such that $\delta_i' = \delta_i\bar{y}$, $1 \leq i \leq m$.

If $\bar{y} = y + (\Delta)_r$, $(y\varepsilon N)$, then $\delta'_i = \delta_i y$, $(1 \leq i \leq m)$

<u>Proof</u> This is a straightforward application of the Density theorem for 2-primitive near-rings (Theorem 3.1.15)

3.4 Let $M_o = \mathcal{M}_G(\Gamma)$

Let $M = \{m\varepsilon M_o \mid \Delta m \subseteq \Delta$, and $\delta m\bar{g} = \delta\bar{g}m, \forall\delta\varepsilon\Delta, \forall\bar{g}\varepsilon\bar{G}\}$.

If $m_1, m_2 \varepsilon M$ then $\delta(m_1 - m_2) = \delta m_1 - \delta m_2 \ \varepsilon\Delta$, $\forall\,\delta\varepsilon\Delta$.

$\delta(m_1 - m_2)\bar{g} = \delta m_1\bar{g} - \delta m_2\bar{g} = \delta\bar{g}(m_1 - m_2)$, $\forall\,\delta\varepsilon\Delta$, $\forall\bar{g}\varepsilon\bar{G}$.

$\delta m_1 m_2 \ \varepsilon\Delta, \forall\delta\varepsilon\Delta$. $\delta(m_1m_2)\bar{g} = (\delta m_1)\bar{g}m_2 = \delta\bar{g}m_1m_2$, $\forall\,\delta\varepsilon\Delta$, $\forall\,\bar{g}\varepsilon\bar{G}$.

Thus M is a subnear-ring of M_o.

If N_R is the set of right multiplications of Γ by elements of N then $N \cong N_R$ as near-rings. If $n'\varepsilon N_R$ then $\gamma n' = \gamma n$ for some $n\varepsilon N$, $\forall\ \gamma\varepsilon\Gamma$. $\Delta n' \subseteq \Delta$, $(\delta\bar{g})n' = \delta\bar{g}n = \delta\bar{g}(n + (\Delta)_r)$

$= \delta(n + (\Delta)_r)\bar{g} = \delta n\bar{g} = \delta n'\bar{g}$ Thus $N_R \subseteq M$.

3.5 Since we have d.c.c. on right ideals both in N and $N/_{(\Delta)_r}$, by 1.2, we see that there must be a finite number of equivalence classes on G with respect to $\sim$ and on Δ with respect to $\sim^*$. The non-zero equivalence classes will be referred to respectively as the orbits induced on C by G and on Δ by $\bar{G}$. If there are h orbits on C induced by G and t orbits on Δ induced by $\bar{G}$, we will find representatives of each orbit. Let these be $\gamma_1, \ldots, \gamma_h$ on C and $\delta_1, \ldots, \delta_t$ on Δ.

Thus $C = \bigcup_{i=1}^{h} \gamma_i G$ and $\Delta = \bigcup_{j=1}^{t} \delta_j \bar{G}$

Thus $\Gamma = \{0\} \cup (\bigcup_{i=1}^{h} \gamma_i G) \cup (\bigcup_{j=1}^{t} \delta_j \bar{G})$.

If $m \in M$, then pick any $0 \neq \gamma \in \Gamma$. Either $\gamma \in C$ or $\gamma \in \Delta$ and so

either $\gamma = \gamma_i g$ for some $g \in G$ and $i \in \{1, \ldots, h\}$

or $\gamma = \delta_j \bar{g}$ for some $\bar{g} \in G$ and $j \in \{1, \ldots, t\}$

Then γm equals $\gamma_i mg$ or $\delta_j m\bar{g}$ and consequently if we know the values of $\gamma_1 m, \ldots, \gamma_h m, \delta_1 m, \delta_2 m, \ldots, \delta_t m$, we can then determine the value of γm for an arbitrary $\gamma \in \Gamma$.

$(0.m = 0)$.

3.6 <u>Theorem.</u> $N \cong M$ as near-rings.

<u>Proof</u>. We already know that $N \cong N_R \subseteq M$

Let $m \in M$, put $\gamma_i m = \gamma_i'$. $1 \leq i \leq h$ and

$\delta_j m = \delta_j'$, $1 \leq j \leq t$.

From theorem 3.3. $\exists y \in N$ s.t. $\delta_j y = \delta_j'$, $1 \leq j \leq t$.

Let $\gamma_i y = \gamma_i''$ for $1 \leq i \leq h$.

By theorem 2.4 $\exists$ $x\varepsilon(\Lambda)_r$ such that

$\gamma_i x = \gamma_i' - \gamma_i''$, $1\leq i\leq h$.

Put $n = x + y$

then $\gamma_i n = \gamma_i x + \gamma_i y = \gamma_i' - \gamma_i'' + \gamma_i'' + \gamma_i'' = \gamma_i'$, $1\leq i\leq h$

and $\delta_j n = \delta_j x + \delta_j y = 0 + \delta_j'$, $1\leq j\leq t$

Thus the right multiplication by n of elements of Γ is equivalent as a mapping to m.

Thus $m = n'$ where $n' : \Gamma\to\Gamma$ defined by

$\gamma n' = \gamma n \ \forall \gamma\varepsilon\Gamma$. $n'\varepsilon N_R$

Thus $M \subseteq N_R$ and so $M = N_R \cong N$.

3.7 To summarize, if N is a 0-primitive and not 2-primitive near-ring with identity and d.c.c. on right ideals such that the set Δ of non-generators of Γ is an N-module of type 2, and $N/_{(\Delta)_r}$ is not a ring then N is (isomorphic to) the near-ring of all mappings of Γ into itself; commuting with all the N-automorphisms of Γ; which take the group Δ into itself and commute with the centralizer in $N/_{(\Delta)_r}$ of Δ.

Naturally it is quite possible that in a finite near-ring both these centralizers are just the zero endomorphism. Then the near-ring is the set of all zero-preserving maps of Γ into itself that take Δ into itself.

Obviously N will be a subnear-ring of the near-ring of mappings of Γ into itself commuting with the centralizer in N of Γ.

§4. The case when $N/(\Delta)_r$ is a ring.

In this section $N/(\Delta)_r$ will be assumed to be a ring; in particular it is a primitive, artinian ring. It has a faithful, (ring) $N/(\Delta)_r$-module Δ, which must therefore be an abelian additive group

4.1 Proposition. $C_{N/(\Delta)_r}(\Delta)$ is a division ring, and Δ is a finite dimensional vector space over $C_{N/(\Delta)_r}(\Delta) = D$

(Jacobson [1]. Chapter 2.)

4.2 Theorem. If $\delta_1, \delta_2, \ldots \delta_m \epsilon \Delta$ and are linearly independent with respect to D, and $\delta_1', \delta_2', \ldots, \delta_m' \epsilon \Delta$

Then $\exists \bar{y} \epsilon N/(\Delta)_r$ such that

$$\delta_i' = \delta_i y \quad . \quad 1 \leq i \leq m.$$

If $\bar{y} = y + (\Delta)_r$ for some $y \epsilon N$, then

$\delta_i' = \delta_i \bar{y} = \delta_i(y + (\Delta)_r) = \delta_i y \quad 1 \leq i \leq m.$

Proof. This is an application of the density theorem for rings (Jacobson [1]. P. 28)

4.3 Let $M_o = \mathcal{M}_G(\Gamma)$

If $M = \{m \epsilon M_o \mid \Delta m \subseteq \Delta, \ \delta m d = \delta d m, \forall \delta \epsilon \Delta, \forall d \epsilon \bar{D}$ and $(\delta' + \delta'')m = \delta' m + \delta'' m, \forall \delta', \delta'' \epsilon \Delta\}$

then M is a subnear-ring of M_o

If N_R is set of right multiplications of Γ by elements of N

$N_R \subseteq M$ if we can show that $(\delta + \delta')n = \delta n + \delta' n \;\; \forall n \in N, \forall \delta, \delta' \in \Delta$.

Now if $n \in (\Delta)_r$ then $(\delta + \delta')n = \delta n + \delta' n$.

If $n \notin (\Delta)_r$, put $\bar{n} = n + (\Delta)_r$.

Then $(\delta + \delta')(n + (\Delta)_r) = \delta(n + (\Delta)_r) + \delta'(n + (\Delta)_r)$ since

Δ is a ring module with respect to $N/(\Delta)_r$.

thus $(\delta + \delta')n = \delta n + \delta' n$.

Hence $N_R \subseteq M$.

4.4 We have d.c.c. on right ideals of N and of $N/(\Delta)_r$.

Hence we have a finite number of equivalence classes on C and Δ is a finite dimensional vector space over D. Suppose we have h equivalence classes on C and the dimension of Δ with respect to D is t.

Then $C = \bigcup_{i=1}^{h} \gamma_i G$ and $\Delta = \bigoplus_{j=1}^{t} \delta_j D$.

for suitable orbit representatives γ_i of C and D-basis δ_j of Δ. Clearly if $m \in M$ then when $\gamma_1 m, \ldots, \gamma_h m, \delta_1 m, \ldots, \delta_t m$ are known, then the action of m on an arbitrary element of Γ is determined.

4.5 Theorem. $N \cong M$ as near-rings.

Proof. We show that $M \subseteq N_R$.

Let $m \in M$ and put $\delta_j' = \delta_j m$, $1 \leq j \leq t$

and $\gamma_i' = \gamma_i m$. $1 \leq i \leq h$.

Clearly by what has preceeded we can find an $n \in N$ such that

$\gamma_i' = \gamma_i n = \gamma_i m$, $1 \leq i \leq h$

and $\delta_j' = \delta_j n = \delta_j m$, $1 \leq j \leq t$

Hence $M \subseteq N_R$ so $M = N_R$.

4.6 If N is a 0-primitive, and not 2-primitive, near-ring with identity and d.c.c. on right ideals such that the set Δ of non-generators of Γ is an N-module of type 2 and $N/(\Delta)_r$ is a ring, then N is (isomorphic to) the near-ring of all mappings of Γ into itself commuting with all the N-automorphisms of Γ; which take the group Δ into itself and are homomorphisms on Δ commuting with the centralizer of Δ in $N/(\Delta)_r$.

Here again N is a subnear-ring of $M_0 = \gamma\gamma\gamma_G(\Gamma)$ where $G = Aut_N(\Gamma)$

§5. The converse and final classification

If we denote by m^Δ the restriction of the map $m : \Gamma \to \Gamma$ to the subset Δ we can rewrite the results of the previous two sections.

5.1 Thoeren If N is 0-primitive, but not 2-primitive, with identity and d.c.c. on right ideals, satisfies 2.1,

then $N \simeq \{n \in \gamma\gamma\gamma_G(\Gamma) \mid m^\Delta \in \gamma\gamma\gamma_{\bar{G}}(\Delta)\}$

if $N/(\Delta)_r$ is not a ring.

$N = \{m \varepsilon \gamma\gamma\gamma_{G}(\Gamma) \mid n^{\Delta} \varepsilon \operatorname{Hom}_{D}(\Delta)\}$

of $N/_{(\Delta)_r}$ is a ring.

Where $G = \operatorname{Aut}_N(\Gamma)$, $\bar{G} = \operatorname{Aut}_{N/(\Delta)_r}(\Delta)$ and $D = (0) \cup \bar{G}$.

We now investigate the validity of the converse.

5.2 Theorem. If Γ is an additive group and $(0) \neq \Delta$ a subgroup of Γ. Suppose $\bar{G}$ is a group of regular automorphisms of Δ which induce a finite number of orbits on Δ, and G is a group of automorphisms of Γ such that

(i) $\{\Gamma, \Delta, G\}$ are compatible.

(ii) each element of G is regular on $\Gamma \setminus \Delta$

(iii) G induces a finite number of orbits on $\Gamma \setminus \Delta$

(iv) the restriction g^{Δ} of any $g \varepsilon G$, to Δ, is in $\bar{G}$.

Then the near-ring

$N = \{m \varepsilon \gamma\gamma\gamma_G(\Gamma) \mid m^{\Delta} \varepsilon \gamma\gamma\gamma_{\bar{G}}(\Delta)\}$

is 0-primitive, is not 2-primitive, has an identity and d.c.c. on right ideals.

Proof. If Δ is not normal in Γ then Δ is not an N-submodule of Γ.

If $\exists\ \gamma \varepsilon C = \Gamma \setminus \Delta, \delta \varepsilon \Delta$, such that $\gamma + \delta \notin \gamma G$, then if $\gamma + \delta \varepsilon \gamma_{\delta} G$ where $\gamma_{\delta} G \cap \gamma G = \phi$, some $\gamma_{\delta} \varepsilon (\Gamma \setminus \Delta)$.

define $n \varepsilon N$ by $(\gamma + \delta) g n = (\gamma + \delta) g$

$$\gamma' n = 0 \quad \text{all } \gamma' \varepsilon \Gamma \setminus (\gamma + \delta) G$$

then $\gamma n = 0$ since $[(\gamma + \delta)G] \cap \gamma G = \phi$

so $(\gamma + \delta)n - \gamma n = \gamma + \delta \notin \Delta$

If Δ is normal in Γ and $(\gamma_1 + \delta_1) \varepsilon \gamma_1 G$

where γ_1 is any element of C and δ_1 is any element of Δ,

suppose $\exists\ \gamma' \varepsilon C$ such that

$\gamma' g' - \gamma' \notin \Delta$ for some $g' \varepsilon G'$ (see 2.7 (iii))

then $\exists\ \gamma_2 \varepsilon C$, $\delta_2 \varepsilon \Delta$ such that $\gamma_2 + \delta_2 = \gamma_2 g'$.

Define $n_2 \varepsilon N$ by n_2: $\gamma_2 g \to \gamma' g\ \forall\ g \varepsilon G$

$$n_2: \gamma_0 \to 0 \quad \forall\ \gamma_0 \varepsilon \Gamma \setminus \gamma_2 G$$

then $(\gamma_2 + \delta_2)n_2 - \gamma_2 n = \gamma_2 g' n_2 - \gamma_2 n_2 = \gamma_2 n_2 g' - \gamma_2 n_2 = \gamma' g' - \gamma' \notin \Delta$

Hence if $\{\Gamma, \Delta, G\}$ is compatible Δ is not an N-submodule of Γ.

If $\gamma_3 \varepsilon C$ and $\gamma_3' \varepsilon \Gamma$ then define

n_3: $\gamma_3 g \to \gamma_3' g, \forall\ g \varepsilon G$

n_3: $\gamma_0 \to 0, \forall\ \gamma_0 \varepsilon \Gamma \setminus \gamma_3 G$.

Then $\gamma_3 n_3 = \gamma'_3$ and so $\gamma_3 N = \Gamma$.

If $0 \neq \delta_4 \varepsilon \Delta$ and $\delta'_4 \varepsilon \Delta$, define

n_4: $\delta_4 \bar{g} \to \delta_4' \bar{g}\ \forall\ \bar{g} \varepsilon \bar{G}$

n_4: $\gamma_4 \to 0\ \forall\ \gamma_4 \varepsilon \Gamma \setminus \delta_4 \bar{G}$.

Then $n_4 \varepsilon N$ and $\delta_4 n_4 = \delta'_4$ so $\Delta = \delta_4 N$

Thus Δ is an N-module of type 2, and Γ is an N-module of type 0,

Hence N is 0-primitive, we show that N has d.c.c. on right ideals.

$$= N \Big/ (\bar{\gamma}_1)_r + N \Big/ (\bar{\gamma}_2)_r \quad \text{(as N-modules)}$$

suppose $x \in N \Big/ (\bar{\gamma}_1)_r \cap N \Big/ (\bar{\gamma}_2)_r$ then $x \varepsilon (\bar{\gamma}_1)_r \Big/ (\bar{\gamma}_1)_r \cap (\bar{\gamma}_2)_r$

$$\cap \; (\bar{\gamma}_2)_r \Big/ (\bar{\gamma}_1)_r \cap (\bar{\gamma}_2)_r$$

so $x = \alpha + (\bar{\gamma}_1)_r \cap (\bar{\gamma}_2)_r$ where $\alpha \varepsilon (\bar{\gamma}_1)_r$

$= \beta + (\bar{\gamma}_1)_r \cap (\bar{\gamma}_2)_r$ where $\beta \varepsilon (\bar{\gamma}_2)_r$

so $\alpha - \beta \;\varepsilon\; (\bar{\gamma}_1)_r \cap (\bar{\gamma}_2)_r$, then $\alpha - \beta \varepsilon (\bar{\gamma}_1)_r$ so $\beta \varepsilon (\bar{\gamma}_1)_r$

hence $\beta \varepsilon (\bar{\gamma}_1)_r \cap (\bar{\gamma}_2)_r$ and $x = 0$.

Thus $N \Big/ (\bar{\gamma}_1)_r \cap (\bar{\gamma}_2)_r = N \Big/ (\bar{\gamma}_1)_r \oplus N \Big/ (\bar{\gamma}_2)_r$.

In the natural way we can show by induction that

$$N \Big/ (C)_r = N \Big/ (\bar{\gamma}_1)_r \cap \dots \cap (\bar{\gamma}_h)_r \cong N \Big/ (\bar{\gamma}_1)_r \oplus \dots \oplus N \Big/ (\bar{\gamma}_h)_r$$

Each $N \Big/ (\bar{\gamma}_i)_r \cong \mathcal{R}_i$ as N-modules and so

$$N \Big/ (C)_r \cong \bigoplus_{i=1}^{h} \mathcal{R}_i .$$

$(C)_r \neq (0)$ since we may define $n_6 \varepsilon N$ as follows.

$\bar{\gamma}_i n_6 = 0$, $1 \leq i \leq h$

$\bar{\delta}_j n_6 = \bar{\delta}_j$, $1 \leq j \leq t$ Then $n_6 \neq 0$ and $n_6 \varepsilon (C)_r$

$N/_{(\Delta)_r}$ is 2-primitive since Δ is a faithful $N/_{(\Delta)_r}$-module.

If $\delta_6 \epsilon \Delta$ and $(\delta_6)^*_r = \{x \epsilon N/_{(\Delta)_r} \mid \delta_6 x = 0\}$

then $(\delta_6)_r^* = (\delta_6)_r/_{(\Delta)_r}$ is a right ideal of $N/_{(\Delta)_r}$,

maximal as an $(N/_{(\Delta)_r})^+$-subgroup.

If $E_j = \bigcap_{\substack{k=1 \\ k \neq j}}^{t} (\bar{\delta}_k)^*_r$ then $(\bar{\delta}_j)_r^* \cap E_j = \bar{0}$

and so $N/_{(\Delta)_r} = (\bar{\delta}_j)_r^* \oplus E_j$

and clearly in a similar manner to 3.2.6

$N/_{(\Delta)_r} = \bigoplus_{j=1}^{t} E_j$

and each $E_j \cong \Delta$ as $N/_{(\Delta)_r}$-modules.

Thus we have a composition series for $N/_{(\Delta)_r}$.

$N/_{(\Delta)_r} = E_1 \oplus . . \oplus E_t \supset E_1 \oplus . . \oplus E_{t-1} \supset . . . \supset$
$E_1 \oplus E_2 \supset E_1 \supset 0.$

If $E_1 \oplus . . . \oplus E_j = L_j$, $1 \leq j \leq t$

then $L_j/L_{j-1} \cong \Delta$ as $N/_{(\Delta)_r}$-modules.

Each L_j is a right ideal of $N/_{(\Delta)_r}$

$N/(\Delta)_r$ is 2-primitive since Δ is a faithful $N/(\Delta)_r$ -module.

If $\delta_6 \varepsilon \Delta$ and $(\delta_6)^*_r = \{x \varepsilon N/(\Delta)_r \mid \delta_6 x = 0\}$

then $(\delta_6)_r^* = (\delta_6)_r/(\Delta)_r$ is a right ideal of $N/(\Delta)_r$,

maximal as an $(N/(\Delta)_r)^+$-subgroup.

If $E_j = \bigcap_{\substack{k=1 \\ k \neq j}}^{t} (\bar{\delta}_k)^*_r$ then $(\bar{\delta}_j)_r^* \cap E_j = \bar{0}$

and so $N/(\Delta)_r = (\bar{\delta}_j)_r^* \oplus E_j$

and clearly in a similar manner to 3.2.6

$N/(\Delta)_r = \bigoplus_{j=1}^{t} E_j$

and each $E_j \cong \Delta$ as $N/(\Delta)_r$-modules.

Thus we have a composition series for $N/(\Delta)_r$.

$N/(\Delta)_r = E_1 \oplus . . \oplus E_t \supset E_1 \oplus . . \oplus E_{t-1} \supset . . . \supset$
$E_1 \oplus E_2 \supset E_1 \supset 0.$

If $E_1 \oplus . . . \oplus E_j = L_j$, $1 \leq j \leq t$

then $L_j/L_{j-1} \cong \Delta$ as $N/(\Delta)_r$-modules.

Each L_j is a right ideal of $N/(\Delta)_r$

So let $L_j = K_j / (\Delta)_r$ for $1 \leq j \leq t$, K_j is a right ideal of N containing $(\Delta)_r$.

A simple verification shows that $K_j / K_{j-1} \cong \Delta$ as N-modules

Then

$$N \supset K_t \supset K_{t-1} \supset \ldots \supset K_2 \supset K_1 \supset (\Delta)_r \supset \mathcal{R}_2 \oplus \ldots \oplus \mathcal{R}_h \supset \mathcal{R}_3 \oplus \ldots \oplus \mathcal{R}_h \supset \ldots \supset \mathcal{R}_{h-1} \oplus \mathcal{R}_h \supset \mathcal{R}_h \supset 0$$

is a composition series for N, for

$K_1 / (\Delta)_r \cong \Delta$ as N-modules

$\mathcal{R}_i \oplus \ldots \oplus \mathcal{R}_h / \mathcal{R}_{i+1} \oplus \ldots \oplus \mathcal{R}_h \cong \Gamma$ as N-modules

for $i = 1, \ldots, h-1$ and $\mathcal{R}_h \cong \Gamma$ as N-modules.

$N \cong (C)_r \oplus (\Delta)_r$

This completes the proof of the theorem

The corresponding theorem for the other case is as follows.

