Analysis Of Some Deterministic & Stochastic Evolution Equations With Solutions Taking Values In An Infinite Dimensional Hilbert Manifold

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Abstract

The objective of this thesis is threefold:

Firstly, to deal with the deterministic problem consisting of non-linear heat equation of gradient type. It comes out as projecting the Laplace operator with Dirichlet boundary conditions and polynomial nonlinearly of degree $2n-1$, onto the tangent space of a sphere M in a Hilbert space H. We are going to deal with questions of the existence and the uniqueness of a global solution, and the invariance of manifold M i.e. if the suitable initial data lives on M then all trajectories of solutions also belong to M.

Secondly, to generalize the deterministic model to its stochastic version i.e. stochastic non-linear heat equation driven by the noise of Stratonovich type. We are going to show that if the suitable initial data belongs to manifold M then M-valued unique global solution to the generalized stochastic model exists.

Thirdly, to investigate the small noise asymptotics of the stochastic model. A Freidlin-Wentzell large deviation principle is established for the laws of solutions of stochastic heat equation on Hilbert manifold.

Keywords: evolution equations, projections, Hilbert manifold, Stochastic evolution equations, large deviation principle, weak convergence approach.

Contents

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remarkable contribution in bringing me to this stage.

Declaration

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature. The work was done under the guidance of Prof. Zdzisław Brzeźniak, at Mathematics department of University of York. This work has not previously been presented for an award at this, or any other, University.

Introduction

Suppose that H is a Hilbert space and M is its unit sphere. Let f be a vector field on H (possibly only densely defined) such that the initial value problem

$$\begin{aligned} \frac{du}{dt} &= f\left(u(t)\right),\ t \geq 0, \\ u(0) &= x, \end{aligned} \tag{0.0.1}$$

has a unique global solution for every $x \in H$. The semi-flow generated by above initial value problem, denoted by $\left(\varphi(t,x)\right)_{t\geq 0}$, in general does not stay on M even though $x \in M$. The reason for this is that, in general, the vector field f is not tangent to M i.e. does not satisfy the following,

$$f\left(x\right) \in T_xM,\ x \in D(f) \cap M. \tag{0.0.2}$$

However, it is easy to modify f to a new $\widetilde{f}$ such that property (0.0.2) is satisfied. This can be achieved by using a map $\pi : H \to \mathcal{L}(H,H)$

$$\pi(x) = \{H \ni y \mapsto y - \langle x, y\rangle\, x \in H\} \in \mathcal{L}(H,H),\ \text{for every } x \in H.$$

The remarkable property of π is that when $x \in M$, the linear map $\pi(x) : H \to T_xM$ is the orthogonal projection so that vector field $\widetilde{f}$ defined by

$$\widetilde{f} : D(f) \ni x \mapsto \pi(x)\left[f(x)\right] \in H. \tag{0.0.3}$$

It is indeed easy to see that whenever $x \in M \cap D(f)$ we have,

$$\begin{aligned}\left\langle \widetilde{f}(x), x \right\rangle &= \langle \pi(x)\,[f(x)]\,, x\rangle \\ &= \langle f(x) - \langle x, f(x)\rangle\, x, x\rangle \\ &= \langle f(x), x\rangle - \langle x, f(x)\rangle\,\langle x, x\rangle \\ &= \langle f(x), x\rangle - \langle x, f(x)\rangle = 0.\end{aligned}$$

Hence $\widetilde{f}$ satisfies property (0.0.2).

If f is globally defined (i.e. $D(f) = H$) and locally Lipschitz map then $\widetilde{f}$ is also globally defined and locally Lipschitz map. Moreover, the modified equation

$$\begin{aligned}\frac{du}{dt} &= \widetilde{f}\,(u(t))\,,\ t \geq 0, \\ u(0) &= x,\end{aligned} \tag{0.0.4}$$

has a local solution for every $x \in M$. This solution stays on M whenever $x \in M$.

The situation is not so clear when f is only densely defined. There are two examples of such an f in concrete case:

Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded domain with sufficiently smooth boundary, $H = L^2(\mathcal{O})$, $A = -\Delta$ (i.e. negative Laplace operator with Dirichlet boundary condition so that $D(A) = H_0^{1,2}(\mathcal{O}) \cap H^{2,2}(\mathcal{O})$ and $f(u) = -Au$. The first case we will see that

$$\widetilde{f}(u) = -Au + |\nabla u|^2_{L^2(\mathcal{O})}\, u.$$

The second case is when $f(u) = -u^{2n-1}$, in this case one can find that

$$\widetilde{f}(u) = -u^{2n-1} + |u|^{2n}_{L^{2n}}\, u.$$

Roughly speaking, the two major aims of the thesis are to give complete treatment to both examples simultaneously and its generalization to the stochastic case. To be more precise, we prove that our corresponding initial value problems in the deterministic and stochastic cases have the unique global solutions for every $u(0) = x \in H^{1,2}(\mathcal{O})$.

Evolution equations are alternatively called the time-dependent partial differential equations. Several physical process from natural sciences gives rise to the nonlinear evolution equations as their mathematical models to represent them. Some of the well-known evolution equations are, Navier-Stokes Equation from fluid mechanics, the Schrödinger Equation from quantum mechanics, Reaction-diffusion equation to model the biological processes and heat transfer phenomenon, the Black-Scholes equation from finance etc. For a given evolution equation with some suitable initial data, two fundamental questions can be asked. The existence of a global solution to the problem and the study of long-term behaviour of solutions. We will try to address both of the questions for our proposed model. Some part of this work is motivated by P. Rybka [43], in which the author dealt with heat flow projected on a manifold M defined by some integral constraints. In [43], it is proven that solutions to this heat flow converge to a steady state solution as time $t \to \infty$.

The second aim is to study the generalization of deterministic model to stochastic case i.e.

$$\begin{aligned} \frac{du}{dt} &= \widetilde{f}(u(t)) + \text{Noise}, \; t \geq 0, \\ u(0) &= x \in H^{1,2}(\mathcal{O}) \cap M. \end{aligned} \tag{0.0.5}$$

where $f(u) = -Au - u^{2n-1}$. For such stochastic evolution equation on the manifold, fundamentally, we are going to deal with the questions of the existence of the global solution and develop the large deviation principle.

Historically, the development of the Itō calculus and Itô stochastic ordinary differential equations can be traced back to 1940 (see e.g. [24], [25]). In the decades 1960s and 1970s, we can see the emergence of the theory of the Itô stochastic partial differential equations (SPDEs) and its connection with several physical and biological sciences. In particular, the function spaces-valued evolution equations were used to model the bosonic quantum fields, some of cosmological processes and dynamics of the population (for e.g see [41], [31]).

For a long time, the stochastic evolution equation of the form

$$du(t) = A(t, u(t))dt + B(t, u(t))dW(t)$$

, has remained an object of interest for mathematicians. When the drift term satisfies the conditions of monotonicity such equations were first studied by R.Temam and Bensousssan ([36], [35]). Next, Pardoux in [36] developed the theory of stochastic evolution equation of above form, when the drift and diffusion are unbounded nonlinear operators, and showed that such equation is strongly solvable. Rozovskii and Krylov proved that, if the coefficients in drift and diffusion terms of the evolution equation are deterministic then its solution satisfies the Markov property. For the random Lipschitz coefficients, the Itô theorem for the strong solvability of finite dimensional SDEs was generalized. In the same paper a very useful version of Itô Lemma was proved in Hilbert spaces, for the square norm of semi-martingales. To study the stochastic version of our problem we will rely on the Itô Lemma developed by Pardoux in [37] wherein the author studied the class of SPDEs of the form

$$du(t) + A(t, u(t))dt = [B(t, u(t)), dW(t)]\,,\ u(0) = u_{0.}$$

Here A and B are some unbounded partial differential operators in Hilbert spaces, satisfying some coercivity hypothesis. Moreover $(W(t))_{t\geq 0}$ is $\mathbb{R}^d$-valued standard Brownian motion and $[,]$ denotes the scalar product in $\mathbb{R}^d$. Existence and uniqueness of the solution for above class of SPDEs was established in [37].

Finally, some of classical and modern references on stochastic partial differential equations on manifolds, are [6], [7], [9], [10], [11], [15], [18], [22].

Let me briefly provide the lay out of the thesis.

In Chapter 1 we provide all necessary preliminaries to the reader which are important to read rest of the thesis. The preliminaries include definitions and results (without proof) from functional analysis, semigroup theory, dynamical systems, the theory of Hilbert manifolds, measure theory and stochastic analysis. The chapter ends at providing a brief introduction to some important results from Pardoux [37].

Chapter 2 deals with the problem of the existence and uniqueness of a global solution of the constrained problem (0.0.4), for the case when $f(u) = -Au - u^{2n-1}$. To do this, we will introduce an abstract approximate evolution equation and prove the existence and uniqueness of a solution by Banach fixed point theorem in appropriately chosen Banach spaces. Next, using the Kartowski-Zorn Lemma we will prove the existence of a local maximal solution of approximate evolution equation. Then we show that if the energy norm of initial data u_0 is bounded by some constant R, then the solution (local or maximal) of approximate evolution equation is equivalent to the main evolution equation. In Proposition 2.2.11 we prove a sufficient condition for the local mild solution to be the global solution. The Lemma 2.3.3 is devoted to the proof of the invariance of manifold i.e. if the initial data lives in manifold then the solution of projected evolution equation itself lives in the manifold. Finally, we will prove the main result (see Theorem 2.3.5) of chapter i.e. the existence of the unique global solution. The chapter ends at studying some dynamical properties of the global solution.

Chapter 3 is devoted to a study the stochastic generalization of projected heat flow studied in Chapter 2 i.e. constrained problem (0.0.4). More precisely, we are going to study a non-linear parabolic first order in the time (heat) equation on Hilbert Manifold driven by Wiener process of Stratonovich type. Because of the constraint condition given by manifold M, it is natural to consider equations in the Stratonovich form (see also [6]). We prove the existence and the uniqueness of the global mild solution to the described stochastic evolution equation.

We will begin by introducing the main stochastic problem in both the Stratonovich and the Itô forms. As in the deterministic case, a truncated version of the main problem will be introduced. After introducing the main and truncated problem we will prove the estimates for *deterministic (drift)* terms and *stochastic (diffusion)* terms of the main equation in the Itô form. By employing the fixed point argument, we will construct a local mild solution of the approximate equation. After this from this local mild solution we recover the *local mild solution* to the original problem. We show next that approximate evolution equation admits a global solution, which will be useful later to prove the existence and the uniqueness of the local maximal solution of the main problem.

In last part of the chapter the key results about the main problem will be proved. By using the set of previously proven results about approximate evolution equation, we will show that the unique local maximal solution of the main problem exists. Furthermore, an important result about the life span of the maximal solution i.e. *no explosion result* will be proven. We will show an interesting result about the invariance of the manifold i.e. if initial data belongs to the manifold then almost all trajectories of solution belong to the manifold. Chapter 3 will end by proving the existence of global solution to our main stochastic evolution equation. The main tool for this will be Khashminskii test for non-explosion (Theorem III.4.1 of [30] for the finite dimensional case) and indeed an appropriate Itô's formula from [37] (Theorem 1.2).

In the 4th and final chapter, we study the small noise asymptotics by proving the Large deviation principle (LDP). To meet this aim, we will adopt the following weak convergence approach to prove the Large deviations principle.

Let $\{u^\varepsilon\}$ be a family of X-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where X is a Polish space.

Definition 0.0.1. *A function $\mathcal{I} : X \to [0, \infty]$ is called a **rate function** if $\mathcal{I}$ is lower semi-continuous i.e. for each $k \in \mathbb{R}$ the set $\{x \in X : \mathcal{I}(x) \leq k\}$ is closed (or equivalently, the set $\{x \in X : \mathcal{I}(x) > k\}$ is open). A rate function I is called **good rate function** if the level set $\{x \in X : \mathcal{I}(x) \leq k\}$ is compact for each $k \in [0, \infty)$.*

Definition 0.0.2. *(The Large deviation principle) A family $\{u^\varepsilon : \varepsilon > 0\}$ of X-valued random variables, is said to satisfy the **Large deviation principle (LDP)** with the rate function $\mathcal{I}$ if for each Borel subset B of X,*

$$-\inf_{x \in \mathring{B}} \mathcal{I}(x) \leq \liminf_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(u^\varepsilon \in B) \leq \limsup_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{P}(u^\varepsilon \in B) \leq -\inf_{x \in \overline{B}} \mathcal{I}(x),$$

where $\mathring{B}$ and $\overline{B}$ denote the interior and closure of B in X, respectively.

Definition 0.0.3. *(Laplace principle) The family $\{u^\varepsilon\}$ of X-valued random variables is said to satisfy the **Laplace principle (LP)** with the rate function $\mathcal{I}$, if for each real-valued bounded continuous function f defined on X we have:*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log \mathbb{E} \left\{ e^{-\frac{f(u^\varepsilon)}{\varepsilon^2}} \right\} = -\inf_{x \in X} \{f(x) + \mathcal{I}(x)\}.$$

The weak convergence method mainly is based on the equivalence of Laplace principle and Large deviation principle, provided that I is good rate function. This equivalence was formulated in [40] and can also be obtained as the consequence of Varadhan's Lemma [50] and Bryc's converse theorem [4]. Another elementary proof of equivalence can be found in [21] and [20].

Chapter 1

Preliminaries

The prime objective of this chapter is to make the dissertation self-contained and to make it easily accessible for readers. In this dissertation, I will be studying an infinite dimensional Non-linear Heat equation on Hilbert Manifold, deterministic and stochastic both. For deterministic equation, I will be arguing for the existence and uniqueness of Global solution and then the long-term dynamical behavior of the solution, so this legitimizes to provide some preliminaries (definition, results, and remarks) from functional analysis, differential geometry, and infinite dimensional dynamical systems. After dealing with the deterministic equation, I will then move towards the existence and uniqueness of stochastic version of the model and the Large Deviation Principle. Therefore, for the convenience of the reader it will be a good idea to include some of key preliminaries and results from the stochastic analysis. Let us begin with the functional analytic preliminaries.

1.1 Functional Analytic Preliminaries

In this section we aim to provide all those definitions and classic results which will be used later throughout the dissertation. Our main source for the this section is [27].

1.1.1 Normed and Banach spaces

Definition 1.1.1. *A* ***normed space*** *X is a vector space over field K (real or complex) with a norm defined on it. Here a* ***norm*** *on a vector space X is a real-valued function on X whose value at an $x \in X$, is denoted by $\|x\|$ and which has the following properties:*

1. $\|x\| \geq 0$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|\alpha x\| = |\alpha| \, \|x\| \, , \alpha \in K$
4. $\|x + y\| \leq \|x\| + \|y\|$

for all vectors $x, y \in X$.

Definition 1.1.2. *A sequence (x_n) in a normed space X is called* ***Cauchy sequence*** *if for each $\varepsilon > 0$ there is a natural number $N = N(\varepsilon)$ such that*

$$\|x_n - x_m\| < \varepsilon \text{ for all } n, m > N.$$

Definition 1.1.3. *A sequence (x_n) in a normed space X is called* ***convergent*** *if* *there is an* $x \in X$ *such that*

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Definition 1.1.4. *A normed space* X *is called* ***complete normed space*** *or* ***Banach Space*** *if every Cauchy sequence in* X *is convergent.*

Definition 1.1.5. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed spaces and* $f : X \to Y$. *We say* f *is* ***continuous at*** $x \in X$, *if for each sequence* (x_n) *in* X,

$$\|x_n - x\|_X \to 0 \textit{ implies } \|f(x_n) - f(x)\|_Y \to 0.$$

Moreover, if f *is continuous at each point* x *in* X *then* f *is* ***continuous****.*

Another use full observation is the following.

Proposition 1.1.6. *If* $(X, \|\cdot\|)$ *be normed space then the function* $\|\cdot\| : X \to \mathbb{R}$ *is continuous in* X.

Now we provide some important examples of the Banach spaces.

Spaces of continuous function:

Assume that D be a compact subset of $\mathbb{R}^n$. Consider the set,

$$C(D) := \{f : \ f \ \text{ is continuous function on } D\},$$

this set can be given the structure of the Banach space endowed with norm

$$\|f\|_{C(D)} := \sup_{x\in D} |f(x)|.$$

Indeed, a sequence (f_n) converges to f in $C(D)$ if $\sup\limits_{x\in\Omega} |f_n(x) - f(x)| \to 0$, i.e. (f_n) converges uniformly to f in D. It is well known that the uniform limit of a continuous function is continuous, hence $C(D)$ is Banach space.

Furthermore, if D is open then the set

$$C^k(D) = \{f : f\ , f', ..., f^{(k)} \in C(D)\}$$

with norm

$$\|f\|_{C^k(D)} = \|f\|_{C(D)} + \sum_{1 \leq |\alpha| \leq k} \|D^\alpha f\|_{C(D)},$$

where $\alpha := (\alpha_i)_{i=1}^n \in \mathbb{Z}_+^n$, $|\alpha| = \sum_{i=1}^n \alpha_i = n$, and $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_1^{\alpha_2}} ... \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}}$. Then one can show that $C^k(D)$ is Banach space.

1.1.2 Compactness in function spaces: Arzela-Ascoli Theorem

Another important classical result that we need in proving the Large deviation principle, is the apply Arzela-Ascoli Theorem. Before presenting the result let us recall some of important definitions.

Definition 1.1.7. *([27],Theorem 5.18) Let (X, ρ) be a compact metric space i.e. every open cover of X has a finite subcover. A family of functions $\Lambda \subset C(X)$ is called **equicontinuous** if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $f \in \Lambda$,*

$$|f(u) - f(v)| < \varepsilon, \text{ for all } u, v \in X \text{ satisfying } \rho(u, v) < \delta\ .$$

*The family Λ is called **uniformly bounded** or **equivocated** if there is a constant C such that*

$$|f(u)| \leq C, \text{ for all } f \in \Lambda \text{ and for all } u \in X.$$

Example 1.1.8. *Let(X, ρ) be a metric space and let C be a positive constant. Consider the following family of continuous functions on X*

$$\Lambda := \{f \in C(X) : \ f \text{ satisfies } |f(u) - f(v)| \leq C\rho(u, v)\,, \text{ for all } u, v \in X\}\,.$$

Then this family Λ *is equicontinuous. To verify this assertion, consider* $\varepsilon > 0$ *and* $\delta = \varepsilon/C$. *One can see that for any* $v \in B(\delta : x)$ *i.e.* $\rho(u, v) < \delta$ *and for any* $f \in \Lambda$, *we have the following,*

$$|f(u) - f(v)| \leq C\rho(u, v) < C \cdot \delta = C \cdot \varepsilon/C = \varepsilon.$$

Thus Λ *is equicontinuous.*

Definition 1.1.9. *Let* (X, ρ) *be a metric space, a subset* $K \subset X$ *is called precompact (or relatively compact) if the closure* $\overline{K}$ *of* K *is compact in* X.

Now we present the **Arzela-Ascoli Theorem**.

Theorem 1.1.10. *([27], Theorem 5.20) Let* (X, ρ) *be a compact metric space. For a family of functions* $\Lambda \subset C(X)$, *then the following statements are equivalent,*

i) Λ *is relatively compact,*

ii) Λ *is equicontinuous and uniformly bounded.*

The following is an immediate corollary of the last theorem, we can also treat this as another version of Arzela-Ascoli theorem.

Corollary 1.1.11. *([27], Corollary 5.21) Let* (X, ρ) *be a compact metric space. If a sequence of functions* $(f_n) \subset C(X)$ *is equicontinuous and uniformly bounded then this sequence* (f_n) *contains a uniformly convergent subsequence.*

1.1.3 Hilbert spaces

In this dissertation, we are mainly concerned with Hilbert spaces so this would a good idea to review some of basic stuff about Hilbert spaces. Let us begin with defining and recalling some of the basic results about Hilbert spaces.

Definition 1.1.12. *Let X be a linear space over $\mathbb{R}$. An* ***inner product*** *in X is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ that satisfies the following three properties. For all $x, y, z \in X$ and all scalars $\alpha, \beta \in \mathbb{R}$,*

i) $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0$ *if and only if* $x = 0$, *(Positivity)*

ii) $\langle x, y \rangle = \langle y, x \rangle$, *(Symmetry)*

iii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$. *(Collinearity)*

A linear space X endowed with inner product is called an ***inner product space****.*

An inner product also induces a norm, in the following manner,

$$\|x\| := \sqrt{\langle x, x \rangle}, \text{ for all } x \in X. \tag{1.1.1}$$

As an immediate consequence of the definition of inner product space and the norm described in last equation, we can easily prove the following theorem.

Theorem 1.1.13. *If X is an inner product space then:*

i) For any $x, y \in X$, we have the following inequality, called ***Cauchy-Schwartz inequality****,*

$$|\langle x, y \rangle| \leq \|x\| \, \|y\| \, .$$

Equality holds if and only if x and y linearly dependent.

ii) For any $x, y \in X$, we have the following ***Parallelogram Law****:*

$$\|x + y\|^2 + \|x - y\|^2 = 2 \|x\|^2 + 2 \|y\|^2 \, .$$

iii) For $x, y \in X$, we have the following triangle inequality,

$$\|x + y\| \leq \|x\| + \|y\| \, .$$

Definition 1.1.14. *If H be an inner product space then H is called a* ***Hilbert space*** *if it is complete with respect to induced norm* (1.1.1).

Examples of the Hilbert space The following are some of the important examples of Hilbert spaces, few of which we are going to encounter in the rest of chapters.

Example 1.1.15. *$\mathbb{R}^n$ is inner product space with the following inner product,*

$$\langle x, y\rangle_{\mathbb{R}^n} := \sum_{i=1}^{n} x_i y_i, \ \text{where } x = (x_i)_{i=1}^{n} \text{ and } y = (y_i)_{i=1}^{n} \text{ belong to } \mathbb{R}^n.$$

Moreover, $\mathbb{R}^n$ is Hilbert space with the following induced Euclidean norm,

$$\|x\| = \sum_{i=1}^{n} x_i^2.$$

Example 1.1.16. *The space $L^2(\Omega)$ is inner product space with respect to inner product,*

$$\langle u, v\rangle_{L^2(\Omega)} = \int_{\Omega} uv.$$

and Hilbert space under the following induced norm,

$$\|u\|_{L^2(\Omega)} = \int_{\Omega} u^2.$$

Some important theorems about Hilbert spaces The orthogonal projection is going to play an important role in our formulation of main deterministic and stochastic problem. Therefore, it is worth recalling the classical **projection theorem** in Hilbert spaces. The material in this subsection is based on Chapter 21 of [27].

Definition 1.1.17. *If H be a Hilbert space and $x, y \in H$, we say x is **orthogonal** to y if $\langle x, y\rangle = 0$. In general, if $V \subset H$ then*

$$V^{\perp} = \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in V\}.$$

*The set $V^{\perp}$ is called **orthogonal complement** of V.*

Lemma 1.1.18. *If V is closed subspace of a Hilbert space H then orthogonal complement $V^{\perp}$ of V is also closed subspace of H. Moreover*

$$H = V \oplus V^{\perp},$$

where $\oplus$ denotes the direct sum.

Theorem 1.1.19. *(Projection Theorem) Let V be a closed subspace of Hilbert space H. Then for each $x \in H$, there exists unique element $\widehat{x} \in V$ such that*

$$\|\widehat{x} - x\| = \inf_{v \in V} \|v - x\|.$$

Moreover, the following properties hold:

i) $\widehat{x} = x$ iff $x \in V$,

ii) $x - \widehat{x} \in V^{\perp}$, and

$$\|x\|^2 = \|\widehat{x}\|^2 + \|x - \widehat{x}\|^2.$$

Corollary 1.1.20. *If V is the closed subspace H and $\widehat{x}$ is the unique element as described in last Projection theorem then there exists a unique map $\pi : x \longmapsto \widehat{x}$, from H into V, which is linear, bounded and satisfies the following properties:*

$$\begin{aligned} \|\pi\| &= \sup_{x \neq 0} \frac{\|\pi x\|}{\|x\|} = 1 \\ \pi^2 &= \pi \text{ and } \ker \pi = V^{\perp}. \end{aligned}$$

Remark 1.1.21. *The map π described in last corollary is called the* ***orthogonal projection*** *of H onto V.*

The Hilbert spaces that we are going to deal in this dissertation will be separable Hilbert spaces over the field of real numbers. Let us recall definitions and some basic properties of separable Hilbert spaces.

Definition 1.1.22. *A Hilbert space H is called **separable Hilbert space** if it contains a countable dense subset.*

Definition 1.1.23. *An orthonormal basis in a separable Hilbert space H is a sequence $(e_j)_{j=1}^{\infty} \subset H$ such that*

$$\begin{aligned} \langle e_j, e_k \rangle &= \delta_{jk}, \text{ where } j,k \in \mathbb{N} \\ \|e_j\| &= 1, \text{ for all } j \in \mathbb{N}. \end{aligned}$$

Here δ_{jk} denotes the Kronecker delta.

Proposition 1.1.24. *Every separable Hilbert space H admits an orthonormal basis.*

1.1.4 Linear operators, Duality and Weak Convergence

At several instances in the dissertation, we will deal with several kinds of linear operators and dual spaces. Moreover, weak convergence will be an important tool to prove large deviation principle. Therefore, we present these notions and important related results.

Definition 1.1.25. *Let X and Y be normed spaces. A **linear operator** from X into Y is a map $L : X \to Y$ such that for $\alpha, \beta \in \mathbb{R}$ and for $x, y \in X$,*

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y).$$

Definition 1.1.26. *If $L : X \to Y$ is a linear operator, the **kernel** of L, can be defined as pre-image of null vector in Y i.e.*

$$\ker L := \{x \in X : Lx = 0\}.$$

*The **range** of L is the set of all images i.e.*

$$L(X) := \{Lx : x \in X\}.$$

Definition 1.1.27. *Let X and Y be normed spaces. A linear operator $L : X \to Y$ is called bounded if there exists a number C such that*

$$\|Lx\|_Y \leq C \|x\|_X$$

Example 1.1.28. *([52] Example 7.2, 69) i) (Multiplication operator) For any $f \in C[a,b]$, consider the operator M_f on $L^2(a,b)$ defined in the following manner:*

$$(M_f x)(t) := f(t)x(t), \text{ for all } x \in L^2(a,b).$$

One can see that M_f is linear, the boundedness of M_f can be seen in the following manner,

$$\begin{aligned}
\|M_f x\|^2 &= \int_a^b |f(t)x(t)|^2 \, dt \\
&\leq \sup_{t\in(a,b)} |f(t)|^2 \int_a^b |x(t)|^2 \, dt \\
&= \|f\|^2_{C[a,b]} \|x\|^2 = C\|x\|^2, \text{ where } C := \|f\|^2_{C[a,b]} < \infty.
\end{aligned}$$

ii) For $a,b,c,d \in \mathbb{R}$ and continuous $k : [a,b] \times [c,d] \to \mathbb{R}$, define an integral operator $K : L^2(a,b) \to L^2(c,d)$ in the following manner,

$$(Kx)(t) := \int_a^b k(t,s)\, x(s) ds, \ t \in (c,d).$$

Indeed, K is linear. Moreover, using Cauchy-Schwartz inequality we can obtain boundedness of K in the following manner, for fixed $t \in (c,d)$,

$$\begin{aligned}
|(Kx)(t)|^2 &= \left| \int_a^b k(t,s)\, x(s) ds \right|^2, \\
&\leq \left(\int_a^b |k(t,s)|^2 \, ds \right) \left(\int_a^b |x(s)|^2 \, ds \right)
\end{aligned}$$

Integrating both sides on $t \in (c,d)$,

$$\begin{aligned}
\int_c^d |(Kx)(t)|^2 \, dt &\leq \left(\int_a^b |x(s)|^2 \, ds \right) \int_c^d \int_a^b |k(t,s)|^2 \, ds dt \ \text{i.e.} \\
\|Kx\|^2 &\leq \|x\|^2 \|k\|^2_{C([a,b]\times[c,d])} = C\|x\|^2
\end{aligned}$$

where $C := \|k\|^2_{C([a,b]\times[c,d])} < \infty$.

iii) Let X *be a normed space of all polynomials on* $[0,1]$ *with norm* $\|x\| = \max|x(t)|, t \in [0,1]$. *A differentiation operator* T *is defined on* X *by* $Tx(x) = x'(t)$, *where prime denotes the differentiation with respect to* t. *This operator is linear but not bounded. Indeed, let* $x_n(t) = t^n$, *where* $n \in \mathbb{N}$. *Then* $\|x_n\| = 1$ *and* $Tx_n(t) = x'_n(t) = nt^{n-1}$ *so that* $\|Tx_n\| = n$. *Since* $n \in \mathbb{N}$ *is arbitrary, this shows that there is no fixed number* c *such that* $\frac{\|Tx_n\|}{\|x_n\|} \leq c$. *Hence* T *is not bounded.*

Proposition 1.1.29. *([27], Theorem 7.18) Let* X *and* Y *be normed spaces. A linear operator* $L : X \to Y$ *is bounded if and only if it is continuous i.e. for any* $x_n \overset{X}{\to} x$ *implies* $Lx_n \overset{Y}{\to} Lx$.

Definition 1.1.30. *([27], Definition 7.20) By the set* $\mathcal{L}(X,Y)$ *we denote the set of all bounded linear operators from* X *into* Y. *If* $X = Y$ *we will write* $\mathcal{L}(X)$. *On* $\mathcal{L}(X,Y)$, *we can define norm in the following manner, for* $L \in \mathcal{L}(X,Y)$,

$$\|L\| := \sup_{\|x\|=1} \frac{\|Lx\|_Y}{\|x\|_X}.$$

Theorem 1.1.31. *([27], Theorem 7.20) Let* X *and* Y *be Banach spaces. Then* $\mathcal{L}(X,Y)$, *with the norm defined in last equation, is also Banach space.*

Definition 1.1.32. *Let* X *and* Y *be normed spaces. Let* $L \in \mathcal{L}(X,Y)$. *An* $\mathcal{L}(X,Y)$-*valued sequence* (L_n) *of operators is said to:*

a) converge in ***operator norm*** *to* L *if,*

$$\|L_n - L\|_{\mathcal{L}(X,Y)} \to 0 \text{ as } n \to \infty.$$

b) ***strongly converge*** *to* L *iff* $(L_n x)$ *converges strongly in* Y *for each* x *in* X, *i.e.,*

$$\|L_n x - Lx\|_Y \to 0 \text{ as } n \to \infty.$$

Definition 1.1.33. *Let X be a real Banach space. Then a **linear functional** on X is bounded linear mapping from X to $\mathbb{R}$. The **dual space** denoted by X^* is $\mathcal{L}(X, \mathbb{R})$ i.e. set of all linear bounded linear functionals on X.*

Theorem 1.1.34. *([27], Theorem 21.6) (Riesz Representation Theorem) Let H be a Hilbert space. Then H^* is isometrically isomorphic to H. In particular, for every $x \in H$ the linear functional defined by*

$$L_x(y) = \langle x, y \rangle, \text{ for all } y \in H,$$

is bounded with norm $\|L_x\| = \|x\|$. Moreover, for every $L \in H^*$ there exists a unique $u_L \in H$ such that:

$$Lx = \langle u_L, y \rangle, \text{ for all } y \in H.$$

Moreover, $\|L\| = \|u_L\|$.

Definition 1.1.35. *([27], Theorem 21.7) Let H be a Hilbert space. A sequence $(x_n)_{n\in\mathbb{N}} \subset H$ is called **weakly convergent** to $x \in H$ if*

$$\langle x_n, y \rangle \to \langle x, y \rangle, \text{ for all } y \in H.$$

One can see easily that a sequence converges in usual sense also converges weakly because,

$$|\langle x_n, y \rangle - \langle x, y \rangle| \leq \|x_n - x\| \|y\|.$$

Let us state the following simple case of the Banach-Alaoglu Theorem.

Theorem 1.1.36. *([27], Theorem 21.8) Every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in a Hilbert space H has a weakly convergent subsequence.*

Theorem 1.1.37. *([27], Theorem 21.11) Every weakly convergent sequence $(x_n)_{n\in\mathbb{N}} \subset H$ is bounded.*

Definition 1.1.38. *[27] Let X be a normed space over a field K . The* ***double dual*** *of X , denoted by X^{**} and defined by $X^{**} = (X^*)^*$ i.e. dual of the dual of X.*

Definition 1.1.39. *([52], Definition 6.1) Let X be a be a normed space over field K, X^* be its dual and X^{**} be its double dual. Then we have a canonical map $x \to \widehat{x}$ defined by:*

$$\widehat{x}(f) = f(x) \text{ where } f \in X^*$$

gives an isometric linear isomorphism(embedding) from X into X^{**}.
The space X is called ***reflexive*** *if this map is also surjective.*

Closed operators

Definition 1.1.40. *([52], Definition 2.4.1) Let X and Y be normed space and $L : D(T) \to Y$ is a linear operator, $D(L) \subset X$, then L is called* ***closed linear operator*** *if its* ***graph***

$$G(L) = \{(x, y) : x \in D(L), y = Lx\}$$

is closed in the normed space $X \times Y$.

Theorem 1.1.41. *[27] Let X and Y be Banach spaces and $L : D(L) \to Y$ is a closed linear operator, $D(L) \subset X$. If $D(L)$ is closed in X then the operator L is bounded i.e. there exists $c > 0$ such that*

$$\|Lx\|_Y \leq c\,\|x\|_X\,.$$

Theorem 1.1.42. *[27] Let X and Y be normed spaces and $L : D(L) \to Y$ is a linear operator, $D(L) \subset X$.Then L is closed iff it has the following property:*

If $x_n \to x$, where $x_n \in D(L)$, and $Tx_n \to y$, then $x \in D(L)$ and $Lx = y$.

Another useful characterization of closed operators is following:

Theorem 1.1.43. *[27] Let A be a linear operator in Banach space $(X, \|\cdot\|)$. Define norm on $D(A)$ by*

$$\|x\|_A = \|x\| + \|Ax\|, \text{ for } x \in D(A).$$

then $(D(A), \|\cdot\|_A)$ is Banach space iff A is closed.

Adjoint Operators

If X and Y be Hilbert spaces and $L \in \mathcal{L}(X, Y)$ then for fixed $y \in Y$, let us define a map $T_y : X \to \mathbb{R}$ in the following manner,

$$T_y(x) := \langle Lx, y \rangle_Y .$$

Indeed, T_y is linear and in fact bounded i.e. $T_y \in X^*$ because,

$$\begin{aligned} |T_y(x)| &\leq |\langle Lx, y\rangle_Y| \leq \|Lx\|_Y \|y\|_X \leq \|L\|_{\mathcal{L}(X,Y)} \|x\|_X \|y\|_Y \\ \|T_y\| &\leq \|L\|_{\mathcal{L}(X,Y)} \|y\|_Y . \end{aligned}$$

Hence by the Riesz Representation Theorem there exists $w \in X$ depending on y, we denote this by $w = L^*y$ such that

$$T_y(x) := \langle x, L^*y \rangle_X \text{,for all } x \in X \text{ and } y \in Y.$$

Definition 1.1.44. *([52], Definition 7.3) For a linear operator $L \in \mathcal{L}(X, Y)$, its* ***adjoint*** *is an operator $L^* : Y \to X$ that satisfies the identity,*

$$\langle Lx, y \rangle_Y = \langle x, L^*y \rangle_X , \text{ for all } x \in X \text{ and } y \in Y.$$

Example 1.1.45. *i) Recall that from the Riesz representation theorem for each $L \in H^*$ there exist a unique element $u_L \in H$. Hence this naturally gives a way to define a map $R : H^* \to H$ $(L \mapsto u_L)$, called the Riesz map. One can see that R is canonical isometry. We claim that $R^* = R^{-1} : H \to H^*$.*

$$\langle RL, y \rangle_H = \langle L, y \rangle_{H \times H^*} = \left\langle L, R^{-1}y \right\rangle_{H^*} = \langle L, R^*y \rangle_{H^*} .$$

ii) Consider a linear operator $L : L^2(0,1) \to L^2(0,1)$ be the linear map

$$(Lx)(t) = \int_0^t x(s)ds, \text{ where } t \in (0,1).$$

From part ii) of Example 1.1.28 we infer that L is bounded. In order to find L^* consider the following,

$$\langle Lx, y\rangle = \int_0^1 \left[y(t) \int_0^t x(s)ds \right] dt$$

Changing order of integration,

$$\begin{aligned} \langle Lx, y\rangle &= \int_0^1 \left[x(s) \int_t^1 y(t)dt \right] ds \\ &= \langle x, L^* y\rangle \end{aligned}$$

Thus

$$L^* y = \int_t^1 y(t)dt, \text{ where } t \in (0,1).$$

Definition 1.1.46. *([52], Definition 7.17) Let H be a Hilbert space. A densely defined operator $L : D(L) \to H$, with $D(L) \subset H$, is called **self-adjoint** iff $L = L^*$.*

Later in this section we will discuss in detail the Laplace operator, which is an important example of the self-adjoint operator. We will do this because our both deterministic and stochastic equations involve Laplace operator with the Dirichlet boundary condition.

Compact Operators

Compact operators play a significant role in the theory of differential equations. We will also encounter the some compact operators while proving the large deviation principle, therefore in this subsection our aim is to review some of basic and useful results about Compact operators. Let us begin by defining the compact operators.

Definition 1.1.47. *([52], Definition 8.1) Let X and Y be normed spaces. A linear operator $L : X \to Y$ is called* ***compact*** *if for each bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, the sequence $(Lx_n)_{n\in\mathbb{N}}$ has a convergent subsequence in Y.*

Example 1.1.48. *([52], Example 8.2) i) Recall that the dimension of range of linear operator is called its rank. Any bounded linear operator $L : X \to Y$ with finite rank, must be compact. Let me give an informal proof. Since the rank finite i.e. range $L(X)$ must be finite dimensional subspace of Y and it is normed space with the restriction of norm of Y. Since closed bounded sets are compact so for any bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X the image set of sequence $\{Lx_n : n \in \mathbb{N}\}$ is bounded in $L(X)$ and hence $\overline{\{Lx_n : n \in \mathbb{N}\}}$ is compact. Therefore, by Weierstrass theorem $\{Lx_n : n \in \mathbb{N}\}$ contains a limit point, say x, in $L(X)$. Hence there must be subsequence of $(Lx_n)_{n\in\mathbb{N}}$ converging towards x. Thus L is compact.*

ii) If H is an infinite dimensional Hilbert space then identity operator I on H is not compact. Consider an infinite orthonormal sequence (x_n) in H, then for $m \neq n$,

$$\begin{aligned} \|x_n - x_m\|^2 &= \langle x_n - x_m, x_n - x_m\rangle \\ &= \langle x_n, x_n\rangle - \langle x_m, x_n\rangle - \langle x_n, x_m\rangle + \langle x_m, x_m\rangle \\ &= \|x_n\|^2 - 0 + 0 + \|x_m\|^2 = 2. \end{aligned}$$

Last identity shows that distinct terms of sequence are at distance $\sqrt{2}$. Hence $(Ix_n)_{n\in\mathbb{N}} = (x_n)_{n\in\mathbb{N}}$ contains no Cauchy and hence convergent subsequence. Thus I is not compact.

Proposition 1.1.49. *([52] Every compact operator is bounded.*

Proposition 1.1.50. *[52] If X and Y be Banach spaces then $\mathcal{L}C(X,Y)$ set of all compact operators from X into Y is a closed (and hence complete) subspace of $\mathcal{L}(X,Y)$ with operator norm.*

It is obvious that to check that whether a given operator is compact or not by definition is difficult. Sometimes Hilbert-Schmidt operators can rescue in such situation. Definition and some of the useful results about Hilbert-Schmidt operators are given below.

Definition 1.1.51. *([52], Definition 8.5) Let X and Y be the Hilbert spaces. A linear operator $L : X \to Y$ is called **Hilbert-Schmidt** if we can find a complete orthonormal sequence $(e_n)_{n\in\mathbb{N}}$ in X such that $\|A\|_{HS} := \sum_{j=1}^{\infty} \|Ae_j\|^2 < \infty$.*

Example 1.1.52. *([52], Example 8.6) The following Voltera operator L defined on $L^2\,(0,1)$ is Hilbert-Schmidt.*

$$(Lx)\,(t) := \int_0^t x(s)ds, \ \ \text{where } t \in (0,1)\,.$$

In a minute, we are going to present a result that if the kernel of an integral operator is square integrable then the integral operator is compact. Since kernel in the Volterra operator is 1, which is indeed square integrable hence L is Hilbert-Schmidt.

Theorem 1.1.53. *([52], Theorem 8.7) Every Hilbert-Schmidt operator is compact.*

Theorem 1.1.54. *([52], Theorem 8.8) If $k : (c,d) \times (a,b) \to \mathbb{R}$ be a measurable function such that*

$$\int_c^d \int_a^b |k(s,t)|^2\, dsdt < \infty.$$

Then the following integral operator $K : L^2\,(a,b) \to L^2\,(c,d)$ with kernel k

$$(Kx)\,(t) := \int_a^b k\,(t,s)\,x(s)ds, \ \ t \in (c,d)\,.$$

is Hilbert Schmidt and hence compact.

1.1.5 Semigroups of Linear operators

Throughout the section X denotes a Banach space. More over $\|\cdot\|$ would denote norm on $\mathcal{L}\,(X)\,.$

Definition 1.1.55. *([51], Definition 2.1.1) A one-parameter family* $\{T(t), t \geq 0\}$ *of bounded linear operators from* X *into* X *is a* ***semigroup*** *of bounded linear operators on* X *if*

(i) $T(0) = I$, *(*I *is the identity operator on* X*).*

(ii) $T(s+t) = T(s)T(t)$ *for every* $t, s \geq 0$ *(the semigroup property). Here* $T(s)T(t)$ *is composition of functions.*

Definition 1.1.56. *([51], Remark 2.3.2) A semigroup of bounded linear operators,* $T(t)$*, is* ***uniformly continuous*** *if*

$$\lim_{t \to 0} \|T(t) - I\| = 0 \tag{1.1.2}$$

where $\|\cdot\|$ *is norm on* $\mathcal{L}(X, X)$ *as mentioned earlier.*

Definition 1.1.57. *([51], Definition 2.3.1) A semigroup of linear operators* $\{T(t) : t \geq 0\}$ *is called a semigroup of class* C_0*, or* C_0- *semigroup, if for each* $x \in X$*, we have*

$$\|T(t)x - x\| \to 0 \text{ as } t \to 0. \tag{1.1.3}$$

Example 1.1.58. *([51], Example 2.3.1) Let* $C_{b,u}(\mathbb{R})$ *be the space of uniformly continuous bounded real-valued functions on* $\mathbb{R}$*. For* $t \geq 0$*, define an operator* $T(t) : C_{b,u}(\mathbb{R}) \to C_{b,u}(\mathbb{R})$ *as:*

$$(T(t)x)(s) := x(t+s), \text{ for all } x \in C_{b,u}(\mathbb{R}), \ s \geq 0.$$

Then $\{T(t) : t \geq 0\}$ *is a* C_0*-semigroup.*

Theorem 1.1.59. *For* $f \in L^2(\mathbb{R})$, *and* $t > 0$ *define*

$$(T(t)f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y) e^{-\frac{|x-y|^2}{4t}} dy$$

and set $T(0)f = f$*. Then* $\{T(t) : t \geq 0\}$ *is a* C_0*-semigroup of contractions.*

Lemma 1.1.60. *([51], Theorem 2.3.1) Let* $\{T(t), t \geq 0\}$ *be a semigroup on Banach space* X*. There exist constants* $\omega \in \mathbb{R}$ *and* $M \geq 1$ *such that the following holds:*

$$\|T(t)\| \leq Me^{\omega t}\,,\ \text{for}\ 0 \leq t < \infty.$$

Corollary 1.1.61. *([51], Corollary 2.3.1) If* $\{T(t), t \geq 0\}$ *is a semigroup on Banach space* X*, then for each* $f \in X$*, the map* $t \mapsto T(t)f$ *is a continuous function from* $(0, \infty)$ *to* X*.*

Definition 1.1.62. *[51] Let* $\{T(t), t \geq 0\}$ *be* $C_0 -$ *semigroup. In Lemma 1.1.60 if* $\omega = 0$ *then* $\{T(t), t \geq 0\}$ *is called* ***uniformly bounded.***

Definition 1.1.63. *([51], Corollary 2.3.2) Let* $\{T(t), t \geq 0\}$ *be* $C_0 -$ *semigroup. In Lemma 1.1.60 if* $\omega = 0$*,* $M = 1$*, i.e.* $\|T\| \leq 1$*. Such a semigroup* $\{T(t), t \geq 0\}$ *is called* C_0- ***semigroup of contractions.***

1.1.6 Function spaces

In this subsection, my intentions are to introduce all of those function spaces which we are going to encounter throughout the dissertation. In particular, we are going to discuss distributions, some of important Sobolev spaces and Bochner spaces. Let us begin with setting up some notations.

Definition 1.1.64. *An open and connected subset* Ω *of* $\mathbb{R}^n$, $n \in \mathbb{N}$, *is called domain. We denote closure and boundary of* Ω *by* $\overline{\Omega}$ *and* $\partial\Omega$ *respectively.*

Consider the following set of notations:

$$\begin{aligned}
\mathbb{Z}_+^n &: = \{(\alpha_i)_{i=1}^n : \alpha_i \in \mathbb{Z}_+\}, \\
x &: = (x_i)_{i=1}^n \in \mathbb{R}^n, \\
\alpha &: = (\alpha_i)_{i=1}^n \in \mathbb{Z}_+^n, \\
\partial_i u &: = \frac{\partial u}{\partial x_i}, \\
|\alpha| &: = \sum_{i=1}^n \alpha_i, \\
D^\alpha u &: = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}} \\
\nabla u &: = (D_i u)_{i=1}^n \\
|\nabla u| &: = \left(\sum_{i=1}^n |D_i u|^2 \right)^{1/2}.
\end{aligned}$$

We now introduce the space of smooth functions with compact support and convergence in it.

Space of smooth functions with compact support & Distributions

Definition 1.1.65. *[51] (Page 15-16) Let Ω be an open subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$. The following set*

$$supp f := \overline{\{x \in \Omega : f(x) \neq 0\}},$$

*is called the **support** of function f. By $C_0^\infty(\Omega)$ we mean the set of all functions f on Ω which are smooth i.e. infinitely differentiable, $supp f$ is compact included in Ω. Consider a sequence (f_n) in $C_0^\infty(\Omega)$ and $f \in C_0^\infty(\Omega)$, we say that $f_n \to f$ in $C_0^\infty(\Omega)$ if*

i) there exists a compact subset K of Ω such that $supp f_n \subset K$ for all $n \in \mathbb{N}$,

ii) $D^\alpha f_n \to D^\alpha f$ uniformly on Ω, as $n \to \infty$, for all $\alpha \in \mathbb{Z}_+^n$.

By $\mathfrak{D}(\Omega)$ we mean the space $C_0^\infty(\Omega)$ endowed with above mentioned structure of convergence, moreover we denote this convergence by $f_n \overset{\mathfrak{D}(\Omega)}{\to} f$.

Definition 1.1.66. *[51] (Page 16) By a Cauchy sequence in $\mathfrak{D}(\Omega)$ we a sequence (f_n) in $\mathfrak{D}(\Omega)$ if there exists a compact set $K \subset \Omega$ such that supp$f_n \subset K$ and such that, for all $n \in \mathbb{N}$, $\|D^\alpha f_n - D^\alpha f_m\|_{C(\Omega)} \to 0$ as $n, m \to \infty$ for all $\alpha \in \mathbb{Z}_+^n$.*

Definition 1.1.67. *[51] (Page 16) A linear real-valued continuous functional on $\mathfrak{D}(\Omega)$ is called* ***distribution*** *on $\mathfrak{D}(\Omega)$. We denote space of all distributions on $\mathfrak{D}(\Omega)$ by $\mathfrak{D}'(\Omega)$.*

Example 1.1.68. *i) The Dirac delta function $\delta_c : f \mapsto f(c)$ is a distribution for any $c \in \mathbb{R}^n$.*

ii) Assume that f is locally integrable i.e. for each compact $K \subset \mathbb{R}^n$ the integral $\int_K |f| < \infty$. Define $L_f : \mathfrak{D}(\Omega) \to \mathbb{R}$ as:

$$L_f\varphi := \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \text{ for } \varphi \in \mathfrak{D}(\Omega).$$

Now consider a sequence $\varphi_n \overset{\mathfrak{D}(\Omega)}{\to} \varphi$,

$$\begin{aligned} |L_f\varphi_n - L_f\varphi| &= \left|\int_{\mathbb{R}^n} f(x)\left(\varphi_n(x) - \varphi(x)\right)dx\right| \\ &\leq \int_{supp\varphi} |f(x)|\,|\varphi_n(x) - \varphi(x)|\,dx \\ &\leq \sup_x |\varphi_n(x) - \varphi(x)| \int_{supp\varphi} |f(x)|\,dx \\ &= \|\varphi_n - \varphi\| \int_{supp\varphi} |f(x)|\,dx \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence L_f defines a distribution on $\mathfrak{D}(\Omega)$.

Definition 1.1.69. *[51] (Page 16) Let $\alpha \in \mathbb{Z}_+^n$ be a multi-index and $u : \Omega \to \mathbb{R}$ is a locally integrable function . The α-th order* ***derivative of u, in the sense of***

distribution *over on* $\mathfrak{D}(\Omega)$, *is the distribution* $D^\alpha u$ *defined by:*

$$(D^\alpha u)(f) := (-1)^{|\alpha|} \int_\Omega u \cdot D^\alpha f \; d\omega, \text{ for all } f \in \mathfrak{D}(\Omega).$$

Sobolev spaces and their properties

In this subsection, we aim to recall some of Sobolev spaces of prime importance in this dissertation. Before introducing these spaces let me introduce a more sophisticated notion of derivative known as "weak derivative".

Definition 1.1.70. *[27] (Page 266) A function* $f \in L^1_{loc}(\Omega)$ *if and only if* $f \in L^1(\Omega')$, *for every bounded* Ω' *with* $\overline{\Omega'} \subset \Omega$.

Definition 1.1.71. *[27] (Page 266) For* $u \in L^1_{loc}(\Omega)$ *a function* $v \in L^1_{loc}(\Omega)$ *is called the weak derivative of* u *in the direction of* x_i, *where* $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, *if*

$$\int_\Omega u(x)\partial_i \varphi(x)dx = -\int_\Omega v(x)\varphi(x)dx, \text{ for all test functions } \varphi \in C^1_0(\Omega).$$

In this case we write $v = D_i u$.

Example 1.1.72. *[27] (Page 266) For* $\Omega = (-1, 1) \subset \mathbb{R}$,

i) Consider the function $u(x) = |x|$. *We claim the following is equal to the weak derivative of* u,

$$\begin{aligned} v(x) &= 1, \qquad 0 \le x < 1 \\ &= -1, \; -1 < x < 0. \end{aligned}$$

Indeed, if $\varphi \in C^1_0(\Omega)$ *then*

$$\begin{aligned} \int_{-1}^{1} v(x)\varphi(x)\,dx &= -\int_{-1}^{0} \varphi(x)\,dx + \int_0^1 \varphi(x)\,dx \\ &= \int_{-1}^{1} \varphi'(x)\,|x|\,dx. \end{aligned}$$

ii) We now show that not every function in $L^1_{loc}(\Omega)$ has a weak derivative. Consider

$$\begin{aligned} u(x) &= 1,\ 0 < x < 1 \\ &= 0,\ -1 < x < 0, \end{aligned}$$

Apparently, from above definition we expect $\partial u = 0$. But this not the case, consider $\varphi \in C_0^1(\Omega)$ for which $\varphi(0) \neq 0$. For such φ,

$$0 = \int_{-1}^{1} \varphi(x) \cdot 0\ dx = -\int_{-1}^{1} \varphi(x) \cdot u(x)\ dx = -\int_{0}^{1} \varphi'(x) = \varphi(0),$$

which leads to an absurdity, hence the required does not holds for such φ, i.e. weak derivative exists.

Definition 1.1.73. *[27] (Page 267) Let $\alpha \in \mathbb{Z}^n_+$ and $u \in L^1_{loc}(\Omega)$. A function $v \in L^1_{loc}(\Omega)$ is called the α-th weak derivative of u if*

$$\int_\Omega u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega v(x)\varphi(x)dx, \text{ for all test functions } \varphi \in C_0^{|\alpha|}(\Omega).$$

In this case we are going to use symbol $v = D^\alpha u$.

Definition 1.1.74. *[27] (Page 267) Let $k \in \mathbb{N}$, $p \in [1, \infty)$. We say $u \in W^{k,p}(\Omega)$ if and only if for any $\alpha \in \mathbb{Z}^n_+$, satisfying $|\alpha| \leq k$, the weak derivative $D^\alpha u$ (all derivatives up to order k) exists and belongs to $L^p(\Omega)$. Introduce norm on $W^{k,p}(\Omega)$ in the following manner,*

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}.$$

Remark 1.1.75. *Indeed $W^{0,p}(\Omega) = L^p(\Omega)$.*

Theorem 1.1.76. *[27] (Page 270) The Sobolev space $\left(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)}\right)$ is separable Banach space.*

Corollary 1.1.77. *[27] (Page 271) For $p = 2$, the Sobolev space $H^k(\Omega) := W^{k,2}(\Omega)$ is a Hilbert space with the inner product,*

$$\langle u, v \rangle_{H^k(\Omega)} := \int_\Omega \sum_{|\alpha| \leq k} D^\alpha u \cdot D^\alpha v dx.$$

Definition 1.1.78. *[27] (Page 271) By $W_0^{k,p}(\Omega)$ we mean the closure of space $\mathfrak{D}(\Omega)$ in $W^{k,p}(\Omega)$.*

Finally, we will end this section by presenting a well-known inequality, known as Gagliardo-Nirenberg inequality. We are going use consequences of this inequality very frequently throughout the dissertation. Before that let us recall the definition of *embedding operator.*

Definition 1.1.79. *[44] (Page 328) Let X and Y be normed linear spaces.*

*i) X is called **continuously** embedded into Y if and only if the operator $i : X \to Y$ is injective and continuous i.e. there exists constant $c > 0$ such that*

$$\|u\|_X \leq c \|u\|_Y, \text{ for all } u \in X.$$

*In this case we denote this embedding symbolically by $X \hookrightarrow Y$ and operator $i : X \to Y$ is called **embedding operator**.*

*ii) The embedding $X \hookrightarrow Y$ is **compact** if and only the operator $i : X \to Y$ is compact i.e. each bounded sequence (u_n) in X has a subsequence that converges in Y.*

Following inequality is one important inequality that we are going to employ frequently, throughout the dissertation.

Lemma 1.1.80. ***(Gagliardo-Nirenberg inequality)*** *[47] (Page 10) Assume that $r, q \in [1, \infty)$, and $j, m \in \mathbb{Z}$ satisfy $0 \leq j < m$. Then for any $u \in C_0^\infty(\mathbb{R}^n)$*

$$\left|D^j u\right|_{L^p(\mathbb{R}^n)} \leq C \left|D^m u\right|^a_{L^r(\mathbb{R}^n)} |u|^{1-a}_{L^q(\mathbb{R}^n)},$$

where $\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{d}\right) + (1-a)\frac{1}{q}$ *for all* $a \in \left[\frac{j}{m}, 1\right]$. *If* $m - j - \frac{n}{r}$ *is a non-negative integer, then the equality holds only for* $a \in \left[\frac{j}{m}, 1\right)$.

1.1.7 Bochner Spaces

Assume that X denotes a Banach space and $\Omega \subset \mathbb{R}$. Lets us set some important terminology. The stuff in the following subsection is based on Subsection 1.2 of [51].

Definition 1.1.81. *i) A function* $f : \Omega \to X$ *is called* ***Simple*** *function, if there exist* $x_1, x_2, .., x_n \in X$ *and measurable and mutually disjoint subsets* $\Gamma_1, \Gamma_2, ..., \Gamma_k \subset \Omega$ *such that:*

$$f(t) = x_i, \text{ for all } t \in \Gamma_i \text{ and for all } i = 1, 2, ..., k.$$

ii) A function $f : \Omega \to X$ *is called* ***strongly measurable*** *if there exists a sequence of simple function* (f_n) *such that* $f_n \overset{X}{\to} f$ *a.e. on* Ω, *i.e.* $\|f_n(t) - f(t)\|_X \to 0$ *as* $n \to \infty$, *a.e. on* Ω.

Lemma 1.1.82. *Let* $f : \Omega \to X$ *be a strongly measurable function then the function* $\lambda : \Omega \to \mathbb{R}$ $(t \mapsto \|f(t)\|_X)$ *is Lebesgue measurable i.e. for any open set* O *the inverse image* $\lambda^{-1}(O)$ *is open.*

Definition 1.1.83. *A function* $f : \Omega \to X$ *is called* ***Bochner integrable***, *if there exists a sequence of simple functions* (f_n), *such that:*

i) f *is strongly measurable i.e.* $\|f_n(t) - f(t)\|_X \to 0$ *as* $n \to \infty$, *a.a.* $t \in \Omega$.

ii) Also $\int_\Omega \|f_n(t) - f(t)\|_X \, dt \to 0$ *as* $n \to \infty$.

Theorem 1.1.84. *If* $f : \Omega \to X$ *is strongly measurable then* f *is Bochner integrable iff the Lebesgue integral of* $\|f(\cdot)\|_X$ *over* Ω, *is finite.*

Corollary 1.1.85. *If $f \in C\left(\overline{\Omega};X\right)$ then f is Bochner integrable iff the Lebesgue integral of $\|f(\cdot)\|_X$ over Ω, is finite.*

Lemma 1.1.86. *If f is Bochner integrable over Ω, then*

i) $\left\|\int_\Omega f(t)dt\right\|_X \le \int_\Omega \|f(t)\|_X\, dt$,

ii) $\lim\limits_{\substack{|\Gamma|\to 0^+ \\ \Gamma\subset\Omega}} \int_\Gamma f(t)dt = 0 \in X$.

$L^p(\Omega;X)$ Spaces

We have employed subsection 1.3 of [51] for the following subsection.

Definition 1.1.87. *Let X be a Banach space, $1 \le p \le \infty$, $\Omega \subset \mathbb{R}$. By $L^p(\Omega;X)$ we mean the set of all strongly measurable functions $f:\Omega \to X$ that satisfies the following two properties:*

i) For $1 \le p < \infty$,

$$\int_\Omega \|f(t)\|_X^p\, dt < \infty.$$

ii) For $p = \infty$,

$$ess\sup_\Omega \|f(t)\|_X < \infty.$$

Theorem 1.1.88. *The space $L^p(\Omega;X)$ described in last definition is the linear space. By $f_1 = f_2$ we mean $f_1(t) = f_2(t)$, for a.a. $t \in \Omega$. Then $L^p(\Omega;X)$ are Banach spaces with the following described norms,*

$$\begin{aligned} \|f\|_{L^p(\Omega;X)} &= \left(\int_\Omega \|f(t)\|_X^p\, dt\right)^{1/p}, \text{ where } 1 \le p < \infty, \\ \|f\|_{L^\infty(\Omega;X)} &= ess\sup_{t\in\Omega} \|f(t)\|_X < \infty, \text{ when } p = \infty. \end{aligned}$$

Moreover, if X is reflexive (or separable) Banach space then for $1 \le p < \infty$, $L^p(\Omega;X)$ is also reflexive (or separable) Banach space.

Lemma 1.1.89. *i) Let Ω be the bounded interval then* $\left\| \int_{\Omega} f(t)dt \right\|_X \leq \int_{\Omega} \|f(t)\|_X \, dt \leq \|f\|_{L^p(\Omega;X)} |\Omega|^{\frac{p-1}{p}}$,

ii) Let Ω be the bounded interval then $C_0^{\infty}((0,T);X)$ is dense in $L^p(\Omega;X)$, for all $1 \leq p < \infty$.

Now we present some well known key inequalities in L^p-spaces.

Theorem 1.1.90. *i) (Holder inequality) Let $p, q \geq 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$ (and if $p = 1$ then set $q := \infty$ and vice versa). If $u \in L^p(\Omega;X)$ and $v \in L^q(\Omega;X)$ then $uv \in L^1(\Omega;X)$ and*

$$\|uv\|_{L^1(\Omega;X)} \leq \|u\|_{L^p(\Omega;X)} \|v\|_{L^q(\Omega;X)}.$$

ii) (Minkowski's inequality) Let $1 \leq p \leq \infty$. If $u, v \in L^p(\Omega;X)$ then $u + v \in L^p(\Omega;X)$ and

$$\|u+v\|_{L^p(\Omega;X)} \leq \|u\|_{L^p(\Omega;X)} + \|v\|_{L^p(\Omega;X)}.$$

1.1.8 Some important notions and results from PDE theory

Laplace operator with Dirichlet boundary condition

In this dissertation, both deterministic problem and its stochastic generalization, involve Laplace operator with Dirichlet boundary conditions. The aim of this subsection is to recall some basic stuff about the Dirichlet boundary condition in $L^2(\Omega)$ setting.

Definition 1.1.91. *([51], Example 4.1.2) Let Ω be a non-empty and open subset of R^n. Let us consider the operator A on $L^2(\Omega)$, defined by,*

$$\begin{aligned} D(A) &= \left\{ u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega) \right\} \\ Au &= \Delta u, \text{ for each } u \in D(A). \end{aligned}$$

The operator A is called Laplace operator with Dirichlet boundary condition.

Theorem 1.1.92. *([51], Theorem 4.1.2) The linear operator defined above generates a C_0-semigroup of contractions. Moreover, A is self-adjoint, and $\left(D(A), \|\cdot\|_{D(A)}\right)$ is continuously embedded into $H_0^1(\Omega)$. Further, if Ω is bounded with C^1 boundary, then $\left(D(A), \|\cdot\|_{D(A)}\right)$ is compactly embedded into $L^2(\Omega)$.*

Indeed $H_0^1(\Omega) \cap H^2(\Omega) \subset D(A)$. One can see that if Ω is bounded with C^2 boundary then reverse inclusion also holds i.e.

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega).$$

Some important results

For existence and uniqueness of the solution, we will employ the following classical Banach fixed point theorem.

Theorem 1.1.93. *([27],Theorem 4.7) Let (X, d) be a Complete metric space, $K \subset H$ be a closed subset, $f : K \to K$ be a function that satisfy the inequality, for some $0 \leq \alpha < 1$,*

$$d(f(u), f(v)) \leq \alpha d(u, v), \text{ for all } u, v \in H,$$

Then f has uniquely determined fixed point in K i.e. there exists a unique $a \in K$ such that $f(a) = a$.

Lemma 1.1.94. *([49], Lemma III 1.2) Let V, H be two Hilbert spaces, V^* and H^* be corresponding dual spaces. Assume that $V \hookrightarrow H = H^* \hookrightarrow V^*$, where embedding are dense also. If a function u belongs to $L^2(0, T; V)$ and its weak derivative u' belongs to $L^2(0, T; V')$, then u is a.e. equal to an absolutely continuous function from $[0, T]$ into H, and the following equality, which holds in sense of distributions on $(0, T)$:*

$$\frac{d}{dt} \|u(t)\|_H^2 = 2 \langle u', u \rangle.$$

Lemma 1.1.95. *[47] Let T be given with $0 < T \leq \infty$. Suppose that $y(t)$ and $h(t)$ are non-negative continuous functions defined on $[0.T]$ and satisfy the conditions:*

$$\frac{dy}{dt} \leq A_1 y^2 + A_2 + h(t), \tag{1.1.4}$$

$$\int_0^T y(t) \leq A_3 \text{ and } \int_0^T h(t) \leq A_4, \tag{1.1.5}$$

where $A_i, i = 1, 2, 3, 4$. Then for any $r > 0$ with $0 < r < T$, the following estimate hold

$$y(t+r) \leq \left(\frac{A_3}{r} + A_2 r + A_4\right) e^{A_1 A_2},\ t \in [0, T-r).$$

Further, if $T = +\infty$, then

$$\lim_{t \to +\infty} y(t) = 0.$$

Lemma 1.1.96. *[47] (Bellman–Gronwall Inequality) Suppose $\phi \in L^1[a,b]$ satisfies*

$$\phi(t) \leq f(t) + \beta \int_a^t \phi(s) ds,$$

where $f \in L^1[a,b]$ and β is a positive constant, then

$$\phi(t) \leq f(t) + \beta \int_a^t f(s) e^{\beta(t-s)} ds.$$

In particular if $f(t) = \alpha$ (constant) then

$$\phi(t) \leq \alpha e^{\beta(t-a)},\ \text{for all } t \in [a,b].$$

1.2 Geometric Preliminaries

In this section, we aim to give the detailed account of all those geometric preliminaries needed for the dissertation. In particular, our main topic of concern will be the Hilbert manifold.

We first recall some of basic topological definitions, for this purpose assume that X and Y are topological spaces.

Definition 1.2.1. *A function $f : X \to Y$ is called **Homeomorphism** iff f is bijection and bi-continuous i.e. f and f^{-1} are continuous.*

Definition 1.2.2. *A cover $\{U_i\}_{i\in I}$ (not necessarily open) of X is called **Refinement** of another cover $\{V_j\}_{j\in J}$, of X, if for all $i \in I$ there exists $j \in J$ such that $U_i \subset V_j$.*

Definition 1.2.3. *A family $\{U_i\}_{i\in I}$ of subsets of X is called **Locally finite** if for each $x \in X$ there exists a neighborhood N whose intersection with U_i is non-empty for only finitely many i.*

Definition 1.2.4. *A Hausdroff space X is called **Paracompact** if every open cover $\{U_i\}_{i\in I}$ of X has an open, locally finite refinement.*

1.2.1 Hilbert Manifold

We intend to introduce the formal apparatus required to define Hilbert manifold and hence the definition of Hilbert manifold itself. Throughout subsection, we assume that M is paracompact topological space and H is separable Hilbert space. All results and definitions of this subsection are from Chapter 2, Section 2.1 of [33].

Definition 1.2.5. *A **chart** of M with values in H is a pair (O, φ), where O is an open subset of M and $\varphi : O \to \varphi(M)$ is homeomorphism between U and open subset $\varphi(M)$ of H. The set O is called the **domain of chart**.*

Definition 1.2.6. *Let (U, φ) and (V, ψ) be charts of M, taking values in H, are said to be **compatible**, if either U and V are disjoint, or the map*

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

*is C^∞ diffeomorphism i.e. the map $\psi \circ \varphi^{-1}$ is bijection also $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are smooth, between open subsets of $\varphi(U \cap V)$ and $\psi(U \cap V)$. The map $\psi \circ \varphi^{-1}$ is said to be a **transition map**.*

Definition 1.2.7. *A family of charts* $\mathfrak{A} := \{(U_i, \varphi_i)\}_{i\in I}$ *is called* ***Atlas*** *of* M *with values in* H, *if,*

i) $M = \cup_{i\in I} U_i$;

ii) for all distinct $i, j \in I$, *the charts* (U_i, φ_i) *and* (U_j, φ_j) *are compatible.*

Definition 1.2.8. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be the two atlases on* M , *taking values in* H, *and* $\mathfrak{A} \cup \mathfrak{B}$ *is also an atlas then* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* ***equivalent atlases***. *Assume that* Λ *be the set of all atlases on* M. *Define a relation* $\sim$ *on* Λ *as:*

For $\mathfrak{A}, \mathfrak{B} \in \Lambda$, $\mathfrak{A} \sim \mathfrak{B}$ *if and only if* $\mathfrak{A} \cup \mathfrak{B}$ *is also an atlas.*

One can see that the relation $\sim$ *is equivalence relation on* Λ. *Hence this equivalence relation is going to partition the* Λ *into equivalence classes. Each such equivalence class* $\mathcal{C}$ *is called differentiable structure on* M *and pair* $(M, \mathcal{C})$ *is called* ***differentiable manifold modelled on the Hilbert space*** H *i.e.* ***Hilbert manifold.*** *Moreover, if* $\mathcal{C}$ *is differentiable structure on* M, *then the set*

$$\mathfrak{A}_{\mathcal{C}} := \bigcup_{\mathfrak{A}\in\mathcal{C}} \mathfrak{A},$$

is also an atlas of $\mathcal{C}$, *called the* ***maximal atlas*** *of for* $\mathcal{C}$.

