AN INVESTIGATION OF THE FORMATION OF MATHEMATICAL ABSTRACTIONS THROUGH SCAFFOLDING

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Submitted in accordance with the requirements for the degree of Doctor of Philosophy

The University of Leeds
School of Education

July 2005

The candidate confirms that the work submitted is his own and that appropriate credit has been given wherever reference has been made to the work of others.

Acknowledgement

This PhD thesis is a product achieved with the contributions of many. First and foremost, my supervisors, Dr. John Monaghan and Mr. Tom Roper, are two solid researchers who contributed to this thesis from the very start by their guidance, by their careful readings and revisions and by their supports and encouragements. I would like to express my sincere thanks to them both for all they have done.

I would like to thank Prof. Peter Tomlinson for drawing my attention to the issue of consolidation and thank Prof. Phil Scott for his suggestions in my analysis of scaffolding.

I am indebted to my dear friend, Erhan Bingolbali, for his support both intellectually and emotionally.

My thanks also go to Dr. Ali Delice, Erol Karakirik and Kerem Karaagac for their encouragement and academic support. I would like to thank the students who took part in data collection phase of this study.

Turkish Ministry of Education funded this research study and without this I would not be able to complete this thesis.

Last, but not the least, I am truly grateful to all my family members who have always been there for me whenever I needed them. I deeply and sincerely thank them for their patience, prayers and faith in me.

Acknowledgment of authorship

This PhD thesis has led to several conference and journal papers which are already published or in the process of publication. Some of the papers are prepared in collaboration with Dr. John Monaghan or Mr. Tom Roper. Contents of the papers are used in the relevant chapters of this PhD thesis with certain modifications. I hereby would like to acknowledge the authorship and contribution of these academics to the relevant sections of this thesis in which the contents of the jointly produced papers are used. A list of the papers stemming from this study is presented below.

Monaghan, J. and Ozmantar, M.F. (2004), 'Abstraction and consolidation'. In M. J. Høines and A. B. Fuglestad (eds.), *Proceedings of the 28th international conference for the psychology of mathematics education,* vol.3 (pp. 353-360), Bergen, Norway.

Monaghan, J. and Ozmantar, M.F. (in press), 'Abstraction and consolidation'. Paper is accepted for publication in *Educational Studies in Mathematics.*

Ozmantar, M.F and Monaghan, J. (2005), 'Voices in scaffolding mathematical constructions'. Paper is presented in CERME-4 (*Fourth Congress of the European Society for Research in Mathematics Education*), in Sant Feliu de Guixols, Spain, 17-21 February 2005.

Ozmantar, M.F. (2004), 'An empirical investigation of mathematical abstraction through scaffolding'. Paper is presented in II-YERME Summer School, Podebrady, Czech Republic, 23-29 August 2004.

Ozmantar, M.F. (2004), 'Scaffolding, abstraction and emergent goals', *Proceedings of the British Society for Research into Learning Mathematics*, Leeds, 24(2), 83-89.

Ozmantar, M.F. and Roper, T. (2004), 'Mathematical abstraction through scaffolding'. In M. J. Høines and A. B. Fuglestad (eds.), *Proceedings of the 28th international conference for the psychology of mathematics education,* vol.3 (pp. 481-488), Bergen, Norway: Bergen University College.

Ozmantar, M.F. (2005), 'Mathematical abstraction: a dialectical view'. The paper is presented in *Society for Research into Learning Mathematics*, Open University, Milton Keynes, 11 July 2005.

Ozmantar, M.F. (in press), 'Relational identities in peer collaboration: self-perceptions, assumed roles and individual tendencies'. Paper is accepted for publication in *Euro-Asian Journal of Educational Research.*

ABSTRACT

This study takes an activity-theoretic approach to abstraction in context recently proposed by Hershkowitz, Schwarz and Dreyfus (2001, HSD hereafter). Key to HSD's theory of abstraction is the construction of new mathematical knowledge and consolidation of it. In this connection, this study aims to investigate three particular issues: (1) the construction of mathematical knowledge through scaffolding, (2) the nature of the consolidation process and (3) the validity of HSD's abstraction theory.

In order to investigate these issues, a qualitative research design methodology with explanatory and exploratory inquiry purposes was taken. This study employed multiple case study strategy with the purpose of literal and theoretical replications. A number of cases were designed with students working as pairs and individuals such that some of the students worked with the scaffolded help and others without. All participants worked on four days over four sequential tasks connected with the graphs of absolute value functions. Tasks were applied in paper-and-pencil format. The data for this study was composed of the participant's written works and audio records of the sessions.

In relation to the first issue, analysing the students' verbal data suggests certain causative relationships between the scaffolder's interventions and the students' developing constructions. It is also observed that the scaffolder's interventions mediate the students' constructions. Analysis of the data further suggests that construction through scaffolding is a subtle and intricate phenomenon which involves a complex set of social, cultural, historical, contextual and semiotic issues. It is argued, with examples, that scaffolded discourse involves many dynamics such as value judgements, individuals' personal histories, common cultural practices, individuals' emergent goals, voices of absent others and certain patterns of interaction.

Regarding the second issue, the data suggest that newly formed constructions are fragile entities and in need of consolidation. In the course of consolidation, it is observed that earlier constructions are reconstructed, used in a flexible manner and expressed confidently with general mathematical statements. These observations lead to the argument that an abstraction is a consolidated construction that can be used to create new constructions.

With regard to the final issue, on the basis of the students' verbal data, this study provides a critical evaluation of HSD's theory of abstraction by focusing on three key dimensions which characterise it: its epistemological and sociocultural principles, epistemic actions and genesis of an abstraction. Throughout this evaluation some clarifications and amendments are proposed to this theory.

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CHAPTER 1: INTRODUCTION

The idea of abstraction has been the subject of extensive scientific and philosophic discourse for centuries. Abstraction is a term often linked with an empiricist philosophy tracing from the writings of, for example, Aristotle to Hume to Russell. In this tradition, abstraction is viewed as higher-order knowledge which consists of 'classifications' and 'generalisations' arising from the recognition of commonalities isolated in a large number of specific instances. This view often considers abstraction as a developmental process ascending *from the concrete to the abstract.*

In mathematics, the issue of abstraction has long attracted particular attention of many. For example, von Glasersfeld and Richards (1983) trace it back to a von Humboldt manuscript written in 1796. Especially in the mid-20th century, the issue of abstraction has been applied to elementary mathematics learning through the works of such figures as Piaget and Dienes. However, mathematical abstraction seems to have been neglected until the 1990s. But in recent years, especially starting from the mid-1990s – for example Noss and Hoyles' (1996) construct of 'situated abstraction' – mathematical abstraction has re-entered the agenda of many mathematics educators. For instance, in PME 2002, a research forum with the participation of a wide range of researchers (Dreyfus, Gray, Boero, Gravemeijer, Hershkowitz, Schwarz, Sierpinska and Tall, 2002) was organised to discuss different theories and newly emergent ideas on the issue of mathematical abstraction. In PME-2004, there were seven research reports which directly dealt with mathematical abstraction (Ozmantar and Roper, 2004; Monaghan and Ozmantar, 2004; Williams, 2004; Kidron and Dreyfus, 2004; Bikner-Ahsbahs, 2004; Schwarz, Dreyfus, Hadas and Hershkowitz, 2004; Mitchelmore and White, 2004). Further to this, a recent volume of the journal 'Cognitive Science Quarterly' (volume 1 number 2-3) was devoted to the issue of abstraction. There appear two main reasons for a considerable increase, in recent years, in the number of studies dealing with mathematical abstraction.

The first one is that, in contrast to the many other topics in mathematics education, the issue of mathematical abstraction has been dealt with mainly at philosophical and theoretical levels and the number of empirical studies of abstraction remains limited. This is expressed by many stating, for example, that "although there is *little or no empirical* support for [von Glasersfeld's] specific assertions about the progressive abstraction of concepts, he does provide a clear account of how they *might* develop" (Stevenson, 1998, p.94; emphasis added). Further to this, the deficiency and hence necessity of empirical studies on abstraction is stated by von Glasersfeld himself, "the need for an experimental basis for the abstraction of concepts is often overlooked, because of the formalist myth that all that matters in mathematics is the manipulation of symbols" (1996, p.312). Even the theoretical formulations of the development of abstraction are not something upon which an agreement exists, as Greeno (1997, p.13) states, "on the issue of abstraction ... the disagreement ... is about theoretical formulations, rather than being about empirical claims."

The second reason is related to the dissatisfaction of the classical accounts of abstraction and a desire to develop alternative ways to view mathematical abstraction. The classical view attributes three features to abstraction. I will consider these features in greater depth in the next chapter (Chapter 2). However, I here briefly consider these features to give the reader an idea of why new accounts of abstraction have emerged. The first feature is that abstraction is classically seen as a process of decontextualisation. On the basis of the ideas inspired by the situated cognition and sociocultural views of cognitive growth, researchers found classical ideas wanting on the grounds that mathematical abstraction always takes place through social, cultural, historical and contextual forces operating in the learning environments and activities. The second classical feature is that abstraction is viewed as involving generalisation arising from the recognition of commonalities isolated in a large number of specific instances. A problem with this view is that if someone is to recognise commonalities, then surely that person must already have some understanding (which may be rudimentary) of the abstraction. The third classical feature is that abstraction is seen as an ascent from the concrete to the abstract. However, recently it has been proposed that abstraction is not so much an uphill struggle from the concrete to the abstract but rather involves a dialectical relationship between the abstract and the concrete (Davydov, 1990).

These realisations have recently awakened interest in carrying out empirical studies on the issue of abstraction which gave rise to the emergence of several theories, including Noss and Hoyles (1996), van Oers (2001), Ohlsson and Regan (2001) and Hershkowitz et al. (2001). Amongst these, I found the theory developed by Hershkowitz et al. most promising to carry out an empirical investigation of mathematical abstraction. This is because they provide an operational definition and suggest three epistemic actions (recognising, building-with and constructing) which, they argue, can be identifiable in any abstraction process. These authors view abstraction as a vertical reorganisation of previously constructed mathematics into a new mathematical structure. They argue that the construction of new knowledge is central to the formation of mathematical abstractions. The new structure emerging from the epistemic actions needs to be consolidated so that it can be used to form further abstractions. Although Hershkowitz et al. provide some empirical evidence regarding their argument on construction of a new structure through epistemic actions, they *a priori* assume the necessity of consolidation of such knowledge structures. Despite the fact that the authors call for further research to investigate the validity of their theory, many have used it as an analytical tool into the investigation of mathematical constructions through epistemic actions rather than providing a critical evaluation of this theory. Hence my first purpose in doing this research was to investigate the validity of this theory.

Throughout my reading of the empirical studies of abstraction, I came to realise that in these studies researchers claim to gain insight into students' abstraction processes via the interventions and help of a knowledgeable agent e.g. the researcher/interviewer. Hershkowitz et al. (2001), for example, note that the interviewer in their study aims to induce the student to reflect on what she is doing so that she might progress beyond the point that she would have reached without the interviewer. Others argue that the successful completion of an abstraction process is contingent upon providing the student with 'hinting' (Ohlsson and Regan, 2001) and 'shifting the focus of activities' (van Oers, 2001). All three sets of researchers effectively argue that abstraction is not an easy process and that it may be beyond the learners' unassisted efforts. However, impacts of interventions from a knowledgeable agent on the students' abstraction process, to date, have not been investigated. In the literature, such interventions are usually associated with the ideas inspired by the metaphor of scaffolding. The metaphor of scaffolding, in its simplest form, refers to a tutor's role in supporting a 'novice' (learner/student) towards a level of a competence which is not quite available to the novice's unassisted efforts. On the basis of this consideration, my second purpose in doing this research was shaped as an investigation of the role of scaffolding in the process of abstraction.

In chapter 2, I will provide further details on the issues of mathematical abstraction and scaffolding and develop the theoretical framework of this study. At this point it should be noted that these two research foci, as briefly sketched out so far, are formulated into three research questions as follows:

1. How are new mathematical constructions formed through scaffolding?
2. What is the nature of consolidation?
3. To what extent is the theory of abstraction proposed by Hershkowitz et al. valid beyond the cases presented by the original authors?

In order to answer these research questions, I adopted a qualitative research design methodology with explanatory and exploratory inquiry purposes. In this research I employed a multiple case study strategy with the purpose of literal and theoretical replications. I designed a number of cases with students working as pairs and individuals in such a way that some of the students worked with the scaffolded help and others without. All participants worked over four sequential tasks connected with the graphs of absolute value functions. Prior to actual data collection, the four tasks were piloted. The pilot study located task design faults and accordingly the initial four tasks were revised and redesigned. Actual data collection took place in Turkey. During this time, all four redesigned tasks were applied on four consecutive days, there being a one day time interval between two successive task applications. Tasks were applied in paper-and-pencil format. Participants' written works and audio records of the sessions composed the data for this study. Following this, transcriptions of the sessions, obtained in Turkish, were translated into English. In doing so the verbal protocols of the sessions were prepared. The details of this study's methodology are provided in Chapter 3.

In Chapter 4, I provide an overview of the participants' progress over four tasks and discuss some of the problems encountered in the data collection process. Following this, I present and analyse substantial verbal protocols of two girls who worked with scaffolded help. These verbal protocols are used to address the first and third research questions. In my analyses of these verbal protocols, I carry out two separate analyses: one that concerns social process and one that concerns cognitive processes. In the analysis of the social process, I first divide the whole verbal protocol into episodes on the basis of subgoals and focus on the successive utterances of the individuals physically present in the activity, that is, the students and the interviewer/scaffolder. The students' utterances are examined according to their functions into six categories: proposing, explaining, elaborating, quest, agreement and disagreement. The scaffolder's utterances are examined in relation to some scaffolding interventions extracted from the literature. In addition to this, on the basis of examining the relationships between an utterance and the previous ones, an interaction flowchart is generated. In order to analyse the cognitive processes, I drew on the epistemic actions suggested by Hershkowitz et al. (2001), namely, recognising, building-with and constructing.

In Chapter 5, on the basis of my analyses presented in Chapter 4, I attend to the first research question which is concerned with scaffolding and new knowledge construction. In my attempt to answer this research question, I begin by focusing on the scaffolder's assisting interventions which are discussed under two broad categories: instructional and pedagogical. Then I explore why and how the scaffolder's interventions lead the students to the formation of a new construction(s). In response to the 'why' question I suggest causative relationships between the scaffolder's interventions and the students' resulting actions. Regarding the 'how' question I develop the idea of human mediation. On the basis of the idea of human mediation, I argue the inseparability of social and cognitive developments by utilising Bakhtinian notions of 'utterance', 'voice' and 'dialogicality'. These notions are also used to examine the interactions taking place amongst the students and scaffolder in the course of the activities. This interaction is further characterised by focusing on two opposing tendencies which exist in any interaction at varying degrees and with relative importance: intersubjectivity and alterity. Following this, I propose a model of 'emergent goals' which aims to display the dynamic and dialectical interrelationships amongst the students, scaffolder and the tasks in the course of an activity. My final considerations in this chapter concern the transfer of control and regulation from the scaffolder to the students and its implications for the construction of new knowledge structures.

In Chapter 6, I attend to the second research question concerning the consolidation of the newly constructed mathematical structures. In order to answer this question, I draw on the verbal protocols obtained in the main and the pilot study. Through analysing the pilot study students' verbal protocols, it was realised that unless students have an opportunity to consolidate their new construction, they are unlikely to use these new constructions to form further abstractions. The students' verbal protocols generated in the actual data collection corroborate this observation. In order to exemplify the insights into the process of consolidation, the verbal protocols of an individual student working with the scaffolded help are presented in Chapter 6. Based on these verbal protocols, I discuss the initial state of new constructions, changes coming about in the course of consolidation, task design issues in consolidation and some reflections on language development, use of examples and establishment of interconnections.

In Chapter 7, I attend to the third and final research question related to the validity of the abstraction theory proposed by Hershkowitz et al. In order to answer this question, I draw on the verbal protocols presented in Chapter 4 and 6; and consider Hershkowitz et al.'s theory in its entirety. In this connection, I provide a critical evaluation of this theory by focusing my attention on three key dimensions which characterise it: its epistemological and sociocultural principles, epistemic actions and genesis of an abstraction. Throughout my evaluation I suggest some clarifications, amendments and further insights into the issues raised by this theory. This chapter concludes with a set of issues which warrant further research attention.

The final chapter of this thesis, Chapter 8, provides an overview of the study, briefly details the findings in relation to the research questions and explain the contributions that this research makes to the relevant literature.

CHAPTER 2: LITERATURE REVIEW

This chapter is composed of three main sections. In the first section, I review the literature on the issue of mathematical abstraction and describe the model adopted for this study. Following this I account for the rationale behind my motive to investigate the issue of abstraction in relation to scaffolding. In the second section, I provide a literature review on the metaphor of scaffolding and state my understanding of it. The final section suggests a theoretical framework in which this study is grounded.

1. Literature review on mathematical abstraction

In this study, I work within the framework of an abstraction theory proposed by Hershkowitz, Schwarz and Dreyfus (2001), which will be later described in detail. In order to illuminate the rationale behind this preference I provide a literature review on the issue of abstraction in mathematics education. In reviewing the literature on mathematical abstraction, it is possible to distinguish between two broad traditions. The studies in the first one, that I shall call the classical or cognitivist tradition, develop their accounts by essentially focusing on the cognitive aspect of mathematical abstraction, which assumes hierarchically organised one-way progressive development of individual cognitions. However, following the strong emergence of situated cognition and sociocultural theories of cognitive development within the last 25 years or so the classical approach has been criticised on the grounds that it ignores the variation of understanding across settings and that it does not pay sufficient attention to the social and cultural aspects of cognitive growth. As a result, new accounts of mathematical abstraction emerged by taking into consideration social, cultural, historical and contextual forces operating in the course of mathematical abstraction. These studies constitute the second tradition which I shall call the sociocultural tradition.

1.1. Cognitivist views on abstraction

Researchers within the cognitivist tradition tend to see knowledge as residing in individuals' 'minds'. They commonly consider it as decomposable into small units and analyse cognitive performances into complexes of rules with each rule thought of as a component of the total skill. According to this tradition, as Suchman (1987, p.178) points out, "an adequate account of any phenomenon ... is a formal theory that represents just those aspects of the phenomenon that are true regardless of particular circumstances." One of the leading figures in this tradition is Dienes (1963). He describes abstraction as "the extraction of what is common to a number of different situations. It is just another word for the formation of a class, the end-point being the realisation of the attribute or attributes which make elements eligible or not for membership of the class" (p.57). However, although everyday objects are classified by visible appearance or known function, mathematical ideas are classified by deep structure: "abstraction here is essentially the formation of an isomorphism" (p.59). What he means by isomorphism is to discover 'the same type of pattern' amongst different sets of materials, with each set embodying the same concept, e.g. an isomorphism between balancing weights on a balance beam and making rectangles from a given set of unit squares. In a similar vein, Skemp (1986, p.21) also linked abstraction with the act of classifying:

> abstracting is an activity by which we become aware of similarities ... among our experiences. Classifying means collecting together our experiences on the basis of these similarities. An abstraction is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class.

The most influential work within this tradition comes from Piaget. In most of his work, Piaget concentrated on the development of knowledge of children, rarely going beyond adolescence. Piaget's account of abstraction inspired many of the views on abstraction within this tradition. Piaget's writings on abstraction are too numerous to cite here but a summary of his works can be found in Dubinsky (1991). Piaget distinguished between empirical and pseudo-empirical abstraction, both of which are concerned with properties of (or actions on) physical objects, and reflective abstraction which involves interrelationship among actions. Reflective abstraction (Piaget, 1970) does not simply involve the separation and retention of different qualities of objects but rather consists in recognitions of these objects, reflecting on them, projecting them on to the plane of thought and finally integrating this new form into a fresh structure which is a reconstruction of the former ones. Reflective abstraction differs from empirical abstraction in that it is concerned with actions as opposed to objects and it differs from pseudo-empirical abstraction in that it deals not so much with the actions themselves but with the interrelationships amongst actions.

Piaget's idea of reflective abstraction was a starting point for many researchers in the classical approach. For example, many mathematics educators concerned with 'Advanced Mathematical Thinking' have paid substantial attention to the process of abstraction. A subgroup of researchers in this field published a volume (Tall, 1991) which collects together an overview of this approach and provides a helpful starting point for investigating the notion of abstraction. Many of these researchers tend to see abstraction as an essential component in the construction of a mathematical world. In this volume, the idea of reflective abstraction appears to be the essence of many of the models that authors implicitly assume. For example, Dubinsky (1991) elaborates the idea of reflective abstraction by isolating five phases in this process: interiorisation, coordination, encapsulation, generalisation and reversal. Robert and Schwarzenberger (1991) describe abstraction as involving the recognition of objects and properties which not only apply to the objects from which a generalisation is made, but also to any other objects which obey the same properties. Dreyfus (1991) suggests that the process of synthesis and generalisation form a prerequisite basis for abstraction. In his view, abstraction makes a heavier cognitive demand on students than either synthesis or generalisation.

These formulations point to a broader theory that remains fairly dominant in the mathematics education literature. The key component to this theory is the dual nature of mathematical concepts. The argument is that abstract notions can be conceived of in two fundamentally different ways: operationally as processes and structurally as objects. An articulation of this theory comes from Sfard (1991). She argues that the operational conception is the first stage in the acquisition of new mathematical notions, while the structural conception is the more advanced and inherently difficult phase. According to her, despite the essential complementarity of the two conceptions, there is a deep ontological gap between them with the structural conception 'progressing' to the operational conception through three stages, each commencing where the former ends: interiorisation, condensation and reification. In the interiorisation stage the learner becomes acquainted with and skilled in using the processes or operations that are performed on known, lower-level mathematical objects. It is the condensation stage where sequences of operations are thought of as a whole and in the stage of reification 'the new entity' is detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a particular category of concepts.

Even though reification theory seems coherent within itself, a careful analysis of the theory reveals many critical points that need to be addressed (see Confrey and Costa, 1996; Noss and Hoyles, 1996). First, it simply overlooks the social dimension of learning and thus separates mathematical thinking from its origins in social contexts. Second, it is doubtful that this approach is independent of the mathematical 'objects' with which it is concerned. Even though reification theorists believe that the theory holds for every mathematical 'object' equally, there is insufficient evidence to substantiate this claim. In addition, in its current form, the theory of reification leads to the inescapable conclusion that the process of understanding is a hierarchically structured and uphill struggle from lower level to higher level. Furthermore, mathematics is hierarchically categorised into spirals of abstraction which are recursively replicated, the stage of reification being the commencement point of the next cycle. However, it is questionable whether such a strict division is possible for mathematical concepts and structures. For example, consider the idea of the derivative of a function that incorporates within it the idea of a function, the notion of a mapping between functions and graphs, and the concept of a tangent to a curve at a point. According to which criteria will we categorise each term of being lower or higher level mathematical concept? The theory also leads us to conclude that the history of mathematics is a metaphor for a straightforward, purifying progress. However, historically and a more contentious claim, ontogenetically, the evolution of mathematical concepts is not a straightforward one.

There are many other researchers who could be regarded as being within the realm of this tradition (e.g. Mitchelmore and White, 1995, Hiebert and Lefevre, 1986; Gray and Tall, 2002). Instead of giving all of the definitions and theories provided by these researchers, it is more beneficial to consider the main tenets of these theories. As may be seen from the studies presented above, researchers within this paradigm tend to associate abstraction with three essential features: (1) generalisation arising from the recognition of commonalities isolated in a large number of specific instances; (2) an ascent from lower concrete levels to higher levels of abstract thinking; and (3) a process of decontextualisation. I suffice to note these features here but return to them later in order to consider in more details.

Most of the criticisms of a purely cognitivist account of abstraction, which ignores the variation of understandings across settings, come from an epistemological point of view which stresses the social construction of knowledge as well as from a perspective of cognitive development which recognises that contextual factors are crucial to knowledge acquisition (see, for example, Noss and Hoyles, 1996, chapter 2). I now consider these studies' accounts of mathematical abstractions.

1.2. Sociocultural views on abstraction

The sociocultural theories have evolved from, and been influenced by, a variety of different ideas including situated cognition of Lave and Vygotsky's approach to genetic analysis of mental functioning which finds its roots in the interpersonal plane (see Wertsch, 1991). Researchers within this tradition do not treat knowledge as entirely in one's head but rather they place the emphasis on the connection of learning and knowledge to, for example, the context of the learning, social interaction, personal histories, and to tools and artefacts available in a learning situation. For example, Greeno (1997) claims that knowledge is not just 'in the head', if it is to be found there at all, rather knowledge consists in the ways a person interacts with other people and situations. Researchers within this tradition hold the belief that all knowledge and meanings are formed culturally and are shaped further in situations of social interaction between the individuals.

One of the most elaborated works on the claim that knowledge is shaped by setting comes from Lave (1988) and her colleagues. Perhaps the most suggestive and simple finding on which her theoretical ideas are based is her observation that shoppers in a supermarket performed calculations almost always correctly; yet their success fell dramatically when they were asked to perform the 'identical' calculations with paper-and-pencil. Lave's explanation for such inconsistencies is part of a broader analysis in which she contends that the setting itself creates problems and structures its own solutions. The key point of her theory is that people create solutions in the course of action and that these solutions are structured by their activity. In a similar vein Brown, Collins and Duguid (1989) assert that action is grounded in the concrete situations in which it occurs and all knowledge is inextricably a product of the activity and situations in which they are produced. Resnick (1991, p.2) goes as far as to argue that "every cognitive act must be viewed as a specific response to a specific set of circumstances. Only by understanding the circumstances and the participant's construal of the situation can a valid interpretation of the cognitive activity be made."

Researchers within this tradition often point out the importance of context in which learning takes place and challenge the separation of what is learnt from how and where it is learnt. Many contend that the influences of context on one's actions and decisions are important elements in the formation of one's cognition and thus need to be taken into consideration within any study which examines this process. For example, Schoenfeld (1983) argues that the responses of a student to any particular demand must be interpreted in the context of the social environment that generated them. Noss and Hoyles (1996) make a powerful case through their analyses of students' performances and assert that effects of context on students' cognition vary widely ranging from the wording of the problem to the tools and resources that they have at their disposal.

One of the studies of abstraction in this tradition comes from van Oers (1998, 2001). In order to emphasise the social, cultural and contextual forces operating in the course of mathematical abstractions van Oers embraces Leont'ev's (1981) activity theory. According to him, abstraction is an activity, something that people do, a behaviour. Abstraction often involves the manipulation of physical materials and cycles of perceiving to discover new features and conceptual reframing. He defines abstract thinking as a process of contextualising an experience, providing a framework by which particulars become 'a situation' and hence action may be organised, 'ascending to the concrete' (van Oers , 2001, p.301). He provides a definition that, according to him, is free from any reference to hidden entities: abstraction is assuming a point of view for ordering objects. In other words, abstracting means the ability to notice some things while disregarding some others and to do it in a certain systematic way. By this definition, he seems to favour the earlier definitions of abstraction (e.g. Skemp, 1986) but a fundamental difference between van Oers and Skemp is the role of context in which the activity of abstraction occurs.

