Toeplitz and Hankel operators on Hardy spaces of complex domains

Joshua Malcolm Tattersall

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Department of Pure Mathematics

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Abstract

The major focus is on the Hardy spaces of the annulus $\{z : s < |z| < 1\}$, with the measure on the boundary being Lebesgue measure normalised such that each boundary has weight 1. There is also consideration of higher order annuli, the Bergmann spaces and slit domains. The focus was on considering analogues of classical problems in the disc in multiply connected regions.

Firstly, a few factorisation results are established that will assist in later chapters. The Douglas-Rudin type factorisation is an analogue of factorisation in the disc, and the factorisation of H^1 into H^2 functions are analogues of factorisation in the disc, whereas the multiplicative factorisation is specific to multiply connected domains.

The Douglas-Rudin type factorisation is a classical result for the Hardy space of the disc, here it is shown for the domain $\{z : s < |z| < 1\}$. A previous factorisation for H^1 into H^2 functions exists in [4], an improved constant not depending on s is found here.

We proceed to investigate real-valued Toeplitz operators in the annulus, focusing on eigenvalues and eigenfunctions, including for higher order annuli, and amongst other results the general form of an eigenfunction is determined. A paper of Broschinski [10] details the same approach for the annulus $\{z : s < |z| < 1\}$ as here, but does not consider higher genus settings. There exists work such as in [6] and [5] detailing an alternative approach to eigenvalues in a general setting, using theta-functions, and does not detail the eigenfunctions.

After this, kernels of a more general symbol are considered, compared to the disc, and Dyakanov's theorem from the disc is extended for the annulus.

Hankel operators are also considered, in particular with regards to optimal symbols.

Finally, analogues of results from previous chapters are considered in the Bergman space, and the Hardy space of a slit annulus.

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Contents

Chapter 1

Introduction

This thesis will focus mainly on the Hardy spaces H^p for the annulus, mostly in comparison to the corresponding spaces in the disc.

1.1 Literature Review

Starting with the background on Hardy spaces in the disc, we have Duren's book [18] details the H^p spaces for a simply connected region, together with some of the important related spaces, and details some of the important function theory in this area, such as the construction of Blaschke products and existance of boundary values for the functions. The book also details how the H^p spaces can be constructed in the case of multiply connected regions, though it does not proceed further in these spaces. The book [24] is also a useful reference for the H^p spaces for the disc, and as well as detailing the basic function theory, it also details the shift operator in this space.

For the operators on these spaces, the book [17] details Toeplitz operators on the Hardy spaces of simply connected regions quite well, especially with regards to the spectrum, and Nikolski's book [30] also covers Toeplitz operators, as well as providing a useful reference for Hankel operators, including discussion on the Nehari problem.

In moving into the spaces for multiply connected regions, the paper [2] by Abrahamse gives a good deal of information on the H^p spaces here, in particular, the inner and outer factorisation for these spaces is covered, and the associated spaces (of no longer singly valued functions) are described.

Topelitz operators on these spaces are also discussed here, with Abrahamse detailing some of the theory of the essential spectra for these, and the failure of Copburn's lemma (and thus the possiblity for self-adjoint Toeplitz operators to have eigenvalues) is shown. The most important result with regards to the essential spectra here appears to be Abrahamse's reduction theory, which essentially reduces problems regarding the essential spectra in multiply connected domains back to the disc.

The older work of Abrahamse [1] covers some of the ideas in [2], but the later paper has more content, and covers everything in the older paper.

Sarason's work in [38] is also a good reference for the function theory in the H^p spaces of the annulus, in particular it constructs the formula for the character of an outer function with given boundary values, and another for the character of a given inner function.

The book [16] also mentions Toeplitz operators defined on multiply connected regions, mostly following along the same lines as the work in Abrahamse, though a few results are extended slightly, concerning the algebra generated by Toeplitz operators with continuous symbol.

For discussion of the eigenvalues of self-adjoint Toeplitz operators on multiply connected domains, [40] section 4 illustrates the diagonalisation of the Toeplitz operator on the annulus $\{z : s < |z| < 1\}$ whose symbol is the indicator function for the boundary $\{z : |z| = 1\}$, giving the eigenvalues and eigenfunctions for this.

The recent paper [10], and the author's recent (2014) thesis [11] detail the eigenvalues and eigenfunctions for a general self-adjoint Toeplitz operator on this annulus, using the same methods that were arrived at here (though differing in the approach taken to showing that the eigenfunctions will be outer). The thesis [11] also mentions more general multiply connected regions, however only symbols that have constant sign on each boundary are considered, and the general approach here is in looking for characters such that an eigenvalue exists for the Toeplitz operator applied to the space of modulus automorphic functions with character α, which is a different problem than here where the character is fixed (as 0).

[5] [6] also consider eigenvalues for Toeplitz operators in more general regions, with the approach here considering the resolvent operator, with the eigenvalues resulting from the zeros of certain theta functions. The actual eigenfunctions are not considered in these papers, however, and the eigenvalues are not explicitly calculated, though conditions for infinite accumulation in an interval are given.

There appear to be few papers on Hankel operators in an annulus, and most of these appear to deal with Hankel operators on the Bergman space of the annulus, whereas in this thesis the Hankel operators are only considered with regards to the Hardy space on the annulus. The paper [4] does focus on Hankel operators for the Hardy space of the annulus, and includes one of the factorisations that will be presented in the second factor, though the result presented here will improve on the norms. This paper also does not consider optimal symbols, but instead uses the factorisation in consideration of the boundedness of Hankel operators.

With regards to Bergman spaces, the book [23] provides the background for the space. As with the Hardy space, there has been considerable interest in Toeplitz operators here. The survey [33] mentions some of the important problems in this area, such as the still open problem of when a Toeplitz operator is open on the space, and the solved problem of compactness in terms of Berezin transforms. The approach to these for radial Toeplitz operators is also mentioned, pointing to the work of the paper [20], detailing compactness and boundedness of radial Toeplitz operators in the disc.

The paper [22] also considers Toeplitz operators on Bergman spaces, the main results in this connect the reproducing kernels for the Bergman space with Schatten class norms of Toeplitz operators and Carleson measures.

Generalisation of Toeplitz operators on the Bergman space of the disc appears mostly focused on weighted Bergman spaces and polydiscs, rather than to multiply connected regions. However, a few papers do exists in the area, mostly focusing on when a Toeplitz operator is compact, with the strongest result being from the paper [25], which extends the solution of the problem of finding when a Toeplitz operator is compact from the Bergman space of the disc to the Bergman space of multiply connected domains.

For the slit domain, the book [3] provides a detailed study of the invariant and nearly invariant subspaces of the slit disc, with the central results being a complete classification of the invariant and nearly invariant subspaces under the shift.

1.2 Background

The description of the H^p spaces for the disc from in [18], [34], will be outlined here.

Definition 1.2.1 *Take Δ to be the interior of the unit disc- $\Delta = \{z : |z| < 1\}$.*
Let $p \in [1, \infty]$. $H^p(\Delta)$ is defined as the set of complex valued functions f defined on Δ such that f is analytic inside Δ, and for which $\|f\|_p = \lim_{r\to 1} M_p(r, f) < \infty$, where $M_p(r, f) = (\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta)^{\frac{1}{p}}$ for $1 \le p < \infty$, and for $p = \infty$, $M_\infty(r, f) = \max_{0\le\theta<2\pi} |f(re^{i\theta})|$.

These spaces will be Banach spaces for $1 \le p < \infty$, and a Hilbert space when $p = 2$, as shown in [24] and [18].

It can be shown (see [24], p.51) that for any $f \in H^p$ for $p \in [1, \infty]$, well-defined radial limits exist almost everywhere on the boundary, and the behaviour of any H^p function can be recovered from its boundary values. Thus, we have a natural embedding of H^p into L^p.

The inner product for H^2 is $\langle f, g\rangle = \int_{|z|=1} f\bar{g}$, taking the integral with respect to uniform Lebesgue measure on the boundary of the disc.

H^p can be identified with the subset of L^p in which all negative Fourier coefficients vanish.

As a result, $(z^n)_{n=0}^{\infty}$ will be a basis for these spaces for $1 < p < \infty$.
Since z^n are all bounded functions, it follows immediately from this that H^∞ is dense in H^p for all $1 \leq p < \infty$. As well as H^p, there also exists the Nevalinna space $N(\Delta)$.

Definition 1.2.2 $N(\Delta)$ *is the space of functions* f *holomorphic on* Δ *with* $\lim_{r\to\infty} \int_{|z|=r} \log^+ |f(z)||dz| < \infty$.

These are quite important in Hardy space theory. Firstly, we have that $H^p \subset N$ for all p (see [15] p 273), and thus every f in H^p is $\log$-integrable. We also have that for any $f \in N$, there exist $g_1, g_2 \in H^\infty$ with $f = \frac{g_1}{g_2}$.
It follows from these (see [15] theorem 2.11) that we have $\log |f| \in L^1$ whenever $f \in H^p$.

1.2.1 Inner-outer factorisation

An important result in the theory of H^p spaces is the existence of a unique inner-outer factorisation for functions in these spaces. Firstly, note that we have the following restriction on accumulation of zeros at the boundary ([18] p.18):

Theorem 1.2.3 *Let f be holomorphic in the disc, with $f \neq 0$, and let (z_n) be the zeros of f, repeated according to multiplicity. Then* $\lim_{r\to\infty} \int_0^{2\pi} \log |f(re^{i\theta})|d\theta$ *is bounded only if* $\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$.

Since $\int_0^{2\pi} \log |f|d\theta < \infty$ was seen to be a necessary condition for $f \in H^p$, this is a necessary condition on the zero set of an H^p function.

Furthermore, given any set of a_n satisfying the condition $\sum(1 - |a_n|) < \infty$, we have that there exists $B(z) \in H^\infty$ with $B(a_n) = 0\ \forall n$, and $|B(z)| = 1$ a.e. on $\{|z| = 1\}$. The construction is given in [18] p.19, with $B(z) = \Pi_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a_n} z}$.
This is termed a Blaschke product. For any $f \in H^p$, if we set (a_n) to be the zero set of f,

and B to be the Blaschke product with zero set (a_n), we have that $f/B \in H^p$ (see [18] p.20), and $f/B \neq 0$ inside Δ. Furthermore, $\|f/B\|_p = \|f\|_p$, since $|B| = 1$ a.e. on $\partial\Delta$. It follows from the previous theorem that $\sum_{z\in A}(1-|z|) < \infty$ is a necessary condition for A to be the zero set of some H^p function. Since we can construct suitable Blaschke products when this condition holds, it is also sufficient.

We can proceed further and factorise what is left into what is called an outer function and singular inner function.

Definition 1.2.4 [24] *p.61*
An outer function is defined to be $F : \Delta \to \mathbb{C}$ such that

$$F(z) = \lambda \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log f(e^{i\theta})d\theta\right).$$

Where $\lambda \in \mathbb{C}$ a constant, and f is a positive real-valued integrable function.

It follows from the definition that an outer function will have no zeros inside the disc. The following results on outer functions will also be important, a proof of which can be found in [37] 17.16:

Theorem 1.2.5 *Let F be an outer function as defined in 1.2.4 with respect to f, $\log f \in L^1$.*

1. $\lim_{r\to 1}|F(re^{i\theta})| = f(e^{i\theta})$ *a.e.*
2. $F \in H^p(\Delta)$ *if and only if* $f \in L^p(\partial\Delta)$.

We also have that $|F(z)| = |f(z)|$ almost everywhere on the boundary, and that $F \in H^p$ if and only if $f \in L^p$.

A singular inner function is also defined by a boundary integral, this time against a singular measure on the boundary. The following is, once again, from [24]:

Definition 1.2.6 *A function f is said to be a singular inner function if there exists a singular, positive, measure μ on $\partial\Delta$ such that:*
$f(z) = \exp(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta))$.

The factorisation theorem for the disc is as follows:

Theorem 1.2.7 *Let $f \neq 0 \in H^1$. Then there exists a unique factorisation $f = BSF$ where B is a Blaschke product, S is singular inner, and F is an outer function.*

Again, this comes from [24] p.67-68.

1.2.2 Toeplitz Operators

Using the equivalence between H^2 functions and their boundary values, and that H^2 can thus be embedded as a closed subspace of L^2, there exists orthogonal projection P from L^2 onto H^2. Thus, given $\phi \in L^\infty$, we can define a Toeplitz operator T_ϕ as follows:

Definition 1.2.8 $T_\phi f = P(\phi f)$.

In the disc, it can be shown that if ϕ a non-constant real-valued function on the boundary, then the set of eigenvalues for T_ϕ is empty- this follows from Coburn's theorem (see [29])- we will have that either T or T^* is injective, if $T \neq 0$. Since $T = T^*$, it must therefore have no eigenvalues.
It is interesting to consider the matrix with respect to our basis (z^n) for a Toeplitz operator. Since we have $\langle T_\phi z^n, z^m \rangle = \langle \phi z^n, z^m \rangle = \langle \phi z^{n+1}, z^{m+1} \rangle$, we will have that $a_{n,m}$ is constant on the diagonals of fixed $n - m$, which will be the infinite diagonals in our

matrix. As a consequence, we immediately have that no non-trivial compact Toeplitz operators exist.

We can also note that $T_\phi f = 0$ implies that ϕ is log-integrable.

Lemma 1.2.9 *Let* $f \neq 0$, $\phi \in L^\infty(\partial\Delta)$, *and* $T_\phi f = 0$. *Then* $\log|\phi| \in L^1$.

Proof

In the disc, we have that $H^{2\perp} = \bar{z}\bar{H}^2$, and so we have that all functions in $H^{2\perp}$ have $\log$-integrable modulus on the boundary.

Now, if $T_\phi f = 0$, we have that $\phi f \in H^{2\perp}$. Since $|f|$ and $|\phi f|$ are $\log$-integrable, $|\phi|$ must also be $\log$-integrable. $\square$

1.2.3 Spectra

This thesis will spend some time on discussion of spectra of operators, so these will be mentioned here for bounded operators. For T a bounded operator $\sigma(T)$ is said to be the spectrum of T, where:

Definition 1.2.10 $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \textit{has no bounded inverse}\}$.

The spectrum has several subsets that are important in operator theory, generally classified based on why $(T - \lambda I)$ fails to be invertible.

We have the point spectrum $\sigma_p(T) = \{\lambda : (T - \lambda I)$ not injective$\}$.

If $(T - \lambda I)$ is not injective, it follows that λ must be an eigenvalue for T, and so the point spectrum corresponds to the set of eigenvalues.

The essential spectrum of operators will also be considered here. This is taken to be $\sigma_e = \{\lambda \in \mathbb{C} : (T - \lambda I)$ not Fredholm$\}$, where a Fredholm operator is an operator which is invertible in the space of operators quotiented out by the compacts.

An introduction to Fredholm operators can be found in [39] chapter 5. This also mentions essential spectra, briefly.

Some important properties are that T a Fredholm operator is equivalent to T having finite dimensional kernel, finite dimensional co-kernel, and closed range.

If we define the index $Ind(T) = \dim(\ker(T)) - \dim(\text{coker}(T))$ on the Fredholm operators, we will also have $Ind(T_1T_2) = Ind(T_1) + Ind(T_2)$.

We do not necessarily have that σ_e and σ_p are disjoint here- if we have λ is an eigenvalue of infinite multiplicity, then $\lambda \in \sigma_p$ is immediate. $\lambda \in \sigma_e$ will also follow since $T - \lambda I$ has infinite dimensional kernel, and thus is not Fredholm.

For T self-adjoint (and so kernel and co-kernel are both equal in dimension), we have that $\sigma_p(T) \cup \sigma_e(T) = \sigma(T)$, since suppose $0 \notin \sigma_e(T)$, $0 \notin \sigma_p(T)$.

Then, we have that T has zero kernel and cokernel, and closed range. Thus, T is a $1 - 1$ mapping from our Hilbert space into itself, and it follows that it has bounded inverse.

However, this is not necessarily the case for non self-adjoints. Considering the shift operator on the Hardy space of the disc T_z it is immediate that 0 is not an eigenvalue for T_z. We have that $T_zH^2 = zH^2$, is a closed subspace of codimension 1. Thus T_z is Fredholm. Thus $0 \notin \sigma_p(T_z)$, and $0 \notin \sigma_e(T_z)$. Yet $0 \in \sigma(T_z)$ follows as ${T_z}^*$ has a zero-eigenvalue, and is thus not invertible.

There exist further subsets of the spectrum which are of interest in various parts of analysis, such as the residual and continuous spectrum, however only the essential and point spectra are considered in this thesis.

Spectral properties of Toeplitz operators in the disc have long been studied. It is already known, for instance, that $\sigma_p(T_\phi)$ is empty for ϕ a real-valued non-trivial symbol, and that $\sigma_e(T_\phi)$ is the range of ϕ for a continuous symbol ϕ, with the Fredholm index given by the winding number of ϕ (see for example [17] 7.26).

1.2.4 Hardy space of the annulus

Sarason's work [38], as well as [2] provide useful references for the Hardy spaces in multiply connected regions.

We will take the annulus $\mathbb{A}$ to be the set $\{z : s < |z| < 1\}$ where $0 < s < 1$ is some constant.

In this thesis, D will be used to represent a bounded multiply connected region in the plane. We will consider only D for which D is an open, connected domain in the plane bounded by at most finitely many analytic Jordan curves.

In both cases, we take Γ_i to be the connected components of the boundary of our region, with Γ_0 chosen to correspond to the outermost boundary of our region when possible (so in the case of $\mathbb{A}$, $\Gamma_0 = \{z : |z| = 1\}$).

There are a couple of different ways to define the hardy spaces on an annulus, both of which will be detailed for arbitrary multiply connected regions in the plane. In a domain D whose boundaries consist of finitely many analytic Jordan curves, these definitions will in fact be equivalent.

Firstly, we can take the definition of H^p from [18] p179-183, which defines the space as follows:

Definition 1.2.11 *For D an arbitary domain in the complex plane, $H^p(D)$ is the set of analytic functions f such that $|f(z)|^p$ has a harmonic majorant in D.*

In [18], the space is then given norm based on the harmonic majorant- here, we have $\|f\|_p = \inf(u(z_0))^{1/p}$, where z_0 is some fixed arbitrarily chosen point in D, and taking infimum over u harmonic majorants for $|f|^p$. Our inner product for the $p = 2$ case will be $\langle f, g\rangle = \int_{\partial D} f\bar{g}dm$ for some measure on the boundary of D. An explicit calculation of this measure can be found in [2] p263.

To see that this construction gives a Banach space, one can consult [36] p49-50.

For the alternative definition, we can define $E^p(D)$ as the set of functions f which are analytic inside D, and for which there exists a sequence of domains with boundaries $\{\Gamma_n\}$, consiting of a finite number of rectifiable Jordan curves, which eventually enclose each compact subset of D, have bounded length, and $\limsup_{n\to\infty} \int_{\Gamma_n} |f(z)|^p |dz| < \infty$.

It is shown in [18] that these definitions are in fact equivalent for finitely connected domains whose boundary curves are analytic. The approach taken there is that if C_n is the domain enclosed by Γ_n, we have that for every $f \in H^p(D)$, one has $f = f_1 + \ldots + f_n$, with $f_i \in H^p(D_k)$, for either of the definitions, and the same decomposition can be shown for $f \in E^p(D)$ also. The equivalence then follows from the equivalence of E^p and H^p for simply connected domains- see, for example, [37].

Since defining H^p by means of harmonic majorants and by integrals with respect to uniform lebesgue measure are equivalent, it follows that the harmonic measure dm will be equivalent to the uniform measure $|dz|$. More detail on the relationship can be found in [2] p263.

Whilst normally the space will be considered with respect to the uniform lebesgue measure, at stated points the harmonic measure will be used. In this case, we will use P^h to denote orthogonal projections with respect to the harmonic measure, and $\langle ., . \rangle_h$ to denote an inner-product with respect to harmonic measure.

It is useful to consider a covering map from the disc to the annulus, θ. For the annulus $\{z : s < |z| < 1\}$, one can define this as follows:

First, let $f_1 : z \to \frac{z+1}{1-z}$. This is a conformal equivalence from the disc to the right half plane.

Let $f_2 : z \to \frac{-\log s}{\pi} i \log z + \frac{-\log s}{2}$.

This is a conformal equivalence between the right half plane and the strip $\{0 < Re(z) < -\log s\}$.

Let $f_3 : z \to e^{-z}$. This maps the previous strip to the set $\{s < |z| < 1\}$, i.e. to an annulus.

So $\theta = f_3 \circ f_2 \circ f_1$ gives us a covering map from the disc to $\mathbb{A}$.

Since f_2 and f_1 are conformal equivalences, and f_3 is ∞ to 1 with the identity $f_3(z + 2\pi i) = f_3(z)$ generating the automorphism group, we see that there exists a conformal equivalence ψ on the disc generating the group of identities for the map θ, with $\psi(z) = f_1^{-1}(f_2^{-1}(f_2(f_1(z)) + 2\pi i)$.

It is useful to consider how our θ acts on the boundary.

Firstly, note that moving from the disc to the annulus has some pathological points— we have that θ does not extend to the boundary at ± 1. This results from the behaviour of f_3 around the boundary of the strip at $\pm\infty i$ (which $f_2 f_1$ sends ± 1 to). In fact, considering f_3 we can see that $\{z : 0 < Re(z) < -\log s, 2\pi n < Im(z) < 2\pi(n+1)\}$ is mapped $1-1$ into $\mathbb{A}$. So any relatively open region of the boundary of the strip about $\pm\infty i$ is mapped in an $\infty - 1$ manner to the entire boundary of $\mathbb{A}$. It follows therefore that on any arc of $\partial\Delta$ relatively open and containing ± 1, θ is an $\infty - 1$ map onto the boundary of the annulus. These are the only such points in $\partial\Delta$, since $\log(z)$ extends to the boundary of the strip everywhere except at $\pm\infty i$.

This can also be seen from the fact that $\psi^{\pm n}(z) \to \pm 1$ as $n \to \infty$ (since we have ψ can be represented as mapping to the strip, adding $2\pi i$, then mapping back to the disc, and so accumulates at the points in the disc corresponding to $\pm\infty i$ in the strip.

If we take t with $t \neq 0$, $t \neq \pi$, consider the effect of θ on e^{it}.

We have $f_1(e^{it}) = \frac{e^{it}+1}{e^{it}-1} = \frac{2i \sin t}{|e^{it}-1|^2}$.

f_2 takes a log of this, then multiplies and shifts by a constant, so we have $f_2(f_1(e^{it})) = i(\log(\frac{2\sin(t)}{|1-e^{it}|^2}))\frac{-\log s}{\pi} + 0$ when $0 < t < \pi$.

$f_2(f_1(e^{it})) = i(\log(\frac{2\sin(t)}{|1-e^{it}|^2}))\frac{-\log s}{\pi} + (-\log s)$ for $\pi < t < 2\pi$.

Finally, f_3 is exponentiation, so $\theta(e^{it}) = e^{-i(\log(\frac{2\sin(t)}{|1-e^{it}|^2}))\frac{-\log s}{\pi}}$ when $0 < t < \pi$, and $\theta(e^{it}) = se^{i(\log(\frac{2\sin(-t)}{|1-e^{it}|^2}))\frac{-\log s}{\pi}}$ for $\pi < t < 2\pi$.

So we can see once more that as t approaches 0 or π, we are wrapping around the appropriate boundary of the annulus infinitely many times.

It is also clear that Γ_0 lifts to the set of points on the boundary of the disc with positive imaginary part, and Γ_1 to those with negative imaginary part.

If we take a lift $\tilde{f}$ of f an $H^p(D)$ function back to the disc under a covering map θ_D from Δ onto D, we have that $\tilde{f} \in H^p(\Delta)$. This is trivial in the case $p = \infty$. It is proven in [2] p267 that (for the space defined with respect to Harmonic majorants) that if $\tilde{f}$ is the lift to the disc of $f \in H^p(D)$, we have $\|\tilde{f}\|_p = \|f\|_p$ (the proof is to show that if u is a harmonic majorant for $|f|^p$, then lifting u gives a suitable majorant for the disc).
Now, since we are working with uniform measure on the boundary, we will not have that the norms remain equal. However, since the harmonic and lebesgue norms are equivalent, we will have that $\|\tilde{f}\|_p < \infty \Leftrightarrow \|f\|_p < \infty$.
Abrahamse in [2] in fact proves more than that just the lifts of $H^p(D)$ functions are in $H^p(\Delta)$. We in fact have that $L^p(\partial D)$ functions lift to $L^p(\partial\Delta)$ functions.
It follows, for instance, that the functions in $H^p(\mathbb{A})$ will have well-defined radial limits on the boundary of the annulus- one simply has to show that a non-tangential line to the boundary of the annulus is mapped to a set of non-tangential lines on the boundary of the disc under θ^{-1}.
Let $A : [0, 1] \to S$ be our path approaching the boundary of the annulus non-tangentially, and we can assume w.l.o.g. $A(1) = 1$.
Since we approach the boundary of the annulus non-tangentially, we must have that our path is of finite length, and so it can only wind about the annulus at most finitely many times. Since f_1 and f_2 are conformal equivalences, we need only consider f_3. By taking $A'(x) = A(t + (1-t)x)$ for t suitably close to 1, we can assume that $|(A(x) - 1| < \epsilon$, where ϵ is taken so that $\{z : |z-1| < \epsilon\} \bigcap \mathbb{A}$ is simply connected. Now, we have that f_3 restricted to a suitable set is a conformal equivalence between $\{z : |z-1| < \epsilon\} \bigcap \mathbb{A}$ and the appropriate pre-image, so the pre-images of A must approach the boundary non-tangentially.

Since we have that boundary values now exist almost everywhere, we have that H^2 can

be considered as a subspace of the L^2 space of the boundary of the annulus, and it must be closed since H^2 is complete. From this, we can define orthogonal projection onto H^2, and thus a Toeplitz operator can be defined on the annulus by $T_\phi f = P_{H^2(\mathbb{A})}(\phi f)$.
When working with the harmonic measure, we will denote the associated Toeplitz operator $\mathcal{T}_\phi$, with $\mathcal{T}_\phi f = P^h_{H^2}(\phi f)$.

1.2.5 Basis for $H^p(\mathbb{A})$

In the disc, we have that $(z^n)_{n=0}^\infty$ provide an orthonormal basis for $H^2(\Delta)$.
For the annulus $\mathbb{A}$, with normalized lebesgue measure on the boundary, we have a very similar result.

