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Stochastic Analysis with Lévy Noise in the Dual of a Nuclear Space.

by

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To my beloved wife and my baby.

Abstract

In this thesis we introduce a new theory of stochastic analysis with respect to Lévy processes in the strong dual of a nuclear space.

First we prove some extensions of the regularization theorem of Itô and Nawata to show conditions for the existence of continuous and càdlàg versions to cylindrical and stochastic processes in the dual of a nuclear space. Sufficient conditions for the existence of continuous and càdlàg versions taking values in a Hilbert space continuously included on the dual space are also provided. Then, we apply these results to prove the Lévy-Itô decomposition and the Lévy-Khintchine formula for Lévy processes taking values in the dual of a complete, barrelled, nuclear space.

Later, we introduce a theory of stochastic integration for operator-valued processes taking values in the strong dual of a quasi-complete, bornological, nuclear space with respect to some classes of cylindrical martingale-valued measures. The stochastic integrals are constructed by means of an application of the regularization theorems. In particular, this theory allows us to introduce stochastic integrals with respect to Lévy processes via Lévy-Itô decomposition. Finally, we use our theory of stochastic integration to study stochastic evolution equations driven by cylindrical martingale-valued measure noise in the dual of a nuclear space. We provide conditions for the existence and uniqueness of weak and mild solutions. Also, we provide applications of our theory to the study of stochastic evolution equations driven by Lévy processes.

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My mouth shall speak wisdom; the meditation of my heart shall be understanding.
PSALM 49:3.

Yours, Lord, is the greatness and the power and the glory and the majesty and the splendour, for everything in heaven and earth is yours. Yours, Lord, is the kingdom; you are exalted as head over all.
1 CHRONICLES 29:11.

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Contents

Introduction

The aim of this work is to introduce a new theory of stochastic analysis with respect to Lévy processes in the strong dual of a nuclear space.

Apart from Banach spaces, nuclear spaces are the most important class of locally convex spaces used in functional analysis. They have many useful properties and they constitute a class of infinite dimensional spaces which also shares many properties of finite dimensional spaces. For example, they satisfy the Heine-Borel property, i.e. bounded subsets are precompact.

Nuclear spaces were introduced in 1951 by A. Grothendieck in [35] and were further developed by him in [36]. Grothendieck defined these class of spaces by means of his theory of tensor products of locally convex spaces. However, it is hardly an exaggeration to say that much of the true power behind the theory of nuclear spaces was better understood thanks to the characterization of nuclear spaces in terms of summable and absolute summable families of operators due to A. Pietsch [86]. In this thesis we will utilize a characterization of nuclear spaces in terms of a family of Hilbertian semi-norms generating its locally convex topology and an associated family of Hilbert spaces related to each other by means of some Hilbert-Schmidt operators (see Trèves [99]).

The importance of the nuclear spaces in the theory of probability is manifest in the problem of the existence of Radon measure extensions for cylindrical measures defined on the dual of a nuclear space. Indeed, this relation was clarified with the celebrated work of R. A. Minlos who in 1958-9 (see [66]) proved that an analogue of Bochner's theorem that characterizes the Fourier transform of a finite Borel measure holds in the dual of a (barrelled) nuclear space. Several monographs devote large sections to the study of cylindrical measures on duals of nuclear spaces. For example Gel'fand and Vilenkin [31] and Schwartz [95].

Stochastic analysis in duals of a nuclear space experienced a period of intensive activity during the 1980s and 1990s. Some of the pioneering work was carried our by K. Itô [41], [44], [42], [43], A. S. Üstünel [101], [102], [103], [105], [106], [108], I. Mitoma [67], [68], [69], [71], [72], [73], [75], and by G. Kallianpur and his collaborators [49], [51], [53], [56], [55]. However, as in many other branches of mathematics there are a large number of authors that contributed to its development, we cite for example J. Xiong [118], S. Ramaswamy [89], V. Pérez-Abreu and C. Tudor [79], [80], [81], [82], [84], T. Bojdecki, L. G. Gorostiza and J. Jakubowski [11], [12], [13], [46], [47], J. K. Brooks and his collaborators [17], [18], [19] and H. Körezlioğlu and C. Martias [59], [60], [61].

Much of the motivation behind the development of stochastic analysis on duals of nuclear spaces is its high range of applications. Among some of the most important applications is the modelling of the dynamics of nerve signals. See for example the works of Kallianpur and Wolper [53], Kallianpur, Mitoma and Wolper [51] and Kallianpur *et al.* [56]. In Kallianpur and Xiong [54] one can find also applications to model environmental pollution.

Some other applications are for example to statistical filtering (see Üstünel [104], [106] and D. Ding [27]), chemical kinetics and interacting particles systems (see Bojdecki and Gorostiza [9], Gorostiza and Nualart [34], Hitsuda and Mitoma [38], Kallianpur and Pérez-Abreu [52], Kallianpur and Mitoma [50], Kallianpur and Xiong [55], and Mitoma [70]).

With some few exceptions, much of the works cited above were developed under the hypothesis of the nuclear space being Fréchet or either that its strong dual is also nuclear. Moreover, the stochastic integrals and the noise driving the stochastic differential equations has been considered either with respect to Wiener processes or Poisson random measures, but to the extent of our knowledge, no theory has been developed with respect to the general Lévy process case. This is the main motivation for the development of our theory in this thesis. We are also interested in developing this theory under the weakest possible assumptions on the nuclear space and its strong dual.

In general terms, our contribution to theory of stochastic analysis on nuclear spaces can be divided into four main aspects. First, we will show some extensions of the regularization theorem of Itô and Nawata [44] to the case of cylindrical processes on the dual of a nuclear space. These theorems will be a corner stone for our theory of stochastic analysis. Our second main contribution is the proof of a Lévy-Itô decomposition for Lévy process taking values in the dual of a complete, barrelled, nuclear space. The third is the introduction of a new theory of stochastic integration with respect to some classes of cylindrical martingale-valued measures on the dual of a nuclear space. This theory allow us to introduce stochastic integrals with respect to Lévy process by means of the Lévy-Itô decomposition. Finally, our last main contribution is the application of our theory of stochastic integration to model stochastic evolution equations on the dual of a nuclear space. Contrary to what can be found in the literature, we will consider semi-linear equations driven by multiplicative noise.

This thesis is organized as follows:

Chapter 1 is devoted to the introduction to the main tools that we will need on the subsequent chapters. First we review the basic properties of classes of locally convex spaces encountered on this thesis. We focus our attention on those concepts related to nuclear spaces and their strong duals. Later, we review basic concepts of cylindrical and stochastic processes on the dual of a nuclear space. Then, we proceed to prove our extensions of the regularization theorem. We finalize this chapter by studying martingales.

In Chapter 2 we study basic properties of Lévy processes in the dual of a nuclear space. The proof of the Lévy-Itô decomposition is going to take most of our effort in this chapter. As a corollary we will proof the Lévy-Khintchine formula for the characteristic function of any Lévy processes.

The aim of Chapter 3 is to develop the theory of stochastic integration. First, we introduce the class of cylindrical martingale-valued measures that will be the integrators of our integrals. Then, we develop the stochastic integration theory in two steps. The first is a theory of weak stochastic integration with respect to cylindrical martingale-valued measures. For the second stage, we use the regularization theorems of Chapter 1 and the weak stochastic integral to introduce a theory of strong stochastic integration, i.e. we define stochastic integrals for some families of operator-valued processes with respect to the cylindrical martingale-valued measures. Applications to define stochastic integrals with respect to Lévy process will be given.

Finally, in Chapter 4 we apply our theory of stochastic integration to study stochastic evolution equations driven by cylindrical martingale-valued measures. We start by introducing some notions of deterministic integration for random integrands. Then, we introduce the class of stochastic evolution equations that we are going to consider in this thesis. In particular we will focus on the study of equivalence between weak and mild solutions, and we consider conditions on the coefficients for the existence and uniqueness of these types of solutions. We finalize this chapter with an example of an application to stochastic evolution equations driven by Lévy noise.

Notation and Useful Facts

In this thesis $\mathbb{N}$, $\mathbb{Z}$ $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural, integers, rational, real and complex numbers respectively. Denote $\mathbb{R}_+ = [0, \infty)$. For any $n \in \mathbb{N}$, $\mathbb{R}^n$ is the n-dimensional Euclidean space.

For $a, b \in \mathbb{R}$, we will use $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. If I is a countable set, we denote by δ_{ij} the **Kronecker delta** for $i, j \in I$, i.e. $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

For any two sets A and B, we denote by $A \cup B$, $A \cap B$ and $A \setminus B$ the union, the intersection and the complement of B in A respectively. When we are considering a subset U of a given set E, we write $U^c = E \setminus U$. If A is a finite set, we denote by $\#A$ the number of elements of A.

If $\mathcal{A}$ is a collection of subsets of a set S, we denote by $\sigma(\mathcal{A})$ the **σ-algebra generated** by $\mathcal{A}$. If $(\Omega, \mathcal{F})$ and $(S, \mathcal{S})$ are measurable spaces and $Y : \Omega \to S$ is a $\mathcal{F}/\mathcal{S}$ measurable map, the **σ-algebra generated** by Y is $\sigma(Y) = \{Y^{-1}(B) : B \in \mathcal{S}\}$.

We denote by $\mathbb{1}_A(\cdot)$ the indicator function of the set A, defined by $\mathbb{1}_A(x) = 1$ for $x \in A$ and $\mathbb{1}_A(x) = 0$ for $x \notin A$.

Let $\mathcal{T}_1$, $\mathcal{T}_2$ be any two topologies on a set X. If $\mathcal{T}_1$ is contained in $\mathcal{T}_2$ (i.e. if any element of $\mathcal{T}_1$ is also an element of $\mathcal{T}_2$), we denote this fact by: $\mathcal{T}_1 \subseteq \mathcal{T}_2$. In this case we say that $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$, and that $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$.

If U is a subset of a topological space (X, τ) we denote by $\overline{U}$ its closure and by $\mathring{U}$ its interior.

For a topological space (X, τ), we denote by $\mathcal{B}(X)$ the Borel σ-algebra of X. It is the smallest σ-algebra of subsets of X which contains all the open sets. $\mathbb{R}$ and $\mathbb{R}^d$ will be always assumed to be equipped with their Borel σ-algebra. A measure μ on $(X, \mathcal{B}(X))$ is called a **Borel measure**.

The **Dirac measure** on X for a given $x \in X$ will be denoted by δ_x and is defined by $\delta_x(A) = \mathbb{1}_A(x)$, for any $A \subseteq X$.

For two Borel measures μ and ν on a topological vector space X, denote by $\mu * \nu$ their **convolution**. Recall that $\mu * \nu(A) = \int_{X \times X} \mathbb{1}_A(x + y)\, \mu(dx)\nu(dy)$, for any $A \in \mathcal{B}(X)$. Denote $\nu^{*n} = \nu * \cdots * \nu$ (n-times) and we use the convention $\nu^0 = \delta_0$.

Let (S, Σ, μ) be a measure space. For $1 \le p < \infty$, $L^p(S, \Sigma, \mu)$ is the usual space of (equivalence classes of) real-valued measurable functions that agree almost everywhere with respect to μ and for which $||f||_p := \left(\int_S |f(x)|^p \mu(dx)\right)^{\frac{1}{p}} < \infty$ for all $f \in L^p(S, \Sigma, \mu)$. It is a Banach space with respect to the norm $||\cdot||_p$ and for $p = 2$ it is a Hilbert space with respect to the inner product $\langle f\, ,\, g\rangle_2 := \left(\int_S f(x)g(x)\mu(dx)\right)^{\frac{1}{2}} < \infty$ for all $f, g \in L^2(S, \Sigma, \mu)$.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and (S, Σ) be a measurable space. A $\mathscr{F}/\Sigma$-

measurable map $X : \Omega \to S$ will be called a S**-valued random variable**. In this thesis we will only consider **Borel random variables**, i.e. S will be a topological space and $\Sigma = \mathcal{B}(S)$.

Let J be $\mathbb{R}_+$ or $[0,T]$ for $T > 0$. A S**-valued process** is a collection $X = \{X_t\}_{t\in J}$ of S-valued random variables. We say that X is **continuous** (respectively **càdlàg**) if for $\mathbb{P}$-a.e. $\omega \in \Omega$, the **sample paths** $t \mapsto X_t(w) \in S$ of X are continuous (respectively right-continuous with left limits).

Let X be a real-valued random variable. If X is integrable, i.e. $X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$, we define its expectation to be

$$\mathbb{E}X = \int_\Omega X(\omega)\mathbb{P}(d\omega).$$

We say that the random variable is p-integrable $(1 \leq p < \infty)$ if $X \in L^p(\Omega, \mathscr{F}, \mathbb{P})$. In this case, the p-moment of X is $\mathbb{E}X^p$.

Unless otherwise stated, throughout this document we will only consider vector spaces over a field $\mathbb{K}$, which will always be $\mathbb{R}$ or $\mathbb{C}$. Usually we denote a vector space by E. If S is a subset of E, $\operatorname{span}\{S\}$ denotes the linear span of S.

If A and B are subsets of E, let $A+B := \{x+y : x \in A, y \in B\}$, $\lambda A := \{\lambda x : x \in A\}$ where $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$), and $A + y := A + \{y\}$ for $y \in E$.

Let A and B be subsets of E. We say that A **absorbs** B if there exist some $\eta_0 \in \mathbb{K}$ such that $B \subseteq \eta A$ whenever $|\eta| \geq |\eta_0|$. A subset U of E is called **absorbing** if U absorbs every finite subset of E. A subset C of E is **balanced** if $\alpha C \subseteq C$ whenever $\alpha \in \mathbb{K}$, $|\alpha| \leq 1$. A subset D of E is said to be **convex** if $x, y \in D$ implies that $\lambda x + (1-\lambda)y \in D$ for all $0 < \lambda < 1$.

Chapter 1

Probabilities on the Dual of a Nuclear Spaces

The main purpose of this chapter is to introduce the main concepts of probability on nuclear spaces that we will need on this thesis. This chapter is divided into two main sections. In the first, we review some concepts of locally convex spaces, linear operators and of nuclear spaces that will be used throughout this thesis. In the second section we start by reviewing basic properties of cylindrical and stochastic processes in the dual of a nuclear space. Then, we show some new results on the existence of continuous and càdlàg versions for cylindrical and stochastic process taking values in the dual of a nuclear space. Finally, we apply these results to the study of martingales taking values in the dual of a nuclear space.

§ 1.1 Review of Locally Convex Spaces

In this section we give a brief presentation of those concepts on locally convex spaces which are used on this thesis. For a more detailed treatment the reader is referred to Schaefer [93], Trèves [99], Jarchow [48] or Narici and Beckenstein [77].

1.1.1 Semi-norms

Let E be a vector space over a field $\mathbb{K}$, which will always be $\mathbb{R}$ or $\mathbb{C}$. A non-negative real-valued function p on E having the properties:

$$p(x+y) \leq p(x) + p(y), \quad p(\alpha x) = |\alpha| \, p(x), \quad \forall \, x, y \in E, \, \alpha \in \mathbb{K},$$

is called a **semi-norm** on E and a **norm** if it additionally satisfies: $x \neq 0$, implies $p(x) > 0$.

Let p be a semi-norm on E. The set $B_p(r) = \{x \in E : p(x) \leq r\}$ for $r > 0$, is called the **closed ball of radius** r of p. In the case $r = 1$ we call it the **closed unit ball** of p. The closed ball $B_p(r)$ is a convex, balanced and absorbing subset of E (see Notation and Useful Facts).

A semi-norm (respectively a norm) p is called **Hilbertian** if $p(x)^2 = Q(x,x)$, for all $x \in E$, where Q is a symmetric, non-negative bilinear form (respectively inner product) on $E \times E$.

Let p be a semi-norm on E and let $N(p) = \{x \in E : p(x) = 0\}$ be its null space. Let E_p be the Banach space that corresponds to the completion of the normed space $(E/N(p), \tilde{p})$ (where $\tilde{p}(x + N(p)) = p(x)$ for each $x \in E$). The quotient map $E \to E/N(p)$ has an unique linear extension $i_p : E \to E_p$ called the **inclusion or canonical map**. If p is Hilbertian then E_p is a Hilbert space with inner product induced by p in the obvious way. If the Banach space E_p is separable, then we will say that the semi-norm p is **separable**.

Let q be another semi-norm on E for which $p \leq Cq$, for some $C > 0$. In this case, $N(q) \subseteq N(p)$. Moreover, the canonical map from $E/N(q)$ into $E/N(p)$ is linear and continuous, and therefore it has a unique continuous extension $i_{p,q} : E_q \to E_p$, which is called again **canonical**. Furthermore, we have the following relation between canonical maps: $i_p = i_{p,q} \circ i_q$.

1.1.2 Locally Convex Spaces

A vector space E equipped with a topology $\mathcal{T}$ such that the addition and scalar multiplication are continuous is called a **topological vector space** and the topology $\mathcal{T}$ is called a **vector topology** for E.

For a topological vector space $(E, \mathcal{T})$, the topology $\mathcal{T}$ is completely determined by a local base of neighborhoods of zero. This is because the continuity of the addition operation implies that a neighborhood base of any element of E can be obtained by translation of a neighborhood base of zero in E (see Chapter 3 of Trèves [99]).

If K is a convex, balanced and absorbing subset of E, the **Minkowski functional** p_K of K, given by

$$p_K(x) = \inf\{\lambda > 0; \, x \in \lambda K\}, \quad \forall \, x \in E, \tag{1.1}$$

is a semi-norm on E. Moreover, p_K is continuous if and only if K is a neighborhood of zero in E. Furthermore, if K is closed then $K = B_{p_K}(1)$ (see Chapter 5 of of Narici and Beckenstein [77]).

A subset B of a topological vector space $(E, \mathcal{T})$ is said to be **bounded** if for any neighborhood of zero $U \subseteq E$, there exist some $\alpha > 0$ such that $B \subseteq \alpha U$. Finite unions, multiples by scalars and closures of bounded subsets are all bounded (see Chapter 1 of Schaefer [93]). Moreover, every finite subset is bounded and also any compact subset is bounded (see Proposition 14.1 of Trèves [99] p.137). If a semi-norm p on E is bounded on every bounded subset of E, we say that p is **locally bounded**.

A topological vector space $(E, \mathcal{T})$ is called a **locally convex space** if there exists a family $\{p_\alpha\}_{\alpha \in A}$ of semi-norms on E such that the collection of all the sets of the form:

$$\bigcap_{j=1,\dots,n} B_{p_{\alpha_j}}(r_j) = \{x \in E : p_{\alpha_j}(x) \leq r_j, \, j = 1, \dots, n\},$$

where $n \in \mathbb{N}$, $r_j > 0$, $\alpha_j \in A$ $(j = 1, \dots, n)$, is a local base of convex, closed, balanced, neighborhoods of zero in E. In that case, we say that the family of semi-norms $\{p_\alpha\}_{\alpha \in A}$ **generates** the topology $\mathcal{T}$. Furthermore, if this family of semi-norms satisfies the **separation condition**, i.e for any $x_0 \neq 0$, there exists $\alpha \in A$ such that $p_\alpha(x_0) \neq 0$, then the topology $\mathcal{T}$ is Hausdorff.

Now we introduce some important classes of locally convex spaces.

A **barreled space** E is a locally convex space for which every lower semicontinuous semi-norm on E is continuous. The locally convex spaces that are also Baire spaces are barrelled spaces (see Result 7.1, Chapter 2 of Schaefer [93], p.60).

A locally convex space is called **pseudo-metrizable** if its topology is generated by a countable family of semi-norms. A **pre-Fréchet space** is a Hausdorff locally convex space E that is pseudo-metrizable. Hence, every pre-Fréchet space is metrizable (see Theorem 1, Section 2.8 of Jarchow [48], p.40). A complete pre-Fréchet space is called a **Fréchet space**. Any Banach space is a Fréchet space. Moreover, from *Baire's category theorem* (see Theorem 4.4.10 of Narici and Beckenstein [77], p.82-3) any complete pseudo-metrizable space is a Baire space, therefore any Fréchet space is barrelled.

A **bornological space** is a locally convex space E for which every locally bounded semi-norm on E is continuous. Pre-Fréchet spaces are examples of bornological spaces (see Result Result 8.1, Chapter 2 of Schaefer [93], p.61).

There are examples of bornological spaces that are not barrelled and of barrelled spaces that are not bornological (see Schaefer [93], p.63 for references), however every quasi-complete bornological space is barrelled (see the Corollary of Result 8.4, Chapter 2 of Schaefer [93], p.63 for a proof). Recall that a locally convex space is **quasi-complete** if every bounded and closed subset is complete. Clearly, any complete locally convex space is quasi-complete.

1.1.3 Projective and Inductive Topologies

We start by defining projective limits. Let E be a vector space and let $\{(E_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in A}$ be a family of locally convex spaces. For each $\alpha \in A$, let Γ_α be a family of semi-norms generating the topology $\mathcal{T}_\alpha$ on E_α and let f_α be a linear map from E into E_α. The family $\{(E_\alpha, \mathcal{T}_\alpha, f_\alpha) : \alpha \in A\}$ is called a **projective system** on E. The **projective topology** $\mathcal{T}_p$ on E with respect to the projective system $\{(E_\alpha, \mathcal{T}_\alpha, f_\alpha) : \alpha \in A\}$ is the coarsest (weakest) locally convex topology on E with respect to which each of the mappings f_α is continuous. A family of semi-norms generating the projective topology is $\{p_\alpha \circ f_\alpha : p_\alpha \in \Gamma_\alpha, \alpha \in A\}$. The space $(E, \mathcal{T}_p)$ is called the **projective limit** of the family $\{E_\alpha\}_{\alpha\in A}$ determined by the mappings f_α and we denoted it by $\mathrm{proj}_{\alpha\in A} E_\alpha$.

Now we define inductive limits. Let E be a vector space and let $\{(E_\alpha, \mathcal{T}_\alpha)\}_{\alpha\in A}$ be a family of locally convex spaces. Suppose for each $\alpha \in A$ that f_α is a linear map from E_α into E and that the linear span of $\bigcup_{\alpha\in A} f_\alpha(E_\alpha)$ is E. The family $\{(E_\alpha, \mathcal{T}_\alpha, f_\alpha) : \alpha \in A\}$ is called an **inductive system** on E. The **inductive topology** $\mathcal{T}_i$ on E with respect to the inductive system $\{(E_\alpha, \mathcal{T}_\alpha, f_\alpha) : \alpha \in A\}$ is the finest (strongest) locally convex topology on E with respect to which each of the mappings f_α is continuous. A local base of neighborhoods of zero for $\mathcal{T}_i$ is the family $\mathfrak{U}$ of all the convex, balanced, absorbing subsets of E such that for each $\alpha \in A$, $U \in \mathfrak{U}$, $f_\alpha^{-1}(U)$ is a neighborhood of zero in $(E_\alpha, \mathcal{T}_\alpha)$. The space $(E, \mathcal{T}_i)$ is called the **inductive limit** of the family $\{E_\alpha\}_{\alpha\in A}$ determined by the mappings f_α and we denoted it by $\mathrm{ind}_{\alpha\in A} E_\alpha$.

A locally convex space that is (isomorphic to) the inductive limit of a family of Banach spaces is called **ultrabornological**. An ultrabornological space is both bornological and barrelled (see Chapter 13 of Jarchow [48]). Conversely, any sequentially complete, Hausdorff, bornological space is ultrabornological (see Theorem 13.2.12 of Narici and Beckenstein [77]). As we shall see in Chapters 1, 3 and 4, ultrabornological spaces play a fundamental role in this thesis and the main reason for that is because a very general version of the closed graph theorem holds in these spaces (see Theorem 1.1.3).

1.1.4 Linear Operators between Topological Vector Spaces

Let E and F be two topological vector spaces. We denote by $\mathcal{L}(E,F)$ the collection of all the continuous linear maps T on E into F with $\mathrm{Dom}(T)=E$.

Let X, Y be topological spaces and let $T:\mathrm{Dom}(T)\subseteq X\to Y$. If the *graph* $\{(x,Tx):x\in\mathrm{Dom}(T)\}$ of T is a closed (respectively sequentially closed) subspace of $X\times Y$, then we say that T is a **closed map** (respectively a sequentially closed map). Clearly, any closed map is sequentially closed. The converse is also true when both X and Y are first countable. A closed linear map between two topological vector spaces will be called a **closed linear operator**.

A very useful criteria for a map to be closed (or sequentially closed) is given in the following result. For a proof see Theorem 14.1.1 of Narici and Beckenstein [77], p.460.

Theorem 1.1.1. *Let X and Y be topological spaces and $T:Dom(T)\subseteq X\to Y$. T is a closed (respectively sequentially closed) map if an only if for any net (respectively for any sequence) $\{x_\alpha\}_\alpha\subseteq Dom(T)$,*

$$\lim x_\alpha = x \text{ and } \lim Tx_\alpha = y \quad \text{imply that} \quad x\in Dom(T) \text{ and } Tx=y. \tag{1.2}$$

Let E, F be topological vector spaces. If F is Hausdorff, any continuous map $T:E\to F$ is closed. The converse is not true in general even if T is linear. However, the following two results give conditions on E and F for this to be true. For a proof of the first see Theorem 14.3.4 of Narici and Beckenstein [77], p.465-6 and for the second see Theorem 2, Section 5.4 of Jarchow [48], p.94.

Theorem 1.1.2 (Closed graph theorem)**.** *Let E be a Baire topological vector space and let F be a complete, metrizable topological vector space. Then, every closed linear operator $T:E\to F$ is continuous.*

Theorem 1.1.3 (Closed graph theorem)**.** *Let E be the inductive limit of a family of metrizable, Baire, topological vector spaces spaces (e.g if E is a ultrabornological space) and let F be a complete, metrizable, topological vector space. Then, every sequentially closed linear operator $T:E\to F$ is continuous.*

1.1.5 Dual Topologies, Operators and Reflexivility

Let $(E,\mathcal{T})$ be a topological vector space. The (topological) **dual space** of E is the set E' of all $\mathcal{T}$-continuous linear maps from E into $\mathbb{K}$ (i.e. $\mathbb{R}$ or $\mathbb{C}$). For any $x\in E$ and any $f\in E'$, we shall denote by $f[x]$ the value of f at the point x.

For $A\subseteq E$, we define its **polar** as $A^0=\{f\in E' : \sup_{x\in A}|f[x]|\leq 1\}\subseteq E'$. The set A^0 is a convex, balanced subset of E' and if A is bounded, A^0 is also absorbing (see Proposition 19.1. of Trèves [99] p.196). Also, if $A\subseteq B$ then $B^0\subseteq A^0$; furthermore $(cA)^0=(1/c)A^0$ for $c>0$. Moreover, for any $A,B\subseteq E$, $(A\cup B)^0=A^0\cap B^0$.

Now we introduce vector topologies on E'. Let $\mathfrak{B}$ be a family of all the bounded subsets of E. For each $B\in\mathfrak{B}$ let $\eta_B:E'\to\mathbb{R}_+$ given by

$$\eta_B(f):=p_{B^0}(f)=\sup_{\psi\in B}|f[\psi]|, \quad \forall\, f\in E'.$$

where p_{B^0} is the Minkowski functional of the polar B^0 of B. For each $B\in\mathfrak{B}$, η_B is a semi-norm on E' and the family $\{\eta_B : B\in\mathfrak{B}\}$ satisfies the separation condition (see Section 1.1.2). Therefore, the family $\{\eta_B : B \in \mathfrak{B}\}$ generates a unique Hausdorff locally convex topology β on E' that we will call the **strong topology**. We denote by E'_β the space (E', β) and we call it the **strong dual** of E.

If instead $\mathfrak{B}$ is the family of all the finite subsets of E, then the Hausdorff locally convex topology σ generated by the family of semi-norms $\{\eta_B : B \in \mathfrak{B}\}$ is called the **weak topology**. We denote the space (E', σ) by E'_σ. In general, $\sigma \subseteq \beta$.

Of great importance in this thesis is the concept of reflexive spaces. Let $(E, \mathcal{T})$ be a Hausdorff locally convex space. The space E can be identified with a proper subspace of its **bidual** $E'' = (E'_\beta)'$ by means of the **canonical embedding** from E into E'', i.e. the map $E \ni x \to \widehat{x} \in E''$ given by $\widehat{x}(f) = f[x]$ for each $f \in E'$. We say that E is **semi-reflexive** when the canonical embedding is surjective, i.e. if $E = E''$. If furthermore the canonical embedding is a topological isomorphism (for the topological vector space structures) of $(E, \mathcal{T})$ into the **strong bidual** $(E'_\beta)'_\beta$, then E is said to be **reflexive**. In particular, every Hilbert space is reflexive.

Now, let E and F be topological vector spaces and let $T : \mathrm{Dom}(T) \subseteq E \to F$ be a linear operator. If $\mathrm{Dom}(T)$ is dense in E (we say that T is **densely defined**), we define the **dual or adjoint operator** of T as the map $T' : F' \to E'$ defined by

$$\mathrm{Dom}(T') = \big\{ g \in F' \,:\, \exists\, f \in E' \text{ s.t. } g[Tx] = f[x],\, \forall\, x \in \mathrm{Dom}(T) \big\}$$

and

$$T'g = f,\, \forall g \in \mathrm{Dom}(T'),$$

i.e. the dual operator satisfies

$$T'g[x] = g[Tx], \quad \forall\, x \in \mathrm{Dom}(T),\, g \in \mathrm{Dom}(T'). \tag{1.3}$$

Is clear from the definition that T' is a linear operator. If $T \in \mathcal{L}(E, F)$ and both E and F are locally convex, then $T' \in \mathcal{L}(F'_\beta, E'_\beta)$ (see Proposition 19.5 of Trèves [99], p.199).

Let E be a locally convex space and let p be a continuous semi-norm on E. Let E_p be as defined in Section 1.1.1. The canonical inclusion $i_p : E \to E_p$ is continuous. Also, the set $K = B_p(1)^0$ is a closed, bounded, convex, balanced subset of E' equipped with any topology between σ and β. Moreover, one can prove (see Chapter 47 of Trèves [99]) that the dual space E'_p of E_p corresponds to the linear subspace $\bigcup_{n \in \mathbb{N}} nK$ of E' generated by K and equipped with the norm defined by the Minkowski functional p_K, i.e. the dual norm p' on E'_p is given by

$$p'(f) = p_{B_p(1)^0}(f) := \sup\{ |f[x]| : x \in E,\, p(x) \leq 1 \}, \quad \forall\, f \in E'_p. \tag{1.4}$$

The space E'_p is a Banach space and the dual operator i'_p corresponds to the canonical inclusion from E'_p into E'_β.

Of frequent use will be the following inequality that follows from (1.4)

$$|f[x]| \leq p'(f)p(x), \quad \forall\, x \in E_p,\, f \in E'_p. \tag{1.5}$$

1.1.6 Nuclear Spaces

We now introduce the most important class of locally convex spaces for this thesis. For a review of relevant facts about Hilbert-Schmidt operators the reader if referred to Appendix B.

Definition 1.1.4. A (Hausdorff) locally convex space $(\Phi, \mathcal{T})$ is **nuclear** if there exist a family of Hilbertian semi-norms $\{p_\alpha\}_{\alpha \in A}$ generating $\mathcal{T}$ such that for each $\alpha \in A$, there exists some $\beta \in A$, such that $p_\alpha \leq p_\beta$ and $i_{p_\alpha, p_\beta} : \Phi_{p_\beta} \rightarrow \Phi_{p_\alpha}$ is Hilbert-Schmidt.

An equivalent and very useful characterization of nuclear spaces is the following:

Proposition 1.1.5. *A locally convex space Φ is nuclear if and only if its topology is generated by a family of Hilbertian semi-norms and for each continuous Hilbertian semi-norm p on Φ, there exists another continuous Hilbertian semi-norm q on Φ, such that $p \leq q$ and the canonical inclusion $i_{p,q} : \Phi_q \rightarrow \Phi_p$ is Hilbert-Schmidt.*

Remark 1.1.6. *An important consequence of the nuclearity of the space Φ is that for every continuous Hilbertian semi-norm p on Φ, the Banach space Φ_p is separable (see Proposition 4.4.9 of Pietsch [86], p. 82) . If moreover p is Hilbertian, then both Φ_p and Φ'_p are separable and therefore it is always possible to find a complete orthonormal system $\{\phi_j^p\}_{j \in \mathbb{N}}$ of Φ_p that is contained in Φ.*

The class of barrelled nuclear spaces and its strong dual will play an special role in this thesis. The next result contains some of their more important properties. For proofs see the results on nuclear spaces and reflexive spaces in Chapters III and IV of Schaefer [93], and Chapter 50 of Trèves [99].

Theorem 1.1.7. *Let Φ be a barrelled nuclear space. Then,*

*(1) Let $\mathcal{K} = \{B_p(1)^0 : p$ is a continuous Hilbertian semi-norm on $\Phi\}$. Each member of the family $\mathcal{K}$ is a closed, bounded, convex, balanced subset of Φ'_β. Moreover, $\mathcal{K}$ is a **fundamental system of balanced subsets** of Φ'_β, i.e. every bounded subset of Φ'_β is contained in a suitable member of $\mathcal{K}$.*

(2) If Φ is additionally quasi-complete (e.g. if it is complete), then Φ and Φ'_β are both reflexive and every closed, bounded subset of them is also compact.

Remark 1.1.8. *If Φ is a Banach space that is also nuclear then it is necessarily of (algebraic) finite dimension. See Corollary 2, Chapter 2 of Trèves [99].*

Let Φ be a nuclear space. We now introduce the concept of countably Hilbertian topologies on Φ. It will play a key role in our study of Φ'_β-valued random variables in Section 1.2.

Let $\{p_n\}_{n \in \mathbb{N}}$ be an increasing sequence of continuous Hilbertian semi-norms on Φ. Then, for every $n \in \mathbb{N}$, Φ_{p_n} is a separable Hilbert space (see Remark 1.1.6) and the canonical inclusion $i_{p_n} : \Phi \rightarrow \Phi_{p_n}$ is linear and continuous. Moreover, for $m \geq n$ the canonical inclusion $i_{p_n, p_m} : \Phi_{p_m} \rightarrow \Phi_{p_n}$ is linear and continuous. Let θ denotes the completion of the locally convex topology on Φ generated by the semi-norms $\{p_n\}_{n \in \mathbb{N}}$. We call θ the **countably Hilbertian topology on Φ determined by the semi-norms** $\{p_n\}_{n \in \mathbb{N}}$ and we denote by Φ_θ the space (Φ, θ).

In general, Φ_θ is a separable, complete, pseudo-metrizable space and hence Baire. Moreover, the dual Φ'_θ of Φ_θ satisfies

$$\Phi'_\theta = \bigcup_{n \in \mathbb{N}} \Phi'_{p_n}, \tag{1.6}$$

It is very important the fact that the topology θ is weaker than the nuclear topology on Φ. This is satisfied because every neighborhood of zero with respect to the topology

θ is a neighboorhood of zero with respect to the nuclear topology on Φ. Therefore, the canonical inclusion $i_\theta : \Phi \to \Phi_\theta$ is linear and continuous and for each $n \in \mathbb{N}$, the map i_{p_n} is linear and continuous from Φ_θ into Φ_{p_n}. By duality, the above implies that the dual map $i'_\theta : (\Phi'_\theta, \beta_\theta) \to \Phi'_\beta$ is linear and continuous and that for each $n \in \mathbb{N}$, the map i'_{p_n} from Φ'_{p_n} into $(\Phi'_\theta, \beta_\theta)$ is linear and continuous, where β_θ denotes the strong topology on Φ'_θ.

1.1.6.1 Examples of Nuclear Spaces

Example 1.1.9. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Then, $\mathbb{K}^{\mathbb{N}}$ equipped with the product topology is a separable, complete, bornological, barrelled nuclear space. See Kalton [57].

Example 1.1.10. Let X be a non-empty open subset of $\mathbb{R}^d$. Let $\mathcal{C}^\infty(X)$ be the space of all functions $f : X \to \mathbb{C}$ such that f is infinitely differentiable. Let $\{K_j\}_{j\in\mathbb{N}}$ be an increasing sequence of compact subsets of X with non-empty interior such that $X = \bigcup_{j\in\mathbb{N}} K_j$ (see Lemma 10.1 of Trèves [99], p.87). For every $n \in \mathbb{N}$, define a semi-norm p_n on $\mathcal{C}^\infty(X)$ by

$$p_n(f) = \sup_{|\alpha|\leq n} \sup_{x\in K_n} |D^\alpha f(x)|, \quad \forall f \in \mathcal{C}^\infty(X),$$

where $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\alpha = (\alpha_1, \dots, \alpha_d)$ is a d-tuple of non-negative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $D^\alpha f(x) = \partial^{|\alpha|} f(x)/\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}$. The space $\mathcal{C}^\infty(X)$ equipped with the topology generated by the family of semi-norms $\{p_n(\cdot) : n \in \mathbb{N}\}$ is a Fréchet nuclear space (see Example 28.9 of Meise and Vogt [65] p.349).

Example 1.1.11. Let U be a non-empty open subset of $\mathbb{R}^d$. A real-valued function $f \in \mathcal{C}^2(U)$ is said to be *harmonic* on U if for each ball $B(x,\epsilon) = \{y \in \mathbb{R}^d : ||x-y|| < \epsilon\} \subseteq U$, f satisfies the mean value property

$$f(x) = \frac{1}{r^d V_d} \int_{B(x,r)} f(y)dy$$

Here V_d is the volume of the d-dimensional unit ball.

The set $\mathcal{H}(U)$ of all harmonic functions on U is a linear space under the usual operations of pointwise sum and multiplication by scalar of functions. Consider the family of semi-norms $||\cdot||_K$ on $\mathcal{H}(U)$ given by

$$||f||_K = \sup_{x\in K} |f(x)|, \quad \forall f \in \mathcal{H}(U),$$

where K is a compact subset of $\mathbb{R}^d$ contained in U. Equipped with the topology generated by this family of semi-norms, $\mathcal{H}(U)$ is a Fréchet nuclear space (see Theorem 6.3.3 of Pietsch [86], p.103-4).

Example 1.1.12. Let $\mathcal{S}(\mathbb{R}^d)$ be the space of all $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that for any d-tuple of non-negative integers $\alpha = (\alpha_1, \dots, \alpha_d)$ and $n \in \mathbb{N} \cup \{0\}$, $\lim_{||x||\to\infty} ||x||^n |D^\alpha f(x)| = 0$. The space $\mathcal{S}(\mathbb{R}^d)$ equipped with the topology generated by the family $\{||\cdot||_{m,n} : m, n \in \mathbb{N}\}$ of semi-norms

$$||f||_{m,n} = \sup_{x\in\mathbb{R}^d} \sup_{|\alpha|\leq m} (1+|x|)^n |D^\alpha f(x)| < \infty, \quad \forall m, n \in \mathbb{N},$$

is a Fréchet nuclear space (see Example IV, Section 9, Chapter 10 and the Corollary of Theorem 51.5 of Trèves [99], p.92,530) and is called the *(Schwartz) space of rapidly decreasing functions.* Note that the topology in $\mathcal{S}(\mathbb{R}^d)$ is strictly finer than the subspace topology induced by $\mathcal{C}^\infty(\mathbb{R}^d)$. The dual space $\mathcal{S}'(\mathbb{R}^d)$ is called the *space of tempered distributions.*

§ 1.2 Cylindrical and Stochastic Processes in the Dual of a Nuclear Space

Assumption 1.2.1. *Throughout this section and unless otherwise specified* Φ *will denote a nuclear space over* $\mathbb{R}$.

Assumption 1.2.2. $(\Omega, \mathscr{F}, \mathbb{P})$ *will denote a complete probability space equipped with a filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *that satisfies the* ***usual conditions****, i.e. it is right continuous and* $\mathcal{F}_0$ *contains all sets of* $\mathcal{F}$ *of* $\mathbb{P}$*-measure zero. All our random variables are defined on* $(\Omega, \mathscr{F}, \mathbb{P})$*, unless otherwise stated. The space* $L^0(\Omega, \mathscr{F}, \mathbb{P})$ *of real-valued random variables defined on* $(\Omega, \mathscr{F}, \mathbb{P})$ *will be always assumed to be equipped with the topology of convergence in probability.*

In this section we review some concepts of cylindrical and stochastic processes in Φ'_β that we will use throughout this thesis. We start by studying Borel and cylindrical measures.

A Borel measure μ on Φ'_β is called a **Radon measure** if for every $\Gamma \in \mathcal{B}(\Phi'_\beta)$ and $\epsilon > 0$, there exist a compact set $K_\epsilon \subseteq \Gamma$ such that $\mu(\Gamma \backslash K_\epsilon) < \epsilon$. Equivalently, a Borel measure is a Radon measure if and only if (i) μ is **inner regular**, i.e. if for every $\Gamma \in \mathcal{B}(\Phi'_\beta)$ and $\epsilon > 0$, there exist a closed set $A_\epsilon \subseteq \Gamma$ such that $\mu(\Gamma \backslash A_\epsilon) < \epsilon$, and if (ii) μ is **tight**, i.e. if for every $\epsilon > 0$, there exist a compact subset $K_\epsilon \subseteq \Phi'_\beta$ such that $\mu(\Phi'_\beta \backslash K_\epsilon) < \epsilon$. In general not every Borel measure is Radon, however, when Φ is a Fréchet nuclear space or a countable inductive limit of Fréchet nuclear spaces, then every Borel measure on Φ'_β is a Radon measure (see Corollary 1.3 of Dalecky and Fomin [21], p.11).

We denote by $\mathfrak{M}^b_R(\Phi'_\beta)$ and by $\mathfrak{M}^1_R(\Phi'_\beta)$ the spaces of all bounded Radon measures and of all Radon probability measures on Φ'_β.

We will need the following terminology in Section 2.2.2. A subset $M \subseteq \mathfrak{M}^b_R(\Phi'_\beta)$ is called **uniformly tight** if

(1) $\sup\{\mu(\Phi'_\beta) : \mu \in M\} < \infty$,

(2) For all $\epsilon > 0$ there exist a compact $K \subseteq \Phi'_\beta$ such that $\mu(K^c) < \epsilon$ for all $\mu \in M$.

A subset $M \subseteq \mathfrak{M}^b_R(\Phi'_\beta)$ is called **shift tight** if for every $\mu \in M$ there exists $f_\mu \in \Phi'_\beta$ such that $\{\mu * \delta_{f_\mu} : \mu \in M\}$ is uniformly tight (see section Notations and Useful Facts for the definition of convolution of measures).

Now we proceed to introduce the concept of cylindrical measures on Φ'. We start defining cylindrical sets. For any $n \in \mathbb{N}$ and any $\phi_1, \dots, \phi_n \in \Phi$, we define a linear map $\pi_{\phi_1,\dots,\phi_n} : \Phi' \to \mathbb{R}^n$ by

$$\pi_{\phi_1,\dots,\phi_n}(f) = (f[\phi_1], \dots, f[\phi_n]), \quad \forall\, f \in \Phi'. \tag{1.7}$$

The map $\pi_{\phi_1,\dots,\phi_n}$ is clearly linear, moreover it is weakly continuous (and hence strongly continuous). Let M be a subset of Φ. A subset of Φ' of the form

$$\mathcal{Z}\left(\phi_1, \dots, \phi_n; A\right) = \left\{ f \in \Phi' : (f[\phi_1], \dots, f[\phi_n]) \in A \right\} = \pi^{-1}_{\phi_1,\dots,\phi_n}(A) \tag{1.8}$$

where $n \in \mathbb{N}$, $\phi_1, \dots, \phi_n \in M$ and $A \in \mathcal{B}(\mathbb{R}^n)$ is called a **cylindrical set** based on M. The set of all the cylindrical sets based on M is denoted by $\mathcal{Z}(\Phi', M)$. It is an algebra but if M is a finite set then it is a σ-algebra. The σ-algebra generated by $\mathcal{Z}(\Phi', M)$ is denoted by $\mathcal{C}(\Phi', M)$ and it is called the **cylindrical σ-algebra** with respect to (Φ', M). If $M = \Phi$, we write $\mathcal{Z}(\Phi') = \mathcal{Z}(\Phi', \Phi)$ and $\mathcal{C}(\Phi') = \mathcal{C}(\Phi', \Phi)$.

If we consider Φ' equipped with the strong topology β, one can easily see from (1.8) that $\mathcal{Z}(\Phi'_\beta) \subseteq \mathcal{B}(\Phi'_\beta)$. Therefore, $\mathcal{C}(\Phi'_\beta) \subseteq \mathcal{B}(\Phi'_\beta)$. In general this inclusion is strict but if Φ is separable (for example if it is a Fréchet nuclear space) then $\mathcal{C}(\Phi'_\beta) = \mathcal{B}(\Phi'_\beta)$ (see Lemma 4.1 in Mitoma, Okada and Okazaki [76]).

A function $\mu : \mathcal{Z}(\Phi') \to [0, \infty]$ is called a **cylindrical measure** on Φ', if for each finite subset $M \subseteq \Phi'$ the restriction of μ to $\mathcal{C}(\Phi', M)$ is a measure. A cylindrical measure μ is called **finite** if $\mu(\Phi') < \infty$ and a **cylindrical probability measure** if $\mu(\Phi') = 1$. Other equivalent definitions of cylindrical sets and cylindrical measures can be found for example in Exposé No.1 of Badrikian [7].

The restriction of any finite Borel measure on Φ'_β to the cylindrical σ-algebra $\mathcal{C}(\Phi')$ defines a finite cylindrical measure on Φ'. In general it is not true that any cylindrical measure on Φ' can be extended to be a Borel measure on Φ'_β. However, if such an extension exists it must be unique (e.g. see Section 7.12 of Bogachev [8]). Sufficient conditions for a cylindrical probability measure on Φ to define a Radon probability measure on Φ'_β are given by the Minlos theorem (Theorem 1.2.3) in terms of the continuity of its characteristic function that we define as follows.

Let μ be a finite cylindrical measure on Φ'. The complex-valued function $\widehat{\mu} : \Phi \to \mathbb{C}$ defined by

$$\widehat{\mu}(\phi) = \int_{\Phi'} e^{if[\phi]} \mu(df) = \int_{-\infty}^{\infty} e^{iz} \mu_\phi(dz), \quad \forall \phi \in \Phi,$$

where for each $\phi \in \Phi$, $\mu_\phi := \mu \circ \pi_\phi^{-1}$, is called the **characteristic function** or **Fourier transform** of μ. If μ and ν are two finite Radon measures on Φ'_β and $\widehat{\mu} = \widehat{\nu}$, then one has $\mu = \nu$. Moreover, $\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}$ where recall that $\mu * \nu$ denotes the convolution of μ and ν.

Recall that $F : \Phi \to \mathbb{C}$ is said to be **positive definite** if $\sum_{j,k=1}^{n} F(\phi_j - \phi_k)\alpha_j \overline{\alpha}_k \geq 0$, $\forall n \in \mathbb{N}$, $\phi_1, \dots, \phi_n \in \Phi$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. From the Bochner theorem, it follows that a complex-valued function $F : \Phi \to \mathbb{C}$ is the characteristic function of a cylindrical probability measure if and only if F is positive definite, continuous on finite dimensional subspaces of Φ and $F(0) = 1$. The following fundamental result was shown by R. A. Minlos in 1958-9 (see [66]) for the case of countably Hilbertian nuclear spaces. For a proof of the general case see Theorem 1.3, Chapter III of Dalecky and Fomin [21].

Theorem 1.2.3 (Minlos theorem). *Let Φ be a nuclear space. For a function F : $\Phi \to \mathbb{C}$ to be the characteristic function of a Radon probability measure on Φ'_β it is sufficient, and necessary if Φ is barrelled, that it be positive definite, continuous at zero and satisfies $F(0) = 1$.*

Remark 1.2.4. *An analogue of the Minlos theorem holds on any locally convex space E for a complex-valued function F defined on E that is positive definite, satisfying $F(0) = 1$, and that is continuous with respect to the so called Sazonov or Hilbert-Schmidt topology (see Section VI.4.2 of Vakhania, Tarieladze, Chobanyan [109] or Section 6.10 of Bourbaki [16]). The Hilbert space case was shown by Sazonov in 1958 (see [92]). In 1959, Kolmogorov (see [58]) extended the Sazonov topology to countably Hilbertian* *spaces and pointed out the connection between the works of Minlos and Sazonov. For that reason the Sazonov topology in the more general case of multi-Hilbertian spaces is often referred as the Kolmogorov topology. The extension to the case of Badrikian spaces (that is a class of locally convex spaces that generalizes both the Hilbert spaces and the dual of a barrelled nuclear space) was carried out by Badrikian in 1967 (see [6]). Using a different approach, Wu proved a generalization of the results of Minlos and Sazonov for the characteristic functional of a Radon probability measure in the weak dual of a multi-Hilbertian space (see [113]) and for the characteristic functional of a* σ*-concentrated Radon probability measure in the weak dual of a locally convex space (see [116]).*

Next we review some basic concepts of random variables and measures on Φ'_{β}. Let X be a Φ'_{β}-valued random variable (i.e. $X : \Omega \rightarrow \Phi'_{\beta}$ is $\mathscr{F}/\mathcal{B}(\Phi'_{\beta})$-measurable). The **distribution** μ_X of X is defined by $\mu_X(\Gamma) = \mathbb{P}(X \in \Gamma)$, $\forall\, \Gamma \in \mathcal{B}(\Phi'_{\beta})$, and is a Borel probability measure on Φ'_{β}.

Two Φ'_{β}-valued random variables X and Y are said to be **equivalent** if $\mathbb{P}(\omega \in \Omega : X(\omega) = Y(\omega)) = 1$. In such a case each of these random variables is said to be a **version** or **modification** of the other.

A net $\{X_i\}_{i \in I}$ of Φ'_{β}-valued random variables **converges almost surely** to some Φ'_{β}-valued random variable X (we write $X_i \overset{as}{\rightarrow} X$) if for each bounded subset $B \subseteq \Phi$, $\lim_{i \in I} \sup_{\phi \in B} \left| X_i(\omega)[\phi] - X(\omega)[\phi] \right| = 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

On the other hand, a net $\{X_i\}_{i \in I}$ of Φ'_{β}-valued random variables **converges in probability** to some Φ'_{β}-valued random variable X (we write $X_i \overset{\mathbb{P}}{\rightarrow} X$) if for each $\epsilon > 0$ and each bounded subset $B \subseteq \Phi$, $\lim_{i \in I} \mathbb{P}\left(\omega \in \Omega : \sup_{\phi \in B} \left| X_i(\omega)[\phi] - X(\omega)[\phi] \right| \geq \epsilon \right) = 0$.

As the Hausdorff locally convex topology on Φ'_{β} is an uniform structure (in the topological sense), then if a sequence $\{X_n\}_{n \in \mathbb{N}}$ of Φ'_{β}-valued random variables converges almost surely to some Φ'_{β}-valued random variable X, then it also converges in probability (Proposition 1, Chapter V, Part 2 of Schwartz [95], p.248) to X. This result is false in the case of convergence of nets.

Now we proceed to introduce the most important class of Φ'_{β}-valued random variables utilized in this thesis.

Definition 1.2.5. A Φ'_{β}-valued random variable X is called **regular** if there exists an increasing sequence $\{p_n\}_{n \in \mathbb{N}}$ of continuous Hilbertian semi-norms on Φ such that $\mathbb{P}\left(X \in \bigcup_{n \in \mathbb{N}} \Phi'_{p_n}\right) = 1$.

Our definition of regular random variable has the following implication: *If* X *and* $\{p_n\}_{n \in \mathbb{N}}$ *are as in Definition 1.2.5, and if* θ *is the countably Hilbertian topology on* Φ *determined by the semi-norms* $\{p_n\}_{n \in \mathbb{N}}$ *(see Section 1.1.6), then* (1.6) *implies that* $\mathbb{P}(X \in \Phi'_{\theta}) = 1$.

Remark 1.2.6. *In [43], Itô defined a* σ*-concentrated* Φ'*-valued random variable to be a* $\mathscr{F}/\mathcal{C}(\Phi')$*-measurable map* $X : \Omega \rightarrow \Phi'$ *for which there exists a countably Hilbertian topology on* Φ *such that* $\mathbb{P}(X \in \Phi'_{\theta}) = 1$. *If* Φ *satisfies that* $\mathcal{C}(\Phi') = \mathcal{B}(\Phi'_{\beta})$ *(e.g. if* Φ *is a Fréchet nuclear space), then the definitions of* σ*-concentrated and regular random variables coincide. We chose to utilize the terminology of* regular *random variable to emphasize its connection with the regularization theorem of Itô and Nawata in [44] (see Theorem 1.2.14 below).*

An important consequence of the definition of a regular random variable is the following.

Proposition 1.2.7. *Let X be a Φ'_{β}-valued regular random variable and let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of continuous Hilbertian semi-norms as in Definition 1.2.5. Let θ be the countably Hilbertian topology on Φ determined by the semi-norms $\{p_n\}_{n\in\mathbb{N}}$. Then, X has a $(\Phi'_{\theta},\beta_{\theta})$-valued version.*

Proof. Let $\Omega_{\theta}=\{\omega\in\Omega : X(\omega)\in\Phi'_{\theta}\}$. By our hypothesis $\mathbb{P}(\Omega_{\theta})=1$. Let $\tilde{X}$ be given by $\tilde{X}(\omega)=X(\omega)$ if $\omega\in\Omega_{\theta}$ and $\tilde{X}(\omega)=0$ if $\omega\in\Omega^{c}_{\theta}$. Now we show that $\tilde{X}$ is a Φ'_{θ}-valued random variable.

First, as Φ_{θ} is separable (see Section 1.1.6), then $\mathcal{C}(\Phi'_{\theta})=\mathcal{B}((\Phi'_{\theta},\beta_{\theta}))$ (see Lemma 4.1 in Mitoma, Okada and Okazaki [76]). Therefore, it is sufficient (and necessary) to show that $\tilde{X}$ is $\mathcal{F}/\mathcal{C}(\Phi'_{\theta})$-measurable. Let Z be a cylindrical subset of Φ'_{θ}. Then, $i'_{\theta}Z\in\mathcal{C}(\Phi'_{\beta})$ and hence $i'_{\theta}Z\in\mathcal{B}(\Phi'_{\beta})$, where recall that i_{θ} is the canonical inclusion from Φ into Φ_{θ}. Moreover, as X is a Φ'_{β}-valued random variable and from the definition of $\tilde{X}$ it follows that $\tilde{X}^{-1}(Z)=X^{-1}(i'_{\theta}Z)\cap\Omega_{\theta}\in\mathcal{F}$. Therefore, as the cylindrical subsets of Φ'_{θ} generates the cylindrical σ-algebra $\mathcal{C}(\Phi'_{\theta})$, then $\tilde{X}$ is $\mathcal{F}/\mathcal{C}(\Phi'_{\theta})$-measurable. Consequently, $\tilde{X}$ is a Φ'_{θ}-valued random variable that is a version of X. □

Definition 1.2.8. If X is a Φ'_{β}-valued random variable, for $\phi\in\Phi$ we denote by $X[\phi]$ the real-valued random variable defined by $X[\phi](\omega):=X(\omega)[\phi]$, for all $\omega\in\Omega$.

Proposition 1.2.9. *Let X be a Φ'_{β}-valued regular random variable and let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of continuous Hilbertian semi-norms as in Definition 1.2.5. Let θ be the countably Hilbertian topology on Φ determined by the semi-norms $\{p_n\}_{n\in\mathbb{N}}$. Then, the map $X:\Phi\rightarrow L^{0}(\Omega,\mathscr{F},\mathbb{P})$ given by $\phi\mapsto X[\phi]$ is linear and θ-continuous.*

Proof. The map $X:\Phi\rightarrow L^{0}(\Omega,\mathscr{F},\mathbb{P})$ is clearly linear. To prove its continuity, let $\tilde{X}$ be a Φ'_{θ}-valued random variable that is a version of X. Such a version exists by Proposition 1.2.7. Then, $X=i'_{\theta}\tilde{X}$ $\mathbb{P}$-a.e., where recall that i_{θ} is the canonical inclusion from Φ into Φ_{θ} and it is linear and continuous. Now, as the space Φ_{θ} is a separable pseudo-metrizable space (see Section 1.1.6), then by standard arguments it can be show that the map $\tilde{X}:\Phi_{\theta}\rightarrow L^{0}(\Omega,\mathscr{F},\mathbb{P})$ given by $\phi\mapsto\tilde{X}[\phi]$ is linear and continuous.

On the other hand, the relation $X=i'_{\theta}\tilde{X}$ $\mathbb{P}$-a.e. implies that the maps X and $\tilde{X}$ satisfy $X[\phi]=\tilde{X}[i_{\theta}\,\phi]$, for all $\phi\in\Phi$. Therefore, the continuity of $\tilde{X}:\Phi_{\theta}\rightarrow L^{0}(\Omega,\mathscr{F},\mathbb{P})$ and of $i_{\theta}:\Phi\rightarrow\Phi_{\theta}$ implies that the map $X:\Phi\rightarrow L^{0}(\Omega,\mathscr{F},\mathbb{P})$ is θ-continuous. □

The following result establish a characterization of regular random variables.

Theorem 1.2.10. *Let Φ be a nuclear space. For a Φ'_{β}-valued random variable to be regular it is necessary, and sufficient if Φ is barrelled, that its distribution be a Radon probability measure.*

Proof. If X is regular, then Proposition 1.2.9 shows that the map $X:\Phi\rightarrow L^{0}(\Omega,\mathscr{F},\mathbb{P})$ given by $\phi\mapsto X[\phi]$ is linear and θ-continuous. But as the topology θ is weaker than the nuclear topology on Φ then the map X is also continuous on Φ. This in turn implies that the characteristic function of X is continuous. Then, Minlos theorem shows that the distribution μ_X of X is a Radon probability measure.

Conversely, assume that Φ is barrelled and that μ_X is a Radon measure. Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive real numbers converging to zero. For every $n\in\mathbb{N}$, because μ_X is tight there exists a compact subset K_n of Φ'_{β} such that $\mu_X(K_n)>1-\epsilon_n$.

On the other hand, as Φ is barrelled it follows from Theorem 1.1.7(1) that there exists a continuous Hilbertian semi-norm p_n on Φ such that $K_n \subseteq B_{p_n}(1)^0$. Then, as $B_{p_n}(1)^0$ is the unit ball of the Hilbert space Φ'_{p_n}, it follows that

$$\mathbb{P}(X \in \Phi'_{p_n}) = \mu_X(\Phi'_{p_n}) \geq \mu_X(B_{p_n}(1)^0) \geq \mu_X(K_n) > 1 - \epsilon_n.$$

Then, as $\epsilon_n \to 0$ it follows from the above inequality that $\mathbb{P}\left(X \in \bigcup_{n\in\mathbb{N}} \Phi'_{p_n}\right) = 1$. Finally, by defining $\varrho_n^2 = \sum_{j=1}^n p_n^2$ for each $n \in \mathbb{N}$, then $\{\varrho_n\}_{n\in\mathbb{N}}$ is an increasing sequence of continuous Hilbertian semi-norms on Φ such that $\bigcup_{n\in\mathbb{N}} \Phi'_{\varrho_n} = \bigcup_{n\in\mathbb{N}} \Phi'_{p_n}$. Therefore, $\mathbb{P}\left(X \in \bigcup_{n\in\mathbb{N}} \Phi'_{\varrho_n}\right) = 1$ and hence X is a regular random variable. □

Remark 1.2.11. *Let Φ be a Fréchet nuclear space or a countable inductive limit of Fréchet nuclear spaces. Then every Φ'_β-valued random variable Y is regular. This is due to the fact that the distribution of Y is a Radon measure, and as Φ is barrelled it follows from Theorem 1.2.10 that Y is regular.*

Some other useful properties of regular random variables are given below. We have not been able find then in the literature.

Proposition 1.2.12. *Let X, Y be Φ'_β-valued random regular variables. Then, $X = Y$ $\mathbb{P}$-a.e. if and only if for all $\phi \in \Phi$, $X[\phi] = Y[\phi]$ $\mathbb{P}$-a.e.*

Proof. The necessity is clear. Assume X and Y are regular and let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of Hilbertian semi-norms on Φ such that Definition 1.2.5 is satisfied for both X and Y. If θ is the countably Hilbertian topology on Φ determined by the semi-norms $\{p_n\}_{n\in\mathbb{N}}$, then $\mathbb{P}(\Omega_\theta) = 1$, where $\Omega_\theta = \{\omega \in \Omega : X(\omega) \in \Phi'_\theta, Y(\omega) \in \Phi'_\theta\}$. Because Φ_θ is separable, there exists a countable subset $\{\phi_j : j \in \mathbb{N}\}$ of Φ that is dense in Φ_θ. For every $j \in \mathbb{N}$, it follows from our hypothesis that $\mathbb{P}(\Omega_j) = 1$, where $\Omega_j = \{\omega \in \Omega : X(\omega)[\phi_j] = Y(\omega)[\phi_j]\}$. Let $\Gamma = \Omega_\theta \cap \bigcap_{j\in\mathbb{N}} \Omega_j$. Then, we have $\mathbb{P}(\Gamma) = 1$. Fix $\omega \in \Gamma$ and let $\phi \in \Phi$. Then, there exists a sequence $\{\phi_{j_k}\}_{k\in\mathbb{N}} \subseteq \{\phi_j : j \in \mathbb{N}\}$ that converges to ϕ in Φ_θ. Therefore, as $X(\omega), Y(\omega) \in \Phi'_\theta$ and because $X(\omega)[\phi_{j_k}] = Y(\omega)[\phi_{j_k}]$ for all $k \in \mathbb{N}$, it follows that

$$X(\omega)[\phi] = \lim_{k\to\infty} X(\omega)[\phi_{j_k}] = \lim_{k\to\infty} Y(\omega)[\phi_{j_k}] = Y(\omega)[\phi].$$

As the above is true for any $\phi \in \Phi$, it follows that $X(\omega) = Y(\omega)$ for every $\omega \in \Gamma$. Therefore, $X = Y$ $\mathbb{P}$-a.e. □

Proposition 1.2.13. *Let $X^1, \dots, X^k$ be Φ'_β-valued regular random variables. Then, $X^1, \dots, X^k$ are independent if and only if for every $n \in \mathbb{N}$ and $\phi_1, \dots, \phi_n \in \Phi$, the $\mathbb{R}^n$-valued random variables $(X^1[\phi_1], \dots, X^1[\phi_n])$, ..., $(X^k[\phi_1], \dots, X^k[\phi_n])$ are independent.*

Proof. The necessity follows from the independence of the Φ'_β-valued random variables $X^1, \dots, X^k$ and the fact that $\mathcal{C}(\Phi'_\beta) \subseteq \mathcal{B}(\Phi'_\beta)$. For the sufficiency, we will show the case $k = 2$ to simplify the exposition, the general case can be proved using similar arguments.

First, as X^1 and X^2 are regular, it follows from Proposition 1.2.7 that there exists a countably Hilbertian topology θ on Φ, weaker than the nuclear topology, and two Φ'_θ-valued random variables $\tilde{X}^1$ and $\tilde{X}^2$ such that $X^1 = i'_\theta \tilde{X}^1$ and $X^2 = i'_\theta \tilde{X}^2$ $\mathbb{P}$-a.e. Now, because the map i'_θ is continuous and hence measurable (recall the dual of a continuous linear operator is continuous and $i_\theta : \Phi \to \Phi_\theta$ is continuous because θ is weaker than the nuclear topology), to show that X^1 and X^2 are independent it is sufficient to show that $\tilde{X}^1$ and $\tilde{X}^2$ are independent as Φ'_θ-valued random variables.

To do this, note that as $\mathcal{C}(\Phi'_\theta) = \mathcal{B}(\Phi'_\theta)$, then it is enough to show that

$$\mathbb{P}\left(\tilde{X}^1 \in Z_1, \tilde{X}^2 \in Z_2\right) = \mathbb{P}\left(\tilde{X}^1 \in Z_1\right)\mathbb{P}\left(\tilde{X}^2 \in Z_2\right), \quad \forall Z_1, Z_2 \in \mathcal{Z}(\Phi'_\theta), \tag{1.9}$$

where $\mathcal{Z}(\Phi'_\theta)$ denotes the collection of all cylindrical subsets of Φ'_θ. In effect, let $m, n \in \mathbb{N}$, $\phi_1, \dots, \phi_m$, $\varphi_1, \dots, \varphi_n \in \Phi$ and $B_1 \in \mathcal{B}(\mathbb{R}^m)$, $B_2 \in \mathcal{B}(\mathbb{R}^n)$. Then, $Z_1 = Z(\phi_1, \dots, \phi_m; B_1) \cap \Phi'_\theta$ and $Z_2 = Z(\varphi_1, \dots, \varphi_n; B_2) \cap \Phi'_\theta$ are cylindrical subsets of Φ'_θ. Now, if $A_1 = B_1 \times \mathbb{R}^n$ and $A_2 = \mathbb{R}^m \times B_2$, then one can easily check that $Z_1 = Z(\phi_1, \dots, \phi_m, \varphi_1, \dots, \varphi_n; A_1) \cap \Phi'_\theta$ and $Z_2 = Z(\phi_1, \dots, \phi_m, \varphi_1, \dots, \varphi_n; A_2) \cap \Phi'_\theta$. Therefore, from our hypothesis of independence of the $\mathbb{R}^{m+n}$-valued random variables $(X^j[\phi_1], \dots, X^j[\phi_m], X^j[\varphi_1], \dots, X^j[\varphi_n])$, $j = 1, 2$, and the fact that $X^j = i'_\theta \tilde{X}^j$ $\mathbb{P}$-a.e. $j = 1, 2$, then we have

$$\begin{aligned} \mathbb{P}\left(\tilde{X}^j \in Z_j; j = 1, 2\right) &= \mathbb{P}\left((X^j[\phi_1], \dots, X^j[\phi_m], X^j[\varphi_1], \dots, X^j[\varphi_n]) \in A_j; j = 1, 2\right) \\ &= \prod_{j=1,2} \mathbb{P}\left((X^j[\phi_1], \dots, X^j[\phi_m], X^j[\varphi_1], \dots, X^j[\varphi_n]) \in A_j\right) \\ &= \prod_{j=1,2} \mathbb{P}\left(\tilde{X}^j \in Z_j\right). \end{aligned}$$

Now because $\mathcal{Z}(\Phi'_\theta) = \mathcal{Z}(\Phi'_\beta) \cap \Phi'_\theta$, the above shows that (1.9) is satisfied and therefore $\tilde{X}^1$ and $\tilde{X}^2$ are independent, then so are X^1 and X^2, as described above. □

A **cylindrical random variable** in Φ' (or a *linear random functional* on Φ) is a linear map $X : \Phi \to L^0(\Omega, \mathscr{F}, \mathbb{P})$. Note that the linearity of X means that for each $\lambda_1, \lambda_2 \in \mathbb{R}$, $\phi_1, \phi_2 \in \Phi$,

$$X(\lambda_1 \phi_1 + \lambda_2 \phi_2) = \lambda_1 X(\phi_1) + \lambda_2 X(\phi_2), \quad \mathbb{P}\text{-a.e.}, \tag{1.10}$$

where it is important to stress that the exceptional $\mathbb{P}$-null set for which (1.10) holds might depend on λ_1, λ_2, ϕ_1, ϕ_2.

Two cylindrical random variables X and Y in Φ' are said to be equivalent if for any $n \in \mathbb{N}$, and any $\phi_1, \dots, \phi_n \in \Phi$, the $\mathbb{R}^n$-valued random variables $(X(\phi_1), \dots, X(\phi_n))$ and $(Y(\phi_1), \dots, Y(\phi_n))$ define the same distribution on $\mathbb{R}^n$.

There is a one to one relationship between (equivalence classes of) cylindrical random variables and cylindrical probability measures on Φ'. Indeed, let X be a cylindrical random variable in Φ'. A cylindrical probability measure μ_X on Φ' can be defined by the following prescription: if $Z = \mathcal{Z}(\phi_1, \dots, \phi_n; A)$ is a cylindrical set, for $\phi_1, \dots, \phi_n \in \Phi$ and $A \in \mathcal{B}(\mathbb{R}^n)$, let

$$\mu_X(Z) := \mathbb{P}\left((X[\phi_1], \dots, X[\phi_n]) \in A\right) = \mathbb{P} \circ X^{-1} \circ \pi^{-1}_{\phi_1, \dots, \phi_n}(A). \tag{1.11}$$

Conversely, if μ is a cylindrical probability measure on Φ', there exists a (unique up to equivalence) cylindrical random variable X_μ in Φ' satisfying (1.11) (see Chapter V, Par II of Schwartz [95], p.256-8). We say that X_μ is *associated* with μ.

If $X : \Phi \to L^0(\Omega, \mathscr{F}, \mathbb{P})$ is a cylindrical random variable in Φ', the **characteristic function** of X is defined to be the characteristic function $\widehat{\mu}_X : \Phi \to \mathbb{C}$ of its associated cylindrical probability measure μ_X. From (1.11), we have $\widehat{\mu}_X(\phi) = \mathbb{E}\left(e^{iX(\phi)}\right)$, $\forall \phi \in \Phi$.

Let X be a cylindrical random variable. We say that X is **n-integrable** if it has finite n-th moments, i.e. if $\mathbb{E}\left(|X(\phi)|^n\right) < \infty$, $\forall \phi \in \Phi$. When $n = 1$, we just say that X is **integrable** and if $n = 2$ we say that it is **square integrable**. If X is integrable and $\mathbb{E}\left(X(\phi)\right) = 0$, $\forall \phi \in \Phi$, then we say that X has **zero mean**.

If X is a cylindrical random variable in Φ', a Φ'_β-valued random variable Y is said to be a Φ'_β-valued **version** of X if for all $\phi \in \Phi$, $X(\phi)(\omega) = Y(\omega)[\phi]$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Furthermore, if X is regular we say that it is a **regular version** of Y.

It is clear that any Φ'_β-valued random variable X defines a cylindrical random variable $\tilde{X}$ in Φ' by means of the prescription $\tilde{X}(\phi) = X[\phi]$, for all $\phi \in \Phi$ (see Definition 1.2.8). Conversely, if X is a cylindrical random variable in Φ' it is not true in general that the sample functional $\omega \mapsto X(\omega) = \{X(\phi)(\omega) : \phi \in \Phi\}$ defines a Φ'_β-valued random variable. However, the next result shows a sufficient condition for this.