Example 1.2.9. *i) Every Hilbert space* H *has a canonical structure of Hilbert manifold modelled on itself. An atlas for this structure is given by only chart* (H, i_H), *where* i_H *is identity map.*

ii) If O *be an open subset of Hilbert Manifold* M *then* A *inherits the structure of Hilbert Manifold in the following manner. If* $\mathfrak{A} = \{(U_i, \varphi_i)\}_{i\in I}$ *denotes an atlas on* M *then* $\mathfrak{A}|_O = \{(U_i \cap O, \varphi_i|_{U_i\cap O})\}_{i\in I}$ *is induced atlas on* A.

1.2.2 Tangent space on Hilbert Manifold M

Assume that M be Hilbert manifold modelled on Hilbert space H and let $\mathfrak{A}$ be the maximal atlas of the structure. Consider an arbitrary point $m \in M$ and

$\mathfrak{A}_m = \{(U_i, \varphi_i)\}_{i\in I}$ be the set of charts in $\mathfrak{A}$ containing m. Set $\Lambda_m := \mathfrak{A}_m \times H$ and define a relation $\sim$ on Λ_m, in the following manner.

For (U, φ, u) and (V, ψ, v), we say

$$(U, \varphi, u) \sim (V, \psi, v) \text{ if and only if } \left[\left(\psi \circ \varphi^{-1}\right)' (\varphi (m))\right] (u) = v.$$

The map $(\psi \circ \varphi^{-1})' (\varphi (m)) : H \to H$ denotes the differential of $(\psi \circ \varphi^{-1})$ at point $\varphi (m)$. One can see that $\sim$ equivalence relation on Λ_m.

Definition 1.2.10. *Let $\sim$ be the above described equivalence relation on Λ_m then **tangent space** T_mM at point m, is the quotient set:*

$$T_mM := \Lambda_m/ \sim= \{[m]_\sim : m \in M\}.$$

1.3 Dynamical systems preliminaries

In chapter 3 of this dissertation, we will be dealing with the long-term behavior of the solution of the deterministic problem and we will show that solution converges to steady state solution as time $t \to \infty$. To do this one should need to know the following dynamical system preliminaries. The **dynamical system** that we are going to consider will be the semigroups defined on Hilbert spaces. The reference for this section is Section 1 & Chapter 1 of [49].

Definition 1.3.1. *Let H be a Hilbert space. A family $\{S(t) : t \geq 0\}$ of operators from H into H, that evolve in time is called **Semigroup** if it satisfy:*

$$\begin{aligned} S(0) &= I, \text{ where } I \text{ is identity operator on } H, \\ S(t+s) &= S(t)S(s), \text{ for all } s, t \geq 0, \end{aligned}$$

Moreover, $u(t) := S(t)u_0$ is continuous in t and u_0.

Definition 1.3.2. *For $u_0 \in H$, by an **orbit/trajectory** starting from u_0 the set $\bigcup\limits_{t\geq 0} S(t)u_0$, equivalently $\{u(t) = S(t)u_0 : t \geq 0\}$.*

Definition 1.3.3. *A set $A \subset H$, is called **invariant** under $S(\cdot)$ if $S(t)A = A$, for any $t \geq 0$.*

Definition 1.3.4. *A continuous $u(\cdot) : \mathbb{R} \to H$ is **global solution** for $S(\cdot)$, if it satisfies $S(t)u(s) = S(t+s)u_0$, for all $t \geq 0$ and $s \in \mathbb{R}$.*

Lemma 1.3.5. *A set A is invariant under $S(\cdot)$ if and only if it consists of collection of orbits of global solution.*

Definition 1.3.6. *For $u_0 \in H$ (or $A \subset H$), the ω-**limit set** of $u_0 \in H$ (or $A \subset H$) can be defined as,*

$$\omega(u_0) := \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s} S(t)u_0}, \quad \left(\text{or } \omega(A) := \bigcap_{s\geq 0}\overline{\bigcup_{t\geq s} S(t)A}\right).$$

Lemma 1.3.7. *If $u_0 \in H$ then $\varphi \in \omega(A)$ if and only if there exists a sequence of elements (φ_n) in A and sequence $t_n \to +\infty$ such that*

$$S(t_n)\varphi_n \to \varphi \text{ as } n \to \infty.$$

Definition 1.3.8. *A point $u_0 \in H$ is called **fixed/equilibrium or stationary point** of $\{S(t) : t \geq 0\}$ if and only if*

$$S(t)u_0 = u_0, \text{ for all } t \geq 0.$$

Lemma 1.3.9. *Assume that $\emptyset \neq A \subset H$ and for some $t_0 \geq 0$, the set $\bigcup\limits_{t\geq t_0} S(t)A$ is relatively compact in H. Then $\omega(A)$ is non-empty, compact and invariant.*

1.4 Miscellaneous Preliminaries

In this section we aim to provide some of fundamentals from classical set theory, real analysis and measure theory.

1.4.1 Equivalence relations, Partially ordered set and Kurtowski-Zorn's Lemma

Main objective of this subsection is to introduce the Kurtowski-Zorn's Lemma, which will play a significant role in the existence of a local maximal solution of our deterministic Heat equation on Hilbert-Manifold. Let us begin by recalling fundamentals of relation theory. The primary source for this subsection is Chapter 1 of [32].

Definition 1.4.1. *Let A and B be two sets. By a* ***relation*** *(or* ***binary relation****) $\sim$, from a set A into B, is a subset of $A \times B$. Suppose R is a relation from A into B, if $(a, b) \in A \times B$ we denote this by aRb or $R(a) = b$. Moreover, the* ***domain,*** *denoted by $\mathcal{D}(R)$, and the* ***range*** *or* ***image****, denoted by $\mathcal{I}(R)$, of relation $\sim$ can be defined in the following manner,*

$$\begin{aligned} \mathcal{D}(R) \ &: \ = \{x : x \in A \textit{ and there exists } y \in B \textit{ such that } (x, y) \in R\}, \\ \mathcal{I}(R) \ &: \ = \{y : y \in B \textit{ and there exists } x \in A \textit{ such that } (x, y) \in R\}. \end{aligned}$$

Example 1.4.2. *Let X be a non-empty set. Let R be set of all ordered pairs $(U, V) \in X \times X$ such that $U \subseteq V$ i.e.*

$$R := \{(U, V) : (U, V) \in X \times X \textit{ and } U \subseteq V\},$$

defines a binary relation.

Definition 1.4.3. *Let R be a binary relation on set Ξ. Then R is called:*

i) ***Reflexive****, if for all $x \in \Xi$ we have xRx.*

ii) ***Symmetric****, if for all $x, y \in \Xi$, xRy implies yRx,*

iii) ***Transitive****, if for all $x, y, z \in \Xi$, xRy and yRz implies xRz.*

Definition 1.4.4. *Let E be a relation on Ξ. Then E is called Equivalence relation on Ξ if R is reflexive, symmetric, transitive relations.*

Example 1.4.5. *Let X be a non-empty set. Consider $\sim$ be relation on X defined as, $U \sim V$ if and only if $U \subseteq V$ then $\sim$ is:*

i) Reflexive, because for any $U \subseteq X$, trivially $U \subseteq U$ i.e. $U \sim U$.

ii) Transitive, because for any $U, V, W \subseteq X$ if $U \sim V$ and $V \sim W$ i.e. $U \subseteq V \subseteq W$ implies $U \subseteq W$ i.e. $U \sim W$.

But $\sim$ is not symmetric because $U \subseteq V$ does not necessarily imply $V \subseteq U$ i.e.$U \sim V$ does not necessarily imply $V \sim U$. Thus $\sim$ is not equivalence relation.

Definition 1.4.6. *Let E be an equivalence on set Ξ. For all $x \in E$, by $[x]$ we mean the set*

$$[x] = \{y \in A : yEx\}$$

*called the **equivalence class** determined by x.*

Following is an important theorem about the fundamental properties of equivalence classes.

Theorem 1.4.7. *Let E be the equivalence relation on Ξ. Then for all $x, y \in \Xi$:*

i) $[x] \neq \emptyset$,

ii) if $y \in [x]$ then $[x] = [y]$,

iii) either $[x] \cap [y] = \emptyset$ or $[x] = [y]$,

iv) $\Xi = \bigcup\limits_{x \in \Xi} [x]$.

Definition 1.4.8. *Let Ξ be $\mathcal{P}$ be non-empty collection of subsets of Ξ. We say that $\mathcal{P}$ is called partition of Ξ if the following properties hold:*

i) for all $B, C \in \mathcal{P}$ either $B = C$ or $B \cap C = \emptyset$,

ii) $\Xi = \bigcup\limits_{B \in \mathcal{P}} B$.

Theorem 1.4.9. *Let E be the equivalence relation on Ξ. Then*

$$\mathcal{P} = \{[x] : x \in A\},$$

defines a partition of Ξ.

Definition 1.4.10. *A relation $\preceq$ on Ξ is called* ***partial order*** *if $\preceq$ is:*

i) $\preceq$ is reflexive i.e. $x \preceq x$ for all $x \in \Xi$,

ii) $\preceq$ is antisymmetric i.e. for all $x, y \in \Xi$ if $x \preceq y$ and $y \preceq x$ then $x = y$,

iii) $\preceq$ is transitive i.e. for all $x, y, z \in \Xi$ if $x \preceq y$ and $y \preceq z$ implies $x \preceq z$.

In this case $(\Xi, \preceq)$ is called ***Partially ordered set (or POSET)****.*

Example 1.4.11. *Let X be a non-empty set. Consider $\preceq$ a relation on X defined as, $U \preceq V$ if and only if $U \subseteq V$ then $\preceq$ is:*

i) Reflexive, because for any $U \subseteq X$, trivially $U \subseteq U$ i.e. $U \preceq U$,

ii) Antisymmetric, because $U \preceq V$ and $V \preceq U$ i.e. $U \subseteq V \subseteq U$ implies $U = V$,

iii) Transitive, because for any $U, V, W \subseteq X$ if $U \preceq V$ and $V \preceq W$ i.e. $U \subseteq V \subseteq W$ implies $U \subseteq W$ i.e. $U \preceq W$.

Hence $\preceq$ is partial order on X and $(X, \preceq)$ is a POSET.

Definition 1.4.12. *A partially ordered set $(\Xi, \preceq)$ is called* ***chain*** *or* ***linearly ordered set*** *if for all $x, y \in \Xi$ either $x \preceq y$ or $y \preceq x$.*

Definition 1.4.13. *Let $(\Xi, \preceq)$ be a Poset and $B \subseteq \Xi$ then:*

i) an element $u \in \Xi$ is called ***upper bound*** *of B if $x \preceq u$ for all $x \in B$.*

ii) an element $l \in \Xi$ is called ***least upper bound*** *of B if the following two are true:*

a) l is upper bound of B,

b) if $c \in \Xi$ is upper bound of B then $l \preceq c$.

Definition 1.4.14. *Let $(\Xi, \preceq)$ be a Poset. An element $m \in \Xi$ is called **maximal element** of A if there is no element $x \in \Xi$ such that $m \preceq x$ and $m \neq x$.*

Following the main result of this subsection i.e. Kurtowski-Zorn's Lemma.

Lemma 1.4.15. *(Kurtowski-Zorn's Lemma) If every chain in a Poset $(\Xi, \preceq)$ has an upper bound in Ξ then Ξ contains a maximal element.*

1.4.2 Limit theorem in $\mathbb{R}$

In this subsection, we recall one of the basic limit theorem from real analysis, which we are going to use at several instances throughout the dissertation.

Definition 1.4.16. *[27] A sequence of real numbers $(a_n)_{n\in N}$ is called **monotonically increasing** iff $a_n \leq a_{n+1}$, for all $n \in N$. Similarly it is called monotonically **decreasing sequence** iff $a_n \leq a_{n+1}$, for all $n \in N$.*

Lemma 1.4.17. *[27] A bounded monotonically increasing (resp. decreasing) sequence $(a_n)_{n\in N}$ converges to $\sup\limits_{n\in N}\{a_n\}$ (resp. $\inf\limits_{n\in N}\{a_n\}$).*

1.4.3 Measure Theoretic preliminaries

In this subsection, we review some of the basic notions from Measure theory. Our main reference for this is the Chapter 1 of [42].

Definition 1.4.18. *A collection $\mathcal{F}$ of subsets of X is said to be σ**-algebra** in X if $\mathcal{F}$ has the following properties:*

a) $X \in \mathcal{F}$,

b) If $A \in \mathcal{F}$, then $A^c = X \backslash A \in \mathcal{F}$,

c) If $A_n \in \mathcal{F}$, $n = 1, 2, ...,$ then $\bigcup\limits_{k=1}^{\infty} A_k \in \mathcal{F}$.

*If $\mathcal{F}$ is a σ-algebra in X then $(X, \mathcal{F})$ is called a **measurable space** and members of $\mathcal{F}$ are called the **measurable sets** in X.*

Definition 1.4.19. *If X is measurable space, Y is a topological space, and f is a mapping from X into Y, then f is said to be **measurable function** if for every open set A in Y, $f^{-1}(A)$ is a measurable set in X.*

Theorem 1.4.20. *Let $\sum$ be a collection of subsets of X, then there exists a smallest σ-algebra $\mathcal{F}$ in X such that $\sum \subset \mathcal{F}$. This σ-algebra is denoted by $\sigma\left(\sum\right)$ and called σ-algebra generated by $\sum$.*

Definition 1.4.21. *Assume that (X, τ) is a topological space,by above theorem there exists a smallest σ-algebra $\mathcal{B}$ such that $\tau \subset \mathcal{B}$. The space (X, τ) is called **Borel σ-algebra** and elements of $\mathcal{B}$ are called **Borel sets** of X.*

Remark 1.4.22. *Closed sets are Borel because $\mathcal{B}$ contains all open sets and being σ-algebra contains the complements of open sets. Moreover since $\mathcal{B}$ is σ-algebra so it contains countable union of closed sets and countable intersection of open sets and so these are Borel, denote F_δ by countable union of closed sets G_δ by the countable intersection of open sets. Since $\mathcal{B}$ is σ-algebra in X so $(X, \mathcal{B})$ can be treated as the measurable space. A function $f : X \to Y$, where Y is topological space, is called a **Borel measurable function** if inverse image of each open set in Y, is a Borel set in X .*

Theorem 1.4.23. *Suppose $\mathcal{F}$ be a σ-algebra in X and Y be a topological space. Let f be a map from X into Y, then the following holds:*

a) If $\sum = \{E \subset Y : f^{-1}(E) \in \mathcal{F}\}$ then $\sum$ is σ-algebra in Y.

b) The inverse image of Borel set in Y, under a Borel measurable map f, is a measurable set in X.

c) If $Y = \mathbb{R}$ and $f^{-1}(\alpha, \infty) \in \mathcal{F}$, for every $\alpha \in \mathbb{R}$, then f is measurable.

d) If f is measurable, if Z is a topological space and if $g : Y \to Z$ is Borel measurable then $g \circ f$ is also measurable.

Theorem 1.4.24. *Let $f_n : X \to \mathbb{R}$ be a measurable function, for each $n = 1, 2, 3...$, then $\sup_{n \geq 1} f_n$ and $\lim_{n\to\infty} \sup f_n$ are also measurable.*

Definition 1.4.25. *Let $(X, \mathcal{F})$ be the measurable space. A map $\mu : \mathcal{F} \to [0, \infty]$ is called* ***measure*** *on X if:*

i) $\mu(\emptyset) = 0$,

ii) Any countable family $(A_i)_{i\in\mathbb{N}} \subset \mathcal{F}$ of pair-wise disjoint sets, satisfies the following σ-additivity property,

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

In this case $(X, \mathcal{F}, \mu)$ is called ***measure space****. The measure μ is called* ***finite*** *if $\mu(X) < \infty$ and is called probability measure if $\mu(X) = 1$.*

Theorem 1.4.26. *Let μ be a positive measure on the σ-algebra $\mathcal{F}$ then:*

a) $\mu(\emptyset) = 0$,

b) If $A_1, A_2, ..., A_n \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$, for $i \neq j$, $i, j = 1, 2, ..., n$, then

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i),$$

c) If $A_1, A_2, ... \in \mathcal{F}$ and $A_i \subset A_{i+1}$, for all $i \in \mathbb{N}$, then

$$\mu(A_n) \to \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \quad \text{as } n \to \infty,$$

d) If $A_1, A_2, ... \in \mathcal{F}$ and $A_i \supset A_{i+1}$, for all $i \in \mathbb{N}$, then

$$\mu(A_n) \to \mu\left(\bigcap_{i=1}^{\infty} A_i\right) \quad \text{as } n \to \infty.$$

and $\mu(A_1) < \infty$.

Definition 1.4.27. *Let $(X, \mathcal{F}, \mu)$ be the measure space. A measurable function $\phi : X \to [0, \infty)$ is called **simple function** if there exist real valued sequence $(\phi_i)_{i=1}^n$ such that ϕ can be written in form:*

$$\phi = \sum_{i=1}^{n} \phi_i 1_{A_i},$$

where $A_i := \phi^{-1}(\phi_i)$. also $1_{A_i}(x) = 1$ if $x \in A_i$ and $1_{A_i}(x) = 0$ if $x \in X \backslash A$.

For $E \in \mathcal{F}$ we define,

$$\int_E \phi d\mu := \sum_{i=1}^{n} \phi_i \mu(A_i \cap E).$$

Note that if for some i, $\phi_i = 0$ and $\mu(A_i \cap E) = \infty$ then we set $\phi_i \mu(A_i \cap E) := 0$.

*Finally, if $f : X \to [0, \infty]$ be a measurable function and $E \in \mathcal{F}$, we define **Lebesgue integral** of f over E, is number in $[0, \infty]$ as,*

$$\int_E f d\mu := \sup_{0 \le s \le f} \int_E \phi d\mu.$$

Theorem 1.4.28. ***(Monotone Convergence Theorem)*** *Let (f_n) be a sequence of Lebesgue measurable functions on Ω, satisfying*

a) $0 \le f_1 \le f_2 \le ... \le \infty$

b) $f_n \to f$ point-wise as $n \to \infty$.

Then f is measurable and $\int_\Omega f_n d\mu \to \int_\Omega f d\mu$ as $n \to \infty$.

Theorem 1.4.29. *If $f_n : \Omega \to [0, \infty]$ is measurable for all $n \in \mathbb{N}$, and $f(x) = \sum_n f_n(x)$, for all $x \in \Omega$, then*

$$\int_\Omega f d\mu = \sum_n \int_\Omega f_n d\mu$$

Theorem 1.4.30. ***(Fatou's Lemma)*** *If $f_n : \Omega \to [0, \infty]$ is measurable for all $n \in \mathbb{N}$, then*

$$\int_\Omega \left(\lim_{n\to\infty} \inf f_n \right) d\mu \le \lim_{n\to\infty} \inf \int_\Omega f_n d\mu$$

$$\lim_{n\to\infty} \sup \int_\Omega f_n d\mu \le \int_\Omega \left(\lim_{n\to\infty} \sup f_n \right) d\mu$$

Theorem 1.4.31. *If $f \in L^1(\mu)$ i.e. $\int_E f d\mu < \infty$, then*

$$\left| \int_\Omega f d\mu \right| \leq \int_\Omega |f| \, d\mu.$$

Theorem 1.4.32. ***(Lebesgue Dominated Convergence Theorem)*** *Suppose (f_n) be the sequence of complex measurable functions on Ω such that $f_n \to f$ pointwise as $n \to \infty$. If there exists a function $g \in L^1(\mu)$ such that $|f_n| \leq g$ on Ω then $f \in L^1(\mu)$ and*

$$\lim_{n\to\infty} \int_\Omega |f_n - f| \, d\mu = 0 \text{ also } \lim_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu.$$

Definition 1.4.33. *[46] Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces. Then*

$$\mathcal{A} \otimes \mathcal{B} := \sigma(\mathcal{A} \times \mathcal{B}) = \sigma(\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}).$$

*is called a **product σ-algebra**, and $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ is the **product of measurable space**.*

1.5 Stochastic Preliminaries

This section is very important as we aim to provide preliminaries from stochastic analysis.

For the section we fix the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us begin by reviewing the basic definitions related to stochastic processes that we are going to encounter frequently, throughout the dissertation. All results and definitions of this section we refer to Chapter 1 of [39].

Definition 1.5.1. *A **continuous-time stochastic process** $(X_t)_{t\in\mathbb{T}}$ is a family of H-valued random variables indexed by time t, where H denotes a measurable space, moreover we treat either $\mathbb{T} := [0, T]$ or $\mathbb{T} := [0, \infty)$. For each $\omega \in \Omega$, the map $X(\omega) : t \in \mathbb{T} \to X_t(\omega)$ is called **path (or trajectory)** of the process for event ω, moreover for each $t \in \mathbb{T}$ the map $\omega \mapsto X_t(\omega)$ is a random variable.*

Definition 1.5.2. *A **filtration** on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{T}}$ of σ-fields of $\mathcal{F}$ such that, for all $s \leq t$ in $\mathbb{T}$, we have $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathbb{F}$. A process $(X_t)_{t\in\mathbb{T}}$ is called **Adapted** w.r.t $\mathbb{F}$, if for all $t \in \mathbb{T}$, X_t is $\mathcal{F}_t$-measurable. Further, the process $(X_t)_{t\in\mathbb{T}}$ is called is called **Predictable**, w.r.t filtration $\mathcal{F}$, if the map $(t, \omega) \mapsto X_t(\omega)$ is measurable on $\mathbb{T} \times \Omega$ equipped with σ-field generated by the $\mathbb{F}$-adapted and continuous processes. Moreover, $(X_t)_{t\in\mathbb{T}}$ is called **progressively measurable**, w.r.t $\mathcal{F}$, if for all $t \in \mathbb{T}$, the mapping $(s, \omega) \mapsto X_s(\omega)$ is measurable on $[0, t] \times \Omega$ equipped with the product σ-field $\mathbb{B} \otimes \mathcal{F}_t$.*

Remark 1.5.3. *i) Any progressively measurable process adapted and measurable on $\mathbb{T} \times \Omega$ equipped with the product σ-field $\mathbb{B}(\mathbb{T}) \otimes \mathcal{F}$.*

ii) Any continuous and adapted process X is predictable.

Lemma 1.5.4. *Limits of progressively measurable processes are progressively measurable. Moreover, if the processes $(X_t)_{t\in\mathbb{T}}$ is adapted with right continuous paths, then it is progressively measurable.*

Definition 1.5.5. *A random variable $\tau : \Omega \to [0, \infty]$ i.e. a random time, is a stopping time (w.r.t filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in\mathbb{T}}$) if for all $t \in \mathbb{T}$,*

$$\{\tau \leq t\} := \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

*A stopping time τ is called **accessible (or predictable)** if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that: almost surely we have the following:*

i) $\lim_{n\to\infty} \tau_n = \tau$,

ii) on $\{\tau > 0\}$ we have $\tau_n < \tau$, for all $n \in \mathbb{N}$.

We say that sequence $(\tau_n)_{n\in\mathbb{N}}$ approximates (announces) τ.

Remark 1.5.6. *i) One can easily see that if τ and σ are two stopping times then $\tau \wedge \sigma$, $\tau \vee \sigma$ and $\tau + \sigma$ are also stopping times.*

ii) Every stopping time τ is $\mathcal{F}_\tau$-measurable, where $\mathcal{F}_\tau$ is the following σ-field

$$\mathcal{F}_\tau = \{B : B \cap \{\tau \le t\} \in \mathcal{F}_t, \text{ for all } t \in \mathbb{T}\}.$$

iii) A r.v ξ is $\mathcal{F}_\tau$-measurable if and only if for all $t \in \mathbb{T}$, $\xi 1_{\{\tau \le t\}}$ is $\mathcal{F}_t$-measurable.

Proposition 1.5.7. *Let $(X_t)_{t\in\mathbb{T}}$ be a progressively measurable process, and τ a stopping time. Then $X_\tau 1_{\{\tau \le t\}}$ is $\mathcal{F}_\tau$-measurable and stopped process $X_{t\wedge\tau}$ is also progressively measurable.*

Proposition 1.5.8. *Let $(X_t)_{t\in\mathbb{T}}$ be a progressively measurable process and Γ be an open subset of $\mathbb{R}^d$. The hitting time of Γ, given by:*

$$\sigma_\Gamma := \inf\{t \ge 0 : X_t \in \Gamma\}, \textit{with } \inf\emptyset = \infty,$$

is a stopping time. Further, if $\Gamma \in \mathcal{B}(\mathbb{T})$ (Borel σ-algebra) then σ_Γ is still a stopping time.

Theorem 1.5.9. *(Section theorem) Let $(X_t)_{t\in\mathbb{T}}$ and $(Y_t)_{t\in\mathbb{T}}$ be two progressively measurable processes. Assume that for any stopping time τ, we have*

$$X_\tau = Y_\tau \text{ a.s. on } \{\tau < \infty\}.$$

Then, the two processes $(X_t)_{t\in\mathbb{T}}$ and $(Y_t)_{t\in\mathbb{T}}$ are indistinguishable i.e.

$$\mathbb{P}(\{X_t = Y_t, \text{ for all } t \in \mathbb{T}\}) = 1.$$

Definition 1.5.10. *(Standard Brownian Motion) A **Standard $d-$dimensional Brownian motion** on $\mathbb{T}$ is a continuous process valued in $\mathbb{R}^d$, $(W_t)_{t\in\mathbb{T}} = \left(W_t^1, W_t^2, ..., W_t^d\right)_{t\in\mathbb{T}}$ such that:*

i) $W_0 = 0$,

ii) For all $0 \le s < t$ in $\mathbb{T}$, the increment in $W_t - W_s$ is independent of $\sigma(\{W_u, u \le s\})$ and follows a centred Gaussian distribution with variance-covariance matrix $(t-s)I_d$.

Remark 1.5.11. *One obvious deduction is that,* $i = 1, 2, ..., d$, *coordinates* $(W_t^i)_{t\in\mathbb{T}}$, *of* $d-$*dimensional Brownian motion, is a real-valued independent standard Brownian motion. Converse is also true i.e. if* $(W_t^i)_{t\in\mathbb{T}}$, *for each* $i = 1, 2, ..., d$, *be the standard real-valued Brownian motion then* $(W_t)_{t\in\mathbb{T}} = \left(W_t^1, W_t^2, ..., W_t^d\right)_{t\in\mathbb{T}}$ *is standard* $d-$*dimensional Brownian motion on* $\mathbb{T}$.

Now we present definition standard vectorial Brownian motion w.r.t a certain filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$.

Definition 1.5.12. *A* $d-$***dimensional Brownian motion*** *on* $\mathbb{T}$ *w.r.t* $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$, *is a continuous* $\mathbb{F}$*-adapted process valued in* $\mathbb{R}^d$, $(W_t)_{t\in\mathbb{T}} = \left(W_t^1, W_t^2, ..., W_t^d\right)_{t\in\mathbb{T}}$ *such that:*

i) $W_0 = 0$,

ii) For all $0 \leq s < t$ *in* $\mathbb{T}$, *the increment in* $W_t - W_s$ *is independent of* $\mathcal{F}_s$ *and follows a centred Gaussian distribution with variance-covariance matrix* $(t-s)I_d$.

Definition 1.5.13. *An adapted process* $(X_t)_{t\in\mathbb{T}}$, *taking values in measurable space* (H, μ), *is called* H*-valued* ***martingale*** *if it is integrable i.e.* $\mathbb{E}\left(|X_t|\right) = \int |X_t|\, d\mu < \infty$, *and*

$$\mathbb{E}\left(X_t|\mathcal{F}_s\right) = X_s \text{ a.s., for all } 0 \leq s \leq t \text{ and } s, t \in \mathbb{T}.$$

Theorem 1.5.14. *(Optional sampling Theorem) Let* σ *and* τ *be* $\mathbb{T}$*-valued bounded stopping times such that* $\sigma \leq \tau$. *If* $(X_t)_{t\in\mathbb{T}}$ *is the martingale with right continuous paths then,*

$$\mathbb{E}\left(X_\tau|\mathcal{F}_\sigma\right) = X_\sigma \text{ a.s..}$$

Corollary 1.5.15. *Let* $X = (X_t)_{t\in\mathbb{T}}$ *be an adapted process with right-continuous paths. We have the following,* *i) X is martingale iff for any $\mathbb{T}$-valued bounded stopping times σ and τ we have $X_\tau \in L^1$ and*

$$\mathbb{E}(X_\tau) = X_0 \ a.s. \ .$$

ii) If X is martingale and τ is a stopping time, then the stopped process $X^\tau = (X_{t\wedge\tau})_{t\in\mathbb{T}}$ is also a martingale.

Theorem 1.5.16. *(Doob inequalities) Let $X = (X_t)_{t\in\mathbb{T}}$ be a non-negative martingale with right-continuous paths. Then for each stopping time $\mathbb{T}$-valued stopping time τ, we have,*

$$\begin{aligned} \mathbb{P}\left(\sup_{0\le t\le\tau}|X_t| \ge \lambda\right) &\le \frac{\mathbb{E}(|X_\tau|)}{\lambda}, \text{ for all } \lambda \ge 0, \\ \mathbb{E}\left(\sup_{0\le t\le\tau}|X_t|\right)^p &\le \left(\frac{p}{p-1}\right)^p \mathbb{E}(|X_\tau|^p), \text{ for all } p \ge 1. \end{aligned}$$

Theorem 1.5.17. *(Burkholder-Davis-Gundy) For any $1 \le p < \infty$ there exist positive constants c_p and C_p such that, for all continuous martingales X with $X_0 = 0$ and stopping times τ, the following inequality holds.*

$$\mathbb{E}\left[\langle X\rangle_\tau^{p/2}\right] \le \mathbb{E}\left(\sup_{0\le t\le\tau}|X_t|\right)^p \le C_p\mathbb{E}\left[\langle X\rangle_\tau^{p/2}\right].$$

Where $\langle X\rangle$ denotes the Quadratic variation of X.

1.5.1 Some Stochastic PDE results from Pardoux [37]

In this last section of preliminaries chapter, we intend to present the existence and uniqueness results for the most general form of the stochastic partial differential equation, studied in [37]. Moreover, we also present the corresponding version of Itô Lemma that we are going to use at several places in the dissertation. Let me begin by setting up some notion. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and

$(W_t)_{t\geq 0}$ the $d-$dimensional standard Brownian motion. Let $(V, \|\cdot\|)$ and $(H, |\cdot|_H)$ be two separable Hilbert spaces, such that

$$V \subset H \subset V',$$

where $(V', \|\cdot\|^*)$ be dual of V and also the above inclusions are dense.By $\langle , \rangle$ we mean the duality product between V and V', further by $(,)$ scalar product in H, and $[,]$ denotes the scalar product in $\mathbb{R}^d$. Finally, the norm on H^d will be denoted by:

$$|u|_{H^d} = \left(\sum_{i=1}^{d} |u_i|_H^2 \right)^{1/2}.$$

Hilbert space-valued Stochastic Integrals and Itô Lemma

In this subsection we are up to present the stochastic integral taking values in Hilbert space and hence the Itô Lemma. For this purpose, we are going to employ the notation introduced above.

For a Hilbert space X, by $M^2(0, T; X)$ we mean the space of all of X-valued measurable processes, $(u(t))_{t\in[0,T]}$, which satisfy the following two conditions:

i) $u(t)$ is $\mathcal{F}_t$ measurable a.e. in $t \in [0, T]$,

ii) $\mathbb{E} \int_0^T |u(t)|_X^2 \, dt < \infty$.

In particular, X can be taken as $\mathbb{R}^d$, H, H^d, V, V^d and V'.

Proposition 1.5.18. *[37] The space $M^2(0, T; X)$ is a closed subspace of $L^2(\Omega \times (0, T), dP \otimes dt; X)$ and hence complete.*

For $\varsigma \in M^2(0, T; H^d)$ and any $h \in H$, let us define a map $h \longmapsto \int_0^t [(h, \varsigma(s)), dWs]$, from H into $L^2(\Omega)$, where $[,]$ denotes the scalar product in $\mathbb{R}^d$. Indeed, this map is linear. Using this map we can define the H-valued random variable $\int_0^t [\varsigma(s), dWs]$, in the following manner:

$$\left(h, \int_0^t [\varsigma(s), dW_s]\right) := \int_0^t [(h, \varsigma(s)), dW_s], \text{ for all } h \in H.$$

Proposition 1.5.19. *[37] The process $M_t := \int_0^t [\varsigma(s), dW_s]$ is Continuous H-martingale and it satisfies the following:*

$$\begin{aligned} |M_t|^2 - \int_0^t |\varsigma(s)|^2 ds &= 2\int_0^t [(M_s, \varsigma(s)), dW_s], \\ \mathbb{E}|M_t|^2 &= \mathbb{E}\left(\int_0^t |\varsigma(s)|^2 ds\right), \text{ for } t \in [0,T]\ . \end{aligned}$$

1.5.2 A generalized version of the Itô Lemma

Following is the version of Itô Lemma that we are going to incorporate at several places in dissertation.

Theorem 1.5.20. *[37] Let $u \in M^2(0,T;V)$, $u_0 \in H$, $v \in M^2(0,T;V')$ and $\varsigma \in M^2\left(0,T;H^d\right)$, all these satisfy*

$$u(t) = u_0 + \int_0^t v(s)ds + \int_0^t [\varsigma(s), dW_s].$$

Moreover, assume that $\psi : H \to \mathbb{R}$ be twice differentiable functional, which satisfies the following:

i) The maps ψ, ψ' and ψ'' are locally bounded.,

ii) the maps ψ and ψ' are continuous on H,

iii) for all $Q \in \mathcal{L}^1(H)$, $Tr(Q \circ \psi)$ is a continuous functional H.

iv) if $u \in V, \psi'(u) \in V$ then the map $u \longmapsto \psi'(u)$ is continuous from V (with the strong topology), into V endowed with the weak topology.

v) there exists k s.t.

$$\|\psi'(u)\| \leq k(1 + \|u\|), \text{ for all } u \in V.$$

Then for $t \in [0,T]$*:*

$$\begin{aligned}\psi(u(t)) - \psi(u_0) &= \int_0^t \langle v(s), \psi'(u(s))\rangle\, ds + \int_0^t [(\psi'(u(s)), \varsigma(s)), dW(s)] \\ &+ \frac{1}{2}\sum_{i=1}^{d}\int_0^t (\psi''(u(s))\,\varsigma_i(s), \varsigma_i(s))\, ds,\end{aligned}$$

1.5.3 Existence and Uniqueness result

For $f \in M^2(0,T;V')$, $g \in M^2\left(0,T;H^d\right)$ and $u \in M^2(0,T;V)$, the most generalized form of the problem considered in Pardoux [37] is the following:

$$\begin{aligned} du(t) + (A(t)u(t) + f(t))dt &= [B(t)u(t) + g(t), dW_s] \\ u(0) &= u_0 \in H. \end{aligned}$$

Theorem 1.5.21. *[37] There exists a unique solution of above described problem, which also satisfies:*

i) $u \in L^2(\Omega; C(0,T;H))$,

ii)

$$\begin{aligned} &|u(t)|^2 - |u_0|^2 + 2\int_0^t \langle Au(s) + f(s), u(s)\rangle\, ds \\ =& \int_0^t [(Bu(s) + g(s), u(s)), dW(s)] + \int_0^t |Bu(s) + g(s)|^2\, ds. \end{aligned}$$

Chapter 2

Global solution of non-linear Heat equation on Hilbert Manifold

In this chapter, we are concerned with the problem of existence and uniqueness of a local, local maximal and global solution for the nonlinear heat flow equation projected on a manifold (Hilbert) M. In the first section, we are going to begin with some motivational comments and then we will be setting up some notation related to spaces, manifold, and operators, which we are going to deal with later in this chapter. Later, in the first section we will introduce the deterministic constrained problem (abstract and particular projected evolution equation) of our interest, we will end the first section by setting up necessary assumptions and providing the definition of the solution. In the second section, we aim to construct the local mild and local maximal solution of main evolution equation. To do this, we first work on an approximate evolution equation and obtain its solution, with the aid of Banach fixed point theorem. Next, using the Kartowski-Zorn Lemma we aim to obtain the local maximal solution of approximate evolution equation. Then we are going to show that if we can find a constant R such that the energy norm of initial data u_0 is bounded by R, then the solution (local or maximal) of approximate evolution equation is equivalent to the solution (local and maximal) of main evolution equation. The third and last section of this chapter begins with finding a sufficient condition for the local mild solution to be the global solution. Then we are going to prove invariance of manifold i.e. if the initial data lives in manifold then the solution, of projected evolution equation, itself lives in the manifold. Finally, the section and chapter end at the proof of global solution. One interesting fact about the projected flow is that it possesses a gradient flow structure.

2.1 Introduction, Main Problem and Motivation

Rybka in [43] has considered the heat equation in $L^2(\mathcal{O})$ projected on a manifold M, where

$$M = \left\{ u \in L^2(\mathcal{O}) \cap C(\mathcal{O}) : \int_\Omega u^k(x)dx = C_k, k = 1, 2, ..., N \right\},$$

and $\mathcal{O}$ be a bounded, connected region in $\mathbb{R}^d$. Rybka has shown that solution to this problem converges to a steady state as a time of motion. Our approach to tackle the problem is absolutely different from the approach of Rybka [43].

2.1.1 Notation

Let us set some notation that we are going to follow not only in this chapter but throughout the dissertation.

Assumption 2.1.1. *We assume that* $(E, |\cdot|_E)$, $(V, \|\cdot\|)$, $(H, |\cdot|_H)$ *are Banach spaces such that*

$$E \hookrightarrow V \hookrightarrow H,$$

and the embeddings are dense and continuous.

Remark 2.1.2. *In our motivating application, we will consider the following choice of space*

$$\begin{aligned} E &= D(A), \\ V &= H_0^{1,2}(\mathcal{O}), \\ H &= L^2(\mathcal{O}), \end{aligned}$$

where $\mathcal{O} \subset \mathbb{R}^d$ *for* $d \geq 1$, *is a bounded domain with sufficiently smooth boundary.* A *be the Laplace operator with Dirichlet boundary conditions, defined by*

$$\begin{aligned} D(A) &= H_0^{1,2}(\mathcal{O}) \cap H^{2,2}(\mathcal{O}), \\ Au &= -\Delta u, \ u \in D(A). \end{aligned} \tag{2.1.1}$$

It is well known that, see [51] (Theorem 4.1.2, page 79), that A *is a self-adjoint positive operator in* H *and that* $V = D(A^{1/2})$, *and*

$$\|u\|^2 = \left|A^{1/2}u\right|_H^2 = \int_{\mathcal{O}} |\nabla u(x)|^2 \, dx.$$

Moreover,

$$E \subset V \subset H \subset V' \equiv H^{-1}(\mathcal{O}),$$

and inclusion are continuous and dense. Hence E, V *and* H *satisfy Assumption 2.1.1.*

2.1.2 Manifold and Projection

The version of Hilbert manifold we are going to deal with, is the following submanifold M of a Hilbert space H (with inner product denoted by $\langle \cdot, \cdot \rangle$),

$$M = \left\{ u \in H : |u|_H^2 = 1 \right\}.$$

Moreover the tangent space, at a point u in H, is of form,

$$T_u M = \{ v : \langle u, v \rangle = 0 \}.$$

Let $\pi_u : H \to T_u M$ be orthogonal projection of H onto tangent space M then immediately we have the following lemma.

Lemma 2.1.3. *Let $\pi_u : H \to T_u M$ be orthogonal projection then $\pi_u(v) = v - \langle u, v \rangle u$, where $v \in H$.*

We aim to study the projection of Laplace operator and polynomial non-linearity of degree $2n - 1$.

Let us pick a $u \in E$. Using the last lemma we calculate an explicit expression for projection of $\Delta u - u^{2n-1}$ under π_u. The below given calculation using integration by parts, cf. [3] (corollary 8.10, page-82),

$$\begin{aligned} \pi_u \left(\Delta u - u^{2n-1} \right) &= \Delta u - u^{2n-1} - \left\langle \Delta u - u^{2n-1}, u \right\rangle u \\ &= \Delta u - u^{2n-1} + \langle -\Delta u, u \rangle u + \left\langle u^{2n-1}, u \right\rangle u \\ &= \Delta u - u^{2n-1} + \langle \nabla u, \nabla u \rangle u + \left\langle u^{2n-1}, u \right\rangle u \\ &= \Delta u + \left(\|u\|^2 + |u|_{L^{2n}}^{2n} \right) u - u^{2n-1} \end{aligned} \tag{2.1.2}$$

We now proceed towards introducing the evolution equation that arise from above mentioned projection.

2.1.3 Statement of main problem

Let the spaces E, V and H be the Hilbert spaces as in Remark 2.1.2. The following is the main deterministic evolution equation that we are going to deal in this chapter.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \pi_u(\Delta u - u^{2n-1}) = \Delta u + \left(\|u\|^2 + |u|_{L^{2n}}^{2n}\right) u - u^{2n-1}, \\ u(0) &= u_0, \end{aligned} \tag{2.1.3}$$

where n is a natural number (or, more generally, a real number bigger than $\frac{1}{2}$), and $u_0 \in V \cap M$.

Assume that E, V, H be the abstract spaces satisfying Assumption 2.1.1. Then we can also treat above mentioned evolution equation as a special case of the following evolution equation (of parabolic type) with abstract F

$$\begin{aligned} \frac{\partial u}{\partial t}(t) + Au(t) &= F(u(t)), \ t \geq 0, \\ u(0) &= u_0. \end{aligned} \tag{2.1.4}$$

Here A is self-adjoint operator, F is map from V into H is locally Lipschitz and satisfies a certain symmetric estimate, which we are going to describe in next section.

Remark 2.1.4. *We will prove existence of the local and local maximal solution in abstract spaces E, V, H satisfying Assumption 2.1.1. For the existence of global solutions we will employ spaces E, V and H described in Remark 2.1.2.*

2.1.4 Solution space, assumptions, definition of the solution

We now introduce the most important Banach space that we are going to deal throughout the dissertation. Assume that E, V and H are the abstract Banach space as described in Assumption 2.1.1. For any $T \geq 0$, let us denote

$$X_T := L^2(0,T;E) \cap C([0,T];V).$$

It can be easily shown that $\left(X_T, |\cdot|_{X_T}\right)$ is Banach space with norm

$$|u|_{X_T}^2 = \sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T |u(t)|_E^2 \, dt, \; u \in X_T.$$

Now we state some of important assumptions which we are going to use in the upcoming sections.

Assumption 2.1.5. *Let $E \subset V \subset H$ satisfy Assumption 2.1.1. Assume that $S(t), t \in [0,\infty)$, is an analytic semigroup of bounded linear operators on H, such that there exist positive constants C_1 and C_0:*

i) For every $T > 0$ and $f \in L^2(0,T;H)$ a function $u = S * f$ defined by

$$u(t) = \int_0^T S(t-r)f(r)dt, \; t \in [0,T]$$

belongs to X_T and

$$|u|_{X_T} \leq C_1 |f|_{L^2(0,T;H)} \tag{2.1.5}$$

Note that $S* : L^2(0,T;H) \to X_T$ ($f \longmapsto S * f$) is a linear map and in the view of (2.1.5) *it is also bounded.*

ii) For every $T > 0$ and every $u_0 \in V$, a function $u = S(\cdot)u_0$ defined by

$$u(t) = S(t)u_0, \; t \in [0,T]$$

belongs to X_T and satisfies

$$|u|_{X_T} \leq C_0 \|u_0\|. \tag{2.1.6}$$

Now we introduce an auxiliary function which will be used later for truncation of norm of the solution. Let $\theta : \mathbb{R}^+ \to [0,1]$ be a non increasing smooth function with compact support such that

$$\inf_{x\in\mathbb{R}_+} \theta'(x) \geq -1,\ \theta(x) = 1 \text{ iff } x \in [0,1] \text{ and } \theta(x) = 0 \text{ iff } x \in [2,\infty). \tag{2.1.7}$$

For $n \geq 1$ set $\theta_n(\cdot) = \theta\left(\frac{\cdot}{n}\right)$. We have the following easy Lemma about θ as consequence of previous description.

Lemma 2.1.6. *([13], page 57) If $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non decreasing function, then for every $x, y \in \mathbb{R}$,*

$$\theta_n(x)h(x) \leq h(2n),\ \ |\theta_n(x) - \theta_n(y)| \leq \frac{1}{n}|x-y|. \tag{2.1.8}$$

Next we are going to define that what we mean by local, local maximal and global solution of problem (2.1.3).

Definition 2.1.7. *For $u_0 \in V$, a function $u : [0, T_1) \to V$ is called **local mild** solution to problem* (2.1.4) *with initial data u_0 if the following conditions are satisfied,*

i) for all $t \in [0, T_1)$, $u|_{[0,t)} \in X_t$,

ii) for all $t \in [0, T_1)$,

$$u(t) = S(t)u_0 + \int_0^t S(t-r)F(u(r))\,dr.$$

*A local mild solution $(u(t), t \in [0, T_1))$ is called a **maximal** solution if for any other local mild solution $\left(\widehat{u}(t), t \in [0, \widehat{T}_1)\right)$ such that:*

i) $\widehat{T}_1 \geq T_1$,

ii) the restriction of $\widehat{u}$ to $[0, T_1)$ agrees with u implies $\widehat{T}_1 = T_1$.

*A local maximal solution $(u(t), t \in [0, T_1))$ is called a **global** solution in $T_1 = \infty$.*

2.2 Existence and Uniqueness of local and local maximal solutions

In order to prove the existence and the uniqueness of local mild and local maximal solutions to our abstract problem (2.1.4) (in particular projected constrained Problem (2.1.3), we are first going to study the existence and uniqueness of solutions to approximate abstract evolution equation given below. Let us assume T as some positive real number. We are interested in proving the existence and the uniqueness a local mild and local maximal solution u^n to the following evolution equation:

$$\begin{aligned} u^n(t) &= S(t)u_0 + \int_0^t S(t-r)\theta_n\left(|u^n|_{X_r}\right) F\left(u^n(r)\right) dr,\ t \in [0,T], \quad (2.2.1) \\ u^n(0) &= u_0,\ \text{where } u_0 \in V. \end{aligned}$$

All the results proven in this section will be in abstract E, V and H spaces satisfying Assumption 2.1.1. For the existence of a local solution of (2.2.1), we will construct a globally Lipschitz. map from X_T into $L^2(0,T;H)$ and then by using this globally Lipschitz map, we will construct a contraction and hence the existence and uniqueness of local mild solution are guaranteed by Banach fixed point theorem. Then using Zorn's lemma on the set of Local mild solutions, we will imply the existence of a local maximal solution of (2.2.1).

2.2.1 Important Estimates

In this subsection we are going to treat E, V and H in Remark 2.1.2. The aim of this subsection is to show that the nonlinear part of our projected heat flow problem

(2.1.3) i.e. the function $F : V \to H$ defined by:

$$F(u) := \|u\|^2 u - u^{2n-1} + u\,|u|_{L^{2n}}^{2n}, \tag{2.2.2}$$

is locally Lipschitz and satisfies a certain symmetric estimate. Recall the following well known Gagliardo-Nirenberg-Sobolev inequality.

Lemma 2.2.1. *[47] Assume that $r, q \in [1, \infty)$, and $j, m \in \mathbb{Z}$ satisfy $0 \leq j < m$. Then for any $u \in C_0^\infty(\mathbb{R}^d)$*

$$\left|D^j u\right|_{L^p(\mathbb{R}^d)} \leq C\,|D^m u|_{L^r(\mathbb{R}^d)}^a\,|u|_{L^q(\mathbb{R}^d)}^{1-a}, \tag{2.2.3}$$

where $\frac{1}{p} = \frac{j}{d} + a\left(\frac{1}{r} - \frac{m}{d}\right) + (1-a)\frac{1}{q}$, for all $a \in \left[\frac{j}{m}, 1\right]$. If $m - j - \frac{d}{r}$ is a non-negative integer, then the equality (2.2.3) holds only for $a \in \left[\frac{j}{m}, 1\right)$.

Observe that our projected heat flow problem (2.1.3) involves L^{2n} norm, therefore at several instances throughout this section and dissertation we will use the following particular case of Gagliardo-Nirenberg-Sobolev inequality.

For our case we choose $r = q = 2$, $j = 0$, $m = 1$, $d = 2$, and $p = 2n$, so

$$\begin{aligned} \frac{1}{p} &= \frac{0}{2} + a\left(\frac{1}{2} - \frac{1}{2}\right) + (1-a)\frac{1}{2}, \\ \frac{1}{2n} &= (1-a)\frac{1}{2}, \\ \frac{1}{n} &= 1 - a \text{ or } a = 1 - \frac{1}{n}. \end{aligned}$$

Plugging values of r, q, j, m, d and p in inequality (2.2.3) we get, with $a = 1 - \frac{1}{n}$,

$$|u|_{L^{2n}(\mathbb{R}^2)} \leq C\,|\nabla u|_{L^2(\mathbb{R}^2)}^a\,|u|_{L^2(\mathbb{R}^2)}^{1-a}, \ u \in H_0^{1,2}(\mathcal{O}).$$

As $H = L^2(\mathcal{O})$ and $V = H_0^{1,2}(\mathcal{O})$ (i.e.$\|u\| = |\nabla u|_{L^2(\mathbb{R}^2)}$) so above inequality can be re written as,

$$|u|_{L^{2n}(\mathbb{R}^2)} \leq C\,\|u\|^a\,|u|_H^{1-a}. \tag{2.2.4}$$

Remark 2.2.2. *i) From Remark 2.1.2 we know that embedding $V \hookrightarrow H$ is compact i.e. there exists $c > 0$ such that*

$$|u|_H \leq c\,\|u\|\,,\ u \in V.$$

Hence inequality (2.2.4)*, simplifies to*

$$|u|_{L^{2n}(\mathbb{R}^2)} \leq C\,\|u\|\,,\ u \in V, \tag{2.2.5}$$

where $C := cC$. The last inequality reflects the fact that $V \hookrightarrow L^{2n}(\mathbb{R}^2)$ continuously, where $n \in \mathbb{N}$.

Before proving the main result of this subsection, consider two useful Lemmas.

Lemma 2.2.3. *If $a, b \geq 0$ then*

$$(a^n - b^n) \leq n a^{n-1}(a-b)\,. \tag{2.2.6}$$

Proof. Indeed, for $a = b$ required result holds trivially. Now assume the case $a \neq b$. Let us begin with the following observation,

$$a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}b + ... + b^{n-1}\right).$$

Consider the case when $a > b$, then from equation above

$$\begin{aligned} a^n - b^n &= (a-b)\left(a^{n-1} + a^{n-2}b + ... + b^{n-1}\right), \\ &< (a-b)\left(a^{n-1} + a^{n-2}a + ... + a^{n-1}\right), \\ &= (a-b)\left(a^{n-1} + a^{n-1} + ... + a^{n-1}\right), \\ &= n a^{n-1}(a-b). \end{aligned}$$

Now consider the case $a < b$ i.e. $\frac{b}{a} > 1$ then,

$$\begin{aligned} a^n - b^n &= (a-b)\left(a^{n-1} + a^{n-2}b + ... + b^{n-1}\right), \\ \frac{a^n - b^n}{a-b} &= a^{n-1} + a^{n-2}b + ... + b^{n-1}, \\ &= a^{n-1}\left(1 + \frac{b}{a} + ... + \left(\frac{b}{a}\right)^n\right) > a^{n-1}(1+1+..+1) = na^{n-1}, \\ \frac{a^n - b^n}{a-b} &> na^{n-1}. \end{aligned}$$

As $a < b$ so $a - b < 0$ so on multiplying $a - b$ on both sides of above inequality

$$(a^n - b^n) \leq na^{n-1}(a-b).$$

Thus the inequality holds in all cases. This completes the proof. ■

Lemma 2.2.4. *For real number a, b and $n \in \mathbb{N}$,*

$$|a^n - b^n| \leq \frac{n}{2}|a-b|\left(|a|^{n-1} + |b|^{n-1}\right). \tag{2.2.7}$$

Proof. Since

$$\begin{aligned} |a^n - b^n| &= \left|(a-b)\left(a^{n-1} + a^{n-2}b + ... + b^{n-1}\right)\right|, \\ &\leq |a-b|\left(\sum_{i=1}^{n}|a|^{n-i}|b|^{i-1}\right). \end{aligned} \tag{2.2.8}$$

Using young's inequality

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}, \text{ where } q = \frac{p}{p-1},$$

for $x = |a|^{n-i}$, $y = |b|^{i-1}$, $p = \frac{n-1}{n-i}$ and $q = \frac{n-1}{i-1}$ we

$$\begin{aligned}
|a|^{n-i}|b|^{i-1}, &\leq \frac{\left(|a|^{n-i}\right)^{\frac{n-1}{n-i}}}{\frac{n-1}{n-i}} + \frac{\left(|b|^{i-1}\right)^{\frac{n-1}{i-1}}}{\frac{n-1}{i-1}}, \\
\sum_{i=1}^{n} |a|^{n-i}|b|^{i-1} &\leq \frac{1}{n-1}\left(|a|^{n-1}\sum_{i=1}^{n}(n-i) + |b|^{n-1}\sum_{i=1}^{n}(i-1)\right), \\
&= \frac{1}{n-1}\left(|a|^{n-1}\sum_{i=1}^{n}(n-i) + |b|^{n-1}\sum_{i=1}^{n}(i-1)\right), \\
&= \frac{1}{n-1}\left(\frac{n(n-1)}{2}|a|^{n-1} + \frac{n(n-1)}{2}|b|^{n-1}\right), \\
&= \frac{n}{2}\left(|a|^{n-1} + |b|^{n-1}\right). \qquad (2.2.9)
\end{aligned}$$

Using inequality (2.2.9) in (2.2.8) we get

$$|a^n - b^n| \leq \frac{n}{2}|a-b|\left(|a|^{n-1} + |b|^{n-1}\right).$$

■

Following is the main result of this subsection.

Lemma 2.2.5. *Consider a map $F : V \to H$ defined by $F(u) = \|u\|^2 u - u^{2n-1} + u\,|u|_{L^{2n}}^{2n}$. Then there exists a constant $C > 0$ such that the map F satisfies:*

$$|F(u) - F(v)|_H \leq G(\|u\|, \|v\|)\,\|u - v\| \qquad (2.2.10)$$

Where $G : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a bounded and symmetric polynomial map. In fact one can take

$$G(r,s) := C\left[\left(r^2 + s^2\right) + (r+s)^2\right] + C_n\left[\begin{array}{c} \left(\frac{2n-1}{2}\right)\left(r^{2n-1} + s^{2n-1}\right)(r+s) \\ + \left(r^{2n} + s^{2n}\right) + \left(r^{2n-2} + s^{2n-2}\right) \end{array}\right].$$

Proof. Set $F(u) = \|u\|^2 u - u^{2n-1} + u\,|u|_{L^{2n}}^{2n} =: F_1(u) - F_2(u) + F_3(u)$. We will now find the estimate for each of F_1, F_2 and F_3.

Let us begin with considering estimate for F_1. Let us fix $u, v \in V$. Then using triangle inequality,

$$\begin{aligned}
|F_1(u) - F_1(v)|_H &= \left|\|u\|^2 u - \|v\|^2 v\right|_H \\
&= \left|\|u\|^2 u - \|u\|^2 v + \|u\|^2 v - \|v\|^2 v\right|_H \\
&= \left|\|u\|^2 (u-v) + \left(\|u\|^2 - \|v\|^2\right) v\right|_H \\
&\leq \|u\|^2 |u-v|_H + (\|u\| + \|v\|)(\|u\| - \|v\|)|v|_H \\
&\leq \left(\|u\|^2 + \|v\|^2\right)|u-v|_H \\
&\quad + (\|u\| + \|v\|)\|u-v\|(|u|_H + |v|_H).
\end{aligned}$$

Since embedding $V \hookrightarrow H$ is continuous so there exists C such that $|u|_H \leq C\|u\|$ for all $u \in V$. We infer that ,

$$|F_1(u) - F_1(v)|_H \leq C\left[\left(\|u\|^2 + \|v\|^2\right) + (\|u\| + \|v\|)^2\right]\|u-v\|. \tag{2.2.11}$$

Now consider F_3. Again fix $u, v \in V$. Then

$$\begin{aligned}
|F_3(u) - F_3(v)|_H &= \left|u\,|u|_{L^{2n}}^{2n} - v\,|v|_{L^{2n}}^{2n}\right|_H \\
&= \left|u\,|u|_{L^{2n}}^{2n} - u\,|v|_{L^{2n}}^{2n} + u\,|v|_{L^{2n}}^{2n} - v\,|v|_{L^{2n}}^{2n}\right|_H \\
&\leq \left|u\left(|u|_{L^{2n}}^{2n} - |v|_{L^{2n}}^{2n}\right)\right|_H + \left|(u-v)\,|v|_{L^{2n}}^{2n}\right|_H \\
&= |u|_H \left||u|_{L^{2n}}^{2n} - |v|_{L^{2n}}^{2n}\right| + |u-v|_H\,|v|_{L^{2n}}^{2n}.
\end{aligned}$$

Using Lemma 2.2.3, and inequality (2.2.5) we get

$$\begin{aligned}
|F_3(u)-F_3(v)|_H &\le \left(\frac{2n-1}{2}\right)\left(|u|_{L^{2n}}^{2n-1}+|v|_{L^{2n}}^{2n-1}\right)|u|_H\,|u-v|_{L^{2n}} \\
&\quad +C^{2n}\,|u-v|_H\,\|v\|^{2n} \\
&\le \left(\frac{2n-1}{2}\right)C^{2n+1}\left(\|u\|^{2n-1}+\|v\|^{2n-1}\right)\|u\|\,\|u-v\| \\
&\quad +C^{2n+1}\,\|u-v\|\,\|v\|^{2n} \\
&= C_n\left[\begin{array}{c}\left(\frac{2n-1}{2}\right)\left(\|u\|^{2n-1}+\|v\|^{2n-1}\right)\left(\|u\|+\|v\|\right) \\ +\left(\|u\|^{2n}+\|v\|^{2n}\right)\end{array}\right] \\
&\quad \cdot\|u-v\| \qquad (2.2.12)
\end{aligned}$$

where $C_n := C^{2n+1}$.

Now consider F_2. Let us again fix $u,v\in V$. In the following chain of inequalities we are going to use inequality (2.2.5), Cauchy-Schwartz inequality and the continuity of embedding of $V\hookrightarrow L^4(D)$, with $C_n=\left(\frac{2n-2}{2}\right)$,

$$\begin{aligned}
|F_2(u)-F_2(v)|_H^2 &= \left|u^{2n-1}-v^{2n-1}\right|_H^2 \le \int_D \left|u^{2n-1}(x)-v^{2n-1}(x)\right|^2 dx \\
&\le \int_D\left(\frac{2n-2}{2}\,|u(x)-v(x)|\left(|u(x)|^{2n-2}+|v(x)|^{2n-2}\right)\right)^2 dx \\
&= \left(\frac{2n-2}{2}\right)^2\int_D\left(|u(x)|^{2n-2}+|v(x)|^{2n-2}\right)^2|u(x)-v(x)|^2\,dx \\
&\le \left(\frac{2n-2}{2}\right)^2\left(\int_D\left(|u(x)|^{2n-2}+|v(x)|^{2n-2}\right)^4 dx\right)^{1/2} \\
&\quad \left(\int_D|u(x)-v(x)|^4\,dx\right)^{1/2} \\
|F_2(u)-F_2(v)|_H &\le C\left(\int_D\left(|u(x)|^{2n-2}+|v(x)|^{2n-2}\right)^4 dx\right)^{\frac14}|u-v|_{L^4} \\
&\le CC_n\left(\int_D\left(|u(x)|^{2n-2}+|v(x)|^{2n-2}\right)^4 dx\right)^{\frac14}\|u-v\|.
\end{aligned}$$

Next we going to use the Miknkowski inequality

$$\left(\int_D (f(x)+g(x))^4 dx\right)^{1/4} \le \left(\int_D (f(x)^4 dx\right)^{1/4} + \left(\int_D g(x)^4 dx\right)^{1/4}$$

for $f(x) = |u(x)|^{2n-2}, g(x) = |v(x)|^{2n-2}$. Moreover, use of the continuity of embedding $V \hookrightarrow L^{8n-8}$, it follows that

$$\begin{aligned} |F_2(u) - F_2(v)|_H &\le CC_n \left[\left(\int_D \left(|u(x)|^{8n-8} \right) dx \right)^{1/4} + \left(\int_D \left(|v(x)|^{8n-8} \right) dx \right)^{1/4} \right] \|u - v\| \\ &\le C \left(|u|_{L^{8n-8}}^{2n-2} + |v|_{L^{8n-8}}^{2n-2} \right) \|u - v\| \\ &\le CC_n c^{2n-1} \left(\|u\|^{2n-2} + \|v\|^{2n-2} \right) \|u - v\| \end{aligned} \tag{2.2.13}$$

where $C_n := CC_n c^{2n-1}$.

Combining inequalities (2.2.11), (2.2.12) and (2.2.13) we get the desired inequality. ∎

2.2.2 Existence and Uniqueness of Local mild solutions of Approximate abstract evolution equation

In this subsection, we intend to prove the existence and uniqueness of a local mild solution to evolution equation (2.2.1). Let us begin by proving the following abstract result.

Proposition 2.2.6. *Let E, V and H satisfy Assumption 2.1.1. Assume that $F : V \to H$ be an abstract map that satisfies the following inequality, for all $u_1, u_2 \in V$*

$$|F(u_1) - F(u_2)|_H \le \|u_1 - u_2\| \, G(\|u_1\|, \|u_2\|), \tag{2.2.14}$$

where $G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be symmetric function and for all $K > 0$ there exists $C = C_K$ such that

$$|G(r, s)| \le C_K \text{ for all } r, s \in [0, K]. \tag{2.2.15}$$

Let θ is as described by (2.1.7) and (2.1.8). Define a map $\Phi^n_{T,F} : X_T \to L^2(0, T; H)$ by,

$$\left[\Phi^n_{T,F}(u)\right](t) = \theta_n(|u|_{X_t}) F(u(t)), \ t \in [0, T].$$

Then $\Phi^n_{T,F}$ is globally Lipschitz and for any $u_1, u_2 \in X_T$, it satisfies,

$$\left|\Phi^n_{T,F}(u_1) - \Phi^n_{T,F}(u_2)\right|_{L^2(0,T;H)} \leq C_n \left|u_1 - u_2\right|_{X_T} T^{\frac{1}{2}}. \tag{2.2.16}$$

where $C_n = (2n)^4 \left|G(2n, 0)\right| + G^2(2n, 2n)$.

Proof. We start by showing that $\Phi^n_{T,F}$ is well-defined. Let $u \in X_T$ then

$$\begin{aligned} \left|\Phi^n_{T,F}(u)\right|^2_{L^2(0,T;H)} &= \left|\theta_n(|u|_{X_t})F(u(t))\right|^2_{L^2(0,T;H)} \\ &= \int_0^T \left|\theta_n(|u|_{X_t})F(u(t))\right|^2_H dt. \end{aligned}$$

Since $|\theta|^2 \leq 1$ so $|\theta_n|^2 \leq 1$, using this and the inequality (2.2.15) we infer that,

$$\begin{aligned} \left|\Phi^n_{T,F}(u)\right|^2_{L^2(0,T;H)} &\leq \int_0^T \left|F(u(t))\right|^2_H dt \\ &\leq \int_0^T \|u(t)\|^2 \left|G(\|u(t)\|, 0)\right|^2 dt. \end{aligned}$$

Since $X_T \subset C([0,T]; V)$ so $\|u(t)\| \leq |u|_{X_T} < \infty$, for all $t \in [0,T]$. Also from (2.2.15) we know that $|G(\|u(t)\|, 0)| \leq C_K$. Using this inference in to the last inequality above it follows that,

$$\left|\Phi^n_{T,F}(u)\right|^2_{L^2(0,T;H)} \leq \int_0^T |u|^2_{X_T} C_K^2 dt = |u|^2_{X_T} C_K^2 T < \infty.$$

Hence $\Phi^n_{T,F}$ is well-defined.