Theorem 1.2.12 $(\frac{z^n}{\|z^n\|_2} : -\infty < n < \infty)$ *is an orthonormal basis for* $H^2(\mathbb{A})$.

Proof
It is immediate that $z^n \perp z^m$ when $n \neq m$, since we have that $\langle z^n, z^m \rangle = \int_0^1 e^{2i\pi(n-m)t}dt + \int_0^1 s^{n+m}e^{2i\pi(n-m)t}dt$.
Now, given any f holomorphic in $\mathbb{A}$, we have a Laurent series $f = \sum_{-\infty}^{+\infty} a_n z^n$.
So, $\int_{|z|=r} |f|^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 r^{2n}$.
So, for $f \in H^2(\mathbb{A})$, we must have $\sum |a_n|^2(1+s^{2n}) < \infty$. However, $\|z^n\|_2^2 = 1 + s^{2n}$, and thus $\sum_{n=-N}^{N} a_n z^n$ is convergent in H^2, and thus converges to f.
It follows that $(\frac{z^n}{\|z^n\|_2})$ for $-\infty < n < \infty$ gives a basis for $H^2(\mathbb{A})$. $\square$
From this, for instance, it follows that H^∞ is dense in H^p for $1 < p \leq \infty$.
For a more general annulus D of genus g, constructing a basis for $H^2(D)$ is somewhat harder. However, we can use the additive decomposition (previously used in showing the two alternate definitions for H^p were equivalent) to show that $\{z^n, (z-a_i)^{-n} : 0 \leq n < \infty\}$ has dense span in $H^2(D)$, where $a_i \in \mathbb{C}$ are chosen for

$1 \leq a_i \leq g$ such that a_i is enclosed by Γ_i, and so it will follow that H^∞ is dense in H^p. However, even our (z^n) need not be an orthogonal series in $H^2(D)$.

1.2.6 Decomposition of L^2

In the disc, we have $L^2(\partial\Delta) = H^2(\Delta) \oplus \overline{z}\overline{H^2(\Delta)}$, and so $H^{2\perp} = \overline{z}\overline{H^2}$.

In the case of $\mathbb{A}$, if ω is the function with $\omega = 1$ on Γ_0, and $\omega = -1$ on Γ_1, it is easy to see that $\omega\overline{H^2} \perp H^2$, since $\langle f, \omega\overline{g}\rangle = \langle fg, \omega\rangle = 0$.

We also have that $H^2 \oplus \omega\overline{H^2} = L^2$. To do so, note that $z^n + s^{2n}\omega\overline{z}^{-n} = (1 + s^{2n})e^{in\theta}$ on Γ_0, and is 0 on Γ_1. $z^n - \omega z^{-n}$ is $(1 + s^{-n})e^{in\theta}$ on Γ_1 and 0 on Γ_0.

Thus, we have that $H^2 \oplus \omega\overline{H^2}$ is dense in L^2, and since the space is closed, it is L^2. It follows that $H^{2\perp} = \omega H^2$.

We may also wish to consider a similar result in an arbritary annulus D.

We have from [2] Theorem 1.7 that for the Hardy space on D (with Harmonic measure), we have $H^{2\perp} = \overline{\nu}^{-1}\overline{H^2}$ for some function ν, meromorphic on an open set about our region D, with zeros and poles strictly inside D. We have that ν^{-1} is thus continuous, bounded and non-zero on ∂D. Thus, if we are working with uniform Lebesgue measure, we have $H^{2\perp} = \overline{\nu}^{-1}\frac{|dz|}{dm}$, where dm was the previous Harmonic measure.

We know that $|dz|$ and dm are equivalent measures [2], so $\frac{|dz|}{dm}$ will be bounded away from 0 and ∞. It is also shown that this is strictly positive in [2].

One consequence is that $H^{2\perp}$ will consist of functions whose boundary values have log-integrable modulus, since $|\nu|$ is log-integrable on the boundary, as is our change of measure $\frac{|dz|}{dm}$ and H^2 (H^2 having log-integrable boundary values is in [2] Theorem 1.18).

It follows that a necessary condition for $T_\phi f = 0$ will be that $|\phi|$ is $\log$-integrable.

Another decomposition shown in Abrahamse is that $L^2 = H^2 \oplus \overline{H^2} \oplus N$, with N a g-dimensional set, where g is the genus of D.

Again, this is shown with respect to harmonic norm. If we instead use uniform Lebesgue measure, we still have $H^2 + \overline{H^2} + N = L^2$, however orthogonality is no longer preserved.

1.2.7 Inner-outer factorisation in the annulus

Inner-outer factorisation in an annulus works somewhat differently than in the disc. By taking lifts back to the disc, we can produce an inner-outer factorisation in the disc of the lifted function, however there is no guarantee that the new functions from factorisation can be passed back down to the disc (since our covering map from the disc will be an ∞-1 mapping, and so we cannot send a general $H^p(\Delta)$ function back to D).
Considering the case for an annulus first, we have by a result of [38] that if f is an outer function, and $|f|$ is ψ-invariant on the boundary, then f will be a modulus automorphic function, defined as follows:

Definition 1.2.13 *$f \in H^p(\Delta)$ is a modulus automorphic function, and $k \in (0,1]$ its character if we have that $f(\psi \circ z) = e^{2\pi i k} f(z)$ a.e. on $\partial\Delta$.*

We can take an inner-outer factorisation in the disc of $\tilde{f} = \tilde{f}_i \tilde{f}_o$ for an arbitrary function $f \in H^p(\mathbb{A})$, and we will have that $\tilde{f}_o$ will be a modulus invariant function, and thus $\tilde{f}_i$ will also be modulus invariant, with complementary character.
When considering the disc as a covering space of more general regions, we have that the identity group of our covering map θ_D is no longer generated by a single element.
We can still, however, define modulus automorphic functions, as is done in [2] p267.

Definition 1.2.14 *If G is the group of disc automorphisms such that $\theta_D \circ G = \theta_D$, we have that f is a modulus automorphic function if we have $|f \circ S| = |f|$ for all $S \in G$.*

Now, [2] states that if f is modulus automorphic, then we have $f \circ S = \alpha(S) f$ for some $\alpha \in \hat{G}$ the dual group of G. We say that α is the character of f.

If we have f a many-valued function in $H^p(D)$ that lifts to a modulus automorphic function in $H^p(\Delta)$, the character of f is taken to be the character of its lift to Δ.

There are two approaches to defining inner and outer functions in the annulus.

One is to accept many-valued functions that lift to modulus automorphic functions for the inner and outer factorisation, and proceed as before.

The second is to only accept single-valued functions, and instead cancel out the modulus automorphism of the lifts of each part by multiplying by g and g^{-1}, with g an outer (by the former definition) function chosen to have constant modulus on each boundary, and to have suitable character (in the case of $\mathbb{A}$, these will be z^α, with α corresponding to the character of the previous inner-outer factorisations). The requirement for inner functions to have $|f_i| = 1$ on the boundary must then be reduced instead to the requirement for $|f|$ to be constant on each boundary.

In this nomenclature, the functions z^n are both inner and outer for the H^p spaces of $\mathbb{A}$. Functions that are both inner and outer for an annulus are referred to as units.

In this thesis, the first of these approaches will be taken unless stated otherwise- inner and outer components of a function may be considered as multivalued. Considering multiply valued functions whose lifts back to the disc are modulus automorphic in fact generates a family of spaces related to our H^p spaces:

Definition 1.2.15 *$H^p_\alpha(D)$ is the set of modulus-automorphic functions in Δ with character α.*

Further details can be found in [2] p266-267, or in [38]. That these are essentially copies of $H^p(D)$ can be seen from the result $H^p_\alpha(D) = \tilde{s_\alpha} H^p_0(D)$, where s_α is an outer function, with modulus bound away from 0 and ∞ on each boundary, and such that $\tilde{s_\alpha}$ has character α.

The result follows from the observation that multiplying two modulus automorphic functions will result in a function whose character is the sum of the individual characters (shown in, for example, [2]).

Definition 1.2.16 *Let $\tilde{f} \in H^1_\alpha$. Then f is said to be outer/Blaschke/singular inner for D if $\tilde{f}$ is outer/Blaschke/singular inner for $H^1(\Delta)$, where $\tilde{}$ denotes the lift to the disc: $\tilde{f}(z) = f(\theta(z))$, where θ is our covering map from Δ to D.*

Theorem 1.2.17 *Let $f \in H^1(D)$. Then there exists a unique factorisation $f = BSG$, where B is a Blaschke product for D, S is a singular inner function for D, and G is outer for D.*
If α, β, γ are the corresponding characters for B, S, G, then $\alpha + \beta + \gamma \in \mathbb{Z}$.

This is from [2] p268.
For an outer function with given modulus on the boundary, [38] p.35, Theorem 6, gives the following technique to calculate its character (modified by a constant to take characters in the range $[0, 1)$.

Theorem 1.2.18 *Let f be a multiply-valued function such that $\tilde{f} \in H^p_{\alpha_f}(\Delta)$. We have that $\alpha_f = \frac{\int_{\Gamma_0} \log|f| - \int_{\Gamma_1} \log|f|}{\log s}$ (mod 1).*

Corollary 1.2.19 *z^x as a multi-valued function in $\mathbb{A}$ has character x (mod 1).*

1.2.8 Blaschke products

In $\mathbb{A}$, similar conditions on the zeros accumulating at the boundaries hold as to in the disc, though the condition instead becomes $\sum_{z:f(z)=0} \min\{1 - |z|, |z| - s\} < \infty$.
This can be seen by taking lifts back to the disc. If $f \in H^p(\mathbb{A})$, we have that $\tilde{f} \in H^p(\Delta)$, and so we have that $\sum_{z:\tilde{f}(z)=0}(1 - |\tilde{f}|(z)) < \infty$.
Now, consider some region of Δ on which θ gives a $1-1$ mapping from the annulus. Such a region will have a boundary consisting of two arcs on the boundary of the disc,

connected by two paths in the interior of the disc. On this region, our lift θ will only distort distances between the respective boundaries by a finite factor. So, if we restrict to a sequence of zeros accumulating on the outer boundary of the annulus, we must have $\sum(1-|z|) < \infty$ over this sequence.

To show the result for a sequence of zeroes approaching other boundary, apply the conformal equivalence $z \to sz^{-1}$ to $\mathbb{A}$ to swap the interior and outer boundaries.

We can define Blaschke products in the annulus to be those functions whose lifts back to the disc are Blaschke products (all such lifts will be infinite Blaschke products in the disc, since θ is infinity to one).

Consider the Blaschke product in the annulus with a single zero at ω, B_w. It is useful to know what the character of this function is. In order to do so, note that $(z-w) = B_w f_o$, where f_o is an outer function with $|f_o| = |z-w|$ almost everywhere on $\partial\mathbb{A}$. (This follows since we know the zero set of $z-w$ and there is no singular inner component since $|z-w|$ does not decay on any radial limit approaching the boundary.)

Now, since $(z-w)$ is single-valued, B_w has character complementary to f_o, which by Theorem 6 of [38] has character $\frac{\int_{\Gamma_0} \log|z-w| - \int_{\Gamma_1} \log|z-w|}{\log s}$.

Alternatively, [38] provides a construction of the character of an inner function directly. There are two things to note:

1. The character of B_w depends only on $|w|$.

2. The character of B_w lies in the range $(0, \frac{1}{2}]$, attaining $\frac{1}{2}$ only when $|w| = \sqrt{s}$.

So a single-valued Blaschke-product must have at least 2 zeros (with multiplicity) in the annulus, and in the case of a single-valued Blaschke with two zeros, both must lie on the line $|z| = \sqrt{s}$.

It is useful to extend here a classical result from the Hardy space of the disc, that for g outer we have gH^2 is dense in H^2.

Theorem 1.2.20 *Let $g \in H^2(\mathbb{A})$ be outer, and single-valued. Then gH^∞ is dense in H^2.*

Proof

This follows the same way as it does in the disc. We have that $Cl(gH^\infty)$ is invariant under M_z and $M_{z^{-1}}$, and closed by definition. Now, Beurling's theorem has been extended to a general multiply connected region in [35] p161, and thus we have that $Cl(gH^\infty) = \Theta H^2$ for some (single-valued, constant modulus on each boundary) inner function Θ. $g \in \Theta H^2$, so we have Θ divides g. If g is outer, it follows that Θ must be a unit, since the lift of Θ divides an outer function, it is outer, and we have from before that Θ has constant modulus on each boundary). However, any outer function with constant modulus on each boundary is invertible with inverse in H^∞, and so $\Theta H^2 = H^2 = Cl(gH^\infty)$.

□

1.2.9 Reproducing kernels

A Hilbert H consisting of functions on a set X is said to be a reproducing kernel space if, for all $\lambda \in X$, we have there exists $k_\lambda \in H$ for which $\langle f, k_\lambda \rangle = f(\lambda)$ for all $f \in H$. k_λ is referred to as a reproducing kernel for λ.

These can be a useful tool for the study of our space, and it happens that our Hardy spaces are reproducing kernel spaces.

For seperable Hilbert spaces, it is easy to construct our k_λ in terms of a given orthonormal basis by the following well known lemma:

Lemma 1.2.21 *Let H be a separable reproducing kernel Hilbert space of functions on X, $\lambda \in X$, and (e_n) an orthonormal basis for H. Then $k_\lambda = \sum_{i=1}^{\infty} \overline{e_n}(\lambda) e_n$*

Proof

Let $f \in H$, then $f = \sum_n a_n e_n$ for some square summable sequence (a_n).

Now, we have that $\langle f, \sum_{n=1}^{\infty} \overline{e_n}(\lambda)e_n\rangle = \langle \sum_{m=0}^{\infty} a_m e_m, \sum_{n=0}^{\infty} \overline{e_n}(\lambda)e_n\rangle$.
$= \sum_{n=0}^{\infty} a_n e_n(\lambda) = f(\lambda)$.
Thus, k_λ will be a reproducing kernel for H.
□
Applying this lemma, we can construct the well-known reproducing kernels for the disc:

Theorem 1.2.22 *For $H^2(\Delta)$, we have $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$.*

Proof
From Lemma 1.2.21 applied to our basis z^n, we have that $k_\lambda(z) = \sum_{n=0}^{\infty} \overline{\lambda}^n z^n = \frac{1}{1-\bar{\lambda}z}$.
□
In the case of $H^2(\mathbb{A})$, we can apply lemma 1.2.21 with respect to our basis $\frac{z^n}{1+s^2n}$ to obtain $k_\lambda = \sum_{-\infty}^{+\infty} (\frac{\overline{\lambda}^n}{1+s^{2n}}) \frac{z^n}{1+s^{2n}}$.
This does not have such a pleasing simplification as in the disc, but we can see that the series converges for all $\lambda \in \mathbb{A}$, and so we are once again in a reproducing kernel space.
For a more general annulus D, the reproducing kernels are somewhat harder to construct, and typically theta functions are used to do so, however we do know that we have a reproducing kernel space, and that k_λ are $H^\infty(D)$ functions (see [8] for instance).

1.2.10 Vector valued Hardy Spaces

The vector valued Hardy Spaces are a classical extension of the Hardy space, whose elements are vector valued functions on the disc. An introduction to these can be found in [30] 3.11. In this thesis, only finite dimensional vectors will be considered, so one can consider these spaces to be defined as follows:

Definition 1.2.23 *For $n \in \mathbb{Z}$, $n \geq 1$, and taking $\|f\|_p = sup_{0<r<1}(\int_0^{2\pi} \|f(re^{i\theta})\|^p d\theta)^{1/p}$ for $f \in Hol(\Delta, \mathbb{C}^n)$, we define the vector valued Hardy spaces as*

$H^p(\Delta, \mathbb{C}^n) = \{f : f \in Hol(\Delta, \mathbb{C}^n), \|f\|_p < \infty\}$.

We will have once again that boundary values exist almost everywhere, and that f tends to its boundary values $a.e.$ along non-tangential paths. We will thus be able to define Toepltiz operators, whose symbols in this setting will be matrices as opposed to scalars in the scalar valued case.

In order to extend this to a multiply-connected region D, it helps to consider defining this space in terms of harmonic majorants instead. We have from [27] p3 that we can define our H^2 space as follows:

Definition 1.2.24

$$H^2(\Delta, \mathbb{C}^n) = \{f : f \in Hol(D, \mathbb{C}^n), \|f\|_2 < \infty]\},$$

where $\|f\|_2 = \inf\{\nu(z_0)^{\frac{1}{2}} : \nu$ *a harmonic majorant for* $\|f\|^2\}$*, and* z_0 *some arbitrarily chosen point in* D*.*

We will once again be able to lift back to the disc under a suitable covering map. In particular, since we already showed that the lifts under θ of non-tangential paths approaching the boundary of $\mathbb{A}$ we will have that limits as we approach the boundary are defined a.e. and will lie in L^2, so we can discuss Toeplitz operators for the vector-valued Hardy space on $\mathbb{A}$.

1.2.11 Bergman spaces

As well as the Hardy spaces, we can also define the Bergman spaces on Δ , $A^p(\Delta)$.

Definition 1.2.25 *For* $1 < p < \infty$*, we have* $A^p(\Delta) = \{f : f \in Hol(\Delta, \mathbb{C}) : \|f\|^p_{A^p} = \int_\Delta |f(z)|^2 dA(z) < \infty\}$*, where* A *is Lebesgue area measure.*

[23] provides a good reference to these spaces. Now, we can no longer recover a function in this space by its boundary values, but instead we have that $A^p(\Delta)$ a closed subspace of $L^p(\Delta)$, (the limit of any A^p function is in L^p by definition, and holomorphic in the interior of Δ since we have almost uniform convergence), so we can define Toeplitz operators on the Bergman spaces with symbols a function in $L^\infty(\Delta)$.

One key difference is the lack of inner-outer factorisations. Blaschke products fail to be so useful in Bergman spaces since we can no longer remove a Blaschke product whilst preserving norm. Moreover, we no longer have the same conditions on how zeros may accumulate on the boundary, and an analogue of the necessary and sufficient conditions that we have in the annulus has yet to be found. In the Bergman space, for instance, if we have a set z_n of zeros of a Bergman space function lie on some radial line in the disc, we have $\sum 1-|z_n| < \infty$ as in the case of the Hardy space. However, it is quite possible for $\sum 1-|z_n| > \infty$ for the set of zeros of a Bergman space function. (See [23], chapter 4).

We have that A^2 is a reproducing kernel space, with $k_\lambda(z) = \frac{1}{(1-\bar{\lambda}z)^2}$.

1.2.12 Slit disc

We have G the slit disc is the set $G = \{z : |z| < 1, z \notin [0,1)\}$.

A full description of how to define the Hardy spaces on G can be found in [3]. The Hardy space here will be taken as what is referred to as E^2 and the Hardy-Smirnov class in [3]- in slit domains, we no longer necessarily have that this is equivalent to the space one obtains when dealing with harmonic majorants.

As in [3] p11, $E^2(G)$ is the spaces for which $\sup_n \int_{\gamma_n} |f|^2 ds < \infty$, with ds arc-length measure, and (γ_n) a sequence of rectifiable Jordan curves, which eventually contain every compact subset of our domain.

An application of a theorem originally attributed to Keldysh and Lavarentiv, which can be found in [3] page 11, details that if ϕ_G is a conformal map from the disc to G, then $f \in E^2$ if and only if $\sup_{0<r<1} \int_{\phi(\{|z|=r\})} |f|^2 ds < \infty$.

We can thus take $\|f\|_{E^2(G)} = (\sup_{0<r<1} \int_{\phi(\{|z|=r\})} |f|^2 ds)^{\frac{1}{2}}$.
It will follow from this that $f \in E^2(G) \Leftrightarrow (f \circ \phi_G)(\phi_G')^{\frac{1}{2}} \in H^2(\Delta)$.
We can also see from this, (and since ϕ_G' extends a.e. to the boundary, as will be shown in chapter 5) that $f \in E^2(G)$ has boundary values a.e. (considering limits from above and below on the slit seperately), and $\|f\|_{E^2(G)} = \int_{\partial\Delta} |f \circ \phi_G|^2 |\phi_G'| = \int_{\partial G} |f|^2 |dz|$, (Where we take the integral over the boundary to cover the slit twice, once with limits from above, once with limits from below) for $f \in E^2$, and the inner product can be written similarly.

1.2.13 Hankel operators

Hankel operators have long been studied in the disc, defined as follows:

Definition 1.2.26 *For $\phi \in L^\infty$, Γ_ϕ is defined to be the operator from $H^2 \to \bar{H}^2$ defined as $\Gamma_\phi f = P_{\bar{H}^2}\phi f$.*

[31] provides a useful introduction to the Hankel operators, and covers important results such as Nehari's theorem, the Nehari extension problem, and Nevalinna-pick interpolation. A Hankel operator will, if we represent it as a matrix with respect to our z^n basis, have constant value along the finite length diagonals, $a_{n,m}$ with $n + m$ constant, in contrast to the result with Toeplitz operators constant along diagonals with $n - m$ constant (shown in [31] p30). Now, as opposed to Toeplitz operators, we no longer immediately have $\|\Gamma_\phi\| = \|\phi\|_\infty$, nor do we even have that the symbol of the operator is unique, for it is immediate that $\Gamma_f = 0$ whenever f is an analytic function.
The problem of finding a symbol of optimal norm is a classic problem in the Hardy space of the disc. It turns out that there will always exist an optimal symbol ϕ for which $\|\phi\|_\infty = \|\Gamma_\phi\|$, and this is easy to construct for a Toeplitz operator achieving its norm:

Theorem 1.2.27 *Sarason's solution to the Nehari problem*
Suppose that Γ *is a Hankel operator on the disc, with* $\|\Gamma f\|_2 = \|\Gamma\|_\infty \|f\|_2$.*Then we have that* $\phi = \frac{\Gamma f}{f}$ *is an optimal symbol for* Γ *with* $\|\phi\|_\infty = \|\Gamma\|$.

This can be seen in [31]. The theorem is quite powerful- while a Hankel operator need not achieve its norm on H^2, there are interesting classes of Hankel operators which will. Hankel operators with a symbol polynomial in $\bar{z}$, for instance— representing these as a matrix, only finitely many entries are non-zero. So, using finite dimensional linear algebra, it is easy to construct a function where our Hankel operator achieves its norm.

For multiply connected regions, there are two different ways that a Hankel operator can be defined, as can be found in [30].
One is to take $\Gamma_\phi f = P_{H_2^\perp} \phi f$, which is the definition that will be considered in this thesis. The other is to take $\Gamma_\phi f = P_{H^2} \phi \bar{f}$.
These parallel the two approaches that can be taken for Hankel operators defined on the Bergman spaces, which are referred to as the big and little Hankel operators. These act rather differently since the L^2 for Bergman spaces is a far larger space than A^2.
In comparison, for the Hardy spaces, $L^2 = H^2 + \overline{H^2} + N$ where N has dimension equal to the genus of our annulus, as mentioned earlier.

Chapter 2

Factorisations

This section will discuss a few factorisation results in the annulus. One of the results will be an analogue of a factorisation result from the disc— the result that an L^1 function can be factorised as $f\bar{g}$ with $f, g \in H^2$ whenever it is log-integrable, which comes from [9]. The next will be a multiplicative analogue of the standard additive decomposition, and the final one an extension of the $H^1 = H^2H^2$ factorisation in the disc.

2.1 Douglas-Rudin in the annulus

The following result is known in the Hardy space of the unit disc:

Theorem 2.1.1 *Let $f \in L^\infty(\partial\Delta)$, $log|f| \in L^1(\partial\Delta)$, and $\epsilon > 0$.*
Then there exist $g, h \in H^\infty(\Delta)$ such that $f = g\bar{h}$, with

$$\|f\|_\infty \leq \|g\|_\infty \|h\|_\infty < (1+\epsilon)\|f\|_\infty.$$

The proof of this, which will be outlined here, is that shown in [14]. This section shall focus on showing that a version of this also holds in the annulus, and the proof will be given afterwards.

A few preliminary results are shown in [14] with the theme of approximating functions with the argument of the ratio of two Blaschke products. The first important one is Theorem 3.2.5. of [14].

Lemma 2.1.2 *Let* $\epsilon, \theta, \delta \in (0, \pi)$, $\eta \in (0, \theta/\pi)$, $\alpha \in (\epsilon - \pi, \pi - \epsilon)$, *and* $\Omega = \{e^{iw} : -\theta < w < \theta\}$.

Then there exist finite Blaschke products B_1 *and* B_2 *with simple zeroes such that:*

$$m(\{z \in \Omega : |\alpha - \arg(B_1(z)/B_2(z))| > \epsilon\}) < \eta,$$

with $-\delta < \arg(B_1(z)/B_2(z)) < \delta$ *for all* $z \in \partial\Delta\backslash\Omega$,

and $\sum_{z:B_1(z)B_2(z)=0}(1 - |z|) \leq 2m(\Omega)\log(\frac{4\pi}{\epsilon})$.

This is then used to show that one can approximate measurable functions with such ratios:

Lemma 2.1.3 *Let* $\epsilon, \theta, \delta \in (0, \pi), \nu \in (0, 1)$, Ω *a relatively open subset of* $\partial\Delta$, *and* $\psi : \Omega \to (-\pi, \pi]$ *be a measurable function. Then there exist finite Blaschke products* B_1 *and* B_2, *with simple zeroes, such that:*

$$m(\{z \in \Omega : |\psi(z) - \arg(B_1(z)/B_2(z))| > \epsilon\}) < \eta,$$

while $-\delta < \arg(B_1(z)/B_2(z)) < \delta$ *for all* $z \in \partial\Delta\backslash\Omega$,

and

$$\sum_{\{z:B_1(z)B_2(z)=0\}} (1 - |z|) \leq 2m(\Omega)\log(\frac{12\pi}{\epsilon}).$$

Lemma 2.1.3 is then strengthened to the following:

Lemma 2.1.4 *Let $\epsilon \in (0, \pi]$, $A \subset \partial\Delta$ measurable, and $\phi : \partial\Delta \to (-\pi, \pi]$ a measurable function vanishing outside A. Then there exist Blaschke products B_1 and B_2 such that*

$$\|\phi - \arg(B_1/B_2)\|_\infty < \epsilon,$$

with $\sum_{\{z:B_1(z)B_2(z)=0\}}(1 - |z|) \leq 2m(A)\log(\frac{50\pi}{\epsilon})$.