Based on the assumption that activity is the real context for human actions, he states, "we can acknowledge that this activity must be constructed from the concrete situation" (van Oers, 2001, p.288). This is the process that he calls contextualisation. It is an abstracting process of making sense of a concrete situation and translating it into a particular (but still abstract) socio-cultural activity from which new actions can emerge. According to van Oers, abstract thinking is natural and need not be taught by transmitting abstractions; the understanding develops interactively, in a discursive process by which meaning is negotiated accordingly; students should be given 'a perspective on where they are going' by having a role in the contextualisation process itself. He clearly states that in order for an abstraction to be achieved, students need to be directed in the given situation as a part of the situation by being involved in the activity. He verbalises this belief as "it is the expert (teacher) who discursively focuses pupils on particular and increasingly 'isolated' aspects of the situation and helps the children in the construction of new mental objects (abstractions), that provide the means for seeing various things as related and thus to ascend from the abstract to the concrete" (van Oers, 2001, p.301). In a similar vein, Ohlsson and Regan (2001, p.32) also claim that "to operate beyond the boundary of current knowledge, one has to be guided by something in addition to the facts and skills already acquired, this could be a very delicate hinting."

Another study from within this tradition is that of Noss and Hoyles (1996) who use the term 'situated abstraction' to focus attention on the specific features of the situation, particularly, on the linguistic and conceptual resources available for expressing mathematically within these specific features. In their view, situated abstraction describes "how learners construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed" (p.122). The idea of 'webbing' conveys "the presence of a structure that learners can draw upon and reconstruct for support" (p.108). When students progress through a series of activities (in a social context, in the presence of tools), they learn to attune practices from previous contexts to new ones. Therefore, according to Noss and Hoyles, students do not detach from concrete referents at all. On the contrary, there is a process of 'webbing' which refers to a structure which enables learners to make use of the previous constructions and to reconstruct these constructions for support.

A structure in Noss and Hoyles' view varies widely from, for example, an understanding of a mathematical idea (e.g. the idea of tangent, perpendicular lines) to the properties of computational settings (e.g., specific icons, particular experience in using, say, Logo and Cabri, opening the appropriate menu item in a software). This structure helps students connect to previous similar activities and draw on the tools that they have at their disposal to construct new mathematical knowledge. In their study, they illustrate several computational environments in which abstraction is situated within the conceptual resources that students have at their disposal to form abstractions within the situation. However, they do not simply imply that knowledge structures are strictly bound into the situation. Rather, they are mainly concerned with 'meanings' and see abstracting as a way of layering meanings on each other, connecting between ways of knowing and seeing, rather than as a way of replacing one kind of meaning with another. Situated abstraction is a process as well as an object, abstracting *in situ*, abstracting in a domain. Thus situated abstraction is not a thing on its own, it is simultaneously an articulation, a statement and a (re)thinking-in-progress. The main difficulty involved in the idea of situated abstraction is that it does not clearly state how students construct new knowledge through webbing and thus the link(s) between webbing and the construction of new knowledge remains vague.

In my opinion, a clear formulation of the link between the construction of new structures, and the tools and contextual variables comes from Hershkowitz, Schwarz and Dreyfus (2001). These authors set out a dialectical materialist and microgenetic account where the genesis of an abstraction develops from an undeveloped initial entity, through the use of mediational means and social interaction. Specifically Hershkowitz et al. view abstraction as an activity of vertically reorganising previously constructed mathematical knowledge into a new mathematical structure. The term 'activity' is used in the sense of activity theory (Leont'ev, 1981) which stresses that actions occur in a social and historical context. Attainment of the reorganisation of mathematical knowledge requires actions on mental or material objects and such reorganisation is called 'vertical'. Through these actions mathematical elements are combined together, structured, organised and developed into more formal elements. 'Structure' is used in the sense of Davydov (1990), that the development of an abstraction from an undeveloped initial entity involves establishing a structure that ultimately results in a differentiated and structured entity. The term 'structure' is further used (see Dreyfus and Tsamir, 2004) as a generic term for abstracted structures, methods, strategies and concepts.

Hershkowitz et al.'s new structures arise in an activity from three epistemic actions (i.e. actions related to the acquisition of knowledge): recognising, building-with and constructing (referred to as RBC hereafter and Hershkowitz et al.'s model/theory will be referred to as 'RBC theory of abstraction'). Recognising a familiar mathematical structure occurs when a learner realises a structure which is inherent in a given mathematical situation. Building-with consists of combining existing structural elements to meet a goal such as solving a problem or justifying a statement. Constructing is the rarest but most important action. It consists of assembling knowledge artefacts to produce a new structure which becomes familiar to the learner. Distinguishing features of the constructing and building-with actions are the 'novelty' and 'motive' driving the activity. In building-with, students are not enriched with novel, more complex structures as these actions respond to an extraneous goal such as justifying a statement and the goal is achieved by using previously acquired structures. On the other hand, constructing actions are related to the reorganisation of the previously acquired structures which bring about emergence of a novel structure. In this process, construction of a new structure is often the goal of the activity; and even if it is not, it is indispensable for the achievement of the goal. The theory claims that RBC actions are dynamically nested: that building-with actions are nested in constructing actions and recognising actions are nested in building-with actions and in constructing actions. These three epistemic actions are subjective. For example, a student can only recognise a structure that (s)he has constructed in an earlier activity. Consequently, the personal history of the student determines which structures can be recognised, and whether a certain task leads to building-with or to constructing. A structure that can be recognised by a student may have to be constructed as a new structure by another.

In Hershkowitz et al.'s view, the genesis of an abstraction is seen as passing through three stages: (a) the need for a new structure; (b) the construction of a new abstract entity through nested recognising and building-with epistemic actions with extant structures; and (c) the consolidation of the abstract entity/structure which involves recognising it and building-with it in further activities. These authors in a companion paper (Dreyfus, Hershkowitz and Schwarz, 2001) investigate the distribution of the process of abstraction in the context of peer interaction with the intention of elucidating the abstraction as an activity in socially rich environments. Based on their data, they show that abstraction occurs by means of interactive social, not only mental, processes. Furthermore, their observations clearly suggest that peer collaboration provides a context that influences individual experiences and contributions.

Having reviewed sociocultural views on mathematical abstraction, I now turn my attention to the three main features attributed to the process of abstraction by the classical views.

1.3. Revising the three features associated with abstraction by classical view

As mentioned before, classical views associate abstraction with three main features: (1) a generalisation on the basis of the recognition of commonalities; (2) an ascent from the concrete to the abstract; and (3) a process of decontextualisation. I will now briefly evaluate these features in the light of the recent studies which renew our understandings of mathematical abstraction.

1.3.1. Abstraction as a generalisation on the basis of the recognition of commonalities?

In the classical tradition, abstraction is viewed as related to generalisation. For example, Dreyfus (1991) regards generalisation as the identification of commonalties amongst particulars which expand domains of validity of a concept and claims that generalisation forms a prerequisite basis for abstraction. In a similar vein, Dienes (1963, p.57) writes that abstraction involves "the extraction of what is common to a number of different situations." In fact these arguments find their roots in Aristotelian empiricist accounts which dominated Western mathematics education writings for much of the 20th century. Aristotelian empiricists view abstraction as involving generalisation arising from the recognition of commonalities isolated in a large number of specific instances.

Ohlsson and Lehtinen (1997) read the classical Aristotelian view of generalisation as an analysis in terms of applicability. They claim that, in the classical view, 'to be general' means 'to apply to a large proportion of instances' in some set and hence levels of generality are distinguished on the basis of the number of instances. Ohlsson and Lehtinen find this view wanting with regard to the Zeitgeist of contemporary cognitive research on distributed cognition, apprenticeship learning, case-based reasoning and situated cognition which all agree that high level cognition depends on domain-specific knowledge which is highly particular. They also find this view wanting with regard to history of: Newton's equation $F = m \text{ x } a$, Darwin's theory of evolution and the development of the derivative of a function. On the basis of these considerations, they make a powerful case and argue that there is no intrinsic relationship between abstraction and generalisation because "abstraction is a property of knowledge structure and that property is unrelated to the number of well-fitting instances" (ibid., p.45).

Since abstractions are believed to be produced from particular instances, the classical view considers the particulars as epistemologically more basic than abstractions. However, this idea has recently been criticised by many and new models has been proposed (van Oers, 2001; Hershkowitz et al., 2001; Ohlsson and Lehtinen, 1997). For example, Ohlsson and Lehtinen claim, "people experience particulars as similar precisely to the extent that, and because, those particulars are recognised as instances of the same abstraction" (p.41). Therefore, "in order to recognise an object as an instance of an abstraction, the knower must already possess that abstraction" (p.41). Thus the problem with the classical view of abstraction becomes clear that if someone is to recognise commonalities, then surely that person must already have some understanding (which may be rudimentary) of the abstraction. Thus abstractions beget recognitions rather than vice versa.

1.3.2. Abstraction as an ascent from the concrete to the abstract?

According to the classical view, abstraction is an individual process which is a transition and development of mathematical concepts from the concrete to abstract. Thus abstraction is perceived as an end point of cognitive development that is achieved only after a long time and usually with hard work. The classical view sets the concrete and abstract as bipolar opposites and assumes an ascending from the former to the latter (see van Oers, 2001 for a critique).This perception is mainly motivated by the Piagetian view of cognitive development in which children become progressively detached from the world of concrete objects and local contingencies and gradually ascend to the level of abstract thought (see Noss and Hoyles, 1996 for more on this). A just evaluation of this view lies in our answer to the essential question, what do we mean by the concrete and abstract?

Now should we accept a consideration of the abstract as removed or divorced from reality and/or meaning (Mason, 1989, p.2)? If so, the abstract becomes 'meaning-less'. Or should we accept Piaget's division that 'logical' and 'mental' structures are abstract, whereas physical knowledge – the knowledge based on experience in general – is concrete (Piaget, 1970). If such a division is accepted, then there is little chance to regard any mathematical structure as concrete, much less to discuss the relationship between the abstract and concrete in abstraction of mathematical structures which are in fact 'logical' and 'mental' structures and thus by all means abstract. Further to this, such a division trivialises the role of physical knowledge, i.e. knowledge which is immediately available to the senses, in the course of an abstraction. This is also recognised by Noss and Hoyles (1996, p.45):

> a standard description of the difference between thinking in lived-in experience and mathematical thinking is that the former is concrete, the latter is abstract. There is no doubt which way the hierarchy sits: the history of Western thought has privileged the latter at the expense of the former. Abstract is general, decontextualised, intellectually demanding; concrete is particular, context-bound, intellectually trivial.

Noss and Hoyles make a powerful case and provide evidence for the necessity and importance of experience-based knowledge – the concrete – in the course of an abstraction by drawing on empirical data and on the works of such authors as Ackerman (1991) and Wilensky (1991).

Ackerman's and Wilensky's works originate from an appraisal of Piaget's view of cognitive development which they found wanting and attempted to develop an alternative view of the relation between the abstract and concrete. In this respect, Ackermann (1991) examines the location of intellectual development within rather than beyond the concrete. The relationship between the concrete and abstract is exemplified by a re-analysis of Piaget's water experiments in which Ackermann highlights the coordination of local knowledge in building abstractions and argues that the production of abstract laws flows from the regularities that children find out in specific contexts while simultaneously applying general rules in order to make sense of local situations. For Ackermann, abstract and concrete are two sides of the same coin.

In Wilensky's (1991) view, whether something is abstract or concrete is not an inherent property of the thing but rather "a property of a person's relationship to an object [i.e., a concept, idea, piece of knowledge]" (p.198). Wilensky takes the argument to its conclusion and writes that if the concreteness is a relational property then

> the more connections we make between an object and other objects, the more concrete it becomes for us. The richer the set of representations of the object, the more ways we have of interacting with it, the more concrete it is for us (p.198).

Here Wilensky focuses on the interaction of knowledge (objects and relations) and the individual, and approaches the concrete (or the abstract) in terms of one's 'closeness' and relationship with the object. He clearly views this relation as related to the establishment of connections: the more connections one establishes, the more concrete it becomes.

Ohlsson and Lehtinen (1997) also point out the difficulty involved in the view of abstraction as a flow from the concrete to abstract. The term 'concrete' in Ohlsson and Lehtinen's account, although not clearly stated, might be considered as "particular and specific ... practical and narrative" (p.37). However, 'abstract' is not so much concerned with particular instances, yet it is not general either, as they argue that there is no intrinsic relationship between the abstract(ion) and the general(isation). Rather, being abstract is a matter of complexity as manifested in terms of the depth of an idea in the sense of having some "other ideas as parts" (p.42). Ohlsson and Lehtinen separate

> the property of being abstract from the process of abstracting. The latter does not move from the concrete to the abstract because it operates on ideas that are abstract to begin with. The movement is across levels of complexity, not level of abstraction or generality (ibid., p.43).

These authors reject the necessity of the 'concrete' in the process of abstraction. In Ohlsson and Lehtinen's account "the abstract has the primacy over the concrete" (p.41) precisely because "the specific features of their [concrete ideas'] components conflict with each other. Abstract ideas on the other hand do not carry such baggage and so can be combined more easily. The cognitive function of abstraction is to facilitate the assembly of larger, more complex knowledge structures" (p.43). The result of an abstraction is therefore more complex and always abstract. So the concrete in Ohlsson and Lehtinen's approach is seen a source of creating conflicting features which inhibit abstraction.

However, Davydov (1990) argues such conflicting features are necessary and further that they constitute the initial form of abstractions. In his 'method of ascent', Davydov argues that it is the "disclosure of contradictions between the aspects of a relationship that is established in an initial abstraction" and "it is of theoretical importance to find and designate these contradictions" (p.291). Thus he views the genesis of an abstraction starting from "a simple, undifferentiated, undeveloped form" (p.278), which need not be internally and externally consistent. The development of abstraction progresses from analysis, at the initial stage, to synthesis. It ends with a consistent and highly structured final form. This process, described by Davydov as the "ascent to the concrete, from undeveloped to the developed" is accomplished "through real interconnections among phenomena" (pp.302-303). These real interconnections can be achieved by means of 'theoretical thought' rather than 'empirical thought'. In Davydov's view, empirical thought is concerned with establishing "particular connections and relationships" (p.253) which can be expressed verbally "as the results of sensory observations" (p.255) (e.g., observing similarities and differences between things). However, establishment of real interconnections requires theoretical thought which he describes as "an idealisation of the basic aspect of practical activity involving objects and of the reproduction in that activity of the universal forms of things, their measures, and their laws" (p.249). For Davydov, theoretical thought is necessary to establish internal links or "essential relationships [which] cannot be observed directly by the senses, since they are not given in available, established, resultative, and disassociated being" (p.255).

On the basis of these considerations, I suggest that the abstract is concerned with complexity in the sense of Ohlsson and Lehtinen (i.e. in terms of the depth of an idea in the sense of having some other ideas as parts), goes beyond the particular instances and is related to theoretic thought in the sense of Davydov. The concrete on the other hand is concerned with particular instances and experiences and is often related to empirical thought in the sense of Davydov. The critical point is that the development occurring through abstraction is not from the concrete to the abstract but, rather, a dialectical *to and fro* between the concrete and the abstract. As Noss and Hoyles (1996, p.44) put it, the abstraction process can be conceived of not so much as a step upwards ascending process but rather as an "intertwining of theories, experiences and previously disconnected fragments of knowledge". In line with this, my own position is that the concrete and abstract are dialectically interrelated rather than detached in the course of abstraction and they both are necessary for individuals to appreciate one another.

1.3.3. Abstraction as a process of decontextualisation?

The classical view tends to see abstraction as a process of decontextualisation. The underlying assumption of this tendency lies in the belief that "knowledge acquired in 'context-free' circumstances is supposed to be available for general application in all contexts" (Lave, 1988, p. 9). The classical view hypothesises a mechanism by which the essence can be extracted from an object, either by stripping off accidental, irrelevant features or by directly focusing on the essence (prototype, scheme) (van Oers, 2001). In other words, abstraction is a process of extricating the mathematics from the problem, removing it from action to recognition (see Vergnaud, 1987 for more on this). This is the transition from knowledge in action to the reflective knowledge or reflective abstraction (Piaget, 1970). The idea of decontextualisation is more clearly and strongly articulated by Laborde (1989, p.35) who asserts that "the usual formulations of mathematical discourse require a certain level of acquisition of mathematical objects and relations: they must be sufficiently decontextualised and detached from students' action."

I believe that consideration of abstraction as a decontextualisation process is an inappropriate one (and indeed open to criticisms) and that context is indispensable for the formation of mathematical abstractions. My use of the term 'context' here in its simplest form concerns the social and cultural environment which involves individual's social and personal histories, objects of the real world, mental and material tools and artefacts. As discussed so far, the process of abstraction does not grow up free from the context which contributes to the constitution of the 'meaning' for individual actions. If we equate abstraction to decontextualisation, then this faces us with an assumption that understanding and development of new meanings can be achieved without relying on the specificities of the context in which individuals' learning activities take place. Having realised this, van Oers (1998, p.135) comments that "the notion of 'decontextualisation' is a poor concept that provides little explanation for the developmental process toward meaningful abstract thinking." In a similar vein, Noss and Hoyles (1996, p.21) also see the difficulty involved in the idea of abstraction as a process of decontextualisation and write

> where can meaning reside in a decontextualised world? If meanings reside only within the world of real objects, then mathematical abstraction involves mapping meaning from one world to another, meaningless, world – certainly no simple task even for those with the capacity to do it. If meaning has to be generated from within mathematical discourse without recourse to real referents, is this not inevitably impossible for most learners?

These authors make a compelling case and show that the specificities of the context such as the tools and artefacts that the students have at their disposal, the students' previous actions and presentation of the tasks are all part of the students' abstraction process. Therefore they argue that all abstractions are situated in the sense that they are formed in specific contexts. This view also shared by Hershkowitz et al. (2001) who refer to abstraction as an activity, an interactive process, involving tools, language and procedures. Indeed, inspired by activity theory the authors remind us that any particular action a person may undertake is a part of a greater activity, and occurs in a social and historical context, is inseparable from its activity-related goals and can be truly meaningful, and thus interpretable, only within the context of this activity. Hence they call their theory 'abstraction in context' thereby rejecting the notion of abstraction as a process of decontextualisation.

1.4. Why RBC theory of abstraction?

As mentioned before, in this study I work within the framework of Hershkowitz et al.'s (2001) RBC theory of abstraction. There are four main reasons for this preference. First of all, this theory, in my opinion, reflects a renewed understanding of mathematical abstractions in that three main features associated with abstraction by the classical view are reinterpreted in this theory. More specifically, regarding the first feature attributed to abstraction by the classical view (i.e. a generalisation on the basis of recognition of commonalities), Hershkowitz et al's RBC theory rightly suggests that recognition is possible if a structure is already abstracted. Therefore, they argue that recognising is an epistemic action occurring in the process of abstraction and that it is the recognition of already formed abstractions (not that of what is going to be abstracted) that is required to form new abstractions. Regarding the second feature (i.e. abstraction as an ascent from the concrete to the abstract), RBC theory adopts Davydov's dialectical view of the abstract and concrete in the course of abstraction. Regarding the third feature (i.e. abstraction as a process of decontextualisation), as already noted, RBC theory suggests that abstractions take place in context and find it irrelevant to associate abstraction with decontextualisation.

Secondly, this theory provides an operational definition which allows us to carry out empirical investigations of mathematical abstractions by means of investigating three epistemic actions (i.e. recognising, building-with and constructing – RBC). These epistemic actions provide us with an analytical tool to examine the students' abstraction process. Thirdly, this theory directs our attention to the critical issues in relation to abstraction such as construction and consolidation of new mathematical structures which require further investigation. Finally, the RBC authors call for further research into the validity of their theory. Following its publication, RBC theory attracted the attention of many who 'partly' validated this theory (e.g., Bikner-Ahsbahs, 2004; Kidron and Dreyfus, 2004; Schwarz, Dreyfus, Hadas and Hershkowitz, 2004; Stehlikova, 2003; Tabach, Hershkowitz and Schwarz, 2001; Tsamir and Dreyfus, 2002; Williams, 2002, 2003, 2004; Wood and McNeal, 2003). This theory is 'partly' validated because most of the above-cited studies use RBC theory as an analytic tool into their investigation of abstraction process by focusing on (and identifying) the three epistemic actions and thus providing empirical evidence for the occurrence of epistemic actions; yet to date there does not appear any study which critically evaluates RBC theory by considering it as a whole i.e. by considering its epistemological and sociocultural principles, epistemic actions and genesis of abstraction as proposed by Hershkowitz et al. Thus the first aim of this study is to investigate the validity of RBC theory by considering it as a whole, not solely focusing on the epistemic actions as many other follow-up studies do.

Throughout my reading Hershkowitz et al.'s (2001) study along with the others such as Dubinsky and Lewin (1986), Mason (1989), Ohlsson and Regan (2001) and van Oers (2001), I came to realise that these studies collect their data in environments where a 'knowledgeable' agent (interviewer/teacher/tutor) assisted the students who consequently achieved the intended abstractions through this assistance. For example, in Hershkowitz et al.'s study, the interviewer assisted the student by compelling her to explain what she was doing (and why) and by inducing her to reflect on what she was doing so as to help her progress beyond the point she would have reached without the interviewer. In a similar vein, Dubinsky and Lewin (1986, pp.66-68), in their investigation of a student's abstraction process of mathematical induction, note "the atmosphere of the interview was generally supportive with free use of prompting" and in the course of interview, asking the prompting questions "gradually leads the child to progressively construct the appropriate concept." Others also point out that a successful completion of the abstraction process is contingent upon providing the students with 'hinting' (Ohlsson and Regan, 2001), 'a delicate shift of attention' (Mason, 1989) and shifting the focus of activities (van Oers, 2001).

I realised through my reading of these studies that despite certain differences in interpretations of what mathematical abstraction is and involves, there appears an agreement that abstraction is a hard task for students to achieve. Further to this, interventions from a knowledgeable agent which provide students with purposeful help and regulate them towards the achievement of mathematical abstraction are an important aspect of many studies. I also realised that the impact of such interventions on the students' performances have not been studied in relation to formation of mathematical abstractions. These realisations prompted me to focus on the idea of scaffolding in connection with the formation of mathematical abstraction. As a result, the second aim of this study is shaped as an investigation of the role of scaffolding in the process of abstraction. In the next section I will review the literature on the metaphor of scaffolding and state my own understanding of it. Following this I will sketch out this study's theoretical framework in which these two aims can be realised.

2. The metaphor of scaffolding

In this section, I review the literature on the metaphor of scaffolding in four parts. I begin with a consideration of the origin of the metaphor of scaffolding and its theoretical roots. Following this, I briefly review scaffolding-related research carried out in different contexts, paying particular attention to the difficulties involved in employing scaffolding in classroom contexts. Then I review the forms of assistance which could be considered within the realm of scaffolding. Finally, on the basis of the relevant literature, I propose a definition of scaffolding which is used in this study.

2.1. Origin of the metaphor of scaffolding

The metaphor of scaffolding is first used in a paper by Wood, Bruner and Ross (1976) when describing an adult tutor's role in assisting some children (aged 3, 4 and 5 years) to build a particular three-dimensional pyramid structure that requires a degree of skill which is initially beyond them. They argue that the adult intervention in the form of scaffolding "enables a child or novice to solve a problem, carry out a task or achieve a goal which would be beyond his unassisted efforts" (p.90). In their account the process of scaffolding involves "the adult 'controlling' those elements of the task that are initially beyond the learner's capacity, thus permitting him to concentrate upon and complete only those elements that are within his range of competence" (p.90).

Although Wood et al.'s initial use of this metaphor was largely pragmatic and not based on a particular theory, in subsequent discussions it was linked to Vygotsky's (1978) notion of the zone of proximal development (ZPD). In this respect Bruner (1985), one of the inventors of this metaphor, referring back to the Wood et al.'s paper, says, "a study...[that] I am only beginning to understand" and writes of the implications of an adult "acting as a support for the child's foray into the zone of proximal development (p.29)". Vygotsky (1978, p.86) defined the ZPD as

> the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers.

In neo-Vygotskian discussions, the term 'problem solving' has been interpreted as 'performance' as a more general statement of the ZPD (a more detailed discussion can be found in Rogoff and Wertsch, 1984). Thus, generally speaking, the notion of the ZPD refers to the difference between children's assisted achievement and their unassisted performance, which is, for Vygotsky, of fundamental importance for their learning and development. Although Vygotsky's principal discussion in relation to the ZPD is centred on children, identical processes could be seen functioning in the learning of adults (Tharp and Gallimore, 1988). Therefore, unless otherwise stated, I will employ the terms 'students' or 'learners' to indicate that it is the children as well as adults to whom assistance can be provided.

In most of the later studies inspired by Wood et al's (1976) initial work, the metaphor of scaffolding is often associated with the notion of the ZPD (e.g., Hobsbaum, Peters and Sylva, 1996; Pea, 2004). In an attempt to illuminate the commonalties between these two, Scott (1997) identifies four characteristic features of scaffolding which are consistent with the theoretical underpinnings of the ZPD and which are also compatible with the principles embodied in Wood et al.'s (1976) original use of scaffolding.

The first characteristic feature common to both the ZPD and scaffolding is that they apply to learning a specific skill. The ZPD plots the difference between what a learner is capable of achieving with and without assistance, regarding that particular skill. Likewise, scaffolding embraces interaction between a tutor and learner by concentrating on the learner developing some specific skills which are initially beyond his/her unassisted effort.

Secondly, the learner's limited understanding of the situation and thus (temporary) dependence on the other-regulation is also common to both the ZPD and scaffolding. Wertsch (1979) argues that in the initial stages of the ZPD, the learner is likely to have a very limited understanding of the main activity, task or the target competence to be achieved. At this stage he/she is very unlikely to view the target skill and the goal of the activity in the way that a tutor does. Through the conversation during the task performance, however, the learner gradually comes to understand the meaning of the activity, how its parts are related to one another and the intended skill. In scaffolded learning, by giving direct assistance, the tutor can handle difficult aspects of a task and organise the learner's involvement with features of the activity that are within their grasp. In doing so, the tutor acts as 'a vicarious form of consciousness' "until such a time as the learner is able to master his own action through his own consciousness and control" (Bruner, 1985, p.24).

The third characteristic is the support provided by the tutor. Learning in the ZPD includes the tutor in supporting the learner's progress between current and potential levels of performance. Throughout scaffolding, in interacting with the learner, the tutor becomes aware of and responsive to existing modes of (and any changes in) a learner's thinking, in supporting the development of the target competence.