Let us fix $u_1, u_2 \in X_T$. Set

$$\tau_i = \inf\left\{t \in [0,T] : |u_i|_{X_t} \geq 2n\right\}, i = 1, 2.$$

WLOG we can assume that $\tau_1 \leq \tau_2$. Consider

$$\begin{aligned} \left|\Phi^n_{T,F}(u_1) - \Phi^n_{T,F}(u_2)\right|_{L^2(0,T;H)} &= \left[\int_0^T \left|\Phi^n_{T,F}(u_1(t)) - \Phi^n_{T,F}(u_2(t))\right|^2_H dt\right]^{\frac{1}{2}}, \\ &= \left[\int_0^T \left|\theta_n(|u_1|_{X_t})F(u_1(t)) - \theta_n(|u_2|_{X_t})F(u_2(t))\right|^2_H dt\right]^{\frac{1}{2}}. \end{aligned}$$

For $i = 1, 2$, $\theta_n(|u_i|_{X_t}) = 0$ and for $t \geq \tau_2$

$$\begin{aligned}
\left|\Phi^n_{T,F}(u_1) - \Phi^n_{T,F}(u_2)\right|_{L^2(0,T;H)} &= \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})F(u_1(t)) - \theta_n(|u_2|_{X_t})F(u_2(t))\right|^2_H\right]^{\frac{1}{2}}, \\
&= \left[\int_0^{\tau_2} \left|\begin{array}{c} \theta_n(|u_1|_{X_t})F(u_1(t)) - \theta_n(|u_1|_{X_t})F(u_2(t)) \\ +\theta_n(|u_1|_{X_t})F(u_2(t)) - \theta_n(|u_2|_{X_t})F(u_2(t)) \end{array}\right|^2_H\right]^{\frac{1}{2}}, \\
&= \left[\int_0^{\tau_2} \left|\begin{array}{c} \theta_n(|u_1|_{X_t})\left(F(u_1(t)) - F(u_2(t))\right) + \\ \left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) F(u_2)(t) \end{array}\right|^2_H\right]^{\frac{1}{2}}.
\end{aligned}$$

Using Minkowski's inequality we infer that

$$\begin{aligned}
\left|\Phi^n_{T,F}(u_1) - \Phi^n_{T,F}(u_2)\right|_{L^2(0,T;H)} &\leq \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})\left(F(u_1(t)) - F(u_2(t))\right)\right|^2_H dt\right]^{\frac{1}{2}} \\
&+ \left(\int_0^{\tau_2} \left|\left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) \cdot F(u_2(t))\right|^2_H dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.2.17}$$

Set

$$\begin{aligned}
A &:= \left[\int_0^{\tau_2} \left|\left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) F(u_2(t))\right|^2_H\right]^{\frac{1}{2}} dt \\
B &:= \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})\left(F(u_1(t)) - F(u_2(t))\right)\right|^2_H\right]^{\frac{1}{2}} dt.
\end{aligned}$$

Hence the inequality (2.2.17) can be rewritten as

$$|\Phi_T(u_1) - \Phi_T(u_2)|_{L^2(0,T;H)} \leq A + B. \tag{2.2.18}$$

Since the function θ_n is Lipschitz we infer that

$$\begin{aligned}
A^2 &= \int_0^{\tau_2} \left|\left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) F(u_2(t))\right|_H^2 dt \\
&\leq 4n^2 \int_0^{\tau_2} \left||u_1|_{X_t} - |u_2|_{X_t}\right|_H^2 |F(u_2(t))|_H^2 dt \\
&\leq 4n^2 \int_0^{\tau_2} \left||u_1|_{X_t} - |u_2|_{X_t}\right|_H^2 |F(u_2(t))|_H^2 dt \\
&\leq 4n^2 \int_0^{\tau_2} |u_1 - u_2|_{X_t}^2 |F(u_2(t))|_H^2 dt \\
&\leq 4n^2 |u_1 - u_2|_{X_T}^2 \int_0^{\tau_2} |F(u_2(t))|_H^2 dt. \qquad (2.2.19)
\end{aligned}$$

Next we want to estimate the integral the last inequality. By use of inequality (2.2.14)

$$\begin{aligned}
\int_0^{\tau_2} |F(u_2(t))|_H^2 dt &\leq \int_0^{\tau_2} \|u_2(t)\|^2 |G(\|u_2(t)\|, 0)|^2 dt \\
&\leq \sup_{t\in[0,\tau_2)} \|u_2(t)\|^2 \int_0^{\tau_2} |G(\|u_2(t)\|, 0)|^2 dt.
\end{aligned}$$

Since $|u_2|_{X_{\tau_2}}^2 = \sup_{t\in[0,\tau_2]} \|u_2(t)\|^2 + \int_0^{\tau_2} |u_2(t)|_E^2$ therefore $\sup_{t\in[0,\tau_2]} \|u_2(t)\|^2 \leq |u_2|_{X_{\tau_2}}^2$ $\leq (2n)^2$. Thus the last inequality takes the following form

$$\int_0^{\tau_2} |F(u_2(t))|_H^2 dt \leq (2n)^2 \int_0^{\tau_2} |G(\|u_2(t)\|, 0)|^2 dt = (2n)^2 |G(2n, 0)|^2 \tau_2.$$

Using the last inequality in (2.2.19) we get

$$\begin{aligned}
A^2 &\leq (2n)^4 |G(2n, 0)|^2 \tau_2 |u_1 - u_2|_{X_T}^2 \\
&\leq (2n)^4 |G(2n, 0)|^2 |u_1 - u_2|_{X_T}^2 T \\
A &\leq A_n |u_1 - u_2|_{X_T} T^{\frac{1}{2}}, \qquad (2.2.20)
\end{aligned}$$

where $A_n = (2n)^4 |G(2n, 0)|^2$. Since $\theta_n(|u_1|_{X_t}) = 0$ for $t \geq \tau_1$ and $\tau_1 \leq \tau_2$, we have

$$\begin{aligned}
B &= \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t}) (F(u_1(t)) - F(u_2(t)))\right|_H^2 dt\right]^{\frac{1}{2}} \\
&= \left[\int_0^{\tau_1} \left|\theta_n(|u_1|_{X_t}) (F(u_1(t)) - F(u_2(t)))\right|_H^2 dt\right]^{\frac{1}{2}}.
\end{aligned}$$

Also since $\theta_n(|u_1|_{X_t}) \le 1$ for $t \in [0, \tau_1)$ we infer that

$$B \le \left[\int_0^{\tau_1} |(F(u_1(t)) - F(u_2(t)))|_H^2 \, dt \right]^{\frac{1}{2}}.$$

Using inequality (2.2.14)

$$\begin{aligned} B^2 &\le \int_0^{\tau_1} \left[\|u_1(t) - u_2(t)\| \, G\left(\|u_1(t)\|, \|u_2(t)\|\right)\right]^2 dt \\ &\le \sup_{t\in[0,\tau_1)} \|u_1(t) - u_2(t)\|^2 \int_0^{\tau_1} \left[G\left(\|u_1(t)\|, \|u_2(t)\|\right)\right]^2 dt. \qquad (2.2.21) \end{aligned}$$

Using the fact that, $\sup_{t\in[0,\tau_1)} \|u_1(t) - u_2(t)\|^2 \le |u_1 - u_2|_{X_{\tau_1}}^2$, and using $\sup_{t\in[0,\tau_1)} \|u_i(t)\|^2 \le |u_i|_{X_{\tau_i}} \le 2n$, $i = 1, 2$, the last inequality takes form

$$\begin{aligned} B^2 &\le |u_1 - u_2|_{X_{\tau_1}}^2 \, G^2\left(2n, 2n\right) \int_0^{\tau_1} dt \\ &\le \tau_1 G^2\left(2n, 2n\right) |u_1 - u_2|_{X_T}^2 \\ &\le B_n \, |u_1 - u_2|_{X_T}^2 \, T, \end{aligned}$$

where $B_n^2 = G^2\left(2n, 2n\right)$. Thus

$$B \le B_n T^{\frac{1}{2}} \, |u_1 - u_2|_{X_T}$$

Using the last inequality together with inequality (2.2.20) in (2.2.18), we get

$$\left|\Phi_{T,F}^n(u_1) - \Phi_{T,F}^n(u_2)\right|_{L^2(0,T;H)} \le (A_n + B_n) \, |u_1 - u_2|_{X_T} T^{\frac{1}{2}} = C_n \, |u_1 - u_2|_{X_T} T^{\frac{1}{2}},$$

where $C_n := (A_n + B_n)$. This completes the proof of the theorem. ■

Now we will prove the main result of this subsection i.e. the existence and the uniqueness of local mild solution to the approximate evolution equation (2.2.1).

Proposition 2.2.7. *Let E, V and H satisfy Assumption 2.1.1. Assume that Assumptions 2.1.5 are satisfied. Let us consider a map $\Psi_{T,F}^{n,u_0} : X_T \to X_T$ defined by*

$$\Psi_{T,F}^{n,u_0}(u) = Su_0 + S * \Phi_{T,F}^n(u), \qquad (2.2.22)$$

where $\Phi^n_{T,F}$ *is as described in Proposition 2.2.6 and* $u_0 \in V$*. Then there exists* $T_0 > 0$ *such that for all* $T \in [0, T_0)$ *,* $\Psi^{n,u_0}_{T,F}$ *is strict contraction. In particular, for all* $T \in [0, T_0)$ *there exists* $u \in X_T$*, such that* $\Psi^{n,u_0}_{T,F}(u) = u$*.*

Proof. The map $\Psi^{n,u_0}_{T,F}$ defined by formula (2.2.22). Indeed, for $u \in X_T$,

$$\begin{aligned}\left|\Psi^{n,u_0}_{T,F}(u)\right|_{X_T} &= \left|Su_0 + S * \Phi^n_{T,F}(u)\right|_{X_T} \\ &\leq |Su_0|_{X_T} + \left|S * \Phi^n_{T,F}(u)\right|_{X_T},\end{aligned}$$

Using the inequalities (2.1.6) and (2.1.5) from Assumption 2.1.5 we infer that

$$\left|\Psi^{n,u_0}_{T,F}(u)\right|_{X_T} \leq C_0 \|u_0\| + \left|\Phi^n_{T,F}(u)\right|_{L^2(0,T;H)}.$$

Next using the inequality (2.2.16) we infer that,

$$\left|\Psi^{n,u_0}_{T,F}(u)\right|_{X_T} \leq C_0 \|u_0\| + C_n T^{1/2} |u|_{X_T} < \infty.$$

Let us fix $u_1, u_2 \in X_T$. Then consider the following,

$$\begin{aligned}\left|\Psi^{n,u_0}_{T,F}(u_1) - \Psi^{n,u_0}_{T,F}(u_2)\right|_{X_T} &= \left|Su_0 + S * \Phi^n_{T,F}(u_1) - Su_0 - S * \Phi^n_{T,F}(u_2)\right|_{X_T} \\ &\leq \left|S * \Phi^n_{T,F}(u_1) - S * \Phi^n_{T,F}(u_2)\right|_{X_T} \\ &= \left|S * \left(\Phi^n_{T,F}(u_1) - \Phi^n_{T,F}(u_2)\right)\right|_{X_T}\end{aligned}$$

Next using inequality (2.1.6) with $u = S * \left(\Phi^n_{T,F}(u_1) - \Phi^n_{T,F}(u_2)\right)$ and $f = \Phi^n_{T,F}(u_1) - \Phi^n_{T,F}(u_2)$, we infer that

$$\left|\Psi^{n,u_0}_{T,F}(u_1) - \Psi^{n,u_0}_{T,F}(u_2)\right|_{X_T} \leq C_1 \left|\Phi^n_{T,F}(u_1) - \Phi^n_{T,F}(u_2)\right|_{L^2(0,T;H)}.$$

By using inequality (2.2.16) from Proposition 2.2.6 we infer that

$$\left|\Psi^n_{T,u_0}(u_1) - \Psi^n_{T,u_0}(u_2)\right|_{X_T} \leq C_1 C_n |u_1 - u_2|_{X_T} T^{\frac{1}{2}}.$$

This shows that $\Psi^{n,u_0}_{T,F}$ is globally Lipschitz. Observing that C_1 and C_n are independent of T. We can reduce T in such a way that $C_1 C_n T^{\frac{1}{2}} < 1$. Hence there exists $T_0 := \frac{1}{(C_1C_n)^2} < \infty$ such that $\Psi^{n,u_0}_{T,F}$ is strict contraction for all $T \in [0, T_0)$. Consequently by Banach fixed point theorem, for all $T \in [0, T_0)$ there exists $u \in X_T$, such that $\Psi^{n,u_0}_{T,F}(u) = u$. This completes the proof. ∎

2.2.3 Local Maximal solution of approximate evolution equation

In this subsection, we intend to prove the existence of local maximal solution of approximate evolution equation (2.2.1), through the Kurtowski-Zorn's Lemma. Further, we are going to show that if energy norm of initial data i.e. $\|u_0\|$ finite then the local mild solution (resp. maximal) solution to approximate evolution equations 2.2.1 is equivalent to the local mild solution (resp. maximal) of the main evolution equation (2.1.4).

Let Ξ be set of all local mild solutions constructed in last subsection. Let $u_1, u_2 \in \Xi$ be defined on $[0, \tau_1)$ and $[0, \tau_2)$ respectively.

Define an order " $\preceq$ "on Ξ by

$$u_1 \preceq u_2 \text{ iff } \tau_1 \leq \tau_2 \text{ and } u_2|_{[0,\tau_1)} = u_1.$$

The following result is about the existence of local maximal solution evolution equation (2.2.1) has been proven.

Lemma 2.2.8. *If Ξ and $\preceq$ are described above then Ξ contains a maximal element.*

Proof. In order to show that Ξ contains a maximal element, we are going to use Kuratowski-Zorn Lemma i.e. we are going to show that Ξ is a partially ordered set and for every increasing chain in Ξ there exists an upper bound.

Consider three arbitrary elements $u_1, u_2, u_3 \in \Xi$, where u_i is local solution on $[0, \tau_i)$, $i = 1, 2, 3$, of approximate equation 2.2.1. In order to see that Ξ is partially ordered set we are going to verify that relation $\preceq$ is reflexive, antisymmetric and transitive.

Reflexivity: Clearly $u_1 \preceq u_1$ because $\tau_1 = \tau_1$ and $u_1|_{[0,\tau_1)} = u_1$. Hence $\preceq$ is reflexive.

Antisymmetry: Let $u_1 \preceq u_2$ and $u_2 \preceq u_1$ then $\tau_1 \leq \tau_2$ and $\tau_2 \leq \tau_1$ so $\tau_1 = \tau_2$. Moreover $u_2|_{[0,\tau_1)} = u_1$ and $u_1|_{[0,\tau_2)} = u_2$. Hence

$$u_1 = u_2|_{[0,\tau_1)} = u_2|_{[0,\tau_2)} = u_2.$$

This shows that $\preceq$ is antisymmetric.

Transitivity: Let $u_1 \preceq u_2$ and $u_2 \preceq u_3$ so $\tau_1 \leq \tau_2$ and $\tau_2 \leq \tau_3$ therefore $\tau_1 \leq \tau_3$. Also moreover $u_2|_{[0,\tau_1)} = u_1$ and $u_3|_{[0,\tau_2)} = u_2$. Therefore

$$u_3|_{[0,\tau_1)} = u_2|_{[0,\tau_1)} = u_1.$$

and hence $\preceq$ is transitive.

Hence $(\Xi, \preceq)$ is partially ordered set.

Now let $u_1 \preceq u_2 \preceq u_3 ...$ be an increasing chain in Ξ, where $u_i : [0, \tau_i) \to X_T$ for all $i \in \mathbb{N}$. We show that this sequence has an upper bound.

Set $\tau = \sup_{i \in \mathbb{N}} \tau_i$. Define $u : [0, \tau) \to X_T$ by

$$u|_{[0,\tau_i)} = u_i.$$

Indeed $u \in \Xi$ and each $[0, \tau_i) \subset [0, \tau)$, for all $i \in \mathbb{N}$. Moreover the last equation implies $u_i \preceq u$, for all $i \in \mathbb{N}$. Therefore the chain has an upper bound u in Ξ.

Thus by Kuratowski-Zorn lemma Ξ has a maximal element. This completes the proof. ■

Remark 2.2.9. *The maximal solution that we obtained as a consequence of last lemma is a local maximal solution of* (2.2.1)*.*

Proposition 2.2.10. *For a given $R > 0$, there exists T_0, depending on R, such that for every $u_0 \in V$ satisfying $\|u_0\| \leq R$, there exists a unique local solution $u : [0, \tau) \to V$ of the abstract problem* (2.1.4).

Proof. Take and fix $R > 0$ and $u_0 \in V$ satisfying $\|u_0\| \leq R$. Recall the map $\Psi^{n,u_0}_{T,F} : X_T \to X_T$ given by:

$$\Psi^{n,u_0}_{T,F}(u) = Su_0 + S * \Phi^n_F(u).$$

By Proposition 2.2.6, we know that this map $\Psi^{n,u_0}_{T,F}$ satisfies

$$\left|\Psi^{n,u_0}_{T,F}(u_1) - \Psi^{n,u_0}_{T,F}(u_2)\right|_{X_T} \leq C_1 C_n T^{\frac{1}{2}} |u_1 - u_2|_{X_T}.$$

Now since $\Psi^{n,u_0}_{T,F}(0) = Su_0$, by assumption (2.1.6) we have

$$\left|\Psi^{n,u_0}_{T,F}(0)\right|_{X_T} = |Su_0|_{X_T} \leq C_0 \|u_0\| \leq C_0 R.$$

Observe that the right hand side in the last inequality does not depends on the n. Choose r in such a way that is satisfies

$$r \geq \frac{1}{1/2} C_0 R = 2C_0 R.$$

Let us choose a natural number $n = \lfloor 2C_1 R \rfloor + 1$, where C_1 is as in assumption (2.1.5), such that $n \geq r$. If T is reduced sufficiently such that $T < \frac{1}{(2C_1C_n)^2} =: T_0$ to satisfy

$$\left|\Psi^{n,u_0}_{T,F}(u_1) - \Psi^{n,u_0}_{T,F}(u_2)\right|_{X_T} \leq C_1 C_n T^{\frac{1}{2}} |u_1 - u_2|_{X_T} \leq \frac{1}{2} |u_1 - u_2|_{X_T},$$

then $\Psi^{n,u_0}_{T,F}$ is strict contraction for all $T \leq T_0$. Then by the Banach fixed point theorem we can find a unique $u \in X_T$ which is fixed point of $\Psi^{n,u_0}_{T,F}$ i.e. $\Psi^{n,u_0}_{T,F}(u) = u$. Therefore u satisfies,

$$u(t) = S(t)u_0 + \int_0^t S(t-r)\left[\theta_n\left(|u|_{X_r}\right) F(u(r))\right] dr, \text{ where } t \in [0, T_0]. \quad (2.2.23)$$

Moreover, using the inequalities (2.1.6) and (2.1.5) in Assumption 2.1.5 we infer that the fixed point u satisfies,

$$\begin{aligned}|u|_{X_T} &= \left|\Psi^{n,u_0}_{T,F}(u)\right|_{X_T} \leq |Su_0|_{X_T} + |S * \Phi^n_F(u)|_{X_T} \\ &\leq C_0 \|u_0\| + C_1 |\Phi^n_F(u)|_{X_T} = C_0 R + C_1 C_n |u|_{X_T} T^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$\begin{aligned}|u|_{X_T} &\leq C_0 R + \frac{1}{2}|u|_{X_T} \\ |u|_{X_T} &\leq 2C_0 R \leq r.\end{aligned}$$

But since

$$|u|^2_{X_T} = \sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T |u(t)|^2_E \, ds,$$

we infer that,

$$\sup_{t\in[0,T]} \|u(t)\|^2 \leq r^2.$$

Hence $\|u(t)\| \leq r,$ for all $t \in [0, T_0]$.
In particular, $\|u(t)\| \leq r \leq n$, for all $t \in [0, \tau)$. Therefore by definition of θ_n,

$$\theta_n\left(|u|_{X_t}\right) = 1, \text{ for } t \in [0, T_0].$$

Thus using last equation in equation (2.2.23), we obtain

$$u(t) = S(t)u_0 + \int_0^t S(t-r)F(u(r))dr, \text{ where } t \in [0, T_0].$$

Hence u is a local solution of (2.1.4) on $[0, T_0]$. This completes the proof. ■

We are going to end this subsection and section by proving a sufficient condition for a local mild solution to be the global solution.

Proposition 2.2.11. *If u is maximal local solution of main abstract evolution equation* (2.1.4) *on* $[0,\tau)$ *and there exists R such that*

$$\sup_{t\in[0,\tau)} \|u(t)\| \leq R.$$

then $\tau = \infty$.

Proof. Suppose contrary that $\tau < \infty$. Let R be the positive constant such that every $u_0 \in V$ satisfy $\|u_0\| \leq R$. By proposition 2.2.10 there exists a $T_0 > 0$, depends on R. Take $t_0 < \tau$ such that $\tau - t < \frac{T_0}{2}$. Now since $\|u(t_0)\| \leq R$, then by Proposition 2.2.10 there exists unique solution v on $[t_0, t_0 + T_0]$ such that $v(t_0) = u(t_0)$. So on the interval $[t_0, \tau)$ by the uniqueness of solution we must have $v = u$. Set $t_0 + T_0 = \widetilde{\tau}$. Define $z : [0, \widetilde{\tau}] \to V$ in the following manner,

$$\begin{aligned} z(t) &= u(t),\ t \in [0, t_0] \\ &= v(t), t \in [t_0, \widetilde{\tau}) . \end{aligned}$$

We claim that u is no more a maximal solution i.e. $u \preceq z$ and $u \neq z$.

Recall that domain of u is $[0, \tau)$ and that of z is $[0, \widetilde{\tau}]$, also $[0, \tau) \subset [0, \widetilde{\tau}]$ hence $u \neq z$. Next we are going to show that z is local mild solution. Let us begin by proving that $z \in X_{\widetilde{\tau}}$. By using the definition of z consider

$$\begin{aligned}
|z|^2_{X_{\widetilde{\tau}}} &= \sup_{t\in[0,\widetilde{\tau}]} \|z(t)\|^2 + \int_0^{\widetilde{\tau}} |z(t)|^2_E \, dt \\
&\leq \sup_{t\in[0,t_0]} \|z(t)\|^2 + \sup_{t\in[t_0,\widetilde{\tau}]} \|z(t)\|^2 \\
&\quad + \int_0^{t_0} |z(t)|^2_E \, dt + \int_{t_0}^{\widetilde{\tau}} |z(t)|^2_E \, dt. \\
&= \sup_{t\in[0,t_0]} \|u(t)\|^2 + \sup_{t\in[t_0,\widetilde{\tau}]} \|v(t)\|^2 \\
&\quad + \int_0^{t_0} |u(t)|^2_E \, dt + \int_{t_0}^{\widetilde{\tau}} |v(t)|^2_E \, dt \\
&= \sup_{t\in[0,t_0]} \|u(t)\|^2 + \int_0^{t_0} |u(t)|^2_E \, dt \\
&\quad \sup_{t\in[t_0,\widetilde{\tau}]} \|v(t)\|^2 + \int_{t_0}^{\widetilde{\tau}} |v(t)|^2_E \, dt \\
&= |u|^2_{X_{t_0}} + |v|^2_{X_{[t_0,\widetilde{\tau}]}} < \infty.
\end{aligned}$$

Finally to prove that z is local mild solution it remains to show that z satisfies the following integral equation,

$$z(t) = S(t)z(0) + \int_0^t S(t-r)F\left(z(r)\right) dr, \text{ for all } t \in [0, \widetilde{\tau}] .$$

Since $z(t) = u(t)$, for all $t \in [0, t_0]$ and u being solution on $[0, \tau)$ (containing $[0, t_0]$) satisfies the above integral equation so does z.

Next consider z on $[t_0, \widetilde{\tau}]$. By definition of z we know that $z(t) = v(t)$ for all $t \in [t_0, \widetilde{\tau}]$, where v is the solution of (2.1.4) on $[t_0, \widetilde{\tau}]$. Therefore

$$z(t) = v(t) = S(t_0)v\left(t_0\right) + \int_{t_0}^t S(t_0 - r)F\left(v(r)\right) dr, \ t \in [t_0, \widetilde{\tau}] .$$

Since $u(t_0) = v(t_0)$, and using semigroup property,

$$\begin{aligned} z(t) &= S(t-t_0)\left[S(t_0)u(t_0) + \int_0^{t_0} S(t_0-r)F(u(r))\,dr\right] \\ &\quad + \int_{t_0}^{t} S(t-r)F(v(r))\,dr \\ &= S(t-t_0)S(t_0)u(t_0) + \int_0^{t_0} S(t-t_0)S(t_0-r)F(z(r))\,dr \\ &\quad + \int_{t_0}^{t} S(t-r)F(z(r))\,dr \\ &= S(t)z(0) + \int_0^{t_0} S(t-t_0)S(t_0-r)F(z(r))\,dr + \int_{t_0}^{t} S(t-r)F(z(r))\,dr \\ &= S(t)z(0) + \int_0^{t} S(t-r)F(z(r))\,dr, \text{ for all } t \in [0,\widetilde{\tau}]. \end{aligned}$$

Thus z satisfies the required integral equation and hence z is solution to evolution equation (2.1.4) on $[0,\widetilde{\tau}]$.

So far we have shown that $u \preceq z$ and $u \neq z$ and z is solution of evolution equation (2.1.4) on $[0,\widetilde{\tau}]$. This clearly is the contradiction to the maximality of u. Hence our assumption that $\tau < \infty$ is absurd. Thus $\tau = \infty$. ■

2.3 Invariance and Global solution

Throughout this section we assume that E, V and H are the spaces as described in Remark 2.1.2 i.e. $E = D(A)$, $V = D\left(A^{1/2}\right)$ and $H = L^2(\mathcal{O})$, where $A = -\Delta$ with $D(A) = H_0^{1,2}(\mathcal{O}) \cap H^{2,2}(\mathcal{O})$. Following is an important remark that gives us a way to get global solution out of local maximal solution.

Now we mention an important [49] (Lemma 1.2, Chapter 3).

Lemma 2.3.1. *Let V, H and V' be three Hilbert spaces with V' being the dual space of V and each included and dense in the following one*

$$V \hookrightarrow H \cong H' \hookrightarrow V'.$$

If an abstract function u belongs to $L^2(0,T;V)$ and its weak derivative $\frac{\partial u}{\partial t}$ belongs to $L^2(0,T;V')$ then u is a.e. equal to a function continuous from $[0,T]$ into H and we have the following equality:

$$|u(t)|^2 = |u(0)|^2 + 2\int_0^t \langle u'(s), u(s)\rangle \, ds, \ t \in [0,T].$$

Lemma 2.3.2. *If $u \in X_T = L^2(0,T;E) \cap C([0,T];V)$ is a solution to (2.1.3) then $\frac{\partial u}{\partial t} \in L^2(0,T;H)$.*

Proof. In order to show that $\frac{\partial u}{\partial t} \in L^2(0,T;H)$, let us recall that key evolution equation (2.1.3) of our concern

$$\frac{\partial u}{\partial t} = \Delta u + |\nabla u|_H^2 \, u - u^{2n-1} + u \, |u|_{L^{2n}}^{2n}. \tag{2.3.1}$$

Also recall that norm on X_T can be given as,

$$|u|_{X_T}^2 = \sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T |u(t)|_E^2 \, dt.$$

In order to show that $\frac{\partial u}{\partial t} \in L^2(0,T;H)$, it is sufficient to show that each term on the right hand side of equation (2.3.1), belongs to $L^2(0,T;H)$.

Consider the first term of equation (2.2.4). Since $u \in X_T$

$$\int_0^T |\Delta u(t)|_H^2 \, dt = \int_0^T |u(t)|_E^2 \, dt \le |u|_{X_T}^2 < \infty.$$

therefore $\Delta u \in L^2(0,T;H)$, where $E = D(A)$ and $H = L^2(D)$.

Consider the second term i.e.

$$I_1 := |\nabla u|_H^2 \, u = \|u\|^2 \, u.$$

To see that $I_1 \in L^2(0,T;H)$ we consider,

$$\int_0^T |I_1(t)|_H^2 \, dt = \int_0^T \left| \|u(t)\|^2 \, u(t) \right|_H^2 dt \le \sup_{t\in[0,T]} \|u(t)\|^4 \int_0^T |u(t)|_H^2 \, dt. \tag{2.3.2}$$

Since embedding is continuous $E \hookrightarrow V \hookrightarrow H$ so there exists C such that $|u(t)|_H \le C\,|u(t)|_E$. we infer that

$$\int_0^T |u(t)|_H^2\,dt \le C^2 \int_0^T |u(t)|_E^2\,dt \le C^2\,|u|_{X_T}^2 < \infty.$$

Moreover, since

$$\sup_{t\in[0,T]} \|u(t)\|^4 \le \left(\sup_{t\in[0,T]} \|u(t)\|^2\right)^2 \le |u|_{X_T}^4 < \infty.$$

By last two inferences it follows that right hand side of inequality (2.3.2) is finite and consequently the left hand side, hence $I_1 \in L^2(0,T;H)$.

Now consider the fourth term of evolution equation (2.3.1),

$$I_2 := u\,|u|_{L^{2n}}^{2n}$$

To see that $I_2 \in L^2(0,T;H)$. By using the Holder inequality,

$$\begin{aligned}\int_0^T |I_2(t)|_H^2\,dt &= \int_0^T \left|u(t)\,|u(t)|_{L^{2n}}^{2n}\right|_H^2 dt = \int_0^T |u(t)|_H^2\,|u(t)|_{L^{2n}}^{2n}\,dt \\ &\le \left(\int_0^T |u(t)|_H^4\,dt\right)^{\frac{1}{2}} \left(\int_0^T |u(t)|_{L^{2n}}^{4n}\,dt\right)^{\frac{1}{2}}. \end{aligned} \tag{2.3.3}$$

By GN inequality (2.2.4), $|u|_H \le C\,\|u\|^{\frac{1}{2}}\,|u|_E^{\frac{1}{2}}$

$$\begin{aligned}\int_0^T |u(t)|_H^4\,dt &\le C^4 \int_0^T \|u(t)\|^2\,|u(t)|_E^2\,dt \\ &\le \sup_{t\in[0,T]} \|u(t)\|^2 \int_0^T |u(t)|_E^2\,dt < \infty \\ &\le |u|_{X_T}^2\,|u|_{X_T}^2 < \infty. \end{aligned} \tag{2.3.4}$$

Moreover, since $V \hookrightarrow L^{2n}$ so there exists $C > 0$ such that $|u|_{L^{2n}} \le C\,\|u\|$. Now consider the integral in inequality (2.3.3),

$$\begin{aligned}
\int_0^T |u(t)|_{L^{2n}}^{4n}\, dt &\leq C^{4n}\int_0^T \|u(t)\|^{4n}\, dt \\
&\leq C^{4n}\sup_{t\in[0,T]}\|u(t)\|^{4n}\int_0^T dt \\
&\leq C^{4n}\left(\sup_{t\in[0,T]}\|u(t)\|^2\right)^{2n} T \\
&\leq C^{4n}\,|u|_{X_T}^{4n}\, T < \infty. \qquad (2.3.5)
\end{aligned}$$

Hence using inequalities (2.3.4) and (2.3.5) in (2.3.3), it follows that $I_2 \in L^2(0,T;H)$.

Consider the third term of evolution equation (2.3.1) i.e. $I_3 := u^{2n-1}$. Using fact that $V \hookrightarrow L^{4n-2}$, we can see that for $t \in [0,T]$,

$$\begin{aligned}
|I_3(t)|_H^2 &= \int_D (u(t,x))^{4n-2}\, dx \\
&= |u(t)|_{L^{4n-2}}^{4n-2} \\
&\leq C^{4n-2}\|u(t)\|^{4n-2} \\
\int_0^T |I_3(t)|_H^2\, dt &\leq C^{4n-2}\int_0^T \|u(t)\|^{4n-2}\, dt \\
&\leq C^{4n-2}\sup_{t\in[0,T]}\|u(t)\|^{4n-2}\int_0^T dt \\
&\leq C^{4n}\left(\sup_{t\in[0,T]}\|u(t)\|^2\right)^{2n-1} T \\
&\leq C^{4n}\,|u|_{X_T}^{4n-2}\, T < \infty.
\end{aligned}$$

Hence $I_3 \in L^2(0,T;H)$.

Thus all terms of evolution equation (2.3.1) belong to $L^2(0,T;H)$ and hence $\frac{\partial u}{\partial t} \in L^2(0,T;H)$. By this we are done with the proof. ■

The following proposition is a crucial proposition about the invariance of our manifold M, which is key to obtaining global solution.

Proposition 2.3.3. *If $u(t), t \in [0,\tau)$ with $|u_0|_H^2 = 1$, is the local mild solution satisfies main problem* (2.1.4) *then*

$$u\,(t) \in M \;\; i.e. \;\; |u(t)|_H^2 = 1, \;\; \text{for all } t \in [0,\tau).$$

Proof. In order to prove the invariance of manifold we begin with applying Lemmas 2.3.1 and 2.3.2. For $t \in [0,\tau)$, consider the following chain of equations,

$$\begin{aligned}
\frac{1}{2}\left(|u(t)|_H^2 - 1\right) &= \frac{1}{2}\left(|u_0|_H^2 - 1\right) + \int_0^t \langle u'(s), u(s)\rangle_H \, ds \\
&= \int_0^t \langle u'(s), u(s)\rangle_H \, ds \\
&= \int_0^t \left\langle \Delta u(s) + |\nabla u(s)|_H^2 \, u(s) - u(s)^{2n-1} + u(s)\,|u(s)|_{L^{2n}}^{2n}, u(s)\right\rangle_H ds
\end{aligned}$$

$$\begin{aligned}
= \;& \int_0^t \langle \Delta u(s), u(s)\rangle_H \, ds + \int_0^t \left\langle |\nabla u(s)|_H^2 \, u(t), u(s)\right\rangle_H ds \\
& - \int_0^t \left\langle u^{2n-1}(s), u(s)\right\rangle_H ds + \int_0^t \left\langle u(s)\,|u(s)|_{L^{2n}}^{2n}, u(s)\right\rangle_H
\end{aligned}$$

$$\begin{aligned}
=\;& -\int_0^t |\nabla u(s)|_H^2 \, ds + \int_0^t |\nabla u(s)|_H^2 \, \langle u(s), u(s)\rangle_H \, ds \\
& - \int_0^t \left\langle u^{2n-1}(s), u(s)\right\rangle_H ds + \int_0^t |u(s)|_{L^{2n}}^{2n} \langle u(s), u(s)\rangle_H \, ds \\
=\;& \int_0^t |\nabla u(s)|_H^2 \, |u(s)|_H^2 \, ds - \int_0^t |\nabla u(s)|_H^2 \, ds \\
& - \int_0^t \int_D \left| u^{2n-1}(s,x) u(s,x)\right|^2 dx ds + \int_0^t |u(s)|_{L^{2n}}^{2n} \langle u(s), u(s)\rangle_H \, ds \\
=\;& \int_0^t |\nabla u(s)|_H^2 \left(|u(s)|_H^2 - 1\right) ds - \int_0^t |u(s)|_{L^{2n}}^{2n} \, ds + \int_0^t |u(s)|_{L^{2n}}^{2n} \, |u(s)|_H^2 \, ds
\end{aligned}$$

or

$$\frac{1}{2}\left(|u(t)|_H^2 - 1\right) = \int_0^t |\nabla u(s)|_H^2 \left(|u(s)|_H^2 - 1\right) ds + \int_0^t |u(s)|_{L^{2n}}^{2n} \left(|u(s)|_H^2 - 1\right) ds$$

Set

$$\phi(\cdot) := |u(\cdot)|_H^2 - 1.$$

Then

$$\phi(t) = 2\int_0^t \left(|\nabla u(s)|_H^2 + |u(s)|_{L^{2n}}^{2n}\right)\phi(s)ds.$$

Or in stronger form

$$\frac{d\phi(t)}{dt} = 2\left(|\nabla u(t)|_H^2 + |u(t)|_{L^{2n}}^{2n}\right)\phi(t)$$

Since $\left(|\nabla u\,(t)|_H^2 + |u\,(t)|_{L^{2n}}^{2n}\right)_{t\in[0,\tau)}$ polynomial so it must be continuous in time therefore

$$\phi(t) = \phi(0)e^{2\left(|\nabla u(t)|_H^2 + |u(t)|_{L^{2n}}^{2n}\right)}$$

$$\phi(t) = 0 \text{ i.e. } |u(t)|_H^2 = 1, \text{for all } t \in [0,\tau).$$

This completes the proof. ∎

In the next lemma we are going to show that energy is of C_2-class.

Lemma 2.3.4. *The map $\psi : V \to \mathbb{R}$ defined by*

$$\psi(u) = \frac{1}{2}|\nabla u|_{L^2(\mathcal{O})}^2 + \frac{1}{2n}|u|_{L^{2n}(\mathcal{O})}^{2n}, \; n \in \mathbb{N}. \tag{2.3.6}$$

is of C^2-class.

Proof. In order to compute the first order derivative we are going to use the following definition of Fréchet derivative of Ψ at $u \in V$

$$d_u\psi(h) := \lim_{t\to 0}\frac{\psi(u+th) - \psi(u)}{t},$$

where $d_u\psi : V \to \mathbb{R}$ is linear with respect to h. Consider

$$\begin{aligned}
\psi(u+th) &= \frac{1}{2}\left|\nabla u + th\right|^2_{L^2(\mathcal{O})} + \frac{1}{2n}\left|u+th\right|^{2n}_{L^{2n}(\mathcal{O})} \\
&= \frac{1}{2}\int_D \left(\nabla u(x) + t\nabla h(x)\right)^2 dx + \frac{1}{2n}\int_D \left(u(x)+th(x)\right)^{2n} \\
&= \frac{1}{2}\int_D \left(\nabla u(x)\right)^2 dx + \frac{t^2}{2}\int_D \left(\nabla h(x)\right)^2 dx + \frac{2t}{2}\int_D \nabla u(x)\nabla h(x)dx \\
&\quad + \frac{1}{2n}\int_D \left(u(x)\right)^{2n} + \frac{2nt}{2n}\int_D \left(u(x)\right)^{2n-1} h(x) \\
&\quad + \frac{(2n-1)}{2n}t^2\int_D \left(u(x)\right)^{2n-2}\left(h(x)\right)^2 + ... \\
&= \psi(u) + \frac{t^2}{2}\int_D \left(\nabla h(x)\right)^2 dx + t\int_D \nabla u(x)\nabla h(x)dx \\
&\quad + \frac{1}{2n}\int_D \left(u(x)\right)^{2n} dx + t\int_D u(x)^{2n-1}h(x)dx \\
&\quad + \frac{(2n-1)}{2n}t\int_D \left(u(x)\right)^{2n-2}\left(h(x)\right)^2 dx + ...
\end{aligned}$$

$$\begin{aligned}
\frac{\psi(u+th)-\psi(u)}{t} &= \frac{t}{2}\int_D \left(\nabla h(x)\right)^2 dx + \int_D \nabla u(x)\nabla h(x)dx \\
&\quad + \int_D u(x)h(x)^{2n-1} + \frac{(2n-1)}{2n}t\int_D \left(u(x)\right)^{2n-2}\left(h(x)\right)^2 + ... \\
\lim_{t\to 0}\frac{\psi(u+th)-\psi(u)}{t} &= \int_D \nabla u(x)\nabla h(x)dx + \int_D \left(u(x)\right)^{2n-1} h(x)
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
\lim_{t\to 0}\frac{\psi(u+th)-\psi(u)}{t} &= -\int_D \Delta u(x)h(x)dx + \int_D \left(u(x)\right)^{2n-1} h(x) \\
&= \left\langle -\Delta u + u^{2n-1}, h\right\rangle \text{ for all } h \in V.
\end{aligned}$$

We can see that the operator $d_u\psi$ is linear. Next we are going to show that $d_u\psi$ is bounded. We are going to use integration by parts, Cauchy-Schwartz inequality, and continuity of embedding $V \hookrightarrow L^{2n}$ in the following chain of inequalities. For all

$h \in V$ consider,

$$\begin{aligned} |d_u\psi(h)| &= \left|\left\langle -\Delta u + u^{2n-1}, h\right\rangle\right| \\ &= \left|\int_D \nabla u(x)\nabla h(x)dx + \int_D u^{2n-1}(x)h(x)dx\right| \\ &\leq \left|\int_D \nabla u(x)\nabla h(x)dx\right| + \left|\int_D u^{2n-1}(x)h(x)dx\right| \\ &\leq |\nabla h|_H |\nabla u|_H + |u|_{L^{2n}}^{2n-1} |h|_{L^{2n}} \\ &\leq \|u\| \|h\| + c^{2n-1} \|u\|^{2n-1} \|h\| \\ &= \left(\|u\| + c^{2n-1} \|u\|^{2n-1}\right) \|h\| = C \|h\|, \end{aligned} \tag{2.3.7}$$

where $C = \|u\| + c^{2n-1} \|u\|^{2n-1} < \infty$. Hence $d_u\psi$ is bounded.

Let us compute the second order Frechet derivative at a point. For $h_1, h_2 \in V$, we want to compute the following limit,

$$\lim_{t\to 0} \frac{\psi(u + t(h_1 + h_2)) - \psi(u + th_1) - \psi(u + th_2) + \psi(u)}{t^2}.$$

To compute this limit let us begin with computing $\psi(u + t(h_1 + h_2))$. By definition of ψ,

$$\psi(u + t(h_1 + h_2)) = \frac{1}{2}\|u + t(h_1 + h_2)\|^2 + \frac{1}{2n}|u + t(h_1 + h_2)|_{L^{2n}}^{2n} \tag{2.3.8}$$

To make our life easy let us compute the above to norms separately, begin with

$$\begin{aligned} &\tfrac{1}{2}\|u + t(h_1 + h_2)\|^2 \\ &= \frac{1}{2}\int_D (\nabla u(x) + t(\nabla h_1(x) + \nabla h_2(x)))^2 dx \\ &= \frac{1}{2}\int_D (\nabla u(x))^2 dx + \frac{t^2}{2}\int_D (\nabla h_1(x) + \nabla h_2(x))^2 dx \\ &\quad + t\int_D \nabla u(x)(\nabla h_1(x) + \nabla h_2(x)) dx \\ &= \frac{1}{2}\|u\|^2 + \frac{t^2}{2}\int_D (\nabla h_1(x))^2 dx + \frac{t^2}{2}\int_D (\nabla h_2(x))^2 dx \\ &\quad t^2\int_D \nabla h_1(x)\nabla h_2(x)dx + t\int_D \nabla u(x)(\nabla h_1(x) + \nabla h_2(x)) dx \\ &= \frac{1}{2}\|u\|^2 + \frac{t^2}{2}\|h_1\|^2 + \frac{t^2}{2}\|h_2\|^2 + t^2\langle h_1, h_2\rangle_V + t\langle u, h_1 + h_2\rangle_V. \end{aligned} \tag{2.3.9}$$

Next

$$\begin{aligned}
&\tfrac{1}{2n}\left|u+t\left(h_1+h_2\right)\right|_{L^{2n}}^{2n}\\
= &\ \frac{1}{2n}\int_D \left(u(x)+t\left(h_1(x)+h_2(x)\right)\right)^{2n} dx\\
= &\ \frac{1}{2n}\int_D u(x)^{2n}dx+\frac{2nt}{2n}\int_D u(x)^{2n-1}\left(h_1(x)+h_2(x)\right)dx\\
&+\frac{(2n-1)\,t^2}{2n}\int_D u(x)^{2n-2}\left(h_1(x)+h_2(x)\right)^2 dx+...\\
= &\ \frac{1}{2n}\left|u\right|_{L^{2n}}^{2n}+t\left\langle u^{2n-1},h_1+h_2\right\rangle+\frac{(2n-1)\,t^2}{2n}\int_D u(x)^{2n-2}h_2(x)^2dx\\
&+\frac{(2n-1)\,t^2}{2n}\int_D u(x)^{2n-2}h_2(x)^2dx+\frac{(2n-1)\,t^2}{n}\int_D u(x)^{2n-2}h_1(x)h_2(x)dx+...\\
= &\ \frac{1}{2n}\left|u\right|_{L^{2n}}^{2n}+t\left\langle u^{2n-1},h_1+h_2\right\rangle+\frac{(2n-1)\,t^2}{2n}\left\langle u^{2n-2},{h_1}^2\right\rangle\\
&+\frac{(2n-1)\,t^2}{2n}\left\langle u^{2n-2},{h_2}^2\right\rangle+\frac{(2n-1)\,t^2}{n}\left\langle u^{2n-2},h_1h_2\right\rangle+... \qquad (2.3.10)
\end{aligned}$$

Substituting (2.3.10) and (2.3.9) into (2.3.8) we get,

$$\begin{aligned}
\psi\left(u+t\left(h_1+h_2\right)\right) = &\ \psi\left(u\right)+\frac{t^2}{2}\left\|h_1\right\|^2+\frac{t^2}{2}\left\|h_2\right\|^2+t^2\left\langle h_1,h_2\right\rangle_V+t\left\langle u,h_1+h_2\right\rangle_V\\
&+t\left\langle u^{2n-1},h_1+h_2\right\rangle+\frac{(2n-1)\,t^2}{2n}\left\langle u^{2n-2},{h_1}^2\right\rangle\\
&+\frac{(2n-1)\,t^2}{2n}\left\langle u^{2n-2},{h_2}^2\right\rangle+\frac{(2n-1)\,t^2}{n}\left\langle u^{2n-2},h_1h_2\right\rangle+... \qquad (2.3.11)
\end{aligned}$$

Next we intend to compute in the similar way $\psi\left(u+th_1\right)$. Consider

$$\begin{aligned}
&\psi\left(u+th_1\right)\\
=\;& \frac{1}{2}\left\|u+th_1\right\|^2+\frac{1}{2n}\left|u+th_1\right|_{L^{2n}}^{2n}\\
=\;& \frac{1}{2}\left\|u\right\|^2+\frac{t^2}{2}\int_D\left(\nabla h_1(x)\right)^2dx+t\int_D\nabla u(x)\nabla h_1(x)dx\\
&+\frac{1}{2n}\int_D u(x)^{2n}dx+\frac{2nt}{2n}\int_D u(x)^{2n-1}h_1(x)dx\\
&+\frac{\left(2n-1\right)t^2}{2n}\int_D u(x)^{2n-2}h_1(x)^2dx+...\\
=\;& \frac{1}{2}\left\|u\right\|^2+\frac{t^2}{2}\left\|h_1\right\|^2+t\left\langle u,h_1\right\rangle_V\\
&+\frac{1}{2n}\left|u\right|_{L^{2n}}^{2n}+t\left\langle u^{2n-1},h_1\right\rangle+\frac{\left(2n-1\right)t^2}{2n}\left\langle u^{2n-2},{h_1}^2\right\rangle+...\\
=\;& \psi\left(u\right)+\frac{t^2}{2}\left\|h_1\right\|^2+t\left\langle u,h_1\right\rangle_V+t\left\langle u^{2n-1},h_1\right\rangle+\frac{\left(2n-1\right)t^2}{2n}\left\langle u^{2n-2},{h_1}^2\right\rangle+...
\end{aligned} \tag{2.3.12}$$

In the precise same manner we can also compute

$$\begin{aligned}
&\psi\left(u+th_2\right)\\
&=\psi\left(u\right)+\frac{t^2}{2}\left\|h_2\right\|^2+t\left\langle u,h_2\right\rangle_V+t\left\langle u^{2n-1},h_2\right\rangle+\frac{\left(2n-1\right)t^2}{2n}\left\langle u^{2n-2},{h_2}^2\right\rangle+...
\end{aligned} \tag{2.3.13}$$

Using equations (2.3.11),(2.3.12) and (2.3.13) it follows that,

$$\begin{aligned}
&\psi\left(u+t\left(h_1+h_2\right)\right)-\psi\left(u+th_1\right)-\psi\left(u+th_2\right)+\psi\left(u\right)\\
&\qquad=t^2\left\langle h_1,h_2\right\rangle_V+\frac{\left(2n-1\right)t^2}{n}\left\langle u^{2n-2},h_1h_2\right\rangle+....
\end{aligned}$$

Dividing both sides by t^2 and taking limit $t\to 0$ we infer that

$$\begin{aligned}
&\lim_{t\to 0}\frac{\psi\left(u+t\left(h_1+h_2\right)\right)-\psi\left(u+th_1\right)-\psi\left(u+th_2\right)+\psi\left(u\right)}{t^2}\\
=\;&\left\langle h_1,h_2\right\rangle_V+\frac{\left(2n-1\right)}{n}\left\langle u^{2n-2},h_1h_2\right\rangle.
\end{aligned}$$

Thus the second order derivative $d_u^2\psi : V\times V\to\mathbb{R}$ can be give in the following duality,

$$\begin{aligned} d_u^2\psi\,(h_1,h_2) &\equiv \langle \psi''\,(u)\,h_1,h_2\rangle \\ &= \langle h_1,h_2\rangle_V + \frac{(2n-1)}{n}\left\langle u^{2n-2},h_1h_2\right\rangle. \end{aligned} \tag{2.3.14}$$

We can see the above defined operator is linear and also it is bounded because for $(h_1,h_2)\in V\times V$,

$$\begin{aligned} \left|d_u^2\psi\,(h_1,h_2)\right| &= \left|\langle h_1,h_1\rangle_V + \frac{(2n-1)}{n}\left\langle u^{2n-2},h_1h_2\right\rangle\right| \\ &\leq |\langle h_1,h_1\rangle_V| + \frac{(2n-1)}{n}\left|\left\langle u^{2n-2},h_1h_2\right\rangle\right| \\ &\leq \|h_1\|\,\|h_2\| + \frac{(2n-1)}{n}\left|u^{2n-2}\right|_H |h_1h_2|_H \\ &= \|h_1\|\,\|h_2\| + \frac{(2n-1)}{n}|u|_{L^{4n-4}}^{2n-2}\left(\int_D h_1^2(x)h_2^2(x)dx\right)^{1/2} \\ &\leq \|h_1\|\,\|h_2\| + \frac{(2n-1)}{n}|u|_{L^{4n-4}}^{2n-2} \\ &\quad\cdot\left(\int_D h_1^4(x)dx\right)^{1/4}\left(\int_D h_2^4(x)dx\right)^{1/4} \\ &= \|h_1\|\,\|h_2\| + \frac{(2n-1)}{n}|u|_{L^{4n-4}}^{2n-2}|h_1|_{L^4}|h_2|_{L^4} \\ &\leq \|h_1\|\,\|h_2\| + c^{2n-2}\frac{(2n-1)}{n}\|u\|^{2n-2}\|h_1\|\,\|h_2\| \\ &\leq \left(1 + c^{2n}\frac{(2n-1)}{n}\|u\|^{2n-2}\right)\|h_1\|\,\|h_2\| = C\,\|h_1\|\,\|h_2\|. \end{aligned}$$

where $C = 1 + c^{2n}\frac{(2n-1)}{n}\|u\|^{2n-2} < \infty$. Hence $d_u^2\psi$ is bounded bilinear form. ■

Following is the main result of this section, comprising of proof of a global solution to the projected constrained problem (2.1.3).

Theorem 2.3.5. *If u is the local maximal solution, of main problem* (2.1.3)*, defined on* $[0,\tau)$*, with initial data* $u_0\in V\cap M$ *, then*

$$\|u(t)\| \leq 2\psi(u_0),\ \textit{for all } t\in[0,\tau),$$

and $\tau = \infty$, where ψ is the energy defined by (2.3.6). *In particular u is global solution.*

Proof. We begin with application of Lemmas 2.3.1 and 2.3.2. Consider the following,

$$
\begin{aligned}
&\tfrac{1}{2}\left\|u(t)\right\|^2 - \tfrac{1}{2}\left\|u_0\right\|^2 = \\
&= \int_0^t \left\langle \nabla u(s), \nabla \frac{du}{dt}(s) \right\rangle ds \\
&= \int_0^t \left\langle -\Delta(s), \frac{du}{dt}(s) \right\rangle, \text{(here we have used integration by parts.)} \\
&= -\int_0^t \left\langle \frac{du}{dt}(s), \frac{du}{dt}(s) \right\rangle ds + \int_0^t \left\langle \frac{du}{dt}(s) - \Delta(s), \frac{du}{dt}(s) \right\rangle ds \\
&= -\int_0^t \left|\frac{du}{dt}(s)\right|_H^2 ds + \int_0^t \left\langle \left(\|u(s)\|^2 + |u(s)|_{L^{2n}}^{2n}\right) u(s) - u^{2n-1}(s), \frac{du}{dt}(s) \right\rangle ds \\
&= -\int_0^t \left|\frac{du}{dt}(s)\right|_H^2 ds + \int_0^t |\nabla u(s)|_H^2 \left\langle u(s), \frac{du}{dt}(s) \right\rangle ds - \int_0^t \left\langle u^{2n-1}(s), \frac{du}{ds}(s) \right\rangle ds \\
&\quad + \int_0^t |u(s)|_{L^{2n}}^{2n} \left\langle u(s), \frac{du}{dt}(s) \right\rangle ds, \text{ for all } t \in [0,\tau).
\end{aligned}
$$

Since $u_0 \in M$ so by Lemma 2.3.3 $u(t) \in M$, for all $t \in [0,\tau)$. Moreover, since $\frac{du}{dt} = \pi_u\left(\Delta u - u^{2n-1}\right) \in T_u M$, hence $\left\langle u(t), \frac{du}{dt}(t) \right\rangle = 0$, for all $t \in [0,\tau)$. Therefore the last equation reduces to

$$
\begin{aligned}
\frac{1}{2}\left\|u(t)\right\|^2 - \frac{1}{2}\left\|u_0\right\|^2 &= -\int_0^t \left|\frac{du}{dt}(s)\right|_H^2 ds - \int_0^t \left\langle u^{2n-1}(s), \frac{d}{ds}u(s) \right\rangle ds \\
&= -\int_0^t \left|\frac{du}{dt}(s)\right|_H^2 ds - \int_0^t \frac{d}{ds}\left\langle u^{2n-1}(s), u(s) \right\rangle ds \\
&= -\int_0^t \left|\frac{du}{dt}(s)\right|_H^2 ds - \frac{1}{2n}\left|u(t)\right|_{L^{2n}}^{2n} + \frac{1}{2n}\left|u_0\right|_{L^{2n}}^{2n} \quad (2.3.15) \\
\psi(u(t)) - \psi(u_0) &= -\int_0^t \left|\frac{du}{dt}(s)\right|_H^2 ds, \text{ for all } t \in [0,\tau). \quad (2.3.16)
\end{aligned}
$$

where

$$
\psi(u) = \frac{1}{2}\left\|u\right\| + \frac{1}{2n}\left|u\right|_{L^{2n}(\mathcal{O})}^{2n}.
$$

From equation (2.3.16) it clearly follows ψ is non increasing function i.e. $\psi(u(t)) \leq \psi(u_0)$, for all $t \in [0, \tau)$. Thus in particular we infer that

$$\|u(t)\| \leq 2\psi(u(t)) \leq 2\psi(u_0), \text{ for all } t \in [0, \tau).$$

Hence by Remark 2.2.11 we get $\tau = \infty$ i.e. u is global solution. ■

2.4 Some auxiliary results

Proposition 2.4.1. *Assume we are in framework of Theorem 2.3.5. If u be global solution then the orbit $\{u(t), t \geq 1\}$ is pre-compact in V.*

Proof. In order to prove that $\{u(t), t \geq 1\}$ precompact, it is sufficient to prove that $\{u(t), t \geq 1\}$ is bounded in $D(A^\alpha)$, for $\alpha > \frac{1}{2}$, where $A = -\Delta$. By application of A^α to $u(t)$ gives the following variation of constant formula,

$$A^\alpha u(t) = A^\alpha e^{-At} u_0 + \int_0^t A^\alpha e^{-A(t-s)} F(u(s)) ds$$

where $F(u) = \|u\|^2 u - u^{2n-1} + u\,|u|_{L^{2n}}^{2n} := F_1(u) - F_2(u) + F_3(u)$. Taking H norm on both sides

$$\begin{aligned} |A^\alpha u(t)|_H &= \left| A^\alpha e^{-At} u_0 + \int_0^t A^\alpha e^{-A(t-s)} F(u(s)) ds \right|_H \\ &\leq \left| A^\alpha e^{-At} u_0 \right|_H + \left| \int_0^t A^\alpha e^{-A(t-s)} F(u(s)) ds \right|_H \\ &\leq \left| A^\alpha e^{-At} u_0 \right|_H + \int_0^t \left| A^\alpha e^{-A(t-s)} F(u(s)) \right|_H ds \end{aligned}$$

By Proposition 1.4.3 of [23] (or Theorem 6.13, inequality 6.25) of [38] , we conclude that

$$|A^\alpha u(t)|_H \leq M_\alpha t^{-\alpha} e^{-\delta t} |u_0|_H + \int_0^t M_\alpha (t-s)^{-\alpha} e^{-\delta(t-s)} |F(u(s))|_H ds$$

But since $u_0 \in M$ i.e. $|u_0|_H = 1$ so

$$|A^\alpha u(t)|_H \le M_\alpha t^{-\alpha} e^{-\delta t} + M_\alpha \int_0^t \frac{e^{-\delta(t-s)}}{(t-s)^\alpha} |F(u(s))|_H \, ds \qquad (2.4.1)$$

We compute the bound for the term $|F(u)|_H$, i.e.

$$\begin{aligned} |F(u)|_H &= \left| \|u\|^2 u - u^{2n-1} + u\, |u|_{L^{2n}}^{2n} \right|_H \\ &\le \|u\|^2 |u|_H + \left|u^{2n-1}\right|_H + |u|_{L^{2n}}^{2n} |u|_H \end{aligned} \qquad (2.4.2)$$

Recall that

$$\psi(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2n}|u|_{L^{2n}}^{2n}.$$

In Theorem 2.3.5 we proved that ψ is increasing function with respect to time. Hence it follows that

$$\begin{aligned} \|u\|^2 &\le 2\psi(u) \le 2\psi(u_0) \\ \text{and } |u|_{L^{2n}}^{2n} &\le 2n\psi(u) \le 2n\psi(u_0) \end{aligned} \qquad (2.4.3)$$

Also $u_0 \in M$ then by Lemma 2.3.3 if follows that $u(t) \in M$ i.e. $|u(t)|_H = 1$, for all $t \ge 0$. Hence invariance of manifold and inequalities (2.4.3) into inequality (2.4.3), we get

$$\begin{aligned} |F(u)|_H &\le 2\psi(u_0) + \left|u^{2n-1}\right|_H + 2n\psi(u_0) \\ &= 2(1+n)\psi(u_0) + \left|u^{2n-1}\right|_H. \end{aligned} \qquad (2.4.4)$$

Now consider

$$\left|u^{2n-1}\right|_H = \left(\int_D u(x)^{4n-2} dx\right)^{\frac{1}{2}} = |u|_{L^{4n-2}}^{2n-1}.$$

Using the continuity of embedding $V \hookrightarrow L^{4n-2}$,

$$\left|u^{2n-1}\right|_H = |u|_{L^{4n-2}}^{2n-1} \le c^{2n-1}\|u\|^{2n-1} \le 2^{2n-1} c^{2n-1} \psi(u_0)^{2n-1}$$

Using above inequality in inequality (2.4.3), it follows that

$$|F(u)|_H \leq 2(1+n)\psi(u_0) + 2^{2n-1}c^{2n-1}\psi(u_0)^{2n-1} =: L < \infty.$$

Using last inequality in (2.4.1),

$$\begin{aligned} |u(t)|_{D(A^\alpha)} &= |A^\alpha u(t)|_H \leq M_\alpha t^{-\alpha}e^{-\delta t} + M_\alpha L \int_0^t \frac{e^{-\delta(t-s)}}{(t-s)^\alpha} ds \\ &\leq M_\alpha t^{-\alpha}e^{-\delta t} + M_\alpha L \int_0^\infty \frac{e^{-\delta(t-s)}}{(t-s)^\alpha} ds \\ &= M_\alpha t^{-\alpha}e^{-\delta t} + M_\alpha L\Gamma(1-\alpha) := K_\alpha < \infty \end{aligned}$$

Hence $\{u(t), t \geq 1\}$ is bounded in $D(A^\alpha)$, for $\alpha > \frac{1}{2}$. Thus orbit $\{u(t), t \geq 1\}$ is pre-compact in V, this completes the proof of lemma. ■

Corollary 2.4.2. *The ω-limit set $\omega(u_0) = \bigcap_{r\geq 1} \overline{\{u(t) : t \geq r\}}$ exists and is compact in V.*

Proof. From last corollary, $\{u(t) : t \geq r\}$ is precompact in V for all $r \geq 1$. Since closure of precompact is also precompact so $\overline{\{u(t) : t \geq r\}}$ is also precompact for all $r \geq 1$. Further $\overline{\{u(t) : t \geq r\}}$ is closed and hence complete in V norm therefore $\overline{\{u(t) : t \geq r\}}$ being precompact and complete $\overline{\{u(t) : t \geq r\}}$ it follows that $\overline{\{u(t) : t \geq r\}}$ is compact for all $r \geq 1$, in V. Thus $\omega(u_0)$ being decreasing intersection of non-empty compact sets in V, is non-empty and compact in V. ■

We now going to make an interesting observation that our global solution u is a gradient flow and (V, S, ψ) is gradient system. Recall, the following definitions of Lyapunov functional, gradient system and gradient flow.

Definition 2.4.3. *[47] Suppose V is a complete metric space and $S(t)$ is a non-linear C_0-semigroup defined on H. A continuous function $\psi : V \to \mathbb{R}$ is called Lyapunov function w.r.t $S(t)$ if the following two conditions are satisfied:*

i) For any $x \in V$, $\psi(S(t)x)$ is a monotone non-increasing in t,

ii) ψ is bounded below i.e. there exists a constant C such that $\psi(x) \geq C$, for all $x \in V$.

Definition 2.4.4. *[47] Suppose V is a complete metric space and $S(t)$ is a non-linear C_0-semigroup defined on V. Moreover, assume that continuous function $\psi : H \to \mathbb{R}$ be a Lyapunov function w.r.t $S(t)$. Then non-linear Semigroup, or more precisely system $(V, S(t), \psi)$ is called gradient system if the following are satisfied:*

i) For any $x \in V$, there is $t_0 > 0$ such that $\cup_{t \geq t_0} S(t)x$ is relatively compact in V.

ii) If for $t > 0$, $\psi(S(t)x) = \psi(x)$ then x is fixed point of semigroup $S(t)$.

Accordingly, the orbit $u(t)$ is called a gradient flow.

Theorem 2.4.5. *(V, S, ψ) is gradient system, where $V = D(A^{1/2})$, $S(t)$ is C_0-semigroup and $\psi : V \to \mathbb{R}$ is the energy defined by (2.3.6).*

Proof. We begin by showing that $\psi : V \to \mathbb{R}$ is Lyapunov function. We know Theorem 2.3.5 that ψ in non-increasing. Moreover since $\psi(u) = \frac{1}{2} |\nabla u|^2_{L^2(\mathcal{O})} + \frac{1}{2n} |u|^{2n}_{L^{2n}(\mathcal{O})} \geq 0$, hence ψ is Lyapunov function. It remains to verify the condition (i) and (ii) Definition 2.4.4. The Condition (i) have been already established in Corollary 2.4.1. Let us prove condition (ii) now. From the end of proof of Theorem 2.3.5 we know that,

$$\begin{aligned} \left|\frac{du}{dt}\right|^2_H &= -\frac{d}{dt}\psi(u) \\ \text{or} \int_0^t \left|\frac{du}{ds}\right|^2_H ds &= \psi(u_0) - \psi(u(t)) \end{aligned} \tag{2.4.5}$$

If there is $t_0 \geq 0$ such that,

$$\psi(u(t_0)) = \psi(S(t)u_0) = \psi(u_0)$$

then for $0 \leq t \leq t_0$, from equation (2.4.5),

$$\begin{aligned} \int_0^t \left| \frac{du}{ds} \right|_H^2 ds &= 0 \\ \left| \frac{du}{dt} \right|_H^2 &= 0 \\ \frac{du}{dt}(t) &= 0. \\ u(t) = u(0) &= \text{constant}. \end{aligned}$$

Therefore u_0 is fixed point of semigroup S and thus $(V, S(t), \psi)$ is gradient system.

■

Chapter 3

Stochastic Heat Flow equation on Hilbert Manifold: Existence and Uniqueness of Global Solution

The aim of this chapter is to study the stochastic generalization of projected heat flow studied in chapter 2. More precisely, we are going to study a nonlinear parabolic first order in time (heat) equation on Hilbert Manifold driven by Wiener noise of Stratonovich type. We devote this chapter to existence and uniqueness of the global solution to described stochastic evolution equation.

The first section has been devoted to setting up notations related to spaces, manifold and operators, and certain essential assumptions. Next we will introduce the main stochastic problem of our concern in this chapter, along with its approximated version, which are going to solve first. In the end of section **1** we will provide the definitions of Local mild, maximal and the global solution to our main problem.

In the section **2** we are primarily concerned with studying the *truncated stochastic evolution equation.* In the first part of the section, we prove the estimates for *deterministic (drift)* terms and *stochastic (diffusion)* terms of our main stochastic evolution equation. Next, by employing fixed point argument we will construct the local mild solution of approximate evolution equation, after that from this constructed local mild solution we recover the *local mild solution* to the original problem. The second section is going to end at constructing the global solution of approximate evolution equation.

The section **3**, mainly consists of proof of existence and uniqueness of *Maximal local solution* to original stochastic evolution equation.

The chapter ends at section **4**. The chapter begins with proof of the very important *no explosion result.* Further, we will show the invariance of manifold i.e. if initial data belongs to manifold then almost all trajectories of solution belong to the manifold. Afterward, we proceed towards proving some of the important estimates, which will be used for proof for a global solution. The section and chapter end by providing proof of global solution to our main stochastic evolution equation. The main tool for proving the global solution will be Khashminskii test for non-explosion and indeed appropriate Itô's formula.

3.1 Notation, Assumptions, Estimates and

Introduction to main problem

The aim of the section is, firstly, to setup all important notations, function spaces, manifold and state all necessary assumptions. Secondly to state the main constrained stochastic evolution equation of our concern in this chapter.

3.1.1 Preliminary Notation

Let us begin by introducing some important notation and function spaces which we are going to use in this and then subsequent chapters of the dissertation.

Assumption 3.1.1. *We assume that* $(E, |\cdot|_E)$, $(V, \|\cdot\|)$, $(H, |\cdot|_H)$ *are abstract Banach spaces such that*

$$E \hookrightarrow V \hookrightarrow H, \tag{3.1.1}$$

and the embeddings are dense continuous.

Remark 3.1.2. *In our motivating application, we will consider the following choice of space*

$$\begin{aligned} E &= D(A), \\ V &= H_0^{1,2}(\mathcal{O}), \\ H &= L^2(\mathcal{O}) \end{aligned}$$

where $D \subset \mathbb{R}^d$ *for* $d \geq 1$, *is a bounded domain with sufficiently smooth boundary and* A *be the Laplace operator with Dirichlet boundary conditions, defined by*

$$\begin{aligned} D(A) &= H_0^{1,2}(\mathcal{O}) \cap H^{2,2}(\mathcal{O}) \\ Au &= -\Delta u, \ u \in D(A). \end{aligned} \tag{3.1.2}$$

It is well known that, (cf. [51], Theorem 4.1.2, page 79), that A is a self-adjoint positive operator in H and that $V = D(A^{1/2})$, and

$$\|u\|^2 = \left|A^{1/2}u\right|_H^2 = \int_D |\nabla u(x)|^2 \, dx$$

Moreover $V' = H^{-1}(\mathcal{O})$ and the condition (3.1.1) holds true. Hence the spaces E, V and H satisfies Assumption 3.1.1.

3.1.2 Manifold and Projection

The version of Hilbert manifold we are going to deal with, is the following submanifold M of a Hilbert space H (with inner product denoted by $\langle \cdot, \cdot \rangle$),

$$M = \left\{u \in H : |u|_H^2 = 1\right\}.$$

Moreover the tangent space, at a point u in H, is of form,

$$T_uM = \{v : \langle u, v\rangle = 0\}.$$

Let $\pi_u : H \to T_uM$ be orthogonal projection of H onto tangent space M then we have the following lemma.

Lemma 3.1.3. *Let $\pi_u : H \to T_uM$ be orthogonal projection then $\pi_u(v) = v - \langle u, v\rangle u$, where $v \in H$.*

We aim to study a stochastic evolution equation with drift term consisting of the projection of difference of the Laplace operator with Dirichlet boundary condition and the polynomial nonlinearity of degree $2n - 1$.