The approximations are then used to show Theorem 3.2.8 in [14], namely,

Theorem 2.1.5 *Let $\phi \in L^\infty(\partial\Delta, (-\pi, \pi])$ and $\epsilon > 0$. Then there exist Blaschke products B_1 and B_2 such that*

$$\|H(\phi - \arg(B_1/B_2))\|_\infty < \epsilon,$$

H denoting Hilbert transform.

The Douglas-Rudin factorisation then appears from Theorem 2.1.5 as Corollary 3.2.9 of [14], as follows:

Proof

Let v be the outer function with

$$v(z) = \exp(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(e^{it})|\frac{e^{it}+z}{e^{it}-z}dt),$$

i.e. $|v| = |f|$ a.e.

Then $v \in H^\infty$, and f/v is unimodular. Let $f/v = e^{i\phi}$, $\phi \in L^\infty(\partial\Delta, (-\pi, \pi])$.

Theorem 2.1.5 shows that given any ϕ, we have Blaschke products in the disc, B_1 and B_2, and $\gamma \in L_\infty$ such that $\gamma = \phi - \arg(B_1/B_2)$, and $\|H(\gamma)\|_\infty < \log(1+\epsilon)$, where H is the Hilbert transform.

Thus, letting $F = \frac{1}{2}(\gamma + iH(\gamma))$, then $F \in H^\infty$, and $\exp(i\phi) = B_1\exp(iF)\overline{B_2\exp(-iF)}$. So if $g = vB_1\exp(iF), h = B_2\exp(-iF)$, then $f = g\bar{h}$, $\|g\|_\infty < \|f\|_\infty\sqrt{1+\epsilon}$, and $\|h\|_\infty < \sqrt{1+\epsilon}$. $\Box$

Adaptation to the annulus

The following analogue of the result can be shown in the annulus:

Theorem 2.1.6 *Let* $f \in L^\infty(\partial\mathbb{A})$ *be such that* $\log|f| \in L^1(\partial\mathbb{A})$. *Then, there exists* $g, h \in H^\infty(\mathbb{A})$ *with* $f = g\bar{h}$.

The proof of this will follow most of the method used in [14] for the disc, but several adaptations need to be made when working with the Hardy spaces for an annulus.
The following preliminary result is needed on outer functions with modulus close to 1.

Lemma 2.1.7 *Suppose that* $f \in H^2(\mathbb{A})$ *is outer,* $1 - \epsilon < |f| < 1$ *on* $\partial\mathbb{A}$*, and* $\tilde{f}(0) > 0$*, with* $\tilde{f}$ *being the lift to* $H^2(\Delta)$ *under* θ *(*θ *defined in section 1.12).*
Then $\|1 - f\|_2^2 < 2\epsilon$.

Proof
Since taking lifts back to Δ preserves the given conditions, it is enough to show this is true for $f \in H^2(\Delta)$, as $H^2(\mathbb{A})$ is equivalent to the subspace of lifts in $H^2(\Delta)$.
Since f is outer, and $1 - \epsilon < |f| < 1$ on $\partial\mathbb{A}$, we have that $1 - \epsilon < |f(z)| < 1 \; \forall z \in \Delta$.
Now, $\langle 1-f, 1-f\rangle = 1+\|f\|^2-2Re(\langle f, 1\rangle) = 1+\|f\|^2-2Re(f(0)) \leq 1+1-2(1-\epsilon) \leq 2\epsilon$. $\Box$
We start with the following lemma:

Lemma 2.1.8 *Let*

$$\epsilon, t, \delta \in (0, \pi), \nu \in (0, t/\pi), \eta \in (0, 1), \alpha \in (\epsilon - \pi, \pi - \epsilon), \Omega = \{e^{i\omega} : -t < \omega < t\}.$$

Then we have finite Blaschke products B_1*,* $B_2 \in H^2(\mathbb{A})$ *with simple zeroes, such that* $m(\{|z \in \Omega : |\alpha - \arg(B_1/B_2)| > \epsilon\}) < \nu, -\delta < \arg(B_1/B_2) < \delta \; \forall z \in \partial\mathbb{A} \setminus \Omega$ *except* *on a set of measure at most* η.

Also, $\sum_{\{z:B_1(z)B_2(z)=0\}}(1-|z|) \leq 2m(\Omega)\log(\frac{4\pi}{\epsilon})$.

Proof

Let B_1' and B_2' be the restriction to the annulus of the Blaschke products for the corresponding theorem for the disc (Lemma 2.1.7). These will not be Blaschke products in $H^2(\mathbb{A})$— their modulus will be strictly less than 1 on Γ_1.

B_1'/B_2' takes values as required on Γ_0, by construction. Consider B_1'/B_2' on Γ_1.

We have that B_1' is a Blaschke product in the disc with zeros at $z \in Z_1$, where $Z_1 = \{r^{1/N}e^{2\pi ik/N} : 0 \leq k \leq N-1\} \cap \{r^{1/N}e^{iw} : -\beta < w < \beta\}$ for some constants r, N, β, as in [14].

Now B_2' has zero set $Z_2 = \{e^{i\alpha/N}z : z \in Z_1\}$.

Then $|e^{i\alpha/N}\frac{(w-z)(1-\overline{we^{i\alpha/N}}z)}{(1-\bar{w}z)(we^{i\alpha/N}-z)} - 1| = |\frac{e^{i\alpha/N}(w-z)(1-\overline{we^{i\alpha/N}}z)-(1-\bar{\omega}z)(\omega e^{i\alpha/N}-z)}{(1-\bar{w}z)(we^{i\alpha/N}-z)}| = |\frac{z(1-e^{i\alpha/N})(1-|\omega|^2)}{(1-\bar{w}z)(we^{i\alpha/N}-z)}| = |s|^2|\frac{(1-|w|^2)(1-e^{i\alpha/N})}{(\bar{z}-s^2\bar{w})(z-\omega e^{i\alpha/N})|}$

Now, we have that $(1-|w|^2) = (1-r^{2/N}) \leq 2N^{-1}log(1/r)$, $|1-e^{i\alpha/N}| \leq \alpha/N$, as was the case in the disc. Also, we have that $|z-s^2\omega| \geq (s-s^2)$ and $|z-e^{i\alpha/N}| \geq (1-s)$, for $z \in \Gamma_1$.

Thus, we have that $|e^{i\alpha/N}\frac{(w-z)(1-\overline{we^{i\alpha/N}}z)}{(1-\bar{w}z)(we^{i\alpha/N}-z)} - 1| \leq A\alpha N^{-2}\log(1/r)$ Since B_1'/B_2' is the product of N such terms, we have that on these boundaries, $|B_1'/B_2' - 1| \leq (1 + AN^{-2}\log(1/r))^N - 1) = O(1/N)$ for large N. However, in this construction, N can be chosen to be arbitrarily large, so we can make B_1'/B_2' arbitrarily close to 1 on Γ_1.

Since a bound on $\sum_{z\in Z_i}(1-|z|)$ independent of N from the proof in [14] was used, this does not affect our bound on for $\sum_{B_1(z)B_2(z)=0}(1-|z|)$. Now, the B_1', B_2' constructed are not inner functions for the annulus, however, letting $B_1' = f_i f_o, B_2' = g_i g_o$, inner-outer factorisations into (potentially multiply-valued) functions, we have that $B_1'/B_2' = f_i f_o/(g_i g_o) = (f_o/g_o)f_i\bar{g}_i$.

Since $|B_1'/B_2'| - 1$ can be made arbitrarily small, we have that $|f_o/g_o| - 1$ can be made arbitrarily small. However, since $|g_o| = |B_2'|$, we have that $|g_o|$ is continuous, bounded

away from 0 and infinity, thus $f_o/g_o \in H^2$ and is another outer function.
Since $|f_o/g_o|$ is close to 1 everywhere, we have from Lemma 2.1.7 that $\|f_o/g_o - 1\|$ can be made arbitrarily close to 0, and thus have argument close to 0 except on a set of small measure. We must choose N sufficiently large that this set has measure less than η. (Since we do not necessarily have approximation in ∞-norm, we no longer have B_1/B_2 has small argument outside Ω in our final result, as opposed to the case for the disc in [14]).

Now, f_i and g_i obtained need not be single valued, however this can be avoided— the zero sets of f_i and g_i coincide with those of B_1' and B_2' by construction, and the zero set of B_2' is a rotation of the zero set of B_1'.
In $H^2(\mathbb{A})$, the character of a Blaschke product with zero at z depends only on $|z|$, and so f_i and g_i will have the same character, call it κ.
Then, let $B_1 = f_i K$, and $B_2 = g_i K$, where K is a finite Blaschke product with character $-\kappa$ (the existence of such a K is proven in [2] p269). Then $B_1 \bar{B}_2 = f_i \bar{g}_i$, and both have character 0, i.e. they are single-valued.
□
From this, we proceed to the following refinement, which is an analogue of Lemma 3.2.6 from [14]:

Lemma 2.1.9 *Let $\epsilon, \delta \in (0, \pi), \nu, \eta \in (0, 1), \Omega$ relatively open, and $\Omega \subset \theta^{-1}(\Gamma_0)$ and $\phi : \partial\Delta \to (-\pi, \pi]$ a ψ-invariant measurable function with support in Ω. Then there exist finite, $\psi-$ invariant Blaschke products with $m(\{z \in \Omega : |\phi - \arg(B_1/B_2)| > \epsilon\}) < \nu$, with $-\delta < \arg(B_1/B_2) < \delta$ outside Ω except on a set of measure η, and $\sum_{z:B_1(z)B_2(z)=0}(1 - |z|) \leq 2m(\Omega) \log \frac{12\pi}{\epsilon}$.*

Essentially, this says that if one takes a function on the outer boundary of the annulus and lift back to the disc, then one can approximate with Blaschke products that are also lifts of Blaschke products from the annulus. Unlike in the case of the disc, we no longer have $\arg(B_1/B_2)$ small everywhere outside Ω, since our previous approximation in 2.1.8 no longer had this property (as a result of having to multiply by an outer function lying close to 1 in norm).

Proof

As in [14], simply take an appropriate step function approximating the function that we wish to target (using lifts of the functions in 2.1.8), and factorise this as a product of functions of the type dealt with in Lemma 2.1.8. The result then follows. □

Now, since this holds for the outer boundary, it does so after application of conformal equivalences. So, for Γ_i, we can apply a conformal equivalence to swap Γ_i and Γ_0, apply the result, then move back via the conformal equivalence. Thus we can drop the requirement that Ω has support on $\theta^{-1}(\Gamma_0)$.

So, we have

Lemma 2.1.10 *Let* $\epsilon, \delta \in (0, \pi), \nu, \eta \in (0, 1), \Omega$ *relatively open and* $\phi : \partial\Delta \to (-\pi, \pi]$ *a* ψ*-invariant measurable function. Then there exist finite,* $\psi-$*invariant Blaschke products with* $m(\{z \in \Omega : |\phi - \arg(B_1/B_2)| > \epsilon\}) < \nu$*,*
while $-\delta < \arg(B_1/B_2) < \delta$ *outside* Ω *except on a set of measure* η*, and* $\sum_{z:B_1(z)B_2(z)=0}(1 - |z|) \leq 2m(\Omega) \log \frac{12\pi}{\epsilon}$.

From this lemma, we can produce the following analogue of [14] Theorem 3.2.7:

Lemma 2.1.11 *Let* $\epsilon \in (0, \pi]$*,* $\Omega \subset \partial\Delta$ *and* ϕ *a* ψ*-invariant measurable function with support* Ω*. Then there exist* ψ*-invariant Blaschke products* B_1 *and* B_2 *with* $\|\phi - \arg(B_1/B_2)\|_\infty < \epsilon$*,* $\sum_{z:B_1(z)B_2(z)=0}(1 - |z|) < 2m(\Omega) \log(\frac{50\pi}{\epsilon})$.

The proof is mostly the same as in the disc from [14], except for use of Borel-Cantelli since our approximations are weaker than in [14].

Proof

Let (ν_k) and (δ_k) be sequences with $2\sum_{k=1}^{\infty}\nu_k \leq m(\Omega)\log(\frac{25}{24})$ and $\sum_{k=1}^{\infty}\delta_k < \epsilon/4$.

Let Ξ_1 be an open set (on the boundary) containing Ω, $m(\Xi_1) < m(\Omega)+\nu_0$, and let ψ_1 be defined by:

$\psi_1(z) = \phi(z) - \epsilon/2$ if $\phi(z) > 0$,

$\psi_1(z) = \phi(z) + \epsilon/2$ for $\phi(z) \leq 0$.

Then, we construct inductively a sequence of open sets Ξ_k, Ω_k, measurable functions ψ_k, and Blaschke products $B_1^{(k)}$, $B_2^{(k)}$ such that the following hold:

1. $$\sum_{\{z:B_1^{(k)}B_2^{(k)}=0\}}(1-|z|) \leq 2m(\Xi_k)\log(\frac{48\pi}{\epsilon}).$$

2. $$m(\{z\in\Xi_k^c : |\arg(B_1^k/B_2^k)| > \delta_k\}) < \nu_k.$$

3. Ξ_k contains
$$\Omega_{k+1} = \{z\in\Omega_k : |\psi_k(z) - \arg(B_1^k/B_2^k(z))| > \epsilon/4\}\bigcup\{z\in\Omega_k^c : |\arg(B_1/B_2)| > \delta_k\}.$$
and also
$$m(\Xi_k) < m(\Omega_k)+\nu_k < 2\nu_k.$$

4. $$\psi_{k+1} = \phi(z) - \epsilon/2 - \sum_{j=1}^{k}\arg(B_1^{(j)}) - \arg(B_2^{(j)}) \text{ for } \phi(z) > 0.$$
$$= \phi(z) + \epsilon/2 - \sum_{j=1}^{k}\arg(B_1^{(j)}) - \arg(B_2^{(j)}) \text{ for } \phi(z) \leq 0.$$

If we let l be a positive integer, and suppose we have found $\Xi_{k+1}, \psi_{k+1}, B_1^{(k)}, B_2^{(k)}$ for $k < l$.

Then, applying Lemma 2.1.10 with $\nu = \nu_l$, $\delta = \delta_l$, $\Omega = \Xi_l$ and $\psi = \psi_l|_\Omega$, $\epsilon = \epsilon/4$, we obtain B_1 and B_2 ψ-invariant Blaschke products with

$$m(\{z \in \Xi_l : |\psi(z) - \arg(B_1^{(l)}/B_2^{(l)})| > \epsilon/4\}) < \nu_l, \text{ and } |\arg(B_1^l/B_2^l))| < \delta$$

outside Ω_k except on a set of measure $< \nu$,

and $\sum_{B_1(z)B_2(z)=0}(1 - |z|) \leq 2m(\Omega)\log(\frac{48\pi}{\epsilon})$.

Now, we have that 1, 2, 3, 4 are all satisfied for $k = l$, hence by induction we can make the construction for all integers.

We have that $\sum_{k=1}^{\infty}\sum_{B_1^{(k)}(z)B_2^{(k)}(z)=0}(1 - |z|) \leq \sum_{k=1}^{\infty} 2m(\Xi_k)\log(\frac{48\pi}{\epsilon})$.

Since $m(\Xi_k) < 2\nu_k$, this series is summable, so $B_1 = \Pi_{k=1}^{\infty} B_1^{(k)}$ and $B_2 = \Pi_{k=1}^{\infty} B_2^{(k)}$ exist, and are Blaschke products, with $\sum_{B_1(z)B_2(z)=0} d(z, \partial\mathbb{A}) \leq 2m(\Omega)\log(\frac{50\pi}{\epsilon})$.

To show that we have $\|\phi - \arg(B_1/B_2)\|_\infty < \epsilon$, note that as $m(\Omega_k)$ is summable, we have that $m\{z : z \in \Omega_k \text{ infinitely often}\} = 0$ by the Borel-Cantelli Lemma.

Thus, we have that, aside from a null set, for each $z \in \partial\Delta$ we have there exists an l for which $z \in \Omega_l$ and $z \notin \Omega_k \forall k > l$.

Thus, $\psi_{l+1}(z) \leq \epsilon/4$, and since $|\arg(B_1^k/B_2^k)| < \delta_k$ all $k > l$, and $\sum \delta_k < \epsilon/4$, we have that $\psi_k(z) \leq \epsilon/2 \forall k > l$.

It thus follows that $\|\phi - \arg(B_1/B_2)|\|_\infty < \epsilon$, and $\sum(1 - |z|) < 2m(\Omega)\log(\frac{50\pi}{\epsilon})$. □

We then have that the following analogue of [14] Theorem 3.2.8 holds:

Theorem 2.1.12 *Let $\phi \in L^\infty(\partial\Delta)$ be ψ-invariant, and $\epsilon > 0$.*

Then there exist ψ-invariant Blaschke products B_1 and B_2 such that

$\|H(\phi - \arg(B_1/B_2))\|_\infty < \epsilon$.

where H is the Hilbert transform.

Proof

Let δ be constant depending on ϵ, $\phi_0 = \phi$, $h_1 = 0$. Then, by the previous lemma, we have ϕ_1, and Blaschke products $B_1^{(1)}, B_2^{(1)}$, $\|\phi_1\|_\infty \leq \frac{\delta}{2}$ and $\phi_1 = \phi_0 - Re(h_1) - \arg(\frac{B_1^{(1)}}{B_2^{(1)}})$, with $B_1^{(1)}$ and $B_2^{(1)}$ ψ-invariant Blaschke products, $\sum_{B_1^{(1)}(z)B_2^{(1)}(z)=0}(1-|z|) \leq \log(\frac{100\pi}{\delta})$.
Suppose for $k \geq 1$, we have ϕ_k ψ-invariant, with $\|\phi_k\|_\infty \leq 2^{-k}\delta$.
Then, by Corollary 3.2.4 from [14], we have that there exists h_{k+1} with
$\|\phi_k - Re(h_{k+1})\|_2 \leq 2^{-k}\|\phi_k\|_\infty \leq 2^{-2k}\delta$.
and $\|h_{k+1}\|_\infty \leq c\log(2^k)\|\phi_k\|_\infty \leq \frac{ck\log 2}{2^k}\delta$.
We need to adapt this so that the h_k obtained is modulus-invariant. Let f be a real-valued bounded ψ-invariant function, and g_1 and g_2 analytic functions such that $f = g_1 + \bar{g}_2$. Since $f \in H^\infty$, these will be BMOA functions, and thus lie in H^1.
Then we must have that g_1=g_2+A, where A a constant, since $f = \bar{f}$ thus $g_1+\bar{g}_2 = \bar{g}_1+g_2$, thus $g_1 - g_2 = \overline{g_1 - g_2}$, and so $g_1 - g_2 = A$
So, $(f - A) = g + \bar{g}$. Since the left hand-side is ψ-invariant, we have that $Re(g)$ is invariant, and thus g must be ψ-invariant (otherwise, $g - \psi \circ g$ would be in H^p and have vanishing real part on the boundary).
Now, applying [14] Lemma 3.2.2 to deal with boundedness, the construction involves multiplying by an outer function whose modulus on the boundary will be invariant, and thus this yields a modulus-invariant function h_{k+1}. Suppose this has character $-\beta_{k+1}$, and since h_k approximates an invariant function arbitrarily well for large k, we must have that the β_k are tending to 0. So, replacing h_{k+1} with $e_{\alpha_{k+1}}h_{k+1}$, with e_{α_k} the lift to the disc of the function z^{α_k} in the annulus, where $\alpha_k = \beta_k + m$, with $m \in \mathbb{Z}$ chosen so that $-\frac{1}{2} \leq \alpha_k \leq \frac{1}{2}$, we recover an invariant function. Since $-\frac{1}{2} \leq \alpha_k \leq \frac{1}{2}$, we at most increase by a constant multiple of s^{-1} for $\|e_{\alpha_{k+1}}h_{k+1}\|_\infty$. We need the following to show that we still approximate in L_2 as well:

Lemma 2.1.13 *$\|\phi_k - Re(h_{k+1}e_{\alpha_{k+1}})\|_2 \leq 2^{-2k}E\delta$ for E some constant depending only on the inner radius of our annulus $\mathbb{A}$.*

Proof

Since we have that $\|h_{k+1}-e_{\alpha_{k+1}}h_{k+1}\|_2 = \|h_{k+1}(1-e_{\alpha_{k+1}})\|_2 \leq \|h_{k+1}\|_\infty \|(1-e_{\alpha_{k+1}})\|_2$, and calculate the integral on the strip $S = \{z : 0 < Re(z) < -logs\}$ (choosing the map from disc to strip so that composition with e^{-z} will return our map from the disc to annulus θ) with measure on the boundary $|dz|\frac{1}{cosh(Im(z))}$ (which is an equivalent norm to lebesgue measure in the disc on applying our conformal equivalence, by [7]. Since $z \to e^{-z}$ is our map from the strip to the annulus, we have that our e_{α_k} lifts to $e^{-z\alpha_k}$.
So $\|1 - e_{\alpha_{k+1}}\|_2^2 \leq \sum_{n=1}^{\infty}(\sup_{2\pi n \leq \theta \leq 2\pi(n+1)} |1 - e^{i\alpha_k\theta}|^2 + |1 - s^{\alpha_k}e^{i\alpha_k\theta}|^2)\mu(A_n)$, where A_n is the set $\{it : 2n\pi < t < 2(n+1)\pi\} \cup \{-\log s + it : 2n\pi < t < 2(n+1)\pi\}$.
Now, $|1 - s^{\alpha_k}e^{i\alpha_k\theta}|^2 \leq |1 - e^{i\alpha_k\theta}|^2 + |1 - s^{\alpha_k}|^2$, so we have that:
$\|1 - e_{\alpha_{k+1}}\|_2^2 \leq 2\sum_{n=1}^{\infty}(\sup_{2\pi n \leq \theta \leq 2\pi(n+1)} |1 - e^{i\alpha_k\theta}|^2\mu(A_n)) + (1 - s^{\alpha_k})^2$.
Using the approximation $|1 - e^{ix}| \leq |x|$ (which follows since the length of the chord between 1 and e^{ix} on the unit disc is bounded by the length of the arc between them) :
$\|1 - e_{\alpha_{k+1}}\|_2^2 \leq \sum_{n=1}^{\infty} \pi^2(s + 1)^2\alpha_k^2\frac{1}{\cosh(2\pi ns)} + |1 - s^{\alpha_k}|^2$.
Since $\sum_{n=1}^{\infty}(2\pi(n + 1))^2\frac{1}{\cosh(2\pi n)} = D$ for D some real constant, it follows that $\|1 - z^{\alpha_k}\|_2^2 \leq \alpha_k^2 D_1 + (1 - s^{\alpha_k})^2 \leq \alpha_k^2 C$ for some real constant C depending only on our inner radius.

To get bounds for α_k, suppose we have that s_k is the $H^2(\Delta)$ function such that $\phi_k = s_k + \bar{s_k}$ and $s_k(0)$ is real.
Then (see [14] Lemma 3.2.1 and corollary 3.2.4), we obtain h_k by multiplying s_k by the outer function $\tilde{G_k}$, the lift of $G_k \in H^2(\mathbb{A})$ to the disc where $|G_k| = 1$ on $F_{2^k} = \{z : z \in \partial\mathbb{A}, |s_k(\theta^{-1}(z))| \leq 2^k\}$, and $|G_k| = \frac{2^k}{|s_k|}$ on $E_{2^k} = {F_{2^k}}^c$.
Since s_k was invariant, we only need to know the character of G_k, and $\alpha_k = \frac{(\int_{|z|=1}\log|G_k| - \int_{|z|=s}\log|G_k|)}{\log s}$.
Thus $|\alpha_k| \leq \frac{1}{|\log s|}\int_{|s_k|\geq\log(2^k)} |\log(\frac{\log(2^k)}{|s_k|})| \leq \frac{1}{-\log s}\int_{|s_k|\geq\log(2^k)} |\log(|s_k|)| \leq \frac{1}{-\log s}\sum_{k\leq t<\infty}(t + 1)\mu(\{z : 2^t < |s_k(z)| < 2^{t+1}\})$.
Now, we have $\mu(z : 2^t < |s_k(z)| < 2^{t+1}) \leq \frac{1}{2^{2t}}\|s_k\|_2^2$.

Thus $|\alpha_k| \leq \frac{1}{-2\pi \log s} \|s_k\|_2^2 \sum_{t=k}^{\infty} \frac{(t+1)}{2^{2t}}$.

Now, since $\|s_k\|_2^2 \leq \|\phi_k\|_2^2 \leq \|\phi_k\|_\infty^2$ (For the first inequality, if $\phi_k = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{a_n} \bar{z}^n$, we have $s_k = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n z^n$, and so the result follows), and the series $\sum_{t=k}^{\infty} \frac{(t+1)}{2^{2t}}$ has a bound independent of k, we have that $|\alpha_k| \leq E(2^{-2k})\delta$, where E is a constant depending only on our inner radius.

It will follow that $\|\phi_k - Re(h_{k+1}e_{k+1})\|_2 \leq \|\phi_k - Re(h_{k+1})\|_2 + \|h_{k+1} - h_{k+1}e_{k+1}\|_2 \leq E2^{-2k}\delta$ for some constant E depending only on inner radius (since δ will later be chosen to be small, we can assume $\delta < 1$ and so $\delta^2 < \delta$). $\square$

The rest of the proof of this theorem proceeds without further need for modification from that in [14] Theorem 3.2.8.

Letting $g_{k+1} = h_{k+1} z^{\alpha_{k+1}}$, we have that:

$$\|\phi_k - Re(g_{k+1})\|_2 \leq 2^{-2k} E\delta$$
$$\|g_{k+1}\|_\infty \leq \frac{ck \log 2}{2^k}\delta.$$

Letting $\Delta_k = \{z \in \partial\Delta : |\phi_k - Re(g_{k+1})| > 2^{-(k+2)}\delta\}$, we have that this will be ψ-invariant, and thus $f_k = (\phi_k - Re(h_{k+1}z^{\alpha_{k+1}}))\chi_{\Delta_k}$ will be ψ-invariant, and $m(\Delta_k) \leq \frac{(2^{-2k}E\delta)^2}{2^{-2(k+2)}\delta^2} \leq \frac{16E^2k^2}{2^{2k}}$.