Theorem 1.2.14 (Regularization theorem)**.** *Let $X : \Phi \to L^0(\Omega, \mathscr{F}, \mathbb{P})$ be a cylindrical random variable in Φ' such that map $X : \Phi \to L^0(\Omega, \mathscr{F}, \mathbb{P})$ is continuous. Then, X has a unique (up to equivalence) Φ'_β-valued regular version.*

The regularization theorem was firstly proved by Itô and Nawata [44] (see also Theorem 2.3.2 of Itô [43]). An alternative proof can be found in Ramaswamy [89]. In Section 1.2.1 we will show a more general version of the regularization theorem.

Now we proceed to study Φ'_β-valued cylindrical and stochastic processes. Let $J = \mathbb{R}_+$ or $J = [0, T]$ for some $T > 0$. Let $X = \{X_t\}_{t \in J}$ be an Φ'_β-valued stochastic processes, i.e. X_t is a Φ'_β-valued random variable for each $t \in J$. We say that X is **regular** if X_t is regular, $\forall t \in J$.

Let $X = \{X_t\}_{t \in J}$ and $Y = \{Y_t\}_{t \in J}$ be Φ'_β- valued stochastic processes. Y is said to be a **version** or a **modification** of X if for each $t \in J$, X_t and Y_t are equivalent random variables. If furthermore $\mathbb{P}\left(\{\omega \in \Omega : X_t(\omega) = Y_t(\omega), \forall t \in J\}\right) = 1$, we say that X and Y are **indistinguishable**.

A Φ'_β-valued stochastic process X is **continuous** (respectively **càdlàg**) if for $\mathbb{P}$-a.e. $\omega \in \Omega$, the **sample path** $t \mapsto X_t(w) \in \Phi'_\beta$ of X is continuous (respectively right-continuous with left limits).

The Φ'_β- valued stochastic processes $X^j = \{X^j_t\}_{t \in J}$, $j = 1, \dots, k$, are said to be **independent** if for all $(n_1, \dots, n_k) \in \mathbb{N}^k$ and $(t_{j,1}, \dots, t_{j,n_j}) \in \mathbb{R}^{n_j}_+$, for $j = 1, \dots k$, the σ-algebras $\sigma\left(X^j_{t_{j,1}}, \dots, X^j_{t_{j,n_j}}\right)$, $j = 1, \dots, k$, are independent.

Some important properties of Φ'_β- valued processes are given below.

Proposition 1.2.15. *Let $X = \{X_t\}_{t \in J}$ and $Y = \{Y_t\}_{t \in J}$ be Φ'_β- valued regular stochastic processes such that for each $\phi \in \Phi$, $X[\phi] = \{X_t[\phi]\}_{t \in J}$ is a version of $Y[\phi] = \{Y_t[\phi]\}_{t \in J}$. Then X is a version of Y. Furthermore, if X and Y are right-continuous then they are indistinguishable processes.*

Proof. Fix $t \in J$. Then, as for each $\phi \in \Phi$, $X_t[\phi] = Y_t[\phi]$ $\mathbb{P}$-a.e., then Proposition 1.2.12 shows that $X_t = Y_t$ $\mathbb{P}$-a.e. Therefore, X is a version of Y. Now, assume that both X and Y are right-continuous. Let Ω_X and Ω_Y denote respectively the sets of all $\omega \in \Omega$ such that the maps $t \mapsto X_t(\omega)$ and $t \mapsto Y_t(\omega)$ are right-continuous. Let $\Gamma_{X,Y} = \{\omega \in \Omega : X_t(\omega) = Y_t(\omega), \forall t \in \mathbb{Q}_+\}$, where $\mathbb{Q}_+ = \mathbb{Q} \cap J$. Then, $\mathbb{P}(\Omega_X \cap \Omega_Y \cap \Gamma_{X,Y}) = 1$ and by the right-continuity of X and Y and the denseness of $\mathbb{Q}_+$ in J, it follows by a standard argument (see e.g Protter [88], p.4) that $X_t(\omega) = Y_t(\omega)$ for all $t \in J$, for each $\omega \in \Omega_X \cap \Omega_Y \cap \Gamma_{X,Y}$. Thus, X and Y are indistinguishable. $\square$

Proposition 1.2.16. *Let $X^1 = \left\{X^1_t\right\}_{t \in J}$, ..., $X^k = \left\{X^k_t\right\}_{t\in J}$ be Φ'_β- valued regular processes. Then, $X^1, \dots, X^k$ are independent if and only if for all $n \in \mathbb{N}$ and $\phi_1, \dots, \phi_n \in \Phi$, the $\mathbb{R}^n$-valued processes $\{(X^j_t[\phi_1], \dots, X^j_t[\phi_n]) : t \in J\}$, $j = 1, \dots, k$, are independent.*

Proof. The necessity follows from the independence of the Φ'_β-valued regular processes $X^1, \dots, X^k$ and the fact that $\mathcal{C}(\Phi'_\beta) \subseteq \mathcal{B}(\Phi'_\beta)$. As in Proposition 1.2.13 we will only show the case $k = 2$ as the general case can be proved using similar arguments.

Let $m, n \in \mathbb{N}$ and $t_1, \dots, t_m, s_1, \dots, s_n \in \mathbb{R}_+$. As X^1 and X^2 are regular processes, then the random variables $X^1_{t_1}, \dots, X^1_{t_m}$, $X^2_{s_1}, \dots, X^2_{s_n}$ are all regular. Therefore, from Proposition 1.2.7 there exists a countably Hilbertian topology θ on Φ, weaker than the nuclear topology, and Φ'_θ-valued random variables $\tilde{X}^1_{t_1}, \dots, \tilde{X}^1_{t_m}$ and $\tilde{X}^2_{s_1}, \dots, \tilde{X}^2_{s_n}$ such that $X^1_{t_j} = i'_\theta \tilde{X}^1_{t_j}$ and $X^2_{s_l} = i'_\theta \tilde{X}^2_{s_l}$ $\mathbb{P}$-a.e. for $j = 1, \dots, m$ and $l = 1, \dots, n$. Then, as in the proof of Proposition 1.2.7 to prove that the random vectors $(X^1_{t_1}, \dots, X^1_{t_m})$ and $(X^2_{s_1}, \dots, X^2_{s_n})$ are independent it is enough to show that the random vectors $(\tilde{X}^1_{t_1}, \dots, \tilde{X}^1_{t_m})$ and $(\tilde{X}^2_{s_1}, \dots, \tilde{X}^2_{s_n})$ are independent. Because, $\mathcal{C}(\Phi'_\theta) = \mathcal{B}(\Phi'_\theta)$, a sufficient condition for this is that

$$
\begin{aligned}
&\mathbb{P}\left(\tilde{X}^1_{t_1} \in Z^1_1, \dots, \tilde{X}^1_{t_m} \in Z^1_m, \tilde{X}^2_{s_1} \in Z^2_1, \dots, \tilde{X}^2_{s_n} \in Z^2_n\right) \\
&\qquad = \mathbb{P}\left(\tilde{X}^1_{t_1} \in Z^1_1, \dots, \tilde{X}^1_{t_m} \in Z^1_m\right) \mathbb{P}\left(\tilde{X}^2_{s_1} \in Z^2_1, \dots, \tilde{X}^2_{s_n} \in Z^2_n\right), \quad (1.12)
\end{aligned}
$$

for any cylindrical subsets $Z^1_1, \dots, Z^1_m, Z^2_1, \dots, Z^2_n$ of Φ'_θ. The proof that (1.12) holds can be carried out following similar arguments to those used in the proof of Proposition 1.2.7 and from our hypothesis of the independence of the processes $\{(X^1_t[\phi_1], \dots, X^1_t[\phi_r]) : t \in J\}$ and $\{(X^2_t[\phi_1], \dots, X^2_t[\phi_r]) : t \in J\}$ for all $r \in \mathbb{N}$ and $\phi_1, \dots, \phi_r \in \Phi$. Therefore, X^1 and X^2 are independent processes. □

We say that $X = \{X_t\}_{t\in J}$ is a **cylindrical process** in Φ' if X_t is a cylindrical random variable, for each $t \in J$.

A Φ'_β-valued processes $Y = \{Y_t\}_{t\in J}$ is said to be a Φ'_β-valued **version** of the cylindrical process $X = \{X_t\}_{t\in J}$ on Φ' if for each $t \in J$, Y_t is a Φ'_β-valued version of X_t.

Clearly, any Φ'_β-valued stochastic processes $X = \{X_t\}_{t\in J}$ defines a cylindrical process under the prescription: $X[\phi] = \{X_t[\phi]\}_{t\in J}$, for each $\phi \in \Phi$. We will say that it is the **cylindrical process determined** by X.

A cylindrical processes $X = \{X_t\}_{t\in J}$ is said to be **n-th integrable** is X_t is n-th integrable for each $t \in J$. Similarly, X is said to have **zero-mean** if each X_t have zero mean. The same definitions apply to Φ'_β-valued stochastic processes .

1.2.1 Existence of Continuous and Càdlàg Versions.

Notation 1.2.17. Throughout this thesis $C_T(\mathbb{R})$ and $D_T(\mathbb{R})$ will denote respectively the space of continuous and càdlàg real-valued processes defined in $[0, T]$, both spaces are considered equipped with the topology of uniform convergence in probability on $[0, T]$[1].

[1]For the case of the space $D_T(\mathbb{R})$, we have deliberately chose that our random variables have paths in the space of real-valued càdlàg functions on $[0, T]$ equipped with the topology of uniform convergence rather than the Skorohod topology. The reason for doing this is that the first of these topologies is stronger than the other (see e.g. Parthasarathy [78]) and also because this way we can have that in all the results in this section the proofs for the càdlàg case are very similar to the continuous case.

In this section we establish several results that show sufficient conditions for the existence of a continuous or càdlàg regular version for cylindrical processes on Φ' and for Φ'_β-valued regular stochastic processes. These results will have several applications in this thesis. For example, in Chapter 2 we use them to show the existence of càdlàg versions for Φ'_β-valued Lévy processes and they are also of great importance in the proof of the Lévy-Itô decomposition. In Chapter 3 we use them as a key tool to construct the stochastic integral and in Chapter 4 we utilize them to construct a deterministic integral for random integrands that will be necessary to give a proper meaning to the solutions of stochastic evolution equations. We hope that further applications may emerge in the future.

It is very important to stress that all the results in this section are either completely new or are generalizations of previous results. However, because the proofs of some of our results are highly technical, for the convenience of the reader we are going to state these result without proofs. Full proofs can be found in Appendix A.

The most important result of this section is the following extension of Theorem 1.2.14 which will be called the regularization theorem. All the results in this section will follow from this theorem.

Theorem 1.2.18 (Regularization Theorem). *Let $(\Phi, \mathcal{T})$ be a nuclear space and let $X = \{X_t\}_{t\in[0,T]}$ be a cylindrical process in Φ' such that for each $\phi \in \Phi$, the real-valued process $X(\phi) := \{X_t(\phi)\}_{t\in[0,T]}$ has a continuous (respectively càdlàg) version. Suppose that the linear mapping from Φ into $C_T(\mathbb{R})$ (respectively $D_T(\mathbb{R})$) given by $\phi \mapsto X(\phi)$ is continuous. Then, there exists a countably Hilbertian topology θ on Φ determined by an increasing sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ of continuous Hilbertian semi-norms on Φ, and a $(\Phi'_\theta, \beta_\theta)$-valued continuous (respectively càdlàg) process $Y = \{Y_t\}_{t\in[0,T]}$, such that for every $\phi \in \Phi$, $Y[\phi]$ is a version of $X(\phi)$. In particular, Y is a Φ'_β-valued continuous (respectively càdlàg) version of X that is unique up to indistinguishable versions.*

The above regularization theorem was firstly proved by Mitoma in [72] under the assumption that Φ is a nuclear Fréchet space and that X is a Φ'_β-valued process such that for each $\phi \in \Phi$, $X[\phi]$ has a càdlàg version. With the same assumption on X but using different methodologies than those used by Mitoma, Fouque [30] extended the regularization theorem to the case where Φ is an inductive limit of a countable family of Fréchet nuclear spaces and Fernique [29] extended it to the case where Φ is the strong dual of a Fréchet nuclear space or the strong dual of the inductive limit of a countable family of Féchet nuclear spaces. In all the above cases the continuity of the map $\phi \mapsto X[\phi]$ is a consequence of the assumption that X is a Φ'_β-valued process (Proposition 1.2.21 below is a generalization of this fact). Finally, using arguments similar to those used by Mitoma, Martias [63] carried out the extension of the regularization theorem to the case of Φ being a separable nuclear space and of X of the form $X : [0,T] \times \Omega \to \Phi'$, such that for each $\phi \in \Phi$, $X[\phi]$ is a real-valued process with a càdlàg version and assuming that the map $\phi \mapsto X[\phi]$ is continuous. Note that the assumptions on Φ and X in Theorem 1.2.18 are weaker than the assumptions in all the previously cited works and therefore Theorem 1.2.18 constitutes a generalization of all the other results encountered in the literature.

One can easily see that Theorem 1.2.18 implies Theorem 1.2.14. Indeed, our proof of Theorem 1.2.18 is a generalization of the proof of Itô and Nawata (see [44]) for Theorem 1.2.14.

As a corollary of the proof of Theorem 1.2.18 (see Appendix A) we obtain the following important result that establishes conditions for the existence of a continuous or a càdlàg version taking values in one of the Hilbert spaces Φ'_q.

Corollary 1.2.19. *Let $(\Phi, \mathcal{T})$ be a nuclear space and let $X = \{X_t\}_{t\in[0,T]}$ be a cylindrical process in Φ' such that for each $\phi \in \Phi$, the real-valued process $X(\phi) := \{X_t(\phi)\}_{t\in[0,T]}$ has a continuous (respectively càdlàg) version. Suppose that there exists a continuous Hilbertian semi-norm p on Φ such that the linear mapping from Φ into $C_T(\mathbb{R})$ (respectively $D_T(\mathbb{R})$) given by $\phi \mapsto X(\phi)$ is p-continuous. Then, there exists a continuous Hilbertian semi-norm ϱ on Φ, $p \leq \varrho$, such that $i_{p,\varrho}$ is Hilbert-Schmidt and a Φ'_ϱ-valued continuous (respectively càdlàg) process $Y = \{Y_t\}_{t\in[0,T]}$, such that for every $\phi \in \Phi$, $Y[\phi]$ is a version of $X(\phi)$. Moreover, Y is unique up to indistinguishable versions in Φ'_β.*

Now, let $X = \{X_t\}_{t\in[0,T]}$ be a Φ'_β-valued process such that for each $\phi \in \Phi$ the real-valued process $X[\phi] = \{X_t[\phi]\}_{t\in[0,T]}$ has a continuous version. If we do not assume any additional property on the space Φ, in general one may not expect that the map $\phi \mapsto X[\phi]$ from Φ into $C_T(\mathbb{R})$ is continuous. However, the next very useful result shows that this is always satisfied if the process X is regular. This result is very important because from Theorem 1.2.10 any stochastic process with Radon measures taking values in the strong dual of a barrelled nuclear space is regular, and most of the nuclear spaces encountered in applications are barrelled (see e.g. Section 1.1.6.1).

Theorem 1.2.20. *Let $(\Phi, \mathcal{T})$ be a nuclear space and let $X = \{X_t\}_{t\in[0,T]}$ be a Φ'_β-valued regular stochastic process such that for each $\phi \in \Phi$, the real-valued process $X[\phi] = \{X_t[\phi]\}_{t\in[0,T]}$ has a continuous (respectively càdlàg) version. Then, the linear mapping X from Φ into $C_T(\mathbb{R})$ (respectively $D_T(\mathbb{R})$) given by $\phi \mapsto X[\phi]$ is continuous.*

Proof. Without loss of generality we assume that each $X[\phi] = \{X_t[\phi]\}_{t\in[0,T]}$ has a continuous version. The proof is identical in the case of càdlàg versions.

Let D be a countable dense subset of $[0, T]$. For every $t \in D$, let $\{p_{t,n}\}_{n\in\mathbb{N}}$ be a sequence of continuous Hilbertian semi-norms such that Definition 1.2.5 is satisfied for X_t. For $t \in D$, let θ_t be the countably Hilbertian topology on Φ determined by the semi-norms $\{p_{t,n}\}_{n\in\mathbb{N}}$.

Let ϑ be the countably Hilbertian topology on Φ determined by the semi-norms $\{p_{t,n} : n \in \mathbb{N}, t \in D\}$. Then Φ_ϑ is a complete, separable, pseudo-metrizable Baire space. Moreover, by its definition is clear that the topology ϑ is finer than the topology θ_t for every $t \in D$.

Now we show that $\phi \mapsto X[\phi]$ as a map from Φ_ϑ into $C_T(\mathbb{R})$ is sequentially closed. Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence converging to ϕ in Φ_ϑ and assume that there exists some $Y \in C_T(\mathbb{R})$ such that $\sup_{t\in D} |X_t[\phi_n] - Y| \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$. We have to prove that $X[\phi] = Y$.

First, for every $t \in D$ it follows from Proposition 1.2.9 that the map $X_t : \Phi_{\theta_t} \to L^0(\Omega, \mathscr{F}, \mathbb{P})$ is continuous and because the topology θ_t is weaker than ϑ, then X_t is also continuous as a map from Φ_ϑ into $L^0(\Omega, \mathscr{F}, \mathbb{P})$. Then, as $\{\phi_n\}_{n\in\mathbb{N}}$ converges to ϕ in Φ_ϑ the continuity of X_t implies that the sequence of random variables $\{X_t[\phi_n]\}_{n\in\mathbb{N}}$ converges in probability to $X_t[\phi]$.

On the other hand, the condition $\sup_{t\in D} |X_t[\phi_n] - Y| \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$ implies that for every $t \in D$, the sequence of random variables $\{X_t[\phi_n]\}_{n\in\mathbb{N}}$ converges in probability to Y_t. Therefore, by uniqueness of limits in $L^0(\Omega, \mathscr{F}, \mathbb{P})$ it follows that $X_t[\phi] = Y_t$

$\mathbb{P}$-a.e. for every $t \in D$. But as the real-valued processes $X[\phi] = \{X_t[\phi]\}_{t\in[0,T]}$ and $Y = \{Y_t\}_{t\in[0,T]}$ are continuous, then they are indistinguishable and hence $X[\phi] = Y$ in $L^0(\Omega, \mathscr{F}, \mathbb{P})$. Hence, it follows from Theorem 1.1.1 that $\phi \mapsto X[\phi]$ as a map from Φ_ϑ into $C_T(\mathbb{R})$ is sequentially closed.

Now, as Φ_ϑ is first countable, it follows that $\phi \mapsto X[\phi]$ as a map from Φ_ϑ into $C_T(\mathbb{R})$ is indeed a closed operator and hence from the closed graph theorem (Theorem 1.1.2) it is also continuous. Then, as the canonical inclusion map $i_\vartheta : \Phi \to \Phi_\vartheta$ is linear and continuous, it follows that $\phi \mapsto X[\phi]$ as a map from Φ into $C_T(\mathbb{R})$ is continuous. □

One can prove the following result in a similar way as in the proof of Theorem 1.2.20 by using the version of the closed graph theorem for ultrabornological spaces (Theorem 1.1.3).

Proposition 1.2.21. *Let Ψ be an ultrabornological space and let $X = \{X_t\}_{t\in[0,T]}$, be a cylindrical process in Ψ' such that for each $\psi \in \Psi$, the real-valued process $X(\psi) := \{X_t(\psi)\}_{t\in[0,T]}$ has a continuous (respectively càdlàg) version. Suppose that for every $t \in [0,T]$ the linear mapping $X_t : \Psi \to L^0(\Omega, \mathscr{F}, \mathbb{P})$ is continuous. Then, the linear mapping from Ψ into $C_T(\mathbb{R})$ (respectively $D_T(\mathbb{R})$) given by $\psi \mapsto X(\psi)$ is continuous.*

Now we proceed to apply the previous results to provide conditions for the existence of continuous and càdlàg versions for stochastic processes taking values in the dual of a nuclear space.

Theorem 1.2.22. *Let $(\Phi, \mathcal{T})$ be a nuclear space and let $X = \{X_t\}_{t\geq 0}$ be a Φ'_β-valued regular process such that for each $\phi \in \Phi$, the real-valued process $X[\phi] = \{X_t[\phi]\}_{t\geq 0}$ has a continuous (respectively càdlàg) version. Then, there exists a countably Hilbertian topology θ_X on Φ determined by an increasing sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ of continuous Hilbertian semi-norms on Φ, and a $(\Phi'_{\theta_X}, \beta_{\theta_X})$-valued continuous (respectively càdlàg) process $Y = \{Y_t\}_{t\geq 0}$, such that for every $\phi \in \Phi$, $Y[\phi]$ is a version of $X[\phi]$. In particular, Y is a Φ'_β-valued continuous (respectively càdlàg) version of X that is unique up to indistinguishable versions.*

Proof. Without loss of generality we assume that each $X[\phi] = \{X_t[\phi]\}_{t\geq 0}$ has a continuous version. The càdlàg version case follows from the same arguments.

Let $\{T_k\}_{k\in\mathbb{N}}$ be an increasing sequence of positive numbers such that $\lim_{k\to\infty} T_k = \infty$. From Theorems 1.2.18 and 1.2.20, for each $n \in \mathbb{N}$ there exists a countably Hilbertian topology θ_k on Φ determined by an increasing sequence $\{q_{k,n}\}_{n\in\mathbb{N}}$ of continuous Hilbertian semi-norms on Φ, and a Φ'_{θ_k}-valued continuous process $Y^{(k)} = \{Y^{(k)}_t\}_{t\in[0,T_k]}$, such that for every $\phi \in \Phi$, $\{Y^{(k)}_t[\phi]\}_{t\in[0,T_k]}$ is a version of $\{X_t[\phi]\}_{t\in[0,T_k]}$.

Without loss of generality we can assume that for every $k \in \mathbb{N}$, $q_{k,n} \leq q_{k+1,n}$, for all $n \in \mathbb{N}$. This implies that for every $k \in \mathbb{N}$, the topology θ_{k+1} is weaker than θ_k and therefore that the canonical inclusion $i_{\theta_k,\theta_{k+1}} : \Phi_{\theta_{k+1}} \to \Phi_{\theta_k}$ is linear and continuous. Moreover, note that for each $k \in \mathbb{N}$, because for every $\phi \in \Phi$, $Y^{(k)}_t[\phi] = X_t[\phi] = Y^{(k+1)}_t[\phi]$ $\mathbb{P}$-a.e. for $t \in [0,T_k]$, Proposition 1.2.15 shows that $\{i'_{\theta_k,\theta_{k+1}} Y^{(k)}_t\}_{t\in[0,T_k]}$ and $\{Y^{(k+1)}_t\}_{t\in[0,T_k]}$ are indistinguishable processes in $(\Phi'_{\theta_{k+1}}, \beta_{\theta_{k+1}})$.

Now, let θ_X be the completion of the locally convex topology on Φ generated by the family of semi-norms $\{q_{k,n} : k, n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, let $\varrho_n^2 = \sum_{j=1}^n q_{j,j}^2$. Then $\{\varrho_n\}_{n\in\mathbb{N}}$ is an increasing sequence of continuous Hilbertian semi-norms on Φ. Moreover, from the properties: $q_{k,n} \leq q_{k+1,n}$ and $q_{k,n} \leq q_{k,n+1}$ that are valid for all

$n,k \in \mathbb{N}$, and from the definition of the semi-norms ϱ_n, it can be proved that the families of semi-norms $\{q_{k,n} : k,n \in \mathbb{N}\}$ and $\{\varrho_n\}_{n\in\mathbb{N}}$ generate the same topology on Φ. This implies that θ_X is the countably Hilbertian topology generated by the semi-norms $\{\varrho_n\}_{n\in\mathbb{N}}$.

Note that as for each $k \in \mathbb{N}$ the topology θ_X is by definition finer than θ_k, then the space Φ_{θ_X} is continuously embedded in Φ_{θ_k}, and hence by duality we have that $(\Phi'_{\theta_k}, \beta_{\theta_k})$ is continuously embedded in $(\Phi'_{\theta_X}, \beta_{\theta_X})$. Then, if we take $Y = \{Y_t\}_{t\geq 0}$ defined by the prescription $Y_t = Y_t^{(k)}$ if $t \in [0,T_k]$, it follows from the corresponding properties of the processes $Y^{(k)}$ that Y is a $(\Phi'_{\theta_X}, \beta_{\theta_X})$-valued continuous process such that for every $\phi \in \Phi$, $Y[\phi]$ is a version of $X[\phi]$.

Now, as the topology θ_X is weaker than the nuclear topology on Φ, then the canonical inclusion $i_{\theta_X} : \Phi \to \Phi_{\theta_X}$ is linear and continuous. Then, the dual operator $i'_{\theta_X} : (\Phi'_{\theta_X}, \beta_{\theta_X}) \to \Phi'_\beta$ is linear and continuous and therefore it follows that Y is a Φ'_β-valued continuous version of X. Proposition 1.2.15 shows that Y defined this way is unique up to indistinguishable processes on Φ'_β. □

Remark 1.2.23. *If Φ is a Fréchet nuclear space or a countable inductive limit of Fréchet nuclear spaces then every Φ'_β-valued process is regular (see Remark 1.2.11). Therefore, if X is a Φ'_β-valued process such that for each $\phi \in \Phi$, the real-valued process $X[\phi]$ has a continuous (respectively càdlàg) version, then it follows from Theorem 1.2.22 that X has a Φ'_β-valued continuous (respectively càdlàg) version. As discussed above this is exactly the result obtained by Mitoma in [72], Fouque [30] and Fernique [29].*

Now we consider conditions for the existence of continuous and càdlàg versions with finite moments on some Hilbert space Φ'_p contained on the dual space Φ'_β. For the proof we will need the following terminology. For $n \in \mathbb{N}$, we denote by $C^n_T(\mathbb{R})$ (respectively $D^n_T(\mathbb{R})$) the linear space of all the continuous (respectively càdlàg) processes satisfying $\mathbb{E}\sup_{t\in[0,T]} |Z_t|^n < \infty$. It is a Banach space equipped with the norm $||Z||_{n,T} = \left(\mathbb{E}\sup_{t\in[0,T]} |Z_t|^n\right)^{1/n}$.

Theorem 1.2.24. *Let $(\Phi, \mathcal{T})$ be a nuclear space and let $X = \{X_t\}_{t\in[0,T]}$ be a Φ'_β-valued regular process such that for each $\phi \in \Phi$, the real-valued process $X[\phi] = \{X_t[\phi]\}_{t\in[0,T]}$ has a continuous (respectively càdlàg) version. Assume that there exists $n \in \mathbb{N}$ such that $\mathbb{E}\left(\sup_{t\in[0,T]} |X_t[\phi]|^n\right) < \infty$, $\forall \phi \in \Phi$. Let $p : \Phi \to \mathbb{R}_+$, given by*

$$p(\phi) = \left(\mathbb{E}\left(\sup_{t\in[0,T]} |X_t[\phi]|^n\right)\right)^{1/n}, \quad \forall\, \phi \in \Psi.$$

Then, p is a continuous semi-norm on Φ. Moreover, there exists a continuous Hilbertian semi-norm q on Φ, $p \leq q$ (such that $i_{p,q}$ is Hilbert-Schmidt if $n = 2$), and there exists a Φ'_q-valued continuous (respectively càdlàg) version $\tilde{Y} = \{\tilde{Y}_t\}_{t\in[0,T]}$ of X satisfying $\mathbb{E}\left(\sup_{t\in[0,T]} q'(\tilde{Y}_t)^n\right) < \infty$.

Proof. We prove the continuous case as the càdlàg case follows from similar arguments. We start by checking that p defines a semi-norm on Φ.

First, note that the map from Φ into $C^n_T(\mathbb{R})$ given by $\phi \mapsto X[\phi]$ is linear. Then, because $||\cdot||_{n,T}$ its a norm on $C^n_T(\mathbb{R})$ and $p(\phi) = ||X[\phi]||_{n,T}$ for all $\phi \in \Phi$, it follows that p is also a norm.

To prove that p is continuous, one can use the closed graph theorem and similar arguments to those used in the proof of Theorem 1.2.20 to show that the map $\phi \mapsto X[\phi]$ from Φ into $C^n_T(\mathbb{R})$ is continuous. But because $p(\psi) = ||X[\phi]||_{n,T}$, for all $\phi \in \Phi$, then the continuity of the maps $\phi \mapsto X[\phi]$ and $Z \mapsto ||Z||_{n,T}$ implies that p is continuous.

Now, from the Markov and Jensen inequalities, for any $\epsilon > 0$ we have

$$\mathbb{P}\left(\omega \in \Omega : \sup_{t\in[0,T]} |X_t(\omega)[\phi]| > \epsilon\right) \leq \frac{1}{\epsilon}\left(\mathbb{E}\left(\sup_{t\in[0,T]} |X_t[\phi]|^n\right)\right)^{1/n} = \frac{1}{\epsilon}p(\phi), \quad \forall \phi \in \Phi.$$

Therefore, the map $X : \Phi \to C_T(\mathbb{R})$ given by $\psi \mapsto X[\phi] := \{X_t[\phi]\}_{t\in[0,T]}$ is p-continuous. Now, as the topology on Φ is generated by a family of Hilbertian semi-norms and because the semi-norm p is continuous, there exists a continuous Hilbertian semi-norm ϱ on Φ such that $p \leq \varrho$. This implies that the map $X : \Phi \to C_T(\mathbb{R})$ is ϱ-continuous. Then, it follows from Corollary 1.2.19 that there exists a continuous Hilbertian semi-norm r on Φ, $\varrho \leq r$ such that $i_{r,\varrho}$ is Hilbert-Schmidt and such that X has a Φ'_r-valued continuous version $Y = \{Y_t\}_{t\in[0,T]}$.

The next step is to prove that we can find another Hilbert space in which Y takes values, and furthermore has finite n-moments on it uniformly on $[0,T]$.

Let q be a continuous Hilbertian semi-norm on Φ such that $r \leq q$ and $i_{r,q}$ is Hilbert-Schmidt. Then, the dual operator $i'_{r,q} : \Phi'_r \to \Phi'_q$ is Hilbert-Schmidt and hence from Theorem B.0.18 there exists a constant $C > 0$, and a Radon probability measure ν on the unit ball $B^*_r(1)$ of Φ_r (equipped with the weak topology) such that,

$$q'(i'_{r,q}f) \leq C \cdot \left(\int_{B^*_r(1)} |f[\phi]|^n \, \nu(d\phi)\right)^{1/n}, \quad \forall\, f \in \Phi'_r. \tag{1.13}$$

As Y is a Φ'_r-valued continuous process, then $\phi \mapsto Y[\phi]$ is a continuous and linear map from Φ_r into $C^n_T(\mathbb{R})$. Therefore, it follows from (1.13) that,

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\in D} q'(i'_{r,q}Y_t)^n\right) &\leq C^n \, \mathbb{E} \sup_{t\in D} \int_{B^*_r(1)} |Y_t[\phi]|^n \, \nu(d\phi) \\
&\leq C^n \int_{B^*_r(1)} ||Y[\phi]||^n_{c,n,T} \; \nu(d\phi) \\
&\leq C^n \, ||Y||^n_{\mathcal{L}(\Phi_r, C^n_T(\mathbb{R}))} < \infty.
\end{aligned}$$

Hence, $\tilde{Y} = \{\tilde{Y}_t\}_{t\in[0,T]}$, defined by $\tilde{Y}_t = i'_{r,q}Y_t$, for every $t \in [0,T]$, is a Φ'_q-valued continuous version of X satisfying $\mathbb{E}\left(\sup_{t\in D} q'(\tilde{Y}_t)^n\right) < \infty$. □

The following specialized version of Theorem 1.2.24 will be of great importance in our study of Lévy processes in Chapter 2. In particular, it will be a key tool in the proof of the Lévy-Itô decomposition in Section 2.2.3.

Theorem 1.2.25. *Let $(\Phi, \mathcal{T})$ be a nuclear space and let $X = \{X_t\}_{t\geq 0}$ be a Φ'_β-valued regular process such that for each $\phi \in \Phi$, the real-valued process $X[\phi] = \{X_t[\phi]\}_{t\geq 0}$ has a continuous (respectively càdlàg) version. Assume that there exists some $n \in \mathbb{N}$ and a continuous Hilbertian semi-norm ϱ on Φ such that for each $T > 0$ there exists some $C(T) > 0$ such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t[\phi]|^n\right) \leq C(T)\varrho(\phi)^n, \quad \forall\, \phi \in \Phi. \tag{1.14}$$

Then, there exists a continuous Hilbertian semi-norm q on Φ, $\varrho \leq q$, such that $i_{\varrho,q}$ is Hilbert-Schmidt and there exists a Φ'_q-valued continuous (respectively càdlàg) version $\tilde{Y} = \{\tilde{Y}_t\}_{t \geq 0}$ of X satisfying $\mathbb{E}\left(\sup_{t \in [0,T]} q'(\tilde{Y}_t)^n\right) < \infty$, for all $T > 0$.

Proof. As before, we will prove the continuous case as the càdlàg case follows similarly. Fix $T > 0$ for the moment. Note that (1.14) is equivalent to $p \leq C(T)^{1/n}\varrho$, where p is the semi-norm defined in Theorem 1.2.24. Therefore, the map from Φ into $C_T(\mathbb{R})$ given by $\phi \mapsto X[\phi]$ is ϱ-continuous. Hence, it follows from Theorem 1.2.24 that there exists a continuous Hilbertian semi-norm q on Φ (only depending on ϱ), such that $\varrho \leq q$ and $i_{\varrho,q}$ is Hilbert-Schmidt, and a Φ'_q-valued continuous version $Y^T = \{Y^T_t\}_{t \in [0,T]}$ of $\{X_t\}_{t \in [0,T]}$ satisfying $\mathbb{E}\left(\sup_{t \in [0,T]} q'(Y^T_t)^n\right) < \infty$.

Let $\{T_n\}_{n \in \mathbb{N}}$ an increasing sequence of positive numbers such that $\lim_{n \to \infty} T_n = \infty$. For each $n \in \mathbb{N}$, let $Y^{T_n} = \{Y^{T_n}_t\}_{t \in [0,T_n]}$ as defined above. Note that for each $n \in \mathbb{N}$, the Φ'_q-valued continuous processes $\{Y^{T_n}_t\}_{t \in [0,T_n]}$ and $\{Y^{T_{n+1}}_t\}_{t \in [0,T_n]}$ are indistinguishable. Therefore, if we define $\tilde{Y} = \{\tilde{Y}_t\}_{t \geq 0}$ by the prescription $\tilde{Y}_t = Y^{T_n}_t$ if $t \in [0, T_n]$, then $\tilde{Y}$ is a Φ'_q-valued continuous version of X satisfying $\mathbb{E}\left(\sup_{t \in [0,T]} q'(\tilde{Y}_t)^n\right) < \infty$, for all $T > 0$. □

1.2.2 Martingales in the Strong Dual of a Nuclear Space

In this section we study some properties of Φ'_β-valued martingales and local martingales.

Definition 1.2.26. A **cylindrical martingale** (respectively a **cylindrical local martingale**) on Φ' is a cylindrical process $M = \{M_t\}_{t \geq 0}$ such that for each $\phi \in \Phi$, the real-valued process $M(\phi) = \{M_t(\phi)\}_{t \geq 0}$ is a $\{\mathcal{F}_t\}$-adapted martingale (respectively a $\{\mathcal{F}_t\}$-adapted local martingale).

A Φ'_β-valued stochastic process is a **martingale** (respectively a **local martingale**) if it is regular and the associated cylindrical process is a cylindrical martingale (respectively a cylindrical local martingale).

The following result contains some of the basic properties of Φ'_β-valued martingales.

Theorem 1.2.27. *Let Φ be a nuclear space and let $M = \{M_t\}_{t \geq 0}$ be a Φ'_β-valued martingale. Then, M has a Φ'_β-valued càdlàg version $\tilde{M} = \{\tilde{M}_t\}_{t \geq 0}$ such that:*

(1) For each $T > 0$ there exist some continuous Hilbertian semi-norm p_T on Φ such that $\{\tilde{M}_t\}_{t \in [0,T]}$ is a Φ'_{p_T}-valued zero-mean càdlàg martingale.

(2) If additionally M is n-th integrable, for $n \in \mathbb{N}$, then one can choose p_T such that $\tilde{M}$ also satisfies that $\mathbb{E}\left(\sup_{t \in [0,T]} p'_T(\tilde{M}_t)^n\right) < \infty$.

(3) If moreover for some $n \in \mathbb{N}$, $\sup_{t \geq 0} \mathbb{E}\left(|M_t[\phi]|^n\right) < \infty$, for each $\phi \in \Phi$, then there exist some continuous Hilbertian semi-norm q on Φ such that $\tilde{M}$ is a Φ'_q-valued zero-mean càdlàg martingale satisfying $\sup_{t \geq 0} \mathbb{E}\left(q'(\tilde{M}_t)^n\right) < \infty$.

If for each $\phi \in \Phi$ the real-valued process $\{M_t[\phi]\}_{t \geq 0}$ has a continuous version, then M has a Φ'_β-valued continuous version $\hat{M} = \{\hat{M}_t\}_{t \geq 0}$ such that it satisfies $(1) - (3)$ above replacing the property càdlàg by continuous.

Proof. The results follows from an application of Doob's inequality, Theorems 1.2.24 and 1.2.25, and the fact that each real-valued martingale has a càdlàg version. □

Remark 1.2.28. *The results in Theorem 1.2.27 were originally proven by Mitoma (see [67]) for the case of martingales in the strong dual of a nuclear Fréchet space. Note that we have been able to extend these results to any nuclear space.*

Let $T > 0$ and $n \in \mathbb{N}$. We denote by $\mathcal{M}^n_T(\Phi'_\beta)$ (respectively by $\mathcal{M}^{n,loc}_T(\Phi'_\beta)$) the linear space of all the equivalence classes of Φ'_β-valued càdlàg (respectively locally) zero-mean n-th integrable martingales defined on $[0,T]$. We introduce some vector topologies on the spaces $\mathcal{M}^n_T(\Phi'_\beta)$ and $\mathcal{M}^{n,loc}_T(\Phi'_\beta)$, but before we do this we review some basic properties of the space of Banach space-valued n-th integrable martingales and of the space of real-valued locally n-th integrable martingales.

Let $(E, ||\cdot||_E)$ be a separable Banach space. We denote by $\mathcal{M}^n_T(E)$ the linear space of (equivalence classes of) E-valued zero-mean n-th integrable càdlàg martingales defined on $[0,T]$. It is a Banach space equipped with the norm $||\cdot||_{\mathcal{M}^n_T(E)}$ defined by

$$||M||_{\mathcal{M}^n_T(E)} = \left(\mathbb{E} \sup_{t\in[0,T]} ||M_t||^n_E \right)^{\frac{1}{n}}, \quad \forall M \in \mathcal{M}^n_T(E). \tag{1.15}$$

For a proof of the square integrable case see Proposition 3.9 of Da Prato and Zabczyk [20], p.79. The proof therein extends easily to the n-th integrable case.

We denote by $\mathcal{M}^n(E)$ the linear space of (equivalence classes of) E-valued zero-mean n-th integrable càdlàg martingales defined on $[0,\infty)$. As for each $M = \{M_t\}_{t\geq 0} \in \mathcal{M}^n(E)$, we have $\{M_t\}_{t\in[0,T]} \in \mathcal{M}^n_T(E)$, for all $T > 0$, then there exist a canonical inclusion j_K of the space $\mathcal{M}^n(E)$ into the space $\mathcal{M}^n_K(E)$, for $K \in \mathbb{N}$. Therefore, we can equip $\mathcal{M}^n(E)$ with the projective limit topology determined by the projective system $\{(\mathcal{M}^n_K(E), j_K) : K \in \mathbb{N}\}$ (see Section 1.1.3). Then, equipped with these topology, $\mathcal{M}^n(E)$ is a Fréchet space and a family of semi-norms generating its topology is $\{||j_K(\cdot)||_{\mathcal{M}^n_K(E)}\}_{K\in\mathbb{N}}$. In particular, convergence in $\mathcal{M}^n(E)$ is then equivalent to convergence in the space $L^n(\Omega, \mathcal{F}, \mathbb{P}; E)$ (see Definition C.0.23) uniformly on compact intervals of $[0,\infty)$.

Now, we employ the notation $\mathcal{M}^{n,loc}_T(\mathbb{R})$ to denote the linear space of (equivalence classes of) locally zero-mean n-th integrable càdlàg martingales on $[0,T]$. We equip this space with the vector topology $\mathcal{T}_{n,loc}$ generated by the local base of neighbourhoods of zero $\{O_{\epsilon,\delta} : \epsilon > 0, \delta > 0\}$, where $O_{\epsilon,\delta}$ is given by

$$O_{\epsilon,\delta} = \left\{ M \in \mathcal{M}^{n,loc}_T(\mathbb{R}) : \mathbb{P}\left(\omega \in \Omega : \sup_{t\in[0,T]} |M_t(\omega)|^n > \epsilon \right) < \delta \right\}. \tag{1.16}$$

Hence, under the topology $\mathcal{T}_{n,loc}$, a sequence $\{M^{(k)}\}_{k\in\mathbb{N}}$ converges to M in $\mathcal{M}^{n,loc}_T(\mathbb{R})$ if and only if

$$\sup_{t\in[0,T]} \left| M^{(k)}_t - M_t \right|^n \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \to \infty. \tag{1.17}$$

Moreover, equipped with the topology $\mathcal{T}_{n,loc}$ the space $\mathcal{M}^{n,loc}_T(\mathbb{R})$ is complete and metrizable (see Section 9.1 of Skorohod [96]).

Remark 1.2.29. *The space $(\mathcal{M}^{n,loc}_T(\mathbb{R}), \mathcal{T}_{n,loc})$ is not locally convex in general. In particular, if $\mathbb{P}$ is an atomless measure (see Definition 1.12.7 of Bogachev [8], p.55), then every convex neighbourhood of zero is identical to $\mathcal{M}^{n,loc}_T(\mathbb{R})$. This can be proven following similar arguments to those in Remarque 1 of Badrikian [7], p.2.*

Now we return to our problem of how to introduce some topologies on the space $\mathcal{M}^n_T(\Phi'_\beta)$. To do this, we will need the following result:

Proposition 1.2.30. *Let Φ be a nuclear space and let $n \in \mathbb{N}$. The mapping from $\mathcal{M}^n_T(\Phi'_\beta)$ into $\mathcal{L}(\Phi, \mathcal{M}^n_T(\mathbb{R}))$ given by*

$$M \mapsto \left(\phi \mapsto M[\phi] = \{M_t[\phi]\}_{t\in[0,T]}\right), \tag{1.18}$$

is a linear isomorphism.

Proof. First we check that the map (1.18) is well-defined. Let $M \in \mathcal{M}^n_T(\Phi'_\beta)$. By definition M is a cylindrical martingale and hence the map $\phi \mapsto M[\phi]$ is linear. Moreover, it follows from Theorem 1.2.24 and (1.15) (with $E = \mathbb{R}$) that the map $\phi \mapsto ||M[\phi]||_{\mathcal{M}^n_T(\mathbb{R})}$ is continuous. This in particular implies that the linear map $\phi \mapsto M[\phi]$ is continuous and hence belongs to $\mathcal{L}(\Phi, \mathcal{M}^n_T(\mathbb{R}))$.

Is clear that the map defined in (1.18) is linear. Moreover, it is also injective because it is linear and its kernel only contains the zero vector of $\mathcal{M}^n_T(\Phi'_\beta)$.

The map is also surjective. This is because if $A \in \mathcal{L}(\Phi, \mathcal{M}^n_T(\mathbb{R}))$, then A defines a cylindrical process in Φ' such that for each $\phi \in \Phi$, $A\phi = \{(A\phi)_t\}_{t\in[0,T]} \in \mathcal{M}^n_T(\mathbb{R})$ and such that it is continuous as a map from Φ into $D_T(\mathbb{R})$. This later fact is a consequence of the fact that $\mathcal{M}^n_T(\mathbb{R})$ is continuously embedded in $D_T(\mathbb{R})$ and the continuity of A. Then, Theorem 1.2.18 implies that there exists a Φ'_β-valued regular càdlàg process $M = \{M_t\}_{t\in[0,T]}$ such that for each $t \in [0,T]$, $\mathbb{P}$-a.e. we have $M_t[\phi] = (A\phi)_t$, for all $\phi \in \Phi$. This therefore implies that M belongs to $\mathcal{M}^n_T(\Phi'_\beta)$. Hence, the map given in (1.18) is a linear isomorphism. □

Now, to introduce a topology on $\mathcal{M}^n_T(\Phi'_\beta)$ we identify this space with the space $\mathcal{L}(\Phi, \mathcal{M}^n_T(\mathbb{R}))$ by means of the isomorphism given in (1.18). Recall that the topology of bounded (respectively simple) convergence on $\mathcal{L}(\Phi, \mathcal{M}^n_T(\mathbb{R}))$ is the locally convex topology generated by the following family of semi-norms:

$$A \to \sup_{\phi\in B} ||A\phi||_{\mathcal{M}^n_T(\mathbb{R})},$$

where B runs over the bounded (respectively finite) subsets of Φ.

Then, if we identify each M in $\mathcal{M}^n_T(\Phi'_\beta)$ with the corresponding element $\phi \mapsto M[\phi]$ in $\mathcal{L}(\Phi, \mathcal{M}^n_T(\mathbb{R}))$, we can introduce on $\mathcal{M}^n_T(\Phi'_\beta)$ the **topologies of simple and bounded convergence**. A family of semi-norms generating the topology of bounded (respectively simple) convergence on $\mathcal{M}^n_T(\Phi'_\beta)$ is then given by

$$M \to \sup_{\phi\in B} ||M[\phi]||_{\mathcal{M}^n_T(\mathbb{R})} = \sup_{\phi\in B} \mathbb{E}\left(\sup_{t\in[0,T]} |M_t[\phi]|^n\right)^{\frac{1}{n}}, \tag{1.19}$$

where B runs over the bounded (respectively finite) subsets of Φ. The next result follows from the corresponding properties of the topologies of bounded and simple convergence of the space $\mathcal{L}(\Phi, \mathcal{M}^n_T(\mathbb{R}))$. See Section 6, Chapter 39 of Kothë [62].

Proposition 1.2.31. *Let Φ be a barrelled nuclear space. Then, the space $\mathcal{M}^n_T(\Phi'_\beta)$ is quasi-complete equipped with either the topology of bounded convergence or the topology of simple convergence. If additionally Φ is bornological, then $\mathcal{M}^n_T(\Phi'_\beta)$ is complete equipped with the topology of bounded convergence.*

Chapter 2

Lévy Processes in Duals of Nuclear Spaces

In this chapter we introduce Lévy process taking values in Φ'_{β}, which is the most important class of stochastic processes in our study. The chapter is divided into two sections.

In the first section, we introduce Lévy processes and establish some of their basic properties. In particular, we show the existence of a càdlàg version taking values in the dual of a countably Hilbertian space which is continuously included in Φ'_{β}. Basic properties of Wiener and compound Poisson processes are also studied. In the second section we establish the Lévy-Itô decomposition for any Φ'_{β}-valued Lévy processes. As a corollary, we prove the Lévy-Khintchine formula for the characteristic function of Φ'_{β}-valued Lévy processes.

§ 2.1 Lévy Processes: Basic Properties.

Assumption 2.1.1. *Throughout this section and unless otherwise specified* Φ *will be a nuclear space over* $\mathbb{R}$.

Definition 2.1.2. A Φ'_{β}-valued regular process $L = \{L_t\}_{t \geq 0}$ is called a **Lévy process** if it satisfies:

(1) $L_0 = 0$ a.s.
(2) L has **independent increments**, i.e. for any $n \in \mathbb{N}$, $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ the Φ'_{β}-valued random variables $L_{t_1}, L_{t_2} - L_{t_1}, \dots, L_{t_n} - L_{t_{n-1}}$ are independent.
(3) **stationary increments**, i.e. for any $0 \leq s \leq t$, $L_t - L_s$ and L_{t-s} are identically distributed.
(4) L is **stochastically continuous**, i.e. for all $t \geq 0$, $X_s \xrightarrow{\mathbb{P}} X_t$ as $s \to t$.

Remark 2.1.3. *If* Φ *is a barrelled nuclear space, it follows from Theorem 1.2.10 that the assumption of being a regular process on the definition of Lévy processes can be equivalently replaced by the assumption that for every* $t \geq 0$ *the distribution* μ_{L_t} *of* L_t *is a Radon measure.*

Following the definition given in Applebaum and Riedle [4] for cylindrical Lévy processes in Banach spaces, we introduce the following definition.

Definition 2.1.4. A cylindrical process $L = \{L_t\}_{t \geq 0}$ in Φ' is said to be a **cylindrical Lévy process** if for every $n \in \mathbb{N}$ and $\phi_1, \dots, \phi_n \in \Phi$, the $\mathbb{R}^n$-valued process $L(\phi_1, \dots, \phi_n) = \{(L_t(\phi_1), \dots, L_t(\phi_n))\}_{t \geq 0}$ is a Lévy process.

Lemma 2.1.5. *A Φ'_β-valued Lévy process L determines a cylindrical Lévy process.*

Proof. Let $n \in \mathbb{N}$ and $\phi_1, \dots, \phi_n \in \Phi$. The fact that $L(\phi_1, \dots, \phi_n)$ is a $\mathbb{R}^n$-valued Lévy process follows from standard arguments from the corresponding properties of L. In particular, for the independent increments we can use Proposition 1.2.13. ☐

The following result is a converse of the above lemma.

Lemma 2.1.6. *Let L be a Φ'_β-valued regular càdlàg process such that it determines a cylindrical Lévy process. Then, L is a Φ'_β-valued Lévy process.*

Proof. First, note that as for every $\phi \in \Phi$ we have $L_0[\phi] = 0$ $\mathbb{P}$-a.e., then Proposition 1.2.12 shows that $L_0 = 0$ $\mathbb{P}$-a.e.

To prove that L has independent and stationary increments let $n \in \mathbb{N}$ and $\phi_1, \dots, \phi_n \in \Phi$. Because L is a cylindrical Lévy process, then $\{(L_t[\phi_1], \dots, L_t[\phi_n])\}_{t \geq 0}$ has independent and stationary increments. Therefore, Propositions 1.2.12 and 1.2.13 imply that L has independent and stationary increments.

Finally, to prove the stochastic continuity, observe that as L has (a.s.) right-continuous paths, then for any $s \geq 0$ we have $L_s \overset{\mathbb{P}}{\to} L_t$ (in Φ'_β) as $s \to t$, $t \leq s$. Hence, L is right-stochastically continuous.

To prove the left-stochastic continuity, note that for any $0 \leq s \leq t$, the stationary increments of L, implies that

$$\mathbb{P}\left(\sup_{\phi \in B} \left| (L_t - L_s)[\phi] \right| > \epsilon \right) = \mathbb{P}\left(\sup_{\phi \in B} \left| L_{t-s}[\phi] \right| > \epsilon \right), \tag{2.1}$$

for any $\epsilon > 0$ and any bounded subset B of Φ. Then, as $t - s \to 0$ as $s \to t$, $0 < s \leq t$, the right-stochastic continuity and (2.1) implies the left-stochastic continuity of L. Therefore, L is a Lévy process. ☐

Theorem 2.1.7. *Let $L = \{L_t\}_{t \geq 0}$ be a cylindrical Lévy process in Φ'. Suppose that for every $t \geq 0$, the linear map $L_t : \Phi \to L^0(\Omega, \mathscr{F}, \mathbb{P})$ is continuous. Then, there exists a countably Hilbertian topology θ_L on Φ determined by an increasing sequence $\{\varrho_n\}_{n \in \mathbb{N}}$ of continuous Hilbertian semi-norms on Φ, and a $(\Phi'_{\theta_L}, \beta_{\theta_L})$-valued càdlàg process $\hat{L} = \{\hat{L}_t\}_{t \geq 0}$, such that for every $\phi \in \Phi$, $\hat{L}[\phi]$ is a version of $L(\phi)$. Moreover, $\hat{L}$ is a Φ'_β-valued Lévy process.*

Proof. First, as for every $t \geq 0$ the linear map $L_t : \Phi \to L^0(\Omega, \mathscr{F}, \mathbb{P})$ is continuous, it follows from the regularization theorem (Theorem 1.2.14) that there exists a Φ'_β-valued regular process $\overline{L} = \{\overline{L}_t\}_{t \geq 0}$, such that for every $\phi \in \Phi$ and $t \geq 0$, $L_t(\phi) = \overline{L}_t[\phi]$ $\mathbb{P}$-a.e. Therefore, for every $\phi_1, \dots, \phi_n \in \Phi$ the $\mathbb{R}^n$-valued process $\{(\overline{L}_t[\phi_1], \dots, \overline{L}_t[\phi_n])\}_{t \geq 0}$ is a version of $\{(L_t(\phi_1), \dots, L_t(\phi_n))\}_{t \geq 0}$, and because this last is a $\mathbb{R}^n$-valued Lévy process, it follows that $\{(\overline{L}_t[\phi_1], \dots, \overline{L}_t[\phi_n])\}_{t \geq 0}$ is also a $\mathbb{R}^n$-valued Lévy process (see Lemma 1.4.8 of Applebaum [3], p.67). Then, $\overline{L}$ determines a cylindrical Lévy process in Φ'.

Now, for each $\phi \in \Phi$, the fact that $\overline{L}[\phi]$ is a Lévy process implies that it has a càdlàg version (see Theorem 2.1.8 of Applebaum [3], p.87). Then, because $\overline{L}$ is a regular process, it follows from Theorem 1.2.22 that there exists a countably Hilbertian topology θ_L on Φ determined by an increasing sequence $\{\varrho_n\}_{n \in \mathbb{N}}$ of continuous Hilbertian semi-norms on Φ, and a $(\Phi'_{\theta_L}, \beta_{\theta_L})$-valued càdlàg process $\hat{L} = \{\hat{L}_t\}_{t\geq 0}$, such that for every $\phi \in \Phi$, $\hat{L}[\phi]$ is a version $\overline{L}[\phi]$. This in particular implies that for every $\phi \in \Phi$, $\hat{L}[\phi]$ is a version of $L(\phi)$. Finally, as $\hat{L}$ is a Φ'_{β}-valued càdlàg process that is also a cylindrical Lévy process, it follows from Lemma 2.1.6 that $\hat{L}$ is a Lévy process. □

We now apply the previous results to show the existence of càdlàg versions for Φ'_{β}-valued Lévy processes.

Theorem 2.1.8. *Let $L = \{L_t\}_{t\geq 0}$ be a Φ'_{β}-valued Lévy process. Then, there exists a countably Hilbertian topology θ_L on Φ determined by an increasing sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ of continuous Hilbertian semi-norms on Φ, and a $(\Phi'_{\theta_L}, \beta_{\theta_L})$-valued càdlàg process $\hat{L} = \{\hat{L}_t\}_{t\geq 0}$, such that for every $\phi \in \Phi$, $\hat{L}[\phi]$ is a version of $L[\phi]$. Moreover, $\hat{L}$ is a Φ'_{β}-valued Lévy process.*

Proof. First, note that as L is a regular process, then by Proposition 1.2.9 for every $t \geq 0$ the map $L_t : \Phi \to L^0(\Omega, \mathscr{F}, \mathbb{P})$ given by $\phi \to X_t[\phi]$ is linear and continuous. Second, as L is a Lévy process it follows from Lemma 2.1.5 that L determines a cylindrical process. Then, Theorem 2.1.7 shows the existence of the process $\hat{L}$ satisfying the required properties. □

Corollary 2.1.9. *If $L = \{L_t\}_{t\geq 0}$ is a Φ'_{β}-valued càdlàg Lévy process, there exists an increasing sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ of continuous Hilbertian semi-norms on Φ such that $\mathbb{P}\left(L_t \in \bigcup_{n\in\mathbb{N}} \Phi'_{\varrho_n}, \forall\, t \geq 0\right) = 1$.*

Proof. First, it follows from Theorem 2.1.8 that there exists a countably Hilbertian topology θ_L on Φ determined by an increasing sequence $\{\varrho_n\}_{n\in\mathbb{N}}$ of continuous Hilbertian semi-norms on Φ, and a $(\Phi'_{\theta_L}, \beta_{\theta_L})$-valued càdlàg version $\hat{L}$ of L. As L is also càdlàg, it follows from Proposition 1.2.15 that L and $\hat{L}$ are indistinguishable process. Therefore, $\mathbb{P}(L_t \in \Phi'_{\theta_L}, \forall\, t \geq 0) = 1$. But from the definition of the dual of a countably Hilbertian space (see (1.6)) we have that $\mathbb{P}\left(L_t \in \bigcup_{n\in\mathbb{N}} \Phi'_{\varrho_n}, \forall\, t \geq 0\right) = 1$. □

For the remain of this thesis we always make the following assumptions:

Assumption 2.1.10. *If $L = \{L_t\}_{t\geq 0}$ is a Φ'_{β}-valued Lévy process we assume that:*

- *L is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$,*
- *$L_t - L_s$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t$.*
- *L is a càdlàg process.*

Some basic properties of Φ'_{β}-valued Lévy processes are summarized in the following result. The proof can carried out following similar arguments to those in the real-valued case. See for example Applebaum [3].

Theorem 2.1.11. *The property of being a Φ'_{β}-valued Lévy process is preserved under modifications, finite sums of independent processes and uniform limits in probability in compact subsets of $[0, \infty)$.*

2.1.1 Wiener and Compound Poisson Processes

Now we will introduce two special classes of Φ'_{β}-valued Lévy processes: the Wiener and the compound Poisson processes. As it will be seen in Section 2.2 when we shall prove the Lévy-Itô decomposition, Wiener and compound Poisson processes play a fundamental role in the study of the paths of a Lévy process.

Definition 2.1.12. A Φ'_β-valued continuous Lévy process $W = \{W_t\}_{t\geq 0}$ is called a Φ'_β-valued **Wiener process**.

Some basic properties of Φ'_β-valued Wiener processes are collected in the following result. See Theorem 2.7.1 of Itô [43] for a proof. A Φ'_β-valued process $G = \{G_t\}_{t\geq 0}$ is called **Gaussian** if for any $n \in \mathbb{N}$ and any $\phi_1, \dots, \phi_n \in \Phi$, $\{(G_t[\phi_1], \dots, G_t[\phi_n]) \, : \, t \geq 0\}$ is a Gaussian process on $\mathbb{R}^n$.

Theorem 2.1.13. *Let $W = \{W_t\}_{t\geq 0}$ be a Φ'_β-valued Wiener process. Then, W is Gaussian and hence square integrable. Moreover, there exists $\mathfrak{m} \in \Phi'_\beta$ and a continuous Hilbertian semi-norm $\mathcal{Q}$ on Φ, called respectively the* ***mean*** *and the* ***covariance functional*** *of W, such that*

$$\mathbb{E}\left(W_t[\phi]\right) = t\mathfrak{m}[\phi], \quad \forall\, \phi \in \Phi, \, t \geq 0. \tag{2.2}$$

$$\mathbb{E}\left(\left(W_t - t\mathfrak{m}\right)[\phi]\left(W_s - s\mathfrak{m}\right)[\varphi]\right) = (t \wedge s)\mathcal{Q}(\phi, \varphi), \quad \forall\, \phi, \varphi \in \Phi, \, s, t \geq 0. \tag{2.3}$$

where in (2.3) $\mathcal{Q}(\cdot, \cdot)$ corresponds to the continuous, symmetric, non-negative bilinear form on $\Phi \times \Phi$ associated to $\mathcal{Q}$. Furthermore, the characteristic function of W is given by

$$\mathbb{E}\left(e^{iW_t[\phi]}\right) = \exp\left(it\mathfrak{m}[x] - \frac{t}{2}\mathcal{Q}(\phi)^2\right), \quad \text{for each } t \geq 0, \, \phi \in \Phi. \tag{2.4}$$

The following result due to Itô (see [41]) provides the existence of a Φ'_β-valued Wiener processes.

Theorem 2.1.14. *Given $\mathfrak{m} \in \Phi'_\beta$ and a continuous Hilbertian semi-norm $\mathcal{Q}$ on Φ, there exists a Φ'_β-valued Wiener process $W = \{W_t\}_{t\geq 0}$ such that $\mathfrak{m}$ and $\mathcal{Q}$ are the mean and covariance functional of W. Moreover, such a process is unique in distribution.*

Now we proceed to study the basic properties of compound Poisson processes.

Definition 2.1.15. Suppose that $\{Z_n \, : \, n \in \mathbb{N}\}$ is a sequence of independent and identically distributed Φ'_β-valued regular random variables with common distribution μ and let $\pi = \{\pi_t \, : \, t \geq 0\}$ be a Poisson process with intensity $a > 0$ that is independent of all the Z_ns. Then, the Φ'_β-valued stochastic process $L = \{L_t\}_{t\geq 0}$ defined by

$$L_t = \begin{cases} 0, & \text{if } \pi_t = 0; \\ \sum_{j=1}^{\pi_t} Z_j, & \text{otherwise;} \end{cases} \tag{2.5}$$

is called a **compound Poisson process** with associated measure μ.

Definition 2.1.16. If $L = \{L_t\}_{t\geq 0}$ is an integrable Φ'_β-valued compound Poisson process, the stochastic process $\widetilde{L}$ given by $\widetilde{L}_t[\phi] := L_t[\phi] - \mathbb{E}\left(L_t[\phi]\right)$ for each $t \geq 0$ and each $\phi \in \Phi$, is called a **compensated compound Poisson process**.

Some properties of compound and compensated compound Poisson process are summarized in the following result. They can be proved using similar arguments to those in Chapter 4 of Peszat and Zabczyk [85], who work with Lévy processes in Hilbert spaces.

Theorem 2.1.17. *Let $L = \{L_t\}_{t\geq 0}$ be an Φ'_β-valued compound Poisson process. Then, with the notation of Definition 2.1.15 we have that:*

(1) L is a càdlàg Lévy process and its distribution is given by:

$$\mathbb{P}\left(L_t \in \Gamma\right) = e^{-at} \sum_{k=0}^{\infty} \frac{(at)^k}{k!} \mu^{*k}\left(\Gamma\right), \quad \forall\, t \geq 0, \forall\, \Gamma \in \mathcal{B}(\Phi'_\beta). \tag{2.6}$$

where $\mu^{*k} = \mu * \cdots * \mu$ (k-times) for $k \in \mathbb{N}$ and $\mu^{*0} = \delta_0$. Moreover, the characteristic function of each L_t for $t \geq 0$ is given by

$$\mathbb{E}\left[e^{iL_t[\phi]}\right] = \exp\left\{ t \int_{\Phi'_\beta} \left(e^{if[\phi]} - 1\right) a\mu(df) \right\}, \quad \forall \phi \in \Phi. \tag{2.7}$$

(2) If L is integrable, we have

$$\mathbb{E}\left(L_t[\phi]\right) = t \int_{\Phi'_\beta} f[\phi] a\mu(df), \quad \forall\, t \geq 0,\, \phi \in \Phi. \tag{2.8}$$

In such case, $\widetilde{L}$ is a zero-mean càdlàg martingale and a Lévy process. Moreover, the characteristic function of $\widetilde{L}_t$ for each $t \geq 0$ is given by

$$\mathbb{E}\left(e^{i\widetilde{L}_t[\phi]}\right) = \exp\left\{ t \int_{\Phi'_\beta} \left(e^{if[\phi]} - 1 - if[\phi]\right) a\mu(df) \right\}, \quad \forall\, \phi \in \Phi. \tag{2.9}$$

If furthermore L is square integrable, we have that

$$\mathbb{E}\left(\left|\widetilde{L}_t[\phi]\right|^2\right) = t \int_{\Phi'_\beta} \left|f[\phi]\right|^2 a\mu(df), \quad \forall t \geq 0,\, \phi \in \Phi. \tag{2.10}$$

§ 2.2 The Lévy-Itô Decomposition.

In this section we will show that if Φ is a complete, barrelled, nuclear space, any Φ'_β-valued Lévy process can be decomposed as the sum of four components (see Theorem 2.2.13): a deterministic linear term (drift), a Wiener process (the continuous part), a compound Poisson process (large jumps part) and a compensated compound Poisson process (small jumps part). Such a decomposition is usually known as the Lévy-Itô decomposition.

The Lévy-Itô decomposition for Lévy processes taking values in the strong dual of a nuclear space Ψ was firstly studied by Üstünel in [105]. In this work, the nuclear space Ψ is assumed to be separable, complete and bornological[1] and the strong dual Ψ'_β is assumed to be Suslin and nuclear. The Lévy-Itô decomposition was shown for additive processes, i.e. for Ψ'_β-valued processes X, with $X_0 = 0$ $\mathbb{P}$-a.e., independent increments and such that the characteristic function of X is continuous on $\mathbb{R}_+ \times \Psi$. A key ingredient in the proof of Üstünel is the fact that the measure μ_{X_t} is infinitely divisible and hence it satisfies the Lévy-Khintchine formula for infinitely divisible measures proved by Dettweiler in [23] in the context of a complete locally convex space. Based on the result of Üstünel, in [83] V. Pérez Abreu, A. Rocha Arteaga and C. Tudor proved a special version of the Lévy-Khintchine formula for additive processes taking values

[1]Indeed Üstünel did not explicitly assumed that the space Ψ is bornological, but this is implicit in his proof.

in a cone $C' \subseteq \Psi'_{\beta}$, where Ψ is a special class of Fréchet nuclear spaces previously considered by Kallianpur and Xiong [54].

For our proof of the Lévy-Itô decomposition, we will employ a different approach than that used by Üstünel in [105], which is based on the characterization of Lévy measures obtained by A. Tortrat [98] (see Theorem 2.2.6) and the use of the Poisson integrals defined by the Poisson random measure associated to a Lévy process. This can be thought of as an infinite dimensional version of the approach used by Applebaum in [3] for the $\mathbb{R}^n$-case.

One of the main advantages of our proof of the Lévy-Itô decomposition is that we only require the space Φ to be complete, barrelled and nuclear (no need for Φ be separable nor bornological) and no assumptions are made on the strong dual (no need for Φ'_{β} to be Suslin nor nuclear). Therefore, we are in a more general situation than in Üstünel [105]. Also, as a corollary of our proof we will obtain the Lévy-Khintchine formula (see Theorem 2.2.14) for the characteristic function of any Φ'_{β}-valued Lévy process.

Assumption 2.2.1. *Throughout this section* Φ *will be a complete, barrelled, nuclear space over* $\mathbb{R}$. $L = \{L_t\}_{t \geq 0}$ *will be a* Φ'_{β}*-valued Lévy process satisfying Assumptions 2.1.10. Also,* $\Omega_L \subseteq \Omega$ *is a set with* $\mathbb{P}(\Omega_L) = 1$ *and such that for each* $\omega \in \Omega_L$ *the map* $t \mapsto L_t(\omega)$ *is càdlàg in* Φ'_{β}.

2.2.1 Poisson Random Measures and Poisson Integrals.

In this section we review basic properties of the Poisson random measure defined by a Lévy process. The Poisson integrals associated to it play a key role in our proof of the Lévy-Itô decomposition. We refer the reader to Section 9, Chapter 1 of Ikeda and Watanabe [40] for the general properties of random measures that we will use in this section. We will not provide proofs for the results in this section as they follow from essentially the same arguments to that in other contexts. The reader is referred to for example Chapter 2 of Applebaum [3] for proofs in the $\mathbb{R}^n$ case or to Peszat and Zabczyk [85] for the Hilbert space case.

We define by $\Delta L_t := L_t - L_{t-}$ the **jump** of the process L at the time $t \geq 0$. Note that $\Delta L = \{\Delta L_t\}_{t \geq 0}$ is an $\{\mathcal{F}_t\}$-adapted Φ'_{β}-valued regular stochastic process.

We say that a set $A \in \mathcal{B}(\Phi'_{\beta} \setminus \{0\})$ is **bounded below** if $0 \notin \overline{A}$, where $\overline{A}$ is the closure of A. Then, A is bounded below if and only if A^c is contained in the complement of a neighborhood of zero. We denote by $\mathcal{A}$ the collection of all the subsets of $\Phi'_{\beta} \setminus \{0\}$ that are bounded below. Clearly, $\mathcal{A}$ is a ring.

For any $A \in \mathcal{B}(\Phi'_{\beta} \setminus \{0\})$ and $0 \leq t < \infty$ define

$$N(t,A)(\omega) = \# \{0 \leq s \leq t \, : \, \Delta L_s(\omega) \in A\} = \sum_{0 \leq s \leq t} \mathbb{1}_A \left(\Delta L_s(\omega) \right), \quad \text{if } \omega \in \Omega_L \quad (2.11)$$

and $N(t,A)(\omega) = 0$ if $\omega \in \Omega_L^c$.

As L has càdlàg paths for each $\omega \in \Omega_L$, $\Delta L_t \neq 0$ for at most a countable number of $t \geq 0$ and thus $A \mapsto N(t,A)(\omega)$ is a counting measure on $\left(\Phi'_{\beta} \setminus \{0\}, \mathcal{B}(\Phi'_{\beta} \setminus \{0\}) \right)$. Then, $\Delta L = \{\Delta L_t\}_{t \geq 0}$ is a stationary Poisson point processes on $\left(\Phi'_{\beta} \setminus \{0\}, \mathcal{B}(\Phi'_{\beta} \setminus \{0\}) \right)$ and $N = \{N(t,A) \, : \, t \geq 0, A \in \mathcal{B}(\Phi'_{\beta} \setminus \{0\})\}$ is the **Poisson random measure** associated to ΔL with respect to the ring $\mathcal{A}$, i.e. N satisfies:

(1) If $A \in \mathcal{A}$, then $\{N(t,A)\}_{t\geq 0}$ is a Poisson process with intensity $\mathbb{E}\left(N(1,A)\right)$.
(2) If $A_1, \dots, A_m$ bounded below and are disjoint, and if $s_1, \dots, s_m$ are distinct and each $s_i \geq 0$, then the random variables $N(s_1, A_1), \dots, N(s_m, A_m)$ are independent.