Let us pick a $u \in E$. Using the last lemma, we calculate an explicit expression for projection of $\Delta u - u^{2n-1}$ under π_u. The below-given calculation using integration by parts, cf. [3] (Corollary 8.10, page-82), to get

$$\begin{aligned}
\pi_u\left(\Delta u - u^{2n-1}\right) &= \Delta u - u^{2n-1} - \left\langle \Delta u - u^{2n-1}, u\right\rangle u \\
&= \Delta u - u^{2n-1} + \langle -\Delta u, u\rangle u + \left\langle u^{2n-1}, u\right\rangle u \\
&= \Delta u - u^{2n-1} + \langle \nabla u, \nabla u\rangle u + \left\langle u^{2n-1}, u\right\rangle u \\
&= \Delta u + \left(\|u\|^2 + |u|_{L^{2n}}^{2n}\right) u - u^{2n-1}. \qquad (3.1.3)
\end{aligned}$$

3.1.3 Statement of main and approximate stochastic evolution equation

Let the spaces E, V and H be the Hilbert spaces as in Remark 3.1.2. The following is the main stochastic evolution equation that we are going to study in running and the subsequent chapters.

$$\begin{aligned}
du &= \pi_u\left(\Delta u - u^{2n-1}\right) dt + \sum_{j=1}^{N} B_j(u) \circ dW_j, \qquad (3.1.4)\\
&= (\Delta u + F(u))\, dt + \sum_{j=1}^{N} B_j(u) \circ dW_j, \\
u(0) &= u_0,
\end{aligned}$$

where the map $F : V \to H$ is defined by $F(u) := \|u\|^2 u - u^{2n-1} + u\,|u|_{L^{2n}}^{2n}$, with n being a natural number (or, more generally, a real number bigger than $\frac{1}{2}$). Further, for fixed elements $f_1, f_2, ...f_N$ from V, the map $B_j : V \to V$ is defined by

$$B_j(u) := \pi_u(f_j) = f_j - \langle f_j, u\rangle u, \; j = 1, 2, 3..N. \qquad (3.1.5)$$

Note that noise term in above stochastic partial differential equation involves the noise of stratonovich type. This is because of the constraint condition given by manifold M; it is natural to consider equations in the Stratonovich form (see also

[6]). To write the equation (3.1.4) in Itô form we can write the Stratonovich term in the following manner,

$$B_j(u) \circ dW_j = B_j(u)dW_j + \frac{1}{2}d_uB_j(B_j(u))dt,$$

hence the equation (3.1.4) can be rewritten in the following the Itô form,

$$\begin{aligned} du &= \left[\Delta u + F(u) + \frac{1}{2}\sum_{j=1}^{N}\kappa_j(u)\right]dt + \sum_{j=1}^{N} B_j(u)dW_j, \qquad (3.1.6) \\ u(0) &= u_0, \end{aligned}$$

where

$$\kappa_j(u) \equiv d_uB_j\left(B_j\left(u\right)\right), \text{for all } u \in H \text{ and } j = 1, 2, ..., N. \qquad (3.1.7)$$

We introduce now an auxiliary function which will be used later for truncation of the norm of the solution. Let $\theta : \mathbb{R}^+ \to [0,1]$ be a non-increasing smooth function with compact support such that

$$\inf_{x\in\mathbb{R}_+} \theta'(x) \geq -1,\ \theta(x) = 1 \text{ iff } x \in [0,1] \text{ and } \theta(x) = 0 \text{ iff } x \in [2,\infty). \qquad (3.1.8)$$

For $n \geq 1$ set $\theta_n(\cdot) = \theta\left(\frac{\cdot}{n}\right)$. We have the following easy Lemma about θ as consequence of previous description.

Lemma 3.1.4. *([13], page 57) If $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a non decreasing function, then for every $x, y \in \mathbb{R}$,*

$$\theta_n(x)h(x) \leq h(2n),\ \ |\theta_n(x) - \theta_n(y)| \leq \frac{1}{n}|x-y|. \qquad (3.1.9)$$

In order to prove the existence and uniqueness of local mild and local maximal solutions to our original problem (3.1.6), we first we obtain existence and uniqueness of the solution of below given approximate evolution equation. Let us assume that

T is a positive constant. We are interested in proving the existence and uniqueness of global solution u^n to the following integral equation:

$$\begin{aligned} u^n(t) &= S(t)u_0 + \int_0^t S(t-r)\theta_n\left(|u^n|_{X_r}\right) F\left(u^n(r)\right) dr \qquad (3.1.10) \\ & \frac{1}{2}\sum_{j=1}^{N}\int_n^t S(t-r)\theta_n\left(|u^n|_{X_r}\right)\kappa_j(u^n)dr \\ & +\sum_{j=1}^{N}\int_0^t S(t-r)\left(|u^n|_{X_r}\right) B_j\left(u^n(r)\right) dW_j(r),\ t\in[0,T]. \qquad (3.1.11) \end{aligned}$$

3.1.4 Solution spaces, assumptions and definition of Solution

By $\mathcal{L}(X,Y)$ we mean the space of all bounded linear operators from Banach X to the Banach space Y.

For any $b > a \geq 0$, let us denote $L^2(a,b;X)$ spaces of all equivalence classes of measurable functions u defined on $[a,b]$, taking values in a separable Banach spaces X such that:

$$|u|_{L^2(a,b;X)} := \left(\int_a^b |u(t)|_X^2\, dt\right)^{1/2} < \infty.$$

For $b > a \geq 0$ we set

$$X_{a,b} := L^2(a,b;E) \cap C([a,b];V),$$

then it can be shown easily that $\left(X_{a,b}, |\cdot|_{X_{a,b}}\right)$ is Banach space with norm,

$$|u|_{X_{a,b}}^2 = \sup_{t\in[a,b]} \|u(t)\|^2 + \int_a^b |u(t)|_E^2\, dt.$$

For $a = 0$ and $b = T > 0$ we are going to write $X_T := X_{0,T}$. Note that the map $t \mapsto |u|_{X_t}$ is increasing function.

Let $\left(\Omega, \mathbb{F}, \left(\mathcal{F}_t\right)_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space consisting of probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and filtration $\left(\mathcal{F}_t\right)_{t\geq 0}$ contained in $\mathbb{F}$. The filtered space satisfies the usual conditions i.e. the following conditions are satisfied:

i) $(\Omega, \mathbb{F}, \mathbb{P})$ is complete,

ii) The σ-algebras $\mathcal{F}_t$ contains the all sets in $\mathbb{F}$ with probability 0,

iii) The filtration is right continuous.

Assume that $N \in \mathbb{N}$. Let $\left(W_j(t)\right)_{j=1}^{N}, t \geq 0$ be the $\mathbb{R}^N$-valued $\mathbb{F}$-Wiener process. Throughout this dissertation, and in particular this chapter, we denote $M^2(X_T)$, the space of all E-valued progressively measurable processes u such that all trajectories of u almost surely belong to X_T. The norm on $M^2(X_T)$ is as under:

$$|u|^2_{M^2(X_T)} = \mathbb{E}\left(|u|^2_{X_T}\right) = \mathbb{E}\left(\sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T |u(t)|^2_E \, dt\right) < \infty. \tag{3.1.12}$$

Further we have the following main assumptions,

Assumption 3.1.5. *Let $E \subset V \subset H$ satisfy assumption 3.1.1. Assume that $S(t), t \in [0, \infty)$, is an analytic semigroup of bounded liner operators on space H, such that there exist positive constants C_0, C_1 and C_2:*

i) For every $T > 0$ and $f \in L^2(0, T; H)$ a function $u = S * f$ defined by:

$$u(t) = \int_0^T S(t-r)f(r)dt, \ t \in [0, T]$$

belongs to X_T and satisfies

$$|u|^2_{X_T} \leq C_1 |f|^2_{L^2(0,T;H)} \tag{3.1.13}$$

Note that $S* : f \longmapsto S * f$ is a linear and bounded map from $L^2(0, T; H)$ into X_T.

ii) For each $T > 0$, and every process $\xi \in M^2(0, T; H)$ a process $u = S\Diamond\xi$ defined by:

$$u(t) = \int_0^T S(t-r)\xi(r)dW(r), \ t \in [0, T]$$

belongs to $M^2(X_T)$ *and satisfies*

$$|u|^2_{M^2(X_T)} \leq C_2 |\xi|^2_{M^2(0,T;V)}. \tag{3.1.14}$$

Note that $S\diamondsuit : \xi \longmapsto S\diamondsuit\xi$ *is a linear and bounded map from* $M^2(0,T;H)$ *into* $M^2(X_T)$.

iii) For every $T > 0$ *and every* $u_0 \in V$, *a function* $u = Su_0$ *defined by:*

$$u(t) = S(t)u_0, \ t \in [0,T]$$

belongs to X_T *and satisfies*

$$|u|_{X_T} \leq C_0 \|u_0\|. \tag{3.1.15}$$

Recall the definition of accessible stopping time.

Definition 3.1.6. *An stopping time* τ *is called accessible if there exists an increasing sequence* $(\tau_m)_{m\in\mathbb{N}}$ *of stopping times such that, on set* $\{\tau > 0\}$, $\tau_m < \tau$ *and* $\lim_{m\to\infty} \tau_m = \tau$. *Such a sequence* $(\tau_m)_{m\in\mathbb{N}}$ *is called an approximating sequence for* τ.

Let us now define that what we mean by a local mild, local maximal solution and global solution of our main problem (3.1.6).

Definition 3.1.7. *(Local Mild Solution) Assume that we have been given,* V*-valued* F_0*-measurable random variable,* u_0 *with* $\mathbb{E}\,\|u_0\|^2 < \infty$.

A ***local mild solution*** *to problem* (3.1.6) *is a pair* (u,τ) *such that:*

i) τ *is an accessible stopping time,*

ii) $u : [0,\tau) \times \Omega \mapsto V$ *is an admissible process,*

iii) There exists an approximating sequence $(\tau_m)_{m\in\mathbb{N}}$ *of finite stopping times such that* $\tau_m < \tau$ *with* $\lim_{m\to\infty} \tau_m = \tau$. *For* $m \in \mathbb{N}$ *and* $t \geq 0$, *we have*

$$|u|^2_{X_{t\wedge\tau_m}} = \mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_m]} \|u(s)\|^2 + \int_0^{t\wedge\tau_m} |u(s)|^2_E\right) < \infty,$$

and

$$\begin{aligned} u(t\wedge\tau_m) &= S(t\wedge\tau_m)u_0+\int_0^{t\wedge\tau_m} S(t\wedge\tau_m-r)F\left(u(r)\right)dr \\ &+\frac{1}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_m} S(t\wedge\tau_m-r)\kappa_j\left(u\left(r\right)\right)dr \\ &+\sum_{j=1}^{N}\int_0^{t\wedge\tau_m} S(t\wedge\tau_m-r)B_j\left(u(r)\right)dW_j(r), \mathbb{P}\text{-}a.s., \end{aligned} \tag{3.1.16}$$

where B_j *and* κ_j *are as defined in* (3.1.5) *and* (3.1.7) *respectively.*

Definition 3.1.8. *(Local Maximal & Global Solution) Let* (u,τ_∞) *be a local solution to the problem* (3.1.6) *such that*

$$\lim_{t\to\tau_\infty}|u|_{X_t}=\infty\ \mathbb{P}\text{-}a.s.\ \ on\{\omega\in\Omega:\tau_\infty(\omega)<\infty\}\ \ a.s.\ .$$

Then (u,τ_∞) *is called* ***local maximal solution****. If* $\tau_\infty<\infty$ *with the positive probability, then* τ_∞ *is called the explosion time. Moreover, we are going to say that, a local maximal solution* (u,τ_∞) *is unique if for any other local maximal solution* (v,σ_∞) *we have* $\tau_\infty=\sigma_\infty$ *and* $u=v$ *on* $[0,\tau_\infty)$ $\mathbb{P}$*-a.s. . Finally, a local solution* (u,τ_∞) *is called* ***global solution*** *iff* $\tau_\infty=\infty$*.*

3.1.5 Existence and Uniqueness of Local Mild Solution

In order to prove the existence and uniqueness of local mild to our main problem (3.1.6), first we are going to study the existence and uniqueness of the solution to approximate evolution equation (3.1.11). Let us fix T as the some positive real number. We are interested in proving the existence and uniqueness solution u^n to equation (3.1.11). All the results proven in this section will be in abstract E,V and H spaces satisfying Assumption 3.1.1. We will see in this section that existence and uniqueness of the Local solution to approximate equation (3.1.11) enable us to construct the local mild solution to original problem (3.1.6). Moreover, in next section we are going to construct a global solution to approximate evolution equation (3.1.11). This global solution enables us to prove the existence of a unique local maximal solution to main problem (3.1.6).

For the existence of the local solution to approximate equation (3.1.11), we will construct three globally Lipschitz maps by truncating the functions F, κ and B, involved in drift and diffusion of problem (3.1.6). Then by using these three globally Lipschitz maps, we will construct a contraction and hence the existence and uniqueness of local solution are guaranteed by Banach fixed point theorem. Next we will see that for measurable and V-valued square integrable data u_0, the solution Local solution of approximate equation agrees with the Local mild solution of our main problem (3.1.6). In the next subsection, we will show that approximate equation admits a global solution u^n. The section ends at the show that global solution u^n allows us to construct a unique local maximal solution of main problem (3.1.6).

3.1.6 Important Estimates

The aim of the subsection is to show that non-linear functions F, κ and B, involved in drift and diffusion terms of main stochastic evolution equation (3.1.6) are locally Lipschitz and satisfy the symmetric estimates.

Recall the following well known Gagliardo-Nirenberg-Sobolev inequality from Chapter 1.

Lemma 3.1.9. *[47] Assume that $r, q \in [1, \infty)$, and $j, m \in \mathbb{Z}$ satisfy $0 \leq j < m$.*

Then for any $u \in C_0^\infty\left(\mathbb{R}^d\right)$

$$\left|D^j u\right|_{L^p\left(\mathbb{R}^d\right)} \leq C\left|D^m u\right|^a_{L^r\left(\mathbb{R}^d\right)} |u|^{1-a}_{L^q\left(\mathbb{R}^d\right)}, \tag{3.1.17}$$

where $\frac{1}{p} = \frac{j}{d} + a\left(\frac{1}{r} - \frac{m}{d}\right) + (1-a)\frac{1}{q}$ *for all* $a \in \left[\frac{j}{m}, 1\right]$*. If* $m - j - \frac{d}{r}$ *is a nonnegative integer, then the equality* (3.1.17) *holds only for* $a \in \left[\frac{j}{m}, 1\right)$*.*

Observe that our projected heat flow problem (3.1.6) involves L^{2n} norm, therefore at several instances throughout this chapter we will need the following particular case of Gagliardo-Nirenberg-Sobolev inequality.

For our case we choose $r = q = 2, j = 0, m = 1, d = 2$, and $p = 2n$, so

$$\begin{aligned} \frac{1}{p} &= \frac{0}{2} + a\left(\frac{1}{2} - \frac{1}{2}\right) + (1-a)\frac{1}{2} \\ \frac{1}{2n} &= (1-a)\frac{1}{2} \\ \frac{1}{n} &= 1 - a \text{ or } a = 1 - \frac{1}{n}. \end{aligned}$$

Plugging values of r, q, j, m, d and p in inequality (3.1.17) we get

$$\begin{aligned} |u|_{L^{2n}(\mathbb{R}^2)} &\leq C\left|\nabla u\right|^a_{L^2(\mathbb{R}^2)} |u|^{1-a}_{L^2(\mathbb{R}^2)} \\ |u|_{L^{2n}(\mathbb{R}^2)} &\leq C\left\|u\right\|^a |u|^{1-a}_H, \text{where } a = 1 - \frac{1}{n}. \end{aligned} \tag{3.1.18}$$

Remark 3.1.10. *i) From Remark 3.1.2 we know that embedding* $V \hookrightarrow H$ *i.e. there exists* $c > 0$ *such that*

$$|u|_H \leq c\left\|u\right\|$$

hence inequality (3.1.18)*, simplifies to*

$$|u|_{L^{2n}(\mathbb{R}^2)} \leq C_*\left\|u\right\|. \tag{3.1.19}$$

where $C_* := cC$*. The last inequality reflects the fact that* $V \hookrightarrow L^{2n}\left(\mathbb{R}^2\right)$*, where* $n \in \mathbb{N}$*.*

Recall the following Lemma 2.1 from chapter 1, which provides provide an estimate for the non-linear term F, that appears in drift term of stochastic equation (3.1.6).

Lemma 3.1.11. *Consider map $F : V \to H$ is the map defined by $F(u) = \|u\|^2 u - u^{2n-1} + u\,|u|_{L^{2n}}^{2n}$. Then there exists a constant $C > 0$ such that*

$$|F(u) - F(v)|_H \leq G(\|u\|, \|v\|)\, \|u - v\|, \;\; u, v \in V, \tag{3.1.20}$$

where $G : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a bounded and symmetric map, defined as

$$\begin{aligned} G(r, s) \;\; : \;\; &= C^2 \left(r^2 + s^2\right) + (r + s)^2 + C^{2n+1} \left[\begin{array}{c} \binom{2n-1}{2} \left(r^{2n-1} + s^{2n-1}\right)(r + s) \\ + \left(r^{2n} + s^{2n}\right) \end{array} \right] \\ &+ C^{2n-1} \binom{2n-2}{2} \left(r^{2n-2} + s^{2n-2}\right). \end{aligned}$$

Next lemma gives the estimate for the diffusion (stochastic) term of stochastic equation (3.1.6).

Proposition 3.1.12. *i) For $f \in H$, consider the map $B : H \to H$, defined by*

$$B(u) = f - \langle f, u \rangle\, u. \tag{3.1.21}$$

Then for all $u_1, u_2 \in H$, the map B is locally Lipschitz i.e.

$$|B(u_1) - B(u_2)|_H \leq |f|_H \left(|u_1|_H + |u_2|_H\right) |u_1 - u_2|_H \tag{3.1.22}$$

ii) If an addition $f \in V$, then such that,

$$\|B(u_1) - B(u_2)\| \leq \|f\| \left(\|u_1\| + \|u_1\|\right) \|u_1 - u_2\|. \tag{3.1.23}$$

Proof. Let us fix $u_1, u_2 \in H$, we have the following chain of equation/inequalities

$$\begin{aligned}
|B(u_1) - B(u_2)|_H &= |f - \langle f, u_1\rangle u_1 - f + \langle f, u_2\rangle u_2|_H \\
&= |\langle f, u_1\rangle u_1 - \langle f, u_1\rangle u_2 + \langle f, u_1\rangle u_2 - \langle f, u_2\rangle u_2|_H \\
&= |\langle f, u_1\rangle| \, |u_1 - u_2|_H + |\langle f, u_1\rangle - \langle f, u_2\rangle| \, |u_2|_H \\
&= |\langle f, u_1\rangle| \, |u_1 - u_2|_H + |\langle f, u_1 - u_2\rangle| \, |u_2|_H \\
&\leq |f|_H \, |u_1|_H \, |u_1 - u_2|_H + f_{jH} \, |u_1 - u_2|_H \, |u_2|_H \\
&= |f|_H \left(|u_1|_H + |u_2|_H\right) |u_1 - u_2|_H \, . \qquad (3.1.24)
\end{aligned}$$

This completes the proof.

ii) Proof goes exactly on same lines of part i). ■

In the following lemma, we are going to compute the Frećhet derivative $d_u B$, which we will use subsequent lemma.

Lemma 3.1.13. *If $B : H \to H$ be the map defined in Lemma 3.1.12 with $f \in H$. For $u \in H$, the Frećhet derivative $d_u B$, exists and satisfies,*

$$d_u B\,(h) := -\langle f, u\rangle h - \langle f, h\rangle u, \text{ for all } h \in H. \qquad (3.1.25)$$

Proof. Consider the following expression

$$\begin{aligned}
B\,(h + u) - B\,(u) &= f - \langle f, u + h\rangle (u + h) - f + \langle f, u\rangle u \\
&= \langle f, u\rangle u - (\langle f, u\rangle + \langle f, h\rangle) u - (\langle f, u\rangle + \langle f, h\rangle) h \\
&= \langle f, u\rangle u - \langle f, u\rangle u - \langle f, h\rangle u - \langle f, u\rangle h - \langle f, h\rangle h \\
&= -\langle f, h\rangle u - \langle f, u\rangle h + o(|h|_H)
\end{aligned}$$

$$\lim_{|h|_H \to 0} \frac{|B\,(h + u) - B\,(u) - (\langle f, h\rangle u + \langle f, u\rangle h)|}{|h|_H} \leq \lim_{|h|_H \to 0} \frac{o(|h|_H)}{|h|_H} \to 0$$

and hence the Frećhet derivative $d_u B$ satisfies (3.1.25). This completes the proof.

■

In the next Lemma we are going to prove that Frechet derivative $\kappa(\cdot) \equiv d_u B(B(\cdot))$, is locally Lipschitz and hence satisfies some useful estimate.

Proposition 3.1.14. *Assume the framework of Lemma 3.1.13. If κ is as defined in equation (3.1.7) Then*

i) For all $u_1, u_2 \in H$, the following inequality holds,

$$|\kappa(u_1) - \kappa(u_2)|_H \le |f|_H^2 \left[2 + |u_1|_H^2 + |u_2|_H^2 + (|u_1|_H + |u_2|_H)^2\right] |u_1 - u_2|_H .$$

ii) For all $u_1, u_2 \in V$, the following inequality holds,

$$\|\kappa(u_1) - \kappa(u_2)\| \le C \|f\|^2 \left[2 + \|u_1\|^2 + \|u_2\|^2 + (\|u_1\| + \|u_2\|)^2\right] \|u_1 - u_2\| .$$

Proof. Let us choose and fix $u_1, u_2 \in H$. Then we have the following chain of equations/inequalities

$$\begin{aligned}
|\kappa(u_1) - \kappa(u_2)|_H &= |d_{u_1} B(B(u_1)) - d_{u_2} B(B(u_2))|_H \\
&= |-\langle f, u_1\rangle B(u_1) - \langle f, B(u_1)\rangle u_1 + \langle f, u_2\rangle B(u_2) + \langle f, B(u_2)\rangle u_2|_H \\
&= |\langle f, u_2\rangle B(u_2) - \langle f, u_1\rangle B(u_1) + \langle f, B(u_2)\rangle u_2 - \langle f, B(u_1)\rangle u_1|_H \\
&= \left| \begin{array}{c} \langle f, u_2\rangle B(u_2) - \langle f, u_2\rangle B(u_1) + \langle f, u_2\rangle B(u_1) - \langle f, u_1\rangle B(u_1) \\ + \langle f, B(u_2)\rangle u_2 - \langle f, B(u_2)\rangle u_1 + \langle f, B(u_2)\rangle u_1 - \langle f, B(u_1)\rangle u_1 \end{array} \right|_H \\
&= \left| \begin{array}{c} \langle f, u_2\rangle (B(u_2) - B(u_1)) + (\langle f, u_2\rangle - \langle f, u_1\rangle) B(u_1) \\ + \langle f, B(u_2)\rangle (u_2 - u_1) + (\langle f, B(u_2)\rangle - \langle f, B(u_1)\rangle) u_1 \end{array} \right|_H \\
&\le |\langle f, u_2\rangle| \, |B(u_2) - B(u_1)|_H + |\langle f, u_2 - u_1\rangle| \, |B(u_1)|_H \\
&\quad + |\langle f, B(u_2)\rangle| \, |u_1 - u_2|_H + |\langle f, B(u_2) - B(u_2)\rangle| \, |u_1|_H
\end{aligned}$$

By the Cauchy Schwartz inequality,

$$\begin{aligned}
|\kappa(u_1) - \kappa(u_2)|_H &\le |B(u_2) - B(u_1)|_H |f|_H |u_2|_H + |f|_H |u_1 - u_2|_H |B(u_1)|_H \\
&\quad + |f|_H |u_1 - u_2|_H |B(u_2)|_H + |B(u_2) - B(u_1)|_H |f|_H |u_1|_H \\
&= |f|_H (|B(u_1)|_H + |B(u_2)|_H) |u_1 - u_2|_H \\
&\quad + |f|_H (|u_1|_H + |u_2|_H) |B(u_2) - B(u_1)|_H
\end{aligned}$$

Using the definition of B, inequality (3.1.24) and the Cauchy Schwartz inequality in last inequality,

$$\begin{aligned} |\kappa(u_1) - \kappa(u_2)|_H &\leq |f|_H \left(|f - \langle f, u_1\rangle u_1|_H + |f - \langle f, u_2\rangle u_2|_H\right) |u_1 - u_2|_H \\ &\quad + |f|_H^2 \left(|u_1|_H + |u_2|_H\right)^2 |u_1 - u_2|_H \\ &\leq |f|_H \left(|f|_H + |f|_H |u_1|_H^2 + |f|_H + |f|_H |u_2|_H^2\right) |u_1 - u_2|_H \\ &\quad + |f|_H^2 \left(|u_1|_H + |u_2|_H\right)^2 |u_1 - u_2|_H \\ &= |f|_H^2 \left[2 + |u_1|_H^2 + |u_2|_H^2 + \left(|u_1|_H + |u_2|_H\right)^2\right] |u_1 - u_2|_H \end{aligned}$$

This completes the proof of part i).

ii) On the same lines of part i). ■

Remark 3.1.15. *From last proposition we know that map κ, given by equation 3.1.25, satisfies*

$$|\kappa(u_1) - \kappa(u_2)|_H \leq |f|_H^2 \left[2 + |u_1|_H^2 + |u_2|_H^2 + \left(|u_1|_H + |u_2|_H\right)^2\right] |u_1 - u_2|_H .$$

Now since the embedding $V \hookrightarrow H$ is continuous so there exists a constant c such that $|u|_H \leq c\,\|u\|$, hence there exists a constant C such that the last inequality becomes

$$|\kappa(u_1) - \kappa(u_2)|_H \leq C\,\|f\|^2 \left[2 + \|u_1\|^2 + \|u_2\|^2 + \left(\|u_1\| + \|u_2\|\right)^2\right] \|u_1 - u_2\| . \tag{3.1.26}$$

3.1.7 Existence and Uniqueness of Local mild solution of stochastic evolution equations

In this subsection, we aim to prove existence and uniqueness of local mild solutions of truncated (3.1.11) and original evolution equations (3.1.6). In the last subsection we saw that maps F and κ are locally Lipschitz and satisfy some symmetric estimates, next abstract result is going to help us to assert that truncated F and κ are globally Lipschitz.

Proposition 3.1.16. *Assume that Banach spaces E, V and H satisfy Assumption 3.1.1. Assume that $Z : V \to H$ be a function such that $Z(0) = 0$ and satisfies the following inequality, for all $u_1, u_2 \in V$,*

$$|Z(u_1) - Z(u_2)|_H \leq G(\|u_1\|, \|u_2\|) \|u_1 - u_2\|. \tag{3.1.27}$$

where $G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a symmetric function bounded on compacts i.e. for all $K > 0$ there exists $C = C_K$ such that

$$|G(s,t)| \leq C_K \text{ for all } s, t \in [0, R]. \tag{3.1.28}$$

Assume that $\theta : \mathbb{R}_+ \to [0, 1]$ is a non-increasing smooth function satisfying (3.1.8) and (3.1.9). Define a map $\Phi^n_{T,Z} : X_T \to L^2(0, T; H)$ by

$$\left(\Phi^n_{T,Z} u\right)(t) = \theta_n(|u|_{X_t}) Z(u(t)), \text{ where } t \in [0, T] \text{ and } X_T = L^2(0, T; E) \cap C(0, T; V).$$

Then $\Phi^n_{T,Z}$ is globally Lipschitz. Moreover there exists $D_n > 0$ such that, for all $u_1, u_2 \in X_T$

$$\left|\Phi^n_{T,Z}(u_1) - \Phi^n_{T,Z}(u_2)\right|_{L^2(0,T;H)} \leq D_n |u_1 - u_2|_{X_T} T^{\frac{1}{2}}, \tag{3.1.29}$$

where $D_n := 2|G(2n, 0)| + G(2n, 2n)$.

Proof. We start by showing that $\Phi^n_{T,F}$ is well-defined. Let $u \in X_T$ then

$$\begin{aligned} \left|\Phi^n_{T,Z}(u)\right|^2_{L^2(0,T;H)} &= \left|\theta_n(|u|_{X_t}) Z(u(t))\right|^2_{L^2(0,T;H)} \\ &= \int_0^T \left|\theta_n(|u|_{X_t}) Z(u(t))\right|^2_H dt. \end{aligned}$$

Since $|\theta|^2 \leq 1$ so $|\theta_n|^2 \leq 1$, we infer that

$$\begin{aligned} \left|\Phi^n_{T,Z}(u)\right|^2_{L^2(0,T;H)} &\leq \int_0^T |Z(u(t))|^2_H dt \\ &\leq \int_0^T \|u(t)\|^2 |G(\|u(t)\|, 0)|^2 dt. \end{aligned}$$

Since $X_T \subset C\left([0,T]\,;V\right)$ so $\|u(t)\| \le |u|_{X_T} =: K < \infty$, for all $t \in [0,T]$. Also from (3.1.28) $|G\left(\|u(t)\|,0\right)| \le C_K$. It follows from above last inequality

$$\begin{aligned}\left|\Phi^n_{T,Z}(u)\right|^2_{L^2(0,T;H)} &\le \int_0^T \|u\left(t\right)\|^2 \left|G\left(\|u\left(t\right)\|,0\right)\right|^2 dt \\ &\le \int_0^T K^2 C_K^2 dt = K^2 C_K^2 T < \infty.\end{aligned}$$

and hence $\Phi^n_{T,Z}$ is well-defined. Let us fix $u_1, u_2 \in X_T$. Set

$$\tau_i = \inf\left\{t \in [0,T] : |u_i|_{X_t} \ge 2n\right\}, i = 1,2.$$

WLOG we can assume that $\tau_1 \le \tau_2$. Consider

$$\begin{aligned}\left|\Phi^n_{T,Z}(u_1) - \Phi^n_{T,Z}(u_2)\right|_{L^2(0,T;H)} &= \left[\int_0^T \left|\Phi^n_{T,Z}(u_1(t)) - \Phi^n_{T,Z}(u_2(t))\right|^2_H dt\right]^{\frac{1}{2}}, \\ &= \left[\int_0^T \left|\theta_n(|u_1|_{X_t})Z(u_1(t)) - \theta_n(|u_2|_{X_t})Z(u_2(t))\right|^2_H dt\right]^{\frac{1}{2}}.\end{aligned}$$

For $i = 1,2$, $\theta_n(|u_i|_{X_t}) = 0$ and for $t \ge \tau_2$

$$\begin{aligned}\left|\Phi^n_{T,Z}(u_1) - \Phi^n_{T,Z}(u_2)\right|_{L^2(0,T;H)} &= \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})Z(u_1(t)) - \theta_n(|u_2|_{X_t})Z(u_2(t))\right|^2_H\right]^{\frac{1}{2}}, \\ &= \left[\int_0^{\tau_2} \left|\begin{array}{c}\theta_n(|u_1|_{X_t})Z(u_1(t)) - \theta_n(|u_1|_{X_t})Z(u_2(t)) \\ +\theta_n(|u_1|_{X_t})Z(u_2(t)) - \theta_n(|u_2|_{X_t})Z(u_2(t))\end{array}\right|^2_H\right]^{\frac{1}{2}}, \\ &= \left[\int_0^{\tau_2} \left|\begin{array}{c}\theta_n(|u_1|_{X_t})\left(Z(u_1(t)) - Z(u_2(t))\right) \\ +\left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) Z(u_2)(t)\end{array}\right|^2_H\right]^{\frac{1}{2}}.\end{aligned}$$

Using Minkowski's inequality we infer that

$$\begin{aligned}\left|\Phi^n_{T,Z}(u_1) - \Phi^n_{T,Z}(u_2)\right|_{L^2(0,T;H)} &\le \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})\left(Z(u_1(t)) - Z(u_2(t))\right)\right|^2_H dt\right]^{\frac{1}{2}} \\ &\quad + \left[\int_0^{\tau_2} \left|\left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) Z(u_2(t))\right|^2_H dt\right]^{\frac{1}{2}}.\end{aligned} \tag{3.1.30}$$

Set

$$\begin{aligned} A &: = \left[\int_0^{\tau_2} \left|\left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) Z(u_2(t))\right|_H^2\right]^{\frac{1}{2}} dt, \\ B &: = \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})\left(Z(u_1(t)) - Z(u_2(t))\right)\right|_H^2\right]^{\frac{1}{2}} dt. \end{aligned}$$

and hence the inequality (3.1.30) can be rewritten as

$$\left|\Phi^n_{T,Z}(u_1) - \Phi^n_{T,Z}(u_2)\right|_{L^2(0,T;H)} \leq A + B. \tag{3.1.31}$$

Since the function θ_n is Lipschitz we infer that

$$\begin{aligned} A^2 &= \int_0^{\tau_2} \left|\left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) Z(u_2(t))\right|_H^2 dt \\ &\leq 4n^2 \int_0^{\tau_2} \left||u_1|_{X_t} - |u_2|_{X_t}\right|_H^2 |Z(u_2(t))|_H^2 dt \\ &\leq 4n^2 \int_0^{\tau_2} \left||u_1|_{X_t} - |u_2|_{X_t}\right|_H^2 |Z(u_2(t))|_H^2 dt \\ &\leq 4n^2 \int_0^{\tau_2} |u_1 - u_2|_{X_t}^2 |Z(u_2(t))|_H^2 dt \\ &\leq 4n^2 |u_1 - u_2|_{X_T}^2 \int_0^{\tau_2} |Z(u_2(t))|_H^2 dt. \end{aligned} \tag{3.1.32}$$

Next we want to estimate the integral the last inequality. By use of inequality (3.1.27)

$$\begin{aligned} \int_0^{\tau_2} |Z(u_2(t))|_H^2 dt &\leq \int_0^{\tau_2} \|u_2(t)\|^2 \left|G\left(\|u_2(t)\|, 0\right)\right|^2 dt \\ &\leq \sup_{t\in[0,\tau_2)} \|u_2(t)\|^2 \int_0^{\tau_2} \left|G\left(\|u_2(t)\|, 0\right)\right|^2 dt. \end{aligned}$$

Since $|u_2|_{X_{\tau_2}}^2 = \sup_{t\in[0,\tau_2]} \|u_2(t)\|^2 + \int_0^{\tau_2} |u_2(t)|_E^2$ therefore $\sup_{t\in[0,\tau_2]} \|u_2(t)\|^2 \leq |u_2|_{X_{\tau_2}}^2 \leq (2n)^2$. Thus the last inequality takes the following form

$$\int_0^{\tau_2} |Z(u_2(t))|_H^2 dt \leq (2n)^2 \int_0^{\tau_2} \left|G\left(\|u_2(t)\|, 0\right)\right|^2 dt = (2n)^2 \left|G\left(2n, 0\right)\right|^2 \tau_2$$

Using the last inequality in (3.1.32) we get

$$\begin{aligned} A^2 &\leq (2n)^4 \left|G(2n,0)\right|^2 \tau_2 \left|u_1-u_2\right|^2_{X_T} \\ &\leq (2n)^4 \left|G(2n,0)\right|^2 \left|u_1-u_2\right|^2_{X_T} T \\ A &\leq A_n \left|u_1-u_2\right|_{X_T} T^{\frac{1}{2}}, \end{aligned} \tag{3.1.33}$$

where $A_n = (2n)^4 \left|G(2n,0)\right|^2$. Since $\theta_n(|u_1|_{X_t}) = 0$ for $t \geq \tau_1$ and $\tau_1 \leq \tau_2$, we have

$$\begin{aligned} B &= \left[\int_0^{\tau_2} \left|\theta_n(|u_1|_{X_t})\left(Z(u_1(t)) - Z(u_2(t))\right)\right|_H^2 dt\right]^{\frac{1}{2}} \\ &= \left[\int_0^{\tau_1} \left|\theta_n(|u_1|_{X_t})\left(Z(u_1(t)) - Z(u_2(t))\right)\right|_H^2 dt\right]^{\frac{1}{2}}. \end{aligned}$$

Also since $\theta_n(|u_1|_{X_t}) \leq 1$ for $t \in [0,\tau_1)$ we infer that

$$B \leq \left[\int_0^{\tau_1} \left|\left(Z(u_1(t)) - Z(u_2(t))\right)\right|_H^2 dt\right]^{\frac{1}{2}}.$$

Using inequality (3.1.27)

$$\begin{aligned} B^2 &\leq \int_0^{\tau_1} \left[\|u_1(t)-u_2(t)\| \, G\left(\|u_1(t)\|, \|u_2(t)\|\right)\right]^2 dt \\ &\leq \sup_{t\in[0,\tau_1)} \|u_1(t)-u_2(t)\|^2 \int_0^{\tau_1} \left[G\left(\|u_1(t)\|, \|u_2(t)\|\right)\right]^2 dt. \end{aligned} \tag{3.1.34}$$

Using the fact that, $\sup_{t\in[0,\tau_1)} \|u_1(t)-u_2(t)\|^2 \leq |u_1-u_2|^2_{X_{\tau_1}}$, and using $\sup_{t\in[0,\tau_1)} \|u_i(t)\|^2 \leq |u_i|_{X_{\tau_i}} \leq 2n$, $i = 1,2$, the last inequality takes form

$$\begin{aligned} B^2 &\leq |u_1-u_2|^2_{X_{\tau_1}} G^2(2n,2n) \int_0^{\tau_1} dt \\ &\leq \tau_1 G^2(2n,2n) \left|u_1-u_2\right|^2_{X_T} \\ &\leq B_n \left|u_1-u_2\right|^2_{X_T} T, \end{aligned}$$

where $B_n^2 = G^2(2n,2n)$. Thus

$$B \leq B_n T^{\frac{1}{2}} \left|u_1-u_2\right|_{X_T}$$

Using the last inequality together with inequality (3.1.33) in (3.1.31), we get

$$\left|\Phi^n_{T,Z}(u_1) - \Phi^n_{T,Z}(u_2)\right|_{L^2(0,T;H)} \leq (A_n + B_n) \left|u_1 - u_2\right|_{X_T} T^{\frac{1}{2}} = D_n \left|u_1 - u_2\right|_{X_T} T^{\frac{1}{2}},$$

where $D_n := (A_n + B_n)$. This completes the proof of the theorem. ∎

Corollary 3.1.17. *Assume all assumptions of the Proposition 3.1.16 and the map F be map as defined in the Lemma 3.1.11. Recall that $X_T = L^2(0,T;E) \cap C(0,T;V)$. Let $\Phi^n_{T,F} : X_T \to L^2(0,T;H)$ be a map defined by,*

$$\Phi^n_{T,F}(u(t)) := \theta_n(|u|_{X_t}) F(u(t)).$$

Then there exists $C_n > 0$ such that,

$$\left|\Phi^n_{T,F}(u_1) - \Phi^n_{T,F}(u_2)\right|_{L^2(0,T;H)} \leq C_n \left|u_1 - u_2\right|_{X_T} T^{\frac{1}{2}}, \text{ where } u_1, u_2 \in V. \quad (3.1.35)$$

Proof. The result follows directly from Proposition 3.1.16. Indeed, by Lemma 3.1.11 F satisfies the inequality (3.1.27) i.e.

$$|F(u) - F(v)|_H \leq G(\|u\|, \|v\|) \|u - v\|,$$

where $G : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a bounded and symmetric map, defined as

$$\begin{aligned} G(r,s) \;:\; &= C^2 \left(r^2 + s^2\right) + (r+s)^2 + C^{2n+1} \left[\begin{array}{c} \left(\frac{2n-1}{2}\right) \left(r^{2n-1} + s^{2n-1}\right)(r+s) \\ + \left(r^{2n} + s^{2n}\right) \end{array} \right] \\ &+ C^{2n-1} \left(\frac{2n-2}{2}\right) \left(r^{2n-2} + s^{2n-2}\right). \end{aligned}$$

Take $Z \equiv F$. Clearly the map G is symmetric and being polynomial G is bounded. Hence the required inequality follows from the Proposition 3.1.16. ∎

Before mentioning next two results probably it will be better to recall some definition.

For a given $f \in V$, recall that B is a vector field on H defined by

$$B(u) = f - \langle f, u \rangle u, \text{ for all } u \in H.$$

Recall from Proposition 3.1.12 the Frećhet derivative of map B at u i.e. $\kappa \equiv d_u B$, can be given as

$$\kappa(h) = -\langle f, u \rangle h - \langle f, h \rangle u, \text{ for all } h \in H.$$

And obviously $X_T = L^2(0, T; E) \cap C(0, T; V)$.

Corollary 3.1.18. *Let B and κ be maps described above and κ satisfies the inequality* (3.1.14). *Define that $\Lambda_{\kappa,T} : X_T \to L^2(0, T; H)$ by,*

$$[\Lambda_{\kappa,T}(u)](t) = \theta_n\left(|u|_{X_t}\right) \kappa(u(t)), t \in [0, T].$$

Then $\Lambda_{\kappa,T}$ is globally Lipschitz and moreover there exists $K_n > 0$ such that,

$$|\Lambda_{\kappa,T}(u_1) - \Lambda_{\kappa,T}(u_2)|_{L^2(0,T;H)} \leq K_n |u_1 - u_2|_{X_T} T^{1/2}, \; u_1, u_2 \in X_T. \quad (3.1.36)$$

Proof. The result follows directly from Proposition 4.2.10. From inequality (3.1.26) we know that

$$|\kappa(u) - \kappa(v)|_H \leq G(\|u\|, \|v\|) \|u - v\|$$

Where $G : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a bounded and symmetric map, defined as

$$G(r, s) = C \|f\|^2 \left[2 + r^2 + s^2 + (r + s)^2\right]$$

Take $Z \equiv \kappa$. Clearly the map G is symmetric and being polynomial G is bounded. Hence the required inequality follows from the Proposition 4.2.10. ■

In the next result, we will show that truncated diffusion term i.e. B in main problem (3.1.6) is globally Lipschitz.

Proposition 3.1.19. *Assume that all assumptions of Proposition 3.1.16. Let us assume that B is defined by* (3.1.21)*. Define a map $\Phi_{B,T} : X_T \to L^2(0,T;V)$ by*

$$[\Phi_{B,T}u](t) = \theta_n\left(|u|_{X_t}\right) B(u(t)),\ t \in [0,T].$$

Then $\Phi_{B,T}$ is globally Lipschitz and there exists $M_n > 0$ such that

$$|\Phi_{B,T}(u_1) - \Phi_{B,T}(u_2)|_{L^2(0,T;V)} \le M_n |u_1 - u_2|_{X_T} T^{\frac{1}{2}},\ u_1, u_2 \in X_T. \tag{3.1.37}$$

Proof. We start by showing that $\Phi_{B,T}$ is well-defined. Pick and fix $u \in X_T$ then

$$\begin{aligned} |(\Phi_{B,T}(u))(t)|^2_{L^2(0,T;V)} &= \left|\theta_n(|u|_{X_t}) B(u(t))\right|^2_{L^2(0,T;V)}, \\ &= \int_0^T \left\|\theta_n(|u|_{X_t}) B(u(t))\right\|^2 dt. \end{aligned}$$

By definition of θ we infer that $|\theta|^2 \le 1$ and so it follows that $|\theta_n|^2 \le 1$. Therefore using inequality (3.1.23) in last equation,

$$\begin{aligned} |\Phi_{B,T}(u)|^2_{L^2(0,T;V)} &\le \int_0^T \|B(u(t))\|^2 dt, \\ &\le C^2 |f|^2_H \int_0^T \|u(t)\|^4 dt. \end{aligned}$$

Since $u \in C([0,T];V)$ there exists $K > 0$ s.t. $\|u(t)\| \le K$ for all $t \in [0,T]$, so last inequality becomes

$$|\Phi_{B,T}(u)|^2_{L^2(0,T;V)} \le C^2 |f|^2_H \int_0^T K^4 dt = C^2 |f|^2_H K^4 T < \infty.$$

Hence $\Phi_{B,T}$ is well-defined. Next we will show that $\Phi_{B,T}$ is globally Lipschitz, for this choose and fix $u_1, u_2 \in X_T$. Set

$$\tau_i := \inf\left\{t \in [0,T] : |u_i|_{X_T} \ge 2n\right\}, i = 1, 2.$$

WLOG we can assume that $\tau_1 \le \tau_2$. Consider

$$\begin{aligned} |\Phi_{B,T}(u_1) - \Phi_{B,T}(u_2)|_{L^2(0,T;V)} &= \left[\int_0^T \|[\Phi_{B,T}(u_1)](t) - [\Phi_{B,T}(u_2)](t)\|^2 dt\right]^{\frac{1}{2}}, \\ &= \left[\int_0^T \left\|\theta_n(|u_1|_{X_t}) B(u_1(t)) - \theta_n(|u_2|_{X_t}) B(u_2(t))\right\|^2 dt\right]^{\frac{1}{2}}. \end{aligned}$$

Since $\theta_n(|u_i|_{X_t}) = 0$ for $t \geq \tau_2$, $i = 1, 2$ we infer that

$$\begin{aligned}
|\Phi_{B,T}(u_1) - \Phi_{B,T}(u_2)|_{L^2(0,T;V)} &= \left[\int_0^{\tau_2} \left\|\theta_n(|u_1|_{X_t})B(u_1(t)) - \theta_n(|u_2|_{X_t})B(u_2(t))\right\|^2\right]^{\frac{1}{2}}, \\
&= \left[\int_0^{\tau_2} \left\| \begin{array}{c} \theta_n(|u_1|_{X_t})B(u_1(t)) - \theta_n(|u_1|_{X_t})B(u_2(t)) \\ +\theta_n(|u_1|_{X_t})B(u_2(t)) - \theta_n(|u_2|_{X_t})B(u_2(t)) \end{array} \right\|^2\right]^{\frac{1}{2}}, \\
&= \left[\int_0^{\tau_2} \left\| \begin{array}{c} \theta_n(|u_1|_{X_t})\left(B(u_1(t)) - B(u_2(t))\right) \\ +\left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) B(u_2)(t) \end{array} \right\|^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Therefore, by using the Minkowski inequality we get,

$$\begin{aligned}
|\Phi_{B,T}(u_1) - \Phi_{B,T}(u_2)|_{L^2(0,T;V)} &\leq \left[\int_0^{\tau_2} \left\|\theta_n(|u_1|_{X_t})\left(B(u_1(t)) - B(u_2(t))\right)\right\|^2 dt\right]^{\frac{1}{2}}, \\
&\quad + \left[\int_0^{\tau_2} \left\|\left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) B(u_2(t))\right\|^2 dt\right]^{\frac{1}{2}}.
\end{aligned} \tag{3.1.38}$$

Set

$$\begin{aligned}
A &:= \left[\int_0^{\tau_2} \left\|\left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) B(u_2(t))\right\|^2\right]^{\frac{1}{2}} dt, \\
B &:= \left[\int_0^{\tau_2} \left\|\theta_n(|u_1|_{X_t})\left(B(u_1(t)) - B(u_2(t))\right)\right\|^2\right]^{\frac{1}{2}} dt.
\end{aligned}$$

With the notation previous inequality reads,

$$|\Phi_{B,T}(u_1) - \Phi_{B,T}(u_2)|_{L^2(0,T;V)} \leq A + B. \tag{3.1.39}$$

Since θ_n is Lipschitz constant so

$$\begin{aligned}
A^2 &= \int_0^{\tau_2} \left\| \left(\theta_n(|u_1|_{X_t}) - \theta_n(|u_2|_{X_t})\right) B(u_2(t)) \right\|^2 dt, \\
&\le \frac{1}{n^2} \int_0^{\tau_2} \left\| \left| |u_1|_{X_t} - |u_2|_{X_t} \right|^2 B(u_2(t)) \right\|^2 dt, \\
&\le \frac{1}{n^2} \int_0^{\tau_2} \left| |u_1|_{X_t} - |u_2|_{X_t} \right|^2 \|B(u_2(t))\|^2 dt, \\
&\le \frac{1}{n^2} \int_0^{\tau_2} |u_1 - u_2|_{X_t}^2 \|B(u_2(t))\|^2 dt, \\
&\le \frac{1}{n^2} |u_1 - u_2|_{X_T}^2 \int_0^{\tau_2} \|B(u_2(t))\|^2 dt. \qquad (3.1.40)
\end{aligned}$$

By use of inequalities (3.1.23),

$$\begin{aligned}
\int_0^{\tau_2} \|B(u_2(t))\|^2 dt &\le C^2 |f|_H^2 \int_0^{\tau_2} \|u_2(t)\|^4 dt, \\
&\le C^2 |f|_H^2 \left(\sup_{t\in[0,\tau_2)} \|u_2(t)\|^2 \right)^2 \int_0^{\tau_2} dt, \\
&= C^2 |f|_H^2 \left(\sup_{t\in[0,\tau_2)} \|u_2(t)\|^2 \right)^2 \tau_2.
\end{aligned}$$

Since $|u_2|_{X_{\tau_2}}^2 = \sup_{t\in[0,\tau_2]} \|u_2(t)\|^2 + \int_0^{\tau_2} |u_2(t)|_E^2$ we infer that $\sup_{t\in[0,\tau_2]} \|u_2(t)\|^2 \le |u_2|_{X_{\tau_2}}^2$, and using fact that $|u_2|_{X_{\tau_2}} \le 2n$, the last inequality takes form

$$\begin{aligned}
\int_0^{\tau_2} \|B(u_2(t))\|^2 dt &\le C^2 |f|_H^2 \left(\sup_{t\in[0,\tau_2)} \|u_2(t)\|^2 \right)^2 \tau_2 \\
&\le C^2 |f|_H^2 \left(|u_2|_{X_{\tau_2}}^2 \right)^2 \tau_2 \\
&\le (2n)^4 C^2 |f|_H^2 \tau_2
\end{aligned}$$

Using last inequality in (3.1.40),

$$\begin{aligned}
A^2 &\le \frac{1}{n^2} |u_1 - u_2|_{X_T}^2 (2n)^4 C^2 |f|_H^2 \tau_2 \le 4n^2 C^2 |f|_H^2 |u_1 - u_2|_{X_T}^2 T \\
A &\le A_n |u_1 - u_2|_{X_T} T^{\frac{1}{2}}, \text{ where } A_n = 2nC|f|_H \qquad (3.1.41)
\end{aligned}$$

Similarly because, for $\theta_n(|u_1|_{X_t}) = 0$ for $t \geq \tau_1$ and $\tau_1 \leq \tau_2$, we have

$$\begin{aligned} B &= \left[\int_0^{\tau_2} \left\|\theta_n(|u_1|_{X_t})\left(B(u_1(t)) - B(u_2(t))\right)\right\|^2 dt\right]^{\frac{1}{2}}, \\ &= \left[\int_0^{\tau_1} \left\|\theta_n(|u_1|_{X_t})\left(B(u_1(t)) - B(u_2(t))\right)\right\|^2 dt\right]^{\frac{1}{2}}. \end{aligned}$$

Since $\theta_n(|u_1|_{X_t}) \leq 1$ for $t \in [0, \tau_1)$

$$B \leq \left[\int_0^{\tau_1} \left\|\left(B(u_1(t)) - B(u_2(t))\right)\right\|^2 dt\right]^{\frac{1}{2}}.$$

Therefore, using inequality (3.1.23) we infer that,

$$\begin{aligned} B^2 &\leq C^2 |f|_H^2 \int_0^{\tau_1} \left[\left(\|u_1(t)\| + \|u_1(t)\|\right)\|u_1(t) - u_2(t)\|\right]^2 dt, \\ &\leq C^2 |f|_H^2 \sup_{t\in[0,\tau_1)} \|u_1(t) - u_2(t)\|^2 \int_0^{\tau_1} \left(\|u_1(t)\| + \|u_2(t)\|\right)^2 dt, \\ &\leq C^2 |f|_H^2 \sup_{t\in[0,\tau_1)} \|u_1(t) - u_2(t)\|^2 \sup_{t\in[0,\tau_1)} \left(\|u_1(t)\| + \|u_2(t)\|\right)^2 \int_0^{\tau_1} dt \end{aligned}$$

again use of Cauchy Schwartz inequality and $\sup_{t\in[0,\tau_1)} \|u_1(t) - u_2(t)\|^2 \leq |u_1 - u_2|_{X_{\tau_1}}^2$, and using $\sup_{t\in[0,\tau_1)} \|u_i(t)\|^2 \leq |u_i|_{X_{\tau_i}}^2 \leq (2n)^2$, $i = 1, 2$, the last inequality takes form

$$\begin{aligned} B^2 &\leq C^2 |f|_H^2 |u_1 - u_2|_{X_{\tau_1}}^2 (2n + 2n)^2 \int_0^{\tau_1} dt, \\ &\leq (4n)^2 C^2 |f|_H^2 |u_1 - u_2|_{X_T}^2 \tau_1, \\ &\leq (4n)^2 C^2 |f|_H^2 |u_1 - u_2|_{X_T}^2 T \\ &\leq B_n^2 |u_1 - u_2|_{X_T}^2 T, \text{ where } B_n = 4nC |f|_H, \\ B &\leq B_n |u_1 - u_2|_{X_T} T^{\frac{1}{2}}. \end{aligned} \tag{3.1.42}$$

Using last inequality together with (3.1.41) in (3.1.42),we get

$$|\Phi_{B,T}(u_1) - \Phi_{B,T}(u_2)|_{L^2(0,T;V)} \leq (A_n + B_n)|u_1 - u_2|_{X_T} T^{\frac{1}{2}} = M_n |u_1 - u_2|_{X_T} T^{\frac{1}{2}}.$$

where $M_n := (A_n + B_n) = 2nC|f|_H + (4nC|f|_H)$. This completes the proof of the theorem. ■

Next we prove one of the key results of this subsection i.e. existence and uniqueness of the local solution to approximate evolution equation (3.1.11).

Proposition 3.1.20. *Assume that assumptions of Proposition 3.1.16 as well as Assumption 3.1.5. For given* $f_1, f_2, ..., f_N \in V$ *and* $u_0 \in V$*. Define a map* $\Psi^n_{T,u_0} : M^2(X_T) \to M^2(X_T)$ *defined by:*

$$\Psi^n_{T,u_0}(u) = Su_0 + S * \Phi^n_{T,F}(u) + \frac{1}{2}\sum_{j=1}^{N} S * \Lambda_{\kappa_{j,T}}(u) + \sum_{j=1}^{N}\left(S\diamondsuit\Phi_{B_{j,T}}(u)\right). \quad (3.1.43)$$

where B_j *and* κ_j*, are as defined as* (3.1.21) *and* (3.1.25) *respectively,* $j = 1, 2, ..., N$*. Then there exists* $C(n) > 0$ *such that*

$$\left|\Psi^n_{T,u_0}(u_1) - \Psi^n_{T,u_0}(u_2)\right|_{M^2(X_T)} \leq C(n)\left|u_1 - u_2\right|_{M^2(X_T)} T^{1/2}.$$

Moreover, there exists $T_0 > 0$ *such that for all* $T \in [0, T_0)$ *,* Ψ^{n,u_0}_T *is strict contraction. In particular, for all* $T \in [0, T_0)$ *there exists* $u \in X_T$*, such that* $\Psi^{n,u_0}_T(u) = u$*.*

Proof. Let begin with proving that Ψ^n_{T,u_0} is well-defined. For $u \in M^2(X_T)$, using triangle inequality consider,

$$\begin{aligned}
\left|\Psi^n_{T,u_0}(u)\right|_{M^2(X_T)} &= \left|Su_0 + S * \Phi^n_{T,F}(u) + \frac{1}{2}\sum_{j=1}^{N} S * \Lambda_{\kappa_{j,T}}(u) + \sum_{j=1}^{N}\left(S\diamondsuit\Phi_{B_{j,T}}(u)\right)\right|_{M^2(X_T)} \\
&\leq |Su_0|_{M^2(X_T)} + \left|S * \Phi^n_{T,F}(u)\right|_{M^2(X_T)} + \frac{1}{2}\sum_{j=1}^{N}\left|S * \Lambda_{\kappa_{j,T}}(u)\right|_{M^2(X_T)} \\
&\quad + \sum_{j=1}^{N}\left|S\diamondsuit\Phi_{B_{j,T}}(u)\right|_{M^2(X_T)} \\
&= \sqrt{\mathbb{E}\,|Su_0|^2_{X_T}} + \sqrt{\mathbb{E}\left|S * \Phi^n_{T,F}(u)\right|^2_{X_T}} + \frac{1}{2}\sum_{j=1}^{N}\sqrt{\mathbb{E}\left|S * \Lambda_{\kappa_{j,T}}(u)\right|^2_{X_T}} \\
&\quad + C_2\sum_{j=1}^{N}\left|S\diamondsuit\Phi_{B_{j,T}}(u)\right|_{M^2(X_T)}.
\end{aligned}$$

From Assumptions 3.1.5 and inequalities (3.1.35), (3.1.37) and (3.1.36) it follows that,

$$\begin{aligned}
\left|\Psi^n_{T,u_0}(u)\right|_{M^2(X_T)} &\leq C_0\sqrt{\mathbb{E}\left\|u_0\right\|^2} + C_1\sqrt{\mathbb{E}\left|\Phi^n_{T,F}(u)\right|^2_{L^2(0,T;H)}} + \frac{C_1}{2}\sum_{j=1}^{N}\sqrt{\mathbb{E}\left|\Lambda_{\kappa_{j,T}}(u)\right|^2_{L^2(0,T;H)}} \\
&\quad +C_2\sum_{j=1}^{N}\sqrt{\left|\Phi_{B_{j,T}}(u)\right|^2_{M^2(0,T;V)}} \\
&= C_0\sqrt{\mathbb{E}\left\|u_0\right\|^2} + C_1\sqrt{\mathbb{E}\left|\Phi^n_{T,F}(u)\right|^2_{L^2(0,T;H)}} + \frac{C_1}{2}\sum_{j=1}^{N}\sqrt{\mathbb{E}\left|\Lambda_{\kappa_{j,T}}(u)\right|^2_{L^2(0,T;H)}} \\
&\quad C_2\sum_{j=1}^{N}\sqrt{\mathbb{E}\left|\Phi_{B_{j,T}}(u)\right|^2_{L^2(0,T;V)}} \\
&\leq C_0\sqrt{\mathbb{E}\left\|u_0\right\|^2} + C_1C_nT^{1/4}\sqrt{\mathbb{E}\left|u\right|^2_{X_T}} + \frac{C_1}{2}T^{1/4}\sum_{j=1}^{N}K_{n,j}\sqrt{\mathbb{E}\left|u\right|^2_{X_T}} \\
&\quad +C_2T^{1/4}\sum_{j=1}^{N}M_{n,j}\sqrt{\mathbb{E}\left|u\right|^2_{X_T}} \\
&= C_0\sqrt{\mathbb{E}\left\|u_0\right\|^2} + C_1C_nT^{1/4}\left|u\right|_{M^2(X_T)} + \frac{C_1}{2}K_nT^{1/4}\left|u\right|_{M^2(X_T)}\sum_{j=1}^{N}K_{n,j} \\
&\quad +C_2T^{1/4}\left|u\right|_{M^2(X_T)}\sum_{j=1}^{N}M_{n,j} \\
&< \infty.
\end{aligned}$$

Hence Ψ_T is well-defined.

Let us choose and fix $u_1, u_2 \in M^2(X_T)$. Using triangle inequality consider the following,

$$\left|\Psi^n_{T,u_0}(u_1) - \Psi^n_{T,u_0}(u_2)\right|_{M^2(X_T)}$$

$$= \left|\begin{array}{c} S*\left(\Phi^n_{T,F}(u_1) - \Phi^n_{T,F}(u_2)\right) + \frac{1}{2}\sum_{j=1}^{N} S*\left(\Lambda_{\kappa_{j,T}}(u_1) - \Lambda_{\kappa_{j,T}}(u_2)\right) \\ +\sum_{j=1}^{N} S\diamondsuit\left(\Phi_{B_{j,T}}(u) - \Phi_{B_{j,T}}(u)\right) \end{array}\right|_{M^2(X_T)},$$

$$\leq \quad \left|S * \left(\Phi^n_{T,F}\left(u_1\right) - \Phi^n_{T,F}\left(u_2\right)\right)\right|_{M^2(X_T)} + \frac{1}{2}\sum_{j=1}^{N}\left|S * \left(\Lambda_{\kappa_{j,T}}(u_1) - \Lambda_{\kappa_{j,T}}(u_2)\right)\right|_{M^2(X_T)}$$

$$+\sum_{j=1}^{N}\left|S\diamondsuit\left(\Phi_{B_{j,T}}\left(u\right) - \Phi_{B_{j,T}}\left(u\right)\right)\right|_{M^2(X_T)}.$$

From Assumptions 3.1.5, it follows that

$$\leq \quad C_1\sqrt{\mathbb{E}\left|\Phi^n_{T,F}(u_1) - \Phi^n_{T,F}(u_2)\right|^2_{L^2(0,T;H)}} + \frac{C_1}{2}\sum_{j=1}^{N}\sqrt{\mathbb{E}\left|\Lambda_{\kappa_{j,T}}(u_1) - \Lambda_{\kappa_{j,T}}(u_2)\right|^2_{L^2(0,T;V)}}$$

$$+C_2\sum_{j=1}^{N}\sqrt{\mathbb{E}\left|\Phi_{B_{j,T}}\left(u_1\right) - \Phi_{B_{j,T}}\left(u_2\right)\right|^2_{L^2(0,T;H)}}.$$

Using inequalities (3.1.35), (3.1.37) and 3.1.36 we get,

$$\leq \quad C_1C_nT^{\frac{1}{4}}\sqrt{\mathbb{E}\left|u_1 - u_2\right|^2_{X_T}} + T^{\frac{1}{4}}\sum_{j=1}^{N}M_{n,j}\sqrt{\mathbb{E}\left|u_1 - u_2\right|^2_{X_T}}$$

$$+C_2T^{\frac{1}{4}}\sum_{j=1}^{N}K_{n,j}\sqrt{\mathbb{E}\left|u_1 - u_2\right|^2_{X_T}}$$

$$= \quad C_1C_nT^{\frac{1}{4}}\left|u_1 - u_2\right|_{M^2(X_T)} + T^{\frac{1}{4}}\left|u_1 - u_2\right|_{M^2(X_T)}\sum_{j=1}^{N}M_{n,j}$$

$$+C_2T^{\frac{1}{4}}\left|u_1 - u_2\right|_{M^2(X_T)}\sum_{j=1}^{N}K_{n,j}$$

$$= \quad C(n)T^{\frac{1}{4}}\left|u_1 - u_2\right|_{M^2(X_T)}$$

where

$$C(n) = \left[C_1C_n + \frac{C_1}{2}\sum_{j=1}^{N}K_{n,j} + C_2\sum_{j=1}^{N}M_{n,j}\right].$$

Now we can see that $C(n)$ is independent of time, also we can reduce T in such a way that $C(n)T^{\frac{1}{4}} < 1$. Hence $\Psi^{n,u_0}_{T,F}$ is strict contraction for all $T \in [0, T_0)$ and consequently by Banach fixed point theorem, for all $T \in [0, T_0)$ there exists $u \in X_T$, such that $\Psi^{n,u_0}_{T,F}\left(u\right) = u$.

■

Before proceeding further, let us recall an important observation from [8] (Page 145-147). We are going to use it at several instances.

Remark 3.1.21. *Assume that ψ be a Hilbert space H-valued process such that*

$$\int_0^t \|S(t-r)\psi(r)\|_{HS}^2 ds < \infty, \text{ for all } t \geq 0 \ \mathbb{P}\text{-a.s.},$$

where $(S_t)_{t\geq 0}$ is analytic semigroup of bounded on H and $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. The stochastic convolution

$$I(t) = \int_0^t S(t-r)\psi(r)dW(r), \ t \geq 0,$$

is well defined. If τ is some stopping time of our interest then consider the following stopped process

$$I(t\wedge\tau) = \int_0^{t\wedge\tau} S(t\wedge\tau - r)\psi(r)dW(r), \ t \geq 0.$$

Observe that integrand in above stopped process is not adapted and progressively measurable, hence the integral above does not make sense. To tackle this issue let us consider the following integral

$$I_\tau(t) = \int_0^t S(t-r)\left(1_{[0,\tau)}\psi(r\wedge\tau)\right)dW(r), \ t \geq 0.$$

It was shown in the Lemma A.1 of [8] (Page 146) that if we assume further that I and I_τ have continuous paths almost surely, then

$$S(t - t\wedge\tau) I_\tau(t\wedge\tau) = I_\tau(t), \text{for all } t \geq 0 \ \mathbb{P}\text{-a.s.}$$

In particular the integral $I(t\wedge\tau)$ make sense in the following manner,

$$I(t\wedge\tau) = I_\tau(t\wedge\tau), \ \text{for all } t \geq 0 \ \mathbb{P}\text{-a.s.}$$

In the following result, we intend to prove the existence of the unique local mild solution to main problem (3.1.6). One can observe that from the following result we can estimate from below the length of existence time interval with a lower bound, which depends on the second moment of V-norm initial data, on a large subset of Ω whose probability does not depend upon the moment. In next section, we will prove that X_T-norm of intimal data converges to ∞ and t converges to lifespan, provided that life span is finite.

Proposition 3.1.22. *If $R > 0$ and $\varepsilon > 0$, then there exists a number $T^*(\varepsilon, R) > 0$, such that for every $\mathcal{F}_0$-measurable V-valued random variable u_0 satisfying $\mathbb{E}\ \|u_0\|^2 \le R^2$, there exists a unique local solution $(u(t), t < \tau)$ of 3.1.6 such that $\mathbb{P}(\tau \ge T^*) \ge 1 - \varepsilon$.*

Proof. Take and fix $R > 0$ and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $n^2 \ge \frac{4C_0^2 R^2}{\varepsilon}$, where C_0 is as in assumption (3.1.14). Let u_0 be $\mathcal{F}_0$-measurable V-valued random variable satisfying $\mathbb{E}\|u_0\|^2 \le R^2$. For all $n \in \mathbb{N}$ and $T > 0$ we consider a map $\Psi^n_{T,u_0} : M^2(X_T) \to M^2(X_T)$ given by:

$$\Psi^n_{T,u_0}(u) = Su_0 + S * \Phi^n_{T,F}(u) + \frac{1}{2}\sum_{j=1}^{N} S * \Lambda_{\kappa_{j,T}}(u) + \sum_{j=1}^{N}\left(S\diamondsuit\Phi_{B_{j,T}}(u)\right).$$

Now since $\Psi^n_{T,u_0}(u_0) = Su_0$ so using assumption (3.1.14). we have

$$\left|\Psi^n_{T,u_0}(u_0)\right|_{M^2(X_T)} = \mathbb{E}\,|Su_0|^2_{X_T} \le C_0\mathbb{E}\,\|u_0\|^2 \le C_0 R.$$

Moreover the map Ψ^n_{T,u_0} also satisfies
$\left|\Psi^n_{T,u_0}(u)\right|_{M^2(X_T)} =$

$$\left|Su_0 + S * \Phi^n_{T,F}(u) + \frac{1}{2}\sum_{j=1}^{N} S * \Lambda_{\kappa_{j,T}}(u) + \sum_{j=1}^{N}\left(S\diamondsuit\Phi_{B_{j,T}}(u)\right)\right|_{M^2(X_T)}.$$

Using triangle inequality,

$$\begin{aligned}
\left|\Psi^n_{T,u_0}(u)\right|_{M^2(X_T)} &\leq \left|Su_0\right|_{M^2(X_T)} + \left|S*\Phi^n_{T,F}(u)\right|_{M^2(X_T)} + \frac{1}{2}\sum_{j=1}^{N}\left|S*\Lambda_{\kappa_{j,T}}(u)\right|_{M^2(X_T)} \\
&\quad + \sum_{j=1}^{N}\left|S\diamondsuit\Phi_{B_{j,T}}(u)\right|_{M^2(X_T)} \\
&= \sqrt{\mathbb{E}\left|Su_0\right|^2_{X_T}} + \sqrt{\mathbb{E}\left|S*\Phi^n_{T,F}(u)\right|^2_{X_T}} + \frac{1}{2}\sum_{j=1}^{N}\sqrt{\mathbb{E}\left|S*\Lambda_{\kappa_{j,T}}(u)\right|^2_{X_T}} \\
&\quad + \sum_{j=1}^{N}\sqrt{\mathbb{E}\left|S\diamondsuit\Phi_{B_{j,T}}(u)\right|^2_{X_T}}.
\end{aligned}$$

Using Assumptions 3.1.5,

$$\begin{aligned}
\left|\Psi^n_{T,u_0}(u)\right|_{M^2(X_T)} &\leq C_0\sqrt{\mathbb{E}\left\|u_0\right\|^2} + C_1\sqrt{\mathbb{E}\left|\Phi^n_{T,F}(u)\right|^2_{L^2(0,T;H)}} \\
&\quad + \frac{C_1}{2}\sum_{j=1}^{N}\sqrt{\mathbb{E}\left|\Lambda_{\kappa_{j,T}}(u)\right|^2_{L^2(0,T;H)}} \\
&\quad + C_2\sum_{j=1}^{N}\sqrt{\mathbb{E}\left|\Phi_{B_j,T}(u)\right|^2_{L^2(0,T;V)}}.
\end{aligned}$$

Next Using inequalities (3.1.35), 3.1.37 and 3.1.36 we get,

$$\begin{aligned}
&\leq C_0R + C_n\sqrt{\mathbb{E}\left|u\right|^2_{X_T}}T^{\frac{1}{2}} + \frac{C_1T^{\frac{1}{2}}}{2}\sum_{j=1}^{N}K_{n,j}\sqrt{\mathbb{E}\left|u\right|^2_{X_T}} + C_2T^{\frac{1}{2}}\sum_{j=1}^{N}M_{n,j}\sqrt{\mathbb{E}\left|u\right|^2_{X_T}} \\
&= C_0R + K_nT^{\frac{1}{2}}\left|u\right|_{M^2(X_T)}, \qquad (3.1.44)
\end{aligned}$$

where $K_n := C_n + \frac{C_1}{2}\sum_{j=1}^{N}K_{n,j} + C_2\sum_{j=1}^{N}M_{n,j}$. The above inequality holds for all $u \in M^2(X_T)$, and for all $T > 0$, $n \in \mathbb{N}$.