We can, by lemma 2.1.11 construct B_1^{k+1}/B_2^{k+1} ψ-invariant, with

$$\|f_k - \arg(B_1^{k+1}/B_2^{k+2})\|_\infty < 2^{-(k+2)},$$
$$\sum_{B_1(z)B_2(z)=0} (1 - |z|) \leq 2m(\Delta_k) \log(\frac{50\pi}{2^{-(k+2)}\delta}).$$

Take $\phi_{k+1} = \phi_k - Re(g_{k+1}) - \arg(B_1^{k+1}/B_2^{k+1})$.

Then, by evaluating on Δ_k and Δ_k^c separately, we can see that $\|\phi_{k+1}\|_\infty \leq 2^{-(k+1)}\delta$.

We have that $\sum_{k=0}^{\infty} \|\phi_k - \arg(B_1^{k+1}/B_2^{k+1}) - \phi_{k+1}\|_\infty = \sum_{k=0}^{\infty} \|Re(g_{k+1})\|_\infty \leq \sum_{k=0}^{\infty} \|g_{k+1}\|_\infty \leq \epsilon$ for suitably small δ.

Thus $\sum_{k=0}^{\infty} \phi_k - \arg(B_1^{k+1}/B_2^{k+1}) - \phi_{k+1} = \phi_0 - \arg(\Pi_{k=0}^{\infty} B_1^{k+1}/B_2^{k+1})$.

Also, $\|H(\phi_0 - \arg(\Pi_{k=0}^{\infty} B_1^{k+1}/B_2^{k+1}))\|_\infty = \|H(\sum_{k=0}^{\infty} \phi_k - \arg(B_1^{k+1}/B_2^{k+1}) -$

$\phi_{k+1})\|_\infty = \|H(\sum Re(g_{k+1}))\|_\infty \leq \sum \|Im(g_{k+1})\|_\infty \leq \sum \|g_{k+1}\|_\infty < \epsilon$. And, we also have $\sum_{k=0}^{\infty} \sum_{B_1^{k+1}(z)B_2^{k+1}(z)=0}(1-|z|) < \infty$. Thus the products of the ψ-invariant Blaschke factors converge to suitable ψ-invariant Blaschke products B_1 and B_2. □

From this, we can show the main theorem, which now proceeds in the same way as the the proof in [14] follows from Theorem 3.2.8. in that book:

Theorem 2.1.14 *Whenever $f \in L^\infty(\partial\mathbb{A})$ with $\log|f| \in L^1$, we have $f = g\bar{h}$ for some $g, h \in H^\infty(\mathbb{A})$.*

Proof

First, lift to the disc so that we have a ψ-invariant boundary function in the disc, factor out an (invariant) outer function so that we can restrict to the case of factorising a unimodular function (to deal with this outer function having non-integer character, use the fact that $z^\alpha s_i \bar{z^\alpha} \bar{s}_i = |z|^{2\alpha}$ in the annulus and taking lifts back to the disc, we can adjust for this (where s_i is some inner function with the right character to make $z^\alpha s_i$ single valued).

So $f = e^{i\phi}$, with ϕ real valued. Lifting to the disc, we can produce B_1 and B_2 invariant such that $\|H(\phi - \arg(B_1/B_2)\|_\infty < \log(1+\epsilon)$.

Then, have that $F = \frac{1}{2}(\gamma + iH(\gamma)) \in H^\infty$ where $\gamma = \phi - \arg(B_1/B_2)$, and $\exp(i\phi) = B_1 \exp(iF)\bar{B}_2\overline{\exp(-iF)}$.

Since γ is ψ-invariant, so is F (since otherwise, $F - F \circ \psi$ would be non-constant, analytic with constant real part) and thus the decomposition is in terms of invariant functions, which can then be passed down to the annulus to obtain our factorisation. □

The proof here does not seem to extend to a higher genus annulus, the main problem being that Lemma 2.1.8 appears difficult to adapt to have B_i single-valued for the higher genus annulus.

2.2 Multiplicative factorisation

There is a well known additive decomposition for $H^2(\mathbb{A})$ into $H^2(\Delta) \oplus H^2((s\Delta)^c)$.
If $f = \sum_{-\infty}^{\infty} a_n z^n$, then $f = f_1 + f_2$ with $f_1 = \sum_0^{\infty} a_n z^n$, and $f_2 = \sum_{-\infty}^{-1} a_n z^n$.
It is immediate that $f_1 \in H^2(\Delta)$, $f_2 \in H^2((s\Delta)^c)$, and $f_1 \perp f_2$.
The following gives a multiplicative decomposition which somewhat parallels this.

Theorem 2.2.1 *Let $f \in H^p(\mathbb{A})$, $p \in [1, \infty]$. Then there exists f_1, f_2 with $f = f_1 f_2$, $f_1 \in H^p(\Delta)$, and $f_2 \in H^p((s\Delta)^c)$, such that $f = f_1 f_2$.*

Proof

First, consider the case when $|f| > 0$ in $\mathbb{A}$. Since f is analytic and non-zero on $\mathbb{A}$, it has well-defined winding number about 0. Call this n_f. Then, letting $f' = fz^{-n_f}$, f' has winding number 0 about 0, and thus we have a well-defined logarithm in $\mathbb{A}$.
Let $\log f' = \sum_{-\infty}^{\infty} a_n z^n$ be the power representation of this function, and set $g_1(z) = \sum_0^{\infty} a_n z^n$, $g_2(z) = \sum_{-\infty}^{-1} a_n z^n$. Since $\log f'$ is bounded on $\{z : (s + \epsilon)\} < |z| < 1 - \epsilon\}$ for all $\epsilon > 0$, it follows that g_1 is bounded on $\{z : |z| < 1 - \epsilon\}$, and g_2 bounded on $\{z : |z| > s + \epsilon\}$ for all $\epsilon > 0$.
We have that $f = z^n e^{g_1 + g_2} = z^n e^{g_1} e^{g_2}$, so we have only to verify that $e^{g_1} \in H^p(\Delta)$, and $e^{g_2} \in H^p((s\Delta)^c)$.
g_2 is analytic and non-zero on $\{|z| > s + \epsilon\}$, and hence $|e^{g_2}|$ has non-zero infinum. Letting $A_2 = \inf_{|z|=1} |e^{g_2}| \neq 0$, we have that $\int_{|z|=1} |e^{g_1}|^p \leq \int_{\{|z|=1\}} |f'|^p A_2^{-p} < \infty$.
Hence $e^{g_1} \in H^p(\Delta)$ (since we have a power series consiting of only positive powers of z, we know that that the integral on $|z| = \rho$ is strictly increasing with ρ). By a symmetric argument, $e^{g_2} \in H^p((s\Delta)^c)$.
Thus a function without zeros can be factorised. However, for an arbitrary function f, we can remove a Blaschke factor to deal with the zeros— let B_1 be a Blaschke product in $H^p(\Delta)$ with $B_1(z) = 0 \Leftrightarrow \{f(z) = 0, |z| \geq s^{\frac{1}{2}}\}$, and B_2 a Blaschke product in

$H^p((s\Delta)^c)$ with $B_2(z) = 0 \Leftrightarrow \{f(z) = 0, |z| < s^{\frac{1}{2}}\}$.
Then we have that $f = B_1 B_2 f'$ with $f' \in H^p$, $|f'| > 0$ in $\mathbb{A}$. Letting $f' = f_1 f_2$ from before, we have that $f = (f_1 B_1)(f_2 B_2)$ factorises the function as required. $\square$

2.2.1 Non-uniqueness

This factorisation fails to be unique— if one takes f with finitely many zeros, the Blaschke product factor of f can be included it in either factor, and each will yield a different factorisation (since the method for factorising non-zero functions produces non-zero functions in their respective domains).
Consider the following example:
Take $p(z) = z - \frac{1}{2}$, assuming that $s < \frac{1}{2}$.
To factorise this, we can take a Blaschke product for Δ, zero at $\frac{1}{2}$, or for $(s\Delta)^c$.
The first choice gives the factorisation $p(z) = \frac{z-\frac{1}{2}}{1-\frac{1}{2}z}(1 - \frac{1}{2}z)$, we observe that the second factor already lies in $H^2(\Delta)$, so we are done.

Alternatively, we can take the Blaschke product for $(s\Delta)^c$ with zero at $\frac{1}{2}$ $(= \frac{1-2z}{s-\frac{1}{2s}z})$.
So, we have $z - \frac{1}{2} = \frac{1-2z}{s-\frac{1}{2s}z}(s - \frac{1}{2s}z)$.
$s - \frac{1}{2s}z$ winds about 0 on any loop in our annulus, so take out a factor z. $s - \frac{1}{2s}z = -z(\frac{1}{2s} - sz^{-1})$. But $\frac{1}{2s} - sz^{-1}$ is already in $H^2((s\Delta)^c)$, so we are done. Simplifying each product we have, $z - \frac{1}{2} = (z - \frac{1}{2}).1 = z.(1 - \frac{1}{2}z^{-1})$ are the two different ways of writing the product as $H^2(\Delta)H^2((s\Delta)^c)$, depending on which side the zero goes. These factorisations differ by multiplication of meromorphic functions, and we can in fact show that factorisation is unique up to meromorphic functions.

Theorem 2.2.2 *Suppose that we have f, f_1, f_2, f_3, f_4 with $f \in H^2(\mathbb{A})$, $f_1, f_3 \in H^2(\Delta)$, $f_2, f_4 \in H^2((s\Delta)^c)$, and $f = f_1 f_2 = f_3 f_4$.*
Then we have that f_3/f_1 is meromorphic in $\mathbb{C}$.

Proof

Eugene Shargorodsky suggested the following proof, which is simpler than the original. We have that $\frac{f}{f_1 f_4} = \frac{f_2}{f_4} = \frac{f_3}{f_1}$ on $\mathbb{A}$. Now, we have that $\frac{f_2}{f_4}$ is meromorphic in $(s\Delta)^c$, as it is the ratio of analytic functions in that domain. We also have that $\frac{f_3}{f_1}$ is meromorphic in Δ. Since $\frac{f_2}{f_4} = \frac{f_3}{f_1}$, we can extend to a meromorphic function on $\mathbb{C}$, as required. $\square$

2.2.2 Extension to higher genus setting

We can extend this factorisation as follows:

Theorem 2.2.3 *Let D be a general genus g annulus, consisting of the unit disc Δ with g components A_i removed, with each A_i conformally equivalent to the unit disc. Let $f \in H^p(D)$. Then there exists f_i, $0 \leq i \leq g$, such that $f_0 \in H^p(\Delta)$, and $f_i \in H^p(A_i^c)$.*

Proof

We can prove this by induction. Firstly, note that if we are in the case $g = 1$, we can apply a conformal equivalence to ensure that the inner disc is centred about 0, and then apply the case for $\{z : s < |z| < 1\}$.

Now, suppose D is an annulus of genus $n + 1$ constructed as in the theorem. Let $D' = \{z : s_1 < |z| < 1\}$, where s_1 is chosen so that D' is contained in D. Now, if we have $f \in H^2(D)$, we also have that $f \in H^p(D')$, and applying the $g = 1$ case, we have that $f = f_0 f_1$ with $f_0 \in H^p(\Delta)$, and $f_1 \in H^p(\{z : |z| > s_1\}$.

We can extend f_1 analytically by $f_1 = f/f_0$ to yield $f_1 \in H^p(D \bigcup \Delta^c\})$, with $f = f_0 f_1$. Now, $D \bigcup \Delta^c$ is a genus-n annulus. Through application of a conformal equivalence, we can send this to an annulus consisting of the unit disc with $g - 1$ removed subdiscs, and thus f_1 can be factorised completely. Moving back to the original domain with a conformal equivalence, we will have that

$$f = f_0 f_1 \dots f_g.$$

With $f_i \in H^p(A_i^c)$, $f_0 \in H^p(\Delta)$. $\square$

2.3 Factorisation of H^1 function into H^2 functions

In the disc, the following result is a well known result following from the inner-outer decomposition of functions:

Theorem 2.3.1 *Let $f \in H^1(\Delta)$, then there exist g_1, g_2 with $f = g_1 g_2$, $g_i \in H^2(\Delta)$, and $\|f\|_1 = \|g_1\|_2 \|g_2\|_2$.*

In the annulus, whilst an H^1 function can still be factorised as a product of H^2 functions, the norm of the factorisation is no longer preserved— which is quite important with later consideration of Hankel operators.

Theorem 2.3.2 *Let $f \in H^1(\mathbb{A})$, then there exist g_1, g_2, $f = g_1 g_2$, $g_i \in H^2(\mathbb{A})$, $\|g_1\|_2 \|g_2\|_2 \leq A \|f\|_1$, where $A \in \mathbb{R}$ is a constant.*

This has been shown already in [4] for arbitrary multiply connected regions, with the constant of factorisation depending on the region in question. In the case of an annulus of the form $\{z : s < |z| < 1\}$, this will be improved upon to show that a constant of factorisation independent of s can be obtained.

A proof of the theorem with the constant $A = \frac{1}{s}$ depending on the inner radius of $\mathbb{A}$, s, will first be given. Later, this constant will be made independent of the inner radius of the annulus $\mathbb{A}$. The method is akin to that in the disc, with a small alteration.

Proof

Let $f = f_i f_o$ be the inner outer factorisation of f.

Then let $g_1 = (f_i f_0^{1/2} z^{\alpha/2})$, and $g_2 = (f_0^{1/2} z^{-\alpha/2})$, where $\alpha \in (0, 1)$ is the character of f_o. (The $z^{\alpha/2}$ term appears to ensure that our g_i are still single-valued functions, as $f_0^{\frac{1}{2}}$ will have character $\frac{\alpha}{2}$, thus $f_i f_0^{\frac{1}{2}}$ would have character $-\frac{\alpha}{2}$).

then $f = g_1 g_2$, and $\|g_1\|_2 \|g_2\|_2 \leq s^{-1} \|f\|_1$. $\square$

This illustrates the difficulty that arises for the annulus— in taking square roots for our outer part, we are altering the characters, and so the inclusion of $z^{\frac{1}{2}\alpha}$ is needed to counter this.

The constant in the following proof comes out as $((1+e)(1+2e))^{\frac{1}{2}}$, which is independent of our annulus inner radius.

Proof

Assume w.l.o.g. that $\int_{\Gamma_1} |f| = 1$, and $\int_{\Gamma_0} |f| \geq 1$ (transform to swap boundaries if necessary, then scale). Let $\int_{\Gamma_0} |f| = B$.

Let $f = f_o f_i$ be the inner-outer factorisation of f, and factorise as $f = (g_1 f_i z^x)(g_2 z^{-x})$, where we have that:

g_1 is an outer function,

$|g_1| = |f|^{\frac{1}{2}}$ whenever $|f| > 1$,

$|g_1| = |f|$ when $|f| \leq 1$,

and x is chosen so that $g_1 f_i z^x$ is a single valued function.

We have from [38] that the character of an outer function is given by $\frac{\int_{\Gamma_0} \log |f| - \int_{\Gamma_1} \log |f|}{\log s}$ (mod 1).

It thus follows that the difference between the characters of g_1 and f_o is equal to $\frac{\int_{\Gamma_0} \frac{1}{2} \log^+ |f| - \int_{\Gamma_1} \frac{1}{2} \log^+ |f|}{\log s}$.

Now, since $\int_{\Gamma_1} |f| = 1$, we have that $\int_{\Gamma_1} \log^+ |f| \leq 1$ (since $\log^+ |f| \leq |f|$).

We also have that $\int_{\Gamma_0} \log^+ |f| \leq 1 + \log B$. To see this, consider the problem of finding a function $g \in L^2(\Gamma_0, \mathbb{R})$ to maximise $\int_{\Gamma_0} \log^+ |g|$ subject to the constraint $\int |g| = B$

It is immediate that a maximal function will not take values in the set $(0, 1)$- otherwise, we can create a function g' with $g' = 0$ where $g < 1$, which would decrease $\int g'$ without changing $\int \log^+ g'$.

Secondly, note that a maximal function will be constant on the set $g > 1$— otherwise, choose g' such that $g' = \mathbf{1}_{\{z:g>1\}}(\int_{g>1} \frac{g}{\mu(\{z:g>1\})})$. Then, we will have $\int g = \int g'$, and by Jensen's inequality (and noting that $\log^+ g = \log g$ on $g > 1$), we have that $\int \log^+ g' > \int \log^+ g$.

Thus, we have that the maximal g will be such that $g = 0$ on a set of measure $1 - x$, and $g = B/x$ on a set of measure x. The case $x = 0$ is trivial, and optimising over x we see that $\int \log^+ |f| < \sup_{x\in(0,1)} x \log \frac{B}{x}$.

This is maximised as x approaches 1 for $B > e$, giving an upper bound of $\log B$ in this case. For $1 < B < e$, we can see that 2 is an upper bound, so $1 + \log B$ is an upper bound for all $A > 1$.

Since the character of z^x is x, we have that $|x| \leq \frac{1+\log B}{-2\log s}$, and so $s^{2x} \leq s^{\frac{1+\log B}{\log s}} \leq eB$.

Now, we have that $\|f\|_1 = (B + 1)$.

Also, $\|g_1 f_i z^x\|_2^2 \leq (B + eB)$ (since $|g_1|^2 \leq |f|$), and

$\|g_2 z^{-x}\| \leq ((B + 1) + (1 + 1)(eB))$ (since $|g_2|^2 \leq \max\{1, |f|\}$).

So $\frac{\|g_1 f_i z^x\|_2^2 \|g_2 z^{-x}\|}{\|f\|_1^2} \leq \frac{((1+e)B)(1+(1+2e)B)}{(1+B)^2} \leq ((1 + e)(1 + 2e))$. □

The constant that arises in these factorisations is entirely due to characters— if you permit factorisations of the type $f = g_1 g_2$ with $\tilde{g}_1 \in H^2_\alpha$, $\tilde{g}_2 \in H^2_{-\alpha}$, $\alpha \in [0, 2\pi)$ and H^2_α the set of analytic functions in the annulus of character α (see, e.g. [38]), one can factorise with $\|g_1\|_2\|g_2\|_2 = \|f\|_1$— we can simply take square roots of the outer component of f, now that we are no longer restricted by characters. The same is also true in a higher order annulus, with $\alpha \in \hat{G}$:

Theorem 2.3.3 *Let $f \in H^1(D)$, then there exists g_1, g_2, with $\tilde{g}_1 \in H^2_\alpha$, $\tilde{g}_2 \in H^2_{-\alpha}$ for some α some character, and with $\|g_1\|_2\|g_2\|_2 = \|f\|_1$.*

Proof

This follows straightforwardly from the inner-outer factorisation. Let $f = f_i f_o$. Then take $g_1 = f_i f_o^{\frac{1}{2}}$, and $g_2 = f_o^{\frac{1}{2}}$, taking $f_o^{\frac{1}{2}}$ to be the outer function with modulus on the boundary to be the root of the modulus of f_o.

It is immediate that $f = g_1 g_2$, and that $\|g_1\|_2 = \|g_2\|_2 = \|f\|_1^{\frac{1}{2}}$. If f_o has character κ, $f_i f_o^{\frac{1}{2}}$ will have character $-\frac{\kappa}{2}$, so these are no longer single-valued functions in general. $\square$

Chapter 3

Toeplitz operators on the annulus

3.1 Toeplitz Operators with real-valued symbol

3.1.1 Eigenvalues/eigenvectors of Toeplitz operators with real-valued symbols

The initial motivation at the start of this project was to consider whether a self-adjoint (and thus real valued) Toeplitz operator on the Hardy space of the annulus could have eigenvalues, and as to what they would be. The solution to this problem followed quickly from some of the results in Abrahamse's paper on Toeplitz operators in multiply connected domains [2]. Research in a paper of Broschinski also came to the same conclusions with similar method on the eigenvalues for self-adjoint Toeplitz operators on the annulus in [10], though they did not consider higher order annuli. An older paper by Aryana and Clancey [6] describes the eigenvalues of Toeplitz operators in a general annulus of arbitrary genus, in terms of theta-functions, and a recent paper by Aryana [5] gives more details in the genus-1 case, giving the same theorems on when infinitely many eigenvalues accumulate in the spectrum, but not giving explicit calculation of the eigenvalues. The methods in the Aryana paper were focused on the resolvent of the operators in question, as opposed to the methods here. Whilst Brochinski uses much the same methods, he does not continue to higher genus annuli in his paper. In his recent (2014) thesis [11], he mentions Toeplitz on a more general multiply connected domain, but the results stated are in terms of Toeplitz operators on H^2_α, looking for α that give an eigenvalue given a Toeplitz operator, and so do not immediately provide information on H^2. The results on Toeplitz operators in [11] are also limited to real-valued symbol with constant sign on each boundary component, and properties of the eigenfunctions are also not considered in Brochinski's work on more general domains.

In this chapter, $\mathbb{A}$ will be used to refer to the annulus $\{z : s < |z| < 1\}$, and D will be used to refer to an arbitary finitely-connected domain contained in the disc, whose boundary curves are analytic Jordan curves.

We will take $\mathbb{A}$ and D to have uniform Lebesgue measure on the boundary, unless stated otherwise.

Lemma 3.1.1 *Let ϕ be a bounded real-valued symbol of a Toeplitz operator on the Hardy space of the annulus $\mathbb{A}$:=$\{z : s < |z| < 1\}$.*
Let $\Gamma_0 = \{z : |z| = 1\}$, and $\Gamma_1 = \{z : |z| = s\}$.
Let ess $\inf \phi|_{\Gamma_0} = a$; ess $\sup \phi|_{\Gamma_0} = b$.
Let ess $\inf \phi|_{\Gamma_1} = c$; ess $\sup \phi|_{\Gamma_1} = d$.
Then we have that:

i $\sigma_e(T_\phi) = [a, b] \bigcup [c, d]$.

ii *If $a \leq b \leq c \leq d$, then $\sigma_p \subset [b, c]$, and if $c \leq d \leq a \leq b$, then $\sigma_p \subset [d, a]$.*
otherwise, $\sigma_p = \emptyset$.

Proof

(i) follows from a corollary of Abrahamse's reduction theorem, Corollary 3.2 of [2].

Theorem 3.1.2 *(Abrahamse) The operator T_ϕ is Fredholm if and only if T_{ϕ_i} is Fredholm for $i = 0, \ldots n$, and so $\sigma_e(T_\phi) = \bigcup_{i=0}^n \sigma_e(T_{\phi_i})$.*

(T_{ϕ_i} refers to the Toeplitz operator on the Hardy space of the complement of the area enclosed by Γ_i for $i \geq 1$, and on the disc for $i = 0$, with ϕ_i being the restriction of ϕ to Γ_i).
The essential spectrum of a self-adjoint Toeplitz operator on a disc, or any other connected space, is the essential range of its symbol. Thus (i) follows.□
To prove (ii), we first need the following lemma on the eigenfunctions with eigenvalue 0 of a self-adjoint Toeplitz operator:

Lemma 3.1.3 *If ϕ is a real-valued L^∞ function with $T_\phi f = 0$ then $\phi|f|^2 \in H_\infty^\perp \bigcap \overline{H_\infty^\perp}$.*

Proof
$T_\phi f = 0 \Rightarrow \langle \phi f, g \rangle = 0$ for all $g \in H^2$.
Thus $\langle \phi f, gf \rangle = 0$ for all $g \in H^\infty$, and so $\langle \phi f \bar{f}, g \rangle = 0$ for all $g \in H^\infty$.
Thus we have that $\phi|f|^2 \in H_\infty^\perp$.
However, $\phi|f|^2$ is real-valued, so on taking conjugates we have orthogonality to $\overline{H_\infty^\perp}$ also.□

Note that this result will work on the Hardy space of any domain. In particular, if we consider the Hardy space of the disc, this shows that $\phi|f|^2$ is perpendicular to both the analytic and anti-analytic functions, and thus we have no eigenvalues when ϕ is non-zero. Whilst it is not immediate that ${H_\infty}^\perp$ is contained in L^1 (as ${L_\infty}^\perp$ is not L^1 in general), $\phi|f|^2$ will always be in L^1, so we can take intersection with L^1 in any case.
Now, for the Hardy space of analytic functions on the annulus under consideration, we have that the space of functions orthogonal to both analytics and anti-analytic functions is a 1-dimensional space, spanned by the function ω, where $\omega|_{\Gamma_0} = 1$, and $\omega|_{\Gamma_1} = -1$.

Lemma 3.1.4 *If h is such that $\langle h, f \rangle = 0$ for all $f \in H^\infty$,and $\langle h, \bar{f} \rangle = 0$ for all $f \in H^\infty$, then $h = k\omega$ for some $k \in \mathbb{R}$.*

Proof

Considering orthogonality against z^n and $\bar{z}^{-n}$ respectively:

$$\int_{\Gamma_0} h(e^{i\theta})e^{in\theta} + s^n \int_{\Gamma_1} h(se^{i\theta})e^{in\theta} = 0$$

and

$$\int_{\Gamma_0} h(e^{i\theta})e^{in\theta} + s^{-n} \int_{\Gamma_1} h(se^{i\theta})e^{in\theta} = 0.$$

Taking the difference, when $n \neq 0$, we have that all Fourier coefficient apart from the zeroth are 0 on Γ_1, i.e. h is a constant on Γ_1, and the same follows for the Fourier coefficients on Γ_0. Thus $h = k$ on Γ_0, and $h = l$ on Γ_1, where k and l are constants.
Since $\langle h, 1\rangle = 0$, we must have that $l = -k$, i.e. $h = k\omega$. □

Combining these two lemmas, we have that $T_\phi f = 0 \Rightarrow \phi|f|^2 = \lambda\omega$.
This can be improved upon as follows:

Theorem 3.1.5 *If $\phi \in L^\infty(\partial\mathbb{A})$ is real-valued, and $f \in H^2$, $f \neq 0$, then $T_\phi f = 0 \Leftrightarrow f^{-1} \in H^2$, and $\phi|f|^2 = k\omega$, with $k \in \mathbb{C}$.*

Proof

To show the $\Rightarrow$ direction of the statement, we have from the previous that $\phi|f|^2 = k\omega$ on the boundary of the annulus. Thus we have $|f|$ is bounded away from 0 (since ϕ is bounded), and so $f^{-1} \in L_2$.
We have that $L_2(\mathbb{A}) = H^2 \oplus \bar{H}^{2'} \oplus \langle\omega\rangle$, and so $f^{-1} = h_1 + \overline{h_2} + \mu\omega$, $\mu \in \mathbb{C}$ (where $H^{2'}$ is taken to be the space quotiented out by the constant functions).
By assumption $\langle T_\phi f, g\rangle = 0 \forall g \in H^2$.
Then

$$\langle T_\phi f, g\rangle = \langle \phi f \overline{f f^{-1}}, g\rangle = \langle k\omega, (h_1 + \overline{h_2} + \mu\omega)g\rangle.$$

We have that $k \neq 0$ if $\phi \neq 0$. Taking $g = 1$, we see that $\mu = 0$, so $f^{-1} = h_1 + \bar{h_2}$.
We have that $\langle\omega, h_1 g\rangle = 0\ \forall g \in H^2$ (since ω is orthogonal to all H^1 functions) , and so

$\langle k\omega, \overline{h_2}g\rangle = 0\ \forall g$.