Finally, the support and regulation gradually fades away when the learner makes progress towards the target competence. Tharp and Gallimore (1988) argue that in the ZPD, there occurs a steadily declining plane of adult responsibility for task performance and a reciprocal increase in the learner's proportion of responsibility. This indicates a transition from other-regulation to self-regulation. However, as Moll (1990) pointed out, this transition of other to self-regulation from joint to independent problem solving, does not simply happen automatically or by chance, but rather requires very detailed and delicate teaching interactions on the part of the tutor. Similarly, scaffolding involves a progressive withdrawal of assistance, a gradual 'handover' (Bruner, 1983) of responsibility from tutor to learner.

2.2. Scaffolding in different contexts

I hitherto considered scaffolding in relation to the ZPD. I will now briefly review scaffolding-related research carried out in different contexts, paying particular attention to the difficulties involved in employing scaffolding in classroom contexts. An awareness of these difficulties is important as they influenced my decision to collect data in one-to-one dyads and small groups rather than in classroom contexts.

Initial contributions to the idea of scaffolding came essentially from workers in two fields: it is explored in the domain of developmental psychology in the studies of parent-child interactions (e.g. Greenfield, 1984; Hodapp, Goldfield, and Boyatzis, 1984) and in the anthropologic studies of how craft skills (like weaving, making a basket, or putting away the shopping) (e.g., Rogoff and Gardner, 1984; Rogoff, 1990) are passed on from an expert to a novice. Findings of these studies became prevalent in the related literature and there appear many attempts to utilise the insights gathered from these studies into the investigation of one-to-one tutoring, especially in the cognitive psychology (e.g., Chi, Siler, Jeong, Yamauchi and Hausmann, 2001; Graesser, Person and Magliano, 1995). Findings of the cognitive psychologists along with the others were used relatively successfully, though not fully, in the studies of artificial intelligence where some intelligent tutorial systems are developed in supporting individual learning activities (e.g., Guzdial, 1994; Tabak, 2004).

Despite the relative success in the aforementioned fields, the idea of scaffolding erects formidable challenges when it comes to the classroom contexts. For example, the number of students involved is rather greater in classrooms than research studies where a tutor needs to deal with one or two students at a time. Due to the large number of students, teachers are quite likely to have difficulties in making appropriate judgements about the students' individual ZPDs. The difficulties involved in implementing scaffolding in classrooms become even greater when we consider the diversity of the students' actual development levels and the pace at which they learn. Further to this, some (Mercer, 1995; Collins, Brown and Newman, 1989) also draw attention to the danger in making simplistic comparisons between what parents or craft experts do and what teachers have to do to scaffold learners by pointing out that teachers and students are working in classes which are rather different contexts than the ones where parents and young children or experts and their apprentices interact. Here the argument is that in the school context the primary concern of a teacher is to teach and the relationships are inevitably more fragmented (Mercer, 1995); however, in the context of, for example, craft expert the problems and tasks arise not from pedagogical and didactical concerns, but from the demands of workplace (Collins et al., 1989)

These concerns drive some to develop alternative conceptions to scaffolding. For example, Mercer (1995) developed the concept of 'guided construction of knowledge', Collins et al. (1989) 'cognitive apprenticeship', Turnbull, Turnbull, Shank and Leal (1999) 'instructional scaffolding', and Tharp and Gallimore (1988) 'assisted performance'. These studies attempt to operationalise the insights gathered from research on scaffolding within the formal educational settings.

Considering the difficulties, I feel, like Pea (2004), it is still premature to extend scaffolding considerations to the level of a whole classroom of learners. However, despite these challenges, the metaphor of scaffolding is an inspirational one for both research studies and classroom contexts in terms of the nature of assistance that is desirable in supporting the learner(s) towards the achievement of a level of understanding which is not readily available to the learner's unassisted efforts. But essential questions here are: can any kind of assistance be referred to as scaffolding? If not, what kind of assistance can be considered as scaffolding? What are the forms of these assistances? I will attend to these questions in the next section.

2.3. Forms of assistance in scaffolding

In this section I review the forms of assistance which could be considered within the realm of scaffolding. In addition, I will consider how these assistances are to be provided in the course of scaffolding by drawing mainly on the Wood's (1991, 2001) 'contingency principle' and Scott's

(1997, 1998) 'action cycle model'. These considerations will then be used to provide a definition of scaffolding adopted in this study.

There appear two broad categories of assistance provided to a learner in the course of scaffolding. The assistance in the first category is concerned with the preparation of the teaching-learning activities. Rogoff (1990, p.86) calls this sort of assistance 'structuring situations' when referring to the choice of puzzles and selection of appropriate tools and materials. Likewise Scott (1998, p.70) employs the term 'instructional means' when speaking of the assistance related to "teaching activities which are planned ahead of the instruction." It is noteworthy here that little research attention has been directed towards this kind of assistance in the scaffolding literature.

The second category involves the tutors' direct interventions in the course of an activity. This is what Scott (1998) calls 'pedagogical means' and the scaffolding literature is, by and large, concerned with this kind of assistance. Pedagogical means occur in the conduct of a teaching-learning activity and more specifically involve the tutor's spontaneous responses to the learner's performance. In their initial work, Wood et al. (1976) describe six functions of the tutor in scaffolding, each of which can be considered as a different form of pedagogical assistance: enlisting the learner's adherence to the requirements of the task, reducing the degrees of freedom, maintaining the learner's direction in pursuit of a particular objective, marking critical features of a task, controlling the learner's frustration and demonstrating solutions to a task. In the later studies, there appear many other different forms of assistance within the realm of scaffolding, including, for example, explaining (Anghileri, 2002), inviting learners to participate (Hogan and Pressley, 1997), questioning (Graesser et al., 1995), describing the problem (McArthur, Stasz and Zmuidzinas, 1990), hinting (Bliss, Askew and Macrae, 1996), clarification of direction, purpose and expectation (McKenzie, 1999).

However, a single-minded focus on forms of assistance or the role of tutor may suggest that scaffolding proceeds from tutor to learner whose main objective is to follow the given instructions as "a diligent apprentice to an all-powerful master" (Daniels, 2001, p.65). In fact this was the concern of many (e.g., Harste, Woodward and Burke, 1984; Searle, 1984; Leseman and Sijsling, 1996) who argue that scaffolding promotes a conception of adult-child interaction as one-sided in nature in favour of the adult and thus neglecting the importance of the perspectives brought to the activity by the child.

These concerns and criticisms did not go unnoticed and indeed led to further clarifications of the metaphor of scaffolding. Later interpretations of this metaphor suggested that a tutor's role in scaffolding is not that of getting a learner to follow his/her instructions towards the completion of a task through step-by-step demonstrations and/or explanations. Instead it is repeatedly noted, with some variation in emphasis and terminology, that the learner should be actively involved in task completion along with the tutor who needs to be 'contingent' in his/her assisting (e.g., Hammond, 2001; Wood and Wood, 1996a). In this respect, when explaining how scaffolding works, Hammond (ibid., p.60), for example, writes

> knowing when and how to intervene is what scaffolding is all about. It is about the teacher taking an informed and active role in guiding students' learning as they come to terms with new ideas and concepts. ... Scaffolding is far more than 'helping out' so that a student can complete a task. It requires the teacher to act contingently, using a variety of strategies, so that students can gain understanding and confidence to work independently in applying new learning.

Two particular aspects of Hammond's argument here are important. The first is that scaffolding is not simply concerned with a task completion through some assistance. But rather it is concerned with assisting the student to gain an understanding and confidence towards independent work on a task. The second is that in order for a tutor to assist in this manner, he/she needs to act contingently by "taking an informed and active role" in guiding the student. But what does it mean, 'a tutor acting contingently'? And how can a tutor take an 'active and informed' role? The answer to the first question is related to Wood's (1991, 2001) 'contingency principle'. Regarding the second question, I will try to provide an answer by invoking Scott's (1997, 1998) 'action cycle model'.

The 'contingency principle' is seen as an essential quality of scaffolding (Mercer, 1995) and it refers to two complementary provisions of support depending on the student's performance: augmentation of assistance when the students are having difficulties and thus likely to fail; and withdrawal of assistance if the students progress successfully. Thus the amount and specificity of an assistance should be contingent upon the status of the student's preceding actions (Wood, 2001). The importance of the contingency principle lies in the assumption that it helps to ensure that the demands placed on the student are likely "neither to be too complex, producing defeat, nor too simple, generating boredom or distraction (Wood, 1991, p.108)." Although it is difficult for a tutor to act contingently all the time, there is evidence in the literature that, when used appropriately, this principle brings about both short-term and long-term benefits in terms of students' learning and motivation (e.g., Pratt and Savoty-Levine, 1998).

Regarding the tutor's active and informed role in scaffolding, this is again closely linked to the contingency principle and Scott's action cycle model provides a good deal of insight in this respect. In connection with scaffolding, Scott (1997) considers tutor's responsiveness to student learning as a matter at the heart of this metaphor. In his account, tutor's responsiveness indicates that as learning proceeds and progress is made towards the learning goal (changing the current level of performance) then the nature of the tutor's assistance is modified appropriately. He further develops the idea of 'responsiveness' by focusing on three components: monitoring – tutor monitors the present performance of the learner; analysing – tutor analyses the nature of any difference between present performance and the target performance; assisting – tutor responds with an appropriate intervention to support the learner in progressing from present to target performance. He suggests an 'action cycle' for these three components (Scott, 1998) as shown in Figure 2.1.

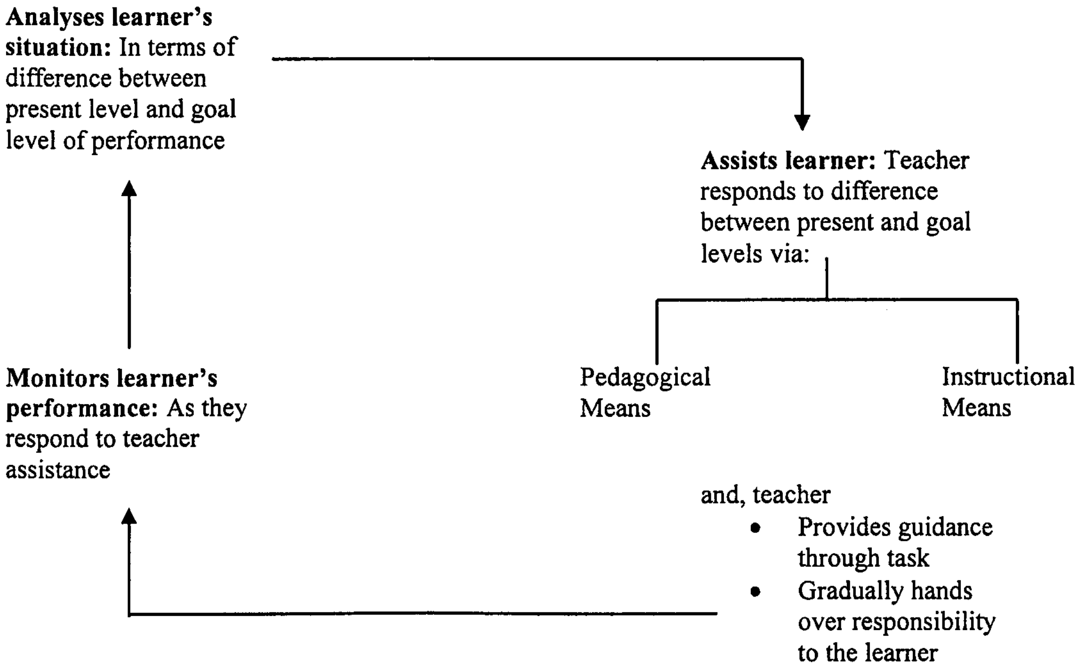


Figure 2.1. Cycle of responsiveness in scaffolding

This figure shows how a tutor moves around the components of the cycle in order to scaffold the learning: monitoring the learner's performance; analysing how that level of performance relates to the goal level; taking action to assist the learner towards the goal; monitoring the learner's new performance and so on. In order to assist the learner in moving towards the target performance, the tutor might make use of pedagogical and/or instructional means, which are described at the beginning of this section. When the learner makes progress towards the learning goal, the level of assistance is decreased and the responsibility is eventually handed over to the learner. If however the learner's performance suggests that he/she needs more assistance then the amount of assistance is increased accordingly.

The action cycle model of scaffolding is useful to display, in an operational way, the active, systematic and dynamic structure of scaffolding. When viewed from viewpoint of action cycle model, Hammond's (2001) argument that a tutor takes an active and informed role in guiding students suggests that the tutor's assisting interventions should be informed by his constant monitoring and analysing of the students' performance. It is through these monitoring and analysing actions that a tutor may act contingently.

2.4. A definition of scaffolding

In this section I propose a definition of scaffolding which will be used in this study. Considering the literature reviewed so far, we can identify six key principles of scaffolding which are similar to the ones identified by Love, (2002); Maybin, Mercer and Stierer, (1992); Stone, (1998). These principles are as follows:

1. Scaffolding involves a tutor and student(s) who are working together on a particular task towards the student's development of a competence.
2. In the course of scaffolding, the tutor intervenes to support the student by qualitatively different means of assistance.
3. The given support is augmented or reduced depending on the student's developing competence (contingency principle).
4. The tutor takes an informed role by monitoring the student's situation and analysing his/her current performance in relation to goal level of performance.
5. In the course of scaffolding the student is actively involved in the learning task and the tutor makes a particular effort to ensure this.
6. In scaffolding, the tutor intends for the student to develop a specific skill, grasp a particular concept, or achieve a particular level of understanding which is not readily achievable by the student's unassisted efforts.

On the basis of these principles, I view scaffolding as an asymmetric collaboration between a ('more knowledgeable') tutor (teacher/parent/peer) and ('novice') student(s) (child/learner) *towards* a successful completion of a task within the student's ZPD for which the tutor provides assistance which is augmented or reduced depending on the student's progress.

This definition integrates all six principles extracted above, though not all of them explicitly appear in the wordings. Therefore, I will briefly elaborate on the definition in relation to these principles.

To begin with, the term 'asymmetric' is used to indicate the fact that there is an unequal distribution of, at least, knowledge and power between the parties (tutor-student) involved in scaffolding. In order to emphasise this asymmetry I used the terms in brackets 'more knowledgeable' and 'novice'. This asymmetry could bring about a productive interaction (productivity owing to the tutor's broader perspective on the learning task) as well as oppression of the student and, even perhaps, an imposition of the tutor's own perspective on the student. Surely scaffolding desires productivity and not imposition which could result in a mode of interaction where the task is completed by the student following the tutor's step-by-step instructions and/or explanations.

In order to emphasise the productivity, I employed the term 'collaboration' which indicates that it is not only the tutor who is responsible for task completion but also the student who is to be actively involved in the learning task. By referring to scaffolding as 'an asymmetrical collaboration', my intention is to emphasise that scaffolding is a joint activity in which the tutor takes an informed role and often sets up a role for the student to play, thus making an effort to get the student's active involvement.

The next expression is that of 'towards successful completion of a task'. By this expression I have two intentions. The first is related to the outcomes of scaffolding. By successfully completing the task in collaboration with the tutor, the student develops a particular skill, a new understanding or grasps a particular concept. However, there are studies providing evidence that although teachers employ scaffolding in their instructions, the students may fail to benefit (see Bliss et al., 1996 for some of the reasons). This pinpoints my second intention that scaffolding does not guarantee these outcomes but desires them. In order to underscore this fact I employed the seemingly unimportant preposition 'towards' which underlines that even if success is not guaranteed, the tutor should have an intention of success on the part of learner and exert himself/herself for this.

The expression 'within the student's ZPD' indicates that task completion and outcomes are not readily available from the student's unassisted efforts. In this sense this expression is particularly important to demonstrate the relationship between scaffolding and the ZPD.

The expression 'tutor provides assistance' is used in connection with two broad categories of assistance as discussed in the previous subsection and more specifically suggests that in scaffolding a student's work, the tutor employs different means of assistance. However, this assistance is tailored according to the needs of the student. To do this, the tutor needs to monitor and analyse the learner's current progress in relation to the task's demands, as suggested by Scott's action cycle model.

Finally the assistance is 'augmented or reduced depending on the student's progress' refers to the Wood's contingency principle. An important point to be made here is that the tutor, in supporting the student, does not need to wait until the student completely fails in progressing on the task. This might have a negative effect on the student who may refuse to carry on working even with the tutor's assistance. Therefore, when the tutor feels a need, on the basis of his/her monitoring and analysing actions, then support is to be provided. However, when the student progresses then the assistance should be reduced and even removed to hand the responsibility over to the student. This expression thus also points out the tutor's efforts to sustain the student's developing competence towards independent task mastery.

At this juncture an important terminological notation is necessary. In the context of my study, I prefer the term 'scaffolder' to 'tutor' in describing the tutor's role in supporting the students' works. This is first because the tutor in this study is the same person as the researcher writing these lines and knows what it means to scaffold. Secondly, the tutor in my study had considerable experience in scaffolding students' work and was surely concerned with and had a clear intention to scaffold the participating students' work. Thirdly, the tutor was successful in his scaffolding. Therefore, I use the term scaffolder rather than tutor in describing the function of the interviewer in this study.

3. Theoretical framework of the study

As mentioned earlier, this study has two particular aims: to investigate the validity of Hershkowitz et al.'s (2001) RBC theory of abstraction and to investigate the role of scaffolding in the process of abstraction. In order to realise these aims I need a framework in which RBC theory and the notion of scaffolding could be combined and elaborated together. Development of such a framework is of critical importance to guide the data analysis and to explore the implications of the analysis. Towards this direction, I found useful the Vygotskian notion of the zone of proximal development (ZPD) to develop a theoretical framework in which both RBC theory and scaffolding can be situated. The connections between the ZPD and scaffolding have already been discussed in the scaffolding literature review (see section 2.1. above). Therefore, my main concern in this section will be to situate RBC theory within the ZPD. To this aim, I will first give a detailed account of the ZPD with specific reference to Tharp and Gallimore's (1988) study. Then I describe how RBC theory of abstraction fits within the ZPD. In the final part of this section, I provide my research questions.

3.1. The zone of proximal development

The notion of the ZPD is a by-product of Vygotsky's attempt to create a psychology whose aims and objectives are to understand the social origins of human consciousness. He claims, "the first problem [of psychology] is to show how the individual response emerges from the forms of collective life" (1981b, p.165). By employing the ZPD Vygotsky tries to explain individual's cognitive development through interacting with the more capable others in a culture. However, his writings on this notion took place at different times and in relation to different issues. In this connection, Wells (1999) distinguishes two versions of the ZPD within the Vygotsky's original writings. One version appears in *Mind in Society* (Vygotsky, 1978) where he views the ZPD as the difference between actual and potential development level. This view of ZPD reflects Vygotsky's emphasis on dynamic assessment of individual's intellectual abilities rather than static means such as IQ tests. Wells points out that the second version of the ZPD is presented in *Thinking and Speech* (Vygotsky, 1987, chapter 6) in relation to Vygotsky's consideration of the development of scientific concepts in childhood. Here Vygotsky's emphasis is on instruction rather than assessment:

> Instruction is only useful when it moves ahead of development. When it does, it impels or awakens a whole series of functions that are in a stage of maturation lying in the zone of proximal development. ... Instruction would be completely unnecessary if it merely utilised what had already matured in the developmental process, if it were not itself a source of development (1987, p.212).

Here in this quotation Vygotsky views a 'good' instruction as that which takes place within the ZPD and which leads to new development on the part of individuals. In either version of the

ZPD, Vygotsky is concerned with the difference between individuals' assisted achievement and their unassisted performance. The importance that he attributes to more knowledgeable or capable members of a culture for a realisation of this achievement is all too apparent throughout his writings.

Subsequent to its introduction to the Western world, the notion of ZPD has undergone a development process at both the theoretical and the empirical level; a list of extensions to the ZPD over the years can be found in Wells (1999, p.333) (see also, for example, Wertsch and Hickman, 1987; McLane, 1987; Goldstein, 1999). Later interpretations of this notion make it fairly clear that the ZPD is not a 'fixed' attribute of an individual (Wells, 1999) nor is it a physical space in the sense of the individual's equipment (Meira and Lerman, 2001). Instead it stands out as a symbolic space for the potential for new learning which is expanded or contracted depending on, for example, the nature of the specific activity, on the mode of social interactions and on the kinds of communicative processes utilised (see McLane, 1987). Furthermore, although the initial considerations of the ZPD focused on (more capable and novice) dyads in face-to-face interaction, this view is later extended to include all participants in collaborative communities of practice, on the basis of the argument that the ZPD constitutes a potential for learning that is created in the interaction amongst participants as they engage in a particular activity together (see Wells, 1999, chapter 10 for more on this). Based on an analysis of Vygotsky's original writings and later interpretations, Tharp and Gallimore (1988) provide an insightful account of the ZPD. Although these authors published their work in 1988, their account, in my opinion, is still one of the best elaborations of the ZPD available to date. It was through reading these authors that I realised how the notion of the ZPD accommodates both RBC theory and scaffolding. So I will next elaborate on these authors' account of the ZPD.

3.2. Genesis of performance through and beyond the ZPD

Tharp and Gallimore (1988) discuss the genesis of performance progressing through and beyond the ZPD. By focusing on the relationship between self- and social-control, they develop a model which represents individuals' progression in relation to the ZPD at four stages, which is given below (Figure 2.2.). I will now consider each of these stages.

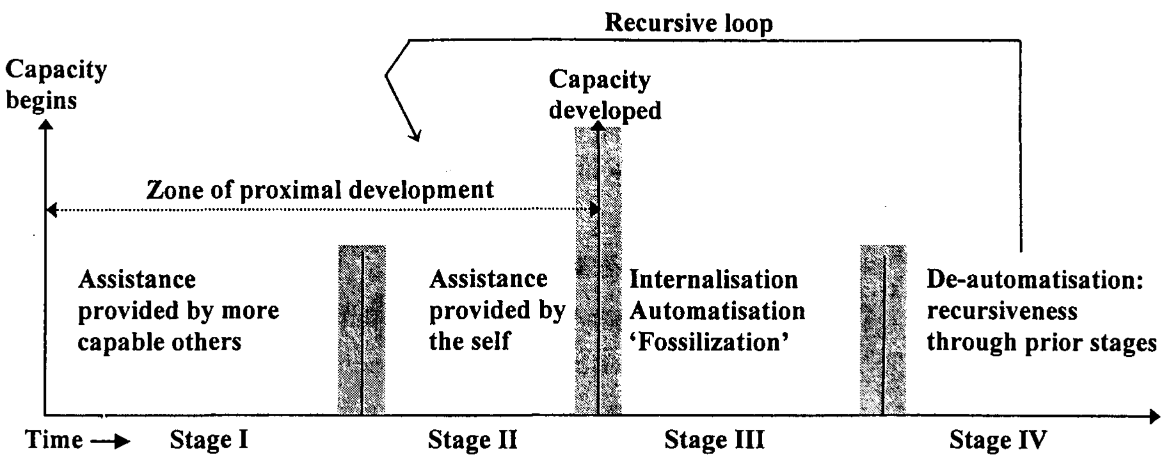


Figure 2.2. Genesis of performance capacity: progression through the ZPD and beyond (Tharp and Gallimore, 1988, p.35).

Stage I: *Where performance is assisted by more capable others:* Before learners function independently without needing any assistance, they are dependent on adults or more capable peers for an outside regulation of the task performance. The amount and type of assistance and outside regulation that a learner needs are contingent upon his/her age and the nature of the task. During the initial period of this stage, the learner may have a very limited understanding of the situation, the task or the goal to be achieved; at this level more capable others (a teacher, a knowledgeable person, a peer) offer directions or modelling. It is this stage where assistance from more capable others is most valuable and necessary. The learner's developing competence reflects itself through a gradual transition of regulation from others to self.

Stage II: *Where performance is assisted by the self:* In this stage the learner takes over responsibility and carries out a task without needing as much assistance from the more capable others as he/she does in stage I. What was guided by the other is now beginning to be guided and directed by the self. However this self-guidance does not mean that the performance is fully developed. In other words, at some points, even if occasionally, the learner may need others' regulation until he/she achieves development of the full competence.

Stage III: *Where performance is developed, automatised, and 'fossilised':* In this stage performance is fully developed and assistance from the more capable others and from the self is no longer needed to carry out a task. The learner can execute the task smoothly and competently. Any instruction from others could even be disruptive and irritating. It is in this stage where the learner has developed a competency which Vygotsky calls the 'fruits' of development (1978, p.86), and describes it as 'fossilised', pointing to its distance from the social forces of change.

Stage IV: *Where de-automatisation of performance leads to recursion back through the ZPD:* At any time of ontogenetical development (development of an individual over his/her lifetime), it is quite likely that one can no longer do what one could formerly do for some reasons such as environmental changes. In addition, during a period of difficulty, individuals do not have to solely rely on internal mediations to overcome these difficulties and/or find solutions. They can ask for help when stuck. In that case a cycle of self-assistance to other assistance occurs for enhancement, improvement and maintenance of performance.

3.3. Situating RBC theory of abstraction within the ZPD

Having reviewed Tharp and Gallimore's four-stage model of the ZPD, I now turn my attention to RBC theory of abstraction. In Hershkowitz, Schwartz and Dreyfus' (2001) (referred to HSD hereafter) view, abstraction is seen as "an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure" (p.202). This statement encapsulates the epistemological principles and sociocultural underpinnings of RBC theory.

Therefore, the terms used in HSD's account will be scrutinised to explicate how it is related to the ZPD.

The term 'activity' in HSD's account is used in the sense of activity theory to indicate that abstraction involves "a chain of actions undertaken by an individual or a group and driven by a motive that is specific to a context" (HSD, p.202). By referring to abstraction as an activity, they claim that abstraction is an interactive process, involving tools, language and procedures. Indeed, inspired by activity theory the authors remind us that any particular action a person may undertake is a part of a greater activity, and occurs in a social and historical context, is inseparable from its activity-related goals and can be truly meaningful, and thus interpretable, only within the context of this activity. In doing so, they regard abstraction not as 'an objective, universal process' rather it is a subjective process which finds its origins in its social contexts and thus point to the socio-historical aspect of abstraction.