For any $u_1, u_2 \in M^2(X_T)$ from Proposition 1.19, we know that

$$\left|\Psi^n_{T,u_0}(u_1) - \Psi^n_{T,u_0}(u_2)\right|_{M^2(X_T)} \leq C_nT^{1/2}\left|u_1 - u_2\right|_{M^2(X_T)}.$$

Choose $T_1 > 0$ such that $C_nT_1^{\frac{1}{2}} < \frac{1}{2}$. This will make Ψ^n_{T,u_0}, for all $T \leq T_1$, is $\frac{1}{2}$-strict contraction and hence there exists a unique fixed point $u^n \in M^2(X_{T_1})$ such that

$\Psi^n_{T,u_0}(u^n) = u^n$. Choose $T_2 > 0$ such that $K_n T^{\frac{1}{2}} \leq \frac{1}{2}$. Now take u in particular as a fixed point u^n of $\Psi^n_{T_1,u_0}$ then from last inequality (3.1.44), we infer that

$$|u^n|_{M^2(X_T)} \leq C_0 R + \frac{|u^n|_{M^2(X_T)}}{2}.$$

Hence

$$|u^n|_{M^2(X_T)} \leq 2C_0 R.$$

Clearly from last inequality it follows that for every $T \leq T_1 \wedge T_2$, the range of Ψ^n_{T,u_0} is a subset of the ball of radius $2C_0R$, centered at 0, in $M^2(X_T)$. If we set $T^* = T_1 \wedge T_2$. then from last inequality we infer that, infer that

$$|u^n|_{M^2(X_{T^*})} \leq 2C_0 R. \tag{3.1.45}$$

Consider the stopping time τ_n defined by,

$$\tau_n = \inf \left\{ t \in [0, T^*], |u^n|_{X_t} \geq n \right\}.$$

As $u^n(0) = u_0$ so observe that

$$\begin{aligned} \text{if } \|u_0(\omega)\| &\geq n \text{ then } \tau_n = 0 \text{ and} \\ \text{if } \|u_0(\omega)\| &< n \text{ then } \tau_n \in (0, T^*]. \end{aligned}$$

Now we claim that (u^n, τ_n) is local mild solution. Then u^n as a fixed point it satisfies

$$\begin{aligned} u^n(t) &= S(t)u_0 + \int_0^t S(t-r)\theta_n\left(|u^n|_{X_r}\right) F(u^n(r))\, dr \\ &+ \frac{1}{2}\sum_{j=1}^{N} \int_0^t S(t-r)\theta_n\left(|u^n|_{X_r}\right) \Lambda_{\kappa_{j,T}}(u(r))\, dr \\ &+ \sum_{j=1}^{n} \int_0^t S(t-r)\theta_n\left(|u^n|_{X_r}\right) B(u^n(r))\, dW_j(r), \ \mathbb{P}\text{-a.s.} \end{aligned}$$

Observe that processes involved in the both sides of above equality are continuous, so this equality still holds when the deterministic time t is replaced by $t \wedge \tau_n$. The stopped equation can be given as, for $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned} u^n(t\wedge\tau_n) &= S(t)u_0 + \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\theta_n\left(|u^n|_{X_r}\right) F(u^n(r))\,dr \\ &+\frac{1}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\theta_n\left(|u^n|_{X_r}\right)\Lambda_{\kappa_{j,T}}(u(r))\,dr \\ &+\sum_{j=1}^{N}\int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\theta_n\left(|u^n|_{X_r}\right) B(u^n(r))\,dW_j(r),\ \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.1.46}$$

We claim that

$$\theta_n\left(|u^n|_{X_r}\right) = 1,\ \text{for all } r \in [0, t\wedge\tau_n]. \tag{3.1.47}$$

In order to do so, let $r \in [0, t\wedge\tau_n]$, where $t \in [0,T]$, therefore $r \leq \tau_n$. Now since

$$\begin{aligned} |u^n|_{\tau_n} &= n,\ \text{if } \tau_n < T \\ &\leq n,\ \text{if } \tau_n = T, \end{aligned}$$

and the map $r \longmapsto |u^n|_r$ increasing, we infer that $|u^n|_r \leq |u^n|_{\tau_n} \leq n$, for all $r \leq \tau_n$. Hence by definition of θ_n it follows that

$$\theta_n\left(|u^n|_{X_r}\right) = 1,\ \text{for all } r \in [0, t\wedge\tau_n].$$

Keeping in view the Remark 3.1.21 and using (3.1.47) into (3.1.46), we infer that,

$$\begin{aligned} u^n(t\wedge\tau_n) &= S(t)u_0 + \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)F(u^n(r))\,dr \\ &+\frac{1}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\Lambda_{\kappa_{j,T}}(u(r))\,dr \\ &+\sum_{j=1}^{N}\int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)B(u^n(r))\,dW_j(r), \mathbb{P}\text{-a.s.} \end{aligned}$$

for all $t \in [0, T^*)$. Therefore, (u^n, τ_n) is local mild solution to our main problem (3.1.6). By the definition of stopping time τ_n,

$$\{\omega \in \Omega : \tau_n(\omega) \le T^*(\omega)\} \subset \left\{\omega \in \Omega : |u^n|_{X_{T^*(\omega)}} \ge n\right\}.$$

Indeed $\tau_n \le T^*$ iff there exists $t \le T^*$ such that $|u^n|_{X_t} \ge n$. Since $s \mapsto |\cdot|_{X_s}$ increasing so $|u^n|_{X_{T^*}} \ge |u^n|_{X_t} \ge n$, for all $t \le T^*$, hence the above inclusion holds. By using the Chebyshev inequality and inequality (3.1.45), moreover using choice of n in the beginning of the proof i.e. $n^2 \ge \frac{4C_0^2 R^2}{\varepsilon}$, we get

$$\begin{aligned}
\mathbb{P}\{\tau_n \le T^*\} &\le \mathbb{P}\{|u^n|_{T^*} \ge n\} \\
&\le \frac{\mathbb{E}\,|u^n|^2_{X_{T^*}}}{n^2} \\
&= \frac{|u^n|^2_{M^2(X_{T^*})}}{n^2} \\
&\le \frac{4C_0^2 R^4}{n^2} \le \varepsilon,
\end{aligned}$$

Equivalently

$$\mathbb{P}\{\tau_n \ge T^*\} < 1 - \varepsilon.$$

If we set $\tau = \tau_n$ and $u = u^n$ then we have the required result. This completes the proof. ■

3.1.8 Global solution to Approximate Evolution equation

Let us begin by defining a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ in the following manner,

$$\tau_n := \inf\left\{t \in [0, T] : |u|_{X_t} \ge n\right\} \wedge T.$$

Theorem 3.1.23. *Suppose Assumptions 3.1.2 and assumptions of Proposition 3.1.20 hold and $(\tau_n)_{n\in\mathbb{N}}$ be the above sequence of stopping time. Then for each*

$n \in \mathbb{N}$, the truncated evolution equation (3.1.11) *admits a unique global solution $u^n \in M^2(X_T)$. Moreover (u^n, τ_n) is local mild solution to the main problem* (3.1.6).

Proof. Let us fix $n \in \mathbb{N}$, $T \in (0,\infty)$ and $u_0 \in L^2(\Omega,\mathbb{P};V)$. Let Ψ^n_{T,u_0} be a map on $M^2(X_T)$ defined by,

$$\Psi^n_{T,u_0}(u) = Su_0 + S * \Phi^n_{T,F}(u) + \frac{1}{2}\sum_{j=1}^{N} S * \Lambda_{\kappa_j,T}(u) + \sum_{j=1}^{N}\left(S\Diamond\Phi_{B_j,T}(u)\right).$$

From Proposition 3.1.20 we know Ψ^n_{T,u_0} maps $M^2(X_T)$ into $M^2(X_T)$ and it is globally Lipschitz. Moreover, for sufficiently small T Ψ^n_{T,u_0} is strict contraction. Hence we can find $\eta := \eta(n) > 0$ such that Ψ^n_{T,u_0} is a $\frac{1}{2}$-contraction. Let $(\eta_k)_{k\in\mathbb{N}\cup\{0\}}$ be a sequence of times defined by $\eta_k = k\eta$, where $k \in \mathbb{N}\cup\{0\}$. Since $\Psi^n_{\eta,u_0} : M^2(X_\eta) \to M^2(X_\eta)$ is a $\frac{1}{2}$ -contraction, there exists a unique $u^{n,1} \in M^2\left(X_{[0,\eta]}\right) := M^2(X_\eta)$ such that

$$\begin{aligned} u^{n,1} &= \Psi^n_{\eta,u_0}\left(u^{n,1}\right) \\ &= Su_0 + S * \Phi^n_{\eta,F}(u^{n,1}) + \frac{1}{2}\sum_{j=1}^{N} S * \Lambda_{\kappa_j,\eta}(u^{n,1}) + \sum_{j=1}^{N}\left(S\Diamond\Phi_{B_{j,\eta}}\left(u^{n,1}\right)\right). \end{aligned}$$

As $u^{n,1} \in M^2(X_\eta)$ and $u^{n,1}$ is F_t-measurable so by definition of $M^2(X_\eta)$ we infer that $u^{n,1}(\eta) \in L^2(\Omega,\mathbb{P};V)$.

Now by replacing u_0 by $u^{n,1}(\eta)$ by repeating the same argument as above we can find $u^{n,2} \in M^2\left(X_\eta, \mathcal{F}_{\eta_1}\right)$ such that

$$\begin{aligned} u^{n,2} &= \Psi^n_{\eta,u^{n,1}(\eta_1)}\left(u^{n,2}\right) \\ &= Su^{n,1}(\eta_1) + S * \Phi^n_{\eta,F}(u^{n,2}) + \frac{1}{2}\sum_{j=1}^{N} S * \Lambda_{\kappa_j,\eta}(u^{n,2}) + \sum_{j=1}^{N}\left(S\Diamond\Phi_{B_{j,\eta}}\left(u^{n,2}\right)\right). \end{aligned}$$

In the same fashion, arguing inductively one can find sequence $\left(u^{n,k}\right)_{k=1}^{\infty}$ such that

$u^{n,k} \in M^2\left(X_\eta, \mathcal{F}_{\eta_{k-1}}\right)$ which satisfies

$$\begin{aligned} u^{n,k} &= \Psi^n_{\eta, u^{n,k-1}\left(\eta_{k-1}\right)}\left(u^{n,k}\right) \\ &= Su^{n,k}\left(\eta_{k-1}\right) + S * \Phi^n_{\eta,F}(u^{n,k}) + \frac{1}{2}\sum_{j=1}^{N} S * \Lambda_{\kappa_{j,\eta}}(u^{n,k}) + \sum_{j=1}^{N}\left(S\Diamond\Phi_{B_{j,\eta}}\left(u^{n,k}\right)\right). \end{aligned}$$

Let us define a process u^n in the following manner:

$$\begin{aligned} u^n(t) &: = u^{n,1}(t), \qquad \text{for } t \in [0, \eta_1) \\ &: = u^{n,2}(t), \quad \text{for } t \in [\eta_1, \eta_2) \\ &\quad ... \\ &: = u^{n,k}(t), \text{ for } t \in [\eta_k, \eta_{k+1}) \text{ and } k = \left[\frac{T}{\eta_k}\right] + 1. \end{aligned}$$

We claim that for $u^n \in M^2(X_T)$ such that $\Psi^n_{T,u_0}(u^n) = u^n$.

$$\begin{aligned} |u^n|_{M^2(X_T)} &= \mathbb{E}\left(\sup_{t\in[0,T]} \|u^n(t)\|^2 + \int_0^T |u^n(t)|^2_E\, dt\right) \\ &\le \mathbb{E}\left[\sum_k\left(\sup_{t\in[\eta_k,\eta_{k+1})}\left\|u^{n,k}(t)\right\|^2 + \int_{\eta_k}^{\eta_{k+1}}\left|u^{n,k}(t)\right|^2_E dt\right)\right] \\ &= \sum_k \mathbb{E}\left(\sup_{t\in[\eta_k,\eta_{k+1})}\left\|u^{n,k}(t)\right\|^2 + \int_{\eta_k}^{\eta_{k+1}}\left|u^{n,k}(t)\right|^2_E dt\right) \\ &= \sum_k \left|u^{n,k}\right|_{M^2\left(X_{[\eta_k,\eta_{k+1})}\right)} < \infty. \end{aligned}$$

Hence $u^n \in M^2(X_T)$. Next we aim to show that $\Psi^n_{T,u_0}(u^n) = u^n$ i.e. the following evolution equation is satisfied for all $t \in [0, T]$,

$$\begin{aligned} u^n(t) &= S(t)u_0 + \int_0^t S(t-r)\theta_n\left(|u^n|_{X_r}\right) F\left(u^n(r)\right) dr \\ &\quad + \frac{1}{2}\sum_{j=1}^{N}\int_0^t S(t-r)\theta_n\left(|u^n|_{X_r}\right)\Lambda_{\kappa_{j,T}}\left(u^n(r)\right) dr \\ &\quad + \sum_{j=1}^{N}\int_0^t S(t-r)\theta_n\left(|u^n|_{X_r}\right)\Phi_{B_{j,T}}\left(u^n(r)\right) dW_j(r), \mathbb{P}\text{-a.s, } t \in [0,T]. \end{aligned} \tag{3.1.48}$$

By definition $u^n(t) = u^{n,1}(t)$, for all $t \in [0, \eta_1)$ and since $u^{n,1}$ satisfies above equation on $[0, \eta_1)$ so does u^n.

Since $u^{n,1}(\eta_1^-) = u^{n,2}(\eta_1^+)$, using semigroup property the following chain of equations hold for all $t \in [\eta_1, \eta_2)$,

$$
\begin{aligned}
u^n(t) &= S(t-\eta_1)\left[\begin{array}{c} S(\eta_1)u_0 + \int_0^{\eta_1} S(\eta_1 - r)\theta_n\left(|u^{n,1}|_{X_r}\right) F\left(u^{n,1}(r)\right) dr \\ +\frac{1}{2}\sum_{j=1}^N \int_0^{\eta_1} S(\eta_1 - r)\theta_n\left(|u^{n,1}|_{X_r}\right) \Lambda_{\kappa_{j,T}}\left(u^{n,1}(r)\right) dr \\ \sum_{j=1}^N \int_0^{\eta_1} S(\eta_1 - r)\theta_n\left(|u^{n,1}|_{X_r}\right) \Phi_{B_{j,T}}\left(u^{n,1}(r)\right) dW_j(r) \end{array}\right] \\
&\quad + \int_{\eta_1}^t S(t-r)\theta_n\left(\left|u^{n,2}\right|_{X_r}\right) F\left(u^{n,2}(r)\right) dr \\
&\quad + \frac{1}{2}\sum_{j=1}^N \int_{\eta_1}^t S(t-r)\theta_n\left(\left|u^{n,2}\right|_{X_r}\right) \Lambda_{\kappa_{j,T}}\left(u^{n,2}(r)\right) dr \\
&\quad + \sum_{j=1}^N \int_{\eta_1}^t S(t-r)\theta_n\left(\left|u^{n,2}\right|_{X_r}\right) \Phi_{B_{j,T}}\left(u^{n,2}(r)\right) dW_j(r), \mathbb{P}\text{-a.s,}
\end{aligned}
$$

$$
\begin{aligned}
&= S(t)u_0 + \int_0^{\eta_1} S(t-r)\theta_n\left(|u^n|_{X_r}\right) F\left(u^n(r)\right) dr + \int_{\eta_1}^t S(t-r)\theta_n\left(|u^n|_{X_r}\right) F\left(u^n(r)\right) dr \\
&\quad + \frac{1}{2}\sum_{j=1}^N \int_0^{\eta_1} S(\eta_1 - r)\theta_n\left(|u^n|_{X_r}\right) \Lambda_{\kappa_{j,T}}\left(u^n(r)\right) dr \\
&\quad + \frac{1}{2}\sum_{j=1}^N \int_0^{\eta_1} S(\eta_1 - r)\theta_n\left(|u^n|_{X_r}\right) \Lambda_{\kappa_{j,T}}\left(u^n(r)\right) dr \\
&\quad + \sum_{j=1}^N \int_0^{\eta_1} S(\eta_1 - r)\theta_n\left(|u^n|_{X_r}\right) \Phi_{B_{j,T}}\left(u^n(r)\right) dW_j(r) \\
&\quad + \sum_{j=1}^N \int_{\eta_1}^t S(t-r)\theta_n\left(|u^n|_{X_r}\right) \Phi_{B_{j,T}}\left(u^n(r)\right) dW_j(r), \mathbb{P}\text{-a.s.}
\end{aligned}
$$

$$\begin{aligned} u^n(t) &= S(t)u_0 + \int_0^t S(t-r)\theta_n\left(|u^n|_{X_r}\right)F\left(u^n(r)\right)dr \\ &+ \frac{1}{2}\sum_{j=1}^N \int_0^t S(\eta_1 - r)\theta_n\left(|u^n|_{X_r}\right)\Lambda_{\kappa_{j,T}}\left(u^n\left(r\right)\right)dr \\ &+ \sum_{j=1}^N \int_0^t S(\eta_1 - r)\theta_n\left(|u^n|_{X_r}\right)\Phi_{B_{j,T}}\left(u^n\left(r\right)\right)dW_j(r), \mathbb{P}\text{-a.s,} \end{aligned}$$

for all $t \in [\eta_1, \eta_2)$. In the similar manner we can show that u^n satisfies (3.1.48) for all $t \in [\eta_k, \eta_{k+1})$, for all $k \in \mathbb{N}$.

Thus, u^n is global solution of truncated evolution equation (3.1.11). We now move towards proving the uniqueness of global solution. Let (v, τ) be another global solution of truncated evolution equation (3.1.11). We claim that $\mathbb{P}\left(u^n = v\right) = 1$ on $(0, \tau]$. Set $\beta_k = \tau \wedge \eta_k$, where (η_k) is as in first paragraph of proof. Clearly $\beta_k \to \tau$, almost surely, as $k \to \left[\frac{T}{\eta}\right]$. By contraction argument and uniqueness

$$\begin{aligned} \mathbb{P}\left(u^n\left(t\right) = v(t)\right) &= 1, \text{for all} t \in \ (0, \tau \wedge \eta]\,, \\ \text{and } \mathbb{P}\left(u^n\left(t\right) = v(t)\right) &= 1, \text{for all } t \in \ (0, \beta_k]\,. \end{aligned}$$

Proceeding limit $k \to \infty$, it follows that

$$\mathbb{P}\left(u^n(t) = v(t)\right) = 1, \text{ for all } t \in (0, \tau]\,.$$

Next we aim to show that (u^n, τ_n) is local mild solution to the main problem 3.1.6.

Let us begin by observing that processes involved in the both sides of equality (3.1.48) are continuous, so the equality (3.1.48) still holds when the deterministic time t is replaced by $t \wedge \tau_n$. The stopped equation can be given as, for $t \in [0, T]$,

$\mathbb{P}$-a.s.,

$$\begin{aligned} u^n(t\wedge\tau_n) &= S(t\wedge\tau_n)u_0 + \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\theta_n\left(|u^n|_{X_r}\right)F\left(u^n(r)\right)dr \\ &+\frac{1}{2}\sum_{j=1}^N \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\theta_n\left(|u^n|_{X_r}\right)\Lambda_{\kappa_{j,T}}\left(u^n(r)\right)dr \\ &+\sum_{j=1}^N \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\theta_n\left(|u^n|_{X_r}\right)\Phi_{B_{j,T}}\left(u^n(r)\right)dW_j(r). \end{aligned} \tag{3.1.49}$$

We claim that

$$\theta_n\left(|u^n|_{X_s}\right) = 1, \text{for all } s\in[0,t\wedge\tau_n]. \tag{3.1.50}$$

In order to do so, let $s\in[0,t\wedge\tau_n]$, where $t\in[0,T]$, therefore $s\le\tau_n$. Now since

$$\begin{aligned} |u^n|_{\tau_n} &= n, \text{ if } \tau_n < T \\ &\le n, \text{ if } \tau_n = T, \end{aligned}$$

and the map $s\longmapsto |u^n|_s$ increasing, we infer that $|u^n|_s \le |u^n|_{\tau_n} \le n$, for all $s\le\tau_n$. Hence by definition of θ_n it follows that

$$\theta_n\left(|u^n|_{X_s}\right) = 1, \text{ for all } s\in[0,t\wedge\tau_n].$$

Hence using previous observation the integrand of Riemann integrals of equation (3.1.49),

$$\theta_n\left(|u|_{X_r}\right)F\left(u(r)\right) = F\left(u(r)\right), \text{ for } r\in[0,t\wedge\tau_n] \tag{3.1.51}$$

and

$$\theta_n\left(|u|_{X_r}\right)\Lambda_{\kappa_{j,T}}\left(u\left(r\right)\right) = \Lambda_{\kappa_{j,T}}\left(u\left(r\right)\right), \text{ for } r\in[0,t\wedge\tau_n],\ t\in[0,T]. \tag{3.1.52}$$

In the view of Remark 3.1.21 and (3.1.50), we conclude that for any $t\in[0,T]$,

$$\int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\left[\theta_n\left(|u|_{X_r}\right)\Phi_{B_{j,T}}\left(u\left(r\right)\right)\right]dW(r) = \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\Phi_{B_{j,T}}\left(u\left(r\right)\right)dW_j(r), \tag{3.1.53}$$

Using the (3.1.51) (3.1.52) and (3.1.53) into (3.1.49) it follows that, $\mathbb{P}$-a.s., $t \in [0,T]$

$$\begin{aligned} u(t\wedge\tau_n) &= S(t\wedge\tau_n)u_0 + \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)F\left(u(r)\right)dr \\ &+\frac{1}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\Lambda_{\kappa_{j,T}}\left(u\left(r\right)\right)dr \\ &+\sum_{j=1}^{N}\int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\Phi_{B_{j,T}}\left(u\left(r\right)\right)dW_j(r). \end{aligned}$$

This proves that (u, τ_n) is local solution of main problem (3.1.6). This completes the proof. ■

3.2 Construction of Local Maximal Solution

In this subsection, we are going to show that the existence global solution of the truncated evolution equation (3.1.11) enables us to construct the local maximal solution of main problem (3.1.6).

Theorem 3.2.1. *Suppose Assumptions 3.1.2 and assumptions of Proposition 3.1.20 and 3.1.22 hold. Then there exists a unique local maximal solution (u, τ_∞) to main problem* (3.1.6).

Proof. Let us fix $T > 0$. We aim to show that there exists a solution (u, τ_∞) to main problem (3.1.6) that satisfies Definition 3.1.8. Let us recall from Theorem 3.1.23 that there exists a unique global solution u^n of approximate equation (3.1.11).

We begin by constructing a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ in the following manner.

$$\tau_n := \inf\left\{t \in [0,T] : |u^n|_{X_t} \geq n\right\} \wedge T, \text{ where } n \in \mathbb{N}. \tag{3.2.1}$$

Let us fix natural numbers k and n such that $k > n$ and

$$\tau_{n,k} := \inf\left\{t \in [0,T] : \left|u^k\right|_{X_t} \geq n\right\} \wedge T. \tag{3.2.2}$$

We claim that

$$\tau_{n,k} \leq \tau_k.$$

To prove above inequality we are going to show that

$$\left\{t \in [0,T] : \left|u^k\right|_{X_t} \geq n\right\} \supset \left\{t \in [0,T] : \left|u^k\right|_{X_t} \geq k\right\}, \tag{3.2.3}$$

holds.

Take $r \in \left\{t \in [0,T] : \left|u^k\right|_{X_t} \geq k\right\}$ then

$$\left|u^k\right|_{X_r} \geq k.$$

but $k > n$ implies that

$$\left|u^k\right|_{X_r} > n.$$

Therefore

$$r \in \left\{t \in [0,T] : \left|u^k\right|_{X_t} \geq n\right\}.$$

Hence the inclusion (3.2.3) is true. Taking infimums on both sides of (3.2.3) we infer that,

$$\inf \left\{t \in [0,T] : \left|u^k\right|_{X_t} \geq n\right\} \leq \inf \left\{t \in [0,T] : \left|u^k\right|_{X_t} \geq k\right\},$$

and so

$$\inf \left\{t \in [0,T] : \left|u^k\right|_{X_t} \geq n\right\} \wedge T \leq \inf \left\{t \in [0,T] : \left|u^k\right|_{X_t} \geq k\right\} \wedge T$$

Hence

$$\tau_{n,k} \leq \tau_k. \tag{3.2.4}$$

Next we claim that $\left(u^k, \tau_{n,k}\right)$ is local solution equation (3.1.11) and going to satisfy the Definition 3.1.7. Recall the fact that $\left(u^k, \tau_k\right)$ is local solution to equation

(3.1.11) (See Theorem 3.1.23). Consider the following, for all $t \in [0,T]$,

$$\begin{aligned} |u|^2_{X_{t\wedge\tau_{n,k}}} &= \mathbb{E}\left(\sup_{s\in\left[0,t\wedge\tau_{n,k}\right]} \|u(s)\|^2 + \int_0^{t\wedge\tau_{n,k}} |u(s)|^2_E\right) \\ &\leq \mathbb{E}\left(\sup_{s\in[0,t\wedge\tau_k]} \|u(s)\|^2 + \int_0^{t\wedge\tau_k} |u(s)|^2_E\right) \\ &= |u|^2_{X_{t\wedge\tau_k}} < \infty. \end{aligned}$$

Further, we can repeat the argument from the last part of proof of Theorem 3.1.23 to see that $\left(u^k, \tau_{n,k}\right)$ satisfies ii) of Definition 3.1.7.

But since (u^n, τ_n) is the unique solution (3.1.11) proved in Theorem 3.1.23 so we

$$u^k(t) = u^n(t) \text{ a.s. for all } t \in [0, \tau_{n,k} \wedge \tau_n]. \tag{3.2.5}$$

Next we claim that

$$u^k(t) = u^n(t) \text{ a.s. for all } t \in [0, \tau_n]. \tag{3.2.6}$$

To see this we need to see two cases:

Case a) When $\tau_{n,k}(\omega) \geq \tau_n(\omega)$, for $\omega \in \Omega$. Then $\tau_{n,k}(\omega) \wedge \tau_n(\omega) = \tau_n(\omega)$, for $\omega \in \Omega$. Hence (3.2.6) holds trivially.

Case b) When $\tau_{n,k}(\omega) < \tau_n(\omega)$, for $\omega \in \Omega$. Then since the map $t \longmapsto |u^n|_{X_t}$ is increasing therefore,

$$|u^n|_{X_{\tau_{n,k}(\omega)}} < |u^n|_{X_{\tau_n(\omega)}}, \text{ for } \omega \in \Omega$$

Now as $\tau_n(\omega) \leq T$ therefore

$$|u^n|_{X_{\tau_n(\omega)}} \leq n, \text{for } \omega \in \Omega. \tag{3.2.7}$$

But since $\tau_{n,k}(\omega) < \tau_n(\omega) \leq T$ i.e. we infer that

$$\left|u^k\right|_{X_{\tau_{n,k}(\omega)}} = n, \text{ for } \omega \in \Omega.$$

Using (3.2.5) and (3.2.7) it follows that

$$|u^n|_{X_{\tau_{n,k}(\omega)}} = n \geq |u^n|_{X_{\tau_n(\omega)}}\,, \text{ for } \omega \in \Omega.$$

But this is a contradiction to $|u^n|_{X_{\tau_{n,k}(\omega)}} < |u^n|_{X_{\tau_n(\omega)}}$. Hence Case b) is not possible so the Case a) is true. Thus claim (3.2.6) is true.

From the inequality $\tau_{n,k} \geq \tau_n$ above and (3.2.4) it also follows that for all $k > n$, $\tau_n \leq \tau_{n,k} \leq \tau_k$.

Thus the sequence $(\tau_n)_{n\in\mathbb{N}}$ being bounded and increasing sequence of real numbers and hence has a limit $\tau_\infty := \lim\limits_{n\to\infty} \tau_n = \sup\limits_{n\in\mathbb{N}} \tau_n$ a.s.

Now let us define a stochastic processes $(u(t))_{t\in[0,\tau_\infty)}$ in the following manner:

$$u(t) := u^n(t)\,, \text{ for all } t \in [\tau_{n-1}, \tau_n] \text{ and } n \in \mathbb{N}, \tag{3.2.8}$$

where $\tau_0 = 0$.

It follows from (3.2.6) and (3.2.8) that,

$$u(t \wedge \tau_n) = u^n(t \wedge \tau_n) \text{ a.s. for all } t \in [0, T]\,. \tag{3.2.9}$$

Next we are going to prove that

$$\theta_n\left(|u|_{X_s}\right) = 1, \text{ for all } s \in [0, t\wedge\tau_n]. \text{ for all } t \in [0, T] \tag{3.2.10}$$

In order to do so, let $s \in [0, t \wedge \tau_n]$, where $t \in [0, T]$, therefore $s \leq \tau_n$. Now since

$$\begin{aligned} |u^n|_{\tau_n} &= n, \text{ if } \tau_n < T \\ &\leq n, \text{ if } \tau_n = T, \end{aligned}$$

and the map $s \longmapsto |u^n|_s$ increasing, we infer that $|u^n|_s \leq |u^n|_{\tau_n} \leq n$, for all $s \leq \tau_n$. Hence by definition of θ_n it follows that

$$\theta_n\left(|u|_{X_s}\right) = 1, \text{ for all } s \in [0, t\wedge\tau_n], \text{ for all } t \in [0, T]\,.$$

Now (u^n, τ_n) is solution to (3.1.11) so it must satisfies the following evolution equation for $t \in [0,T]$,

$$\begin{aligned} u^n(t\wedge\tau_n) &= S(t\wedge\tau_n)u_0 + \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\theta_n\left(|u^n|_{X_r}\right)F\left(u^n(r)\right)dr \\ &+ \frac{1}{2}\sum_{j=1}^N \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\theta_n\left(|u^n|_{X_r}\right)\Lambda_{\kappa_{j,T}}\left(u^n\left(r\right)\right)dr \\ &+ \sum_{j=1}^N \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\theta_n\left(|u^n|_{X_r}\right)\Phi_{B_{j,T}}\left(u^n\left(r\right)\right)dW(r), \mathbb{P}\text{-a.s.} \end{aligned}$$

Using (3.2.10), (3.2.9) and Remark 4.2.55 into last equation we get, for all $t \in [0,T]$:

$$\begin{aligned} u(t\wedge\tau_n) &= S(t\wedge\tau_n)u_0 + \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)F\left(u(r)\right)dr \\ &+ \frac{1}{2}\sum_{j=1}^N \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\Lambda_{\kappa_{j,T}}\left(u\left(r\right)\right)dr \\ &+ \sum_{j=1}^N \int_0^{t\wedge\tau_n} S(t\wedge\tau_n - r)\Phi_{B_{j,T}}\left(u(r)\right)dW(r), \mathbb{P}\text{-a.s. }. \end{aligned} \tag{3.2.11}$$

This proves that (u, τ_n) is local solution of main problem (3.1.6). Observe that all processes $F, \Lambda_{\kappa_{j,T}}$ and $\Phi_{B_{j,T}}$ are have almost surely continuous trajectories so it follows from [37] (cf. Theorem 1.4) that above evolution equation (3.2.11) also holds in strong form i.e.

$$\begin{aligned} u(t\wedge\tau_n) &= u_0 + \int_0^{t\wedge\tau_n} F\left(u(r)\right)dr \\ &+ \frac{1}{2}\sum_{j=1}^N \int_0^{t\wedge\tau_n} \Lambda_{\kappa_{j,T}}\left(u\left(r\right)\right)dr \\ &+ \sum_{j=1}^N \int_0^{t\wedge\tau_n} \Phi_{B_{j,T}}\left(u(r)\right)dW(r), \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.2.12}$$

Next we are going to show that local solution (u, τ_∞) satisfies the Definition 3.1.8 of local maximal solution. We can formulate this part of proof in two cases i.e. the case when $\tau_\infty = T$ and $\tau_\infty < T$. In the case of $\tau_\infty = T$ the conclusion is trivial. Let us concentrate on the case when $\tau_\infty < T$.

Let us begin with fact that $\tau_\infty = \lim_{n\to\infty} \tau_n = \sup_{n\in\mathbb{N}} \tau_n$ and so $\tau_n \le \tau_\infty$. Also since $s \longmapsto |u^n|_s$, it follows that

$$|u|_{X_{\tau_n}} \le |u|_{X\tau_\infty}, \text{ for all } n \in \mathbb{N},$$

or

$$\lim_{n\to\infty} |u|_{X_{\tau_n}} \le \lim_{t\to\tau_\infty} |u|_{X_t}. \tag{3.2.13}$$

Moreover using (3.2.9), we infer that

$$\lim_{n\to\infty} |u^n|_{X_{\tau_n}} \le \lim_{n\to\infty} |u|_{X_{\tau_n}} \quad . \tag{3.2.14}$$

Since $\tau_n \le \tau_\infty < T$ so it follows that

$$|u^n|_{X_{\tau_n}} = n. \tag{3.2.15}$$

Combining inferences (3.2.13), (3.2.14) and (3.2.15) we have the following chain on $\{\tau_\infty(\omega) < T\}$,

$$\lim_{t\to\tau_\infty} |u|_{X_t} \ge \lim_{n\to\infty} |u|_{X_{\tau_n}} \ge \lim_{n\to\infty} |u^n|_{X_{\tau_n}} = \lim_{n\to\infty} n \to \infty.$$

Hence the Definition 3.1.8 of local maximal solution for (u, τ_∞).

Thus (u, τ_∞) is a local maximal solution to main problem 3.1.6.

Now we prove the uniqueness of local maximal solution. Assume that there is another (v, σ_∞) be another maximal local solution and $(\sigma_n)_{n\ge 0}$ a sequence of stopping times converges to σ_∞ defined by,

$$\sigma_n = \inf\left\{t \in [0,T] : |v|_{X_t} \ge n\right\} \wedge \sigma_\infty \wedge T.$$

By the following same set of arguments as above, we can see that,

$$u(t) = v(t), \text{ for all } t \in [0, \tau_n \wedge \sigma_n] \text{ a.s. } .$$

Taking limits $n \to \infty$ in last equation,

$$u(t) = v(t), \text{ for all } t \in [0, \tau_\infty \wedge \sigma_\infty] \text{ a.s. } .$$

We claim that $\tau_\infty = \sigma_\infty$ a.s. If this claim is not true then either $\tau_\infty < \sigma_\infty$ or $\tau_\infty > \sigma_\infty$ i.e. either the following is true

$$\lim_{t \to \sigma_\infty} \left|1_{\{\tau_\infty < \sigma_\infty\}} v\right|_{X_t} = \lim_{n \to \infty} \left|1_{\{\tau_\infty < \sigma_\infty\}} v\right|_{X_{\sigma_n}} = \lim_{n \to \infty} \left|1_{\{\tau_\infty < \sigma_\infty\}} u\right|_{X_{\tau_n}} = \infty, \quad (3.2.16)$$

or

$$\lim_{t \to \tau_\infty} \left|1_{\{\tau_\infty > \sigma_\infty\}} u\right|_{X_t} = \lim_{n \to \infty} \left|1_{\{\tau_\infty > \sigma_\infty\}} u\right|_{X_{\tau_n}} = \lim_{n \to \infty} \left|1_{\{\tau_\infty > \sigma_\infty\}} v\right|_{X_{\sigma_n}} = \infty. \quad (3.2.17)$$

The equation (3.2.16) is a contradiction to fact that u is maximal so does not explode before time τ_∞. Similarly equation (3.2.17) is contradiction to the fact that v is maximal so does not explode before time τ_∞. Thus both cases $\sigma_\infty < \sigma_\infty$ and $\tau_\infty > \sigma_\infty$ leads to contradiction so hence $\tau_\infty = \sigma_\infty$ is true and we achieved the uniqueness of maximal solution. This completes the proof of theorem. ■

3.3 No explosion and Global Solution

Finally, in this section we are going to prove the *no explosion result* and then using this result we will prove the existence of a unique global solution to the main problem (3.1.6).

3.3.1 No Explosion result

Theorem 3.3.1. *For every V-valued and $\mathcal{F}_0-$ measurable initial data u_0 satisfying $\mathbb{E}\ \|u_0\|^2 < \infty$. There exists the unique local maximal solution (u, τ_∞) to the main* *problem 3.1.6. Moreover,*

$$\mathbb{P}\left(\{\tau_\infty < \infty\} \cap \left\{\sup_{t\in[0,\tau_\infty)} \|u(t)\| < \infty\right\}\right) = 0$$

and on $\{\tau_\infty < \infty\}$,

$$\limsup_{t\to\tau_\infty} \|u(t)\| = \infty, \ \ a.s.$$

Proof. The existence of unique maximal (u, τ_∞) solution is guaranteed by Theorem 3.2.1.

To prove the second statement we argue by the contradiction. Suppose there exists $\varepsilon > 0$ such that

$$\mathbb{P}\left(\{\tau_\infty < \infty\} \cap \left\{\sup_{t\in[0,\tau_\infty)} \|u(t)\| < \infty\right\}\right) = 4\varepsilon.$$

For $R > 0$, let us define an stopping time $\sigma_R := \inf\{t \in [0, \tau_\infty) : \|u(t)\| \geq R\}$ and $\widetilde{\Omega} = \{\sigma_R = \tau_\infty = \infty\}$. Set

$$\widetilde{\Omega}_R = \{\tau_\infty < \infty\} \cap \{\|u(t)\| \leq R \text{ for all } t \in [0, \tau_\infty)\}.$$

We claim that

$$\widetilde{\Omega} = \bigcup_{R=1}^{\infty} \widetilde{\Omega}_R.$$

The inclusion $\widetilde{\Omega} \supset \bigcup_{R=1}^{\infty} \widetilde{\Omega}_R$ is obvious. For the reverse inclusion, if $\omega \in \widetilde{\Omega}$ then $\tau_\infty(\omega)$ and $\sup_{t\in[0,\tau_\infty)} \|u(t)\|$ are both finite i.e. there exists natural number n such that

$$\|u(t)\| \leq n, \text{ for } t \in [0, \tau_\infty],$$

this implies that $\omega \in \widetilde{\Omega}_n \subset \bigcup_{R=1}^{\infty} \widetilde{\Omega}_R$. Hence $\widetilde{\Omega} = \bigcup_{R=1}^{\infty} \widetilde{\Omega}_R$. Clearly $\widetilde{\Omega}_R \subseteq \widetilde{\Omega}_{R+1}$, for every R. Hence by the σ-additivity,

$$\lim_{R\to\infty} \mathbb{P}\left(\widetilde{\Omega}_R\right) = \mathbb{P}\left(\bigcup_{R=1}^{\infty} \widetilde{\Omega}_R\right) = \mathbb{P}\left(\widetilde{\Omega}\right).$$

Hence we find $R > 0$ such that,

$$\mathbb{P}\left(\widetilde{\Omega}_R\right) \geq 3\varepsilon. \tag{3.3.1}$$

For this R and ε let us choose $T^*(\varepsilon, R)$ as in Proposition 3.1.22. Put $\alpha := \frac{T^*(\varepsilon,R)}{2}$

.

Recall that $\tau_\infty = \lim_{n\to\infty} \tau_n = \sup_{n\in\mathbb{N}} \tau_n$ a.s. where

$$\tau_n = \inf\left\{t \in [0, \tau_\infty) : |u|_{X_t} \geq n\right\}.$$

Set $\Omega_n := \left\{\omega \in \widetilde{\Omega} : \tau_\infty(\omega) - \tau_n(\omega) < \alpha\right\}$. We claim that

$$\widetilde{\Omega}_R = \bigcup_{n=1}^{\infty} \Omega_n.$$

The inclusion $\widetilde{\Omega}_R \supset \bigcup_{n=1}^{\infty} \Omega_n$ is obvious. For the reverse inclusion, if $\omega \in \widetilde{\Omega}_R$ then $\tau_\infty(\omega) < \infty$ and

$$\|u(t)\| \leq R, \text{ for } t \in [0, \tau_\infty].$$

Since $\lim_{n\to\infty} \tau_n(\omega) = \tau_\infty(\omega) < \infty$, there exists a natural number m such that $\tau_\infty(\omega) - \tau_m(\omega) < \alpha$. This implies that $\omega \in \Omega_m \subset \bigcup_{n=1}^{\infty} \Omega_n$, hence $\widetilde{\Omega}_R \subset \bigcup_{n=1}^{\infty} \Omega_n$. Since $(\tau_n)_{n\in\mathbb{N}}$ is increasing i.e. for all $n \in \mathbb{N}, \tau_n \leq \tau_{n+1}$, and therefore $\Omega_n \subset \Omega_{n+1}$. Hence,

$$\lim_{n\to\infty} \mathbb{P}(\Omega_n) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} \Omega_n\right) = \mathbb{P}\left(\widetilde{\Omega}_R\right).$$

Thus we can find $\delta > 0$ and a natural number m such that

$$\mathbb{P}(\Omega_m) \geq (1-\delta)\mathbb{P}\left(\widetilde{\Omega}_R\right), \tag{3.3.2}$$

If we choose $\delta = \frac{1}{3}$ we infer from (3.3.1) that

$$\mathbb{P}(\Omega_m) \geq \frac{2}{3}\mathbb{P}\left(\widetilde{\Omega}_R\right) \geq 2\varepsilon. \tag{3.3.3}$$

Set $T_0 = \tau_m$ and

$$\begin{aligned} v_0 &= u(T_0) \text{ on } \Omega_m, \\ &= 0 \text{ other wise.} \end{aligned}$$

As $T_0 = \tau_m < \tau_\infty$ and $\|u(t)\| < R$, for all $t \in [0, \tau_\infty)$, so in particular for $t = T_0$, we have

$$\mathbb{E}\|v_0\|^2 = \mathbb{E}\|u(T_0)\|^2 \leq R^2.$$

By Theorem 3.1.22 there exists a unique solution v to the problem 3.1.6 with initial condition v_0 on the interval $[T_0, T_0 + T_1)$, where T_1 is the lifespan of solution. Further,

$$\mathbb{P}(T_1 \geq T^*(\varepsilon, R)) \geq 1 - \varepsilon. \tag{3.3.4}$$

By Theorem 3.2.1 v is the local maximal solution of the problem 3.1.6 with the initial condition v_0.

Set $\widehat{\Omega} := \Omega_m \cap \{T_1 \geq T^*(\varepsilon, R)\}$. It follows from (3.3.3) and (3.3.4) that

$$\begin{aligned} \mathbb{P}\left(\widehat{\Omega}\right) &= \mathbb{P}(\Omega_m \cap \{T_1 \geq T^*(\varepsilon, R)\}) \\ &= \mathbb{P}(\Omega_m) + \mathbb{P}(\{T_1 \geq T^*(\varepsilon, R)\}) - \mathbb{P}(\Omega_m \cup \{T_1 \geq T^*(\varepsilon, R)\}) \\ &\geq 2\varepsilon + 1 - \varepsilon - 1 \\ &= \varepsilon > 0. \end{aligned}$$

Now define a process z in the following manner,

$$z(t,\omega) = \begin{cases} u(t,\omega), \text{ if } \omega \in \left(\widehat{\Omega}\right)^c \text{ and } t \in [0, \tau_\infty), \\ v(t,\omega), \text{ if } \omega \in \widehat{\Omega} \text{ and } t > T_{0,} \\ u(t,\omega), \text{ if } \omega \in \widehat{\Omega} \text{ and } t \in [0, T_0]. \end{cases}$$

Indeed, the process z defined above is local solution to the problem 3.1.6 with the initial condition u_0. Keeping in view the inequality (3.3.4) and the fact that the map

$t \to |u|_{X_t}$ is increasing. It follows that the process z satisfies the following chain of inequalities,

$$\mathbb{E}\left(|z|_{X_{\tau_\infty + \frac{1}{2}T^*(\varepsilon,R)}} \cdot 1_{\widehat{\Omega}}\right) \leq \mathbb{E}\left(|z|_{X_{T_0+T^*(\varepsilon,R)}} \cdot 1_{\widehat{\Omega}}\right),$$

$$\begin{aligned}
&\leq \mathbb{E}\left(|u|_{X_{T_0}} \cdot 1_{\widehat{\Omega}} + \left|1_{[T_0,T_0+T^*(\varepsilon,R)]}v\right|_{X_{T_0+T^*(\varepsilon,R)}} \cdot 1_{\widehat{\Omega}}\right), \\
&\leq \mathbb{E}\left(|u|_{X_{T_0}} \cdot 1_{\widehat{\Omega}}\right) + \mathbb{E}\left(\left|1_{[T_0,T_0+T^*(\varepsilon,R)]}v\right|_{X_{T_0+T^*(\varepsilon,R)}} \cdot 1_{\widehat{\Omega}}\right), \\
&\leq \mathbb{E}\left(|u|_{X_{T_0}} \cdot 1_{\Omega_m}\right) + \mathbb{E}\left(|v|_{X_{[T_0,T_0+T^*(\varepsilon,R)]}} \cdot 1_{\widehat{\Omega}}\right). \qquad (3.3.5)
\end{aligned}$$

Recall that $T_0 := \tau_m = \inf\left\{t \in [0,\tau_\infty) : |u|_{X_t} \geq m\right\}$. Clearly $T_0 < \tau_\infty < \infty$ on $\Omega_m := \left\{\omega \in \widetilde{\Omega} : \tau_\infty(\omega) - T_0(\omega) < \alpha\right\}$ hence it follows that $|u|_{X_{T_0}} = m$ on Ω_m. Thus

$$\mathbb{E}\left(|u|_{X_{T_0}} \cdot 1_{\Omega_m}\right) = m\mathbb{E}\left(1_{\Omega_m}\right) = m\mathbb{P}\left(\Omega_m\right) \leq m < \infty. \qquad (3.3.6)$$

Also recall that $v(\cdot)$ is solution such that $v(t) \in M^2\left(X_{T_0+T^*(\varepsilon,R)}\right)$, for all $t \in [T_0, T_0+T_1)$ and its life span T_1 satisfies

$$\mathbb{P}\left(T_0+T_1 > T_0+T^*(\varepsilon,R)\right) \geq \varepsilon$$

we infer that

$$\mathbb{E}\left(\left|1_{[T_0,T_0+T^*(\varepsilon,R)]}v\right|_{X_{T_0+T^*(\varepsilon,R)}}\right) \leq \mathbb{E}\left(\left|1_{[T_0,T_0+T^*(\varepsilon,R)]}v\right|^2_{X_{T_0+T^*(\varepsilon,R)}}\right) < \infty.$$

Thus

$$\mathbb{E}\left(|v|_{X_{[T_0,T_0+T^*(\varepsilon,R)]}} \cdot 1_{\widehat{\Omega}}\right) < \infty. \qquad (3.3.7)$$

Using (3.3.6) and (3.3.7) into inequality (3.3.5) we infer that,

$$\mathbb{E}\left(|z|_{X_{\tau_\infty + \frac{1}{2}T^*(\varepsilon,R)}} \cdot 1_{\widehat{\Omega}}\right) < \infty \text{ on } \widehat{\Omega} \subset \{\tau_\infty < \infty\}.$$

The last conclusion above is a clear contradiction to the maximality of u. This completes the proof of theorem. ■

3.3.2 Invariance of Manifold

The objective of this subsection is to prove invariance of manifold i.e. if the initial data belongs to submanifold M then almost all trajectories of solution to main problem (3.1.6) also belong to M. Moreover, we prove a lemma about the energy which is going to be used in proving the existence of the global solution is next subsection. Recall, the following result (Itô Lemma) from [37].

Lemma 3.3.2. *Suppose* $u \in M^2(0,T;E)$, $u_0 \in V$, $v \in M^2(0,T;H)$, $\varphi = (\varphi_i)_{i=1}^d \in M^2\left(0,T;V^d\right)$ *with*

$$u(t) = u_0 + \int_0^t v(s)ds + \sum_{i=1}^d \int_0^t \varphi_i(s)dW_i(s).$$

Let ψ be a functional on V, which is twice differentiable at each point, and satisfies:

i) ψ, ψ' and ψ'' are locally bounded,

ii) ψ and ψ' are continuous on V,

iii) for all $Q \in \mathcal{L}^1(V)$, $Tr\left[Q \circ \psi''\right]$ is continuous functional on V,

iv) If $u \in V$, $\psi'(u) \in V$; the map $u \mapsto \psi'(u)$ is continuous from V (with strong topology), into V endowed with weak topology.

v) there exists k s.t. $\|\psi'(u)\| \le k(1+\|u\|)$, for all $u \in V$.

Then:

$$\begin{aligned}\psi(u(t)) &= \psi(u_0) + \int_0^t \psi'(u(s))(v(s))\,ds + \sum_{i=1}^d \int_0^t \psi'(u(s))(\varphi_i(s))\,dW_i(s) \\ &\quad + \frac{1}{2}\sum_{i=1}^d \int_0^t \psi''(u(s))(\varphi_i(s),\varphi_i(s))\,ds.\end{aligned}$$

Following is an important lemma needed for proving the invariance of manifold.

Lemma 3.3.3. *Consider a map* $\gamma : H \to \mathbb{R}$ *by,*

$$\gamma(u) = \frac{1}{2}|u|_H^2, \text{for all } u \in H.$$

Then the map γ *is of* C^2*-class and for all* $u, h, h_1, h_2 \in H$,

$$\begin{aligned} d_u\gamma(h) &:= \langle u, h\rangle \\ d_u^2\gamma(h_1, h_2) &:= \langle h_1, h_2\rangle. \end{aligned}$$

Moreover, for $f \in V$ *if*

$$B(u) = f - \langle f, u\rangle u$$

and

$$\kappa(u) = -\langle f, B(u)\rangle u - \langle f, u\rangle B(u)$$

then

$$\begin{aligned} \langle \gamma'(u), B(u)\rangle &= \langle u, f\rangle \left(1 - |u|_H^2\right), && (3.3.8) \\ \langle \gamma'(u), \Delta u + F(u)\rangle &= \left(\|u\|^2 + |u)|_{L^{2n}}^{2n}\right)\left(|u|_H^2 - 1\right), && (3.3.9) \\ \langle \gamma'(u), \kappa(u)\rangle &= -|f|_H^2 |u|_H^2 + \langle u, f\rangle^2 \left(2|u|_H^2 - 1\right), && (3.3.10) \\ \gamma''(u)(B(u), B(u)) &= |f|_H^2 + \langle u, f\rangle^2 \left(|u|_H^2 - 2\right). && (3.3.11) \end{aligned}$$

Proof. Let us begin with calculating the first and second order Fréchet derivatives of γ. For any u and $h \in H$, let us calculate the following limit

$$\begin{aligned} \lim_{t\to 0}\frac{\gamma(u+th) - \gamma(u)}{t} &= \lim_{t\to 0}\frac{\frac{1}{2}|u+th|_H^2 - \frac{1}{2}|u|_H^2}{t} \\ &= \lim_{t\to 0}\frac{1}{2}\frac{|u|_H^2 + t^2|h|_H^2 + t\langle u, h\rangle - \frac{1}{2}|u|_H^2}{t} \\ &= \lim_{t\to 0}\frac{1}{2}\frac{t^2|h|_H^2 + t\langle u, h\rangle}{t} \\ &= \langle u, h\rangle, \text{ for all } h \in H. \end{aligned}$$

Thus we have the first order Fréchet derivative $d_u\gamma : H \to \mathbb{R}$, which can be described as the following duality,

$$d_u\gamma(h) \equiv \left\langle \gamma^{'}(u), h \right\rangle = \langle u, h\rangle, \text{ for all } h \in H. \tag{3.3.12}$$

Clearly, $d_u\gamma$ is linear. Moreover to prove continuity it is sufficient to prove boundedness. So for $h \in H$, observe that,

$$|d_u\gamma(h)| = |\langle u, h\rangle| \leq |u|_H\, |h|_H = M\, |h|_H\,,$$

where $M = |u|_H < \infty$.

Now lets turn towards computing second order derivative, for any u, h_1 and $h_2 \in H$,

$$\lim_{t\to 0}\frac{\gamma'(u+th) - \gamma'(u)}{t} = \lim_{t\to 0}\frac{u+th-u}{t} = h.$$

Hence for every $u \in H$, we have the map $\gamma''(u) : H \to H$,

$$\gamma''(u, h) := \gamma''(u)\, h = h, \text{ for all } h \in H.$$

Thus the second order derivative $d_u^2\gamma : H \times H \to \mathbb{R}$ can be given as,

$$d_u^2\gamma(h_1, h_2) \equiv \langle \gamma''(u)\, h_1, h_2\rangle = \langle h_1, h_2\rangle, \text{ for all } h_1, h_2 \in H. \tag{3.3.13}$$

Clearly $d_u^2\gamma$ is bilinear, moreover it is bounded since,

$$\left|d_u^2\gamma(h_1, h_2)\right| = |\langle h_1, h_2\rangle| \leq |h_1|_H\, |h_2|_H\,.$$

Now to obtain equality (3.3.8), for $u \in H$,

$$\begin{aligned}
\langle \gamma'(u), B(u)\rangle &= \langle u, f - \langle f, u\rangle\, u\rangle\,, \\
&= \langle u, f\rangle - \langle f, u\rangle \langle u, u\rangle\,, \\
&= \langle u, f\rangle \left(1 - |u|_H^2\right).
\end{aligned}$$

To get equality (3.3.9) using integration by parts consider the following,

$$\begin{aligned}
\langle \gamma'(u), \Delta u + F(u) \rangle &= \langle u, \Delta u + F(u) \rangle \\
&= \left\langle u, \Delta u + \left(\|u\|^2 + |u|_{L^{2n}}^{2n} \right) u - u^{2n-1} \right\rangle \\
&= \langle u, \Delta u \rangle + \left(\|u\|^2 + |u|_{L^{2n}}^{2n} \right) \langle u, u \rangle - \left\langle u, u^{2n-1} \right\rangle \\
&= -\|u\|^2 - |u|_{L^{2n}}^{2n} + \left(\|u\|^2 + |u|_{L^{2n}}^{2n} \right) |u|_H^2 \\
&= \left(\|u\|^2 + |u|_{L^{2n}}^{2n} \right) \left(|u|_H^2 - 1 \right), \text{ where } u \in E = D(A).
\end{aligned}$$

To prove the equality (3.3.10) for an arbitrary $u \in H$, consider,

$$\begin{aligned}
\langle \gamma'(u), \kappa(u) \rangle &= \langle u, - \langle f, B(u) \rangle u - \langle f, u \rangle B(u) \rangle, \\
&= - \langle f, B(u) \rangle \langle u, u \rangle - \langle f, u \rangle \langle u, B(u) \rangle, \\
&= - \langle f, f - \langle f, u \rangle u \rangle \langle u, u \rangle, \\
&\quad - \langle f, u \rangle \langle u, f - \langle f, u \rangle u \rangle, \\
&= - \left(|f|_H^2 - \langle u, f \rangle^2 \right) |u|_H^2, \\
&\quad - \langle u, f \rangle \left(\langle u, f \rangle - \langle u, f \rangle |u|_H^2 \right), \\
&= - |f|_H^2 |u|_H^2 + \langle u, f \rangle^2 |u|_H^2, \\
&\quad - \langle u, f \rangle^2 + \langle u, f \rangle^2 |u|_H^2, \\
&= - |f|_H^2 |u|_H^2 + \langle u, f \rangle^2 \left(2 |u|_H^2 - 1 \right), \text{ where } u \in H.
\end{aligned}$$

In order to prove the last required equality (3.3.11), consider

$$\begin{aligned}
\gamma''(u)(B(u), B(u)) &= \langle f - \langle f, u \rangle u, f - \langle f, u \rangle u \rangle, \\
&= \langle f, f \rangle - \langle f, u \rangle^2 - \langle f, u \rangle^2 + \langle f, u \rangle^2 \langle u, u \rangle \\
&= |f|_H^2 + \langle u, f \rangle^2 \left(|u|_H^2 - 2 \right), \text{ where } u \in H.
\end{aligned}$$

We are done with the proof. ■

We will now prove now main invariance result of this subsection. For this we will use the following stopping time,

$$\tau_k := \inf \{t \in [0,T] : \|u(t)\| \geq k\}, \text{ where } k \in \mathbb{N}. \tag{3.3.14}$$

Proposition 3.3.4. *Assume that we are in the framework of Lemma 3.3.3. If* $u_0 \in M$ *then* $u(t \wedge \tau_k) \in M$*, for all* $t \in [0,T]$.

Proof. Let us choose and fix $u(0) = u_0 \in V \cap M$, and $t \in [0,T]$. Our intentions are to apply the Itô Lemma, (Lemma 3.3.2), to the map $\gamma : H \ni u \mapsto \frac{1}{2}|u|_H^2 \in \mathbb{R}$. For the convenience let us recall the main evolution equation under consideration,

$$du(t) = \left[\Delta u(t) + F(u(t)) + \frac{1}{2}\sum_{j=1}^{N}\kappa_j(u(t))\right]dt + \sum_{j=1}^{N} B_j(u(t))\, dW_j(t), \ \mathbb{P}\text{-a.s..}$$

where F, B_j and κ_j are as defined in equations (3.1.11) and (3.1.12), respectively. For stopping time τ_k described by equation (3.3.14) let us apply Itô's Lemma to the process $\gamma(u(t \wedge \tau_k))$,. It follows that,

$$\begin{aligned}
\gamma(u(t\wedge\tau_k)) - \gamma(u_0) &= \sum_{j=1}^{N}\int_0^{t\wedge\tau_k} \langle \gamma'(u(s)), B_j(u(s))\rangle\, dW_j(s) \\
&\quad + \int_0^{t\wedge\tau_k} \langle \gamma'(u(s)), \Delta u(s) + F(u(s))\rangle\, ds \\
&\quad + \frac{1}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_k} \langle \gamma'(u(s)), \kappa_j(B_j(u(s)))\rangle\, ds \\
&\quad + \frac{1}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_k} \gamma''(u(s))\left(B_j(u(s)), B_j(u(s))\right) ds, \ \mathbb{P}\text{-a.s.}
\end{aligned}$$

Substituting equations (3.3.8), (3.3.9), (3.3.10) and (3.3.11) with $u = u(s)$ into last equation and using fact that $|u_0|_H^2 = 1$ we get

$$\begin{aligned}\frac{1}{2}\left(|u\,(t\wedge\tau_k)|_H^2-1\right) &= \sum_{j=1}^{N}\int_0^{t\wedge\tau_k}\langle u\,(s)\,,f_j\rangle\left(|u(s)|_H^2-1\right)dW_j\,(s)\\ &\quad+\int_0^{t\wedge\tau_k}\left(\|u(s)\|^2+|u(s)|_{L^{2n}}^{2n}\right)\left(|u(s)|_H^2-1\right)ds\\ &\quad+\frac{1}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_k}\left[-\,|f_j|_H^2\,|u(s)|_H^2+\langle u\,(s)\,,f_j\rangle^2\left(2\,|u(s)|_H^2-1\right)\right]ds\\ &\quad+\frac{1}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_k}\left[|f_j|_H^2+\langle u\,(s)\,,f_j\rangle^2\left(|u(s)|_H^2-2\right)\right]ds,\ \mathbb{P}\text{-a.s.}\end{aligned}$$

Combining all three Riemann integrals of last equation,

$$\begin{aligned}|u\,(t\wedge\tau_k)|_H^2-1 &= \sum_{j=1}^{N}\int_0^{t\wedge\tau_k}2\,\langle u\,(s)\,,f_j\rangle\left(|u(s)|_H^2-1\right)dW_j\,(s)\\ &\quad+\int_0^{t\wedge\tau_k}\left[\begin{array}{c}2\left(\|u(s)\|^2+|u(s)|_{L^{2n}}^{2n}\right)\\ -\,|f_j|_H^2+3\,\langle u\,(s)\,,f_j\rangle^2\end{array}\right]\left(|u(s)|_H^2-1\right)ds,\ \mathbb{P}\text{-a.s.}\end{aligned} \tag{3.3.15}$$

To simplify argument we treat $N = 1$, and for $t \geq 0$ define the following functions,

$$\begin{aligned}\varphi(t) &: = |u\,(t\wedge\tau_k)|_H^2-1\\ \alpha(t) &: = 2\,\langle u\,(t\wedge\tau_k)\,,f_1\rangle\\ \beta(t) &: = 2\left(\|u(t\wedge\tau_k)\|^2+|u(t\wedge\tau_k)|_{L^{2n}}^{2n}\right)-|f_j|_H^2+3\,\langle u\,(t\wedge\tau_k)\,,f_1\rangle^2\\ F(t,\varphi(t)) &: = \alpha(t)\varphi(t)\\ G(t,\varphi(t)) &: = \beta(t)\varphi(t)\end{aligned}$$

The last equation (3.3.15) can be rewritten as, for $t \geq 0$

$$\begin{aligned}\varphi(t) &= \int_0^{t\wedge\tau_k}F(s,\varphi(s))dW_1(s)+\int_0^{t\wedge\tau_k}G(s,\varphi(s))ds \qquad (3.3.16)\\ \text{and }\varphi(0) &: = |u\,(0)|_H^2-1=0\end{aligned}$$

To get the desired result it is sufficient to show existence and uniqueness of lastly presented problem, and for this it is enough to that F and G are Lipschitz in the second argument (See Theorem 7.7, [5]). For $x, y \in \mathbb{R}$, $t \geq 0$ and $\omega \in \Omega$,

$$\begin{aligned} |F(t,x) - F(t,y)| &= |\alpha(t,\omega)x - \alpha(t,\omega)y| = |\alpha(t,\omega)|\,|x-y| \\ \text{and } |G(t,x) - G(t,y)| &= |\beta(t,\omega)x - \beta(t,\omega)y| = |\beta(t,\omega)|\,|x-y| \end{aligned}$$

Hence, to show that F and G are Lipschitz it is only needed to show that maps α and β are bounded. Let us begin with α, using Cauchy-Schwartz inequality it follows that,

$$|\alpha(t,\omega)| \leq 2\,|\langle u\,(t \wedge \tau_k, \omega)\,, f_1\rangle| \leq 2\,|u\,(t \wedge \tau_k, \omega)|_H\,|f_1|_H$$

$f_1 \in H$ so $|f_1|_H < \infty$, also using continuity of embedding $V \hookrightarrow H$ and definition of τ_k it follows that $|u\,(t \wedge \tau_k, \omega)|_H \leq C\,\|u\,(t \wedge \tau_k, \omega)\| \leq Ck$, where $k \in \mathbb{N}$. Hence from last inequality we achieve the bounded ness of α.