5.3 <u>Theorem</u> Let Γ be an additive group, Δ an abelian subgroup of Γ. Suppose Δ is a vector space of finite dimension over a division ring D. Let G be a group of automorphisms of Γ, regular on $\Gamma \setminus \Delta$, with restriction $G^\Delta = \{g^\Delta \mid g \in G\} \subseteq D$, and such that G induces a finite number of orbits on $\Gamma \setminus \Delta$.

If $\{\Gamma, \Delta, G\}$ is compatible, then the near-ring

$$N = \{n \in \mathcal{M}_G(\Gamma) \mid n^\Delta \in \mathrm{Hom}_D(\Delta)\}$$

is 0-primitive and not 2-primitive and has an identity and d.c.c. on right ideals. Also $N/(\Delta)_r$ is a ring.

5.4. If a near-ring N is of the form indicated in the hypothesis of Theorem 5.2 and is not of the form indicated in Theorem 5.3, then we can say that $N/(\Delta)_r$ is not a ring. The confusion arises here in the fairly trivial cases of Δ being a 1-dimensional vector space over a division ring D where $D \setminus \{0\} = \bar{G}$.

5.5. We have completed the classification of this class of 0-primitive near-rings.

5.6. Finally we calculate the radical $J_2(N)$ of near-rings of this type (see 2.1).

Notice that $N = A_1 \oplus \ldots \oplus A_h \oplus B_1 \oplus \ldots \oplus B_t$

where $A_i \cong \Gamma$ as N-modules $1 \le i \le h$,

and $B_j \cong \Delta$ as N-modules $1 \le j \le t$.

Recall that $J_2(N) = \bigcap$ (of all 2-modular right ideals of N.)

And a right ideal K of a near-ring N with identity is 2-modular if N/K is an N-module of type 2.

We notice that if $K_j = \left(\bigoplus_{i=1}^{h} A_i\right) \oplus \left(\bigoplus_{\substack{\ell=1 \\ \ell \neq j}}^{t} B_\ell\right)$, $1 \le j \le t$

then K_j is 2-modular. Hence $J_2(N) \subseteq \bigcap_{j=1}^{t} K_j \subseteq \bigoplus_{i=1}^{h} A_i = (\Delta)_r$.

Thus $J_2(N) \subseteq (\Delta)_r$.

Let K be a 2-modular right ideal

$K = N \cap K = (A_1 \cap K) \oplus \ldots \oplus (A_h \cap K) \oplus (B_1 \cap K) \oplus \ldots \oplus (B_t \cap K)$

Then $A_i \cap K = (0)$ or A_i , $1 \leq i \leq h$

$B_j \cap K = (0)$ or B_j , $1 \leq j \leq t$.

Suppose $A_i \cap K = (0)$ for some $1 \leq i \leq h$,

then $K \oplus A_i = N$ and then $N/_K \simeq \Gamma$ and Γ is not of type 2,

so we have a contradiction.

Thus $A_i \cap K = A_i$, $1 \leq i \leq h$

i.e. $A \oplus \ldots \oplus A_h \subseteq K$ i.e. $(\Delta)_r \subseteq K$.

Hence $J_2(N) = (\Delta)_r$.

The J_2 radical is simply the annihilator of the N-subgroup Δ, the set of non-generators of Γ.

Clearly $J_2(N)$ may contain idempotent elements for example the mapping e which is the identity on $C = \Gamma \setminus \Delta$, but which annihilates Δ.

Then $e^2 = e \varepsilon (\Delta)_r = J_2(N)$.

5.7. <u>Remark</u> $(\Delta)_r \cdot (\Delta)_r = (\Delta)_r$

for if $x \varepsilon (\Delta)_r$ then $ex = x \varepsilon (\Delta)_r \cdot (\Delta)_r$

and thus $(\Delta)_r \subseteq (\Delta)_r \cdot (\Delta)_r$.

Chapter 5. 0-primitive near-rings with identity and d.c.c. on right ideals(II).

This chapter will generalize considerably the results of the previous chapter. We will still make assumptions on the nature of Δ, the set of non-generators of the faithful, type 0, N-module. Instead of considering the case when Δ is an N-module of type 2, we will deal with a more general situation, when Δ is a union of a finite number of N-modules of type 2 having only zero in common. If there is only one of these N-modules of type 2. then we have the situation in chapter 4.

1. The general situation.

We assume that N is 0-primitive, with identity, 1, d.c.c. on right ideals and $J_2(N) \neq (0)$

If Γ is a faithful N-module, put $\Delta = \Gamma \setminus C$ where $C=\{\gamma\epsilon\Gamma \mid \gamma N = \Gamma\}$.

Δ is the set of non-generators of Γ.

We suppose that $\Delta = \bigcup_{\lambda=1}^{p} \Delta_\lambda$; where $\Delta_\lambda \cap \Delta_\mu = (0)$ for $\lambda \neq \mu$; and each Δ_λ is a N-module $(\lambda = 1, \ldots, p)$, of type 2.

Clearly $(\Delta)_r = \bigcap_{\lambda=1}^{p} (\Delta_\lambda)_r$ and for each $\delta_\lambda \epsilon \Delta_\lambda$ with $\delta_\lambda \neq 0$

$N/(\delta_\lambda)_r \cong \Delta_\lambda$ as N-modules $(\lambda = 1, \ldots, p)$.

1.1 <u>Proposition</u> If $\gamma \epsilon C$ then $(\Delta_\lambda)_r \not\subseteq (\gamma)_r$, $1 \leq \lambda \leq p$.

<u>Proof</u> Assume $(\Delta_\lambda)_r \subseteq (\gamma)_r$, then $\Gamma = \gamma N$ and $\Gamma(\Delta_\lambda)_r = \gamma N(\Delta_\lambda)_r \subseteq \gamma(\Delta_\lambda)_r = (0)$

Hence $(\Delta_\lambda)_r \subseteq (\Gamma)_r = (0)$ which implies that Δ_λ is a faithful N-module of type 2, i.e. N is 2-primitive and so $J_2(N) = (0)$ which is a contradiction. This holds for all $1 \leq \lambda \leq p$.

1.2 Proposition. If $\gamma \in C$ then for $1 \leq n \leq p$ $(\Delta_1)_r \cap \ldots \cap (\Delta_n)_r \nsubseteq (\gamma)_r$

Proof. We proceed by induction on n.

If $n = 1$ then from 1.1 $(\Delta_1)_r \nsubseteq (\gamma)_r$

Assume it is true for $n = k$, so $(\Delta_1)_r \cap \ldots \cap (\Delta_k)_r \nsubseteq (\gamma)_r$

Also $(\Delta_{k+1})_r \nsubseteq (\gamma)_r$ by 1.1

hence $(\Delta_1)_r \cap \ldots \cap (\Delta_k)_r \cap (\Delta_{k+1})_r \nsubseteq (\gamma)_r$ by 4.1.2

since N is not a ring. The induction follows.

(Clearly it does not matter what order we take the Δ_i in.)

1.3 Corollary. $(\Delta)_r \neq (0)$. and $(\Delta)_r \nsubseteq (\gamma)_r$ for any $\gamma \in C$

1.4. Proposition If $(\Delta_i)_r \subseteq (\Delta_j)_r$ for some $i \neq j$ then

$(\Delta_i)_r = (\Delta_j)_r$.

Proof. $N/_{(\Delta_i)_r}$ is a 2-primitive near-ring, for

Δ_i is an $N/_{(\Delta_i)_r}$-module in the usual way and is faithful and type 2.

Thus $N/_{(\Delta_i)_r}$ is simple (In 3.3.1 we do not need d.c.c. on

$(N/_{(\Delta_i)_r})^+$-subgroups for this part).

Hence $(\Delta_j)_r/_{(\Delta_i)_r} = (\bar{0})$ or $N/_{(\Delta_i)_r}$. The latter is impossible

and so $(\Delta_j)_r = (\Delta_i)_r$.

1.5. It may be that there exists Δ_i and Δ_j such that

$(\Delta_i)_r = (\Delta_j)_r$, $(i \neq j)$ Clearly $N/_{(\Delta_i)_r} = N/_{(\Delta_j)_r}$

and we put a relation on the set of indices

$I = \{1, 2, \ldots, p\}$ of the Δ's.

1.6 Definition. Let $i, j \in I$ then we define the relation R by $iRj \Leftrightarrow (\Delta_i)_r = (\Delta_j)_r$.

Clearly this is an equivalence relation and we partition I into the equivalence classes,

$$I = \Big(\bigcup_{\alpha\in A} P\alpha\Big) \cup \Big(\bigcup_{\beta\in B} P'_\beta\Big)$$

in such a way that the subsets $\{P'_\beta : \beta\in B\}$ consist of only one element each. (A and B are subsets of I) Naturally either A or B may be empty.

1.7 Let I' be a set consisting of a representative from each equivalence class on I. Then if $i,j \in I'$, $i \neq j$, $(\Delta_i)_r \neq (\Delta_j)_r$

1.8 Definition. Put $G = \mathrm{Aut}_N(\Gamma)$

$$\bar{G}_i = \mathrm{Aut}_{N/(\Delta_i)_r}(\Delta_i) \qquad 1 \leq i \leq p.$$

As automorphism groups, the $\bar{G}_i$ are regular on Δ_i, $1 \leq i \leq p$; and G is regular on $\Gamma \setminus \Delta = C$.

Put relations on C, Δ_i $1 \leq i \leq p$ as follows.

If $\gamma, \gamma' \in C$ then $\gamma \sim \gamma'_1 \iff \gamma'_1 = \gamma g$

for some $g \in G$.

$\delta_i, \delta_i' \in \Delta_i$ then $\delta_i R_i \delta'_i \iff \delta_i' = \delta_i \bar{g}_i$ for some

$\bar{g}_i \in \bar{G}_i$, where $1 \leq i \leq p$.

It may be easily checked that these p + 1 relations are all equivalence relations on their respective sets.

1.10. Proposition If $\gamma,\gamma'\in G$ then $\gamma\sim\gamma' \Leftrightarrow (\gamma)_r = (\gamma')_r$

If $\delta_i,\delta'_i\in\Delta_i$ then

$$\delta_i R_i \delta_i' \Leftrightarrow (\delta_i)_r \big/ (\Delta_i)_r = (\delta_i')_r \big/ (\Delta_i)_r \qquad 1\leq i\leq p$$

1.11 Proposition If $(\Delta_i)_r \neq (\Delta_j)_r$, then for all $\delta_j\in\Delta_j$, $\delta_j \neq 0$ $(\Delta_i)_r \nsubseteq (\delta_j)_r$.

Proof. If $(\Delta_i)_r \subseteq (\delta_j)_r$ then $\delta_j(\Delta_i)_r = 0$

Now $\Delta_j = \delta_j N$

so $\Delta_j (\Delta_i)_r = \delta_j N.(\Delta_i)_r \subseteq \delta_j(\Delta_i)_r = 0$

i.e. $(\Delta_i)_r \subseteq (\Delta_j)_r \Rightarrow (\Delta_i)_r = (\Delta_j)_r$ by 1.4.

1.12 Proposition If $i\in I'$ and J is any subset of $I' \setminus \{i\}$ then $\bigcap_{j\in J} (\Delta_j)_r \nsubseteq (\delta_i)_r$ for any $0 \neq \delta_i\in\Delta_i$.

Proof Suppose that $\exists (0 \neq) \delta_i\in\Delta_i$

and $\bigcap_{j\in J} (\Delta_j)_r \subseteq (\delta_i)_r$ Then $\delta_i(\bigcap_{j\in J} (\Delta_j)_r) = (0)$

$$\Delta_i(\bigcap_{j\in J} (\Delta_j)_r) = \delta_i N (\bigcap_{j\in J} (\Delta_j)_r) \subseteq \delta_i(\bigcap_{j\in J} (\Delta_j)_r) = (0)$$

Thus $\bigcap_{j\in J} (\Delta_j)_r \subseteq (\Delta_i)_r$

Hence $\prod_{j\varepsilon J} (\Delta_j)_r \subseteq \bigcap_{j\varepsilon J} (\Delta_j)_r \subseteq (\Delta_i)_r$

Thus $(\Delta_i) . \left[\prod_{j\varepsilon J} (\Delta_j)_r \right] = (0)$

and this can only imply that $\Delta_i . (\Delta_j)_r = (0)$ for some $j\varepsilon J$, for otherwise $\Delta_i (\Delta_j)_r = \Delta_i$

(We are really just using the fact that $(\Delta_i)_r$ is a 2-primitive ideal).

Thus $(\Delta_j)_r \subseteq (\Delta_i)_r$ for some $j\varepsilon J$, and this is a contradiction by 1.11.

1.13. <u>Theorem</u>. Let $i\varepsilon I'$, and suppose $N/(\Delta_i)_r$ is not a ring and $\delta_{i1}, \ldots, \delta_{ik} \varepsilon \Delta_i$ such that $\delta_{i\ell} R_i \; \delta_{im} \Rightarrow \ell = m$.

If $\delta_i{'}_1, \ldots, \delta'_{ik}$ are arbitrary elements of Δ_i then $\exists \; x_i \varepsilon \bigcap_{j\varepsilon J_i} (\Delta_j)_r = X_i$, where $J_i = I' \setminus \{i\}$ and such that $\delta_{i\ell} \, x_i = \delta'_{i\ell}$ for all $\ell = 1, \ldots, k$.

<u>Proof</u>. Since $N/(\Delta_i)_r$ is 2-primitive, $\exists \; y_i \varepsilon N/(\Delta_i)_r$ such that $\delta_{i\ell} y_i = \delta_i{'}_\ell$ for $\ell = 1, \ldots, k$. by 3.1.12.

Now $X_i \nsubseteq (\Delta_i)_r$ and so $X_i + (\Delta_i)_r = N$. ($(\Delta_i)_r$ is maximal as an ideal.) Let $y_i = t_i + (\Delta_i)_r$

then $t_i = x_i + s_i$ for some $s_i \varepsilon (\Delta_i)_r$, $x_i \varepsilon X_i$.

Then $\delta'_{i\ell} = \delta_{i\ell} y_i = \delta_{i\ell}(x_i + s_i + (\Delta_i)_r) = \delta_{i\ell} x_i$, $\ell = 1, \ldots k$.

<u>Theorem 1.14</u> If $i \varepsilon I'$ and $N/_{(\Delta_i)_r}$ is a ring and $\delta_{i1}, \ldots, \delta_{ik} \varepsilon \Delta_i$ are linearly independent over $\bar{G}_i$. Let $\delta'_{i1}, \ldots, \delta'_{ik} \varepsilon \Delta_i$ be arbitrary. Then $\exists\, x_i \varepsilon X_i = \bigcap_{j \varepsilon J_i} (\Delta_j)_r$, where $J_i = I' - \{i\}$ such that $\delta'_{i\ell} = \delta_{i\ell} x_i \quad \ell = 1, \ldots, k$.

<u>Proof</u>. This is similar to 1.13, and uses the density theorem for primitive rings.

<u>1.15 Theorem</u> Let $\gamma_1, \ldots, \gamma_q \varepsilon C$ such that $\gamma_k \sim \gamma_j \Rightarrow k = j$. If $\gamma_1', \ldots, \gamma_q'$ are arbitrary in Γ, there is

$$x \varepsilon (\Delta)_r = \bigcap_{i \varepsilon I} (\Delta_i)_r = \bigcap_{i \varepsilon I'} (\Delta_i)_r$$

such that $\gamma_k' = \gamma_k x$ for $k = 1, \ldots, q$.

<u>Proof</u>. We have that $(\Delta)_r \nsubseteq (\gamma)_r$ for any $\gamma \varepsilon C$. By induction on q. If $q = 1$. $\Gamma = \gamma_1(\Delta)_r$ and $\gamma_1' = \gamma_1 x$ for some $x \varepsilon (\Delta)_r$. Assume the result is true for $q = s > 1$ Then $(\gamma_2)_r \cap \ldots \cap (\gamma_s)_r \cap (\Delta)_r \nsubseteq (\gamma_{s+1})_r$ since we can put $\gamma'_2 = \ldots = \gamma_s' = 0$ and $\gamma'_{s+1} \neq 0$ and use the inductive hypothesis.

Clearly $(\gamma_1)_r \nsubseteq (\gamma_{s+1})_r$ and since N is not a ring by 4.1.2. $(\gamma_1)_r \cap (\gamma_2)_r \cap \ldots \cap (\gamma_s)_r \cap (\Delta)_r \nsubseteq (\gamma_{s+1})_r$.

By the inductive hypothesis $\exists\ x'\epsilon(\Delta)_r$ such that

$\gamma_1 x' = \gamma_1', \ldots, \gamma_s x' = \gamma_s'$.

$\exists\ y\epsilon(\gamma_1)_r \cap (\gamma_2)_r \cap \ldots \cap (\gamma_s)_r \cap (\Delta)_r$ s.t.

$\gamma_{s+1} y = \gamma'_{s+1} - \gamma_{s+1} x'$

Then put $x = y + x'\epsilon(\Delta)_r$

then $\gamma_1 x = \gamma_1 y + \gamma_1 x' = \gamma_1'$ (by definition of x)

$$\gamma_2 x = \gamma_2'$$
$$\vdots$$
$$\gamma_s x = \gamma_s'$$
$$\gamma_{s+1} x = \gamma'_{s+1}.$$

1.16. Now we look at the relation between Δ_i and Δ_j when $i,j\epsilon I$ and $(\Delta_i)_r = (\Delta_j)_r$.

Then $N/(\Delta_i)_r = N/(\Delta_j)_r$ is a 2-primitive near-ring.

Δ_i and Δ_j are faithful $N/(\Delta_i)_r$-modules of type 2. By Theorem 3.6.1, since $N/(\Delta_i)_r$ is 2-primitive with identity and d.c.c. on right ideals $\Delta_i \cong \Delta_j$ as $N/(\Delta_i)_r$-modules.

Let $\psi\colon \Delta_i \to \Delta_j$ be this isomorphism.

If $\delta_i\epsilon\Delta_i$, $n\epsilon N$ then

$$\begin{aligned}(\delta_i n)\psi &= \delta_i(n + (\Delta_i)_r)\psi \\ &= (\delta_i\psi)(n + (\Delta_i)_r) \\ &= (\delta_i\psi)(n + (\Delta_j)_r) \quad \text{as } (\Delta_i)_r = (\Delta_j)_r \\ &= \delta_j n \quad \text{where } \delta_j = \delta_i\psi\end{aligned}$$

Hence $(\delta_i n)\psi = (\delta_i\psi)n$

and $\Delta_i \cong \Delta_j$ as N-modules.

<u>Theorem</u> If $(\Delta_i)_r = (\Delta_j)_r$ then $\Delta_i \cong \Delta_j$ as N-modules.

1.17 <u>Definition</u> If $i\varepsilon I$ put $C_i = \Gamma \setminus (\Delta_i)$

Since Δ, Δ_i are not N-submodules of Γ, $(i\varepsilon I)$, we have.

1.18 <u>Theorem</u> If N is of the type described at the beginning of §1 then $\{\Gamma, \Delta, G\}$ is compatible in the sense of 4.2.7. and $\{\Gamma, \Delta_i, G\}$ is compatible for all $i\varepsilon I = \{1, \ldots, p\}$ (see 4.2.7 also).

1.19 Put $I' = \tilde{\mathfrak{I}}_1 \cup \tilde{\mathfrak{I}}_2$

where $i\varepsilon \tilde{\mathfrak{I}}_1 \Leftrightarrow i\varepsilon I'$ and $N/_{(\Delta_i)_r}$ is not a ring.

$i\varepsilon \tilde{\mathfrak{I}}_2 \Leftrightarrow i\varepsilon I'$ and $N/_{(\Delta_i)_r}$ is a ring.

If $j\varepsilon I \setminus I'$ then $\exists$ $k\varepsilon I'$ such that $\Delta_j \cong \Delta_k$ as N-modules, so $\exists$ is an N-isomorphism $\psi_{kj}: \Delta_k \to \Delta_j$.

This is true for each $j\varepsilon I \setminus I'$, and we will, in future, assume, whenever $j\varepsilon I \setminus I'$, knowledge of Δ_k and ψ_{kj}.

Define M_o as being the near-ring $\mathcal{M}_G(\Gamma)$.

$$\text{Put } M = \{m\varepsilon M_o \mid m|_{\Delta_i} \varepsilon\, \mathcal{M}_{\bar{G}_i}(\Delta_i) \text{ for all } i\varepsilon \tilde{\mathfrak{I}}_1;$$
$$m|_{\Delta_i} \varepsilon\, \mathrm{Hom}_{\bar{G}_i}(\Delta_i) \text{ for all } i\varepsilon \tilde{\mathfrak{I}}_2;$$
$$\text{and for } j\varepsilon I \setminus I',\ \delta_j m = \delta_j \psi_{kj}^{-1} m \psi_{kj},\ \forall\, \delta_j \varepsilon \Delta_j \text{ where } k\varepsilon I'\ \}$$

Put $N_R = \{m \varepsilon M_o \mid \gamma m = \gamma n \;\forall \gamma \varepsilon \Gamma,\; n \varepsilon N\}$

so N_R is the near-ring of right multiplications by elements of N, on Γ. We will show that $N_R \subseteq M$.

Let $n_R \varepsilon N_R$ so that $n_R : \Gamma \to \Gamma$ defined by

$\gamma n_R = \gamma n, \;\forall \gamma \varepsilon \Gamma$, for some $n \varepsilon N$.

Let $i \varepsilon \widetilde{\mathcal{S}}_1$ and $\delta_i \varepsilon \Delta_i$. Then

$\delta_i n_R \bar{g}_i = \delta_i n \bar{g}_i = \delta_i (n + (\Delta_i)_r) \bar{g}_i = \delta_i \bar{g}_i n, \forall \bar{g}_i \varepsilon \bar{G}_i$

Thus $n_R \mid_{\Delta_i} \varepsilon \mathcal{M}_{\bar{G}_i}(\Delta_i)$.

If $i \varepsilon \widetilde{\mathcal{S}}_2$ and $\delta_i, \delta'_i \varepsilon \Delta_i$ then

$(\delta_i + \delta_i')n_R = (\delta_i + \delta_i')(n + (\Delta_i)_r) = \delta_i n + \delta_i' n$ and so

$n_R \mid_{\Delta_i} \varepsilon \operatorname{Hom}_{\bar{G}_i}(\Delta_i)$.