Activity theory evolved from the writings of Vygotsky, was developed by Leont'ev (1981) and later further developed by, chiefly, Engestrom and his colleagues (see the volume edited by Engestrom, Miettinen and Punamaki, 1999). When the ZPD is examined carefully together with these later modifications and extensions it can be seen that HSD's consideration of abstraction as an activity within its sociocultural context resonates deeply with the ZPD. To begin with the ZPD is not a 'fixed' quality of individuals; rather, it is subjective in the sense that potential learning within the ZPD depends on the individuals' personal histories – actual development level – as well as on the nature of interaction between the participants while they engage in a particular activity together. Surely learning within the ZPD, like HSD's abstraction, is an interactive process. Further to this, learning and development in the ZPD depends on the 'mediational means' (Wertsch, 1998) which involves both technical and psychological tools (e.g., computer software, language and a mathematical formula; see Vygotsky (1981a) for more on tools). In this connection, Wells (1999, p.318) notes that the ZPD is "created in the interaction between the student and the co-participants in an activity, including the available tools and the selected practices, and depends on the nature and quality of that interaction."

The next term in HSD's account of abstraction is 'previously constructed mathematics', which connotes two points: "first, that outcomes of previous processes of abstraction may be used during the present abstraction activity and, second, that the present activity starts from an initial unrefined form of abstraction" (HSD, p.202). They argue that these two points demonstrate the recursive character of abstraction in the sense that an abstraction involves the earlier abstractions.

Regarding these two points, the 'previously constructed structures' can be viewed as residing in the learner's actual developmental level. Vygotsky writes about the actual development level that it is "the level of development of a child's mental functions that has been established as a result of certain already completed developmental cycles" (1978, p.85). Vygotsky's writings lack specificity with regard to how the child's mental functions are established in the intramental plane and thus become a part of the repertoire of his/her actual development. Nevertheless, in connection with these functions, he at least suggests that "if a child can do such-and-such independently, it means that the functions for such-and-such have matured in her" (ibid., p.86). This clearly suggests that to Vygotsky the indication of whether an individual has completed the development of a mental function depends on whether this individual can act independently in relation to this mental function. If so, then this mental function has matured in this person and thus become part of this individual's repertoire in the actual development level.

When we return to HSD's first point (i.e. use of earlier abstractions), an outcome of previous abstraction could be considered as a specific form of functions that Vygotsky considered as the end product of developmental cycles. In addition, although HSD do not explicitly discuss this, it is obvious that in order to use outcomes of previous abstractions the student should be able to identify them when encountered and to operate on them with some degree of independence. This is also compatible with the Vygotsky's emphasis on individuals acting independently in connection to already matured mental functions. Regarding HSD's second point (i.e. abstraction commencing as an unrefined form), an initial unrefined form of abstraction can be said to exist within the learner's actual developmental level in the sense that prerequisite knowledge artefacts necessary for a formation of abstraction have been matured and thus are already part of the learner's repertoire at that level. If they were not, there would be no possibility of talking about the formation of an abstraction, which requires these prerequisite knowledge artefacts, because 'previously constructed mathematics' would then not exist and thus of course not be available.

The final term 'vertically reorganising into a new structure' in HSD's account refers to the "establishment of mathematical connections, [which] includes highly mathematical actions like (a) making a new hypothesis and (b) inventing or reinventing a mathematical generalisation, a proof, or a new strategy for solving a problem" (HSD, p.202). This reorganisation comes about by structuring, and developing mathematical elements into other more abstract and formal elements than the original ones. In addition, the word 'new' is used to express that, as a result of abstraction, participants in the activity perceive something that was previously inaccessible to them.

Whilst 'previously constructed mathematics' can be considered within a student's actual developmental level, vertical reorganisation of these structures into a new one occurs through the learner's ZPD or within the potential development level. Vygotsky argues that new functions are in the process of maturation in this level. Within his/her ZPD, the learner establishes mathematical connections and integrates available mathematical structures. The refined form of structure is obtained through a gradual progress within the student's ZPD. However, when the student starts to operate in his/her ZPD, he/she need not be aware of the internal links or reasons within the initial structure, or of articulation of the structure in relation to already constructed ones. In a nutshell, for a formation of a new abstraction, a vertical reorganisation of previous abstraction(s) is required in the learner's ZPD.

Now we are faced with a difficulty here. Acceptance of my view that 'vertical reorganisation' can only take place within the potential development level necessitates that abstraction can only be achievable with the assistance of more capable others. This is precisely because the very definition of 'potential development level' argues that this level is determined 'through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p.86). How can I then explain some individual cases who can achieve abstraction by working 'alone'? Surely it is not difficult to imagine a learner working alone on a task, thinking hard and achieve an abstraction. In fact literature has already reported such cases within the framework of RBC theory (e.g., Kidron and Dreyfus, 2004). Although this issue will be fully elaborated on the basis of the empirical data in Chapter 5 (section 4), I need to note here that the assistance within the ZPD is not limited to the assistance from the physically present (more capable) others. At this juncture, in order to give an idea of the phenomenon that I have in mind, I suffice to quote from Wells (1999, p.331) that

> the sources of guidance and assistance for learning [in the ZPD] are not limited to human participants who are physically present in the situation; absent participants, whose contributions are recalled from memory or encountered in semiotic artefacts, such as books, maps, diagrams and works of art, can also function as significant [i.e. more capable] others in the ZPD.

3.4. The genesis of abstraction through and beyond the ZPD

Thus far I have focused on HSD's account of abstraction in relation to the ZPD. I now turn to HSD's account of the genesis of abstraction and consider it with regard to Tharp and Gallimore's (1988) model of ZPD at four stages. Before doing this however I should point out that HSD, following Davydov (1990), distinguishes between 'everyday' and 'scientific' concepts and RBC theory is interested in the abstraction of scientific concepts in the domain of mathematics. According to Davydov, everyday concepts are usually acquired through empirical thought which interconnects features of reality (for example, by observing similarities of and differences between things). However, scientific concepts are in his account considered as more complex, general and abstract. These concepts are acquired through theoretical thought which develops comprehensive interrelationships and mental transformations amongst the objects of the world. HSD's theory of abstraction involves empirical as well as theoretical thought.

HSD (p.218) hypothesise that tracing the genesis of an abstraction passes through three stages: (1) a need for a new structure; (2) the constructing of a new abstract entity in which recognising and building-with already existing structures are nested dialectically; and (3) the consolidation of the abstract entity facilitating one's recognising it with increased ease and building-with it in further activities. I will consider each of these three stages within the model of the ZPD as proposed by Tharp and Gallimore. In this respect, I demonstrate the genesis of abstraction in

Figure 2.3. below. This figure, in fact, reflects some slight modifications to RBC theory. I will touch on these when explaining this figure and fully elaborate them later on the basis of empirical data in Chapter 7.

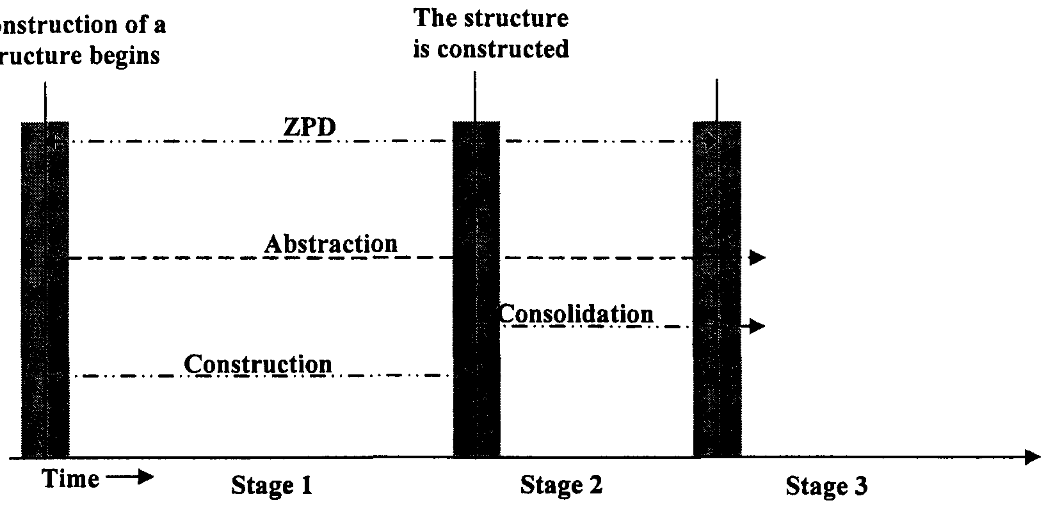


Figure 2.3. The genesis of abstraction through and beyond the ZPD.

1. A need for a new structure: In HSD's theory, abstraction commences with a need for a new structure. If there is no need of a new structure, then it is almost impossible for one to attempt to make an abstraction. This need could occur in the conduct of an activity as a result of, for instance, the context of problem, the demands of the tasks and/or interaction with the others present (or absent) during the activity.

2. Constructing a new abstract entity/structure: This is the second stage of the genesis of abstraction in HSD's theory. In HSD's theory, 'constructing' refers to an epistemic action which is seen as assembling knowledge artefacts to produce a new structure. As a result of this constructing, in HSD's theory, a new abstract entity/structure, which was initially inaccessible, emerges. However, in my analysis of the verbal protocols generated for this study, I observed constructing actions occurring as 'little', yet strongly interrelated, segments or pieces in the students' utterances. In this sense a series of constructing actions were needed for the emergence of a new structure. I will discuss these observations later on the basis of empirical data in Chapter 7 (section 2). However, here I wish to differentiate between constructing which refers to an epistemic action in the sense of HSD and 'construction' which refers to the emergence of a new structure through constructing actions. Throughout the rest of my writings I will employ the terms 'formation of a construction' and 'construction of a (mathematical) structure', each of which indeed refers to what I call 'construction' in the above-sense.

In Figure 2.3. construction is depicted to take place in stage I of the ZPD as a part of abstraction, the other part is consolidation (see below). This depiction has two particular aims. The first is that construction of a structure is possible if and only if prerequisite knowledge structures (i.e. previously constructed mathematics) are available to a learner. For this to be the case, these prerequisite structures need to be part of the learner's repertoire (i.e. within the realm of actual development level). For example, in order for one to form the construction of the notion of derivative, some knowledge of functions, of a mapping between functions and graphs and of the limit concept should be available to this person for otherwise a vertical reorganisation of these structures cannot be achievable. Therefore, in placing construction within the ZPD, my intention is to emphasise that construction of a new structure should be within the range of the learner's competence level if it is to be achieved. The second is that a learner needs assistance from some sources to achieve a new construction and to use it independently. Although I do not confine the sources of assistance to the physically present participants, my particular concern here in this stage will be with the assistance of a more capable adult, in the sense of scaffolding, during the formation of a new construction.

3. Consolidation of the new structure: The construction of a new structure begins at the initial period of stage I and ends in later period of this stage. Following the formation of a new construction, a learner becomes acquainted with a new structure. Yet this does not mean that the new structure is fully developed in that it needs consolidating. HSD's original study and a companion paper (Dreyfus, Hershkowitz and Schwarz, 2001) are mainly interested in, and provide evidence with regard to the first and second stages of abstraction. They *a priori* assume the necessity of consolidation for a new structure and therefore talk about the consolidation of an abstraction. This issue requires further research which I aimed to do in my study. Towards this direction, my analysis of the data generated for this study suggests that it is the new construction(s) which is (not an abstraction) in need of consolidation. I will elaborate on this in Chapter 7 (section 3.2.) on the basis of student protocols. But I here note that the very definition of abstraction proposed by HSD requires the use of previously constructed mathematics (recognising and building-with). However, the data collected for this study suggested that students were unlikely to use the new structures unless they were consolidated. Thus abstraction requires establishment of a sufficient familiarity of a new structure to be used in further abstractions.

In order to reflect this observation, in the above Figure 2.3., I depicted consolidation taking place in stage II of the ZPD and ending in stage III. This reflects my conviction that consolidation follows construction and that a consolidated construction can be used when required; when it becomes an abstraction. Further to this, abstraction is depicted to cover both stage I and II, i.e. both construction and consolidation are constitutive parts of abstraction. However, by considering consolidation within stage II, my intention is not to argue that during consolidation the learner does not need any assistance; quite the contrary they might need some assistance but they are relatively more self-regulative in comparison with their performance in stage I of the ZPD. The issue of consolidation is an important one upon which HSD call for further research and is one of the foci of this study.

At this point it should be noted that the moment of a transition from one stage to another in the ZPD is difficult to determine precisely. In addition, these transitions themselves are not immediate but gradual. Likewise, although in theory it is possible, and indeed feasible, to talk about a start and end point for the construction and consolidation, in practice it is hardly possible to determine these moments. In fact, it is not necessary to determine the exact moments when a construction is formed and a consolidation is started. Therefore, in the above figure, the straight lines that stand for separation between the stages are shaded to indicate that transition occurs somewhere around those lines rather than at a specific point and that this transition, either from other- to self-regulation or from construction to consolidation, is a gradual shift.

As a final point, stage IV of Tharp and Gallimore's treatment of the ZPD is not immediately relevant to the purpose of this study. In the context of RBC theory, this stage may refer to a deformation of abstraction for some reasons in the long term such as failure to remember as a result of not practising a structure for a long time, amnesia or environmental changes. It is not difficult to find some people who used to be good at, for example, derivative at the university years, but years after the graduation, are unable to use it when needed (a nice illustration with this regard can be found in Hughes, Monaghan, Shingadia and Vaughan, in press). In that case it is sensible to think that they need assistance from the others to regain the control of or brush up this structure. However, an investigation of this aspect of abstraction is beyond the scope of this study and indeed is not amongst the aims.

3.5. Research questions

In the light of given literature and the issues raised hitherto, research questions for this study are formulated as follows:

Research Question 1: How are new mathematical constructions formed through scaffolding?

In connection with this question I explore issues stemming from the analyses of students' verbal data, related to the scaffolding and construction of a new structure. These are, more specifically: the nature and occurrence of assisting interventions; relationships between assisting interventions and formation of new constructions; the nature of social interaction between the scaffolder and student(s) in the course of construction; the issue of intersubjectivity and alterity within the interaction; emergent goals and their relationships with the constructions; and handover of the responsibility and its implications for the constructions. Each of these issues requires further literature review which I prefer to present immediately before the related sections of my discussions rather than here to help reader follow the arguments more easily.

Research Question 2: What is the nature of consolidation?

In connection with this research question, I will try to answer the questions: What is the initial state of new constructions? What changes may occur during the consolidation? If and how consolidated constructions are used in further abstractions? However in my attempt to provide answers to these questions, I will not be concerned with the assistance of others or the issue of scaffolding in the course of consolidation. This is not because the issue of assistance and scaffolding is irrelevant to consolidation but because this issue is my explicit focus regarding the first research question and discussions provided in this respect will shed light on the role of assistance and scaffolding in relation to consolidation.

Research Question3: To what extent is the RBC theory of abstraction valid beyond the cases presented by the original authors?

In my considerations hitherto, I already suggested some modifications to RBC theory of abstraction. However, I will elaborate on these modifications and re-evaluate RBC theory on the basis of the students' verbal protocols generated for this study. In this respect I will make critical reflections on this theory by focusing on epistemic actions, the genesis of abstractions, and the epistemological and sociocultural principles as proposed by HSD.

CHAPTER 3: METHODOLOGY AND DESIGN OF THE STUDY

In this chapter I attend to the methodology of the research, design of the study, data collection tools and procedures, and data analysis methods. In this connection, broadly speaking, I focus my attention on six main issues: the nature of the study; case study approach adopted in this research and the design of the cases together with the rationale behind this design; data collection methods and instruments; piloting of the instruments, the insights gained from the pilot study and the subsequent revisions introduced into the instruments in the light of these new insights; actual data collection process and the verbal protocols as data which were used in this study to answer the research questions; and the verbal data analyses procedures.

1. What is the nature of this study?

In relation to the nature of research, the literature suggests two broad traditions: qualitative and quantitative. Qualitative inquiry is often called 'interpretative' research (Williams, 2002) since qualitative researchers "study things in their natural settings, attempting to make sense of, or to interpret, phenomena in terms of the meanings people bring to them" (Denzin and Lincoln, 2000, p.3). There are many different paradigms within the qualitative tradition such as phenomenology, hermeneutics, social constructionism, critical theory, postmodernism and feminist approaches. There are some differences in these paradigms in terms of their approach to the issue of interpretation and understanding of human action (see Schwandt, 2000; Lincoln and Guba, 2000). Nevertheless, these paradigms appear to be unified in their sharp contrast to what is known broadly as the 'positivistic approach'. The positivistic approach is quantitative in its outlook though it at times utilises qualitative data. The qualitative tradition can be best regarded as non-positivistic though it at times uses quantitative data. However, the primacy attributed to qualitative and quantitative data differs in these two traditions.

Further to this, the qualitative/interpretative and quantitative/positivist traditions sharply differ in their assumptions related, at least, to the nature of reality, to epistemology in terms of the relationship of the researcher to the phenomena being researched, to the role of values in a study (axiology) and to the process of research (methodology) (see Creswell, 1998). With regard to the nature of reality, positivists presume a stable, unchanging reality which is "out there to be studied, captured, and understood" (Denzin and Lincoln, 2000, p.9) whereas qualitative researchers view reality as subjective and multi-dimensional, constantly changing due to social, cultural and historical influences (Flick, 1998). In terms of epistemology, positivists consider language and reason as instruments of control in discovering and ordering the reality of the world by an autonomous, disengaged, cognising agent (researcher) (Schwandt, 2000; see also Woolgar, 1996 for a critique). However, the qualitative tradition promotes a view of researcher who should lessen the distance between him/herself and that of being researched and thus acting an instrument of data collection by becoming an 'insider' (Cresswell, 1998).

As to axiology, the qualitative tradition acknowledges the value-laden nature of research and existence of biases. The argument here is that the researcher is always understanding and interpreting in the light of his/her anticipatory prejudgements and prejudices, which themselves are changing in the course of history and which are not necessarily arbitrary and distortive but will be different depending on one's changing horizons and on the different questions he/she learns to ask (Bernstein, 1983). Positivists, however, by advocating statistical measurements on the basis of large numbers of randomly selected cases argue that their work is done from within a value-free framework, free of individual bias and subjectivity (Denzin and Lincoln, 2000). This position also underlies the positivists' methodology that aims to develop methods to analyse relationships between variables, not processes, mainly in terms of causes and effects by operationalising theoretical relations, measuring and quantifying phenomena in terms of amount, intensity or frequency, and thus aiming to allow the generalisation of findings (Flick, 1998). In contrast to this, qualitative researchers often work with particulars and details, provide 'thick descriptions' of the context of the study (Lincoln and Guba, 1979/2000), uses emerging design and employ inductive logic.

Returning to the question posed in the title of this section, the nature of a study/inquiry is strongly dependent on the research questions and purposes. Looking into this study's research questions, it can be realised that one needs to investigate such issues/processes as the construction of new mathematical knowledge in a scaffolded interaction, the nature of interaction between the students and scaffolder, the dynamics involved in such interaction and the nature of newly constructed knowledge in relation to consolidation. In order to study and gain insights into these issues, one needs to carry out, at least, a concurrent inspection of the students' and scaffolder's actions, interactions, interventions and verbal statements; to observe the development of the students' understandings over some activities; to analyse these inductively by focusing on the meaning (real or perceived) of the individuals; and interpret and describe these processes. In order to attain this objective, a qualitative approach was the obvious choice.

With regard to the purposes of a study, Robson (1993) suggests three types of research: exploratory, descriptive and explanatory. One's aims in exploratory research are to find out what is happening, to seek new insights and to assess phenomena in a new light. The researcher conducting a descriptive study aims to portray an accurate profile of persons, events or situations. An explanatory study seeks an explanation of a situation or problem usually in the form of causal relationships. Robson (ibid., p.42) notes that a study might be concerned with more than one purpose, possibly all three, but often one will be the dominant. Once again considering the issues/processes that this study sets out to investigate, this study is mainly explanatory in nature as I aim to explain, for example, how the scaffolder's interventions lead the students to achieve new mathematical constructions and the nature of consolidation of the newly constructed structures. This study has also an exploratory flavour as it aims to explore the data to seek new insights into RBC theory of abstraction.

In summary, this study holds a non-positivistic view of scientific research with a qualitative inquiry and interpretative approach to the phenomena under investigation. It is mainly explanatory in nature but also aims to explore the data to seek new/further insights into the issues raised by RBC theory. Within the qualitative tradition, there are many different methods used to obtain rich, descriptive and contextually situated data such as biography, phenomenological study, grounded theory study, ethnography and case study. For this research, the case study method was preferred, which I will attend to next.

2. The case study method

Case studies are one of the specific forms of inquiry used most commonly within the qualitative research tradition. Robson (1993, p. 146) describes case study as "a strategy for doing research which involves an empirical investigation of a particular contemporary phenomenon within its real life context using multiple sources of evidence". The case study method is seen particularly useful to investigate a unit (case) or set of units over a period of time through detailed, in-depth data collection in a specific context (Creswell, 1998) and thus to achieve a deep understanding of what is happening in that context (Robson, 1993). Further to this, it is deemed as a valuable method to refine a theory and suggest complexities for further investigation, as well as help establish the limits of generalisability (Stake, 2000). As my intention was to further examine RBC theory, to gain insights into the issue of knowledge construction with the aid of a scaffolder and to achieve a detailed understanding of the process of consolidation, the case study approach was thought to be an appropriate strategy for this study.

Essential to a case study is to identify the main unit of analysis (i.e. case or the phenomenon being studied) and the kind of the case (Creswell, 1998). The main unit of analysis or the case being studied in this study is the formation of a mathematical abstraction. However, there were two subunits of analyses within this main unit: formation of a new construction (with the aid of scaffolded help) and consolidation of this construction. Yin (1998) calls this kind of research embedded case study, i.e. one or more subunits of analyses are embedded into the main unit. Further to this, in this study it was decided to use more than one case to investigate the mathematical abstractions. This is what Yin (ibid.) calls 'multiple case study strategy'. Contrary to the common perception, this strategy is not used for the purpose of gathering a 'sample' of cases in order to make generalisations to some population (Robson, 1993) but rather used to achieve 'literal replications' and/or 'theoretical replications' (Yin, 1998). Multiple cases are designed to 'replicate' each other so that corroboratory evidence from two or more cases might be produced – literal replication. Multiple cases can also be designed to cover different theoretical conditions so that some contrasting results, but for predictable reasons, might be produced – theoretical replication. Roughly speaking, in this study I worked with two broad groups of students: those working with the scaffolded help and those without so as to contrast their performances in gaining insights into the influence of scaffolded help. Further to this I designed several cases from each group to produce corroboratory evidence in relation to the observed trends. Therefore, this study employed a multiple case study strategy with the purpose of literal and theoretical replications. I suffice to note this here but later return to the design issues in greater detail.

The case study approach is viewed as a challenging form of inquiry due to the fact that case studies and their outcomes are often multifaceted and difficult to capture adequately on account of the range of dimensions involved in a case (Robson, 1993). So the matter of concern here was with what issues/dimensions/facets to focus, to privilege and to follow in more depth while designing and analysing the case studies. In this respect, case study experts point out the importance of a structured approach in the sense of developing/using a theory (or theories – especially rival ones) in articulating the research questions and in guiding the design and analysis phases of a case study (Creswell, 1998; Yin, 1998). This study was relatively well structured from the very start in that there were certain conceptualisations (e.g., scaffolding) and theories (the ZPD and RBC theory) guiding my design and analyses of the cases. The assistance that the structured nature of this study provided me with is rather telling and will be quite clear in my following writings on the design and analyses issues. However, one obvious drawback of this structured nature could be such that it may obstruct the researcher's openness to the new emergent (and perhaps unanticipated) themes. Yet it was my intention in this study to seek new and emergent themes to evaluate critically RBC theory.

3. Validity, reliability and generality

The case study research is often criticised on the grounds of validity, reliability and generality of the findings (see Gomm, Hammersley and Foster, 2000). In fact this criticism is directed towards the qualitative tradition in general. The terms reliability and validity are psychometric terms and originally developed within the quantitative tradition. Reliability in its simplest form is concerned with the consistency and stability of the findings which correspond to those found by another researcher following the same procedure, i.e. replicability. Validity on the other hand, albeit it comes in many different forms such as content, construct, internal and external validity, is mainly concerned with the success of an instrument in measuring correctly what it is designed to measure (see Frankfort-Nachmias and Nachmias, 1996 for more on the issues of reliability and validity).

As might be seen from those simple descriptions of the terms validity and reliability, they assume a stable and consistent (i.e. unchanging) view of reality or the phenomena being investigated which can be inferred in one 'correct way' as is the case, say, with the law of gravity in physics. However, in the qualitative tradition, these two terms make little sense since qualitative researchers hold the belief that there is no one 'correct' interpretation of the reality which is indeed dynamic and in constant change. However, in the qualitative tradition these two terms are usually intertwined but the treatments are inconsistent, if not conflicting. For example, authors such as Wolcott (1990) suggests that "validity neither guides nor informs" his work (p.136). He does not reject validity but finds this irrelevant to his (or more generally speaking a qualitative researcher's) goal to identify "critical elements" and write "plausible interpretations from them" (p.146). He argues that validity creates a distraction for his work of understanding what is really going on. Others contend that qualitative research is open to criticisms for its failure to "adhere to canons of reliability and validity" (LeCompte and Goetz, 1982, p.31). They attempt to develop parallel qualitative equivalents of these terms. Still others such as Richardson (1994) offer a reconceptualisation in terms of the crystal metaphor. She challenges the positivistic image of validity as a rigid and fixed object and uses the metaphor of 'crystallisation' in connection with validity and reliability in the qualitative research:

> the central image is the crystal, which combines symmetry substances, transmutations, and multidimensionalities, and angles of approach. ... Crystallisation, without losing structure, deconstructs the traditional idea of "validity" (we feel how there is no single truth, we see how texts validate themselves); and crystallisation provides us with a deepened, complex, thoroughly partial understanding of the topic. Paradoxically, we know more and doubt what we know (ibid., p.522).

Richardson's idea of crystallisation recognises the many facets of any given approach to the social world, which is rather complex as a fact of life. Still other researchers suggest alternative terms to validity and reliability. For example in their seminal work, as an alternative concept to validity and reliability, Lincoln and Guba (1985) pose "trustworthiness" which contains four aspects: credibility, transferability, dependability and conformability.

With respect to this issue, my own position is that abandoning validity and reliability issues runs the risk of an accusation of producing 'invalid' and 'unreliable' findings. However, these terms simply cannot be applied to the qualitative research in their traditional forms as employed in psychometry to assess the quality. Any piece of qualitative work often engages in complex (and subjective) descriptions involving a myriad of interconnected variables that, as rightly contended by Schofield (2000), replicating and validating it on a piece-by-piece basis would be a major undertaking. Nevertheless, if the terms validity and reliability are taken to mean establishing some standards to ensure the rigor and quality of a piece of research, then surely this is necessary. In this respect I prefer the terms 'verification' (Creswell, 1998) and 'trustworthiness' (Lincoln and Guba, 1985) to validity and reliability in order to underline the importance and necessity of the efforts on the part of the (qualitative) researcher(s) to ensure that the findings which have been produced are a *credible* depiction, theorisation or explanation of the phenomenon studied. Towards this direction, Creswell (1998) on the basis of a literature review, suggests some of the most commonly used verification strategies to ensure the quality of qualitative research: triangulation, peer review or debriefing, negative case analysis, clarifying researcher bias, member checks, thick (rich) descriptions and external audits. He insists that a qualitative researcher must at least engage in two of these strategies. Morse, Barret, Mayan, Olson and Spiers (2002) make a strong case that these strategies are of little use if employed as *post hoc* evaluations of the findings and they advocate an evaluation process with these strategies all the way through the research from design to analysis to reporting. With this in mind, I will later return to these strategies extracted by Creswell and explain which ones were used in the course of this study to ensure quality and rigor.