Let ν be the Borel measure on Φ'_β defined by $\nu(\{0\}) = 0$ and

$$\nu(\Gamma) = \mathbb{E}\left(N(1,\Gamma)\right), \quad \forall\, \Gamma \in \mathcal{B}\left(\Phi'_\beta \setminus \{0\}\right) \tag{2.12}$$

For each $A \in \mathcal{A}$, $\nu\big|_A \in \mathfrak{M}^b_R(\Phi'_\beta)$ and moreover $\mathbb{E}\left(N(t,\Gamma)\right) = t\nu(\Gamma)$ for all $\Gamma \in \mathcal{B}\left(\Phi'_\beta \setminus \{0\}\right)$. The measure ν is called the **intensity measure** of N.

Let $\mathcal{I}$ be the ring comprising finite unions of sets of the form $I \times A$, where $A \in \mathcal{A}$ and I is a finite union of intervals. We define the **compensated Poisson random measure** as the random measure $\widetilde{N}$ on $\left(\Phi'_\beta \setminus \{0\}, \mathcal{I}\right)$ given by

$$\widetilde{N}(t,A) = N(t,A) - t\nu(A), \quad \forall\, t \geq 0,\, A \in \mathcal{A}. \tag{2.13}$$

For each $A \in \mathcal{A}$, $\{\widetilde{N}(t,A)\}_{t\geq 0}$ is a real-valued, zero-mean càdlàg martingale and so $\widetilde{N} = \{\widetilde{N}(t,A) : t \geq 0,\, A \in \mathcal{A}\}$ is a σ-additive independently scattered martingale-valued measure on $\left(\Phi'_\beta \setminus \{0\}, \mathcal{I}\right)$.

Now we review the properties of Poisson integrals with respect to the Poisson random measure N associated to the Lévy process L. If $A \in \mathcal{A}$, then for any $t \geq 0$, define the **Poisson integral** based on A to be the Φ'_β-valued random variable denoted by $\int_A f N(t,df)$ and defined by

$$\int_A f N(t,df)(\omega)[\phi] = \sum_{0\leq s\leq t} \Delta L_s(\omega)[\phi] \mathbb{1}_A\left(\Delta L_s(\omega)\right), \quad \forall \omega \in \Omega,\, \phi \in \Phi. \tag{2.14}$$

In some occasions, we will employ the shorter notation $J^A = \{J^A_t\}_{t\geq 0}$ to denote the process defined in (2.14). It is a compound Poisson process, where the independent and identically distributed random variables satisfying (2.5) have common distribution $\nu\big|_A$, where ν is the intensity measure associated to the Poisson random measure N. It follows from Theorem 2.1.17 that

$$\mathbb{P}\left(J^A_t \in \Gamma\right) = e^{-at} \sum_{k=0}^{\infty} \frac{(at)^k}{k!} (\nu\big|_A)^{*k}(\Gamma), \quad \forall\, t \geq 0, \forall\, \Gamma \in \mathcal{B}(\Phi'_\beta). \tag{2.15}$$

Its characteristic function is given by

$$\mathbb{E}\left(\exp\left\{ i \int_A f N(t,df)[\phi] \right\}\right) = \exp\left\{ t \int_A \left(e^{if[\phi]} - 1\right) \nu(df) \right\}, \quad \forall\, \phi \in \Phi.$$

Moreover, if $\int_A |f[\phi]|\, \nu(df) < \infty$, for each $\phi \in \Phi$, then

$$\mathbb{E}\left(\int_A f N(t,df)[\phi]\right) = t \int_A f[\phi] \nu(df), \quad \forall\, \phi \in \Phi, \tag{2.16}$$

Furthermore, if $\int_A |f[\phi]|^2\, \nu(df) < \infty$, for each $\phi \in \Phi$, then

$$\mathrm{Var}\left(\int_A f N(t,df)[\phi]\right) = t \int_A |f[\phi]|^2\, \nu(df), \quad \forall\, \phi \in \Phi. \tag{2.17}$$

Remark 2.2.2. *It is important to stress the fact that the Poisson integral exists even if $A \in \mathcal{B}(\Phi'_\beta \setminus \{0\})$ is not necessarily bounded below but if instead it only satisfies that $\nu\big|_A \in \mathfrak{M}^b_R(\Phi'_\beta)$.*

Now, if $\int_A |f[\phi]| \, \nu(df) < \infty$, for each $\phi \in \Phi$, we define the **compensated Poisson integral** $\int_A f \widetilde{N}(t, df)$ on $A \in \mathcal{A}$, for $t \geq 0$ by

$$\int_A f \widetilde{N}(t, df)(\omega)[\phi] = \int_A f N(t, df)(\omega)[\phi] - t \int_A f[\phi] \nu(df), \quad \forall\, \omega \in \Omega,\, \phi \in \Phi. \tag{2.18}$$

We will use the shorter notation $\widetilde{J}^A = \{\widetilde{J}^A_t\}_{t \geq 0}$ for the process defined in (2.18). It is a compensated compound Poisson process, where the independent and identically distributed random variables satisfying (2.5) have common distribution $\nu\big|_A$, and the characteristic function of $\int_A f \widetilde{N}(t, df)$ for each $t \geq 0$ is given by

$$\mathbb{E}\left(\exp\left\{ i \int_A f \widetilde{N}(t, df)[\phi] \right\} \right) = \exp\left\{ t \int_A \left(e^{i f[\phi]} - 1 - i f[\phi] \right) \nu(df) \right\}, \quad \forall\, \phi \in \Phi. \tag{2.19}$$

Moreover, if $\int_A |f[\phi]|^2 \, \nu(df) < \infty$, for each $\phi \in \Phi$, then

$$\mathbb{E}\left(\left| \int_A f \widetilde{N}(t, df)[\phi] \right|^2 \right) = t \int_A |f[\phi]|^2 \, \nu(df), \quad \forall\, \phi \in \Phi. \tag{2.20}$$

Some other important properties of Poisson integrals are summarized in the following result.

Theorem 2.2.3.

(1) Let $A_1, A_2 \in \mathcal{A}$ be disjoint. Then, the processes J^{A_1} and J^{A_2} are independent. If moreover $\int_{A_i} |f[\phi]| \, \nu(df) < \infty$, for all $\phi \in \Phi$, $i = 1, 2$, then $\widetilde{J}^{A_1}$ and $\widetilde{J}^{A_2}$ are independent.

(2) For any $A \in \mathcal{A}$, $L - J^A = \left\{ L_t - J^A_t \right\}_{t \geq 0}$ is a Φ'_β-valued Lévy process. Moreover, the processes $L - J^A$ and J^A are independent.

Proof. (1) For the first part, let $\phi_1, \dots, \phi_n \in \Phi$. Then, it follows from (2.14) that the $\mathbb{R}^n$-valued processes $(J^{A_1}[\phi_1], \dots, J^{A_1}[\phi_n])$ and $(J^{A_2}[\phi_1], \dots, J^{A_2}[\phi_n])$ are compound Poisson processes whose jumps occurs at distinct times for each $\omega \in \Omega$ due to the fact that A_1 and A_2 are disjoint. Then, the same arguments of the proof of Theorem 2.4.6 of Applebaum [3] p.116 show that the processes $(J^{A_1}[\phi_1], \dots, J^{A_1}[\phi_n])$ and $(J^{A_2}[\phi_1], \dots, J^{A_2}[\phi_n])$ are independent. Then, as the processes $J^{A_1}, \dots, J^{A_k}$ are regular it follows from Proposition 1.2.16 that they are independent. The second part follows from the first part.

(2) For any $A \in \mathcal{A}$, the same arguments to those used in Theorem 37, Chapter 1 of Protter [88] p.27, for the case of $\mathbb{R}^n$-valued Lévy processes shows that $L - J^A$ is a Φ'_β-valued Lévy process. To prove the independence of $L - J^A$ and J^A, let $\phi_1, \dots, \phi_n \in \Phi$. As $\left((L - J^A)[\phi_1], \dots, (L - J^A)[\phi_n] \right)$ and $\left(J^A[\phi_1], \dots, J^A[\phi_n] \right)$ are $\mathbb{R}^n$-valued processes Lévy processes that have their jumps at distinct times for each $\omega \in \Omega$, the same arguments of the proof of Lemma 7.9 and Theorem 7.12 of Medvegyev [64] p.468-71 show that the processes $\left((L - J^A)[\phi_1], \dots, (L - J^A)[\phi_n] \right)$ and $\left(J^A[\phi_1], \dots, J^A[\phi_n] \right)$ are independent. Then, the independence of $L - J^A$ and J^A follows from Proposition 1.2.16 as both $L - J^A$ and J^A are regular processes. □

2.2.2 The Lévy Measure of a Lévy process

In this section we will prove that the intensity measure ν of the Poisson random measure N associated to the Lévy process L is a Lévy measure on Φ'_{β}. This result will be fundamental for our proof of the Lévy-Itô decomposition.

We start by recalling the concept of Poisson measures. Let $G \in \mathfrak{M}^{b}_{R}(\Phi'_{\beta})$. The measure $e(G) \in \mathfrak{M}^{1}_{R}(\Phi'_{\beta})$ defined by

$$e(G)(\Gamma) = e^{-G(\Phi'_{\beta})} \sum_{k=0}^{\infty} \frac{1}{k!} G^{*k}(\Gamma), \quad \forall \, \Gamma \in \mathcal{B}(\Phi'_{\beta}),$$

is called a **Poisson measure**. We call G the **Poisson exponent** of $e(G)$. It is clear that

$$\widehat{e(G)}(\phi) = \exp\left[-(\widehat{G}(0) - \widehat{G}(\phi)) \right], \quad \forall \, \phi \in \Phi.$$

We adopt the following general definition of Lévy measures from Dettweiler [23].

Definition 2.2.4. A Borel measure λ on Φ'_{β} is called a **Lévy measure** if $\lambda(\{0\}) = 0$ and if there exist some increasing family $\{G_i\}_{i \in I} \subseteq \mathfrak{M}^{b}_{R}(\Phi'_{\beta})$ such that:

(1) $\lambda(\Gamma) = \sup_{i \in I} G_i(\Gamma)$, for each $\Gamma \in \mathcal{B}(\Phi'_{\beta})$,
(2) the family $\{e(G_i)\}_{i \in I}$ is shift tight (see Section 1.2).

For a proof of the following result, see Lemma 1.4 of Dettweiler [23].

Proposition 2.2.5. *Let λ be a Lévy measure. Then, for each neighborhood of zero $U \subseteq \Phi'_{\beta}$, $\lambda\big|_{U^c} \in \mathfrak{M}^{b}_{R}(\Phi'_{\beta})$.*

The following very important result, due to A. Tortrat (see Section III of [98]), characterizes the Lévy measures on Φ'_{β}. It does not hold in general for any locally convex space, for example there are examples of Banach spaces where it does not hold (see Dettweiler [24]). Our assumption that Φ is complete, barrelled and nuclear is fundamental for its validity.

Theorem 2.2.6. *A Borel measure λ on Φ'_{β} is a Lévy measure if and only if there exists a continuous Hilbertian semi-norm ρ on Φ such that*

$$\int_{B_{\rho'}(1)} \rho'(f)^2 \lambda(df) < \infty, \quad \text{and} \quad \lambda\big|_{B_{\rho'}(1)^c} \in \mathfrak{M}^{b}_{R}(\Phi'_{\beta}), \tag{2.21}$$

where $B_{\rho'}(1) := B_{\rho}(1)^0 = \{ f \in \Phi'_{\beta} : \rho'(f) \leq 1 \}$ is a compact, convex, balanced subset of Φ'_{β}. In particular every Lévy measure on Φ'_{β} is σ-finite.

We are ready to show the main result of this section.

Theorem 2.2.7. *The intensity measure ν of a Φ'_{β}-valued Lévy process L is a Lévy measure on Φ'_{β}.*

Proof. For each $A \in \mathcal{A}$, let $\nu_A := \nu\big|_{A}$. We know from the properties of ν that $\nu_A \in \mathfrak{M}^{b}_{R}(\Phi'_{\beta})$, for all $A \in \mathcal{A}$. Now consider on $\mathcal{A}$ the order relationship given by the inclusion of sets. As $\Phi'_{\beta} \setminus \{0\} = \bigcup_{A \in \mathcal{A}} A$, it follows that $\nu = \sup_{A \in \mathcal{A}} \nu_A$. Our objective is to show that the family of Poisson measures $\{e(\nu_A)\}_{A \in \mathcal{A}}$ is shift tight.

To do this, first note that from Theorem 2.2.3(2), for each $A \in \mathcal{A}$, the processes $L - J^{A}$ and J^{A} are independent. Therefore, for each $t \geq 0$ we have

$$\mu_{L_t} = \mu_{L_t - J^{A}_{t}} * \mu_{J^{A}_{t}}, \quad \forall \, A \in \mathcal{A}. \tag{2.22}$$

Now, as for each $t \geq 0$, μ_{L_t} is a Radon probability measure, and hence tight (Proposition 1.2.10), the relationship given in (2.22) implies that $\{\mu_{J^{A}_{t}}\}_{A \in \mathcal{A}}$ is shift tight (see Theorem of Heyer [37], p.41, the arguments there for probability measures on Banach spaces holds also in our context). Now, it follows from (2.15) that if we take $t = 1$ we have $\mu_{J^{A}_{t}} = e(\nu_{A})$, for all $A \in \mathcal{A}$. Then, our arguments above implies that $\{e(\nu_{A})\}_{A \in \mathcal{A}}$ is shift tight and therefore ν is a Lévy measure. □

Notation 2.2.8. From now, the intensity measure ν of the Lévy process L will be called the **Lévy measure** of L.

2.2.3 The Lévy-Itô Decomposition.

Our main objective of this section is to prove Theorem 2.2.13, which is the Lévy-Itô decomposition. A key step of the proof is the properties of the Lévy measure.

Let ν be the Lévy measure of L. According to Theorems 2.2.6 and 2.2.7, there exists a continuous Hilbertian semi-norm ρ on Φ such that

$$\int_{B_{\rho'}(1)} \rho'(f)^{2} \nu(df) < \infty, \quad \text{and} \quad \nu\big|_{B_{\rho'}(1)^{c}} \in \mathfrak{M}^{b}_{R}(\Phi'_{\beta}), \tag{2.23}$$

where $B_{\rho'}(1) := B_{\rho}(1)^{0} = \{ f \in \Phi'_{\beta} : \rho'(f) \leq 1 \}$ is a compact, convex, balanced subset of Φ'_{β}.

Theorem 2.2.9. *There exists a Φ'_{β}-valued zero-mean, square integrable, càdlàg Lévy process $M = \{M_t\}_{t \geq 0}$ such that for all $t \geq 0$, it has characteristic function given by*

$$\mathbb{E}\left(e^{i M_t[\phi]} \right) = \exp \left\{ t \int_{B_{\rho'}(1)} \left(e^{i f[\phi]} - 1 - i f[\phi] \right) \nu(df) \right\}, \quad \forall \, \phi \in \Phi, \tag{2.24}$$

and second moments given by

$$\mathbb{E}\left(|M_t[\phi]|^{2} \right) = t \int_{B_{\rho'}(1)} |f[\phi]|^{2} \, \nu(df), \quad \forall \, \phi \in \Phi. \tag{2.25}$$

Moreover, there exists a continuous Hilbertian semi-norm q on Φ, $\rho \leq q$, such that $i_{\rho,q}$ is Hilbert-Schmidt and for which M is a Φ'_{q}-valued zero-mean, square integrable, càdlàg Lévy process with second moment given by

$$\mathbb{E}\left(q'(M_t)^{2} \right) = \int_{B_{\rho'}(1)} q'(f)^{2} \nu(df), \quad \forall \, t \geq 0. \tag{2.26}$$

Proof. Let $\mathfrak{B}$ be a local base of closed neighborhoods of zero for Φ'_{β}. Let $\mathcal{A}_{\rho'}$ denotes the collection of all sets of the form $V \cap B_{\rho'}(1)$, where $V^{c} \in \mathfrak{B}$. Is clear that $\mathcal{A}_{\rho'} \subseteq \mathcal{A}$ (see Section 2.2.1). Moreover, as $\Phi'_{\beta} \setminus \{0\} = \bigcup_{V \in \mathfrak{B}} V^{c}$ (this follows because Φ'_{β} is Hausdorff) then we have $B_{\rho'}(1) \setminus \{0\} = \bigcup_{A \in \mathcal{A}_{\rho'}} A$.

Fir an arbitrary $A \in \mathcal{A}_{\rho'}$. It follows from (2.23) that

$$\int_A |f[\phi]|^2 \, \nu(df) \leq \rho(\phi)^2 \int_A \rho'(f)^2 \nu(df) \leq \rho(\phi)^2 \int_{B_{\rho'}(1)} \rho'(f)^2 \nu(df) < \infty, \quad \forall \, \phi \in \Phi. \tag{2.27}$$

Therefore, the compensated Poisson integral $\widetilde{J}^A$ is a square integrable compensated compound Poisson processes with characteristic function given by (2.19) and second moments given by (2.20). From Doob's inequality, (2.20) and (2.27), for every $T > 0$ we have

$$\mathbb{E} \left(\sup_{t \in [0,T]} \left| \widetilde{J}^A_t[\phi] \right|^2 \right) \leq 4T \mathbb{E} \left(\left| \widetilde{J}^A_T[\phi] \right|^2 \right) \leq C(T) \rho(\phi)^2, \quad \forall \, \phi \in \Phi,$$

where $C(T) = 4T \int_{B_{\rho'}(1)} \rho'(f)^2 \nu(df) < \infty$. Then, from Theorem 1.2.25, there exists a continuous Hilbertian semi-norm q on Φ, $\rho \leq q$, such that $i_{\rho,q}$ is Hilbert-Schmidt and for which $\widetilde{J}^A$ possesses a Φ'_q-valued càdlàg version that is also a zero-mean, square integrable Lévy process. We denote this version again by $\widetilde{J}^A$. Let $\{\phi^q_j\}_{j \in \mathbb{N}} \subseteq \Phi$ be a complete orthonormal system of Φ_q. Then, from Fubini's theorem, Parseval's identity and (2.20), for every $t \geq 0$ we have

$$\mathbb{E} \left(q'(\widetilde{J}^A_t)^2 \right) = \sum_{j=1}^{\infty} \mathbb{E} \left(\left| \widetilde{J}^A_t[\phi^q_j] \right|^2 \right) = t \sum_{j=1}^{\infty} \int_A \left| f[\phi^q_j] \right|^2 \nu(df) = t \int_A q'(f)^2 \nu(df). \tag{2.28}$$

Now, consider on $\mathcal{A}_{\rho'}$ the order induced by the inclusion of sets. Our next objective is to show that for every $T > 0$ the net $\{\{\widetilde{J}^A_t\}_{t \in [0,T]} : A \in \mathcal{A}_{\rho'}\}$ converges in $\mathcal{M}^2_T(\Phi'_q)$. To do this, we will show that for a fixed $T > 0$, $\{\{\widetilde{J}^A_t\}_{t \in [0,T]} : A \in \mathcal{A}_{\rho'}\}$ is a Cauchy net in $\mathcal{M}^2_T(\Phi'_q)$, then convergence follows by completeness of this space.

Fix an arbitrary $T > 0$. First observe that if $A_1, A_2 \in \mathcal{A}_{\rho'}$, $A_1 \subseteq A_2$, then from Doob's inequality, the definition of compensated Poisson integral and (2.28) we have

$$\mathbb{E} \left(\sup_{t \in [0,T]} q'(\widetilde{J}^{A_1}_t - \widetilde{J}^{A_2}_t)^2 \right) = 4 \mathbb{E} \left(q'(\widetilde{J}^{A_2 \setminus A_1}_T)^2 \right) = 4T \int_{A_2 \setminus A_1} q'(f)^2 \nu(df). \tag{2.29}$$

Therefore, if we can show that

$$\lim_{A \in \mathcal{A}_{\rho'}} \int_A q'(f)^2 \nu(df) = \int_{B_{\rho'}(1)} q'(f)^2 \nu(df) < \infty, \tag{2.30}$$

then (2.29) and (2.30) show that $\{\widetilde{J}^A\}_{A \in \mathcal{A}_{\rho'}}$ is a Cauchy net on $\mathcal{M}^2_T(\Phi'_q)$.

To prove (2.30), first note that the continuity of the map $i'_{\rho,q}$ implies that the set $B_{\rho'}(1)$ is bounded in Φ'_q, and hence q' is bounded on $B_{\rho'}(1)$. Also, note that as ν is a Borel measure on $B_{\rho'}(1)$, and $B_{\rho'}(1)$ is a Suslin set (it is the image under the continuous map i'_ρ of the unit ball of the separable Hilbert space Φ'_ρ), then ν is a Radon measure on $B_{\rho'}(1)$ (see Theorem 7.4.3 of Bogachev [8], p.85, Vol II). Moreover, because we have that $B_{\rho'}(1) \setminus \{0\} = \bigcup_{A \in \mathcal{A}_{\rho'}} A$ and because ν is a Radon probability measure on $B_{\rho'}(1)$ such that $\nu(\{0\}) = 0$, we have that $\nu(B_{\rho'}(1)) = \lim_{A \in \mathcal{A}_{\rho'}} \nu(A)$ (see Propositions 7.2.2 and 7.2.5 of Bogachev [8], p.74-5, Vol II). Therefore, from all the above we have

$$\begin{aligned} \lim_{A \in \mathcal{A}_{\rho'}} \left| \int_{B_{\rho'}(1)} q'(f)^2 \nu(df) - \int_A q'(f)^2 \nu(df) \right| &\leq \lim_{A \in \mathcal{A}_{\rho'}} \int_{B_{\rho'}(1) \setminus A} q'(f)^2 \nu(df) \\ &\leq \sup_{f \in B_{\rho'}(1)} q'(f)^2 \lim_{A \in \mathcal{A}_{\rho'}} \mu(B_{\rho'}(1) \setminus A) = 0, \end{aligned}$$

and hence (2.30) is valid.

Then, for any $T>0$, $\{\{\widetilde{J}^A_t\}_{t\in[0,T]} : A \in \mathcal{A}_{\rho'}\}$ is a Cauchy net on $\mathcal{M}^2_T(\Phi'_q)$. This in turn implies (see Section 1.2.2) that $\{\widetilde{J}^A : A \in \mathcal{A}_{\rho'}\}$ converges in $\mathcal{M}^2(\Phi'_q)$. Therefore, there exists some $M=\{M_t\}_{t\geq 0}$ that is a Φ'_q-valued zero-mean, square integrable, càdlàg martingale and such that the net $\{\widetilde{J}^A : A \in \mathcal{A}_{\rho'}\}$ converges to M in $L^2\left(\Omega, \mathcal{F}, \mathbb{P}; \Phi'_q\right)$ uniformly on compact intervals of $[0,\infty)$. This uniform convergence, (2.28) and (2.30) implies that M satisfies (2.26). Moreover, viewing M as a Φ'_β-valued processes it is also a Φ'_β-valued, zero-mean, square integrable, càdlàg martingale.

To prove (2.24) and (2.25), let $\phi \in \Phi$ arbitrary but fixed. From (1.5) applied to ρ and a basic estimate of the complex exponential function we have

$$\left| e^{if[\phi]} - 1 - if[\phi] \right| \leq \frac{|f[\phi]|^2}{2} \leq \frac{\rho(\phi)^2 \rho'(f)^2}{2} \leq \frac{\rho(\phi)^2}{2} < \infty, \quad \forall\, f \in B_{\rho'}(1).$$

Therefore, the functions $f \mapsto (e^{if[\phi]} - 1 - if[\phi])$ and $f \mapsto |f[\phi]|^2$ are bounded on $B_{\rho'}(1)$. Then, using similar arguments to those used to prove (2.30) we can show that

$$\lim_{A \in \mathcal{A}_{\rho'}} \int_A |f[\phi]|^2 \,\nu(df) = \int_{B_{\rho'}(1)} |f[\phi]|^2 \,\nu(df), \tag{2.31}$$

and

$$\lim_{A \in \mathcal{A}_{\rho'}} \int_A (e^{if[\phi]} - 1 - if[\phi]) \nu(df) = \int_{B_{\rho'}(1)} (e^{if[\phi]} - 1 - if[\phi]) \nu(df). \tag{2.32}$$

On the other hand, for any $A \in \mathcal{A}_{\rho'}$ and $T>0$, (1.5) applied to q implies that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left| M_t[\phi] - \widetilde{J}^A_t[\phi] \right|^2 \right) \leq q(\phi)^2\, \mathbb{E}\left(\sup_{t\in[0,T]} q'(M_t - \widetilde{J}^A_t)^2 \right). \tag{2.33}$$

Therefore, the fact that $\{\widetilde{J}^A : A \in \mathcal{A}_{\rho'}\}$ converges to M in $\mathcal{M}^2(\Phi'_q)$ and (2.33), implies that $\{\widetilde{J}^A[\phi] : A \in \mathcal{A}_{\rho'}\}$ converges to $M[\phi]$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ uniformly on compact intervals of $[0,\infty)$. This convergence together with (2.20) and (2.31) implies (2.25). Furthermore, as for each $t \geq 0$, $\left\{\widetilde{J}^A_t[\phi] : A \in \mathcal{A}_{\rho'}\right\}$ converges to $M_t[\phi]$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, then the net of characteristics functions $\{\mathbb{E}\left(\exp\left(i\widetilde{J}^A_t[\phi]\right)\right) : A \in \mathcal{A}_{\rho'}\}$ converges to the characteristic function $\mathbb{E}\left(\exp\left(iM_t[\phi]\right)\right)$ of M. Then, (2.19) and (2.32) implies (2.24). Finally, as $\mathcal{M}^2(\Phi'_q)$ is metrizable (see Section 1.2.2), we can choose a subsequence $\{\widetilde{J}^{A_n} : n \in \mathbb{N}\}$ that converges to M in $\mathcal{M}^2(\Phi'_q)$. Then, $\{\widetilde{J}^{A_n} : n \in \mathbb{N}\}$ converges to M in $L^2\left(\Omega, \mathcal{F}, \mathbb{P}; \Phi'_q\right)$ uniformly on compact intervals of $[0,\infty)$ and because each $\widetilde{J}^{A_n}$ is a Φ'_q-valued Lévy process, this implies that M is also a Φ'_q-valued Lévy process. This last fact implies that M is also a Φ'_β-valued Lévy process. ☐

Notation 2.2.10. We will denote the process $M=\{M_t\}_{t\geq 0}$ defined in Theorem 2.2.9 by $\left\{\int_{B_{\rho'}(1)} f\widetilde{N}(t,df) : t \geq 0\right\}$.

The following is a consequence of the fact that $\nu\big|_{B_{\rho'}(1)^c} \in \mathfrak{M}^b_R(\Phi'_\beta)$ (see Section 2.2.1 and see Remark 2.2.2).

Proposition 2.2.11. *The Φ'_{β}-valued process $\left\{\int_{B_{\rho'}(1)^c} f N(t,df) : t \geq 0\right\}$ defined by*

$$\int_{B_{\rho'}(1)^c} f N(t,df)(\omega)[\phi] = \sum_{0 \leq s \leq t} \Delta L_s(\omega)[\phi] \mathbb{1}_{B_{\rho'}(1)^c}(\Delta L_s(\omega)), \quad \forall \omega \in \Omega,\ \phi \in \Phi. \tag{2.34}$$

is a compound Poisson process. Moreover, $\forall \phi \in \Phi$, $t \geq 0$,

$$\mathbb{E}\left(\exp\left\{ i \int_{B_{\rho'}(1)^c} f N(t,df)[\phi] \right\}\right) = \exp\left\{ t \int_{B_{\rho'}(1)^c} \left(e^{if[\phi]} - 1 \right) \nu(df) \right\}. \tag{2.35}$$

Now, define the process $Y = \{Y_t\}_{t \geq 0}$ by

$$Y_t = L_t - \int_{B_{\rho'}(1)^c} f N(t,df), \quad \forall\, t \geq 0. \tag{2.36}$$

From Theorem 2.2.3(2) (that is still valid thanks to (2.23)), it follows that Y is a càdlàg Lévy process independent of $\left\{\int_{B_{\rho'}(1)^c} f N(t,df) : t \geq 0\right\}$. Moreover, from the definition of the Poisson integral (2.34), for any $0 \leq s < t$,

$$Y_t - Y_s = L_t - L_s - \sum_{s < u \leq t} \Delta L_u \mathbb{1}_{B_{\rho'}(1)^c}(\Delta L_u).$$

Therefore, $\sup_{t \geq 0} \rho'(\Delta Y_t(\omega)) \leq 1$ for each $\omega \in \Omega$. This in particular implies that for each $\phi \in \Phi$, the real-valued process $Y[\phi]$ satisfies, $\sup_{t \geq 0} |\Delta Y_t[\phi](\omega)| \leq \rho(\phi) < \infty$ for each $\omega \in \Omega$, thus $Y[\phi]$ has bounded jumps and consequently Y has finite moments to all orders (see Theorem 2.4.7 of Applebaum [3], p.118-9). Moreover, the stationary increments of Y implies that for each $\phi \in \Phi$, the map $t \mapsto \mathbb{E}(Y_t[\phi])$ is additive and measurable. Therefore, there exists some $\mathfrak{m} \in \Phi'_{\beta}$ such that $\mathbb{E}(Y_t[\phi]) = t\mathfrak{m}[\phi]$, for all $\phi \in \Phi$, $t \geq 0$.

Now, consider the process $Z = \{Z_t\}_{t \geq 0}$ given by

$$Z_t = Y_t - t\mathfrak{m}, \quad \forall\, t \geq 0. \tag{2.37}$$

From the properties of Y and the definition of $\mathfrak{m}$, Z is a zero-mean càdlàg Lévy process with moments to all orders and with jumps satisfying $\sup_{t \geq 0} \rho'(\Delta Z_t(\omega)) \leq 1$ for each $\omega \in \Omega$.

Theorem 2.2.12. *The Φ'_{β}-valued stochastic process $W = \{W_t\}_{t \geq 0}$ defined by*

$$W_t = Z_t - \int_{B_{\rho'}(1)} f \widetilde{N}(t,df), \quad \forall\, t \geq 0, \tag{2.38}$$

is a Φ'_{β}-valued Wiener process with mean-zero and covariance functional $\mathcal{Q}$ (as defined in Theorem 2.1.13). Moreover, there exists a continuous Hilbertian semi-norm p on Φ, $\mathcal{Q} \leq p$, such that $i_{\mathcal{Q},p}$ is Hilbert-Schmidt and W is a mean-zero Wiener process on Φ'_p.

Proof. Clearly W is càdlàg, has zero-mean and square moments. To prove it is a Lévy process, let $\mathcal{A}_{\rho'}$ be as defined in the proof of Theorem 2.2.9, and let $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}_{\rho'}$ be such that $\{\widetilde{J}^{A_n} : n \in \mathbb{N}\}$ converges to M in $\mathcal{M}^2(\Phi'_q)$. Then, from an estimate similar to (2.33) it follows that $\{Z - \widetilde{J}^{A_n} : n \in \mathbb{N}\}$ converges to $Z - M$ in probability in Φ'_{β} uniformly in compact intervals of $[0,\infty)$. By Theorem 2.2.3(2), (2.36), and (2.37), $Z - \widetilde{J}^{A_n}$ is a Φ'_{β}-valued Lévy process for each $n \in \mathbb{N}$, and then it follows that $W := Z - M$ is a Φ'_{β}-valued Lévy process (Theorem 2.1.11).

Then, to conclude that W is a Wiener process, we just need to prove that it has a continuous version as this imply that W is itself continuous (i.e. has an indistinguishable continuous version). From Theorem 1.2.22, all that we require is to show that for any $\phi \in \Phi$, the real-valued process $W[\phi] = \{W_t[\phi]\}_{t \geq 0}$ has a continuous version. We proceed in a similar way as in the proof of Proposition 6.2 of Riedle and van Gaans [90], who considered the Banach space case.

First, consider $\phi \in \Phi$ such that $\rho(\phi) = 1$. As $Z[\phi]$ defines a real-valued càdlàg Lévy process (see Lemma 2.1.5) it has a corresponding Lévy-Itô decomposition (see Theorem 2.4.16 of Applebaum [3], p.126) given by

$$Z_t[\phi] = b_{\phi}t + \sigma^2_{\phi}(W_{\phi})_t + \int_{\{|y|\leq 1\}} y\widetilde{N}_{\phi}(t,dy) + \int_{\{|y|>1\}} yN_{\phi}(t,dy)$$

where $b_{\phi} \in \mathbb{R}$, $\sigma^2_{\phi} \in \mathbb{R}_+$, W_{ϕ} is a standard real-valued Wiener process, N_{ϕ} is the Poisson random measure of $Z[\phi]$ and $\widetilde{N}_{\phi}$ its compensated Poisson random measure. All the random components of the decomposition are independent. For a set $C \in \mathcal{B}(\mathbb{R})$ that is bounded below we have that

$$N_{\phi}(t,C)(\omega) = \sum_{0\leq s\leq t} \mathbb{1}_C\left(\Delta Z_s(\omega)[\phi]\right) = \sum_{0\leq s\leq t} \mathbb{1}_{\mathcal{Z}(\phi;C)}\left(\Delta Z_s(\omega)\right) = N_Z\left(t, \mathcal{Z}(\phi;C)\right)(\omega),$$

where $\mathcal{Z}(\phi;C) := \{f \in \Phi' : f[\phi] \in C\}$, and N_Z denotes the Poisson random measure associated to Z. Note that $\mathcal{Z}(\phi;C)$ is a cylindrical set and consequently belongs to $\mathcal{B}(\Phi'_{\beta})$. Moreover, as C is bounded below in $\mathcal{B}(\mathbb{R})$, it follows that $\mathcal{Z}(\phi;C)$ is bounded below in $\mathcal{B}(\Phi'_{\beta})$. To see why this is true, let π_{ϕ} be given by (1.7). Then, by (1.8) and the continuity of π_{ϕ} it follows that $\overline{\mathcal{Z}(\phi;C)} = \overline{\pi^{-1}_{\phi}(C)} \subseteq \pi^{-1}_{\phi}(\overline{C})$. Hence, if $0 \in \overline{\mathcal{Z}(\phi;C)}$ then $0 \in \pi^{-1}_{\phi}(\overline{C})$, and consequently $0 \in \overline{C}$. But this contradicts the fact that C is bounded below. Therefore, $\mathcal{Z}(\phi;C)$ is bounded below.

Now, let $C = [-1,1]^c$ and $D = \{f \in \Phi' : |f[\phi]| \leq 1\}$. We then have that $D = \mathcal{Z}(\phi;C)^c$ and because $\phi \in B_{\rho}(1)$, it follows that $B_{\rho'}(1) \subseteq D$. Now, because the jumps of Z satisfy $\sup_{t\geq 0} \rho'(\Delta Z_t(\omega)) \leq 1$ for each $\omega \in \Omega$, the support of $N_Z(t,\cdot)$ is in $B_{\rho'}(1)$ for each $t \geq 0$, and consequently the support of $\widetilde{N}_Z(t,\cdot)$ is also in $B_{\rho}(1)$ for $t \geq 0$. Since $B_{\rho'}(1) \subseteq D$, it follows that

$$\int_D f\widetilde{N}_Z(t,df)[\phi] = \int_{B_{\rho'}(1)} f\widetilde{N}_Z(t,df)[\phi] + \int_{D\backslash B_{\rho'}(1)} f\widetilde{N}_Z(t,df)[\phi] = \int_{B_{\rho'}(1)} f\widetilde{N}_Z(t,df)[\phi]$$

and

$$\int_{D^c} fN_Z(t,df)[\phi] = 0.$$

Moreover, $\widetilde{N}_Z$ coincides with $\widetilde{N}$ in $B_{\rho'}(1)$, so we have that

$$\begin{aligned} Z_t[\phi] &= b_\phi t + \sigma^2_\phi (W_\phi)_t + \int_{\{|y|<1\}} y \widetilde{N}_\phi(t,dy) + \int_{\{|y|\geq 1\}} y N_\phi(t,dy) \\ &= b_\phi t + \sigma^2_\phi (W_\phi)_t + \int_{D} f \widetilde{N}_Z(t,df)[\phi] + \int_{D^c} f N_Z(t,df)[\phi] \\ &= b_\phi t + \sigma^2_\phi (W_\phi)_t + \int_{B_{\rho'}(1)} f \widetilde{N}_Z(t,df)[\phi] \\ &= b_\phi t + \sigma^2_\phi (W_\phi)_t + \int_{B_{\rho'}(1)} f \widetilde{N}(t,df)[\phi] \end{aligned}$$

Now, taking expectations we obtain that for every $t \geq 0$,

$$0 = \mathbb{E} Z_t[\phi] = b_\phi t + \sigma^2_x \mathbb{E}\left((W_x)_t \right) + \mathbb{E}\left(\int_{B_{\rho'}(1)} f \widetilde{N}(t,df)[\phi] \right) = b_\phi t$$

consequently $b_\phi = 0$. We obtain $W_t[\phi] = Z_t[\phi] - \int_{B_{\rho'}(1)} f \widetilde{N}(t,df)[\phi] = \sigma^2_\phi (W_\phi)_t$ and so $W[\phi]$ has continuous sample paths. The same representation holds for arbitrary $\phi \in \Phi$, as can be seen by replacing ϕ with $\phi / \rho(\phi)$ in the argument just given. Therefore, $W[\phi]$ is continuous for all $\phi \in \Phi$. Then, Theorem 1.2.22 implies that W has a continuous version and therefore W is itself a continuous process. By definition this implies that W is a Wiener process. The fact that it has mean-zero and covariance functional $\mathcal{Q}$ follows from Theorem 2.1.13. Finally, the existence of a continuous Hilbertian semi-norm p on Φ, $\mathcal{Q} \leq p$, such that $i_{\mathcal{Q},p}$ is Hilbert-Schmidt and W is a mean-zero Wiener process on Φ'_p is a consequence of Theorem 1.2.25 and (2.3). □

We are ready for the main result of this section.

Theorem 2.2.13 (Lévy-Itô decomposition)**.** *Let $L = \{L_t\}_{t \geq 0}$ be a Φ'_β-valued Lévy process. Then, for each $t \geq 0$ it has the following representation*

$$L_t = t\mathfrak{m} + W_t + \int_{B_{\rho'}(1)} f \widetilde{N}(t,df) + \int_{B_{\rho'}(1)^c} f N(t,df) \tag{2.39}$$

where

(1) $\mathfrak{m} \in \Phi'_\beta$,
(2) *ρ is a continuous Hilbertian semi-norm on Φ such that the Lévy measure ν of L satisfies* (2.23) *and $B_{\rho'}(1) := \{ f \in \Phi'_\beta : \rho'(f) \leq 1 \}$ is a compact, convex, balanced subset of Φ'_β,*
(3) *$\left\{ \int_{B_{\rho'}(1)^c} f N(t,df) : t \geq 0 \right\}$ is a compound Poisson process with characteristic function given by* (2.35),
(4) *$\{W_t\}_{t \geq 0}$ is a Φ'_β-valued mean zero Wiener process with covariance functional $\mathcal{Q}$,*
(5) *$\left\{ \int_{B_{\rho'}(1)} f \widetilde{N}(t,df) : t \geq 0 \right\}$ is a Φ'_β-valued zero-mean, square integrable, càdlàg Lévy process with characteristic function given by* (2.24) *and second moments given by* (2.25).

All the random components of the decomposition (2.39) *are independent.*

Moreover, there exists a continuous Hilbertian semi-norm q on Φ, with $\rho \leq q$ and $\mathcal{Q} \leq q$, such that the inclusions $i_{\rho,q}$ and $i_{\mathcal{Q},q}$ are Hilbert-Schmidt, and such that W is *a* Φ'_q*-valued zero-mean Wiener process and* $\left\{ \int_{B_{\rho'}(1)} f \widetilde{N}(t,df) : t \geq 0 \right\}$ *is a* Φ'_q*-valued zero-mean, square integrable, càdlàg Lévy process.*

Proof. The decomposition (2.39) and the properties of its components follows from Theorems 2.2.9 and 2.2.12, Proposition 2.2.11, (2.36) and (2.37). Now we prove the independence of the components in (2.39).

For any $\phi_1, \dots, \phi_n \in \Phi$, by considering the Lévy-Itô decomposition of the $\mathbb{R}^n$-valued Lévy process $\{(L_t[\phi_1], \dots, L_t[\phi_n])\}_{t \geq 0}$, it follows that the $\mathbb{R}^n$-valued processes $\{(W_t[\phi_1], \dots, W_t[\phi_n])\}_{t\geq 0}$, $\left\{ \left(\int_{B_{\rho'}(1)} f \widetilde{N}(t,df)[\phi_1], \dots, \int_{B_{\rho'}(1)} f \widetilde{N}(t,df)[\phi_n] \right) : t \geq 0 \right\}$, and $\left\{ \left(\int_{B_{\rho'}(1)^c} f N(t,df)[\phi_1], \dots, \int_{B_{\rho'}(1)^c} f N(t,df)[\phi_n] \right) : t \geq 0 \right\}$ are independent. Then, Proposition 1.2.16 shows that the processes $\{W_t\}_{t\geq 0}$, $\left\{ \int_{B_{\rho'}(1)} f \widetilde{N}(t,df) : t \geq 0 \right\}$ and $\left\{ \int_{B_{\rho'}(1)} f N(t,df) : t \geq 0 \right\}$ are independent. □

As an important by-product of the Lévy-Itô decomposition and more specifically of its proof, we obtain a Lévy-Khintchine theorem for the characteristic function of any Φ'_β-valued Lévy process.

Theorem 2.2.14 (Lévy-Khintchine theorem for Φ'_β-valued Lévy processes)**.**

(1) If $L = \{L_t\}_{t\geq 0}$ *is a* Φ'_β*-valued Lévy process, there exist* $\mathfrak{m} \in \Phi'_\beta$*, a continuous Hilbertian semi-norm* $\mathcal{Q}$ *on* Φ*, a Lévy measure* ν *on* Φ'_β *and a continuous Hilbertian semi-norm* ρ *on* Φ *such that* ν *satisfies* (2.23)*; such that for each* $t \geq 0$*,* $\phi \in \Phi$*,*

$$\mathbb{E}\left(e^{iL_t[\phi]} \right) = e^{t\eta(\phi)}, \quad \text{with}$$
$$\eta(\phi) = i\mathfrak{m}[\phi] - \frac{1}{2}\mathcal{Q}(\phi)^2 + \int_{\Phi'_\beta} \left(e^{if[\phi]} - 1 - if[\phi] \mathbb{1}_{B_{\rho'}(1)}(f) \right) \nu(df). \tag{2.40}$$

(2) Conversely, let $\mathfrak{m} \in \Phi'_\beta$*,* $\mathcal{Q}$ *be a continuous Hilbertian semi-norm on* Φ*, and* ν *be a Lévy measure on* Φ'_β *satisfying* (2.23) *for a continuous Hilbertian semi-norm* ρ *on* Φ*. There exists a* Φ'_β*-valued Lévy process* $L = \{L_t\}_{t\geq 0}$*, unique up to equivalence in distribution, whose characteristic function is given by* (2.40)*.*

Proof. If L is a Φ'_β-valued Lévy process then (2.40) follows from the independence of the random components of the decomposition (2.39), (2.4) (recall here that W has mean zero and covariance functional $\mathcal{Q}$), (2.24) and (2.35).

For the converse, let ν be a Lévy measure on Φ'_β with a continuous Hilbertian semi-norm ρ on Φ satisfying (2.23). As ν is σ-finite (Theorem 2.2.6), there exist a stationary Poisson point processes $p = \{p(t)\}_{t\geq 0}$ on $(\Phi'_\beta, \mathcal{B}(\Phi'_\beta))$ with associated Poisson random measure R, p and R unique up to equivalence in distribution, such that ν is the intensity measure of p (see Theorem 9.1, Chapter 1 of Ikeda and Watanabe [40] p.44. See also Proposition 19.4 of Sato [91] p.122).

Note that in the proof of Theorem 2.2.9, we only used the fact that the Lévy measure ν of a Lévy process L satisfies (2.21), and that the Poisson integral with respect to the Poisson random measure N of L exists and satisfies the properties given in Section 2.2.1. Since we can define Poisson integrals with respect to the Poisson measure R of ν satisfying the same properties as the Poisson integral defined in Section 2.2.1 (see Proposition 19.4 of Sato [91] p.123), and ν satisfies (2.21), we can replicate the arguments in the proof of Theorem 2.2.9 to conclude that there exist a Φ'_β-valued

Lévy process $\widetilde{M} = \{\widetilde{M}_t\}_{t\geq 0}$, with characteristic function given by (2.24) (with N replaced by R). Similarly, the compound Poisson process $\widetilde{J} = \{\widetilde{J}_t\}_{t\geq 0}$, given by $\widetilde{J}_t = \int_{B_{\rho'}(1)} fR(t,df)$ as given in (2.34) (with N replaced by R) exists and satisfies (2.35). The processes $\widetilde{M}$ and $\widetilde{J}$ are independent and are uniquely determined up to equivalence in distribution by ν and ρ (and hence by R and p).

Moreover, from Theorem 2.1.14, there exists a Φ'_β-valued Wiener process $\widetilde{W} = \{\widetilde{W}_t\}_{t\geq 0}$, unique up to equivalence in distribution, such that $\mathfrak{m}$ and $\mathcal{Q}$ are the mean and the covariance functional of $\widetilde{W}$ respectively. Hence, W has characteristic function given by (2.4). We can assume without loss of generality that $\widetilde{W}$ and R are independent. Therefore, $\widetilde{W}$, $\widetilde{M}$ and $\widetilde{J}$ are independent Φ'_β-valued Lévy process. Hence, if we define $L = \{L_t\}_{t\geq 0}$, where for each $t \geq 0$,

$$L_t = \widetilde{W}_t + \widetilde{M}_t + \widetilde{J}_t$$

then L is an Φ'_β-valued Lévy process (Theorem 2.1.11), it is unique up to equivalence in distribution, and for each $t \geq 0$, L_t has characteristic function given by (2.40). □

Definition 2.2.15. If L is an Φ'_β-valued Lévy process where for each $t \geq 0$, L_t has characteristic function (2.40), then the members of the array $(\mathfrak{m}, \mathcal{Q}, \nu, \rho)$ are called **characteristics of the Lévy process** L.

In view of Theorem 2.2.14(2), the characteristics determine uniquely (up to equivalence in distribution) the Lévy process L.

To finish this section we briefly mention the connection of the result obtained in Theorem 2.2.14 with the study of infinitely divisible measures on Φ'_β. A measure $\mu \in \mathfrak{M}^1_R(\Phi'_\beta)$ is called **infinitely divisible** if for every $n \in \mathbb{N}$ there exist a **n-th root** of μ, i.e. a measure $\mu_n \in \mathfrak{M}^1_R(\Phi'_\beta)$ such that $\mu = \mu_n * \cdots * \mu_n$ (n-times). The following result establish the connection between Φ'_β-valued Lévy processes and infinitely divisible measures on Φ'_β. We do not include here a proof of this result but we would like to point out that the proof can be carried out using similar arguments to those used in the proof of Theorem 7.10 of Sato [91] p.35, and it is based on the connection of infinitely divisible measures and (continuous) convolution semigroups of Radon probability measures on Φ'_β (see Lemme 13 of Tortrat [97]).

Theorem 2.2.16. *(1) If $L = \{L_t\}_{t\geq 0}$ is an Φ'_β-valued Lévy process, then μ_{L_t} is infinitely divisible for all $t \geq 0$.*

(2) Conversely, if μ is an infinitely divisible measure on Φ'_β, there exist some Φ'_β-valued Lévy process $L = \{L_t\}_{t\geq 0}$ such that $\mu_{L_1} = \mu$.

From Theorems 2.2.14 and 2.2.16 we can provide a "probabilistic proof" of the following very important result proved by Dettweiler in [23] (see Satz 2.5, Chapter 1).

Theorem 2.2.17 (Lévy-Khintchine formula). *A measure $\mu \in \mathfrak{M}^1_R(\Phi'_\beta)$ is infinitely divisible if and only if there exists $\mathfrak{m} \in \Phi'_\beta$, a continuous Hilbertian semi-norm $\mathcal{Q}$ on Φ, a Lévy measure ν on Φ'_β and a continuous Hilbertian semi-norm ρ on Φ such that ν satisfies (2.23); i.e. the characteristic function of μ satisfies the following formula for every $\phi \in \Phi$:*

$$\widehat{\mu}(\phi) = \exp\left[i\mathfrak{m}[\phi] - \frac{1}{2}\mathcal{Q}(\phi,\phi) + \int_{\Phi'_\beta} \left(e^{if[\phi]} - 1 - if[\phi]\mathbb{1}_{B_{\rho'}(1)}(f) \right) \nu(df) \right].$$

where $\mathcal{Q}$ and ν are uniquely determined by μ and $\mathfrak{m}$ is also determined by ρ.

Chapter 3

Stochastic Integration in Duals of Nuclear Spaces

Let Φ and Ψ be nuclear spaces. The aim of this chapter is to introduce a new theory of stochastic integration of operator-valued processes with domain in Φ'_{β} and range in Ψ'_{β}, with respect to a class of cylindrical martingale-valued measures on Φ'.

Stochastic integration in the dual of a nuclear space has been considered by many authors, using various approaches. For example, in [103] Üstünel defined weak stochastic integrals for Φ-valued processes with respect to semimartingales defined in the dual of a complete, bornological nuclear space Φ whose strong dual Φ'_{β} is also nuclear. An extension of the work of Üstünel to Φ'_{β}-valued stochastic integrals was carried out by Brooks and Kozinski [18]. Assuming the same conditions on the space Ψ, Korezlioglu and Marthias [61] introduced a theory of strong stochastic integration of operator-valued processes with respect to Φ'_{β}-valued square integrable martingales.

Many other authors also introduced stochastic integrals in the case where the space Φ is nuclear and Fréchet. For example, stochastic integrals for operator-valued processes on Φ'_{β} with respect to Φ'_{β}-valued Wiener processes have been considered by Pérez-Abreu [80] and Ding [26]. Stochastic integrals with respect to Φ'_{β}-valued martingale measures were studied by Xie [117]. Also, stochastic integrals with respect to Poisson random measures have been used as driving the noise of a stochastic differential equation, see for example Kallianpur and Wolper [53] and Kallianpur and Xiong [54]. Also, Wu [114] used nonstandard analysis to define a hyperfinite representation of stochastic integrals for operator-valued processes with respect to a Wiener process in the space $\mathcal{D}'$ of distributions.

On the other hand, assuming only that Φ is nuclear and Ψ is a multi-Hilbertian space (that is a locally convex space generated by a family of separable semi-norms), in [42] and [43], Itô introduced a theory of stochastic integration with respect to Wiener process of operator-valued processes which map Φ'_{β} into Ψ'_{β}. This theory was later simplified by Bojdecki and Jakubowski in [12] and extended in [13] (see also [14], [45]) to the case where the integrator belongs to some classes of Gaussian processes with independent increments (also called inhomogeneous or generalized Wiener processes). Compared with the previous works, the stochastic integration developed in [13] is currently the theory that works under the most general conditions regarding both the integrator and the class of integrands.

However, in all the works cited above the stochastic integral was constructed by means to similar arguments to those used in the theory of stochastic integration in Hilbert spaces, either by using the fact that the integrator has a version taking values in a Hilbert space contained in the dual of the nuclear space, or by assuming that the integrands takes values in such a Hilbert space. Furthermore, no stochastic integration theory has been developed to cover the case of integration for operator-valued processes with respect to general Lévy processes. It is because of this that the main goal of this chapter is to cover the Lévy case, and also the more general case of cylindrical martingale-valued noise. In Chapter 4 we will apply this theory to the study of stochastic partial differential equations in the dual of a nuclear space.

In particular, our theory of stochastic integration extends the theory of Bojdecki and Jakubowski in two directions. The first is that our class of cylindrical martingale-valued measures generalizes the class of integrators in [13]. The second is that our stochastic integrals are defined for a more general class of integrands. Here it is important to remark that with respect to the theory in [13] we have imposed the additional assumption that the space Ψ is ultrabornological. This assumption is not very restrictive as many of the most important examples of nuclear spaces satisfy it (e.g. see the examples in Section 1.1.6.1). Indeed this assumption has been used by Jakubowski in later applications of the stochastic integral defined in [13] to the study of stochastic differential equations (see [45], [46], [47]). We hope that this hypothesis could be weakened in further developments of the theory.

The chapter is divided into three sections. In the first we introduce the class of cylindrical martingale-valued measures that will serve as integrators. Our cylindrical-martingale valued measures are defined on $\mathcal{B}(\mathbb{R}_+)\times\mathcal{R}$, where $\mathcal{R}$ is a ring generating the Borel σ-algebra of a topological space U. This later will be called the "jumps space". The covariance of our cylindrical martingale-valued measures is determined by some family of continuous Hilbertian semi-norms $\{q_{r,u}\}_{r\geq 0, u\in U}$ defined on Φ. In particular, the square integrable part of the Lévy-Itô decomposition of a Ψ'_β-valued Lévy process defines a cylindrical martingale-valued measure having this covariance structure.

In the second section we develop the theory of weak stochastic integration. In this theory, the integrands are families of random variables $X=\{X(r,u): r\geq 0, u\in U\}$ taking values in a family of Hilbert spaces determined by the semi-norms $\{q_{r,u}\}_{r,u}$. The weak stochastic integral is then a mapping that assigns to each of these families a real-valued martingale that we will call the weak stochastic integral of the family.

The third section is devoted to developing the theory of strong stochastic integration. Contrary to the case of weak stochastic integration, for the strong stochastic integration the class of integrands are families $R=\{R(r,u): r\geq 0, u\in U\}$ of operator-valued random variables taking values in Ψ'_β, with domains being a subspace of Φ'_β depending on both r and u. The strong integral mapping will assign to each of these families a Ψ'_β-valued martingale that we will call the strong stochastic integral of the family. What is quite specific to our theory of strong stochastic integration is that the weak integral will serve as a building block for the construction of the strong integral by means of the use of the regularization theorems of Section 1.2.1. Applications to the definition of stochastic integrals with respect to Φ'_β-valued Lévy process will be given.

§ 3.1 Cylindrical Martingale-Valued Measures

Assumption 3.1.1. *Throughout this chapter Φ is a locally convex space and Ψ is a quasi-complete, bornological, nuclear space, both defined over $\mathbb{R}$.*

In this section we introduce the concept of cylindrical martingale-valued measures on Φ'. We will follow a definition similar to the one introduced by Applebaum in [2] and Riedle and van Gaans [90] for the cases of a martingale-valued measures defined on separable Hilbert and Banach spaces respectively.

Definition 3.1.2. Let U be a topological space and consider a ring $\mathcal{R} \subseteq \mathcal{B}(U)$ that generates $\mathcal{B}(U)$. A **cylindrical martingale-valued measure** on $\mathbb{R}_+ \times \mathcal{R}$ is a collection $M = (M(t,A) : t \geq 0, A \in \mathcal{R})$ of cylindrical random variables on Φ' such that:

(1) $\forall\, A \in \mathcal{R},\ M(0,A)(\phi) = 0$ $\mathbb{P}$-a.s., $\forall \phi \in \Phi$.
(2) $\forall t \geq 0,\ M(t,\emptyset)(\phi) = 0$ $\mathbb{P}$-a.s. $\forall \phi \in \Phi$ and if $A, B \in \mathcal{R}$ are disjoint then

$$M(t, A \cup B)(\phi) = M(t,A)(\phi) + M(t,B)(\phi)\, \mathbb{P}\text{-a.s.}, \quad \forall \phi \in \Phi.$$

(3) $\forall\, A \in \mathcal{R},\ (M(t,A) : t \geq 0)$ is a zero-mean square integrable càdlàg cylindrical martingale.
(4) For disjoint $A, B \in \mathcal{R}$, $\mathbb{E}\left(M(t,A)(\phi) M(s,B)(\varphi)\right) = 0$, for each $t, s \geq 0$, $\phi, \varphi \in \Phi$.

Moreover, we say that M has **independent increments** if whenever $0 \leq s \leq t$, $M((s,t],A)(\phi) := (M(t,A) - M(s,A))(\phi)$ is independent of $\mathcal{F}_s$, for all $A \in \mathcal{R}$, $\phi \in \Phi$.

Now, in order to construct stochastic integrals with respect to the cylindrical martingale-valued measure M, we will need to impose some conditions on it. More specifically, we are interested in cylindrical martingale-valued measures whose covariance structure satisfies the properties listed below.

Definition 3.1.3. A cylindrical martingale-valued measure M on $\mathbb{R}_+ \times \mathcal{A}$ with independent increments is said to be **nuclear** if for each $A \in \mathcal{R}$ and $0 \leq s < t$,

$$\mathbb{E}\left(|M((s,t],A)(\phi)|^2 \right) = \int_s^t \int_A q_{r,u}(\phi)^2 \mu(du) \lambda(dr), \quad \forall\, \phi \in \Phi. \tag{3.1}$$

where

(1) μ is a σ-finite measure on $(U, \mathcal{B}(U))$ satisfying $\mu(A) < \infty$, $\forall\, A \in \mathcal{R}$,
(2) λ is a σ-finite measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, finite on bounded intervals,
(3) $\{q_{r,u} : r \in \mathbb{R}_+,\, u \in U\}$ is a family of continuous Hilbertian semi-norms on Φ, such that the map $(r,u) \mapsto q_{r,u}(\phi,\varphi)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)/\mathcal{B}(\mathbb{R}_+)$-measurable for each ϕ, φ in Φ. Here, $q_{r,u}(\cdot,\cdot)$ denotes the positive, symmetric, bilinear form associated to the Hilbertian semi-norm $q_{r,u}$.

To the extent of our knowledge, the concept of nuclear cylindrical martingale-valued measures introduced above has never been used in the literature. The next examples shows that it generalizes some other classes of Φ'_{β}-valued processes.

Example 3.1.4. Let Φ be a nuclear space. The following class of Φ'_{β}-valued processes was studied by Bojdecki and Gorostiza in [9] (for $\Phi = \mathcal{S}'(\mathbb{R}^d)$) and was used by Bojdecki and Jakubowski in [12] as integrators for their stochastic integrals.

A Φ'_{β}-valued continuous zero-mean Gaussian process $W = \{W_t\}_{t \geq 0}$ is called a **generalized Wiener process** if

(1) W is $\{\mathcal{F}_t\}$-adapted,
(2) $W_t - W_s$ is independent of $\mathcal{F}_s$, for $0 \leq s \leq t$,

(3)

$$\mathbb{E}\left(W_t[\phi]W_s[\varphi]\right) = \int_0^{t\wedge s} q_r(\phi,\varphi)dr, \quad \forall\, t,s \in R_+,\, \phi \in \Phi. \tag{3.2}$$

where $\{q_r : r \in \mathbb{R}_+\}$ is a family of continuous Hilbertian semi-norms on Φ, such that the map $r \mapsto q_r(\phi,\varphi)$ is Borel measurable and bounded on finite intervals, for each ϕ, φ in Φ. As in Definition 3.1.3, $q_r(\cdot,\cdot)$ denotes the positive, symmetric, bilinear form associated to the Hilbertian semi-norm q_r.

One can easily note from Theorem 2.1.13 that any Φ'_β-valued Wiener process W is a generalized Wiener process and that if $\mathcal{Q}$ is the covariance functional of W, one has (3.2) with $q_r = \mathcal{Q}$, for all $r \in \mathbb{R}_+$.

Is easy to see from the definition of W and from Definition 3.1.3 that if we take M given by

$$M(t,A) = W_t\delta_0(A), \quad \forall\, t \in \mathbb{R}_+,\, A \in \mathcal{B}(\{0\}), \tag{3.3}$$

then M defines a cylindrical martingale-valued measure with independent increments. Moreover, for each $0 \leq s \leq t$, we have:

$$\mathbb{E}\left(\left|M((s,t],\{0\})[\phi]\right|^2\right) = \int_s^t q_r(\phi)^2 dr, \quad \forall\, \phi \in \Phi. \tag{3.4}$$

Now, with respect to the notation in Definition 3.1.3 we have:

- $U = \{0\}$, $\mathcal{R} = \mathcal{B}(\{0\})$ and $\mu = \delta_0$.
- $\lambda = \text{Leb}$, where Leb is the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$.
- $q_{r,0} = q_r$, where $\{q_r : r \in \mathbb{R}_+\}$ is a family of continuous Hilbertian semi-norms on Φ satisfying the properties given above in Definition 3.1.3(3).

From this and (3.2) is clear that all the conditions in Definition 3.1.3 are satisfied. Hence, any generalized Wiener process gives rise to a nuclear martingale-valued measure.

Example 3.1.5. Let Φ be a complete, barrelled nuclear space. Let L be a Φ'_β-valued Lévy process with Lévy measure ν and Poisson random measure N. Let ρ be a continuous Hilbertian semi-norm on Φ such that the Lévy measure ν of L satisfies (see (2.23)):

$$\int_{B_{\rho'}(1)} \left|f[\phi]\right|^2 \nu(df) < \infty, \quad \forall\, \phi \in \Phi. \tag{3.5}$$

where recall that $B_{\rho'}(1) = \{f \in \Phi'_\beta : \rho'(f) \leq 1\}$ is a compact, convex, balanced subset of Φ'_β.

Let $U = B_{\rho'}(1)$ and $\mathcal{R} = \{A \in \mathcal{B}(B_{\rho'}(1)) : 0 \notin \overline{A}\}$. Then, $\mathcal{R}$ is the ring of subsets of $B_{\rho'}(1)$ that are bounded below. Let M be given by

$$M(t,A) = \int_A f\widetilde{N}(t,df), \qquad \text{for } t \geq 0,\, A \in \mathcal{R}. \tag{3.6}$$

From (3.5) it follows that for each $A \in \mathcal{R}$ the compensated Poisson integral in the right hand side of (3.6) is a square integrable compensated compound Poisson process. Hence, for each fixed $A \in A$, $\{M(t,A)\}_{t\geq 0}$ is a zero-mean square integrable Lévy process and therefore M satisfies Definition 3.1.2(3).

Moreover, the properties of Poisson integrals (see Section 2.2.1) shows that M indeed satisfies all the other properties listed in Definition 3.1.2. Therefore, M is a Φ'_β-valued martingale-valued measure with independent increments. We proceed to prove it is also nuclear.

To do this, note that from the second moments of the Poisson integrals (2.20) and (3.6), for each $0 \leq s \leq t$, $A \in \mathcal{R}$ we have:

$$\mathbb{E}\left(|M((s,t],A)[\phi]|^2 \right) = (t-s) \int_A |f[\phi]|^2 \, \nu(df)dr, \quad \forall \, \phi \in \Phi. \tag{3.7}$$

With respect to the notation in Definition 3.1.3 we take:

- $\mu = \nu\big|_{B_{\rho'}(1)}$.
- $\lambda = \text{Leb}$, where Leb is the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$.
- $\{q_{r,f} : r \in \mathbb{R}_+, f \in B_{\rho'}(1)\}$ is such that $q_{r,f} = q_f$, for each $r \in \mathbb{R}_+$ and $f \in B_{\rho'}(1)$, where:

$$q_f(\phi) = |f[\phi]|, \quad \forall \, \phi \in \Phi. \tag{3.8}$$

Now we proceed to verify that the conditions in Definition 3.1.3 are satisfied with the above choose of μ, λ and $\{q_f\}_{f \in U}$.

First, is well-known that the Lebesgue measure is finite in every bounded interval. Now, we know from Section 2.2.1 that ν is a σ-finite measure on $\mathcal{B}(\Phi'_\beta)$ satisfying $\nu(A) < \infty$, for every $A \subseteq \mathcal{A}$ (recall that $\mathcal{A}$ is the ring of subsets of Φ'_β that are bounded below). Hence, it follows that μ is σ-finite on $\mathcal{B}(U)$ and moreover $\mu(A) < \infty$, for every $A \in \mathcal{R}$.

Now, for every $f \in U$, is clear that q_f is a continuous semi-norm on Φ. To prove that it is Hilbertian, note that $Q_f : \Phi \times \Phi \to \mathbb{R}$, given by $Q_f(\phi, \varphi) = f[\phi]f[\varphi]$, for all $\phi, \varphi \in \Phi$ is a positive, symmetric bilinear form on $\Phi \times \Phi$ and that one has $q_f(\phi) = Q_f(\phi, \phi)^{1/2}$, for each $\phi \in \Phi$.

Moreover, for each ϕ, φ in Φ the map $f \to Q_f(\phi, \varphi)$ is continuous on U and hence from (3.8) one can conclude that $f \to q_f(\phi, \varphi)$ is $\mathcal{B}(U)/\mathcal{B}(\mathbb{R}_+)$-measurable.

To comply with the notation of Definition 3.1.3 we can take $\{q_{r,f} : r \in \mathbb{R}_+, f \in U\}$ to be such that $q_{r,f} := q_f$, for each $r \in \mathbb{R}_+$, $f \in U$. Hence, all the conditions in Definition 3.1.3 are satisfied and therefore M is nuclear.

We can obtain new cylindrical martingale-valued measures from old ones by mean of the following concept of independence.

Definition 3.1.6. Let N_1, N_2 be two cylindrical martingale-valued measures on $\mathbb{R}_+ \times \mathcal{R}$. We say that N_1 and N_2 are **independent** if for all $A, B \in \mathcal{R}$ and all ϕ, $\varphi \in \Phi$, the real valued processes $\{N_1(t,A)(\phi)\}_{t \in [0,T]}$ and $\{N_2(t,B)(\varphi)\}_{t \geq 0}$ are independent.

Proposition 3.1.7. *Let N_1, N_2 be two independent cylindrical martingale-valued measures on $\mathbb{R}_+ \times \mathcal{R}$. Let $M = (M(t,A) : r \geq 0, A \in \mathcal{R})$ be given by the prescription:*

$$M(t,A) := N_1(t,A) + N_2(t,A), \quad \forall \, t \in \mathbb{R}_+, \, A \in \mathcal{R}. \tag{3.9}$$

Then, M is a cylindrical martingale-valued measure on $\mathbb{R}_+ \times \mathcal{R}$.

Moreover, if N_1 and N_2 are nuclear, each with covariance structure as in (3.1) determined by the family $\{p^j_{r,u}\}_{r,u}$ of continuous Hilbertian semi-norms on Φ and measures $\lambda_j = \lambda$, $\mu_j = \mu$, for $j = 1, 2$, all of them satisfying the conditions given in Definition 3.1.3, then M is also nuclear, with covariance structure determined by λ, μ and the family of continuous Hilbertian semi-norms $\{q_{r,u}\}_{r,u}$ satisfying:

$$q_{r,u}(\phi)^2 = p^1_{r,u}(\phi)^2 + p^2_{r,u}(\phi)^2, \quad \forall \, r \geq 0, \, u \in U, \, \phi \in \Phi. \tag{3.10}$$

Proof. Is easy to see from (3.9) that M satisfies properties (1)-(3) of Definition 3.1.2. Moreover, property (4) of Definition 3.1.2 follows from a simple calculation and the corresponding property of N_1 and N_2, together with the fact that they are independent and the fact that the martingales they define have zero mean (Definition 3.1.2(2)). Hence, M is a cylindrical martingale-valued measure on $\mathbb{R}_+ \times \mathcal{R}$.

Now, assume N_j is nuclear with $\{p^j_{r,u}\}_{r,u}$, λ_j and μ_j satisfying the properties given in the assumptions of the proposition, for $j=1,2$. For each $r \geq 0$, $u \in U$, it is clear from the fact that each of the semi-norms $p^1_{r,u}$ and $p^2_{r,u}$ are continuous and Hilbertian (see Definition 3.1.3(3)) that each $q_{r,u}$ satisfying (3.10) is a continuous Hilbertian semi-norm on Φ. Moreover, (3.10) and the parallelogram law implies that:

$$q_{r,u}(\phi,\varphi) = p^1_{r,u}(\phi,\varphi) + p^2_{r,u}(\phi,\varphi), \quad \forall\, \phi,\, \varphi \in \Phi.$$

This, and the corresponding properties of $\{p^1_{r,u}\}_{r,u}$ and $\{p^2_{r,u}\}_{r,u}$ imply that the map $(r,u) \mapsto q_{r,u}(\phi,\varphi)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)/\mathcal{B}(\mathbb{R}_+)$-measurable for each ϕ, φ in Φ. Thus, the family of semi-norms $\{q_{r,u}\}_{r,u}$ satisfies Definition 3.1.3(3).

Now we proceed to prove that M is nuclear. Let $A \in \mathcal{R}$, $0 \leq s < t$ and $\phi \in \Phi$ be arbitrary but fixed. Note that the independence of N_1 and N_2, and the fact that the martingales they define have zero mean implies that

$$\mathbb{E}\left(N_1((s,t],A)[\phi]\, N_2((s,t],A)[\phi]\right) = \mathbb{E}\left(N_1((s,t],A)[\phi]\right) \mathbb{E}\left(N_2((s,t],A)[\phi]\right) = 0.$$

Hence, from (3.1) applied to each N_j, (3.9), (3.10), and the above, we have that

$$\begin{aligned}
\mathbb{E}\left(|M((s,t],A)[\phi]|^2\right) &= \mathbb{E}\left(|N_1((s,t],A)[\phi] + N_2((s,t],A)[\phi]|^2\right) \\
&= \mathbb{E}\left(|N_1((s,t],A)[\phi]|^2\right) + \mathbb{E}\left(|N_2((s,t],A)[\phi]|^2\right) \\
&\quad +2\,\mathbb{E}\left(N_1((s,t],A)[\phi]\, N_2((s,t],A)[\phi]\right) \\
&= \int_s^t \int_A p^1_{r,u}(\phi)^2 \mu(du)\lambda(dr) + \int_s^t \int_A p^2_{r,u}(\phi)^2 \mu(du)\lambda(dr) \\
&= \int_s^t \int_A q_{r,u}(\phi)^2 \mu(du)\lambda(dr)
\end{aligned}$$

Thus, M is nuclear. □

As an application of the above result, we have the following example that relates our concept of cylindrical nuclear martingale-valued measures to square integrable Lévy processes.

Example 3.1.8. Let Φ be a complete, barrelled nuclear space. Let L be a Φ'_β-valued Lévy process with Lévy-Itô decomposition given by (2.39). Hence, W is a Φ'_β-valued Wiener process with covariance functional $\mathcal{Q}$, N is the Poisson random measure of L, ν is the Levy measure of L and ρ is a continuous Hilbertian semi-norm on Φ such that the Lévy measure ν of L satisfies (2.23). Note that (2.23) implies (3.5).