If $j \varepsilon I \setminus I'$, then $\delta_j n_R = \delta_k \psi_{kj} n = \delta_k n \psi_{kj}' = \delta_j \psi_{kj}^{-1} . n \psi_{kj}$

where $k \varepsilon I$ and $\delta_j = \delta_k \psi_{kj}$; $\delta_j \varepsilon \Delta_j$ and $\delta_k \varepsilon \Delta_k$.

Thus $n_R \varepsilon M$ i.e. $N_R \subseteq M$.

1.20. N has d.c.c. on right ideals and so the groups Δ_i, $i \varepsilon \widetilde{\mathcal{S}}_1$, have only a finite number of equivalence classes under the $\bar{G}_i$. Let these have representatives $\delta_{i1}, \ldots, \delta_{ik_i}, 0$, respectively. The groups $\Delta_i, i \varepsilon \widetilde{\mathcal{S}}_2$ are finite dimensional $\bar{G}_i$ vector spaces. Let their bases be $\delta_{i1}, \ldots, \delta_{ik_i}$ respectively.

C has a finite number of orbits under G and we will denote their representatives by $\gamma_1, \ldots, \gamma_q \varepsilon C$.

If $m \varepsilon M$ then the action of m on the whole of Γ is completely determined by a knowledge of the action of m on the various orbit representatives,

$\gamma_1, \ldots, \gamma_q; \quad \delta_{i1}, \ldots, \delta_{ik_i}, \; i\varepsilon I'$

So m is determined by a knowledge of a finite number of elements of Γ.

We will now proceed to show that $M \leq N_R$

1.21. If $(\Delta_i)_r = (\Delta_j)_r$ then $\overline{G}_j = \psi_{ij}^{-1}.\overline{G}_i\psi_{ij}$

where ψ_{ij} is the N-isomorphism $\psi_{ij}:\Delta_i \to \Delta_j$

<u>Proof</u> If $\overline{g}_j \varepsilon \overline{G}_j = Aut_{N/(\Delta_i)_r}(\Delta_j)$

then $\alpha = (\psi_{ij}\overline{g}_j.\psi_{ij}^{-1}) : \Delta_i \to \Delta_i$ and is a group homomorphism

$\delta_i\alpha = 0 \Rightarrow \delta_i\psi_{ij}\overline{g}_j\psi_{ij}^{-1} = 0 \Rightarrow \delta_i = 0$ then α is 1 - 1.

α is clearly onto and

$$\delta_i(n + (\Delta_i)_r)\ \alpha = \delta_i n\alpha = \delta_i n\psi_{ij}\overline{g}_j\psi_{ij}^{-1} = \delta_i\psi_{ij}\overline{g}_j.\psi_{ij}^{-1}.n$$
$$= \delta_i\alpha(n + (\Delta_i)_r)$$

Thus $\alpha\varepsilon\overline{G}_i$ and so $\overline{g}_j \; \varepsilon \; \psi_{ij}^{-1}.\overline{G}_i.\psi_{ij}$.

If $\beta\varepsilon\psi_{ij}^{-1}\,\overline{G}_i\psi_{ij}$ then $\psi_{ij}\beta\psi_{ij}^{-1} \; \varepsilon \; \overline{G}_i$,

and a similar process reveals that $\beta \; \varepsilon \; \overline{G}_j$

Clearly the two multiplicative groups $\overline{G}_i$ and $\overline{G}_j$ are isomorphic as groups.

Also $\overline{G}_i = \psi_{ij}.\overline{G}_j.\psi_{ij}^{-1}$

1.22 Proposition Let Δ_1 and Δ_2 be N-modules such that there is an N-isomorphism ψ_{12}: $\Delta_1 \to \Delta_2$. If $\overline{G}_1$ and $\overline{G}_2$ are $\mathrm{Aut}_{N/(\Delta_1)_r}(\Delta_1)$, $\mathrm{Aut}_{N/(\Delta_2)_r}(\Delta_2)$ respectively, and if

$$\Delta_1^* = \bigcup_{\lambda=1}^{k_1} \delta_{1\lambda}\overline{G}_1 \text{ , } \Delta_2^* = \bigcup_{\mu=1}^{k_2} \delta_{2\mu}\overline{G}_2 \text{ where } \Delta_i^* = \Delta_i \setminus \{o\}, \ (i=1,2),$$

$\delta_{1\lambda}\overline{G}_1 \cap \delta_{1\lambda'}\overline{G}_1 = \phi$, $\lambda \neq \lambda'$

$\delta_{2\mu}\overline{G}_2 \cap \delta_{2\mu'}\overline{G}_2 = \phi$, $\mu \neq \mu'$

then $k_1 = k_2$.

Proof We can assume that $k_2 \leq k_1$

Then if $\mu \varepsilon \{1, \ldots, k_2\}$

$\delta_{2\mu} = \delta_{1\lambda}\overline{g}_1\psi_{12}$ for some $\lambda \varepsilon \{1, \ldots, k_1\}$, $\overline{g}_1 \varepsilon \overline{G}_1$.

If $k_2 < k_1$ then $\lambda' \varepsilon \{1, \ldots, k_1\}$

such that $\delta_{2\mu} \neq \delta_{1\lambda'}\overline{G}_1\psi_{12}$ for any $\mu \varepsilon \{1, \ldots, k_2\}$

Now $\delta_{1\lambda'}\psi_{12} \varepsilon \Delta_2$ so

$\delta_{1\lambda'}\psi_{12} = \delta_{2\mu'}\overline{g}_2$ for some $\mu' \varepsilon \{1, \ldots, k_2\}$, $\overline{g}_2 \varepsilon \overline{G}_2$

now $\delta_{2\mu'} = \delta_{1\lambda}\overline{g}_1\psi_{12}$ for some $\lambda \varepsilon \{1, \ldots, k_1\}$, $\overline{g}_1 \varepsilon \overline{G}_1$

so $\delta_{1\lambda'}\psi_{12} = \delta_{1\lambda}\overline{g}_1\psi_{12}\overline{g}_2$

i.e. $\delta_{1\lambda'} = \delta_{1\lambda}\overline{g}_1\psi_{12}\overline{g}_2\psi_{12}^{-1}$

now $\overline{g}_2 = \psi_{12}^{-1}\overline{g}_1'\psi_{12}$ for some $\overline{g}_1' \varepsilon \overline{G}_1$

thus $\delta_{1\lambda'} = \delta_{1\lambda}\overline{g}_1\overline{g}_1'$

which contradicts the hypothesis.

Thus $k_1 = k_2$.

1.23 Let $m \in M$. We then, by 1.13 and 1.14, can find an $x \in N$ such that

$$\delta_{ij}x = \delta_{ij}m \quad \forall\ j = 1, \ldots, k_i, \ \forall\ i \in I'.$$

By 1.15 $\exists\ y \in (\Delta)_r$ such that

$$\gamma_\ell y = \gamma_\ell m - \gamma_\ell x \qquad \forall\ \ell = 1, \ldots, q.$$

Put $n = y + x$. Then $n \in N$

and $\gamma_\ell n = \gamma_\ell m - \gamma_\ell x + \gamma_\ell x = \gamma_\ell m$ $\quad \ell = 1, \ldots, q$

$$\delta_{ij}n = \delta_{ij}m \quad , \qquad j = 1, \ldots, k_i, \ i \in I'.$$

If $t \in I \setminus I'$, then $\exists\ s \in I'$ such that

$\Delta_t \cong \Delta_s$ as N-modules under the isomorphism

$\psi_{st} : \Delta_s \to \Delta_t$.

If $\delta_t \in \Delta_t$ then

$$\delta_t m = \delta_t \psi_{st}^{-1} m \psi_{st}.$$

Now $\delta_t . \psi_{st}^{-1} . m = \delta_t . \psi_{st}^{-1} . n$

so $\delta_t m = \delta_t . \psi_{st}^{-1} . n\ \psi_{st} = \delta_t n . \psi_{st}^{-1} . \psi_{st} = \delta_t n$.

Thus m acts on the group Γ in exactly the same way as the right multiplication mapping n_R. Put $N_R = \{n_R : n \in N\}$.

Then $N_R = M$

1.24 The choice of I' was not unique, necessarily, and we now show that, choosing different representatives of the equivalence classes on I under the relation R, gives us the same near-ring as 1.23. We choose a particular I', then $I = \mathfrak{S}_1 \cup \mathfrak{S}_2 \cup (I \sim I')$. Let us choose I_1' as follows. Pick any $i \varepsilon \tilde{\zeta}_1$ such that $(\Delta_i) \approx (\Delta_j)$ for some $j \varepsilon\ I \setminus I'$. (If this is not possible we can choose an $i \varepsilon \tilde{\zeta}_2$ and proceed similarly).

Now put $I_1' = (I' \setminus \{i\}) \cup \{j\}$

Then $I = \tilde{\zeta}_1' \cup \tilde{\zeta}_2 \cup (I \setminus I_1')$ where $\tilde{\zeta}_1' = (\tilde{\zeta}_1 \setminus \{i\}) \cup \{j\}$.

Put N = M as defined in 1.19.

$$\text{Put } N_1 = \{m \varepsilon M_o \mid m|_{\Delta_\rho} \varepsilon \gamma\gamma_{\overline{G}_\rho}(\Delta_\rho) \text{ for all } \rho \varepsilon \tilde{\zeta}_1';$$

$$m|_{\Delta_\rho} \varepsilon \mathrm{Hom}_{\overline{G}_\rho}(\Delta_\rho) \text{ for all } \rho \varepsilon \tilde{\zeta}_2;$$

$$\text{for } \sigma \varepsilon I \setminus I_1', \quad \delta_\sigma m = \delta_\sigma \psi_{k\sigma}^{-1}.m\psi_{k\sigma}, \forall \delta_\sigma \varepsilon \Delta_\sigma \text{ , where } k \varepsilon I_1'\}$$

Let $m \varepsilon M = N$.

We must show that

$$m|_{\Delta_j} \varepsilon \gamma\gamma_{\overline{G}_j}(\Delta_j)$$

and $\delta_i m = \delta_i . \psi_{ji}^{-1}.m\psi_{ji} \qquad \forall \delta_i \varepsilon \Delta_i$.

We already know that

$m|_{\Delta_i} \varepsilon \gamma\gamma_{\overline{G}_i}(\Delta_i)$ and $\delta_j m = \delta_j \psi_{ji} m \psi_{ji}^{-1}$ from the definition of N.

Clearly $m : \Delta_j \to \Delta_j$. Let $\delta_j \varepsilon \Delta_j$, $\overline{g}_j \varepsilon\ \overline{G}_j$ then

$\delta_j m \overline{g}_j = \delta_j \psi_{ji} m \psi_{ji}^{-1}.\overline{g}_j$. Now $\overline{g}_j = \psi_{ji} \overline{g}_i\ \psi_{ji}^{-1}$

for some $\overline{g}_i \varepsilon \overline{G}_i$.

$$\begin{aligned} \text{Thus} \quad \delta_j m \overline{g}_j &= \delta_j \psi_{ji} m \overline{g}_i . \psi_{ji}^{-1} \\ &= \delta_j \psi_{ji} \overline{g}_i\ m . \psi_{ji}^{-1} \\ &= \delta_j \overline{g}_j \psi_{ji} m . \psi_{ji}^{-1} = (\delta_j \overline{g}_j) m. \end{aligned}$$

Thus $m|_{\Delta_j} \in \mathcal{M}_{\overline{G}_j}.(\Delta_j)$.

Now for any $\delta_i \epsilon \Delta_i$, $\exists\, \delta_j \epsilon \Delta_j$ s.t. $\delta_i = \delta_j \psi_{ji}$

$$\text{so} \quad \delta_i \psi_{ji}^{-1}.m\, \psi_{ji} = \delta_j m \psi_{ji}$$

$$= \delta_j \psi_{ji} m \psi_{ji}^{-1} \psi_{ji}$$

$$= \delta_i m$$

Thus $m \epsilon N_1$. $\quad N \subseteq N_1$.

And symmetry proves that $N = N_1$.

It is clear similar processes will show that whatever the choice of I', the near-ring obtained in 1.23 will not differ.

1.25 <u>Theorem</u> Let N be 0-primitive with identity and d.c.c. on right ideals and $J_2(N) \neq (0)$. If Γ is the faithful N-module of type 0 and the set Δ of non-generators of Γ , is the union of a finite number of N-modules Δ_i, $(i \epsilon I)$, of type 2 such that $\Delta_i \cap \Delta_j = (0)$ for $i \neq j$.

Then if I' is chosen with respect to the relation R of 1.6, and $I' = \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_2$ as in 1.19, then

$$N \cong \{ m \epsilon \mathcal{M}_G(\Gamma) \mid m|_{\Delta_i} \epsilon\, \mathcal{M}_{\overline{G}_i}(\Delta_i) \text{ all } i \epsilon \tilde{\mathcal{S}}_1;\ m|_{\Delta_i} \epsilon\, \mathrm{Hom}_{\overline{G}_i}(\Delta_i) \text{ all } i \epsilon \tilde{\mathcal{S}}_2$$

$$\text{and for } j \epsilon I \setminus I', \text{ any } \delta_j \epsilon \Delta_j,\ \delta_j m = \delta_j \psi_{kj}^{-1}.m\psi_{kj} \text{ for a } k \epsilon I' \}$$

as near-rings: where $G = \mathrm{Aut}_N(\Gamma)$, $G_i = \mathrm{Aut}_{N/(\Delta_i)_r}(\Delta_i)$,$i \epsilon I'$.

$\psi_{kj} : \Delta_k \to \Delta_j$ is an N-isomorphism.

§2 The converse and final classification

We now try and build up from suitable systems of groups and their mappings, all 0-primitive near-rings, of the form studied in §1.

2.1 Given three mutually disjoint finite sets, $\tilde{\mathfrak{S}}_1, \tilde{\mathfrak{S}}_2, \tilde{\mathfrak{S}}_3$ of positive integers, suppose Γ is an additive group and Δ_i; $i \in \tilde{\mathfrak{S}}_1$ a finite collection of subgroups, and Δ_i; $i \in \tilde{\mathfrak{S}}_2$ a finite collection of abelian subgroups, and Δ_i; $i \in \tilde{\mathfrak{S}}_3$ a finite collection of subgroups, each isomorphic to one of the Δ_i, $i \in \tilde{\mathfrak{S}}_1 \cup \tilde{\mathfrak{S}}_2$; and such that $\Delta_i \cap \Delta_j = (0)$ for $i \neq j$, $i,j \in \tilde{\mathfrak{S}}_1 \cup \tilde{\mathfrak{S}}_2 \cup \tilde{\mathfrak{S}}_3 = I$

Suppose for each $i \in \tilde{\mathfrak{S}}_1$; $\overline{G}_i$ represents a group of regular automorphisms of Δ_i, and for each $i \in \tilde{\mathfrak{S}}_2$; $\overline{G}_i$ represents a divisionring over which Δ_i is a vector space.

Suppose G is a group of automorphisms which are regular on $C = \Gamma \setminus \{\bigcup_{i \in I} (\Delta_i)\}$ and such that if $g \in G$ then $g|_{\Delta_i} \in \overline{G}_i$, all $i \in \tilde{\mathfrak{S}}_1 \cup \tilde{\mathfrak{S}}_2$.

If $\{\Gamma, \Delta, G\}$ is compatible, where $\Delta = \bigcup_{i \in I} \Delta_i$, and $\{\Gamma, \Delta_i, G\}$ are compatible for $i \in I$

then the near-ring

$N = \{m \in \mathcal{M}_G(\Gamma) \mid m|_{\Delta_i} \in \mathcal{M}_{\overline{G}_i}(\Delta_i)$ for $i \in \tilde{\mathfrak{S}}_1$;

$m|_{\Delta_i} \in \mathrm{Hom}_{\overline{G}_i}(\Delta_i)$ for $i \in \tilde{\mathfrak{S}}_2$;

for $j \in \tilde{\mathfrak{S}}_3$, $\delta_j \in \Delta_j$, then $\delta_j m = \delta_j . \psi_{kj}^{-1} . m . \psi_{kj}$

where ψ_{kj} is the isomorphism from Δ_k to Δ_j for a $k \in \tilde{\mathfrak{S}}_1 \cup \tilde{\mathfrak{S}}_2$.$\}$

is 0-primitive, such that Δ_i are N-modules of type 2 for $i \in I$, $N/(\Delta_i)_r$ is not a ring for $i \in \tilde{\zeta}_1$, $N/(\Delta_i)_r$ is a ring for $i \in \tilde{\zeta}_2$ and if $j \in \tilde{\zeta}_3$ then $(\Delta_k)_r = (\Delta_j)_r$ {where $k \in \tilde{\zeta}_1 \cup \tilde{\zeta}_2$ and $\Delta_k \cong \Delta_j$ as groups.} and Δ_k and Δ_j are in fact isomorphic N-modules. N has an identity.

2.2 Let the group G induce a finite number of orbits on C, and the groups G_i ; $i \in \tilde{\zeta}_1$ induce a finite number of orbits on Δ_i ; and the vector spaces Δ_i, $i \in \tilde{\zeta}_2$, have finite $\overline{G}_i$- dimension. Then N has d.c.c. on right ideals.

<u>Proof</u> We note first that if $y \in \tilde{\zeta}_3$ and $f \in \tilde{\zeta}_1$ such that $\psi_{fy} : \Delta_f \to \Delta_y$ is an isomorphism, then if $\Delta_f = \bigcup_{k=1}^{t} \delta_{fk} \overline{G}_f \cup \{0\}$

$$\Delta_y = \bigcup_{k=1}^{t} (\delta_{fk}\psi_{fy})\psi_{fy}^{-1}\overline{G}_f\psi_{fy} \cup \{0\}$$

i.e. Δ_y has a finite number of orbits induced by the automorphism group $\overline{G}_y = \psi_{fy}^{-1} \overline{G}_i \psi_{fy}$.

If $\psi_{fy} : \Delta_f \to \Delta_y$ and $f \in \tilde{\zeta}_2$ then

$$\Delta_f = \bigoplus_{k=1}^{t} \delta_{fk} \overline{G}_f$$

and $\Delta_j = \bigoplus_{k=1}^{t} (\delta_{fk}\psi_{fy})\psi_{fy}^{-1}\overline{G}_f\psi_{fy}$, i.e. Δ_y is $(\psi_{fy}^{-1}.\overline{G}_f.\psi_{fy})$ - finite dimensional.

Now let $I = \{1, \ldots, p\}$, and define for any $\theta \in I \setminus (\tilde{\zeta}_1 \cup \tilde{\zeta}_2)$

$\overline{G}_\theta = \psi_{\ell\theta}^{-1}\overline{G}_\ell\psi_{\ell\theta}$; where $\ell \in \tilde{\zeta}_1 \cup \tilde{\zeta}_2$.

For each $i \in I$, let $\delta_{i1}, \ldots, \delta_{i\kappa_i}$ be orbit representatives or basis with respect to the $\overline{G}_i$, $i \in I$,

Let $\tilde{\zeta}_1 \cup \tilde{\zeta}_2 = \{1, \ldots, q\}$

Define $E_{1j} = \bigcap_{\substack{\ell \neq j \\ \ell=1}}^{k_1} (\delta_{1\ell})_r \Big/ (\Delta_1)$ for all $j\varepsilon\{1, \ldots, k_1\}$

$E_{2j} = \bigcap_{\substack{\ell \neq j \\ \ell=1}}^{k_2} (\delta_{2\ell})_r \cap (\Delta_1)_r \Big/ (\Delta_1)_r \cap (\Delta_2)_r$ for all $j\varepsilon\{1, \ldots l_2\}$

$\vdots$

$E_{qj} = \left(\bigcap_{\substack{\ell \neq j \\ j=1}}^{k_q} (\delta_{p\ell})_r \cap (\Delta_1)_r \cap (\Delta_2)_r \cap \ldots \cap (\Delta_{q-1})_r \right) \Big/ (\Delta_1)_r \cap \ldots \cap (\Delta_q)_r.$

The E_{ij} are all right ideals of the near-rings $N \big/ (\Delta_1)_r \cap \ldots \cap (\Delta_i)_r$

and each E_{ij} corresponds to a right ideal of the near-ring N lying between $(\Delta_1)_r \cap \ldots \cap (\Delta_i)_r$ and $(\Delta_1)_r \cap \ldots \cap (\Delta_{i-1})_r$ for $i \neq 1$ and the E_{1j} corresponds to one lying between $(\Delta_1)_r$ and N.

Clearly $\left(N \big/ (\Delta_1)_r \cap \ldots \cap (\Delta_i)_r \right) \Big/ \left((\delta_{ij})_r \big/ (\Delta_1)_r \cap \ldots \cap (\Delta_i)_r \right) \cong$

$N \big/ (\delta_{ij})_r \cong \Delta_i$ as N-modules.

We can find an element $x\varepsilon E_{ij}$ such that $\delta_{ij}x \neq 0$ for $j = 1, \ldots, k_i$, $i\varepsilon\{1, \ldots, q\}$. Thus $E_{ij} \nsubseteq (\delta_{ij})_r$ for $j = 1, \ldots, k_{ij}; i = 1, \ldots, q$.

Thus $\delta_{ij} E_{ij} = \Delta_i$ in the obvious way.

And so $E_{ij} \cong \Delta_i$ as an $N \big/ (\Delta_1)_r \cap \ldots \cap (\Delta_i)_r$ - module and in fact as an N-module, with the natural definition of

$e_{ij}.n = e_{ij}.(n + (\Delta_1)_r \cap \ldots \cap (\Delta_i)_r)$ all $e_{ij} \in E_{ij}$, $n \in N$.

Now $$\frac{(\Delta_1)_r \cap \ldots \cap (\Delta_{i-1})_r}{(\delta_{ij})_r \cap (\Delta_1)_r \cap \ldots \cap (\Delta_{i-1})_r} \cong \frac{(\delta_{ij})_r + (\Delta_1)_r \cap \ldots \cap (\Delta_{i-1})_r}{(\delta_{ij})_r}$$

$$= N/(\delta_{ij})_r \cong \Delta_i \text{ as N-modules,}$$

For clearly $(\Delta_1)_r \cap \ldots \cap (\Delta_{i-1})_r \not\subseteq (\delta_{ij})_r$ all $j = 1, \ldots, k_i$ and $i = 1, \ldots, q$.