I now turn my attention to the issue of generalisation. The researchers in the classical scientific (i.e. positivist) view hold the belief that generalisations are the main aim of science and what is worthwhile in doing science is to produce generalisations (see Lincoln and Guba, 1979/2000). To achieve generalisations researchers often work with large (and often randomly selected) samples that, they hold, enable them to generalise the findings (usually statistical inferences) to a population. From this standpoint, case studies (in fact, most qualitative research) deal with rather small samples, with (limited) particulars, and thus they constitute a poor basis for generalisation. However appealing it might be, the issue of generalisation is at best a matter of controversy. What do we mean by general and generalisation? If one takes an empiricist, Aristotelian view of generalisation, then to be general means to be true of each and all members in some large set and generalisation is defined as summaries of recurring features of experience (see Ohlsson and Lehtinen, 1997). Now the question here is: how large should the sample be to make this sort of generalisation? However large the sample might be, how can one be absolutely certain that observed trend holds for *each and all* of the members of a population to which the trend is generalised? Could that be ever possible considering the generaliser's experience with a limited number of particulars not with the *each and all* of the members of the universe? If one however is permissive to certain exceptions, how many exceptions would not do any harm to the generalised phenomenon?

Lincoln and Guba (1979/2000) examine Kaplan's (1964) 'nomic' or 'nomological' generalisations which maintain a view of generalisation as "truly universal, unrestricted as to time and space" (Kaplan, ibid., p.91). These authors make a powerful case against Kaplan's view of generalisation. They argue that this view of generalisation assumes an either-or proposition which discards varieties of human actions and activities. They go on to say that this view leads into a deterministic and mechanistic view of social world and they question if a phenomenon can exist independent of time and context. In a similar vein, Donmoyer (1990/2000) scrutinises Thorndike's (1910) view of generalisation which assumes the existence of an absolutist law-like nature for human beings whose intellect, character and behaviour can be determined and explained by cause and effect relationships with the same surety as in the physical sciences. Donmoyer makes a compelling case against this view and challenges it on the grounds of the complexity of the social phenomena, the social purposes and the influence of different paradigms in which, he claims, *a priori* assumptions such as that of Thorndike's determine what the 'facts' are rather than being determined by the 'facts' themselves.

Personally I am certainly uncomfortable with deterministic views of generalisation promoted by Aristotelian empiricist views such as Kaplan and Thorndike. But I am not completely against the idea of generalisation either. There are some other views of generalisation such as that of 'naturalistic' generalisation by Stake (1978) 'holographic' generalisation by Lincoln and Guba (1979/2000), 'moderatum' generalisation by Williams (2002) that I found more plausible. Common to these views of generalisations are flexibility and variability, and the realisations of the influences of the researcher's beliefs and values on the generalisations. Further to this, these accounts of generalisations are relativist in nature and could be, in my opinion, best described as theoretical inferences, working hypotheses, or perhaps in one word 'theorising'. Yet, I believe that the merit of a study does not necessarily lie in its ability to offer generalisations, however this conception might be described, but rather in whether or not the work communicates or 'says' something to us, whether or not the work contributes to our understanding of important educational questions, based on "how we conceptualise our reality and our images of the world" (Denzin and Lincoln, 2000, p.11). Therefore in this study my main concern is not with making generalisations but with theorising and hypothesising, with substantial care, towards an enrichment of the theoretical understandings of RBC theory, the ZPD, consolidation and scaffolding with new depths and dimensions.

4. Design of the case studies

In this section I outline how the case studies were designed for this study. In order to answer the research questions, I had to make observations with regard to some issues: the process of new mathematical knowledge/structure construction, the influence of the scaffolded help on this process, epistemic actions and the consolidation process. However there were some apparent methodological difficulties in attaining these observations. For example, Hershkowitz et al. (2001) argue that construction is a rare event and it might even occur while a student is working 'alone' on his/her own. Surely similar comments can be easily made for the process of consolidation. Therefore the case studies needed to be designed in such a way that would enable me to overcome these difficulties, achieve the necessary observations and draw appropriate inferences.

Regarding the construction and consolidation of new structures, many suggest that these events could possibly come out through rich social contexts e.g. group working or tutor guided inquiry, which also help these events become observable (Dreyfus et al., 2001; Dreyfus and Tsamir, 2004). In small groups, students often need to verbalise and explain to one another their thoughts, reasoning and the actions that they undertake in the course of a problem solving activity. The verbalisations may aid (force) students to reflect on their thoughts as well as on what the others are saying. In addition, there is a large body of research evidence that suggests that asking new questions, creating new problems, arguing the controversies amongst ideas and trying to justify what one claims are likely to lead in the construction of knowledge (e.g., Webb, 1991; Miller, 1987; Schwarz, Neuman and Biezuner, 2000; Vries, Lund and Baker, 2002). Further to this Dreyfus et al. (2001) argue that in the environments where interaction is part of an activity, participants' verbalisations may attest to epistemic actions, thus making them observable. Therefore in order to make the necessary observations, I decided to have some students work in pairs. Pair working has been adopted as a specific form of group working because it has certain advantages as a research instrument. For instance, students in pair working are less likely to have difficulties in getting organised and reaching an agreement than learners in larger groups (Artzt and Newman, 1990).

In order to investigate the role of scaffolded help on the construction process, I decided to organise multiple case studies to achieve theoretical replications (Yin, 1998). For this reason I needed some students working in pairs with scaffolded help and some others working without. In doing so I expected that performance of those working within the non-scaffolded environments would inform and enrich my understanding of the knowledge construction in relation to scaffolded help. I took one step further towards theoretical replications by including some individual students working on their own and working with the scaffolded help. The reason was because I wished to observe the individuals who would be working with neither the scaffolder's nor a partner's help. As a counterpart of these students, I also included some individuals working with the scaffolded help. The expectation here was that some comparisons between performances of pairs-with-scaffolder and individuals-with-scaffolder might contribute to the understanding of the scaffolded help in connection with the new construction. In summary, to achieve theoretical replications this study employed four groups of students: individuals and pairs working with and without scaffolded help (see Table 3.1. below).

In order to collect corroboratory evidence in relation to possible observations, it was decided to carry out literal replications (Yin, 1998) and hence I designed more than one case from each group of students. In relation to the number of cases, Robson (1993) argues that if an observed trend is corroborated by three cases, then this produces compelling evidence for the verification of the observation. In a similar vein, Creswell (1998) notes that a more accepted trend amongst case study researchers is to work with three or four cases if, of course, possible. In this study, I initially decided to work with three cases within each groups of students. In actual data collection, however, there occurred an unexpected problem with one of the pairs working with scaffolded help (that is, one of the partners in a pair was rather reluctant to continue to work for some reasons not immediately apparent to me, though she did complete the study; see Chapter 4 section 1 for more on this) and therefore one more case was included into the study. The summary of the cases with regard to different groups of students is presented in the Table 3.1. below.

<table>
<tr><td colspan="12">Group 1: Three individuals working on their own without scaffolded help.</td></tr>
<tr><td colspan="4">☺</td><td colspan="4">☺</td><td colspan="4">☺</td></tr>
<tr><td colspan="12">Group 2: Three individuals working on the tasks with scaffolded help.</td></tr>
<tr><td colspan="4">☺ with SC</td><td colspan="4">☺ with SC</td><td colspan="4">☺ with SC</td></tr>
<tr><td colspan="12">Group 3: Three pairs working on their own without scaffolded help.</td></tr>
<tr><td colspan="4">☺ ☺</td><td colspan="4">☺ ☺</td><td colspan="4">☺ ☺</td></tr>
<tr><td colspan="12">Group 4: Four pairs working with scaffolded help (SC).</td></tr>
<tr><td colspan="3">☺ ☺ with SC</td><td colspan="3">☺ ☺ with SC</td><td colspan="3">☺ ☺ with SC</td><td colspan="3">☺ ☺ with SC</td></tr>
</table>

Table 3.1. Design of the cases in relation to the different groups. SC – Scaffolded help.

5. Data collection methods

In order to answer the research questions, it was necessary to investigate students' verbalisations of their thoughts and their successive behaviours, performances and actions while working on some tasks. Considering the design of the cases, as immediately detailed above, there emerged three different forms of data collection methods: think-aloud protocols for the group 1, scaffolded interviews for the group 2 and 4 and pair working for the group 3 and 4. In this section, these methods are presented and a discussion is provided with respect to the usage of these methods. The data obtained from the participants will be called 'verbal protocols'. Strictly speaking, verbal protocols refer to the detailed records of behaviour (usually in verbal form) that result from instructions to 'think-aloud' during a task (Gilhooly and Green, 1996). However, throughout this study I use the terms 'verbal data', 'verbal protocols' and 'protocol data' interchangeably when referring to the students' detailed concurrent verbalisations generated during their work on some task whether these verbalisations occurred under think aloud instructions or with group working instructions or with scaffolded help. In order to differentiate between verbal data in the sense used in the literature and used in my sense, I prefer to call the former 'think aloud' protocols.

5.1. Think aloud protocols

Think aloud analysis is a methodology frequently used in cognitive psychology and education. This method involves an individual who is working on a task(s) on his/her own under thinking-aloud instructions. The strength of the method is seen in its potential to uncover the cognitive processes of an individual problem solver by using verbal reports (Gilhooly and Green, 1996). The underlying assumption in the use of think-aloud is that it brings thoughts into consciousness, making the ideas verbal if needed and then verbalising them (Ericsson & Simon, 1980). Think-aloud protocols can be generated either concurrently with the task performance or retrospectively after the task performance (Ericsson and Simon, 1993). In this study a concurrent think-aloud protocol method was employed for the students in the group 1.

The method is sensitive to several variables, some of which are the instructions research participants receive, the text types used in the experiment, the context in which the tasks are presented and the participants' ability to verbalise their thoughts (Ericsson and Simon, 1993). Schoenfeld (1985a) argues that instructions for any think-aloud task should be focused in relation to the research purposes. The purpose of working with individuals in the group 1 was to observe how far an individual can go on his/her own, and without any help through scaffolding or peer assistance during the application of the tasks. For this purpose, the individuals involved in think-aloud work were instructed as follows:

1. Translate your thoughts, e.g. ideas and images, into words and say them aloud.
2. Verbalise aloud all the steps that you go through when solving problems. Do not censor. No thought or step is too small, easy, obvious, or unimportant to verbalise.

3. Verbalise all the thinking you do before you start to solve the problem, e.g. what you are going to do.
4. Verbalise all thoughts during the problem solving.

Training the participants for the think-aloud sessions is seen as of crucial importance (Green and Gilhooly, 1996). Participants may vary in their ability to verbalise, and even for those who are competent in mathematics, thinking and speaking at the same time is an unusual task. However, the literature suggests that participants often seem to understand the requirement and comply with it well especially if some warm up exercises are done (Wijgh, 1995). Further to this, in the training session(s), Ericsson and Simon (1993) suggest providing students with some simple arithmetic exercises for which it is comparatively easy to do thinking-aloud is useful. Hence it was decided to give the students of the group 1 a training session in which instructions were conveyed to the students and simple think-aloud exercises were provided (see Appendix 1). Moreover, it was thought that the individuals involved in think-aloud sessions for this study might have some problems to get along with the tasks (NB: tasks are detailed below) and thus might get increasingly frustrated and perhaps irritated so that they would refuse to continue to work. In order to prevent this possible frustration and irritation, it was also decided to give some hints to the students if they encounter serious problems in carrying out the tasks. These hints are presented in Appendix 2.

5.2. Scaffolded interviews

Robson (1993, p. 228) describes interviews as a kind of conversation; a conversation with a purpose. According to Cohen and Manion (1994, p. 271), this conversation is the one that is "initiated by the interviewer for the specific purpose of obtaining research-relevant information and focused by him on content specified by research objectives of systematic description, prediction or explanation". Although the method of 'scaffolded interview' used in this study has some certain commonalties with these descriptions, it is not really an 'interview' in the form of the ones described in and used widely by the research methodology literature. Conventionally, an interview is often viewed as lying along a continuum between the structured and unstructured interviews (see Dyer 1995), though it comes in many different types such as individual, group and telephone interviews (Robson, 1993). Interviews are seen as one way of gathering information from the interviewees and it might be a bias to influence the interviewees towards a certain direction. However, scaffolded interviews are designed precisely because the interviewer/scaffolder has an explicit agenda to influence the interviewees/students during their work on the task especially when the students, for example, get stuck, feel uncertain or are not on the right track. Nevertheless, this influence should not be construed as a kind of imposition but rather as that which results from some form of negotiation (recall the very definition of scaffolding adopted for this study). In this sense, scaffolded interview used in this study could be described as a kind of conversation with a purpose, the purpose involving the scaffolder's didactic intentions; this conversation is aimed to be generated through a collaboration between the scaffolder and student(s) during a task performance.

Scaffolded interviews were used to collect data from the group 2 and 4. In these interviews, the intention was to give the students explicit and purposeful help towards a successful completion of the tasks. In this connection, it was decided, for example, that the interviewer/scaffolder would ask the student(s) to clarify or explain his/her actions taken in the course of the activities, give direction if needed and intervene if necessary. However, in the course of the scaffolded interviews, the intention was not to take over the students' autonomy. Quite the contrary, the scaffolder (i.e. the author) aimed to support the students' autonomy by means of: (1) prompting the students to explain what they were doing and why they were doing it; (2) encouraging the students to reflect on the problems that they were solving and to analyse the contribution of their actions to their solutions; (3) attempting to obtain the students' active involvement; (4) trying to avoid 'unnecessary' explanations and interventions; (5) paying attention to the students' perspectives before and while intervening.

5.3. Pair working groups

Students in groups 3 and 4 worked as pairs. The aim of having students work together as pairs was to obtain data with respect to students' construction and consolidation processes. To achieve this aim, it was ideal that the students should work together in harmony and synchronization; that is, both of the students in each pair should participate in the activities, verbalise their thoughts, and ask questions and clarifications to each other. However, simply placing two students in a group and requesting them to participate in the activities does not guarantee that they would do so. Especially if the students have never worked in groups or do not know what to do in group-working-problem-solving activities. Then it would be a disaster for both students and the generated data. Therefore, it was decided to give the students working as pairs a training session in which the expectations were clearly communicated and a relatively easy task was given to them (see Appendix 1). On the basis of an examination of the research on collaborative and cooperative work (e.g., Artzt and Newman, 1990; Ross and Raphael, 1990; Robertson, Davidson and Dees, 1994), the following instructions were given to the students:

1. Think along with your partner about the solution of the tasks. Follow every step of your partner and make sure you understand them.
2. Have your partner identify and define important terms, variables, rules and procedures. If your partner skips over a step without explanation, ask him/her to explain what you think is missing.
3. Do not work the problem out independently. Listen to and work along with your partner.
4. Never let your partner get ahead of you. Whenever necessary, ask your partner to wait so that you can check a procedure or computation and catch up with the work.
5. Check your partner's actions at every step. Do not wait for the answer. Check everything – each computation, diagram or procedure.

6. If you find an error in your partner's action and/or thinking, do not avoid correcting and discussing it.

6. Data collection instruments and piloting

In this study, four sequential tasks were prepared on the graphs of linear absolute value functions, $y=|f(x)|$, $y=f(|x|)$ and $y=|f(|x|)|$, in order to collect data from the students. In addition, a diagnostic test was designed to select the appropriate sample to work on these tasks for the study. All the tasks and diagnostic test were piloted prior to the main data collection. Piloting is often simply a check that instruments and procedures fit a study's purpose(s). As it happened, in the case of this study, piloting located task design[1] faults that were instructive with regard to my understanding of consolidation, construction and abstraction. Before the piloting, I was quite perplexed with the differentiation between construction and abstraction, the perplexity partly stemming from Hershkowitz et al.'s (2001) inconsistent use of these terms (see Chapter 7). However, analysis of the pilot verbal data and later the actual data provided convincing evidence that there are subtle differences between these two. Before proceeding, I feel a clarification in the way that I will use the terms 'construction' and 'abstraction' is necessary for the reader to follow my subsequent writings more easily. The term abstraction involves construction (of a new structure) but is not equivalent to constructions. A construction, I argue, becomes an abstraction following its consolidation (see Chapter 7 section 2 and 3 for more on this).

Following the pilot study I changed the organisation and the structure of the tasks in the light of new understandings gained from the analysis of the piloting data. In order to navigate the reader through my own development and through the evolution of the tasks from the pilot study to the actual study, this section is structured as follows. First the initial four tasks will be discussed with the rationale behind the organisation and design of the tasks. Then, the initial preparation of the diagnostic test will be explained. Following this, the pilot study will be described in some detail and the pilot study students' work on the initial tasks, which led to some revisions on the initial tasks and diagnostic test, will briefly be considered. Finally the revised diagnostic test and revised tasks, which were used in actual data collection, will be described, paying particular attention to the one developed to consolidate the new constructions.

6.1. Initial four tasks

In order to answer the research questions it was essential to observe the construction and consolidation of some new structures. Further to this, it was my desire to see if and how the newly constructed structures were used in further abstractions. Therefore it was necessary to design tasks which comply with at least three conditions, they must (1) involve a new

[1] I regard the issue of task design, in mathematics educations research and in instruction, as an underdeveloped area of major importance (see also Artigue, 2002). It emerged that task design is also crucial in considerations of abstraction and consolidation. The issue of task design will be the focus of attention on several occasions throughout the rest of this study.

construction(s); (2) provide opportunities for consolidation of the construction(s); and (3) allow the students to use the new constructions in the formation of new abstractions.

With these three conditions in mind, I had to prepare several sequential tasks. The mathematical content of the tasks was, to some extent, unimportant as long as it's content was new to the students taking part in this study and involved new constructions. I decided upon graphical representation of the composition of linear functions with the absolute value function. Linear absolute value functions was chosen as an area of mathematics where sequential tasks could be designed which could allow the students to construct new structures and use these new structures in the formation of further abstractions and as a topic which would be intellectually challenging but doable.

With the above-noted conditions in mind, four sequential tasks were initially designed (see Appendix 3). The organisation of the first three tasks was identical apart from the mathematical focus i.e. $y=|f(x)|$, $y=f(|x|)$ and $y=|f(|x|)|$ (referred to as $|f(x)|$, $f(|x|)$ and $|f(|x|)|$ hereafter). The mathematical focus of the first, second and third tasks was for students to draw/sketch the graphs of the linear absolute value functions $|f(x)|, f(|x|)$ and $|f(|x|)|$ by making use of the graph of $f(x)$. Each of these tasks had five questions. The first three questions of these tasks presented functional equations and asked students to draw or comment on the respective absolute value functions. Question 4 presented Cartesian graphs without functional equations and asked students to draw the respective absolute function graphs. Question 5 asked students to find/describe method(s) to obtain the absolute function graph from the linear function graph. In the first and second tasks it was expected that students would draw/sketch the graphs of $|f(x)|$ and $f(|x|)$ from their knowledge of the graph of $f(x)$ and of the absolute values of numbers. It was then expected that students would use this new knowledge to construct the graph of $|f(|x|)|$ in the third task. Question 4 of the first, second and third tasks was designed to provide an opportunity for consolidation. Two other opportunities for consolidation were envisaged: the third task (on $|f(|x|)|$) was intended to build upon the structures constructed in the first and second tasks (i.e. $|f(x)|$ and $f(|x|)$) and thus consolidate these structures; the fourth task had six questions designed to consolidate all of the constructions in the first, second and third tasks. It was thought that analyses of the students' performance on the fourth task would allow me to gain insights into the consolidation process.

The study was not designed as research into students' understanding of absolute value linear function topic *per se* but, as this was a vehicle for studying abstraction, it is relevant to review studies in this area, which I do in the next paragraph.

Surprisingly there was little literature on students' understanding of absolute values and virtually nothing on absolute value functions. Most of what I found was quite old and was more or less equally divided between reports in professional journals for teachers and short research reports. Most of the difficulties reported in the literature are concerned with algebraic aspects of absolute values. For instance, the introduction of the formal definition, i.e. $|x|=x$ if $x\geq 0$ and $|x|=-x$ if $x<0$, creates some difficulties for students (Parish, 1992). Students can find it difficult to accept that $|x|$ can be $-x$ (Perrin-Glorian, 1995) and see this as contradictory to their perception of an absolute value as a positive number (Sink, 1979). Applying the formal definition of an absolute value to solve equations and inequalities involving letters are also reported to create difficulties for some students (Chiarugi, Grazia & Furinghetti, 1990). Solutions of these equations or inequalities require students to remove the absolute value and distinguish between the $x\geq 0$ and $x<0$ cases. Students often make mistakes in breaking equations and inequations down into cases (Arcidiacono, 1983) and in using logical connectives, 'if...then' (Chiarugi et al., 1990) and 'and...or' (Parish, 1992), in solution strategies. The only paper that I found which concerns absolute value functions is Eisenberg and Dreyfus (1994) and this is only a minor focus of the paper. They report that their students generally find it easier to transform the graph of $f(x)$ into the graph of $y=|f(x)|$ than into the graph of $y=f(|x|)$.

6.2. Selection of the sample and the diagnostic test

In this study, I employed a purposeful sampling strategy (Miles and Huberman, 1994) which is commonly used in the qualitative research tradition and which requires specification of certain sampling criteria depending on the research objectives. Creswell (1998) insists that the rationale behind the each criterion should be made clear by the researcher. In this study, it was decided to select the students on two criteria: (1) that they had the prerequisite knowledge necessary to complete the tasks and were thus capable of achieving the intended abstractions; and (2) that they had not previously encountered the topic area and had not formed the intended abstractions. These two conditions were necessary since if the students have already formed the intended abstractions, then there will be virtually no opportunity to observe their abstraction process. In other words they would not need to construct new structures regarding the graph of absolute value linear functions. On the other hand if they have not had the prerequisite knowledge necessary to cope with the tasks, then they are unlikely to proceed and make new constructions. In addition, in order to achieve literal and theoretical replications of the cases, it was necessary to make some comparative investigations amongst the performance of the differently organised students (see Table 3.1.). To do so successfully, students needed to be selected from amongst those whose knowledge levels were as close to each other as possible or else it might become questionable to make some comparisons amongst the cases.

It was decided to collect data from the Turkish students as I found this more convenient. With respect to the selected topic, the only appropriate sample could be drawn from Turkish Year 10 students (16 – 18 years old). This is because application of absolute values in the algebraic domain appears in Year 9 and graphs of absolute value functions are studied in Year 11 of the Turkish National Curriculum. Thus it was decided to select Year 10 students for the study. Nevertheless, I had to make sure that the selected students did not have the intended abstractions. Hence a diagnostic test was prepared to select the students. This test was to be prepared in such a way that should diagnose whether students had all the prerequisite knowledge necessary for a successful completion of the tasks. It was also crucial that this test should reveal whether students had the intended abstractions. To prepare a diagnostic test with these qualities, the prepared tasks were investigated thoroughly to determine and categorise the prerequisite knowledge for a student to accomplish the intended abstractions. This investigation led to a decomposition of the topic of graphs of absolute value linear functions (see Appendix 4) on the basis of which the diagnostic test was produced.

The initial diagnostic test was composed of 10 questions as given in Appendix 5. Of these questions, the initial 8 questions aimed to discover whether the students had the prerequisite knowledge to achieve the intended abstractions. This part included more than one question from each prerequisite knowledge category in order to reduce the possibility of answering the questions by chance and thus to increase the possibility of getting the intended appropriate sample. Moreover, in some questions, students were asked to show all their work on the questions as these, I thought, could be necessary in the decision-making process for the selection of a student. Appendix 6 presents the reasons for preparation of the each question in the diagnostic test. The last two questions, question 9 and 10, were two straightforward evaluations of the expected constructions of the first and second tasks. In question 9 and 10, two specific equations of $|f(x)|$ and $f(|x|)$ were presented together with Cartesian grids and the students were asked to obtain these graphs and show all their work on these questions. It was decided to select the students from amongst those who would answer all the first eight questions correctly and who would answer neither of the last two questions correctly.

6.3. Piloting of the initial four tasks

A pilot study on a small number from a representative sample of the population to be researched in order to find out possible problems and difficulties is always advised. In the case of this study, piloting turned out to be critically important and alerted me to some serious problems and deficiencies and at the same time it contributed to my understanding of the issue of abstraction and consolidation. The initial four tasks were first pre-piloted on six fellow PhD students and on three British students doing A-Level. In the light of this pre-pilot, I made a number of changes and executive decisions in the tasks. For example, at the beginning it was not determined whether the questions in the tasks should be prepared in an open or closed ended format, after the pre-piloting I decided that the questions should be presented in open-ended formats. Although the pre-piloting was certainly beneficial to try out the tasks, the sample was quite inappropriate to make realistic inferences as to the feasibility of the tasks. Consequently it was decided to carry out pilot study on some Turkish students.

These initial four tasks were formally piloted, i.e. all data collection procedures strictly attended to, on five Turkish students and it took four weeks to complete. These five students were selected from 13 volunteers, as these five were the only students who met the two diagnostic test criteria described above. Four of these students worked in two pairs and one worked alone on the tasks over four consecutive days. One of the pairs worked with the help of scaffolder;

and the other pair and the individual student worked on these tasks without the scaffolder's help. The individual student did not achieve any of the expected constructions. The students working in pairs with and without scaffolded help achieved the intended constructions, $|f(x)|$ and $f(|x|)$, of the first and second tasks. They could not, however, make use of this knowledge to obtain the construction of $|f(|x|)|$. Further to this, they made virtually no progress on the fourth task, which was prepared for the purpose of consolidation.

In order to give the reader an idea of the pairs' performance over the four tasks, I will briefly present some excerpts from the scaffolded pair's work. At the end of the first and second tasks these students (M&Y) were asked to briefly explain how to obtain the graph of, respectively, $|f(x)|$ and $f(|x|)$ given the graph of $f(x)$. With regard to $|f(x)|$, they explained (I refers to the scaffolder/interviewer):

> **M:** when a line [a graph of $f(x)$] is given, then to use this umm by using this... intersection of [the] x [axis] with the umm line... I mean from that point...
> **Y:** negative values [of y]... they are under here [the x-axis]
> **M:** from that point we will take the symmetry in the x line
> **I:** x line?
> **M:** yeah I mean up there I mean towards upwards
> **Y:** from the point of intersection, negative values [of y]
> **M:** negative values [of y] are taken symmetrically in the x-axis.
> **Y:** uh huh yes!