Let $U = B_{\rho'}(1)$, $\mathcal{R} = \{A \in \mathcal{B}(B_{\rho'}(1)) : 0 \notin \overline{A}\} \cup \{0\}$, and $M = (M(t,A) : r \geq 0, A \in \mathcal{R})$ be given by

$$M(t,A) = W_t \delta_0(A) + \int_{A\setminus\{0\}} f \widetilde{N}(t,df), \quad \text{for } t \geq 0,\, A \in \mathcal{R}. \tag{3.11}$$

As for every $A \in \mathcal{R}$, the Wiener process W is independent of the compound Poisson integral $\left\{ \int_{A\setminus\{0\}} f \widetilde{N}(t, df) : t \geq 0 \right\}$, then the nuclear cylindrical martingale-valued measures defined in Example 3.1.4 (for the Wiener case) and Example 3.1.5 are independent in the sense of Definition 3.1.6. Therefore, it follows from Proposition 3.1.7 that M defined in (3.11) is also a nuclear cylindrical martingale-valued measures. Moreover, for each $0 \leq s \leq t$, $A \in \mathcal{R}$,

$$\mathbb{E}\left(|M((s,t], A)[\phi]|^2 \right) = (t-s) \left[\mathcal{Q}(\phi)^2 + \int_{A\setminus\{0\}} |f[\phi]|^2 \, \nu(df) \right], \quad \forall \, \phi \in \Phi. \tag{3.12}$$

In particular, with respect to the notation in Definition 3.1.3 we have:

- $\mu = \delta_0 + \nu\big|_{B_{\rho'}(1)}$.
- $\lambda = \text{Leb}$, where Leb is the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$.
- $\{q_{r,f} : r \in \mathbb{R}_+, f \in B_{\rho'}(1)\}$ is such that $q_{r,f} = q_f$, for each $r \in \mathbb{R}_+$ and $f \in B_{\rho'}(1)$, where:

$$q_f(\phi) = \begin{cases} \mathcal{Q}(\phi), & \text{if } f = 0, \\ |f[\phi]|, & \text{if } f \in B_{\rho'}(1) \setminus \{0\}. \end{cases} \tag{3.13}$$

We will call M defined in (3.11) a **Lévy martingale-valued measure**.

Additionally to the properties of the family of semi-norms $\{q_{r,u} : r \in \mathbb{R}_+, \, u \in U\}$ given in Definition 3.1.3, we will assume they satisfy the following:

Assumption 3.1.9. *For each $T > 0$ there exists a countable subset D of Φ that is dense in $\Phi_{q_{r,u}}$ for each $r \in [0,T]$, $u \in U$.*

Sufficient conditions for the validity of the above assumption are given below:

Proposition 3.1.10. *Suppose that either of the following conditions is satisfied:*

(1) Φ is separable,

(2) Φ is barrelled and for all $T > 0$ the mapping $(r,u) \mapsto q_{r,u}(\phi)$ is bounded on $[0,T] \times U$.

Then, Assumption 3.1.9 is satisfied.

The proof of *(1)* is an immediate consequence of the fact that by definition Φ is dense in each $\Phi_{q_{r,u}}$ (see Section 1.1.1). For a proof of *(2)* see Theorem 4.2 of Bojdecki and Jakubowski [12].

Remark 3.1.11. *The nuclear cylindrical martingale-valued measures defined in Example 3.1.4 (assuming Φ barreled) and Example 3.1.8 satisfy the conditions of Proposition 3.1.10(2) and hence they satisfy the Assumption 3.1.9.*

For the next result we need the definition of the predictable σ-algebra. Let $\Omega_\infty = [0,\infty) \times \Omega$, and denote by $\mathcal{P}_\infty$ the σ-algebra generated by the subsets of Ω_∞ of the form:

$$]s,t] \times F, \quad 0 \leq s < t < \infty, \, F \in \mathcal{F}_s, \text{ and } \{0\} \times F, \, F \in \mathcal{F}_0$$

$\mathcal{P}_\infty$ is called the **predictable σ-algebra** and its elements are **predictable sets**. For any $T > 0$, the restriction of the σ-algebra $\mathcal{P}_\infty$ to $[0,T] \times \Omega$ will be denoted by $\mathcal{P}_T$.

Proposition 3.1.12. *Let $\{q_{r,u}\}$ satisfy Assumption 3.1.9. Let the functions $(r,\omega,u) \mapsto f(r,\omega,u) \in \Phi_{q_{r,u}}$ and $(r,\omega,u) \mapsto g(r,\omega,u) \in \Phi_{q_{r,u}}$ be such that for each $\phi \in \Phi$, the functions $(r,u,\omega) \mapsto q_{r,u}(f(r,\omega,u),\phi)$ and $(r,\omega,u) \mapsto q_{r,u}(g(r,\omega,u),\phi)$ are $\mathcal{P}_T \otimes \mathcal{B}(U)/\mathcal{B}(\mathbb{R}_+)$-measurable. Then, the function $(r,\omega,u) \mapsto q_{r,u}(f(r,\omega,u), g(r,\omega,u))$ is $\mathcal{P}_T \otimes \mathcal{B}(U)/\mathcal{B}(\mathbb{R}_+)$-measurable.*

The proof of the above proposition follows from the same arguments to those in the proof of Proposition 1.8 of Bojdecki and Jakubowski [13].

Notation 3.1.13. Throughout this chapter and unless otherwise stated, M will denote a nuclear cylindrical martingale valued measure on $\mathbb{R}_+ \times \mathcal{R}$ and satisfying (3.1) for μ, λ and $\{q_{r,u}\}$ as in Definition 3.1.3. Also, the family of semi-norms $\{q_{r,u}\}$ satisfy Assumption 3.1.9. Furthermore we consider some $T > 0$ arbitrary but fixed.

§ 3.2 The Weak Stochastic Integral

In this section we construct and study the basic properties of the weak stochastic integral. The integral will be defined by following an Itô approach, that is, we first define the integral for a class of simple integrands and then we extend it to a larger class of integrands with finite second moments by means of an isometry. A further extension for integrands with almost sure finite moments will be given. Also a stochastic Fubini theorem will be proven.

3.2.1 The Weak Stochastic Integral for Integrands with Square Moments

We start by introducing the space of integrands. Recall that Φ is only assumed to be a locally convex space.

Definition 3.2.1. Let $\Lambda^2_w(M;T)$ denote the collection (of equivalence classes) of families $X = \{X(r,\omega,u) : r \in [0,T], \omega \in \Omega, u \in U\}$ of Hilbert space-valued maps satisfying the following conditions:

(1) $X(r,\omega,u) \in \Phi_{q_{r,u}}$, for all $r \in [0,T]$, $\omega \in \Omega$, $u \in U$,
(2) X is $q_{r,u}$**-predictable**, i.e. for each $\phi \in \Phi$, the mapping $[0,T] \times \Omega \times U \to \mathbb{R}_+$ given by $(r,\omega,u) \mapsto q_{r,u}(X(r,\omega,u),\phi)$ is $\mathcal{P}_T \otimes \mathcal{B}(U)$-measurable.
(3)
$$\mathbb{E} \int_0^T \int_U q_{r,u}(X(r,u))^2 \mu(du) \lambda(dr) < \infty. \tag{3.14}$$

Remark 3.2.2. *Note that the integrand in* (3.14) *is well defined. This because if X satisfies conditions* (1) *and* (2) *of Definition 3.2.1, then Proposition 3.1.12 guaranties that the map $(r,\omega,u) \mapsto q_{r,u}(X(r,\omega,u))^2 = q_{r,u}(X(r,\omega,u), X(r,\omega,u))$ is $\mathcal{P}_T \otimes \mathcal{B}(U)$-measurable.*

In view of (3.14), we can define the inner product $\langle \cdot \, , \cdot \rangle_{w,M;T}$ on $\Lambda^2_w(M;T)$ by:

$$\langle X \, , Y \rangle_{w,M;T} = \mathbb{E} \int_0^T \int_U q_{r,u}(X(r,u), Y(r,u)) \mu(du) \lambda(dr), \tag{3.15}$$

for each $X, Y \in \Lambda^2_w(M;T)$ and the corresponding norm $||\cdot||_{w,M;T}$ is given by

$$||X||^2_{w,M;T} = \mathbb{E}\int_0^T \int_U q_{r,u}(X(r,u))^2 \mu(du)\lambda(dr), \tag{3.16}$$

for each $X \in \Lambda^2_w(M;T)$.

When there is no necessity to give emphasis to the dependence of the space $\Lambda^2_w(M;T)$ with respect to M, we will denote $\Lambda^2_w(M;T)$, $\langle\cdot,\cdot\rangle_{w,M;T}$ and $||\cdot||_{w,M;T}$ by $\Lambda^2_w(T)$, $\langle\cdot,\cdot\rangle_{w,T}$, and $||\cdot||_{w,T}$ respectively. We will keep using the shorter notation for the remainder of this section.

With some minor changes, the proof of the following proposition can be carried out following similar arguments to those in the proof of Proposition 2.4 of Bojdecki and Jakubowski [13].

Proposition 3.2.3. *$\Lambda^2_w(T)$ equipped with the inner product $\langle\cdot,\cdot\rangle_{w,T}$ is a Hilbert space.*

Now, we define a class of simple families of random variables contained in $\Lambda^2_w(T)$.

Definition 3.2.4. Let $\mathcal{S}_w(T)$ be the collection of all the families $X = \{X(r,\omega,u) : r \in [0,T], \omega \in \Omega, u \in U\}$ of Hilbert space valued maps of the form:

$$X(r,\omega,u) = \sum_{i=1}^n \sum_{j=1}^m \mathbb{1}_{]s_j,t_j]}(r)\, \mathbb{1}_{F_j}(\omega)\, \mathbb{1}_{A_i}(u)\, i_{q_{r,u}}\phi_{i,j}, \tag{3.17}$$

for all $r \in [0,T]$, $\omega \in \Omega$, $u \in U$, where $m, n \in \mathbb{N}$, and for $i = 1,\dots,n$, $j = 1,\dots,m$, $0 \leq s_j < t_j \leq T$, $F_j \in \mathcal{F}_{s_j}$, $A_i \in \mathcal{R}$ and $\phi_{i,j} \in \Phi$. Moreover, recall that for each $r \in [0,T]$, $u \in U$, $i_{q_{r,u}} : \Phi \to \Phi_{q_{r,u}}$ is the canonical inclusion.

It is clear that $\mathcal{S}_w(T)$ is a vector space. We will show that it is a subspace of $\Lambda^2_w(T)$. Let $X \in \mathcal{S}_w(T)$ be given by (3.17). We will assume without loss of generality that it additionally satisfies:

$$\text{for } k \neq j, \quad]s_k,t_k]\cap]s_j,t_j] \neq \emptyset \quad \Rightarrow \quad]s_k,t_k] =]s_j,t_j] \text{ and } F_k \cap F_j = \emptyset. \tag{3.18}$$

It is clear from the simple form of X that it satisfies properties (1) and (2) of Definition 3.2.1. Moreover, note that from (3.1), (3.17) and (3.18), we have that

$$\begin{aligned} ||X||^2_{w,T} &= \mathbb{E}\int_0^T \int_U q_{r,u}(X(r,u))^2 \mu(du)\lambda(dr) \\ &= \sum_{i=1}^n \sum_{j=1}^m \mathbb{P}(F_j) \int_{s_j}^{t_j} \int_{A_i} q_{r,u}(\phi_{i,j})^2 \mu(du)\lambda(dr) \\ &= \sum_{i=1}^n \sum_{j=1}^m \mathbb{P}(F_j)\mathbb{E}\left(|M((s_j,t_j],A_i)(\phi_{i,j})|^2\right) < \infty. \end{aligned} \tag{3.19}$$

Note that in the above calculation we use the simple fact that for every $r \in [0,T]$, $u \in U$, $q_{r,u}(i_{q_{r,u}}\phi) = q_{r,u}(\phi)$, for every $\phi \in \Phi$.

Therefore $\mathcal{S}_w(T)$ is a subspace of $\Lambda^2_w(T)$. Our next objective is to show that $\mathcal{S}_w(T)$ is dense in $\Lambda^2_w(T)$. This is carried out in the next proposition.

Proposition 3.2.5. *$S_w(T)$ is dense in $\Lambda^2_w(T)$.*

Proof. Let $C_w(T)$ be the collection of all families of Hilbert space valued maps $Y = \{Y(r,\omega,u) : r \in [0,T], \omega \in \Omega, u \in U\}$ of the simple form

$$Y(r,\omega,u) = \mathbb{1}_{]s,t]}(r)\,\mathbb{1}_F(\omega)\,\mathbb{1}_A(u)\,i_{q_{r,u}}\phi, \quad \forall\, t \in [0,T],\ \omega \in \Omega,\ u \in U, \tag{3.20}$$

where $0 \leq s < t \leq T$, $F \in \mathcal{F}_s$, $A \in \mathcal{R}$ and $\phi \in \Phi$.

Is clear from (3.17) and (3.20) that $C_w(T)$ spans $S_w(T)$. Our objective is then to prove that the only element of $\Lambda^2_w(T)$ that is orthogonal to $C_w(T)$ is the zero family (to be precise, its equivalence class). This will imply that $S_w(T)$ is dense in $\Lambda^2_w(T)$.

To do this, let $X \in \Lambda^2_w(T)$. If $Y \in S_w(T)$ is of the form (3.20), then we have from (3.15) that

$$\langle X\,,\,Y\rangle_{w,T} = \int_F \int_s^t \int_A q_{r,u}(X(r,\omega,u), i_{q_{r,u}}\phi)\mu(du)\lambda(dr)\mathbb{P}(d\omega). \tag{3.21}$$

Assume that $X \in C_w(T)^{\perp}$, where $C_w(T)^{\perp}$ denotes the orthogonal complement of $C_w(T)$ in $\Lambda^2_w(T)$. Hence, it follows from (3.21) that X satisfies:

$$\int_F \int_s^t \int_A q_{r,u}(X(r,\omega,u), i_{q_{r,u}}\phi)\mu(du)\lambda(dr)\mathbb{P}(d\omega) = 0, \tag{3.22}$$

for all $0 \leq s < t \leq T$, $F \in \mathcal{F}_s$, $A \in \mathcal{R}$ and $\phi \in \Phi$.

Moreover, as $\mathcal{P}_T \otimes \mathcal{B}(U)$ is generated by the family of all subsets of $[0,T] \times \Omega \times U$ of the form $G =]s,t] \times F \times A$, where $0 \leq s < t \leq T$, $F \in \mathcal{F}_s$, $A \in \mathcal{R}$; then (3.22) and the Fubini theorem implies that $q_{r,u}(X(r,\omega,u), i_{q_{r,u}}\phi) = 0$ $\lambda \otimes \mathbb{P} \otimes \mu$-a.e., for all $\phi \in \Phi$. Furthermore, as for each $(r,u) \in [0,T] \times U$, $i_{q_{r,u}}(\Phi)$ is dense in $\Phi_{q_{r,u}}$, then it follows that $X(r,\omega,u) = 0$ $\lambda \otimes \mathbb{P} \otimes \mu$-a.e. Thus, $C_w(T)^{\perp} = 0$ and hence $S_w(T)$ is dense in $\Lambda^2_w(T)$. □

Now we define the weak stochastic integral for the elements of $\mathcal{S}_w(T)$.

Definition 3.2.6. Let $X \in \mathcal{S}_w(T)$ be of the form (3.17) and satisfying (3.18). We define

$$I^w_T(X) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbb{1}_{F_j} M((s_j,t_j], A_i)(\phi_{i,j}). \tag{3.23}$$

It is easy to see from the finite-additivity of M on $\mathcal{R}$ and the linearity on Φ of $M(t,A)$ for any $t \geq 0$, $A \in \mathcal{R}$, that the weak stochastic integral $I^w_T(X)$ is independent (up to modifications) of the representation of $X \in \mathcal{S}_w(T)$ (i.e. of the expression of X as in (3.17)).

Further properties of the weak stochastic integral are given in the following result:

Theorem 3.2.7. *For every $X \in \mathcal{S}_w(T)$,*

$$\mathbb{E}\left(I^w_T(X)\right) = 0, \quad \mathbb{E}\left(\left|I^w_T(X)\right|^2\right) = ||X||^2_{w,T}\,. \tag{3.24}$$

Moreover, the map $I^w_T : \mathcal{S}_w(T) \to L^2(\Omega, \mathscr{F}, \mathbb{P})$, $X \mapsto I^w_T(X)$, is a linear isometry.

Proof. Let $X \in \mathcal{S}_w(T)$ be of the form (3.17) and satisfying (3.18). From the definition of $I^w_T(X)$ in (3.23), the independent increments of M and Definition 3.1.2(3), we have

$$\mathbb{E}\left(I^w_T(X)\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbb{P}(F_j)\mathbb{E}\left(M((s_j,t_j], A_i)(\phi_{i,j})\right) = 0.$$

To prove (3.24), first note that for each $i, k = 1, \ldots, n,\ j, l = 1, \ldots, m,\ i \neq k,\ j \neq l$,

$$\mathbb{E}\left(M((s_j, t_j], A_i)(\phi_{i,j}) \cdot M((s_l, t_l], A_k)(\phi_{k,l})\right) = 0.$$

This follows from the orthogonality of M on the ring $\mathcal{A}$ (Definition 3.1.2(4)) and the fact that any real-valued martingale has orthogonal increments.

Therefore, if follows from the above orthogonality property, (3.19) and (3.23) that

$$\begin{aligned}
\mathbb{E}\left(|I_T^w(X)|^2\right) &= \sum_{i,k=1}^{n}\sum_{j,l=1}^{m} \mathbb{E}\left(\mathbb{1}_{F_j} M((s_j, t_j], A_i)(\phi_{i,j}) \mathbb{1}_{F_l} M((s_l, t_l], A_k)(\phi_{k,l})\right) \qquad (3.25)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbb{P}(F_j)\mathbb{E}\left(|M((s_j, t_j], A_i)(\phi_{i,j})|^2\right)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbb{P}(F_j)\int_{s_j}^{t_j}\int_{A_i} q_{r,u}(\phi_i)^2 \mu(du)\lambda(dr)\\
&= ||X||_{w,T}^2 .
\end{aligned}$$

Thus, we have showed (3.24). The linearity of the map $I_T^w : \mathcal{S}_w(T) \rightarrow L^2(\Omega, \mathscr{F}, \mathbb{P})$ follows from the properties (2) and (3) of M in Definition 3.1.2. Finally, that I_T^w is an isometry is a consequence of (3.24). □

Now, from Proposition 3.2.5 and Theorem 3.2.7, the map I_T^w extends to a linear isometry from $\Lambda_w^2(T)$ into $L^2(\Omega, \mathscr{F}, \mathbb{P})$, that we still denote by I_T^w. Moreover, from (3.24) we have the following **Itô isometry**

$$\mathbb{E}\left(|I_T^w(X)|^2\right) = ||X||_{w,T}^2 , \quad \forall\, X \in \Lambda_w^2(T). \qquad (3.26)$$

For $0 \leq t \leq T,\ X \in \Lambda_w^2(T)$, it is clear that $\mathbb{1}_{[0,t]}X \in \Lambda_w^2(T)$ and hence we can define a real-valued process $I^w(X) = \{I_t^w(X)\}_{t \geq 0}$ by means of the prescription

$$I_t^w(X) := I_T^w(\mathbb{1}_{[0,t]}X), \quad \forall\, t \in [0, T]. \qquad (3.27)$$

The process $I^w(X)$ will be called the **weak stochastic integral** of X and sometimes we denote it by $\left\{\int_0^t \int_U X(r, u)M(dr, du) : t \in [0, T]\right\}$. Some of the properties of the weak stochastic integral are given in the following theorem.

Theorem 3.2.8. *For each $X \in \Lambda_w^2(T)$, $I^w(X) = \{I_t^w(X)\}_{t \in [0,T]}$ is a real-valued zero-mean, square integrable, càdlàg martingale with second moments given by*

$$\mathbb{E}\left(|I_t^w(X)|^2\right) = \mathbb{E}\int_0^t \int_U q_{r,u}(X(r, u))^2 \mu(du)\lambda(dr), \quad \forall\, t \in [0, T]. \qquad (3.28)$$

Moreover, $I^w(X)$ is mean square continuous and it has a predictable version. Furthermore, the mapping $I^w : \Lambda_w^2(T) \rightarrow \mathcal{M}_T^2(\mathbb{R})$ given by $X \mapsto I^w(X) = \{I_t^w(X)\}_{t \in [0,T]}$ is linear and continuous.

Proof. Let $X \in \mathcal{S}_w(T)$ be of the form (3.17) and satisfying (3.18). From (3.23) and (3.27) we have

$$I_t^w(X) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbb{1}_{F_j} M((s_j \wedge t, t_j \wedge t], A_i)(\phi_{i,j}), \quad \forall t \in [0, T].$$

Then, from the independent increments of M, Definition 3.1.2(3) and (3.24) it follows that that $I^{w}(X)$ is a real-valued zero-mean, square integrable, càdlàg martingale, i.e. $I^{w}(X) \in \mathcal{M}^{2}_{T}(\mathbb{R})$. Moreover, similar calculations to those used in (3.19) and (3.25) shows that $I^{w}(X)$ satisfies (3.28).

Now we prove that for any $X \in \Lambda^{2}_{w}(T)$ we have $I^{w}(X) \in \mathcal{M}^{2}_{T}(\mathbb{R})$. Let $X \in \Lambda^{2}_{w}(T)$. By Proposition 3.2.5 there exists a sequence $\{X_k\}_{k\in\mathbb{N}} \subseteq \mathcal{S}_w(T)$ that converges to X in $\Lambda^{2}_{w}(T)$ as $k \to \infty$. Now, because for every $k \in \mathbb{N}$ we have $I^{w}(X_k) \in \mathcal{M}^{2}_{T}(\mathbb{R})$, it follows from the linearity of the map I^{w}_{t} for each $t \in [0,T]$, Doob's inequality and (3.24) that for any $k, l \in \mathbb{N}$,

$$\begin{aligned} ||I^{w}(X_k) - I^{w}(X_l)||^{2}_{\mathcal{M}^{2}_{T}(\mathbb{R})} &= \mathbb{E}\left(\sup_{t\in[0,T]} |I^{w}_{t}(X_k - X_l)|^{2} \right) \\ &\leq 4T\,\mathbb{E}\left(|I^{w}_{T}(X_k - X_l)|^{2} \right) \\ &= 4T\,||X_k - X_l||^{2}_{w,T}\,. \end{aligned} \tag{3.29}$$

But as the sequence $\{X_k\}_{k\in\mathbb{N}}$ converges in $\Lambda^{2}_{w}(T)$, then (3.29) implies that the sequence of integral processes $\{I^{w}(X_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{M}^{2}_{T}(\mathbb{R})$. Then, because the space $\mathcal{M}^{2}_{T}(\mathbb{R})$ is complete, the sequence $\{I^{w}(X_k)\}_{k\in\mathbb{N}}$ converges to some $H = \{H_t\}_{t\in[0,T]}$ in $\mathcal{M}^{2}_{T}(\mathbb{R})$. This last fact in particular implies that for every $t \in [0,T]$, $\{I^{w}_{t}(X_k)\}_{k\in\mathbb{N}}$ converges to H_t in $L^{2}(\Omega, \mathscr{F}, \mathbb{P})$.

On the other hand, as $\{X_k\}_{k\in\mathbb{N}}$ converges to X in $\Lambda^{2}_{w}(T)$, it follows from (3.26) that for every $t \in [0,T]$, $\{I^{w}_{t}(X_k)\}_{k\in\mathbb{N}}$ converges to $I^{w}_{t}(X)$ in $L^{2}(\Omega, \mathscr{F}, \mathbb{P})$. Therefore, by uniqueness of limits in $L^{2}(\Omega, \mathscr{F}, \mathbb{P})$ we have $H_t = I^{w}_{t}(X)$ $\mathbb{P}$-a.e. for each $t \in [0,T]$. Then, because $H \in \mathcal{M}^{2}_{T}(\mathbb{R})$ it follows that $I^{w}(X) = \{I^{w}_{t}(X)\}_{t\in[0,T]} \in \mathcal{M}^{2}_{T}(\mathbb{R})$. Moreover, because each X_k satisfies (3.28), the fact that $\{I^{w}_{t}(X_k)\}_{k\in\mathbb{N}}$ converges to $I^{w}_{t}(X)$ in $L^{2}(\Omega, \mathscr{F}, \mathbb{P})$ implies that X also satisfies (3.28).

To prove the mean square continuity property, note that if $X \in \Lambda^{2}_{w}(T)$, then it follows from (3.28) that for any $0 \leq s \leq t \leq T$ we have:

$$\mathbb{E}\left(|I^{w}_{t}(X) - I^{w}_{s}(X)|^{2} \right) = \mathbb{E} \int_{s}^{t} \int_{U} q_{r,u}(X(r,u))^{2} \mu(du)\lambda(dr) \leq ||X||^{2}_{w,T}\,,$$

and hence from an application of the dominated convergence theorem we have

$$\mathbb{E}\left(|I^{w}_{t}(X) - I^{w}_{s}(X)|^{2} \right) \to 0 \quad \text{as } s \to t, \text{ or } t \to s.$$

Thus, $I^{w}(X)$ is mean square continuous. Now, as $I^{w}(X)$ is $\mathcal{F}_t$-adapted and stochastically continuous it has a predictable version (see Proposition 3.21 of Peszat and Zabczyk [85], p.27).

Now, the map $I^{w} : \Lambda^{2}_{w}(T) \to \mathcal{M}^{2}_{T}(\mathbb{R})$ given by $X \mapsto I^{w}(X) = \{I^{w}_{t}(X)\}_{t\in[0,T]}$ is well-defined. Moreover, the linearity for each $t \in [0,T]$ of the map $I^{w}_{t} : \Lambda^{2}_{w}(T) \to L^{2}(\Omega, \mathscr{F}, \mathbb{P})$ implies that the map I^{w} is linear. Finally, for every $X \in \Lambda^{2}_{w}(T)$ it follows from Doob's inequality and (3.26) that we have

$$||I^{w}(X)||^{2}_{\mathcal{M}^{2}_{T}(\mathbb{R})} = \mathbb{E}\left(\sup_{t\in[0,T]} |I^{w}_{t}(X)|^{2} \right) \leq 4T\,\mathbb{E}\left(|I^{w}_{T}(X)|^{2} \right) = 4T\,||X||^{2}_{w,T}\,. \tag{3.30}$$

Then, the continuity of the map I^{w} follows from (3.30). □

Definition 3.2.9. We call the map I^w defined in Theorem 3.2.8 the **weak stochastic integral** mapping.

Proposition 3.2.10. *If for each $A \in \mathcal{R}$ and $\phi \in \Phi$, the real-valued process $(M(t,A)(\phi) : t \geq 0)$ is continuous, then for each $X \in \Lambda^2_w(T)$ the stochastic integral $I^w(X)$ is a continuous process.*

Proof. The result follows clearly from the definition of $I^w(X)$ for $X \in \mathcal{S}_w(T)$ and this can be extended by the denseness of $\mathcal{S}_w(T)$ in $\Lambda^2_w(T)$ to any $X \in \Lambda^2_w(T)$. ☐

Example 3.2.11. Let Φ be a barrelled nuclear space and W be a generalized Wiener process in Φ'_β. Let M be the cylindrical martingale-valued measure defined in Example 3.1.4 by (3.3). Then, the space $\Lambda^2_w(T)$ is the collection (of equivalence classes) of families $X = \{X(r,\omega,0) : r \in [0,T], \omega \in \Omega, 0 \in \Phi'_\beta\}$ of Hilbert space-valued maps satisfying the conditions of Definition 3.2.1 with respect to the family of semi-norms $\{q_{r,0}\}$ defined in Example 3.1.4. In particular, the condition (3.14) takes the form

$$\mathbb{E}\int_0^T q_r(X(r,0))^2 dr < \infty.$$

We denote by $\{\int_0^t X(r,0)dW(r) : t \in [0,T]\}$ the weak stochastic integral with respect to M and in view of Proposition 3.2.10 it is a continuous process.

Example 3.2.12. Let Φ be a complete, barrelled nuclear space and M be the Lévy martingale-valued measure defined in Example 3.1.5 by (3.6). Then, the space $\Lambda^2_w(T)$ is the collection (of equivalence classes) of families $X = \{X(r,\omega,f) : r \in [0,T], \omega \in \Omega, f \in B_{\rho'}(1)\}$ of Hilbert space-valued maps satisfying the conditions of Definition 3.2.1 with respect to the family of semi-norms $\{q_{r,f}\}$ defined in (3.8). In particular, the condition (3.14) takes the form

$$\mathbb{E}\int_0^T \int_{B_{\rho'}(1)} |f[X(r,f)]|^2 \, \nu(df)dr < \infty.$$

We denote by $\{\int_0^t \int_{B_{\rho'}(1)} X(r,f)\widetilde{N}(dr,df) : t \in [0,T]\}$ the weak stochastic integral with respect to M.

Example 3.2.13. Let Φ be a complete, barrelled nuclear space and M be the Lévy martingale-valued measure defined in Example 3.1.8 by (3.11). Then, the space $\Lambda^2_w(T)$ is the collection (of equivalence classes) of families $X = \{X(r,\omega,f) : r \in [0,T], \omega \in \Omega, f \in B_{\rho'}(1)\}$ of Hilbert space-valued maps satisfying the conditions of Definition 3.2.1 with respect to the family of semi-norms $\{q_{r,f}\}$ defined in (3.13). In particular, the condition (3.14) takes the form

$$\mathbb{E}\int_0^T \left(\mathcal{Q}(X(r,0))^2 + \int_{B_{\rho'}(1)\setminus\{0\}} |f[X(r,f)]|^2 \, \nu(df) \right) dr < \infty.$$

Moreover, from Examples 3.2.11 and 3.2.12, and from the properties of the weak stochastic integral that we will show below in Proposition 3.2.16, for all $t \in [0,T]$ we have

$$\int_0^t \int_{B_{\rho'}(1)} X(r,f)M(dr,df) = \int_0^t X(r,0)dW(r) + \int_0^t \int_{B_{\rho'}(1)} X(r,f)\widetilde{N}(dr,df),$$

3.2.2 Properties of the Weak Stochastic Integral

In this section we prove some properties of the weak stochastic integral. The following result can be proven using similar arguments as those in the proof of Lemma 4.9 of Da Prato and Zabczyk [20], p.94-5.

Proposition 3.2.14. *Let* $X \in \Lambda^2_w(T)$ *and* σ *be an* $\{\mathcal{F}_t\}$*-stopping time such that* $\mathbb{P}(\sigma \leq T)=1$. *Then,* $\mathbb{P}$*-a.e.*

$$I^w_t(\mathbb{1}_{[0,\sigma]}X) = I^w_{t\wedge\sigma}(X), \quad \forall\, t \in [0,T]. \tag{3.31}$$

Proposition 3.2.15. *Let* $0 \leq s_0 < t_0 \leq T$ *and* $F_0 \in \mathcal{F}_{s_0}$. *Then, for every* $X \in \Lambda^2_w(T)$, $\mathbb{P}$*-a.e. we have*

$$I^w_t(\mathbb{1}_{]s_0,t_0]\times F_0}X) = \mathbb{1}_{F_0}\left(I^w_{t\wedge t_0}(X) - I^w_{t\wedge s_0}(X)\right), \quad \forall\, t \in [0,T]. \tag{3.32}$$

Proof. Let X be of the simple form:

$$X(r,\omega,u) = \mathbb{1}_{]s_1,t_1]}(r)\,\mathbb{1}_{F_1}(\omega)\,\mathbb{1}_A(u)\,i_{q_{r,u}}\phi, \quad \forall\, r \in [0,T],\, \omega \in \Omega,\, u \in U, \tag{3.33}$$

where $0 \leq s_1 < t_1 \leq T$, $\mathcal{F}_1 \in \mathcal{F}_{s_1}$, $A \in \mathcal{R}$ and $\phi \in \Phi$.
Then, for such simple X one can easily see that we have:

$$\begin{aligned}\mathbb{1}_{]s_0,t_0]\times F_0}X &= \mathbb{1}_{]s_0,t_0]\cap]s_1,t_1]}\mathbb{1}_{F_0\cap F_1}\mathbb{1}_A i_{q_{r,u}}\phi \\ &= \mathbb{1}_{]s_0\vee s_1, t_0\wedge t_1]}\mathbb{1}_{F_0\cap F_1}\mathbb{1}_A i_{q_{r,u}}\phi.\end{aligned}$$

Hence, $\mathbb{1}_{]s_0,t_0]\times F_0}X$ belongs to $S_w(T)$. This is because if $]s_0,t_0]\cap]s_1,t_1] \neq \emptyset$, then $s_0 \vee s_1 < t_0 \wedge t_1$ and $F_0 \cap F_1 \in \mathcal{F}_{s_0\vee s_1}$.
Now, from the definition of the weak stochastic integral for simple integrands (see Definition 3.2.6) and the fact that $]s_0,t_0]\cap]s_1,t_1] \neq \emptyset$ if and only if $s_1 < t_0$ and $s_0 < t_1$, it follows from tedious, but straightforward calculations that for every $t \in [0,T]$, we have

$$\begin{aligned}&I^w_t(\mathbb{1}_{]s_0,t_0]\times F_0}X) \\ &= \mathbb{1}_{F_0\cap F_1}M\left(((s_0\vee s_1)\wedge t, (t_0\wedge t_1)\wedge t], A\right)[\phi] \\ &= \mathbb{1}_{F_0}\mathbb{1}_{F_1}\left(M\left(((t_0\wedge s_1)\wedge t, (t_0\wedge t_1)\wedge t], A\right)[\phi] - M\left(((s_0\wedge s_1)\wedge t, (s_0\wedge t_1)\wedge t], A\right)[\phi]\right) \\ &= \mathbb{1}_{F_0}\left(I^w_{t\wedge t_0}(X) - I^w_{t\wedge s_0}(X)\right).\end{aligned}$$

Hence, (3.32) is satisfied for X of the simple form (3.33).
The linearity of the integral implies that (3.32) is valid for any $X \in S_w(T)$. Moreover, by the density of $S_w(T)$ in $\Lambda^2_w(T)$ and the continuity of the weak stochastic integral mapping I^w (Theorem 3.2.8), it follows that (3.32) is satisfied for every $X \in \Lambda^2_w(T)$. □

Now we are going to study the behaviour of the weak stochastic integral with respect to a Φ'_β-nuclear martingale-valued measure that is defined as the "sum" of two independent Φ'_β-martingale-valued measures (see Proposition 3.1.7).

Proposition 3.2.16. *Let* N_1, N_2 *be two independent nuclear* Φ'_β*-valued martingale-valued measures on* $\mathbb{R}_+ \times \mathcal{R}$, *each with covariance structure as in* (3.1) *determined by the family* $\{p^j_{r,u}\}_{r,u}$ *of continuous Hilbertian semi-norms on* Φ *and measures* $\lambda_j = \lambda$, $\mu_j = \mu$, *for* $j = 1,2$*; all of them satisfying the conditions given in Definition 3.1.3.*

Let M be the nuclear Φ'_{β}-valued martingale-valued measure on $\mathbb{R}_+ \times \mathcal{R}$ defined by N_1 and N_2 as in Proposition 3.1.7. Let $\{q_{r,u}\}_{r,u}$ be the family of semi-norms determining its covariance structure (3.1)*.*

Assume $X \in \Lambda^2_w(M;T)$. Then,

(1) For each $j=1,2$, $\{i_{p^j_{r,u},q_{r,u}} X(r,\omega,u) : r \in [0,T], \omega \in \Omega, u \in U\} \in \Lambda^2_w(N_j;T)$, where for each $r \in [0,T]$ and $u \in U$, $i_{p^j_{r,u},q_{r,u}}$ denotes the inclusion map from $\Phi_{q_{r,u}}$ into $\Phi_{p^j_{r,u}}$.

(2) $\mathbb{P}$-a.e., for all $t \in [0,T]$ we have,

$$\int_0^t \int_U X(r,u) M(dr,du) = \int_0^t \int_U i_{p^1_{r,u},q_{r,u}} X(r,u) N_1(dr,du) + \int_0^t \int_U i_{p^2_{r,u},q_{r,u}} X(r,u) N_2(dr,du). \tag{3.34}$$

Proof. We follow ideas from the proof of Proposition 3.7 of Bojdecki and Jakubowski [13]. First, from (3.10) we have that for every $r \in [0,T]$, $u \in U$, $p^j_{r,u} \leq q_{r,u}$, for each $j=1,2$. Hence, the inclusions $i_{p^j_{r,u},q_{r,u}}$ are well-defined and moreover are linear and continuous.

To prove (1), fix for the moment $j=1,2$. We have to show that $\{i_{p^j_{r,u},q_{r,u}} X(r,\omega,u) : r \in [0,T], \omega \in \Omega, u \in U\}$ belongs to $\Lambda^2_w(N_j;T)$. First, note that as $i_{p^j_{r,u},q_{r,u}} \in \mathcal{L}(\Phi_{q_{r,u}}, \Phi_{p^j_{r,u}})$, then $i_{p^j_{r,u},q_{r,u}} X(r,\omega,u) \in \Phi_{p^j_{r,u}}$ for each $r \in [0,T]$, $\omega \in \Omega$, $u \in U$.

Our next objective is to prove that for every $\phi \in \Phi$, the map

$$(r,\omega,u) \mapsto p^j_{r,u}\left(i_{p^j_{r,u},q_{r,u}} X(r,\omega,u), \phi\right)$$

is $\mathcal{P}_T \otimes \mathcal{B}(U)$-measurable. Let $\theta_{q_{r,u}}$ be the isometry between $\Phi_{q_{r,u}}$ and $\Phi'_{q_{r,u}}$ given by the Riesz representation theorem, i.e. $\theta_{q_{r,u}}(\phi) = q_{r,u}(\cdot,\phi)$. Similarly we have the isometry $\theta_{p^j_{r,u}}$ between $\Phi_{p^j_{r,u}}$ and $\Phi'_{p^j_{r,u}}$

Then, using the definition of $\theta_{q_{r,u}}$ and $\theta_{p^j_{r,u}}$, and the definition of dual operator applied to $i_{p^j_{r,u},q_{r,u}}$ and $i'_{p^j_{r,u},q_{r,u}}$, for every $\phi \in \Phi$ and $\varphi \in \Phi_{q_{r,u}}$ we have:

$$\begin{aligned} p^j_{r,u}\left(i_{p^j_{r,u},q_{r,u}} \varphi, \phi\right) &= \theta_{p^j_{r,u}} \phi \left[i_{p^j_{r,u},q_{r,u}} \varphi\right] \\ &= i'_{p^j_{r,u},q_{r,u}} \theta_{p^j_{r,u}} \phi \left[\varphi\right] \\ &= q_{r,u}\left(\varphi, \theta^{-1}_{q_{r,u}} i'_{p^j_{r,u},q_{r,u}} \theta_{p^j_{r,u}} \phi\right). \end{aligned}$$

Now, because $(r,u) \mapsto q_{r,u}\left(\varphi, \theta^{-1}_{q_{r,u}} i'_{p^j_{r,u},q_{r,u}} \theta_{p^j_{r,u}} \phi\right)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(U)/\mathcal{B}(\mathbb{R}_+)$-measurable, from an application of Proposition 3.1.12 it follows that the map

$$(r,\omega,u) \mapsto p^j_{r,u}\left(i_{p^j_{r,u},q_{r,u}} X(r,\omega,u), \phi\right)$$

is $\mathcal{P}_T \otimes \mathcal{B}(U)$-measurable.

Finally, as for all $r \geq 0$, $u \in U$, (3.10) extends to any $\phi \in \Phi_{q_{r,u}}$ to give

$$q_{r,u}(\phi)^2 = p^1_{r,u}(i_{p^1_{r,u},q_{r,u}} \phi)^2 + p^2_{r,u}(i_{p^2_{r,u},q_{r,u}} \phi)^2,$$

we have

$$q_{r,u}(X(r,\omega,u))^2 = p^1_{r,u}(i_{p^1_{r,u},q_{r,u}} X(r,\omega,u))^2 + p^2_{r,u}(i_{p^2_{r,u},q_{r,u}} X(r,\omega,u))^2.$$

Therefore,

$$\mathbb{E}\int_0^T\int_U q_{r,u}(X(r,u))^2\mu(du)\lambda(dr)<\infty.$$

implies

$$\mathbb{E}\int_0^T\int_U p^j_{r,u}(i_{p^j_{r,u},q_{r,u}}X(r,u))^2\mu(du)\lambda(dr)<\infty,$$

for each $j=1,2$. Hence, we have proved (1).

To prove (2). Let X be of the simple form:

$$X(r,\omega,u)=\mathbb{1}_{]s_0,t_0]}(r)\,\mathbb{1}_F(\omega)\,\mathbb{1}_A(u)\,i_{q_{r,u}}\phi,\quad \forall t\in[0,T],\,\omega\in\Omega,\,u\in U,$$

where $0\leq s_0<t_0\leq T$, $F\in\mathcal{F}_{s_0}$, $A\in\mathcal{R}$ and $\phi\in\Phi$. From the simple form of X, the definition of M (see (3.9)) and the definition of the weak stochastic integral (see (3.23)), we have for every $t\in[0,T]$,

$$\begin{aligned}\int_0^t\int_U X(r,u)M(dr,du) &= \mathbb{1}_F M((s_0\wedge t,t_0\wedge t],A)[\phi]\\ &= \mathbb{1}_F N_1((s_0\wedge t,t_0\wedge t],A)[\phi]+\mathbb{1}_F N_2((s_0\wedge t,t_0\wedge t],A)[\phi]\\ &= \int_0^t\int_U i_{p^1_{r,u},q_{r,u}}X(r,u)N_1(dr,du)\\ &\quad+\int_0^t\int_U i_{p^2_{r,u},q_{r,u}}X(r,u)N_2(dr,du).\end{aligned}$$

Therefore, X satisfies (3.34).

The result extends by the linearity of the weak stochastic integral to all the elements in $S_w(M;T)$. Furthermore, it extends to $\Lambda^2_w(M;T)$ by denseness. □

3.2.3 An Extension of The Class of Integrands

A third step in our construction of the weak stochastic integral is to extend it to a class of families of random variables with only almost sure second moments. More specifically, we want to extend our theory to the following class of integrands:

Definition 3.2.17. Let $\Lambda^{2,loc}_w(M;T)$ denote the collection (of equivalence classes) of families $X=\{X(r,\omega,u):r\in[0,T],\omega\in\Omega,u\in U\}$ of Hilbert space-valued maps satisfying the following conditions:

(1) $X(r,\omega,u)\in\Phi_{q_{r,u}}$, for all $r\in[0,T]$, $\omega\in\Omega$, $u\in U$,

(2) X is $q_{r,u}$**-predictable**, i.e. for each $\phi\in\Phi$, the mapping $[0,T]\times\Omega\times U\to\mathbb{R}_+$ given by $(r,\omega,u)\mapsto q_{r,u}(X(r,\omega,u),\phi)$ is $\mathcal{P}_T\otimes\mathcal{B}(U)$-measurable.

(3)
$$\mathbb{P}\left(\omega\in\Omega:\int_0^T\int_U q_{r,u}(X(r,\omega,u))^2\mu(du)\lambda(dr)<\infty\right)=1. \quad (3.35)$$

As before, we will sometimes denote $\Lambda^{2,loc}_w(M;T)$ by $\Lambda^{2,loc}_w(T)$ when is clear to which cylindrical martingale-valued measure M we are referring.

One can easily check that the space $\Lambda^{2,loc}_w(T)$ is a linear space. We equip this space with the vector topology $\mathcal{T}^M_{2,loc}$ generated by the local base of neighbourhoods of zero $\{\Gamma_{\epsilon,\delta}:\epsilon>0,\delta>0\}$, where $\Gamma_{\epsilon,\delta}$ is given by

$$\Gamma_{\epsilon,\delta}=\left\{X\in\Lambda^{2,loc}_w(T):\mathbb{P}\left(\omega\in\Omega:\int_0^T\int_U q_{r,u}(X(r,\omega,u))^2\mu(du)\lambda(dr)>\epsilon\right)\leq\delta\right\}.$$

Hence, under the topology $\mathcal{T}^{M}_{2,loc}$, a sequence $\{X^{(n)}\}_{n\in\mathbb{N}}$ converges to X in $\Lambda^{2,loc}_{w}(T)$ if and only if

$$\int_0^T \int_U q_{r,u}(X_n(r,u) - X(r,u))^2 \mu(du)\lambda(dr) \overset{\mathbb{P}}{\to} 0, \quad \text{as } n \to \infty. \tag{3.36}$$

Proposition 3.2.18. *The space* $(\Lambda^{2,loc}_{w}(T), \mathcal{T}^{M}_{2,loc})$ *is a complete, metrizable topological vector space.*

Proof. On $\Lambda^{2,loc}_{w}(T)$, we introduce the translation invariant metric d_Λ given by

$$d_\Lambda(X,Y) = \mathbb{E}\left[G\left(\int_0^T \int_U q_{r,u}(X(r,u) - Y(r,u))^2 \mu(du)\lambda(dr)\right)\right], \tag{3.37}$$

for all $X, Y \in \Lambda^{2,loc}_{w}(T)$, where $G : \mathbb{R} \to \mathbb{R}$ is given by $G(x) = \frac{x}{1+x}$, for each $x \in \mathbb{R}$. It is clear that d_Λ is well-defined due to (3.35).

Let $X \in \Lambda^{2,loc}_{w}(T)$ and $\epsilon > 0$. Because G is increasing and from Markov's inequality we have

$$\begin{aligned}
&\mathbb{P}\left(\int_0^T \int_U q_{r,u}(X(r,u))^2 \mu(du)\lambda(dr) > \epsilon\right) \\
&\leq \frac{1+\epsilon}{\epsilon}\mathbb{E}\left[G\left(\int_0^T \int_U q_{r,u}(X(r,u)))^2 \mu(du)\lambda(dr)\right)\right] = \frac{1+\epsilon}{\epsilon} d_\Lambda(X,0)
\end{aligned} \tag{3.38}$$

On the other hand, because the function G is bounded by 1, we have

$$\begin{aligned}
d_\Lambda(X,0) &= \mathbb{E}\left[G\left(\int_0^T \int_U q_{r,u}(X(r,u))^2 \mu(du)\lambda(dr)\right)\right] \\
&\leq \frac{\epsilon}{1+\epsilon}\mathbb{P}\left(\int_0^T \int_U q_{r,u}(X(r,u))^2 \mu(du)\lambda(dr) < \epsilon\right) \\
&\quad + \mathbb{P}\left(\int_0^T \int_U q_{r,u}(X(r,u))^2 \mu(du)\lambda(dr) > \epsilon\right) \\
&\leq \epsilon + \mathbb{P}\left(\int_0^T \int_U q_{r,u}(X(r,u))^2 \mu(du)\lambda(dr) > \epsilon\right)
\end{aligned} \tag{3.39}$$

Then, it follows from (3.38) and (3.39) that d_Λ generates a vector topology equivalent to $\mathcal{T}^{M}_{2,loc}$. Therefore, $(\Lambda^{2,loc}_{w}(T), \mathcal{T}^{M}_{2,loc})$ is a metrizable topological vector space. The proof of the completeness can be carried out by following similar arguments to those used in the proof of Proposition 2.4 of Bojdecki and Jakubowski [13]. □

Remark 3.2.19. *In general the space* $\Lambda^{2,loc}_{w}(T)$ *is not locally convex. This fact will have important consequences for the construction of the strong stochastic integral (see Remark 3.3.35). Indeed, if* $\mathbb{P}$ *is an atomless measure (see Definition 1.12.7 of Bogachev [8], p.55) we can show that every convex neighbourhood of zero is identical to* $\Lambda^{2,loc}_{w}(T)$*, and hence* $\Lambda^{2,loc}_{w}(T)$ *is not locally convex. To prove this, we will adapt the arguments used in Remarque 1 of Badrikian [7], p.2.*

Assume V *is a convex neighbourhood of zero of* $\Lambda^{2,loc}_{w}(T)$*. Then, there exist some* $\epsilon, \delta > 0$ *such that* $\Gamma_{\epsilon,\delta} \subseteq V$*. Let* A_δ *given by*

$$A_\delta = \left\{X \in \Lambda^{2,loc}_{w}(T) : \mathbb{P}\left(\omega \in \Omega : \int_0^T \int_U q_{r,u}(X(r,\omega,u))^2 \mu(du)\lambda(dr) > 0\right) \leq \delta\right\}.$$

Then, $A_{\delta} \subseteq \Gamma_{\epsilon,\delta} \subseteq V$ *.*

As $\mathbb{P}$ *is atomless, there exist* $n \in \mathbb{N}$ *and pairwise disjoint subsets* $\Omega_1, \cdots, \Omega_n \in \mathcal{F}$ *such that* $\mathbb{P}(\Omega_i) \leq \delta$ *and* $\Omega = \bigcup_{i=1}^{n} \Omega_i$ *(see Theorem 1.12.9 of Bogachev [8], p.55).*

Now, for each $i = 1, \dots, n$ *, let*

$$\Lambda_i = \left\{ X \in \Lambda_w^{2,loc}(T) : \int_0^T \int_U q_{r,u}(X(r,\omega,u))^2 \mu(du)\lambda(dr) = 0, \ \text{if } \omega \notin \Omega_i \right\}.$$

As $\mathbb{P}(\Omega_i) \leq \delta$ *, then we have* $\Lambda_i \subseteq A_{\delta}$ *. Moreover, note that* $\sum_{i=1}^{n} \Lambda_i = \Lambda_w^{2,loc}(T)$ *and that for each* $a > 0$ *,* $a\Lambda_i = \Lambda_i$ *.*

Hence, for any $a_1, \dots, a_n$ *such that* $a_i > 0$ *and* $\sum_{i=1}^{n} a_i = 1$ *, then*

$$\sum_{i=1}^{n} a_i \Lambda_i = \sum_{i=1}^{n} \Lambda_i = \Lambda_w^{2,loc}(T),$$

and because $\Lambda_i \subseteq A_{\delta}$ *, for all* $i = 1, \dots, n$ *, then the convex hull of* A_{δ} *(see Schaefer [93], p.39) is equal to* $\Lambda_w^{2,loc}(T)$ *. This implies that the convex hull of* V *is equal to* $\Lambda_w^{2,loc}(T)$ *, because* $A_{\delta} \subseteq V$ *. Now, since* V *is convex this implies* $V = \Lambda_w^{2,loc}(T)$ *.*

The extension of the weak stochastic integral to the elements of $\Lambda_w^{2,loc}(T)$ will be provided by the following result.

Theorem 3.2.20. *Let* $X \in \Lambda_w^{2,loc}(T)$ *. Then,*

(1) There exists an increasing sequence $\{\tau_n\}_{n \in \mathbb{N}}$ *of* $\{\mathcal{F}_t\}$ *-stopping times satisfying* $\lim_{n \to \infty} \tau_n = T$ *(*$\mathbb{P}$*-a.e.) and such that for each* $n \in \mathbb{N}$ *,* $\mathbb{1}_{[0,\tau_n]} X \in \Lambda_w^2(T)$ *.*

(2) There exists a unique càdlàg real-valued locally zero-mean square integrable martingale $\hat{I}^w(X) = \{\hat{I}_t^w(X)\}_{t \in [0,T]}$ *such that for any sequence of* $\{\mathcal{F}_t\}$ *-stopping times* $\{\sigma_n\}_{n \in \mathbb{N}}$ *satisfying* $\lim_{n \to \infty} \sigma_n = T$ *(*$\mathbb{P}$*-a.e.) and* $\mathbb{1}_{[0,\sigma_n]} X \in \Lambda_w^2(T)$ *for each* $n \in \mathbb{N}$ *, the process* $\hat{I}^w(X)$ *satisfies:*

$$\hat{I}_{t \wedge \sigma_n}^w(X) = I_t^w(\mathbb{1}_{[0,\sigma_n]} X), \quad \forall\, t \in [0,T], \tag{3.40}$$

for all $n \in \mathbb{N}$ *, where the process on the right-hand side of* (3.40) *is the weak stochastic integral of* $\mathbb{1}_{[0,\sigma_n]} X$ *.*

Proof. To prove *(1)*, for each $n \in \mathbb{N}$ define τ_n by

$$\tau_n(\omega) = \inf \left\{ t \in [0,T] : \int_0^t \int_U q_{r,u}(X(r,\omega,u))^2 \mu(du)\lambda(dr) \geq n \right\}, \quad \forall\, \omega \in \Omega, \tag{3.41}$$

with the convention $\inf \emptyset = 0$. Then, $\{\tau_n\}_{n \in \mathbb{N}}$ is an increasing sequence of $\{F_t\}$-stopping times satisfying $\lim_{n \to \infty} \tau_n = T$, $\mathbb{P}$-a.e. The proof that $\mathbb{1}_{[0,\tau_n]} X \in \Lambda_w^2(T)$, for all $n \in \mathbb{N}$ follows from standard arguments (e.g see the proof of Proposition 2.3.8 of Prévôt and Röckner [87]).

To prove *(2)*. Let $\{\sigma_n\}_{n \in \mathbb{N}}$ be a sequence of stopping times satisfying the assumptions of the statement. Such a sequence exists by part *(1)*.

Now, define $\hat{I}^w(X) = \{\hat{I}_t^w(X)\}_{t \in [0,T]}$ by means of the following prescription: for $t \in [0,T]$, let

$$\hat{I}_t^w(X) = I_t^w(\mathbb{1}_{[0,\sigma_n]} X), \tag{3.42}$$

where $n \in \mathbb{N}$ is such that $\sigma_n \geq t$. Notice that if $m \geq n$ is such that $\sigma_m \geq t$, then it follows from Proposition 3.2.14 that $\mathbb{P}$-a.e.

$$I^w_{t\wedge\sigma_n}(\mathbb{1}_{[0,\sigma_m]}X) = I^w_t(\mathbb{1}_{[0,\sigma_n]}(\mathbb{1}_{[0,\sigma_m]}X)) = I^w_t(\mathbb{1}_{[0,\sigma_n]}X). \tag{3.43}$$

Therefore the definition (3.42) is consistent. Moreover, it follows from (3.42) and (3.43) that $\hat{I}^w(X)$ satisfies (3.40).

The fact that $\hat{I}^w(X)$ is a càdlàg real-valued locally zero-mean square integrable martingale follows from (3.40) and Theorem 3.2.8.

Finally, let $\{\theta_n\}_{n\in\mathbb{N}}$ is another sequence of stopping times satisfying the properties of the statement. A similar argument to that used to obtain (3.43) shows that the definition (3.42) given with respect to the sequence $\{\theta_n\}_{n\in\mathbb{N}}$ leads to an indistinguishable processes. This proves the uniqueness of $\hat{I}^w(X)$, and its independence of the sequence of stopping times satisfying (3.40). □

Definition 3.2.21. For every $X \in \Lambda^{2,loc}_w(T)$, we will call the process $\hat{I}^w(X)$ given in Theorem (3.2.20) the **weak stochastic integral** of X. We will sometimes denote the process $\hat{I}^w(X)$ by $\left\{\int_0^t \int_U X(r,u)M(dr,du) : t \in [0,T]\right\}$.

The property (3.40) allow us to "transfer" the properties of the weak stochastic integral for integrands in $\Lambda^2_w(T)$ (see Section 3.2.2) to those in $\Lambda^{2,loc}_w(T)$. We summarize this in the following result:

Proposition 3.2.22. *Let $X \in \Lambda^{2,loc}_w(T)$. Then, all the assertions in Propositions 3.2.14, 3.2.15 and 3.2.16 are valid for the weak stochastic integral $\hat{I}^w(X)$ of X.*

As was shown for the weak stochastic integral for integrands in $\Lambda^2_w(T)$, we can also prove that the **extended weak stochastic integral** map $\hat{I}^w : \Lambda^{2,loc}_w(T) \to \mathcal{M}^{2,loc}_T(\mathbb{R})$, $X \mapsto \hat{I}^w(X)$, is linear and continuous, where we recall that $\mathcal{M}^{2,loc}_T(\mathbb{R})$ is the space of all locally zero-mean square integrable càdlàg martingales (see Section 1.2.2).

The linearity of the map $\hat{I}^w$ follows from (3.40) and the corresponding linearity of the map $I^w : \Lambda^2_w(T) \to \mathcal{M}^2_T(\mathbb{R})$. The continuity follows from the following estimate that can by proved by similar arguments to those used in the proof of Proposition 4.16 of Da Prato and Zabczyk [20], p.104-5.

Proposition 3.2.23. *Assume $X \in \Lambda^{2,loc}_w(T)$. Then, for arbitrary $a > 0$, $b > 0$,*

$$\mathbb{P}\left(\sup_{t\in[0,T]}\left|\hat{I}^w_t(X)\right| > a\right) \leq \frac{b}{a^2} + \mathbb{P}\left(\int_0^T \int_U q_{r,u}(X(r,u))^2\mu(du)\lambda(dr) > b\right).$$

Proposition 3.2.24. *The extended weak stochastic integral mapping $\hat{I}^w : \Lambda^{2,loc}_w(T) \to \mathcal{M}^{2,loc}_T(\mathbb{R})$ is linear and continuous.*

Proof. As the map $\hat{I}^w$ is linear, we need only to show its continuity. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence converging to X in $\Lambda^{2,loc}_w(T)$. As both $\Lambda^{2,loc}_w(T)$ and $\mathcal{M}^{2,loc}_T(\mathbb{R})$ are metrizable, it is sufficient to prove that $\{\hat{I}^w(X_n)\}_{n\in\mathbb{N}}$ converges to $\hat{I}^w(X)$ in $\mathcal{M}^{2,loc}_T(\mathbb{R})$.

Let $\epsilon, \delta > 0$. As $\{X_n\}_{n\in\mathbb{N}}$ converges to X in $\Lambda^{2,loc}_w(T)$, then there exists some $N_{\epsilon,\delta} \in \mathbb{N}$ such that for all $n \geq N_{\epsilon,\delta}$,

$$\mathbb{P}\left(\int_0^T \int_U q_{r,u}(X(r,u) - X_n(r,u))^2\mu(du)\lambda(dr) > \frac{\delta\epsilon^2}{2}\right) \leq \frac{\delta}{2}. \tag{3.44}$$

By linearity of the integral map, Proposition 3.2.23 and (3.44), for all $n \geq N_{\epsilon,\delta}$, we have

$$\mathbb{P}\left(\sup_{t \in [0,T]} \left| \hat{I}^w_t(X) - \hat{I}^w_t(X_n) \right| > \epsilon \right)$$
$$\leq \frac{\delta}{2} + \mathbb{P}\left(\int_0^T \int_U q_{r,u}(X(r,u) - X_n(r,u))^2 \mu(du)\lambda(dr) > \frac{\delta\epsilon^2}{2} \right) \leq \delta.$$

And hence (see (1.17)) $\{\hat{I}^w(X_n)\}_{n\in\mathbb{N}}$ converges to $\hat{I}^w(X)$ in $\mathcal{M}^{2,loc}_T(\mathbb{R})$. □

3.2.4 The Stochastic Fubini Theorem

The final topic in our study of the properties of the weak stochastic integral is the stochastic Fubini theorem that we introduce and prove below. It will be of great importance in Chapter 4 where we will study the solutions of stochastic partial differential equations driven by a cylindrical martingale-valued measure. We start by describing the class of integrands for which the theorem is valid.

Definition 3.2.25. Let $(E, \mathcal{E}, \varrho)$ be a σ-finite measure space. We denote by $\Xi^{1,2}_\omega(T,E)$ the linear space of all (equivalence classes of) families $X = \{X(r,\omega,u,e) : r \in [0,T], \omega \in \Omega, u \in U, e \in E\}$ of Hilbert space-valued maps satisfying the following conditions:

(1) $X(r,\omega,u,e) \in \Phi_{q_{r,u}}$, $\forall r \in [0,T]$, $\omega \in \Omega$, $u \in U$, $e \in E$.
(2) The map $[0,T] \times \Omega \times U \times E \to \mathbb{R}_+$ given by $(r,\omega,u,e) \mapsto q_{r,u}(X(r,\omega,u,e),\phi)$ is $\mathcal{P}_T \otimes \mathcal{B}(U) \otimes \mathcal{E}$-measurable, for every $\phi \in \Phi$.
(3)

$$\int_E ||X(\cdot,\cdot,\cdot,e)||_{w,T}\, \varrho(de) = \int_E \left(\mathbb{E} \int_0^T \int_U q_{r,u}(X(r,u,e))^2 \mu(du)\lambda(dr) \right)^{\frac{1}{2}} \varrho(de) < \infty.$$

Is easy to see that $\Xi^{1,2}_\omega(T,E)$ equipped with the norm $|||\cdot|||_{w,T,E}$ given by

$$|||X|||_{w,T,E} = \int_E ||X(\cdot,\cdot,\cdot,e)||_{w,T}\, \varrho(de), \quad \forall\, X \in \Xi^{1,2}_\omega(T,E), \tag{3.45}$$

is a Banach space.
We will denote by $\Xi^{2,2}_\omega(T,E)$ the subspace of $\Xi^{1,2}_\omega(T,E)$ of all $X = \{X(r,\omega,u,e) : r \in [0,T], \omega \in \Omega, u \in U, e \in E\}$ satisfying:

$$\int_E ||X(\cdot,\cdot,\cdot,e)||^2_{w,T}\, \varrho(de) = \int_E \left(\mathbb{E} \int_0^T \int_U q_{r,u}(X(r,u,e))^2 \mu(du)\lambda(dr) \right) \varrho(de) < \infty.$$

One can easily prove that the space $\Xi^{2,2}_\omega(T,E)$ is a Hilbert space equipped with the Hilbertian norm $|||\cdot|||_{w,T,E}$ given by

$$|||X|||^2_{w,2,T,E} = \int_E ||X(\cdot,\cdot,\cdot,e)||^2_{w,T}\, \varrho(de), \quad \forall\, X \in \Xi^{2,2}_\omega(T,E).$$

Remark 3.2.26. *Properties (1)-(3) of Definition 3.2.25 together with the (deterministic) Fubini's Theorem imply that the map* $e \mapsto ||X(\cdot,\cdot,\cdot,e)||^2_{w,T}$ *is* $\mathcal{E}$*-measurable. Hence, the map* $e \mapsto X(\cdot,\cdot,\cdot,e) \in \Lambda^2_w(T)$ *is* $\mathcal{E}/\mathcal{B}(\Lambda^2_w(T))$*-measurable. Thus,* $\Xi^{1,2}_w(T,E)$ *is a subspace of* $L^1(E,\mathcal{E},\varrho;\Lambda^2_w(T))$ *and* $\Xi^{2,2}_w(T,E)$ *is a subspace of* $L^2(E,\mathcal{E},\varrho;\Lambda^2_w(T))$.

The following result will be an important ingredient for the proof of the stochastic Fubini's Theorem.

Lemma 3.2.27. *Let* $X \in \Xi_w^{1,2}(T)$. *There exists a sequence* $\{X_n\}_{n\in\mathbb{N}} \subseteq \Xi_w^{2,2}(T)$ *such that* ϱ*-a.e.* $||X_n(\cdot,\cdot,\cdot,e)||_{w,T} \leq ||X_{n+1}(\cdot,\cdot,\cdot,e)||_{w,T}$, $\forall n \in \mathbb{N}$, *and*

$$\lim_{n\to\infty} |||X - X_n|||_{w,T,E} = 0.$$

Proof. First, from Definition 3.2.25(3), there exist some $E_0 \subseteq E$ with $\varrho(E \setminus E_0) = 0$ such that $\forall e \in E_0$, $||X(\cdot,\cdot,\cdot,e)||_{w,T} < \infty$.

Let $\{G_n\}_{n\in\mathbb{N}}$ be an increasing sequence on $\mathcal{E}$ such that $E_0 = \bigcup_{n\in\mathbb{N}} G_n$ and such that $\forall n \in \mathbb{N}$, $\varrho(G_n) < \infty$. For each $n \in \mathbb{N}$, let $X_n = \{X_n(r,\omega,u,e)\}$ be the bounded family of random variables defined by:

$$\begin{aligned} X_n(r,\omega,u,e) &= \frac{nX(r,\omega,u,e)}{||X(\cdot,\cdot,\cdot,e)||_{w,T}} \mathbb{1}_{\{e\in G_n:||X(\cdot,\cdot,\cdot,e)||_{w,T}>n\}}(e) \\ &\quad + X(r,\omega,u,e)\mathbb{1}_{\{e\in G_n:||X(\cdot,\cdot,\cdot,e)||_{w,T}\leq n\}}(e). \end{aligned} \tag{3.46}$$

As the mapping $e \mapsto \mathbb{1}_{G_n}(e)\,||X(\cdot,\cdot,\cdot,e)||_{w,T}$ is $\mathcal{E}$-measurable, then the properties (1)-(3) of Definition 3.2.25 for X implies that X_n satisfies properties (1) and (2) of Definition 3.2.25. Moreover, (3.46) implies that

$$\int_E ||X_n(\cdot,\cdot,\cdot,e)||^2_{w,T}\,\varrho(de) \leq n^2\varrho(G_n) < \infty,$$

and therefore $X_n \in \Xi_w^{2,2}(T,E)$.

Now, from the fact that $E_0 = \bigcup_{n\in\mathbb{N}} G_n$ and that $\{G_n\}_{n\in\mathbb{N}}$ is increasing, it follows that

$$\lim_{n\to\infty} \mathbb{1}_{E_0\setminus G_n}(e) = 0, \quad \forall e \in E. \tag{3.47}$$

Similarly, from the definition of E_0 we have,

$$\lim_{n\to\infty} \mathbb{1}_{\{e\in E_0:||X(\cdot,\cdot,\cdot,e)||_{w,T}>n\}}(e) = 0, \quad \forall e \in E. \tag{3.48}$$

Hence, (3.45), (3.46), (3.47), (3.48), Definition 3.2.25 (3) and the dominated convergence theorem implies that:

$$\begin{aligned} &\lim_{n\to\infty} |||X - X_n|||_{w,T,E} \\ &= \lim_{n\to\infty} \int_E ||X(\cdot,\cdot,\cdot,e) - X_n(\cdot,\cdot,\cdot,e)||_{w,T}\,\varrho(de) \\ &= \lim_{n\to\infty} \int_{E_0} \mathbb{1}_{E_0\setminus G_n}(e)\,||X(\cdot,\cdot,\cdot,e)||_{w,T}\,\varrho(de) \\ &\quad + \lim_{n\to\infty} \int_{E_0} \mathbb{1}_{\{e\in G_n:||X(\cdot,\cdot,\cdot,e)||_{w,T}>n\}}(e) \left|1 - \frac{n}{||X(\cdot,\cdot,\cdot,e)||_{w,T}}\right| ||X(\cdot,\cdot,\cdot,e)||_{w,T}\,\varrho(de) \\ &\leq \lim_{n\to\infty} \int_{E_0} \mathbb{1}_{E_0\setminus G_n}(e)\,||X(\cdot,\cdot,\cdot,e)||_{w,T}\,\varrho(de) \\ &\quad + 2\lim_{n\to\infty} \int_{E_0} \mathbb{1}_{\{e\in E_0:||X(\cdot,\cdot,\cdot,e)||_{w,T}>n\}}(e)\,||X(\cdot,\cdot,\cdot,e)||_{w,T}\,\varrho(de) \\ &= 0. \end{aligned}$$

Finally, the fact that for every $e \in E_0$, $||X_n(\cdot,\cdot,\cdot,e)||_{w,T} \leq ||X_{n+1}(\cdot,\cdot,\cdot,e)||_{w,T}$, $\forall n \in \mathbb{N}$, follows from (3.46). □

A proof of the following result can be carried out using similar arguments to those in the proof of Proposition 3.3.7.

Lemma 3.2.28. *Let $S_w(T,E)$ denotes the collection of all families $X=\{X(r,\omega,u,e): r\in[0,T],\omega\in\Omega,u\in U,e\in E\}$ of Hilbert space-valued maps of the form:*

$$X(r,\omega,u,e)=\sum_{l=1}^{p}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbb{1}_{]S_j,t_j]}(r)\,\mathbb{1}_{F_j}(w)\,\mathbb{1}_{A_i}(u)\,\mathbb{1}_{D_l}(e)\,i_{q_{r,u}}\phi_{i,j,l}, \tag{3.49}$$

for all $r\in[0,T]$, $\omega\in\Omega$, $u\in U$, $e\in E$, where $m,\ n,\ p\in\mathbb{N}$, and for $l=1,\dots,p$, $i=1,\dots,n$, $j=1,\dots,m$, $0\leq s_j<t_j\leq T$, $F_j\in\mathcal{F}_j$, $A_i\in\mathcal{R}$, $D_l\in\mathcal{E}$ and $\phi_{i,j,l}\in\Psi$. Then, $S_w(T,E)$ is dense in $\Xi^{2,2}_w(T,E)$.

We are ready to proof the stochastic Fubini theorem. Relevant properties of the Bochner integral can be consulted on Appendix C.

Theorem 3.2.29 (Stochastic Fubini's Theorem)**.** *Let $X\in\Xi^{1,2}_w(T,E)$. Then,*

(1) For a.e. $(r,\omega,u)\in[0,T]\times\Omega\times U$, the mapping $E\ni e\mapsto X(r,\omega,u,e)\in\Phi_{q_{r,u}}$ is Bochner integrable. Moreover, the family

$$\int_E X(\cdot,\cdot,\cdot,e)\,\varrho(de)=\left\{\int_E X(r,\omega,u,e)\,\varrho(de): r\in[0,T],\omega\in\Omega,u\in U\right\},$$

is an element of $\Lambda^2_w(T)$.

(2) The mapping $E\ni e\mapsto I^w(X(\cdot,\cdot,\cdot,e))\in\mathcal{M}^2_T(\mathbb{R})$ is Bochner integrable. Moreover,

$$\left(\int_E I^w(X(\cdot,\cdot,\cdot,e))\,\varrho(de)\right)_t=\int_E I^w_t(X(\cdot,\cdot,\cdot,e))\,\varrho(de),\quad\forall\, t\geq 0.$$

(3) The following equality holds $\mathbb{P}$-a.e.

$$I^w_t\left(\int_E X(\cdot,\cdot,\cdot,e)\,\varrho(de)\right)=\int_E I^w_t(X(\cdot,\cdot,\cdot,e))\,\varrho(de),\quad\forall\, t\in[0,T]. \tag{3.50}$$

Proof. Assume $X\in\Xi^{1,2}_w(T,E)$. For convenience we divide the proof in three parts.

Proof of (1).