Thus $(\delta_{ij})_r \cap (\Delta_1)_r \cap \ldots \cap (\Delta_{i-1})_r$ is a max. right ideal of the near-ring $(\Delta_1)_r \cap \ldots \cap (\Delta_{i-1})_r$

so $$\frac{(\delta_{ij})_r \cap (\Delta_1)_r \cap \ldots \cap (\Delta_{i-1})_r}{(\Delta_1)_r \cap \ldots \cap (\Delta_i)_r} + E_{ij} = \frac{(\Delta_1)_r \cap \ldots \cap (\Delta_{i-1})_r}{(\Delta_1)_r \cap \ldots \cap (\Delta_i)_r}$$

and in fact the sum is direct.

In the same way as 4.5.2.

$$\frac{(\Delta_1)_r \cap \ldots \cap (\Delta_{i-1})_r}{(\Delta_1)_r \cap \ldots \cap (\Delta_i)_r} = \bigoplus_{j=1}^{k_i} E_{ij}.$$

Put $F_{i\ell} = \dotplus_{j=1}^{\ell} E_{ij}$ for $\ell = 1, \ldots, k_i$

then $F_{i1} \subseteq F_{i2} \subseteq F_{i3} \subseteq \ldots \subseteq F_{ik_i}$ and each $F_{i\ell}$ is a rt. ideal of

$N/(\Delta_1)_r \ldots (\Delta_i)_r$.

Thus suppose that for $\ell = 1, \ldots, k_i$

$F_{i\ell} = H_{i\ell}/(\Delta_1)_r \cap \ldots \cap (\Delta_i)_r$. Then the $H_{i\ell}$ are right ideals of

of N and

$(\Delta_1)_r \cap \ldots \cap (\Delta_i)_r \subset H_{i1} \subset H_{i2} \subset \ldots \subset H_{ik_i} \subseteq (\Delta_1)_r \cap \ldots \cap (\Delta_{i-1})_r$

all $i = 1, \ldots, q$.

Now $H_{i\ell+1}/H_{i\ell} \cong \Delta_i$ as N-modules $1 \leq \ell \leq q - 1$.

Put $(\Delta)_r = \bigcap_{i=1}^{q} (\Delta_i)_r$ and for $1 \leq t \leq k$

define $P_t = \bigcap_{\substack{s \neq t \\ s=1}}^{k} (\gamma_s)_r \cap (\Delta)_r$ where k is the number of orbits induced

on C by G.

In a similar way to 4.5.2.

$$N/(C)_r \cong \bigoplus_{t=1}^{k} P_t.$$

We now have a composition series for N,

$N \supset H_{11} \supset H_{12} \supset \ldots \supset H_{1k_1} \supset H_{21} \supset \ldots \supset H_{2k_2} \supset H_{31} \supset \ldots \supset H_{q1} \supset \ldots \supset$
$H_{qk_q} \supset Q_1 \supset \ldots \supset Q_k \supset 0$ where $Q_\ell = \bigoplus_{t=1}^{\ell} P_t$,

In this series each factor group is an N-module of type 0.

Thus N possesses d.c.c. on right ideals.

---

§3. An Example

3.1 We give a typical example of the type of near-ring that arises. For convenience, we restrict ourselves to the finite case.

Let Γ be any group and Δ_1 and Δ_2 subgroups such that $\Delta_1 \cong \Delta_2$ as groups. Let Δ_3 be an abelian subgroup of prime power order, such that $\Delta_1 \cap \Delta_2 = \Delta_2 \cap \Delta_3 = \Delta_1 \cap \Delta_3 = (0)$. In our notation, $\mathfrak{Z}_1 = \{1\}$, $\mathfrak{Z}_2 = \{3\}$. ψ_{12} is the isomorphism from Δ_1 to Δ_2. Let $|\Delta_1| > 2$.

Put $N = \{m \varepsilon \gamma\gamma\gamma_o(\Gamma) \mid m|_{\Delta_1} \varepsilon \gamma\gamma\gamma_o(\Delta_1);\ m|_{\Delta_3} \varepsilon \mathrm{Hom}_{Z_p}(\Delta_3);$

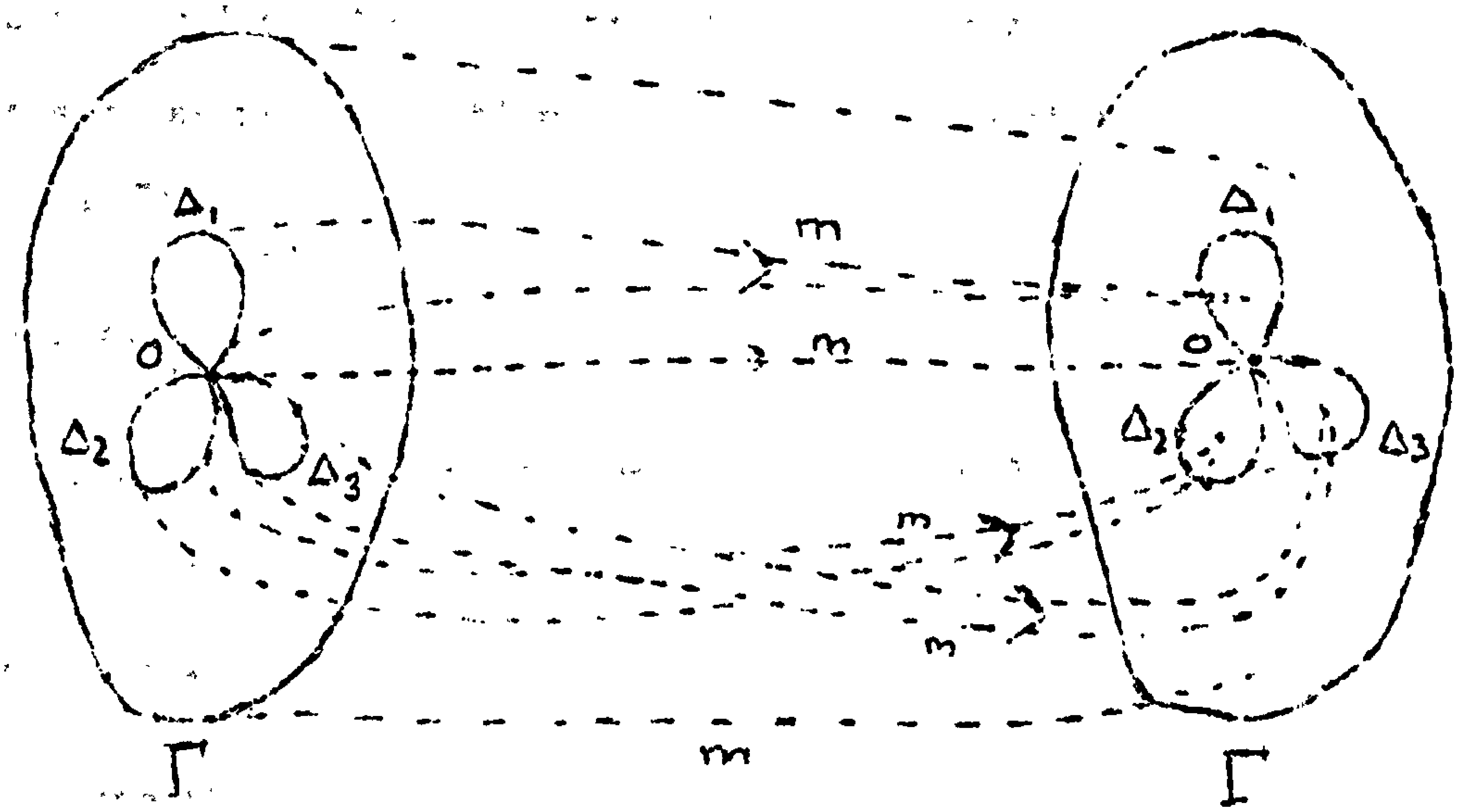
for $\delta_2 \varepsilon \Delta_2,\ \delta_2 m = \delta_2 \psi_{12}^{-1} m \psi_{12}\ \}$

where $p^r = |\Delta_3|$ and Z_p = the field of p elements.

Then N is a near-ring of the type considered in section 2.

It is 0-primitive with $\Delta_1 \cup \Delta_2 \cup \Delta_3$ the set of non-generators of Γ, and $N/(\Delta_1)_r$ not a ring and $N/(\Delta_3)_r$ is a ring.

We can illustrate the near-ring as follows, where m is a typical element



The mappings of N take all elements of $\Gamma \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3)$ into Γ, all elements of Δ_1 into Δ_1, Δ_2 into Δ_2, Δ_3 into Δ_3 satisfying the provisos of the definition.

If $N_1 = \{m \varepsilon \gamma\gamma\gamma_o(\Gamma) \mid m|_{\Delta_i} \varepsilon\ \gamma\gamma\gamma_o(\Delta_i);\ i = 1, 2, 3\}$

then $N \subseteq N_1 \subseteq \gamma\gamma\gamma_o(\Gamma)$.

Clearly N_1 is 0-primitive but $N_1/(\Delta_3)_r$ is not a ring unless $|\Delta_3| = 2$, and Δ_1 and Δ_2 are not isomorphic N_1- modules. $\mathcal{M}_0(\Gamma)$ is 2-primitive. All these near-rings have identities.

§4. The radical $J_2(N)$

The radical $J_2(N)$ of a near-ring of the type discussed in this chapter is an ideal of the near-ring different from zero.

In a similar way to section 4.5.6. we can show that

$$J_2(N) = \bigcap_{i\varepsilon I} \bigcap_{j=1}^{k_i} (\delta_{ij})_r$$

Thus $J_2(N) = (\Delta)_r$

Here again $J_2(N)$ will possess an idempotent element different from zero and $J_2(N).\ J_2(N) = J_2(N)$.

Since $J_2(N) = (\Delta)_r$ and $J_2(N/J_2(N)) = (0)$

by 2.I.4, we see that $J_2(N/(\Delta)_r) = (0)$.

$N/(\Delta)_r$ has d.c.c. on right ideals and an identity.

$$N/(\Delta)_r = N/\bigcap_{i\varepsilon I'}(\Delta_i)_r = \oplus\ N/(\Delta_i)_r .$$

and the $N/(\Delta_i)_r$ are 2-primitive near-rings (some may be rings), which illustrates Thm. 3.4.4.

Chapter 6. Prime Ideals

We have already introduced the idea of primitive ideals and we shall now study the possibilities of generalisations of the concept of a prime ideal.

§1. Preliminary results

1.1 Proposition (Betsch) An ideal P of a near-ring N is a ν-primitive ideal if and only if there is an N-module M of type ν with

$(M)_r = P$. Here $\nu = 0, 1$ or 2.

Proof If M is an N-module of type ν and $P = (M)_r$ then M is an $N_{/P}$-module and is faithful as such. Thus $N_{/P}$ is ν-primitive, hence P is ν-primitive. If P is ν-primitive, then there is a faithful $N_{/P}$-module M. If $n \in N$, define

$m.n = m(n + P) \in M.$

Thus M is a N-module of type ν and

$(M)_r = \{n \in N \mid Mn = 0\} = \{n \in N \mid M(n + P) = 0\}$
$= P.$

1.2 Definitions

An ideal P of a near-ring N ($\neq N$) is 2-prime if for any N± subgroups K_1, K_2; $K_1 K_2 \subseteq P \Rightarrow K_1 \subseteq P$ or $K_2 \subseteq P$. An ideal P ($\neq N$) is 1-prime if for any right ideals R_1, R_2; $R_1 R_2 \subseteq P \Rightarrow R_1 \subseteq P$ or $R_2 \subseteq P$. An ideal P ($\neq N$) is 0-prime if for any ideals

I_1, I_2; $I_1 I_2 \subseteq P \Rightarrow I_1 \subseteq P$ or $I_2 \subseteq P$.

A right ideal S($\neq N$) is a 1-prime right ideal if for any right ideals R_1, R_2; $R_1 R_2 \subseteq S \Rightarrow R_1 \subseteq S$ or $R_2 \subseteq S$.

1.3 Remark It is clear that a 2-prime ideal is 1-prime and 0-prime and a 1-prime ideal is 0-prime. However a 0-prime ideal need not be a 1-prime as we shall see shortly. (3.8).

1.4 Proposition The following statements are equivalent.

(i) P is a 2-prime ideal of N

(ii) If K_1 is an $N_\pm$ subgroup and S is an ideal of N then

$K_1 S \subseteq P \Rightarrow K_1 \subseteq P$ or $S \subseteq P$.

Proof (i) $\Rightarrow$ (ii). S is an N±subgroup and so this is straightforward.

(ii) $\Rightarrow$ (i). Let K_1, K_2 be N±subgroups and $K_1, K_2 \subseteq P$.

Assume $K_1 \nsubseteq P$, then $K_2 \subseteq (P:K_1) = \{n \varepsilon N \mid K_1 n \subseteq P\} = R$, say

Then R is an ideal of N, for if

$r \varepsilon R$, n_1, $n \varepsilon N$, then $\forall$ $k. \varepsilon K_1$,

$k_1(-n + r + n) = -k_1 n + k_1 r + k_1 n \ \varepsilon\ P$ as P is an ideal

$k_1[(n + r)\ n_1 - nn_1] = (k_1 n + k_1 r)\ n_1 - k_1 n.n_1 \ \varepsilon\ P$. Also $K_1 nr \subseteq K_1 r \subseteq P$.

Thus $K_1 R \subseteq P$, $K_1 \nsubseteq P \Rightarrow R \subseteq P$.

Hence $K_2 \subseteq R \subseteq P$.

1.5 Proposition The following statements are equivalent

(i) P is an 0-prime ideal of N

(ii) If I_1 is an ideal, R a right ideal, then

$I_1 R \subseteq P \Rightarrow I_1 \subseteq P$ or $R \subseteq P$.

Proof We need only remark that -

$(P:I_1) = \{n \varepsilon N \mid I_1 n \subseteq P\}$ is an ideal.

1.6 __Proposition__ Suppose P is a 2-prime ideal and $X_1, \ldots, X_n$ are N-subgroups.

Then $X_1 \ldots X_n \subseteq P \Rightarrow X_i \subseteq P$ for some $i \in \{1, \ldots, n\}$.

__Proof__ Let $X_1 \ldots X_n \subseteq P$ and $X_1 \nsubseteq P$. Then

$$X_2 \ldots X_n \subseteq (P:X_1) \text{ which is an ideal.}$$

Now $X_1 . (P:X_1) \subseteq P$ so $(P:X_1) \subseteq P$ as P is 2-prime and using 1.4.

Thus $X_2 \ldots X_n \subseteq P$. We repeat the procedure if $X_2 \nsubseteq P$, and eventually $X_i \subseteq P$ for some $i \in \{1, \ldots, n\}$.

1.7 __Proposition__ If P is a 1-prime ideal and $X_1, \ldots, X_n$ are rt. ideals then $X_1 \ldots X_n \subseteq P \Rightarrow X_i \subseteq P$ for some $i \in \{1, \ldots, n\}$

1.8 __Proposition__ If P is a 0-prime ideal and $X_1, \ldots, X_n$ are all ideals, then, $X_1 \ldots X_n \subseteq P \Rightarrow X_i \subseteq P$ for some $i \in \{1, \ldots, n\}$.

1.9 __Proposition__ If R_1 is a 1-prime right ideal and

$X_1, \ldots, X_n$ are right ideals, then

$X_1 \ldots X_n \subseteq R_1 \Rightarrow X_i \subseteq R_1$ for some $i \in \{1, \ldots, n\}$.

1.10 Clearly we now have the situation where a 2-prime ideal will contain all the nilpotent N-groups, a 1-prime ideal will contain all the nilpotent right ideals, a 0-prime ideal will contain all the nilpotent ideals of N.

§2. __The relation between ν-prime and ν-primitive ideals__

2.1 __Proposition__ A ν-primitive ideal is ν-prime ($\nu = 0, 1, 2$).

__Proof__ $\nu = 2$. If P is a 2-primitive ideal, then

$\exists$ M, an N-module of type 2 with $(M)_r = P$.

Let K_1, K_2 be N^+-groups and $K_1K_2 \subseteq P$.

Then $MK_1K_2 = 0$

Assume $K_1 \nsubseteq P$ then $MK_1 \neq 0$ so $MK_1 = M$. But

$MK_1K_2 = MK_2 = 0 \Rightarrow K_2 \subseteq (M)_r = P$.

$\underline{\nu = 1}$ If P is a 1-primitive ideal and $P = (M)_r$ where M is an N-module of type 1.

Clearly if R_1, R_2 are rt. ideals with $R_1R_2 \subseteq P$ and $R_1 \nsubseteq P$

Then $MR_1R_2 = MR_2 = 0 \Rightarrow R_2 \subseteq P$.

$\underline{\nu = 0}$ If P is a 0-primitive ideal and $P = (M)_r$ for an N-module M of type 0. Then if I_1, I_2 are ideals with $I_1I_2 \subseteq P$ but $I_1 \nsubseteq P$. Then $MI_1 \neq 0$ so for some $m \in M$, $M = mN$. $0 \neq MI_1 = mNI_1 \subseteq mI_1$. Thus mI_1 is an N-submodule of M and non-zero. $mI_1 = M$. But $mI_1I_2 = 0 = MI_2$

Thus $I_2 \subseteq P$.

2.2 We now study near-rings N with descending chain condition on it. N±subgroups.

2.3 Proposition If N has d.c.c. on it N±subgroups then a 2-prime ideal P is 2-primitive.

Proof Pass to $\overline{N} = N_{/P}$. This has d.c.c. on rt. $\overline{N}$± subgroups.

Choose a minimal one $M \neq (\overline{0})$.

Then if $(\overline{M})_r$ is the rt-annihilator of M in $\overline{N}$,

$M.(\overline{M})_r = (\overline{0}) \Rightarrow M = (\overline{0})$ or $(\overline{M})_r = (\overline{0})$ since $(\overline{0})$ is a 2-prime ideal of $\overline{N}$. Thus P is a 2-primitive ideal since $\overline{N}$ is a 2-primitive near-ring.

2.4 Proposition (Laxton, Ramakotaiah). If N has d.c.c. on rt. N±subgroups then a 0-prime ideal P is 0-primitive.

<u>Proof</u> Pass to $\overline{N} = N/_P$. $\overline{N}$ possesses no non-zero nilpotent ideals, and so $J_0(\overline{N}) = (\overline{0})$.

By 2.5.10, $J_0(\overline{N}) = (D(\overline{N}):\overline{N})_L = \{\overline{n}\in\overline{N} \mid \overline{N}\overline{n} \subseteq D(\overline{N})\} = (\overline{0})$

If $\overline{N}$ has no 0-modular right ideals, then $D(\overline{N}) = \overline{N}$ and this contradicts $(D(\overline{N}):\overline{N}) = (\overline{0})$. $(\overline{0}) = J_0(\overline{N}) = \bigcap(\Gamma)_r$ for all $\overline{N}$-modules Γ of type 0. $J_0(\overline{N}) = \bigcap_{i=1}^{p} (\overline{\Gamma}_i)_r$ for some finite subset of the set of all $\overline{N}$-modules of type 0.

Now $(\overline{\Gamma}_1)_r \cdot (\overline{\Gamma}_2)_r \cdot \ldots \cdot (\overline{\Gamma}_p)_r \subseteq J_0(\overline{N}) = (\overline{0})$

so $(\overline{\Gamma}_i)_r = (\overline{0})$ for some $\overline{N}$-modules $\overline{\Gamma}_i$ of type 0.

Thus $\overline{N}$ is 0-primitive and hence P is a 0-primitive ideal.

2.5 <u>Remark</u> It is not known whether a 1-prime ideal is 1-primitive under suitable chain conditions.

§3 <u>ν-prime near-rings</u>

Elementwise characterizations of ν-prime ideals are not very satisfactory. We exhibit one result.

3.1 <u>Proposition</u> Let N be a near-ring with a right identity e.

Then P is a 2-prime ideal $\Leftrightarrow$ $aNb \subseteq P \Rightarrow a\varepsilon P$ or $b\varepsilon P$.

<u>Proof</u> (i) $\Rightarrow$. P is 2-prime. Then $aNb \subseteq P \Rightarrow$

$aNbN \subseteq P$ so $aN \subseteq P$ or $bN \subseteq P$

i.e. $ae = a\varepsilon P$ or $b\varepsilon P$.

(ii) $\Leftarrow$. Let $X_1 \cdot X_2 \subseteq P$; X_1, X_2 rt. N+-subgroups.

Suppose $X_2 \nsubseteq P$. $\exists\, x_2 \varepsilon X_2$ s.t. $x_2 \notin P$,

now $x_1 N x_2 \subseteq X_1 X_2 \subseteq P$, $\forall\, x_1 \varepsilon X_1$

Thus $x_1 \subseteq P$. Hence $X_1 \subseteq P$.

3.2 Remarks It is interesting to study near-rings which are υ-prime (υ = 0, 1, 2) where a υ-prime near-ring is simply a near-ring such that the zero ideal is a υ-prime ideal. Recall that a famous theorem due to Goldie in ring theory gives necessary and sufficient conditions for a prime ring to have an artinian primitive classical ring of quotients. We could ask now, whether a υ-prime near-ring may have a υ-primitive near-ring with d.c.c. on right ideals, as a near-ring of quotients. This seems a difficult problem and first we must establish that non-trivial 2-prime near-rings exist, for clearly the first case to look at is when υ= 2.

3.3 Example Let (Z, +) be the additive group of integers.

Put (S,.) the multiplicative semigroup of non-negative integers. Thus

$S = (0) \cup P$, P the positive integers.

Put $N = \mathcal{M}_S(Z)$.

Then $Z = (0) \cup 1.P \cup (-1).P$

Clearly each mapping n of N is determined when we know

$1.n$ and $(-1)n$.

Suppose $n_1 N n_2 = (0)$ and $n_1 \neq 0$, $n_2 \neq 0$.

If $1n_1 \neq 0$, $1n_2 \neq 0$, $(-1)n_1 \neq 0$, $(-1)n_2 \neq 0$

then $1(n_1 n_2) = (1n_1)n_2 \neq 0$.

If $1n_2 = 0$ and $1n_1 = z_1 \neq 0$, $-1n_2 \neq 0$

then if $z_1 > 0$ put $n \in N$ where $n: 1 \to -1$, $n: -1 \to 1$

then $1n_1 n n_2 = z_1 n n_2 = -z_1 n_2 = (-1)n_2 . z_1 \neq 0$.