Dropping the k ($k \neq 0$ unless $\phi = 0$, the trivial case), we have that $\langle \omega h_2, g\rangle = 0 \forall g \in H^2$, that is to say, $T_\omega h_2 = 0$.

Noting that $T_\omega = -2(T_{\mathbf{1}_{\Gamma_1}} - \frac{1}{2}I)$, we have that $T_{\mathbf{1}_{\Gamma_1}} h_2 = -\frac{1}{2}h_2$.

However, the eigenvalues and eigenfunctions for $T_{\mathbf{1}_{\Gamma_1}}$ are described in [40], and we thus have that $h_2 = A$ for some constant A. However, we quotient out by the constant functions, so $A = 0$. Thus $f^{-1} = h_1$, i.e. $f^{-1} \in H^2$.

For the $\Leftarrow$ direction of the statement:

If f is invertible in H^2, and $\phi|f|^2 = k\omega$, then $f^{-1} \in H^\infty$, since ϕ is a bounded function.

Now $\langle \phi f, g\rangle = \langle \phi f\bar{f}, f^{-1}g\rangle = 0\ \forall g \in H^\infty$, since multiplication by H^∞ functions maps H^∞ to itself, and $\phi f\bar{f}$ is orthogonal to the analytics by assumption.

Since H^∞ is dense in H^2, we thus have $T_\phi f = 0$.$\square$

Since $T_1 = I$, we have the following:

Theorem 3.1.6

$$\textit{For } f \neq 0, \phi \neq 0, T_\phi f = \lambda f \iff f^{-1} \in H^2 \textit{ and } |f|^2(\phi - \lambda) = k\omega \textit{ some } k \in \mathbb{R}.$$

Thus, the problem reduces to finding when f exists with the appropriate modulus on the boundary. Since f invertible immediately implies that f is an outer function, and outer functions with modulus bounded away from 0 are invertible, we can restrict our attention to the problem of finding outer functions with the appropriate modulus on the boundary.

Considering the space of modulus automorphic functions, there will exist a modulus automorphic function with given boundary values (provided that $\int_{\partial A} \log|f|$ is well-defined and bounded), and its character, κ, is given by

$$\kappa = \frac{1}{2\pi \log s}[\int_0^{2\pi} \log|f(e^{it})|dt - \int_0^{2\pi} \log|f(se^{it})|dt].$$

This is proven in [38], p35 (Since here we are consider character in $[0,1)$, the extra 2π factor appears).

So, we have:

Lemma 3.1.7 *λ is an eigenvalue for T_ϕ if and only if $\kappa_\lambda \in \mathbb{Z}$, where*

$$\kappa_\lambda = \frac{1}{2\pi\log s}\int_0^{2\pi} \log|\sqrt{\frac{1}{\phi(e^{it})-\lambda}}| - \log|\sqrt{\frac{1}{\lambda-\phi(se^{it})}}|dt$$
$$= \frac{1}{4\pi\log s}\int_0^{2\pi} -\log|(\phi(e^{it})-\lambda)| + \log|(\lambda-\phi(se^{it}))|dt.$$

In this case the eigenfunction is given by the unique (up to a constant factor) outer function f in the annulus with boundary values given by $|f|^2 = \pm\frac{\omega}{\phi-\lambda}$, choosing sign appropriately. Such an eigenfunction can be constructed by constructing the outer function in the strip $\{0 < Re(z) < 1\}$ (or disc) with appropriate modulus on the boundary, given by an appropriate integration on the boundary. The outer functions whose modulus on the boundary is invariant are all modulus invariant inside the disc/strip. When the conditions are met, they will be actually invariant, rather than merely modulus invariant, and so can be passed back to the annulus, giving the eigenfunction with eigenvalue λ by the previous results.

So, combining the results, we can describe the eigenvalues and eigenfunctions for the self-adjoint Toeplitz operators on the annulus fully as follows:

Theorem 3.1.8 *For ϕ, a real-valued symbol, we have $\lambda \in \sigma_p(T_\phi)$ if and only if*

$$\sup\phi|_{\Gamma_0} \leq \lambda \leq \inf\phi|_{\Gamma_1}(\text{ or } \sup\phi|_{\Gamma_1} \leq \lambda \leq \inf\phi|_{\Gamma_0}) \text{ and } \kappa_\lambda \in 2\pi\mathbb{Z}.$$

Furthermore, if these conditions hold, λ is an eigenvalue of multiplicity 1, f is the associated eigenfunction if f is a constant multiple of the unique outer function such that $|f|^2(\phi-\lambda) = \pm\omega$, choosing sign appropriately.

Proof

If we are not in either of the cases $\phi|_{\Gamma_0} \leq \lambda \leq \phi|_{\Gamma_1}($ or $\phi|_{\Gamma_1} \leq \lambda \leq \phi|_{\Gamma_0})$ then we have that the sign of $\phi - \lambda$ is not constant on some boundary, and thus $(\phi - \lambda)|f|^2 \neq k\omega$ for any k, a contradiction by Theorem 3.1.6.

Supposing now this condition does hold, then $T_\phi f = \lambda f$ if and only if f is an outer function with $|f| = \sqrt{k\frac{w}{\phi-\lambda}}$.

Since the outer function with given modulus on the boundary is unique up to constant multiples, we have that f is uniquely determined up to multiplication by constants, i.e. the eigenvalue has multiplicity one.

Finally, by the result of [38] p.35, we have that such an f is a single-valued function on the annulus if and only if κ_λ is an integer, since otherwise the appropriate modulus automorphic function will not correspond to a single valued function in the annulus. □

Theorem 3.1.9 *Let ϕ be a real-valued bounded function on our boundary, as before. Assuming that* $\sup \phi|_{\Gamma_0} < \inf \phi|_{\Gamma_1}$*, we have that infinitely many eigenvalues accumulate at* $\sup \phi|_{\Gamma_0}$ *if and only if* $\int_{\Gamma_0} \log|(\sup \phi|_{\Gamma_0} - \phi)| = \infty$*, and infinitely many accumulate at* $\inf \phi_{\Gamma_1}$ *if and only if* $\int_{\Gamma_1} \log|(\phi - \inf \phi|_{\Gamma_1})| = \infty$.

Proof

Note that κ_λ is a strictly increasing (decreasing) function in the case $a \leq b \leq c \leq d$ $(c \leq d \leq a \leq b)$, for $b \leq \lambda \leq c$ Assuming we are in the case $a \leq b \leq c \leq d$, we thus have that if $\log|\phi(e^{it}) - b|$ is not integrable, $b \neq c$, then one must have infinitely many eigenvalues accumulating at b, since $-\int_{\Gamma_0} \log|\phi(e^{it}) - \lambda| \to \infty$ as $\lambda \downarrow b$, and so $\kappa_\lambda \to -\infty$ thus there are infinitely many $\kappa_\lambda \in 2\pi\mathbb{Z}$ by continuity in λ.

In this case, b can not be an eigenvalue since any $f \in H^2$ has $\int_{\partial\mathbb{A}} \log|f| < \infty$. If f were to be an eigenfunction with eigenvalue b, then we would have $|f|^2(\phi - b) = C\omega$, and thus $|f|(z) = C|\phi(z) - b|^{-\frac{1}{2}}$ for $z \in \partial\mathbb{A}$, contradicting the requirement of log integrability.

If $\log|\phi(e^{it}) - b|$ is an integrable function, we cannot have eigenvalues accumulating at this point since κ_λ is strictly decreasing and bounded below by its value at b, so it can pass through only finitely many integer values between b and k, for any $b < k < c$ (likewise eigenvalues accumulate at c if and only if $\log(c - \phi(se^{it}))$ is not integrable, regardless of the behaviour at b.

If the log integral at $\lambda = b$ does exist, it is possible for b to be an eigenvalue if and only if κ_b is well-defined and an integer (even in the case $b = c$, provided the log integrals on both boundaries are finite as well as κ_b being an integer. As, for example, occurs at 0 in the case of $\phi|_{\Gamma_0} = -\phi|_{\Gamma_1}$ for any ϕ such that $\phi|_{\Gamma_0} \geq 0, \log\phi|_{\Gamma_0}$ integrable, and essinf $\phi|_{\Gamma_0} = 0$).

Thus one has infinitely many eigenvalues in the case where either $(b - \phi|_{\Gamma_0})$ or $(\phi|_{\Gamma_1} - c)$ are not log integrable, and $b \neq c$.

By using the conformal equivalence $z \to sz^{-1}$, or by repeating the above argument with some relabelling, it can be seen that in the case that $c \leq d \leq a \leq b$, we instead have accumulation of eigenvalues at $\sup\phi_{\Gamma_1}$ if and only if $\int_{\Gamma_1}(\sup\phi|_{\Gamma_1} - \phi) = \infty$, and similarly for accumulation at $\inf\phi|_{\Gamma_0}$. $\square$

3.1.2 Toeplitz operators on vector valued Hardy spaces

The analogue of a Toeplitz operator on the space of vector valued analytic function on the disc is multiplication by a matrix followed by compression to the Hardy space.

In this situation, the self-adjoint operators will be those for which the representing matrix is Hermitian.

In this setting we have that an extension of Lemma 3.1.3 still holds:

Lemma 3.1.10 *If ϕ is a bounded Hermitian symbol, $f \neq 0$, $\phi \neq 0$, then$T_\phi f = 0 \Rightarrow f^\dagger\phi f \in W$, where $W = H_\infty^\perp \bigcap \overline{H_\infty}^\perp$, and $\dagger$ denotes the conjugate transpose.*

Proof

Let $g \in H^\infty$, and f be such that $T_\phi f = 0$.

Then $\langle f^\dagger \phi f, g\rangle = \langle \sum_j \bar{f}_j (\phi f)_j, g\rangle = \sum_j \langle (\phi f)_j, g f_j\rangle = 0$ (since $(\phi f)_j \in H_2^\perp$.)

Now $\langle f^\dagger \phi f, \bar{g}\rangle = 0$ follows since ϕ is Hermitian. $\square$

However, this result is not so useful in the context of vector valued Toeplitz operators. Whilst a necessary condition, there appears to be no obvious way to make this sufficient. Consider for example $\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Here $0 \notin \sigma_p(T_\phi)$, yet for any f of the form $\begin{pmatrix} ig \\ g \end{pmatrix}$ for $g \in H^2$, the conditions of the lemma are met.

Whilst these difficulties occur in attempting to approach the point spectra of operators here, the essential spectrum is easier to deduce from the results attained in the scalar case, in the case where we have the components of the symbol to be continuous functions. The following result is similar to one proven for the polydisc in [16] p44.

Theorem 3.1.11 *Let ϕ be a continuous, hermitian symbol. Then we have:*
$\lambda \in \sigma_e(T_\phi) \Leftrightarrow 0 \in \sigma_e(T_{\det(\phi - \lambda I)})$.

Proof

Let $A \cong B$ be taken to mean $A = B + K$, with K a compact operator, i.e. $\cong$ is equality modulo the compacts.

We require the following well known lemma on congruence of Toeplitz operators, which can be found in [16] p38 for (scalar) Toeplitz operators on a multiply connected domain.

Lemma 3.1.12 $T_{\phi_1} T_{\phi_2} \cong T_{\phi_1 \phi_2}$ *for ϕ_1 and ϕ_2 continuous.*

It suffices prove the theorem in the case $\lambda = 0$.

The result follows from [28] Theorem 1.1, which shows invertibility of a matrix is equivalent to invertibility of the determinant under certain conditions:

Theorem 3.1.13 *Let K be some ring, $a_{mk} \in K$ (for $1 \leq m \leq n, 1 \leq k \leq n$), and suppose $[a_{mk}, a_{pq}] = 0$ all m, k, p, q, where $[a, b]$ denotes the commutator $[a, b] = ab - ba$. Then, for $a = [a_{mk}] \in K^{n \times n}$, we have a is invertible if and only if $\det a$ is invertible.*

Now, to apply this theorem here, we take K to be the Calkin algebra for $H^2(D)$, and apply to the matrix a_{mk}, where a_{mk} is T_{mk} modulo the compact operators.

We have that the condition on the commutators is met, by application of Lemma 3.1.12.

The result follows. □

From this we can deduce when a given vector-valued Toeplitz operator will be diagonalizable, as this condition is equivalent to the essential spectrum consisting of discrete points for a self-adjoint operator on a separable Hilbert space.

Restricting to the case where $n = 2$, and supposing we have that $\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \bar{\phi}_{12} & \phi_{22} \end{pmatrix}$,

then we have $\det(\phi - \lambda I) = \lambda^2 - (\phi_{11} + \phi_{22})\lambda + \phi_{11}\phi_{22} - |\phi_{12}|^2$.

The essential spectrum for the Toeplitz operator with this symbol consists of the λ for which this is zero at some point on ∂D, that is

$$\lambda_{\pm} = ((\phi_{11} + \phi_{22}) \pm \sqrt{(\phi_{11} + \phi_{22})^2 + 4|\phi_{12}|^2 - 4\phi_{11}\phi_{22})})/2$$

is the condition for λ to be in the essential spectrum.

So a symbol with continuous values is diagonalizable if and only if $\lambda_{\pm}$ are both constant on each boundary of D. Thus $(\phi_{11} + \phi_{22})$ is constant on each boundary, as is $(\phi_{11} - \phi_{22})^2 + 4|\phi_{12}|^2$.

As an example, we have that in the case where D is the unit disc, T_ϕ, with $\phi = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ is diagonalizable, since it has essential spectrum $\{0, 1\}$.

Theorem 3.1.11 cannot be extended to the case of a Toeplitz operator with symbol with discontinuous boundary values.

Consider the operator with symbol $\phi = \begin{pmatrix} \mathbf{1}_A & 0 \\ 0 & \mathbf{1}_{A^C} \end{pmatrix}$ where A is a subset of one component of the boundary, with $0 < m(A) < 1$.

This has $\det(\phi - \lambda I) = \lambda^2 - \lambda$, and 0 is in the essential spectrum of $T_{\lambda^2 - \lambda}$ if and only if $\lambda = 0$ or 1.

However, T_ϕ has essential spectrum $[0, 1]$ (since its symbol is diagonal, simply consider the essential spectrum of $T_{\mathbf{1}_A}$ and $T_{\mathbf{1}_{A^c}}$, which is known to be $[0, 1]$ for Toeplitz operators on the disc). Thus the result cannot be extended to operators with discontinuous symbols.

3.1.3 Toeplitz operators on higher genus annuli

This section will focus on scalar-valued functions once again, this time considering how these results change when rather than considering Toeplitz operators on the boundary of the set $\{z : s \leq |z| \leq 1\}$, instead considering a region D with g removed components.

The result of Lemma 3.1.3 still holds in this setting, however the space ${H_\infty}^\perp \bigcap \overline{H_\infty}^\perp$ is larger— for an annulus of genus g, this space will be g dimensional (shown by Abrahamse in [2], which also gives a basis for this space in the case of harmonic, rather than uniform measure— Theorem 1.6 of [2]).

One consequence is that our eigenfunctions need no longer be invertible, instead we have the following:

Lemma 3.1.14 *Let D be our genus-g annulus. Suppose ϕ is a bounded, real valued function defined on ∂D, $0 \notin \sigma_e(T_\phi), T_\phi f = 0$ for some $f \in H^2(D)$.*

Then f has at most $g - 1$ zeros in the interior of D.

Proof

We need the following lemma about the Fredholm index of a Toeplitz operator:

Lemma 3.1.15 *Suppose that T_ϕ is Fredholm, with Fredholm index n. Then $T_{(z-a)\phi}$ has Fredholm index $n-1$ when $a \in \mathbb{A}$.*

Proof

We have that T_{z-a} is Fredholm (by Abrahamse's reduction theorem, for instance, and using continuity on the boundary), with index -1 when $a \in D$ (the Fredholm index for Toeplitz operator with bounded analytic symbol is given in [2] p278 as the number of zeros inside D).

Now, $T_{(z-a)\phi} = T_{z-a}T_\phi$, so $Ind(T_{(z-a)\phi}) = Ind(T_{z-a}) + Ind(T_\phi) = n-1$. □

Now, suppose we have a real-valued symbol ϕ, f and $a_i \in D$ such that $T_\phi f = 0$, and $f = (z-a_1)\ldots(z-a_g)h$.

Then we have that $T_{\phi(z-a_1)\ldots(z-a_n)}h = 0$, and this operator will have Fredholm index $-n$ (since T_ϕ has Fredholm index 0 from its self-adjointness).

Thus $T_{\overline{\phi(z-a_1)\ldots(z-a_n)}}$ has kernel of dimension at least $n+1$.

However, it is an old result from Abrahamse [2] that if $T_\phi f = 0$ and $T^*_\phi h = 0$, then $\phi f\bar{h} \in W$, with W the space orthogonal to the bounded analytics and anti-analytic functions in D.

Since W is g-dimensional (from [2]), we must therefore have $n \leq g-1$.

□

Lemma 3.1.16 *Let D be a genus-2 annulus. For all a in D there exists a Toeplitz operator T_ϕ, such that $(z-a)$ lies in the kernel of T_ϕ.*

We first need to establish a preliminary condition:

Lemma 3.1.17 *Suppose that $f = (z-a)h$, where h is invertible in H^2, and $\phi|f|^2 \in W$. Then we have that $T_\phi f = 0 \Leftrightarrow \langle \phi f, 1\rangle = 0$.*

Proof

We have that $T_\phi f = 0 \Leftrightarrow \langle \phi f, g\rangle = 0 \; \forall g \in H^\infty$ (since H^∞ dense in H^2).

Since $f = (z-a)h$, and $\phi|f|^2 \in W$, we have that $\langle \phi f, (z-a)g\rangle = \langle \phi|f|^2, gh^{-1}\rangle = 0$ $\forall g \in H^\infty$.

If $\langle \phi f, 1\rangle = 0$, then $\langle \phi f, g\rangle = 0 \; \forall g \in H^\infty$, since $g = \frac{g-g(a)}{z-a}(z-a) + g(a)$, and so: $\langle \phi f, g(a)\rangle = g(a)\langle \phi f, 1\rangle = 0$, and $\langle \phi f, \frac{g-g(a)}{z-a}(z-a)\rangle = \langle \phi f, (z-a)h\rangle = 0$ since $h = \frac{g-g(a)}{z-a} \in H^\infty$. $\square$

Now, we can prove the initial lemma:

Proof

Let w_1 and w_2 be two basis vectors for our space W, and consider $\phi = \frac{xw_1+yw_2}{|z-a|^2}$, with x and y to be determined later.

Then it follows immediately that $\phi|z-a|^2 \in W$.

Given $\phi = \frac{xw_1+yw_2}{|z-a|^2}$ we have two linear degrees of freedom in choosing ϕ, and the requirement $\langle \phi(z-a), 1\rangle = 0$ is a linear function of ϕ, we can always find non-trivial choice of x and y such that the condition $\langle \phi f, 1\rangle = 0$ is met.

$\square$

It can thus be seen that eigenfunctions exist which are no longer invertible.

It will be proven later that if some function in the kernel of a (not-necessarily self-adjoint) Toeplitz operator has non-trivial singular inner component, then this kernel must be infinite-dimensional, and so for the g-dimensional annulus we can restrict our attention to functions $(z-a_1)\dots(z-a_i)h$ with h invertible and $i \leq g$.

A special case

Whilst the description of the space W is considerably more complicated in even the $g = 2$ case than it was for the simple annulus with both circles centred at 0, some of the behaviour of the eigenvalues can be worked out quite easily for a particular operator on a genus-2 annulus with certain symmetry, and assuming we define our norm by integrals with respect to the harmonic measure. So, recall, we will denote our Toeplitz operators in this setting by $\mathcal{T}$.

Consider an annulus of genus 2, with inner radii of removed circles s, centres at $\pm r$.
Then, we have $W = \langle \omega_1, \omega_2 \rangle$, with $\omega_i \in L^\infty(D)$ defined to be the derivative with respect to the normal of the harmonic extension to D of the indicator function for Γ_i (as in [2]) for $i = 0, 1, 2$. (ω_0 is not needed in the basis for W since $\omega_0 + \omega_1 + \omega_2 = 0$ follows since this will be the derivative of the harmonic extension of the constant function on the boundary of D, which is clearly 0, since the harmonic extension of a constant is constant).
By symmetry under the map $z \to -z$, which maps Γ_1 to Γ_2 and vice-versa, we must have that $\omega_1(z) = \omega_2(-z)$, as well as $\omega_0(z) = \omega_0(-z)$.
So, if $\int_{\Gamma_0} \omega_0 = A$, then $\int_{\Gamma_0} \omega_1 = \int_{\Gamma_0} \omega_2 = -A/2$,
and $\int_{\Gamma_1} \omega_0 = \int_{\Gamma_2} \omega_0 = -A/2$, and letting $\int_{\Gamma_1} \omega_1 = B$, we have $\int_{\Gamma_1} \omega_2 = A/2 - B$.

Lemma 3.1.18 *Given real-valued $\phi \in L^\infty$, with ϕ bounded away from 0, there exists invertible $f \in H^2$ such that $\mathcal{T}_\phi f = 0$ if and only if there exist $a_0, a_1, a_2 \in \mathbb{C}$ such that:*

$$\int_{\Gamma_j'} \frac{\partial}{\partial n}(\frac{1}{2}\log(\widehat{\sum a_i\omega_i})) - \frac{\partial}{\partial n}(\widehat{\frac{1}{2}\log\phi})|dz| \in 2\pi\mathbb{Z}$$

for $0 \leq j \leq 2$,
with $\hat{s}$ denoting the harmonic extension of a boundary function s, and Γ_j' a loop homotopic to Γ_j. If so, f is the outer function such that $\phi|f|^2 = \sum a_i\omega_i$.
If one has suitable convergence at the boundary, then we can simply consider Γ_j instead

of Γ_j'.

A single redundant condition exists since Γ_0 *is homotopic to* $\sum_{i=1}^{2} -\Gamma_i$, *(since the integrals come from* [26] *and correspond to the change in argument of our multi-valued function when traversing the contour, and so are constant when distorting the contour by a continuous deformation.)*

Proof

We have that f is a 0-eigenfunction for $\mathcal{T}_\phi$ if $\phi|f|^2 \in W$ and f is invertible in H^2: $\langle \mathcal{T}_\phi f, g\rangle_h = \langle \phi f, g\rangle_h = \langle \phi|f|^2, fg\rangle = 0$ for all $g \in H^\infty$ (since $\phi|f|^2$ orthoognal to the analytics), and we have $fH^\infty = H^\infty$ (since f invertible).

So, it is enough to seek for $\phi|f|^2 = \sum a_i\omega_i$ for some a_i with f an outer function. If we take u to be the harmonic extension of $\log((\frac{\sum a_i\omega_i}{\phi})^{\frac{1}{2}})$ from the boundary to D, v to be its conjugate, and $f = e^{u+iv}$, we want f to be single-valued, and so v must have period conjugate to 0 mod 2π about each component of boundary. From Khavinson's work on conjugate functions in multiply connected domains [26], if u is a harmonic function in D with harmonic conjugate v, v has period around Γ_j of $\Delta_{\Gamma_j} = -\int_{\Gamma_j} \frac{\partial u}{\partial n}|dz|$ (with the derivative normal to the boundary). The result follows. $\square$

Now, taking our symbol $\phi = \mathbf{1}_{\Gamma_0}$, it follows from the definition of ω_i that for $0 < \lambda < 1$, $\frac{\partial}{\partial n}\widehat{\log(\phi - \lambda)} = \log(1-\lambda)\omega_0 + \log(\lambda)\omega_1 + \log(\lambda)\omega_2 + i\pi(\omega_1 + \omega_2)$.

Taking the result of the lemma, and evaluating on Γ_0 and Γ_1, we have:

$$(\int_{\Gamma_0} \frac{\partial}{\partial n}(\frac{1}{2}\widehat{\log(\sum a_i\omega_i)})|dz| - \frac{1}{2}(\log(1-\lambda)A - (\log\lambda)A) + i\pi A) \in 2\pi\mathbb{Z},$$

$$(\int_{\Gamma_1} \frac{\partial}{\partial n}(\frac{1}{2}\widehat{\log(\sum a_i\omega_i)})|dz| - \frac{1}{2}(\frac{-A}{2}\log(1-\lambda) + (\log\lambda)B + (\frac{A}{2} - B)\log\lambda) + i\pi\frac{A}{2}) \in 2\pi\mathbb{Z}.$$

Let $C = \int_{\Gamma_0} \frac{\partial}{\partial n}(\frac{1}{2}\widehat{\log(\sum a_i\omega_i)})|dz|$, $D = \int_{\Gamma_1} \frac{\partial}{\partial n}(\frac{1}{2}\widehat{\log(\sum a_i\omega_i)})|dz|$, then:

$$(C - i\pi A) - A(\log\frac{1-\lambda}{\lambda}) \in 2\pi\mathbb{Z},$$

$$(D - i\pi A/2) + \frac{A}{2}(\log\frac{1-\lambda}{\lambda}) \in 2\pi\mathbb{Z}.$$

So, we have that $C + 2D \in 2\pi\mathbb{Z}$, and C and D depend only on a_i in $\omega|f|^2 = \sum a_i\omega_i$.
Since $\omega_0 + \omega_1 + \omega_2 = 0$, we can assume that $a_2 = 0$, and since scaling f by a constant has trivial effect, we can assume $|a_0|^2 + |a_1|^2 = 1$, and $a_1 \in \mathbb{R}$ and can consider a_0 as our only independent complex variable. Thus if we we fix an n and seek a_i such that $C + 2D = 2\pi n$, then we have a finite set of a_i which solve this. Taking one such set of a_i, and letting $E = (C + i\pi A)$, then eigenvalues corresponding to outer eigenfunctions exist when $E - A\log(\frac{1-\lambda}{\lambda}) = 2\pi m$, $m \in \mathbb{Z}$. So, we have that

$$\lambda_m = \frac{1}{1 + e^{\frac{E-2\pi m}{A}}}$$

parametrises the set of solutions with the chosen a_i.
In comparison, the eigenvalues in the one-holed annulus for ϕ the indicator of one component of the boundary are given in [40] to be $\frac{s^{2n}}{(1+s^{2n})}$.