First, from Definition 3.2.25(ii) the mapping $(r,\omega,u,e)\mapsto q_{r,u}(X(r,\omega,u,e))$ is $P_T\otimes U\otimes\mathcal{E}$-measurable, then from the Minkowski inequality for integrals (see Theorem 13.14 of Schilling [94], p. 130) it follows that:

$$\begin{aligned}&\int_{[0,T]\times\Omega\times U}\left(\int_E |q_{r,u}(X(r,\omega,u,e))|\,\varrho(de)\right)^2(\lambda\otimes\mathbb{P}\otimes\mu)(d(r,\omega,u)) \qquad (3.51)\\ &\leq\left(\int_E\left(\int_{[0,T]\times\Omega\times U}q_{r,u}(X(r,\omega,u,e))^2(\lambda\otimes\mathbb{P}\otimes\mu)(d(r,\omega,u))\right)^{\frac{1}{2}}\varrho(de)\right)^2\\ &=|||X|||^2_{W,T,E}<\infty\end{aligned}$$

Therefore, it follows from (3.51) that

$$\int_E q_{r,u}(X(r,\omega,u,e))\varrho(de)<\infty,\quad\text{for}\quad\lambda\otimes\mathbb{P}\otimes\mu\text{-a.e.}(r,\omega,u)\in[0,T]\times\Omega\times U. \tag{3.52}$$

Now, as for fixed (r,ω,u) the map $e \mapsto q_{r,u}(X(r,\omega,u,e))$ is $\mathcal{E}$-measurable, then the map $e \mapsto X(r,\omega,u,e)$ is $\mathcal{E}/\mathcal{B}(\Phi_{q_{r,u}})$-measurable. Then, because the Hilbert space $\Phi_{q_{r,u}}$ is separable it follows that the map $e \mapsto X(r,\omega,u,e)$ is strongly measurable. Moreover, (3.52) implies that for almost every (r,ω,u) the map $e \mapsto X(r,\omega,u,e)$ is Bochner integrable.

To prove the second statement, let $\Gamma_0 \subseteq [0,T]\times\Omega\times U$ be such that (3.52) is satisfied. Then, for every $(r,\omega,u)\in\Gamma_0$ the Bochner integral $\int_E X(r,\omega,u,e)\varrho(de) \in \Phi_{q_{r,u}}$ exists. For $(r,\omega,u)\Gamma_0^c$ we define $\int_E X(r,\omega,u,e)\varrho(de) = 0$. Then, the family $\int_E X(\cdot,\cdot,\cdot,e)\varrho(de)$ defined such way satisfies Definition 3.2.25(i).

Now, from the properties of the Bochner integral for $(r,\omega,u)\in\Gamma_0$, we have

$$q_{r,u}\left(\int_E X(r,\omega,u,e)\varrho(de),\phi\right) = \int_E q_{r,u}(X(r,\omega,u,e)),\phi)\varrho(de), \quad \forall \phi\in\Phi. \tag{3.53}$$

Then, as the map $(r,\omega,u,e)\mapsto q_{r,u}(X(r,\omega,u,e)),\phi)$ is $\mathcal{P}_T\otimes\mathcal{B}(U)\otimes\mathcal{E}$-measurable and ϱ-integrable for each $\phi\in\Phi$ (and for all $(r,\omega,u)\in\Gamma_0$), then by Fubini's theorem and (3.53) the map $(r,\omega,u)\mapsto q_{r,u}\left(\int_E X(r,\omega,u,e)\varrho(de),\phi\right)$ is $\mathcal{P}_T\times\mathcal{B}(U)$-measurable, for every $\phi\in\Phi$. Hence Definition 3.2.25(ii) is satisfied.

Finally, to prove that $\left|\left|\int_E X(\cdot,\cdot,\cdot,e)\varrho(de)\right|\right|_{w,T}<\infty$, first note that from our definition of $\int_E X(\cdot,\cdot,\cdot,e)\varrho(de)$ and the properties of the Bochner integral, we have that:

$$q_{r,u}\left(\int_E X(r,\omega,u,e)\varrho(de)\right) \leq \int_E q_{r,u}(X(r,\omega,u,e)\varrho(de), \quad \forall (r,\omega,u). \tag{3.54}$$

Hence, from (3.51), (3.54) and Fubini's theorem, it follows that

$$\begin{aligned}
&\mathbb{E}\int_0^T \int_U q_{r,u}\left(\int_E X(\cdot,\cdot,\cdot,e)\varrho(de)\right)^2 \mu(du)\lambda(dr) \qquad (3.55)\\
&\leq \mathbb{E}\int_0^T \int_U \left|\int_E q_{r,u}(X(r,\omega,u,e))\varrho(de)\right|^2 \mu(du)\lambda(dr)\\
&\leq \int_{[0,T]\times\Omega\times U}\left(\int_E |q_{r,u}(X(r,\omega,u,e))|\,\varrho(de)\right)^2 (\lambda\otimes\mathbb{P}\otimes\mu)(d(r,\omega,u))\\
&\leq |||X|||^2_{w,T,E}<\infty.
\end{aligned}$$

Thus, $\left|\left|\int_E X(\cdot,\cdot,\cdot,e)\varrho(de)\right|\right|_{w,T}<\infty$, and hence $\int_E X(\cdot,\cdot,\cdot,e)\varrho(de)\in\Lambda^2_w(T)$.

Proof of (2).

First, note that from Definition 3.2.25(iii), there exists some $E_1\subseteq E$ such that $\varrho(E\setminus E_1)=0$ and such that $||X(\cdot,\cdot,\cdot,e)||_{w,T}<\infty$, $\forall\, e\in E_1$.

Hence, by redefining a version of X to be equal to X whenever $e\in E_1$ and to be 0 whenever $e\in E\setminus E_1$, if we call this version again by X, we have that for every $e\in E$, $X(\cdot,\cdot,\cdot,e)\in\Lambda^2_w(T)$. Therefore, for every $e\in E$ the stochastic integral $I^w(X(\cdot,\cdot,\cdot,e))\in\mathcal{M}^2_T(\mathbb{R})$ exists.

To prove that the map $e\mapsto I^w(X(\cdot,\cdot,\cdot,e))$ is strongly measurable, we will show that there exists a sequence of simple maps from E into $\mathcal{M}^2_T(\mathbb{R})$ such that this sequence converges to $e\mapsto I^w(X(\cdot,\cdot,\cdot,e))$ ϱ-a.e.

To do this, note that by an application of Lemmas 3.2.27 and 3.2.28, there exists a sequence $\{X_k\}_{k\in\mathbb{N}}$ of families of the simple form (3.49) and such that

$$\lim_{k\to\infty}|||X-X_k|||_{w,T,E}=0. \tag{3.56}$$

Note that if X_k is of the form (3.49), then for $e \in E$ its stochastic integral takes the form:

$$I^w_t(X_k(\cdot,\cdot,\cdot,e)) = \sum_{l=1}^{p} M^{(l)}_t \mathbb{1}_{D_l}(e), \quad \forall t \in [0,T],$$

where for each $l = 1, \cdots, p$, it follows from Definition 3.2.6 that

$$M^{(l)}_t(\omega) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbb{1}_{F_j}(\omega)\, M((s_j \wedge t, t_j \wedge t], A_i)[\phi_{i,j,l}], \quad \forall t \in [0,T], \, \omega \in \Omega,$$

(to simplify the notation, above we have omitted the dependence on k of the components of (3.49)) In particular, each $M^{(l)}$ is an element of $\mathcal{M}^2_T(\mathbb{R})$. Therefore, for each $k \in \mathbb{N}$ the map $e \mapsto I^w(X_k(\cdot,\cdot,\cdot,e))$ from E into $\mathcal{M}^2_T(\mathbb{R})$ is simple. Moreover, from the linearity of the weak stochastic integral, Doob's inequality, (3.30) and (3.56), it follows that:

$$\begin{aligned}
&\lim_{k\to\infty} \int_E ||I^w(X(\cdot,\cdot,\cdot,e)) - I^w(X_k(\cdot,\cdot,\cdot,e))||_{\mathcal{M}^2_T(\mathbb{R})}\, \varrho(de) \qquad (3.57)\\
&\leq 2\sqrt{T} \lim_{k\to\infty} \int_E ||I^w_T(X(\cdot,\cdot,\cdot,e) - X_k(\cdot,\cdot,\cdot,e))||_{L^2(\Omega,\mathscr{F},\mathbb{P})}\, \varrho(de)\\
&= 2\sqrt{T} \lim_{k\to\infty} \int_E ||X(\cdot,\cdot,\cdot,e) - X_k(\cdot,\cdot,\cdot,e)||_{w,T}\, \varrho(de)\\
&= 2\sqrt{T} \lim_{k\to\infty} |||X - X_k|||_{w,T,E}\,.
\end{aligned}$$

Then, it follows from (3.57) and a standard use of the Chebyshev inequality and the Borel-Cantelli lemma that there exists a set $E_2 \subseteq E$ with $\varrho(E \setminus E_2) = 0$ and a subsequence $\{X_{k_q}\}_{q\in\mathbb{N}}$ such that

$$\lim_{q\to\infty} \left|\left|I^w(X(\cdot,\cdot,\cdot,e)) - I^w(X_{k_q}(\cdot,\cdot,\cdot,e))\right|\right|_{\mathcal{M}^2_T(\mathbb{R})} = 0, \quad \forall e \in E_2.$$

In particular, this implies that the map $e \mapsto I^w(X(\cdot,\cdot,\cdot,e))$ is strongly measurable. Moreover, by a similar calculation to that in (3.57) we have

$$\int_E ||I^w(X(\cdot,\cdot,\cdot,e))||_{\mathcal{M}^2_T(\mathbb{R})}\, \varrho(de) \leq 2\sqrt{T}\, |||X|||_{w,T,E} < \infty,$$

then the mapping $e \mapsto I^w(X(\cdot,\cdot,\cdot,e))$ is Bochner integrable and furthermore,

$$\int_E I^w(X(\cdot,\cdot,\cdot,e))\varrho(de) = \lim_{q\to\infty} \int_E I^w(X_{k_q}(\cdot,\cdot,\cdot,e))\varrho(de), \qquad (3.58)$$

where the limit is taken in $\mathcal{M}^2_T(\mathbb{R})$.

Finally, the property

$$\left(\int_E I^w(X(\cdot,\cdot,\cdot,e))\varrho(de)\right)_t = \int_E I^w_t(X(\cdot,\cdot,\cdot,e))\varrho(de), \quad \forall t \in [0,T],$$

follows from (3.58) and the fact that this is satisfied for every X_{k_q} due to their simple form.

Proof of (3).

In the proofs of (1) and (2), we proved that the integrals on both sides of (3.50) are well-defined elements of $\mathcal{M}^2_T(\mathbb{R})$.

Let $\{X_{k_q}\}_{q\in\mathbb{N}}$ be the sequence to simple families as defined in the proof of part 2. For each $q \in \mathbb{N}$, the simple form of X_{k_q} (see (3.49)) implies that it satisfies (3.50).

Now, from Doob's inequality, (3.30), (3.55) and (3.56) (there with k replaced by k_q), and the linearity of both the weak stochastic integral and the Bochner integral, it follows that

$$\begin{aligned}
&\left\| I^w\left(\int_E X(\cdot,\cdot,\cdot,e)\varrho(de)\right) - I^w\left(\int_E X_{k_q}(\cdot,\cdot,\cdot,e)\varrho(de)\right)\right\|_{\mathcal{M}^2_T(\mathbb{R})} \qquad (3.59)\\
&= \left\| I^w\left(\int_E (X(\cdot,\cdot,\cdot,e) - X_{k_q}(\cdot,\cdot,\cdot,e))\varrho(de)\right)\right\|_{\mathcal{M}^2_T(\mathbb{R})}\\
&\leq 2\sqrt{T}\left\|\int_E (X(\cdot,\cdot,\cdot,e) - X_{k_q}(\cdot,\cdot,\cdot,e))\varrho(de)\right\|_{w,T}\\
&\leq 2\sqrt{T}\,\left|\left|\left|X - X_{k_q}\right|\right|\right|_{w,T,E} \to 0, \quad \text{as } q\to\infty.
\end{aligned}$$

On the other hand, by (3.58) we have

$$\left\|\int_E I^w(X(\cdot,\cdot,\cdot,e))\varrho(de) - \int_E I^w(X_{k_q}(\cdot,\cdot,\cdot,e))\varrho(de)\right\|_{\mathcal{M}^2_T(\mathbb{R})} \to 0, \quad \text{as } q\to\infty \quad (3.60)$$

Then, it follows from (3.59) and (3.60), and the fact that (3.50) is valid for each X_{k_q}, that the processes $I^w\left(\int_E (X(\cdot,\cdot,\cdot,e)\varrho(de)\right)$ and $\int_E I^w(\cdot,\cdot,\cdot,e)\varrho(de)$ are equal as elements of $\mathcal{M}^2_T(\mathbb{R})$ and hence this implies that X satisfies (3.50). ☐

Remark 3.2.30. *Note that* (3.50) *can be also written in the more familiar manner:*

$$\int_E\left(\int_0^t\int_U X(r,u,e)M(dr,du)\right)\varrho\,(de) = \int_0^t\int_U\left(\int_E X(r,u,e)\,\varrho\,(de)\right)M(dr,du),$$

for $t\in[0,T]$.

§ 3.3 The Strong Stochastic Integral

In this section we introduce and study basic properties of the strong stochastic integral. Recall that Φ is a locally convex space and Ψ is a quasi-complete, bornological, nuclear space.

The main reason why we want Ψ to be quasi-complete and bornological is because this implies it is ultrabornological (see Section 1.1.3). In this case we will be able to use the version of the closed graph theorem for sequentially closed maps given in Theorem 1.1.3. We will see that this is a fundamental tool in our development of the strong stochastic integral.

Now we proceed to describe the construction and main properties of the strong stochastic integral. As mentioned early in the introduction, the integrands are families of continuous linear operators taking values in Ψ'_β and whose domain is a Hilbert subspace of Φ'_β depending on both the jump space and the time variables. The strong stochastic integral assigns to each of these families a Ψ'_β-valued regular process.

Contrary to the weak stochastic integral, the strong integral is not defined by means of an isometry. However, the weak integral acts as a building block in the construction of the strong integral and hence the Itô isometry is playing an indirect role. In effect, the main idea behind the construction of the strong integral is to choose a well behaved class of integrands that map the space Ψ into the space of weak integrands (i.e. into $\Lambda^{2,loc}_{w}(T)$). The strong integral is then defined by means of a regularization procedure using the continuity of the weak integral mapping. The strong integral is a Ψ'_{β}-valued martingale or a local martingale.

As the reader can see, this is a completely new approach that exploits the powerful regularization results valid for cylindrical processes taking values in the strong dual of a nuclear space. In comparison with the previous work done by Itô [43] and Bojdecki and Jakubowski [12, 13], there are at least three benefits of this approach:

(1) The integrator is very general; it is a cylindrical martingale-valued measure in the dual of a locally convex space and with some special covariance structure. This in particular covers the cases of Φ'_{β}-valued square integrable martingales and of generalized Wiener processes that are used as integrators in [13].
(2) The class of integrands is more general than the class defined in [43] and [12, 13] in two senses. First, we do not require our integrands to be families of Hilbert-Schmidt maps from some Hilbert spaces continuously included in Φ'_{β} into a fixed Hilbert space continuously included in Ψ'_{β} (see Remark 3.3.5). As described before, our integrands are only required to be families of continuous linear operators and their range takes values in Ψ'_{β} (see Definition 3.3.1).
Second, we require very weak moment conditions for our integrands. In particular, the existence of some weak (almost sure) square moments is enough (see Definition 3.3.32). These conditions are implied by the stronger conditions satisfied by the integrands in [43] and [12, 13].
(3) Although we are working with a more general theory of stochastic integration, our proofs and arguments are not more complicated than in [43] and [12, 13]. Indeed, as we will see in this section, many of the properties of the strong integral follow from very straightforward arguments and the corresponding properties of the weak integral.

We proceed to provide the details of construction of the strong stochastic integral. Throughout this section we denote the space $\Lambda^{2}_{w}(M;T)$ by $\Lambda^{2}_{w}(T)$. Recall that $\mathcal{M}^{2}_{T}(\Psi'_{\beta})$ denotes the space of all the Φ'_{β}-valued càdlàg zero-mean square integrable martingales defined on $[0,T]$. Properties of the elements of $\mathcal{M}^{2}_{T}(\Psi'_{\beta})$ and topologies on this space were studied in Section 1.2.2.

3.3.1 The Space of Strong Integrands

We start by introducing the class of strong integrands.

Definition 3.3.1. Let $\Lambda^{2}_{s}(\Psi,M;T)$ denote the collection (of equivalence classes) of families $R=\{R(r,\omega,u): r\in[0,T], \omega\in\Omega, u\in U\}$ of operator-valued maps satisfying the following conditions:

(1) $R(r,\omega,u)\in\mathcal{L}(\Phi'_{q_{r,u}},\Psi'_{\beta})$, for all $r\in[0,T]$, $\omega\in\Omega$, $u\in U$,
(2) R is $q_{r,u}$**-predictable**, i.e. for each $\phi\in\Phi$, $\psi\in\Psi$, the mapping $[0,T]\times\Omega\times U\to\mathbb{R}_{+}$ given by $(r,\omega,u)\mapsto q_{r,u}(R(r,\omega,u)'\psi,\phi)$ is $\mathcal{P}_{T}\otimes\mathcal{B}(U)$-measurable.
(3)
$$\mathbb{E}\int_{0}^{T}\int_{U} q_{r,u}(R(r,u)'\psi)^{2}\mu(du)\lambda(dr)<\infty, \quad \forall\,\psi\in\Psi. \tag{3.61}$$

Remark 3.3.2. *Note that from Definition 3.3.1(1), for* $r \in [0,T]$, $\omega \in \Omega$, $u \in U$, *the dual operator* $R(r,\omega,u)'$ *of* $R(r,\omega,u)$ *satisfies* $R(r,\omega,u)' \in \mathcal{L}(\Psi, \Phi_{q_{r,u}})$ *(here we are implicitly using the fact that* Ψ *is reflexive; see Theorem 1.1.7(2)). In particular, this implies* $R(r,\omega,u)'\psi \in \Phi_{q_{r,u}}$, *for all* $\psi \in \Psi$. *This last fact, Definition 3.3.1(2) and Remark 3.2.2 imply that the map* $(r,\omega,u) \mapsto q_{r,u}(R(r,\omega,u)'\psi)^2 = q_{r,u}(R(r,\omega,u)'\psi, R(r,\omega,u)'\psi)$ *is* $\mathcal{P}_T \otimes \mathcal{B}(U)$*-measurable and hence the integral in* (3.61) *is well defined.*

When there is no necessity to give emphasis to the dependence of the space $\Lambda^2_s(\Psi, M; T)$ with respect to Ψ and M, we denote this space by $\Lambda^2_s(T)$. We will use this shorter notation for the rest of this section.

It is easy to check that $\Lambda^2_s(T)$ is a linear space. Now we introduce a class of subspaces of $\Lambda^2_s(T)$ that will help us to have a better understanding of its inner structure.

Definition 3.3.3. Let p be a continuous Hilbertian semi-norm on Ψ. Let $\Lambda^2_s(p,T)$ denote the collection (of equivalence classes) of families $R = \{R(r,\omega,u) : r \in [0,T], \omega \in \Omega, u \in U\}$ of operator-valued maps satisfying the following conditions:

(1) $R(r,\omega,u) \in \mathcal{L}_2(\Phi'_{q_{r,u}}, \Psi'_p)$, for all $r \in [0,T]$, $\omega \in \Omega$, $u \in U$,

(2) R is $q_{r,u}$**-predictable**, i.e. for each $\phi \in \Phi$, $\psi \in \Psi$, the mapping $[0,T] \times \Omega \times U \to \mathbb{R}_+$ given by $(r,\omega,u) \mapsto q_{r,u}(R(r,\omega,u)'\psi, \phi)$ is $\mathcal{P}_T \otimes \mathcal{B}(U)$-measurable.

(3)
$$\mathbb{E} \int_0^T \int_U ||R(r,u)||^2_{\mathcal{L}_2(\Phi'_{q_{r,u}}, \Psi'_p)} \, \mu(du)\lambda(dr) < \infty. \tag{3.62}$$

Remark 3.3.4. *Similar arguments to those in Remark 3.3.2, and the definition of the Hilbert-Schmidt norm shows that the map* $(r,\omega,u) \mapsto ||R(r,\omega,u)||^2_{\mathcal{L}_2(\Phi'_{q_{r,u}}, \Psi'_p)} = ||R(r,\omega,u)'||^2_{\mathcal{L}_2(\Psi_p, \Phi_{q_{r,u}})}$ *is* $\mathcal{P}_T \otimes \mathcal{B}(U)$*-measurable and hence the integral in* (3.62) *is well defined.*

Remark 3.3.5. *The space* $\Lambda^2_s(p,T)$ *for a fixed continuous Hilbertian semi-norm* p *on* Ψ *corresponds to an extension of the class of integrands considered by Bojdecki and Jakubowski in [13] to integrands depending also on the jump space variable* u.

The proof of the following result can be carried out similarly to the proof of Proposition 2.4 of Bojdecki and Jakubowski [13].

Proposition 3.3.6. *Let* p *be a continuous Hilbertian semi-norm on* Ψ. *Then, the space* $\Lambda^2_s(p,T)$ *is a Hilbert space when equipped with the inner product corresponding to the Hilbertian norm* $||\cdot||_{s,p,T}$ *given by*

$$||R||^2_{s,p,T} = \mathbb{E} \int_0^T \int_U ||R(r,u)||^2_{\mathcal{L}_2(\Phi'_{q_{r,u}}, \Psi'_p)} \, \mu(du)\lambda(dr), \quad \forall\, R \in \Lambda^2_s(p,T). \tag{3.63}$$

Proposition 3.3.7. *If* p *and* q *are continuous Hilbertian semi-norms on* Ψ *such that* $p \leq q$, *then* $i'_{p,q}\Lambda^2_s(p,T) \subseteq \Lambda^2_s(q,T)$, *i.e. for each* $R = \{R(r,\omega,u)\} \in \Lambda^2_s(p,T)$, *we have* $i'_{p,q}R = \{i'_{p,q}R(r,\omega,u)\} \in \Lambda^2_s(q,T)$.

Proof. As $p \leq q$, then $\Psi'_p \subseteq \Psi'_q$ and the inclusion $i'_{p,q} : \Psi'_p \to \Psi'_q$ is linear and continuous. Therefore, if $R \in \Lambda^2_s(p,T)$, it is easy to check that $i'_{p,q}R$ belongs to $\Lambda^2_s(q,T)$. So, is clear that $i'_{p,q}R$ satisfies condition (2) in Definition 3.3.3. It also satisfies condition (1). This follows from the fact that for each $(r,\omega,u) \in [0,T] \times \Omega \times U$,

$R(r,\omega,u) \in \mathcal{L}_2(\Phi'_{q_{r,u}}, \Psi'_p)$ implies $i'_{p,q}R(r,\omega,u) \in \mathcal{L}_2(\Phi'_{q_{r,u}}, \Psi'_q)$. Finally, condition (3) is satisfied because from (3.63) it follows that

$$\left|\left|i'_{p,q}R\right|\right|_{s,q,T} \leq \left|\left|i'_{p,q}\right|\right|_{\mathcal{L}(\Psi'_p,\Psi'_q)} \left|\left|R\right|\right|_{s,p,T} < \infty.$$

Hence, $i'_{p,q}R \in \Lambda^2_s(q,T)$. □

Proposition 3.3.8. *For every continuous Hilbertian semi-norm p on Ψ, we have $i'_p\Lambda^2_s(p,T) \subseteq \Lambda^2_s(T)$, i.e. for each $R = \{R(r,\omega,u)\} \in \Lambda^2_s(p,T)$, we have $i'_pR = \{i'_pR(r,\omega,u)\} \in \Lambda^2_s(T)$.*

Proof. Let p be a continuous Hilbertian semi-norm on Ψ and let $R \in \Lambda^2_s(p,T)$. It is clear from Definition 3.3.3 and the fact that the inclusion $i'_p : \Psi'_p \to \Psi'_\beta$ is linear and continuous, that the family i'_pR satisfies the properties (1) and (2) of Definition 3.3.1. Moreover, from Proposition B.0.17, we have

$$\begin{aligned} q_{r,u}(R(r,\omega,u)'i_p\psi) &\leq p(i_p\psi)\left|\left|R(r,\omega,u)'\right|\right|_{\mathcal{L}(\Psi_p,\Phi_{q_{r,u}})} \\ &\leq p(i_p\psi)\left|\left|R(r,\omega,u)\right|\right|_{\mathcal{L}_2(\Phi'_{q_{r,u}},\Psi'_p)}, \end{aligned} \tag{3.64}$$

for every $(r,\omega,u) \in [0,T] \times \Omega \times U$, and all $\psi \in \Psi$. Then it follows from (3.64) and (3.62) that

$$\begin{aligned} &\mathbb{E}\int_0^T \int_U q_{r,u}(R(r,u)'i_p\psi)^2 \mu(du)\lambda(dr) \\ &\qquad \leq p(i_p\psi)^2\, \mathbb{E}\int_0^T \int_U \left|\left|R(r,u)\right|\right|^2_{\mathcal{L}_2(\Phi'_{q_{r,u}},\Psi'_p)} \mu(du)\lambda(dr) < \infty. \end{aligned}$$

for all $\psi \in \Psi$. Hence, the fact that $(i'_pR(r,\omega,u))' = R(r,\omega,u)'i_p$ for every (r,ω,u), and the above inequality implies that i'_pR satisfies (3.61). Thus, $i'_pR \in \Lambda^2_s(T)$. □

Our next objective is to show that the spaces $\Lambda^2_s(T)$ and $\mathcal{L}(\Psi, \Lambda^2_w(T))$ are isomorphic (Theorem 3.3.10), where $\mathcal{L}(\Psi, \Lambda^2_w(T))$ is the linear space of continuous and linear operators from Ψ into $\Lambda^2_w(T)$. This fact will have two consequences. The first, that we can equip $\Lambda^2_s(T)$ with a locally convex topology induced by the operator topology on $\mathcal{L}(\Psi, \Lambda^2_w(T))$. The second, as a consequence of the proof of Theorem 3.3.10 we will show (see Corollary 3.3.12) that to every $R \in \Lambda^2_s(T)$ we can associate some $\tilde{R} \in \Lambda^2_s(p,T)$ such that $R = i'_p\tilde{R}$ $\lambda \otimes \mathbb{P} \otimes \mu$-a.e. This will have important implications in our construction of the strong stochastic integral in Section 3.3.2.

For the proof of Theorem 3.3.10 will need the following result.

Lemma 3.3.9. *Let $S \in \mathcal{L}(\Psi, \Lambda^2_w(T))$. Then, $q : \Psi \to \mathbb{R}_+$ given by*

$$q(\psi) = \left|\left|S\psi\right|\right|_{w,T} = \left(\mathbb{E}\int_0^T \int_U q_{r,u}(S\psi)^2 \mu(du)\lambda(dr)\right)^{1/2}, \quad \forall\, \psi \in \Psi.$$

is a continuous Hilbertian semi-norm on Ψ. Moreover, there exists a continuous Hilbertian semi-norm p on Ψ, $q \leq p$ such that $i_{q,p}$ is Hilbert-Schmidt and for which the map S has an extension $\tilde{S}$ such that $\tilde{S} \in \mathcal{L}_2(\Psi_p, \Lambda^2_w(T))$.

Proof. It is easy to see that $q : \Psi \to \mathbb{R}_+$ is a Hilbertian semi-norm on Ψ. Now, as $\left|\left|\cdot\right|\right|_{w,T}$ generates the topology on $\Lambda^2_w(T)$, it follows by definition that the map

$X \mapsto ||X||_{w,T}$ from $\Lambda^2_w(T)$ into $\mathbb{R}_+$ is continuous. Then because $S \in \mathcal{L}(\Psi, \Lambda^2_w(T))$ and $q(\psi) = ||S\psi||_{w,T}$ for all $\psi \in \Psi$, it follows that $q : \Psi \to \mathbb{R}_+$ is continuous.

Now, as Ψ is a nuclear space, there exists a continuous Hilbertian semi-norm p on Ψ such that $q \leq p$ and $i_{q,p}$ is Hilbert-Schmidt. Hence, $||S\psi||_{w,T} \leq p(\psi)$, for all $\psi \in \Psi$ and therefore S is p-continuous. As Ψ is dense in Ψ_p, it follows that S has an extension $\tilde{S}$ such that $\tilde{S} \in \mathcal{L}(\Psi_p, \Lambda^2_w(T))$.

Moreover, $\tilde{S}$ is Hilbert-Schmidt. This is because if $\{\psi^p_j\}_{j\in\mathbb{N}} \subseteq \Psi$ is a complete orthonormal system in Ψ_p, then

$$\left|\left|\tilde{S}\right|\right|^2_{\mathcal{L}_2(\Psi_p,\Lambda^2_w(T))} = \sum_{j=1}^{\infty} \left|\left|\tilde{S} i_p \psi^p_j\right|\right|^2_{w,T} = \sum_{j=1}^{\infty} q(i_p\psi^p_j)^2 = ||i_{q,p}||^2_{\mathcal{L}_2(\Psi_p,\Psi_q)} < \infty.$$

Thus, $\tilde{S} \in \mathcal{L}_2(\Psi_p, \Lambda^2_w(T))$. □

Now we prove that $\Lambda^2_s(T)$ and $\mathcal{L}(\Psi, \Lambda^2_w(T))$ are isomorphic.

Theorem 3.3.10. *The mapping $\Delta : \Lambda^2_s(T) \to \mathcal{L}(\Psi, \Lambda^2_w(T))$ given by*

$$R \mapsto \left(\psi \mapsto R'\psi := \{R(r,\omega,u)'\psi : r \in [0,T], \omega \in \Omega, u \in U\}\right), \tag{3.65}$$

is an isomorphism.

Proof. We divide the proof into two steps.

Step 1. *For every $R \in \Lambda^2_s(T)$, the map $\psi \mapsto R'\psi$ is an element of $\mathcal{L}(\Psi, \Lambda^2_w(T))$. Hence, the map Δ given by* (3.65) *is well-defined and linear.*

Let $R \in \Lambda^2_s(T)$. First, note that from Definition 3.3.1 and Remark 3.3.2, for every $\psi \in \Psi$, the family $R'\psi$ given by (3.65) satisfies the conditions of Definition 3.2.1, and hence it is an element of $\Lambda^2_w(T)$. Therefore, the map $\psi \mapsto R'\psi$ from Ψ into $\Lambda^2_w(T)$ is well-defined. Moreover, it is also linear as one can easily see from the linearity of each operator $R(r,\omega,u)' \in \mathcal{L}(\Psi, \Phi_{q_{r,u}})$.

To prove that $\psi \mapsto R'\psi$ is also continuous, we will prove firstly that it is a sequentially closed operator. In such a case, from the fact that Ψ is ultrabornological and $\Lambda^2_w(T)$ is a Hilbert space, the closed graph theorem (Theorem 1.1.3) implies that it is continuous.

We will prove that $\psi \mapsto R'\psi$ is sequentially closed by proving that it satisfies the characterization given in Theorem 1.1.1. Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence in Ψ converging to some $\psi \in \Psi$ and let $X \in \Lambda^2_w(T)$ be such that $\{R'\psi_n\}_{n\in\mathbb{N}}$ converges to X in $\Lambda^2_w(T)$, i.e. we have

$$\lim_{n\to\infty} \left|\left|R'\psi_n - X\right|\right|^2_{w,T} = \lim_{n\to\infty} \mathbb{E}\int_0^T \int_U q_{r,u}(R(r,u)'\psi_n - X(r,u))^2 \mu(du)\lambda(dr) = 0. \tag{3.66}$$

We need to prove that $X = R'\psi$.

First, note that as for each $(r,\omega,u) \in [0,T]\times\Omega\times U$, we have $R(r,\omega,u)' \in \mathcal{L}(\Psi, \Phi_{q_{r,u}})$, then it follows that for each (r,ω,u), $\{R(r,\omega,u)'\psi_n\}_{n\in\mathbb{N}}$ converges to $R(r,\omega,u)'\psi$ in $\Phi_{q_{r,u}}$ as $n \to \infty$. It follows from this, Fatou's lemma and (3.66), that we have

$$\begin{aligned}
\left|\left|R'\psi - X\right|\right|^2_{w,T} &= \mathbb{E}\int_0^T \int_U q_{r,u}(R(r,u)'\psi - X(r,u))^2 \mu(du)\lambda(dr) \\
&= \mathbb{E}\int_0^T \int_U \lim_{n\to\infty} q_{r,u}(R(r,u)'\psi_n - X(r,u))^2 \mu(du)\lambda(dr) \\
&\leq \liminf_{n\to\infty} \mathbb{E}\int_0^T \int_U q_{r,u}(R(r,u)'\psi_n - X(r,u))^2 \mu(du)\lambda(dr) = 0
\end{aligned}$$

Therefore, we have $X = R'\psi$. Thus, $\psi \mapsto R'\psi$ is sequentially closed and by the closed graph theorem this implies that it is continuous. Hence, $\psi \mapsto R'\psi$ belongs to $\mathcal{L}(\Psi, \Gamma^2_w(T))$. This in particular implies that the mapping Δ is well-defined.

Finally, the fact that Δ is linear follows easily from (3.65) and the fact that for any $a \in \mathbb{R}$, $R, S \in \Lambda^2_s(T)$, for every $(r,\omega,u) \in [0,T]\times\Omega\times U$, it follows that $aR(r,\omega,u)' + S(r,\omega,u)' = (aR(r,\omega,u) + S(r,\omega,u))'$.

Step 2. *The mapping* Δ *given by* (3.65) *is invertible.*

We start by proving that Δ is injective. As it is linear, it is sufficient (and necessary) to prove that $\text{Ker}(\Delta) = \{0\}$.

Let $R \in \Lambda^2_s(T)$ be such that $\Delta(R) = 0$. Then, $R(r,\omega,u)'\psi = 0$, for all $(r,\omega,u) \in [0,T]\times\Omega\times U$ and all $\psi \in \Psi$. Therefore, $R = 0$. Thus, $\text{Ker}(\Delta) = \{0\}$.

Now, to prove that Δ is surjective, let $S \in \mathcal{L}(\Psi, \Lambda^2_w(T))$. From Lemma 3.3.9, there exists a continuous Hilbertian semi-norm p on Ψ for which S has an extension $\tilde{S}$ such that $\tilde{S} \in \mathcal{L}_2(\Psi_p, \Lambda^2_w(T))$.

Moreover, as $\tilde{S}$ is Hilbert-Schmidt by Proposition B.0.16 there exists an orthonormal system $\{\psi^p_j\}_{j\in J}$ in Ψ_p, an orthonormal system $\{X_j\}_{j\in J}$ in $\Lambda^2_w(T)$ and a sequence of positive numbers $\{\gamma_j\}_{j\in J}$ satisfying $\sum_{j\in J}\gamma_j^2 < \infty$, with $J \subseteq \mathbb{N}$, such that $\tilde{S}$ admits the representation:

$$\tilde{S}\psi = \sum_{j\in J} \gamma_j \, p(\psi, \psi^p_j) X_j, \quad \forall\, \psi \in \Psi_p. \tag{3.67}$$

Choose a complete orthonormal system $\{\psi^p_j\}_{j\in\mathbb{N}}$ which is an extension of the orthonormal system $\{\psi^p_j\}_{j\in J}$. Then, from (3.67) we have

$$\tilde{S}\psi^p_j = \gamma_j X_j \text{ if } j \in J, \quad \text{and} \quad \tilde{S}\psi^p_j = 0 \text{ if } j \in \mathbb{N}\setminus J. \tag{3.68}$$

Now, from Parseval's identity and the fact that $\tilde{S} \in \mathcal{L}_2(\Psi_p, \Lambda^2_w(T))$ it follows that

$$\mathbb{E}\int_0^T \int_U \sum_{j\in\mathbb{N}} q_{r,u}((\tilde{S}\psi^p_j)(r,\omega,u))^2 \mu(du)\lambda(dr) = ||\tilde{S}||^2_{\mathcal{L}_2(\Psi_p,\Lambda^2_w(T))} < \infty. \tag{3.69}$$

Then, it follows from (3.68) and (3.69) that there exists $\Gamma \subseteq [0,T]\times\Omega\times U$, such that $(\lambda\otimes\mathbb{P}\otimes\mu)(\Gamma) = 1$ and

$$\sum_{j\in J} \gamma_j^2 q_{r,u}(X_j(r,\omega,u))^2 = \sum_{j\in\mathbb{N}} q_{r,u}((\tilde{S}\psi^p_j)(r,\omega,u))^2 < \infty, \quad \forall\,(r,\omega,u) \in \Gamma. \tag{3.70}$$

Let $F = \{F(r,\omega,u) : r \in [0,T], \omega \in \Omega, u \in U\}$, where for every $\psi \in \Psi$,

$$F(r,\omega,u)\psi = \begin{cases} (\tilde{S}\psi)(r,\omega,u), & \forall\,(r,\omega,u) \in \Gamma, \\ 0, & \forall\,(r,\omega,u) \in \Omega\setminus\Gamma. \end{cases} \tag{3.71}$$

Our objective is to prove that the family F satisfies the following properties:

(a) $F(r,\omega,u) \in \mathcal{L}_2(\Psi_p, \Phi_{q_{r,u}})$, for all $(r,\omega,u) \in [0,T]\times\Omega\times U$,

(b) The map $(r,\omega,u) \mapsto q_{r,u}(F(r,\omega,u)\psi, \phi)$ is $\mathcal{P}_T\otimes\mathcal{B}(U)$-measurable, for each $\phi \in \Phi$, $\psi \in \Psi$,

(c) $\mathbb{E}\int_0^T \int_U ||F(r,u)||^2_{\mathcal{L}_2(\Psi_p,\Phi_{q_{r,u}})} \mu(du)\lambda(dr) < \infty$.

To prove (a), first note that from (3.67) and (3.71), $F(r,\omega,u)$ is a linear operator from Ψ into $\Phi_{q_{r,u}}$, for all $(r,\omega,u)\in[0,T]\times\Omega\times U$.

Fix $(r,\omega,u)\in\Gamma$. Then, from (3.67), (3.71), the Cauchy-Schwarz inequality and Parseval's identity, it follows that for all $\psi\in\Psi_p$ we have

$$\begin{aligned} q_{r,u}(F(r,\omega,u)\psi)^2 &= q_{r,u}\left(\sum_{j\in J}\gamma_j\, p(\psi,\psi_j^p)X_j(r,\omega,u)\right)^2 \\ &\leq \left(\sum_{j\in J}p(\psi,\psi_j^p)^2\right)\cdot\left(\sum_{j\in J}\gamma_j^2\, q_{r,u}\left(X_j(r,\omega,u)\right)^2\right) \\ &\leq C\,p(\psi)^2 \end{aligned}$$

where $C=\sum_{j=1}^{\infty}\gamma_j^2 q_{r,u}\left(X_j(r,\omega,u)\right)^2<\infty$ by (3.70). Then, $F(r,\omega,u)$ is a continuous operator from Ψ_p into $\Phi_{q_{r,u}}$. Moreover, because $\{\psi_j^p\}_{j\in\mathbb{N}}$ is a complete orthonormal system in Ψ_p, (3.68), (3.70) and (3.71) show that $||F(r,\omega,u)||_{\mathcal{L}_2(\Psi_p,\Phi_{q_{r,u}})}<\infty$ and hence $F(r,\omega,u)\in\mathcal{L}_2(\Psi_p,\Phi_{q_{r,u}})$. As for $(r,\omega,u)\in\Omega\setminus\Gamma$ we have $F(r,\omega,u)=0$, from the above it follows that $F(r,\omega,u)\in\mathcal{L}_2(\Psi_p,\Phi_{q_{r,u}})$ for all $(r,\omega,u)\in[0,T]\times\Omega\times U$ and therefore we have proved (a).

For (b), fix $\psi\in\Psi$ and $\phi\in\Phi$. From (3.67) and (3.71), for all $(r,\omega,u)\in\Gamma$ we have

$$q_{r,u}(F(r,\omega,u)\psi,\phi)=\sum_{j\in J}\gamma_j\, p(\psi,\psi_j^p)q_{r,u}(X_j(r,\omega,u),\phi). \tag{3.72}$$

As for each $j\in\mathbb{N}$, $X_j\in\Lambda^2_w(T)$, then the map $(r,\omega,u)\mapsto q_{r,u}(X_j(r,\omega,u),\phi)$ is $\mathcal{P}_T$-measurable (see Definition 3.2.1). Putting this together with (3.72) implies that the map $(r,\omega,u)\mapsto q_{r,u}(F(r,\omega,u)\psi,\phi)$ is $\mathcal{P}_T$-measurable. So we have proved (b).

Finally, (3.69) and (3.71) imply that

$$\begin{aligned} &\mathbb{E}\int_0^T\int_U ||F(r,u)||^2_{\mathcal{L}_2(\Psi_p,\Phi_{q_{r,u}})}\,\mu(du)\lambda(dr) \\ &\qquad\qquad = \mathbb{E}\int_0^T\int_U\sum_{j\in J}q_{r,u}((\tilde{S}\psi_j^p)(r,u))^2\mu(du)\lambda(dr)<\infty. \end{aligned}$$

This proves (c).

Define $R=\{R(r,\omega,u):r\in[0,T],\omega\in\Omega,u\in U\}$ to be given by:

$$R(r,\omega,u)=F(r,\omega,u)',\quad\forall\,(r,\omega,u)\in[0,T]\times\Omega\times U. \tag{3.73}$$

then from the properties (a)-(c) above it follows that $R\in\Lambda^2_s(p,T)$ (see Definition 3.3.3) and hence by Proposition 3.3.8 we have $i'_pR\in\Lambda^2_s(T)$. Moreover, as $\tilde{S}$ is an extension of S, from (3.71) for every $\psi\in\Psi$ and $(r,\omega,u)\in\Gamma$, we have that

$$(i'_pR(r,\omega,u))'\psi=F(r,\omega,u)i_p\psi=(\tilde{S}i_p\psi)(r,\omega,u)=(S\psi)(r,\omega,u), \tag{3.74}$$

and then from (3.65) it follows that $S=\Delta(i'_pR)$. Therefore, the map Δ is surjective and hence it is an isomorphism. □

Corollary 3.3.11. *For every continuous semi-norm p on Ψ, the mapping Δ given by (3.65) defines an isometric isomorphism from $\Lambda^2_s(p,T)$ into $\mathcal{L}_2(\Psi_p,\Lambda^2_w(T))$.*

Proof. With some very light modifications on the arguments of Step 1 in the proof of Theorem 3.3.10, we can show that for $R \in \Lambda^2_s(p,T)$, we have $\Delta(R) \in \mathcal{L}(\Psi_p, \Lambda^2_w(T))$. Moreover, if $\{\psi_j\}_{j\in\mathbb{N}}$ is a complete orthonormal system in Ψ_p, then Fubini's theorem, (3.16) and the fact that $R \in \Lambda^2_s(p,T)$ implies that:

$$\begin{aligned} ||\Delta(R)||^2_{\mathcal{L}_2(\Psi_p,\Lambda^2_w(T))} &= \sum_{j=1}^{\infty} \left|\left|\Delta(R)\psi^p_j\right|\right|^2_{w,T} \qquad (3.75)\\ &= \sum_{j=1}^{\infty} \mathbb{E}\int_0^T \int_U q_{r,u}(R(r,u)'\psi^p_j)^2 \mu(du)\lambda(dr) \\ &= \mathbb{E}\int_0^T \int_U \sum_{j=1}^{\infty} q_{r,u}(R(r,u)'\psi^p_j)^2 \mu(du)\lambda(dr) \\ &= \mathbb{E}\int_0^T \int_U ||R(r,u)'||^2_{\mathcal{L}_2(\Psi_p,\Phi_{q_{r,u}})} \mu(du)\lambda(dr) \\ &= ||R||^2_{s,p,T} < \infty, \end{aligned}$$

and therefore $\Delta(R) \in \mathcal{L}_2(\Psi_p, \Lambda^2_w(T))$. Note that (3.75) indeed implies that Δ is an isometry from $\Lambda^2_s(p,T)$ into $\mathcal{L}_2(\Psi_p, \Lambda^2_w(T))$.

Finally, the arguments of Step 2 in the proof of Theorem 3.3.10 show that Δ is an isomorphism from $\Lambda^2_s(p,T)$ into $\mathcal{L}_2(\Psi_p, \Lambda^2_w(T))$. □

Corollary 3.3.12. *Let $R \in \Lambda^2_s(T)$. There exists a continuous Hilbertian semi-norm p on Ψ and $\tilde{R} \in \Lambda^2_s(p,T)$ such that $R(r,\omega,u) = i'_p\tilde{R}(r,\omega,u)$, for $\lambda\otimes\mathbb{P}\otimes\mu$-a.e. $(r,\omega,u) \in [0,T]\times\Omega\times U$.*

Moreover, if $H(\Psi)$ denotes the collection of all the continuous Hilbertian semi-norms on Ψ, then

$$\Lambda^2_s(T) = \bigcup_{p\in H(\Psi)} i'_p\Lambda^2_s(p,T).$$

Proof. First, from Step 1 of the proof of Theorem 3.3.10 we have that $\psi \mapsto R'\psi$ given in (3.65) is an element of $\mathcal{L}(\Psi, \Lambda^2_w(T))$. Then it follows from Step 2 of the proof of Theorem 3.3.10 that there exists a continuous Hilbertian semi-norm p on Ψ and there exists $\tilde{R}$ in $\Lambda^2_s(p,T)$ such that for $\lambda\otimes\mathbb{P}\otimes\mu$-a.e. $(r,\omega,u) \in [0,T]\times\Omega\times U$, $R(r,\omega,u)'\psi = (i'_p\tilde{R}(r,\omega,u))'\psi$ (note that this is (3.74) with S replaced by the map $\psi \mapsto R'\psi$).

To prove the second statement, note that as a consequence of the first statement we have $\Lambda^2_s(T) \subseteq \bigcup_{p\in H(\Psi)} i'_p\Lambda^2_s(p,T)$. Now, from Proposition 3.3.8 we have that $i'_p\Lambda^2_s(p_\gamma,T) \subseteq \Lambda^2_s(T)$, for each $p \in H(\Psi)$. Then, $\bigcup_{p\in H(\Psi)} i'_p\Lambda^2_s(p,T) \subseteq \Lambda^2_s(T)$. □

Now we proceed to introduce vector topologies on the space $\Lambda^2_s(T)$ of strong integrands. Note that as from Theorem 3.3.10 the spaces $\Lambda^2_s(T)$ and $\mathcal{L}(\Psi, \Lambda^2_w(T))$ are isomorphic (as vector spaces), then a natural way to introduce a topology on $\Lambda^2_s(T)$ is to equip it with one of the known topologies on $\mathcal{L}(\Psi, \Lambda^2_w(T))$ of uniform convergence in some families of bounded subsets of Ψ (see Section 1.1.5).

Recall that the topology of bounded (respectively simple) convergence on $\mathcal{L}(\Psi, \Lambda^2_w(T))$ is the locally convex topology generated by the following family of semi-norms:

$$S \to \sup_{\psi\in B} ||S\psi||_{w,T},$$

where B runs over the bounded (respectively finite) subsets of Ψ.

Then, by identifying each element R of $\Lambda^2_s(T)$ with the unique element $(\psi \mapsto R'\psi)$ in $\mathcal{L}(\Psi, \Lambda^2_w(T))$ given by (3.65), we introduce on $\Lambda^2_w(T)$ the **topologies of simple and bounded convergence**. A family of semi-norms generating the topology of bounded (respectively simple) convergence on $\Lambda^2_s(T)$ is then given by

$$R \to \sup_{\psi \in B} \left|\left| R'\psi \right|\right|_{w,T} = \sup_{\psi \in B} \left(\mathbb{E} \int_0^T \int_U q_{r,u}(R(r,u)'\psi)^2 \mu(du)\lambda(dr) \right)^{\frac{1}{2}}, \tag{3.76}$$

where B runs over the bounded (respectively finite) subsets of Ψ.

Proposition 3.3.13. *The space $\Lambda^2_s(T)$ is complete equipped with the topology of bounded convergence and quasi-complete equipped with the topology of simple convergence.*

Proof. The assertion follow from the corresponding properties of the topologies of bounded and simple convergence of the space $\mathcal{L}(\Psi, \Lambda^2_w(T))$. See Section 6, Chapter 39 of Kothë [62]. □

From Proposition 3.3.8, the spaces $\Lambda^2_s(p,T)$, where p ranges over the continuous Hilbertian semi-norms p on Ψ, are linear subspaces of $\Lambda^2_s(T)$. The following result shows that the Hilbert topology on each space $\Lambda^2_s(p,T)$ (see Proposition 3.3.8) is finer than the subspace topology induced on them by the topologies of simple and bounded convergence on $\Lambda^2_s(T)$.

Proposition 3.3.14. *Let p be a continuous Hilbertian semi-norm on Ψ. Let $\Lambda^2_s(T)$ be equipped with either the topology of simple or the topology of bounded convergence. Then, the inclusion map $i'_p : \Lambda^2_s(p,T) \to \Lambda^2_s(T)$, $R \mapsto i'_p R$, is linear and continuous.*

Proof. The linearity of the inclusion map is evident. To prove its continuity, let B be any bounded subset of Ψ. As p is continuous, there exists $C > 0$ such that $B \subseteq C B_p(1)$. Then, for any $R \in \Lambda^2_s(p,T)$ we have from (3.63), (3.64) and (3.76) that,

$$\begin{aligned} \sup_{\psi \in B} \left|\left| R' i_p \psi \right|\right|^2_{w,T} &\leq C^2 \sup_{\psi \in B_p(1)} \mathbb{E} \int_0^T \int_U q_{r,u}(R(r,u)' i_p \psi)^2 \mu(du)\lambda(dr) \\ &\leq C^2 \left(\sup_{\psi \in B_p(1)} p(\psi)^2 \right) \mathbb{E} \int_0^T \int_U \left|\left| i'_p R(r,u) \right|\right|^2_{\mathcal{L}_2(\Phi'_{q_{r,u}}, \Psi'_p)} \mu(du)\lambda(dr) \\ &= C^2 \left|\left| i'_p R \right|\right|^2_{s,p,T}. \end{aligned}$$

Then, the inclusion map $i'_p : \Lambda^2_s(p,T) \to \Lambda^2_s(T)$ is continuous. □

There exists an alternative way to introduce a vector topology on $\Lambda^2_s(T)$. Let $H(\Psi)$ denotes the collection of all the continuous Hilbertian semi-norms on Ψ, then from Corollary 3.3.12 we have $\Lambda^2_s(T) = \bigcup_{p \in H(\Psi)} i'_p \Lambda^2_s(p,T)$. Therefore, due to Propositions 3.3.7 and 3.3.8 the family $\{(\Lambda^2_s(p,T), i'_p) : p \in H(\Psi)\}$ is an inductive system on $\Lambda^2_s(T)$ and hence we can equip this space with the **inductive limit topology** with respect to this system (see Section 1.1.3). A local base of neighborhoods of zero for this topology is the collection of all convex, balanced, absorbing subsets U of $\Lambda^2_s(T)$ such that for each $p \in H(\Psi)$, $(i'_p)^{-1}(U)$ is a neighborhood of zero in $\Lambda^2_s(p,T)$.

Proposition 3.3.15. *Equipped with the inductive topology the space $\Lambda^2_s(T)$ is ultra-bornological and if Ψ is Fréchet then $\Lambda^2_s(T)$ is also complete. Moreover, the inductive limit topology on $\Lambda^2_s(T)$ is finer than the topologies of simple and bounded convergence*

Proof. The first assertion follows because $\Lambda^2_s(T)$ equipped with the inductive limit topology is by definition the inductive limit of the Hilbert spaces $\Lambda^2_s(p,T)$. If Ψ is Fréchet then the inductive limit can be equivalently defined with respect to an increasing sequence of continuous semi-norms $\{p_n\}_{n\in\mathbb{N}}$. But as each $\Lambda^2_s(p_n,T)$ is complete, this implies that $\Lambda^2_s(T)$ is also complete (see result 6.6, Chapter II of Schaeffer, p.59). Finally, by definition the inductive limit topology is the finest locally convex topology on $\Lambda^2_s(T)$ such that the inclusion map $i'_p : \Lambda^2_s(p,T) \to \Lambda^2_s(T)$ is continuous for all $p \in H(\Psi)$. Hence, Proposition 3.3.14 implies that the inductive limit topology on $\Lambda^2_s(T)$ is finer than the topologies of simple and bounded convergence. □

3.3.2 The Strong Stochastic Integral: Construction and Basic Properties

In this section we construct the strong stochastic integral for integrands in $\Lambda^2_s(T)$ and study some of its basic properties. We start by showing the existence of the strong integral for elements of the space $\Lambda^2_s(p,T)$.

Theorem 3.3.16. *Let $R \in \Lambda^2_s(p,T)$, where p is a continuous Hilbertian semi-norm on Ψ. Then, there exists a unique (up to indistinguishable versions) Ψ'_p-valued, zero-mean, square integrable, càdlàg martingale $I^s(R) = \{I^s_t(R)\}_{t\in[0,T]}$, such that for all $\psi \in \Psi_p$, $\mathbb{P}$-a.e.*

$$I^s_t(R)[\psi] = I^w_t(R'\psi), \quad \forall\, t \in [0,T], \tag{3.77}$$

where the stochastic process on the right-hand side of (3.77) *corresponds to the weak stochastic integral of $R'\psi \in \Lambda^2_w(T)$ defined in* (3.65). *Moreover, for all $t \in [0,T]$,*

$$\mathbb{E}\, p'(I^s_t(R))^2 = \mathbb{E}\int_0^t \int_U ||R(r,u)||^2_{\mathcal{L}_2(\Phi'_{q_{r,u}},\Psi'_p)}\, \mu(du)\lambda(dr). \tag{3.78}$$

Furthermore, $I^s(R)$ is mean-square continuous and has a predictable version.

Proof. First, from Corollary 3.3.11 we have $\Delta(R) \in \mathcal{L}_2(\Psi_p, \Lambda^2_w(T))$. Then, $\Delta(R)$ has a representation

$$\Delta(R)\psi = \sum_{j\in J} \gamma_j\, p(\psi,\psi^p_j) X_j, \quad \forall\, \psi \in \Psi_p. \tag{3.79}$$

where $\{\psi^p_j\}_{j\in J}$ and $\{X_j\}_{j\in J}$ are orthonormal systems in Ψ_p and $\Lambda^2_w(T)$ respectively, $\{\gamma_j\}_{j\in J}$ is a sequence of positive numbers satisfying $\sum_{j\in J}\gamma_j^2 < \infty$ and $J \subseteq \mathbb{N}$.
Let $\{\psi^p_j\}_{j\in\mathbb{N}}$ be a complete orthonormal system which is an extension of the orthonormal system $\{\psi^p_j\}_{j\in J}$. Then, from (3.79) we have

$$\Delta(R)\psi^p_j = \gamma_j X_j \text{ if } j \in J, \quad \text{and} \quad \Delta(R)\psi^p_j = 0 \text{ if } j \in \mathbb{N}\setminus J. \tag{3.80}$$

Now, for each $j \in \mathbb{N}$, let f^p_j be given by $f^p_j[\cdot] = p(\cdot,\psi^p_j)$. Then, $\{f^p_j\}_{j\in\mathbb{N}}$ is a complete orthonormal system in Ψ'_p that is in duality with $\{\psi^p_j\}_{j\in\mathbb{N}}$, i.e. $f^p_j[\psi^p_i] = \delta_{ij}$, for each $i,j \in \mathbb{N}$. Moreover, it follows from (3.79) and the continuity of the weak integral mapping $I^w : \Lambda^2_w(T) \to \mathcal{M}^2_T(\mathbb{R})$ (Theorem 3.2.8), that for each $t \in [0,T]$, $I^w \circ \Delta(R)$ has the representation:

$$(I^w \circ \Delta(R)\psi)_t = I^w_t(\Delta(R)\psi) = \sum_{j\in J} \gamma_j\, f^p_j[\psi] I^w_t(X_j), \quad \forall\, \psi \in \Psi_p,\, t \in [0,T]. \tag{3.81}$$

Now, note that from Fubini's theorem and (3.30), we have

$$\mathbb{E}\left(\sup_{t\in[0,T]}\sum_{j\in J}\gamma_j^2\,|I_t^w(X_j)|^2\right)\leq\sum_{j\in J}\gamma_j^2\,\mathbb{E}\left(\sup_{t\in[0,T]}|I_t^w(X_j)|^2\right)\leq 4T\sum_{j\in J}\gamma_j^2\,||X_j||_{w,T}^2<\infty, \tag{3.82}$$

where recall $||X_j||^2_{w,T}=1$, for all $j\in\mathbb{N}$ and $\sum_{j=1}^{\infty}\gamma_j^2<\infty$. Hence, it follows from the above that the set $\Omega_0=\left\{\omega\in\Omega:\sup_{t\in[0,T]}\sum_{j\in J}\gamma_j^2\,|I_t^w(X_j)|^2<\infty\right\}$ is such that $\mathbb{P}(\Omega_0)=1$.

Let $I^s(R)=\{I_t^s(R)\}_{t\in[0,T]}$ be defined for each $t\in[0,T]$ by:

$$I_t^s(R)(\omega)=\begin{cases}\sum_{j\in J}\gamma_j\,f_j^p I_t^w(X_j)(\omega), & \text{if }\omega\in\Omega_0,\\ 0, & \text{if }\omega\in\Omega\setminus\Omega_0.\end{cases} \tag{3.83}$$

From the definition of Ω_0 it follows that the sum in (3.83) converges in Φ'_p for every $\omega\in\Omega_0$. Hence, $I^s(R)$ is a Φ'_β-valued process. Moreover, from Parseval's identity, (3.82) and (3.83), we have

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]}p'(I_t^s(R))^2\right) &= \mathbb{E}\left(\sup_{t\in[0,T]}\sum_{k=1}^{\infty}\left|\sum_{j\in J}\gamma_j\,f_j^p[\psi_k^p]I_t^w(X_j)\right|^2\right)\\ &= \mathbb{E}\left(\sup_{t\in[0,T]}\sum_{j\in J}\gamma_j^2\,|I_t^w(X_j)|^2\right)<\infty.\end{aligned} \tag{3.84}$$

Therefore, $I^s(R)$ is square integrable. Moreover, from (3.83) and the fact that each $I^w(X_j)\in\mathcal{M}_T^2(\mathbb{R})$, it follows that $I^s(R)$ is also a Ψ'_p-valued zero-mean càdlàg martingale.

Now, for any $\psi\in\Psi_p$ it follows from (3.81) and (3.83) that for every $\omega\in\Omega_0$ we have

$$I_t^s(R)(\omega)[\psi]=\sum_{j\in J}\gamma_j\,f_j^p[\psi]I_t^w(X_j)(\omega)=I_t^w(\Delta(R)\psi)(\omega),\quad\forall\,t\in[0,T].$$

Then we have proved (3.77). The condition (3.77) and Proposition 1.2.15 shows that $I^s(R)$ is the unique (up to indistinguishable versions) Φ'_p-valued process satisfying the conditions on the statement of the theorem.

To prove (3.78), let $t\in[0,T]$. Then, from Parseval's identity, Fubini's theorem, (3.28) and (3.77) we have

$$\begin{aligned}\mathbb{E}\,p'(I_t^s(R))^2 &= \sum_{j=1}^{\infty}\mathbb{E}\left[\left|I_t^s(R)[\psi_j^p]\right|^2\right]\\ &= \sum_{j=1}^{\infty}\mathbb{E}\left[\left|I_t^w(R'\psi_j^p)\right|^2\right]\\ &= \sum_{j=1}^{\infty}\mathbb{E}\int_0^t\int_U q_{r,u}(R(r,u)'\psi_j^p)^2\mu(du)\lambda(dr)\\ &= \mathbb{E}\int_0^t\int_U ||R(r,u)||^2_{\mathcal{L}_2(\Phi'_{q_{r,u}},\Psi'_p)}\,\mu(du)\lambda(dr).\end{aligned}$$

This proves (3.78). Now, to show that $I^{s}(R)$ is mean-square continuous, note that from (3.78) it follows that for any $0 \leq s \leq t \leq T$ we have:

$$\mathbb{E}\left(p'(I^{s}_{s}(R)-I^{s}_{t}(R))^{2}\right) = \mathbb{E}\int_{s}^{t} ||R(r,u)||^{2}_{\mathcal{L}_{2}(\Phi'_{q_{r,u}},\Psi'_{p})}\, \mu(du)\lambda(dr) \leq ||R||^{2}_{s,p,T},$$

and hence from an application of the dominated convergence theorem we have

$$\mathbb{E}\left(p'(I^{s}_{s}(R)-I^{s}_{t}(R))^{2}\right) \to 0 \quad \text{as } s\to t, \text{ or } t \to s.$$

Thus, $I^{s}(R)$ is mean square continuous. Finally, as $I^{s}(R)$ is a Ψ'_{p}-valued, $\mathcal{F}_{t}$-adapted and stochastically continuous process it has a predictable version (see Proposition 3.21 of Peszat and Zabczyk [85], p.27). □

We now show the existence of the strong stochastic integral for elements of $\Lambda^{2}_{s}(T)$.

Theorem 3.3.17. *Let $R \in \Lambda^{2}_{s}(T)$. Then, there exists a unique (up to indistinguishable versions) Ψ'_{β}-valued, zero-mean, square integrable càdlàg martingale $I^{s}(R) = \{I^{s}_{t}(R)\}_{t\in[0,T]}$, such that for all $\psi \in \Psi$, $\mathbb{P}$-a.e.*

$$I^{s}_{t}(R)[\psi] = I^{w}_{t}(R'\psi), \quad \forall\, t \in [0,T], \tag{3.85}$$

where the stochastic process in the right-hand side of (3.85) corresponds to the weak stochastic integral of $R'\psi \in \Lambda^{2}_{w}(T)$ defined in (3.65).

Moreover, there exist a continuous Hilbertian semi-norm p on Ψ and $\tilde{R} \in \Lambda^{2}_{s}(p,T)$ such that $R(r,\omega,u) = i'_{p}\tilde{R}(r,\omega,u)$, for $\lambda\otimes\mathbb{P}\otimes\mu$-a.e. $(r,\omega,u) \in [0,T]\times\Omega\times U$ and such that $I^{s}_{t}(R) = i'_{p}I^{s}_{t}(\tilde{R})$ for all $t \in [0,T]$, where $I^{s}(\tilde{R})$ is the Ψ'_{p}-valued process defined in Theorem 3.3.16.

Proof. First, it follows from Corollary 3.3.12 that there exists a continuous Hilbertian semi-norm p on Ψ and $\tilde{R} \in \Lambda^{2}_{s}(p,T)$ such that $R(r,\omega,u) = i'_{p}\tilde{R}(r,\omega,u)$, for $\lambda\otimes\mathbb{P}\otimes\mu$-a.e. $(r,\omega,u) \in [0,T]\times\Omega\times U$.

Let $I^{s}(\tilde{R})$ be the Ψ'_{p}-valued process defined in Theorem 3.3.16. Define $I^{s}(R) = \{I^{s}_{t}(R)\}_{t\in[0,T]}$ for every $t \in [0,T]$ by $I^{s}_{t}(R) = i'_{p}I^{s}_{t}(\tilde{R})$. Then, from the properties of the process $I^{s}(\tilde{R})$ it follows that $I^{s}(R)$ is a Ψ'_{β}-valued, zero-mean, square integrable càdlàg martingale.

Let $\psi \in \Psi$. From the fact that i'_{p} is the dual operator of i_{p}, (3.65), (3.77) and the fact that $R = i'_{p}\tilde{R}$ $\lambda\otimes\mathbb{P}\otimes\mu$-a.e., it follows that $\mathbb{P}$-a.e. for all $t \in [0,T]$ we have

$$I^{s}_{t}(R)[\psi] = i'_{p}I^{s}(\tilde{R})[\psi] = I^{s}(\tilde{R})[i_{p}\psi] = I^{w}_{t}(\tilde{R}'i_{p}\psi) = I^{w}_{t}((i'_{p}\tilde{R})'\psi) = I^{w}_{t}(R'\psi).$$

This proves (3.85). Finally, (3.85) and Proposition 1.2.15 shows that $I^{s}(R)$ is the unique (up to indistinguishable versions) Φ'_{β}-valued process satisfying the conditions on the statement of the theorem. □

Proposition 3.3.18. *If for each $A \in \mathcal{R}$ and $\phi \in \Phi$, the real-valued process $(M(t,A)(\phi) : t \geq 0)$ is continuous, then for each $R \in \Lambda^{2}_{s}(p,T)$, the Ψ'_{p}-valued process $I^{s}(R)$ defined in Theorem 3.3.16 is continuous. Similarly, for each $P \in \Lambda^{2}_{s}(T)$ the Ψ'_{β}-valued process $I^{s}(P)$ defined in Theorem 3.3.17 is continuous.*

Proof. Let $R \in \Lambda^{2}_{s}(p,T)$. With the notation of the proof of Theorem 3.3.16, our assumption implies that for each $j \in J$, $I^{w}(X_{j})$ is a continuous process (see Proposition 3.2.10). Then, it follows from (3.83) that $I^{s}(R)$ is continuous.

If $P \in \Lambda^2_s(T)$ it follows from the second part of Theorem 3.3.17 that there exists a continuous Hilbertian semi-norm p on Ψ and $\tilde{P} \in \Lambda^2_s(p,T)$ such that $I^s(P)$ and $i'_p I^s(\tilde{P})$ are indistinguishable processes. But as $I^s(\tilde{P})$ is a Ψ'_p-valued continuous process and $i'_p : \Psi'_p \to \Psi'_\beta$ is continuous it follows that $I^s(P)$ is continuous. □

We are ready to define the strong stochastic integral.

Definition 3.3.19.

(1) Let p be a continuous Hilbertian semi-norm on Ψ. For $R \in \Lambda^2_s(p,T)$ let $I^s(R)$ be the Ψ'_p-valued process defined in Theorem 3.3.16. We call $I^s(R)$ the **strong stochastic integral** of R.

(2) For $R \in \Lambda^2_s(T)$ let $I^s(R)$ be the Ψ'_β-valued process defined in Theorem 3.3.17. We call $I^s(R)$ the **strong stochastic integral** of R. We will sometimes denote the stochastic integral $I^s(R)$ of R by $\left\{ \int_0^t \int_U R(r,u) M(dr,du) : t \in [0,T] \right\}$. The map $I^s : \Lambda^2_s(T) \to \mathcal{M}^2_T(\Psi'_\beta)$ given by $R \mapsto I^s(R)$, will be called the **strong integral mapping**.

Proposition 3.3.20. *For every continuous Hilbertian semi-norm p on Ψ and $R \in \Lambda^2_s(p,T)$, the two definitions of the strong stochastic integral given in Definition 3.3.19 are consistent, i.e. $I^s_t(i'_p R) = i'_p I^s_t(R)$ for all $t \in [0,T]$.*

Proof. Let $I^s(R)$ be the strong stochastic integral of R as given in Definition 3.3.19(1). From Proposition 3.3.8 it follows that $i'_p R \in \Lambda^2_s(T)$ and hence it has a strong stochastic integral $I^s(i'_p R)$ as given in Definition 3.3.19(2). But from the second part of Theorem 3.3.17 it follows that $I^s_t(i'_p R) = i'_p I^s_t(R)$ for all $t \in [0,T]$. □

Now we proceed to study some properties of the strong integral mapping.

Proposition 3.3.21. *The strong integral mapping $I^s : \Lambda^2_s(T) \to \mathcal{M}^2_T(\Psi'_\beta)$ is linear.*

Proof. Let $a \in \mathbb{R}$, $P, R \in \Lambda^2_s(T)$. As $aP + R \in \Lambda^2_s(T)$, from Theorem 3.3.17 the strong stochastic integral $I^s(aP+R)$ of $aP+R$ satisfies that $\forall\, \psi \in \Psi$, $\mathbb{P}$-a.e.

$$I^s_t(aP+R)[\psi] = I^w_t((aP+R)\psi), \quad \forall\, t \in [0,T].$$

Now, by the linearity of the weak integral and (3.85) for both P and R, for every $\psi \in \Psi$, $\mathbb{P}$-a.e. it follows that

$$I^s_t(aP+R)[\psi] = I^w_t((aP+R)\psi) = aI^w_t(P\psi) + I^w_t(R\psi) = aI^s_t(P)[\psi] + I^s_t(R)[\psi],$$

for all $t \in [0,T]$. Therefore, for each $\psi \in \Psi$ the process $I^s(aP+R)[\psi]$ is a version of $(aI^s(P) + J^s(R))[\psi]$. Then, it follows from Proposition 1.2.15 that $I^s(aP+R)$ and $aI^s(P) + I^s(R))$ are indistinguishable Ψ'_β-valued processes. Therefore, the map I^s is linear. □

Corollary 3.3.22. *Let p be a continuous Hilbertian semi-norm on Ψ. The strong integral mapping I^s restricts to a continuous and linear operator from $\Lambda^2_s(p,T)$ into $\mathcal{M}^2_T(\Psi'_p)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \Lambda^2_s(p,T) & \xrightarrow{\; i'_p \;} & \Lambda^2_s(T) \\ \Big\downarrow I^s & & \Big\downarrow I^s \\ \mathcal{M}^2_T(\Psi'_p) & \xrightarrow{\; i'_p \;} & \mathcal{M}^2_T(\Psi'_\beta) \end{array}$$

Proof. The fact that the strong integral mapping I^s restricts to a linear operator from $\Lambda^2_s(p,T)$ into $\mathcal{M}^2_T(\Psi'_p)$ such that the diagram above commutes is a consequence of Propositions 3.3.20 and 3.3.21.

Finally, from Doob's inequality, (1.15), (3.63) and (3.78), it follows that

$$||I^s(R)||_{\mathcal{M}^2_T(\Psi'_p)} \leq 2\sqrt{T}\,||R||_{s,p,T}\,, \quad \forall\, R \in \Lambda^2_s(p,T).$$

Therefore the the strong integral mapping is continuous as an operator from $\Lambda^2_s(p,T)$ into $\mathcal{M}^2_T(\Psi'_p)$. □

The next result shows that the strong integral map is also continuous from $\Lambda^2_s(T)$ into $\mathcal{M}^2_T(\Psi'_\beta)$. We will need the topologies on $\mathcal{M}^2_T(\Psi'_\beta)$ defined in Section 1.2.2 and the topologies on $\Lambda^2_s(T)$ defined in Section 3.3.1.