If $z_1 < 0$ put $n' \in N$ where $n' =$ identity.

Then $(1)n_1 n_2 = z_1 n_2 = (-1n_2)z_1 \neq 0$.

Similarly if $(-1)n_2 = 0$, and if $(1)n_1 = 0$ then $-1n_1 \neq 0$

and so $(-1)(n_1 N n_2) \neq 0$ for all non-zero $n_2 \in N$.

Thus for any $n_1, n_2 \in N$,

$n_1 N n_2 = 0 \Rightarrow n_1 = 0$ or $n_2 = 0$.

Thus N is a 2-prime near-ring, and we shall see later that N possesses a near-ring of right quotients in a natural way.

3.4 We finally define a near-ring to be <u>υ-semi-prime</u> ($\upsilon = 0, 1, 2$) if $\bigcap_{P_\upsilon \in \mathcal{B}_\upsilon} P_\upsilon = (0)$ where $\mathcal{B}_\upsilon$ is the set of all υ-prime ideals.

<u>Example</u> Let N be a υ-prime near-ring and R a prime ring then $N \oplus R$ is a υ-semi-prime near-ring ($\upsilon = 0, 1, 2$)

For $0 \oplus R$ is a υ-prime ideal and also $N \oplus 0$ is a υ-prime ideal and thus $(0) = 0 \oplus 0 = (N \oplus 0) \cap (0 \oplus R)$.

3.5 <u>Proposition</u> If N is υ-semi-prime then N possesses no non-zero nilpotent: ideals (for $\upsilon = 0$)

: rt. ideals (for $\upsilon = 1$)

: rt. $N \pm$ subgroups (for $\upsilon = 2$)

<u>Proof</u> Any nilpotent ideal is contained in every 0-prime ideal and thus in the intersection of all of these, i.e. the zero ideal. Similarly for cases $\upsilon = 1$, $\upsilon = 2$

3.6 <u>Theorem</u> If N has d.c.c. on rt. N± subgroups then N is ν-semi-prime if and only if $J_\nu(N) = (0)$ for $\nu = 0, 2$.

N is 1-semiprime $\Rightarrow D(N) = (0)$.

<u>Proof</u> $J_\nu(N) = (0) \Leftrightarrow$ N contains no non-zero nilpotent ideals for $\nu = 0$ and N± gps. for $\nu = 2$. $D(N) = (0) \Leftrightarrow$ N contains no non-zero nilpotent right ideals. $J_2(N) = (0) \Rightarrow (0) = \cap P_2$, where P_2 is taken over all the 2-primitive ideals of N. These are 2-prime, so

$J_2(N) = (0) \Rightarrow$ N is 2-semi-prime.

Also $J_1(N) = (0) \Rightarrow$ N is 1-semi-prime

$J_0(N) = (0) \Rightarrow$ N is 0-semi-prime.

3.7 <u>Theorem</u> If N has d.c.c. on rt. N± subgroups then

$D(N) = (0) \Leftrightarrow \cap R_1 = (0)$ where R_1 is taken over all the 1-prime right ideals.

<u>Proof</u> Clearly if $\cap R_1 = (0)$ then N has no non-zero nilpotent right ideals and so $D(N) = (0)$. Let $D(N) = (0)$. We show that a 0-modular right ideal is a 1-prime right ideal.

Let R be a 0-modular right ideal and $e \varepsilon N$ s.t. $en - n \varepsilon R$, $\forall n \varepsilon N$.

$\Gamma = N^+ \setminus R$ is an N-module of type 0 and $e + R$ is an N-generator for Γ.

i.e. $(e + R)N = \Gamma$. If $\bar{0}$ is the zero of $N^+ \setminus R$.

Suppose X_1, X_2 are right ideals with $X_1 X_2 \subseteq R$ but $X_1 \subsetneq R$.

Then $(e + R)X_1 X_2 = \bar{0}$ and $(e + R)x_1 \neq \bar{0}$ some $x_1 \varepsilon X_1$.

Thus $(e + R)X_1 = \Gamma$ and so $(e + R)X_1 X_2 = \Gamma X_2 = \bar{0} \Rightarrow X_2 \subseteq R$

3.8 We give an example of a near-ring N, which is 0-prime but not 1-prime.

Let N be a near-ring of the form examined in 4.5.2.

Keeping the same notation.

Let $X = (C)_r.(\Delta)_r$.

Suppose $x \varepsilon X$ and $x = y.z$ where $y \varepsilon (C)_r$, $z \varepsilon (\Delta)_r$.

Then for any $\gamma \varepsilon C$,

$$\gamma x = \gamma yz = 0.z = 0.$$

and for any $\delta \varepsilon \Delta$,

$$\delta x = \delta yz \varepsilon \Delta z = 0$$

Thus $x = 0$ and hence $X = (0)$

We have $(C)_r.(\Delta)_r = (0)$, where the $(C)_r$ and $(\Delta)_r$ are two right ideals which are non-zero.

Thus N is not 1-prime. However N is 0-primitive and so is 0-prime.

Chapter 7. Right Orders in 2-primitive near-rings with identity and d.c.c. on right ideals

We remarked in the previous chapter that is may be possible to find necessary and sufficient conditions for a υ-prime near-ring to have a near-ring of right quotients which is a υ-primitive near-ring with identity and d.c.c. on right ideals. We will consider an example in this chapter which indicates that the case for $\upsilon = 2$ may prove fruitful.

§1 Right orders in a near-ring

1.1 Definitions

(a) An element x of a near-ring N is called a regular element of N if for $n_1, n_2, n_3, n_4 \in N$

$n_1 x = n_2 x \Rightarrow n_1 = n_2$

and $x n_3 = x n_4 \Rightarrow n_3 = n_4$.

(b) A regular element x in a near-ring N with identity 1, is invertible if there exists a $y \in N$ such that

$xy = yx = 1$. Clearly y must be regular and invertible. We will denote y, when it exists by x^{-1}.

(c) A near-ring N has a near-ring of right quotients Q if

(i) $N \subseteq Q$ and Q has an identity.

(ii) If x is a regular element of N then

$y \in Q$ s.t. $xy = yx = 1_Q$. We write $y = x^{-1}$.

(iii) If $q \in Q$ then $q = n x^{-1}$ where $n \in N$ and x is a regular element of N.

(d) A near-ring N is a rt. order in a near-ring Q, if Q is a near-ring of right quotients of N.

1.2 <u>Remark</u> If N is a rt. order in Q then N satisfies the following property:-

Let $n_1, x_1 \in N$, with x_1 regular, then $\exists\ n_2, x_2 \in N$, with x_2 regular, such that $x_1 n_2 = n_1 x_2$. The proof is identical to the equivalent statement in ring theory.

However it is not known whether we may conclude that the converse is true.

1.3 <u>Examples</u>

(1) Clearly if F is a near-field then F is its own near-ring of right quotients.

(2) If R is a ring which is a rt. order in a ring S, and F is a near-field, then $N = R \oplus F$ is a rt. order in the near-ring $Q = S \oplus F$.

1.4 We could define equally well a near-ring of left quotients of a near-ring N. This is done in an analogous manner. We choose to work with right quotient near-rings because the structure of 2-primitive near-rings is on the right. Nothing is known, yet, about 'left-primitive near-rings', or simple near-rings with left-sided chain conditions.

§2 <u>The construction of an example of a right order in a 2-primitive near-ring</u>

We introduce a class of examples which will prove interesting.

2.1 Let S be a multiplicative semigroup.

S is <u>left cancellative</u> if $ss_1 = ss_2 \Rightarrow s_1 = s_2$ for $s, s_1, s_2 \varepsilon S$. S is <u>right cancellative</u> if

$s_1 s = s_2 s \Rightarrow s_1 = s_2$ for $s, s_1, s_2 \varepsilon S$.

S is <u>left reversible</u> if $s_1 S \cap s_2 S \neq \phi$, $\forall s_1, s_2 \varepsilon S$.

S is <u>right reversible</u> if $Ss_1 \cap Ss_2 \neq \phi$, $\forall s_1, s_2 \varepsilon S$.

2.2 <u>Theorem</u> If S is a left and right cancellative semigroup and S is right reversible then S may be embedded in a group G of left quotients.

Then $\forall s \varepsilon S$, s^{-1} exists in G and $\forall\ g \varepsilon G$,

$g = s_1^{-1} s$ for suitable $s_1, s \varepsilon S$.

2.3 This is a well known theorem in semigroup theory, and the reader is inviteád to compare this with the similar theorem in ring theory concerning integral domains being left orders in division rings under a similar condition to the right reversibility condition.

2.4 <u>Theorem</u> Suppose Γ is an additive group, and S is a multiplicative semigroup of endomorphisms of $(\Gamma, +)$, which includes the identity endomorphism, but not the zero endomorphism. Suppose that S is left and right cancellative, left and right reversible, and, when $s \varepsilon S$, and $\gamma s = 0$ then $\gamma = 0$.

Then S has a group G of left quotients, and G acts as a group of automorphisms on an additive group Δ.

<u>Proof</u> We consider the cartesian product $\Gamma \times S$.

Let $(\gamma, s), (\gamma_1, s_1) \in \Gamma \times S$, and define a relation $\sim$ by

$(\gamma, s) \sim (\gamma_1, s_1) \Leftrightarrow \exists a_1, b_1 \varepsilon S$ such that

$sb_1 = s_1a_1$ and $\gamma b_1 = \gamma_1 a_1$.

To show that this is well defined and an equivalence relation we proceed as follows.

Suppose $sb_2 = s_1a_2$ for $a_2, b_2 \varepsilon S$.

Then $\exists x_1, y_1 \varepsilon S$ s.t. $a_1x_1 = a_2y_1$

so $sb_1x_1 = s_1a_1x_1 = s_1a_2y_1 = sb_2y_1 \Rightarrow b_2y_1 = b_1x_1$

Then $\gamma_1a_2y_1 = \gamma b_1x_1 = \gamma b_2y_1 = \gamma_1a_1x_1$

Thus $(\gamma b_2 - \gamma_1a_2)y_1 = 0$

giving $\gamma b_2 = \gamma_1 a_2$.

In the usual way we show that $\sim$ is an equivalence relation.

For example, if $(\gamma, s) \sim (\gamma_1, s_1)$ and $(\gamma_1, s_1) \sim (\gamma_2, s_2)$,

we have a_1, b_1, a_2, b_2 such that

$sb_1 = s_1a_1$, $\gamma b_1 = \gamma_1a_1$, $s_1b_2 = s_2a_2$, $\gamma_1b_2 = \gamma_2a_2$.

Now there are $x, y \varepsilon S$ such that

$a_1y = b_2x$. Put $a_3 = a_2x$, $b_3 = b_1y$.

Then $sb_3 = sb_1y = s_1a_1y = s_1b_2x = s_2a_2x = s_2a_3$

and $\gamma b_3 = \gamma b_1y = \gamma_1a_1y = \gamma_1b_2x = \gamma_2a_2x = \gamma_2a_3$

Thus $(\gamma, s) \sim (\gamma_2, s_2)$ and $\sim$ is an equivalence relation on the set $\Gamma \times S$.

We partition $\Gamma \times S$ into equivalence classes.

If $(\gamma, s) \varepsilon E$, an equivalence class of $\Gamma \times S$,

We write $E = \gamma/_s$

Put $\Delta = \{\gamma/_s \mid (\gamma, s)$ are representatives of the equivalence classes on $\Gamma \times S$ under $\sim\}$.

We denote the group of left quotients of the semigroup S by G.

We shall show that Δ is an additive group and G acts on Δ as a group of automorphisms.

Let $\gamma/_s, \gamma'/_{s'} \in \Delta$, and define

$$\gamma/_s + \gamma'/_{s'} = (\gamma a + \gamma' b)/_m$$

where $m = s'b = sa$ for $a, b \in S$.

This operation is well defined.

For suppose $(\gamma_1', s'_1) \sim (\gamma', s')$

and $(\gamma_1, s_1) \sim (\gamma, s)$

Consider $\gamma_1/_{s_1} + \gamma'_1/_{s'_1}$ Let $\alpha, \beta \in S$ st.

$s\alpha = s_1\beta$ and $\gamma\alpha = \gamma_1\beta$

If $\lambda, \mu \in S$ s.t. $s'\lambda = s'_1\mu$ and $\gamma'\lambda = \gamma'_1\mu$

then $\gamma/_s + \gamma'/_{s'} = (\gamma a + \gamma' b)/_{sa}$, $sa = s'b$

$$\gamma_1/_{s_1} + \gamma'_1/_{s'_1} = (\gamma_1 x + \gamma'_1 y)/_{s_1 x}$$ where $x, y \in S$ such that

$s_1 x = s'_1 y$.

Choose $e, f \in S$ with $s_1 xf = sae$

Then $sae = s'be$, $s_1 xf = s'_1 yf$

$\therefore s'be = s'_1 yf$

Now $\exists\, k, h \in S$ s.t. $aek = \alpha h$

$saek = s\alpha h = s_1\beta h = s_1 xfk$

thus $\beta h = xfk$

$\gamma aek = \gamma\alpha h = \gamma_1\beta h = \gamma_1 . xfk$

then $(\gamma ae - \gamma_1 xf)k = 0 \Rightarrow \gamma ae = \gamma_1 xf$

Also $\gamma' be\ell = \gamma'\lambda m$ where $\ell, m \varepsilon S$ such that

$be\ell = \lambda m$

Now $s' be\ell = sae\ell = s_1' yf\ell = s'\lambda m$

$s_1' yf\ell = s_1' \mu m \Rightarrow yf\ell = \mu m$

And $\gamma' be\ell = \gamma'\lambda m = \gamma_1'\mu m = \gamma_1' yf\ell$

giving $\gamma' be = \gamma_1' yf$.

As e and f are endomorphisms of Γ,

$(\gamma a + \gamma' b)e = \gamma ae + \gamma' be$

$= \gamma_1 xf + \gamma_1' yf = (\gamma_1 x + \gamma_1' y)f$.

But $sae = s_1 xf$

Hence $(\gamma a + \gamma' b, sa) \sim (\gamma_1 x + \gamma_1' y, s_1 x)$.

Let $\frac{\gamma}{b} + \frac{\gamma_1}{d} = (\gamma x + \gamma_1 y) /_{dy}$ where $dy = bx$

and $\frac{\gamma}{b} + \frac{\gamma_1}{d} = (\gamma x' + \gamma_1 y') /_{dy'}$ where $dy' = bx'$

we must show that the two expressions are equivalent.

Choose p and $q \varepsilon S$ s.t. $dyq = dy'p$ then $yq = y'p$

$dy'p = bx'p = bxq$ and

$x'p = xq$.

Thus $(\gamma x + \gamma_1 y)q = \gamma xq + \gamma_1 yq = \gamma x'p + \gamma_1 y'p = (\gamma x' + \gamma_1 y')p$

so $(\gamma x + \gamma_1 y, dy) \sim (\gamma x' + \gamma_1 y', dy')$

Addition in Δ is well defined.

It is easily verified that Δ is an additive group using similar techniques.

If $\gamma/_s \varepsilon \Delta$, $g \varepsilon G$ we define

$$(\gamma/_s)g = {}^{\gamma u_1}/_{s_1 b_1} \quad \text{where } g = {}^r/_{s_1}$$

and $u_1, b_1 \varepsilon S$ are such that $su_1 = rb$.

This is also a well-defined operation. We show that g is an automorphism of $(\Delta, +)$.

Let $\gamma/_s$, $\gamma'/_{s'} \varepsilon \Delta$, $g = r/_{s_1}$

then $(\gamma/_s + \gamma'/_{s'})(r/_{s_1}) = ((\gamma a + \gamma' b)/_{sa})(r/_{s_1})$ where $sa = s'b$.

so $(\gamma/_s + \gamma'/_{s'})(r/_{s_1}) = (\gamma a + \gamma' b)u_1/_{s_1 b_1}$

$$= (\gamma a u_1 + \gamma' b u_1)/_{s_1 b_1} \quad \text{where } sau_1 = rb_1$$

Now $(\gamma/_s)(r/_{s_1}) = \gamma u_2/_{s_1 b_2}$ where $su_2 = rb_2$

and $(\gamma'/_{s'})(r/_{s_1}) = \gamma' u_3/_{s_1 b_3}$ where $s'u_3 = rb_3$

then $(\gamma/_s)(r/_{s_1}) + (\gamma'/_{s'})(r/_{s_1}) = (\gamma u_2 c + \gamma' u_3 d)/_{s_1 b_2 c}$

where $s_1 b_2 c = s_1 b_3 d$. i.e. $b_2 c = b_3 d$.

Choose $x, y \varepsilon S$ s.t. $s_1 b_2 cx = s_1 b_1 y$

Then $b_2 cx = b_1 y = b_3 dx$

and $su_2 cx = rb_2 cx = rb_3 dx = rb_1 y = sau_1 y$

thus $u_2 cx = au_1 y$

hence $\gamma u_2 cx = \gamma a u_1 y$.

Also $s'u_3 dx = rb_3 dx = s'bu_1 y$ so

$$u_3 dx = bu_1 y \text{ and}$$

$$\gamma' u_3 dx = \gamma' b u_1 y$$

Then $(\gamma a u_1 + \gamma' b u_1)y = \gamma a u_1 y + \gamma' b u_1 y = (\gamma u_2 c + \gamma' u_3 d)x$

$$\text{i.e. } (\gamma a u_1 + \gamma' b u_1, s_1 b_1) \sim (\gamma u_2 c + \gamma' u_3 d, s_1 b_2 c)$$

Thus G is a group of endomorphisms on Δ.

Suppose now that

$$(\gamma_{/s})(r_{/s_1}) = 0 \quad \text{for } \gamma_{/s} \in \Delta,\ r_{/s_1} \in G$$

then $\gamma u_{2/s_1 b_2} = 0$ where $su_2 = rb_2$

i.e. $\gamma u_2 = 0$ which implies $\gamma = 0$.

Then $r_{/s_1}$ is a monomorphism.

Let $\gamma/_s \in \Delta$, $r_{/s_1} \in G$, we will find $\bar{\gamma}_{/\bar{b}} \in \Delta$

such that $\gamma_{/s} = (\bar{\gamma}_{/\bar{b}})(r_{/s_1})$

Pick $v_1, w_1 \in S$ s.t. $sv_1 = s_1 w_1$

Put $\bar{\gamma} = \gamma v_1$, $\bar{b} = rw_1$

$$\text{then } \left(\bar{\gamma}_{/\bar{b}}\right)\left(r_{/s_1}\right) = \left(\gamma v_{1/rw_1}\right)\left(r_{/s_1}\right)$$

$$= \gamma v_1 a_{1/s_1 c_1} \qquad \text{where } rw_1 a_1 = rc_1$$

i.e. $w_1 a_1 = c_1$ so

$$\left(\bar{\gamma}_{/\bar{b}}\right)\left(r_{/s_1}\right) = \gamma v_1 a_{1/s_1 w_1 a_1} = \gamma v_1 a_{1/sv_1 a_1} = \gamma_{/s}$$

and $(\gamma v_1 a_1, sv_1 a_1) \sim (\gamma, s)$

Thus G is a group of automorphisms of Δ.

We have thus obtained from an additive group Γ with a semigroup S of endomorphisms of Γ, satisfying certain conditions, a 'larger' additive group Δ and a group G of automorphisms of Δ which contains a 'copy' of S.

We now ask what conditions make all elements of G, regular automorphisms on Δ. This is answered in our next Lemma.

2.5 Lemma In the terminology of 2.4, G is a group of regular automorphisms of Δ if and only if for every $0 \neq \gamma \in \Gamma$,

$$\gamma s_1 = \gamma s_2 \Rightarrow s_1 = s_2. \quad (s_1, s_2 \in S)$$

Proof. Suppose G is a group of regular autos.

Let $\gamma \in \Gamma$; s, $s_2 \in S$ and $\gamma s = \gamma s_2$ with $s \neq s_2$.

Then $(\gamma/1)(s/s_2) = \gamma a/s_2 b$

where $1.a = sb$ so

$$(\gamma/1)(s/s_2) = \gamma sb/s_2 b = \gamma s_2 b/s_2 b = \gamma.$$

$s/s_2 \in G$ and we have found an element $\gamma \in \Delta$ and $s/s_2 \in G$ such that

$(\gamma)(s/s_2) = \gamma$

thus $\gamma = 0$ must follow, since s/s_2 is regular.

Conversely suppose $\gamma/s \in \Delta$, $r/s_1 \in G$ and $\gamma \neq 0$ and

$(\gamma/s)(r/s_1) = \gamma/s$

Then $\gamma/s = \gamma u_1/s_1 b_1$ where $rb_1 = su_1$

$\exists$ $u_2, b_2 \in S$ s.t.

$\gamma u_2 = \gamma u_1 b_2$

and $su_2 = s_1 b_1 b_2$ since

$(\gamma, s) \sim (\gamma u_1, s_1 b_1)$.

Now $u_2 = u_1 b_2$ as $\gamma \neq 0$

so $su_2 = su_1 b_2 = s_1 b_1 b_2 \Rightarrow su_1 = s_1 b_1 = rb_1$

and thus $s_1 = r$

This means that $r_{/s_1}$ is the identity automorphism of Δ.

Thus G is a group of regular automorphisms of Δ .

2.6 Lemma With the same notation as 2.4 and 2.5,

If $\Gamma = \{0\} \cup \{\bigcup_{i=1}^{\rho} \gamma_i S\}$ for suitable $\gamma_1, \ldots, \gamma_\rho \epsilon \Gamma$,

with $\gamma_i S \cap \gamma_j S = \phi$ for $i \neq j$

then Δ has ρ orbits under the action of G.

Proof Let $0 \neq \delta \epsilon \Delta$. Then $\delta = \gamma_{/s}$ for some $\gamma \epsilon \Gamma$, $s \epsilon S$

and $\gamma = \gamma_i s_i$ for some $s_i \epsilon S$ and $i \leq i \leq \rho$

$$\delta = (\gamma_i s_i)_{/s} = \gamma_i \cdot (s_{i/s}) \;\epsilon \bigcup_{i=1}^{\rho} \gamma_i G$$

Thus $\Delta = \{\bigcup_{i=1}^{\rho} \gamma_i G\} \cup \{0\}$

Suppose $\delta' \neq 0$ s.t. $\delta' \epsilon \gamma_i G \cap \gamma_j G$ where $i \neq j$.

let $\delta' = \gamma_i(r_{/s}) = \gamma_j(y_{/z})$ where $r, s, y, z \epsilon S$.