At the end of the second task they explained:

> **M:** when a [graph of a] function is given, absolute value of this function [i.e. $f(|x|)$]... err to find this function... this function's intersection point... from the point of intersection, a line drawn parallel to the x-axis... we took the symmetry in the line...
> **Y:** for the negative x, it is for the negative x! The symmetry is for the negative x!
> **M:** the symmetry is in this parallel line... negative x are symmetric... the reason for the symmetry is... umm the reason is from negative number to positive number... I mean it's like symmetry...
> **Y:** it is symmetric... we take its symmetry...
> **M:** I mean the symmetry of –4 is here ... -4 becomes +4 in the symmetry... it is the same thing ... same as absolute value of –4, it is +4. So they are symmetric.

M&Y's accounts and their written work suggested that they achieved the intended constructions of the first and second tasks. However, M&Y could not achieve the construction of the third task, i.e. to develop a method(s) to obtain the graph of $|f(|x|)|$ given the graph of $f(x)$, despite the scaffolder's assistance. They did recognise some aspects of their constructions of $|f(x)|$ and $f(|x|)$ but were not able to use them. For instance:

> **M:** this cannot take any negative x value in here [referring to $|f(|x|)|=|(2|x|-6)|$]
> **Y:** cannot take negative?
> **M:** no! Look there are whole lot of absolute values covering x ... if x takes negative, absolute value makes it positive...
> **I:** but the graph of $f(|x|)$ took the negative value! Do you remember what we did yesterday?

M: but we were taking symmetry… we had to make it positive! For example –4, -3, -2… we took them symmetry to make positive.. I mean +4, +3 and +2.
Y: yeah it was symmetry for the negative… it was also symmetry on the first day [referring to their work on $|f(x)|$] it was positive too. Umm symmetries are different though…

Apparently M&Y recognised the symmetry aspects of their earlier constructions but seemed to be confused between the differentiation of the graphs of $|f(x)|$ and $f(|x|)$. The above excerpt continued for quite a while and the scaffolder had to remind the students what they did on the first two tasks. They got confused and decided to substitute for different values of x into the given equation. The students were overwhelmed by the number of symmetric relationships in the graph of $|f(|x|)|=|(2|x|-6)|$ which was a W-shaped graph. At the end of their work on the fourth task M&Y concluded that:

I: what would you tell to one asking about how to draw the graph of $|f(|x|)|$?
M: a function umm from the intersection point of the y-axis
Y: a [graph of linear] function is taken symmetrically towards the positive direction I mean from within the area where the x values are negative towards the positive direction!
I: you told me before many symmetries.
Y: Then that means negative things I mean negative x and y values need to be made positive thus we had to take symmetry…
M: I think in the given equation of a function we just substitute and find the points and then combine them… this is the way I can tell.

As this final statement clearly demonstrates, M&Y failed to achieve the intended construction of the third task. Even worse was that they could not make much progress in the fourth task which was supposed to provide consolidation opportunities and help the students use their earlier constructions with an increasing ease and with greater flexibility. Thus the consolidation task did not work at all as intended. Following the analyses of the pilot study students' verbal data, the tasks and underlying assumptions were revisited. I will now discuss the changes introduced into the initial four tasks in the next section.

6.4. Revised four tasks

Pilot study students' responses and verbalisations drew my attention to the organisation of the tasks and the opportunities for consolidation. As mentioned above, in the initial form of the four tasks (see Appendix 3) there were three consolidation opportunities envisioned: (1) question 4 of the first, second and third tasks; (2) the third task was intended to build upon the structures constructed in the first and second tasks and thus consolidate these structures; (3) the fourth task designed to consolidate all of the constructions in the first, second and third tasks. Pitfalls in these three opportunities for the consolidation are discussed now.

Question 4 of the first, second and third tasks

Prior to piloting, it was thought that question 4 of the first, second and third tasks (see Appendix 3) would provide an opportunity for students to consolidate their new construction but protocols demonstrated that this question played an important role in the construction itself. The reason seems to me that this question presents graphs without functional equations and this appeared to aid (force) students to develop a method to obtain the intended graphs by analysing the graphs alone (there was no opportunity to substitute numbers for variables and plot points). Therefore, this question did not help students to consolidate but rather played a key role for the construction of a new structure.

The third task

It was expected that the third task, concerned with $|f(|x|)|$, would help students consolidate the new constructions, $|f(x)|$ and $f(|x|)$. Students did recognise these constructions but were unable to use them. This suggests that without consolidation these constructions are unlikely to be used in further abstractions. This is compatible with the genesis of abstraction proposed by RBC theory, that structures are used in further abstractions after consolidation.

The fourth task

The problem, amongst others, with this task was that it involved a question about $|f(|x|)|$ which has not yet been constructed by the students at the time they worked on it. It was incapable of providing opportunities for consolidation as it fell short in capturing and highlighting the aspects of the new constructions which needed to be handled attentively. Further to this, in the first, second and third tasks, the students were required to obtain the intended graphs of $|f(x)|$, $f(|x|)$ and $|f(|x|)|$ by using the equations and graphs of $f(x)$ but in the fourth task students were given a graph and expected to find its equation. The assumption here was that when students were working on a graph to find its equation, they would need to reflect on their earlier constructions and that this in turn would create opportunities to use these constructions in the course of this activity. However, I could not at the time appreciate the import of the 'direction of learning' (equation to graph) that I was encouraging in the first three tasks and the difficulties that a change in the 'direction of learning' (graph to equation) might involve. There are unanswered questions here: Does such a change in the direction require a new construction? Do such changes in direction still create difficulties for the students even if they consolidated the new constructions? Answers to such questions are unclear. What appears to be clear, however, is that such changes in direction are difficult when new structures have not been consolidated. This task may be more 'difficult' as it required students to construct a new way to use emergent structures rather than providing opportunities for consolidation. This suggests to me that the difference in the level of difficulty between the consolidation tasks and original tasks should not be large. In a nutshell, the fourth task was unable to create opportunities for consolidation of the new structures.

My analysis of the problems of the initial task with regard to consolidation opportunities suggested that the consolidation task, the initial fourth task, should become the third task, amended to consolidate only the constructions $|f(x)|$ and $f(|x|)$ and that the initial third task becomes the fourth task. Hence I completely changed the consolidation task but kept the tasks related to $|f(x)|$, $f(|x|)$ and $|f(|x|)|$. However, there were some slight changes in the wordings and in some of the specific equations of the selected functions of these tasks. For example, the equation of $y=|(2|x|-6)|$ changed into $y=|(|x|-2)|$ as the grid provided to the students was unsuitable in that students needed to add some new lines to draw the graph and this created some confusion. There were some other minor changes which I will not report here but these can be easily seen by a simple comparison of the piloted version of the tasks (see Appendix 3) and the final version of the tasks used in the actual data collection (see Appendix 7). In the next section I detail the preparation of the amended consolidation task.

6.5. Preparation of the consolidation task

The extant literature gives very little help as to how to prepare a consolidation task. Although the RBC model of abstraction emphasises the necessity of consolidation after a construction, suggestions as to what activities might lead to consolidation remain imprecise. Dreyfus and Tsamir (2004) claim that consolidation is likely to occur through building-with actions, where students recognise and use the new structure. Thus, problem situations that generate a need for recognising and using the new structure may provide ways for the consolidation. They also posit that discussions about the new structure, examining it from different perspectives and reflecting on it, can provide opportunities for consolidation. Further to this the pilot study demonstrated that students get confused about the different ways to obtain the graph of $|f(x)|$ and $f(|x|)$. These considerations informed the redesign of the third task.

The new consolidation task was composed of five questions (see Appendix 7 task 3). Question 1 presented a linear function with the equation and asked students to obtain the graph of $|f(x)|$ and $f(|x|)$. Students were free to choose whichever method they wanted to use to answer the question. The aim of this question was to see the state of the new structures and observe how students would obtain the graphs. Question 2 asked students to tell the ways about how to obtain the graphs of $|f(x)|$ and $f(|x|)$ given the graph of $f(x)$ with general mathematical terms. The aim of this question was to impel students to verbalise a method(s) for the graphs of $|f(x)|$ and $f(|x|)$. These two questions were a brief repetition of the first and second tasks and also aimed to remind students of what they have done in these tasks.

In questions 3 and 4 a hypothetical situation was depicted where three imaginary students made claims as to how to obtain the graph of, respectively, $|f(x)|$ and $f(|x|)$, given the graph of $f(x)$. All of the claims in these questions were different and incorrect (though they were designed to be 'intelligently incorrect'!). The students were asked to analyse, examine, give examples when appropriate and react to each claim. The aim was to engage students in discussion and for students to justify and clarify their ideas. The claims presented in these questions were inspired by students' misconceptions and mistakes observed in the pilot study students' verbal data. As an illustration of this point, the given claims in the question 4 will be succinctly detailed. As mentioned earlier the students in the pilot became confused about the difference in obtaining the graph of $|f(x)|$ and $f(|x|)$. Therefore, the first imaginary student proposed a way to obtain the graph of $|f(x)|$ rather than the graph of $f(|x|)$. The aim here was to push students to realise the differences between these two graphs. The second imaginary student's claim was expected to force students to realise the symmetry of the graph of $f(|x|)$ in the y-axis and thus establish this aspect. The final imaginary student's claim was expected to aid (force) students to establish that the graph of $f(x)$ at the positive values of x remains unchanged in the graph of $f(|x|)$. At the end of the third and fourth questions students were asked to explain the method(s) to obtain the graph of $f(|x|)$ from the graph of $f(x)$.

In question 5 students were presented six different graphs and asked to identify if they could be the graph of $|f(x)|$ and/or $f(|x|)$. Question 5 aimed to focus on the difference between $|f(x)|$ and $f(|x|)$ from graphic considerations alone. In the piloting, it was observed that the students tended to perceive symmetry as the only determination of the graph of $|f(x)|$ and $f(|x|)$ regardless of the place of it. In order to highlight this flaw involved in this perception, some of these graphs included symmetries but they were neither the graph of $|f(x)|$ nor of $f(|x|)$. The aim here was to alert students that symmetric relationship is not the only determining property of the graphs of $|f(x)|$ and $f(|x|)$.

6.6. Final version of the tasks as used in the main study

The final version of the tasks again involved four sequential tasks (see Appendix 7). The first and second tasks aimed to construct a method(s) to draw/sketch the graphs of, respectively, $|f(x)|$ and $f(|x|)$ given the graph of $f(x)$. The third task was included to consolidate the constructions of the first and second tasks. The fourth task aimed to construct a method(s) to obtain the graph of $|f(|x|)|$ from the graph of $f(x)$. I have already detailed the aims/goals of the questions in the consolidation task. In this section, I briefly elaborate on the aims/goals of the questions presented in the first, second and fourth tasks which were organised in the same way and involved five questions.

Question 1 presented an equation of the object graph (i.e. $|f(x)|$, $f(|x|)$ or $|f(|x|)|$) which students were asked to obtain and comment on any patterns or symmetries in the graph. The aim was for the students to realise some symmetric relationships. Question 2 asked the students to report on any pattern between the object graph and the graph of $f(x)$. The aim here was to prompt the students to establish some (initial) interrelationships between the object graph and the graph of $f(x)$. In question 3, a graph of $f(x)$ was given to the students who were asked to obtain the object graph by using this given graph of $f(x)$. The aim was to impel the students to clarify and further develop the initial relationships observed/discovered in the first two questions. Question 4 presented students with four linear graphs, without graphical equations, on Cartesian grids and asked them to obtain the object graphs by using these linear graphs. This question had two particular aims: first to force students to develop a method to obtain the object graph without using the equation but rather with only the graphic representation of $f(x)$; and second to provide the students with opportunities to validate their method if they have already come up with one. Presenting four graphs was also thought to help the students practise their constructions in this question. In the final question, the students were asked to explain how to sketch the object graph by using the graphical representation of a linear function.

6.7. Piloting of the diagnostic test

As noted above, in the pilot study, 13 students took the diagnostic test with 10 questions. The condition for the students to be selected for this study was to get the initial 8 questions correct and the last two questions (9 and 10) incorrect. Although the diagnostic test largely worked as intended, there were some slight problems observed in the last two questions. In these questions the students were presented with two equations of, respectively, $|f(x)|$ and $f(|x|)$ and expected to obtain their graphs. However, some of the students in the pilot attempted to solve these questions by substitution on the basis of their knowledge of sketching the linear graphs. I thought that students in the main study, while working on the diagnostic test, might do the same and this situation might influence their work on the actual tasks. In order to prevent this, these two questions were revised. Instead of giving two equations of $|f(x)|$ and $f(|x|)$, I presented two linear graphs and asked the students sketch the graphs of $|f(x)|$ and $f(|x|)$ by using given linear graphs. Apart from these changes, there were some other slight changes in the wording of the initial eight questions. I do not report them here but interested readers can easily see those changes by a simple comparison between the revised forms of the diagnostic test presented in Appendix 8 with the initial diagnostic test presented in Appendix 5.

7. Data collection process for the main study

Actual data collection for the main study took place in Turkey over a period of eleven weeks. In order to select the appropriate sample, the (revised) diagnostic test was given to 134 Year 10 (16-18 years old) students drawn from two school districts. Both of these schools select their students on the basis of a national examination taken at Year 8 (14-16 years old). Amongst 134 students, 24 fully met the two diagnostic test criteria. Two of these students did not want to take part in the study. Initially 18 students were selected from amongst the 22 students and arranged such that 12 worked as pairs and 6 works as individually. Half of the pairs and individuals worked with the scaffolded help and others without. However, as the data collection proceeded, there occurred a problem with one of the pairs working with scaffolded help (see Chapter 4 section 1 for more on this) and therefore I decided to include one more pair who worked with scaffolded help. Thus the actual data collection involved 20 students: 12 girls and 8 boys. In the course of organisation of the students, every student was asked if they wanted to work in a pair or individually and those wishing to work in pairs were given opportunity to decide with whom they wanted to be paired. Thus in the final organisation, most of the pairs were close friends or at least wanted to work together rather than individually. The aim was to create a relaxed working environment for the partners. Each of the students' time was compensated by £10 by which the students' full participation over the four tasks was assured.

The revised four tasks were given to the students in paper-and-pencil format: one task per day without any time limitations, 30-60 minutes for each task, all tasks completed within one week over four consecutive days, there being a one-day time interval between two successive tasks. The students were also provided with rulers, different coloured-pens, spare sheets which included Cartesian grids. Prior to their work on the tasks, I explained to the participating students about the purpose of the study and communicated the instructions with which they were expected to comply (see sections 5.1. and 5.3. above). All of the students were given a simple warm-up task (see Appendix 1) which involved four questions. The first three questions were related to definition and simple questions concerning the notion of absolute value. The fourth question was used to exercise the instructions as to how to work together in the case of pairs and as to how to do thinking-aloud for those working individually without scaffolded help. There were no time constraints on the part of the students to complete the tasks. While working on the tasks, the students were located in a spare room in their respective schools and care was taken to prevent any outside interruption during the sessions. In order to obtain the verbal protocols from the students, all of the sessions were audio-recorded and the students' written responses were also retained for data analyses.

8. Preparation of the verbal data (verbal protocols) for the analysis

The data for this study consisted of students' written work during and audiotapes of the sessions and were prepared for the analysis in three steps. First of all, speakers' (interviewer/student) conversations in the verbal protocols were transcribed verbatim, paying careful attention to the accuracy and the speaker sequence. Transcripts were parsed into turns, each defined as segments of speaker's continuous speech. If an interruption stopped the speaker from speaking, then the turn was considered over, even if the content of the turn was resumed later. If the speaker did not stop talking even though someone else was speaking, then all of the content was considered to be part of that same turn. Attentive responses, such as 'yes', 'umm', and so on, were also considered as turns. Speaker utterances were numbered with each uninterrupted utterance by one speaker being assigned a natural number. Three dots (...) were sometimes used to indicate either the speaker paused or spoke inaudibly.

Secondly, as the students' utterances were connected to their written responses, e.g., graphs and computations, it was necessary to carry out a simultaneous examination of the written responses and transcriptions. As a result of this examination, written responses were inserted into the transcripts as appropriate. Some comments were inserted in square brackets to allow a reader to follow the conversation amongst the participants.

Finally, the verbal protocols, originally produced in Turkish, were later translated into English. During the translation, careful attention was paid to find equivalent words and phrases. The intention was to ensure that content of the original Turkish utterances was retained through translation. Thus two principles guiding translation were: (1) that the English should be clear; and (2) faithfulness to the original intent, e.g. wording as near as possible to the original Turkish.

9. Verbal protocol analysis procedure

In order to propose some answers to the first research question concerning the construction of new knowledge and scaffolding, I mainly focused my attention upon the verbal data generated within the scaffolded environments, though non-scaffolded students' work also provided some insights in relation to this research question. For the second research question dealing with the issue of consolidation, I decided to exemplify my observations with an individual student's verbal data to avoid having to deal with the complexities of the social interactions which will be done in relation to the first research question. The third research question aims to evaluate RBC theory for which I will draw upon the data presented for the first and second research questions. In this section, I will detail the data analysis procedure followed for the scaffolded pairs who provided more complex and fruitful data than the individuals and thus were used to answer the first research question.

In order to analyse the scaffolded pairs' knowledge construction, it was necessary to investigate the verbal protocols in terms of concurrent cognitive and social processes. To this extent, two different and independent analyses of the protocols were carried out: one that analysed cognition and one that analysed interaction. My efforts to investigate the protocols from these two perspectives, however, should not be construed as a position embracing a Cartesian dualistic view (see Nunez, Edwards, and Matos, 1999) as I do not wish to draw a major theoretical line or distinction between cognitive and social processes. Writers such as Rogoff (1995) take considerable pains to emphasise the lack of difference between the social and cognitive processes. In order to avoid any hint of dualistic position, she asserts that cognitive development is social development. She writes, "the specific processes by which they [children] communicate and share in decision making are the substance of cognitive development" (ibid., p.151). While my position is close to that of Rogoff's, I find it convenient to focus my investigation on two concurrent analyses of social and cognitive processes. Yet I do not wish to artificially divide the unified, whole developmental process. For the convenience of my study and discussion, I focus upon each one of these processes separately yet I have to acknowledge the integrated nature of the social and cognitive processes in the formation of mathematical abstractions. The integrated nature of these two processes will be elaborated and developed in the following chapters. In this section, I will lay out the coding and analyses procedures for the social and cognitive processes.

9.1. Procedure to analyse the social processes

Analysis of the social aspect of the protocols requires an examination of the verbal data from the lenses of scaffolder and students as their utterances are qualitatively different from each other in terms of at least occurrence and motive. When a student gives an explanation, this may serve as a means, for instance, to understand his/her own action, to convince (or share his/her understanding with) his/her partner (or scaffolder). However, when the scaffolder gives an explanation, it is to assist the students, give them direction or regulate their actions. Therefore, students and scaffolder's utterances were examined under separate categories as if they were isolated. This artificial isolation was necessary for the clarity and convenience of my discussion and should not be construed as a major theoretical division of a unified interaction. In order to explore the participants' (students and scaffolder) influences on one another during their interaction, an 'interaction flowchart' was developed. I will later describe how this flowchart was prepared and used to analyse the conversational organisation of the participants. However, before doing this, I will first explain the categorisation of the students' utterances and then detail the procedure followed to analyse the scaffolder's interventions (utterances).

9.1.1. Procedure for coding the students' utterances

It is a common practice in the analyses of verbal protocols to create some coding categories and investigate the data to seek for certain patterns and relationships (Green and Gilhooly, 1986). Deciding upon the kind of categories depends strongly on the purpose of the study and the research questions. My aim was to examine the scaffolder's interventions and occurrence of certain types of utterances on the part of students as a result of these interventions. I was looking into the verbal protocols to see if the utterances resulting from the scaffolder's interventions can be related to new knowledge construction, which could be viewed as a kind of new learning. With this in mind, I examined the extant literature (e.g., Dreyfus et al., 2001; Baker, 2002; Cobo and Fortuny, 2000; Soller, 2001) concerned with social interaction in cooperative and/or collaborative learning to come up with certain interaction categories associated with new learning or knowledge construction. As a result I came up initially with eight categories of students' utterances according to their functions: proposing, explaining, assessing, elaborating, query, consulting, agreement and disagreement. I now briefly detail the rationale behind each of these categories.

The category of proposing was included because research literature (e.g., Barron, 2000 and 2003) suggests strong correlations between the quality of the problem-solving outcomes of the collaborative groups and the quality of group members' interaction in terms of generating and documenting proposals and evaluating their aptness to the solution. As to the explaining category, there is a clear trend of research findings that giving task related explanation improves learning and understanding (Chi, DeLeeuw, Chi and LaVancher 1994; Webb, 1991). Thus I wanted to see if the scaffolder's interventions aid (force) students to produce explanations. As to the assessment category, research findings, such as that of Goos and Galbraith (1996) and Goss, Galbraith and Renshaw (2002), provide convincing evidence that assessing the usefulness, aptness, accuracy and soundness of a proposal, solution strategy/plan or a result positively influences students' success in problem-solving activities. Elaborating was used in the sense of execution of a proposal, a strategy or a plan. The intention was to see if and how the scaffolder's interventions regulate the students' elaborations. The final four categories, i.e. query, consulting, agreement and disagreement, were thought of as components of argumentative discourse. The literature provide strong evidence that when engaged in argumentative discourse, the students become obliged to render their understandings explicit, reflect upon them, communicate them, and uncover some hidden difficulties involved in their understandings, which are perhaps revised (Miller, 1987; Vries et al., 2002). It was my intention to examine if the scaffolder leads the students into argumentation which reflects itself through these four categories.

In order to develop a coding scheme on the basis of these categories, following the stages in development of content analysis schemes suggested by Pilkington (2001), I first provided an initial definition for each category. Then I applied this scheme to some protocol data to test the applicability of the coding scheme. In my application, I realised that this scheme was a polythetic one in which an utterance can be assigned to more than one category (Graesser and Person, 1994). Further to this, some of the utterances did not fall into any of the categories. Subsequently, I gave some training to a fellow PhD student so as to test the inter-coder reliability of the coding scheme. The inter-coder reliability is defined as the percentage of the total number of codings on which two coders are agreed (Green and Gilhooly, 1996). We coded some segments from the transcriptions generated within the scaffolded environment. The initial inter-coder reliability was 63% which was unacceptably low. Although there were some problems with the specification of every category, there appeared two main difficulties. The first was that assessment statements were often regarded as explaining statements by the other coder. I shared the other coder's concern and was convinced that the assessment category should be merged with the explaining category. The second was that the difference between consulting and query statements was also not clear. In order to increase the reliability percentage, I tried to further clarify and specify each category paying particular attention to the consulting and query categories so that they could be differentiated. In addition to these changes, I also decided to add a new category used by Dreyfus et al. (2001) called 'attention' which refers to those utterances which exhibit some interest to the presence of other participants (e.g., "wait!", "Yes!"). Thus the second version of the coding scheme was developed with eight categories: proposing, explaining, elaborating, query, consulting, agreement, disagreement and attention.

Following these changes, the second version of the scheme was also tested to find out the inter-coder reliability which was calculated as 74%. Although this percentage was better than the first one, it was also regarded as low. The main problems were related to the differentiation between the consulting and query categories, to the new 'attention' category and to the differentiation between explaining and elaborating category. The second coder argued that many of the utterances fell into the 'attention' category as the participants were listening to and attending to each other. Subsequently, a third version of the coding scheme was prepared. In this version, consulting and query categories merged together under the label of 'quest'. The category of 'attention' was excluded. The differences between explaining and elaborating were also further emphasised. This third version was once again tested for the inter-coder reliability which was found 81% and this percentage is considered as high (Ericsson and Simon, 1993). Thus the final version of the coding scheme used in this study consisted in six categories: proposing, explaining, elaborating, quest, agreement and disagreement. Description of each category is given below. Please note that the coding scheme will be illustrated later (see section 9.1.3. below) and so I will not provide examples for the categories for now.

1. **Proposing:** Utterances in this category involve suggesting a plan, strategy, procedure, hypothesis, idea, solution, pattern or relationship stemming from an observation and/or mathematical reasoning.

2. **Elaborating:** This category includes articulations of what has been done or is being done to continue or develop an idea. It often comes about when students are executing a strategy, a plan or a procedure and when they carry out some computations.

3. **Explaining:** This category consists in students' verifications, justifications and/or clarifications of proposals, their comments on the usefulness, appropriateness, relevancy, accuracy and/or rationality of a proposal, of a solution strategy/plan or of a result.

4. **Quest:** Utterances in this category involve students' queries for clarification, justification and/or verification of a proposing statement (see category 1 above). In this sense, a quest might be directed to one's own action. This category also involves utterances asking a partner's reaction and opinion (i.e. consulting).

5. **Agreement:** This category includes utterances in which a student accepts (or at least appears to accept) or makes a concession about his/her partner's proposal, explanation and/or elaboration.

6. **Disagreement:** This category involves utterances in which a student has a motive to oppose his/her partner's proposals, explanations and/or elaborations. An important note for categories 5 and 6 utterances is that agreements or disagreements are not always overtly expressed. For the coding of these two categories, however, I focused on those utterances where agreement or disagreement is openly expressed in some linguistic form such as "yes", "OK", "no", "I don't think so".

9.1.2. Procedure to analyse the scaffolder's interventions

The scaffolder's interventions in this study were examined with regard to the action cycle model proposed by Scott (1997, 1998). As detailed earlier this model is composed of three actions: monitoring, analysing and assisting (or handing-over). The scaffolder's monitoring and analysing actions could become discernible by means of investigating diagnostic interventions such as "what do you mean?" "how are you planning to draw this graph?", "tell me more" and so on. Nonetheless, the actions that the scaffolder undertakes to monitor the students' work and progress and thus analyse their performance are not always observable through his interventions. He may not intervene but still be monitoring and analysing on the basis of students' verbal and non-verbal interactions. This situation creates a methodological problem in terms of substantiation of the inferences about scaffolder's decisions in relation to monitoring and analysing. I tried to overcome this problem by drawing circumstantial inferences about the scaffolder's interventions related to assisting. In other words, an 'hermeneutical' analysis (inferential comments) of the scaffolder's interventions in relation to his assisting interventions was part of the analysis.