Proposition 3.3.23. *Let $\Lambda^2_s(T)$ and $\mathcal{M}^2_T(\Psi'_\beta)$ be equipped with either the topology of simple or the topology of bounded convergence. Then, the map $I^s : \Lambda^2_s(T) \rightarrow \mathcal{M}^2_T(\Psi'_\beta)$ is continuous.*

Proof. Let B be any bounded subset of Ψ. For any $R \in \Lambda^2_s(T)$, it follows from (3.30), (3.76) and (3.85) that

$$\sup_{\psi \in B} ||I^s(R)[\psi]||^2_{\mathcal{M}^2_T(\mathbb{R})} = \sup_{\psi \in B} \left|\left|I^w(R'\psi)\right|\right|^2_{\mathcal{M}^2_T(\mathbb{R})} \leq 4T \sup_{\psi \in B} \left|\left|R'\psi\right|\right|^2_{w,T}.$$

And hence I^s is continuous for $\Lambda^2_s(T)$ and $\mathcal{M}^2_T(\Psi'_\beta)$ equipped with either the topology of simple or of bounded convergence. □

Corollary 3.3.24. *Let $\Lambda^2_s(T)$ be equipped with the inductive limit topology and $\mathcal{M}^2_T(\Psi'_\beta)$ be equipped with either the topology of simple or the topology of bounded convergence. Then, the map $I^s : \Lambda^2_s(T) \rightarrow \mathcal{M}^2_T(\Psi'_\beta)$ is linear and continuous.*

Proof. Because the inductive limit topology on $\Lambda^2_s(T)$ is finer than the topologies of simple and bounded convergence (Proposition 3.3.15), then Proposition 3.3.23 implies that the strong integral map is also continuous for $\Lambda^2_s(T)$ equipped with the inductive limit topology. □

We finish this section with some important examples of applications of our theory of strong stochastic integration.

Example 3.3.25. Let Φ be a barrelled nuclear space and W be a generalized Wiener process in Φ'_β. Let M be the cylindrical martingale-valued measure defined in Example 3.1.4 by (3.3). Then, the space $\Lambda^2_s(T)$ is the collection (of equivalence classes) of families $R = \{R(r,\omega,0) : r \in [0,T], \omega \in \Omega, 0 \in \Psi'_\beta\}$ of operator-valued maps satisfying the conditions of Definition 3.3.1 with respect to the family of semi-norms $\{q_{r,0}\}$ defined in Example 3.1.4. From Example 3.2.11 it follows that the condition (3.61) takes the form

$$\mathbb{E}\int_0^T q_r(R(r,0)'\psi)^2 dr < \infty, \quad \forall\, \psi \in \Psi.$$

We denote by $\{\int_0^t R(r,0)dW(r) : t \in [0,T]\}$ the strong stochastic integral with respect to M; in view of Proposition 3.3.18 it is a continuous process.

Example 3.3.26. Let Φ be a complete, barrelled nuclear space and M be the Lévy martingale-valued measure defined in Example 3.1.5 by (3.6). Then, the space $\Lambda^2_s(T)$ is the collection (of equivalence classes) of families $R = \{R(r,\omega,f) : r \in [0,T], \omega \in \Omega, f \in B_{\rho'}(1)\}$ of operator-valued maps satisfying the conditions of Definition 3.3.1 with respect to the family of semi-norms $\{q_{r,f}\}$ defined in (3.8). From Example 3.2.12 it follows that the condition (3.61) takes the form

$$\mathbb{E}\int_0^T \int_{B_{\rho'}(1)} \left| f[R(r,f)'\psi]\right|^2 \nu(df) dr < \infty, \quad \forall \psi \in \Psi.$$

We denote by $\{\int_0^t \int_{B_{\rho'}(1)} R(r,f)\widetilde{N}(dr,df) : t \in [0,T]\}$ the strong stochastic integral with respect to M.

Example 3.3.27. Let Φ be a complete, barrelled nuclear space and M be the Lévy martingale-valued measure defined in Example 3.1.8 by (3.11). Then, the space $\Lambda^2_s(T)$ is the collection (of equivalence classes) of families $R = \{R(r,\omega,f) : r \in [0,T], \omega \in \Omega, f \in B_{\rho'}(1)\}$ of operator-valued maps satisfying the conditions of Definition 3.3.1 with respect to the family of semi-norms $\{q_{r,f}\}$ defined in (3.13). From Example 3.2.13 it follows that the condition (3.61) takes the form

$$\mathbb{E}\int_0^T \left(\mathcal{Q}(R(r,0)'\psi)^2 + \int_{B_{\rho'}(1)\setminus\{0\}} \left| f[R(r,f)'\psi]\right|^2 \nu(df) \right) dr < \infty, \quad \forall \psi \in \Psi.$$

Moreover, from Examples 3.3.25 and 3.3.26, and from the properties of the strong stochastic integral that we will show below in Proposition 3.3.31, for all $t \in [0,T]$ we have

$$\int_0^t \int_{B_{\rho'}(1)} R(r,f)M(dr,df) = \int_0^t R(r,0)dW(r) + \int_0^t \int_{B_{\rho'}(1)} R(r,f)\widetilde{N}(dr,df).$$

3.3.3 Some Properties of the Strong Stochastic Integral

In this section we prove some of the basic properties of the strong stochastic integral. Thanks to the relation between the strong and weak stochastic integrals given in (3.85), we will see that most of the properties of the weak integral can be transferred to the strong integral.

Proposition 3.3.28. *Let Υ be a quasi-complete, bornological, nuclear space and let $S \in \mathcal{L}(\Psi'_\beta, \Upsilon'_\beta)$. Then, for each $R \in \Lambda^2_s(\Psi, M; T)$, we have $S \circ R := \{S \circ R(r,\omega,u) : r \in [0,T], \omega \in \Omega, u \in U\} \in \Lambda^2_s(\Upsilon, M; T)$, and moreover $\mathbb{P}$-a.e., we have*

$$I^s_t(S \circ R) = S\left(I^s_t(R)\right), \quad \forall t \in [0,T]. \tag{3.86}$$

Proof. We have to prove that $S \circ R \in \Lambda^2_s(\Upsilon, M; T)$.

First, since for each $(r,\omega,u) \in [0,T] \times \Omega \times U$, we have $R(r,\omega,u) \in \mathcal{L}(\Phi'_{q_{r,u}}, \Psi'_\beta)$ and $S \in \mathcal{L}(\Psi'_\beta, \Upsilon'_\beta)$, it follows that $S \circ R(r,\omega,u) \in \mathcal{L}(\Phi'_{q_{r,u}}, \Upsilon'_\beta)$.

Now, let $\phi \in \Phi$ and $\upsilon \in \Upsilon$. As $S'\upsilon \in \Psi$, Definition 3.3.1(2) applied to R implies that the mapping $[0,T] \times \Omega \times U \to \mathbb{R}_+$ given by

$$(r,\omega,u) \mapsto q_{r,u}((S \circ R(r,\omega,u))'\upsilon, \phi) = q_{r,u}(R(r,\omega,u)'S'\upsilon, \phi),$$

is $\mathcal{P}_T \otimes \mathcal{B}(U)$-measurable.

Finally, as $S'\upsilon \in \Psi$ for every $\upsilon \in \Upsilon$, Definition 3.3.1(3) applied to R implies that

$$\mathbb{E}\int_0^T \int_U q_{r,u}((S\circ R(r,u))'\upsilon)^2 \mu(du)\lambda(dr) = \mathbb{E}\int_0^T \int_U q_{r,u}(R(r,u)'S'\upsilon)^2 \mu(du)\lambda(dr) < \infty,$$

for every $\upsilon \in \Upsilon$. Therefore, $S \circ R \in \Lambda^2_s(\Upsilon, M; T)$.

Now, note that (3.85) implies that for all $\upsilon \in \Upsilon$, for $\mathbb{P}$-a.e. $\omega \in \Omega$ we have

$$I^s_t(S\circ R)(\omega)[\upsilon] = I^w_t(R' \circ S'\upsilon)(\omega) = I^s_t(R)(\omega)[S'\upsilon] = S\left(I^s_t(R)(\omega)\right)[\upsilon], \quad \forall\, t \in [0,T],$$

where in the last step we have applied the definition of the dual operator S'. Therefore, we have that for all $\upsilon \in \Upsilon$, $I^s(S\circ R)[\upsilon] = S\left(I^s(R)\right)[\upsilon]$ are indistinguishable processes. Then, Proposition 1.2.15 shows that the Ψ'_β-valued processes $I^s(S\circ R)$ and $S\left(I^s(R)\right)$ are indistinguishable. This shows (3.86). □

Proposition 3.3.29. *Let $0 \leq s_0 < t_0 \leq T$ and $F_0 \in \mathcal{F}_{s_0}$. Then, for every $R \in \Lambda^2_s(T)$, $\mathbb{P}$-a.e. we have*

$$I^s_t(\mathbb{1}_{]s_0,t_0]\times F_0} R) = \mathbb{1}_{F_0}\left(I^s_{t\wedge t_0}(R) - I^s_{t\wedge s_0}(R)\right), \quad \forall\, t \in [0,T]. \tag{3.87}$$

Proof. Let $R \in \Lambda^2_s(T)$. Then, it is easy to see that $\mathbb{1}_{]s_0,t_0]\times F_0} R \in \Lambda^2_s(T)$ and hence its strong stochastic integral exists.

Now, let $\psi \in \Psi$. It follows from Theorem 3.3.10 that $R'\psi \in \Lambda^2_w(T)$. Then, from Proposition 3.2.15 there exists $\Gamma_\psi \subseteq \Omega$, such that $\mathbb{P}(\Gamma_\psi) = 1$ and for each $\omega \in \Gamma_\psi$,

$$I^w_t(\mathbb{1}_{]s_0,t_0]\times F_0} R'\psi)(\omega) = \mathbb{1}_{F_0}\left(I^w_{t\wedge t_0}(R'\psi)(\omega) - I^w_{t\wedge s_0}(R'\psi)(\omega)\right), \quad \forall\, t \in [0,T]. \tag{3.88}$$

On the other hand, it follows from (3.85) that there exists $\Omega_\psi \subseteq \Omega$, with $\mathbb{P}(\Omega_\psi) = 1$, such that for each $\omega \in \Omega_\psi$, we have

$$I^s_t(\mathbb{1}_{]s_0,t_0]\times F_0} R)(\omega)[\psi] = I^w_t(\mathbb{1}_{]s_0,t_0]\times F_0} R'\psi)(\omega), \quad \forall\, t \in [0,T], \tag{3.89}$$

$$I^s_{t\wedge t_0}(R)(\omega)[\psi] - I^s_{t\wedge s_0}(R)(\omega)[\psi] = I^w_{t\wedge t_0}(R'\psi)(\omega) - I^w_{t\wedge s_0}(R'\psi)(\omega), \quad \forall\, t \in [0,T]. \tag{3.90}$$

Let $\Theta_\psi = \Gamma_\psi \cap \Omega_\psi$. Then, $\mathbb{P}(\Theta_\psi) = 1$. Moreover, from (3.88), (3.89) and (3.90), for every $\omega \in \Theta_\psi$ it follows that

$$I^s_t(\mathbb{1}_{]s_0,t_0]\times F_0} R)(\omega)[\psi] = I^s_{t\wedge t_0}(R)(\omega)[\psi] - I^s_{t\wedge s_0}(R)(\omega)[\psi], \quad \forall\, t \in [0,T].$$

Then, for every $\psi \in \Psi$, $I^s(\mathbb{1}_{]s_0,t_0]\times F_0} R)[\psi]$ and $I^s_{\cdot\wedge t_0}(R)[\psi] - I^s_{\cdot\wedge s_0}(R)[\psi]$ are indistinguishable processes. But as the Ψ'_β-valued processes $I^s(\mathbb{1}_{]s_0,t_0]\times F_0} R)$ and $I^s_{\cdot\wedge t_0}(R) - I^s_{\cdot\wedge s_0}(R)$ as regular and càdlàg, it follows from Proposition 1.2.15 that they are indistinguishable. This shows (3.87). □

Proposition 3.3.30. *Let $R \in \Lambda^2_s(T)$ and σ be an $\{\mathcal{F}_t\}$-stopping time such that $\mathbb{P}(\sigma \leq T) = 1$. Then, $\mathbb{P}$-a.e.*

$$I^s_t(\mathbb{1}_{[0,\sigma]} R) = I^s_{t\wedge\sigma}(R), \quad \forall\, t \in [0,T]. \tag{3.91}$$

Proof. The proof follows from Proposition 3.2.14, Theorem 3.3.10 and similar arguments to those used in Proposition 3.3.29. □

Proposition 3.3.31. *Let N_1, N_2 be two independent nuclear Φ'_β-valued martingale-valued measures on $\mathbb{R}_+ \times \mathcal{R}$, each with covariance structure as in* (3.1) *determined by the family $\{p^j_{r,u}\}_{r,u}$ of continuous Hilbertian semi-norms on Φ and measures $\lambda_j = \lambda$, $\mu_j = \mu$, for $j = 1, 2$; all of them satisfying the conditions given in Definition 3.1.3.*

Let M be the nuclear Φ'_β-valued martingale-valued measure on $\mathbb{R}_+ \times \mathcal{R}$ defined by N_1 and N_2 as in Proposition 3.1.7. Let $\{q_{r,u}\}_{r,u}$ be the family of semi-norms determining the covariance structure (3.1).

Assume $R \in \Lambda^2_s(M;T)$. Then,

(1) For each $j = 1, 2$, $\{R(r, \omega, u)i'_{p^j_{r,u},q_{r,u}} : r \in [0, T], \omega \in \Omega, u \in U\} \in \Lambda^2_s(N_j;T)$, where for each $r \in [0, T]$ and $u \in U$, $i'_{p^j_{r,u},q_{r,u}}$ denotes the inclusion map from $\Phi'_{p^j_{r,u}}$ into $\Phi'_{q_{r,u}}$.

(2) $\mathbb{P}$-a.e., for all $t \in [0, T]$ we have,

$$\begin{aligned}\int_0^t \int_U R(r,u) M(dr,du) &= \int_0^t \int_U R(r,u) i'_{p^1_{r,u},q_{r,u}} N_1(dr,du) \\ &\quad + \int_0^t \int_U R(r,u) i'_{p^2_{r,u},q_{r,u}} N_2(dr,du).\end{aligned} \tag{3.92}$$

Proof. As in the proof of Proposition 3.2.16 we have from (3.10) that for every $r \in [0, T]$, $u \in U$, $p^j_{r,u} \leq q_{r,u}$, for each $j = 1, 2$. Hence, the inclusions $i'_{p^j_{r,u},q_{r,u}}$ are well-defined and are linear and continuous.

The proof of (1) follows from similar arguments to those in the proof of Proposition 3.2.16, using $i'_{p^j_{r,u},q_{r,u}} \in \mathcal{L}(\Phi'_{p^j_{r,u}}, \Phi'_{q_{r,u}})$.

The proof of (3.92) now follows from an application of the same arguments used in the proof of Propositions 3.3.29, using (3.85), and the fact that from Proposition 3.2.16 we have that for every $\psi \in \Psi$, $\mathbb{P}$-a.e. for all $t \in [0, T]$,

$$\begin{aligned}\int_0^t \int_U R(r,u)'\psi M(dr,du) &= \int_0^t \int_U i_{p^1_{r,u},q_{r,u}} R(r,u)'\psi N_1(dr,du) \\ &\quad + \int_0^t \int_U i_{p^2_{r,u},q_{r,u}} R(r,u)'\psi N_2(dr,du).\end{aligned}$$

and the fact that the dual operator of $i_{p^j_{r,u},q_{r,u}} R(r,u)'$ is $R(r,u) i'_{p^j_{r,u},q_{r,u}}$, $j = 1, 2$. ☐

3.3.4 Extension of the Class of Integrands

We now proceed to extend the strong stochastic integral to a larger class of integrands. The strong stochastic integral is defined by means of the regularization theorem.

Definition 3.3.32. Let $\Lambda_s(\Psi, M;T)$ denote the collection (of equivalence classes) of families $R = \{R(r, \omega, u) : r \in [0, T], \omega \in \Omega, u \in U\}$ of operator-valued maps satisfying the following conditions:

(1) $R(r, \omega, u) \in \mathcal{L}(\Phi'_{q_{r,u}}, \Psi'_\beta)$, for all $r \in [0, T]$, $\omega \in \Omega$, $u \in U$,

(2) R is $q_{r,u}$**-predictable**, i.e. for each $\phi \in \Phi$, $\psi \in \Psi$, the mapping $[0, T] \times \Omega \times U \to \mathbb{R}_+$ given by $(r, \omega, u) \mapsto q_{r,u}(R(r, \omega, u)'\psi, \phi)$ is $\mathcal{P}_T \otimes \mathcal{B}(U)$-measurable.

(3) For every $\psi \in \Psi$,

$$\mathbb{P}\left(\omega \in \Omega : \int_0^T \int_U q_{r,u}(R(r,\omega,u)'\psi)^2 \mu(du)\lambda(dr) < \infty\right) = 1. \tag{3.93}$$

Remark 3.3.33. *The class $\Lambda_{s}(\Psi, M; T)$ generalizes considerably the class of extended stochastic integrands in Bojdecki and Jakubowski [13] (see Definition 2.6 there). Indeed, to the extent of our knowledge $\Lambda_{s}(\Psi, M; T)$ is one of the largest classes of integrands considered in the literature of stochastic integration in duals of nuclear spaces.*

Again, when it is not necessary to give emphasis to the dependence of the space $\Lambda_{s}(\Psi, M; T)$ with respect to Ψ and M, we denote this space by $\Lambda_{s}(T)$. One can easily check that $\Lambda_{s}(T)$ is a linear space. Moreover, $\Lambda^{2}_{s}(T) \subseteq \Lambda_{s}(T)$.

We proceed to construct the strong stochastic integral for the integrands belonging to $\Lambda_{s}(T)$. We start with the following result that is the analogue of Theorem 3.3.10 for the elements of $\Lambda_{s}(T)$.

Theorem 3.3.34. *The mapping $\Delta' : \Lambda_{s}(T) \rightarrow \mathcal{L}(\Psi, \Lambda^{2,loc}_{w}(T))$ given by*

$$R \mapsto \left(\psi \mapsto R'\psi := \{ R(r,\omega,u)'\psi : r \in [0,T], \omega \in \Omega, u \in U \} \right), \tag{3.94}$$

is an injective linear operator.

Proof. The proof follows from similar arguments to those used in the proof of Theorem 3.3.10 and hence we will mention only the main points.

First, note that for every $R \in \Lambda^{2}_{s}(T)$ the properties listed in Definition 3.3.32 imply that the map $\psi \mapsto R'\psi$ from Ψ into $\Lambda^{2,loc}_{w}(T)$ is well-defined. Moreover, we can easily see that it is also linear; indeed this follows from the linearity of each operator $R(r,\omega,u)' \in \mathcal{L}(\Psi, \Phi_{q_{r,u}})$.

We need to prove that $\psi \mapsto R'\psi$ is also continuous. First, we can show that $\psi \mapsto R'\psi$ is sequentially closed, this by following similar arguments to those used in Step 2 of the proof of Theorem 3.3.10 but with the norm $||\cdot||_{w,T}$ there being replaced by the metric d_{Λ} defined in (3.37). Then, the closed graph theorem (Theorem 1.1.3) shows that $\psi \mapsto R'\psi$ is continuous. Therefore the mapping Δ' is well-defined. The proof that Δ' is linear and injective is exactly as in the proof of Theorem 3.3.10. □

Remark 3.3.35. *We do not know if the map Δ' defined in Theorem 3.3.34 is surjective. This is because as the space $\Lambda^{2,loc}_{w}(T)$ is not in general a Hilbert space or even a Banach space (indeed is not in general locally convex; see Remark 3.2.19), it is not clear how the arguments used in Step 2 of the proof of Theorem 3.3.10 can be modified for elements of $\mathcal{L}(\Psi, \Lambda^{2,loc}_{w}(T))$.*

We are ready to prove the existence of the extension of the strong stochastic integral to the elements of $\Lambda_{s}(T)$. This is carried out in the following result.

Theorem 3.3.36. *Let $R \in \Lambda_{s}(T)$. There exist a unique (up to indistinguishable versions) Ψ'_{β}-valued càdlàg locally zero-mean square integrable martingale $\hat{I}^{s}(R) = \{\hat{I}^{s}_{t}(R)\}_{t \in [0,T]}$, such that for all $\psi \in \Psi$, $\mathbb{P}$-a.e.*

$$\hat{I}^{s}_{t}(R)[\psi] = \hat{I}^{w}_{t}(R'\psi), \quad \forall \, t \in [0,T]. \tag{3.95}$$

where for each $\psi \in \Psi$, the stochastic process in the right-hand side of (3.95) *corresponds to the weak stochastic integral of $R'\psi \in \Lambda^{2,loc}_{w}(T)$.*

Proof. Let $R \in \Lambda_{s}(T)$. From the continuity of the extended weak integral map (Proposition 3.2.24) and Theorem 3.3.34, it follows that the map $\hat{I}^{w} \circ \Delta'(R) : \Psi \rightarrow \mathcal{M}^{2,loc}_{T}(\mathbb{R})$ is linear and continuous. As $\mathcal{M}^{2,loc}_{T}(\mathbb{R})$ is continuously and linearly injected in $D_T(\mathbb{R})$, then $\hat{I}^w \circ \Delta'(R) = \{\hat{I}^w_t \circ \Delta'(R)\}_{t\in[0,T]}$ is a cylindrical process in Ψ' such that the map $\psi \mapsto \hat{I}^w \circ \Delta'(R)$ from Ψ into $D_T(\mathbb{R})$ is continuous. Then, it follows from the regularization theorem (Theorem 1.2.18) that there exists a Ψ'_β-valued regular càdlàg process $\hat{I}^s(R) = \{\hat{I}^s_t(R)\}_{t\in[0,T]}$, such that for every $\psi \in \Psi$ the real-valued process $\hat{I}^s(R)[\psi]$ is a version of $\hat{I}^w \circ \Delta'(R)(\psi) = \hat{I}^w(R'\psi)$. But as for every $\psi \in \Psi$, the processes $\hat{I}^s(R)[\psi]$ and $\hat{I}^w \circ \Delta'(R)(\psi) = \hat{I}^w(R'\psi)$ are both càdlàg then they are indistinguishable. This shows (3.95). Moreover, as for each $\psi \in \Psi$, $\hat{I}^w(R'\psi) = \{\hat{I}^w_t(R'\psi)\}_{t\in[0,T]}$ is a càdlàg real-valued locally zero-mean square integrable martingale, (3.95) implies that $\hat{I}^s(R)$ is also a Ψ'_β-valued locally zero-mean square integrable martingale. Finally, the uniqueness of $\hat{I}^s(R)$ up to indistinguishable versions is a consequence of (3.95) and Proposition 1.2.15. □

Definition 3.3.37. For every $R \in \Lambda_s(T)$, we will call the process $\hat{I}^s(R)$ given in Theorem (3.3.36) as the **strong stochastic integral** of R. We will sometimes denote the process $\hat{I}^s(R)$ by $\left\{\int_0^t \int_U R(r,u) M(dr,du) : t \in [0,T]\right\}$.

From (3.95) and the properties of the weak stochastic integral for integrands in $\Lambda^{2,loc}_w(T)$ (see Proposition 3.2.22) we can show that the properties of the stochastic integral for integrands in $\Lambda^2_s(T)$ (see Section 3.3.3) are also satisfied for the strong stochastic integral for integrands in $\Lambda_s(T)$. We summarize this in the following result:

Proposition 3.3.38. *Let $R \in \Lambda_s(T)$. Then, all the assertions in Propositions 3.3.28, 3.3.29, 3.3.30 and 3.3.31 are true for the strong stochastic integral $\hat{I}^s(R)$ of R.*

The map $\hat{I}^s : \Lambda_s(T) \to \mathcal{M}^{2,loc}_T(\Psi'_\beta)$ given by $R \mapsto \hat{I}^s(R)$, will be called the **extended strong integral mapping**. Here recall that $\mathcal{M}^{2,loc}_T(\Psi'_\beta)$ denotes the space of Ψ'_β-valued càdlàg locally zero-mean square integrable martingales. By using (3.95) and the same arguments on the proof of Proposition 3.3.21 we can show the following result.

Proposition 3.3.39. *The extended strong integral mapping $\hat{I}^s : \Lambda_s(T) \to \mathcal{M}^{2,loc}_T(\Psi'_\beta)$ is linear.*

Chapter 4

Stochastic Evolution Equations in Duals of Nuclear Spaces

In this chapter we will apply the theory of stochastic integration introduced in Chapter 3 to the study of some classes of stochastic evolution equations taking values in the dual of a nuclear space Ψ and driven by cylindrical martingale-valued measure noise.

Stochastic evolution equations in the dual of a nuclear space has been considered by many authors. For example, Bojdecki and Gorostiza [9], [10], Bojdecki and Jakubowski [15], Dawson and Gorostiza [22], Ding [26], Fernández and Gorostiza [28], Gorostiza [33], Hitsuda and Mitoma [38], Ito [43], Kallianpur and Pérez-Abreu [52], Mitoma [74], Pérez-Abreu and Tudor [84], Üstünel [100], [107], Wu [115] and Walsh [111].

In all of these works, only equations with additive Wiener or square integrable martingale noise on the dual of a nuclear Fréchet space have been considered. The only exception is [26] where multiplicative noise with respect to Wiener processes is also studied. The class of stochastic evolution equations considered in this chapter generalizes all the above works (see (4.10)). We will consider both mild and weak solutions to these equations.

This chapter is organized as follows. In Section 4.1 we will introduce some results to define deterministic integrals for stochastic integrands. These classes of integrals will be necessary to provide an adequate definition for the deterministic integral occurring within mild solutions to our equations. In Section 4.2 we will give a detailed description of the class of stochastic evolution equations studied in this chapter. We will also provide details of the definitions of weak and mild solutions. Sufficient conditions for the equivalence between weak and mild solutions will be show in Section 4.3. Properties of the stochastic convolution will be studied in Section 4.4. Finally, in Section 4.5 we show the existence and uniqueness of weak and mild solutions under some Lipschitz and growth conditions.

§ 4.1 A Regularization Theorem for Deterministic Integrals

Throughout this section Ψ will be an ultrabornological nuclear space over $\mathbb{R}$.

In this section the objective is to introduce a new theory of regularization results for deterministic integrals of random integrands taking values in Ψ'_{β}. The reason why we need this theory will be clear in Section 4.2 where we define the deterministic convolution of a C_0-semigroup on Ψ'_{β} with a random function taking values on Ψ'_{β}.

The deterministic integral developed in this section can also be viewed as a substitute for the Bochner integral that is normally used to define integrals as described above in the context of a Banach space (see e.g. Da Prato and Zabczyk [20]).

For the proof of the next theorem we will need to recall some properties of absolutely continuous functions. For $t > 0$, let AC_t denotes the linear space of all absolutely continuous functions on $[0,t]$. It is well-known (see Theorem 5.3.6 of Bogachev [8], p.339, Vol I) that $G \in AC_t$ if and only if there exists an integrable function g defined on $[0,t]$ such that:

$$G(s) = G(0) + \int_0^s g(r)dr, \quad \forall s \in [0,t]. \tag{4.1}$$

The space AC_t is a Banach space equipped with the norm $||\cdot||_{AC_t}$ given by:

$$||G||_{AC_t} = |G(0)| + \int_0^t |g(r)|\, dr,$$

for $G \in AC_t$ with g satisfying (4.1).

Theorem 4.1.1. *Let $T > 0$ and let $X : [0,T] \times [0,T] \times \Omega \to \Psi'$ be such that*

(1) For each $t \in [0,T]$, the map $(r,\omega) \mapsto \mathbb{1}_{[0,t]}(r)\, X(t,r,\omega)[\psi]$ is $\mathcal{P}_t$-measurable, for all $\psi \in \Psi$.

(2) For each $t \in [0,T]$,

$$\mathbb{P}\left(\omega \in \Omega : \int_0^t |X(t,r,\omega)[\psi]|\, dr < \infty\right) = 1, \quad \forall\, \psi \in \Psi.$$

Then, there exists a Ψ'_β-valued regular process $\left\{\int_0^t X(t,r)dr : t \in [0,T]\right\}$ satisfying: for every $t \in [0,T]$ and $\psi \in \Psi$, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\left(\int_0^t X(t,r)dr\right)(\omega)[\psi] = \int_0^t X(t,r,\omega)[\psi]dr, \tag{4.2}$$

where for each $t \in [0,T]$ and every $\psi \in \Psi$, the integral on the right hand side of (4.2) is the Lebesgue integral of the real-valued function $r \mapsto X(t,r,\omega)[\psi]$ on $[0,t]$, that is defined for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Proof. For every $t \in [0,T]$, $\psi \in \Psi$, let $\Omega_{t,\psi} = \{\omega \in \Omega : \int_0^t |X(t,r,\omega)[\psi]|\, dr < \infty\}$. Then, from property *(2)* it follows that $\mathbb{P}(\Omega_{t,\psi}) = 1$.

Now for every $t \in [0,T]$, let $Z_t : \Psi \to L^0(\Omega, \mathscr{F}, \mathbb{P})$ be given for each $\psi \in \Psi$ by

$$Z_t(\psi)(\omega) := \begin{cases} \int_0^t X(t,r,\omega)[\psi]dr, & \text{for } \omega \in \Omega_{t,\psi}, \\ 0, & \text{elsewhere.} \end{cases} \tag{4.3}$$

Property *(1)* above and the definition of $\Omega_{t,\psi}$ imply that for each $\psi \in \Psi$, $Z_t(\psi) \in L^0(\Omega, \mathscr{F}, \mathbb{P})$ and hence Z_t is well-defined. Moreover, it is clear that Z_t is a cylindrical random variable. To prove the theorem, we need to show that each map $Z_t : \Psi \to L^0(\Omega, \mathscr{F}, \mathbb{P})$ is continuous. This is because in that case, from the regularization theorem (Theorem 1.2.14) there exists a Ψ'_β-valued regular random variable $\int_0^t X(t,r)dr$ that is a version of Z_t. This together with (4.3) implies (4.2).

Now we prove that $Z_t : \Psi \to L^0(\Omega, \mathscr{F}, \mathbb{P})$ is continuous. To do this we will need some preparations. First, note that from conditions *(1)* and *(2)* of X and a consequence of Fubini's theorem we have:

(a) $\forall \psi \in \Psi$, for all $\omega \in \Omega_{t,\psi}$, $\left\{ \int_0^s X(t,r,\omega)[\psi]dr : s \in [0,t] \right\} \in AC_t$
(b) $\forall \psi \in \Psi$, the map $\omega \mapsto \int_0^t |X(t,r,\omega)[\psi]| \, dr$ is $\mathcal{F}_t$-measurable.

Let $J_t : \Psi \mapsto L^0(\Omega, \mathcal{F}, \mathbb{P}; AC_t)$ given for every $\psi \in \Psi$ by

$$J_t(\psi)(\omega)(s) = \begin{cases} \int_0^s X(t,r,\omega)[\psi]dr, & \text{for } \omega \in \Omega_{t,\psi},\ s \in]0,t], \\ 0, & \text{elsewhere.} \end{cases} \tag{4.4}$$

To show that J_t is well-defined, note that from (a) above and (4.4) we have $J_t(\psi)(\omega) \in AC_t$ for all $\omega \in \Omega$. Moreover, (b) above shows that

$$\omega \mapsto ||J_t(\psi)(\omega)||_{AC_t} = \int_0^t |X(t,r,\omega)[\psi]| \, dr$$

is $\mathcal{F}_t$-measurable and hence is $\mathcal{F}$-measurable. Therefore, it follows that for each $\psi \in \Psi$, $J_t(\psi)$ is an AC_t-valued random variable. Therefore J_t is well-defined. It is also clear that J_t is linear.

Define the map $\Gamma_t : L^0(\Omega, \mathcal{F}, \mathbb{P}; AC_t) \to L^0(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\Gamma_t(Y)(\omega) = Y(\omega)(t), \quad \forall Y \in L^0(\Omega, \mathcal{F}, \mathbb{P}; AC_t). \tag{4.5}$$

The map Γ_t is clearly linear. Moreover, by (4.3), (4.4) and (4.5) it follows that $Z_t = \Gamma_t \circ J_t$. Therefore, to prove that Z_t is continuous, it is sufficient to prove that both J_t and Γ_t are continuous. We proceed to do this.

Claim 1: *The map J_t is continuous.*

We will prove first that J_t is sequentially closed. Let $\{\psi_n\}_{n \in \mathbb{N}}$ be a sequence converging to ψ in Ψ and let $Y \in L^0(\Omega, \mathcal{F}, \mathbb{P}, AC_t)$ be such that $||J_t(\psi_n) - Y||_{AC_t} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$. We have to prove that $J_t(\psi) = Y$.

Let $g : \Omega \to L^1([0,t], \mathcal{B}([0,t]), \text{Leb})$ be such that $\forall \omega \in \Omega$,

$$Y(\omega)(s) = Y(\omega)(0) + \int_0^s g(\omega)(r)dr, \quad \forall s \in [0,t]. \tag{4.6}$$

Such a g exists because $Y \in L^0(\Omega, \mathcal{F}, \mathbb{P}, AC_t)$.

Because $||J_t(\psi_n) - Y||_{AC_t} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$, there exist a subsequence $\{\psi_{n_k}\}_{k \in \mathbb{N}}$ and a subset Ω_0 of Ω with $\mathbb{P}(\Omega_0) = 1$ such that for each $\omega \in \Omega_0$:

$$\lim_{k \to \infty} ||J_t(\psi_{n_k})(\omega) - Y(\omega)||_{AC_t} = 0. \tag{4.7}$$

Note that (4.7) and the fact that $J_t(\psi_{n_k})(\omega)(0) = 0$, $\forall k \in \mathbb{N}$, $\omega \in \Omega$ implies that $Y(\omega)(0) = 0$, $\forall \omega \in \Omega_0$.

Now, the continuity of $X(t,r,\omega)$ on Ψ for each $(r,\omega) \in [0,t] \times \Omega$ implies that

$$\lim_{k \to \infty} X(t,r,\omega)[\psi_{n_k}] = X(t,r,\omega)[\psi], \quad \forall (r,\omega) \in [0,t] \times \Omega.$$

This, together with (4.7) and Fatou's Lemma implies that for every $\omega \in \Omega_0$ we have:

$$\begin{aligned} ||J_t(\psi)(\omega) - Y(\omega)||_{AC_t} &= \int_0^t |X(t,r,\omega)[\psi] - g(\omega)| \, dr \\ &= \int_0^t \lim_{k \to \infty} |X(t,r,\omega)[\psi_{n_k}] - g(\omega)| \, dr \\ &\leq \liminf_{k \to \infty} \int_0^t |X(t,r,\omega)[\psi_{n_k}] - g(\omega)| \, dr \\ &= \lim_{k \to \infty} ||J_t(\psi_{n_k})(\omega) - Y(\omega)||_{AC_t} = 0 \end{aligned}$$

Therefore, J_t is sequentially closed. Now, as Ψ is ultrabornological and AC_t is a Banach space, the closed graph theorem (Theorem 1.1.3) shows that J_t is continuous. This proves Claim 1.

Claim 2: The map Γ_t is continuous.

First, note that as the map Γ_t is linear, we only need to prove its continuity at zero. Let $Y \in L^0(\Omega, \mathcal{F}, \mathbb{P}; AC_t)$ and let $g : \Omega \to L^1([0,t], \mathcal{B}([0,t]), \text{Leb})$ satisfying (4.6). Then, from the definition of Γ_t and of the norm $||\cdot||_{AC_t}$ we have that for every $\omega \in \Omega$,

$$|\Gamma_t(Y)(\omega)| = |Y(\omega)(t)| \leq |Y(\omega)(0)| + \int_0^t |g(\omega)(r)|\, dr = ||Y(\omega)||_{AC_t}.$$

Therefore, for every $\epsilon > 0$, the above inequality implies that

$$\mathbb{P}\left(\omega \in \Omega : |\Gamma_t(Y)(\omega)| > \epsilon\right) \leq \mathbb{P}\left(\omega \in \Omega : ||Y(\omega)||_{AC_t} > \epsilon\right),$$

and this is sufficient to prove the continuity of Γ_t at zero and hence it is continuous. This proves Claim 2.

Then, from Claims 1 and 2 we have that Z_t is continuous. From our arguments at the beginning of the proof, this completes the proof of the theorem. □

Corollary 4.1.2. *Let $X : [0,T] \times [0,T] \times \Omega \to \Psi'$ be such that*

(1) For each $t \in [0,T]$, the map $(r,\omega) \mapsto X(t,r,\omega)[\psi]$ is $\mathcal{P}_t$-measurable, for all $\psi \in \Psi$.

(2) There exists $n \in \mathbb{N}$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t X(t,r)[\psi] dr\right|^n\right] < \infty, \quad \forall\, \psi \in \Psi$$

(3) For every $\psi \in \Psi$, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the map $t \mapsto \int_0^t X(t,r,\omega)[\psi] dr$ is continuous.

Then, there exist a continuous Hilbertian semi-norm q on Ψ and a Ψ'_q-valued, $\{\mathcal{F}_t\}$-adapted and continuous process $\left\{\int_0^t X(t,r)dr : t \in [0,T]\right\}$ with

$$\mathbb{E}\left[\sup_{t\in[0,T]} q'\left(\int_0^t X(t,r)dr\right)^n\right] < \infty. \tag{4.8}$$

and satisfying: for each $\psi \in \Psi$, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\left(\int_0^t X(t,r)dr\right)(\omega)[\psi] = \int_0^t X(t,r,\omega)[\psi] dr, \quad \forall\, t \in [0,T], \tag{4.9}$$

where for each $t \in [0,T]$ and every $\psi \in \Psi$, the integral on the right hand side of (4.9) is the Lebesgue integral of the real-valued function $r \mapsto X(t,r,\omega)[\psi]$ on $[0,t]$, that is defined for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Moreover, the process $\left\{\int_0^t X(t,r,\omega)dr : t \in [0,T], \omega \in \Omega\right\}$ has a predictable version.

Proof. First, note that the property *(2)* of this corollary implies the property *(2)* of Theorem 4.1.1. Therefore, from Theorem 4.1.1 there exists a Ψ'_β-valued regular process $\left\{\int_0^t X(t,r)dr : t \in [0,T]\right\}$ satisfying (4.2) $\mathbb{P}$-a.e. $\omega \in \Omega$ for all $\psi \in \Psi$.

Now, the properties *(2)* and *(3)* of this corollary and Theorem 1.2.24 imply the existence of a continuous Hilbertian semi-norm q on Ψ and of a Ψ'_q-valued continuous version of $\left\{\int_0^t X(t,r,\cdot)dr : t \in [0,T]\right\}$ (which we denote by the same notation) satisfying (4.8). For fixed $\psi \in \Psi$, the fact that the processes in both sides of (4.2) are continuous implies that they are indistinguishable (as each one is a version of the other), therefore this implies that (4.9) is valid. Also, note that as in the proof of Theorem 4.1.1 the property *(1)* of this corollary implies that for each $t \in [0,T]$ and $\psi \in \Psi$, the map $\omega \mapsto \int_0^t X(t,r,\omega)[\psi]dr$ is $\mathcal{F}_t$-measurable. Therefore, as $\mathcal{C}(\Psi'_q) = \mathcal{B}(\Psi'_q)$, the process $\left\{\int_0^t X(t,r,\cdot)dr : t \in [0,T]\right\}$ is $\{\mathcal{F}_t\}$-adapted. Moreover, as this process is also continuous and Ψ'_q is a separable Hilbert space, then it has a predictable version (see Proposition 3.21 of Peszat and Zabczyk [85], p.27). □

§ 4.2 Stochastic Evolution Equations: The General Setting

Note 4.2.1. From now on we will make an intensive use of the properties of C_0-semigroups in a nuclear space and its strong dual. For a review of the relevant facts about the theory of C_0-semigroups in locally convex spaces the reader is referred to Appendix D.

In this section we will introduce the general model of stochastic evolution equations in the dual of a nuclear space driven by a nuclear cylindrical martingale-valued measure. Let Φ be a locally convex space and Ψ be a quasi-complete, bornological, nuclear space, both defined over $\mathbb{R}$. Let U be a topological space. We are concerned with the following class of stochastic evolution equations

$$dX_t = (A'X_t + B(t,X_t))dt + \int_U F(t,u,X_t)M(dt,du), \quad \text{for } t \geq 0, \tag{4.10}$$

where we will assume the following:

Assumption 4.2.2.

(A1) *A is the infinitesimal generator of a $(C_0,1)$-semi-group $\{S(t)\}_{t\geq 0}$ on Ψ.*

(A2) *M is a nuclear cylindrical martingale-valued measure on $\mathbb{R}_+ \times \mathcal{R}$, where $\mathcal{R}$ is a ring $\mathcal{R} \subseteq \mathcal{B}(U)$ that generates the Borel σ-algebra $\mathcal{B}(U)$ of the topological space U, and the covariance of M is determined by the measure $\lambda = Leb$ on $\mathbb{R}_+$, a σ-finite Borel measure μ on U, and the semi-norms $\{q_{r,u} : r \in \mathbb{R}_+, u \in U\}$; all satisfying the conditions in Definition 3.1.3 and Assumption 3.1.9.*

(A3) *$B : \mathbb{R}_+ \times \Psi' \to \Psi'$ is such that the map $(r,g) \mapsto B(r,g)[\psi]$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\Psi'_\beta)$-measurable, for every $\psi \in \Psi$.*

(A4) *$F = \{F(r,u,g) : r \in \mathbb{R}_+, u \in U, g \in \Psi'\}$ is such that*

- *(a) $F(r,u,g) \in \mathcal{L}(\Phi'_{q_{r,u}}, \Psi'_\beta)$, $\forall r \geq 0$, $u \in U$, $g \in \Psi'$.*
- *(b) The mapping $(r,u,g) \mapsto q_{r,u}(F(r,u,g)'\phi, \psi)$ is $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(U)\otimes\mathcal{B}(\Psi'_\beta)$-measurable, for every $\phi \in \Phi$, $\psi \in \Psi$.*

Note that Ψ being reflexive (Theorem 1.1.7(2)), assumption (A1) implies that A' (the dual operator of A) is the infinitesimal generator of the dual semi-group $\{S(t)'\}_{t\geq 0}$ and this last is a C_0-semigroup on Ψ'_β (see Theorem D.2.7).

Remark 4.2.3. *It is well known that the solutions of stochastic evolutions equations in infinite dimensional spaces are not in general càdlàg, for that reason instead of considering equations with left limits in the right hand side of* (4.10) *we require only* *that our solution be predictable (see Definitions 4.2.5 and 4.2.7). For a more detailed discussion on this the reader is referred to Section 9.2.1 of Peszat and Zabczyk [85].*

Remark 4.2.4. *The use of* $(C_0, 1)$*-semi-groups for the study of stochastic evolution equations in duals of nuclear spaces has its origins in the work of Kallianpur and Pérez-Abreu [52] where they considered such semigroups on a nuclear Fréchet space. Indeed, the authors considered the more general context of* $(C_0, 1)$*-reversed evolution systems. Again in the framework of nuclear Fréchet spaces, Ding [26] also used* $(C_0, 1)$*-semi-groups to study stochastic evolution equations. He assumed that the dual semigroup is* $(C_0, 1)$*, with a more restrictive hypothesis that there exists a family of Hilbertian semi-norms generating the nuclear topology on* Ψ'_β *such that these semi-norms satisfy the conditions of Theorem D.2.4.*

We are interested in to studying weak and mild solutions to (4.10). The precise formulation of these types of solutions is given below.

Definition 4.2.5. A Ψ'_β-valued regular and predictable process $X = \{X_t\}_{t\geq 0}$ is called a **weak solution** to (4.10) if

(a) For every $t > 0$, X, B and F satisfy the following conditions:

$$\mathbb{P}\left(\omega \in \Omega : \int_0^t |X_r(\omega)[\psi]| \, dr < \infty\right) = 1, \quad \forall \psi \in \Psi.$$

$$\mathbb{P}\left(\omega \in \Omega : \int_0^t |B(r, X_r(\omega))[\psi]| \, dr < \infty\right) = 1, \quad \forall \psi \in \Psi.$$

$$\mathbb{P}\left(\omega \in \Omega : \int_0^t \int_U q_{r,u}(F(r, u, X_r(\omega))'\psi)^2 \mu(du) dr < \infty\right) = 1, \quad \forall \psi \in \Psi.$$

(b) For every $\psi \in \text{Dom}(A)$ and every $t \geq 0$, $\mathbb{P}$-a.e.

$$\begin{aligned} X_t[\psi] &= X_0[\psi] + \int_0^t (X_r[A\psi] + B(r, X_r)[\psi]) dr \\ &+ \int_0^t \int_U F(r, u, X_r)'\psi M(dr, du), \end{aligned} \tag{4.11}$$

where the first integral in the right-hand side of (4.11) is a Lebesgue integral that is defined for each $\psi \in \Psi$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. The second integral in the right-hand side of (4.11) is the weak stochastic integral of $F'\psi = \{F(r, u, X_r(\omega))'\psi : r \in [0, t], \omega \in \Omega, u \in U\} \in \Lambda^{2,loc}_w(t)$, and is well-defined for all $\psi \in \Psi$.

Proposition 4.2.6. *The assumptions (A1)-(A4) together with the conditions (a) of Definition 4.2.5 are sufficient to guarantee the existence of all the integrals in* (4.11).

Proof. We start with the deterministic integral. Fix $\psi \in \Psi$. The fact that X is predictable together with (A3), implies that the map $(r, \omega) \mapsto (X_r(\omega)[A\psi] + B(r, X_r(\omega))[\psi])$ is $\mathcal{P}_\infty$-measurable. Then, condition (a) of Definition 4.2.5 implies that the above map is Lebesgue integrable on $[0, \infty)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Now we prove that the stochastic integral is well-defined. To do this, fix $\psi \in \Psi$. Then, the fact that X is predictable together with (A4) implies that $F(r, u, X_r)'\psi \in \Phi'_{q_{r,u}}$, for each $r \geq 0$, $\omega \in \Omega$ and $u \in U$, and that the map $(r, \omega, u) \mapsto q_{r,u}(F(r, u, X_r)'\psi, \phi)$ is $\mathcal{P}_\infty \otimes \mathcal{B}(U)$-measurable, for every $\phi \in \Phi$, $\psi \in \Psi$. The above properties and condition (a) of Definition 4.2.5 imply that $\{F(r, u, X_r(\omega))'\psi : r \in [0, t], \omega \in \Omega, u \in U\} \in \Lambda^{2,loc}_w(t)$ (see Definition 3.2.17) and hence from Theorem 3.2.20 the weak stochastic integral $\int_0^t \int_U F(r, u, X_r)'\psi M(dr, du)$ exists for every $t \geq 0$. □

Definition 4.2.7. A Ψ'_{β}-valued regular and predictable process $X = \{X_t\}_{t \geq 0}$ is called a **mild solution** to (4.10) if

(a) For every $t \geq 0$, for all $\psi \in \Psi$,

$$\mathbb{P}\left(\omega \in \Omega : \int_0^t \left| S(t-r)' B(r, X_r(\omega))[\psi] \right| dr < \infty \right) = 1.$$

$$\mathbb{P}\left(\omega \in \Omega : \int_0^t \int_U q_{r,u}(F(r,u,X_r(\omega))' S(t-r)\psi)^2 \mu(du) dr < \infty \right) = 1.$$

(b) For every $t \geq 0$, $\mathbb{P}$-a.e.

$$X_t = S(t)' X_0 + \int_0^t S(t-r)' B(r, X_r) dr + \int_0^t \int_U S(t-r)' F(r,u,X_r) M(dr, du), \tag{4.12}$$

where the first integral at the right-hand side of (4.12) is a Ψ'_{β}-valued regular, $\{\mathcal{F}_t\}$-adapted process $\left\{ \int_0^t S(t-r)' B(r, X_r) dr : t \geq 0 \right\}$ such that for all $t \geq 0$ and $\psi \in \Psi$, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\left(\int_0^t S(t-r)' B(r, X_r(\omega)) dr \right) [\psi] = \int_0^t S(t-r)' B(r, X_r(\omega))[\psi] dr, \tag{4.13}$$

where for each $t \geq 0$, $\psi \in \Psi$, the integral on the right-hand side of (4.13) is the Lebesgue integral of the function $\mathbb{1}_{[0,t]}(\cdot)\, S(t-\cdot)' B(\cdot, X_{\cdot}(\omega))[\psi]$ defined on $[0,t]$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. The second integral at the right-hand side of (4.12) is the strong stochastic integral of $\{ \mathbb{1}_{[0,t]}(r)\, S(t-r)' F(r,u,X_r(\omega)) : r \in [0,t], \omega \in \Omega, u \in U \}$.

Proposition 4.2.8. *The assumptions (A1)-(A4) together with the conditions (a) of Definition 4.2.7 are sufficient to guarantee the existence of all the integrals in* (4.12).

Proof. We start with the existence of the process $\left\{ \int_0^t S(t-r)' B(r, X_r) dr : t \geq 0 \right\}$. Fix $t \geq 0$. We need to show that the conditions (1)-(2) of Theorem 4.1.1 are satisfied for the map $X : [0,t] \times [0,t] \times \Omega \to \Psi'$ given by

$$X(s,r,\omega) = \mathbb{1}_{[0,s]}(r)\, S(s-r)' B(r, X_r(\omega)), \quad \text{for } (s,r,\omega) \in [0,t] \times [0,t] \times \Omega. \tag{4.14}$$

From the arguments on the proof of Proposition 4.2.6 it follows that the map $(r,\omega) \mapsto B(r, X_r(\omega))[\psi]$ is $\mathcal{P}_{\infty}$-measurable, for every $\psi \in \Psi$. Then, for any $s \in [0,t]$, the continuity of the map $r \mapsto \mathbb{1}_{[0,s]}(r)\, S(s-r)\psi$, implies that the map

$$(r,\omega) \mapsto X(s,r,\omega) = \mathbb{1}_{[0,s]}(r)\, B(r, X_r(\omega))[S(s-r)\psi],$$

is $\mathcal{P}_s$-measurable, for all $\psi \in \Psi$. Hence, X satisfies the condition (1) of Theorem 4.1.1. On the other hand, the condition (a) of Definition 4.2.7 is exactly the condition (2) of Theorem 4.1.1 for X defined by (4.14). Therefore, Theorem 4.1.1 implies the existence of the process $\left\{ \int_0^t S(t-r)' B(r, X_r) dr : t \geq 0 \right\}$ satisfying the conditions of Definition 4.2.7.

For the stochastic integral, we have to check that for each $t \geq 0$, the integrand is an element of $\Lambda_s(t)$ (Definition 3.3.32).

Fix $t \geq 0$. Let $R = \{R(r,\omega,u)\}$ be given by

$$R(r,\omega,u) = S(t-r)' F(r,u,X_r(\omega)), \quad \forall\, r \in [0,t],\, \omega \in \Omega,\, u \in U.$$

It is clear that $R(r,\omega,u) \in \mathcal{L}(\Phi'_{q_{r,u}}, \Psi'_{\beta})$, for each $r \in [0,t]$, $\omega \in \Omega$, $u \in U$.

Next, we need to check that R is $q_{r,u}$-predictable (see Definition 3.3.32). Recall from Remark 4.2.6 that the map $(r,\omega,u) \mapsto q_{r,u}(F(r,u,X_r)'\varphi, \phi)$ is $\mathcal{P}_\infty \otimes \mathcal{B}(U)$-measurable, for every $\phi \in \Phi$, $\varphi \in \Psi$. Then, by the continuity of the map $r \mapsto S(t-r)\psi$ for $r \in [0,t]$ and fixed $\psi \in \Psi$, it follows that the map

$$(r,\omega,u) \mapsto q_{r,u}(R(r,\omega,u)'\psi, \phi) = q_{r,u}(F(r,u,X_r(\omega))'S(t-r)\psi, \phi),$$

defined on $[0,t] \times \Omega \times U$ is $\mathcal{P}_t \otimes \mathcal{B}(U)$-measurable for each $\psi \in \Psi$.

Finally, condition (a) of Definition 4.2.7 implies that R satisfies (3.93). Therefore, $R \in \Lambda_s(t)$ and hence Theorem 3.3.36 shows the existence of the stochastic integral $\int_0^t \int_U S(t-r)'F(r,u,X_r)M(dr,du)$. Moreover, from (3.95) the following holds for all $\psi \in \Psi$, $t \in [0,T]$, $\mathbb{P}$-a.e.

$$\int_0^t \int_U S(t-r)'F(r,u,X_r)M(dr,du)[\psi] = \int_0^t \int_U F(r,u,X_r)'S(t-r)\psi M(dr,du). \tag{4.15}$$

□

§ 4.3 Equivalence Between Mild and Weak Solutions

In this section we provide sufficient conditions for the equivalence between mild and weak solutions. The main result of this section is the following:

Theorem 4.3.1. *Let $X = \{X_t\}_{t \geq 0}$ be a Ψ'_β-valued regular and predictable process and assume that for every $T > 0$, X, B and F satisfy:*

$$\mathbb{E} \int_0^T |X_r[\psi]|\, dr < \infty, \quad \forall\, \psi \in \Psi. \tag{4.16}$$

$$\mathbb{E} \int_0^T |B(r,X_r)[\psi]|\, dr < \infty, \quad \forall \psi \in \Psi. \tag{4.17}$$

$$\mathbb{E} \int_0^T \int_U q_{r,u}(F(r,u,X_r)'\psi)^2 \mu(du)dr < \infty, \quad \forall \psi \in \Psi. \tag{4.18}$$

Then, X is a weak solution to (4.10) *if and only if it is a mild solution to* (4.10).

For our proof we benefit from ideas taken from the proofs in Da Prato and Zabczyk [20] and Peszat and Zabczyk [85] of the equivalence between weak and mild solutions on separable Hilbert spaces, and the proof of Gorajski [32] of equivalence between mild and weak solutions of some classes of stochastic evolution equations in UMD Banach spaces.

To prove Theorem 4.3.1 we will need to make some technical preparations that for convenience of the reader we have decided to distribute into the following three lemmas.

Lemma 4.3.2. *Let $X = \{X_t\}_{t \geq 0}$ be a Ψ'_β-valued regular and predictable process and assume that F satisfies* (4.18). *For every $\psi \in Dom(A)$ and $t > 0$, the following identities holds $\mathbb{P}$-a.e.*

$$\begin{aligned}\int_0^t \left(\int_0^s \int_U F(r,u,X_r)'S(t-s)A\psi M(dr,du) \right) ds \\ = \int_0^t \int_U F(r,u,X_r)'S(t-r)\psi M(dr,du) - \int_0^t \int_U F(r,u,X_r)'\psi M(dr,du)\end{aligned} \tag{4.19}$$

$$\int_0^t \left(\int_0^s \int_U S(s-r)' F(r,u,X_r) M(dr,du)[A\psi] \right) ds$$
$$= \int_0^t \int_U F(r,u,X_r)' S(t-r)\psi M(dr,du) - \int_0^t \int_U F(r,u,X_r)'\psi M(dr,du) \quad (4.20)$$

Proof. Fix $\psi \in \text{Dom}(A)$ and $t > 0$. To show that the above equalities holds, we will make use of the stochastic Fubini theorem (Theorem 3.2.29) applied to the following families of Hilbert-space valued maps:

$$Y_1(r,\omega,u,s) = \mathbb{1}_{[0,s]}(r)\, F(r,u,X_r(\omega))' S(t-s)A\psi,$$
$$Y_2(r,\omega,u,s) = \mathbb{1}_{[0,s]}(r)\, F(r,u,X_r(\omega))' S(s-r)A\psi.$$

for $r \in [0,t]$, $\omega \in \Omega$, $u \in U$ and $s \in [0,t]$.

First, we need to verify that both Y_1 and Y_2 belong to $\Xi^{1,2}_w(t,[0,t])$ (see Definition 3.2.25) for $E = [0,t]$, $\mathcal{E} = \mathcal{B}([0,t])$, $\varrho = \text{Leb}$.

We start by proving that Y_1 satisfies the conditions of Definition 3.2.25. First, as $S(t-s)A\psi \in \Psi$, $\forall\, s \in [0,t]$, by (A4)(a) it follows that $Y_1(r,\omega,u,s) \in \Phi_{q_{r,u}}$ for $(r,\omega,u,s) \in [0,t] \times \Omega \times U \times [0,t]$.

Now, let $\phi \in \Phi$. From the strong continuity of the semigroup $\{S(t)\}_{t\geq 0}$ it follows that the map $[0,t] \ni s \mapsto S(t-s)A\psi \in \Psi$ is continuous and therefore Borel measurable. This fact together with (A4)(b) and the predictability of X implies that the mapping

$$(r,\omega,u,s) \mapsto \mathbb{1}_{[0,s]}(r)\, q_{r,u}(F(r,u,X_r(\omega))' S(t-s)A\psi, \phi),$$

is $\mathcal{P}_t \otimes \mathcal{B}(U) \otimes \mathcal{B}([0,t])$-measurable.

For the final part, note that (A4), the predictability of X and (4.18) implies that $\{F(r,u,X_r(\omega)) : r \in [0,t], \omega \in \Omega, u \in U\} \in \Lambda^2_s(t)$. Then, from Theorem 3.3.17 there exists a continuous Hilbertian semi-norm p on Ψ and $\tilde{F}_X = \{\tilde{F}_X(r,\omega,u) : r \in [0,t], \omega \in \Omega, u \in U\} \in \Lambda^2_s(p,t)$ such that $F(r,u,X_r(\omega)) = i'_p \tilde{F}_X(r,\omega,u)$, for $\text{Leb} \otimes \mathbb{P} \otimes \mu$-a.e. $(r,\omega,u) \in [0,t] \times \Omega \times U$. Moreover, as $\tilde{F}_X \in \Lambda^2_s(p,t)$ then (see (3.62))

$$\left|\left| \tilde{F}_X \right|\right|^2_{s,p,t} = \mathbb{E} \int_0^t \int_U \left|\left| \tilde{F}_X(r,u)' \right|\right|^2_{\mathcal{L}_2(\Psi_p,\Phi_{q_{r,u}})} \mu(dr)dr < \infty. \quad (4.21)$$

Now, as $\{S(t)\}_{t\geq 0}$ is a $(C_0,1)$-semigroup on Ψ and p is a continuous semi-norm on Ψ, from Theorem D.2.4 there exists a continuous semi-norm q on Ψ, $p \leq q$ and there exists a C_0-semigroup $\{S_q(t)\}_{t\geq 0}$ on the Banach space Ψ_q such that

$$S_q(t) i_q \varphi = i_q S(t)\varphi, \quad \forall\, \varphi \in \Psi,\, t \geq 0. \quad (4.22)$$

Moreover, there exist $M_q \geq 1$, $\theta_q \geq 0$ such that

$$q(S_q(t) i_q \varphi) \leq M_q e^{\theta_q t} q(i_q \varphi), \qquad \forall\, \varphi \in \Psi,\, t \geq 0. \quad (4.23)$$

Note that as $p \leq q$ the canonical inclusion $i_{p,q} : \Psi_q \to \Psi_p$ is a continuous and linear operator.

Now, from the fact that $\tilde{F}_X \in \Lambda^2_s(p,t)$ and from (4.22), it follows that for $\text{Leb} \otimes \mathbb{P} \otimes \mu \otimes \text{Leb}$-a.e. (r,ω,u,s),

$$\begin{aligned} Y_1(r,\omega,u,s) &= \mathbb{1}_{[0,s]}(r)\, F(r,u,X_r(\omega))' S(t-s)A\psi \\ &= \mathbb{1}_{[0,s]}(r)\, \tilde{F}_X(r,\omega,u)' i_p S(t-s)A\psi \\ &= \mathbb{1}_{[0,s]}(r)\, \tilde{F}_X(r,\omega,u)' i_{p,q} S_q(t-s) i_q A\psi. \end{aligned}$$

Therefore from (4.23) it follows that for $\text{Leb} \otimes \mathbb{P} \otimes \mu \otimes \text{Leb}$-a.e. (r, ω, u, s),

$$\begin{aligned} &q_{r,u}(Y_1(r,\omega,u,s))^2 \\ &\le \mathbb{1}_{[0,s]}(r) \left|\left|\tilde{F}_X(r,\omega,u)'\right|\right|^2_{\mathcal{L}_2(\Psi_p,\Phi_{q_{r,u}})} ||i_{p,q}||^2_{\mathcal{L}(\Psi_q,\Psi_p)} M_q^2 e^{2\theta_q(t-s)} q(i_q A\psi)^2. \end{aligned}$$

From this last inequality and (4.21), it follows that

$$\begin{aligned} \int_0^t ||Y_1(\cdot,\cdot,\cdot,s)||_{w,t}\, ds &= \int_0^t \left(\mathbb{E} \int_0^t \int_U q_{r,u}(Y_1(r,\omega,u,s))^2 \mu(du) dr \right)^{1/2} ds \\ &\le M_q e^{\theta_q T} q(i_q A\psi) \, ||i_{p,q}||_{\mathcal{L}(\Psi_q,\Psi_p)} \left|\left|\tilde{F}_X\right|\right|_{s,p,t} < \infty. \end{aligned}$$

Therefore, Y_1 satisfies all the conditions of Definition 3.2.25 and hence $Y_1 \in \Xi^{1,2}_w(t,[0,t])$. By similar reasoning we find that Y_2 also satisfies all the conditions of Definition 3.2.25 and hence we have $Y_2 \in \Xi^{1,2}_w(t,[0,t])$.

We now prove (4.19) and (4.20). First, note that for all $r \in [0,t]$, $u \in U$, from a change of variable (Proposition D.1.2), Proposition D.1.2(3) and Theorem D.2.5(2), the following identity holds $\mathbb{P}$-a.e.

$$\begin{aligned} \int_0^t \mathbb{1}_{[0,s]}(r) \, F(r,u,X_r)' S(t-s) A\psi ds &= \int_0^{t-r} F(r,u,X_r)' S(s) A\psi ds \qquad (4.24) \\ &= F(r,u,X_r)' \int_0^{t-r} S(s) A\psi ds \\ &= F(r,u,X_r)'(S(t-r)\psi - \psi). \end{aligned}$$

Similarly, for all $r \in [0,t]$, $u \in U$, from Proposition D.1.2(3) and Theorem D.2.5(2) we have $\mathbb{P}$-a.e.

$$\begin{aligned} \int_0^t \mathbb{1}_{[0,s]}(r) \, F(r,u,X_r)' S(s-r) A\psi ds &= F(r,u,X_r)' \int_r^t S(s-r) A\psi ds \quad (4.25) \\ &= F(r,u,X_r)'(S(t-r)\psi - \psi). \end{aligned}$$

To prove (4.19), note that from the stochastic Fubini theorem applied to Y_1, (4.24) and the linearity of the weak stochastic integral, we have $\mathbb{P}$-a.e.

$$\begin{aligned} &\int_0^t \left(\int_0^s \int_U F(r,u,X_r)' S(t-s) A\, \psi M(dr,du) \right) ds \\ &= \int_0^t \int_U \left(\int_0^t \mathbb{1}_{[0,s]}(r) \, F(r,u,X_r)' S(t-s) A\, \psi ds \right) M(dr,du) \\ &= \int_0^t \int_U F(r,u,X_r)' S(t-r) \psi M(dr,du) - \int_0^t \int_U F(r,u,X_r)' \psi M(dr,du). \end{aligned}$$

Thus, we showed (4.19). Similarly, to prove (4.20), from (4.15) (where ψ is there replaced by $A\psi$), the stochastic Fubini theorem applied to Y_2, (4.25) and the linearity of the weak stochastic integral, we have $\mathbb{P}$-a.e.

$$\begin{aligned}
&\int_0^t \left(\int_0^s \int_U S(s-r)' F(r,u,X_r) M(dr,du)[A\psi] \right) ds \\
&= \int_0^t \left(\int_0^s \int_U F(r,u,X_r)' S(s-r) A\, \psi M(dr,du) \right) ds \\
&= \int_0^t \int_U \left(\int_0^t \mathbb{1}_{[0,s]}(r)\, F(r,u,X_r)' S(s-r) A\, \psi ds \right) M(dr,du) \\
&= \int_0^t \int_U F(r,u,X_r)' S(t-r) \psi M(dr,du) - \int_0^t \int_U F(r,u,X_r)' \psi M(dr,du).
\end{aligned}$$

Hence, we have showed (4.20). □

Lemma 4.3.3. *Let X be a Ψ'_β-valued regular and predictable process and assume that X and B satisfy* (4.16) *and* (4.17)*. Then, for each $T>0$ there exists a continuous Hilbertian semi-norm ϱ on Ψ such that*

$$\mathbb{E} \int_0^T \varrho'(X_r) dr < \infty, \tag{4.26}$$

$$\mathbb{E} \int_0^T \varrho'(B(r,X_r)) dr < \infty. \tag{4.27}$$

Proof. Let $T>0$. We start by showing the existence of the semi-norm for X.
Let $\sigma(\cdot) = \frac{1}{T}\text{Leb}(\cdot)$, where Leb denotes the Lebesgue measure on $[0,T]$. Then $([0,T]\times \Omega, \mathcal{P}_T, \sigma \otimes \mathbb{P})$ is a complete probability space. The predictability of X implies that the map $X_T : \Psi \to L^1([0,T]\times\Omega, \mathcal{P}_T, \sigma\otimes\mathbb{P})$, given by

$$X_T(\psi)(r,\omega) = T X_r(\omega)[\psi], \quad \forall \psi \in \Psi,\ (r,\omega) \in [0,T]\times\Omega, \tag{4.28}$$

is well defined and is linear. An application of Fatou's lemma shows that it is sequentially closed and then the closed graph theorem (Theorem 1.1.3) implies that it is also continuous. Then, the regularization Theorem (Theorem 1.2.14) shows that X_T possesses a version, that we denote again by X_T, which is a Ψ'_β-valued regular random variable defined on the probability space $([0,T]\times\Omega, \mathcal{P}_T, \sigma\otimes\mathbb{P})$. Note that (4.16) and (4.28) show that X_T satisfies:

$$\int_{[0,T]\times\Omega} |X_T(r,\omega)[\psi]| \, (\sigma\otimes\mathbb{P})(d(r,\omega)) = \mathbb{E} \int_0^T |X_r[\psi]| \, dr < \infty, \quad \forall \psi\in\Psi. \tag{4.29}$$

Now, from (4.29) and Theorem 1.2.24 (by identifying the random variable X_T with the Ψ'_β-valued regular process $\{X_T(t)\}_{t\in[0,1]}$ given for each $t\in[0,1]$, by $X_T(t)(r,\omega) = X_T(r,\omega)$, for all $(r,\omega)\in[0,T]\times\Omega$) there exists a continuous Hilbertian semi-norm p on Ψ such that X_T has a Ψ'_p-valued version, again denoted by X_T, that is defined on the probability space $([0,T]\times\Omega, \mathcal{P}_T, \sigma\otimes\mathbb{P})$. Moreover, $\int_{[0,T]\times\Omega} p'(X_T(r,\omega))\sigma(dr)\otimes\mathbb{P} < \infty$. But, as X_T is a version of the map defined in (4.28), then the above integrability property and (4.28) implies that $\mathbb{E}\int_0^T p'(X_r)dr < \infty$.
Now, following the same arguments as above but considering the map $B_T : \Psi \to L^1([0,T]\times\Omega, \mathcal{P}_T, \sigma\otimes\mathbb{P})$, given by

$$B_T(\psi)(r,\omega) = T B(r, X_r(\omega))[\psi], \quad \forall \psi\in\Psi,\ (r,\omega)\in[0,T]\times\Omega, \tag{4.30}$$

we see that there exists some continuous Hilbertian semi-norm q on Ψ such that $\mathbb{E}\int_0^T q'(B(r,X_r))dr < \infty$ is satisfied.

Finally, choose ϱ such that $p \leq \varrho$, $q \leq \varrho$. Then, from (4.28) and (4.30) we have

$$\begin{aligned}\mathbb{E}\int_0^T \varrho'(X_r)dr &= \int_{[0,T]\times\Omega} \varrho'(i'_{p,\varrho}X_T(r,\omega))(\sigma\otimes\mathbb{P})(d(r,\omega)) \\ &\leq ||i'_{p,\varrho}||_{\mathcal{L}(\Psi'_p,\Psi'_\rho)} \int_{[0,T]\times\Omega} p'(X_T(r,\omega))(\sigma\otimes\mathbb{P})(d(r,\omega)) < \infty,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\int_0^T \varrho'(B(r,X_r))dr &= \int_{[0,T]\times\Omega} \varrho'(i'_{q,\varrho}B_T(r,\omega))(\sigma\otimes\mathbb{P})(d(r,\omega)) \\ &\leq ||i'_{q,\varrho}||_{\mathcal{L}(\Psi'_q,\Psi'_\rho)} \int_{[0,T]\times\Omega} q'(B_T(r,\omega))(\sigma\otimes\mathbb{P})(d(r,\omega)) < \infty.\end{aligned}$$

So we have proved (4.26) and (4.27). □

Lemma 4.3.4. *Let X be a Ψ'_β-valued regular and predictable process and assume that X and B satisfy (4.16) and (4.17). For each $\psi \in Dom(A)$ and $t > 0$, the following equalities holds $\mathbb{P}$-a.e.*

$$\int_0^t \left(\int_0^s X_r[AS(t-s)A\,\psi]dr\right)ds = \int_0^t X_r[S(t-r)A\,\psi]dr - \int_0^t X_r[A\,\psi]dr, \quad (4.31)$$

$$\int_0^t \left(\int_0^s B(r,X_r)[S(t-s)A\,\psi]dr\right)ds = \int_0^t B(r,X_r)[S(t-r)\psi]dr - \int_0^t B(r,X_r)[\psi]dr, \quad (4.32)$$

$$\int_0^t \left(\int_0^s S(s-r)'B(r,X_r)dr[A\,\psi]\right)ds = \int_0^t B(r,X_r)[S(t-r)\psi]dr - \int_0^t B(r,X_r)[\psi]dr. \quad (4.33)$$

Proof. Fix $\psi \in \mathrm{Dom}(A)$ and $t \geq 0$. We start by showing (4.31). First, we need to prove that the integrals exist. Note that the predictability of X and the strong continuity of the semigroup $\{S(t)\}_{t\geq 0}$ implies that all the integrands in (4.31) are $\mathcal{P}_t$-measurable.