$\exists \alpha, \beta \epsilon S$ s.t. $s\alpha = z\beta = m$ (say)

$$\delta' = (\gamma_i r)_{/s} = (\gamma_j y)_{/z}$$

$$\delta' m = ((\gamma_i r)_{/s})(m_{/1})$$

$$= \gamma_i r u_{1/b_1} \text{ where } u_1, b_1 \epsilon S \text{ and}$$

$$m b_1 = s u_1$$

i.e. $s\alpha b_1 = s u_1$ so $u_1 = \alpha b_1$

$$\delta' m = (\gamma_i r \alpha b_1)_{/ 1} = \gamma_i r \alpha$$

$\delta' m = \gamma_j y u_{2/b_2}$ where $u_2, b_2 \epsilon S$ and

$m b_2 = z u_2$ i.e.

$z\beta b_2 = z u_2$ so $\beta b_2 = u_2$

$\delta' m = \gamma_j y \beta b_{2/b_2} = \gamma_j y \beta$

Thus $\delta' m \in \gamma_i S \cap \gamma_j S$ which is a contradiction.

2.7 Theorem Let $N = \gamma\gamma\gamma_S\ (\Gamma)$

and $Q = \gamma\gamma\gamma_G\ (\Delta)$

where $\Gamma = \{0\} \cup \{\bigcup_{i=1}^{\rho} \gamma_i S\}$

$$\Delta = \{0\} \cup \{\bigcup_{i=1}^{\rho} \gamma_i G\}$$

Where Γ, S, G, Δ satisfy all the requirements of 2.4, 2.5, 2.6.

Then we can embed N in Q as near-rings.

Proof If $n \in N$ and if the $\gamma_i n$, $i = 1, \ldots, \rho$ are known, then γn is known for all $\gamma \in \Gamma$.

For if $\gamma \neq 0$, then $\gamma = \gamma_i s$ for some $s \in S$ and $i \in \{1, \ldots, \rho\}$ and

$$\gamma n = \gamma_i sn = (\gamma_i n)s.$$

If $n \in N$, define $\overline{n}: \Delta \to \Delta$ by

$$[\gamma_i(r_{/s})]\overline{n} = (\gamma_i n)(r_{/s})$$

where $i \in \{1, \ldots, \rho\}$, and $0.\overline{n} = 0$.

Let $\delta \in \Delta$, $c_{/d} \in G$ then

$$(\delta.(c_{/d}))\overline{n} = [(\gamma_i).(a_{/b}).(c_{/d})]\overline{n}$$

where $\gamma_i(a_{/b}) = \delta$ for some $i \in \{1, \ldots \rho\}$ and $a, b \in S$.

Put $e_{/f} = (a_{/b})(c_{/d})$.

Then $(\delta g)\overline{n} = (\gamma_i(e_{/f}))\overline{n}$ where $g = c_{/d}$

$$= (\gamma_i n)(e_{/f})$$
$$= (\gamma_i n)(a_{/b}).(c_{/d})$$
$$= [\gamma_i\ (a_{/b})]\bar{n}\ (c_{/d})$$
$$= \delta\bar{n}\ (c_{/d})$$

And thus $n \in Q$.

We embed N in Q in the natural way,

let $\xi : N \to Q$ be defined by

$$n\xi = \bar{n} \quad \forall n \in N. \quad \text{Let } n, n_1 \in N$$

Then $(n + n_1)\xi = \overline{n + n_1}$

and $\delta\overline{(n + n_1)} = (\gamma_i(a_{/b}))\ (\overline{n + n_1})$ where $0 \neq \delta = \gamma_i(a_{/b}) \in \Delta$

$$= [\gamma_i(n + n_1)]\ (a_{/b})$$
$$= [\gamma_i n + \gamma_i n_1]\ (a_{/b})$$
$$= \gamma_i n(a_{/b}) + \gamma_i n_1(a_{/b}) \qquad ((a_{/b}) \text{ is auto.})$$
$$= \delta\bar{n} + \delta\bar{n}_1$$
$$= \delta(\bar{n} + \bar{n}_1)$$

Thus $(n + n_1)\xi = n\xi + n_1\xi$

$$(nn_1)\xi = \overline{nn_1}$$

so $\delta(\overline{nn_1}) = (\gamma_i(a_{/b}))(\overline{nn_1})$

$= \gamma_i nn_1.(a_{/b})$. Let $\gamma_i nn_1 = \gamma_j r_1$, $j \in \{1, \ldots, \rho\}$, $r_1 \in S$

$$= (\gamma_j r_1)(a_{/b})$$

$\delta\bar{n}.\bar{n}_1 = [\gamma_i(a_{/b})]\ \bar{n}.\ \bar{n}_1 = [\gamma_i n.(a_{/b})]\ \bar{n}_1$

$= [\gamma_k r_2(a_{/b})]\bar{n}_1$ where $\gamma_i n = \gamma_k r_2$

and $k \in \{1, \ldots, \rho\}$, $r_2 \in S$.

Hence $\delta\bar{n}.\bar{n}_1 = [\gamma_k(r_2 a_{/b})]\bar{n}_1 = \gamma_k n_1(r_2 a_{/b})$.

Let $\gamma_k n_1 = \gamma_\ell r_3$ for some $\ell \in \{1, \ldots, \rho\}$ $r_3 \in S$.

Then $\gamma_i nn_1 = \gamma_j r_1 = (\gamma_k r_2)n_1 = \gamma_\ell r_3 r_2$

and $\delta(\overline{nn_1}) = \gamma_\ell r_3 r_2(a_{/b}) = \gamma_j r_1(a_{/b}) = \delta(\overline{nn_1})$.

Thus $(nn_1)\xi = \overline{nn_1} = \overline{n}.\overline{n_1} = (n\xi)(n_1\xi)$, and ξ is a near-ring homomorphism.

If $n \in \ker \xi$, i.e. $n\xi = 0 = \overline{n}$,

then $\Delta\overline{n} = 0$.

So $\forall \delta \in \Delta$, $\delta\overline{n} = [\gamma_i(a_{/b})]\overline{n} = (\gamma_i n)(a_{/b}) = 0$

In particular $\gamma_i n = 0$, $\forall\ i \in \{1, \ldots, \rho\}$

and n is the zero mapping on Γ.

Thus $n = 0$.

ξ is a monomorphism and we can embed N in Q. In future we will assume that $N \subseteq Q$.

2.8 Remark Q is a 2-primitive near-ring with identity and d.c.c. on right ideals by construction (Ch. 3).

2.9 Definition. With the above notation, let $I = \{1, \ldots, \rho\}$.

Suppose n is any element of N.

For any $k \in I$, define

$$I_k(n) = \{i \in I \mid \gamma_i n \in \gamma_k S\}$$

It is clear that $I_k(n)$ may be empty and that if $\ell \in I$ and $k \neq \ell$, then

$$I_k(n) \cap I_\ell(n) = \phi$$

2.10 Lemma If n is a regular element of N, then the set $\{I_k(n); k \in I\}$ is a permutation of I.

<u>Proof</u> Suppose $k\epsilon I$ and $I_k(n) = \phi$

Define n_1 such that $\gamma_i n_1 = \gamma_i$. $i\epsilon I$, $i \neq k$

$$\gamma_k n_1 = 0$$

Then $\gamma_i n1 = \gamma_i nn_1$ $\quad i \neq k$

$$\gamma_k n1 = \gamma_j s = \gamma_j sn_1 = \gamma_k nn_1$$

where $\gamma_k n = \gamma_j s$ some $j\epsilon I \setminus \{k\}$ and $s\epsilon S$.

Thus $n1 = nn_1 \Rightarrow n_1 = 1$ a contradiction.

Thus $I_k(n) \neq \phi$ for all $k\epsilon I$.

Since I is a finite set, the set $\{I_k(n);\ k\epsilon I\}$ is a permutation of the set $\{1, 2, \ldots, \rho\} = I$

(identifying singleton sets and their elements).

2.11 <u>Theorem</u> If n is a regular element of N, then $\bar{n}$ is invertible in Q, i.e. $\exists q\epsilon Q$ s.t.

$$\bar{n}q = q\bar{n} = 1_Q.$$

<u>Proof</u> Let $\gamma_i n = \gamma_{j_i} \cdot s_i$, $j_i \epsilon I$, $s_i \epsilon S$, for $i = 1, \ldots, \rho$.

Then $\{j_1, \ldots, j_\rho\}$ is a permutation of $\{1, \ldots, \rho\} = I$

Let this permutation be π, i.e.

$$j_i = \pi(i). \quad , \quad i\epsilon I. \quad (\pi \text{ is 1-1 and onto}).$$

Each element s_i, $i\epsilon I$, has an inverse in G, let it be s_i^{-1}. $(i\epsilon I)$.

Define $q:\Delta \to \Delta$ by

$$\begin{cases} 0q = 0, \\ (\gamma_{\pi(i)}\, g_i)q = \gamma_i s_i^{-1} g_i \quad \text{for } i\epsilon I,\ g_i\epsilon G. \end{cases}$$

Let $\delta\epsilon\Delta$, $g\epsilon G$, and $\delta = \gamma_{\pi(i)}\, g_1$ some $i\epsilon I$, $g_1\epsilon G$.

Then $(\delta g)q = (\gamma_{\pi(i)}g_1 g)q$

$$= \gamma_i s_i^{-1}(g_1 g)$$

$$= (\gamma_i s_i^{-1} g_1)g$$

$$= (\gamma_{\pi(i)} g_1)q.g$$

$$= \delta qg$$

Thus $q \varepsilon Q$.

Let $0 \neq \delta' \varepsilon \Delta$, and $\delta' = \gamma_i g_i = \gamma_i(r_{i/t_i})$

where $i \varepsilon I$ and r_i, t_i ε S.

Then $\delta'\bar{n}\, q = (\gamma_i(r_{i/t_i}))\bar{n}q = (\gamma_i n)(r_{i/t_i})q$

$$= (\gamma_{\pi(i)}\, s_i)(r_{i/t_i})q$$

$$= (\gamma_i s_i^{-1} s_i)(r_{i/t_i}) = \gamma_i r_{i/t_i} = \delta'$$

Thus $\bar{n}q = 1_Q$

$\delta' q\bar{n} = (\gamma_i(r_{i/t_i}))q\bar{n} = (\gamma_{\pi(K)}(r_{i/t_i}))q\bar{n}$

for some $K \varepsilon I$, $\pi(K) = i$.

$\delta' q\bar{n} = (\gamma_K s_K^{-1}(r_{i/t_i}))\bar{n} = (\gamma_K n\, s_K^{-1})(r_{i/t_i})$

Now $\gamma_K n = \gamma_{\pi(K)} s_K$ and thus

$\delta' q\bar{n} = (\gamma_{\pi(K)}\, s_K s_K^{-1})(r_{i/t_i})$

$$= \gamma_i(r_{i/t_i}) = \delta'$$

This means that $q\bar{n} = 1 = \bar{n}q$.

Thus we can 'invert' all the regular elements of N, in Q. We will now investigate the form an arbitrary element of Q takes. This gives us:-

2.12 Theorem If x is an arbitrary non-zero element of Q then there is Θ, $n_1 \varepsilon N$ with Θ regular in N such that

$x = n_1\Theta^{-1}$, where Θ^{-1} is the inverse in Q of the element Θ.

We first need the following Lemma.

2.13 <u>Lemma</u> Let $r_1, \ldots, r_\sigma, t_1, \ldots, t_\sigma \in S$ then $\exists\, m \in S, h_1, \ldots, h_\sigma \in S$ such that

$$mr_i = h_i t_i \text{ for } i = 1, \ldots, \sigma.$$

<u>Proof</u> We proceed by induction on σ.

The case $\sigma = 1$, clearly $\exists\, m, h_1 \in S$ such that

$$mr_1 = h_1 t_1.$$

We assume the truth of the lemma for sets of τ elements of S where $1 < \tau \leq \sigma - 1$

So given $r_1, \ldots, r_\tau, t_1, \ldots, t_\tau \in S$,

$\exists\, m', h'_1, \ldots, h'_\tau \in S$

s.t. $m'r_j = h'_j t_j$ for $j = 1, \ldots, \tau$.

$$\exists\, \alpha_{\tau+1}, \beta_{\tau+1} \in S \text{ s.t.}$$

$$\alpha_{\tau+1} r_{\tau+1} = \beta_{\tau+1} t_{\tau+1}$$

$\exists\, z, w \in S$ s.t. $zm' = w\alpha_{\tau+1}$

Putting $m = zm'$, $h_{\tau+1} = w\beta_{\tau+1}$, $h_j = zh'_j$, $j = 1, \ldots, \tau$,

$$mr_j = zm'r_j = zh_j t_j = h_j t_j \text{ for } j = 1, \ldots \tau,$$

$$mr_{\tau+1} = zm'r_{\tau+1} = w\alpha_{\tau+1} r_{\tau+1} = w\beta_{\tau+1} t_{\tau+1} = h_{\tau+1} t_{\tau+1}$$

$\therefore \quad mr_K = h_K t_K$ for all $K = 1, \ldots, \tau + 1$.

The principle of induction gives the result

<u>Proof</u> of 2.12. Let $x \in Q$, $x \neq 0$. Put $X = \{\alpha \in I \mid \gamma_\alpha x = 0\}$

Suppose $x : \gamma_i \to \gamma_{j_i} g_i$, $i \in I \setminus X$, $j_i \in I$, $g_i \in G$

$$x : \gamma_\alpha \to 0 \quad \text{for } \alpha \in X.$$

For any $K \in I$; put

$I_K^*(x) = \{i \in I \ / \ \gamma_i x \varepsilon \gamma_K G\}$ we have $I_K^*(x)$ consisting of ℓ_K elements viz:, $I_K^*(x) = \{ \rho_1^K , \ldots , \rho_{\ell_K}^K \}$ or ϕ .

$I = X \cup \{ \bigcup_{K=1}^{\rho} I_K^*(x)\}$

Assume $\rho_\lambda^K \varepsilon I_K(x)$, for some $k \varepsilon I$

then $\gamma_{\rho_\lambda}{}^K . x = \gamma_K . g_{\rho_\lambda}{}^k = (\gamma_K)(r_{\rho_\lambda}{}^K)/(t_{\rho_\lambda}{}^K)$ $(\lambda = 1, \ldots , \ell_K)$

from the lemma

$\exists m_K, h_{\rho_1}{}^K , \ldots , h_{\rho_{\ell_K}}{}^K \varepsilon S$ s.t.

$(m_K)(r_{\rho_K}{}^\lambda) = (h_{\rho_\lambda}{}^K)(t_{\rho_\lambda}{}^K)$ for all $\lambda = 1, \ldots , \ell_K$

We define a mapping $n_1 : \Gamma \to \Gamma$ by

$(\gamma_{\rho_\lambda}{}^K) . n_1 = \gamma_K . (h_{\rho_\lambda}{}^K)$ $\quad \lambda = 1, \ldots , \ell_K, \quad k = 1, \ldots , \rho$

$0 . n_1 = 0$

$\gamma_\alpha . n_1 = 0$ for $\alpha \varepsilon X$.

It may be seen easily that $n_1 \varepsilon N$.

Let $I'' = \{ j \varepsilon I \mid \gamma_i x \varepsilon \gamma_j G$ for some $i \varepsilon I \}$.

Thus I'' is the set of indices whose associated orbits appear in the image of x in Δ.

Define $\Theta : \Gamma \to \Gamma$ by

$\gamma_t \Theta = \gamma_t m_t \quad \forall t \varepsilon I''$

$\gamma_i \Theta = \gamma_i \quad \forall i \notin I'', (i \varepsilon I)$

$0 . \Theta = 0.$

$\Theta \varepsilon N$ and Θ is a regular element of N and so Θ^{-1} exists in Q, from its construction

$\Theta^{-1} : \gamma_t g \to \gamma_t m_t^{-1} g$ for $t \varepsilon I''$ (The m_t are defined above)

$\Theta^{-1} : \gamma_i g \to \gamma_i g$ for $i \varepsilon I \setminus I''$

Let $y = n\theta^{-1}$

for $i \in I_K^*(x)$, $\gamma_i y = \gamma_i n_1 \theta^{-1} = (\gamma_K h_i)\theta^{-1} = \gamma_K m_K^{-1} h_i$

as $k \in I''$.

Now $m_K r_i = {}_1 t_i$ and so in G

$$r_i t_i^{-1} = m_K^{-1} h_i$$

Thus $\gamma_j y = \gamma_K r_i t_i^{-1} = \gamma_i x$

If $j \in X$, then $\gamma_j y = \gamma_j n_1 \theta^{-1} = 0$

as $\gamma_j x = 0$,

$y = n_1 \theta^{-1} = x$, θ regular in N

Thus Q is a near-ring of right quotients of N.

We summarize in the following theorem.

2.14 Theorem Let Γ be an additive group and S a multiplicative semigroup, with identity, of monomorphisms of $(\Gamma,+) \to (\Gamma,+)$, satisfying :-

(i) S is left and right cancellative

(ii) S is left and right reversible

(iii) Whenever $0 \neq \gamma \in \Gamma$, then $\gamma S_1 = \gamma S_2 \Rightarrow S_1 = S_2$, $(S_1, S_2 \in S)$

(iv) $\Gamma = \{0\} \cup \left\{ \bigcup_{i=1}^{\rho} \gamma_i S \right\}$ for some $\gamma_1, \ldots, \gamma_\rho \in \Gamma$, such that

$$\gamma_i S \cap \gamma_j S = \phi \text{ if } i \neq j$$

Then the near-ring $N = \gamma\gamma\gamma_S(\Gamma)$ has a near-ring of right quotients Q, which is 2-primitive with identity and has d.c.c. on right ideals. In other words N is a right order in Q.

2.15 Proposition N is a 2-prime near-ring, where N is as studied in 2.14.

Proof Assume that $k, k' \in N$ and $kNk' = 0$ with $k \neq 0$ and $k' \neq 0$.

$\exists\, i,j\varepsilon I$ s.t. $\gamma_i k = \gamma_j s$ for some $s\varepsilon S$.

$\exists\, t,r\varepsilon I$ s.t. $\gamma_t k' = \gamma_r s'$ for some $s'\varepsilon S$.

Let $n\varepsilon N$ be defined by

$$n : \gamma_j \to \gamma_t$$
$$n : \gamma_\ell \to 0 \qquad \forall \ell\varepsilon I,\ \ell \neq j.$$

Then $\gamma_i knk' = (\gamma_j s)nk' = ((\gamma_j n)s)k' = (\gamma_t s)k' = (\gamma_t k')s$

$$= \gamma_r s's \neq 0$$

This is clearly a contradiction, and so either k or k' =0.

We will see in the next section that N also possesses certain finiteness conditions. Unlike the last proposition, however, the proof of these does not depend on the special construction of this section and are valid for arbitrary right orders in 2-primitive near-rings with d.c.c. on right ideals and identity. We shall study these in section 3.

We now introduce specific examples of the above construction.

2.16 Examples

(i) Put $\Gamma = < Z, + >$, the additive group of integers.

S = semigroup of positive integers under multiplication

Then $Z = \{0\} \cup 1.S \cup (-1).S$

S has a group of quotients G, the group of rational positive numbers under multiplication. The conditions of Theorem 2.14 are satisfied and the near-ring

$N = \Gamma\Upsilon_S(Z+)$ is a right order in the near-ring

$Q = \gamma\gamma\gamma_G(\Delta)$ which is 2-primitive, has identity and d.c.c. on right ideals.

Here Δ is the additive group of rational numbers, and clearly

$\Delta = \{0\} \cup 1.G \cup (-1).G$

(ii) Let F be the field of real numbers. Put $R = F[x]$, the polynomials in x. Write $F' = \{f \varepsilon F \mid f > 0\}$.

Put S = {set of all $r \varepsilon R$ such that the leading term of r is in F'}

S is left and right cancellative and left and right reversible (in fact commutative). The elements of S act as endomorphisms on F [x] and are in fact monomorphisms.

$rs_1 = rs_2 \Rightarrow s_1 = s_2$ if $0 \neq r \varepsilon R$ and $s_1, s_2 \varepsilon S$.

$R = \{0\} \cup (1)S \cup (-1)S$.

Thus we can apply Theorem 2.14.

§3 Arbitrary right orders in near-rings with d.c.c. on right ideals and an identity

Here N is a right order in Q a near-ring with identity and d.c.c. on right ideals. Thus every regular element of N has an inverse in Q and $q \varepsilon Q$ then $\Rightarrow$ $q = nr^{-1}$ where r is regular in N and $n, r \varepsilon N$.

3.1 Proposition If Q has d.c.c. on right annihilators, then so does N. (A right annihilator is a right ideal of Q which annihilates some non-zero subset of Q on the right)

Proof Suppose $(Z_1)_r \subsetneq (Z_2)_r \subset \ldots\ldots$ is a chain of right annihilators in N (Z_i are subsets of N)

Put $(Z_i)_r = X$. Then clearly

$Z_i XQ = 0 \Rightarrow XQ \subseteq$ right annihilator of Z_i in Q.

Let $Z_i q = 0$, $q \epsilon Q$

then $q = nr^{-1}$, $n, r \epsilon N$, r regular in N.

so $Z_i nr^{-1} = 0 \Rightarrow Z_i n = 0 \Rightarrow n \epsilon (Z_i)_r = X$.

Thus $q \epsilon XQ$.

Then the right annibilator of Z_i in Q is XQ.

Thus $(Z_1)_r Q \subset (Z_2)_r Q$ . . . . .

is a chain of right annibilators in Q, and must terminate, $\exists K$ s.t.

$(Z_K)_r Q = (Z_{K+1})_r Q$

Let $x \epsilon (Z_{K+1})_r$,

$x.1 \epsilon (Z_K)_r Q$

so $x = tnr^{-1}$ some $t \epsilon (Z_K)_r$, $n, r \epsilon N$, r regular.

$Z_K x = Z_K tnr^{-1} = 0$

$x \epsilon (Z_K)_r$

Thus $(Z_{K+1})_r = (Z_K)_r$ for some integer k.