3.2 Kernels of Toeplitz operators

We will consider once again the Hardy space, inner product and associated projections with respect to uniform Lebesgue measure. When considering Toeplitz operators with more general, not necessarily real-valued, symbols, some results were found when considering the kernel of a Toeplitz operator on the annulus.
Once again our annulus $\mathbb{A}$ is taken to be $\{z : s < |z| < 1\}$.

Theorem 3.2.1 *Let f in H^2 be continuous and non-zero on the boundary of $\mathbb{A}$, ϕ be continuous and non-zero on the boundary of $\mathbb{A}$, and f in the kernel of T_ϕ.*

Then we have that $\dim\ker(T_\phi) \geq \max(n, 1)$*, where n is the number of zeros of f inside of $\mathbb{A}$.*

Proof

We have that T_ϕ is Fredholm, and [38] gives the Fredholm index to be $i(T_\phi) = n(\phi_{\Gamma_1}, 0) - n(\phi|_{\Gamma_0}, 0)$, that is, the difference of the winding number about 0 on the inner and outer boundaries.

By definition, $i(T_\phi) = \dim\ker T_\phi - \dim\ker T_\phi^*$.

If f is in the kernel of T, and g in the kernel of T^*, then $\phi f \bar{g} = \omega$, (shown in [2]), and thus both kernels can only be non-zero if they are one-dimensional (since W is one-dimensional), and in this case we have index 0.

Thus, if $\ker T_\phi \neq 0$, then $i(T_\phi) \geq 0$, and if $i(T_\phi) > 0$, we have $i(T_\phi) = \dim\ker(T_\phi)$. If $T_\phi f = 0$, we have that $T_{\phi f} 1 = 0$, hence $i(T_{\phi f}) \geq 0$, however,

$$i(T_{\phi f}) = n(\phi f_{\Gamma_1}, 0) - n(\phi f|_{\Gamma_0}, 0) = n(\phi_{\Gamma_1}, 0) - n(\phi|_{\Gamma_0}, 0) + n(f|\Gamma_1, 0) - n(f|\Gamma_0, 0)$$

$= i(T_\phi) - n.$

Thus $i(T_\phi) \geq n$, i.e. we have a kernel with at least dimension n. $\square$

If f is non-zero and continuous on the outer boundary, we can construct a Toeplitz operator whose kernel contains f and has dimension n. Considering $\phi = \frac{\omega}{f}$, we have by computing winding numbers, $i(T_{\omega/f}) = n$, thus our kernel has dimension n if $n > 0$, dimension 1 if $n = 0$, and f lies in the kernel since $T_{\frac{\omega}{f}} f = P_{H^2}\omega = 0$.

Infinite-dimensional kernel given singular inner factors

In the disc, it is trivial to show that if $T_{\bar{f}}$ is an anti-analytic Toeplitz operator, with f containing a non-trivial singular inner factor, we have that $\ker T_f$ is infinite dimensional. In the setting of the annulus, it appears considerably more difficult to prove— the complication is caused by the fact that we can no longer take arbitrary powers of a singular inner function and still obtain single valued functions.

The following proof works for an arbitrary annulus, not just the single-holed case.

Theorem 3.2.2 *Let $f \in H^2(D)$ be a function with $f = f_i f'$ where f_i is a singular inner function, $f_i \neq 1$, and $f' \in H^2$.*
Then $\ker T_{\bar{f}}$ is infinite-dimensional.

Proof

Let s_o be an outer function with constant modulus on each boundary, and complementary character to f_i (existence is given by [2], p269 proposition 1.15. In the case of the annulus $\{s < |z| < 1\}$ we can take $s_o = z^\alpha$ some $\alpha \in [0, 1)$).

Since $T_{\bar{f}} = T_{\overline{f' s_o^{-1}}} T_{\overline{f_i s_o}}$, it is enough to show that $\ker T_{\overline{f_i s_o}}$ has infinite dimension.
To do so, first we show that $0 \in \sigma_e(T_{\overline{f_i s_o}})$. Use the multiplicative factorisation to factor $f_i s_0 = k_1 k_2$ with $k_1 \in H^\infty(\Delta)$, and $k_2 \in H^\infty(D \bigcup \Delta^c)$.
Proceed using Abrahamse's reduction theorem ([2] p.282)- $\sigma_e(T_\phi) = \bigcup \sigma_e(T_{\phi|_{\Gamma_i}})$.
We have that $T_{\overline{f_i s_o}|\Gamma_0} = T_{\bar{k_1}|_{\Gamma_0}} T_{\bar{k_2}|_{\Gamma_0}}$. Now, since k_2 is analytic on an open set around Γ_0, it is continuous on Γ_0, and since $|f_i s_o| = 1 = |k_1||k_2|$, with $|k_1|$ bounded, we have k_2 is non-zero, and thus $T_{\bar{k_2}|_{\Gamma_0}}$ is Fredholm invertible. Since we have that T_{k_1} is an anti-analytic Toeplitz operator with non-trivial singular inner component, it is well known that the kernel must be infinite-dimensional in the Hardy space on the unit disc, and so is non-Fredholm. Since the product of a Fredholm operator with a non-Fredholm operator is non-Fredholm, we have that $0 \in \sigma_e(T_{\overline{f_i s_o}})$.
Now, if an operator has finite-dimensional kernel, finite-dimensional co-kernel, and closed range, it must be Fredholm, thus if $T_{\overline{f_i s_o}}$ has finite-dimensional co-kernel and closed range, it must then have infinite-dimensional kernel.
Being an anti-analytic Toeplitz operator, the co-kernel must be trivial, so it remains only to show the range is closed. The range will in fact be H^2:
We have that $T_{\bar{s_o}\bar{f_i}}(s_o f_i)g = P_{H^2}(|s_o|^2 g) = T_{|s_o|^2} g$.
Now, we have by our choice of s_o that $|s_o|^2$ is a strictly positive constant on each boundary. Thus, by the result on essential spectrum, $T_{|s_o|^2}$ will be a Toeplitz operator whose essential spectrum consists of discrete points, all of which are positive. Furthermore,

its eigenvalues will be strictly positive. So, it is an invertible operator, and thus given any h, we can find g such that $T_{|s_o|^2}g = h$, and so $T_{\bar{s_o}\bar{f_i}}(s_o f_i g) = h$. Thus the range is closed, and therefore $T_{\bar{s_o}\bar{f_i}}$ has infinite-dimensional kernel, as does $T_{\bar{f}}$.

□

The only results used here were our factorisation (which still holds on changing to an equivalent norm), and the reduction theorem, which also remains true changing to equivalent norms, so this will still hold taking Toeplitz operators defined on our space with equivalent norms, i.e. it will hold true with harmonic norm on our boundary.
We can proceed from here to show a further result on kernels of self-adjoint Toeplitz operators. The proof requires use of harmonic measure on the boundary of our domain D.

Theorem 3.2.3 *In the Hardy space of the annulus, with norms and inner product given by integration with respect to a harmonic measure (as is the case in the construction of $H^2(D)$ by harmonic majorants), if ϕ is real-valued, $\phi \in L^\infty(\partial D)$, and $\mathcal{T}_\phi f = 0$, then f has trivial singular inner component.*

We first need to establish a preliminary result on the kernel that was proven to be infinite dimensional in Theorem 3.2.2

Lemma 3.2.4 *Let $\mathcal{T}_\phi$ be a Toeplitz operator where ϕ is anti-analytic with non-trivial singular inner component. Then there exists a sequence $(s_j)_{j=1}^\infty$ such that $s_j \perp s_k$ for $j \neq k$, $s_j \in H^\infty$, and $\mathcal{T}_\phi s_j = 0$.*

Once this lemma is established, we can prove our theorem as follows:

Proof

Since we are using harmonic measure here, there exists ν such that $H_2^\perp = \bar{\nu}^{-1}\bar{H}^2$, and ν is meromorphic on some open set containing the closure of Δ, with n zeros and 1 pole ([2] p263). Thus $\nu = \frac{(z-a_1)...(z-a_n)}{(z-b_1)} v_o$, where $v_o \in H^\infty$, $v_o^{-1} \in H^\infty$.

Suppose $\mathcal{T}_\phi f = 0$ with ϕ real-valued and f having non-trivial singular inner component. Suppose $f = f' g_o$, where f' has constant modulus on each boundary, and g_o outer, and suppose $f = f_i f_0$ is an inner-outer factorisation of f. Let (s_j) be the bounded sequence of orthogonal functions from Lemma 3.2.4, $\mathcal{T}_{\bar{f}'} s_j = 0$. Now, we have that $\bar{f}' s_j = \bar{\nu}^{-1} \bar{t_j}$.
We have that $\phi f \in \bar{\nu}^{-1} \bar{H}^2$, thus $\phi f \bar{t_j} \in \bar{\nu}^{-1} \bar{H}^2$.
Now, $f \bar{t_j} = f \bar{\nu} \bar{f}' s_j = f_i f_o g_o \bar{\nu} \bar{f_i} \bar{f_o} s_j$,
$= f_o g_o |f_o|^2 \frac{\overline{(z-a_1)\ldots(z-a_n)}}{\overline{(z-b_1)}} \bar{\nu_o} s_j$.
So we have that

$$P^h_{H^2}(\phi |f_o|^2 \overline{(z-a_1)\ldots(z-a_n)} f_o g_o \frac{1}{\overline{(z-b_1)}} s_j \bar{\nu_0})$$
$$= \mathcal{T}_{\bar{\nu_0}} \mathcal{T}_{\overline{(z-a_1)\ldots(z-a_n)}} \mathcal{T}_{\frac{\phi |f_o|^2}{(|z-b_1|^2)}} f_o g_o (z-b_1) s_j = 0$$

using $\frac{1}{\overline{z-b_1}} = \frac{z-b_1}{|z-b_1|^2}$, and the identities $\mathcal{T}_{\bar{h}} \mathcal{T}_\phi = \mathcal{T}_{\bar{h}\phi}$ for $h \in H^\infty$. By our choice of s_j, the $f_o g_o (z-b_1) s_j$ are a linear independent sequence. Since $\mathcal{T}_{\bar{h}}$ has 0-kernel for h an outer function (follows simply from hH^∞ dense for h outer), and $\mathcal{T}_{\overline{(z-a_1)\ldots(z-a_n)}}$ has n-dimensional kernel, it follows that $\mathcal{T}_{\frac{\phi |f_o|^2}{|z-b_1|^2}}$ has infinite dimensional kernel. However, $\mathcal{T}_{\frac{\phi |f_o|^2}{|z-b_1|^2}}$ is a self-adjoint Toeplitz operator, and thus has finite dimensional kernel. Thus we arrive at a contradiction from our assumptions, thus any f in the kernel of a self-adjoint Toeplitz operator $\mathcal{T}$ has trivial singular inner component.

□

It remains to prove the lemma:

Proof

Let $f = uf''$, where $u \in H^\infty(D)$ has non-trivial singular inner component, $|u|$ bounded away from 0 and ∞ on each boundary, $u \neq 0$ in D, and $f'' \in H^2(D)$. Let $u = u_i u_o$ be the inner-outer factorisation of u. Since $\mathcal{T}_{\bar{f}} = \mathcal{T}_{\overline{f''}} \mathcal{T}_{\bar{u}}$, it is enough to establish the lemma for $\mathcal{T}_u$.
We have that $\ker \mathcal{T}_{\bar{u}} = K_u = H^2 \ominus uH^2$.
To find the required sequence, we must consider the reproducing kernels for our spaces. Let k_λ be the reproducing kernels for $H^2(D)$, f_λ for uH^2 and g_λ for K_u. If we can show

that g_λ are bounded, the lemma will follow. We have that $k_\lambda \in H^\infty(\mathbb{A})$ for all λ— see, for example, Ball and Clancey paper on reproducing kernels [8].
Thus, we have that g_λ bounded follows from f_λ being bounded.

Suppose that we have some λ for which f_λ is unbounded, let $f = f_\lambda$. Let (z_n) be a sequence for which $f(z_n) \to \infty$.
Since u is a bounded function, we have $|(f/u)(z_n)| \to \infty$ also.
Let B_1 be a finite Blaschke product such that B_1 has the same character as u_i, and that $B_1(\lambda) \neq 0$.
We have that $|(f_\lambda u_o B_1/u)(z_n)| \to \infty$ still, and thus $|\langle \frac{f_\lambda u_o B_1}{u}, k_{z_n}\rangle_h| \to \infty$.
We have that $|u_i/B_1| = 1$ so $|\langle \frac{u_i}{B_1}\frac{f_\lambda u_o B_1}{u_o u_i}, k_{z_n}\frac{u_i}{B_1}\rangle_h| \to \infty$.
Thus $|\langle f_\lambda, k_{z_n}\frac{u_i}{B_1}\rangle_h| \to \infty$.
Now, were we to have that B_1 was a factor of k_{z_n}, we would have $k_{z_n}\frac{u_i}{B_1} \in uH^2$, and so a contradiction would follow since f_λ is a reproducing kernel there— we would have:
$\langle k_{z_n}\frac{u_i}{B_1}, f_\lambda\rangle_h = k_{z_n}(\lambda)\frac{u_i}{B_1}(\lambda) = \bar{k_\lambda}(z_n)\frac{u_i}{B_1}(\lambda)$, and we have already that k_λ is bounded, giving a contradiction.
Since we cannot assume this, suppose that B_1 has zeroes at $\omega_1, \omega_2, \ldots \omega_m$, and that these are distinct points.
We seek to find ${}_{z_n}a_1, \ldots {}_{z_n}a_m$ such that $k_{z_n}(\omega_i)u_i(\omega_i) + \sum_{j=1}^m {}_{z_n}a_j u^j(\omega_i) = 0$ for each i.
This is the problem, given $b_i = k_{z_n}(\omega_i)u_i(\omega_i)$, of finding a polynomial P_m which takes the values b_i at the points $u(\omega_i)$.
Now, we require $u(\omega_i)$ to be a distinct set. Supposing they are not, then, we can replace the factorisation $f = uf''$ with $f = (u(z-t)^N)(f''(z-t)^{-N})$, where $t \notin D$, and $N \in \mathbb{Z}$. $u(z-t)^N$ will still be single-valued, H^∞, non-zero in D and the character of u_i is unchanged, and $f''(z-t)^{-N}$ also in H^∞, so we can apply the previous parts of the proof, with ω_i are unchanged, and there will always exist a choice of N and t such that $u(\omega_i)(\omega_i - t)^N$ are distinct points (for instance, if we choose t such that $|\omega_i - t|$ are unique, we can then choose N such that $|(\omega_i - t)^N u(\omega_i)|$ are all unique, and it follows

that $(\omega_i - t)^N u(\omega_i)$ are unique).

Thus, we can assume without loss of generality that $u(\omega_i)$ are distinct points.

Then, since ω_i are fixed, and the b_i lie in a bounded set (since $k_z(\omega) = \overline{k_\omega(z)}$ is bounded over all z for fixed ω), we can solve this with a universal bound on ${}_{z_n}a_i$ independent of z_n.

We thus have that $\langle \frac{k_{z_i}u_i}{B_1}, f_\lambda \rangle_h = \langle \frac{k_{z_n}u_i - \sum_{z_n} a_j u^j}{B_1} + \frac{\sum_{z_n} a_j u^j}{B_1}, f_\lambda \rangle_h$.

Since the ${}_{z_n}a_j$ are bounded over all z_n, we have

$$\|\frac{\sum_{j=1}^m {}_{z_n}a_j u^j}{B_1}\|_2 \leq \|\frac{\sum_{j=1}^m {}_{z_n}a_j u^j}{B_1}\|_\infty < A_1$$

is a universal bound over all z_n.

We also have that $\frac{k_{z_n}u_i - \sum_{z_n} a_j u^j}{B_1} \in uH^2$, so that

$$\langle \frac{k_{z_n}u_i - \sum_{z_n} a_j u^j}{B_1}, f_\lambda \rangle_h = (\frac{k_{z_n}u_i - \sum_{z_n} a_j u^j}{B_1})(\lambda),$$

which is bounded over all z_n since $k_{z_n}(\lambda)$ and ${}_{z_n}a_j$ are.

Thus, we have that $f_\lambda(z_n)$ must be bounded.

Since $f_\lambda \in H^\infty$, we have $g_\lambda \in H^\infty$.

Since the reproducing kernels have dense closed span, and we have already established that the space is infinite dimensional, we can pick out a sequence with no linear dependences. Applying Gram-Schmidt we can produce a suitable orthonormal sequence.

□

We would like to show this result also holds for Toeplitz operators defined on $H^2(D)$ with respect to uniform lebesgue measure on D.

In fact, this follows directly from the result with harmonic measure.

Note that if we consider T_ϕ a Toeplitz operator symbol ϕ with respect to projection defined on the space with uniform Lebesgue measure, and take $\mathcal{T}_\phi$ to be a Toeplitz operator with respect to harmonic measure, we have that $T_{\phi \frac{dm}{|dz|}} f = \mathcal{T}_\phi f$ for all functions $f \in H^2$. ($\frac{dm}{|dz|}$ is bounded since our measures are equivalent, and if $\langle ., . \rangle$ is the inner product with respect to uniform lebesgue measure, and $\langle ., . \rangle_h$ with respect to harmonic measure, we have:

$\langle T_{\phi\frac{dm}{|dz|}} f, g\rangle = \int_{\partial D} \phi \frac{dm}{|dz|} f\bar{g}|dz| = \int_{\partial D} \phi f\bar{g} dm = \langle \phi f\bar{g} dm\rangle_h = \langle \mathcal{T}_\phi, g\rangle_h)$.
Now, by [2] p263 (3), we have that $\frac{dm}{|dz|}$ is positive, and since our measures are absolutely continuous, it is bounded away from 0 and ∞.
It follows that we can extend the previous theorem to the space with Lebesgue measure.

Theorem 3.2.5 *Considering the norm and projections on H^2 with respect to uniform lebesgue measure, we have that if $\phi \in L^\infty(\partial D)$ real-valued, $T_\phi f = 0$, then f has trivial singular inner component.*

Proof
Let $\phi' = \phi\frac{|dz|}{dm}$. We will have that ϕ' is real-valued, and bounded, and $T_\phi f = 0$ if and only if $\mathcal{T}_{\phi'} f = 0$. The result thus follows from the case with respect to harmonic measure. □
We shall consider an annulus to have uniform Lebesgue norm on the boundary for the remainder of this thesis. Combining this result, Lemma 3.1.14, and Lemma 3.1.3 we can state the following on the eigenfunctions of Toeplitz operators with real valued symbol:

Theorem 3.2.6 *Considering Toeplitz operators in $H^2(D)$ with respect to uniform lebesgue measure, let $\phi \in L^\infty(D)$ be a real-valued function, and suppose we have $f \in H^2(D)$ such that $T_\phi f = \lambda f$ some $\lambda \in \mathbb{R}$. Then we have that $f = f_o B$ where B is a finite Blaschke product, of order at most $g-1$, where g is the genus of the annulus D, and f_o is an outer function with $\phi|f|^2 \in H_\infty^\perp \bigcap \bar{H_\infty}^\perp$.*

3.2.1 Minimal kernels

The paper [12] discusses minimal kernels of Toeplitz operators containing a given function in the setting of the Hardy space of the disc, and the following results concern attempts to reproduce some of these results in the annulus.

Theorem 3.2.7 *If $f = f_o f_i$ is the inner outer factorisation of $f \in H^2(\mathbb{A})$, then the symbol of a Toeplitz operator with minimal kernel containing f is $\psi = \omega \bar{f}_i \frac{\bar{f}_o}{f_o} \bar{z}^{-\alpha}$, with α being the character of f_o.*

(In comparison, the corresponding symbol for an operator with minimal kernel in the setting of the disc is known to be $\bar{f}_i \frac{\bar{f}_o}{f_o}$.)

Proof

Let $f = f_i f_o$ be an H^2 function with α the character of f_o, and ϕ some symbol such that $T_\phi f_i f_o = 0$.

Then, we must have that $\phi f_i f_o = \omega \overline{g_i g_o}$.

Let v be an outer function that $|\phi| = v$ on Γ_0, and $|\phi| = Av$ on Γ_1. Then we have that $T_\phi = T_{\bar{v}} T_{\phi'}$, with $|\phi'|$=1 on Γ_0, $|\phi'| = A$ on Γ_1 Since $\ker T_{\bar{v}} = \{0\}$, we have $\ker T_\phi = \ker T_{\phi'}$, so we may assume w.l.o.g.that $|\phi| = 1$ on Γ_0, and $|\phi| = A$ on Γ_1 for some A.

So $|g_o| = |f_o|$ on Γ_0, $|g_o| = A|f_o|$ on Γ_1, that is, we have $\beta \in \mathbb{R}$ such that $|f_o z^\beta| = |g_o|$ (on the boundary).

Since modulus on the boundary determines an outer function up to constant factors, we can take $g_o = f_o z^\beta$. We will thus have the character of g_0 is the character of f_o plus β, i.e. $\alpha + \beta$.

So, we have that $\phi = \omega \frac{\bar{f}_o}{f_o} \overline{g_i f_i} \bar{z}^\beta$.

Now, suppose that h lies in the kernel of T_ψ.

Then $\omega \bar{f}_i \frac{\bar{f}_o}{f_o} \bar{z}^{-\alpha} h = \omega \overline{l_o l_i}$. Thus h=$\overline{l_o l_i} f_i \frac{f_o}{\bar{f}_o} \bar{z}^\alpha$,

and so ϕh=$\overline{l_o l_i} f_i \frac{f_o}{\bar{f}_o} \bar{z}^\alpha \omega \frac{\bar{f}_o}{f_o} \overline{g_i f_i} \bar{z}^\beta = \omega \overline{l_o l_i g_i z^{\beta+\alpha}}$. Since g_o has character $\alpha + \beta$, g_i must have character $(-(\alpha + \beta))$, and so $l_o l_i g_i z^{\beta+\alpha}$ is single valued.

It follows that $\phi h \in \omega \bar{H}^2$, thus $T_\phi h = 0$.

Hence, $\ker T_\phi \supset \ker T_\psi$, so ψ is the symbol with minimal kernel. $\square$

This can be extended to cover an arbitrary g-holed annulus.

Lemma 3.2.8 *If we have ϕ and f such that $T_\phi f = 0$, then $\ker T_\phi \supset \ker T_\psi$, where ψ is the function $\psi = \frac{\overline{f_i f_o}}{f_o}\bar{l}_f \bar{v}$, with l_f the unit, i.e. outer function with constant modulus on each boundary, such that character$(l_f) = -$character(f_o), and ν the measure such that $H_2^\perp = \bar{\nu}\bar{H}^2$.*

Proof

First, given any ϕ, if ϕ is not log integrable, it has zero kernel, and the result is trivial (since ϕf is not log integrable, whereas all of $H_2^\perp$ is. Otherwise, we have that there exists an outer function s such that $|\phi/s|$ is constant on each boundary (the existence of such an s follows from the existence of a (multiple-valued) outer function with given boundary values, and the work in [2]).

Then, we have that $\phi = \bar{s}\phi'$, and so $T_\phi = T_{\bar{s}} T_{\phi'}$. Since $T_{\bar{s}}$ has zero kernel, we can restrict attention to $T_{\phi'}$, i.e. we can assume without loss of generality that our symbol has constant modulus on each boundary. We can then proceed as follows:

$$\phi f_i f_o = \bar{\nu}\bar{g}_i\bar{g}_o, \text{ with } |g_o| = |\phi|\frac{|f_o|}{|\nu|}.$$

So, $g_o = f_o l_\phi \mu$, where μ is the outer function such that $|\mu| = |\nu|^{-1}$, and l_ϕ is the outer function with modulus equal to that of ϕ on each boundary (which will be constant and non-zero on each boundary).

Substituting this into the formula for ϕ, we obtain

$$\phi = \overline{\nu g_i f_i l_\phi \mu}\frac{\bar{f}_o}{f_o}.$$

Suppose that we have h such that $T_\psi h = 0$, then we must show that $T_\phi h = 0$.

Since $T_\psi h = 0$, we have that $\psi h_i h_o = \bar{\nu}\bar{m}_i\bar{m}_o$, so:

$$\begin{aligned}\phi h_i h_o &= \overline{\nu g_i f_i l_\phi \mu}\frac{\bar{f}_o}{f_o} h_i h_o \\ &= \psi \bar{l}_f^{-1} \bar{g}_i \bar{l}_\phi h_i h_o \\ &= \bar{\nu}\bar{m}_i\bar{m}_o \bar{l}_f^{-1}\bar{l}_\phi \bar{g}_i.\end{aligned}$$

Since l_f is outer, with constant non-zero modulus on each boundary, we have that l_f^{-1} is also H^∞, and so $T_\phi h = 0$, as required. □

The next problem to consider is what the minimal kernel is for a Toeplitz operator containing a set of functions, which is a harder problem.

The following is a start to showing whether the existence of a non-trivial kernel containing a pair of functions exists reduces to showing this in the disc.

Lemma 3.2.9 *For $f_1, f_2 \in H^2(\mathbb{A})$, there exists non-trivial ϕ such that $T_\phi f_1 = 0$, $T_\phi f_2 = 0$ only if there exists $\phi' \in L_\infty(\partial\Delta)$ such that $T'_{\phi'}\tilde{f}_i = 0$ for each i, where tildes denote lifts back to the disc, and T' indicates that the Toeplitz operator acts on the Hardy space for the disc.*

Proof

By assumption, we have $\phi f_1 = \omega \bar{g}_1$ and $\phi f_2 = \omega \bar{g}_2$ for some g_1,g_2.

We have that $\bar{z}(\widetilde{\omega\phi})\tilde{f}_1 = \bar{z}\tilde{\bar{g}}_1$, and the same for $\tilde{f}_2$, so we have $\bar{z}(\widetilde{\omega\phi})$ is a symbol for a Toeplitz operator whose kernel contains $\tilde{f}_1$ and $\tilde{f}_2$. □

So, if f_1 and f_2 lie in a non-trivial Toeplitz kernel in the annulus, their exists a non-trivial Toeplitz kernel in the disc containing both $\tilde{f}_1$ and $\tilde{f}_2$. Thus we have a necessary condition for two functions in the Hardy space on the annulus to lie in a non-trivial kernel.