Next, for $t \geq 0$ and $\omega \in \Omega$, again using continuity of embeddings $V \hookrightarrow H$ and $V \hookrightarrow L^{2n}$, and the definition of τ_k, we infer the boundedness of map β in the following manner,

$$\begin{aligned} |\beta(t)| &= \left|2\left(\|u\,(t \wedge \tau_k, \omega)\|^2 + |u\,(t \wedge \tau_k, \omega)|_{L^{2n}}^{2n}\right) - |f_j|_H^2 + 3\,\langle u\,(t \wedge \tau_k)\,, f_1\rangle^2\right| \\ &\leq 2\left(\|u\,(t,\omega)\|^2 + |u\,(t,\omega)|_{L^{2n}}^{2n}\right) + |f_1|_H^2 + 3\,|u\,(t,\omega)|_H^2\,|f_1|_H^2 \\ &\leq 2\left(\|u\,(t,\omega)\|^2 + c^{2n}\,\|u\,(t,\omega)\|^{2n}\right) + (1 + 3\,\|u\,(t,\omega)\|^2)\,|f_1|_H^2 \\ &\leq 2\left(k^2 + c^{2n}k^{2n}\right) + (1 + 3k^2)\,|f_1|_H^2 < \infty. \end{aligned}$$

Hence the unique solution v of linear equation (3.3.16) exists and since $\varphi(t) = 0$ for all $t \in [0,T]$ also satisfies (3.3.16), so by uniqueness $v(t) = \varphi(t) = 0$ for all $t \in [0,T]$, i.e. $|u\,(t \wedge \tau_k)|_H^2 = 1$ for all $t \in [0,T]$. This completes the proof. ■

Remark 3.3.5. *In this remark we will show that the map satisfies* $\gamma : H \ni u \mapsto \frac{1}{2}\,|u|_H^2 \in \mathbb{R}$ *the Itô Lemma 3.3.2. For this we show that it satisfies* *i)-v). Recall from Lemma 3.3.3 that for* $u \in H$*, we calculated* $\gamma^{'}(u) = u$ *and* $\gamma''(u)\,h = h$*, for all* $h \in H$*.*

i) We proved in the Lemma 3.3.3 that γ' *and* γ'' *are bounded hence both are locally bounded.*

ii) We saw in the Lemma 3.3.3 that γ'*,* γ''*exists and hence* γ *and* γ' *are continuous on* H*.*

iii) For every $Q \in \mathcal{L}^1(H)$*,*

$$\begin{aligned} Tr\left[Q \circ \psi''(u)\right] &= \sum_{j=1}^{\infty} \langle Q \circ \psi''(u)\, e_j, e_j \rangle \\ &= \sum_{j=1}^{\infty} \langle Q e_j, e_j \rangle \end{aligned}$$

which is a constant in $\mathbb{R}$*, so the map* $H \ni u \mapsto Tr\left[Q \circ \psi''(u)\right] \in \mathbb{R}$ *is a continuous functional on* H*.*

iv) Next we want to show that for any $u \in H$ *and* $\gamma'(u) \in H$*, the map* $u \mapsto \gamma'(u)$ *is continuous from* H *(with strong topology) into* H *(with weak topology).*

For any $h^* \in H^* = \left(L^2(D)\right)^* = H$*, and* $u \in H$*, the duality product,*

$${}_H\langle \gamma'(u+h) - \gamma'(u), h^* \rangle_H = {}_H\langle u + h - u, h^* \rangle_H = {}_H\langle h, h^* \rangle_H .$$

This shows that γ' *is weakly continuous. Let* τ *and* τ_w *be the strong and weak topologies on* H *respectively. Now the weak continuity of* γ' *implies that for every* $B \in \tau_w$ *we must have* $(\gamma')^{-1}(B) \in \tau_w \subset \tau$ *i.e.* $(\gamma')^{-1}(B) \in \tau$*. Thus we have iv).*

v) The required inequality is trivial as $\gamma^{'}(u) = u$ *for all* $u \in H$ *i.e.*

$$\left|\gamma^{'}(u)\right|_H = |u|_H \leq (1 + |u|_H) .$$

Recall, the spaces $E = D(A)$, $V = H_0^{1,2}(\mathcal{O})$, $H = L^2(\mathcal{O})$ from Remark 3.1.5. Now let us define the energy functional $\psi : V \to \mathbb{R}$, in the following manner

$$\psi(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2n}|u|_{L^{2n}}^{2n}, \text{ for all } u \in V.$$

Lemma 3.3.6. *The energy function $\psi : V \to \mathbb{R}$ defined above is of C^2-class and for all $u, h, h_1, h_2 \in V$*

$$\langle \psi'(u), h\rangle \equiv d_u\psi(h) = \langle u, h\rangle_V + \left\langle u^{2n-1}, h\right\rangle = \left\langle -\Delta u + u^{2n-1}, h\right\rangle, \tag{3.3.17}$$

$$\langle \psi''(u)\, h_1, h_2\rangle \equiv d_u^2\psi(h_1, h_2) = \langle h_1, h_2\rangle_V + \frac{(2n-1)}{n}\left\langle u^{2n-2}, h_1 h_2\right\rangle. \tag{3.3.18}$$

Moreover, for $f \in V$ if

$$B(u) = f - \langle f, u\rangle\, u$$

and

$$\kappa(u) = -\langle f, B(u)\rangle\, u - \langle f, u\rangle\, B(u)$$

then

$$\langle \psi'(u), \Delta u + F(u)\rangle = -\left|\pi_u\left(\Delta u - u^{2n-1}\right)\right|_H^2, \tag{3.3.19}$$

$$\begin{aligned}\langle \psi'(u), B(u)\rangle &= \langle u, f\rangle_V + \left\langle u^{2n-1}, f\right\rangle \\ &\quad - \langle f, u\rangle\left(\|u\|^2 + |u|_{L^{2n}}^{2n}\right)\end{aligned} \tag{3.3.20}$$

$$\begin{aligned}\langle \psi'(u), \kappa(u)\rangle &= \left(\|u\|^2 + |u|_{L^{2n}}^{2n}\right)\left[2\langle f, u\rangle^2 - |f|_H^2\right] \\ &\quad - \langle f, u\rangle\left[\langle u, f\rangle_V + \left\langle u^{2n-1}, f\right\rangle\right].\end{aligned} \tag{3.3.21}$$

$$\langle \psi''(u)\, B(u), B(u)\rangle = \|B(u)\|^2 + \frac{(2n-1)}{n}\left\langle u^{2n-2}, (B(u))^2\right\rangle. \tag{3.3.22}$$

Proof. The proof that the energy functional ψ is of C^2 has been already done in Chapter 2 Lemma 2.3.4. Here we only focus on computing the required duality products i.e. required equalities (3.3.19)-(3.3.22).

Let us begin with recalling the equation (3.1.3) i.e.

$$\pi_u\left(\Delta u - u^{2n-1}\right) = \Delta u + F(u),$$

where $\pi_u : H \to T_u M$ is orthogonal projection. Now using (3.3.17) and integration by parts,

$$\begin{aligned}
\langle \psi'(u), \Delta u + F(u) \rangle &= \langle u, \Delta u + F(u) \rangle_V + \left\langle u^{2n-1}, \Delta u + F(u) \right\rangle \\
&= \left\langle -\Delta u, \pi_u \left(\Delta u - u^{2n-1} \right) \right\rangle + \left\langle u^{2n-1}, \pi_u \left(\Delta u - u^{2n-1} \right) \right\rangle \\
&= -\left\langle \Delta u - u^{2n-1}, \pi_u \left(\Delta u - u^{2n-1} \right) \right\rangle \\
&= -\left\langle \Delta u - u^{2n-1}, \pi_u \left(\Delta u - u^{2n-1} \right) \right\rangle \\
&= -\left\langle \pi_u \left(\Delta u - u^{2n-1} \right), \pi_u \left(\Delta u - u^{2n-1} \right) \right\rangle \\
&= -\left| \pi_u \left(\Delta u - u^{2n-1} \right) \right|_H^2
\end{aligned}$$

above is required equality (3.3.19). Next,

$$\begin{aligned}
\langle \psi'(u), B(u) \rangle &= \langle u, B(u) \rangle_V + \left\langle u^{2n-1}, B(u) \right\rangle, \\
&= \langle u, f - \langle f, u \rangle u \rangle_V + \left\langle u^{2n-1}, f - \langle f, u \rangle u \right\rangle, \\
&= \langle u, f \rangle_V - \langle f, u \rangle \langle u, u \rangle_V + \left\langle u^{2n-1}, f \right\rangle - \langle f, u \rangle \left\langle u^{2n-1}, u \right\rangle, \\
&= \langle u, f \rangle_V + \left\langle u^{2n-1}, f \right\rangle - \langle f, u \rangle \left(\|u\|^2 + |u|_{L^{2n}}^{2n} \right),
\end{aligned}$$

which is required equality (3.3.20). For the equality (3.3.21),

$$\begin{aligned}
\langle \psi'(u), \kappa_j(u) \rangle &= \langle u, \kappa_j(u) \rangle_V + \left\langle u^{2n-1}, \kappa_j(u) \right\rangle \\
&= \langle u, -\langle f_j, B_j(u) \rangle u - \langle f_j, u \rangle B_j(u) \rangle_V \\
&\quad + \left\langle u^{2n-1}, -\langle f_j, B_j(u) \rangle u - \langle f_j, u \rangle B_j(u) \right\rangle
\end{aligned}$$

$$\begin{aligned}
= {} & -\langle f_j, B_j(u) \rangle \langle u, u \rangle_V - \langle f_j, u \rangle \langle u, B_j(u) \rangle_V \\
& -\langle f_j, B_j(u) \rangle \left\langle u^{2n-1}, u \right\rangle - \langle f_j, u \rangle \left\langle u^{2n-1}, B_j(u) \right\rangle
\end{aligned}$$

$$\begin{aligned}
= {} & -\langle f_j, B_j(u) \rangle \left(\|u\|^2 + |u|_{L^{2n}}^{2n} \right) - \langle f_j, u \rangle \left(\langle u, B_j(u) \rangle_V + \left\langle u^{2n-1}, B_j(u) \right\rangle \right) \\
= {} & -\langle f_j, f_j - \langle f_j, u \rangle u \rangle \left(\|u\|^2 + |u|_{L^{2n}}^{2n} \right) - \langle f_j, u \rangle \\
& \cdot \left(\langle u, f_j - \langle f_j, u \rangle u \rangle_V + \left\langle u^{2n-1}, f_j - \langle f_j, u \rangle u \right\rangle \right)
\end{aligned}$$

$$= -\left(\langle f_j, f_j\rangle - \langle f_j, u\rangle^2\right)\left(\|u\|^2 + |u|_{L^{2n}}^{2n}\right) - \langle f_j, u\rangle\left[\langle u, f_j\rangle_V - \langle f_j, u\rangle\langle u, u\rangle_V\right]$$
$$- \langle f_j, u\rangle\left[\left\langle u^{2n-1}, f_j\right\rangle - \langle f_j, u\rangle\left\langle u^{2n-1}, u\right\rangle\right]$$

$$\langle\psi'(u), \kappa_j(u)\rangle = \left(\|u\|^2 + |u|_{L^{2n}}^{2n}\right)\left[2\langle f_j, u\rangle^2 - |f_j|_H^2\right] - \langle f_j, u\rangle\left[\langle u, f_j\rangle_V + \left\langle u^{2n-1}, f_j\right\rangle\right].$$

Next lets go for the last equality (3.3.21). Using (3.3.18),

$$\begin{aligned}\langle\psi''(u) B_j(u), B_j(u)\rangle &= \langle B_j(u), B_j(u)\rangle_V + \frac{(2n-1)}{n}\left\langle u^{2n-2}, (B_j(u))^2\right\rangle \\ &= \|B_j(u)\|^2 + \frac{(2n-1)}{n}\left\langle u^{2n-2}, (B_j(u))^2\right\rangle.\end{aligned}$$

This completes the proof. ■

3.3.3 Proof of Global solution

Recall the stopping time,

$$\tau_k := \inf\{\in [0, T] : \|u(t)\| \geq k\}, \text{where } k \in \mathbb{N}.$$

Next we are going to prove the existence of unique global solution to our original problem (3.1.6).

Theorem 3.3.7. *Suppose we are in Assumptions 3.1.5 and framework of Lemma 3.3.6. Then for every $\mathbb{F}_0$ -measurable V-valued square integrable random variable $u_0 \in M$ there exists a unique global solution to the main problem* (3.1.6)*.*

Proof. Let us by recalling from Lemma 3.2.1 and Proposition 3.3.1 there exists a unique local maximal solution $u = (u(t), t \in [0, \tau))$ to problem (3.1.6), which also satisfies $\lim_{t\to\tau} \|u(t)\| = \infty$, $\mathbb{P}-a.s.$ on $\{\tau < \infty\}$.

We going to develop argument similar to the proof of Theorem 1.1 of [8] (page 7), based on Khashminskii test for non-explosion (See Theorem III.4.1 of for the finite-dimensional case). To prove that $\tau = \infty$, $\mathbb{P}$-a.s., it is sufficient to prove the following:

i) $\psi \geq 0$ on V,

ii) $q_R := \inf_{\|u\|\geq R} \psi(u) \to \infty$ as $R \to \infty$,

iii) $\psi(u(0)) < \infty$

iv) For $t > 0$, there exists a constant $C_t > 0$ such that

$$\mathbb{E}\left(\psi\left(u(t \wedge \tau_k)\right)\right) \leq C_t, \text{ where } k \in \mathbb{N}.$$

We are going to use the following inequalities at several instances in proof.

$$\begin{aligned} \|u\|^2 &\leq 2\psi\left(u\right) \\ |u|_{L^{2n}}^{2n} &\leq 2n\psi\left(u\right) \end{aligned} \tag{3.3.23}$$

Lets begin proving conditions i)-iv).

i) By definition $\psi(u) = \frac{1}{2}\|u\|^2 + \frac{1}{2n}|u|_{L^{2n}}^{2n} \geq 0$ is obvious.

ii) If $u \in V$ such that $\|u\| \geq R$ then by (3.3.23)

$$\psi\left(u\right) \geq \frac{1}{2}\|u\|^2 \geq \frac{R^2}{2} \to \infty \text{ as } R \to \infty.$$

iii) This is also easy as u_0 is V-valued square integrable and continuity of embedding $V \hookrightarrow L^{2n}$, it follows that

$$\psi(u_0) = \frac{1}{2}\|u_0\|^2 + \frac{1}{2n}|u_0|_{L^{2n}}^{2n} \leq \frac{1}{2}\|u_0\|^2 + \frac{c^{2n}}{2n}\|u_0\|^{2n} < \infty.$$

iv) Finally, to get desired inequality in required condition iv) we will use Itô Lemma 3.3.2.

Application of the Itô Lemma to the process $\left(\psi\left(u\left(t \wedge \tau_k\right)\right)\right)_{t\in[0,T]}$, gives us the following

$$\begin{aligned}\psi\left(u\left(t\wedge\tau_k\right)\right)-\psi\left(u_0\right) &= \sum_{j=1}^{N}\int_0^{t\wedge\tau_k}\left\langle\psi'\left(u(s)\right),B_j\left(u\left(s\right)\right)\right\rangle dW_j\left(s\right)\\ &+\int_0^{t\wedge\tau_k}\left\langle\psi'\left(u(s)\right),\Delta u(s)+F(u(s))\right\rangle ds\\ &+\frac{1}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_k}\left\langle\psi'\left(u(s)\right),\kappa_j(u\left(s\right))\right\rangle ds\\ &+\frac{1}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_k}\left\langle\psi''\left(u(s)\right)B_j\left(u\left(s\right)\right),B_j\left(u\left(s\right)\right)\right\rangle ds,\\ &= \sum_{j=1}^{N}I_{1,j}+I_2+I_3+\sum_{j=1}^{N}I_{4,j},\mathbb{P}\text{-a.s.},t\in[0,T]\,. \quad (3.3.24)\end{aligned}$$

Next we are going to deal with each integral in above sum.

Let us begin with I_1.We intend to show that I_1 is a martingale. In order to show that $I_{1,j}$ is martingale, it is sufficient to show that

$$\mathbb{E}\left(\int_0^{T\wedge\tau_k}\left\langle\psi'\left(u(s)\right),B_j\left(u\left(s\right)\right)\right\rangle^2 ds\right)<\infty.$$

Let us verify the above condition. Using (3.3.17) and elementary inequality $(a+b)^2\leq 2\left(a^2+b^2\right)$, we infer that

$$\begin{aligned}&\mathbb{E}\left(\int_0^{T\wedge\tau_k}\left\langle\psi'\left(u(s)\right),B_j\left(u\left(s\right)\right)\right\rangle^2 ds\right)\\ =&\ \mathbb{E}\left(\int_0^{T\wedge\tau_k}\left(\left\langle u\left(s\right),B_j\left(u\left(s\right)\right)\right\rangle_V+\left\langle u^{2n-1},B_j\left(u\left(s\right)\right)\right\rangle\right)^2 ds\right)\\ =&\ \mathbb{E}\left(\int_0^{T\wedge\tau_k}\left\langle u\left(s\right),B_j\left(u\left(s\right)\right)\right\rangle_V^2 ds\right)+\mathbb{E}\left(\int_0^{T\wedge\tau_k}\left\langle u^{2n-1},B_j\left(u\left(s\right)\right)\right\rangle^2 ds\right)\\ &+2\mathbb{E}\left(\int_0^{T\wedge\tau_k}\left\langle u\left(s\right),B_j\left(u\left(s\right)\right)\right\rangle_V\left\langle u^{2n-1},B_j\left(u\left(s\right)\right)\right\rangle ds\right)\\ \leq&\ 2\mathbb{E}\left(\int_0^{T\wedge\tau_k}\left\|u\left(s\right)\right\|^2\left\|B_j\left(u\left(s\right)\right)\right\|^2 ds\right)+2\mathbb{E}\left(\int_0^{T\wedge\tau_k}\left|u\left(s\right)^{2n-1}\right|_H^2\left|B_j\left(u\left(s\right)\right)\right|_H^2 ds\right)\end{aligned}$$

(3.3.25)

Now consider

$$\left|u^{2n-1}\right|_H^2 = \int_D u^{4n-2}(x)ds = |u|_{L^{4n-2}}^{4n-2} \tag{3.3.26}$$

Using fact that embedding $V \hookrightarrow L^{4n-2}$ is continuous so there exists constant c such that $|u|_{L^{4n-2}} \leq c\,\|u\|$. Using this fact in above equation we get

$$\left|u^{2n-1}\right|_H^2 \leq c^{4n-2}\,\|u\|^{4n-2}. \tag{3.3.27}$$

Using inequality (3.3.27) and (3.1.23) into (3.3.25) we infer that,

$$\mathbb{E}\left(\int_0^{T\wedge\tau_k} \left\langle \psi'\left(u(s)\right), B_j\left(u\left(s\right)\right)\right\rangle^2 ds\right)$$

$$\leq 2C^2\left\|f_j\right\|^2 \mathbb{E}\left(\int_0^{T\wedge\tau_k} \left\|u\left(s\right)\right\|^4 ds\right) + 2c^{4n-2}C^2\left\|f_j\right\|^2 \mathbb{E}\left(\int_0^{T\wedge\tau_k} \left\|u(s)\right\|^{4n} ds\right)$$

By definition of τ_k,we know that $\|u\left(s\right)\| \leq k$, for all $s \leq \tau_k$. Using this fact along with $f_j \in V$ into last inequality we infer that

$$\begin{aligned}\mathbb{E}\left(\int_0^{T\wedge\tau_k} \left\langle \psi'\left(u(s)\right), B_j\left(u\left(s\right)\right)\right\rangle^2 ds\right) &\leq 2k^4C^2\left\|f_j\right\|^2\left(T\wedge\tau_k\right) + 2k^{4n}c^{4n-2}C^2\left\|f_j\right\|^2\left(T\wedge\tau_k\right)\\ &< \infty.\end{aligned}$$

Hence the Itô integral $I_{1,j}$ is martingale and hence

$$\mathbb{E}\left(I_{1,j}\right) = 0. \tag{3.3.28}$$

Next consider the integral I_2. Using equation (3.3.19) it follows that, for $t \in [0,T]$,

$$\begin{aligned} I_2 &= \int_0^{t\wedge\tau_k} \left\langle \psi'\left(u(s)\right), \Delta u(s) + F(u(s))\right\rangle ds \\ &= -\int_0^{t\wedge\tau_k} \left|\pi_u\left(\Delta u(s) - u(s)^{2n-1}\right)\right|_H^2 ds \end{aligned} \tag{3.3.29}$$

Let us turn towards the third integral I_3 and simplify its integrand. Using (3.3.21), continuity of embedding $V \hookrightarrow H$ and (3.3.26),

$$\begin{aligned}
\langle \psi'(u), \kappa_j(u) \rangle &= \left(\|u\|^2 + |u|_{L^{2n}}^{2n}\right)\left[2\langle f_j, u\rangle^2 - |f_j|_H^2\right] - \langle f_j, u\rangle \left[\langle u, f_j\rangle_V + \left\langle u^{2n-1}, f_j\right\rangle\right] \\
&\leq \left(\|u\|^2 + |u|_{L^{2n}}^{2n}\right)\left[2|f_j|_H^2 |u|_H^2 - |f_j|_H^2\right] + |f_j|_H |u|_H \left[\|u\| + |f_j|_H \left|u^{2n-1}\right|_H\right] \\
&\leq \|f_j\|^2 \left(\|u\|^2 + |u|_{L^{2n}}^{2n}\right)\left[2|u|_H^2 - 1\right] \\
&\quad + c^2 \|f_j\|^2 \left[\|u\|^2 + \left|u^{2n-1}\right|_H |u|_H\right]. \qquad (3.3.30)
\end{aligned}$$

Let us settle the term $|u^{2n-1}|_H$ in the above inequality. Consider,

$$\left|u^{2n-1}\right|_H^2 = \int_D u(x)^{4n-2} dx$$

Apply the Holder inequality for $p = \frac{2n}{4n-2}$ and $q = \frac{n}{n-1}$, it follows that,

$$\begin{aligned}
\left|u^{2n-1}\right|_H^2 &\leq \left(\int_D \left(u(x)^{4n-2}\right)^{\frac{2n}{4n-2}} dx\right)^{\frac{4n-2}{2n}} \left(\int_D 1dx\right)^{\frac{1-n}{n}} = \widetilde{C}^2 |u|_{L^{2n}}^{4n-2} \\
\left|u^{2n-1}\right|_H &\leq \widetilde{C} |u|_{L^{2n}}^{2n-1} \leq \widetilde{C} \max\left\{|u|_{L^{2n}}^{2n}, 1\right\} \leq \widetilde{C}\left(|u|_{L^{2n}}^{2n} + 1\right).
\end{aligned}$$

where $\widetilde{C}^2 := \left(\int_D 1dx\right)^{\frac{1-n}{n}} < \infty$. Using the last inequality into (3.3.30),

$$\langle \psi'(u), \kappa_j(u) \rangle \leq c^2 \|f_j\|^2 \left(\|u\|^2 + |u|_{L^{2n}}^{2n}\right)\left[2|u|_H^2 - 1\right] + c^2 \|f_j\|^2 \left[\|u\|^2 + \widetilde{C}\left(|u|_{L^{2n}}^{2n} + 1\right)|u|_H\right]$$

Using the above estimate into I_3 we get,

$$\begin{aligned}
\int_0^{t\wedge\tau_k} \langle \psi'(u(s)), \kappa_j(u(s)) \rangle ds &\leq \int_0^{t\wedge\tau_k} c^2 \|f_j\|^2 \left(\|u(s)\|^2 + |u(s)|_{L^{2n}}^{2n}\right)\left[2|u(s)|_H^2 - 1\right] \\
&\quad + c^2 \|f_j\|^2 \left[\|u(s)\|^2 + \widetilde{C}\left(|u(s)|_{L^{2n}}^{2n} + 1\right)|u(s)|_H\right] ds
\end{aligned}$$

Using the fact of invariance i.e. $u(t) \in M$ for all $t \in [0, T]$ into above inequality we infer that

$$\begin{aligned}
&\int_0^{t\wedge\tau_k} \langle \psi'(u(s)), \kappa_j(u(s)) \rangle ds \\
&\leq \int_0^{t\wedge\tau_k} \left[c^2 \|f_j\|^2 \left(\|u(s)\|^2 + |u(s)|_{L^{2n}}^{2n}\right) + c^2 \|f_j\|^2 \left[\|u(s)\|^2 + \widetilde{C}\left(|u(s)|_{L^{2n}}^{2n} + 1\right)\right]\right] ds \\
&= c^2 \|f_j\|^2 \int_0^{t\wedge\tau_k} \left[2\|u(s)\|^2 + (1 + \widetilde{C})|u(s)|_{L^{2n}}^{2n} + \widetilde{C}\right] ds
\end{aligned}$$

Now using the inequality (3.3.23) in the inequality above we get

$$\begin{aligned}\int_0^{t\wedge\tau_k}\langle\psi'(u(s)),\kappa_j(u(s))\rangle ds &\le c^2\|f_j\|^2\int_0^{t\wedge\tau_k}\left[2\psi(u(s))+2n(1+\widetilde{C})\psi(u(s))+\widetilde{C}\right]ds\\ &\le c^2\|f_j\|^2\left(2+2n(1+\widetilde{C})\right)\int_0^{t\wedge\tau_k}\psi(u(s))ds\\ &\quad +\widetilde{C}c^2\|f_j\|^2(t\wedge\tau_k)\\ &= C_1\int_0^{t\wedge\tau_k}\psi(u(s))ds+C_2(t\wedge\tau_k) \qquad (3.3.31)\end{aligned}$$

where $C_1 := c^2\|f_j\|^2\left(2+2n(1+\widetilde{C})\right)<\infty$ and $C_2 := \widetilde{C}c^2\|f_j\|^2<\infty$.

Now turn towards the final integral i.e. $I_{4,j}$. Let us start by considering the integrand of I_4. Using equation (3.3.22)

$$\begin{aligned}\langle\psi''(u)B_j(u),B_j(u)\rangle &= \|B_j(u)\|^2+\frac{(2n-1)}{n}\left\langle u^{2n-2},(B_j(u))^2\right\rangle\\ &\le \|f_j-\langle f_j,u\rangle u\|^2+\frac{(2n-1)}{n}\left\langle u^{2n-2},(B_j(u))^2\right\rangle\\ &\le \left(\|f_j\|+|f_j|_H|u|_H\|u\|\right)^2\\ &\quad +\frac{(2n-1)}{n}\left\langle u^{2n-2},(B_j(u))^2\right\rangle \qquad (3.3.32)\end{aligned}$$

Now let us deal with term $\left\langle u^{2n-2},(B_j(u))^2\right\rangle$ involved in inequality above. Below we have used elementary inequality $(a-b)^2\le 2(a^2+b^2)$. Consider

$$\begin{aligned}\left\langle u^{2n-2},(B_j(u))^2\right\rangle &= \left\langle u^{2n-2},(f_j-\langle f_j,u\rangle u)^2\right\rangle\\ &\le \left\langle u^{2n-2},2\left(f_j^2+\langle f_j,u\rangle^2u^2\right)\right\rangle\\ &= 2\left\langle u^{2n-2},f_j^2\right\rangle+2\langle f_j,u\rangle^2\left\langle u^{2n-2},u^2\right\rangle\\ &= 2\left\langle u^{2n-2},f_j^2\right\rangle+2|f_j|_H^2|u|_H^2|u|_{L^{2n}}^{2n}\\ &= 2\int_D u(x)^{2n-2}f_j(x)^2dx+2|f_j|_H^2|u|_H^2|u|_{L^{2n}}^{2n}\end{aligned}$$

Using the Holders inequality for $p = \frac{n}{n-1}$ and $q = n$, on the first term we infer that,

$$\begin{aligned}
\left\langle u^{2n-2}, (B_j(u))^2 \right\rangle &\leq 2\left(\int_D u(x)^{2n}dx\right)^{\frac{n-1}{n}} \left(\int_D f_j(x)^{2n}dx\right)^{\frac{1}{n}} + 2|f_j|_H^2 |u|_H^2 |u|_{L^{2n}}^{2n} \\
&= 2|u|_{L^{2n}}^{2n-2} |f_j|_{L^{2n}}^2 + 2|f_j|_H^2 |u|_H^2 |u|_{L^{2n}}^{2n} \\
&\leq 2c\|f_j\|^2 |u|_{L^{2n}}^{2n-2} + 2|f_j|_H^2 |u|_H^2 |u|_{L^{2n}}^{2n} \\
&\leq 2c\|f_j\|^2 \max\left\{|u|_{L^{2n}}^{2n}, 1\right\} + 2|f_j|_H^2 |u|_H^2 |u|_{L^{2n}}^{2n} \\
&\leq 2c\|f_j\|^2 \left(|u|_{L^{2n}}^{2n} + 1\right) + 2|f_j|_H^2 |u|_H^2 |u|_{L^{2n}}^{2n} \\
&= \left(2c\|f_j\|^2 + 2|f_j|_H^2 |u|_H^2\right) |u|_{L^{2n}}^{2n} + 2c\|f_j\|^2 .
\end{aligned}$$

Using the last two inequalities into (3.3.32) we get

$$\begin{aligned}
\langle \psi''(u) B_j(u), B_j(u) \rangle &\leq (\|f_j\| + c\|f_j\| |u|_H \|u\|)^2 \\
&\quad + \frac{(2n-1)}{n}\left[\left(2c\|f_j\|^2 + 2|f_j|_H^2 |u|_H^2\right) |u|_{L^{2n}}^{2n} + 2c\|f_j\|^2\right].
\end{aligned}$$

Finally we are position to use above estimate in $I_{4,j}$,

$$\begin{aligned}
I_{4,j} &= \int_0^{t\wedge\tau_k} \langle \psi''(u(s)) B_j(u(s)), B_j(u(s)) \rangle ds \\
&\leq \int_0^{t\wedge\tau_k} \left[\begin{array}{c} \|f_j\|^2 (1 + c|u(s)|_H \|u(s)\|)^2 \\ + \frac{(2n-1)}{n} 2c\|f_j\|^2 \left[\left(1 + |u|_H^2\right) |u|_{L^{2n}}^{2n} + 2c\|f_j\|^2\right] \end{array}\right] ds
\end{aligned}$$

Since almost all trajectories $u(t) \in M$, hence the above inequality can be further simplified to

$$I_{4,j} \leq \int_0^{t\wedge\tau_k} \left[\begin{array}{c} \|f_j\|^2 (1+c)^2 \|u(s)\|^2 \\ + \frac{(2n-1)}{n} 2c\|f_j\|^2 \left[2|u|_{L^{2n}}^{2n} + 2c\|f_j\|^2\right] \end{array}\right] ds$$

Finally use of inequality (3.3.23) gives

$$\begin{aligned}
I_{4,j} &\leq \int_0^{t\wedge\tau_k} \left[\begin{array}{c} 2\|f_j\|^2 (1+c)^2 \psi(u(s)) \\ + \frac{(2n-1)}{n} 2c\|f_j\|^2 \left[2\psi(u(s)) + 2c\|f_j\|^2\right] \end{array}\right] ds \\
&= \int_0^{t\wedge\tau_k} \left[\left[2\|f_j\|^2 (1+c)^2 + 4c\frac{(2n-1)}{n}\|f_j\|^2\right] \psi(u(s)) + \frac{(2n-1)}{n}(2c)^2 \|f_j\|^2\right] ds \\
&= C_3 \int_0^{t\wedge\tau_k} \psi(u(s))ds + C_4(t\wedge\tau_k) \qquad (3.3.33)
\end{aligned}$$

where $C_3 := 2\left\|f_j\right\|^2(1+c)^2+4c\frac{(2n-1)}{n}\left\|f_j\right\|^2<\infty$ and $C_4 := \frac{(2n-1)}{n}(2c)^2\left\|f_j\right\|^2<\infty$.

Now combining the inequalities (3.3.33), (3.3.31), (3.3.29) into (3.3.24) we infer that

$$\begin{aligned}
\psi\left(u\left(t \wedge \tau_k\right)\right)-\psi\left(u_0\right) \quad \leq \quad & \sum_{j=1}^N I_{1,j}-\int_0^{t \wedge \tau_k}\left|\pi_u\left(\Delta u(s)-u(s)^{2n-1}\right)\right|_H^2 ds \\
& +C_1 \int_0^{t \wedge \tau_k} \psi(u(s)) ds+C_2\left(t \wedge \tau_k\right) \\
& +\sum_{j=1}^N\left(C_3 \int_0^{t \wedge \tau_k}\|u(s)\|^2 ds+C_4\left(t \wedge \tau_k\right)\right)
\end{aligned}$$

Since the integrand in the second is non-negative so hence we can drop this term from sum. It follows that,

$$\begin{aligned}
\psi\left(u\left(t \wedge \tau_k\right)\right)-\psi\left(u_0\right) \quad \leq \quad & \sum_{j=1}^N I_{1,j}+C_1 \int_0^{t \wedge \tau_k} \psi(u(s)) ds+C_2\left(t \wedge \tau_k\right) \\
& +NC_3 \int_0^{t \wedge \tau_k} \psi(u(s)) ds+NC_4\left(t \wedge \tau_k\right) \\
= \quad & \sum_{j=1}^N I_{1,j}+C_5 \int_0^{t \wedge \tau_k} \psi(u(s)) ds+C_6\left(t \wedge \tau_k\right),
\end{aligned}$$

where $C_5=NC_3+C_1<\infty$, and $C_6:=NC_4+C_2<\infty$. Taking expectation on both sides and using equation (3.3.28), we infer that

$$\mathbb{E}\left(\psi\left(u\left(t \wedge \tau_k\right)\right)\right) \leq \mathbb{E}\left(\psi\left(u_0\right)\right)+C_6 T+C_5 \mathbb{E}\left(\int_0^{t \wedge \tau_k} \psi(u(s)) ds\right)$$

Thus using the Gronwall lemma we infer that

$$\begin{aligned}
\mathbb{E}\left(\psi\left(u\left(t \wedge \tau_k\right)\right)\right) \quad \leq \quad & \mathbb{E}\left(\psi\left(u_0\right)\right)+C_6 T+\int_0^{t \wedge \tau_k}\left(\mathbb{E}\left(\psi\left(u_0\right)\right)+C_6\left(t \wedge \tau_k\right)\right) \\
\cdot C_5 \exp\left(\int_s^{t \wedge \tau_k} C_5 ds\right) \quad : \quad & = C_t<\infty .
\end{aligned}$$

Hence, the all four condition of Khashminskii test for non-explosion is true. Thus $\tau=\infty$, $\mathbb{P}$-a.s. .

This completes the proof. ■

Chapter 4

Large Deviation Principle

4.1 General introduction to LDP and weak convergence Method

In the modern probability theory, the notion of the theory of large deviation is an attempt to understand the asymptotic behavior of remote tails of families of probability distributions. In the beginning, the theory of large deviations was developed and employed for computing asymptotics of rare i.e. small probability events on an exponential scale. This precise computation of the probabilities of such small events plays a pivotal role in studying several important problems. For instance, in the insurance business, from the perspective of the insurance company, the revenue per month is constant but claims can pop up randomly at any instant in a month. Indeed, for an insurance company to be in the profit over a specific period of time, its total revenue should be greater than the total claims. Therefore, insurance companies must be interested in knowing the premium P over the n months so that total claims should stay less then nP, hence this problem demands to study the asymptotic behavior of small probability events of claims. The answer of this question was developed by Cramér (see [19]), and this answer becomes the first rigorous attempt to develop large deviation theory. Some of the beautiful applications of large deviation theory can also be found in statistical mechanics, thermodynamics, risk management and information theory and quantum mechanics.

4.1.1 An overview of weak convergence method for LDP

The aim of this subsection is to introduce the general framework of weak convergence, which we will use in latter subsections to prove the large deviation principle for our stochastic evolution equation of concern. Let us begin our introduction to weak convergence approach.

Throughout the subsection we are going to treat $\{u^\varepsilon\}$ as family of X-valued random variables defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where X is a Polish space i.e. separable Banach space.

Definition 4.1.1. *A function $\mathcal{I} : X \to [0, \infty]$ is called a* ***rate function*** *if $\mathcal{I}$ is lower semi-continuous i.e. for each $k \in \mathbb{R}$ the set $\{x \in X : \mathcal{I}(x) \leq k\}$ is closed (equivalently, the set $\{x \in X : \mathcal{I}(x) > k\}$ is open). A rate function I is called* ***good rate function*** *if the level set $\{x \in X : \mathcal{I}(x) \leq k\}$ is compact for each finite number*

k.

Definition 4.1.2. *(Large deviation principle) The family $\{u^\varepsilon\}$ of X-valued random variables, is said to satisfy the* ***large deviation principle (LDP)*** *with the rate function $\mathcal{I}$ if for each Borel subset B of X, we have:*

$$-\inf_{x\in \overset{\circ}{B}} \mathcal{I}(x) \leq \liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(u^\varepsilon \in B\right) \leq \limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(u^\varepsilon \in B\right) \leq -\inf_{x\in \overline{B}} \mathcal{I}(x),$$

where $\overset{\circ}{B}$ and $\overline{B}$ denotes the interior and closure of B in X.

Definition 4.1.3. *(Laplace principle) The family $\{u^\varepsilon\}$ of X-valued random variables, is said to satisfy* ***Laplace principle (LP)*** *with the rate function $\mathcal{I}$, if for each real-valued bounded continuous function f defined on X we have:*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{E}\left\{e^{-\frac{f(u^\varepsilon)}{\varepsilon^2}}\right\} = -\inf_{x\in X}\left\{f(x) + \mathcal{I}(x)\right\}.$$

Remark 4.1.4. *The weak convergence method mainly is based on equivalence of the Laplace principle and the large deviation principle, provided that X is Polish space and I is good rate function. This equivalence was formulated in [40] and can also be deduced as a consequence of Varadhan's Lemma [50] and Bryc's converse theorem [4]. Another elementary proof of equivalence can be found in [21] and [20].*

In the light of last remark we now intend to present sufficient conditions to prove the Laplace principle. Suppose that $W = (W_t)_{t\geq 0}$ be Wiener process on a separable Hilbert space Y with respect to complete filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$, i.e. trajectories of W take values in $C\left([0,T]\,; Z\right)$ where Z is another Hilbert space such that embedding of Y into Z is Hilbert-Schmidt. Suppose $g^\varepsilon : C\left([0,T]\,; Z\right) \to X$ is measurable map and $X^\varepsilon = g^\varepsilon\left(W\right)$.

Let

$$\Lambda := \left\{v : v \text{ is } Y\text{-valued } \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]} \text{ predictable process s.t. } \int_0^T \left|v_t\left(\omega\right)\right|_Y^2 dt < \infty \text{ a.s.}\right\},$$

and

$$S_N := \left\{ \phi \in L^2 \left([0,T], Y \right) : \int_0^T |\phi(t)|_Y^2 \, dt \leq N \right\}.$$

The set S_M endowed with the weak topology, is a Polish space. Define

$$\Lambda_N := \{ v \in \Lambda : v(\omega) \in S_N, \, \mathbb{P}\text{-a.s.} \}.$$

One of the crucial step in proving Laplace principle is based on the following variational representation formula obtained in [16]:

$$-\log \mathbb{E}\left\{ e^{-f(W.)} \right\} = \inf_{v \in \Lambda} \mathbb{E}\left(\frac{1}{2} \int_0^T |v(t)|_Y^2 \, dt + f\left(W. + \int_0^{\cdot} v(s) ds \right) \right), \qquad (4.1.1)$$

where f is any bounded Borel measurable function $C([0,T];Z)$ into $\mathbb{R}$. In case of the finite dimensional Brownian motion the formula (4.1.1) was proven in [2]. In [16] the following sufficient conditions, for Laplace principle (equivalently, large deviation principle) of $\{u^\varepsilon\}$ as $\varepsilon \to 0$, were proved.

Condition 4.1.5. *One can find a measurable map $g^0 : C([0,T];Z) \to X$ such that the following two conditions hold:*

A1) *For each finite constant N, the set*

$$K_N = \left\{ g^0 \left(\int_0^{\cdot} \phi(s) ds \right) : \phi \in S_M \right\}$$

is a compact subset of X.

A2) *Let $\{v^\varepsilon : \varepsilon > 0\} \subset \Lambda_M$ for some $M < \infty$. If v^ε converges to v in the sense of distributions as S_M-valued random elements, then $g^\varepsilon \left(W. + \frac{1}{\varepsilon} \int_0^{\cdot} v^\varepsilon(s) \, ds \right)$ converges to $g^\varepsilon \left(\int_0^{\cdot} v^\varepsilon(s) \, ds \right)$ in the sense of distributions as $\varepsilon \to 0$.*

Lemma 4.1.6. *([16], Theorem 4.4) If $\{g^\varepsilon\}$ satisfies A1) and A2) of condition* (4.1.5), *then family $\{u^\varepsilon\}$ satisfies the Laplace principle (and hence LDP) on X with good rate function $\mathcal{I}$ given by:*

$$\mathcal{I}(f) := \inf_{\phi \in L^2([0,T]:Y): f = g^0\left(\int_0^{\cdot} \phi(s) ds\right)} \left\{ \frac{1}{2} \int_0^T |\phi(s)|_Y^2 \, ds \right\}, \ \textit{where } f \in X.$$

Thus, the last lemma provides us a beautiful way to achieve the LDP by just verifying the assumptions A1) and A2) of Condition (4.1.5). Indeed, one of the key advantage to employ this weak convergence method to prove LDP is that one can avoid the exponential probability estimates, which can be possibly complicated in the case of infinite-dimensional models.

4.2 LDP for the Stochastic Heat equation on Hilbert Manifold

Recall that $E = D(A)$, $V = D(A^{1/2})$, $H = L^2(\mathcal{O})$ and the embeddings $E \hookrightarrow V \hookrightarrow H$ are dense and continuous. The primary aim of this section is to prove the large deviation principle for a family of distribution of solutions the following small noise problem.

$$\begin{aligned} du^\varepsilon &= (\Delta u^\varepsilon + F(u^\varepsilon))\,dt + \sqrt{\varepsilon}\sum_{j=1}^{N} B_j(u^\varepsilon)\circ dW_j \\ &= \left[\Delta u^\varepsilon + F(u^\varepsilon) + \frac{\varepsilon}{2}\sum_{j=1}^{N}\kappa_j(B_j(u^\varepsilon))\right]dt + \sqrt{\varepsilon}\sum_{j=1}^{N} B_j(u^\varepsilon)dW_j \\ u^\varepsilon(0) &= u_0, \end{aligned} \tag{4.2.1}$$

where $\varepsilon \in (0,1]$, the map $F : V \to H$ is defined by

$$F(u^\varepsilon) := \|u^\varepsilon\|^2 u^\varepsilon - (u^\varepsilon)^{2n-1} + u^\varepsilon |u^\varepsilon|^{2n}_{L^{2n}},$$

where n is a natural number (or, more generally, a real number bigger than $\frac{1}{2}$). Moreover, for each $j = 1, 2, 3..N$, the map $B_j : V \to V$ defined by,

$$B_j(u^\varepsilon) = f_j - \langle f_j, u^\varepsilon\rangle u^\varepsilon,$$

where $f_1, f_2, ...f_N$ are fixed elements from V. Finally u_0 is $\mathcal{F}_0$-measurable V-valued square integrable random variable which belongs to manifold M.

Let us begin by collecting all necessary notions needed to state our main result. The first such object is to prove the existence of a Borel measurable map $\Im^\varepsilon : C([0,T]:\mathbb{R}) \to X_T$ such that action of $\Im^\varepsilon$ on the Wiener process i.e. $\Im^\varepsilon(W.)$ is a strong solution to (4.2.1).

Since problem (4.2.1) is particular case of the main problem (3.1.6) studied in chapter 3, so from Chapter 3 Theorem 3.3.7 we may infer that, in particular, there exists a unique strong solution $\{u^\varepsilon\}$ of the taking values in $C([0,T];V\cap M)\cap L^2([0,T],E)$ of problem (4.2.1).

Let us begin with an important proposition about proving the path-wise uniqueness of solutions of problem (4.2.1), this will lead us to the existence of our required Borel measurable map.

Proposition 4.2.1. *Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space and with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. Let $u_1, u_2 : [0,T] \to H$ be $\mathbb{F}$-progressively measurable processes such that, for $i = 1, 2$, the paths of u_i lie in $C([0,T];V\cap M)\cap L^2([0,T],E)$ and each u_i satisfies*

$$u_i^\varepsilon(t) = \left[\Delta u_i^\varepsilon + F(u_i^\varepsilon) + \frac{\varepsilon}{2}\sum_{j=1}^N \kappa_j(B_j(u_i^\varepsilon))\right] dt + \sqrt{\varepsilon}\sum_{j=1}^N B_j(u_i^\varepsilon)dW_j, \text{ for all } t\in[0,T], \ \mathbb{P}\text{-a.s.}$$

Then

$$u_1^\varepsilon(\cdot,\omega) = u_2^\varepsilon(\cdot,\omega), \ \mathbb{P}\text{-a.e.}$$

Proof. To show path-wise uniqueness, let u_1^ε and u_2^ε be the two solutions to the problem 4.2.1 i.e.

$$u_i^\varepsilon(t) = u_0 + \int_0^t (\Delta u_i^\varepsilon(s) + F(u_i^\varepsilon(s)))ds + \frac{\varepsilon}{2}\sum_{j=1}^N \int_0^t \kappa_j(u_i^\varepsilon(s))ds + \sqrt{\varepsilon}\sum_{j=1}^N \int_0^t B_j(u_i^\varepsilon(s))dW_j(s).$$

where $i = 1, 2$. Set $Z^\varepsilon = u_1^\varepsilon - u_2^\varepsilon$. The integral equation for Z can be written as,

$$\begin{aligned} Z^\varepsilon(t) &= \int_0^t \Delta Z^\varepsilon(s)ds + \int_0^t [F(u_1^\varepsilon(s)) - F(u_2^\varepsilon(s))]\, ds \\ &\quad + \frac{\varepsilon}{2}\sum_{j=1}^N \int_0^t [\kappa_j(u_1^\varepsilon(s)) - \kappa_j(u_2^\varepsilon(s))]\, ds \\ &\quad + \sqrt{\varepsilon}\sum_{j=1}^N \int_0^t [B_j(u_1^\varepsilon(s)) - B_j(u_2^\varepsilon(s))]\, dW_j(s). \end{aligned}$$

Next we want to apply the Itô Lemma to the process $\frac{1}{2}|Z^\varepsilon(t)|_H^2$, we get

$$\begin{aligned} \frac{1}{2}|Z^\varepsilon(t)|_H^2 &= \int_0^t \langle Z^\varepsilon(s), \Delta Z^\varepsilon(s)\rangle\, ds + \int_0^t \langle Z^\varepsilon(s), F(u_1^\varepsilon(s)) - F(u_2^\varepsilon(s))\rangle\, ds \\ &\quad + \frac{\varepsilon}{2}\sum_{j=1}^N \int_0^t \langle Z^\varepsilon(s), \kappa_j(u_1^\varepsilon(s)) - \kappa_j(u_2^\varepsilon(s))\rangle\, ds \\ &\quad + \sqrt{\varepsilon}\sum_{j=1}^N \int_0^t \langle Z^\varepsilon(s), B_j(u_1^\varepsilon(s)) - B_j(u_2^\varepsilon(s))\rangle\, dW_j(s) \\ &\quad + \frac{\varepsilon}{2}\sum_{j=1}^N \int_0^t |B_j(u_1^\varepsilon(s)) - B_j(u_2^\varepsilon(s))|_H^2\, ds \\ &= I_1(t) + I_2(t) + \frac{\varepsilon}{2}\sum_{j=1}^N I_{3,j}(t) + \sqrt{\varepsilon}\sum_{j=1}^N I_{4,j}(t) \\ &\quad + \frac{\varepsilon}{2}\sum_{j=1}^N I_{5,j}(t). \end{aligned} \tag{4.2.2}$$

Let estimate each of the integral involved in above equation. Let us begin with I_1, using integration by parts,

$$\begin{aligned} I_1(t) &= \int_0^t \langle Z^\varepsilon(s), \Delta Z^\varepsilon(s)\rangle\, ds \\ &= -\int_0^t \langle \nabla Z^\varepsilon(s), \nabla Z^\varepsilon(s)\rangle\, ds \\ &= -\int_0^t \|Z^\varepsilon(s)\|^2\, ds \\ &\le -\frac{1}{2}\int_0^t \|Z^\varepsilon(s)\|^2\, ds \end{aligned} \tag{4.2.3}$$

Next consider I_2,

$$\begin{aligned}
I_2(t) &= \int_0^t \langle Z^\varepsilon(s), F(u_1^\varepsilon(s)) - F(u_2^\varepsilon(s))\rangle \, ds \\
&\leq \int_0^t |Z^\varepsilon(s)|_H \, |F(u_1^\varepsilon(s)) - F(u_2^\varepsilon(s))|_H \, ds \\
&\leq \int_0^t |Z^\varepsilon(s)|_H \, G(\|u_1^\varepsilon(s)\|, \|u_2^\varepsilon(s)\|) \, \|u_1^\varepsilon(s) - u_2^\varepsilon(s)\| \, ds \\
&= \int_0^t G(\|u_1^\varepsilon(s)\|, \|u_2^\varepsilon(s)\|) \, |Z^\varepsilon(s)|_H \, \|Z^\varepsilon(s)\| \, ds
\end{aligned}$$

Application of the Young inequality gives,

$$I_2(t) \leq \frac{1}{2}\int_0^t \|Z^\varepsilon(s)\|^2 \, ds + \frac{1}{2}\int_0^t G(\|u_1^\varepsilon(s)\|, \|u_2^\varepsilon(s)\|)^2 \, |Z^\varepsilon(s)|_H^2 \, ds \tag{4.2.4}$$

Consider $I_{3,j}$,

$$\begin{aligned}
I_{3,j}(t) &= \int_0^t \langle Z^\varepsilon(s), \kappa_j(u_1^\varepsilon(s)) - \kappa_j(u_2^\varepsilon(s))\rangle \, ds \\
&\leq \int_0^t |Z^\varepsilon(s)|_H \, |\kappa_j(u_1^\varepsilon(s)) - \kappa_j(u_2^\varepsilon(s))|_H \, ds \\
&\leq \int_0^t |Z^\varepsilon(s)|_H^2 \, G_1\left(|u_1^\varepsilon(s)|_H, |u_2^\varepsilon(s)|_H\right) ds
\end{aligned} \tag{4.2.5}$$

where

$$G_{1,j}\left(|u_1^\varepsilon(s)|_H, |u_2^\varepsilon(s)|_H\right) := |f_j|_H^2 \left[2 + |u_1^\varepsilon(s)|_H^2 + |u_2^\varepsilon(s)|_H^2 + C\left(|u_1^\varepsilon(s)|_H + |u_2^\varepsilon(s)|_H\right)^2\right].$$

Consider $I_{5,j}$,

$$\begin{aligned}
I_{5,j}(t) &= \int_0^t |B_j(u_1^\varepsilon(s)) - B_j(u_2^\varepsilon(s))|_H^2 \, ds \\
&= C\,|f_j|_H^2 \int_0^t \left(|u_1^\varepsilon(s)|_H + |u_2^\varepsilon(s)|_H\right)^2 |Z^\varepsilon(s)|_H^2 \, ds \\
&= C\,|f_j|_H^2 \int_0^t \left(|u_1^\varepsilon(s)|_H + |u_2^\varepsilon(s)|_H\right)^2 |Z^\varepsilon(s)|_H^2 \, ds.
\end{aligned} \tag{4.2.6}$$

Using (4.2.3)-(4.2.6) in equation 4.2.2 we get,

$$\begin{aligned}
\frac{1}{2}\left|Z^\varepsilon(t)\right|_H^2 &\leq -\frac{1}{2}\int_0^t \left\|Z^\varepsilon(s)\right\|^2 ds + \frac{1}{2}\int_0^t \left\|Z^\varepsilon(s)\right\|^2 ds \\
&+\frac{1}{2}\int_0^t G(\left\|u_1^\varepsilon(s)\right\|, \left\|u_2^\varepsilon(s)\right\|)^2 \left|Z^\varepsilon(s)\right|_H^2 ds \\
&+\frac{\varepsilon}{2}\sum_{j=1}^N \int_0^t \left|Z^\varepsilon(s)\right|_H^2 G_{1,j}\left(\left|u_1^\varepsilon(s)\right|_H^2, \left|u_2^\varepsilon(s)\right|_H^2\right) ds \\
&+\sqrt{\varepsilon}\sum_{j=1}^N I_{4,j}(t) \\
&+\frac{\varepsilon}{2}\sum_{j=1}^N \int_0^t C\left|f_j\right|_H^2 \left(\left|u_1^\varepsilon(s)\right|_H + \left|u_2^\varepsilon(s)\right|_H\right)^2 \left|Z^\varepsilon(s)\right|_H^2 ds.
\end{aligned}$$

Hence above simplifies to

$$\frac{1}{2}\left|Z^\varepsilon(t)\right|_H^2 \leq \int_0^t \varphi(s)\left|Z^\varepsilon(s)\right|_H^2 + \sqrt{\varepsilon}\xi(t),$$

where

$$\begin{aligned}
\varphi(s) \quad &: \quad = G(\left\|u_1^\varepsilon(s)\right\|, \left\|u_2^\varepsilon(s)\right\|)^2 + \frac{\varepsilon}{2}\sum_{j=1}^N G_{1,j}\left(\left|u_1^\varepsilon(s)\right|_H^2, \left|u_2^\varepsilon(s)\right|_H^2\right) \\
&+\frac{\varepsilon}{2}\sum_{j=1}^N C\left|f_j\right|_H^2 \left(\left|u_1^\varepsilon(s)\right|_H + \left|u_2^\varepsilon(s)\right|_H\right)^2.
\end{aligned}$$

and ξ is the following process $\mathbb{R}$-valued process,

$$\xi(t) := \sum_{j=1}^N I_{4,j}(t) = \int_0^t \sum_{j=1}^N \left[B_j(u_1^\varepsilon(t)) - B_j(u_2^\varepsilon(t))\right] dW_j(t).$$

By applying the Itô Lemma to the following $\mathbb{R}$-valued process

$$Y(t) := \left|Z^\varepsilon(t)\right|_H^2 e^{-\int_0^t \varphi(s)ds},\ t \in [0,T],$$

we may infer that,

$$\begin{aligned}
Y(t) &\leq \int_0^t e^{-\int_0^t \varphi(s)ds} d\xi(t) \\
&= \sqrt{\varepsilon}\sum_{j=1}^N e^{-\int_0^t \varphi(s)ds}\left\langle B_j(u_1^\varepsilon) - B_j(u_2^\varepsilon), Z^\varepsilon(s)\right\rangle dW_j(s), t \in [0,T].
\end{aligned}$$

Since $u_1^\varepsilon, u_2^\varepsilon$ and Z are uniformly bounded and B_j is locally Lipschitz so it follows that the right-hand side of the above inequality is an $\mathbb{F}$-martingale. Hence taking expectation into last inequality gives us,

$$\mathbb{E}Y(t) \leq 0.$$

But from definition we see that $Y(t) \geq 0$ and hence it follows that $Y(t) = 0$, for all $t \in [0,T]$, $\mathbb{P}$-a.s. If we substitute $Y = 0$ in definition of Y we infer that

$$\begin{aligned} |Z^\varepsilon(t)|_H^2 &= 0, \text{ for all } t \in [0,T], \mathbb{P} - a.s. \\ \text{i.e. } Z^\varepsilon(t) &= 0, \text{ for all } t \in [0,T], \mathbb{P} - a.s. \end{aligned}$$

This completes the proof of the Theorem ■

Lemma 4.2.2. *Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space and with filtration $\mathbb{F} = (\mathcal{F}_t)$. There exists a Borel measurable map $\Im^\varepsilon : C([0,T] : \mathbb{R}) \to X_T$ such that $\Im^\varepsilon(W.)$ (i.e. action of $\Im^\varepsilon$ on Wiener process) is a strong solution to* (4.2.1).

Proof. Our argument is essentially on same lines of Theorem 4.2 and 4.4 of [14]. Note that small Noise problem (4.2.1) is a particular case of the main Problem 3.1.6. Hence by Theorem 3.3.7 there exists a $C([0,T]; V \cap M) \cap L^2(0,T;E)$-valued unique strong solution u^ε to the problem (4.2.1). Moreover we explicitly shown the pathwise uniqueness in Lemma 4.2.1 for small noise problem. Once we have pathwise uniqueness then the existence of required Borel measurable $\Im^\varepsilon$ is guaranteed Yamada-Watanabe Theorem for mild solutions from [34] such that $\Im^\varepsilon(W.) = u^\varepsilon$ is $C([0,T]; V \cap M) \cap L^2(0,T;E)$-valued strong solution to the problem (4.2.1). ■

Next important notion of studying is the skeleton equation. In order to state our main result, we introduce first the following skeleton/controlled problem associated with equation (4.2.1).

For $h = (h_j)_{j=1}^N \in L^2\left(0,T;\mathbb{R}^N\right)$ consider the following problem,

$$\begin{aligned} \frac{d}{dt}u_h(t) &= \Delta u_h(t) + F\left(u_h(t)\right) + \sum_{j=1}^N B_j\left(u_h(t)\right) h_j\left(t\right), \ t \in [0,T], \quad (4.2.7)\\ u_h\left(0\right) &= u_0, \text{ where } u_0 \in V \cap M. \end{aligned}$$

In the next subsection, we are going to show that for any $h \in L^2\left(0,T;\mathbb{R}^N\right)$ there exists a unique global solution to above problem. Moreover, we will see that if initial data u_0 belongs to manifold M then all trajectories of solution u_h also belong to M.

Next define a map $\Im_h^0 : L^2\left([0,T] : \mathbb{R}^N\right) \to X_T$ in the following manner,

$$\Im_h^0\left(h\right) = u_h, \text{ for all } h \in L^2\left([0,T] : \mathbb{R}^N\right), \tag{4.2.8}$$

where u_h is the solution to the skeleton problem (4.2.7) i.e. for $t \in [0,T]$ it satisfies the following mild form,

$$u_h\left(t\right) = u_0 + \int_0^t \left(\Delta u_h\left(s\right) + F\left(u_h\left(s\right)\right)\right) ds + \sum_{j=1}^N \int_0^t B_j\left(u_h(t)\right) h_j\left(s\right) ds. \tag{4.2.9}$$

In the subsequent subsections we are going to show that $\Im_h^0\left(h\right)$ lies in X_T for each $h \in L^2\left([0,T] : \mathbb{R}^N\right)$ and then we will show that family of laws $\{\mathcal{L}\left(\Im_0^\varepsilon\left(W\right)\right) = u^\varepsilon, \varepsilon \in (0,1]\}$ on X_T satisfies the large deviation principle with the rate function $\mathcal{I} : X_T \to [0,\infty]$ defined by:

$$\mathcal{I}\left(u\right) := \inf\left\{\frac{1}{2}\left|h\right|^2_{L^2(0,T;\mathbb{R}^N)} : h \in L^2\left(0,T;\mathbb{R}^N\right) \text{ and } u = \Im_h^0\left(h\right)\right\}, \ \forall\ u \in X_T. \tag{4.2.10}$$

To prove the large deviation principle we are going to adopt the weak convergence method introduced in previous section. For $K \in (0,\infty)$, set

$$B_K := \left\{h \in L^2\left(0,T;\mathbb{R}^N\right) : \int_0^T \left|h(s)\right|^2_{\mathbb{R}^N} ds \leq K\right\}$$

with the weak topology of $L^2\left(0,T;\mathbb{R}^N\right)$. We are going to also prove that the map $\Im_h^0$ is Borel-measurable. Hence, analogous to Condition (4.1.5), we can state the following two sufficient conditions to obtain the Laplace principle (and hence large deviation principle).

(C$_1$) For each $K\in(0,\infty)$, the set

$$\left\{\Im_h^0(h): h\in L^2\left(0,T;\mathbb{R}^N\right) \text{ and } \int_0^T |h(s)|_{\mathbb{R}^N}^2\,ds\leq K\right\}$$

is compact subset of X_T.

(C$_2$) Let (ε_n) is sequence of real numbers from $(0,1]$ that converges 0 and let $(h_n)_{n\in\mathbb{N}}=\left((h_{n,j})_{j=1}^N\right)_{n\in\mathbb{N}}$ is a sequence of predictable process such that

$$\int_0^T |h_n(s,\omega)|_{\mathbb{R}^N}^2\,ds\leq K,$$

for all $\omega\in\Omega$ and for all $n\in\mathbb{N}$. If $(h_n)_{n\in\mathbb{N}}$ converges in distribution on B_K to h then $\Im_{h_n}^{\varepsilon_n}\left(\left(\varepsilon_n W_j(\cdot)+\int_0^\cdot h_{n,j}(s)\,ds\right)_{j=1}^N\right)$ converges in distribution on X_T to $\Im_h^0(h)$.

Remark 4.2.3. *Recall from the beginning of running subsection (page 78) that $\Im^\varepsilon(W_\cdot)$ is a strong solution to* (4.2.1). *By Girsanov theorem (cf. Theorem of 5.1 [29]) the process $\left(\left(\varepsilon_n W_j+\int_0^\cdot h_{n,j}(s)\,ds\right)_{j=1}^N\right)$ is $\mathbb{F}$-Wiener process on probability space $\left(\Omega,\widetilde{\mathbb{P}}\right)$, where $\widetilde{\mathbb{P}}$ is a certain probability measure. Hence using this conclusion of Girasanov theorem and the uniqueness of solution we are going to interpret each process $y_n(t):=\Im_{h_n}^{\varepsilon_n}\left(\left(\varepsilon_n W_j+\int_0^\cdot h_{n,j}(s)\,ds\right)_{j=1}^N\right)$ as the solution of the following problem, for $t\in[0,T]$,*

$$\begin{aligned}
y_n(t) &= u_0+\int_0^t\left(\Delta y_n(s)+F(y_n(s))\right)ds+\sum_{j=1}^N\int_0^t B_j(y_n(s))\,h_{n,j}(s)\,ds\\
&\quad+\frac{\varepsilon_n}{2}\sum_{j=1}^N\int_0^t \kappa_j(y_n(s))ds+\sqrt{\varepsilon_n}\sum_{j=1}^N\int_0^t B_j(y_n(s))\,dW_j(s),\\
y_n(0) &= u_0.
\end{aligned}$$

Before ending the subsection let us recall some of the important propositions proved in chapter 1 and chapter 3, which we are going to need at several instances later.

Lemma 4.2.4. *Consider a map $F : V \to H$ defined by $F(u) = \|u\|^2 u - u^{2n-1} + u\,|u|^{2n}_{L^{2n}}$. Then the map F satisfies:*

$$|F(u) - F(v)|_H \leq G(\|u\|, \|v\|)\,\|u - v\| \tag{4.2.11}$$

Where $G : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a bounded and symmetric map, given by,

$$G(r,s) \; : \; = C^2\left(r^2+s^2\right) + (r+s)^2 + C^{2n+1}\left[\begin{array}{c}\left(\frac{2n-1}{2}\right)\left(r^{2n-1}+s^{2n-1}\right)(r+s) \\ +\left(r^{2n}+s^{2n}\right)\end{array}\right]$$
$$+C^{2n-1}\left(\frac{2n-2}{2}\right)\left(r^{2n-2}+s^{2n-2}\right).$$

Proposition 4.2.5. *i) For $f \in H$, consider the map $B : H \to H$, defined by*

$$B(u) := f - \langle f, u\rangle\, u.$$

The map B satisfies

$$|B(u_1) - B(u_2)|_H \leq |f|_H\left(|u_1|_H + |u_1|_H\right)|u_1 - u_2|_H\,,\ \forall u_1, u_2 \in H. \tag{4.2.12}$$

ii) For $f \in V$, consider the map $B : V \to V$, defined above, satisfies

$$\|B(u_1) - B(u_2)\| \leq \|f\|\left(\|u_1\| + \|u_1\|\right)\|u_1 - u_2\|\,, \forall u_1, u_2 \in V. \tag{4.2.13}$$

Lemma 4.2.6. *If $B : H \to H$ be the map as defined in Lemma 4.2.5. For $u \in H$, then Fréchet derivative d_uB of map B exists and can be given as,*

$$\kappa(h) \equiv d_uB\,(h) := -\langle f, u\rangle\, h - \langle f, h\rangle\, u, \ \textit{for all } h \in H. \tag{4.2.14}$$

Proposition 4.2.7. *Assume the framework of Lemma 3.1.13. If κ is as defined in equation* (3.1.7) *Then*

i) For all $u_1, u_2 \in H$, the following inequality holds,

$$|\kappa(u_1) - \kappa(u_2)|_H \le |f|_H^2 \left[2 + |u_1|_H^2 + |u_2|_H^2 + (|u_1|_H + |u_2|_H)^2\right] |u_1 - u_2|_H .$$

ii) For all $u_1, u_2 \in V$, the following inequality holds,

$$\|\kappa(u_1) - \kappa(u_2)\| \le \|f\|^2 \left[2 + \|u_1\|^2 + \|u_2\|^2 + (\|u_1\| + \|u_2\|)^2\right] \|u_1 - u_2\| . \tag{4.2.15}$$

Moreover, we are going to use the following version of the Gronwall Lemma and Young's inequality.

Lemma 4.2.8. *(Gronwall's Lemma) Let $F, \alpha, \beta : [0,T] \to \mathbb{R}^+$ be Lebesgue measurable and β be locally integrable such that*

$$\int_0^T \beta(t) F(t) dt < \infty.$$

If

$$F(t) \le \alpha(t) + \int_0^t \beta(s) F(s) dt, \; t \in [0,T]$$

then we have

$$F(t) \le \alpha(t) + \int_0^t \alpha(s)\beta(s) \left(e^{\int_s^t \beta(r)dr}\right) ds, \; t \in [0,T] .$$

In addition if α is non-decreasing then

$$F(t) \le \alpha(t) \exp\left(\int_0^t \beta(r) dr\right), \; t \in [0,T] .$$

Lemma 4.2.9. *(Young's inequality) If $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then for any positive number σ, a and b we have*

$$ab \le \sigma \frac{a^p}{p} + \sigma^{-\frac{q}{p}} \frac{b^q}{q}.$$

4.2.1 Solution of Skeleton Problem

In this subsection, we aim to study invariance of manifold M, the existence, and uniqueness of the global solution to skeleton problem (4.2.7). We are going to adopt the precisely same strategy that we used in chapter 2 to study are the deterministic model. We now introduce the following approximate skeleton problem, for $t \in [0, T]$,

$$\begin{aligned} u_h^n(t) &= S(t)u_0 + \int_0^t S(t-r)\theta_n\left(|u_h^n|_{X_r}\right) F\left(u_h^n(r)\right) dr \qquad (4.2.16) \\ &\quad + \sum_{j=1}^N \int_0^t S(t-r)\theta_n\left(|u_h^n|_{X_r}\right) B_j\left(u_h^n(r)\right) h_j\left(r\right) dr, \\ u_h^n(0) &= u_0, \text{ where } u_0 \in V. \end{aligned}$$

Let us begin by proceeding towards proof of existence of local mild solution. Throughout the subsection we assume that $h = (h_j)_{j=1}^N \in L^2\left(0, T; \mathbb{R}^N\right)$ and there exists positive constant K such that,

$$\int_0^T |h(s)|_{\mathbb{R}^N}^2 \leq K$$

On the same lines of chapter 2, we intend to run a fixed point argument for the existence of local mild solution to approximate skeleton problem 4.2.16. To do so, we need map F and $B_j\left(\cdot\right) h_j$ (involved in to be 4.2.16) locally Lipschitz. For F we already know from Lemma 4.2.4 that it is locally Lipschitz. In the following lemma we show that $B_j\left(\cdot\right) h_j$ is locally Lipschitz and truncated $B_j\left(\cdot\right) h_j$ is globally Lipschitz. Whenever we use j we are going to assume it as $j = 1, 2, 3...N$.

Proposition 4.2.10. *Define that* $\Gamma_{h_j,T} : X_T \to L^2\left(0, T; H\right)$ *by*

$$\left[\Gamma_{h_j,T}(u)\right](t) = \left[\theta_n\left(|u|_{X_t}\right)\right] B_j\left(u\left(t\right)\right) h_j(t), \text{ where } t \in \left[0, T\right].$$

Then there exists $K_n > 0$ such that,

$$\left|\Gamma_{h_j,T}(u_1) - \Gamma_{h_j,T}(u_2)\right|_{L^2(0,T;H)} \leq K_n \left|u_1 - u_2\right|_{X_T} T^{1/2}, \quad \text{where } u_1, u_2 \in X_T. \tag{4.2.17}$$

In particular, $\Gamma_{h_j,T}$ is globally Lipschitz.

Proof. Keeping in view the inequality 4.2.12 we can observe that the map B_j is locally Lipschitz. Since $h_j(t)$ is real number for every $t \in [0,T]$, hence the map $B_j(\cdot)\, h_j$ is indeed locally Lipschitz and satisfies the following inequality, for all $t \in [0,T]$,

$$|B(u_1(t))h_j(t) - B(u_2(t))h_j(t)|_H = |h_j(t)| \, |B(u_1(t)) - B(u_2(t))|_H$$

$$\leq |f|_H \, |h_j(t)| \left(|u_1(t)|_H + |u_1(t)|_H\right) |u_1(t) - u_2(t)|_H \,, \ \forall u_1, u_2 \in H.$$

Thus proof of required inequality directly follows from abstract Proposition 3.1.16 by particularly taking $Z = B(\cdot)h_j$ and $G(r,s) = |f|_H \, |h_j| \, (r+s)$, for all $r, s \geq 0$.