3.2 <u>Proposition</u> If $J_2(Q) = (0)$ and Q has 1 and d.c.c. on right ideals, then every right ideal E of N with the property: that if I is a right ideal of N, then $E \cap I \neq (0)$, $(I \neq 0)$, possesses a regular element.

<u>Proof</u> $Q = a_1 Q \oplus \ldots \oplus a_t Q$ (**) by 2.3.6.

where the $a_i Q$ are right ideals of Q and Q-modules of type 2.

Now $0 \neq a_i \epsilon Q$ so $a_i = n_i r_i^{-1}$, $r_i, n_i \epsilon N$, r_i regular, $(i = 1, \ldots, t)$

thus $n_i = n_i r_i^{-1} r_i = a_i r_i \epsilon N \cap a_i Q$

which shows that $N \cap a_i Q \neq (0)$ $i = 1, \ldots, t$.

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$N \cap a_iQ$ is a right ideal of N, and we have

$$E \cap (N \cap a_iQ) \neq (0)$$

Let $x_i \neq 0$, $x_i \in E \cap N \cap a_iQ$, $x_i = a_iq_i$ for $i = 1, \ldots, t$

Putting $T = x_1Q + \ldots + x_tQ$

$$= a_1q_1Q + \ldots + a_tq_tQ = Q \quad (\text{as } a_iq_iQ \neq (0))$$

Thus $\quad 1 = x_1n_1c_1^{-1} + \ldots + x_tn_tc_t^{-1}$ for $n_1, \ldots, n_t \in N$, $c_1, \ldots, c_t \in N$

and the $c_1, \ldots, c_t$ are regular.

We now show that $\exists$ c regular in N, $n'_1, \ldots, n'_t \in N$ such

that $\quad c = c_in'_i$ for $i = 1, \ldots, t$.

We proceed by induction on the value of t.

As N is a right order in Q, if $n, a \in N$ with a regular, then

$\exists\, m, b \in N$ with b regular such that

$$am = nb.$$

Case $t = 1$ $\exists\, m, b \in N$ s.t. $c_1m = c_1b$ with b regular

since c_1 and b are regular, so is c_1b,

(for $c_1bx = c_1by \Rightarrow bx = by \Rightarrow x = y$, where $x, y \in N$

$xc_1b = yc_1b \Rightarrow xc_1 = yc_1 \Rightarrow x = y$ where $x, y \in N$)

Thus $c_1b = c_1m$ for some $m \in N$.

Assume the induction hypothesis is true for $t = K$.

We can find c' regular in N, $m_1, \ldots, m_K \in N$

such that $c' = c_im_i$. $i = 1, \ldots, K$

$\exists\, x, y \in N$ with x regular such that

$$c'x = c_{K+1}y = c \text{ is regular}$$

Put $n'_i = m_ix$ for $i = 1, \ldots, K$

$n'_{K+1} = y$

Then $c_j n'_j = c$, $j=1, \ldots, K+1$. This proves the statement.

Thus we can find $n'_1, \ldots, n'_t \in N$, c regular in N with

$c_i^{-1} = n'_i c^{-1}$ $\quad i = 1, \ldots, t$.

so $1 = x_1 n_1 n_1' c^{-1} + \ldots + x_t n_t n'_t c^{-1}$

$= a_1 q_1 n_1 n_1' c^{-1} + \ldots + a_t q_t n_t n_t' c^{-1}$

and $c = a_1 q_1 n_1 n_1' + \ldots + a_t q_t n_t n_t'$ $\quad$ as (**) is a direct sum.

$\therefore c = x_1 n_1 n_1' + \ldots + x_t n_t n_t' \in E$

Hence E possesses a regular element.

3.3 Proposition Let c be a regular element of N, then the right N± subgroup cN has the property that if B is any rt. N± group

then $cN \cap B = (0) \Rightarrow B = (0)$.

Proof $Q \subseteq cQ$ since $1 = c.c^{-1} \in cQ$.

Suppose $cN \cap B = (0)$ for some rt. N± group B.

Let $x \in cQ \cap BQ$, then if $x \neq 0$,

$x = cnr^{-1} = jr_1^{-1}$ where $j \in B$, r, r_1 are regular in N.

$\exists\, r_2, n_1, n_2 \in N$, r_2 regular, such that

$r^{-1} = n_1 r_2^{-1}$, $r_1^{-1} = n_2 r_2^{-1}$

Then $x = cnn_1 r_2^{-1} = jn_2 r_2^{-1}$ so $xr_2 = cnn_1 = jn_2$

and $xr_2 \in cN \cap B$. Thus $xr_2 = 0 \Rightarrow x = 0$ as r_2

is regular. Thus $cQ \cap BQ = (0)$

i.e. $Q \cap BQ = (0)$

i.e. $BQ = (0)$

i.e. $B = (0)$

3.4 <u>Remark</u> We could replace the right ideal E of 3.2 by a right N± group F with the property that if B is any N± group and $F \cap B = (0)$ then $B = (0)$. This type of N± group could be called an <u>essential</u> N± group.

These propositions indicate that if essential N± groups exist they contain regular elements, if N is a right order in a J_2-semi-simple near-ring with d.c.c. on right ideals and identity. Conversely, the existence of regular elements in N guarantees a supply of essential N± groups. Finally we prove:

3.5 <u>Proposition</u> N possesses no infinite direct sums of right ideals if $J_2(Q) = (0)$ and Q has d.c.c. on right ideals.

<u>Proof</u> Let $S = \bigoplus_{\alpha \in \Lambda} I_\alpha$ where I_α are non-zero right ideals, Λ is not necessarily finite.

We can assume that S is such that if I is any right ideal of N then $S \cap I = (0) \Rightarrow I = (0)$. (we could otherwise replace S by $S \oplus I$)

S has a regular element $c \in S$

cN is an essential N± group.

Let $c = x_{\alpha_1} + \ldots + x_{\alpha_K}$; where $\alpha_j \in \Lambda$; $x_{\alpha_j} \in I_{\alpha_j}$; $j \in \{1, \ldots, K\}$

$$cN \subseteq I_{\alpha_1} \oplus \ldots \oplus I_{\alpha_K}$$

Suppose $\lambda \in \Lambda$ and $\lambda \notin \{\alpha_1, \ldots, \alpha_K\}$

Then $I_\lambda \cap (I_{\alpha_1} \oplus \ldots \oplus I_{\alpha_K}) = (0)$

so $I_\lambda \cap cN = (0) \Rightarrow I_\lambda = (0)$

Hence $\Lambda = \{\alpha_1, \ldots, \alpha_K\}$ and the result follows.

Chapter 8

Vector Groups and Near Algebras

In the development of the theory of rings, the concept of a vector space over a division ring plays an important part. Many primitive rings can be described as subrings of the ring of linear transformations of a vector space. In this situation a vector space is simply a unital module over the division ring. To a limited extent the picture is similar in the theory of semigroups although this time we are concerned with sets of transformations of what are termed 'vector sets' over a multiplicative group (with zero adjoined in some cases). Near-rings are basically of an asymetric nature and bear the influence of both the above mentioned theories.

§1 Semigroup Operands

An operand with respect to a semigroup is a natural analogue of a module over a ring. We take our definition from Clifford & Preston, Algebraic theory of semigroups, Vol. II (Clifford & Preston [1])

1.1 Let S be a semigroup (written multiplicatively). Then a rt. operand M over S (or a rt. S-system) is simply a set M together with a mapping $(m,s) \to ms$ of $M \times S$ into M such that $$m(s.s_1) = (m.s)s_1 \quad \forall \quad s, s_1 \in S \quad , m \in M.$$

1.2 Given a right S-operand M we define the set of fixed elements of M; $F(M) = \{x \mid x \in M, \ xs = x \ \ \forall s \in S\}$.

This is the set of all the elements of M, which remain fixed under the action of all the elements of S, and usually this will be a trivial set consisting of one element. Since most of our semigroups will possess a zero and a unity element, we usually insist that our operands also possess a unique invariant element, usually written as 0_M. Then if $\bar{0}$ is the zero of the semigroup S, we have

$$m.\bar{0} = 0_M, \qquad \forall\, m \varepsilon M.$$

Clearly $0_M \varepsilon F(M)$ and in fact we usually have $F(M) = \{0_M\}$. If we have a semigroup S and adjoin a zero to it, we will indicate it by $S \cup \{\bar{0}\}$.

1.3 If M, M_1 are two rt. S-operands then we define an <u>S-mapping of M into M_1</u> to be a map ϕ: $M \to M_1$ such that

$$(m.s)\phi = (m.\phi)s. \quad \forall\, m \varepsilon M,\ s \varepsilon S.$$

1.4 An S-operand M is <u>totally irreducible</u> if

(i) $M.S = \{m.s;\ m\varepsilon M,\ s\varepsilon S\} \not\subset FM$.

(ii) Any S-mapping of M into another S-operand M' is either one-one or the set M' has one element only.

1.5 Let S be a semigroup with a zero element 0.

<u>A monomial (row)-matrix A over S</u> is a matrix with elements in S and such that each row contains at most one non-zero element of S. (Clifford & Preston [1] Vol. I, §3.1).

1.6 If M is any S-operand then the semigroup S_1 of the right multiplications of M by elements of S is called the <u>representation of S generated by the S-operand M</u>. (Hoehnke, [1] §1).

1.7 By analogy with Schur's Lemma the set of S-mappings of a totally irreducible S-operand M, if S and M have zero elements, will take the form of a multiplicative group and a zero element. This suggests that the idea of an operand over a group with zero may be fruitful. In fact Hoehnke [1] and Tully [1] have introduced them. Following Hoehnke we call them vector sets.

1.8 A <u>vector set</u> over a multiplicative group G with a zero adjoined, $(G\cup\{\bar{o}\})$ is a set M such that there is a mapping $(m,\gamma) \to m\gamma$ of $M \times (G\cup\{\bar{o}\}) \to M$ such that

(i) $m(\gamma_1\gamma_2) = (m\gamma_1)\gamma_2 \qquad \forall m\varepsilon M, \quad \gamma_1,\gamma_2\varepsilon G\cup\{\bar{o}\}$

(ii) $m.1 = m \qquad \forall\, m\varepsilon M.$

(iii) If $F(M) = \{m\varepsilon M \mid m\gamma = m \quad \forall\, \gamma\varepsilon G\cup\{\bar{o}\}\}$

then $|F(M)| \leq 1$ or $F(M) = M$

(iv) $F(M) = M \Rightarrow |G| = 1$

(v) $\phi \neq F(M) \neq M \Rightarrow [(M\setminus F(M))(G\cup\{\bar{o}\})]\cap F(M) \neq \phi$

(vi) $\forall\delta,\ \gamma\varepsilon G\cup\{\bar{o}\}$, if $m\varepsilon M$

then $m\gamma = m\delta \Rightarrow \gamma = \delta$ or $m\varepsilon F(M)$.

Note that if $\phi \neq F(M) \neq M$ then $|F(M)| = 1, \quad F(M) = M.\bar{o}$.

1.9 <u>Theorem</u> (Hoehnke [1]). Let M be an S-operand and also a vector set over the group with zero $G\cup\{\bar{o}\}$, such that

$(ms)\gamma = (m\gamma)s \qquad \forall\, m\varepsilon M,\ s\varepsilon S,\ \gamma\varepsilon G\cup\{\bar{o}\}$.

Let $F_S(M) = \{x \mid x\varepsilon M,\ xs = x,\ \forall\, s\varepsilon S\}$.

$F_G(M) = \{x \mid x\varepsilon M,\ x\gamma = x,\ \forall\delta\varepsilon\ G\cup\{\bar{o}\}\}$.

Then if $F_G(M) \subseteq F_S(M)$ and $|M| > |F_G(M)| = 1$ then the representation, S_1, of S generated by the S-system M can be interpreted as a semigroup of monomial matrices over $G\cup\{\bar{o}\}$.

We now consider the natural corresponding concept in

§2. Vector Groups

A 2-primitive near-ring with d.c.c. on right ideals and an identity is the set of all mappings of an additive group Γ, which commute with a group of regular automorphisms, G, where G induces a finite number of orbitson Γ. This is the motivation for the following definition.

2.1 A Vector group over a multiplicative group G with a zero adjoined, is an additive group V and a mapping $(v,\gamma) \to v\gamma$ of $V \times (G \cup \{\bar{o}\}) \longrightarrow V$ such that

(i) $v.(g_1 g_2) = (v.g_1).g_2 \quad \forall\, v\varepsilon V,\ g_1,\ g_2\ \varepsilon G \cup \{\bar{o}\}$.

(ii) $v.0 = 0_v$.

(iii) $(v + v_1).g = v.g + v_1 g \quad \forall\, v,\ v_1 \varepsilon V. \quad g\varepsilon\, G$

(iv) $v.1 = v$.

(v) $v.g = v \Rightarrow v = 0$ or $g = 1$

We will sometimes call these G-vector groups for short.

2.2 If V_1 and V_2 are both G-vector groups then a G-transformation of V_1 into V_2 is a mapping $\psi: V_1 \longrightarrow V_2$ with the property that $(v_1\gamma)\psi = (v_1\psi)\gamma\ . \quad \forall\, v_1 \varepsilon V_1,\ \gamma\varepsilon G \cup \{\bar{o}\}$.

A G-homomorphism of V_1 into V_2 is a G-transformation of V_1 into V_2 which is also a group homomorphism of V_1 into V_2.

2.3 The set of G-transformations of a vector group V over G into itself is a 2-primitive near-ring with identity.

2.4 If $g\varepsilon G$ then $V = Vg$ and the elements of G are automorphisms of the additive group V, and in fact G acts as a regular permutation group on the elements of V.

2.5 A vector subgroup of a vector group over $G \cup \{\bar{o}\}$ is a subset which is also a vector group over G in its own right.

2 6 If $vg = vg_1$ for some $v \neq 0$, $g, g_1 \varepsilon G$, then $v = vg_1g^{-1} \Rightarrow$ $g_1 = g$

We can define an equivalence relation ρ on V as follows.

$v_1 \rho v_2 \Leftrightarrow v_2 = v_1 g$ for some $g \varepsilon G$. If we assemble representatives of these equivalence classes we obtain a decomposition of V as follows:

$V = \left(\bigcup_{v_i \varepsilon W} v_i \; G \right) \cup \{o\}$ where the set W is a set of representatives of the non-zero classes of V. Each element of V has a unique representation in the respect that if $v \neq 0$, then $v = v_i g$ for some $v_i \varepsilon W$ and $g \varepsilon G$ and has no other representation in terms of the particular representatives in question.

2.7 Let V,V' be any vector groups over $G \cup \{\bar{o}\}$ then the set $\Psi = \gamma\gamma_G(V, V')$, i.e. the set of C-transformations of $V \to V'$, is a vector group over $G \cup \{\bar{o}\}$.

Proof. We define $(v)(\psi_1 + \psi_2) = (v)\psi_1 + (v)\psi_2$ for $\psi_1, \psi_2 \varepsilon \Psi$ and $v(\psi\gamma) = [(v)\psi]\gamma$ for $\gamma \varepsilon G \cup \{\bar{o}\}$

The conditions of 2.1 are satisfied.

2.8 For any two vector groups V,V' over $G \cup \{\bar{o}\}$, the set

$H = \mathrm{Hom}_G(V, V')$, the set of G-homomorphisms of $V \to V'$

is a vector set over $G \cup \{\bar{o}\}$

Proof. $F(H) = \{h \mid h\gamma = h \;\; \forall \gamma \varepsilon G \cup \{\bar{o}\}\}$

Suppose $\exists \, g \varepsilon G$ with $g \neq 1$. Then for any $h \varepsilon F(H)$, $v \varepsilon V$; $(vh)g = vh$ but V' is a vector group and so $vh = 0$ for all $v \varepsilon V$. Thus h is the zero mapping. $|F(H)| = 1$. If $G = \{1\}$, then $h.\bar{o} = h \Rightarrow h = 0$ and so $F(H) = 0_H$ as before.

Now we have to check that

$$[(H\setminus O_H)(G\cup\{\bar{0}\})]\cap\{O_H\}\neq\phi.$$

In fact the intersection is clearly equal to $\{O_H\}$.

Also, if $hg = hg_1$ for some $g, g_1 \in G$ and $h\in H$ then $h \neq 0 \Rightarrow$ for any $v\in V$:

$(vh)g = (vh)g_1 \Rightarrow g = g_1$. If $hg = h\bar{0} = O_H$ then $h = O_H g^{-1} = O_H$

2.9 Proposition. A vector group is a vector set.

Proof. If V is a vector group over $G\cup\{\bar{0}\}$, then

$F(V) = \{O_V\}$ and so $|F(V)| = 1$.

Also, $[(V\setminus F(V))(G\cup\{\bar{0}\})]\cap F(V) = \{O_V\}\neq\phi$.

Finally if $v\gamma = v\delta$ for some $\gamma, \delta\in G\cup\{\bar{0}\}$.

Let $\gamma\neq 0$, then $v = v\delta\gamma^{-1} \Rightarrow v = 0$ or $\delta = \gamma$.

If $\gamma = 0$ then $v\delta = 0$ so either $\delta = 0$

or $v = v\delta\delta^{-1} = 0.\delta^{-1} = 0$

All the conditions in 1.8 are satisfied.

We finally move to a result on the lines of 1.9.

2.10 Theorem. Let N be a 2-primitive near-ring with identity and d.c.c. on right ideals. If Γ is a faithful N-module of type 2, then each element of N may be represented by a monomial matrix with elements in $Z = \mathrm{Hom}_N(\Gamma)$.

Each matrix will be m x m where m is the number of orbits induced on Γ by the group $(Z\setminus\{0\})$.

Proof. If $\Gamma = \{0\}\cup\gamma_1 G\cup\ldots\cup\gamma_m G$

where $G = Z\setminus\{0\}$ and $\gamma_1, \ldots, \gamma_m$ are orbit representatives in Γ.

Each element $n\in N$ is determined (as a right multiplication of Γ), by its action on the elements

$$\gamma_1, \ldots, \gamma_m.$$

Let $n \in N$ and suppose

$$\gamma_i n = \gamma_{j_i} z_i \qquad i = 1, \ldots, m, \ z_i \in Z, \quad j_i \in \{1, \ldots, m\}.$$

We would then have, corresponding to n, the matrix with zero's everywhere except at the (i, j_i)-th entries where we would have z_i respectively. This is for $i = 1, \ldots, m$.

2.11 Example. If $m = 6$, let $n \in N$ such that

$\gamma_1 n = \gamma_3 g$ $\quad \gamma_2 n = 0,$ $\quad \gamma_3 n = \gamma_4 g_1,$ $\quad \gamma_4 n = 0,$

$\gamma_5 n = \gamma_1$ $\quad \gamma_6 n = \gamma_6.$ The matrix would be written as:-

$$\begin{pmatrix} 0 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Whereas this set of monomial matrices may be anti-isomorphic as a semigroup under multiplication to the multiplicative semigroup of the near-ring N, it would require a good knowledge of the way the orbits on Γ behave with respect to the additive structure of that set before we would be in a position to go any further in the direction of monomial representation of near-rings.

§3 Isomorphism Theorems

We are looking at the uniqueness of the representation of a near-ring as the near-ring of mappings of a vector group commuting with the group of automorphisms.

Following Jacobson [1],

3.1 Definition. Let V be a vector group over a multiplicative group with an adjoined zero $G \cup \{\bar{0}\}$, and V_1 a vector group over $G_1 \cup \{\bar{\bar{0}}\}$ (G_1 is a mult. gp., $\bar{\bar{0}}$ an adjoined zero). A mapping S of V into V_1 is a semi-transformation if and only if there exists an isomorphism σ of G_1 onto G_2 such that for all $v \in V$ and $g \in G$, $(vg)S = (vS)(g\sigma)$, and also if S is such that $0_V.S = 0_{V_1}$. The isomorphism σ will be called the 'isomorphism associated with S.'

3.2 Definition. A semi-linear transformation of a vector group V over $G \cup \{\bar{0}\}$ into a vector group V_1 over $G_1 \cup \{\bar{\bar{0}}\}$, is a semi-transformation which is at the same time a homomorphism of $(V,+)$ into $(V_1,+)$.

3.3 Proposition. Let N be a 2-primitive near-ring with identity and with right ideals which are N-modules of type 2. Then any two faithful N-modules of type 2 are isomorphic.

Proof. Let F be a right ideal which is of type 2 as an N-module. Put Γ as any faithful N-module of type 2. There is $\gamma \in \Gamma$ with $\Gamma = \gamma F$. The mapping $\phi: F \to \Gamma$, where $f\phi = \gamma f$ is an epimorphism of N-modules. Clearly $\ker \phi = (\gamma)_r \cap F = (0)$. Thus ϕ is an isomorphism. Hence every faithful N-module of type 2 is isomorphic to F.

3.4 __Lemma.__ Let Γ be a vector group over a multiplicative group with zero adjoined. $G \cup \{\bar{0}\}$, where G induces a finite number of orbits on Γ. Put $N = \gamma\gamma_G(\Gamma)$. Then the only endomorphisms of $(\Gamma,+)$ which commute with the elements of N are elements of $G \cup \{\bar{0}\}$

__Proof.__ Let H be the set $End_N(\Gamma)$.

Clearly $G \subseteq H$. Pick $h \in H$, $h \neq 0$.

Let $\gamma \in \Gamma$ with $\gamma \neq 0$. Then we assume that γ and γh belong to different orbits of Γ. We choose an $n \in N$ such that $\gamma n = 0$ and $(\gamma h)n \neq 0$.

Then $(\gamma h)n = (\gamma n)h = 0$ produces a contradiction.

Thus γ and γh belong to the same orbit.

Therefore $\gamma h = \gamma g$ for some $g \in G$.

Now $\Gamma = \gamma N$ and so for any $\gamma_1 \in \Gamma$,

$$\gamma_1 h = \gamma n_1 h = \gamma h n_1 = \gamma g n_1 = \gamma n_1 g = \gamma_1 g \quad \text{where } \gamma_1 = \gamma n_1$$

for a suitable $n_1 \in N$.

Thus h and g are the same mapping.

3.5 __Theorem.__ Let N_i, $i = 1, 2$, be the near-ring $\gamma\gamma_{G_i}(\Gamma_i)$ where Γ_i is a vector group over $G_i \cup \{\bar{0}\}$ and G_i induces a finite number of orbits on Γ_i. $(i = 1, 2)$.