Now, let ϱ be a continuous Hilbertian semi-norm on Ψ satisfying the conditions in Lemma 4.3.3 (with $T = t$). As in the proof of Lemma 4.3.2, because $\{S(t)\}_{t\geq 0}$ is a $(C_0,1)$-semigroup on Ψ and ϱ is a continuous semi-norm on Ψ, there exists a continuous semi-norm q on Ψ, $\varrho \leq q$, a C_0-semigroup $\{S_q(t)\}_{t\geq 0}$ on the Banach space Ψ_q satisfying (4.22), and there exist $M_q \geq 1$, $\theta_q \geq 0$ such that $\{S_q(t)\}_{t\geq 0}$ satisfies (4.23). Note that as $\varrho \leq q$ the canonical inclusion $i_{\varrho,q} : \Psi_q \to \Psi_\varrho$ is a continuous and linear operator. Then, from (4.22), (4.23) and (4.26), it follows that

$$\begin{aligned}&\mathbb{E}\int_0^t |X_r[S(t-r)A\psi]|\,dr \quad (4.34)\\ &\leq \mathbb{E}\int_0^t \varrho'(X_r)\varrho(i_\varrho S(t-r)A\psi)dr \\ &\leq \mathbb{E}\int_0^t \varrho'(X_r)\,||i_{\varrho,q}||_{\mathcal{L}(\Psi_q,\Psi_\varrho)}\, q(S_q(t-r)i_qA\psi)dr \\ &\leq M_q e^{\theta_q t} q(i_qA\psi)\,||i_{\varrho,q}||_{\mathcal{L}(\Psi_q,\Psi_\varrho)}\,\mathbb{E}\int_0^t \varrho'(X_r)dr < \infty.\end{aligned}$$

Let $\varphi = A\psi$. In a similar way to before, we have

$$\begin{aligned}
&\mathbb{E}\int_0^t \left(\int_0^s |X_r[AS(t-s)A\psi]|\, dr\right) ds \qquad (4.35)\\
&\leq \mathbb{E}\int_0^t \left(\int_0^s \varrho'(X_r)\varrho(i_\varrho S(t-r)A\varphi) dr\right) ds\\
&\leq \mathbb{E}\int_0^t \left(\int_0^s \varrho'(X_r)\, ||i_{\varrho,q}||_{\mathcal{L}(\Psi_q,\Psi_\varrho)}\, q(S_q(t-r)i_q A\varphi) dr\right) ds\\
&\leq tM_q e^{\theta_q t} q(i_q A\varphi)\, ||i_{\varrho,q}||_{\mathcal{L}(\Psi_q,\Psi_\varrho)}\, \mathbb{E}\int_0^t \varrho'(X_r)dr < \infty,
\end{aligned}$$

Then, from (4.16), (4.34) and (4.35) it follows that all the integrals in (4.31) exist for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Now we proceed to prove that (4.31) holds. First, by the (deterministic) Fubini's theorem (that we can apply due to (4.35)), and then a change of variable we have for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\begin{aligned}
&\int_0^t \left(\int_0^s X_r(\omega)[AS(t-s)A\psi] dr\right) ds \qquad (4.36)\\
&= \int_0^t \left(\int_r^t X_r(\omega)[AS(t-s)A\psi] ds\right) dr\\
&= \int_0^t \left(\int_0^{t-r} X_r(\omega)[AS(s)A\psi] ds\right) dr.
\end{aligned}$$

For a fixed $r \in [0,t]$, from the definition of dual semi-group $\{S(t)'\}_{t\geq 0}$ and dual generator A', Theorem D.1.2(2) and Theorem D.2.5(2), we have for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\begin{aligned}
\int_0^{t-r} X_r(\omega)[AS(s)A\psi] ds) &= \int_0^{t-r} S(s)'A'X_r(\omega)[A\psi] ds \qquad (4.37)\\
&= \left(\int_0^{t-r} S(s)'A'X_r(\omega) ds\right)[A\psi]\\
&= \big(S(t-r)'X_r(\omega) - X_r(\omega)\big)[A\psi].
\end{aligned}$$

and hence, substituting (4.37) into (4.36), we get that (4.31) holds $\mathbb{P}$-a.e.

To prove (4.32) and (4.33), as before, we need to check that all the integrals there exist. First, the predictability of X, the measurability properties of B in Assumption (A3) and the strong continuity of the semi-group $\{S(t)\}_{t\geq 0}$ implies that all the integrands in (4.32) and (4.33) are $\mathcal{P}_t$-measurable (see the proof of Proposition 4.2.8).

Now, following similar arguments to those used in (4.35), and using (4.27), we have

$$\begin{aligned}
&\mathbb{E}\int_0^t \left(\int_0^s |B(r,X_r)[S(t-s)A\psi]|\, dr\right) ds \qquad (4.38)\\
&\leq tM_q e^{\theta_q t} q(i_q A\psi)\, ||i_{\varrho,q}||_{\mathcal{L}(\Psi_q,\Psi_\varrho)}\, \mathbb{E}\int_0^t \varrho'(B(r,X_r))dr < \infty.
\end{aligned}$$

Hence, (4.38) shows that the integrals in the left hand side of (4.32) exists for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Now, for fixed $s \in [0,T]$, from (4.13) and the definition of dual semi-group $\{S(t)'\}_{t\geq 0}$, we have $\mathbb{P}$-a.e.

$$\begin{aligned}\int_0^s S(s-r)'B(r,X_r)dr[A\psi] &= \int_0^s S(s-r)'B(r,X_r)[A\psi]dr \\ &= \int_0^s B(r,X_r)[S(s-r)A\psi]dr.\end{aligned} \tag{4.39}$$

Then, from (4.39), (4.27) and following similar arguments to those used in (4.35) we have

$$\begin{aligned}&\mathbb{E}\int_0^t \left|\int_0^s S(s-r)'B(r,X_r)dr[A\psi]\right| ds \\ &= \mathbb{E}\int_0^t \left|\int_0^s B(r,X_r)[S(s-r)A\psi]dr\right| ds \\ &\leq \mathbb{E}\int_0^t \left(\int_0^s |B(r,X_r)[S(s-r)A\psi]|\, dr\right) ds \\ &\leq tM_q e^{\theta_q t} q(i_q A\psi)\, ||i_{\varrho,q}||_{\mathcal{L}(\Psi_q,\Psi_\varrho)}\, \mathbb{E}\int_0^t \varrho'(B(r,X_r))dr < \infty.\end{aligned} \tag{4.40}$$

Therefore, from (4.40) the integral in the right-hand side of (4.33) exists for $\mathbb{P}$-a.e. $\omega \in \Omega$.

Likewise, following similar arguments to those used in (4.34) and from (4.27), we have

$$\begin{aligned}&\mathbb{E}\int_0^t |B(r,X_r)[S(t-r)\psi]|\, dr \\ &\leq M_q e^{\theta_q t} q(i_q\psi)\, ||i_{\rho,q}||_{\mathcal{L}(\Psi_q,\Psi_\rho)}\, \mathbb{E}\int_0^t \rho'(B(r,X_r))dr < \infty.\end{aligned} \tag{4.41}$$

Therefore, (4.17) and (4.41) shows that the integrals in the right hand side of both (4.32) and (4.33) exists for $\mathbb{P}$-a.e. $\omega \in \Omega$. Hence, all the integrals in (4.32) and (4.33) are well-defined.

The proof of (4.32) follows from similar arguments to those used to prove (4.31) by means of (4.36) and (4.37). The same arguments apply to prove (4.33) by using (4.39). □

After all of the above preparations we are ready to prove Theorem 4.3.1:

Proof of Theorem 4.3.1. Assume X is a weak solution to (4.10). Fix $t \geq 0$. We start by showing that for all $\psi \in \mathrm{Dom}(A)$, the following holds $\mathbb{P}$-a.e.

$$\begin{aligned}&\int_0^t \left(\int_0^s \int_U F(r,u,X_r)'S(t-s)A\psi M(dr,du)\right) ds \\ &= X_0[\psi] - S(t)'X_0[\psi] + \int_0^t X_r[A\psi]ds - \int_0^t B(r,X_r)[S(t-r)\psi]dr + \int_0^t B(r,X_r)[\psi]dr.\end{aligned} \tag{4.42}$$

To do this, note that for fixed $s \in [0,t]$ and $\psi \in \mathrm{Dom}(A)$, $S(t-s)A\psi \in \mathrm{Dom}(A)$ (Theorem D.2.5(1)), hence from the definition of weak solution to (4.10) (where ψ is there replaced by $S(t-s)A\psi$)(see (4.2.5)) we have $\mathbb{P}$-a.e.

$$\begin{aligned}&\int_0^s \int_U F(r,u,X_r)'S(t-s)A\psi M(dr,du) \\ &= (X_s - X_0)[S(t-s)A\psi] - \int_0^s (X_r[AS(t-s)A\psi] + B(r,X_r)[S(t-s)A\psi])dr.\end{aligned} \tag{4.43}$$

Now, integrating both sides of (4.43) on $[0,t]$ with respect to the Lebesgue measure, and then using (4.31) and (4.32), we have $\mathbb{P}$-a.e.

$$\begin{aligned}
&\int_0^t \left(\int_0^s \int_U F(r,u,X_r)' S(t-s) A\psi M(dr,du) \right) ds \qquad (4.44)\\
&= \int_0^t X_s[S(t-s)A\psi] ds - \int_0^t X_0[S(t-s)A\psi] ds \\
&\quad - \int_0^t \left(\int_0^s X_r[AS(t-s)A\psi] dr \right) ds - \int_0^t \left(\int_0^s B(r,X_r)[S(t-s)A\psi] dr \right) ds \\
&= - \int_0^t X_0[S(t-s)A\psi] ds + \int_0^t X_r[A\psi] dr \\
&\quad - \int_0^t B(r,X_r)[S(t-s)\psi] dr + \int_0^t B(r,X_r)[\psi] dr.
\end{aligned}$$

Now, similar calculations to those used in (4.37) (for $r=0$ and for ψ instead of $A\psi$), shows that $\mathbb{P}$-a.e.

$$\int_0^t X_0[S(t-s)A\psi] ds = \int_0^t X_0[S(s)A\psi] ds = (S(t)'X_0 - X_0)[\psi]. \qquad (4.45)$$

And hence from (4.44) and (4.45) we obtain (4.42).

Substituting (4.19) into the definition of weak solution (4.11), and then using (4.42), we get that $\mathbb{P}$-a.e.

$$\begin{aligned}
&X_t[\psi] \qquad (4.46)\\
&= X_0[\psi] + \int_0^t (X_r[A\psi] + B(r,X_r)[\psi]) dr + \int_0^t \int_U F(r,u,X_r)'\psi M(dr,du) \\
&= X_0[\psi] + \int_0^t (X_r[A\psi] + B(r,X_r)[\psi]) dr + \int_0^t \int_U F(r,u,X_r)' S(t-r)\psi M(dr,du) \\
&\quad - \int_0^t \left(\int_0^s \int_U F(r,u,X_r)' S(t-s) A\psi M(dr,du) \right) ds \\
&= X_0[\psi] + \int_0^t (X_r[A\psi] + B(r,X_r)[\psi]) dr + \int_0^t \int_U F(r,u,X_r)' S(t-r)\psi M(dr,du) \\
&\quad - X_0[\psi] + S(t)'X_0[\psi] - \int_0^t X_r[A\psi] dr + \int_0^t B(r,X_r)[S(t-r)\psi] dr - \int_0^t B(r,X_r)[\psi] dr \\
&= S(t)'X_0[\psi] + \int_0^t B(r,X_r)[S(t-r)\psi] dr + \int_0^t \int_U F(r,u,X_r)' S(t-r)\psi M(dr,du).
\end{aligned}$$

Now, substituting (4.13) and (4.15) in (4.46), we get that $\mathbb{P}$-a.e.

$$\begin{aligned}
X_t[\psi] = \Bigg(& S(t)'X_0 + \int_0^t S(t-r)' B(r,X_r) dr \\
& + \int_0^t \int_U S(t-r)' F(r,u,X_r) M(dr,du) \Bigg) [\psi]. \qquad (4.47)
\end{aligned}$$

As (4.47) is valid for all $\psi \in \text{Dom}(A)$ and $\text{Dom}(A)$ is dense in Ψ (Theorem D.2.5(4)), then we have $\mathbb{P}$-a.e.

$$X_t = S(t)'X_0 + \int_0^t S(t-r)' B(r,X_r) dr + \int_0^t \int_U S(t-r)' F(r,u,X_r) M(dr,du)$$

and therefore X is a mild solution to (4.10).

Conversely, assume X is a mild solution to (4.10). Fix $\psi \in \mathrm{Dom}(A)$ and $t \geq 0$. For $s \in [0,T]$, from the definition of mild solution (4.12), where ψ is there replaced by $A\psi$ and t is replaced by s, we have $\mathbb{P}$-a.e.

$$\begin{aligned} X_s[A\psi] &= S(s)'X_0[A\psi] + \int_0^s S(s-r)'B(r,X_r)dr[A\psi] \\ &\quad + \int_0^s \int_U S(t-r)'F(r,u,X_r)M(dr,du)[A\psi]. \end{aligned} \tag{4.48}$$

Then, integrating both sides of (4.48) on $[0,t]$ with respect to the Lebesgue measure, then using (4.20), (4.33) and (4.37) (where $A\psi$ is there replaced by ψ and with $r = 0$), regrouping terms and finally by using (4.46) (that from the arguments above is equivalent to the definition of mild solution), we have $\mathbb{P}$-a.e.

$$\begin{aligned} &\int_0^t X_s[A\psi]ds \\ &= \int_0^t S(s)'X_0[A\psi]ds + \int_0^t \left(\int_0^s S(s-r)'B(r,X_r)dr[A\psi] \right) ds \\ &\quad + \int_0^t \left(\int_0^s \int_U S(s-r)'F(r,u,X_r)M(dr,du)[A\psi] \right) ds \\ &= S(t)'X_0[\psi] - X_0[\psi] + \int_0^t B(r,X_r)[S(t-r)\psi]dr - \int_0^t B(r,X_r)[\psi]dr \\ &\quad + \int_0^t \int_U F(r,u,X_r)'S(t-r)\psi M(dr,du) - \int_0^t \int_U F(r,u,X_r)'\psi M(dr,du) \\ &= S(t)'X_0[\psi] + \int_0^t B(r,X_r)[S(t-r)\psi]dr + \int_0^t \int_U F(r,u,X_r)'S(t-r)\psi M(dr,du) \\ &\quad - X_0[\psi] - \int_0^t B(r,X_r)[\psi]dr - \int_0^t \int_U F(r,u,X_r)'\psi M(dr,du) \\ &= X_t[\psi] - X_0[\psi] - \int_0^t B(r,X_r)[\psi]dr - \int_0^t \int_U F(r,u,X_r)'\psi M(dr,du) \end{aligned}$$

Therefore, we have $\mathbb{P}$-a.e.

$$X_t[\psi] = X_0[\psi] + \int_0^t (X_r[A\psi] + B(r,X_r)[\psi])dr + \int_0^t \int_U F(r,u,X_r)'\psi M(dr,du),$$

and hence X is a weak solution to (4.10). □

§ 4.4 Regularity Properties of the Stochastic Convolution

In this section our main interest is to study some properties of the stochastic convolution process $\left\{ \int_0^t \int_U S(t-r)'R(r,u)M(dr,du) : t \in [0,T] \right\}$ for $R \in \Lambda^2_s(T)$. These properties will play an important role in the study of existence and uniqueness of weak and mild solutions in Section 4.5. Before we present our main result, we will introduce some notation:

Notation 4.4.1. Sometimes, we will denote by $S' * R = \{(S' * R)_t\}_{t \geq 0}$ the stochastic convolution process $\left\{ \int_0^t \int_U S(t-r)'R(r,u)M(dr,du) : t \in [0,T] \right\}$.

Theorem 4.4.2. *Let $R \in \Lambda^2_s(T)$. There exists a continuous Hilbertian semi-norm ϱ on Ψ such that the process $S' * R$ has a Ψ'_ϱ-valued, mean-square continuous, predictable version $\widetilde{S' * R} = \{(\widetilde{S' * R})_t\}_{t \geq 0}$ satisfying*

$$\sup_{t \in [0,T]} \mathbb{E}\left[\varrho'\left((\widetilde{S' * R})_t \right)^2 \right] < \infty. \tag{4.49}$$

Proof. First, it is important to remark that the fact that $R \in \Lambda^2_s(T)$ and by using similar arguments to those in the proof of Proposition 4.2.8 it follows that the stochastic convolution $S' * R$ is well-defined.

Now we prove the existence of a Hilbert space-valued predictable version of the stochastic convolution process. First, as $R \in \Lambda^2_s(T)$, from Theorem 3.3.17 there exists a continuous Hilbertian semi-norm p on Ψ and $\tilde{R} = \{\tilde{R}(r,\omega,u) : r \in [0,T], \omega \in \Omega, u \in U\} \in \Lambda^2_s(p,T)$ such that $R(r,\omega,u) = i'_p \tilde{R}(r,\omega,u)$, for Leb$\otimes\mathbb{P}\otimes\mu$-a.e. $(r,\omega,u) \in [0,T] \times \Omega \times U$. Now, as in the proof of Lemma 4.3.2, because $\{S(t)\}_{t\geq 0}$ is a $(C_0,1)$-semigroup on Ψ and p is a continuous semi-norm on Ψ, there exists a continuous semi-norm q on Ψ, $p \leq q$, and there exists a C_0-semigroup $\{S_q(t)\}_{t \geq 0}$ on the Banach space Ψ_q such that (4.22) holds. Moreover, there exist $M_q \geq 1$, $\theta_q \geq 0$ such that (4.23) holds. Note that as $p \leq q$ the canonical inclusion $i_{p,q} : \Psi_q \to \Psi_p$ is a continuous and linear operator.

Let ϱ be a continuous Hilbertian semi-norm on Ψ such that $q \leq \varrho$. Such a semi-norm ϱ exists because Ψ is nuclear. Note that because $q \leq \varrho$, the inclusion $i_{q,\varrho} : \Psi_\varrho \to \Psi_q$ is a continuous and linear operator. Then, for fixed $t \in [0,T]$ it follows from the above properties that for Leb $\otimes\, \mathbb{P} \otimes \mu$-a.e. (r,ω,u),

$$\mathbb{1}_{[0,t]}(r)\, S(t-r)' R(r,\omega,u) = \mathbb{1}_{[0,t]}(r)\, i'_\varrho\, i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,\omega,u). \tag{4.50}$$

Our objective is then to prove that $\{\mathbb{1}_{[0,t]}(r)\, i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,\omega,u) : r \in [0,T], \omega \in \Omega, u \in U\} \in \Lambda^2_s(\varrho,T)$ for each $t \in [0,T]$.

First, for every $(r,\omega,u) \in [0,T] \times \Omega \times U$, because $\tilde{R}(r,\omega,u) \in \mathcal{L}_2(\Phi'_{q_{r,u}}, \Psi'_p)$, $i'_{p,q} \in \mathcal{L}(\Psi'_p, \Psi'_q)$, $S_q(t-r)' \in \mathcal{L}(\Psi'_q, \Psi'_q)$ and $i'_{q,\varrho} \in \mathcal{L}(\Psi'_q, \Psi'_\varrho)$, it follows that for each (r,ω,u) we have $i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,\omega,u) \in \mathcal{L}_2(\Psi'_\varrho, \Phi'_{q_{r,u}})$.

Second, fix $\psi \in \Psi$ and $\phi \in \Phi$. From the fact that the map

$$(r,\omega,u) \mapsto \mathbb{1}_{[0,t]}(r)\, q_{r,u}(\tilde{R}(r,\omega,u)' S(t-r)\psi, \phi),$$

is $\mathcal{P}_t \otimes \mathcal{B}(U)$-measurable (see the proof of Proposition 4.2.8) and from (4.50) it follows that the map

$$(r,\omega,u) \mapsto \mathbb{1}_{[0,t]}(r)\, q_{r,u}(\tilde{R}(r,\omega,u)'\, i_{p,q}\, S_q(t-r)\, i_{q,\varrho} i_\varrho \psi, \phi),$$

is also $\mathcal{P}_t \otimes \mathcal{B}(U)$-measurable. Finally, for all (r,ω,u),

$$\begin{aligned} &\left|\left| \tilde{R}(r,\omega,u)'\, i_{p,q}\, S_q(t-r)\, i_{q,\varrho} \right|\right|^2_{\mathcal{L}_2(\Psi_\varrho, \Phi_{q_{r,u}})} \\ &\leq \left|\left| \tilde{R}(r,\omega,u)' \right|\right|^2_{\mathcal{L}_2(\Psi_p, \Phi_{q_{r,u}})} ||i_{p,q}||^2_{\mathcal{L}(\Psi_q,\Psi_p)}\, ||S_q(t-r)||^2_{\mathcal{L}(\Psi_q,\Psi_q)}\, ||i_{q,\varrho}||^2_{\mathcal{L}(\Psi_\varrho,\Psi_q)}. \end{aligned}$$

Then, from the above inequality, (4.21) (with $\tilde{F}$ there replaced by $\tilde{R}$) and (4.23), we have

$$\begin{aligned} &\mathbb{E} \int_0^t \int_U \left|\left| \tilde{R}(r,u)'\, i_{p,q}\, S_q(t-r)\, i_{q,\varrho} \right|\right|^2_{\mathcal{L}_2(\Psi_\varrho, \Phi_{q_{r,u}})} \mu(du)\lambda(dr) \\ &\qquad \leq M_q e^{\theta_q t}\, ||i_{p,q}||^2_{\mathcal{L}(\Psi_q,\Psi_p)}\, ||i_{q,\varrho}||^2_{\mathcal{L}(\Psi_\varrho,\Psi_q)} \left|\left| \tilde{R} \right|\right|^2_{s,p,t} < \infty. \end{aligned} \tag{4.51}$$

Then, $\{\mathbb{1}_{[0,t]}(r)\, i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,\omega,u)\}$ satisfies the conditions of Definition 3.3.3 and hence belongs to $\Lambda^2_s(\varrho,t)$. Moreover, from (4.50) and Theorem 3.3.17, for each $t \in [0,T]$ the stochastic integral $\int_0^t \int_U i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,u) M(dr,du)$ is a Ψ'_ϱ-valued $\mathcal{F}_t$-measurable version of the stochastic convolution integral $\int_0^t \int_U S(t-r)' R(r,u) M(dr,du)$. Our next objective is to prove that the Ψ'_ϱ-valued process

$$\left\{ \int_0^t \int_U i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,u) M(dr,du) : t \in [0,T] \right\},$$

is mean square continuous. We will prove the left continuity as the right continuity follows from similar arguments. Let $0 < t \leq T$. Then, from the linearity of the strong stochastic integral and Proposition 3.3.29, for any $0 \leq s < t$ we have

$$\begin{aligned} &\mathbb{E}\Bigg[\varrho\Bigg(\int_0^t \int_U i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,u) M(dr,du) \\ &\qquad - \int_0^s \int_U i'_{q,\varrho}\, S_q(s-r)'\, i'_{p,q}\, \tilde{R}(r,u) M(dr,du)\Bigg)^2\Bigg] \\ &\leq 2\,\mathbb{E}\left[\varrho\left(\int_0^t \int_U \mathbb{1}_{[s,t]}(r)\, i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,u) M(dr,du)\right)^2\right] \\ &\quad + 2\,\mathbb{E}\left[\varrho\left(\int_0^s \int_U i'_{q,\varrho}\left(S_q(t-r)' - S_q(s-r)'\right) i'_{p,q}\, \tilde{R}(r,u) M(dr,du)\right)^2\right] \end{aligned} \tag{4.52}$$

Now, we start with the first term in the right-hand side of the inequality in (4.52). From (3.78) and arguing in a similar way to the derivation of (4.51) we have for any $0 \leq s < t$ that

$$\begin{aligned} &\mathbb{E}\left[\varrho\left(\int_0^t \int_U \mathbb{1}_{[s,t]}(r)\, i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,u) M(dr,du)\right)^2\right] \\ &= \mathbb{E}\int_s^t \int_U \left|\left| i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,u)\right|\right|^2_{\mathcal{L}_2(\Phi'_{q_{r,u}},\Psi'_\varrho)} \mu(du)dr \\ &= \mathbb{E}\int_s^t \int_U \left|\left| \tilde{R}(r,u)'\, i_{p,q}\, S_q(t-r)\, i_{q,\varrho}\right|\right|^2_{\mathcal{L}_2(\Psi_\varrho,\Phi_{q_{r,u}})} \mu(du)dr \\ &\leq M_q e^{\theta_q(t-s)}\, ||i_{p,q}||^2_{\mathcal{L}(\Psi_q,\Psi_p)}\, ||i_{q,\varrho}||^2_{\mathcal{L}(\Psi_\varrho,\Psi_q)} \left|\left|\tilde{R}\right|\right|^2_{s,p,T}, \end{aligned} \tag{4.53}$$

Then, from (4.53) we have

$$\lim_{s\to t-} \mathbb{E}\left[\varrho\left(\int_0^t \int_U \mathbb{1}_{[s,t]}(r)\, i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,u) M(dr,du)\right)^2\right] = 0. \tag{4.54}$$

For the second term in the right-hand side of the inequality in (4.52), proceeding as in (4.53), we can prove that for any $0 \leq s < t$,

$$\begin{aligned} &\mathbb{E}\left[\varrho\left(\int_0^s \int_U i'_{q,\varrho}\left(S_q(t-r)' - S_q(s-r)'\right) i'_{p,q}\, \tilde{R}(r,u) M(dr,du)\right)^2\right] \\ &= \mathbb{E}\int_0^s \int_U \left|\left| \tilde{R}(r,u)'\, i_{p,q}\left(S_q(t-r) - S_q(s-r)\right) i_{q,\varrho}\right|\right|^2_{\mathcal{L}_2(\Psi_\varrho,\Phi_{q_{r,u}})} \mu(du)dr \\ &\leq M_q e^{\theta_q T}\, ||i_{p,q}||^2_{\mathcal{L}(\Psi_q,\Psi_p)}\, ||i_{q,\varrho}||^2_{\mathcal{L}(\Psi_\varrho,\Psi_q)} \left|\left|\tilde{R}\right|\right|^2_{s,p,T} < \infty. \end{aligned} \tag{4.55}$$

Now, let $\{\psi_j^{\varrho}\}_{j\in\mathbb{N}} \subseteq \Psi$ be a complete orthonormal system in Ψ_{ϱ}. For each $j \in \mathbb{N}$, the strong continuity of the semigroup $\{S_q(t)\}_{t\geq 0}$, the continuity of the maps $i_{p,q}$ and of $R(r,\omega,u)'$ (for fixed (r,ω,u)), and the dominated convergence theorem imply that

$$\lim_{s\to t-} \mathbb{E} \int_0^T \int_U \mathbb{1}_{[0,s]}(r)\, q_{r,u} \left(\tilde{R}(r,u)'\, i_{p,q} \left(S_q(t-r) - S_q(s-r)\right) i_{q,\varrho} \psi_j^{\varrho} \right)^2 \mu(du)dr = 0. \tag{4.56}$$

By Fubini's theorem and Parseval's identity we have

$$\begin{aligned} &= \mathbb{E} \int_0^s \int_U \left|\left| \tilde{R}(r,u)'\, i_{p,q} \left(S_q(t-r) - S_q(s-r)\right) i_{q,\varrho} \right|\right|^2_{\mathcal{L}_2(\Psi_{\varrho}, \Phi_{q_{r,u}})} \mu(du)dr \\ &= \sum_{j=1}^{\infty} \mathbb{E} \int_0^T \int_U \mathbb{1}_{[0,s]}(r)\, q_{r,u} \left(\tilde{R}(r,u)'\, i_{p,q} \left(S_q(t-r) - S_q(s-r)\right) i_{q,\varrho} \psi_j^{\varrho} \right)^2 \mu(du)dr. \end{aligned}$$

Hence, from (4.55), (4.56) and the dominated convergence theorem it follows that

$$\lim_{s\to t-} \mathbb{E} \left[\varrho \left(\int_0^s \int_U i'_{q,\varrho} \left(S_q(t-r)' - S_q(s-r)'\right) i'_{p,q}\, \tilde{R}(r,u) M(dr,du) \right)^2 \right] = 0. \tag{4.57}$$

Finally, from (4.52), (4.54) and (4.57), it follows that $\widetilde{S' * R} = \{(\widetilde{S' * R})_t\}_{t\geq 0}$ given by

$$(\widetilde{S' * R})_t = \int_0^t \int_U i'_{q,\varrho}\, S_q(t-r)'\, i'_{p,q}\, \tilde{R}(r,u) M(dr,du), \quad \forall\, t \in [0,T],$$

is mean square continuous. Furthermore, as it is also $\{\mathcal{F}_t\}$-adapted and Ψ'_{ϱ} is a separable Hilbert space, then it has a predictable version (see Proposition 3.21 of Peszat and Zabczyk [85], p.27). Moreover, from (4.53) (taking $s = 0$) we have (4.49). □

§ 4.5 Existence and Uniqueness of Weak and Mild Solutions

In this section we prove the existence and uniqueness of weak and mild solutions to (4.10) under some Lipschitz and growth conditions on the coefficients B and F. The main idea behind the proof is to define a complete locally convex space where the solution will lie, then we define an operator on this space in such a way that any fixed point of it is a mild solution to (4.10). To prove that a fixed point exists and is unique, we will use the Lipschitz and growth conditions on B and F to show that this operator is a contraction and then the fixed point theorem on locally convex spaces will guarantee the existence of a unique fixed point. This proof is inspired by the methods used by Da Prato and Zabczyk [20] and Peszat and Zabczyk [85] for the Hilbert space case.

We will need the following additional assumptions in this section.

Assumption 4.5.1.
(1) Every continuous semi-norm on Ψ'_{β} is separable.
(2) The dual semigroup $\{S(t)'\}_{t\geq 0}$ is a $(C_0, 1)$ semigroup on Ψ'_{β}.

Remark 4.5.2. *A sufficient condition for Assumption 4.5.1(1) is either that Ψ'_{β} is separable or that it is nuclear (see Remark 1.1.6). A sufficient condition for Assumption 4.5.1(2) is that $\{S(t)\}_{t\geq 0}$ is equicontinuous, as in that case $\{S(t)'\}_{t\geq 0}$ is also equicontinuous (Theorem D.2.7).*

Now we proceed to introduce the conditions in the coefficients B and F. Recall from Section 1.1.5 that for each $K \subseteq \Psi$ bounded, $\eta_K : \Psi' \to \mathbb{R}_+$ given by

$$\eta_K(f) := p_{K^0}(f) = \sup_{\psi \in K} |f[\psi]|, \quad \forall\, f \in \Psi',$$

is a continuous semi-norm on Ψ'_β, where p_{K^0} is the Minkowski functional of K^0. Moreover, the family $\{\eta_K : K \subseteq \Psi,\, K \text{ is bounded}\}$ generates the topology on Ψ'_β.

To avoid any confusion with our previous notation, for each $K \subseteq \Psi$ bounded we denote by Ψ'_K the Banach space Ψ'_{η_K}. The canonical inclusion from Ψ'_β into Ψ'_K will be denoted by j_K. If K, D are any bounded subsets of Ψ such that $K \subseteq D$, then we have $\eta_K \leq \eta_D$ and we denote by $j_{K,D}$ the canonical inclusion from Ψ'_D into Ψ'_K. If for $K \subseteq \Psi$ bounded we have that Ψ'_K is a Hilbert space, then we say that K is **Hilbertian**.

The Lipschitz and growth conditions that we assume for our coefficients B and F are the following:

(E1) *For each $K \subseteq \Psi$ bounded and Hilbertian, there exists a function $a_K : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\int_0^T a_K(r)^2 dr < \infty, \quad \forall\, T > 0,$$

such that, for all $r \in \mathbb{R}_+$, $g_1, g_2 \in \Psi'$,

$$\eta_K(B(r,g_1)) \leq a_K(r)(1 + \eta_K(g_1)),$$
$$\eta_K(B(r,g_1) - B(r,g_2)) \leq a_K(r)\eta_K(g_1 - g_2).$$

(E2) *For each $K \subseteq \Psi$ bounded and Hilbertian, there exists a function $b_K : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\int_0^T b_K(r)^2 dr < \infty, \quad \forall\, T > 0,$$

such that, for all $r \in \mathbb{R}_+$, $g_1, g_2 \in \Psi'$,

$$\int_U ||j_K F(r,u,g_1)||^2_{\mathcal{L}_2(\Phi'_{q_{r,u}}, \Psi'_K)}\, \mu(du) \leq b_K(r)(1 + \eta_K(g_1)^2),$$
$$\int_U ||j_K F(r,u,g_1) - j_K F(r,u,g_2)||^2_{\mathcal{L}_2(\Phi'_{q_{r,u}}, \Psi'_K)}\, \mu(du) \leq b_K(r)\eta_K(g_1 - g_2)^2.$$

We establish a key property of the dual semigroup $\{S(t)'\}_{t \geq 0}$ that will be of great importance for our proof of existence and uniqueness of solutions of (4.10).

Lemma 4.5.3. *There exists a non-empty family $\mathcal{K}_H(\Psi)$ of bounded subsets of Ψ, such that for all $K \in \mathcal{K}_H(\Psi)$, Ψ'_K is a separable Hilbert space and there exists a C_0-semigroup $\{S_K(t)\}_{t \geq 0}$ on Ψ'_K such that*

$$S_K(t) j_K f = j_K S(t)' f, \quad \forall\, t \geq 0,\, f \in \Psi'. \tag{4.58}$$

Proof. First, from the fact that $\{S(t)'\}_{t \geq 0}$ is a $(C_0, 1)$-semigroup on Ψ'_β (Assumption 4.5.1) and Theorem D.2.6, there exists a family $\mathcal{K}(\Psi)$ of bounded subsets of Ψ such that the semi-norms $\{\eta_D : D \in \mathcal{K}(\Psi)\}$ generate the topology on Ψ'_β and such that for each $D \in \mathcal{K}(\Psi)$ there exists a C_0-semigroup $\{T_D(t)\}_{t\geq 0}$ on Ψ'_D such that

$$T_D(t) j_D f = j_D S(t)' f, \quad \forall\, t \geq 0,\ f \in \Psi'. \tag{4.59}$$

Fix and arbitrary $D \in \mathcal{K}(\Psi)$. Then, as the Banach space Ψ'_D is separable (Assumption 4.5.1) there exists a separable Hilbert space $(H, ||\cdot||_H)$, a continuous dense embedding $k_{H,D} : \Psi'_D \to H$ and a C_0-semigroup $\{T_H(t)\}_{t\geq 0}$ on H such that

$$T_H(t) k_{H,D} f = k_{H,D} T_D(t) f, \quad \forall\, t \geq 0,\ f \in \Psi'. \tag{4.60}$$

For a proof of this last fact see Theorem 1.3 of van Neerven [110]. From the arguments above it follows that $k_H : \Psi'_\beta \to H$ given by $k_H := k_{H,D} \circ j_D$ is a continuous dense embedding. Moreover, by (4.59) and (4.60) we have for all $t \geq 0,\ f \in \Psi'$,

$$T_H(t) k_H f = T_H(t) k_{H,D} j_D f = k_{H,D} T_D(t) j_D f = k_{H,D} j_D S(t)' f = k_H S(t)' f. \tag{4.61}$$

Denote by B_H the unit ball in H. Then, the continuity of k_H implies that $k_H^{-1}(B_H)$ is a neighborhood of zero in Ψ'_β. Furthermore, as Ψ'_β is reflexive then $K = (k_H^{-1}(B_H))^0$ is a bounded subset of Ψ (see Theorem 5.2, Chapter IV of Schaefer [93], p.141). Then, η_K is a continuous Hilbertian norm on Ψ'_β and $\eta_K(f) = ||k_H f||_H$, for all $f \in \Psi'$. Hence the map k_H defines an isometric isomorphism between the pre-Hilbert spaces (Ψ', η_K) and $(k_H \Psi', ||\cdot||_H)$.

Now, from (4.61) we have for all $t \geq 0$ that $T_H(t)(k_H \Psi') \subseteq k_H \Psi'$. Therefore, each $T_H(t)$ restricts to a continuous and linear operator on $(k_H \Psi', ||\cdot||_H)$. For each $t \geq 0$, let $T_K(t) := T_H(t)\big|_{k_H \Psi'} \circ k_H$. One can see from the fact that the map k_H identifies the spaces (Ψ', η_K) and $(k_H \Psi', ||\cdot||_H)$ that $\{T_K(t)\}_{t\geq 0}$ is a C_0-semigroup on (Ψ', η_K). Moreover, as Ψ'_K is the completion of the space (Ψ', η_K), then each $T_K(t)$ extends to a continuous and linear operator $S_K(t)$ on Ψ'_K. Therefore, $\{S_K(t)\}_{t\geq 0}$ is a C_0-semigroup on Ψ'_K. Finally, (4.61) and the definition of $T_K(t)$ shows that the semigroup $\{S_K(t)\}_{t\geq 0}$ satisfies (4.58). □

Now, for the proof of existence and uniqueness of solutions to (4.10) we will follow a fixed point theorem argument and to do this we will need a class of Ψ'_β-process where the solution will lie. This class is defined as follows:

Definition 4.5.4. Let $T > 0$. We denote by $\mathcal{H}^2(T, \Psi'_\beta)$ the vector space of all the Ψ'_β-valued, regular, predictable processes $X = \{X_t\}_{t\in[0,T]}$ such that for each $K \subseteq \Psi$ bounded,

$$\sup_{t\in[0,T]} \mathbb{E}\left(\eta_K(j_K X_t)^2\right) < \infty.$$

The next result shows that the space $\mathcal{H}^2(T, \Psi'_\beta)$ contains the Ψ'_β-valued processes that are stochastic convolutions. As in Theorem 4.4.2, we denote by $S' * R = \{(S' * R)_t\}_{t\geq 0}$ the stochastic convolution process $\left\{\int_0^t \int_U S(t-r)' R(r,u) M(dr,du) : t \in [0,T]\right\}$.

Theorem 4.5.5. *Let $T > 0$ and let $R \in \Lambda^2_s(T)$. Then, $S' * R \in \mathcal{H}^2(T, \Psi'_\beta)$, and for each $K \in \mathcal{K}_H(\Psi)$, we have*

$$\mathbb{E}\left[\eta_K\left(j_K (S' * R)_t\right)^2\right] \leq M_K^2 e^{2\theta_K t} \mathbb{E} \int_0^t \int_U ||j_K\, R(r,u)||^2_{\mathcal{L}_2(\Phi'_{q_{r,u}}, \Psi'_K)}\, \mu(du) dr, \tag{4.62}$$

for all $t \in [0,T]$, *where* $M_K \geq 1$ *and* $\theta_K \geq 0$ *are such that the* C_0*-semigroup* $\{S_K(t)\}_{t\geq 0}$ *on* Ψ'_K *defined in Lemma 4.5.3 satisfies*

$$\eta_K(S_K(t)'f) \leq M_K e^{\theta_K t}\eta_K(f), \quad \forall f \in \Psi'_K. \tag{4.63}$$

Moreover, the integral in the right-hand side of (4.62) *is always finite*

Proof. To prove the first part, note that from Theorem 4.4.2 there exists a continuous Hilbertian seminorm ϱ on Ψ such that $S'*R$ has a Ψ'_ϱ-valued, mean-square continuous, predictable version $\widetilde{S'*R} = \{(\widetilde{S'*R})_t\}_{t\geq 0}$ satisfying (4.49). Then, because $\mathbb{P}$-a.e. $(S'*R)_t = i'_\varrho(\widetilde{S'*R})_t$, for all $t \in [0,T]$, $(S'*R)$ has a predictable Ψ'_β-valued version. We will identify $S'*R$ with this version.

Let $K \subseteq \Psi$ be bounded. Because the unit ball $B_\varrho(1)$ of ϱ is a neighborhood of zero, there exists some $C>0$ such that $K \leq CB_\varrho(1)$, hence $\eta_K(f) \leq C\varrho'(f)$, for every $f \in \Psi'_\varrho$. Therefore, the inclusion map $j_K \circ i'_\varrho : \Psi'_p \to \Psi'_K$ is linear and continuous. Hence by (4.49) and the fact that $(S'*R)_t = i'_\varrho(\widetilde{S'*R})_t$, $\mathbb{P}$-a.e. for all $t \in [0,T]$, we have

$$\sup_{t\in[0,T]} \mathbb{E}\left[\eta_K(j_K(S'*R)_t)^2\right] \leq \left|\left|j_K i'_\varrho\right|\right|^2_{\mathcal{L}(\Psi'_\varrho,\Psi'_K)} \sup_{t\in[0,T]} \mathbb{E}\left[\varrho'\left((\widetilde{S'*R})_t\right)^2\right] < \infty.$$

Then, as $S'*R$ is a regular process, $S'*R \in \mathcal{H}^2(T,\Psi'_\beta)$.

To prove the second part, let $K \in \mathcal{K}_H(\Psi)$. As $R \in \Lambda^2_s(T)$, from Theorem 3.3.17 there exists a continuous Hilbertian semi-norm q on Ψ and $\widetilde{R} \in \Lambda^2_s(q,T)$ such that $R(r,\omega,u) = i'_q\widetilde{R}(r,\omega,u)$ for $\text{Leb}\otimes\mathbb{P}\otimes\mu$-a.e. (r,ω,u). Then, using (4.58), for all $t\in[0,T]$ and $\text{Leb}\otimes\mathbb{P}\otimes\mu$-a.e. (r,ω,u) we have

$$\mathbb{1}_{[0,t]}(r)\, j_K S(t-r)'R(r,\omega,u) = \mathbb{1}_{[0,t]}(r)\, S_K(t-r) j_K i'_q \widetilde{R}(r,\omega,u). \tag{4.64}$$

Now, denote by Ψ_K the dual of the Hilbert space Ψ'_K. Let j'_K be the dual operator of j_K. Then, j'_K corresponds to the canonical inclusion from Ψ_K into Ψ. Let $\{\psi_j\}_{j\in\mathbb{N}}$ be a complete orthonormal system in Ψ_K. Then, from Parseval's identity, Fubini's theorem, (3.28), (3.85), (4.58), (4.63) and (4.64), we have

$$\begin{aligned}
\mathbb{E}\left[\eta_K\left(j_K(S'*R)_t\right)^2\right] &= \mathbb{E}\sum_{j=1}^\infty \left|j_K(S'*R)_t[\psi_j]\right|^2 \\
&= \sum_{j=1}^\infty \mathbb{E}\left[\left|(S'*R)_t)[j'_K\psi_j]\right|^2\right] \\
&= \sum_{j=1}^\infty \mathbb{E}\int_0^t\int_U q_{r,u}(R(r,u)'S(t-r)j'_K\psi_j)^2\mu(du)dr \\
&= \mathbb{E}\int_0^t\int_U \left|\left|R(r,u)'S(t-r)j'_K\right|\right|^2_{\mathcal{L}_2(\Psi_K,\Phi_{q_{r,u}})}\mu(du)dr \\
&= \mathbb{E}\int_0^t\int_U \left|\left|j_K S(t-r)'R(r,u)\right|\right|^2_{\mathcal{L}_2(\Phi'_{q_{r,u}},\Psi'_K)}\mu(du)dr \\
&= \mathbb{E}\int_0^t\int_U \left|\left|S_K(t-r)'j_K R(r,u)\right|\right|^2_{\mathcal{L}_2(\Phi'_{q_{r,u}},\Psi'_K)}\mu(du)dr \\
&\leq M_K^2 e^{2\theta_K t}\mathbb{E}\int_0^t\int_U \left|\left|j_K R(r,u)\right|\right|^2_{\mathcal{L}_2(\Phi'_{q_{r,u}},\Psi'_K)}\mu(du)dr.
\end{aligned}$$

Hence we have proved (4.62). Finally, note that the integral in the right-hand side of (4.62) is finite because, from the fact that $j_K R(r,\omega,u) = j_K i'_q \widetilde{R}(r,\omega,u)$, for Leb$\otimes\mathbb{P}\otimes\mu$-a.e. (r,ω,u), we have

$$\mathbb{E}\int_0^t \int_U ||j_K R(r,u)||^2_{\mathcal{L}_2(\Phi'_{q_{r,u}},\Psi'_K)} \mu(du)dr \leq ||j_K i'_q||^2_{\mathcal{L}(\Phi'_q,\Psi'_K)} \left|\left|\widetilde{R}\right|\right|^2_{s,q,T} < \infty.$$

□

Now, we need to equip the linear space $\mathcal{H}^2(T,\Psi'_\beta)$ with a locally convex topology. To do this, we will need the following family of Banach spaces (see Da Prato and Zabczyk [20], p.188).

Definition 4.5.6. Let $K\subseteq\Psi$ bounded. Denote by $\mathcal{H}^2(T,\Psi'_K)$ the real vector space of all Ψ'_K-valued predictable process $X=\{X_t\}_{t\in[0,T]}$ such that $\sup_{t\in[0,T]}\mathbb{E}\,\eta_K(X_t)^2<\infty$. The space $\mathcal{H}^2(T,\Psi'_K)$ is a Banach space when equipped with the topology defined by the norm $||\cdot||_{K,T}$ given by $||X||_{K,T}=\sup_{t\in[0,T]}\left(\mathbb{E}\left(\eta_K(X_t)^2\right)\right)^{1/2}$, $\forall X\in\mathcal{H}^2(T,\Psi'_K)$.

The relation between the elements of the space $\mathcal{H}^2(T,\Psi'_\beta)$ and of the Banach spaces $\mathcal{H}^2(T,\Psi'_K)$ is given in the following result.

Lemma 4.5.7. *For $K\subseteq\Psi$ bounded, $j_K\mathcal{H}^2(T,\Psi'_\beta)\subseteq\mathcal{H}^2(T,\Psi'_K)$, i.e. if $X\in\mathcal{H}^2(T,\Psi'_\beta)$, then $j_K X\in\mathcal{H}^2(T,\Psi'_K)$.*

Proof. The result is an immediate consequence of the definition of the spaces $\mathcal{H}^2(T,\Psi'_\beta)$ and $\mathcal{H}^2(T,\Psi'_K)$, and the continuity of the map j_K. □

Theorem 4.5.8. *For every $T>0$, the space $\mathcal{H}^2(T,\Psi'_\beta)$ is a complete, Hausdorff, locally convex space when equipped with the projective topology induced by the projective system $\{(\mathcal{H}^2(T,\Psi'_\beta),\mathcal{H}^2(T,\Psi'_K),j_K):K\in\mathcal{K}_H(\Psi)\}$, where $\mathcal{K}_H(\Psi)$ is the family of bounded Hilbertian subsets of Ψ given in Lemma 4.5.3. In particular, the topology on $\mathcal{H}^2(T,\Psi'_\beta)$ is generated by the family of semi-norms $\{|||\cdot|||_{K,T}:K\in\mathcal{K}_H(\Psi)\}$, given for each $K\in\mathcal{K}_H(\Psi)$ by*

$$|||X|||_{K,T} := ||j_K X||_{K,T} = \sup_{t\in[0,T]}\left(\mathbb{E}\left(\eta_K(j_K X_t)^2\right)\right)^{1/2}, \quad \forall\, X\in\mathcal{H}^2(T,\Psi'_\beta). \tag{4.65}$$

Proof. From Lemma 4.5.7 it follows that $\{(\mathcal{H}^2(T,\Psi'_\beta),\mathcal{H}^2(T,\Psi'_K),j_K):K\in\mathcal{K}_H(\Psi)\}$ is a projective system and therefore we can define the projective limit topology on $\mathcal{H}^2(T,\Psi'_\beta)$. Then it follows by the definition that $\{|||\cdot|||_{K,T}:K\in\mathcal{K}_H(\Psi)\}$ is a family of semi-norms generating the projective topology (see Section 1.1.3). Finally, the fact that these topology is complete and Hausdorff is a consequence of the fact that $\mathcal{H}^2(T,\Psi'_K)$ satisfies these properties (see Results 5.1 and 5.3, Chapter II of Schaefer [93], p.51-2). □

Remark 4.5.9. *Let $K\in\mathcal{K}_H(\Psi)$. For $\upsilon\geq 0$ we define $|||\cdot|||_{\upsilon,K,T}$ by (see Peszat and Zabczyk [85], p. 164)*

$$|||X|||_{\upsilon,K,T} = \sup_{t\in[0,T]}\left(e^{-\upsilon t}\mathbb{E}\left(\eta_K(j_K X_t)^2\right)\right)^{1/2}, \quad \forall\, X\in\mathcal{H}^2(T,\Psi'_\beta). \tag{4.66}$$

It is clear that $|||\cdot|||_{\upsilon,K,T}$ defines a semi-norm on $\mathcal{H}^2(T,\Psi'_\beta)$. Moreover, is not difficult to see that the semi-norms $|||\cdot|||_{\upsilon,K,T}$, $\upsilon\geq 0$ are equivalent.

Theorem 4.5.10. *Assume that conditions (E1) and (E2) hold. Let Z_0 be a Ψ'_β-valued, regular, $\mathcal{F}_0$-measurable, square integrable random variable. Then, there exists a unique (up to modification) mild solution $X=\{X_t\}_{t\geq 0}$ to* (4.10) *with initial condition $X_0=Z_0$. Moreover, for every $T>0$ there exists a continuous Hilbertian semi-norm $q=q(T)$ on Ψ such that $X=\{X_t\}_{t\in[0,T]}$ has a Ψ'_q-valued predictable version $\tilde{X}=\{\tilde{X}_t\}_{t\in[0,T]}$ satisfying $\sup_{t\in[0,T]}\mathbb{E}\left(q'(\tilde{X}_t)^2\right)<\infty$.*

Furthermore, X is also a weak solution to (4.10).

Proof. For $T>0$, let $\mathbb{A}:\mathcal{H}^2(T,\Psi'_\beta)\rightarrow\mathcal{H}^2(T,\Psi'_\beta)$ be the mapping defined by

$$\mathbb{A}(X)=\mathbb{A}_0(X)+\mathbb{A}_1(X)+\mathbb{A}_2(X),\quad \forall\, X\in\mathcal{H}^2(T,\Psi'_\beta),$$

where for each $t\in[0,T]$,

$$\begin{aligned}\mathbb{A}_0(X)_t &:= S(t)'Z_0,\\ \mathbb{A}_1(X)_t &:= \int_0^t S(t-r)'B(r,X_r)dr,\\ \mathbb{A}_2(X)_t &:= \int_0^t\int_U S(t-r)'F(r,u,X_r)M(dr,du).\end{aligned}$$

Our objective is to show that the map $\mathbb{A}$ is a contraction. For convenience we will divide the proof in two steps.

Step 1 *The map $\mathbb{A}$ is well-defined.*

Our task is to verify that the three components of $\mathbb{A}$ are well-defined. To do this, fix $X\in\mathcal{H}^2(T,\Psi'_\beta)$.

For $\mathbb{A}_0$, the strong continuity of the dual semigroup $\{S(t)'\}_{t\geq 0}$ and the fact that Z_0 is a Ψ'_β-valued, regular, $\mathcal{F}_0$-measurable random variable implies that $\{S(t)'Z_0\}_{t\geq 0}$ is a Ψ'_β-valued, regular, $\{\mathcal{F}_t\}$-adapted, continuous process.

Now, we will prove that $\{S(t)'Z_0\}_{t\geq 0}$ has a predictable version. To do this, note that as Z_0 is also square integrable, then Theorem 1.2.24 (by identifying Z_0 with the process $\{Z_0(s)\}_{s\in[0,T]}$ where for each $s\in[0,T]$, $Z_0(s)=Z_0$) implies that there exists a continuous Hilbertian semi-norm p on Ψ, such that Z_0 possesses a Ψ'_p-valued, $\{\mathcal{F}_t\}$-adapted, continuous version $\tilde{Z}_0$ satisfying $\mathbb{E}p'(\tilde{Z}_0)^2<\infty$.

Furthermore, note that as in the proof of Lemma 4.3.2, because $\{S(t)\}_{t\geq 0}$ is a $(C_0,1)$-semigroup on Ψ and p is a continuous semi-norm on Ψ, there exists a continuous semi-norm q on Ψ, $p\leq q$, and there exists a C_0-semigroup $\{S_q(t)\}_{t\geq 0}$ on the Banach space Ψ_q such that (4.22) hold. Moreover, there exist $M_q\geq 1$, $\theta_q\geq 0$ such that (4.23) hold. Note that as $p\leq q$, then the canonical inclusion $i_{p,q}:\Psi_q\rightarrow\Psi_p$ is linear and continuous.

Then, from the above we have that for each $t\in[0,T]$, $\mathbb{P}$-a.e.

$$S(t)'Z_0=i'_qS_q(t)'i'_{p,q}\tilde{Z}_0.$$

Therefore, $\{S_q(t)'i'_{p,q}\tilde{Z}_0\}_{t\geq 0}$ is a Ψ'_q-valued version of $\{S(t)'Z_0\}_{t\geq 0}$. Moreover, as $\{S_q(t)'i'_{p,q}\tilde{Z}_0\}_{t\geq 0}$ is a Ψ'_q-valued, $\{\mathcal{F}_t\}$-adapted, continuous process and Ψ'_q is a separable Banach space, then it has a predictable version (Proposition 3.21 of Peszat and Zabczyk [85], p.27) and hence $\{S(t)'Z_0\}_{t\geq 0}$ has also a predictable version. We will identify $\{S(t)'Z_0\}_{t\geq 0}$ with this version.

Let $K \subseteq \Psi$ be bounded. By similar arguments to those used in Theorem 4.5.5, the map $j_K \circ i'_q : \Psi'_q \to \Psi'_K$ is linear and continuous. Therefore, from the arguments in the above paragraphs we have

$$\begin{aligned} \mathbb{E}\left(\sup_{t\in[0,T]} \eta_K(j_K S(t)' Z_0)^2 \right) &= \mathbb{E}\left(\sup_{t\in[0,T]} \eta_K(j_K i'_q S_q(t)' i'_{p,q} \tilde{Z}_0)^2 \right) \\ &\leq M_q^2 \, e^{2\theta_q T} \left|\left| j_K i'_q \right|\right|^2_{\mathcal{L}(\Psi'_q, \Psi'_K)} \left|\left| i_{p,q} \right|\right|^2_{\mathcal{L}(\Psi_q,\Psi_p)} \mathbb{E} p'(\tilde{Z}_0)^2 < \infty. \end{aligned} \tag{4.67}$$

Hence, $\{S(t)'Z_0\}_{t\geq 0}$ is an element of $\mathcal{H}^2(T, \Psi'_\beta)$ and therefore $\mathbb{A}_0$ is well-defined. Note that because for all $\psi \in \Psi$, $\{\psi\}$ is a bounded and Hilbertian subset of Ψ, then (4.67) implies that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left| S(t)' Z_0[\psi] \right|^2 \right) < \infty, \quad \forall \, \psi \in \Psi.$$

As $\{S(t)'Z_0\}_{t\geq 0}$ is a Ψ'_β-valued, regular, continuous process, Theorem 1.2.24 shows that there exists a continuous Hilbertian semi-norm ϱ_0 on Ψ such that $\{S(t)'Z_0\}_{t\geq 0}$ has a Ψ'_{ϱ_0}-valued continuous version, which we denote again by $\{S(t)'Z_0\}_{t\geq 0}$, such that

$$\mathbb{E} \sup_{t\in[0,T]} \varrho'_0(S(t)'Z_0)^2 < \infty. \tag{4.68}$$

We now prove that $\mathbb{A}_1$ is well-defined. Let $X \in \mathcal{H}^2(T, \Psi'_\beta)$. Our objective is to show that the map $Y : [0,T] \times [0,T] \times \Omega \to \Psi'$ given by $Y(t,r,\omega) = \mathbb{1}_{[0,t]}(r)\, S(t-r)' B(r, X_r)$ satisfies the conditions of Corollary 4.1.2.

Fix $\psi \in \Psi$. Then, the set $K_\psi = \{\psi\}$ is bounded and Hilbertian. Hence, from (E1) there exists a function $a_\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\int_0^T a_\psi(r)^2 dr < \infty$, and such that

$$|B(r,g)[\psi]| \leq a_\psi(r)(1 + |g[\psi]|), \quad \forall r \in [0,T], \, g \in \Psi'.$$

Then, it follows from the above that

$$\begin{aligned} \mathbb{E} \int_0^T |B(r, X_r)[\psi]|^2 \, dr &\leq \mathbb{E} \int_0^T a_\psi(r)^2 (1 + |X_r[\psi]|)^2 dr \\ &\leq 2 \left(1 + \sup_{t\in[0,T]} \mathbb{E} \left| X_t[\psi] \right|^2 \right) \int_0^T a_\psi(r)^2 dr < \infty. \end{aligned}$$

Hence, from similar arguments to those used in Lemma 4.3.3 it follows that there exists a continuous Hilbertian semi-norm ρ on Ψ such that $\mathbb{E} \int_0^T \rho'(B(r, X_r))^2 dr < \infty$.

Now, as in the proof of Lemma 4.3.2, because $\{S(t)\}_{t\geq 0}$ is a $(C_0, 1)$-semigroup on Ψ and p is a continuous semi-norm on Ψ, there exists a continuous semi-norm q on Ψ, $p \leq q$, and there exists a C_0-semigroup $\{S_q(t)\}_{t\geq 0}$ on the Banach space Ψ_q such that (4.22) hold. Moreover, there exist $M_q \geq 1$, $\theta_q \geq 0$ such that $\{S_q(t)\}_{t\geq 0}$ satisfies (4.23). Note that because $p \leq q$, then the canonical inclusion $i_{p,q} : \Psi_q \to \Psi_p$ is linear and continuous.

Then, from the above and following similar arguments to those leading to (4.41) we have

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t S(t-r)'B(r,X_r)[\psi]dr \right|^2 \qquad (4.69)\\
&\leq \mathbb{E} \sup_{t\in[0,T]} \int_0^t |B(r,X_r)[S(t-r)\psi]|^2 \, dr \\
&\leq M_q^2 e^{2\theta_q T} q(i_q\psi)^2 \, ||i_{\rho,q}||^2_{\mathcal{L}(\Psi_q,\Psi_\rho)} \, \mathbb{E} \int_0^T \rho'(B(r,X_r))^2 dr < \infty.
\end{aligned}$$

Our next task is to prove that the map $t \mapsto \int_0^t S(t-r)B(r,X_r)[\psi]dr$ is continuous for $\mathbb{P}$-a.e. $\omega \in \Omega$. To do this, note that from $\mathbb{E}\int_0^T \rho'(B(r,X_r))^2 dr < \infty$ it follows that there exists some $\Omega_0 \subseteq \Omega$, $\mathbb{P}(\Omega_0) = 1$, such that $\int_0^T \rho'(B(r,X_r(\omega)))dr < \infty$ for each $\omega \in \Omega_0$.

Now, for $0 \leq s \leq t \leq T$, we have

$$\begin{aligned}
&\left| \int_0^s S(s-r)'B(r,X_r)[\psi]dr - \int_0^t S(t-r)'B(r,X_r)[\psi]dr \right| \\
&\leq \int_0^T \mathbb{1}_{[0,s]}(r) \left|(S(s-r)' - S(t-r)')B(r,X_r)[\psi]\right| dr \\
&\quad + \int_0^T \mathbb{1}_{[s,t]}(r) \left|S(t-r)'B(r,X_r)[\psi]\right| dr.
\end{aligned}$$

Then, similarly as in (4.69) for all $\omega \in \Omega_0$ we have

$$\begin{aligned}
&\int_0^T \mathbb{1}_{[0,s]}(r) \left|(S(s-r)' - S(t-r)')B(r,X_r)[\psi]\right| dr \\
&\leq 2M_q e^{\theta_q T} q(i_q\psi) \, ||i_{\rho,q}||_{\mathcal{L}(\Psi_q,\Psi_\rho)} \, \mathbb{E} \int_0^T \rho'(B(r,X_r))dr < \infty.
\end{aligned}$$

Therefore the dominated convergence theorem and the strong continuity of the dual semigroup implies that

$$\int_0^T \mathbb{1}_{[0,s]}(r) \left|(S(s-r)' - S(t-r)')B(r,X_r)[\psi]\right| dr \longrightarrow 0, \quad \text{as } s \to t, \text{ or } t \to s.$$

Again, as in (4.69) for all $\omega \in \Omega_0$ we have

$$\begin{aligned}
&\int_0^T \mathbb{1}_{[s,t]}(r) \left|S(t-r)'B(r,X_r)[\psi]\right| dr \\
&\leq M_q e^{\theta_q T} q(i_q\psi) \, ||i_{\rho,q}||_{\mathcal{L}(\Psi_q,\Psi_\rho)} \, \mathbb{E} \int_0^T \rho'(B(r,X_r))dr < \infty,
\end{aligned}$$

and hence the dominated convergence theorem shows that

$$\int_0^T \mathbb{1}_{[s,t]}(r) \left|S(t-r)'B(r,X_r)[\psi]\right| dr \longrightarrow 0, \quad \text{as } s \to t, \text{ or } t \to s.$$

Therefore, the map $t \mapsto \int_0^t S(t-r)'B(r,X_r)[\psi]dr$ is continuous for all $\omega \in \Omega_0$.

Thus, the map $Y : [0,T] \times [0,T] \times \Omega \to \Psi'$ given by

$$Y(t,r,\omega) = \mathbb{1}_{[0,t]}(r) \, S(t-r)'B(r,X_r), \quad \forall\, (t,r,\omega) \in [0,T] \times [0,T] \times \Omega,$$

satisfies the conditions of Corollary 4.1.2. Then, there exists a continuous Hilbertian semi-norm ϱ_1 on Ψ and a Ψ'_{ϱ_1}-valued, predictable, continuous process

$$\left\{ \int_0^t S(t-r)' B(r, X_r) dr : t \in [0,T] \right\}$$

satisfying (4.13) for all $t \in [0,T]$, $\mathbb{P}$-a.e., for all $\psi \in \Psi$, and such that

$$\sup_{t\in[0,T]} \mathbb{E} \left(\varrho_1' \left(\int_0^t S(t-r)' B(r, X_r) dr \right)^2 \right) < \infty. \tag{4.70}$$

Hence, we have that $\left\{ i'_{\varrho_1} \int_0^t S(t-r)' B(r, X_r) dr : t \in [0,T] \right\}$ is an element of $\mathcal{H}^2(T, \Psi'_\beta)$ and therefore $\mathbb{A}_1$ is well-defined.

Finally, to prove that the map $\mathbb{A}_2$ is well-defined, let $X \in \mathcal{H}^2(T, \Psi'_\beta)$. In this case our objective is to prove that $F_X = \{F(r,u,X_r(\omega)) : r \in [0,T], \omega \in \Omega, u \in U\} \in \Lambda^2_s(T)$. Then by Theorem 4.5.5 we find that $S' * F_X \in \mathcal{H}^2(T, \Psi'_\beta)$ and hence the map $\mathbb{A}_2$ is well-defined.

To prove this, let $\psi \in \Psi$. As the set $K_\psi = \{\psi\}$ is bounded and Hilbertian, it follows from (E2) that there exists a function $b_\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\int_0^T b_\psi(r)^2 dr < \infty$, and such that

$$\int_U q_{r,u}(F(r,u,g)'\psi)^2 \mu(du) \leq b_\psi(r)(1 + |g[\psi]|^2), \quad \forall r \in [0,T],\, g \in \Psi'.$$

Therefore, we have

$$\begin{aligned} \mathbb{E} \int_0^T \int_U q_{r,u}(F(r,u,X_r)'\psi)^2 \mu(du) dr &\leq \mathbb{E} \int_0^T b_\psi(r) \left(1 + |X_r[\psi]|^2 \right) dr \\ &\leq \left(1 + \sup_{t\in[0,T]} \mathbb{E}\, |X_t[\psi]|^2 \right) \int_0^T b_\psi(r) dr < \infty. \end{aligned}$$

Then, from the above and the arguments in the proof of Proposition 4.2.6 it follows that $F_X \in \Lambda^2_s(T)$. Thus, the map $\mathbb{A}_2$ is well-defined and this finally shows that the map $\mathbb{A}$ is also well-defined. Moreover, from Theorem 4.5.5 there exists a continuous Hilbertian semi-norm ϱ_2 on Ψ such that $S' * F_X$ has a Ψ'_{ϱ_2}-valued version satisfying

$$\sup_{t\in[0,T]} \mathbb{E} \varrho_2' \left((\widetilde{S' * F_X})_t \right)^2 < \infty. \tag{4.71}$$

Step 2 *The map $\mathbb{A}$ is a contraction.*

Because from Theorem 4.5.8 the topology on $\mathcal{H}^2(T, \Psi'_\beta)$ is generated by the family of semi-norms $\{|||\cdot|||_{K,T} : K \in \mathcal{K}_H(\Psi)\}$, given in (4.65), then by definition the map $\mathbb{A}$ is a contraction if we can show that for every $K \in \mathcal{K}_H(\Psi)$ there exists $0 < C_{K,T} < 1$ such that

$$|||\mathbb{A}X - \mathbb{A}Y|||_{K,T} \leq C_{K,T}\, |||X - Y|||_{K,T}, \quad \forall\, X, Y \in \mathcal{H}^2(T, \Psi'_\beta).$$

However, by Remark 4.5.9 it is equivalent to show that for each $K \in \mathcal{K}_H(\Psi)$ there exists $\upsilon \geq 0$ and a constant $0 < C_{\upsilon,K,T} < 1$ such that

$$|||\mathbb{A}X - \mathbb{A}Y|||_{\upsilon,K,T} \leq C_{\upsilon,K,T}\, |||X - Y|||_{\upsilon,K,T}, \quad \forall\, X, Y \in \mathcal{H}^2(T, \Psi'_\beta). \tag{4.72}$$

where the semi-norm $|||\cdot|||_{v,K,T}$ is given in (4.66).
To do this, we fix $K \in \mathcal{K}_H(\Psi)$ and $X, Y \in \mathcal{H}^2(T, \Psi'_\beta)$. Then, from the definition of the map $\mathbb{A}$ we have:

$$|||\mathbb{A}X - \mathbb{A}Y|||^2_{v,K,T} \leq 2\,|||\mathbb{A}_1 X - \mathbb{A}_1 Y|||^2_{v,K,T} + 2\,|||\mathbb{A}_2 X - \mathbb{A}_2 Y|||^2_{v,K,T}. \tag{4.73}$$

Now, as $K \in \mathcal{K}_H(\Psi)$ it follows from Lemma 4.5.3 that there exists a C_0-semigroup $\{S_K(t)'\}_{t\geq 0}$ on the Hilbert space Ψ'_K satisfying (4.58) and there exist $M_K \geq 1$ and $\theta_K \geq 0$ such that $\{S_K(t)'\}_{t\geq 0}$ satisfies (4.63).
Now, we need to show that for all $t \in [0,T]$, $\mathbb{P}$-a.e.

$$\int_0^t S_K(t-r)' j_K(B(r,X_r) - B(r,Y_r))dr,$$

is well-defined as a Bochner integral in Ψ'_K. First, for every $t \in [0,T]$ the map

$$(r,\omega) \mapsto \mathbb{1}_{[0,t]}(r)\, S_K(t-r)' j_K(B(r,X_r(\omega)) - B(r,Y_r(\omega)) \in \Psi'_K,$$

is predictable, and following similar arguments to those used in the Step 1 we can prove that there exists a continuous Hilbertian semi-norm ρ on Ψ such that

$$\int_0^T \rho'\left(B(r,X_r) - B(r,Y_r)\right) dr < \infty, \mathbb{P} - \text{a.e.}$$

From this last observation, (4.58) and (4.63) it follows that for all $t \in [0,T]$, $\mathbb{P}$-a.e.

$$\begin{aligned}&\int_0^t \eta_K\left(S_K(t-r)' j_K(B(r,X_r) - B(r,Y_r))\right) dr \\ &\leq M_K e^{\theta_K T} \left|\left|j_K i'_\rho\right|\right|_{\mathcal{L}(\Psi'_\rho, \Psi'_K)} \int_0^T \rho'\left(B(r,X_r) - B(r,Y_r)\right) dr < \infty.\end{aligned}$$

Therefore, the Bochner integral $\int_0^t S_K(t-r)' j_K(B(r,X_r) - B(r,Y_r))dr$ exists for all $t \in [0,T]$, $\mathbb{P}$-a.e.
Let $j'_K : \Psi_K \to \Psi$ be the dual operator to j_K. From (4.13), (4.58) and the properties of the Bochner integral, for all $t \in [0,T]$, $\mathbb{P}$-a.e. we have

$$\begin{aligned}&\eta_K\left(j_K \int_0^t S(t-r)' B(r,X_r)dr - j_K \int_0^t S(t-r)' B(r,Y_r)dr\right) \qquad (4.74)\\ &= \sup_{\psi \in K} \left| \int_0^t S(t-r)' B(r,X_r)dr[j'_K\psi] - \int_0^t S(t-r)' B(r,Y_r)dr[j'_K\psi] \right| \\ &= \sup_{\psi \in K} \left| \int_0^t (B(r,X_r) - B(r,Y_r))[S(t-r)j'_K\psi]dr \right| \\ &= \sup_{\psi \in K} \left| \int_0^t S_K(t-r)' j_K(B(r,X_r) - B(r,Y_r))[\psi]dr \right| \\ &= \sup_{\psi \in K} \left| \left(\int_0^t S_K(t-r)' j_K(B(r,X_r) - B(r,Y_r))dr\right)[\psi] \right| \\ &= \eta_K\left(\int_0^t S_K(t-r)' j_K(B(r,X_r) - B(r,Y_r))dr\right).\end{aligned}$$

Then, from (4.58), (4.74) and (E1) we have (recall the definition of $|||\cdot|||_{\upsilon,K,T}$ in (4.66))

$$
\begin{aligned}
&|||\mathbb{A}_1 X - \mathbb{A}_1 Y|||^2_{\upsilon,K,T} \\
&= \sup_{t\in[0,T]} e^{-\upsilon t}\mathbb{E}\left(\eta_K\left(j_K\int_0^t S(t-r)' B(r,X_r)dr - j_K\int_0^t S(t-r)' B(r,Y_r)dr\right)^2\right) \\
&= \sup_{t\in[0,T]} e^{-\upsilon t}\mathbb{E}\left(\eta_K\left(\int_0^t S_K(t-r)' j_K(B(r,X_r) - B(r,Y_r))dr\right)\right)^2 \\
&\leq \sup_{t\in[0,T]} e^{-\upsilon t}\mathbb{E}\left(\int_0^t \eta_K\left(S_K(t-r)' j_K(B(r,X_r) - B(r,Y_r))\right)dr\right)^2 \\
&\leq M_K^2 e^{2\theta_K T}\sup_{t\in[0,T]} e^{-\upsilon t}\mathbb{E}\left(\int_0^t \eta_K\left(j_K(B(r,X_r) - B(r,Y_r))\right)dr\right)^2 \\
&\leq M_K^2 e^{2\theta_K T}\sup_{t\in[0,T]} e^{-\upsilon t}\mathbb{E}\left(\int_0^t a_K(r)\eta_K\left(j_K(X_r - Y_r)\right)dr\right)^2
\end{aligned}
$$

Now, by the Cauchy-Schwarz inequality,

$$
\int_0^t a_K(r)\eta_K\left(j_K(X_r - Y_r)\right)dr \leq \left(\int_0^t a_K(r)^2 dr\right)^{\frac{1}{2}}\left(\int_0^t \eta_K\left(j_K(X_r - Y_r)\right)^2 dr\right)^{\frac{1}{2}}.
$$

Then,

$$
\begin{aligned}
&|||\mathbb{A}_1 X - \mathbb{A}_1 Y|||^2_{\upsilon,K,T} \\
&\leq M_K^2 e^{2\theta_K T}\left(\int_0^T a_K(r)^2 dr\right)\sup_{t\in[0,T]}\int_0^t e^{-\upsilon(t-r)}e^{-\upsilon r}\mathbb{E}\left(\eta_K\left(j_K(X_r - Y_r)\right)^2\right)dr \\
&\leq M_K^2 e^{2\theta_K T}\left(\int_0^T a_K(r)^2 dr\right)\left(\int_0^T e^{-\upsilon r}dr\right)|||X - Y|||^2_{\upsilon,K,T}.
\end{aligned}
$$

Therefore, we have

$$
|||\mathbb{A}_1 X - \mathbb{A}_1 Y|||^2_{\upsilon,K,T} \leq C^{(1)}_{\upsilon,K,T}\,|||X - Y|||^2_{\upsilon,K,T}, \tag{4.75}
$$

where

$$
C^{(1)}_{\upsilon,K,T} = M_K^2 e^{2\theta_K T}\left(\int_0^T a_K(r)^2 dr\right)\left(\int_0^T e^{-\upsilon r}dr\right). \tag{4.76}
$$

Similarly, from the linearity of the stochastic integral, (4.62), (E2) and the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&|||\mathbb{A}_2 X - \mathbb{A}_2 Y|||^2_{\upsilon,K,T} \\
&= \sup_{t\in[0,T]} e^{-\upsilon t}\mathbb{E}\left(\eta_K\left(\int_0^t\int_U S(t-r)'\left(F(r,u,X_r) - F(r,u,Y_r)\right)M(dr,du)\right)^2\right) \\
&\leq M_K^2 e^{2\theta_K T}\sup_{t\in[0,T]} e^{-\upsilon t}\mathbb{E}\int_0^t\int_U ||j_K\left(F(r,u,X_r) - F(r,u,Y_r)\right)||^2_{\mathcal{L}_2(\Phi_{q_{r,u}},\Psi'_K)}\mu(du)dr \\
&\leq M_K^2 e^{2\theta_K T}\sup_{t\in[0,T]} e^{-\upsilon t}\mathbb{E}\int_0^t b_K(r)\eta_K\left(j_K(X_r - Y_r)\right)^2 dr \\
&\leq M_K^2 e^{2\theta_K T}\left(\int_0^T b_K(r)^2 dr\right)\left(\int_0^T e^{-\upsilon r}dr\right)|||X - Y|||^2_{\upsilon,K,T}.
\end{aligned}
$$

Therefore,

$$|||\mathbb{A}_2 X - \mathbb{A}_2 Y|||^2_{\upsilon,K,T} \leq C^{(2)}_{\upsilon,K,T} \, |||X-Y|||^2_{\upsilon,K,T}\,, \tag{4.77}$$

where

$$C^{(2)}_{\upsilon,K,T} = M_K^2 e^{2\theta_K T} \left(\int_0^T b_K(r)^2 dr \right) \left(\int_0^T e^{-\upsilon r} dr \right). \tag{4.78}$$

Then, it follows from (4.73), (4.75), (4.76), (4.77) and (4.78) that (4.72) is satisfied for $C_{\upsilon,K,T}$ given by

$$C_{\upsilon,K,T} = \left[2M_K^2 e^{2\theta_K T} \left(\frac{1-e^{-\upsilon T}}{\upsilon} \right) \left(\int_0^T a_K(r)^2 + b_K(r)^2 dr \right) \right]^{\frac{1}{2}},$$

and then we can take υ sufficiently large such that $C_{\upsilon,K,T}<1$ and consequently $\mathbb{A}$ is a contraction on $\mathcal{H}^2(T,\Psi'_\beta)$.