In fact, the same will work if, rather than a pair, we instead have an arbitrary set of functions lying in a non-trivial Toeplitz kernel in the annulus.
Sufficiency has yet to be ascertained.

3.2.2 Dyakonov's theorem in the annulus

In the paper of Dyakonov [19], the following theorem is proven:

Theorem 3.2.10 *Given $\phi \in L^\infty(\Delta)$, there exists B, b, g with B, b Blaschke products and g with $g, g^{-1} \in H^\infty$, such that*

$$\ker T_\phi = \frac{g}{b}(K_B \bigcap bH^2)$$

(K_B denoting the model space $H^2 \ominus BH^2$).

In the annulus, a similar result holds, though care has to be taken regarding the characters of functions in inner-outer factorisations, since these factorisations are no longer single-valued in the annulus. The theorem in the annulus becomes:

Theorem 3.2.11 *Given $\phi \in L^\infty(\mathbb{A})$, there exists B, b, g, α, β with B, b Blaschke products, g with $g, g^{-1} \in H^\infty$, $\alpha, \beta \in [0, 1]$ such that:*

$$\ker T_\phi = \frac{gz^{-\alpha}}{b}(bz^{\alpha-\beta}H^2 \cap K_{\omega z^\beta B}).$$

(α and β will be chosen such that $\frac{gz^{-\alpha}}{b}, bz^{\alpha-\beta}, \omega z^\beta B$ are all single-valued.

Proof

First assume that ϕ is log-integrable (else the kernel is zero), and that $|\phi| = 1$ on Γ_0, $|\phi| = A$ on Γ_1 (by writing $\phi = \bar{g}\phi'$ for g a suitable outer function).
Then, one has that $\omega\phi = \frac{\bar{g}}{g} z^\alpha b\bar{B}$, with $g, g^{-1} \in H^\infty$, and B, b Blaschke products. (This

follows from Theorem 2.1.6).

Letting β be the character of g, then B has character $-\beta$, and b will have character $\beta - \alpha$.

If $T_\phi f = 0$, we have that $\phi f \in \omega \bar{H}^2$, and thus $\omega\phi f \in \bar{H}^2$.

Then, $\frac{\bar{g}}{g} z^\alpha b \bar{B} f = \bar{s}$, and so $f z^\alpha \frac{1}{g} b = \bar{s} B \frac{1}{\bar{g}}$.

It follows that:

$$\frac{z^\alpha b}{g} f \in b g^{-1} z^\alpha H^2 \cap \frac{B}{\bar{g}} \bar{H}^2.$$

We have that $\frac{1}{g} = \frac{1}{z^\beta} \frac{z^\beta}{g}$, where $\frac{z^\beta}{g}$ is a single-valued analytic and outer. Since $z^{-\beta} g$ and $z^\beta g^{-1}$ are single-valued, lie in H^∞ and have inverse in H^∞, we have $z^{-\beta} g H^\infty = H^\infty = z^\beta g^{-1} H^\infty$, the previous becomes

$$\frac{z^\alpha b}{g} f \in b z^{\alpha-\beta} H^2 \cap \frac{B}{\bar{z^\beta}} \bar{H}^2.$$

Lemma 3.2.12 $\frac{B}{\bar{z^\beta}} \bar{H}^2 = K_{\omega B z^\beta}$.

Proof

Let $f \in B \bar{z}^{-\beta} \bar{H}^2$, i.e. $f = B z^{-\beta} \bar{g}$.

Then $\langle f, \omega B z^\beta s \rangle = \langle B z^{-\beta} \bar{g}, \omega B z^\beta s \rangle = \langle \bar{g}, \omega s \rangle = 0 \forall s \in H^2$.

Conversely, if $f \in H^2 \ominus \omega B z^\beta H^2$, then $\langle f, \omega B z^\beta s \rangle = 0 \forall s \in H^2$, so $\langle \omega \bar{B} \bar{z}^\beta f, s \rangle = 0 \forall s$, hence $\omega \bar{B} \bar{z}^\beta f = \omega \bar{g}$ for some g, hence $f = B \bar{z}^{-\beta} \bar{g}$. Thus $\frac{B}{\bar{z^\beta}} \bar{H}^2 = K_{\omega B z^\beta}$. $\square$

Thus we have that

$$f \in \frac{g z^{-\alpha}}{b} (b z^{\alpha-\beta} H^2 \cap K_{\omega z^\beta B}).$$

$\square$

3.2.3 Intertwining

On the Hardy space of the disc, one can characterise the Toeplitz operators as being those which fulfil a particular intertwining relationship- that T is a Toeplitz operator if $S^*TS = T$, where S is the shift operator (i.e. M_z) on the Hardy space for the disc.
It seems natural to ask if there exists any operator for which the same holds in the annulus, however this is not the case.

Theorem 3.2.13 *There does not exist an operator S such that:*
$S^*TS = T$ is equivalent to T being a Toeplitz operator on the Hardy space of the annulus $\mathbb{A}$.

Proof
To start the proof, we first show that no multiplication operator S has this property:

Lemma 3.2.14 $\forall \mu \in H^\infty(\mathbb{A})$, $\{T : M_{\bar{\mu}} T M_\mu = T\} \neq \{T : T$ *is Toeplitz*$\}$

Proof
Suppose that μ is such that $\{T : M_{\bar{\mu}} T M_\mu = T\} = \{T : T$ is Toeplitz$\}$. Then, applying to $T_1 = I$, we have that $M_{\bar{\mu}} M_\mu = I$, and so $|\mu|^2 = 1$ almost everywhere.
Thus μ is an inner function. It is then clear that $M_{\bar{\mu}} T M_\mu = T$ for all Toeplitz operators T, to prove a contradiction we must show that there exists a non-Toeplitz operator such that this holds.
Define A an operator on the subspace generated by μ in $H^2(\mathbb{A})$ by $A(1) = 1$, $A(\mu^n) = \mu^n$. Then A satisfies the required conditions on the subspace generated by the closed span of μ^n.
However, if A is a Toeplitz operator, we must have that A is uniquely determined— otherwise, we would have a Toeplitz operator T_ϕ with $\phi \neq 0$, and $T_\phi \mu^n = 0$ for all n,

leading to a contradiction as follows:

Suppose $T_\phi \mu^n = 0$, then $T_\phi 1 = 0$, and have that $\omega\phi = \bar{s}$ with s analytic.

However, $T_\phi \mu^n = 0$, so $\bar{s}\mu^n = \bar{t_n}$ for some analytic functions t_n, and so s is divided by arbitrarily large factors of μ. However, this is impossible for s and μ non-zero.

So, we have that the closed span of μ^n for $n \geq 0$ is H^2, as otherwise we could extend A to H^2 whilst preserving $M_{\bar{\mu}} A M_\mu = A$ in at least two different ways- firstly by defining A to be 0 on the orthogonal complement of the span of μ, and secondly by defining A to be the identity on the orthogonal complement of their span.

Now, if μ has singular inner factor or multiple zeros, this span cannot equal H^2, yet there are no single valued inner functions in $H^2(\mathbb{A})$ with only a single zero.

Thus there is no function μ with the required property in $H^\infty(\mathbb{A})$.

□

To extend this to the set of all operators on our Hardy space, note that $S^*IS = I$, so $S^*S = I$, and thus we have that $S^*SS = S$, so S must be a Toeplitz operator if the intertwining defines the Toeplitz operators. To complete the result, we must show it to be analytic.

Given that S is a Toeplitz operator, and letting μ be its symbol, we have from $S^*S = I$ that it must be an isometry (Since $T_{\bar{\mu}} T_\mu = I$)).

We must thus have $|\mu| \leq 1$ a.e., since $\|\mu\|_\infty = \|T_\mu\| = 1$ ([2] p276, theorem 2.11).

Now, $\|T_\mu f\|_2 \leq \|\mu f\|_2$ with equality if and only if μf is analytic. Thus, we have $\mu f \in H^2$ for all $f \in H^2$, and thus μ must be an analytic function. Combined with the lemma, no such function can exist. □

Chapter 4

Hankel Operators

4.1 Nehari for the annulus

This section will outline Hankel operators on multiply-connected domains, and how they differ from the Hankel operators on the disc.

We take our Hardy space $H^2(D)$ as defined in chapter 1, with uniform lebesgue measure on the boundary.

We define a Hankel operator Γ with symbol ϕ on $H^2(D)$ by $\Gamma f = P_{\omega\overline{H^2}}\phi f$.

ϕ is said to be an optimal symbol for Γ if $\|\phi\|_\infty \leq \|\psi\|_\infty$ for any ψ a symbol for Γ.

Theorem 4.1.1 *Let Γ be a Hankel operator, $\|\Gamma\| < \infty$. Then we have that there exists optimal symbol ϕ for Γ, with $\|\Gamma\| \leq \|\phi\|_\infty \leq A\|\Gamma\|$, where A is the constant in Theorem 2.3.2 for the factorisation of H^1 functions as a product of H^2 functions.*

Proof

The proof parallels the proof in the case of the disc, with the major difference being that the $H^1 = H^2H^2$ factorisation now has a constant introduced.

Let $\phi = \sum_{-\infty}^{\infty} a_n z^n + \sum_{-\infty}^{\infty} b_n \omega \overline{z^n}$ be a symbol for our operator, then we have that

$\langle \Gamma z^n, \omega \overline{z^m} \rangle = \langle \phi z^n, \omega \bar{z^m} \rangle = \langle \phi z^{n+m}, \omega \rangle = b_{n+m}(1 + s^{2(m+n)})$.

It follows that we can produce a linear functional α on products of polynomials by $\alpha(f_1 f_2) = \langle \Gamma(f_1 f_2), \omega \rangle = \langle \Gamma(f_2), \omega \bar{f_1} \rangle$, since we have that $|\alpha(f_1 f_2)| \leq \|\Gamma\| \|f_1\|_2 \|f_2\|_2 \leq A \|f_1 f_2\|_1 \Gamma$. We will have that $\|\alpha\| \leq A \|\Gamma\|$.

Since our operator is bounded, we can extend to H^1 by continuity, and then to L^1 by Hahn-Banach, without increasing norm.

We thus have that there exists h such that $\alpha(g) = \int_{\partial \mathbb{A}} g(z) h(z)$.

Taking the expansion $\omega h = \sum_{-\infty}^{\infty} j_i z^i + \sum_{-\infty}^{\infty} k_i \omega \bar{z}^i$ for some (j_i),(k_i) square summable sequences, we have that $k_i(1 + s^{2i}) = \langle \omega h, \omega \bar{z}^i \rangle = \int h(z) z^i = \alpha(z^i) = b_i(1 + s^{2i})$.

So ωh and ϕ differ by an analytic function, hence they are both symbols for the same Hankel operator.

From the integral representation of α, we have that $\|\omega h\|_\infty = \|h\|_\infty = \|\alpha\| \leq C\|\Gamma\|$. $\|h\|_\infty \geq \|\Gamma\|$ follows from being a symbol for Γ. To show that this is optimal, we must show that for any symbol ϕ of Γ, we have $\|\phi\| \geq \|\alpha\|$.

For $f \in H^\infty$, we have $|\alpha(f)| = |\langle \Gamma(f), \omega \rangle| = |\langle \phi f, \omega \rangle| \leq \|\phi\|_\infty |\langle f, \omega \rangle| \leq \|\phi\|_\infty \|f\|_1$.

Thus, $\|\alpha\| \leq \|\phi\|$ for all symbols ϕ of Γ.

□

This gives us upper and lower bounds for the optimal symbol, and shows that a (not necessarily unique) optimal symbol exists.

It is easy to demonstrate a symbol for which the optimal symbol no longer has the same norm as the Hankel operator, in contrast to the case in the disc where an optimal symbol with the same norm always exists.

Theorem 4.1.2 *There exists Hankel operator* Γ*, such that for* h *an optimal symbol,* $\|h\| \neq \|\Gamma\|$.

Proof

Consider Γ defined by the symbol $\phi = \frac{\omega}{z}$.
We have that $\langle \Gamma z^n, \omega \bar{z}^m \rangle = \langle \omega z^{n-1}, \omega \bar{z}^m \rangle = \langle z^{n-1}, \bar{z^m} \rangle = \langle z^{n-1+m}, 1 \rangle$.
Thus $\Gamma z^n = \omega a_n \bar{z}^{1-n}$, with $a_n = \frac{2}{1+s^{2-2n}}$.
Letting $e_n = \frac{z^n}{\|z^n\|_2}$ be the orthonormal basis,

$$\Gamma_\phi e_n = b_n \omega \overline{e_{1-n}} = \frac{2}{(1+s^{2-2n})^{0.5}(1+s^{2n})^{0.5}} \omega \overline{e_{1-n}}.$$

Since the e_n and $\omega \bar{e_n}$ are orthonormal basis for H^2 and $\omega \bar{H}^2$, we see that $\|\Gamma\| = \sup_n b_n$. b_n achieve their maximum at 0 and 1, so $\|\Gamma\| = \frac{\sqrt{2}}{\sqrt{1+s^2}}$.

Suppose that we have symbol Φ for which $\|\Phi\|_\infty = \|\Gamma\|$.
Since we have that Γ achieves its norm at f, where $f(z) = z$, we have that, as is the case in this situation in the disc, that:

$$\|\Gamma\| \|f\|_2 = \|\Gamma_\Phi f\|_2 \leq \|\Phi f\|_2 \leq \|\Phi\|_\infty \|f\|_2.$$

As is the case in this situation on the disc, when $\|\Gamma\| = \|\Phi\|$, we have equality throughout.
So, $\Gamma f = \Phi f$, i.e. $\Phi = \frac{\Gamma f}{f} = \frac{\omega}{z}$.
However, $\|\frac{\omega}{z}\|_\infty = s^{-1} > \frac{\sqrt{2}}{\sqrt{1+s^2}}$, contradicting our assumption.
Thus, there cannot exist a symbol which achieves the norm of our operator. □

4.1.1 Finding optimal symbols

Even in the case of Γ which attain their norm at some $f \in H^2$, we no longer have that $\frac{\Gamma f}{f}$ gives an optimal symbol for our Γ— since we no longer necessarily have that the optimal symbol matches the norm of the operator (though, in the case where they do agree, this argument works as it does in the disc).

By considering Γ as an operator on each member of the family of spaces H^2_α (the sets of modulus automorphic functions with character α), some of this difficulty can be averted.
Let $\Gamma_{\alpha,\phi}\tilde{f} = P_{(H^2_\alpha)^\perp}(\tilde{\phi}\tilde{f})$ be the extension onto H^2_α.
Then, attempting the Nehari theorem once more, letting $\beta(f_1 f_2) = \langle \Gamma f_1 f_2, \omega\rangle$, we have that $|\beta(f)| = \langle \Gamma_\alpha f_1, \omega \bar{f_2}\rangle \leq \|\Gamma_\alpha\| \|f_1\|_2 \|f_2\|_2$.
Since f_1 and f_2 can now be chosen from $H^2_{\frac{\alpha}{2}}$ rather than merely H^2, we can factorise while preserving norms (as in 2.3.3).
Thus $\|\beta\| \leq \sup_{\alpha\in[0,1]} \|\Gamma_\alpha\|$.
Thus, for the α giving the maximal value of $\|\Gamma_\alpha\|$, if we have that Γ_α takes its norm at some function $f \in H^2_\alpha$, we have that $\frac{\Gamma_\alpha f}{f}$ is an optimal symbol for Γ, as in the case of the disc (Sarason's solution of the Nehari problem).
However, the problem of finding which α to consider, and finding a function f at which the symbol achieves its norm, appear difficult.
Whilst it can be shown Γ_k is compact on H^2_k for arbitrary k if and only if Γ_α is compact on H^2_α for all α, and thus we will still have that Γ_α achieves its norm on all these if our original Hankel operator is compact, the problem is to find where it achieves its norm.
In the disc, it is interesting to consider Hankel operators with a finite polynomial symbol. The point at which the norm is attained can then be found easily since the associated Hankel matrix will have only a finite number of non-zero entries, and thus can be diagonalized with elementary techniques to find where the norm is achieved.
However, in the case of the annulus, any non-zero Hankel matrix will have an infinite number of non-zero entries. If one considers the matrix for a Hankel operator, symbol ψ, with respect to the basis e_n and $\omega\bar{e_n}$ defined earlier, we have that $A_{n,m} = \langle \psi k_n z^n, \omega \bar{z}^m k_m\rangle$, with $k_n = \|z^n\|_2^{-1}$.
Thus, $A_{n,m} = k_n k_m \hat{\psi}(n+m)$. Whilst not constant along backwards diagonals due to the $k_n k_m$ term, if any term on a given backwards diagonal is non-zero, all of them are non-zero. In the case of the disc, we would have that n, m range from 0 to ∞, and so these diagonals are finite, but in the annulus they range from $-\infty$ to $+\infty$, so there are

infinitely many terms on these diagonals.
It seems that finding the optimal symbol in this setting is a more difficult problem.

4.2 Compactness

In the case of Hankel operators on the disc, it is known that the compact Hankel operators are those with symbol in $H^\infty + \mathbf{C}$, where $\mathbf{C}$ denotes the space of continuous functions on the boundary. The result can be found with proof in [32] Chapter 1, section 5.
We can show the same is true in the annulus.
First, note that, as in the case of the disc, symbols of a rational function will correspond to compact operators.

Lemma 4.2.1 *If ϕ is a rational function, then Γ_ϕ is of finite rank.*

Proof
Since we have that $\Gamma_{\frac{f}{g}} = \Gamma_{\frac{1}{g}} M_f$, and $\Gamma_{\phi_1+\phi_2} = \Gamma_{\phi_1} + \Gamma_{\phi_2}$, we have that by dealing with partial fractions, it remains only to show that $\Gamma_{\frac{1}{(z-a)^n}}$ is compact for $a \in \mathbb{A}$.
Now, we have that $H^2(\mathbb{A}) = (z-a)^n H^2 \bigoplus ((z-a)^n H^2)^\perp$. $\Gamma_{(z-a)^{-n}}$ is the zero operator on restriction to the first space in the decomposition, and the latter space is finite dimensional. Thus, Γ is a finite rank operator. $\square$

In order to show that the whole of $\mathbf{C} + H^\infty$ gives compact operators, a little more is needed.

Lemma 4.2.2 *If ϕ is of form $\phi = \omega q$, where q is a rational function, then Γ_ϕ is compact.*

Proof
As before, we can split by partial fractions, and it remains only to consider symbols of

form $\frac{\omega}{(z-a)^n}$.

Decomposing H^2 as before, we have that Γ_ϕ is no longer a zero operator on $(z-a)^n H^2$, however, we have on this space that $\Gamma_{\frac{\omega}{(z-a)^n}} = \Gamma_\omega \circ M_{(z-a)^{-n}}$.

We have that Γ_ω is a compact operator

(on H^2 we have that $\Gamma_\omega \frac{z^n}{\|z^n\|} = \sqrt{\frac{1}{(1+s^{-2n})(1+s^{2n})}} \frac{\omega \bar{z}^{-n}}{\|\omega \bar{z}^{-n}\|}$.)

Thus, it follows that $\Gamma_{\frac{\omega}{(z-a)^n}}$ is compact (but will not be of finite rank).

By considering $\Gamma_{q+\omega q}$ where q is a rational function, we can approximate in norm functions whose restriction to the outer boundary is a continuous function, and zero on the inner boundary, and thus Hankel operators with such a symbol are compact. The same holds with the boundaries reversed, and so we have that any continuous function is compact. Since the Hankel operator with analytic symbol is equivalent to the zero operator, we thus have that all functions in $H^\infty + \mathbf{C}$ are symbols for a compact Hankel operator. $\square$

Proving the reverse, that any compact operator has symbol in $H^\infty + \mathbf{C}$, is a little harder.

Lemma 4.2.3 *If Γ_ϕ is compact, then $\phi \in Cl(H^\infty + \mathbf{C})$, with Cl denoting closure in norm (the space will be shown later to in fact be closed.)*

Proof

First, note that $\|\Gamma_\phi M_{B^n}\| \to 0$ if B is a Blaschke product with finitely many zeroes.

Suppose otherwise, then we have f_n with $\|f_n\| = 1$ and $\|\Gamma_\phi B^n f_n\| \geq \epsilon$ infinitely often for some ϵ.

From compactness of Γ, for some subsequence n_i we have $\Gamma_\phi B^{n_i} f_{n_i} \to \omega \bar{g}$, for some g in H^2.

By taking our subsequence to include only n with $\|\Gamma_\phi B^n f_n\| \geq \epsilon$, we have $g \neq 0$.

To show a contradiction, we need to show that $B^{n_i} f_{n_i}$ is tending to 0 weakly. Suppose this is not the case, that we have $g \in H^2$ with $|\langle B^{n_i} f_{n_i}, g\rangle| \geq \epsilon$ for some subsequence of

n_i, which we will call m_i.

Then, letting f' denote the lifts back to the disc, and letting κ represent the appropriate change of measure, we have $|\langle B'^n f'_{m_i}, g'\kappa\rangle_{\mathbb{D}}| \geq \epsilon\forall i$.

Letting $h = P_{H^2}(\kappa g')$, we have $|\langle B'^n f'_{m_i}, h\rangle_{\mathbb{D}}| \geq \epsilon\forall i$.

However, this cannot be as in the Hardy space of the disc, $B^n f_n$ tends to 0 in the weak topology when B is a Blaschke product and $\|f_n\|$ bounded, (this follows from [13] lemma 3.3, which shows that $T_{\overline{B}^n} g$ tends to 0 in norm for any g).

We thus have $\langle B^n f_n, \omega\phi\bar{g}\rangle \to 0$, and so $\langle \Gamma_\phi B^n f_n, \omega\bar{g}\rangle \to 0$, contradicting the assumption $\Gamma_\phi B^{n_i} f_{n_i} \to \omega g$ for $g \neq 0$.

With $\|\Gamma_\phi M_{B^n}\| \to 0$, the rest proceeds as the proof does for compactness in the disc:

We have that $\phi B^n = s_n + k_n$, with $\|s_n\| \to 0$, and $k_n \in H^\infty$.

Thus $\phi = s_n \bar{B}^n + (k_n/B^n)$.

For B_n a finite Blaschke product, we have that $k_n/B^n \in H^\infty + \mathbf{C}$, and thus $\phi \in \overline{H^\infty + \mathbf{C}}$.

If we can show that $H^\infty + \mathbf{C}$ is closed, then the result is complete. □

However, it is an old result that this space is closed in $\mathbb{A}$. It is proven, for example, in [2] Theorem 1.22. An alternative proof will be given here.

Closedness of $H^\infty + \mathbf{C}$

Theorem 4.2.4 *$H^\infty + \mathbf{C}$ is a closed subspace of L^∞.*

Proof

Suppose we have f_n is a Cauchy sequence in $h^\infty + \mathbf{C}$, with $f_n = q_n + c_n$, $q_n \in H^\infty, c_n \in \mathbf{C}$.

Let F denote the limit function in $L^\infty(\mathbb{A})$ Now, use additive decomposition of $H^\infty(\mathbb{A})$, to have $q_n = r_n + s_n$, and $c_n = d_n + e_n$, where $r_n \in H^\infty(\Delta)$, $s_n \in H^\infty((s\Delta)^c)$, $d_n = c_n|_{\Gamma_0}$, $e_n = c_n|_{\Gamma_1}$.

Now, consider f_n on Γ_0. $f_n|_{\Gamma_0}$ will still be Cauchy, and we have that

$f_n|_{\Gamma_0} = r_n|_{\Gamma_0} + s_n|_{\Gamma_0} + d_n|_{\Gamma_0}$.

Now, since $s_n \in H^\infty((s\Delta)^c)$, we have that s_n is continuous on Γ_0.

Thus $f_n|_{\Gamma_0}$ is a sequence in $H^\infty(\Delta) + \mathcal{C}$ converging to $F|_{\Gamma_0}$.

This space is closed in the disc, so $F|_{\Gamma_0}$ also belongs to it.

Therefore we have R and D with $R \in H^\infty(\Delta)$, $D \in \mathcal{C}(\partial\Delta)$, with $F|_{\Gamma_0} = R + D$.

We can do the same on Γ_1 to attain S and E, $S \in H^\infty((s\Delta)^c)$, $E \in \mathcal{C}(\partial s\Delta)$, with $F|_{\Gamma_1} = S + E$.

So $F = R + D + S + E - R|_{\Gamma_1} - S|_{\Gamma_0}$.

Now, we have that $R|_{\Gamma_1}$ is continuous, as is the restriction of S to Γ_0.

Thus $F \in H^\infty + \mathcal{C}(\partial\mathbb{A})$. $\square$

Chapter 5

The Bergman space and the slit disc

Some similar results to those from previous chapters can be extended to other settings, in particular that of the Bergman space on the annulus.

5.1 The Bergman space

Letting $\mathbb{A}$ represent the annulus as before, define the Bergman space of the annulus $A^2(\mathbb{A})$ to be the set of analytic functions on the disc for which $\int_{\mathbb{A}} |f|^2 dA(z) < \infty$, with A the Lebesgue area measure, and norm $\|f\|_2 = \sqrt{\int_{\mathbb{A}} |f|^2 dA(z)}$.
They are described in [23], for instance.

First, note that we have a factorisation result akin to Theorem 2.2.1-

Theorem 5.1.1 *If $f \in A^2(\mathbb{A})$, we have $f = f_1 f_2$ with $f_1 \in A^2(\Delta)$, and $f_2 \in A^2((s\Delta)^c)$.*

Proof

The proof is akin to that in the Hardy space. Suppose first that f has no zeros in $\mathbb{A}$. Then $f = z^n f'$, where f' has 0 winding number about 0, and has no zeros in the annulus, and thus we can take logarithms.

Say, $\log f' = \sum_{-\infty}^{+\infty} a_n z^n = \sum_0^{\infty} a_n z^n + \sum_{-\infty}^{-1} a_n z^n = g_1 + g_2$.

We thus have that $f = e^{\log f'} = e^{g_1}e^{g_2} = f_1 f_2$.

Now, we have that g_1 and thus f_1 is analytic in Δ, similarly f_2 is analytic in $(s\Delta)^c$. We must prove that the norms are finite also.

Since f_1 and $\frac{1}{f_1}$ are continuous and thus bounded in an open set about Γ_1, $f = f_1 f_2$, and f is square integrable, we have that f_2 has bounded square integral in some open set about Γ_1. By continuity, on the complement of any such set in $(s\Delta)^c$, f_2 has bounded square integral, and thus f_2 has bounded square integral in $s(\Delta)^c$, and so $f_2 \in A^2((s\Delta)^c)$. Similarly $f_1 \in A^2(\Delta)$.