Then s is an isomorphism of N_1 onto N_2 if and only if there exists a 1-1 semi-linear transformation S of Γ_1 onto Γ_2 such that $n_1 s = S^{-1} n_1 S$ for all $n_1 \in N_1$.

<u>Proof.</u> Let S be a 1-1 semi-linear transformation of Γ_1 onto Γ_2. If $n_1 \varepsilon N_1$ then $S^{-1}n_1S : \Gamma_2 \longrightarrow \Gamma_2$.

$$(\gamma_2 g_2)S^{-1}n_1S = (\gamma_2 S^{-1}Sg_2)S^{-1}n_1S \qquad (g_2 \varepsilon G_2)$$

$$= (\gamma_2 S^{-1}S)(g_1\sigma)S^{-1}n_1S \quad \text{where } g_2 = g_1\sigma. \quad (g_1 \varepsilon G_1)$$

$$= (\gamma_2 S^{-1}g_1)SS^{-1}n_1S \qquad = (\gamma_2 S^{-1}g_1)n_1S$$

$$= (\gamma_2 S^{-1}n_1g_1)S \qquad = (\gamma_2 S^{-1}n_1S)(g_1\sigma)$$

$$= (\gamma_2 S^{-1}n_1S)g_2$$

Thus $S^{-1}n_1S \varepsilon N_2$. Also $Sn_2S^{-1} \varepsilon N_1 \; \forall \, n_2 \varepsilon N_2$.

We show that $n_1 \rightarrow S^{-1}n_1S$ is an isomorphism of near-rings.

$$n_1 + n_1' \longrightarrow S^{-1}(n_1 + n_1')S$$

now $\gamma_1 S^{-1}(n_1 + n_1')S = \gamma_1 S^{-1}n_1S + \gamma_1 S^{-1}n_1' S$

$$= \gamma_1(S^{-1}n_1S + S^{-1}n_1' S)$$

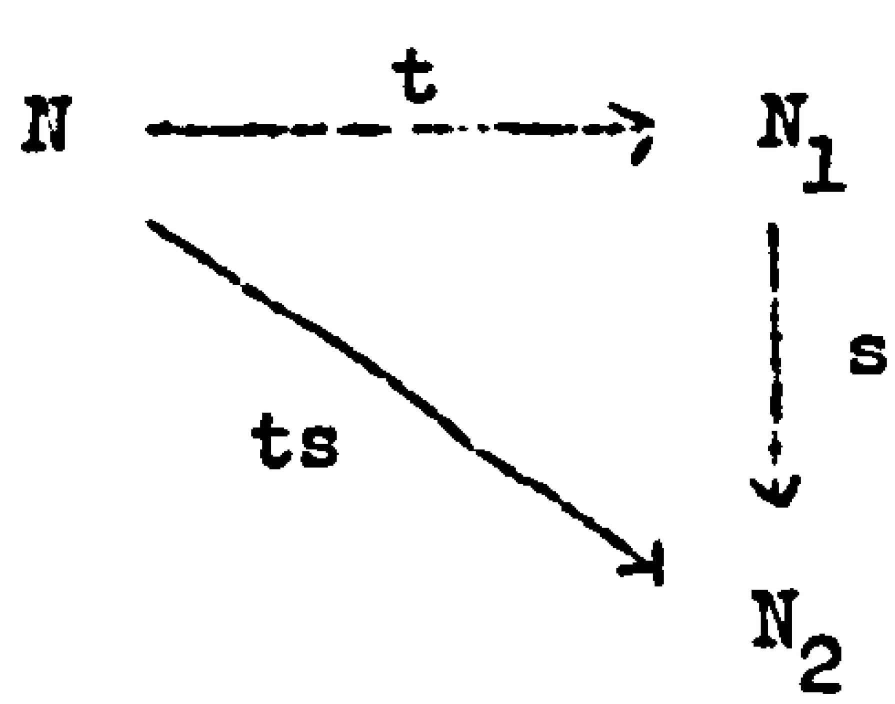
and $\gamma_1 S^{-1}(n_1 . n_1')S = \gamma_1(S^{-1}n_1S)(S^{-1}n_1'S)$

Thus $N_1 \cong N_2$.

Conversely we assume that $N_1 \cong N_2$ under the mapping s.

Let N be an abstract near-ring with the isomorphism $n \rightarrow nt$ of N onto N_1.

Then the mapping $n \rightarrow nts$ is an isomorphism of N onto N_2, i.e.



commutes.

Γ_1 and Γ_2 are N-modules, and in fact are of type 2 and faithful. N has right ideals of type 2 since N_1 has. Thus by 3.3 Γ_1 and Γ_2 are isomorphic N-modules.

Let S be the isomorphism of the N-module Γ_1 onto the N-module Γ_2. Then

$$(\gamma_1 n_1)S = (\gamma_1(nt))S = (\gamma_1 S)nts = (\gamma_1 S)(n_1 s)$$

For all $\gamma_1 \varepsilon \Gamma_1$ and $n_1 \varepsilon N_1$.

Thus if $\gamma_1 = \gamma_2 S^{-1}$ we have

$$\gamma_2 S^{-1} n_1 S = \gamma_2 S^{-1} S n_1 s = \gamma_2 n_1 s$$

hence on Γ_2, $n_1 s = S^{-1} n_1 S$.

Let $g_1' \varepsilon G_1$ then $S^{-1} g_1' S : \Gamma_2 \to \Gamma_2$ and is a homomorphism.

Also $\gamma_2(S^{-1}\bar{n}_1 S)(S^{-1} g_1' S) = \gamma_1 \bar{n}_1 g_1' S = \gamma_1 g_1' \bar{n}_1 S$

$$= \gamma_2(S^{-1} g_1' S)(S^{-1}\bar{n}_1 S) \qquad \forall\ \bar{n}_1 \varepsilon N_1$$

Then by 3.4 $S^{-1} g_1' S \ \varepsilon\ G_2$.

Put $\sigma : G_1 \to G_2$ where $\bar{\bar{g}}_1 \sigma = S^{-1} \bar{\bar{g}}_1 S$ for all $\bar{\bar{g}}_1 \varepsilon G_1$.

Then $(\bar{g}_1 g_1')\sigma = S^{-1} \bar{g}_1 g_1' S = S^{-1} \bar{g}_1 S . S^{-1} g_1' S = \bar{g}_1 \sigma \cdot g_1' \sigma$. $\quad \forall\ \bar{g}_1, g_1' \varepsilon G_1$

Let $g_2' \varepsilon G_2$ then by 3.4 we may consider the mapping $S g_2' S^{-1}$ to be an element of G_1.

Then $\quad S g_2' S^{-1} = g_1''$ for some $g_1'' \varepsilon G_1$ and $g_2' = S^{-1} g_1'' S$ which implies that σ is one too.

Now $S(g_1'\sigma) = S.S^{-1} g_1' S = g_1' S$ we have $(\gamma_1 g_1')S = (\gamma_1 S)(g_1'\sigma)$.

Hence S is a semi-linear transformation with associated isomorphism σ.

3.6 <u>Theorem</u>. Let Γ_1 and Γ_2 be vector groups over $G_1 \cup \{\bar{0}\}$ and $G_2 \cup \{\bar{0}\}$ respectively, and let m_i be the number of orbits induced in Γ_i by G_i, and suppose $m_i < \infty$, $i = 1, 2$.
If there exists a semi-linear transformation S of Γ_1 onto Γ_2 which is also 1:1 then $G_1 \cong G_2$ and $m_1 = m_2$.

Proof Clearly $G_1 \cong G_2$.

Let $\gamma_{11}, \gamma_{12}, \ldots, \gamma_{1m_1}S$ be orbit representatives for Γ_1 over G_1. We now show that $\gamma_{11}S, \gamma_{12}S, \ldots, \gamma_{1m_1}S$ are orbit representatives for Γ_2 over G_2.

Let $\gamma_2 \varepsilon \Gamma_2$ then $\gamma_2 = \gamma_1 S$ for some $\gamma_1 \varepsilon \Gamma_1$.

Let $\gamma_1 = \gamma_{1j} g$ with $g \varepsilon G_1$.

Thus $\gamma_2 = (\gamma_{1j}g)S = (\gamma_{1j}S)(g\sigma)$ (σ the associated isomorphism)

Then γ_2 belongs to the orbit generated by $\gamma_{1j}S$ and G_2.

Hence $\Gamma_2 \subseteq \{OS\} \cup \bigcup_{j=1}^{m_1} (\gamma_{1j}S)G_2$.   (Where O is the zero of Γ_1.)

Suppose now that $\gamma_2' \varepsilon (\gamma_{1j}S)G_2 \cap (\gamma_{1k}S)G_2$

say $\gamma_2' = (\gamma_{1j}S)g_2 = (\gamma_{1k}S)g_2'$.

Then $g_2 = g_1\sigma$ and $g_2' = g_1'\sigma$ for some $g_1, g_1' \varepsilon G_1$

Thus $(\gamma_{1j}S)(g_1\sigma) = (\gamma_{1k}S)(g_1'\sigma)$ i.e. $(\gamma_{1j}g_1)S = (\gamma_{1k}g_1')S$

Applying S^{-1} to both sides give us $\gamma_{1j}g_1 = \gamma_{1k}g_1'$ and so

γ_{ij} and γ_{1k} belong to the same orbit of Γ_1. That is $j = k$.

Hence $\Gamma_2 = \{OS\} \cup \bigcup_{j=i}^{m} (\gamma_{1j}S)G_2$

and $\gamma_{1j}S \cap \gamma_{ik}S = \phi$ if $j \neq k$.

Hence Γ_2 has m_1 orbits induced on it by G_2.

§4

Near Algebras

Some authors have introduced near-algebras as being near-rings and at the same time vector spaces over division rings (with a rule for tying in the two operations)

See Brown . [1] and Yamamuro [1].

From our development we are led to an alternative definition partly because, whereas in ring theory the division rings that appear in algebras often arise naturally from the original ring, for example in the form of the centre or a centralizer of some module, in near-rings these corresponding objects rarely have any additive structure. We usually find ourselves with a multiplicative semigroup.

4.1 Definition. If A is a near-ring and $G \cup \{\bar{0}\}$ a multiplicative group with a zero adjoined, then we call A a(right) G-near-algebra if the following conditions hold:

(i) (A, +) is a G-vector group

(ii) $\forall\, a_1, a_2 \in A, \quad \gamma \in G \cup \{\bar{0}\}$.

$$(a_1 . a_2).\gamma = a_1(a_2\gamma) = (a_1\gamma).a_2.$$

4.2 Let N be any near-ring with identity and let $C = \{x \in N \mid xn = nx \ \forall n \in N\}$ If G is any group contained in $C \setminus \{0\}$, then we can make N into a G-near-algebra if and only if, whenever $n \neq 0$, $n \in N$, then $ng = n \Rightarrow g = 1$. (This restriction is unnecessary if G is a field, for then if $g \neq 1$, then $g - 1$ has an inverse and so whenever $ng = n$, then $n(g - 1) = 0$, and so

$$0 = n(g - 1) = n(g - 1)(g - 1)^{-1} = n = 0.)$$

However perhaps the most important example of a G-near-algebra is the following:-

Let N be a 2-primitive near-ring with identity and d.c.c. on right ideals. Then $N \cong \mathcal{M}_G(\Gamma)$ for suitable groups Γ and G. Then N is a G-near-algebra. (see 2.7)

4.3 Definitions Let A be a G-near-algebra.

(i) A (G,A)-module, M, is an A-module (in the near-ring sense), M is also a G-vector group and

$$(ma)\alpha = m(a\alpha) = (m\alpha)a \quad \forall m \in M,\ a \in A, \alpha \in G \setminus \{0\}.$$

(ii) A (G,A)-subgroup of a (G,A)-module M, is an A-subgroup M', of M, which is also a (G,A)-module

(iii) A (G,A)-submodule of M is an A-submodule, M", which is also a (G,A)-module

(iv) A (G,A)-module M is of type 2 if M contains no non-trivial (G,A)-subgroups and $MA \neq \{0\}$.

(v) A (G,A)-module M is of type 0 if M contains no non-trivial (G,A)-submodules and $\exists\, m \in M$, $m \neq 0$, such that

$$M = mA.$$

(vi) A right ideal of a G-near-algebra A (as a near-algebra rt. ideal) is a (G,A)-submodule of the (G,A)-module A.

(vii) A ν-modular right ideal of a G-near-algebra A is a right ideal R of A (as a G-near-algebra) such that $^{A}/_{R}$ is a (G,A)-module of type ν, where ν = 0,2.

4.4 Proposition. If A is a G-near-algebra, then the set of all 2-modular right ideals of A as a near-ring, is contained in the set of all 2-modular right ideals of A as a G-near-algebra.

Proof. Let F be a 2-modular right ideal in A as a near-ring. We have an $e \in A$ such that

$$ea - a \in F \quad \text{for all } a \in A.$$

We have to show that $F\alpha \subseteq F$ for all $\alpha \in G \setminus \{\bar{0}\}$.

If $\alpha \varepsilon G \cup \{\bar{0}\}$ then $F\alpha$ is a rt. A ±group of the near-ring A. If $F\alpha \not\subseteq F$ then $A = F + F\alpha$ so let $e = b_1 + b_2\alpha$ for $b_1, b_2 \varepsilon F$.

$e^2 = (b_1 + b_2\alpha)e = b_3 + (b_2\alpha)e$ for some $b_3 \varepsilon F$ as F is a right ideal of A. $e^2 = b_3 + b_2 (e\alpha) \varepsilon F$.

Now $e - e^2 \varepsilon F$ and hence $e \varepsilon F \Rightarrow F = A$ which is a contradiction. Hence $F\alpha \subseteq F$ $\forall \alpha \varepsilon \{\bar{0}\} \cup G$.

Thus F is a (G,A)-submodule of the (G,A)-module A. Hence $^A/F$ is a (G,A)-module of type 2.

4.5 Suppose now that F is a 0-modular right ideal of A as a G-near-algebra, then $^A/F$ is a (G,A)-module of type 0. F is a proper modular right ideal of A as a near-ring. F can be embedded in K, a 0-modular right ideal of the near-ring A. Now a 0-modular right ideal of the near-ring A is a 0-modular right ideal of the G-near-algebra A. The proof of this statement is identical to 4.4 except that we notice the fact that the set $F\alpha$ is a right ideal of A, where F is the 0-modular right ideal of A as a near-ring. If $x = (a + f\alpha)a_1 - aa_1$ then $\alpha \neq 0$, (Note $F.0 = 0_A$) $\Rightarrow$

$$x\alpha^{-1} = (a\alpha^{-1} + f)a_1 - (a\alpha^{-1})a_1 \varepsilon F. \quad \text{Hence } x \varepsilon F\alpha.$$

Thus we have

<u>Proposition.</u> The set of 0-modular G-near-algebra right ideals coincides with the set of 0-modular near-ring right ideals (Note that if $F \subseteq K$ and both are 0-modular G-near-algebra rt. ideals, then $K = F$).

4.6 <u>Corollary.</u> $D(A) = D^G(A)$

where $D^G(A) = \bigcap$ all the 0-modular G-near-algebra right ideals (If none exist we put $D^G(A) = A$).

We need not, now, distinguish between the D-radical of A as a near-ring or as a G-near-algebra.

4.7 <u>Proposition.</u> Suppose that A is a G-near-algebra and A has an identity. Every (G,A)-module of type 2 is an A-module of type 2. Any A-module of type 2 can be regarded in one and only one way as a (G,A)-module of type 2.

<u>Proof.</u> If M is a (G,A)-module of type 2 then $M \neq 0$ and for any $0 \neq m \varepsilon M$, $mA = M$.

Hence M is an A-module of type 2.

If M is an A-module of type 2, choose $0 \neq m \varepsilon M$. Then $M = mA$.

If $\alpha \varepsilon G \cup \{\bar{0}\}$ define $(ma)\alpha = m(a\alpha)$ $\forall a \varepsilon A$.

This is well defined for if $mb = 0$, $b \varepsilon A$, then $b \varepsilon (m)_r$, now $(m)_r$ is a 2-modular right ideal of A as a near-ring and so

$(m)_r . \alpha \subseteq (m)_r$ $\forall \alpha \varepsilon G \cup \{0\}$ by 4.4.

Thus $m(b\alpha) = (mb)\alpha = 0$.

M is a (G,A)-module of type 2 defined in a unique way.

4.8 <u>Corollary.</u> Every A-module of type 0 is (G,A)-module of type 0.

4.9 <u>Definition.</u> For any G-near-algebra A we define

$$J_\nu^G(A) = \bigcap_{\Gamma \varepsilon \mathfrak{G}_\nu} (\Gamma)_r \qquad \nu = 0,2$$

where $\mathfrak{G}_\nu$ is the set of all (G,A)-modules of type $\nu = 0,2$.

If $\mathfrak{G}_\nu = \phi$ we define $J_\nu^G(A) = A$.

4.10 Theorem. For any G-near-algebra A,

(i) $J_2(A) = J_2^G(A)$, if A has an identity.

(ii) $J_0(A) = J_0^G(A)$

(iii) $D(A) = D^G(A)$.

Proof. This consists only of proving the converse to corollary 4.8. Let M be a (G,A)-module of type 0. Then

$M = mA$ for some $0 \neq m \in M$.

$(m)_r$ is a right ideal of A as a G-near-algebra and $M \cong A/(m)_r$ as (G,A)-modules. Thus $(m)_r$ is a 0-modular right ideal of A as a G-near-algebra. From 4.5 $(m)_r$ is a 0-modular right ideal of A as a near-ring. Hence $A/(m)_r$ is of type 0 as an A-module. But $M \cong N/(m)_r$ as A-modules and so M is an A-module of type 0.

This proves $J_0(A) \subseteq J_0^G(A)$

4.11 In theorem 4.10 the existence of an identity in A is only required for statement (i). Without this restriction we would only know that $J_2^G(A) \subseteq J_2(A)$.

BIBLIOGRAPHY

Beidleman J.C. [1]. 'Quasi-regularity in near-rings'. Math Z. 89, (1965), 224-229.

[2] 'A radical for near-ring modules'. Michigan Math. J. 12. (1965), 377-383.

[3] 'Distributively-generated near-rings with descending chain condition'. Math Z. 91, (1966), 65-69.

[4] 'Strictly prime distributively-generated near-rings'. Math. Z, 100, (1967), 97-105.

[5] 'Non semi-simple distributively-generated near-rings with descending chain condition'. Math. Ann. 170, (1967), 206-213.

[6] 'On the theory of radicals in d.g. near-rings.' 'I. The primitive radical.' Math. Ann. 173, (1967), 89-101.
'II. The nil radical'. Math. Ann. 173, (1967), 200-218.

Betsch G. [1]. 'Ein Radikal für Fastringe'. Math. Z. 78, (1962), 86-90.

[2]. 'Struktursatze für Fastringe'. Doctoral Dissertation University of Tübingen, 1963.

Blackett D. W. [1]. 'Simple and semi-simple near rings'. Proc. Amer. Soc. 4, (1953), 772-785.

Brown H. [1]. 'Near-algebras'. Illinois J. of Math. 12, (1968), 214-227.

Clifford A. H. & Preston G. B. [1]. 'The Algebraic theory of semigroups. Vol. I, II. Amer Math. Soc. Publications.

Frohlich A. [1]. 'Distributively-generated near-rings.'

'I. Ideal Theory'. Proc. London Math. Soc. (3). 8. (1958), 76-94.

'II. Representation Theory'. Proc. London Math. Soc. (3). 8. (1958), 95-108.

[2]. 'The near-ring generated by the inner automorphisms of a finite simple group'. J. London Math. Soc. 33. (1958), 95-107.

[3]. 'On groups over a d.g. near-ring'

'I. Sum constructions and free R-groups'. Quart. J. Math. Oxford, ser. (2) 11, (1960), 193-210

'II. Categories and functors'. Quart. J. Math. Oxford. Ser(2), 11 (1960), 211-228.

[4] 'Non-abelian homological algebra'.

'I. Derived functors and satellites', Proc. London Math. Soc., (3), 11, (1961), 239-275.

'II. Varieties'. Proc. London Math. Soc., (3) 12, (1962), 1-28.

'III. The functors ext. and tor.' Proc. London Math. Soc., (3), 12, (1962), 739-768.

Goldie A. W. [1]. 'Non-commutative Noetherian rings'. (1967). Notes. York University, Toronto.

Hoehnke H-J. [1]. 'Structure of Semigroups'. Can. J. Math. 18, (1966), 449-491.

Jacobson N. [1]. 'The Structure of Rings'. Amer. Math. Soc. Colloquium Publications.

Laxton R. R. [1]. 'Primitive distributively-generated near-rings'. Mathematika, 8, (1961), 143-158.

[2]. 'A radical and its theory for distributively generated near-rings'. J. London Math. Soc. 38, (1963), 40-49.

[3]. 'Prime ideals and the ideal-radical of a distributively generated near-ring'. Math. Z. 83, (1964), 8-17.

Ramakotaiah D. [1]. 'Radicals for near-rings'. Math. Z. 97, (1967), 45-56.

Tharmaratnam V. [1]. 'Complete primitive distributively generated near-rings'. Quart. J. Math. Oxford Ser. (2). 18, (1967). 293-313.

Thompson J. [1]. 'Finite groups with fixed-point-free automorphisms of prime order' Proc. Nation. Acad. Science. 45. (1959). 578-580.

Tully E. J. [1]. 'Representation of a semigroup by transformations acting transitively on a set'. Amer. J. Math. 83. (1961). 533-541.

Van der Walt A. P. J. [1]. 'Prime ideals and nil radicals in near-rings'. Arch. Math. 15. (196 408-414.

Wielandt H. [1]. 'Über Bereiche aus Gruppenabbildungen . Deutsche Mathematik, 3, (1938), 9-10.

Yamamuro S. [1]. 'On near-algebras of mappings'. Proc. Japan Acad. 41, (1965). 889-892.

[2]. 'A note on d-ideals in near-algebras'. J. Australian Math. Soc. VII. (2). (1967), 135-140

Zassenhaus H. [1]. 'Über endliche Fastkörper'. Abh. Math. Sem. Univ. Hamburg. 11, (1935-36), 187-220.