■

Proposition 4.2.11. *Assume that we are in framework of Proposition 4.2.10 as well as assumptions 2.1.5 of chapter 2 are true. For given $f_1, f_2, ..., f_N \in V$ and $u_0 \in V$. Define a map $\Psi^n_{T,u_0} : X_T \to X_T$ defined by:*

$$\Psi^n_{T,u_0}(u_h) = Su_0 + S * \Phi^n_{T,F}(u_h) + \sum_{j=1}^{N} \left(S * \Gamma_{h_j,T}(u_h)\right). \tag{4.2.18}$$

where $\Gamma_{h_j,T}(\cdot) \equiv B_j(\cdot)h_j$, for $j = 1, 2, ..., N$, and $h = (h_j)_{j=1}^N \in \mathbb{R}^N$. Then there exists $C(n) > 0$ such that

$$\left|\Psi^n_{T,u_0}(u_h) - \Psi^n_{T,u_0}(v_h)\right|_{X_T} \leq C(n) \left|u_h - v_h\right|_{X_T} T^{1/2}.$$

In particular for sufficiently small $T > 0$, the map Ψ^n_{T,u_0} is contraction.

Proof. The proof can be done exactly on the same lines of proof of Proposition 2.2.7. ■

To construct local and local maximal solution of constrained problem 4.2.7, we can follow the precisely same methodology employed in chapter 2 i.e. we can easily prove the analogs of Lemma 2.2.8 and Proposition 2.2.10. We skip their proofs because they should contain precisely same argument. We focus ourselves on providing a complete proof of the invariance, the existence and the uniqueness of the global solution to problem 4.2.7, because the skeleton problem here is not gradient-flow like the deterministic model studied in chapter 2.

Proposition 4.2.12. *If $u_0 \in V \cap M$ and (u_h, τ) be the local maximal solution satisfies skeleton problem* (4.2.7) *then*

$$u_h(t) \in M \ \textit{i.e.} \, |u_h(t)|_H^2 = 1, \ t \in [0, \tau).$$

Proof. We are going to begin with Temmam Lemma III.1.2 of [49]. For $t \in [0, \tau)$, using equation (4.2.7), consider the following,

$$\begin{aligned} \frac{1}{2}\left(|u_h(t)|_H^2 - 1\right) &= \int_0^t \left\langle \frac{du_h}{dt}(s), u_h(s) \right\rangle_H ds \\ &= \int_0^t \langle \Delta u_h(s) + F(u_h(s)), u_h(s) \rangle \, ds \\ &\quad + \sum_{j=1}^N \int_0^t \langle B_j(u_h(s)) h_j(s), u_h(s) \rangle \, ds, \end{aligned} \tag{4.2.19}$$

Consider the computations for $u_h = u_h(t)$,

$$\begin{aligned} \langle u_h, B_j(u_h) h_j \rangle &= \langle u_h, f_j - \langle f_j, u_h \rangle u_h \rangle h_j \\ &= [\langle u_h, f_j \rangle - \langle f_j, u_h \rangle \langle u_h, u_h \rangle] h_j \\ &= -\left(|u_h|_H^2 - 1\right) \langle f_j, u_h \rangle h_j. \end{aligned} \tag{4.2.20}$$

Moreover for $u_h = u_h(t)$,

$$\begin{aligned}
\langle u_h, \Delta u_h + F(u_h) \rangle &= \left\langle u_h, \Delta u_h + \left(\|u_h\|^2 + |u_h|_{L^{2n}}^{2n} \right) u_h - u_h^{2n-1} \right\rangle \\
&= \langle u_h, \Delta u_h \rangle + \left(\|u_h\|^2 + |u_h|_{L^{2n}}^{2n} \right) \langle u_h, u_h \rangle + \left\langle u_h, -u_h^{2n-1} \right\rangle \\
&= -\|u_h\|^2 + \left(\|u_h\|^2 + |u_h|_{L^{2n}}^{2n} \right) |u_h|_H^2 - |u_h|_{L^{2n}}^{2n} \\
&= \left(\|u_h\|^2 + |u_h|_{L^{2n}}^{2n} \right) \left(|u_h|_H^2 - 1 \right). \qquad (4.2.21)
\end{aligned}$$

Using equations (4.2.20) and (4.2.21) into equation (4.2.19) we get

$$\begin{aligned}
\frac{1}{2} \left(|u_h(t)|_H^2 - 1 \right) &= \int_0^t \left(\|u_h(s)\|^2 + |u_h(s)|_{L^{2n}}^{2n} \right) \left(|u_h(s)|_H^2 - 1 \right) ds \\
&\quad - \sum_{j=1}^N \int_0^t h_j(s) \langle u_h(s), f_j \rangle \left(|u_h(s)|_H^2 - 1 \right) ds, \ t \in [0, \tau).
\end{aligned}$$

Set

$$\begin{aligned}
\phi(t) &:= |u(t)|_H^2 - 1, \ t \in [0, \tau), \\
\beta(t) &:= \left[\|u_h(t)\|^2 + |u_h(t)|_{L^{2n}}^{2n} - \sum_{j=1}^N h_j(t) \langle u_h(t), f_j \rangle \right], \ t \in [0, \tau).
\end{aligned}$$

then

$$\phi(t) = 2 \int_0^t \beta(s) \phi(s) ds, \ t \in [0, \tau).$$

or in stronger form

$$\frac{d\phi(t)}{dt} = 2\beta(t)\phi(t), \ t \in [0, \tau).$$

The solution of above differential is form,

$$\phi(t) = \phi(0) \exp\left(\int_0^t \beta(s) ds \right), \ t \in [0, \tau).$$

But since $|u_h(0)|_H^2 = 1$ i.e. $\phi(0) = 0$, hence

$$\begin{aligned}
\phi(t) &= 0 \text{ for all } t \in [0, \tau), \\
\text{or } |u_h(t)|_H^2 &= 1, \text{ for all } t \in [0, \tau).
\end{aligned}$$

The above argument is true provided that $\int_0^t \beta(s)ds < \infty$. To see this consider the following for $t \in [0,\tau)$,

$$\begin{aligned}
\int_0^t \beta(s)ds &= \int_0^t \|u_h(s)\|^2 ds + \int_0^t |u_h(s)|_{L^{2n}}^{2n} ds - \sum_{j=1}^N \int_0^t h_j(t) \langle u_h(t), f_j\rangle ds. \\
&\le \sup_{s\in[0,t]} \|u_h(s)\|^2 \int_0^t ds + \int_0^t |u_h(s)|_{L^{2n}}^{2n} ds + \sum_{j=1}^N \int_0^t h_j(t) \langle u_h(t), f_j\rangle ds \\
&= \sup_{s\in[0,t]} \|u_h(s)\|^2 t + \int_0^t |u_h(s)|_{L^{2n}}^{2n} ds + \sum_{j=1}^N \int_0^t h_j(t) \langle u_h(t), f_j\rangle ds
\end{aligned}$$

Since $u_h \in X_T = C([0,T];V) \cap L^2(0,T;E)$ therefore $\sup_{s\in[0,T]} \|u_h(s)\|^2 < \infty$. Using the continuity of embedding $V \hookrightarrow L^{2n}$ and $V \hookrightarrow H$, the Young inequality on the third integral, the inequality (4.2.1), and the fact that $f_j \in V \subset H$, we get

$$\begin{aligned}
\int_0^t \beta(s)ds &\le \sup_{s\in[0,t]} \|u_h(s)\|^2 t + c^{2n} \int_0^t \|u_h(s)\|^{2n} ds \\
&\quad + \sum_{j=1}^N \frac{1}{2} \int_0^t h_j(s)^2 ds + \sum_{j=1}^N \int_0^t \langle u_h(s), f_j\rangle^2 ds \\
&\le \sup_{s\in[0,t]} \|u_h(s)\|^2 t + c^{2n} \sup_{s\in[0,t]} \|u_h(s)\|^{2n} t \\
&\quad + \sum_{j=1}^N \frac{K_j}{2} + \sum_{j=1}^N |f_j|_H^2 \int_0^t |u_h(s)|_H^2 ds \\
&\le \sup_{s\in[0,t]} \|u_h(s)\|^2 t + c^{2n} \sup_{s\in[0,t]} \|u_h(s)\|^{2n} t + \sum_{j=1}^N \frac{K_j}{2} \\
&\quad + \sum_{j=1}^N |f_j|_H^2 \sup_{s\in[0,t]} |u_h(s)|_H^2 \ t \\
&\le \sup_{s\in[0,t]} \|u_h(s)\|^2 t + c^{2n} \sup_{s\in[0,t]} \|u_h(s)\|^{2n} t + \sum_{j=1}^N \frac{K_j}{2} \\
&\quad + \sum_{j=1}^N c^2 |f_j|_H^2 \sup_{s\in[0,t]} \|u_h(s)\|^2 \ t \\
&< \infty.
\end{aligned}$$

This completes the proof. ■

Proposition 4.2.13. *If $u_0 \in V \cap M$ and (u_h, τ) be the local maximal solution satisfies main problem* (4.2.7) *then there exists a constant* $C(u_0, T, K)$ *such that*

$$\|u_h(t)\| \leq C(u_0, T, K), \text{ for all } t \in [0, \tau),$$

and $\tau = \infty$ i.e. u_h is global solution.

Proof. Let us begin by employing Temmam Lemma III.1.2 of [49]. For all $t \in [0, \tau)$ consider the following chain of equations,

$$\begin{aligned}
\frac{1}{2}\|u_h(t)\|^2 &= \frac{1}{2}\|u_0\|^2 + \int_0^t \left\langle -\Delta u_h(s), \frac{du_h}{dt}(s) \right\rangle ds \\
&= \frac{1}{2}\|u_0\|^2 - \int_0^t \left\langle \frac{du_h}{dt}(s), \frac{du_h}{dt}(s) \right\rangle + \int_0^t \left\langle \frac{du_h}{dt}(s) - \Delta(s), \frac{du_h}{dt}(s) \right\rangle ds
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\|u_0\|^2 - \int_0^t \left|\frac{du_h}{dt}(s)\right|_H^2 ds \\
&+ \int_0^t \left\langle \left(\|u_h(s)\|^2 + |u_h(s)|_{L^{2n}}^{2n}\right) u_h(s) - u_h^{2n-1}(s), \frac{du_h}{dt}(s) \right\rangle ds \\
&+ \sum_{j=1}^N \int_0^t h_j(s) \left\langle B_j(u_h(s)), \frac{du_h}{dt}(s) \right\rangle ds
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\|u_0\|^2 - \int_0^t \left|\frac{du_h}{dt}(s)\right|_H^2 ds \\
&+ \int_0^t \left(\|u_h(s)\|^2 + |u_h(s)|_{L^{2n}}^{2n}\right) \left\langle u_h(s), \frac{du_h}{dt}(s) \right\rangle ds \\
&- \frac{1}{2n}\left\langle u_h^{2n-1}(t), u_h(t) \right\rangle + \frac{1}{2n}\left\langle u_0^{2n-1}(t), u_0(t) \right\rangle \\
&+ \sum_{j=1}^N \int_0^t \left\langle h_j(s)\left(f_j - \langle f_j, u_h(s)\rangle u_h(s)\right), \frac{du_h}{dt}(s) \right\rangle ds
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\|u_h(t)\|^2 - \frac{1}{2}\|u_0\|^2 &= -\frac{1}{2n}|u_h(t)|_{L^{2n}}^{2n} + \frac{1}{2n}|u_0|_{L^{2n}}^{2n} - \int_0^t \left|\frac{du_h}{dt}(s)\right|_H^2 ds \\
&+ \sum_{j=1}^N \int_0^t \left\langle h_j(s)\left(f_j - \langle f_j, u_h(s)\rangle u_h(s)\right), \frac{du_h}{dt}(s) \right\rangle ds
\end{aligned}$$

$$\begin{aligned}\Phi\left(u\left(t\right)\right)-\Phi\left(u_0\right) &= -\int_0^t\left|\frac{du_h}{dt}(s)\right|_H^2 ds+\int_0^t\left(\|u_h(s)\|^2+|u_h(s)|_{L^{2n}}^{2n}\right)\left\langle u_h(s),\frac{du_h}{dt}(s)\right\rangle ds\\ &+\sum_{j=1}^N\int_0^t\left[\left\langle h_j\left(s\right)f_j,\frac{du_h}{dt}(s)\right\rangle+h_j\left(s\right)\left\langle f_j,u_h(s)\right\rangle\left\langle u_h(s),\frac{du_h}{dt}(s)\right\rangle\right]ds,\end{aligned}$$

where $\Phi(u_h):=\frac{1}{2}\left\|u_h\right\|^2+\frac{1}{2n}\left|u_h\right|_{L^{2n}}^{2n}$. Since $u_0\in M$ so by Lemma 4.2.12 $u_h(t)\in M$ and indeed $\frac{du_h}{dt}\in T_uM$ therefore $\left\langle u_h,\frac{du_h}{dt}\right\rangle=0$. Hence last equation reduces to,

$$\Phi\left(u\left(t\right)\right)-\Phi\left(u_0\right)=-\int_0^t\left|\frac{du_h}{dt}(s)\right|_H^2 ds+\sum_{j=1}^N\int_0^t\left\langle h_j\left(s\right)f_j,\frac{du_h}{dt}(s)\right\rangle ds$$

$$\begin{aligned}\Phi\left(u\left(t\right)\right)-\Phi\left(u_0\right) &\leq -\int_0^t\left|\frac{du_h}{dt}(s)\right|_H^2 ds+\sum_{j=1}^N\int_0^t\left|h_j\left(s\right)\right|\left|f_j\right|_H\left|\frac{du_h}{dt}(s)\right|_H ds\\ &= -\int_0^t\left|\frac{du_h}{dt}(s)\right|_H^2 ds+c\sum_{j=1}^N\int_0^t\left|h_j\left(s\right)\right|\left|\frac{du_h}{dt}(s)\right|_H ds\end{aligned}$$

where $c=\max\left\{\left|f_j\right|_H\right\}_{j=1}^N<\infty$. Let us apply the Young inequality on second term right hand side of last inequality, for $p=q=2$, $\sigma=\frac{Nc}{2}$, $a=|h_j|$ and $b=\left|\frac{du_h}{dt}\right|_H$. It follows that

$$\begin{aligned}\Phi(u_h(t))-\Phi(u_0) &\leq -\int_0^t\left|\frac{du_h}{dt}(s)\right|_H^2 ds+c\sum_{j=1}^N\frac{Nc}{2}\int_0^t\frac{\left|h_j\left(s\right)\right|^2}{2}ds\\ &+c\sum_{j=1}^N\frac{2}{Nc}\int_0^t\frac{\left|\frac{du_h}{dt}(s)\right|_H^2}{2}\\ &= -\int_0^t\left|\frac{du_h}{dt}(s)\right|_H^2 ds+\frac{Nc^2}{4}\sum_{j=1}^N\int_0^t\left|h_j\left(t\right)\right|^2 ds\\ &+\frac{N}{N}\int_0^t\left|\frac{du_h}{dt}(t)\right|_H^2 ds\\ &= \frac{Nc^2}{4}\sum_{j=1}^N\int_0^t\left|h_j\left(t\right)\right|^2 s=\frac{Nc^2}{4}K\end{aligned}$$

$$\begin{aligned}\Phi(u_h(t)) &\leq \Phi(u_0)+\frac{Nc^2}{4}K\\ 2\Phi(u_h(t)) &\leq 2\Phi(u_0)+\frac{Nc^2}{2}K.\end{aligned}$$

But since $2\Phi(u_h) = \|u_h\|^2 + \frac{1}{n}|u_h|^{2n}_{L^{2n}} \geq \|u_h\|^2$. It follows from last inequality that

$$\|u_h(t)\|^2 \leq 2\Phi(u_0) + \frac{Nc^2}{2}K =: C(u_0, T, K), \text{ for all } t \in [0, \tau). \tag{4.2.22}$$

Hence the sufficient condition 2.2.11 is satisfied. Thus $\tau = \infty$ i.e. u_h is global solution. ■

4.2.2 Proof of $(\mathbf{C_1})$

Recall that the map $\Im^0_h : L^2(0, T; \mathbb{R}^N) \to X_T$ defined by (4.2.8) i.e. for each $h \in L^2(0, T; \mathbb{R}^N)$, satisfying $\int_0^T |h(s)|^2_{\mathbb{R}^N}\, ds \leq K$,

$$\Im^0_h(h) := u_h$$

where u_h satisfies the equation,

$$u_h(t) = u_0 + \int_0^t (\Delta u_h(s) + F(u_h(s)))\, ds + \sum_{j=1}^N \int_0^t B_j(u_h(s))\, h_j(s)\, ds. \tag{4.2.23}$$

Our main task in this subsection is to prove condition $\mathbf{C}_1\mathbf{)}$ (see page 184) in the form of the following theorem.

Theorem 4.2.14. *For each $K \in (0, \infty)$, the set*

$$\left\{\Im^0_h(h) : h \in L^2(0, T; \mathbb{R}^N) \text{ and } \int_0^T |h(s)|^2_{\mathbb{R}^N}\, ds \leq K\right\}$$

is a compact subset of X_T.

We prove the above result in the form of the following series of Lemmas. Before proving the lemmas let us set some more use full notation. Throughout this subsection we will assume that $(h_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in

$L^2\left(0,T;\mathbb{R}^N\right)$ with limit $h \in L^2\left(0,T;\mathbb{R}^N\right)$. Let us set writing $u_{h_n} := \Im^0_{h_n}(h_n)$, $u_h := \Im^0_h(h)$ and

$$K_n := \int_0^T |h_n(s)|^2_{\mathbb{R}^N}\, ds < \infty, \text{ for all } n \in \mathbb{N}. \tag{4.2.24}$$

Lemma 4.2.15. *There exists constant $C(u_0,T,K)$ such that for all $n \in \mathbb{N}$ and for all $h \in L^2\left(0,T;\mathbb{R}^N\right)$ satisfying $\int_0^T |h(s)|^2_{\mathbb{R}^N}\, ds \leq K$,*

$$\begin{aligned} \sup_{t\in[0,T]} \|u_h(t)\|^2 &\leq C(u_0,T,K), &(4.2.25)\\ \int_0^T |\Delta u_h(s)|^2_H\, ds &\leq C(u_0,T,K). &(4.2.26)\end{aligned}$$

Proof. From Proposition 4.2.13 (see inequality (4.2.22)) we know that there exist a constant $C(u_0,T,K)$ such that

$$\|u_h(t)\|^2 \leq C(u_0,T,K), \text{ for all } t \in [0,T]. \tag{4.2.27}$$

Recall that,

$$u_h(t) = u_0 + \int_0^t (\Delta u_h(s) + F(u_h(s)))\, ds + \sum_{j=1}^N \int_0^t B_j(u_h(s))\, h_j(s)\, ds, \ t \in [0,T].$$

Using Lemma III.2.1 from [49],

$$\begin{aligned} \frac{\|u_h(t)\|^2}{2} &= \int_0^t \langle \Delta u_h(s), -\Delta u_h(s)\rangle\, ds + \int_0^t \langle F(u_h(s)), -\Delta u_h(s)\rangle\, ds \\ &\quad + \sum_{j=1}^N \int_0^t \langle B_j(u_h(s)), -\Delta u_h(s)\rangle\, h_j(s)\, ds \\ &= I_1(t) + I_2(t) + \sum_{j=1}^N I_{3,j}(t), \ t \in [0,T]. &(4.2.28)\end{aligned}$$

Let us compute and estimate each of integral in last sum. Consider the first integral i.e. I_1,

$$\begin{aligned} I_1(t) &= \int_0^t \langle \Delta u_h(s), -\Delta u_h(s)\rangle\, ds \\ &= -\int_0^t |\Delta u_h(s)|^2_H\, ds, t \in [0,T]. &(4.2.29)\end{aligned}$$

Now consider the second integral i.e. I_2 in (4.2.28), for $t \in [0,T]$,

$$\begin{aligned} I_2(t) &= \int_0^t \langle F(u_h(s)), -\Delta u_h(s)\rangle \, ds \\ &\leq \int_0^t |F(u_h(s))|_H \, |\Delta u_h(s)|_H \, ds, t \in [0,T]. \end{aligned}$$

Using inequality (4.2.11), for $t \in [0,T]$,

$$\begin{aligned} I_2(t) &\leq \int_0^t G(\|u_h(s)\|, 0) \, |\Delta u_h(s)|_H \, ds, \\ &= \int_0^t G(\|u_h(s)\|, 0) \, |\Delta u_h(s)|_H \, ds \end{aligned} \tag{4.2.30}$$

Where $G : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a bounded and symmetric map, defined as

$$\begin{aligned} G(r,s) &:= C^2 \left(r^2+s^2\right) + (r+s)^2 + C^{2n+1} \left[\begin{array}{c} \left(\frac{2n-1}{2}\right)\left(r^{2n-1}+s^{2n-1}\right)(r+s) \\ +\left(r^{2n}+s^{2n}\right) \end{array} \right] \\ &+C^{2n-1}\left(\frac{2n-2}{2}\right)\left(r^{2n-2}+s^{2n-2}\right). \end{aligned}$$

From inequality (4.2.27), we infer that there exists a constant $C_1(u_0, T, K,)$ such that

$$G(\|u_h(t)\|, 0) \leq C_1(u_0, T, K), \ \ t \in [0,T].$$

By using this inference into inequality (4.2.30), we get for $t \in [0,T]$,

$$I_2(t) \leq C_1 \int_0^t |\Delta u_h(s)|_H \, ds$$

We are going to apply now the Young inequality (4.2.9) to the integrand in above inequality. Choose $a = C_1$, $b = |\Delta u_h|_H$, $p = q = 2$ and $\sigma = \frac{1}{2}$. Using this into last inequality,

$$\begin{aligned} I_2(t) &\leq 2C_1^2 \int_0^t ds + \frac{1}{4}\int_0^t |\Delta u_h(s)|_H^2 \, ds \\ &\leq CT + \frac{1}{4}\int_0^t |\Delta u_h(s)|_H^2 \, ds, \end{aligned} \tag{4.2.31}$$

where $C := 2C_1^2$.

Now consider the 3rd integral i.e. $I_{3,j}$ of equation (4.2.28). Using Cauchy-Schwartz inequality together with embedding $V \hookrightarrow H$ and (4.2.12), we infer that,

$$\begin{aligned} I_{3,j} &= \int_0^t \langle B_j(u_h(s)), -\Delta u_h(s)\rangle h_j(s)\,ds \\ &\leq \int_0^t |B_j(u_h(s))|_H |\Delta u_h(s)|_H h_j(s)\,ds \\ &\leq \int_0^t c|f_j|_H \|u_h(s)\| \|u_h(s)\| |\Delta u_h(s)|_H h_j(s)\,ds \\ &= \int_0^t c|f_j|_H \|u_h(s)\|^2 |\Delta u_h(s)|_H h_j(s)\,ds \end{aligned} \tag{4.2.32}$$

Set $k := \max\left\{|f_j|_H\right\}_{j=1}^N < \infty$. Using estimate (4.2.27) we infer that there exists a positive constant C_2 such that

$$c|f_j|_H \|u_h(s)\|^2 \leq ckC =: C_2.$$

Therefore the inequality (4.2.32) simplifies to

$$I_{3,j} \leq C_2 \int_0^t |\Delta u_h(s)|\, h_j(s)\,ds.$$

Applying the Young inequality (4.2.9) to the integrand in the last integral, with $a = C_2 h_j$, $b = |\Delta u_n|_H$, $p = q = 2$ and $\sigma = \frac{1}{2N}$, the last inequality becomes

$$\begin{aligned} I_{3,j} &\leq 2NC_2 \int_0^t h_j^2(s)\,ds + \frac{1}{4N}\int_0^t |\Delta u_h(s)|^2\,ds \\ &= C\int_0^t h_j^2(s)\,ds + \frac{1}{4N}\int_0^t |\Delta u_h(s)|^2\,ds,\ 2C_2 =: C. \end{aligned}$$

Taking sum over j on both sides and using (4.2.24),

$$\begin{aligned} \sum_{j=1}^N I_{3,j} &\leq C\sum_{j=1}^N \int_0^t h_j^2(s)\,ds + \sum_{j=1}^N \frac{1}{4N}\int_0^t |\Delta u_h(s)|^2\,ds \\ &= CK + \frac{1}{4}\int_0^t |\Delta u_h(s)|^2\,ds. \end{aligned} \tag{4.2.33}$$

Using inequalities (4.2.29), (4.2.31) and (4.2.33) into we get

$$\begin{aligned}\frac{\|u_h(t)\|^2}{2} \leq & -\int_0^t |\Delta u_h(s)|_H^2 \, ds + CT + \frac{1}{4}\int_0^t |\Delta u_h(s)|_H^2 \, ds \\ & + CK + \frac{1}{4}\int_0^t |\Delta u_h(s)|^2 \, ds.\end{aligned}$$

$$\|u_h(t)\|^2 + \int_0^t |\Delta u_h(s)|_H^2 \, ds \leq 2CT + 2CK =: C(u_0, T, K), \text{ for all } t \in [0,T].$$

In particular,

$$\int_0^T |\Delta u_h(s)|_H^2 \, ds \leq C(u_0, T, K).$$

■

Lemma 4.2.16. *Every subsequence of (u_{h_n}) has a convergent subsequence that converges in $C([0,T]; H)$.*

Proof. We aim to apply Arzelà–Ascoli Theorem (see [27], Corollary 5.21) to get the desired result. From Proposition 4.2.15 we know that there exits a constant $C(u_0, T, K)$ such that,

$$\|u_{h_n}(t)\| \leq C(u_0, T, K), \text{ for all } t \in [0,T]. \tag{4.2.34}$$

From the boundedness of u_{h_n} in V-norm and compactness of embedding $V \hookrightarrow H$, it follows that for each fixed $t \in [0,T]$, the set $\{u_{h_n}(t), n \in \mathbb{N}\}$ is relatively compact in H. Hence by employing the diagonal argument we can find a subsequence (n_k) such that $(u_{h_{n_k}}(t))$ is converges to some point in H , for each rational $t \in [0,T]$. It remains to show that $\{u_{h_n}(t), n \in \mathbb{N}\}$ is a uniformly equi-continuous subset of $C([0,T]; H)$, once we are done with proving this, the required will follow from

Arzelà–Ascoli Theorem. For $s, t \in [0,T]$, satisfying $s \leq t$

$$\begin{aligned} u_{h_n}(t) - u_{h_n}(s) &= u_0 + \int_0^t \left(\Delta u_{h_n}(r) + F\left(u_{h_n}(r)\right)\right) dr \\ &\quad + \sum_{j=1}^N \int_0^t B_j\left(u_{h_n}(r)\right) h_{n,j}(r)\, dr \\ &\quad - u_0 - \int_0^t \left(\Delta u_{h_n}(r) + F\left(u_{h_n}(r)\right)\right) dr \\ &\quad - \sum_{j=1}^N \int_0^t B_j\left(u_{h_n}(s)\right) h_{n,j}(r)\, dr \\ &= \int_s^t \left(\Delta u_{h_n}(r) + F\left(u_{h_n}(r)\right)\right) dr \\ &\quad + \sum_{j=1}^N \int_s^t B_j\left(u_{h_n}(r)\right) h_{n,j}(r)\, dr \end{aligned}$$

Taking H norm on both sides and using triangle inequality,

$$\begin{aligned} \left|u_{h_n}(t) - u_{h_n}(s)\right|_H &\leq \int_s^t \left|\Delta u_{h_n}(r)\right|_H dr + \int_s^t \left|F\left(u_{h_n}(r)\right)\right|_H dr \\ &\quad + \sum_{j=1}^N \int_s^t \left|B_j\left(u_{h_n}(r)\right)\right|_H h_{n,j}(r)\, dr \end{aligned}$$

Using the Young inequality on first integral and inequality (4.2.11), we infer that,

$$\begin{aligned} \left|u_{h_n}(t) - u_{h_n}(s)\right|_H &\leq \frac{1}{2}\int_s^t dr + \int_s^t \left|\Delta u_{h_n}(r)\right|_H^2 dr \\ &\quad + \int_s^t G(\left\|u_{h_n}(r)\right\|, 0) \left\|u_{h_n}(r)\right\| dr \\ &\quad + \sum_{j=1}^N \int_s^t \left|B_j\left(u_{h_n}(r)\right)\right|_H h_{n,j}(r)\, dr. \end{aligned}$$

Using inequalities (4.2.25), (4.2.26) , (4.2.34) and holder inequality afterwards, for $p = q = 2$, into above inequality, we get

$$\begin{aligned} \left|u_{h_n}(t) - u_{h_n}(s)\right|_H &\leq \frac{1}{2}(t-s) + C\left(u_0, T, K\right) + C\left(u_0, T, K\right)(t-s) \\ &\quad + \sum_{j=1}^N \left(\int_s^t h_{n,j}(r)^2\, dr\right)^{1/2} \left(\int_s^t \left|B_j\left(u_{h_n}(r)\right)\right|_H^2 dr\right)^{1/2}, \end{aligned}$$

Finally using inequalities (4.2.13), (4.2.24) and (4.2.25), we conclude that,

$$\begin{aligned} |u_{h_n}(t) - u_{h_n}(s)|_H &\leq \frac{1}{2}(t-s) + C(u_0, T, K) + C(u_0, T, K)(t-s) \\ &+ \sum_{j=1}^{N} K_{n,j}^{\frac{1}{2}} \left(\int_s^t \left(C|f_j|_H \|u_{h_n}(r)\| \right)^2 dr \right)^{1/2}, \\ &\leq \frac{1}{2}(t-s) + C(u_0, T, K) + C(u_0, T, K)(t-s) \\ &+ \sum_{j=1}^{N} K_{n,j}^{\frac{1}{2}} C^2 |f_j|_H (t-s)^{1/2}. \\ \text{where } s,t &\in [0,T]. \end{aligned}$$

Thus $\{u_{h_n}(t), n \in \mathbb{N}\}$ is a uniformly equi-continuous subset of $C([0,T];H)$. This completes the proof. ■

Lemma 4.2.17. *If $\gamma : [0,T] \to H$ be measurable function which satisfies $\int_0^T |\gamma(s)|_H^2 ds < \infty$, then*

$$\sup_{t\in[0,T]} \left| \int_0^t \langle \gamma(s), u_{h_n}(s) - u_h(s) \rangle (h_n(s) - h(s)) ds \right| \to 0 \ \textit{as}\ n \to \infty.$$

Proof. We prove this by contradiction. Assume contrary that there is an $\varepsilon > 0$ and a subsequence (n_k) such that

$$\sup_{t\in[0,T]} \left| \int_0^t \left\langle \gamma(s), u_{h_{n_k}}(s) - u_h(s) \right\rangle (h_{n_k}(s) - h(s)) ds \right| \geq \varepsilon, \text{ for all } k \in \mathbb{N}. \quad (4.2.35)$$

By last Lemma the subsequence $\left(u_{h_{n_k}}\right)$ converges to a point $u \in C([0,T];H)$. Now consider

$$\begin{aligned} & \sup_{t\in[0,T]} \left| \int_0^t \left\langle \gamma(s), u_{h_{n_k}}(s) - u_h(s) \right\rangle (h_{n_k}(s) - h(s)) ds \right| \\ \leq & \sup_{t\in[0,T]} \left| \int_0^t \left\langle \gamma(s), u_{h_{n_k}}(s) - u(s) \right\rangle (h_{n_k}(s) - h(s)) ds \right| \\ & + \sup_{t\in[0,T]} \left| \int_0^t \langle \gamma(s), u(s) - u_h(s) \rangle (h_{n_k}(s) - h(s)) ds \right| \\ : & = A_{n_k} + B_{n_k} \end{aligned} \quad (4.2.36)$$

We claim that $A_{n_k} \to 0$ and $B_{n_k} \to 0$, as $k \to \infty$ and hence we are going to reach at a contradiction to inequality (4.2.35). Consider A_{n_k}, using Cauchy Schwartz inequality, we infer that,

$$\begin{aligned} A_{n_k} &= \sup_{t\in[0,T]} \left| \int_0^t \left\langle \gamma(s), u_{h_{n_k}}(s) - u(s) \right\rangle (h_{n_k}(s) - h(s)) ds \right| \\ &\leq \sup_{t\in[0,T]} \left| \int_0^t |\gamma(s)|_H \left| u_{h_{n_k}}(s) - u(s) \right|_H (h_{n_k}(s) - h(s)) ds \right| \\ &\leq \sup_{t\in[0,T]} \int_0^t |\gamma(s)|_H \left| u_{h_{n_k}}(s) - u(s) \right|_H [h_{n_k}(s) - h(s)] \, ds \\ &\leq \sup_{t\in[0,T]} \left| u_{h_{n_k}}(t) - u(t) \right|_H \sup_{t\in[0,T]} \int_0^t |\gamma(s)|_H [h_{n_k}(s) - h(s)] \, ds \end{aligned}$$

By Holder inequality with $p = q = 2$,

$$A_{n_k} \leq \sup_{t\in[0,T]} \left| u_{h_{n_k}}(t) - u(t) \right|_H \sup_{t\in[0,T]} \left(\int_0^t |\gamma(s)|_H^2 \, ds \right)^{1/2} \sup_{t\in[0,T]} \left(\int_0^t |h_{n_k}(s) - h(s)|^2 \, ds \right)^{1/2}$$

From equation (4.2.24) and assumption $I := \int_0^T |\gamma(s)|_H^2 \, ds < \infty$, we infer that

$$A_{n_k} \leq \sup_{t\in[0,T]} \left| u_{h_{n_k}}(t) - u(t) \right|_H \sqrt{K_n I}.$$

Now since $\left(u_{h_{n_k}}\right)$ converges to a point $u \in C([0,T];H)$ so,

$$A_{n_k} \leq \sqrt{KI} \sup_{t\in[0,T]} \left| u_{h_{n_k}}(t) - u(t) \right|_H \to 0 \text{ as } k \to \infty.$$

To proceed with B_{n_k}, let us define an operator $T_h : L^2\left(0,T;\mathbb{R}^N\right) \to C([0,T];H)$ by,

$$\begin{aligned} (T_h f)(t) \quad &: \;= \int_0^t \mathcal{K}(s,t) f(s) ds, \;\; f \in L^2\left(0,T;\mathbb{R}^N\right), \qquad (4.2.37) \\ \text{where } \mathcal{K}(s,t) &= \langle \gamma(s), u(s) - u_h(s) \rangle, \text{ for } 0 \leq s \leq t \\ &= 0, \text{ for } s > t. \end{aligned}$$

We claim that this operator is compact and to show it we show that it is Hilbert-Schmidt i.e. its kernel satisfies $\int_0^T \int_0^T \mathcal{K}(s,t)^2 dsdt < \infty$. (see Theorem 1.1.54 in

Chapter 1 or ([52], Theorem 8.8). Consider,

$$\begin{aligned}\int_0^T\int_0^T \langle\gamma(s), u(s)-u_h(s)\rangle^2\, dsdt &= \int_0^T dt\int_0^T \langle\gamma(s), u(s)-u_h(s)\rangle^2\, ds\\ &= T\int_0^T \langle\gamma(s), u(s)-u_h(s)\rangle^2\, ds\\ &\leq T\int_0^T |\gamma(s)|_H^2\,|u(s)-u_h(s)|_H^2\, ds\\ &\leq \sup_{r\in[0,t]}|u_h(r)-u(r)|_H^2\int_0^T|\gamma(r)|_H^2\, ds\end{aligned}$$

Since $u \in C([0,T];H)$ and $I := \int_0^t |\gamma(s)|_H^2\, ds < \infty$ from assumption , and $u_h = \Im(h)^0 \in X_T$, it follows that,

$$\int_0^T\int_0^T \langle\gamma(s), u(s)-u_h(s)\rangle^2\, dsdt \leq \sup_{s\in[0,t]}|u_h(s)-u(s)|_H^2\int_0^t|\gamma(s)|_H^2\, ds < \infty.$$

Hence operator T defined in equation (4.2.37) is compact. Recall from the beginning of the section that (h_n) is weakly convergent sequence to h in $L^2\left(0,T;\mathbb{R}^N\right)$. As T is compact so $(Th_n)_{n\in\mathbb{N}}$ converges to Th i.e.

$$\int_0^t \langle\gamma(s), u(s)-u_h(s)\rangle\,(h_{n_k}(s)-h(s))ds \to 0 \text{ as } k\to\infty.$$

Subsequently $B_{n_k}\to 0$ as $k\to\infty$.
Summing the proof by noticing that as $B_{n_k}\to 0$ as $k\to\infty$ and $A_{n_k}\to 0$ as $k\to\infty$ so from inequality (4.2.36),

$$\sup_{t\in[0,T]}\left|\int_0^t \left\langle\gamma(s), u_{h_{n_k}}(s)-u_h(s)\right\rangle(h_{n_k}(s)-h(s))ds\right| \to 0 \text{ as } k\to\infty,$$

which is contradiction to (4.2.35). This completes the proof. ■

Remark 4.2.18. *In the following result we are going to use Lemma 4.2.17 for the particular* $\gamma(u_h) := \Delta B(u_h)$ *where*

$$B(u_h) = f - \langle f, u_h\rangle\, u_h,\ \textit{where } f\in E = D(A).$$

We are going to verify in this remark that $\int_0^T |\gamma(s)|_H^2\, ds < \infty$. *Consider*

$$\begin{aligned}\int_0^T |\gamma(s)|_H^2\, ds &= \int_0^T |\Delta B(u_h(s))|_H^2\, ds \\ &= \int_0^T |\Delta f - \langle f, u_h(s)\rangle \Delta u_h(s)|_H^2\, ds\end{aligned}$$

Using elementary inequality $(a-b)^2 \le 2(a^2+b^2)$, *we infer that,*

$$\begin{aligned}\int_0^T |\gamma(s)|_H^2\, ds &\le 2\int_0^T \left(|\Delta f|_H^2 + \langle f, u_h(s)\rangle^2 |\Delta u_h(s)|_H^2\right) ds \\ &\le 2|f|_E^2 T + 2\int_0^T |f|_H^2 |u_h(s)|_H^2 |\Delta u_h(s)|_H^2\, ds \\ &\le 2|f|_E^2 T + 2|f|_H^2 \int_0^T |u_h(s)|_H^2 |\Delta u_h(s)|_H^2\, ds.\end{aligned}$$

Using the continuity of embedding $V \hookrightarrow H$ *so it follows that,*

$$\begin{aligned}\int_0^T |\gamma(s)|_H^2\, ds &\le 2|f|_E^2 T + 2c|f|_H^2 \int_0^T \|u_h(s)\|^2 |\Delta u_h(s)|_H^2\, ds \\ &\le 2|f|_E^2 T + 2c|f|_H^2 \sup_{s\in[0,T]} \|u_h(s)\|^2 \int_0^T |\Delta u_h(s)|_H^2\, ds\end{aligned}$$

Using the inequalities (4.2.25) *and* (4.2.26) *it follows that,*

$$\int_0^T |\gamma(s)|_H^2\, ds \le 2|f|_E^2 T + 2c|f|_H^2 C(u_0, T, K)^2.$$

Since $f \in E \subset H$ *so hence*

$$\int_0^T |\gamma(s)|_H^2\, ds \le 2|f|_E^2 T + 2c|f|_H^2 C(u_0, T, K)^2 < \infty.$$

Lemma 4.2.19. *If* (h_n) *be a weakly convergent sequence in* $L^2\left(0,T;\mathbb{R}^N\right)$ *with limit* $h \in L^2\left(0,T;\mathbb{R}^N\right)$, *where* (h_n) *and* h *satisfies* (4.2.24). *Then* $(u_{h_n})_{n\in\mathbb{N}}$ *converges to* u_h *in* X_T*-norm as* $n \to \infty$.

Proof. Let us begin by recalling the expression for u_h and u_{h_n}. Recall from (4.2.23),

$$u_h(t) = u_0 + \int_0^t \left(\Delta u_h(s) + F(u_h(s))\right) ds + \sum_{j=1}^N \int_0^t B_j(u_h(s))\, h_j(s)\, ds,\ t \in [0,T].$$

and

$$u_{h_n}(t) = u_0 + \int_0^t \left(\Delta u_{h_n}(s) + F\left(u_{h_n}(s)\right)\right) ds + \sum_{j=1}^N \int_0^t B_j\left(u_{h_n}(s)\right) h_{n,j}(s)\, ds,\ t \in [0,T].$$

Set $u_n := u_{h_n} - u_h$, then using last two equations, for $t \in [0,T]$,

$$\begin{aligned} u_n(t) &= \int_0^t \Delta u_n(s)\, ds + \int_0^t \left[F\left(u_{h_n}(s)\right) - F\left(u_h(s)\right)\right] ds \\ &\quad + \sum_{j=1}^N \int_0^t \left[B_j\left(u_{h_n}(s)\right) h_{n,j}(s) - B_j\left(u_h(s)\right) h_j(s)\right] ds \\ &= \int_0^t \Delta u_n(s)\, ds + \int_0^t \left[F\left(u_{h_n}(s)\right) - F\left(u_h(s)\right)\right] ds \\ &\quad + \sum_{j=1}^N \int_0^t \left[B_j\left(u_{h_n}(s)\right) - B_j\left(u_h(s)\right)\right] h_{n,j}(s)\, ds \\ &\quad + \sum_{j=1}^N \int_0^t B_j\left(u_h(s)\right)\left(h_{n,j}(s) - h_j(s)\right) ds \end{aligned}$$

Using Temmam Lemma III.1.2 of [49],

$$\begin{aligned} \frac{\|u_n(t)\|^2}{2} &= \int_0^t \left\langle \Delta u_n(s), -\Delta u_n(s)\right\rangle ds + \int_0^t \left\langle F\left(u_{h_n}(s)\right) - F\left(u_h(t)\right), -\Delta u_n(s)\right\rangle ds \\ &\quad + \sum_{j=1}^N \int_0^t \left\langle B_j\left(u_{h_n}(s)\right) - B_j\left(u_h(s)\right), -\Delta u_n(s)\right\rangle h_{n,j}(s)\, ds \\ &\quad + \sum_{j=1}^N \int_0^t \left\langle B_j\left(u_h(s)\right), -\Delta u_n(s)\right\rangle \left(h_{n,j}(s) - h_j(s)\right) ds \\ &= I_1(t) + I_2(t) + \sum_{j=1}^N I_{3,j}(t) + \sum_{j=1}^N I_{4,j}(t),\ t \in [0,T]. \end{aligned} \tag{4.2.38}$$

Let us compute and estimate each of integral in last sum. Consider the first integral i.e. I_1,

$$\begin{aligned} I_1(t) &= \int_0^t \left\langle \Delta u_n(s), -\Delta u_n(s)\right\rangle ds \\ &= -\int_0^t \left|\Delta u_n(s)\right|_H^2 ds,\ t \in [0,T]. \end{aligned} \tag{4.2.39}$$

Now consider the second integral i.e. I_2 in (4.2.38), for $t \in [0,T]$,

$$\begin{aligned} I_2(t) &= \int_0^t \langle F(u_{h_n}(s)) - F(u_h(s)), -\Delta u_n(s)\rangle\, ds \\ &\le \int_0^t |F(u_{h_n}(s)) - F(u_h(s))|_H\, |\Delta u_n(s)|_H\, ds, t \in [0,T]. \end{aligned}$$

Using inequality (4.2.11), for $t \in [0,T]$,

$$\begin{aligned} I_2(t) &\le \int_0^t G(\|u_{h_n}(s)\|, \|u_h(s)\|)\, \|u_{h_n}(s) - u_h(s)\|\, |\Delta u_n(s)|_H\, ds, \\ &= \int_0^t G(\|u_{h_n}(s)\|, \|u_h(s)\|)\, \|u_n(s)\|\, |\Delta u_n(s)|_H\, ds \end{aligned} \tag{4.2.40}$$

Where $G : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a bounded and symmetric map, defined as

$$\begin{aligned} G(r,s) \ :&= C^2\left(r^2+s^2\right) + (r+s)^2 + C^{2n+1}\left[\begin{array}{c} \left(\frac{2n-1}{2}\right)\left(r^{2n-1}+s^{2n-1}\right)(r+s) \\ +\left(r^{2n}+s^{2n}\right) \end{array}\right] \\ &+C^{2n-1}\left(\frac{2n-2}{2}\right)\left(r^{2n-2}+s^{2n-2}\right). \end{aligned}$$

From inequality (4.2.25), we infer that there exists a constant $C_1(u_0,T,K)$ such that

$$G(\|u_{h_n}(t)\|, \|u_h(t)\|) \le C_1(u_0,T,K),\ t \in [0,T].$$

By using this inference into inequality (4.2.40), we get for $t \in [0,T]$,

$$I_2(t) \le C_1 \int_0^t \|u_n(s)\|\, |\Delta u_n(s)|_H\, ds$$

We are going to apply now the Young inequality (4.2.9) to the integrand in above inequality. Choose $a = C_1\|u_n\|$, $b = |\Delta u_n|_H$, $p = q = 2$ and $\sigma = \frac{1}{4}$. Using this into last inequality,

$$\begin{aligned} I_2(t) &\le 2C_1^2 \int_0^t \|u_n(s)\|^2\, ds + \frac{1}{4}\int_0^t |\Delta u_n(s)|_H^2\, ds \\ &= C\int_0^t \|u_n(s)\|^2\, ds + \frac{1}{4}\int_0^t |\Delta u_n(s)|_H^2\, ds, \end{aligned} \tag{4.2.41}$$

where $C := 2C_1^2$.

Now consider the 3rd integral i.e. $I_{3,j}$ of equation (4.2.38). Using Cauchy-Schwartz inequality together with (4.2.12)

$$\begin{aligned} I_{3,j} &= \int_0^t \langle B_j(u_{h_n}(s)) - B_j(u_h(s)), -\Delta u_n(s)\rangle h_{n,j}(s)\,ds \\ &\le \int_0^t |B_j(u_{h_n}(s)) - B_j(u_h(s))|_H |\Delta u_n(s)|_H h_{n,j}(s)\,ds \\ &\le \int_0^t C|f|_H(\|u_{h_n}(s)\| + \|u_h(s)\|)\|u_{h_n}(s) - u_h(s)\| h_{n,j}(s)\,ds \\ &= \int_0^t C|f_j|_H(\|u_{h_n}(s)\| + \|u_h(s)\|)\|u_n(s)\| |\Delta u_n(s)| h_{n,j}(s)\,ds \end{aligned} \tag{4.2.42}$$

Set $c := \max\{|f_j|_H\}_{j=1}^N < \infty$. Using estimate (4.2.25) we infer that there exists a positive constant C_2 such that

$$C|f_j|_H(\|u_{h_n}(s)\| + \|u_h(s)\|) \le Cc(C + C) =: C_2.$$

Therefore the inequality (4.2.42) simplifies to

$$I_{3,j} \le C_2 \int_0^t \|u_n(s)\| |\Delta u_n(s)| h_{n,j}(s)\,ds.$$

Applying the Young inequality (4.2.9) to the integrand in the last integral, with $a = C_2\|u_n\| h_{n,j}$, $b = |\Delta u_n|_H$, $p = q = 2$ and $\sigma = \frac{1}{4N}$, the last inequality becomes

$$\begin{aligned} I_{3,j} &\le 2NC_2 \int_0^t \|u_n(s)\|^2 h_{n,j}^2(s)\,ds + \frac{1}{4N}\int_0^t |\Delta u_n(s)|^2\,ds \\ &= C\int_0^t \|u_n(s)\|^2 h_{n,j}^2(s)\,ds + \frac{1}{4N}\int_0^t |\Delta u_n(s)|^2\,ds, \end{aligned}$$

where $2C_2 =: C$. Taking sum over j on both sides

$$\begin{aligned} \sum_{j=1}^N I_{3,j} &\le C\sum_{j=1}^N \int_0^t \|u_n(s)\|^2 h_{n,j}^2(s)\,ds + \sum_{j=1}^N \frac{1}{4N}\int_0^t |\Delta u_n(s)|^2\,ds \\ &= C\sum_{j=1}^N \int_0^t \|u_n(s)\|^2 h_{n,j}^2(s)\,ds + \frac{1}{4}\int_0^t |\Delta u_n(s)|^2\,ds. \end{aligned} \tag{4.2.43}$$

We are going to keep $I_{4,j}$, of equation (4.2.38) as it is, because we intend to apply Lemma 4.2.17 to it.

Using inequalities (4.2.39), (4.2.41) and (4.2.43) into (4.2.38) we get

$$\begin{aligned}\frac{\|u_n(t)\|^2}{2} &\leq -\int_0^t |\Delta u_n(s)|_H^2\, ds + C\int_0^t \|u_n(s)\|^2\, ds + \frac{1}{4}\int_0^t |\Delta u_n(s)|_H^2\, ds \\ &\quad + C\sum_{j=1}^N \int_0^t \|u_n(s)\|^2 h_{n,j}^2(s)\, ds + \frac{1}{4}\int_0^t |\Delta u_n(s)|^2\, ds + \sum_{j=1}^N I_{4,j}(t).\end{aligned}$$

$$\begin{aligned}\|u_n(t)\|^2 + \int_0^t |\Delta u_n(s)|_H^2\, ds &\leq 2\int_0^t \left(C + Ch_{n,j}^2(s)\right)\left(\|u_n(s)\|^2 + \int_0^s |\Delta u_n(r)|_H^2\, dr\right) ds \\ &\quad + 2\sum_{j=1}^N I_{4,j}(t)\end{aligned}$$

Set

$$\begin{aligned}F(t) &:= \|u_n(t)\|^2 + \int_0^t |\Delta u_n(s)|_H^2\, ds \\ \alpha(t) &:= 2\sum_{j=1}^N I_{4,j}(t) \\ \beta(t) &:= 2\left(C + Ch_{n,j}^2(s)\right)\end{aligned}$$

hence the last inequality becomes

$$F(t) \leq \int_0^t 2\beta(s)F(s)ds + \alpha(t)$$

Finally Grownwall Lemma 4.2.8 and equation (4.2.24) we get

$$\begin{aligned}\sup_{t\in[0,T]} \|u_n(t)\|^2 + \int_0^t |\Delta u_n(s)|_H^2\, ds &\leq \alpha(t) + 2\int_0^t \alpha(s)\beta(s)\exp\left(2\int_s^t \beta(r)dr\right) ds \\ &= \alpha(t) + 2\int_0^t \alpha(s)\beta(s)\exp\left(2\int_s^t \left(C + Ch_{n,j}^2(s)\right) dr\right) ds \\ &= \alpha(t) + 2\int_0^t \alpha(s)\beta(s)\exp\left(2C\left(t - s + K_n\right)\right) ds\end{aligned}$$

Taking limit $n \to \infty$, and using Lemma 4.2.17, Remark 4.2.18 $\alpha(t) \to 0$. Moreover, using the Lebesgue dominated convergence theorem (see Theorem 1.4.32), the right hand side of above inequality goes to zero. This completes the proof of Lemma. ■

4.2.3 Proof of ($\mathbf{C_2}$) and Large Deviation Principle

In this subsection we intend to show that the family of laws $\{\mathcal{L}(\Im_0^\varepsilon(W)), \varepsilon \in (0,1]\}$ on X_T, satisfies the large deviation principle with rate function $\mathcal{I} : X_T \to [0,\infty]$ defined as in (4.2.10). For $K \in (0,\infty)$, set

$$B_K := \left\{ h_n \in L^2\left(0,T;\mathbb{R}^N\right) : \int_0^T |h_n(s)|_{\mathbb{R}^N}^2 \, ds \le K \right\}.$$

From Theorem 4.2.14 it follows that the map $\Im_h^0$, defined by equation (4.2.23), is Borel measurable. To prove family of laws $\{\mathcal{L}(\Im_0^\varepsilon(W)), \varepsilon \in (0,1]\}$ on X_T, satisfies the large deviation principle with rate function I, if the following theorem (condition $C2$) is true.

Theorem 4.2.20. *Let (ε_n) and let sequence of real number from $(0,1]$ that converges 0 and if $(h_n)_{n\in\mathbb{N}}$ be sequence of predictable process such that*

$$\int_0^T |h_n(s,\omega)|_{\mathbb{R}^N}^2 \, ds \le K \tag{4.2.44}$$

, for all $\omega \in \Omega$ and for all $n \in \mathbb{N}$. If $(h_n)_{n\in\mathbb{N}}$ converges in distribution on B_K to h then $\Im_{h_n}^{\varepsilon_n}\left(\left(\sqrt{\varepsilon_n}W_j + \int_0^\cdot h_{n,j}(s)\,ds\right)_{j=1}^N\right)$ converges in distribution on X_T to $\Im_h^0(h)$.

We prove the above theorem in the form of the following set of Lemmas. Set $u_n := \Im_{h_n}^0(h_n)$ and $y_n := \Im_{h_n}^{\varepsilon_n}\left(\left(\sqrt{\varepsilon_n}W_j + \int_0^\cdot h_{n,j}(s)\,ds\right)_{j=1}^N\right)$ and $\xi_n := y_n - u_n$.

Let us fix $m \in (\|u_0\|, \infty)$. For each $n \in \mathbb{N}$, define an $(\mathcal{F}_t)$-stopping time,

$$\tau_n(\omega) := \inf\{t \in [0,T] : \|y_n(t,\omega)\| \ge m\} \wedge T, \tag{4.2.45}$$

for all $\omega \in \Omega$.

Proposition 4.2.21. *For $t \in [0,T]$, if $u_0 \in M$ then $y_n(t \wedge \tau_n) \in M$, where τ_n is stopping time described above.*

Proof. We begin by recalling evolution equation satisfied by y_n,

$$\begin{aligned} y_n(t) &= \Im^{\varepsilon_n}\left(\sqrt{\varepsilon_n}W_j + \int_0^t h_{n,j}(s)\,ds\right) \\ &= u_0 + \int_0^t \left(\Delta y_n(s) + F(y_n(s))\right) + \frac{\varepsilon_n}{2}\sum_{j=1}^{N}\int_0^t \kappa_j(y_n(s))ds \\ &\quad + \sum_{j=1}^{N}\int_0^t B_j(y_n(s))\,h_{n,j}(s)\,ds + \sqrt{\varepsilon_n}\sum_{j=1}^{N}\int_0^t B_j(y_n(s))\,dW_j(s),\ t\in[0,T]. \end{aligned}$$

where $u(0) = u_0 \in V \cap M$.

Our intentions are to apply the Itô Lemma to the map $\gamma : H \ni u \mapsto \frac{1}{2}|u|_H^2 \in \mathbb{R}$. For stopping time τ_n described by equation (4.2.45) and $\in [0,T]$, let us apply Itô's Lemma [37] to the process$(\gamma(u(t\wedge\tau_n)))_{t\in[0,T]}$,

$$\begin{aligned} \gamma(u(t\wedge\tau_n)) - \gamma(u_0) &= \int_0^{t\wedge\tau_n} \langle y_n(s), \Delta y_n(s) + F(y_n(s))\rangle\,ds \\ &\quad + \frac{\varepsilon_n}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n} \langle y_n(s), \kappa_j(B_j(y_n(s))\rangle\,ds \\ &\quad + \sum_{j=1}^{N}\int_0^{t\wedge\tau_n} \langle y_n(s), B_j(y_n(s))\rangle\,h_{n,j}(s)\,ds \\ &\quad + \frac{\varepsilon_n}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n} (B_j(y_n(s)), B_j(y_n(s)))\,ds, \\ &\quad + \sqrt{\varepsilon_n}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n} \langle y_n(s), B_j(y_n(s))\rangle\,dW_j(s) \end{aligned} \tag{4.2.46}$$

Let us compute the inner products involved in above explicitly. For the sake of convenience we set $y_n = y_n(s)$.

$$\begin{aligned}
\langle y_n, \Delta y_n + F(y_n)\rangle &= \langle y_n, \Delta y_n\rangle + \langle y_n, F(y_n)\rangle \\
&= -\|y_n\|^2 + \left\langle y_n, \left(\|y_n\|^2 + |y_n|_{L^{2n}}^{2n}\right) y_n - y_n^{2n-1}\right\rangle \\
&= -\|y_n\|^2 + \left(\|y_n\|^2 + |y_n|_{L^{2n}}^{2n}\right)\langle y_n, y_n\rangle - \left\langle y_n, y_n^{2n-1}\right\rangle \\
&= -\|y_n\|^2 + \left(\|y_n\|^2 + |y_n|_{L^{2n}}^{2n}\right)|y_n|_H^2 - |y_n|_{L^{2n}}^{2n} \\
&= \left(|y_n|_H^2 - 1\right)\left(\|y_n\|^2 + |y_n|_{L^{2n}}^{2n}\right). \qquad (4.2.47)
\end{aligned}$$

Next

$$\begin{aligned}
\langle y_n, B_j(y_n)\rangle &= \langle y_n, f_j - \langle f_j, y_n\rangle y_n\rangle, \\
&= \langle y_n, f_j\rangle - \langle f_j, y_n\rangle\langle y_n, y_n\rangle, \\
&= \langle y_n, f_j\rangle\left(1 - |y_n|_H^2\right). \qquad (4.2.48)
\end{aligned}$$

Now using equation (4.2.48),

$$\begin{aligned}
\langle y_n, \kappa_j(y_n)\rangle &= \langle y_n, -\langle f_j, y_n\rangle B_j(y_n) - \langle f_j, B_j(y_n)\rangle y_n\rangle \\
&= -\langle f_j, y_n\rangle\langle y_n, B_j(y_n)\rangle - \langle f_j, B_j(y_n)\rangle\langle y_n, y_n\rangle \\
&= -\langle f_j, y_n\rangle\langle y_n, f_j\rangle\left(1 - |y_n|_H^2\right) - \langle f_j, f_j - \langle f_j, y_n\rangle y_n\rangle |y_n|_H^2 \\
&= \langle y_n, f_j\rangle^2\left(|y_n|_H^2 - 1\right) \\
&\quad - \left[\langle f_j, f_j\rangle - \langle f_j, y_n\rangle\langle f_j, y_n\rangle\right]|y_n|_H^2 \\
&= \langle y_n, f_j\rangle^2\left(|y_n|_H^2 - 1\right) - |f_j|_H^2|y_n|_H^2 + \langle f_j, y_n\rangle^2|y_n|_H^2 \\
&= \langle y_n, f_j\rangle^2\left(2|y_n|_H^2 - 1\right) - |f_j|_H^2|y_n|_H^2 \qquad (4.2.49)
\end{aligned}$$

Finally,

$$\begin{aligned}
\langle B_j(y_n), B_j(y_n)\rangle &= \langle f_j - \langle f_j, y_n\rangle y_n, f_j - \langle f_j, y_n\rangle y_n\rangle, \\
&= \langle f_j, f_j\rangle - \langle f_j, y_n\rangle^2 - \langle f_j, y_n\rangle^2 + \langle f_j, y_n\rangle^2\langle y_n, y_n\rangle \\
&= |f_j|_H^2 + \langle f_j, y_n\rangle^2\left(|y_n|_H^2 - 2\right). \qquad (4.2.50)
\end{aligned}$$

Substituting equations (4.2.47), (4.2.48), (4.2.49) and (4.2.50) into equation (4.2.46) with $y_n = y_n(s)$ and using fact that $|u_0|_H^2 = 1$, it follows that,

$$\begin{aligned}
\frac{1}{2}\left(|y_n(t\wedge\tau_n)|_H^2 - 1\right) &= \int_0^{t\wedge\tau_n}\left(|y_n|_H^2 - 1\right)\left(\|y_n\|^2 + |y_n|_{L^{2n}}^{2n}\right)ds \\
&+\frac{\varepsilon_n}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n}\left[\langle y_n, f_j\rangle^2\left(2|y_n|_H^2 - 1\right) - |f_j|_H^2|y_n|_H^2\right]ds \\
&-\sum_{j=1}^{N}\int_0^{t\wedge\tau_n}\langle y_n, f_j\rangle\left(|y_n|_H^2 - 1\right)h_{n,j}(s)\,ds \\
&+\frac{\varepsilon_n}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n}\left[|f_j|_H^2 + \langle f_j, y_n\rangle^2\left(|y_n|_H^2 - 2\right)\right]ds, \\
&+\sqrt{\varepsilon_n}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n}\langle y_n, f_j\rangle\left(|y_n|_H^2 - 1\right)dW_j(s)
\end{aligned}$$

Combining first, third and second, fourth Riemann integrals into a single Riemann integral of last equation, we get

$$\begin{aligned}
\frac{1}{2}\left(|y_n(t\wedge\tau_n)|_H^2 - 1\right) &= \int_0^{t\wedge\tau_n}\left(|y_n(s)|_H^2 - 1\right)\left(\|y_n(s)\|^2 + |y_n(s)|_{L^{2n}}^{2n} - \langle y_n(s), f_j\rangle h_{n,j}(s)\right)ds \\
&+\frac{\varepsilon_n}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n}\left[\begin{array}{c}\langle y_n(s), f_j\rangle^2\left(2|y_n(s)|_H^2 - 1\right) - |f_j|_H^2|y_n(s)|_H^2 \\ +|f_j|_H^2 + \langle f_j, y_n\rangle^2\left(|y_n|_H^2 - 2\right)\end{array}\right]ds \\
&+\sqrt{\varepsilon_n}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n}\langle y_n, f_j\rangle\left(|y_n|_H^2 - 1\right)dW_j(s).
\end{aligned}$$

Simplifying the integrand of the second integral we get,

$$\begin{aligned}
\frac{1}{2}\left(|y_n(t\wedge\tau_n)|_H^2 - 1\right) &= \int_0^{t\wedge\tau_n}\left(|y_n(s)|_H^2 - 1\right)\left(\|y_n(s)\|^2 + |y_n(s)|_{L^{2n}}^{2n} - \langle y_n(s), f_j\rangle h_{n,j}(s)\right)ds \\
&+\frac{\varepsilon_n}{2}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n}\left[\left(3\langle y_n(s), f_j\rangle^2 - |f_j|_H^2\right)\left(|y_n(s)|_H^2 - 1\right)\right]ds \\
&+\sqrt{\varepsilon_n}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n}\langle y_n, f_j\rangle\left(|y_n|_H^2 - 1\right)dW_j(s).
\end{aligned}$$

Combining the first two Riemann integrals,

$$\begin{aligned}\frac{1}{2}\left(\left|y_n\left(t\wedge\tau_n\right)\right|_H^2-1\right) &= \int_0^{t\wedge\tau_n}\left(\begin{array}{c}\left\|y_n\left(s\right)\right\|^2+\left|y_n\left(s\right)\right|_{L^{2n}}^{2n}-\left\langle y_n(s),f_j\right\rangle h_{n,j}\left(s\right)\\ \frac{\varepsilon_n}{2}\left(3\left\langle y_n\left(s\right),f_j\right\rangle^2-\left|f_j\right|_H^2\right)\end{array}\right)\\ &\quad\cdot\left(\left|y_n\left(s\right)\right|_H^2-1\right)ds\\ &\quad+\sqrt{\varepsilon_n}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n}\left\langle y_n,f_j\right\rangle\left(\left|y_n\right|_H^2-1\right)dW_j\left(s\right).\end{aligned} \tag{4.2.51}$$

To simplify argument we treat $N=1$, and for $t\in[0,T]$ let us define the following functions,

$$\begin{aligned}\varphi(t) &:= \left|y_n\left(t\right)\right|_H^2-1\\ \alpha(t) &:= \left\langle y_n\left(t\right),f_1\right\rangle\\ \beta(t) &:= \left\|y_n\left(t\right)\right\|^2+\left|y_n\left(t\right)\right|_{L^{2n}}^{2n}-\left\langle y_n\left(t\right),f_j\right\rangle h_{n,j}\left(t\right)\\ &\quad+\frac{\varepsilon_n}{2}\left(3\left\langle y_n\left(t\right),f_j\right\rangle^2-\left|f_j\right|_H^2\right)\\ F(t,\varphi(t)) &:= \alpha(t)\varphi(t)\\ G(t,\varphi(t)) &:= \beta(t)\varphi(t)\end{aligned}$$

The last equation (4.2.51) can be rewritten as, for $t\in[0,T]$,

$$\begin{aligned}\frac{\varphi(t)}{2} &= \int_0^{t\wedge\tau_n}F(s,\varphi(s))dW_1(s)+\int_0^{t\wedge\tau_n}G(s,\varphi(s))ds, \\ \text{and } \varphi(0) &:= \left|u\left(0\right)\right|_H^2-1=0\end{aligned} \tag{4.2.52}$$

To get the desired result it is sufficient to show existence and uniqueness of above presented the linear problem, and for this it is enough to that F and G are Lipschitz in the second argument (See Theorem 7.7, [5]). For $x,y\in\mathbb{R}$, $t\in[0,T]$ and $\omega\in\Omega$,

$$\begin{aligned}\left|F(t,x)-F(t,y)\right| &= \left|\alpha(t,\omega)x-\alpha(t,\omega)y\right|=\left|\alpha(t,\omega)\right|\left|x-y\right|\\ \text{and } \left|G(t,x)-G(t,y)\right| &= \left|\beta(t,\omega)x-\beta(t,\omega)y\right|=\left|\beta(t,\omega)\right|\left|x-y\right|\end{aligned}$$

Hence to show that F and G are Lipschitz it is only needed to show that maps α and β are bounded. Let us start with α. Using the Cauchy-Schwartz inequality, we infer that

$$|\alpha(t)| \leq |\langle y_n(t\wedge\tau_n), f_1\rangle| \leq |y_n(t\wedge\tau_n)|_H |f_1|_H, \ t\in[0,T].$$

As $f_1 \in V \subset H$ so $|f_1|_H < \infty$, also using continuity of embedding $V \hookrightarrow H$ and definition of τ_n it follows that

$$|\alpha(t,\omega)| \leq c\,\|y_n(t\wedge\tau_n)\| \leq m\,|f_1|_H, \ t\in[0,T].$$

Hence α is bounded.

Next, for $t\in[0,T]$ and $\omega\in\Omega$, again using continuity of embeddings $V \hookrightarrow H$ and $L^{2n} \hookrightarrow V$, $f_1 \in V \subset H$, for the fixed $t\in[0,T]$ the $|h_{n,j}(t)| < \infty$, and the definition of τ_n, we infer the boundedness of map β in the following manner,

$$\begin{aligned}
|\beta(t)| &= \left| \begin{array}{c} \|y_n(t\wedge\tau_n)\|^2 + |y_n(t\wedge\tau_n)|_{L^{2n}}^{2n} + \varepsilon_n \langle y_n(t\wedge\tau_n), f_1\rangle^2 \\ -\frac{\varepsilon_n}{2}|f_1|_H^2 - \langle y_n(t\wedge\tau_n), f_1\rangle h_{n,j}(t) \end{array} \right| \\
&\leq \|y_n(t\wedge\tau_n)\|^2 + c^{2n}\|y_n(t\wedge\tau_n)\|^{2n} + \varepsilon_n |y_n(t\wedge\tau_n)|_H^2 |f_1|_H^2 \\
&\quad + \frac{\varepsilon_n}{2}|f_1|_H^2 + |\langle y_n(t\wedge\tau_n), f_1\rangle|\,|h_{n,j}(t)| \\
&\leq \|y_n(t\wedge\tau_n)\|^2 + c^{2n}\|y_n(t\wedge\tau_n)\|^{2n} + c^2\varepsilon_n \|y_n(t\wedge\tau_n)\|^2 |f_1|_H^2 \\
&\quad + \frac{\varepsilon_n}{2}|f_1|_H^2 + |y_n(t\wedge\tau_n)|_H\ |f_1|_H^2 |h_{n,j}(t)| \\
&\leq m^2 + c^{2n}m^{2n} + c^2\varepsilon_n m^2 |f_1|_H^2 + \frac{\varepsilon_n}{2}|f_1|_H^2 + m\ |f_1|_H |h_{n,j}(t)|.
\end{aligned}$$

Hence β is bounded.