Now, let f be an arbitrary $A^2(\mathbb{A})$ function. We can assume w.l.o.g that all zeros of f lie in $\{1 > |z| > 1 - \epsilon\} \bigcup \{s < |z| < s + \epsilon\}$ for some ϵ such that these are disjoint sets, since we have finitely many zeros inside any compact set, and if we can factor f/g for g a polynomial, then we can factor f.

Now, we have that f has no zeros on the set $A_2 = \{1 - \epsilon > |z| > s + \epsilon\}$, thus we can apply the previous factorisation to obtain:

$f = f_1 f_2$ with $f_1 \in A_2(\{z : |z| < 1 - \epsilon\})$, and $f_2 \in A_2(\{z : |z| > s + \epsilon)\}$.

By the domain of f_2, we have at most finitely many zeros in the region $\{1 > |z| > 1-2\epsilon\}$. Thus, we can extend f_1 across this region by $f_1 = f/f_2$, and the resultant function will have at most finitely many poles. Similarly, we can extend f_2 to the boundary $|z| = r$ by $f_2 = f/f_1$, and will have at most finitely many poles.

Finally, since $f = f_1 f_2$, and f has no poles in the region, we can remove the poles by factorising instead as $f = \frac{f_1 \Pi(z-a_i)}{\Pi(z-b_i)} \frac{f_2 \Pi(z-b_i)}{\Pi(z-a_i)} = g_1 g_2$, where a_i are the poles of f_1, and b_i poles of f_2.

Now, g_1 is analytic in Δ, g_2 analytic in $(s\Delta)^c$, and by repeating the argument for the case of no zeros, we have that these belong to the appropriate Bergman spaces. □

5.1.1 Toeplitz operators in the Bergman space of the annulus

We have that $A^2(\mathbb{A})$ is a closed subspace of $L^2(\mathbb{A})$, thus we can define a projection operator P from L^2 to A^2, and so define a Toeplitz operator T_ϕ by PM_ϕ for ϕ measurable functions.

In the Bergman space for the disc, it is known that for symbols continuous on the closure of the disc, up to compact operators only behaviour of ϕ on the boundary matters, and the essential spectrum will be the same of that of a Toeplitz operator on the Hardy space with the same boundary values- for example, [21] p16-17.
We can in fact extend this to $\mathbb{A}$:

Theorem 5.1.2 *If U is the isometry from $A^2(\mathbb{A})$ to $H^2(\mathbb{A})$ defined by $U(z^n/\|z^n\|) = z^n/\|z^n\|'$, where $\|.\|'$ is the Hardy norm and $\|.\|$ the Bergman space norm, then we have that for any ϕ symbol continuous on the closure of $\mathbb{A}$, we have:*
$UT_\phi U^* = T'_{\phi|_{\partial\mathbb{A}}} + K$, where K is a compact operator fixed by ϕ, T' denotes a Toeplitz operator on $H^2(\mathbb{A})$, and T denotes a Toeplitz operator on $A^2(\mathbb{A})$.

Proof
We proceed by first proving the result holds for T_z and $T_{z^{-1}}$ From here, we can then extend to symbols in the closure of the set of polynomials in $z, \bar{z}, z^{-1}, \bar{z}^{-1}$, since if the lemma holds for z, it must also hold for $\bar{z}$ by taking adjoints, and by taking powers of each side we have it for $\bar{z}^n$ and z^n all $n \geq 0$. Similarly $\phi = z^{-1}$ will give the result for z^n and $\bar{z}^n$ for all $n < 0$, and if it holds for ϕ_1 and ϕ_2 it holds for $\phi_1 + \phi_2$, etc.
Since the closure of these polynomials generate the continuous functions, we have the required set.
Letting $e_n = z^n/\|z^n\|$, and $e'_n = z^n/\|z^n\|'$, we have that $UT_zU^*e'_n = U^*ze_n = \|z^{n+1}\|/\|z^n\|e'_{n+1}$, and $T'_ze'_n = \|z^{n+1}\|'/\|z^n\|'e'_{n+1}$.
Thus, $\|UT_zU^*e'_n - T'_ze_n\| = \frac{\|z^{n+1}\|}{\|z^n\|} - \frac{\|z^{n+1}\|'}{\|z^n\|'}$.

Now, we have that $\|z^n\|^2 = \int_s^1 r^{2n} r dr = \frac{1-s^{2n+2}}{2n+2}$, and $\|z^n\|'^2 = 1 + s^{2n}$.
Letting $n \to +\infty$, we have that $\frac{\|z^{n+1}\|}{\|z^n\|} \to 1$, $\frac{\|z^{n+1}\|'}{\|z^n\|'} \to 1$.
Letting $n \to -\infty$, we have that $\frac{\|z^{n+1}\|}{\|z^n\|} \to s$, and $\frac{\|z^{n+1}\|'}{\|z^n\|'} \to s$.
Thus, we have that $\|UT_zU^*e'_n - T'_ze'_n\| > \epsilon$ at most a finite number of times. Applying the same to z^{-1} works similarly.
Now, we have that $UT_{z^n}U^* = (UT_zU^*)^n = (T'_{z|\partial\mathbb{A}} + K)^n = T'_{z^n|\partial\mathbb{A}} + S$, where K is the compact operator evaluated before, and S is compact as it is a finite sum of products of a compact operator with bounded operators.
Thus, we can extend to z^n. We can extend from z^{-1} to z^{-n} in the same manner, and taking adjoints we can show this for $\bar{z}^{-n}$ and $\bar{z}^n$.
It follows immediately that the rsult holds when ϕ can be written as a finite polynomial in $z, z^{-1}, \bar{z}, \overline{z^{-1}}$.
We can also extend the result to ϕ which can be uniformly approximated by polynomials in these (which, by the Stone-Weirstrass theorem, is the space of continuous functions on the closure of $\mathbb{A}$):
Let p_n be a sequence of polynomials uniformly approximating ϕ in ∞-norm on $L_\infty(Cl(\mathbb{A}))$ (with Cl denoting closure).
Then, we have that $T_{p_n} \to T_\phi$ in operator norm, and also $T'_{p_n|_\mathbb{A}} \to T'_{\phi|_\mathbb{A}}$ in operator norm.
Thus, $UT_\phi U^* - T'_{\phi|_\mathbb{A}} = \lim_{n\to\infty} K_n$, where $K_n = UT_\phi U^* - T'_{\phi|_\mathbb{A}}$ is compact, and the limit is in operator norm. Thus K will be compact here, and so the result holds for all ϕ continuous on the closure of $\mathbb{A}$. $\square$

Radial Toeplitz operators

We have in the case of a Toeplitz operator on the Bergman space of the disc that if ϕ is a radial symbol, then we have $T_\phi z^n = A_n z^n \; \forall n$ for some $A_n \in \mathbb{C}$, since

$$\langle T_\phi z^n, z^m \rangle = \langle \phi z^n, z^m \rangle = \int_0^1 \int_0^{2\pi} \phi(r,\theta) r^{n+m} r e^{i(n-m)\theta} dr d\theta$$

$$= \int_0^1 r^{n+m+1} \phi(r) \int_0^{2\pi} e^{i(n-m)\theta} d\theta dr.$$

Here, the θ-integral is 0 when $n \neq m$.

The same occurs in the case of the annulus, since:

$\langle T_\phi z^n, z^m \rangle = \langle \phi z^n, z^m \rangle = \int_s^1 \int_0^{2\pi} \phi(r,\theta) r^{n+m} r e^{i(n-m)\theta} dr d\theta$

$= \int_s^1 r^{n+m+1} \phi(r) \int_0^{2\pi} e^{i(n-m)\theta} d\theta dr = 0$ for $n \neq m$,

where s is the inner radius of the annulus.

The paper by S. Grudsky, A. Karapetyants, and N. Vasilevski [20] describes bounded and even compact Toeplitz operators with unbounded radial symbols, and gives some conditions for an operator to be as such.

In the annulus, the condition for the radial Toeplitz operators to be bounded is quite similar to the case in the disc:

Theorem 5.1.3 *If ϕ is a radial symbol for a Toeplitz operator on the Bergman space of the annulus, T_ϕ is bounded if and only if there exists a bounded Toeplitz operator on the Bergman space of the disc whose symbol on the disc whose symbol ϕ' is equal to ϕ in some neighbourhood of 1. The same holds replacing bounded with compact. In other words, the condition to be bounded/compact is simply that the behaviour at each boundary is that of a bounded/compact radial Toeplitz operator on the disc.*

Proof

Let ϕ be a radial symbol. It is immediate that $\langle \phi z^n, z^m \rangle = 0$ for $n \neq m$, thus the operator is diagonal. So, letting a_n be such that $T_\phi z^n = a_n z^n$, we have that T_ϕ is bounded if $\{a_n\}$

is bounded, and compact if $a_n \to 0$ as $n \to \pm\infty$.

Suppose first that ϕ is continuous at s, and that ϕ' is a symbol for a bounded Toeplitz operator on $A^2(\Delta)$ such that $\phi = \phi'|_{[s,1]}$. We show that T_ϕ must now be bounded.

We have that $a_n = \langle \phi' z^n, z^n \rangle_\Delta / \langle z^n, z^n \rangle_\Delta$ are a bounded sequence by assumption.

Looking at the limiting behaviour as $n \to +\infty$, we have that $\int_0^1 \phi'(r) r^{2n+1} dr(2n+2)$ is bounded.

Now, looking at b_n the nth eigenvalue for the Toeplitz operator on the Bergman space of the annulus, we have that:

$b_n = \langle \phi z^n, z^n \rangle_{\mathbb{A}} / \langle z^n, z^n \rangle_{\mathbb{A}}$, and $\langle z^n, z^n \rangle_{\mathbb{A}} = \frac{1}{2n+2}(1 - s^{2n+2}) > \frac{1}{2(2n+2)}$ for all sufficiently large n.

We thus have that $|b_n| = |(\int_s^1 \phi r^{2n+1} dr) / \langle z^n, z^n \rangle| \leq 2(2n+2)|\int_s^1 \phi r^{2n+1} dr| \leq 2|a_n| + 2(2n+2)|\int_0^s \phi' r^{2n+1} dr|$.

However, $|a_n|$ is bounded and the latter integral clearly converges to 0 as $n \to \infty$.

Thus, we have that the eigenvalues corresponding to positive values of n behave appropriately.

Now, since we have that ϕ is approximating C for some constant close to the inner boundary, and we have that the contribution of $\frac{z^{-n}}{\|z^{-n}\|_2}$ on the outer boundary falls exponentially as n tends to ∞, we have that $b_{-n} \to C$ as $n \to \infty$

Thus, we have that T_ϕ is a bounded operator for the Bergman space of the annulus if it is for the disc and it is continuous about s.

In the case $C = 0$, the same argument shows that T_ϕ is compact if $T'_{\phi'}$ is compact. In order to deal with a general symbol, we simply need to show that the corresponding result holds for a symbol continuous at 1 when considering the symbol as the restriction of a bounded symbol on $A^2((s\Delta)^c)$.

From this, the result follows for arbitrary symbols since we can decompose the symbol as $\phi = \phi_1 + \phi_2$ with ϕ_1 continuous at s, and ϕ_2 continuous at 1.

Considering n negative, we have that $b_n = \frac{\int_s^1 \phi r^{2n+1} dr}{\int_s^1 r^{2n+1} dr}$.

Making the substitution $q = \frac{s}{r}$, this becomes $b_n = \frac{\int_s^1 \phi(s/q) q^{-(2n+3)} dq}{\int_s^1 q^{-(2n+3)} dq}$.

Hence the boundedness of b_n for negative n translates to the question of boundedness of b_n for positive n for the symbol $\tilde{\phi}$ with $\tilde{\phi}(r) = \phi(s/r)$. $\square$

In addition to boundedness/compactness, a result in a similar vein can be shown on the Hilbert-Schmidt norms of the diagonal Toeplitz operators (which all radial Toeplitz operators are included in).

A paper of Harper-Smith [22] describes the relation between Hilbert-Schmidt norms for Toeplitz operators on the Bergman space of the disc, and the following shows that the same hold in the annulus.

Theorem 5.1.4 *Suppose that T is a diagonal operator from the Bergman space of the annulus into an arbitrary Hilbert Space.*
Let $T \in \mathcal{B}(A^2(\mathbb{A}), \mathcal{H})$. Then we have $\|T\|_{S_2}^2 = \int_s^1 \int_0^{2\pi} \|Tk_{re^{i\theta}}\|^2 r dr d\theta$, where k_z is the reproducing kernel at z (for the Bergman space here).

Proof

The proof is almost identical to that in the Harper-Smith paper. Let e_n be an arbitrary orthonormal basis. Then we have that:
$\int_s^1 \int_0^{2\pi} \|Tk_{re^{i\theta}}\|^2 r dr d\theta = \sum_{n=0}^{\infty} \int \int |(T^* e_n)(re^{i\theta})|^2 r dr d\theta$,
$= \sum_n \|T^* e_n\|^2 = \|T^*\|_{S_2}^2 = \|T\|_{S_2}^2$.
$\square$

The Harper-Smith paper [22] describes the result in general Schatten classes, and the following will show the proof works in the same way on the annulus. However, the results have also been proven for Bergman spaces on multiply connected domains in [41] by means of conformal equivalences.

Theorem 5.1.5 *If $p > 2$ and $T \in S_p(A^2(\mathbb{A}), \mathcal{H})$, then we have:*

$$\int_{\mathbb{A}} \|T\tilde{k_z}\|^p dm(z) \leq \|T\|_{S_p}^p,$$

where $dm(z) = dA(z)\|k_z\|^2$*, and* $\tilde{k_z} = \frac{k_z}{\|k_z\|}$.
If $1 < p \leq 2$ *and* $\int_{\mathbb{A}} \|T\tilde{k_z}\|^p dm(z) < \infty$*, we have that* $T \in S_p$ *and* $\int_{\mathbb{A}} \|T\tilde{k_z}\|^p dm(z) \geq \|T\|_{S_p}^p$.

Proof

We have that if A is a positive operator, then $Tr(A) = \int_{\mathbb{A}} \langle A\tilde{k_z}, \tilde{k_z}\rangle dm(z)$, with the inner product taken over our Bergman space on $\mathbb{A}$.
This follows identically to the proof in [42] p116:

$$\begin{aligned}
tr(A) &= \sum_{n=1}^{\infty} \langle Ae_n, e_n\rangle = \sum_{n=1}^{\infty} \int_{\mathbb{A}} (Ae_n)(z)\overline{e_n(z)} dA(z) \\
&= \sum_{n=1}^{\infty} \int \langle Ae_n, k_z\rangle \overline{e_n(z)} dA(z) \\
&= \int \langle A \sum_{n=1}^{\infty} e_n \overline{e_n}(z), k_z\rangle dA(z) = \int \langle Ak_z, k_z\rangle dA(z) \\
&= \int_{\mathbb{A}} \langle A\tilde{k_z}, \tilde{k_z}\rangle dm(z).
\end{aligned}$$

From this, and using the fact that for A a positive operator, and f of unit norm, $\langle A^p f, f\rangle \geq \langle Af, f\rangle^p$ for $p \geq 1$, and $\langle A^p f, f\rangle \leq \langle Af, f\rangle^p$ for $p \leq 1$, we have that:

$$\begin{aligned}
\|T\|_{S_p}^p = Tr((T^*T)^{p/2}) &= \int_{\mathbb{A}} \langle (T^*T)^{\frac{p}{2}} \tilde{k_z}, \tilde{k_z}\rangle dm(z) \\
&\geq \int_{\mathbb{A}} (\langle T^*T\tilde{k_z}, \tilde{k_z}\rangle)^{p/2} dm(z), \\
&= \int_{\mathbb{A}} \|T\tilde{k_z}\|^p dm(z).
\end{aligned}$$

For the case $1 < p \leq 2$, the reverse inequality holds since the direction of inequality for $\langle A^p f, f\rangle$ is reversed. □

5.2 Hardy Space of the slit disc

Consider the domain of the slit disc G, the unit disc with the interval $[0, 1)$ removed.
We will work in the space that was defined in the introduction as $E^2(G)$, the Hardy-Smirnov space. The existence of boundary values, and thus toeplitz operators were covered in the introduction.
There exists a conformal mapping ϕ_G between the disc and the slit disc, given in [3] in the appendix, defined as $\phi_G(z) = (w_3(w_2(w_1(-iz)))^2$, where
$w_3(z) = \frac{z-1}{z+1}$, $w_2(z) = \sqrt{z}$, $w_1(z) = i\frac{z+1}{1-z}$.
The book [3] also describes the effect on the boundary here— the arc $\{e^{i\theta} : 0 < \theta < \pi/2\}$ is mapped to the top half of the slit, and the arc $\{e^{i\theta} : -\pi/2 < 0 < \theta\}$ is mapped to the lower half of the slit.
It is useful know the poles and zeros on the boundary of the derivative of our mapping, ϕ'_G.
We have that $\phi'_G(z) = 2(w_3w_2w_1(-iz))w'_3(w_2w_1(-iz))w'_2(w_1(-iz))w'_1(-iz)$.
Considering each factor in turn, $w_3(w_2(w_1(-iz)))$ has no poles on the disc, and has a zero at $z = 1$.
Then $w'_3(z) = \frac{2}{(z+1)^2}$, so $w'_3(w_2(w_1(-iz)))$ has no poles but has a zero of order 1 at $z = i$.
Next, $w'_2(z) = \frac{1}{2}z^{-1/2}$, so $w'_2(w_1(-iz))$ has singularity at $z = -i$, and decays at $z = i$, with the local growth/decay that of $z^{-\frac{1}{2}}$ and $z^{\frac{1}{2}}$ respectively.
Finally, $w'_1(-iz) = \frac{2}{(1+iz)^2}$ has an order 2 pole at $z = i$, and no zeros in the range we are interested in. Thus, putting this all together, we have that ϕ'_G has a zero at $z = 1$ with $\phi'_G(1 - z) = O(z)$ locally, and we have singularities at $\pm i$ with $\phi'_G(z \pm i) = O(z^{-1/2})$.
The function ϕ'_G is important as it is used to normalize the measure when transforming between integrals on the boundary of the slit disc and on the boundary of the disc under the map ϕ_G.

Isomorphism between $E^2(G)$ and $H^2(\Delta)$, and Toeplitz operators

Proposition 5.2.1 *Let μ be the isomorphism between $E^2(G)$ and $H^2(\Delta)$ considered in the introcution, $\mu : f \to F$, with $F(z) = f(\phi_G(z))(\frac{d\phi_G}{dz})^{\frac{1}{2}}(z)$.*
We have that if T'_ψ is a Toeplitz operator on the Hardy space of the slit disc, with symbol ψ, then we have that $T'_{\tilde{\psi}}(\mu f) = T_\psi f$, where $\tilde{\psi}(z) = \psi(\phi_G(z))$.

Proof

This is simply a consequence of the definition of our space and the toeplitz operators on it.

Consider the inner product of a Toeplitz operator applied to μf with μg in the disc.
$\langle \tilde{\psi} \mu f, \mu g \rangle_\Delta = \int_{\partial\Delta} \psi(\phi_G(z)) f((\phi_G(z))) g(\bar{\phi_G}(z)) |\phi'_G(z)||dz|$
$= \int_{\partial G} f(z) g(\bar{z}) \psi(z) |\phi'(\phi_G^{-1}(z))||\phi'(\phi_G^{-1}(z))|^{-1}|dz| = \langle \psi f, g \rangle_G = \langle T_\psi f, g \rangle_G$.
Thus, we have that $\langle T'_{\tilde{\psi}} \mu f, \mu g \rangle_\Delta = \langle T_\psi f, g \rangle_G$ for all $g \in H^2(\Delta)$, and so the result follows.

$\square$

Results about Toeplitz operators in the slit disc can thus be deduced from those in the disc, and vice-versa.

Bibliography

[1] M. B. Abrahamse, *Toeplitz operators in multiply connected regions*, Bull. Amer. Math. Soc. **77** (1971), 449–454.

[2] ———, *Toeplitz operators in multiply connected regions*, Amer. J. Math. **96** (1974), 261–297.

[3] A. Aleman, N. S. Feldman, and W. T. Ross, *The Hardy space of a slit domain*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2009.

[4] J. M. Anderson and D. Khavinson, *Hankel operators in multiply connected domains*, Complex Variables Theory Appl. **13** (1989), no. 1-2, 21–33.

[5] C. P. Aryana, *Self-adjoint Toeplitz operators associated with representing measures on multiply connected planar regions and their eigenvalues*, Complex Anal. Oper. Theory **7** (2013), no. 5, 1513–1524.

[6] C. P. Aryana and K. F. Clancey, *On the existence of eigenvalues of Toeplitz operators on planar regions*, Proc. Amer. Math. Soc. **132** (2004), no. 10, 3007–3018 (electronic).

[7] A. Bakan and S. Kaijser, *Hardy spaces for the strip*, J. Math. Anal. Appl. **333** (2007), no. 1, 347–364.

[8] J. A. Ball and K. F. Clancey, *Reproducing kernels for Hardy spaces on multiply connected domains*, Integral Equations Operator Theory **25** (1996), no. 1, 35–57.

[9] J. Bourgain, *A problem of Douglas and Rudin on factorization*, Pacific J. Math. **121** (1986), no. 1, 47–50.

[10] A. Broschinski, *Eigenvalues of Toeplitz operators on the annulus and Neil algebra*, Complex Anal. Oper. Theory **8** (2014), no. 5, 1037–1059.

[11] A. E. Broschinski, *Eigenvalues of self-adjoint Toeplitz operators with respect to a constrained algebra*, ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–University of Florida.

[12] M. C. Câmara and J. R. Partington, *Near invariance and kernels of Toeplitz operators*, J. Anal. Math. **124** (2014), 235–260.

[13] ———, *Near invariance and kernels of Toeplitz operators*, J. Anal. Math. **124** (2014), 235–260.

[14] I. Chalendar and J. R. Partington, *Modern approaches to the invariant-subspace problem*, Cambridge Tracts in Mathematics, vol. 188, Cambridge University Press, Cambridge, 2011.

[15] J. B. Conway, *Functions of one complex variable. II*, Graduate Texts in Mathematics, vol. 159, Springer-Verlag, New York, 1995.

[16] R. G. Douglas, *Banach algebra techniques in the theory of Toeplitz operators*, American Mathematical Society, Providence, R.I., 1973.

[17] ———, *Banach algebra techniques in operator theory*, Second, Graduate Texts in Mathematics, vol. 179, Springer-Verlag, New York, 1998.

[18] P. L. Duren, *Theory of H^p spaces*, Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London, 1970.

[19] K. M. Dyakonov, *Kernels of Toeplitz operators via Bourgain's factorization theorem*, J. Funct. Anal. **170** (2000), no. 1, 93–106.

[20] S. Grudsky, A. Karapetyants, and N. Vasilevski, *Toeplitz operators on the unit ball in $\mathbb{C}^n$ with radial symbols*, J. Operator Theory **49** (2003), no. 2, 325–346.

[21] S. Grudsky and N. Vasilevski, *Bergman-Toeplitz operators: radial component influence*, Integral Equations Operator Theory **40** (2001), no. 1, 16–33.

[22] Z. Harper and M. P. Smith, *Testing Schatten class Hankel operators, Carleson embeddings and weighted composition operators on reproducing kernels*, J. Operator Theory **55** (2006), no. 2, 349–371.

[23] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, vol. 199, Springer-Verlag, New York, 2000.

[24] K. Hoffman, *Banach spaces of analytic functions*, Dover Publications, Inc., New York, 1988. Reprint of the 1962 original.

[25] M. Jovovic and D. Zheng, *Compact operators and Toeplitz algebras on multiply-connected domains*, J. Funct. Anal. **261** (2011), no. 1, 25–50.

[26] D. Khavinson, *On removal of periods of conjugate functions in multiply connected domains*, Michigan Math. J. **31** (1984), no. 3, 371–379.

[27] Y.-S. Kim, *A Generalization of Beurling's Theorem and Quasi-Inner Functions*, ArXiv Mathematics e-prints (December 2006), available at `arxiv.org/pdf/math/0612790`.

[28] N. Y. Krupnik, *Banach algebras with symbol and singular integral operators*, Operator Theory: Advances and Applications, vol. 26, Birkhäuser Verlag, Basel, 1987.

[29] N. K. Nikolski, *Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer-Verlag, Berlin, 1986.

[30] ______, *Operators, functions, and systems: an easy reading. Vol. 1*, Mathematical Surveys and Monographs, vol. 92, American Mathematical Society, Providence, RI, 2002.

[31] J. R. Partington, *An introduction to Hankel operators*, London Mathematical Society Student Texts, vol. 13, Cambridge University Press, Cambridge, 1988.

[32] V. V. Peller, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[33] A. Perälä, J. Taskinen, and J. A. Virtanen, *New results and open problems on Toeplitz operators in Bergman spaces*, New York J. Math. **17A** (2011), 147–164.

[34] M. Rosenblum and J. Rovnyak, *Hardy classes and operator theory*, Dover Publications, Inc., Mineola, NY, 1997. Corrected reprint of the 1985 original.

[35] H. L. Royden, *Invariant subspaces of $\mathcal{H}^p$ for multiply connected regions*, Pacific J. Math. **134** (1988), no. 1, 151–172.

[36] W. Rudin, *Analytic functions of class H_p*, Trans. Amer. Math. Soc. **78** (1955), 46–66.

[37] ______, *Real and complex analysis*, Third, McGraw-Hill Book Co., New York, 1987.

[38] D. Sarason, *The H^p spaces of an annulus*, Mem. Amer. Math. Soc. No. **56** (1965), 78.

[39] M. Schechter, *Principles of functional analysis*, Second, Graduate Studies in Mathematics, vol. 36, American Mathematical Society, Providence, RI, 2002.

[40] M. Smith, *The spectral theory of Toeplitz operators applied to approximation problems in Hilbert spaces*, Constr. Approx. **22** (2005), no. 1, 47–65.

[41] T. Yu, *Schatten-class Toeplitz operators on bounded multiply-connected domains*, Chinese Ann. Math. Ser. A **26** (2005), no. 3, 361–368.

[42] K. H. Zhu, *Operator theory in function spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 139, Marcel Dekker, Inc., New York, 1990.