However, a question arises here: to what extent can one make convincingly accurate inferences about someone's interventions if these interventions are not manifested through conversation but remain in the 'head' of that person? Although this question might be valid in some situations, in the case of this study it does not create a substantial problem. This is so because in this study the researcher who collected and analysed the data and the scaffolder who contributed to generation of the data were the same person. This situation, to a great extent, reduces the concerns with regard to the level of the accuracy of inferences about how the scaffolder saw and interpreted (in terms of monitoring and analysing) the students' performance. In analysing the verbal data in terms of the scaffolder's monitoring and analysing interventions, a tremendous amount of time was spent on examining the data and recalling the motives behind the interventions occurring within the given context. In addition, the action cycle model suggests that assisting occurs following monitoring and analysing the students' present and target level of performance, though the scaffolder need not be necessarily conscious of his monitoring and analysing actions. Therefore, depending on the nature, amount and type of assistance, reasonably appropriate inferences about the scaffolder's monitoring and analysing actions could be drawn.

In order to analyse the assisting interventions, I examined the extant literature on scaffolding which suggests a number of qualitatively different forms of assistance in relation to scaffolding (see below). These forms of assistance were employed in the analysis to describe the scaffolder's assisting interventions. I will now briefly detail the forms of assistance frequently cited in the literature and used in this study to make sense of the scaffolder's assistance.

- **Reduction in the degrees of freedom** (Wood et al., 1976): This involves simplifying the task by reducing the number of constituent acts required to reach to the solution. Similar kinds of assistance are also referred to as constraining – limiting by Anghileri (2002) and decomposing the task into small units by Rogoff (1990).
- **Marking critical features** (Wood et al., 1976): The scaffolder by variety of means marks or accentuates certain relevant features of the task. It is also referred to as highlighting critical features by Stone (1993).
- **Hinting** (Bliss et al., 1996; Stone, 1993): The scaffolder, rather than explicitly instructing the students, implies a strategy, a solution method etc.
- **Directing the student** (Wood et al., 1976; Tharp and Gallimore, 1988): This involves instructing the student(s) and calling for specific action and direction maintenance.

- **Fill-in-the-blank kind of requests** (Graesser et al., 1995): The scaffolder prompts the student(s) to fill in a word or phrase in a discursive intervention by pausing so as to encourage the student(s) to take an active part in the activity.
- **Providing examples** (Chi et al., 2001): The scaffolder provides examples to point out the discrepancies in the students' explanations, to justify or consolidate the students' thoughts or ideas, or to point at analogical relations.
- **Questioning** (Graesser et al., 1995; Graesser and Person, 1994): The scaffolder asks questions which require the student(s) to apply an idea to a new domain knowledge, to evaluate a claim critically, to make inferences, justifications by reasoning, synthesis of a new idea from multiple information sources. The questions can also be used to elicit explanations (Chi et al., 1994).
- **Summarising** (Graesser et al., 1995): If the scaffolder believes that quality of the students' answer is high or the answer is almost completely correct, then he/she may summarise or recap the answer(s) in order to emphasise the accuracy of the answer(s)and consolidate the students ideas, thoughts etc.
- **Rephrasing students' talks and negotiating meanings** (Anghileri, 2002): In order to highlight the processes involved in solutions, the scaffolder re-describes the students' efforts and clarifies the aspects of the task that are most valuable. He/she rephrases to make ideas clearer without loosing the intended meaning, and negotiates new meanings to establish conceptually valid understanding.
- **Explaining** (Anghileri, 2002): The scaffolder amplifies a process or concept, or elaborates why solutions are incorrect, inaccurate, and/or incomplete.
- **Describing the problem to orient the students to the important features** (McArthur et al., 1990).
- **Comparing the current problem with a previously solved problem** (McArthur et al., 1990).

9.1.3. Interaction flowchart

In order to analyse the social processes of the interaction between the scaffolder and students, an 'interaction flowchart' was constructed to examine the conversational organisation of the verbal protocols. This method was recently used in several studies including Dreyfus et al. (2001), Sfard (2001) and Williams (2003). I found the flowchart analysis particularly useful as it has a potential to reveal certain regularities and is useful to gain insights into how the individuals' interactions evolve, to observe how the regulations and interventions influence the flow of interaction, to evaluate how the individual contributions are interdependent and build upon the previous conversational turns and to make 'systematic' and 'tangible' inferences about the relationship between the occurrence of certain types of utterances (see above section of 9.1.1.) and related cognitive development. These points will be clearer throughout the later chapters. In this section, I will describe the way that an interaction flowchart is constructed. For this purpose, I will analyse a small segment from the verbal protocols of two girls (H&S). This segment is taken from H&S's work on the second task and occurred after H&S sketched a graph of $f(|x|)$ inaccurately. S realised that something was wrong with this graph and they started talking about this. ('I' refers to the interviewer/scaffolder).

98S: Before, we took the symmetry towards y-axis… but here we drew it… I don't know
99H: What do you mean?
100S: I mean, look, it should have been towards here not there
101H: Why should it have been so? It is not necessary… look, these two rays should be symmetric in the line of y=2
102S: Ok, look isn't this line of y=2?
103H: Yes!
104S: If we take the symmetry of that part in the line of y=2
105H: Which part are you saying that we should take symmetry?
106S: That part
107H: No! We shouldn't take this part's symmetry
108I: S says that symmetry of the ray in the left side of the y-axis should be taken according to the line of y=2.

The interaction flowchart of this segment is generated as follows.

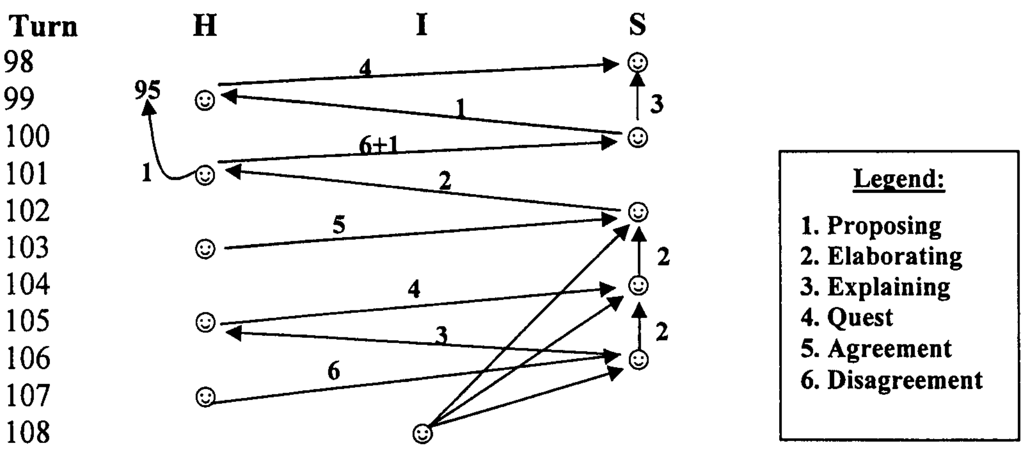


Figure 3.1. An illustration of the interaction flowchart and interaction categories.

The arrows in the flowchart point to the nearest related referent(s) of any specific utterance. For example, the arrow from 99H points to 98S and this arrow is labelled as 'quest' (and designated by number 4, see the legend) because H asked a clarification about what S uttered in 98S. In response to this quest, S in 100S explained what she said in 98 and made a proposal about the direction of the symmetry. Therefore, two arrows emanate from 100H, indicating that this utterance has two referents: one towards 99H as a category 1 arrow and one towards 98S as category 3 arrow. Likewise, 101H has two referents: first H disagreed with S's proposal in 100 and proposed that "… two rays should be symmetric in the line of y=2". So the arrow from 101H to 100S is designated by the numbers 6+1 (disagreement and proposing). H's proposing statement also relied on the graph that they drew in 95H and thus a long arrow from 101 to 95 (designated by the number 1) is depicted.

This disagreement resulted in an argumentation between the students. H tried to defend her proposing statement uttered in 100S and to convince her partner of the inaccuracy of the symmetry in the graph. As a result, in 102S, S started to elaborate H's proposal (101H) of the symmetric line of y=2. Thus, an arrow from 102S to 101H (designated by the number 2) is placed in the diagram, as both of these utterance has 'line of y=2' in common. S's explanation and elaboration continued until the end of this segment but H disagreed with S in 107H (category 6 arrow to 106S). The important point here is that, by disagreeing with S in 107H, H was not satisfied and not convinced by S's explanations and elaborations throughout this episode and in particular 102S, 104S, 106S. This situation can be seen from the flowchart, by implication, by following the related arrows, i.e. 107H→106S→104S→102S. As the obtained graph was erroneous and S rightly claimed that this graph was wrong, in response to H's disagreement, scaffolder felt a need to draw H's attention to this and thus summarised S's explanations and elaborations occurred throughout 106S, 104S and 102S. Therefore, the scaffolder's statement in 108 has all these three referents, which are indicated by three arrows pointing to these three utterances.

9.2. Procedure to analyse the cognitive process

The analysis of verbal protocols in terms of cognitive process was carried out with regard to epistemic actions, that is recognising, building-with and constructing (RBC). For this purpose, I closely followed the methodology of the original study of Hershkowitz et al. (2001) which proposed and formulated these epistemic actions to analyse the cognitive aspect of abstraction process. In addition several other follow-up studies which further detail and apply, and hence contribute to our understanding of, these epistemic actions were also examined. These studies included, for example, Dreyfus et al. (2001), Tsamir and Dreyfus (2002) and Williams (2003). Based on an examination of these studies, detailed definitions and characteristics of these epistemic actions were obtained. These definitions and characteristics are accounted for next and are used to analyse the verbal protocols.

Recognising

In the framework of RBC theory of abstraction, recognising means identifying a mathematical structure that has been constructed earlier, whether in the same activity or earlier. Recognition of a familiar mathematical structure occurs when a student realises that a structure he/she has constructed and possibly used earlier is inherent in a given mathematical situation. Recognising usually occurs as part of an activity with a purpose that goes beyond the act of recognition. Recognition of previously constructed knowledge may occur in at least two cases: (a) by analogy with another object with the same or a similar structure which is already known to the re-cognising subject, (b) by specialisation, i.e. by realising that the object fits a (more general) known (to the subject) class all of whose members have this structure.

Building-With

Building-with consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement. When students are engaged in, for instance, solving a problem, understanding and explaining a situation, or reflecting on a process, they often appeal to known strategies, rules or theorems. To achieve their goal, they recognise structures (from earlier activities) and use them in the course of solution. Hershkowitz et al. (2001) have identified such use and combination of structural elements to achieve a given goal as building-with. When building-with, the students use available structural knowledge to build with it a viable solution to the problem he/she is solving. He/she is not enriched with new, more complex structural knowledge. In building-with structures, the goal is attained by using knowledge that was previously constructed.

Constructing

Constructing consists of assembling knowledge artefacts to produce a new abstract structure(s). New methods, strategies, or concepts can be constructed. The goal of the constructing actions is the construction itself, i.e. the creation of a new cognition. Two main characteristics of construction are novelty and verticality. Hershkowitz et al. (2001, p.212) note that "when a novel structure "enters the mind," it has to be cognised, or pieced together from components" which are already existing knowledge structures and are usually simpler structures. Existing knowledge structures are reorganised vertically in such a way that adds depth to the knowledge.

At this point, it is necessary and beneficial to mention the differences between the building-with and constructing actions. Novelty is one feature that differentiates between building-with and constructing. Another important difference between constructing and building-with lies in the relationship of the action to the motive driving the activity. Building-with actions usually respond to an extraneous goal such as solving a problem; on the other hand, the goal of a constructing action is the construction itself. In building-with structures, the goal is attained by using knowledge that was previously acquired or constructed. In constructing, the process itself, namely the construction or restructuring of knowledge is often the goal of the activity; and even if it is not, it is indispensable for attaining the goal. The goals students have (or are given) thus strongly influence whether they build-with or construct.

Constructing is the most important of these three epistemic actions in the sense that formation of an abstraction strongly depends on the occurrence of constructing actions. By definition, constructing requires student(s) to assemble already constructed knowledge artefacts to create a new one. Such assembly necessitates using, manipulating and combining existing knowledge artefacts. This entails building-with actions and each structure that is used to build-with first needs to be recognised. In this sense, the act of constructing does not merely follow recognition and building-with in a linear fashion but simultaneously requires recognition of and building-with already constructed structures. Hershkowitz et al. (2001) call this dynamic nesting of epistemic actions. In other words, these epistemic actions are dynamically nested in such a way that building-with includes recognising actions, and constructing includes both recognising and building-with actions. This nested relationship amongst the epistemic actions, is reflected in the analysis of cognitive process such that when an utterance is identified as building-with, this utterance is also considered as involving recognising. Likewise, if an utterance (or chain of utterances) is classified as constructing, this utterance(s) is also regarded as involving building-with and recognising.

Please note that in the course of analysis, evidence for the occurrence of these three epistemic actions was sought out with regard to the students' utterances rather than the scaffolder's. That is not to say that the scaffolder's utterances did not have any cognitive element connected to these three epistemic actions or that he did not engage in epistemic actions. Quite the contrary, in order to give appropriate assistance the scaffolder must at least recognise the mathematical structures involved in the task and/or in the students' utterances. He may even engage in some constructing actions related to unknown (to him) properties or aspects of the mathematical structure under scrutiny. However, this study does not aim to uncover or investigate the scaffolder's engagement in epistemic actions and thus this issue will not be a focus of attention.

9.3. Unit of analysis

I have so far detailed and discussed the procedures to analyse the verbal protocols from social and cognitive perspectives. In the next chapter, these two analyses were carried out by examining the transcriptions of the verbal data. These transcriptions were composed of the participants' conversational exchanges as manifested through language and the students' written responses. The essential practical issue with regard to the investigation of the social and cognitive processes was to determine the primary unit of analysis.

Analysing the social process, as detailed hitherto, demanded three different examinations of a single transcription in order to identify: first the students' conversational turns according to their functions in relation to the six categories; second the scaffolder's interventions in terms of monitoring-analysing-assisting; and third the relationship between an utterance and the previous one(s) in order to construct the interaction flowchart. These identifications are concerned with the individual's utterances, which are interdependent. Thus, in order to achieve the intended identifications, one needs to examine more than one utterance at a time because a statement(s) taking place in an utterance is only truly meaningful in relation to previously occurred ones. For example, a disagreement statement can only be meaningful when one knows what is disagreed with. Likewise, a statement can be considered as an elaboration as long as it elaborates something already said or done (e.g., a proposal). In this sense, each utterance comes about as a reaction to or a further action based on what has been said or done (Bakhtin, 1986). This implies that the primary unit of analysis for the examination of social processes is not a single utterance but rather a sequence of the utterances.

In terms of cognitive process analysis, on the basis of what a student says or does, one needs to determine what a student knows and thus recognises and how that knowledge is being used (building-with) to create a new knowledge structure (construction). By implication, recognising actions can be considered as isolated individual incidents which need not necessarily be connected together. Therefore, a recognising action can be observed within (even a part of) a single utterance in isolation from others. Building-with and constructing actions take place within a series of successive utterances depending on the nature of a goal. As mentioned earlier, constructing and building-with actions are closely related to the goal that students have (or are given). As a result, analysing the cognitive process in terms of RBC requires consideration of some successive utterances together with regard to the (sub)goals emerging in the activity. This suggests that primary unit of analysis for RBC is the actions observed in a chain of utterances that are directed towards the fulfilment of a (sub)goal.

The primary unit of analysis should be such that it enables one to carry out the intended examinations feasibly with regard to the two analyses. For this purpose, the unit of analysis was decided as a chain of successive utterances evolving around a common subgoal(s). As each subgoal has distinctive characteristics and thus can be distinguishable from each other, this decision was practical and profitable. However, this brought an additional investigation of the transcripts so as to define subgoals throughout the activity. In order to achieve this, I, following Chi (1997), divided the protocols into segments. Since students were engaged in some specific tasks designed to lead to abstraction, the segmentation was carried out according to the fulfilment of the task goals (i.e. predetermined goals of the questions in the tasks), or according to a re-evaluation of these goals by the scaffolder and/or the students. Both of the analyses, i.e. social and cognitive processes, were carried out separately within each of these segments, which I shall call episodes.

10. Verification strategies employed in the course of analysis

In this section, I detail the verification strategies employed in the course of this study to ensure quality and rigor. There were mainly five such strategies used in this study from design to analysis of the data to reporting the findings: triangulation, negative case analysis, peer review, thick (rich) description and external audits (see Creswell, 1998 for more details).

The first strategy employed in this study was triangulation which is concerned with providing corroboratory evidence to the findings and interpretations by making use of multiple and different sources of information, data collection methods and/or theories. In order to achieve triangulation, this research adopted multiple case study methodology designed in such a way that enables literal and theoretical replications of the cases. A noteworthy advantage of this design was that it provided an opportunity to check the occurrence of certain trends and accuracy of certain observations by analysing multiple and different cases. Further to this, in the course of analysis I drew on the extant literature to make comparisons between my observations and the ones already reported by others.

The second strategy was negative case analysis. This refers to refinement of the hypotheses and interpretations in the light of negative and disconfirming evidence so that all outlier and exceptional observations and interpretations can be eliminated. This was considered one of the most important verification strategy to establish the credibility of the research findings (Robson, 1993). The multiple case study methodology enabled me to carry out negative case analysis strategy as I found a chance, on the basis of analysing several cases, to check the credibility of my expectations, initial hypotheses and initial observations. As a result of investigating the multiple cases I had to change, refine and amend my ideas. I will now illustrate how this strategy was used in my analysis with two particular examples. The first example is that before the actual data collection, based on my analysis of the pilot study data, I envisioned a correlation between the frequency of the students' explanation statements and the achievement of new constructions. Although there were some cases verifying this expectation, there were some others in which the students formed misconstructions despite the fact that they produced a great number of explanation statements. So I realised that it is not necessarily the frequency of the explanation statements but the way that students employ mathematical reasoning on the basis of structural relationships involved in the explanation statements. This realisation, for example, led to a refinement of my initial expectation and indeed informed my understanding of how the scaffolded help induces students to achieve new constructions (see Chapter 5 section 3).

As a second example, in several cases the students working with the scaffolder, I observed some relationships between certain types of scaffolder interventions (e.g., marking critical features and providing examples) and the students' achievement of new construction. However, in my analysis of the other cases I realised that the scaffolder employed the same strategies but they did not result in new constructions on the part of students. As a result of making some comparisons I realised that what matters was not necessarily the type of intervention in a given moment but whether the students saw the motive of this intervention or understood it. Consequently I realised the importance of 'intersubjectivity' and 'alterity' as issues that had to be considered carefully (see Chapter 5 section 5). My analyses towards this direction led me to establish some causative relationships between the scaffolder interventions and the achievement of new constructions and to the idea of mediation of constructions (see Chapter 5 section 3). All of the ideas presented in this study hence were somehow confirmed and informed by the other cases which are not necessarily presented in the following chapters.

The third verification strategy employed in this study was peer review which provides an external check of the research design, analyses and findings much in the same spirit as inter-coder reliability process (Creswell, 1998). This strategy was basically carried out in collaboration with my two supervisors. From the very start of my study, they asked 'hard' questions about the research design, data collection methods and instruments, data analyses methods and my interpretations of the data; they provided me with the opportunity for catharsis by sympathetically listening to my feelings and reasons. It was through my supervisors' challenges and suggestions that I improved the research design, analyses methods and my interpretations. For example, I was initially quite reluctant to do a pilot study as I already did some pre-piloting on some British students. However, my supervisors convinced me of the usefulness of a pilot study. Indeed, as discussed before, the pilot study turned out to be of critical importance for my own development, provided crucial insights into my understanding of abstraction and consolidation and led to important amendments to the data collection instruments. The peer review process was not only used in research design but also in the analysis procedure. In order to analyse the data in relation to my research questions, I developed many different schemes, methods and approaches but some of them did not stand up under the challenge of my supervisors and thus I realised the flaws and difficulties involved in my approaches. For example, I initially thought that students' utterances could be examined on the basis of a differentiation between the cognitive and metacognitive characteristics of the contents. Nevertheless as a consequence of having discussion with my supervisors upon this division, my scheme and my motive for this, I realised that it was not a feasible and profitable approach and gave up this route despite the fact that I spent weeks developing this approach. In addition to this review on the analyses phase of the study, my supervisors also checked my final reports on the issues emerging from the data and read carefully to see if my interpretations could be justifiable by the data that I presented in case of over-interpretations (and under-interpretations).

The fourth strategy used in this study was thick (rich or detailed) descriptions of the research design and rationale, participants and their selection process, underlying assumptions in data collection instruments and in analysis. This was particularly important to allow the reader to make decisions regarding credibility of the findings and make inferences (or judgements) about the applicability of the findings in other situations. In this study, especially in the discussion chapters (of 5, 6 and 7), in order to guide the reader through my line of reasoning and development of my arguments, I provided related literature and detailed the descriptions of the verbal data, as I see it, and thus tried to give the reader a sense of how my observations and interpretations were informed and established.

Finally the strategy of external audits was employed in this study. This strategy requires an external consultant, the auditor, to examine both the process and the product of the interpretations and accounts, assessing their accuracy (Creswell, 1998). The term 'external' is used to emphasise that the auditor should have no direct connection to the research itself as this gives him/her a chance to appraise, as an outsider, whether or not the findings, interpretations and conclusions are supported by the data. This method, like peer review, is quite rigorous but at the same time is quite difficult to perform. In the course of my study, I had a chance to present several papers in British and international conferences in which several reviewers (usually experts in the field) evaluate the papers and comments on them (a list of reviewed papers stemmed from this study is presented on page ii). Further to this, along with these reviewers' comments, the audiences participating my sessions asked 'hard' questions which provided me with some new insights and directions. For example, in one of the research presentation on scaffolding and new construction, one member of the audience pointed out that the scaffolder in my study played a 'leading' role and questioned the effect of this. This comment was valid and I felt a need to examine the verbal protocols from this perspective. This examination led me to realise the cultural dynamics involved in a scaffolding process and the necessity to investigate the influence of the students on the scaffolder's interventions (see Chapter 5 section 4.2.1.).

In addition to this, I found a chance to consult to some experts who were not involved in my research such as Peter Tomlinson, an important figure in educational psychology. Having decided upon the use of four tasks to collect data, I had to make a decision on the time interval necessary between the applications of two successive tasks. To seek some advice, I talked to Peter and he brought the issue of consolidation to my attention. In fact before our meeting, I was aware of the issue of consolidation but was not very much concerned with it. However, following this meeting I realised how important consolidation was and indeed redesigned the initial four tasks. As mentioned earlier these initial four tasks have been reconsidered following the pilot study precisely because of the lack of consolidation opportunities. My intention here is not to go into that process again but to point out that if Peter had not advised me on consolidation, the issue of consolidation perhaps would not have received the attention that it deserves.

As can be seen from these considerations, verification strategies are rather crucial in qualitative research. In the case of this study, these strategies led to a number of amendments which, in my opinion, improved the research itself considerably. However, in line with Morse et al. (2002), I have to point out that these verification strategies need to be employed, whenever possible, from the very start of a qualitative research so that rigor and quality can be ensured.

CHAPTER 4: ANALYSES OF THE VERBAL PROTOCOLS

The data in the main study were collected from 20 students over four tasks: three working individually under think-aloud instructions; three pairs working together on their own; three individuals and four pairs working with the scaffolded help. Thus data collection took place over 52 sessions in total (see Table 4.1. below); a session being the student(s) work on a task. The data for this study were composed of the students' audio-recorded verbalisations and written works during the sessions. Following data collection, audio recordings of all 52 sessions were listened to and the written work of the students was examined to gain an overall view of the data. Later 34 of the session were transcribed: 15 sessions of pairs working with the scaffolder; 10 sessions of individuals working with the scaffolder; 6 sessions of the pairs working together; 3 sessions of the individual doing thinking-aloud. In order to share my observations with my supervisors, transcriptions of the 15 sessions were translated into English: 8 sessions from the scaffolded pairs; 4 sessions from the scaffolded individuals; 3 sessions from the non-scaffolded pairs. However, I did not translate any transcription from the individuals working under think-aloud instructions as they were unsuccessful and did not produce fruitful data (see below). All these transcriptions of the 15 sessions were examined with great details.

In order to answer the first research question concerning the construction of new knowledge through scaffolding, scaffolded pairs and individuals' verbal protocols were examined to seek common patterns and counter examples within them i.e. they were compared and contrasted to find corroboratory or disconfirming evidence for the observed trends. Further to this, I had recourse to the non-scaffolded pairs' work to make some comparisons between the scaffolded and non-scaffolded pairs' performance. In order to answer the second research question concerning the consolidation of new knowledge, I examined the verbal protocols of those students who achieved the construction of the task 1 and 2, then their performances on the task 3 and also their use of the consolidated constructions in the fourth task. My analyses and considerations with regard to these two research questions informed me about the third research question which requires a critical evaluation of RBC theory. In order to exemplify my observations and inferences regarding these three research questions, I will present and analyse some verbal protocols. Before going into the detailed analyses of verbal protocols however I will first provide an overview of the all participating students' performance over four tasks (see Appendix 7).

1. An overview of the participating students' performances

The students' performances were divided into three broad categories: success, progress and failure. These categories were judged differently for task 1, 2 and 4 on one hand and for task 3, the consolidation task, on the other (see Appendix 7). In order for a student or a pair of students to be considered successful in task1, 2 and/or 4, they are expected to suggest, at the end of their work, a *correct*[1] method(s) to obtain the intended graphs by making use of the graph of $f(x)$ other than substitution. The method of substitution on the basis of an equation was used by the students almost exclusively while sketching the graphs. Although this method is certainly a working one to sketch the intended graphs, the students were expected to construct a method which draws on the given graph of $f(x)$ rather than its equation. Therefore the condition for success in these three tasks was judged on the basis of whether the students came up with a correct method which utilises the given graph itself rather than its equation. The students' success on task 3, the consolidation task, was mainly judged on the basis of their responses to question 3, 4 and 5 (see Appendix 7, task 3). If the students have been able to refute all the arguments proposed by three imaginary students in question 3 and 4; and if they have been able to differentiate between the graphs of $|f(x)|$ and $f(|x|)$ for the given graphs in question 5, then these students were considered to have successfully consolidated their constructions of $|f(x)|$ and $f(|x|)$ (see Chapter 6 for more on consolidation).

As to the category of progress in task 1, 2 and 4, if the students have not been able to suggest a method other than substitution, their written work and verbal accounts were examined. If they have sketched at least two intended graphs correctly in a task irregardless of the method employed, then they were regarded as having made some progress. The reason for insisting on at least two correct graphs is because this shows that the student(s) are able to draw the intended graphs by using its equation and this gives the students an opportunity to observe the similarities and differences between the graphs so that a method may be developed. In this sense the students, by sketching at least two graphs, make some progress towards the development of some method. The judgement of progress in task 3 was dependent on whether the students were able to come up with a correct method other than substitution by the end of task 1 and/or 2. If the students have been successful in any of these two tasks, then I looked into their overall performance in task 3. If the students have been able to answer the question 1 and 2 correctly; even if they have not been able to refute all the arguments by three imaginary students in question 3 and/or 4 but at least reject one or more with the right reasoning; *or* if they have been able to show the differences between $|f(x)|$ and $f(|x|)$ for some of the given graphs in question 5, then the students were regarded as making some progress in task 3.