We are ready to show that there exists a unique (up to modification) mild solution to (4.10) satisfying the statements of the theorem.

For a fixed $T>0$, as the map $\mathbb{A}$ is a contraction on $\mathcal{H}^2(T,\Psi'_\beta)$ and this last is a complete, Hausdorff, locally convex space (Theorem 4.5.8), it follows from the fixed point theorem on locally convex spaces (see Theorem 3.5 of Wlondarczyk [112]) that $\mathbb{A}$ has a unique fixed point $X^{(T)} = \{X^{(T)}_t\}_{t\in[0,T]}$ in $\mathcal{H}^2(T,\Psi'_\beta)$. Therefore, $X^{(T)}$ satisfies (4.12) for all $t\in[0,T]$.

Let $\{T_n\}_{n\in\mathbb{N}}$ any sequence of positive real numbers such that $\lim_{n\to\infty} T_n = \infty$ and for each $n\in\mathbb{N}$ let $X^{(T)} = \{X^{(T_n)}_t\}_{t\in[0,T]}$ as above. Let $X=\{X_t\}_{t\geq 0}$ be given for each $t\geq 0$ by $X_t = X^{T_n}_t$ if $T_{n-1}\leq t < T_n$, where we take $T_0=0$. Then, is easy to see that X is well defined and moreover that X is a Ψ'_β-valued, regular, predictable process satisfying (4.12) for all $t\geq 0$. Therefore, X is a mild solution to (4.10) and is unique up to indistinguishable versions.

Now let $T>0$. From Step 1 there exist continuous Hilbertian semi-norms ϱ_0, ϱ_1 and ϱ_2 on Ψ such that the processes $\{S(t)'X_0\}_{t\geq 0}$, $\left\{ \int_0^t S(t-r)'B(r,X_r)dr : t\in[0,T] \right\}$ and $S' * F_X$ have predictable versions taking values in Ψ'_{ϱ_0}, Ψ'_{ϱ_1} and Ψ'_{ϱ_2} respectively, and satisfying (4.68), (4.70) and (4.71). Let q be a continuous Hilbertian seminorm on Ψ such that $\varrho_0\leq q$, $\varrho_1 \leq q$ and $\varrho_2\leq q$, and such that the inclusions $i_{\varrho_0,q}$, $i_{\varrho_1,q}$ and $i_{\varrho_2,q}$ are Hilbert-Schmidt. Then, using similar arguments to those used in the proof of Theorem 1.2.24; in particular using the estimate (1.13) (with r replaced there by ϱ_0, ϱ_1 and ϱ_2) we can show that that the processes $\{S(t)'X_0\}_{t\geq 0}$, $\left\{ \int_0^t S(t-r)'B(r,X_r)dr : t\in[0,T] \right\}$ and $S' * F_X$ have Ψ'_q-valued predictable versions satisfying

$$\mathbb{E} \sup_{t\in[0,T]} q'(S(t)'X_0)^2 < \infty, \quad \sup_{t\in[0,T]} \mathbb{E} q' \left(\int_0^t S(t-r)'B(r,X_r)dr \right)^2 < \infty,$$

$$\text{and} \quad \sup_{t\in[0,T]} \mathbb{E}\, q'\left((\widetilde{S' * F_X})_t \right)^2 < \infty.$$

Now, as X is a mild solution to (4.10), and hence satisfies (4.12), then it follows from the above that $X=\{X_t\}_{t\in[0,T]}$ has a Ψ'_q-valued predictable version $\tilde{X} = \{\tilde{X}_t\}_{t\in[0,T]}$ satisfying $\sup_{t\in[0,T]} \mathbb{E}\left(q'(\tilde{X}_t)^2 \right) < \infty$.

Finally, as X is a mild solution to (4.10), and from the arguments on Step 1 one can check that the conditions of Theorem 4.3.1 are satisfied, so that X is also a weak solution to (4.10). □

We finish this chapter with an application of Theorem 4.5.10 to the existence and uniqueness of weak and mild solutions to stochastic evolution equations driven by Lévy-noise.

Example 4.5.11. Let Φ be a complete, barrelled, nuclear space and Ψ be a complete, bornological, nuclear space such that every continuous semi-norm on Ψ'_{β} is separable. Let L be a Φ'_{β}-valued càdlàg Lévy process with Lévy measure ν and Lévy-Itô decomposition (2.39), that is

$$L_t = t\mathfrak{m} + W_t + \int_{B_{\rho'}(1)} f\widetilde{N}(t,df) + \int_{B_{\rho'}(1)^c} fN(t,df),$$

for all $t \geq 0$, where the components in the above decomposition satisfy the properties given in Theorem 2.2.13.

Assume that L is square integrable, i.e. $\mathbb{E}(|L_t[\phi]|^2) < \infty$, for all $t \geq 0$, $\phi \in \Phi$. Then, it is easy to check that this implies that the compound Poisson process $\left\{\int_{B_{\rho'}(1)^c} fN(t,df) : t \geq 0\right\}$ is square integrable. Then, the Lévy-Itô decomposition of L can be equivalently written as

$$L_t = t\mathfrak{m}' + W_t + \int_{\Phi'} f\widetilde{N}(t,df), \quad \forall\, t \geq 0, \tag{4.79}$$

where

$$\int_{\Phi'} f\widetilde{N}(t,df) := \int_{B_{\rho'}(1)} f\widetilde{N}(t,df) + \int_{B_{\rho'}(1)^c} f\widetilde{N}(t,df), \quad \forall\, t \geq 0,$$

and $\mathfrak{m}'[\phi] = \mathfrak{m}[\phi] + \int_{B_{\rho'}(1)^c} f[\phi]\nu(df)$, for all $\phi \in \Phi$.

In this example our objective is to show the existence of weak and mild solutions to the **Lévy-driven stochastic evolution equation** given by

$$dX_t = (A'X_t + B(t,X_t))dt + F(t,0,X_t)dW_t + \int_{\Phi'} F(t,f,X_t)\tilde{N}(dt,df), \tag{4.80}$$

for all $t \geq 0$, with the initial condition $X_0 = Z_0$, where Z_0 is a Ψ'_{β}-valued, regular, $\mathcal{F}_0$-measurable, square integrable random variable. It follows from (4.79) that equation (4.80) generalizes the case of stochastic evolution equations driven by square integrable Lévy noise.

As in Section 4.2 we will need some assumptions on the coefficients of (4.80). But before this, we need to introduce some notation.

Let $\mathcal{Q}$ be the covariance functional of the Wiener process W. For every $f \in \Phi'$, let $q_f : \Phi \to \mathbb{R}$ be defined for every $\phi \in \Phi$ by

$$q_f(\phi) = \begin{cases} \mathcal{Q}(\phi), & \text{if } f = 0, \\ |f[\phi]|, & \text{if } f \in \Phi' \setminus \{0\}. \end{cases} \tag{4.81}$$

Then, $\{q_f : f \in \Phi'\}$ is a family of continuous Hilbertian semi-norms on Φ.

Assumption 4.5.12.

(L1) *A is the infinitesimal generator of a $(C_0,1)$-semi-group $\{S(t)\}_{t\geq 0}$ on Ψ such that the dual semigroup $\{S(t)\}_{t\geq 0}$ is a $(C_0,1)$-semi-group on Ψ'_β .*

(L2) *$B:\mathbb{R}_+\times\Psi'\to\Psi'$ is such that the map $(r,g)\mapsto B(r,g)[\psi]$ is $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\Psi'_\beta)$-measurable, for every $\psi\in\Psi$.*

(L3) *$F=\{F(r,f,g): r\in\mathbb{R}_+, f\in\Phi', g\in\Psi'\}$ is such that*

(a) $F(r,f,g)\in\mathcal{L}(\Phi'_{q_f},\Psi'_\beta)$, $\forall r\geq 0$, $f\in\Phi'$, $g\in\Psi'$.

(b) The mapping $(r,f,g)\mapsto q_f(F(r,f,g)'\phi,\psi)$ is $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\Phi'_\beta)\otimes\mathcal{B}(\Psi'_\beta)$-measurable, for every $\phi\in\Phi$, $\psi\in\Psi$.

(L4) *For each $K\subseteq\Psi$ bounded and Hilbertian, there exists a function $a_K:\mathbb{R}_+\to\mathbb{R}_+$ satisfying*

$$\int_0^T a_K(r)^2 dr<\infty,\quad \forall T>0,$$

such that, for all $r\in\mathbb{R}_+$, $g_1,g_2\in\Psi'$,

$$\eta_K(B(r,g_1))\leq a_K(r)(1+\eta_K(g_1)),$$
$$\eta_K(B(r,g_1)-B(r,g_2))\leq a_K(r)\eta_K(g_1-g_2).$$

(L5) *For each $K\subseteq\Psi$ bounded and Hilbertian, there exists a function $b_K:\mathbb{R}_+\to\mathbb{R}_+$ satisfying*

$$\int_0^T b_K(r)^2 dr<\infty,\quad \forall T>0,$$

such that, for all $r\in\mathbb{R}_+$, $g_1,g_2\in\Psi'$,

$$||j_K F(r,0,g_1)||^2_{\mathcal{L}_2(\Phi'_Q,\Psi'_K)} + \int_{\Phi'\setminus\{0\}} ||j_K F(r,f,g_1)||^2_{\mathcal{L}_2(\Phi'_{q_f},\Psi'_K)}\,\nu(du)\leq b_K(r)(1+\eta_K(g_1)^2),$$

$$||j_K F(r,0,g_1)-j_K F(r,0,g_2)||^2_{\mathcal{L}_2(\Phi'_Q,\Psi'_K)} + \int_{\Phi'\setminus\{0\}} ||j_K F(r,f,g_1)-j_K F(r,f,g_2)||^2_{\mathcal{L}_2(\Phi'_{q_f},\Psi'_K)}\,\nu(du)\leq b_K(r)\eta_K(g_1-g_2)^2.$$

We say that a Ψ'_β-valued regular and predictable process $X=\{X_t\}_{t\geq 0}$ is a **weak solution** to (4.80) if for every $\psi\in\mathrm{Dom}(A)$ and every $t\geq 0$, $\mathbb{P}$-a.e.

$$\begin{aligned} X_t[\psi] &= X_0[\psi]+\int_0^t (X_r[A\psi]+B(r,X_r)[\psi])dr+\int_0^t F(r,0,X_r)'\psi dW_r \\ &\quad +\int_0^t\int_{\Phi'} F(r,f,X_r)'\psi\tilde{N}(dr,df) \end{aligned}$$

On the other hand, a Ψ'_β-valued regular and predictable process $X=\{X_t\}_{t\geq 0}$ is called a **mild solution** to (4.80) if for every $t\geq 0$, $\mathbb{P}$-a.e.

$$\begin{aligned} X_t &= S(t)'X_0+\int_0^t S(t-r)'B(r,X_r)dr+\int_0^t S(t-r)'F(r,0,X_r)dW_r \\ &\quad +\int_0^t\int_{\Phi'} S(t-r)'F(r,f,X_r)\tilde{N}(dr,df). \end{aligned}$$

Now we proceed to use Theorem 4.5.10 and assumptions (L1) to (L5) to show the existence and uniqueness of weak and mild solutions to (4.80).

Let $M = (M(t,A) : r \geq 0, A \in \mathcal{R})$ be given by

$$M(t,A) = W_t \delta_0(A) + \int_{A \setminus \{0\}} f \widetilde{N}(t, df), \quad \forall\, t \geq 0,\, A \in \mathcal{R}, \tag{4.82}$$

where $\mathcal{R} = \{A \in \mathcal{B}(\Phi'_\beta) : 0 \notin \overline{A}\} \cup \{0\}$. Then, M is a nuclear cylindrical martingale-valued measure on $\mathbb{R}_+ \times \mathcal{R}$, the covariance of M is determined by the measure $\lambda = \text{Leb}$ on $\mathbb{R}_+$, the Lévy measure ν on Φ'_β, and the semi-norms $\{q_f : f \in \Phi'\}$ (this follows from similar arguments to those in Example 3.1.8). Moreover, it follows from (4.82) and Proposition 3.3.31 that the initial value problem (4.80) is equivalent to the following problem

$$dX_t = (A'X_t + B(t,X_t))dt + \int_{\Phi'} F(t,f,X_t) M(dt,df), \tag{4.83}$$

with the initial condition $X_0 = Z_0$.

Note that (L1), (L2) and (L3) together with the properties of M given above implies (A1), (A2), (A3) and (A4) in Section 4.2. Moreover, from (L4) and (L5) it follows that the coefficients B and F satisfy (E1) and (E2) for $U = \Phi'_\beta$ and semi-norms $\{q_f : f \in \Phi'\}$. Then, Theorem 4.5.10 shows that there exists a unique mild (and weak) solution $X = \{X_t\}_{t \geq 0}$ to (4.83), and consequently to (4.80), such that for each $T > 0$ there exists a continuous Hilbertian semi-norm $q = q(T)$ on Ψ for which $X = \{X_t\}_{t \in [0,T]}$ has a Ψ'_q-valued version $\tilde{X} = \{\tilde{X}_t\}_{t \in [0,T]}$ satisfying $\sup_{t \in [0,T]} \mathbb{E}(q'(\tilde{X}_t)^2) < \infty$.

Remark 4.5.13. *The tools developed in this chapter can be also used to show the existence and uniqueness of weak and mild solutions to the following general Lévy-driven stochastic evolution equation*

$$\begin{aligned} dX_t \;&=\; (A'X_t + B(t,X_t))dt + F(t,0,X_t)dW_t \\ &\quad + \int_{B_{\rho'}(1)} F(t,f,X_t)\tilde{N}(dt,df) + \int_{B_{\rho'}(1)^c} F(t,f,X_t)N(dt,df). \end{aligned}$$

To do this, we need to define the stochastic integral and the stochastic convolution with respect to the Poisson random measure N in order to make sense of the last term. These integrals can be defined in terms of the stochastic integration with respect to cylindrical martingale-valued measures developed in Chapter 3. However, as this requires a significant additional amount of work we have decided not to include this here.

Appendix A

Proofs of the Regularization Theorems

In this appendix we provide proofs to the regularization theorems of Section 1.2.1. For the convenience of the reader we also include the statements of the theorems.

Theorem 1.2.18. *Let $(\Phi, \mathcal{T})$ be a nuclear space and let $X = \{X_t\}_{t \in [0,T]}$ be a cylindrical process in Φ' such that for each $\phi \in \Phi$, the real-valued process $X(\phi) := \{X_t(\phi)\}_{t \in [0,T]}$ has a continuous (respectively càdlàg) version. Suppose that the linear mapping from Φ into $C_T(\mathbb{R})$ (respectively $D_T(\mathbb{R})$) given by $\phi \mapsto X(\phi)$ is continuous. Then, there exists a countably Hilbertian topology θ on Φ determined by an increasing sequence $\{\varrho_n\}_{n \in \mathbb{N}}$ of continuous Hilbertian semi-norms on Φ, and a $(\Phi'_\theta, \beta_\theta)$-valued continuous (respectively càdlàg) process $Y = \{Y_t\}_{t \in [0,T]}$, such that for every $\phi \in \Phi$, $Y[\phi]$ is a version of $X(\phi)$. In particular, Y is a Φ'_β-valued continuous (respectively càdlàg) version of X that is unique up to indistinguishable versions.*

Without loss of generality we assume that each $X(\phi) = \{X_t(\phi)\}_{t \geq 0}$ has a continuous version. The proof is identical in the case of càdlàg versions. The following lemma provide the necessary tools to prove the regularization theorem.

Lemma A.0.14. *There exists an increasing sequence of continuous Hilbertian semi-norms $\{q_n\}_{n \in \mathbb{N}}$ on Φ, a subset Ω_Y of Ω such that $\mathbb{P}(\Omega_Y) = 1$ and a sequence of stochastic processes $Y^{(n)} = \{Y^{(n)}_t\}_{t \in [0,T]}$, $n \in \mathbb{N}$, satisfying:*

(1) For each $n \in \mathbb{N}$, $Y^{(n)}$ is a Φ'_{q_n}-valued process such that for every $\phi \in \Phi_{q_n}$, $Y^{(n)}[\phi] = \{Y^{(n)}_t[\phi]\}_{t \in [0,T]}$ is a continuous real-valued process.

(2) For each $\omega \in \Omega_Y$, there exists $N(\omega)$ such that

(a) For all $n \geq N(\omega)$, $\sup_{t \in [0,T]} q'_n(Y^{(n)}_t(\omega)) < \infty$, and

(b) For all $m \geq n \geq N(\omega)$, $Y^{(m)}_t(\omega) = i'_{q_n,q_m} Y^{(n)}_t(\omega)$ for all $t \in [0,T]$.

(3) For every $\phi \in \Phi$, there exists $\Delta_\phi \subseteq \Omega$ with $\mathbb{P}(\Delta_\phi) = 1$ such that for every $\omega \in \Omega_Y \cap \Delta_\phi$ there exists $N(\omega)$ such that for each $n \geq N(\omega)$, $Y^{(n)}_t(\omega)[i_{q_n}\phi] = X_t(\phi)(\omega)$ for all $t \in [0,T]$.

Proof. We follow similar arguments to those used by Itô and Nawata in [44]. Let $\{\epsilon_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$. Let D be a countable dense subset of $[0,T]$. We start by showing the existence of the semi-norms $\{q_n\}_{n \in \mathbb{N}}$.

Fix $n \in \mathbb{N}$. From the continuity of the exponential function there exists a $\delta_n > 0$ such that $\left|1 - e^{ir}\right| \leq \frac{\epsilon_n}{2}$ if $|r| \leq \delta_n$. Now, from the continuity of the map $\phi \mapsto X(\phi) := \{X_t(\phi)\}_{t \in [0,T]}$ from Ψ into $C_T(\mathbb{R})$ there exists a continuous Hilbertian semi-norm p_n on Φ such that

$$\mathbb{P}\left(\omega \in \Omega : \sup_{t \in D} |X_t(\phi)(\omega)| > \delta_n\right) < \frac{\epsilon_n}{4}, \quad \forall\, \phi \in B_{p_n}(1). \tag{A.1}$$

Let $\Upsilon_n = \{\omega \in \Omega : \sup_{t \in D} |X_t(\phi)(\omega)| < \delta_n\}$. Then, if $\phi \in B_{p_n}(1)$ it follows from (A.1) that

$$\begin{aligned} \mathbb{E}\left(\sup_{t \in D}\left|1 - e^{iX_t(\phi)}\right|\right) &\leq \int_{\Upsilon_n} \sup_{t \in D}\left|1 - e^{iX_t(\phi)(\omega)}\right| \mathbb{P}(d\omega) + 2\,\mathbb{P}(\Upsilon_n^c) \\ &\leq \frac{\epsilon_n}{2}\,\mathbb{P}(\Upsilon_n) + \frac{\epsilon_n}{2} \leq \epsilon_n. \end{aligned}$$

On the other hand, because $\sup_{t \in D}\left|1 - e^{iX_t(\phi)}\right| \leq 2$ for any $\phi \in \Phi$, then if $\phi \in B_{p_n}(1)^c$, we have

$$\mathbb{E}\left(\sup_{t \in D}\left|1 - e^{iX_t(\phi)}\right|\right) \leq 2p_n(\phi)^2.$$

Therefore, from the above inequalities we conclude that

$$\mathbb{E}\left(\sup_{t \in D}\left|1 - e^{iX_t(\phi)}\right|\right) \leq \epsilon_n + 2p_n(\phi)^2, \quad \forall\, \phi \in \Phi. \tag{A.2}$$

Note that we can assume without loss of generality that the sequence $\{p_n\}_{n \in \mathbb{N}}$ is increasing. Otherwise we replace p_n by $(p_1^2 + \cdots + p_n^2)^{1/2}$.

Now, as Φ is nuclear, there exists an increasing sequence of continuous Hilbertian semi-norms $\{q_n\}_{n \in \mathbb{N}}$ on Φ such that for each $n \in \mathbb{N}$, $p_n \leq q_n$ and the inclusion i_{p_n,q_n} is Hilbert-Schmidt. Let α be the countably Hilbertian topology on Φ generated by the semi-norms $\{q_n\}_{n \in \mathbb{N}}$. As each Φ_{q_n} is separable, this implies that Φ_α is also separable. Let $B = \{\xi_k\}_{k \in \mathbb{N}}$ be a countable dense subset of Φ_α. For every $n \in \mathbb{N}$, from an application of the Schmidt orthogonalization procedure to B, we can find a complete orthonormal system $\{\phi_j^{q_n}\}_{j \in \mathbb{N}} \subseteq \Phi$ of Φ_{q_n}, such that

$$\xi_k = \sum_{j=1}^{k} a_{j,k,n}\phi_j^{q_n} + \varphi_{k,n}, \quad \forall\, k \in \mathbb{N}, \tag{A.3}$$

with $a_{j,k,n} \in \mathbb{R}$ and $\varphi_{k,n} \in \operatorname{Ker}(q_n)$, for each $j, k \in \mathbb{N}$.

Our first objective is to prove the following: For each $n \in \mathbb{N}$,

$$\mathbb{P}(\Omega_n) \geq 1 - 2\frac{\sqrt{e}}{\sqrt{e} - 1}\epsilon_n, \tag{A.4}$$

where

$$\Omega_n = \left\{\omega \in \Omega : \sup_{t \in D}\sum_{j=1}^{\infty}\left|X_t(\phi_j^{q_n})(\omega)\right|^2 < \infty,\ \sup_{t \in D}\sum_{j=1}^{\infty}|X_t(\varphi_{j,n})(\omega)|^2 = 0\right\}. \tag{A.5}$$

To do this, let $C > 0$. From the inequality: $1 - e^{-r/2} \geq 1 - e^{-1/2} = \frac{\sqrt{e}-1}{\sqrt{e}}$, for $r > 1$, it follows that

$$\mathbb{P}\left(\omega \in \Omega : \sup_{t \in D} \sum_{j=1}^{\infty} \left| X_t(\phi_j^{q_n})(\omega) \right|^2 > C^2 \right)$$

$$\leq \frac{\sqrt{e}}{\sqrt{e}-1} \mathbb{E}\left(1 - \exp\left(-\frac{1}{2C^2} \sup_{t \in D} \sum_{j=1}^{\infty} \left| X_t(\phi_j^{q_n}) \right|^2 \right)\right)$$

$$= \lim_{m \to \infty} \frac{\sqrt{e}}{\sqrt{e}-1} \mathbb{E} \sup_{t \in D} \left(1 - \exp\left(-\frac{1}{2C^2} \sum_{j=1}^{m} \left| X_t(\phi_j^{q_n}) \right|^2 \right)\right) \tag{A.6}$$

Now, setting $\phi = \sum_{j=1}^{m} z_j \phi_j^{q_n}$ for $z_1, \dots, z_m \in \mathbb{R}$, in (A.2) we have

$$\mathbb{E}\left(\sup_{t \in D} \left| 1 - \exp\left(i \sum_{j=1}^{m} z_j X_t(\phi_j^{q_n})\right) \right| \right) \leq \epsilon_n + 2 \sum_{j=1}^{m} z_j^2 p_n(\phi_j^{q_n})^2. \tag{A.7}$$

Integrating both sides of (A.7) with respect to $\prod_{j=1}^{m} N_C(dz_j)$, where N_C is the centered Gaussian measure on $\mathbb{R}$ with variance $1/C^2$, we have

$$\int_{\mathbb{R}^m} \mathbb{E}\left(\sup_{t \in D} \left| 1 - \exp\left(i \sum_{j=1}^{m} z_j X_t(\phi_j^{q_n})\right) \right| \right) \prod_{j=1}^{m} N_C(dz_j) \leq \epsilon_n + \frac{2}{C^2} \sum_{j=1}^{m} p_n(\phi_j^{q_n})^2. \tag{A.8}$$

On the other hand, as $\prod_{j=1}^{m} N_C(dz_j)$ is a Gaussian measure on $\mathbb{R}^m$, for each $t \in [0, T]$ and $\omega \in \Omega$ we have

$$\exp\left(-\frac{1}{2C^2} \sum_{j=1}^{m} \left| X_t(\phi_j^{q_n})(\omega) \right|^2 \right) = \int_{\mathbb{R}^m} \exp\left(i \sum_{j=1}^{m} z_j X_t(\phi_j^{q_n})(\omega)\right) \prod_{j=1}^{m} N_C(dz_j), \tag{A.9}$$

and therefore from (A.9) and Fubini theorem it follows that

$$\mathbb{E} \sup_{t \in D} \left(1 - \exp\left(-\frac{1}{2C^2} \sum_{j=1}^{m} \left| X_t(\phi_j^{q_n}) \right|^2 \right)\right)$$

$$\leq \int_{\mathbb{R}^m} \mathbb{E}\left(\sup_{t \in D} \left| 1 - \exp\left(i \sum_{j=1}^{m} z_j X_t(\phi_j^{q_n})\right) \right| \right) \prod_{j=1}^{m} N_C(dz_j). \tag{A.10}$$

Then, from (A.6), (A.8) and (A.10), it follows that

$$\begin{aligned} \mathbb{P}\left(\omega \in \Omega : \sup_{t \in D} \sum_{j=1}^{\infty} \left| X_t(\phi_j^{q_n})(\omega) \right|^2 > C^2 \right) &\leq \lim_{m \to \infty} \frac{\sqrt{e}}{\sqrt{e}-1} \left(\epsilon_n + \frac{2}{C^2} \sum_{j=1}^{m} p_n(\phi_j^{q_n})^2 \right) \\ &= \frac{\sqrt{e}}{\sqrt{e}-1} \left(\epsilon_n + \frac{2}{C^2} ||i_{p_n,q_n}||^2_{\mathcal{L}_2(\Phi_{q_n}, \Phi_{p_n})}\right), \end{aligned}$$

where recall $||i_{p_n,q_n}||_{\mathcal{L}_2(\Phi_{q_n}, \Phi_{p_n})} < \infty$ as i_{p_n,q_n} is Hilbert-Schmidt. Letting $C \to \infty$, we get

$$\mathbb{P}\left(\omega \in \Omega : \sup_{t \in D} \sum_{j=1}^{\infty} \left| X_t(\phi_j^{q_n})(\omega) \right|^2 < \infty \right) \geq 1 - \frac{\sqrt{e}}{\sqrt{e}-1} \epsilon_n. \tag{A.11}$$

Following the same arguments as above but now changing $\phi_j^{q_n}$ for $\varphi_{j,n}$ and using the fact that $\varphi_{j,n} \in \text{Ker}(p_n)$, for each $j \in \mathbb{N}$ (recall $p_n \leq q_n$), we have that

$$\mathbb{P}\left(\omega \in \Omega : \sup_{t \in D} \sum_{j=1}^{\infty} |X_t(\varphi_{j,n})(\omega)|^2 > 0\right) \leq \frac{\sqrt{e}}{\sqrt{e}-1}\epsilon_n. \tag{A.12}$$

Then, (A.4) follows from (A.11) and (A.12).

The next point in our agenda is to define the set Ω_Y of $\mathbb{P}$-measure 1 and the stochastic processes $\{Y^{(n)}\}_{n\in\mathbb{N}}$ satisfying the properties (1)-(3) of the statement of the Lemma. But before, we set some additional notation.

For each $\phi \in \Phi$, let $X_\phi = \{X_\phi(t,\omega) : t \in [0,T], \omega \in \Omega\}$ be a continuous version of $X(\phi) = \{X_t(\phi)\}_{t\in[0,T]}$. For every $n \in \mathbb{N}$, let $\Gamma_n \subseteq \Omega$ given by

$$\Gamma_n = \left\{\omega : \forall j \in \mathbb{N},\, X_{\phi_j^{q_n}}(t,\omega) = X_t(\phi_j^{q_n})(\omega)\, \forall t \in D, \text{ and } t \mapsto X_{\phi_j^{q_n}}(t,\omega) \text{ is continuous}\right\}. \tag{A.13}$$

Then, for each $n \in \mathbb{N}$ the definition of $X_{\phi_j^{q_n}}$ implies that $\mathbb{P}(\Gamma_n) = 1$. Also, for each $n \in \mathbb{N}$ define $A_n \subseteq \Omega$ by

$$A_n = \left\{\omega : X_t(\xi_k)(\omega) = \sum_{j=1}^{k} a_{j,k,n} X_t(\phi_j^{q_n})(\omega) + X_t(\varphi_{k,n})(\omega),\, \forall k \in \mathbb{N},\, t \in D\right\}. \tag{A.14}$$

For every $n \in \mathbb{N}$, it follows from (A.3) and the linearity of X that $\mathbb{P}(A_n) = 1$. Now, for $n \in \mathbb{N}$ define

$$\Lambda_n = \Omega_n \cap \Gamma_n \cap A_n. \tag{A.15}$$

Then, it follows from (A.4) and the fact that $\mathbb{P}(\Gamma_n) = 1$ and $\mathbb{P}(A_n) = 1$ that

$$\mathbb{P}(\Lambda_n) \geq 1 - 2\frac{\sqrt{e}}{\sqrt{e}-1}\epsilon_n. \tag{A.16}$$

We are ready to define the stochastic processes $\{Y^{(n)}\}_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, let $\{f_j^{q_n}\}_{j\in\mathbb{N}}$ be a complete orthonormal system of Φ'_{q_n} dual to $\{\phi_j^{q_n}\}_{j\in\mathbb{N}}$ (i.e. $f_j^{q_n}[\phi_i^{q_n}] = \delta_{i,j}$). For each $t \in [0,T]$, we define

$$Y_t^{(n)}(\omega) := \begin{cases} \sum_{j=1}^{\infty} X_{\phi_j^{q_n}}(t,\omega) f_j^{q_n}, & \text{for } \omega \in \Lambda_n, \\ 0, & \text{elsewhere.} \end{cases} \tag{A.17}$$

Note that $Y^{(n)} = \{Y_t^{(n)}\}_{t\in[0,T]}$ is a well-defined Φ'_{q_n}-valued stochastic process. This is because if $\omega \in \Lambda_n$, then the infinite sum in (A.17) is convergent, as from (A.5), (A.13) and (A.15) we have:

$$\sup_{t\in[0,T]} q'_n(Y_t^{(n)}(\omega))^2 = \sup_{t\in D} \sum_{j=1}^{\infty} \left|X_{\phi_j^{q_n}}(t,\omega)\right|^2 = \sup_{t\in D} \sum_{j=1}^{\infty} \left|X_t(\phi_j^{q_n})(\omega)\right|^2 < \infty. \tag{A.18}$$

Moreover, it follows from (A.13), (A.15) and (A.17) it follows that for every $\phi \in \Phi_{q_n}$, $Y^{(n)}[\phi]$ is a continuous real-valued process. Therefore, $Y^{(n)}$ satisfies the property (1) in the statement of the Lemma.

Also from (A.13), (A.15) and (A.17) it follows that for each $\omega \in \Lambda_n$, $Y_t^{(n)}(\omega)[i_{q_n}\phi_j^{q_n}] = X_{\phi_j^{q_n}}(t,\omega) = X_t(\phi_j^{q_n})(\omega)$, for all $j \in \mathbb{N}$ and $t \in [0,T]$. Similarly, from the fact that

$q_n(\varphi_{j,n}) = 0$ for all $j \in \mathbb{N}$, and from (1.5) we have $\left| f_j^{q_n}[i_{q_n}\varphi_{j,n}] \right| \leq q_n'(f_j^{q_n}) q_n(i_{q_n}\varphi_{j,n}) = 0$, for all $j \in \mathbb{N}$, and then (A.17) implies that $Y_t^{(n)}(\omega)[i_{q_n}\varphi_{j,n}] = X_t(\varphi_{j,n})(\omega) = 0$, for all $j \in \mathbb{N}$. So, by (A.14) and (A.15) we have

$$\forall\, \omega \in \Lambda_n, \quad Y_t^{(n)}(\omega)[i_{q_n}\xi_k] = X_{\xi_k}(t,\omega) = X_t(\xi_k)(\omega), \quad \forall\, k \in \mathbb{N}, t \in [0,T]. \tag{A.19}$$

Now we are going to show that (A.19) implies that for every $\phi \in \Phi$, $Y_t^{(n)}[i_{q_n}\phi] = X_t(\phi)$ $\mathbb{P}$-a.e. on Λ_n, for all $t \in [0,T]$.

Let $\phi \in \Phi$. Since $B = \{\xi_j\}_{j\in\mathbb{N}}$ is dense in Φ_α, there exists a subsequence $\{\xi_{j_k}\}_{k\in\mathbb{N}} \subseteq B$ that α-converges to ϕ. As α is the countably Hilbertian topology generated by the semi-norms $\{q_n\}_{n\in\mathbb{N}}$, then Φ'_{q_n} is continuously embedded on Φ'_α and therefore $Y_t^{(n)}(\omega) \in \Phi'_\alpha$, for all $t \in [0,T]$ and $\omega \in \Omega$. Therefore, $Y_t^{(n)}(\omega)[i_{q_n}\xi_{j_k}] \to Y_t^{(n)}(\omega)[i_{q_n}\phi]$, as $k \to \infty$, for all $t \in [0,T]$ and $\omega \in \Omega$.

On the other hand, observe that (A.1) implies that X is continuous with respect to the countably Hilbertian topology on Φ generated by the semi-norms $\{p_n\}_{n\in\mathbb{N}}$ and since this topology is weaker than α (because $p_n \leq q_n$, for all $n \in \mathbb{N}$), then it follows that X is α-continuous. Therefore, there exists $\Delta_\phi \subseteq \Omega$ with $\mathbb{P}(\Delta_\phi) = 1$ such that for all $\omega \in \Delta_\phi$, $X_t(\xi_{j_k})(\omega) \to X_t(\phi)(\omega)$, as $k \to \infty$, for all $t \in [0,T]$.

Therefore, (A.19) and the uniqueness of limits implies that

$$\forall\, \omega \in \Lambda_n \cap \Delta_\phi, \quad Y_t^{(n)}(\omega)[i_{q_n}\phi] = X_t(\phi)(\omega), \quad \forall\, t \in [0,T]. \tag{A.20}$$

Our final step is to define the set Ω_Y and to verify that it and the processes $\{Y^{(n)}\}_{n\in\mathbb{N}}$ defined in (A.17) satisfy the conditions (2) and (3) of the statement of the Lemma. First, it follows from (A.16), our assumption that $\sum_{n\in\mathbb{N}} \epsilon_n < \infty$, and the Borel-Cantelli lemma that

$$\mathbb{P}\left(\Omega_Y\right) = 1, \quad \text{where} \quad \Omega_Y := \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} \Lambda_n. \tag{A.21}$$

Let $\omega \in \Omega_Y$. Then, it follows from (A.18) and (A.21) that there exists some $N(\omega)$ such that for all $n \geq N(\omega)$, $\sup_{t\in D} q_n'(Y_t^{(n)}(\omega)) < \infty$. Thus, the property (2)(a) of the statement of the Lemma is satisfied. Moreover, from (A.19) and (A.21) there exists $N(\omega)$ such that for all $m \geq n \geq N(\omega)$, for every $k \in \mathbb{N}$ and $t \in [0,T]$ we have $Y_t^{(m)}(\omega)[i_{q_m}\xi_k] = Y_t^{(n)}(\omega)[i_{q_n}\xi_k]$. But as $B = \{\xi_j\}_{j\in\mathbb{N}}$ is dense Φ_α, and therefore in Φ_{q_m} and in Φ_{q_n}, then it follows that for all $t \in [0,T]$, $Y_t^{(m)}(\omega)[i_{q_m}\phi] = Y_t^{(n)}(\omega)[i_{q_n}\phi]$ for all $\phi \in \Phi$, that is $Y_t^{(m)}(\omega) = i'_{q_n,q_m} Y_t^{(n)}(\omega)$. Hence, the property (2)(b) of the statement of the Lemma is also satisfied.

Finally, the property (3) of the statement of the Lemma is a consequence of (A.20) and (A.21). $\square$

Proof of Theorem 1.2.18. We use similar arguments as those used by Mitoma in [72]. Let $\{q_n\}_{n\in\mathbb{N}}$, Ω_Y and $\{Y^{(n)}\}_{n\in\mathbb{N}}$ be as given in Lemma A.0.14.

For every $n \in \mathbb{N}$, let ϱ_n be a continuous Hilbertian semi-norm on Φ such that $q_n \leq \varrho_n$ and i_{q_n,ϱ_n} is Hilbert-Schmidt. Let θ be the countably Hilbertian topology on Φ generated by the semi-norms $\{\varrho_n\}_{n\in\mathbb{N}}$. The topology θ is then weaker than $\mathcal{T}$ and finer than α.

Let $Y=\{Y_t\}_{t\in[0,T]}$ be defined for each $t\in[0,T]$ by

$$Y_t(\omega):=\begin{cases} i'_{q_n,\varrho_n}Y^{(n)}_t(\omega), & \text{for } \omega\in\Omega_Y,\ n\geq N(\omega),\\ 0, & \text{elsewhere.}\end{cases} \tag{A.22}$$

This stochastic process is well-defined. In effect, as for all $m,n\in\mathbb{N}$, $m\geq n$, $q_n\leq q_m$, $q_n\leq\varrho_n$, and $q_m\leq\varrho_m$, then the following diagram commutes:

$$\begin{array}{ccc} \Phi'_{q_n} & \xrightarrow{i'_{q_n,\varrho_n}} & \Phi'_{\varrho_n} \\ {\scriptstyle i'_{q_n,q_m}}\big\downarrow & & \big\downarrow{\scriptstyle i'_{\varrho_n,\varrho_m}} \\ \Phi'_{q_m} & \xrightarrow[i'_{q_m,\varrho_m}]{} & \Phi'_{\varrho_m} \end{array} \tag{A.23}$$

Then Lemma A.0.14(2)(b) and (A.23) implies that for each $\omega\in\Omega_Y$, if $m\geq n\geq N(\omega)$ we have

$$i'_{q_m,\varrho_m}Y^{(m)}_t(\omega)=i'_{q_m,\varrho_m}\circ i'_{q_n,q_m}Y^{(n)}_t(\omega)=i'_{\varrho_n,\varrho_m}\circ i'_{q_n,\varrho_n}Y^{(n)}_t(\omega),\quad \forall\, t\in[0,T].$$

Therefore, $Y=\{Y_t\}_{t\in[0,T]}$ is well-defined. Moreover, as for each $n\in\mathbb{N}$, Φ'_{ϱ_n} is continuously embedded in $(\Phi'_\theta,\beta_\theta)$ then it follows from (A.22) that Y is a $(\Phi'_\theta,\beta_\theta)$-valued process.

Now we are going to show that Y is continuous. For every $n\in\mathbb{N}$, let $\{\phi^{\varrho_n}_j\}_{j\in\mathbb{N}}\subseteq\Phi$ be a complete orthonormal system in Φ_{ϱ_n}. Fix $\omega\in\Omega_Y$ and let $n\geq N(\omega)$. Then, from the definition of the dual operator i'_{q_n,ϱ_n} of i_{q_n,ϱ_n}, from (1.5) applied to q_n and from Lemma A.0.14(1)-(2)(a), we have

$$\begin{aligned}\sum_{j=1}^{\infty}\sup_{t\in[0,T]}\left|i'_{q_n,\varrho_n}Y^{(n)}_t(\omega)[\phi^{\varrho_n}_j]\right|^2 &\leq \sum_{j=1}^{\infty}\sup_{t\in[0,T]} q'_n(Y^{(n)}_t(\omega))^2 q_n(i_{q_n,\varrho_n}\phi^{\varrho_n}_j)^2 \\ &= \left(\sup_{t\in[0,T]} q'_n(Y^{(n)}_t(\omega))^2\right)||i_{q_n,\varrho_n}||^2_{\mathcal{L}_2(\Phi_{\varrho_n},\Phi_{q_n})}<\infty,\end{aligned} \tag{A.24}$$

where we recall that i_{q_n,ϱ_n} is Hilbert-Schmidt.

Next we prove the right continuity of the map $t\mapsto Y_t(\omega)$ in Φ'_{ϱ_n}. Let $0\leq t<T$. Then, from (A.22), Parseval's identity, (A.24), the dominated convergence theorem and the continuity of each map $t\mapsto Y^{(n)}_t(\omega)[i_{q_n,\varrho_n}\phi^{\varrho_n}_j]$ (Lemma A.0.14(1)), we have

$$\begin{aligned}\lim_{s\to t+}\varrho'_n(Y_t(\omega)-Y_s(\omega))^2 &= \lim_{s\to t+}\sum_{j=1}^{\infty}\left|i'_{q_n,\varrho_n}(Y^{(n)}_t(\omega)-Y^{(n)}_s(\omega))[\phi^{\varrho_n}_j]\right|^2 \\ &= \sum_{j=1}^{\infty}\lim_{s\to t+}\left|(Y^{(n)}_t(\omega)-Y^{(n)}_s(\omega))[i_{q_n,\varrho_n}\phi^{\varrho_n}_j]\right|^2=0.\end{aligned}$$

Therefore, the map $t\mapsto Y_t(\omega)$ is right-continuous in Φ'_{ϱ_n}. The left continuity can be proven similarly. Moreover, as Φ'_{ϱ_n} is continuously included on Φ'_θ then the continuity of $t\mapsto Y_t(\omega)$ in Φ'_{ϱ_n} implies its continuity in Φ'_θ. Hence, the process Y is a Φ'_θ-valued continuous process.

The fact that for every $\phi\in\Phi$, $Y[\phi]$ is a version of $X(\phi)$ is a direct consequence of Lemma A.0.14 and (A.22). Finally, as Φ'_θ is continuously included on Φ'_β, the above properties shows that Y is a Φ'_β-valued continuous version of X.

To prove the uniqueness, notice that if Z is another version of X satisfying the properties of the Theorem, then for every $\phi \in \Phi$, $Y_t[\phi] = X_t(\phi) = Z_t(\phi)$ for all $t \in [0,T]$, and because Y and Z are both continuous then it follows from Proposition 1.2.15 that Y and Z are indistinguishable process. ☐

Corollary 1.2.19. *Let $(\Phi, \mathcal{T})$ be a nuclear space and let $X = \{X_t\}_{t\in[0,T]}$ be a cylindrical process in Φ' such that for each $\phi \in \Phi$, the real-valued process $X(\phi) := \{X_t(\phi)\}_{t\in[0,T]}$ has a continuous (respectively càdlàg) version. Suppose that there exists a continuous Hilbertian semi-norm p on Φ such that the linear mapping from Φ into $C_T(\mathbb{R})$ (respectively $D_T(\mathbb{R})$) given by $\phi \mapsto X(\phi)$ is p-continuous. Then, there exists a continuous Hilbertian semi-norm ϱ on Φ, $p \leq \varrho$, such that $i_{p,\varrho}$ is Hilbert-Schmidt and a Φ'_{ϱ}-valued continuous (respectively càdlàg) process $Y = \{Y_t\}_{t\in[0,T]}$, such that for every $\phi \in \Phi$, $Y[\phi]$ is a version of $X(\phi)$. Moreover, Y is unique up to indistinguishable versions in Φ'_{β}.*

Proof. With the same terminology as in the proof of Lemma A.0.14, the p-continuity of the map $\phi \mapsto X(\phi) := \{X_t(\phi)\}_{t\in[0,T]}$ implies that (A.1) is satisfied for $p_n = p$, for all $n \in \mathbb{N}$. In that case, we would have in (A.4), (A.5), (A.13) and (A.14) that $q_n = q$ for some continuous Hilbertian semi-norm q on Φ, $p \leq q$, and such that $i_{p,q}$ is Hilbert-Schmidt. Therefore, we have from (A.15) and (A.17) that $\Lambda_n = \Lambda_m$ and $Y^{(n)} = Y^{(m)}$ for all $m, n \in \mathbb{N}$. If we choose ϱ, $q \leq \varrho$, such that $i_{q,\varrho}$ is Hilbert-Schmidt, and if in the proof of Theorem 1.2.18 we take $\varrho_n = \varrho$ for all $n \in \mathbb{N}$, then Y defined by (A.22) is a Φ'_{ϱ}-valued continuous processes such that for every $\phi \in \Phi$, the processes $X(\phi)$ and $Y[\phi]$ are indistinguishable. ☐

Appendix B

Basic Properties of Hilbert-Schmidt Operators

In this Appendix we review the basic properties of some important classes of continuous linear operators in Banach and Hilbert spaces.

Let X and Y be two real or complex Banach spaces. A linear operator $T : X \to Y$ is said to be bounded if its operator norm

$$||T||_{\mathcal{L}(X,Y)} := \sup \{||Tx||_Y : ||x||_X \leq 1\},$$

is finite. Denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X into Y. Equipped with the operator norm it is a Banach space. As the dual spaces X' and Y' are Banach spaces, it follows that $\mathcal{L}(Y', X')$ is a Banach space. Moreover, if $T \in \mathcal{L}(X, Y)$ then $T' \in \mathcal{L}(Y', X')$ and furthermore it follows that $||T'||_{\mathcal{L}(Y',X')} = ||T||_{\mathcal{L}(X,Y)}$.

We proceed to review the definition of Hilbert-Schmidt operators and some of their properties. In the following, H and G will represent two Hilbert spaces.

Definition B.0.15. A bounded linear operator $T : H \to G$ is called a **Hilbert-Schmidt** operator if there exists an orthonormal basis $\{h_i\}_{i\in I}$ in H such that

$$\sum_{i\in I} ||Th_i||_G^2 < \infty$$

Theorem B.0.16. *Let $T \in \mathcal{L}(H, G)$. The following conditions are equivalent*

(1) T is a Hilbert-Schmidt operator.

(2) $\sum_{i\in I} ||Th_i||_G^2 < \infty$ for any orthonormal basis $\{h_i\}_{i\in I}$ in H.

(3) T admits the representation

$$Tx = \sum_{j\in J} \lambda_j \langle x\,,\, h_j\rangle_H \, g_j, \quad \forall x \in H$$

where $\{h_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are orthonormal sets in H and G respectively, the λ_j are positive numbers such that $\sum_{j\in J} \lambda_j^2 < \infty$. The index set J is at most countable.

For a proof see Lemma 1 and Theorem 2 of Section 2.2 of Chapter 1 of Gel'fand and Vilenkin [31] p.33-4.

Denote by $\mathcal{L}_2(H,G)$ the space of all Hilbert-Schmidt operators from H to G. It is a Hilbert space equipped with the Hilbertian norm $||\cdot||_{\mathcal{L}_2(H,G)}$ defined by

$$||T||_{\mathcal{L}_2(H,G)} = \sum_{i\in I} ||Th_i||_G^2 , \quad \text{for all } T \in \mathcal{L}_2(H,G) \tag{B.1}$$

where $\{h_i\}_{i\in\mathbb{N}}$ is some orthonormal basis in H. $\mathcal{L}_2(H,G)$ is separable if H and G are separable. By definition $T \in \mathcal{L}(H,G)$ is Hilbert-Schmidt if and only if $||T||_{\mathcal{L}_2(H,G)} < \infty$ and by Theorem B.0.16, $||T||_{\mathcal{L}_2(H,G)}$ is independent of the choice of the orthonormal basis of H. Moreover, one can verify that $||T||_{\mathcal{L}(H,G)} \leq ||T||_{\mathcal{L}_2(H,G)}$.

Proposition B.0.17. *Let $T \in \mathcal{L}_2(H,G)$. Then,*

(1) $T' \in \mathcal{L}_2(G',H')$ and $||T||_{\mathcal{L}_2(H,G)} = ||T'||_{\mathcal{L}_2(G',H')}$.

(2) Let F_1, F_2 be two separable Hilbert spaces. If $S \in \mathcal{L}(F_1,H)$ and $U \in \mathcal{L}(G,F_2)$, then $U \circ T \in \mathcal{L}_2(H,F_2)$ and $T \circ S \in \mathcal{L}_2(F_1,G)$. Moreover,

$$||U \circ T||_{\mathcal{L}_2(H,F_2)} \leq ||T||_{\mathcal{L}_2(H,G)} \, ||U||_{\mathcal{L}(G,F_2)} ,$$

and

$$||T \circ S||_{\mathcal{L}_2(F_1,G)} \leq ||S||_{\mathcal{L}(F_1,H)} \, ||T||_{\mathcal{L}_2(H,G)} .$$

For a proof see Section 2.2, Chapter 1 of Gel'fand and Vilenkin [31].

The following result offers a very useful characterization of Hilbert-Schmidt operators. It is based on the relationship between Hilbert-Schmidt operators and p-summing operators between Hilbert spaces. For details the reader is referred to Theorem 2.12 and Corollary 4.13 of Diestel, Jarchow and Tonge [25], p.44,85.

Theorem B.0.18. *A linear operator $T : H \to G$ is Hilbert-Schmidt if and only if for every $1 \leq p < \infty$ there exists a constant $C > 0$, and a Radon probability measure ν on the unit ball B^* of H' (equipped with the weak topology) such that,*

$$||Tx||_G \leq C \cdot \left(\int_{B^*} |f[x]|^p \, \nu(df) \right)^{1/p} , \quad \forall \, x \in H.$$

Appendix C

The Bochner Integral

In this section we review the construction and basic properties of the Bochner integral. Our exposition is completely based on Chapter 1 of [39]. All the proofs and a very careful exposition of the theory can be found there.

Let $(S, \mathcal{A}, \mu)$ be a measure space and let $(X, ||\cdot||)$ be a real or complex Banach space. A function $f : S \to X$ is called μ-**simple** if it is of the form $f = \sum_{n=1}^{N} \mathbb{1}_{A_n} x_n$, where for every $1 \leq n \leq N$, $x_n \in X$ and the set $A_n \in \mathcal{A}$ satisfies $\mu(A_n) < \infty$.

We say that a function $f : S \to X$ is **strongly μ-measurable** if there exists a sequence of μ-simple functions $f_n : S \to X$ that converges μ-almost everywhere to f. The following result gives sufficient conditions for strongly μ-measurability.

Proposition C.0.19. *If $f : S \to X$ is Borel measurable, the image $f(S) \subseteq X$ of S under f is separable and $\int_S ||f(s)|| \, \mu(ds) < \infty$, then f is strongly μ-measurable.*

Now we proceed to define the Bochner integral. For a μ-simple function $f = \sum_{n=1}^{N} \mathbb{1}_{A_n} x_n$ we define

$$\int_S f(s)\mu(ds) := \sum_{n=1}^{N} \mu(A_n) x_n.$$

One can easily check that this definition does not depend on any particular representation of the function f.

Definition C.0.20. A strongly μ-measurable function $f : S \to X$ is **Bochner integrable with respect to** μ if there exists a sequence of μ-simple functions $f_n : S \to X$ such that

$$\lim_{n\to\infty} \int_S ||f(s) - f_n(s)|| \, \mu(ds) = 0.$$

Let $f : S \to X$ be Bochner integrable with respect to μ and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of simple functions approximating f as in Definition C.0.20. Because for any $m, n \in \mathbb{N}$ we have

$$\begin{aligned} \left|\left| \int_S f_m(s)\mu(ds) - \int_S f_n(s)\mu(ds) \right|\right| &\leq \int_S ||f_m(s) - f_n(s)|| \, \mu(ds) \\ &\leq \int_S ||f(s) - f_m(s)|| \, \mu(ds) + \int_S ||f(s) - f_n(s)|| \, \mu(ds), \end{aligned}$$

then $\left\{\int_S f_n(s)\mu(ds) : n \in \mathbb{N}\right\}$ is a Cauchy sequence on X and because X is complete, this sequence has a limit. This limit is called the **Bochner integral of f with respect to μ** and we denote it by $\int_S f(s)\mu(ds)$, i.e.

$$\int_S f(s)\mu(ds) := \lim_{n\to\infty} \int_S f_n(s)\mu(ds).$$

It is not difficult to see that this definition does not depend on the choice of the approximation sequence for f.

Proposition C.0.21. *A strongly μ-measurable function $f : S \to X$ is Bochner integrable with respect to μ if and only if*

$$\int_S ||f(s)||\, \mu(ds) < \infty,$$

and in this case we have

$$\left|\left|\int_S f(s)\mu(ds)\right|\right| \leq \int_S ||f(s)||\, \mu(ds).$$

Proposition C.0.22. *If $f : S \to X$ is Bochner integrable with respect to μ and T is a continuous and linear operator from X into the Banach space Y, then $Tf : S \to Y$ is Bochner integrable with respect to μ and*

$$T\int_S f(s)\mu(ds) = \int_S Tf(s)\mu(ds).$$

If $f : S \to X$ and $g : S \to X$ are strongly μ-measurable functions, we say that they are **equivalent** if $f = g$ μ-almost everywhere. This defines an equivalence relation on the set of all the strongly μ-measurable functions from S into X.

Definition C.0.23. For $1 \leq p < \infty$, we define the **Bochner space** $L^p(S, \mathcal{A}, \mu; X)$ to be the linear space of all (equivalence classes of) strongly μ-measurable functions $f : S \to X$ for which

$$\int_S ||f(s)||^p\, \mu(ds) < \infty.$$

The space $L^p(S, \mathcal{A}, \mu; X)$ is a Banach space when equipped with the norm

$$||f||_{L^p(S,\mathcal{A},\mu;X)} = \left(\int_S ||f(s)||^p\, \mu(ds)\right)^{1/p}.$$

Appendix D

Semigroups of Linear Operators in Locally Convex Spaces

In this section we will review the basic properties of some types of C_0-semigroups of continuous linear operators on a locally convex space. To do this we will need firstly to recall some properties of the Riemann integral taking values in a locally convex space.

§ D.1 Riemann Integral in Locally Convex Spaces

In this section we summarize the definition and main properties of the Riemann integral in a locally convex space. We base our exposition on Albanese, Bonet and Ricker [1].

Let $[a,b] \subseteq \mathbb{R}$ be a compact interval. Let P be a partition of $[a,b]$, i.e. $P = \{t_k\}_{k=0}^n$ such that $a = t_0 < t_1 < \cdots < t_n = b$, for some $n \in \mathbb{N}$. Set $|P| = \max_{1\le k\le n}\{t_k - t_{k-1}\}$ and define $[P] = \bigtimes_{k=1}^n [t_{k-1}, t_k] \subseteq \mathbb{R}^n$. Denote by $\mathcal{P}([a,b])$ the set of all partitions of $[a,b]$. Define on $\mathcal{D} := \bigcup_{P\in\mathcal{P}([a,b])}\{(P,\xi) : \xi \in [P]\}$ a pre-order $\geq$ as follows: given $P, Q \in \mathcal{P}([a,b])$ and $\xi \in [P]$, $\eta \in [Q]$, say $(P,\xi) \geq (Q,\eta)$ whenever $|P| \leq |Q|$. Hence, $(\mathcal{D}, \geq)$ is a directed set.

Let E be a (Hausdorff) locally convex space. Let $F : [a,b] \to E$ be a bounded function. Given $(P,\xi) \in \mathcal{D}$, with $P = \{t_k\}_{k=0}^n$, $\xi = \{\xi_1, \ldots, \xi_n\}$, then $R(F,P,\xi)$ given by:

$$R(F,P,\xi) = \sum_{k=1}^{n} F(\xi_k)(t_k - t_{k-1}) \in E,$$

is called a **Riemann sum** of F relative to (P,ξ). Then, from the above we have that $\{R(F,P,\xi) : (P,\xi) \in \mathcal{D}\}$ is a net in E. If this net converges to some element of E, then we will say that F is **Riemann integrable** and the limit, which we denote by $\int_a^b F(t)dt$, will be called the **Riemann integral** of F on $[a,b]$.

Denote by $C([a,b],E)$ the vector space of all continuous functions from $[a,b]$ into E. It is a (Hausdorff) locally convex space equipped with the **topology of uniform convergence**. If $\{p_\gamma\}_{\gamma\in\Gamma}$ is a family of semi-norms generating the topology on E, then the family $\{q_\gamma\}_{\gamma\in\Gamma}$ given by

$$q_\gamma(F) = \sup_{t\in[a,b]} p_\gamma(F(t)), \quad \forall\, F \in C^0([a,b],E),$$

generates the topology of uniform convergence on $C([a,b],E)$.

A sufficient condition for the existence of the Riemann integral is the following. For a proof see Albanese, Bonet and Ricker [1].

Theorem D.1.1. *Let E be sequentially complete (e.g E complete) and let $F \in C([a,b],E)$. Then, F is Riemann integrable.*

Let J be an open subset of $\mathbb{R}$. A function $F : J \to E$ is differentiable at $t_0 \in J$ if there exists $F'(t_0) \in E$ satisfying $\lim_{t\to t_0} \frac{F(t)-F(t_0}{t-t_0} = F'(t_0)$, with the limit being taken in the sense of the topology on E. We write $F \in C^1(J,E)$ if F is differentiable at each point $t_0 \in J$ and $F' : J \to E$ is continuous. We denote by $C^1([a,b],E)$ the vector space of all $F : [a,b] \to E$ such that there exists an open $J \subseteq \mathbb{R}$ with $[a,b] \subseteq J$ and some $G \in C^1(J,E)$ such that $G\big|_{[a,b]} = F$.

Some of the basic properties of the Riemann integral are given in the following result.

Proposition D.1.2. *Let E be sequentially complete and $F \in C([a,b],E)$, with $a < b$. Then:*

(1) $\int_a^c F(t)dt + \int_c^b F(t)dt = \int_a^b F(t)dt$, *for all* $c \in]a,b[$

(2) $f\left[\int_a^b F(t)dt\right] = \int_a^b f\left[F(t)\right]dt, \quad \forall f \in E'$.

(3) Let G be a sequentially complete (Hausdorff) locally convex space and let $A \in \mathcal{L}(E,G)$. Then, $\int_a^b AF(t)dt = A\left(\int_a^b F(t)dt\right)$.

(4) $\int_a^b (F\circ h)(t)h'(t)dt = \int_{h(a)}^{h(b)} F(t)dt$, *for all* $h \in C^1(\mathbb{R},\mathbb{R})$.

(5) If $F \in C^1([a,b],E)$, then $F(b) - F(a) = \int_a^b F'(t)dt$.

(6) For every continuous semi-norm p on E, $p\left(\int_a^b F(t)dt\right) \le \int_a^b p(F(t))dt$.

§ D.2 C_0-semigroups on Locally Convex Spaces

Let E be a complete (Hausdorff) locally convex space. In this section we review some basic properties $(C_0,1)$-semigroups of continuous linear operators on E. Our exposition is based on Yosida [119] and Babalola [5].

Definition D.2.1. Let $\{S(t)\}_{t\ge 0}$ be a one-parameter family in $\mathcal{L}(E,E)$. It is called a C_0-semigroup of continuous linear operators (or a C_0-semigroup for short) if it satisfies:

(1) $S(0) = I$, where I is the identity operator on E,

(2) $S(t)S(s) = S(t+s)$, for all $t,s \ge 0$.

(3) $\lim_{t\to s} S(t)x = S(s)x$, for all $s \ge 0$ and any $x \in E$. (Strong continuity)

Definition D.2.2. A C_0-semigroup $\{S(t)\}_{t\ge 0}$ on E is said to be **equicontinuous** if for every continuous semi-norm p on E there exists a continuous semi-norm q on E such that $p(S(t)x) \le q(x)$, for all $t \ge 0$, $x \in X$.

Definition D.2.3. A C_0-semigroup $\{S(t)\}_{t\ge 0}$ on E is said to be $(C_0,1)$**-semigroup** for each continuous semi-norm p on E there exist some $\vartheta_p \ge 0$ and a continuous semi-norm q on E such that $p(S(t)x) \le e^{\vartheta_p t}q(x)$, for all $t \ge 0$, $x \in E$.

Any equicontinuous C_0-semigroup is a $(C_0,1)$-semigroup (this is the case $\vartheta_p = 0$) but the converse is not true in general (see Babalola [5], p.177).

Theorem D.2.4. *A C_0-semigroup $\{S(t)\}_{t\geq 0}$ on E is a $(C_0,1)$-semigroup if and only if there exists a family $\{p_\alpha\}_{\alpha\in\Gamma}$ of semi-norms generating the topology on E such that for each $\alpha\in\Gamma$ there exist some constants $M_\alpha \geq 1$, $\theta_\alpha \geq 0$ such that*

$$p_\alpha(S(t)x) \leq M_\alpha e^{\theta_\alpha t} p_\alpha(x), \quad \text{for all } t \geq 0,\, x \in E.$$

If in particular the semigroup $\{S(t)\}_{t\geq 0}$ is equicontinuous, then one can choose $M_\alpha = 1$, $\theta_\alpha = 0$.

The **infinitesimal generator** A of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on E is defined by

$$Ax = \lim_{h\downarrow 0} \frac{S(h)x - x}{h} \quad \text{(limit in } E\text{)},$$

whenever the limit exists, the domain of A being the set $\mathrm{Dom}(A) \subseteq E$ for which the above limit exists.

In the next result we summarize some of the properties of the generator A and its domain. As we are assuming E is sequentially complete, and because of the continuity of the map $t \mapsto S(t)x$, for each $x \in E$, then the Riemann integral $\int_0^s S(t)x\,dt$ exists for $0 < s < \infty$ and $x \in E$ (Theorem D.1.1). For a proof of the following results see Section 3, Chapter IX of Yosida [119].

Theorem D.2.5. *Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup on E with infinitesimal generator A. Then:*

(1) If $x \in Dom(A)$, then $S(t)x \in Dom(A)$, for all $t \geq 0$ and $S(\cdot) \in C^1([0,\infty), E)$. Moreover, $\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax$, for all $t \geq 0$.
(2) An element $x \in E$ belongs to $Dom(A)$ and $Ax = y$ if and only if $S(t)x - x = \int_0^t S(r)y\,dr$, for all $t \geq 0$.
(3) For every $x \in E$, $\int_a^b S(t)x\,dt \in Dom(A)$ $(0 \leq a < b < \infty)$ and we have $A\left(\int_a^b S(t)x\,dt\right) = S(b)x - S(a)x$.
(4) $Dom(A)$ is dense in E.
(5) If E is barrelled, then A is a closed operator.

One of the most important properties of $(C_0,1)$-semigroups is that they are "compatible" with the family of Banach spaces determined by the semi-norms generating the topology on E given in Theorem D.2.4. More specifically, we have the following very important result:

Theorem D.2.6. *Let $\{S(t)\}_{t\geq 0}$ be a $(C_0,1)$-semigroup on E and let $\{p_\alpha\}_{\alpha\in\Gamma}$ be the family of semi-norms generating the topology on E given in Theorem D.2.4. Then, for each $\alpha \in \Gamma$, there exists a C_0-semigroup $\{S_{p_\alpha}(t)\}_{t\geq 0}$ (C_0-semigroup of contractions if $\{S(t)\}_{t\geq 0}$ is equicontinuous) on the Banach space E_{p_α} such that $\{S_{p_\alpha}(t)\}_{t\geq 0}$ is an extension of $\{S(t)\}_{t\geq 0}$, i.e.*

$$S_{p_\alpha}(t) i_{p_\alpha} x = i_{p_\alpha} S(t)x, \quad \forall\, x \in E,\, t \geq 0,$$

where we recall that $i_{p_\alpha} : E \to E_{p_\alpha}$ is the canonical inclusion of E into E_{p_α}. Moreover, if $\alpha, \beta \in \Gamma$ are such that $\alpha \leq \beta$, then

$$S_{p_\alpha}(t) i_{p_\alpha,p_\beta} x = i_{p_\alpha,p_\beta} S_{p_\beta}(t)x, \quad \forall\, x \in E_{p_\beta},\, t \geq 0,$$

where we recall that $i_{p_\alpha,p_\beta} : E_{p_\beta} \to E_{p_\alpha}$ is the canonical inclusion of E_{p_β} into E_{p_α}.

Furthermore, if A is the infinitesimal generator of $\{S(t)\}_{t\geq 0}$ and for each $\alpha \in \Gamma$, A_{p_α} is the infinitesimal generator of $\{S_{p_\alpha}(t)\}_{t\geq 0}$, then for each $\alpha \in \Gamma$ we have

$$A_{p_\alpha} i_{p_\alpha} x = i_{p_\alpha} A x, \quad \forall\, x \in Dom(A),$$

and if $\alpha, \beta \in \Gamma$ are such that $\alpha \leq \beta$, then

$$A_{p_\alpha} i_{p_\alpha, p_\beta} x = i_{p_\alpha, p_\beta} A_{p_\beta} x, \quad \forall\, x \in Dom(A_{p_\beta}).$$

We finalize this section by studying the properties of the dual semi-group of a C_0-semigroup. In general, if $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup on E then the family $\{S(t)'\}_{t\geq 0}$ of the dual operators is a semigroup of continuous linear operators on E'_β, but in general it is not a C_0-semigroup as the map $t \mapsto S(t)'f$ is not necessarily continuous, for $f \in E'_\beta$ (see Proposition 1, Section 1, Chapter VII of Yosida [119], p.195). Nevertheless, this continuity property is preserved in the case of E being reflexive as the next result shows (see Section 13, Chapter IX of Yosida [119]).

Theorem D.2.7. *Assume E is reflexive and let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup on E with generator A. Then, $\{S(t)'\}_{t\geq 0}$ is a C_0-semigroup on E'_β with generator A'. Moreover, if $\{S(t)\}_{t\geq 0}$ is equicontinuous then $\{S(t)'\}_{t\geq 0}$ is equicontinuous and $R(\lambda, A') = R(\lambda, A)'$ for any $Re(\lambda) > 0$.*

Remark D.2.8. *In general, the dual semigroup of a $(C_0, 1)$-semigroup is not a $(C_0, 1)$-semigroup on E'_β, even if E is reflexive (see Babalola [5]).*

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