This completes the proof. ∎

Lemma 4.2.22. *There exists a constant $C(u_0,T,K)$ such that, for all $n\in\mathbb{N}$ and for all $h_n \in L^2(0,T;\mathbb{R}^N)$ satisfying $\int_0^T |h_n(s)|_{\mathbb{R}^N}^2\,ds \leq K$, we have*

$$\limsup_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \left(\|y_n(t\wedge\tau_n)\|^2\right) + \int_0^{T\wedge\tau_n} |\Delta y_n(s)|_H^2\,ds\right] \leq C(u_0,T,K). \quad (4.2.53)$$

Proof. Before going towards proof of the required inequality let us see that by Definition 4.2.45 of τ_n we have the following inequality,

$$\|y_n(t)\| \leq m, \ 0 \leq t \leq \tau_n. \tag{4.2.54}$$

Let us fix $t \in [0, T]$. Recall that, $\mathbb{P}$-a.s.,

$$\begin{aligned} y_n(t) &= \Im^{\varepsilon_n}\left(\varepsilon_n W_j(t) + \int_0^t h_{n,j}(s)\, ds\right) \\ &= u_0 + \int_0^t (\Delta y_n(s) + F(y_n(s))) + \frac{\varepsilon_n}{2}\sum_{j=1}^N \int_0^t \kappa_j(y_n(s))ds \\ &\quad + \sum_{j=1}^N \int_0^t B_j(y_n(s))\, h_{n,j}(s)\, ds + \sqrt{\varepsilon_n}\sum_{j=1}^N \int_0^t B_j(y_n(s))\, dW_j(s). \end{aligned}$$

Our intentions are to apply Itô Lemma, to process $\left(\|y_n(t\wedge\tau_n)\|^2\right)_{t\in[0,T]}$. For $t \in [0, T]$, by Itô Lemma we have the following

$$\begin{aligned} \|y_n(t\wedge\tau_n)\|^2 &= -2\int_0^{t\wedge\tau_n} \langle \Delta y_n(s), \Delta y_n(s)\rangle\, ds + 2\int_0^{t\wedge\tau_n} \langle -\Delta y_n(s), F(y_n(s))\rangle\, ds \\ &\quad + \varepsilon_n \sum_{j=1}^N \int_0^{t\wedge\tau_n} \langle -\Delta y_n(s), \kappa_j(y_n(s))\rangle\, ds \\ &\quad + 2\sum_{j=1}^N \int_0^{t\wedge\tau_n} \langle -\Delta y_n(s), B_j(y_n(s))\rangle\, h_{n,j}(s)\, ds \\ &\quad + 2\sqrt{\varepsilon_n}\sum_{j=1}^N \int_0^{t\wedge\tau_n} \langle -\Delta y_n(s), B_j(y_n(s))\rangle\, dW_j(s) \\ &\quad + \varepsilon_n \sum_{j=1}^N \int_0^{t\wedge\tau_n} \langle B_j(y_n(s)), B_j(y_n(s))\rangle_V\, ds, \ \mathbb{P}\text{-a.s.} \\ &=: -2I_1 + 2I_2 + \varepsilon_n\sum_{j=1}^N I_{3,j} + 2\sum_{j=1}^N I_{4,j} + 2\sqrt{\varepsilon_n}\sum_{j=1}^N I_{5,j} + \varepsilon_n\sum_{j=1}^N I_{6,j}. \end{aligned} \tag{4.2.55}$$

Let us estimate the each of integral in last equation,

$$\begin{aligned} I_1 &= \int_0^{t\wedge\tau_n} \langle \Delta y_n(s), \Delta y_n(s)\rangle \, ds \\ &= \int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2 \, ds \end{aligned} \tag{4.2.56}$$

Now consider the second integral I_2, using Cauchy-Schwartz, continuity of embedding $V \hookrightarrow H$ and inequalities (4.2.11), (4.2.54) we infer that,

$$\begin{aligned} I_2 &= \int_0^{t\wedge\tau_n} \langle -\Delta y_n(s), F(y_n(s))\rangle \, ds \\ &\le \int_0^{t\wedge\tau_n} |F(y_n(s))|_H \, |\Delta y_n(s)|_H \, ds \\ &\le c\int_0^{t\wedge\tau_n} \|F(y_n(s))\| \, |\Delta y_n(s)|_H \, ds \\ &\le c\int_0^{t\wedge\tau_n} G(\|y_n(s)\|, 0) \, \|y_n(s)\| \, |\Delta y_n(s)|_H \, ds \\ &\le cG(m,0)\, m \int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H \, ds \\ &= C\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H \, ds, \end{aligned} \tag{4.2.57}$$

where $C := cG(m,0)\,m < \infty$ and $G : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a bounded and symmetric map, defined as

$$\begin{aligned} G(r,s) \; &:= C^2\left(r^2+s^2\right) + (r+s)^2 + C^{2n+1}\left[\begin{array}{c} \left(\frac{2n-1}{2}\right)\left(r^{2n-1}+s^{2n-1}\right)(r+s) \\ +\left(r^{2n}+s^{2n}\right) \end{array}\right] \\ &+C^{2n-1}\left(\frac{2n-2}{2}\right)\left(r^{2n-2}+s^{2n-2}\right). \end{aligned}$$

By taking $a = C$, $b = |\Delta y_n(s)|_H$, $p = q = 2$ and $\sigma = \frac{1}{2}$ for the Young inequality (4.2.9) and using it into (4.2.57) we get,

$$I_2 \le C^2\int_0^{t\wedge\tau_n} ds + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2 \, ds.$$

$$\begin{aligned}
&= C^2 (t \wedge \tau_n) + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds \\
&= C^2 (t \wedge \tau_n) + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds \\
&\le C_2 (T \wedge \tau_n) + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds,
\end{aligned}$$

where $C_2 := C^2 < \infty$.

Now consider the third integral $I_{3,j}$, using Cauchy Schwartz inequality and (4.2.15), we infer

$$\begin{aligned}
I_{3,j} &= \int_0^{t\wedge\tau_n} \langle -\Delta y_n(s), \kappa_j(y_n(s))\rangle\, ds \\
&\le \int_0^{t\wedge\tau_n} |\kappa_j(y_n(s))|_H\, |\Delta y_n(s)|_H\, ds \\
&\le \int_0^{t\wedge\tau_n} |f|_H^2 \left[2 + |y_n(s)|_H^2 + |y_n(s)|_H^2\right] |y_n(s)|\, |\Delta y_n(s)|_H\, ds.
\end{aligned}$$

Using continuity of embedding $V \hookrightarrow H$ and (4.2.54) we get

$$\begin{aligned}
I_{3,j} &\le c\,|f|_H^2 \int_0^{t\wedge\tau_n} \left[2 + 2c^2 \|y_n(s)\|^2 +\right] \|y_n(s)\|\, |\Delta y_n(s)|_H\, ds, \\
&\le C\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H\, ds,
\end{aligned}$$

where $C := c\,|f|_H^2\,[2 + 2c^2 m^2]\, m < \infty$. By taking $a = C$, $b = |\Delta y_n(s)|_H$, $p = q = 2$ and $\sigma = \frac{1}{2\varepsilon_n N}$ for the Young inequality (4.2.9), the last inequality becomes,

$$\begin{aligned}
I_{3,j} &\le N\varepsilon_n \int_0^{t\wedge\tau_n} C^2 ds + \frac{1}{4\varepsilon_n N}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds \\
&= N\varepsilon_n C^2 (t\wedge\tau_n) + \frac{1}{4\varepsilon_n N}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds \\
&= \varepsilon_n C\,(T\wedge\tau_n) + \frac{1}{4\varepsilon_n N}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds,
\end{aligned}$$

where $C := NC^2$. Taking sum on j on both sides and multiplying ε_n we get

$$\begin{aligned}
\varepsilon_n \sum_{j=1}^{N} I_{3,j} &\leq \varepsilon_n^2 C \sum_{j=1}^{N} (t \wedge \tau_n) + \frac{1}{4N} \sum_{j=1}^{N} \int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2 \, ds \\
&= N\varepsilon_n^2 C\,(t \wedge \tau_n) + \frac{1}{4} \int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2 \, ds \\
&= \varepsilon_n^2 C_3\,(T \wedge \tau_n) + \frac{1}{4} \int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2 \, ds,
\end{aligned}$$

where $C_3 := NC < \infty$.

Now consider the 4th integral i.e. $I_{4,j}$. Use of Cauchy-Schwartz and inequality (4.2.12), we get

$$\begin{aligned}
I_{4,j} &= \int_0^{t\wedge\tau_n} \langle -\Delta y_n(s), B_j\,(y_n\,(s))\rangle\, h_j^{\varepsilon_n}\,(s)\, ds \\
&\leq \int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H\, |B_j\,(y_n\,(s))|_H\, h_{n,j}\,(s)\, ds \\
&\leq \int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H\, C\, |f|_H\, \|y_n\,(s)\|^2\, h_{n,j}\,(s)\, ds
\end{aligned}$$

Using fact that $f \in V \subset H$ and (4.2.54) , we infer $C\,|f|_H\, \|y_n\,(s)\|^2 \leq CKm^2 =: C$, therefore the last inequality simplifies to the following,

$$I_{4,j} \leq C \int_0^{t\wedge\tau_n} |\Delta y_n\,(s)|_H\, h_{n,j}\,(s)\, ds,$$

By taking $a = Ch_{n,j}, b = |\Delta y_n|_H\,, p = q = 2$ and $\sigma = \frac{1}{4N}$ into the Young inequality and applying to last inequality,

$$\begin{aligned}
I_{4,j} &\leq 2NC^2 \int_0^{t\wedge\tau_n} (h_{n,j}\,(s))^2\, ds + \frac{1}{8N} \int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds \\
&= 2NC^2 K\,(T \wedge \tau_n) + \frac{1}{8N} \int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds.
\end{aligned}$$

In the last first term of inequality we have used (4.2.44). Taking sum on j on both sides and multiplying 2 we get

$$2\sum_{j=1}^{N} I_{4,j} \leq 2N^2C^2K\left(T\wedge\tau_n\right)+\frac{1}{4}\int_0^{t\wedge\tau_n}\left|\Delta y_n(s)\right|_H^2 ds, \qquad (4.2.58)$$
$$= C_4\left(T\wedge\tau_n\right)+\frac{1}{4}\int_0^{t\wedge\tau_n}\left|\Delta y_n(s)\right|_H^2 ds.$$

where $C_4 := 2N^2C^2K < \infty$.

Consider the 5th integral i.e. $I_{5,j}$.

$$I_{5,j} = \int_0^{t\wedge\tau_n}\left\langle -\Delta y_n(s), B_j\left(y_n\left(s\right)\right)\right\rangle dW_j\left(s\right)$$

Taking expectation supremum and taking expectation on both sides $I_{5,j}$

$$\mathbb{E}\left(\sup_{t\in[0,T]} I_{5,j}\right) = \mathbb{E}\left(\sup_{t\in[0,T]}\int_0^{t\wedge\tau_n}\left\langle -\Delta y_n(s), B_j\left(y_n\left(s\right)\right)\right\rangle dW_j\left(s\right)\right)$$

Using the Burkholder-Davis-Gundy inequality $p = 1$ i.e. $\mathbb{E}\left(\sup_{t\in[0,T]}\left|\int_0^t \psi(u(s))dW_j\left(s\right)\right|\right) \leq 3\mathbb{E}\left(\left|\int_0^T \psi(u(s))^2 ds\right|\right)^{1/2}$, (see Theorem 1.1.6, [39]) we infer that

$$\mathbb{E}\left(\sup_{t\in[0,T]} I_{5,j}\right) \leq 3\mathbb{E}\left(\int_0^{T\wedge\tau_n}\left\langle -\Delta y_n(s), B_j\left(y_n\left(s\right)\right)\right\rangle^2 ds\right)^{1/2}$$
$$\leq 3\mathbb{E}\left(\int_0^{T\wedge\tau_n}\left|\Delta y_n(s)\right|_H^2\left|B_j\left(y_n\left(s\right)\right)\right|_H^2 ds\right)^{1/2}$$

Now using inequalities 4.2.13 and (4.2.54) we infer that for $s\leq\tau_n$, we have

$$\left|B_j\left(y_n\left(s\right)\right)\right|_H^2 \leq C\left|f_j\right|_H\left\|y_n\left(s\right)\right\|^2 \leq C\left|f_j\right|_H m =: C_5$$

Hence

$$\mathbb{E}\left(\sup_{t\in[0,T]} I_{5,j}\right) \leq 3C_5\mathbb{E}\left(\int_0^{T\wedge\tau_n}\left|\Delta y_n(s)\right|_H^2 ds\right)^{1/2}$$
$$\leq 3C_5\max\left\{\mathbb{E}\int_0^{T\wedge\tau_n}\left|\Delta y_n(s)\right|_H^2 ds, 1\right\}$$
$$\leq 3C_5\left(1+\mathbb{E}\int_0^{T\wedge\tau_n}\left|\Delta y_n(s)\right|_H^2 ds\right)$$
$$= 3C_5+3C_5\mathbb{E}\int_0^{T\wedge\tau_n}\left|\Delta y_n(s)\right|_H^2 ds. \qquad (4.2.59)$$

Consider the last 6th integral i.e. $I_{6,j}$. Use of inequality (4.2.12), we get

$$\begin{aligned} I_{6,j} &= \int_0^{t\wedge\tau_n} \langle B_j(y_n(s)), B_j(y_n(s))\rangle_V\, ds \\ &= \int_0^{t\wedge\tau_n} \|B_j(y_n(s))\|^2\, ds \\ &\leq C\int_0^{t\wedge\tau_n} |f|_H \|y_n(s)\|^2\, ds \end{aligned}$$

Use of inequality gives (4.2.54),

$$I_{6,j} \leq C|f|_H\, m^2 (T\wedge\tau_n).$$

Taking sum on j on both sides and multiplying ε_n we get

$$\begin{aligned} \varepsilon_n \sum_{j=1}^N I_{6,j} &\leq \varepsilon_n \sum_{j=1}^N C|f|_H\, m^2(T\wedge\tau_n) \\ &= \varepsilon_n NC|f|_H\, m^2(T\wedge\tau_n) \\ &= \varepsilon_n (T\wedge\tau_n)\, C_6, \end{aligned} \tag{4.2.60}$$

where $C_6 := NC|f|_H\, m^2 < \infty$.

Adding inequalities (4.2.56)-(4.2.60) and using it into (4.2.55) we get

$$\begin{aligned} \|y_n(t\wedge\tau_n)\|^2 &\leq -2\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds + C_2(T\wedge\tau_n) + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds \\ &\quad \varepsilon_n C_3(T\wedge\tau_n) + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds \\ &\quad C_4(T\wedge\tau_n) + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds \\ &\quad +2\varepsilon_n \sum_{j=1}^N I_{5,j} + \varepsilon_n(T\wedge\tau_n)\, C_6 \\ &= -\frac{5}{4}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\, ds + (C_2 + \varepsilon_n C_3 + C_4\varepsilon_n C_6)(T\wedge\tau_n) \\ &\quad +\ 2\varepsilon_n \sum_{j=1}^N I_{5,j} \end{aligned}$$

Taking supremum on both sides,

$$\begin{aligned}\sup_{t\in[0,T]} \|y_n(t\wedge\tau_n)\|^2 &\le -\frac{5}{4}\int_0^{t\wedge\tau_n} |\Delta y_n(s)|_H^2\,ds\\ &+ (C_2+\varepsilon_n C_3 + C_4\varepsilon_n C_6)(T\wedge\tau_n)\\ &+ 2\varepsilon_n\sum_{j=1}^N \sup_{t\in[0,T]} I_{5,j}\end{aligned}$$

Taking expectations on both sides we get

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\left(\|y_n(t\wedge\tau_n)\|^2\right) &\le -\frac{5}{4}\mathbb{E}\left(\int_0^{t\wedge\tau_n}|\Delta y_n(s)|_H^2\,ds\right)\\ &+ (C_2+\varepsilon_n C_3 + C_4\varepsilon_n C_6)(T\wedge\tau_n)\\ &+ 2\varepsilon_n\mathbb{E}\left(\sup_{t\in[0,T]} I_{5,j}\right).\end{aligned}$$

Using inequality (4.2.59),

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left(\|y_n(t\wedge\tau_n)\|^2\right) + \left(\frac{5}{4} - N6\varepsilon_n C_5\right)\left(\int_0^{T\wedge\tau_n}|\Delta y_n(s)|_H^2\,ds\right)\right]\\ &\qquad\le (C_2+\varepsilon_n C_3 + C_4 + 6\varepsilon_n C_5 + \varepsilon_n C_6)(T\wedge\tau_n)\end{aligned}$$

Now we can find a natural number n_0 such that $\left(\frac{5}{4} - N6\varepsilon_n C_5\right) \ge \frac{1}{2}$, for all $n \ge n_0$. It follows that,

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left(\|y_n(t\wedge\tau_n)\|^2\right) + \frac{1}{2}\left(\int_0^{T\wedge\tau_n}|\Delta y_n(s)|_H^2\,ds\right)\right]\\ \le\ &\mathbb{E}\left[\sup_{t\in[0,T]}\left(\|y_n(t\wedge\tau_n)\|^2\right) + (1 - N6\varepsilon_n C_5)\left(\int_0^{T\wedge\tau_n}|\Delta y_n(s)|_H^2\,ds\right)\right]\\ \le\ &(C_2+\varepsilon_n C_3 + C_4\varepsilon_n + 6\varepsilon_n C_5 + C_6)(T\wedge\tau_n)(T\wedge\tau_n)\end{aligned}$$

for all $n \ge n_0$. Hence using the inequality above we infer that, for all $n \ge n_0$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(\|y_n(t\wedge\tau_n)\|^2\right) + \int_0^{T\wedge\tau_n}|\Delta y_n(s)|_H^2\,ds\right] \le C,$$

where $C := (C_2+\varepsilon_n C_3 + C_4\varepsilon_n + 6\varepsilon_n C_5 + C_6)(T\wedge\tau_n)$. This completes the proof of the theorem. ■

In the framework of Theorem 4.2.20, we prove the following key Lemma.

Lemma 4.2.23. *Assume that* $m \in (\|u_0\|, \infty)$. *If* τ_n *be the stopping times defined by equation* (4.2.45), *and* $\xi_n := y_n - u_n = \Im^{\varepsilon_n}_{h_n}\left(\left(\varepsilon_n W_j + \int_0^t h_{n,j}(s)\, ds\right)_{j=1}^N\right) - \Im^0_{h_n}$. *Then*

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{t\in[0,T]} \|\xi_n(t\wedge\tau_n)\|^2 + \int_0^{T\wedge\tau_n} |\xi_n(s)|_E^2\, ds\right) = 0$$

Proof. Let us begin by giving expressions for u_n and y_n respectively. For $t \in [0,T]$,

$$\begin{aligned} u_n(t) &= \Im^0_{h_n}(h_n)(t) \\ &= u_0 + \int_0^t (\Delta u_n(s) + F(u_n(s)))\, ds + \sum_{j=1}^N \int_0^t B_j(u_n(s))\, h_{n,j}(s)\, ds \end{aligned}$$

and

$$\begin{aligned} y_n(t) &= \Im^{\varepsilon_n}_{h_n}\left(\left(\varepsilon_n W_j + \int_0^t h_{n,j}(s)\, ds\right)_{j=1}^N\right) \\ &= u_0 + \int_0^t (\Delta y_n(s) + F(y_n(s))) + \frac{\varepsilon_n}{2}\sum_{j=1}^N \int_0^t \kappa_j(y_n(s))ds \\ &+ \sum_{j=1}^N \int_0^t B_j(y_n(s))\, h_{n,j}(s)\, ds + \sqrt{\varepsilon_n}\sum_{j=1}^N \int_0^t B_j(y_n(s))\, dW_j(s), \ \mathbb{P}\text{-a.s.} \end{aligned}$$

As we denote $\xi_n := y_n - u_n$, so by taking difference of last equations, for $t \in [0,T]$,

$$\begin{aligned} \xi_n &= \int_0^t (\Delta\xi_n(s))\, ds + \int_0^t (F(y_n(s)) - F(u_n(s)))\, ds \\ &+ \frac{\varepsilon_n}{2}\sum_{j=1}^N \int_0^t \kappa_j(y_n(s))ds \\ &+ \sum_{j=1}^N \int_0^t [B_j(y_n(s)) - B_j(u_n(s))]\, h_{n,j}(s)\, ds \\ &+ \sqrt{\varepsilon_n}\sum_{j=1}^N \int_0^t B_j(y_n(s))\, dW_j(s), \mathbb{P}\text{-a.s.} \end{aligned} \tag{4.2.61}$$

Our intentions are to apply Itō Lemma, to process $\left(\|\xi_n(t\wedge\tau_n)\|^2\right)_{t\in[0,T]}$.

For $t \in [0,T]$, by Itô Lemma we have the following,

$$\begin{aligned}
\|\xi_n(t\wedge\tau_n)\|^2 &= -2\int_0^{t\wedge\tau_n} \langle \Delta\xi_n(s), \Delta\xi_n(s)\rangle \, ds \\
&\quad +2\int_0^{t\wedge\tau_n} \langle -\Delta\xi_n(s), F\left(y_n\left(s\right)\right) - F\left(u_n\left(s\right)\right)\rangle \, ds \\
&\quad +\varepsilon_n \sum_{j=1}^N \int_0^{t\wedge\tau_n} \langle -\Delta\xi_n(s), \kappa_j(y_n\left(s\right))\rangle \, ds \\
&\quad +2\sum_{j=1}^N \int_0^{t\wedge\tau_n} \langle -\Delta\xi_n(s), B_j\left(y_n\left(s\right)\right) - B_j\left(u_n\left(s\right)\right)\rangle \, h_{n,j}\left(s\right) ds \\
&\quad +2\varepsilon_n \sum_{j=1}^N \int_0^{t\wedge\tau_n} \langle -\Delta\xi_n(s), B_j\left(y_n\left(s\right)\right)\rangle \, dW_j\left(s\right) \\
&\quad +\varepsilon_n^2 \sum_{j=1}^N \int_0^{t\wedge\tau_n} \langle B_j\left(y_n\left(s\right)\right), B_j\left(y_n\left(s\right)\right)\rangle_V \, ds, \ \mathbb{P}\text{-a.s.} \\
&=: -2I_1 + 2I_2 + \varepsilon_n \sum_{j=1}^N I_{3,j} + 2\sum_{j=1}^N I_{4,j} \qquad (4.2.62) \\
&\quad +2\varepsilon_n \sum_{j=1}^N I_{5,j} + \varepsilon_n^2 \sum_{j=1}^N I_{6,j}.
\end{aligned}$$

Before proceeding further let us recall the following inequalities, which we are going to use frequently in rest of proof. From definition of τ_n i.e. (4.2.45), it follows that,

$$\|y_n\left(t\right)\| \le m, \ t \in [0,\tau_n], \qquad (4.2.63)$$

and from Lemma 4.2.15,

$$\|u_n\left(t\right)\| \le C\left(u_0, T, K\right), \ t \in [0,T]. \qquad (4.2.64)$$

Also set

$$\overline{K} := \max\left\{|f_j|_H^2, \|f_j\|^2\right\}_{j=1}^N < \infty. \qquad (4.2.65)$$

Consider the first integral I_1 in last equation,

$$\begin{aligned} I_1 &= \int_0^{t\wedge\tau_n} \langle \Delta\xi_n(s), \Delta\xi_n(s)\rangle \, ds \\ &= \int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds \end{aligned} \tag{4.2.66}$$

Now consider the second integral I_2, using Cauchy-Schwartz and inequalities (4.2.11), (4.2.63) and (4.2.64),

$$\begin{aligned} I_2 &= \int_0^{t\wedge\tau_n} \langle -\Delta\xi_n(s), F(y_n(s)) - F(u_n(s))\rangle \, ds \\ &\leq \int_0^{t\wedge\tau_n} G(\|y_n(s)\|, \|u_n(s)\|) \|\xi_n(s)\| \, |\Delta\xi_n(s)|_H \, ds \\ &\leq \int_0^{t\wedge\tau_n} G(m, C) \|\xi_n(s)\| \, |\Delta\xi_n(s)|_H \, ds \\ &= C\int_0^{t\wedge\tau_n} \|\xi_n(s)\| \, |\Delta\xi_n(s)|_H \, ds, \end{aligned} \tag{4.2.67}$$

where $C := G(m, C)$. Here $G : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a bounded and symmetric map, defined as

$$\begin{aligned} G(r,s) \; : &= C^2\left(r^2+s^2\right) + (r+s)^2 + C^{2n+1}\left[\begin{array}{c} \left(\frac{2n-1}{2}\right)\left(r^{2n-1}+s^{2n-1}\right)(r+s) \\ +\left(r^{2n}+s^{2n}\right) \end{array}\right] \\ &+ C^{2n-1}\left(\frac{2n-2}{2}\right)\left(r^{2n-2}+s^{2n-2}\right). \end{aligned}$$

We are now going to apply the Young inequality (4.2.9) to integrand in inequality (4.2.67). By taking $a = C\|\xi_n\|$, $b = |\Delta\xi_n(s)|_H$, $p = q = 2$ and $\sigma = \frac{1}{2}$ for the Young inequality (4.2.9) and using it into (4.2.67) we get,

$$\begin{aligned} I_2 &\leq C^2\int_0^{t\wedge\tau_n} \|\xi_n(s)\|^2 \, ds + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds \\ &= C_2\int_0^{t\wedge\tau_n} \|\xi_n(s)\|^2 \, ds + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds, \end{aligned}$$

where $C_2 := C^2$.

Now consider the third integral $I_{3,j}$, using Cauchy Schwartz inequality and (4.2.15), we infer

$$\begin{aligned} I_{3,j} &= \int_0^{t\wedge\tau_n} \langle -\Delta\xi_n(s), \kappa_j(y_n(s))\rangle \\ &\leq \int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H \, |\kappa_j(y_n(s))|_H \\ &\leq \int_0^{t\wedge\tau_n} |f|_H^2 \left[2 + |y_n(s)|_H^2 + C\,|y_n(s)|_H^2\right] |y_n(s)|_H \, |\Delta\xi_n(s)|_H \, ds, \end{aligned}$$

Using continuity of the embedding $V \hookrightarrow H$, inequalities (4.2.63) and (4.2.65) we get

$$\begin{aligned} I_{3,j} &\leq \int_0^{t\wedge\tau_n} \overline{K}\left[2 + m^2 + Cm^2\right] m\,|\Delta\xi_n(s)|_H \, ds, \\ &= \int_0^{t\wedge\tau_n} C\,|\Delta\xi_n(s)|_H \, ds \end{aligned}$$

Where $C := \overline{K}\,[2 + m^2 + Cm^2]\,m$. By taking $a = C$, $b = |\Delta\xi_n(s)|_H$, $p = q = 2$ and $\sigma = \frac{1}{2\varepsilon_n N}$ for the Young inequality (4.2.9), the last inequality becomes,

$$\begin{aligned} I_{3,j} &\leq N\varepsilon_n \int_0^{t\wedge\tau_n} C^2 ds + \frac{1}{4\varepsilon_n N}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds \\ &= N\varepsilon_n C^2\,(t\wedge\tau_n) + \frac{1}{4\varepsilon_n N}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds \end{aligned}$$

Taking sum on j on both sides and multiplying by ε_n we get

$$\begin{aligned} \varepsilon_n \sum_{j=1}^N I_{3,j} &\leq N\varepsilon_n C^2 \sum_{j=1}^N (t\wedge\tau_n) + \frac{1}{4\varepsilon_n N}\sum_{j=1}^N \int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds \\ &= N\varepsilon_n^2 C^2\,(t\wedge\tau_n) + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds \\ &= \varepsilon_n^2 C_3\,(t\wedge\tau_n) + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds, \end{aligned} \tag{4.2.68}$$

where $C_3 := NC^2$.

Now consider the 4th integral i.e. $I_{4,j}$. Use of Cauchy-Schwartz and inequality

(4.2.12), we get

$$\begin{aligned} I_{4,j} &= \int_0^{t\wedge\tau_n} \langle -\Delta\xi_n(s), B_j(y_n(s)) - B_j(u_n(s))\rangle h_{n,j}(s)\,ds \\ &\leq \int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H |B_j(y_n(s)) - B_j(u_n(s))|_H h_{n,j}(s)\,ds \\ &\leq \int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H C|f|_H (\|y_n(s)\| + \|u_n(s)\|)\|y_n(s) - u_n(s)\| h_{n,j}(s)\,ds \\ &= \int_0^{t\wedge\tau_n} C|f|_H (\|y_n(s)\| + \|u_n(s)\|)|\Delta\xi_n(s)|_H \|\xi_n(s)\| h_{n,j}(s)\,ds. \end{aligned}$$

Using (4.2.63), (4.2.64) and (4.2.65) we infer $C|f|_H(\|y_n(s)\| + \|u_n(s)\|) \leq C\overline{K}(m + C) =: C$, therefore the last inequality simplifies to the following,

$$I_{4,j} \leq \int_0^{t\wedge\tau_n} C|\Delta\xi_n(s)|_H \|\xi_n(s)\| h_{n,j}(s)\,ds.$$

By taking $a = C\|\xi_n\| h_{n,j}$, $b = |\Delta\xi_n(s)|_H$, $p = q = 2$ and $\sigma = \frac{1}{4N}$ into the Young inequality and applying to last inequality,

$$\begin{aligned} I_{4,j} &\leq 2NC^2 \int_0^{t\wedge\tau_n} \|\xi_n(s)\|^2 (h_{n,j}(s))^2\,ds + \frac{1}{8N}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2\,ds \\ &= 2NC^2 \int_0^{t\wedge\tau_n} \|\xi_n(s)\|^2 (h_{n,j}(s))^2\,ds + \frac{1}{8N}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2\,ds, \end{aligned}$$

Taking sum on j on both sides and multiplying 2 we get

$$2\sum_{j=1}^N I_{4,j} \leq C_4 \sum_{j=1}^N \int_0^{t\wedge\tau_n} \|\xi_n(s)\|^2 (h_{n,j}(s))^2\,ds + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2\,ds,$$

where $C_4 := 4NC^2 < \infty$.

Since $I_{5,j}$ is the Itô integral so we deal it later and keep it as it is.

Consider the last 6th integral i.e. $I_{6,j}$,. Using inequality (4.2.12) it follows that,

$$\begin{aligned} I_{6,j} &= \int_0^{t\wedge\tau_n} \langle B_j(y_n(s)), B_j(y_n(s))\rangle_V \, ds \\ &\leq \int_0^{t\wedge\tau_n} \|B_j(y_n(s))\|^2 \, ds \\ &\leq C\int_0^{t\wedge\tau_n} |f|_H \|y_n(s)\|^2 \, ds \end{aligned}$$

Using inequality (4.2.63) and (4.2.65) into last inequality,

$$I_{6,j} \leq C\overline{K}m\,(t\wedge\tau_n)$$

Taking sum on j on both sides and multiplying ε_n we get

$$\varepsilon_n \sum_{j=1}^{N} I_{6,j} \leq \varepsilon_n \sum_{j=1}^{N} CKm(t\wedge\tau_n) = C_6\varepsilon_n(T\wedge\tau_n), \tag{4.2.69}$$

where $C_6 := CKm$.

Adding inequalities (4.2.66)-(4.2.69) and using it into (4.2.63) we get

$$\begin{aligned} \|\xi_n(t\wedge\tau_n)\|^2 &\leq -2\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds + C_2\int_0^{t\wedge\tau_n} \|\xi_n(s)\|^2 \, ds + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds \\ &\quad +\varepsilon_n^2 C_3\,(t\wedge\tau_n) + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds \\ &\quad +2C_4\sum_{j=1}^{N}\int_0^{t\wedge\tau_n} \|\xi_n(s)\|^2 (h_{n,j}(s))^2 \, ds + \frac{1}{4}\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds \\ &\quad +2\sqrt{\varepsilon_n}\sum_{j=1}^{N} I_{5,j} + C_6\varepsilon_n(T\wedge\tau_n) \\ &\leq -\int_0^{t\wedge\tau_n} |\Delta\xi_n(s)|_H^2 \, ds + 2\sqrt{\varepsilon_n}\sum_{j=1}^{N}\int_0^{t\wedge\tau_n} \langle -\Delta\xi_n(s), B_j(y_n(s))\rangle \, dW_j(s) \\ &\quad +\int_0^{t\wedge\tau_n} \left(C_2 + 2C_4(h_{n,j}(s))^2\right)\|\xi_n(s)\|^2 + \left(C_6\varepsilon_n + \varepsilon_n^2 C_3\right)(T\wedge\tau_n) \end{aligned}$$

$$\begin{aligned}
\|\xi_n(t\wedge\tau_n)\|^2+\int_0^{t\wedge\tau_n}|\Delta\xi_n(s)|_H^2\,ds \ &\leq\ 2\sqrt{\varepsilon_n}\sum_{j=1}^N\int_0^{t\wedge\tau_n}\langle-\Delta\xi_n(s),B_j\left(y_n\left(s\right)\right)\rangle\,dW_j\left(s\right)\\
&\quad+\left(C_6\varepsilon_n+\varepsilon_n^2C_3\right)\left(T\wedge\tau_n\right)\\
&\quad+\int_0^{t\wedge\tau_n}\left(C_2+2C_4\left(h_{n,j}\left(s\right)\right)^2\right)\\
&\quad\cdot\left(\|\xi_n(t\wedge\tau_n)\|^2+\int_0^{s\wedge\tau_n}|\Delta\xi_n(r)|_H^2\,dr\right)ds
\end{aligned}$$

Taking supremum on both sides,

$$\begin{aligned}
&\sup_{t\in[0,T]}\left(\|\xi_n(t\wedge\tau_n)\|^2+\textstyle\int_0^{t\wedge\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)\\
&\leq\ \left(C_6\varepsilon_n+\varepsilon_n^2C_3\right)\left(T\wedge\tau_n\right)\ +2\sqrt{\varepsilon_n}\sum_{j=1}^N\sup_{t\in[0,T]}\int_0^{t\wedge\tau_n}\langle-\Delta\xi_n(s),B_j\left(y_n\left(s\right)\right)\rangle\,dW_j\left(s\right)\\
&\quad+\sup_{t\in[0,T]}\int_0^{t\wedge\tau_n}\left(C_2+2C_4\left(h_{n,j}\left(s\right)\right)^2\right)\cdot\left(\|\xi_n(s\wedge\tau_n)\|^2+\int_0^{s\wedge\tau_n}|\Delta\xi_n(r)|_H^2\,dr\right)ds
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\xi_n(t\wedge\tau_n)\|^2+\textstyle\int_0^{t\wedge\tau_n}|\Delta\xi_n(s)|_H^2\,ds\\
&\leq\ \left(C_6\varepsilon_n+\varepsilon_n^2C_3\right)\left(T\wedge\tau_n\right)+2\sqrt{\varepsilon_n}\sum_{j=1}^N\sup_{t\in[0,T]}\int_0^{t\wedge\tau_n}\langle-\Delta\xi_n(s),B_j\left(y_n\left(s\right)\right)\rangle\,dW_j\left(s\right)\\
&\quad+\int_0^{T\wedge\tau_n}\sup_{s\in[0,T]}\left(C_2+2C_4\left(h_{n,j}\left(s\right)\right)^2\right)\cdot\sup_{s\in[0,T]}\left(\|\xi_n(s\wedge\tau_n)\|^2+\int_0^{s\wedge\tau_n}|\Delta\xi_n(r)|_H^2\,dr\right)ds.
\end{aligned}$$

Taking expectations on both sides we get

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left(\|\xi_n(t\wedge\tau_n)\|^2+\textstyle\int_0^{t\wedge\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)\leq\\
&\left(C_6\varepsilon_n+\varepsilon_n^2C_3\right)\left(T\wedge\tau_n\right)+2\sqrt{\varepsilon_n}C\sum_{j=1}^N\left(\mathbb{E}\sup_{t\in[0,T]}\int_0^{t\wedge\tau_n}\langle-\Delta\xi_n(s),B_j\left(y_n\left(s\right)\right)\rangle\,dW_j\left(s\right)\right)\\
&+\int_0^{t\wedge\tau_n}\left(\sum_{j=1}^N\frac{2C}{N}+2C\left(h_{n,j}\left(s\right)\right)^2\right)\\
&\cdot\mathbb{E}\sup_{t\in[0,T]}\left(\|\xi_n(s\wedge\tau_n)\|^2+\int_0^{s\wedge\tau_n}|\Delta\xi_n(r)|_H^2\,dr\right)ds
\end{aligned}$$

Using the Burkholder inequality for $p=1$,

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left(\|\xi_n(t\wedge\tau_n)\|^2+\int_0^{t\wedge\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)\\
\leq\ & \left(C_6\varepsilon_n+\varepsilon_n^2C_3\right)(T\wedge\tau_n)+12\varepsilon_nC^2\sum_{j=1}^N\mathbb{E}\left(\int_0^{t\wedge\tau_n}\langle-\Delta\xi_n(s),B_j\left(y_n\left(s\right)\right)\rangle^2\,ds\right)\\
&+\int_0^{t\wedge\tau_n}\sup_{s\in[0,T]}\left(\sum_{j=1}^N\frac{2C}{N}+2C\left(h_{n,j}\left(s\right)\right)^2\right)\\
&\cdot\mathbb{E}\sup_{s\in[0,T]}\left(\|\xi_n(s\wedge\tau_n)\|^2+\int_0^{s\wedge\tau_n}|\Delta\xi_n(r)|_H^2\,dr\right)ds\\
\leq\ & \left(C_6\varepsilon_n+\varepsilon_n^2C_3\right)(T\wedge\tau_n)+12\varepsilon_nC^2\sum_{j=1}^N\mathbb{E}\left(\int_0^{t\wedge\tau_n}|-\Delta\xi_n(s)|_H^2\left|B_j\left(y_n\left(s\right)\right)\right|_H^2ds\right)\\
&+\int_0^{t\wedge\tau_n}\sup_{s\in[0,T]}\left(\sum_{j=1}^N\frac{2C}{N}+2C\left(h_{n,j}\left(s\right)\right)^2\right)\\
&\cdot\mathbb{E}\sup_{s\in[0,T]}\left(\|\xi_n(s\wedge\tau_n)\|^2+\int_0^{s\wedge\tau_n}|\Delta\xi_n(r)|_H^2\,dr\right)ds
\end{aligned}$$

Now using inequalities (4.2.13), (4.2.63) and (4.2.65) we infer that for $s\leq\tau_n$, we have

$$\left|B_j\left(y_n\left(s\right)\right)\right|_H^2\leq C\left|f\right|_H\left\|y_n\left(s\right)\right\|^2\leq C\overline{K}m=:C_7$$

Hence the last inequality simplifies to

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left(\|\xi_n(t\wedge\tau_n)\|^2+\int_0^{t\wedge\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)\\
&\leq\ C_6\varepsilon_n(T\wedge\tau_n)+12\varepsilon_nC_7\sum_{j=1}^N\mathbb{E}\left(\sup_{t\in[0,T]}\int_0^{t\wedge\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)\\
&\quad+\int_0^{t\wedge\tau_n}\sup_{s\in[0,T]}\left(\sum_{j=1}^N\frac{2C}{N}+2C\left(h_{n,j}\left(s\right)\right)^2\right)\\
&\quad\cdot\mathbb{E}\sup_{s\in[0,T]}\left(\|\xi_n(s\wedge\tau_n)\|^2+\int_0^{s\wedge\tau_n}|\Delta\xi_n(r)|_H^2\,dr\right)ds
\end{aligned}$$

or

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left(\|\xi_n(t\wedge\tau_n)\|^2+(1-12\varepsilon_nC_7)\int_0^{\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)\right)$$
$$\leq \left(C_6\varepsilon_n+\varepsilon_n^2C_3\right)(T\wedge\tau_n)+\mathbb{E}\int_0^{t\wedge\tau_n}\sup_{s\in[0,T]}\left(\sum_{j=1}^N\frac{2C}{N}+2C\left(h_{n,j}(s)\right)^2\right)$$
$$\cdot\mathbb{E}\sup_{s\in[0,T]}\left(\|\xi_n(s\wedge\tau_n)\|^2+\int_0^{s\wedge\tau_n}|\Delta\xi_n(r)|_H^2\,dr\right)ds$$

Next, we can find a natural number n_0 such that $(1-12\varepsilon_nC_7)\geq\frac{1}{2}$, for all $n\geq n_0$. It follows that for all $n\geq n_0$,

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left(\|\xi_n(t\wedge\tau_n)\|^2+\frac{1}{2}\int_0^{\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)\right)$$
$$\leq \left(C_6\varepsilon_n+\varepsilon_n^2C_3\right)(T\wedge\tau_n)+\mathbb{E}\int_0^{t\wedge\tau_n}\sup_{s\in[0,T]}\left(\sum_{j=1}^N\frac{2C}{N}+2C\left(h_{n,j}(s)\right)^2\right)$$
$$\cdot\mathbb{E}\sup_{s\in[0,T]}\left(\|\xi_n(s\wedge\tau_n)\|^2+\int_0^{s\wedge\tau_n}|\Delta\xi_n(r)|_H^2\,dr\right)ds$$

Multiplying both sides of inequality by 2, we get

$$\mathbb{E}\sup_{t\in[0,T]}\left(\|\xi_n(t\wedge\tau_n)\|^2+\int_0^{t\wedge\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)$$
$$\leq 2\left(C_6\varepsilon_n+\varepsilon_n^2C_3\right)(T\wedge\tau_n)+\int_0^{t\wedge\tau_n}\sup_{s\in[0,T]}\left(\sum_{j=1}^N\frac{4C}{N}+4C\left(h_{n,j}(s)\right)^2\right)$$
$$\cdot\mathbb{E}\sup_{s\in[0,T]}\left(\|\xi_n(s\wedge\tau_n)\|^2+\int_0^{s\wedge\tau_n}|\Delta\xi_n(r)|_H^2\,dr\right)ds,$$

for all $n\geq n_0$. Set

$$F(t) \;:\; =\mathbb{E}\sup_{t\in[0,T]}\left(\|\xi_n(t\wedge\tau_n)\|^2+\int_0^{t\wedge\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)$$
$$\alpha(t) \;:\; =2\left(C_6\varepsilon_n+\varepsilon_n^2C_3\right)(T\wedge\tau_n)$$
$$\beta(t) \;:\; =\sup_{s\in[0,T]}\left(\sum_{j=1}^N\frac{4C}{N}+4C\left(h_{n,j}(s)\right)^2\right)$$

Using this notation the last inequality becomes,

$$F(t)\leq\alpha(t)+\int_0^{t\wedge\tau_n}\beta(s)F(s).$$

Observe that α is non-decreasing and β is non-negative, hence using Gronwall Lemma 4.2.8 and using assumption $\int_0^T h_{n,j}(s)^2\,ds \leq K$,

$$F(t) \leq \alpha(s)e^{\int_0^{t\wedge\tau_n}\beta(r)dr}$$

i.e.

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left(\|\xi_n(t\wedge\tau_n)\|^2 + \int_0^{t\wedge\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)\\
&\leq\ 2\left(C_6\varepsilon_n + \varepsilon_n^2 C_3\right)(T\wedge\tau_n)\exp\left[\int_0^{t\wedge\tau_n}\left(\sum_{j=1}^N \frac{4C}{N} + 4Ch_{n,j}(s)^2\right)\right]\\
&\leq\ 2\left(C_6\varepsilon_n + \varepsilon_n^2 C_3\right)(T\wedge\tau_n)\exp\left[\sum_{j=1}^N \frac{4C}{N}(T\wedge\tau_n) + 4CK\right].
\end{aligned}$$

Passing the limit $n\to\infty$ i.e. $\varepsilon_n \to 0$, we infer that the right hand side of above inequality tends to zero. Thus

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{t\in[0,T]}\|\xi_n(t\wedge\tau_n)\|^2 + \int_0^{t\wedge\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right) = 0\ .$$

This completes the proof. ■

The last Lemma 4.2.20 was one of the key results towards the proving main result of the section Theorem 4.2.20. Before going towards the proof of the main result, let us prove the following important corollary.

Corollary 4.2.24. *Assume that $m \in (\|u_0\|, \infty)$, τ_n be the stopping time defined by equality* (4.2.45) *and $\xi_n := y_n - u_n$ is as described in equation* (4.2.61). *The sequence of X_T-valued process $(\xi_n)_{n\in\mathbb{N}}$ converges in probability to* 0.

Proof. We want show that for all $\delta > 0$,

$$\mathbb{P}\left(\left\{\sup_{t\in[0,T]}\|\xi_n(t)\|^2 + \int_0^T|\Delta\xi_n(s)|_H^2\,ds > \delta\right\}\right) \to 0 \text{ as } n\to\infty.$$

i.e. for all $\delta > 0$ and for all $\varepsilon > 0$ there exists n_0 such that

$$\mathbb{P}\left(\left\{\sup_{t\in[0,T]} \|\xi_n(t)\|^2 + \int_0^T |\Delta\xi_n(s)|_H^2\, ds \geq \delta\right\}\right) < \varepsilon, \text{ for all } n \geq n_0.$$

Recall that, in Lemma 4.2.23 we concluded that,

$$\lim_{n\to\infty} \mathbb{E}\left(\sup_{t\in[0,T]} \|\xi_n(t\wedge\tau_n)\|^2 + \int_0^{\tau_n} |\xi_n(s)|_E^2\, ds\right) = 0.$$

Also recall from 4.2.22 that there exists constant $C(u_0, T, K)$ such that, for all $n \in \mathbb{N}$ and we have

$$\limsup_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \left(\|y_n(t\wedge\tau_n)\|^2\right) + \int_0^{\tau_n} |\Delta y_n(s)|_H^2\, ds\right] \leq C(u_0, T, K).$$

Let $\delta > 0$ and $\varepsilon > 0$. Choose an auxiliary $m > \|u_0\|$ and $n_0 = n_0(\varepsilon, \delta)$ such that,

$$\frac{1}{m}\sup_{n\in\mathbb{N}}\mathbb{E}\left(\sup_{t\in[0,T]} \|y_n(t)\|\right) < \frac{\varepsilon}{2}, \text{ for all } n \geq n_0, \tag{4.2.70}$$

and

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|\xi_n(t\wedge\tau_n)\|^2 + \int_0^{\tau_n} |\xi_n(s)|_E^2\, ds\right) < \frac{\delta\varepsilon}{2}, \text{ for all } n \geq n_0. \tag{4.2.71}$$

Employing the Lemma 4.2.23 and for sufficiently large n consider the following set of inequalities

$$\begin{aligned} &\mathbb{P}\left(\left\{\sup_{t\in[0,T]} \|\xi_n(t\wedge\tau_n)\|^2 + \int_0^T |\Delta\xi_n(s)|_H^2\, ds \geq \delta\right\}\right) \\ \leq\ &\mathbb{P}\left(\left\{\sup_{t\in[0,T]} \|\xi_n(t\wedge\tau_n)\|^2 + \int_0^{\tau_n} |\Delta\xi_n(s)|_H^2\, ds \geq \delta, \tau_n = T\right\}\right) \\ &+\mathbb{P}\left(\left\{\sup_{t\in[0,T]} \|y_n(t)\| \geq m\right\}\right). \end{aligned}$$

Using Chebyshev inequality on both two terms, on right hand side of, last inequality, we get

$$\begin{aligned}
&\mathbb{P}\left(\left\{\sup_{t\in[0,T]}\|\xi_n(t)\|^2+\int_0^T|\Delta\xi_n(s)|_H^2\,ds\geq\delta\right\}\right)\\
\leq\ &\frac{1}{\delta}\mathbb{E}\left(\sup_{t\in[0,T]}\|\xi_n(t\wedge\tau_n)\|^2+\int_0^{\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)\\
&+\frac{1}{m}\mathbb{E}\left(\sup_{t\in[0,T]}\|y_n\left(t\right)\|\right)\\
\leq\ &\frac{1}{\delta}\mathbb{E}\left(\sup_{t\in[0,T]}\|\xi_n(t\wedge\tau_n)\|^2+\int_0^{\tau_n}|\Delta\xi_n(s)|_H^2\,ds\right)\\
&+\frac{1}{m}\sup_{n\in\mathbb{N}}\mathbb{E}\left(\sup_{t\in[0,T]}\|y_n\left(t\right)\|\right)
\end{aligned}$$

Using inequalities (4.2.70)) and (4.2.71) into last inequality,

$$\mathbb{P}\left(\left\{\sup_{t\in[0,T]}\|\xi_n(t)\|^2+\int_0^T|\Delta\xi_n(s)|_H^2\,ds\geq\delta\right\}\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

This completes the proof. ■

Recall that, we shown in the Lemma 4.2.19 that, if $(h_n)_{n\in\mathbb{N}}$ is a sequence in $L^2\left(0,T;\mathbb{R}^N\right)$ that converges weakly to h in $L^2\left(0,T;\mathbb{R}^N\right)$ on B_K then

$$(u_{h_n})_{n\in\mathbb{N}}:=\left(\Im^0_{h_n}\right)_{n\in\mathbb{N}}\text{converges in distribution to } u_h:=\Im^0_h \tag{4.2.72}$$

in X_T-norm. This implies that $\Im^0_h:L^2\left(0,T;\mathbb{R}^N\right)\to X_T$ is Borel measurable.

Next let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence from $(0,1]$ that converges to zero as $n\to\infty$. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of predictable processes that converges in distribution on

$$B_K:=\left\{h\in L^2\left(0,T;\mathbb{R}^N\right):\int_0^T|h\left(s\right)|^2_{\mathbb{R}^N}\,ds\leq K<\infty\right\},$$

to h. We shown in Lemma 4.2.23 and Corollary 4.2.24

$$(y_n-u_n)_{n\in\mathbb{N}}\ \text{converges to 0 in probability,} \tag{4.2.73}$$

as a sequence of random variables in X_T, where $y_n(\cdot) = \Im_{h_n}^{\varepsilon_n}\left(\varepsilon_n W_j + \int_0^{\cdot} h_j^{\varepsilon_n}(s)\,ds\right)$ and $u_n = \Im_{h_n}^0$.

Also recall the following Skorohod Theorem from [28]

Theorem 4.2.25. *([28] , Theorem 3.30, page 56). Let $\xi, \xi_1, \xi_2, \xi_3, \ldots$ be random elements in superable metric space (S, ρ) such that $\xi_n \xrightarrow{d} \xi$. Then on a suitable probability space, there exists some random elements $\eta \stackrel{d}{=} \xi$ and $\eta_n \stackrel{d}{=} \xi_n$, $n \in \mathbb{N}$, with $\eta_n \to \eta$ a.s.*

Finally, we give a proof of main Theorem 4.2.20 of subsection.

Proof. **(of Theorem 4.2.20)** Let us begin by noticing the fact that B_K is a separable metric space so we can employ the Skorohod Theorem. If $(h_n)_{n\in\mathbb{N}}$ converges in distribution to h i.e.

$$\mathcal{L}(h_n) \to \mathcal{L}(h) \text{ on } B_K.$$

By the Skorohod theorem (4.2.25) there exist new probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\right)$ and random elements $\widetilde{h}_n, \widetilde{h} : \widetilde{\Omega} \to B_K$ such that

$$\mathcal{L}\left(\widetilde{h}_n\right) = \mathcal{L}(h_n) \text{ for all } n \in \mathbb{N}, \tag{4.2.74}$$

and

$$\mathcal{L}\left(\widetilde{h}\right) = \mathcal{L}(h), \tag{4.2.75}$$

with

$$\widetilde{h}_n(\widetilde{\omega}) \to \widetilde{h}(\widetilde{\omega}) \text{ on } B_K, \text{ for all } \widetilde{\omega} \in \widetilde{\Omega}. \tag{4.2.76}$$

Next we claim that

$$\mathcal{L}(u_{h_n}) = \mathcal{L}\left(u_{\widetilde{h_n}}\right) \text{ for all } n \in \mathbb{N}\ . \tag{4.2.77}$$

To see the last equality, observe that $u_{h_n} = \Im^0_{h_n} = \Im^0 \circ h_n$ and $u_{\widetilde{h_n}} = \Im^0_{\widetilde{h_n}} = \Im^0 \circ \widetilde{h_n}$, where $\Im^0 : B_K \to X_T$. Let us pick a Borel set B in X_T, and consider the following,

$$\begin{aligned} \mathcal{L}(u_{h_n})(B) &= \mathcal{L}\left(\Im^0 \circ h_n\right)(B) \\ &= \mathbb{P}\left(\left(\Im^0 \circ h_n\right)^{-1}(B)\right) \\ &= \mathbb{P}\left(h_n^{-1} \circ \left(\Im^0\right)^{-1}(B)\right) \\ &= \mathbb{P}\left(h_n^{-1}\left[\left(\Im^0\right)^{-1}(B)\right]\right) \end{aligned}$$

Since B is Borel set and $\Im^0 : B_K \to X_T$ is continuous therefore $\left(\Im^0\right)^{-1}(B)$ is Borel in B_K. Hence the last equation turns into

$$\mathcal{L}(u_{h_n})(B) = \mathcal{L}(h_n)\left(\left(\Im^0\right)^{-1}(B)\right)$$

Using (4.2.75) it follows that

$$\begin{aligned} \mathcal{L}(u_{h_n})(B) &= \mathcal{L}\left(\widetilde{h}_n\right)\left(\left(\Im^0\right)^{-1}(B)\right) \\ &= \mathbb{P}\left(\left(\widetilde{h}_n\right)^{-1}\left[\left(\Im^0\right)^{-1}(B)\right]\right) \\ &= \mathbb{P}\left(\left(\widetilde{h}_n\right)^{-1} \circ \left(\Im^0\right)^{-1}(B)\right) \\ &= \mathbb{P}\left(\left(\Im^0 \circ \widetilde{h}_n\right)^{-1}(B)\right) \\ &= \mathcal{L}\left(\Im^0 \circ \widetilde{h}_n\right)(B) \\ &= \mathcal{L}\left(u_{\widetilde{h_n}}\right)(B), \text{ for all } B. \end{aligned}$$

Hence we are done with proving the equality (4.2.77). By the similar argument we can show that

$$\mathcal{L}(u_h) = \mathcal{L}\left(u_{\widetilde{h}}\right). \tag{4.2.78}$$

Next we are going to show that two convergence results i.e. (4.2.72) and (4.2.73) together imply that y_n converges in distribution to u_h on X_T i.e.

$$\mathcal{L}(y_n) \to \mathcal{L}(u_h) \text{ as } n \to \infty.$$

For a bounded and uniformly continuous function $\psi : X_T \to \mathbb{R}$, we have the following

$$\left|\int \psi d\mathcal{L}\left(y_n\right) - \int \psi d\mathcal{L}\left(u_h\right)\right| =$$

$$\begin{aligned}
&= \left|\int_\Omega \psi\left(y_n\right) dP - \int_{X_T} \psi d\mathcal{L}\left(u_h\right)\right| \\
&= \left|\int_\Omega \psi\left(y_n\right) dP - \int_{X_T} \psi d\mathcal{L}\left(u_{h_n}\right) + \int_{X_T} \psi d\mathcal{L}\left(u_{h_n}\right) - \int_{X_T} \psi d\mathcal{L}\left(u_h\right)\right| \\
&\leq \left|\int_\Omega \psi\left(y_n\right) dP - \int_{X_T} \psi d\mathcal{L}\left(u_{h_n}\right)\right| \\
&\quad + \left|\int_{X_T} \psi(x) d\mathcal{L}\left(u_{h_n}\right) - \int_{X_T} \psi(x) d\mathcal{L}\left(u_h\right)\right| \\
&\leq \left|\int_\Omega \psi\left(y_n\right) dP - \int_\Omega \psi(u_{h_n}) dP\right| \\
&\quad + \left|\int_{X_T} \psi d\mathcal{L}\left(u_{h_n}\right) - \int_{X_T} \psi d\mathcal{L}\left(u_h\right)\right| =: A_n + B_n. \qquad (4.2.79)
\end{aligned}$$

We claim that A_n and B_n both goes to 0 as $n \to \infty$.

Let us begin with A_n. By (4.2.73) we know that $(y_n - u_{h_n})_{n\in\mathbb{N}}$ converges to 0 in probability as $n \to \infty$. But since convergence in probability implies the weak convergence and since ψ is bounded continuous so hence

$$A_n = \left|\int_\Omega \psi\left(y_n\right) dP - \int_\Omega \psi(u_{h_n}) dP\right| \to 0 \text{ as } n \to \infty. \qquad (4.2.80)$$

Next, consider that B_n from (4.2.79). Using (4.2.77) and (4.2.78)

$$\begin{aligned}
B_n &= \left|\int_{X_T} \psi d\mathcal{L}\left(u_{h_n}\right) - \int_{X_T} \psi d\mathcal{L}\left(u_h\right)\right| \\
&= \left|\int_{X_T} \psi d\mathcal{L}\left(u_{\widetilde{h_n}}\right) - \int_{X_T} \psi d\mathcal{L}\left(u_{\widetilde{h}}\right)\right| \\
&= \left|\int_{\widetilde{\Omega}} \psi\left(u_{\widetilde{h_n}(\widetilde{\omega})}\right) d\widetilde{\mathbb{P}} - \int_{\widetilde{\Omega}} \psi\left(u_{\widetilde{h}(\widetilde{\omega})}\right) d\widetilde{\mathbb{P}}\left(\widetilde{\omega}\right)\right| \\
&= \left|\int_{\widetilde{\Omega}} \left[\psi\left(u_{\widetilde{h_n}(\widetilde{\omega})}\right) d\widetilde{\mathbb{P}} - \psi\left(u_{\widetilde{h}(\widetilde{\omega})}\right)\right] d\widetilde{\mathbb{P}}\left(\widetilde{\omega}\right)\right| \qquad (4.2.81)
\end{aligned}$$

Note that $u_h = \Im^0_h = \Im^0 \circ h$. From (4.2.74) and the fact that the map $\Im^0 : B_K \to X_T$

is continuous it follows that for all $\widetilde{\omega} \in \widetilde{\Omega}$,

$$u_{\widetilde{h_n}(\widetilde{\omega})} \to u_{\widetilde{h}(\widetilde{\omega})} \text{ as } n \to \infty.$$

Since ψ is continuous and bounded function it follows that for all $\widetilde{\omega} \in \widetilde{\Omega}$,

$$\left|\psi\left(u_{\widetilde{h_n}(\widetilde{\omega})}\right) - \psi\left(u_{\widetilde{h}(\widetilde{\omega})}\right)\right| \to 0 \text{ and as } n \to \infty.$$

Next from above convergence and application of Lebesgue dominated convergence theorem (see 1.4.32), we infer that,

$$B_n = \left|\int_{\widetilde{\Omega}} \left[\psi\left(u_{\widetilde{h_n}(\widetilde{\omega})}\right) - \psi\left(u_{\widetilde{h}(\widetilde{\omega})}\right)\right] d\widetilde{\mathbb{P}}(\widetilde{\omega})\right| \to 0 \text{ as } n \to \infty. \tag{4.2.82}$$

Thus using convergence (4.2.80) and (4.2.82) into (4.2.79) we get the desired convergence.

This completes the proof of Theorem 4.2.20. ■

Thus as a result of Theorem 4.2.14 and Theorem 4.2.20 we establish the following Large deviation principle.

Theorem 4.2.26. *The family of laws $\{\mathcal{L}(\Im_0^{\varepsilon}(W)) : \varepsilon \in (0,1]\}$ on X_T satisfies the Large deviation principle with rate function I, where I is as defined in equation* (4.2.10).

Bibliography

[1] A. Bensoussan, R. Temam, *Equations aux derives partielles stochastiques non linearies.*, Isr. J. Math., 11, 95-129, 1972.

[2] M. Bouė and P. Dupuis, *A variational representation for certain functionals of Brownian motion.*, Ann. Probab. 26, 1641-1659, 1998.

[3] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations., Springer., 2010.

[4] W. Bryc, *Large deviations by the asymptotic value method.*, Birkhauser, Boston ,vol. 1, pp. 447-472, 1990.

[5] Z. Brzeźniak and T. Zastawniak, Basic Stochastic Process. Springer, 1999.

[6] Z. Brzeźniak and K.D. Elworthy, *Stochastic differential equations on Banach manifolds*, Methods Funct.Anal. Topology 6(1):43-84, 2000.

[7] Z. Brzeźniak and A. Carroll, *Approximations of the Wong–Zakai type for stochastic differential equations in M-type 2 Banach spaces with applications to loop spaces.,* Sémi. de Prob. XXXVII., Springer Lect. Notes in Math., Vol 1832., 251-289., 2003.

[8] Z. Brzeźniak, B. Maslowski, J.Seidler., *Stochastic nonlinear beam equations.*, Prob. Theory & Rel. Fields., Vol.132, Issue 1, 119-149., 2005.

[9] Z. Brzeźniak and M. Ondrejat, *Stochastic Geometric Wave Equations.*, Stoch. Anal. Lect. Series., Sringer-Verlag., Vol 68., 157-188., 2010.

[10] Z. Brzeźniak and M. Ondrejat, *Weak solutions to stochastic wave equations with values in Riemannian manifolds.*, Comm. Part. Diff. Eqs. 36 (9), 1624–1653, 2011.

[11] Z. Brzeźniak, B. Goldys and M. Ondrejat, *Stochastic Geometric Partial Differential Equations*. New Trends in Stoc. Ana. and Related Topics., A Volume in Honour of Professor K. D. Elworthy. Vol. 12. World Sci. Pub. Co.., 1-32., 2012.

[12] Z. Brzeźniak and E. Motyl, *Existence of a martingale solution of Stochastic Naviour-Stokes equation in unbounded 2D and 3D domains.*, Preprint submitted to Jour. of Diff. Eq., March 17, 2014.

[13] Z. Brzeźniak, E. Hausenblas and P. Razafimandimby, *Stochastic Nonparabolic dissipative systems modeling the flow of Liquid Crystals: Strong solution.*, Kurenai., 1875: 41-72, 2014.

[14] Z. Brzeźniak, B. Goldys and T. Jegraj., *Large deviations and transitions between equilibria for Stochastic Landau-Lifshitz equation.*, availble at http://www.maths.usyd.edu.au/u/pubs/publist/preprints/2015/goldys-7.pdf

[15] Z. Brzeźniak, A. Carroll, *The stochastic nonlinear heat equation*, in preparation.

[16] A. Budhiraja and P. Dupuis, *A variational representation for positive functionals of infinite dimensional Brownian motion,* Probab. Math. Statist. 20, 39-61, 2000.

[17] L. Caffarelli and F. Lin, *Nonlocal heat flows preserving the L^2-energy*, Discrete and Continuous Dynamical Systems., Vol 32, 49-64, 2009.

[18] A. Carroll, *The stochastic nonlinear heat equation*. PhD thesis, The University of Hull, 1999.

[19] H. Cramér, *Sur un nouveau théoréme-limite de la théorie des probabilités.*, Actualitiés scientifiques et industrielles., Coloque consacré à la thérie des probabilitiés., Hermann Paris., Vol 3., 5-23, 1938.

[20] A. Dembo and O. Zeitouni, Large deviations techniques and applications, Springer-Verlag, New York, 2000.

[21] P. Dupuis and R. Ellis, A weak convergence approach to the theory of large deviations., Wiley, New York, 1997.

[22] T. Funak, *A stochastic partial differential equation with values in manifold.*, J. Funct. Ana., 109, no. 2, 257-258, 1992.

[23] D. Henry, Geometric theory of semilinear parabolic equations., Lect. Notes in Math. Springer ., vol.840., 1981.

[24] K.Itô, *On a stochastic integral equation*, Proc.Jap. Acad., Vol.22., 1946., 32-35.

[25] K.Itô, *On stochastic differential equation.*, AMS., New York., 1951.

[26] M. Jandoubi, *A Simple Unified Approach to Some Convergence Theorems of L. Simon*, Journal of Functional Analysis 153, 187-202., 1998.

[27] J. Jost, Postmodren Analysis., Springer-Verlag., 1998.

[28] O. Kallenberg, Foundations of modern probability., Springer-Verlag, New York, 2002.

[29] I. Karatzas and S. Shreve, Brownian motion and stochastic calculus, Springer-Verlag, New York, 2005.

[30] R. Z. Khashminskii, *Stability of systems of differential equations under random perturbations of their parameters.*, Nauka, Moskva, 1969 (Russian); Stochastic stability of differential equations, Sijthoff & Noordhoff,Alphen aan den Rijn., 1980.

[31] N. V. Krylov, B. L. Rozovskii., *Stochastic evolution equations.*, Translated from Itogi Naukii Tehniki, Seriya Sovermennye Problemy Matematiki, 14., 71-146.

[32] D. S. Malik, M. K. Sen, J.M. Moderson., Fundamentals of Abstract Algebra., McGraw-Hill Edu., 1997.

[33] A. Masiello, Variational Methods in Lorentzian Geometry., Longman Sci. & Tech., 1994.

[34] M. Ondrejat, *Uniqueness for stochastic evolution equations in Banach spaces*, Dissertationes Math. Rozprawy Mat., 2004,426, 63.

[35] E. Pardoux, *Sur des équations aux dérivées partielles stochastiques monotones.*, C.R. Acad. Sci. Paris Ser. A-B 275., 1975., A101-A103.

[36] E. Pardoux, *Equations aux dérivées partielles stochastiques non linéaires monotones.*, Ph.D. thesis., Universite Paris XI., 1975.

[37] E. Pardoux, *Stochastic differential equations and filtering of diffusion process.*, Stochastics., Vol 3, pp. 127-167, 1979.

[38] A. Pazy, *Semigroups of Linear operator and applications to partial differential equations.*, Springer-verlag., 1983.

[39] H. Pham, Continuous-time Stochastic Control and Optimization with Financial Applications., Sringer-Verlag., 2000.

[40] A. A. Pukhalskii, *On the theory of large deviations*, Theory probab. Appl. 38, 490-497, 1993.

[41] B. L. Rozovskii, *stochastic evolution systems.*, Math. and App., vol. 35., Acad. Pub.Group., Dordrecht., 1990.

[42] W. Rudin, Real and Complex Analysis, 3rd ed., McGraw-Hill,1987.

[43] P. Rybka, *Convergence Of Heat Flow On a Hilbert Manifold.*, Proceedings of the Royal Society of Edinburgh, 136A, 851–862, 2006.

[44] S. Salsa, Partial differential equations in action:From Modeling to theory., Springer-Verlag., 2009.

[45] M. Scheutzow, *A Stochastic Gronwall Lemma.*,arXiv., 2013.

[46] R. L. Schilling, Measures, Integrals and Martingales., Cambridge Uni. Press., 2005.

[47] Z. Songu, Non-Linear evolution equations., Champan & Hall/CRC., 2004.

[48] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics., App. Math.Sciences., Vol 68., 1988.

[49] R. Temam, Navier-Stokes Equations., AMS Chelsa Pub., 2000.

[50] S. R. S. Varadhan, *Asymptotic probabilities and differential equations*, Comm. Pure Appl. Math. 19, 261-286, 1966.

[51] I. I. Vrabie, C_0-semigroups and application., North-Holland Math. Studies 191., 2003.

[52] N. Young, An introduction to Hilbert space., Cambridge Uni. press., 1995.