Regarding the third category, if the students have been neither successful nor making some progress, then they were considered as having failed. The students' performances in terms of success and progress are presented in Table 4.1. below. In this table, two ticks indicate whether the student(s) were successful; one tick whether the student(s) made some progress; and two crosses show that neither success nor progress were observed for the student(s) who therefore failed.

[1] There were some students who came up with an incorrect method which I reported elsewhere (Ozmantar, in press). In such cases, the students were regarded as having failed.

Group 1: Individual students working under think-aloud instructions								
	Task 1		Task 2		Task 3		Task 4	
	PRG	SUC	PRG	SUC	PRG	SUC	PRG	SUC
☺	×	×	×	×	×	×	×	×
☺	✓		✓		×	×	×	×
☺	✓		✓		×	×	×	×
Group 2: Individual students working with the scaffolder								
	Task 1		Task 2		Task 3		Task 4	
	PRG	SUC	PRG	SUC	PRG	SUC	PRG	SUC
☺ with SC		✓✓		✓✓		✓✓		✓✓
☺ with SC		✓✓		✓✓		✓✓		✓✓
☺ with SC		✓✓		✓✓		✓✓		✓✓
Group 3: Pairs working together on their own								
	Task 1		Task 2		Task 3		Task 4	
	PRG	SUC	PRG	SUC	PRG	SUC	PRG	SUC
☺ ☺		✓✓		✓✓	✓		✓	
☺ ☺	×	×		✓✓	✓		×	×
☺ ☺		✓✓	✓		✓		×	×
Group 4: Pairs working with scaffolder								
	Task 1		Task 2		Task 3		Task 4	
	PRG	SUC	PRG	SUC	PRG	SUC	PRG	SUC
☺ ☺ with SC		✓✓*		✓✓*		✓✓*		✓✓*
☺ ☺ with SC		✓✓		✓✓		✓✓		✓✓
☺ ☺ with SC		✓✓		✓✓		✓✓		✓✓
☺ ☺ with SC		✓✓		✓✓		✓✓		✓✓

Table 4.1. Examination of the participating students' performances in terms of success and progress. PRG: Progress; SUC: Success; see below for the ticks with asterisk.

As can be seen from Table 4.1., the individual students working under think-aloud instructions could not achieve the intended constructions, though two of them demonstrated some progress on task 1 and 2. Two of these students even found it difficult to set about task 1. As a result, they were given the hints (see Appendix 2) which asked them to prepare an x, y chart. These hints eventually drew the students' attention to the method of substitution which two of them used with some success and sketched some of the expected graphs accurately. However, none of the students needed the hints prepared for task 2; they started off this task immediately by substituting. These students surely had the prerequisite knowledge structures required to complete the tasks successfully. Then the question of interest is: why could they not have achieved any of the constructions? There appear several reasons for this. First of all, they found the process of thinking-aloud rather difficult despite the fact that they were successful in doing this relatively easily in the training questions (see Appendix 1, question 4). They reported later that they found it difficult to talk and solve the questions at the same time, which was something that they were not used to doing.

Second, as working on these tasks when the students went silent for a while I had to intervene and asked them to think-aloud. They told me that this was distracting their focus of attention and preventing them from making some reflections. Some of the students commented later that sometimes they talked simply because they wished to please me and as a result they lost the track. Third, considering the students' success in doing thinking-aloud on relatively easy tasks and their incapability to do this on the actual tasks which required them to construct a new method suggests that the intended constructions were quite likely to be beyond these individual students' unassisted efforts. This prompted the thought that the think-aloud methods might be useful in investigating the students thought process when working on tasks which reside in their actual development level but not on tasks which reside in their potential development level. The difficulty that these students experienced can, to a certain extent, be attributed to the fact that my tasks required the students to construct a new method and this is rather challenging for them while working under thinking-aloud instructions. That is not to say that if these students were to work in silence without having to talk and think at the same time, then they would be successful. However, in my experience, this think-aloud process seems not to be a particularly useful method to investigate an individual student's construction process.

Regarding group 2 students, all were successful in all the four tasks. They achieved the intended constructions and consolidations and thus formed the intended abstractions. However, one of the students, a female, was quite nervous in her work on all of these four tasks. She was not very talkative and most of the time preferred to work in silence despite the scaffolder's efforts to get her talking. She later commented that she did not feel comfortable because she was afraid to make 'stupid' mistakes which were being recorded and because she was not used to working with a male tutor in one-to-one situation. For mainly these two reasons she said that she did not talk much. Yet her verbalisations and her written work certainly suggested that she was successful in all four tasks. In contrast to this student, the other two, male, students were quite comfortable to work with the scaffolder and they were pleased that their talk was being recorded. All these three students appeared to need the scaffolder's assistance to achieve the constructions.

Regarding group 3 students, none of these pairs achieved the target constructions of all four tasks. One pair achieved the constructions, $|f(x)|$ and $f(|x|)$, of the task 1 and 2 but was unable to construct a method for $|f(|x|)|$ though they made some progress on this. They made some progress in consolidating the constructions of $|f(x)|$ and $f(|x|)$ in task 3 but they could not consolidate them fully in that, at the end of their work on this task, they seemed to be confused about the differentiation between the graphs of $|f(x)|$ and $f(|x|)$ which became especially evident through their work on the question 5 (see Appendix 7, task 3). The second pair was not able to construct a method for $|f(x)|$ and could not even make some progress. But they were successful in their work on task 2 and constructed a method for $f(|x|)$; yet they could not sketch even a single correct graph for task 4. They made some progress on task 3, the consolidation task, with regard to $f(|x|)$; however, they were uncertain about the accuracy of one of the imaginary students' claims (see Appendix 7, task 3, Arzu's claim in question 4). The story for the third pair is rather interesting. In this pair's work, there was a strong asymmetry between the students in that one partner was far more dominant than the other, forcing his understanding on his partner and most of the time he did not even listen to his partner's talk and did not attend to his partner's queries and questions. I have to note that asymmetry was, to some extent, the case in the work of other pairs but not as strong as in this pair's. This pair achieved the construction of $|f(x)|$. They made progress in task 2 in that they drew three correct graphs of $|f(x)|$ but they failed as they came up with a misconstruction for the graphs of $f(|x|)$. Further to this, they could not achieve the construction of $|f(|x|)|$. Interested readers can find more on this pair's work in Ozmantar (in press).

Finally regarding group 4 students, as mentioned earlier, initially I decided to work with three pairs but there occurred some problems with one of the pairs whose work is indicated by ticks with an asterisk in Table 4.1. Before going into details, I have to note that prior to recruiting the students for this study, I asked them all, including this pair, if they would like to take part in the work and explained the expectations such as that they had to work for four successive days without omission and completion of the tasks could take up to an hour and even perhaps more. As a matter of fact some two students did not want to take part and hence they were excluded. However, this pair, two females, wished to work for this study but when they started working on the tasks, as it happened, one of the partner became rather reluctant to work on the tasks and she even created some obstacles for her partner and for the scaffolder. To give the reader an idea of the nature of problem, I present some excerpts from their work on task 2 (see Appendix 7). The excerpt below occurred when these two girls (R&B) were working on the first question of task 2 for which they (but mainly B) sketched the graph of $f(|x|)$ correctly with the help of the scaffolder and afterwards the scaffolder/interviewer (I) suggested that:

I: The question is also asking to make comments on any patterns or symmetries in the graph of $f(|x|)$.
B: Yes, there is symmetry in the y-axis. I mean positive... positive values of x and negative values of x are symmetric...
I: R?
R: What?
I: Do you agree?
R: That's right... Whatever she says...
I: What else can you say?
B: I told everything I see here!
R: And I agree whatever she said here!

They later moved on to question 2 and B suggested some patterns between the graphs of $f(x)$ and of $f(|x|)$. In order to get R's active participation and contribution, the scaffolder directed some questions to R:

I: B, you are talking to R, aren't you? What do you think R?
R: I am not into a mood to talk.
B: Well, I mean... yes... that's what I think!
R: She is telling and talking instead of me!
I: You are supposed to work together, contribute to the solution and try to understand each other.
R: I agree with her on everything she says.

As these two short excerpts suggest, R was rather reluctant to become a part of the activity, was indifferent to the dialogue between the scaffolder and B, and she did not exert herself neither to understand nor to contribute. As a result, the interaction was effectively between the scaffolder and B; and the scaffolder was asking questions to assist B to develop patterns. The interaction pattern in this pair's work and R's attitudes were not much different during their work on task 3 and 4. At the time of data collection and even later when examining this pair's verbal protocols, it did not make much sense to me why R had such negative attitudes and why she was indifferent to the dialogue between the scaffolder and B. Nevertheless due to these problems, during data collection, I decided to include one more pair to this study. In Table 4.1. above, the ticks indicating the success of this pair should be understood as the success of B but not necessarily R. This is because I do not have 'hard' evidence that R also constructed the intended methods along with her partner B, though she joined every session but rather reluctantly.

I did not experience similar problems with the other three pairs. They worked on these tasks and achieved all the expected constructions and consolidations. However, they needed the scaffolder's assistance, albeit in varying degrees, to achieve a successful completion of the tasks. Generally speaking, these three pairs displayed successful collaboration throughout their work. The reasons for this seems to me that first because they were given freedom to select their partners, second because they kept up with the given instructions as to how to work together (see Chapter 3, section 5.3) and third because the scaffolder made a considerable effort to get the active participation and contribution of the all individuals.

2. Analysis of the verbal protocols

In this study, in order to illustrate my observations and findings regarding the first research question, I decided to present verbal protocols of a scaffolded pair's work. For the second research question, I decided to present a scaffolded individual student's work in order to exemplify my observations regarding the issue of consolidation of new constructions. The reason for presenting a scaffolded individual's work is because I wished to give the reader an idea of the scaffolded individual's work and more importantly because I did not wish to deal with the complexities of social interaction which, to some extent, obscure the analysis of consolidation and which I deal with in my analyses and discussions in relation to the first research question. However, it is important to note here that I will not present this individual's verbal protocols in this chapter but rather I will do so in Chapter 6 where I attend to the second research question. In doing so, I hope to help the reader follow the arguments related to consolidation. In my attempt to answer the third and final research question which requires a critical evaluation of RBC theory, I draw on the verbal protocols presented in relation to the first and second research questions. In what follows, I will give some background information about the scaffolded pair, two female students (H&S) and then their verbal protocols will be examined in great detail.

2.1. Background for the verbal protocols

In the rest of this chapter I will present the verbal protocols of H&S who worked with the scaffolder and who successfully achieved all the constructions and consolidations. These students were two 17-year-old girls in Grade 10. They described themselves as 'very good' friends. The school where they attended had a dormitory and they were staying in the same room and even sitting next to each other in the class. They had a habit of working together at in- and out-of-class activities. They wished to work together rather than working alone or with someone else. Thus they were assigned as a pair and worked quite harmoniously during all of the four tasks.

I will particularly focus on H&S's protocols on the second and fourth tasks. There were mainly five reasons to select this particular pair for my analysis. First of all, due to space limitations, some protocols had to be chosen; it is not possible to display all the verbal protocols. Secondly, generally speaking, scaffolded pairs' work produced more fruitful and complex data than that of scaffolded individuals. Thus in selecting this pair's work I am hoping to demonstrate the complexity of new knowledge constructions in socially rich environments. Furthermore, this pair's work provided some nice illustrations of the potential divergences in epistemic actions on the part of students. This increased the complexity of the analyses but at the same time contributed to my understanding of the occurrence of the epistemic actions. Thirdly, in selecting certain pairs it was important that the students needed the scaffolder's assistance to achieve the intended constructions so that the observations made through the analyses of the scaffolded participants could be illustrated. The importance of the scaffolder's assisting interventions in the course of this pair's achievement of the intended constructions of especially the second and fourth tasks (i.e. $f(|x|)$ and $|f(|x|)|$) was all too apparent. In other words, the matters related to scaffolding and construction of new knowledge were clearly apparent in this pair's work. Fourthly, by providing this pair's verbal protocols on two tasks I wished to give the reader an opportunity to check the credibility of my arguments and observations on the basis of more than one case. Finally, these two students were at ease, quite talkative during the activities and also relaxed in their communication with the scaffolder. All these contributed to the production of detailed verbalisations and hence helped me gain a better appreciation of the issue of knowledge construction with a scaffolder's assistance.

In the following protocol excerpts, speaker utterances were numbered with each uninterrupted utterance by one speaker being assigned a natural number. Three dots (...) were used to indicate either that the speaker paused or spoke inaudibly. Some comments were inserted in square brackets to allow the reader to follow the conversation amongst the participants. In the following 'I' refers to the interviewer/scaffolder. Each complete utterance is given a new line number. I provide comments after each episode.

2.2. H&S's verbal protocols on the second task

In this section, I present and analyse H&S's verbal protocols on the second task. Please note that the goals of the questions in this task are detailed in Chapter 3 (section 6.6. on p.57) and the task itself is presented in Appendix 7. The protocol excerpts presented below are divided into seven episodes. In episode 1, H&S suggested an initial method to sketch the graphs of $f(|x|)$ which was applied in episode 2 where the scaffolder drew the students' attention to some problems in this method, though H&S could not realise the problems. In episode 3, they had an argumentation about the accuracy of this method which was fully developed in episode 4 with the assistance from the scaffolder. In episode 5, one of the students, H, developed another method which S was able to develop only later in episode 6. They were confused however about the differences between these two methods which were clarified in episode 7. Before going into the details of their work in episode 1, I briefly review their work until then.

The students obtained the graph of $f(|x|)$ accurately in the first question by substituting different values of x into the given equation of $f(x)$ and thus finding corresponding values of y. In doing so, they found several associated ordered pairs and mapped them onto the Cartesian grid and then united them to sketch the graph of $f(|x|)$. Thus they fulfilled the intended goal of the first question. In the second question, they compared the graphs of $f(|x|)$ with that of $f(x)$ to observe the changes occurring in the graph of $f(x)$ in relation to the graph of $f(|x|)$. They focused solely on the conspicuous features of $f(|x|)$ and recognised that $f(|x|)$ has a line of symmetry and they also reported that the values of y in the graph of $f(|x|)$ take negative values in the interval of $-4<x<4$. Following this, they moved on to the third question for which they once again sketched the intended graph of $f(|x|)$ correctly by substituting (see Table 4.2. – A) and realised that this graph also had a symmetry line. I pick up the students' conversation as they start question 4.

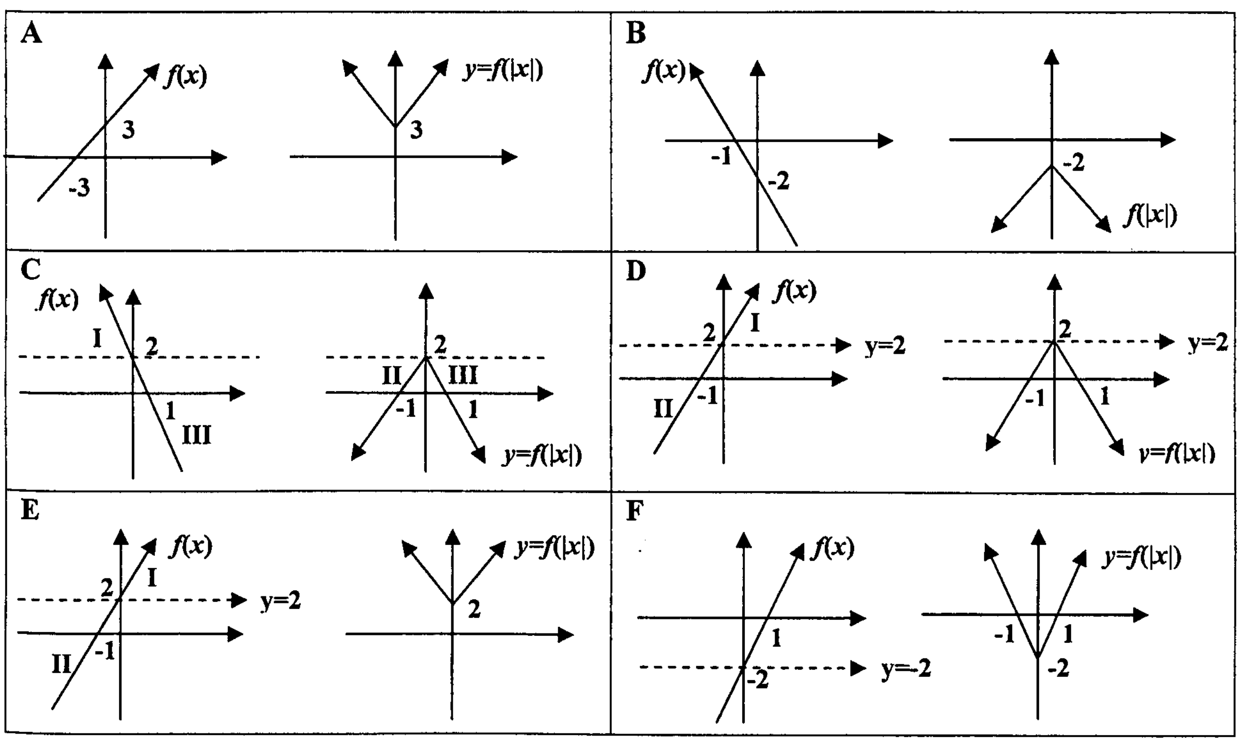


Table 4.2. The graphs that H&S obtained in the second task.

Episode 1

43S: [She reads question 4]. OK, we don't have any equation this time!
44I: OK, let me remind you that before this question you were given the equation and graphs but here in this question you are just presented with the graphs without the equations. Are you planning to find the equation for each of the graphs?
45H: Actually, we don't need to find the equation for every given graph… I think we should find the equation for the first graph and after drawing the graph with this equation we can develop a general pattern to draw the others…
46S: But in the other two graphs…

47H: OK let's find the equation for the first graph by using the intersection points...

The students accurately obtained the equation of the given graph of $f(x)$ in question 4-A (see Appendix, 7) and then by using this equation, they drew the graph of $f(|x|)$ by substituting (see Table 4.2. – B). They started to talk about the graph of $f(|x|)$.

58S: There is something attracting my attention in the last two graphs... that part... how can I say? I mean... the part of the graph until the y-axis [she refers to the part of $f(x)$ on the right of the y-axis] remains the same and afterwards we are taking the symmetry[2] of the remaining part.
59H: You mean when x=0?
60S: Look, for example, it is the same in this graph as well. I mean in this graph the part of $f(x)$ until the y-axis remains the same and then the remaining part is taken symmetry
61H: That means the symmetry is starting from the value of y at x=0? Are you saying that the graph remains the same until x=0?
62S: Yes. For example, the graph takes the value of -2 at x=0 and the graph is left unchanged until the point of (0,-2). But after that a symmetry is taken... I don't know how to say it
63I: You both help each other
64H: Look now... as from the point of (0,-2), the remaining part [she refers to the part on the left side of the y-axis] is taken symmetry in the line of y=-2
65S: Yes it is taken symmetry

This episode is particularly important to show how the interpretation of the questions and perceived goals of these questions lead to new goals. In question 4, four linear graphs without equations were presented to the students. Originally, this question had two predetermined goals depending on students' progress in this task: (1) to force students to develop a general method to draw the graph of $f(|x|)$ on the basis of the graphs obtained in question 1 and 3 as there was no equation readily available and (2) if they have already developed a method then to validate and practice it by working on four additional graphs. As H&S could not develop a method up until question 4, they might be expected to adopt the first goal. Despite the fact that the scaffolder tries to call attention to this (44I), the students (especially H) interpreted this goal differently and decided to find the equation first. Consequently, three new goals different from predetermined ones emerged, that is, (1) to find the equation of $f(x)$ which was represented with a graph, (2) then to draw the graph of $y=f(|x|)$ by substituting which was not overtly stated but obvious from their subsequent actions and finally (3) to try to develop a general method (45H). Fulfilment of the initial two goals were straightforward in the sense that it required H&S to successfully execute the procedure of finding an equation by using x- and y-intercepts and then substituting some values of x into the given equation. In fact they fulfilled these two goals easily and obtained the graph of $f(|x|)$ for the given graph of $f(x)$ in question 4-A (see Table 4.2. – B).

[2] 'Taking the symmetry of' is the verbatim translation of the Turkish term 'Simetriğini almak'. Students are taught to use this term for mathematical reflections. The term could be translated as 'reflecting' and native English speakers may prefer this translation. However, I prefer the verbatim translation as the term 'reflecting' corresponds to the Turkish word 'yansımak' which is usually used in vernacular speech rather than in mathematical discourse.

However, the third goal was relatively difficult to fulfil as it required H&S to determine the common features of the drawn graphs of $f(|x|)$ and assemble them to produce a novel and general method. Between the utterances of 58S and 65S, H&S came up with an initial method, that is, in S's terms, "the part of $f(x)$ until the y-axis remains the same and then the remaining part is taken symmetry" (60S). I term this as the 'reflecting' method which was rather ambiguous as can be seen from the above utterances. The ambiguity was related to insufficient specificity first of the symmetry line, second of "the part of $f(x)$ until the y-axis" and third of "the remaining part"; please realise that the last two terms can be equally taken to mean as parts of the graph of $f(x)$ both at $x<0$ and $x>0$. H&S later moved on to the graph given in question 4-B to apply this method and their related utterances are reproduced below.

Episode 2

68S: ... let's first draw the graph [for the graph of $f(x)$ given in question 4-B] by considering what we've just found out and then control it by substituting, right?
69H: [They now turn to the graph given in 4-B]. So, according to our findings, the graph of $f(x)$ will be the same until the point at x=0.
70S: Uh huh, yeah it is the same
71H: After that what?
72S: For the rest, the symmetry...
73H: We will take the symmetry...
74S: It will be something like this then [she draws the graph, see Table 4.2. – C].
75H: Let's control it now!
76I: I've got to ask a question: which part's symmetry did you take?
77H: In the line of y=2
78S: According to the line... the line is passing when x=0
79H: According to this line [she refers to the line of y=2]
80I: OK I understood according to which line you took the symmetry. But this is not what I was wondering. I want to know which part of $f(x)$ is taken symmetrically?
81H: Oh, it is... we have problems to express it...
82I: OK, you can say for example, the part on the right or left side of the y-axis.
83S: We take the symmetry [of the part of $f(x)$] on the left of the y-axis according to the line of y=2
84H: Yes I agree
85S: The symmetry line is... it passes through the y-axis when x=0
86I: Does this pattern apply to the graphs that you drew earlier?
87H and S: Yes

At the beginning of this episode, the students had two particular goals: (1) to apply the newly developed 'reflecting' method to draw the graph of $f(|x|)$ for question 4-B and (2) to check the accuracy of $f(|x|)$ by substituting (68S). While H&S satisfied the first goal, the scaffolder intervened (76I) before they worked towards the second goal. The scaffolder's intervention at this point of their conversation was related to his awareness of H&S's 'reflecting' method being devoid of necessary specification with regard to which segment of $f(x)$ was reflected and which segment remained the same (i.e. $f(x)$ at $x<0$ or $f(x)$ at $x>0$). The scaffolder did not point out this deficiency between 58 and 65 in order to give the students some autonomy in their work and perhaps he expected them to realise and amend this lack of specificity in their application of this method. H&S applied their 'reflecting' method and obtained the accurate graph (68S-74S and also see Table 4.2. – C). However, they did not realise the aforementioned deficiency in their method and they once again did not explicitly state which part was reflected in concrete terms.

The scaffolder monitored this problem and analysed the students' performance to decide an appropriate assistance so as to draw H&S' attention to this problem. Therefore, in 76, he intervened and explicitly asked the students which part of $f(x)$ was reflected. The students misunderstood the question and talked about the symmetry line (77H-79H). The scaffolder rephrased the question in 80I. However, the students' answers remained rather specific to the graph that they drew. This suggests that H&S could not see the intention of this intervention. Having realised that, the scaffolder attempted to prompt H&S to consider the issue in a more general sense (86I). However, an immediate 'yes' (87) from both of the students suggests that they still did not see the issue from the scaffolder's perspective. As a result, the students failed to clarify which part of $f(x)$ is reflected. As can be seen in the following episodes, this failure posed some serious problems to the students. This episode provides a vivid illustration of the possible discrepancies between the scaffolder's and the students' perspectives.

The following episodes will be examined in terms of both RBC epistemic actions (i.e. R: recognising, B: building-with and C: constructing) and social processes. This examination will be carried out with reference to the interaction table, left hand side of which represents the interaction flowchart while the right hand side represents the epistemic actions identified in the students' utterances. Please note that in the interaction flowchart, students' interaction categories are represented with the numbers 1-6 as follows: **1**: proposing; **2**: elaborating; **3**: explaining; **4**: quest; **5**: agreement; **6**: disagreement.

Episode 3

The students worked on the graph of $f(x)$ given in the question 4-C and drew the graph of $f(|x|)$ by using the 'reflecting' method that they proposed earlier. However they sketched an erroneous graph (see Table 4.2. – D). Having drawn the graph, S realised that something was wrong with this graph and they started talking about this.

> **98S:** Before, we took the symmetry towards the y-axis… but here we drew it… I don't know
> **99H:** What do you mean?
> **100S:** I mean, look, it should have been towards here not there!
> **101H:** Why should it have been so? It is not necessary… look, these two rays [in the graph of $f(|x|)$] should be symmetric in the line of y=2
> **102S:** OK, look isn't this line of y=2?
> **103H:** Yes
> **104S:** If we take the symmetry of that part in the line of y=2
> **105H:** Which part?
> **106S:** That part [part II, see Table 4.2. – D].
> **107H:** No! We shouldn't take the symmetry of this part [part II]
> **108I:** S says that symmetry of the ray in the left side of the y-axis should be taken according to the line of y=2

109S: Yes, I mean shouldn't we take the symmetry of this part? Look at this graph [referring to the graph drawn for the question 4-A, see Table 4.2. – B] the part of $f(x)$ until the y-axis is unchanged and the remaining part is reflected. I mean if I name the rays as I and II [she writes] shouldn't II be reflected in [the line of] y=2 [see Table 4.2. – D].

110H: So are you saying that this graph should be like that? [She refers to the symmetry of II in the line of y=2].

111S: Yes it should be like that and these two rays will be symmetric in the y-axis… All other graphs [of $f(|x|)$] were symmetric in the y-axis… shouldn't it be so here too?

112H: Hang on… wait… I get confused… I couldn't understand… let me think! [They pause for some time and look at the graph that they have drawn]… You are saying that two rays must be symmetric in the y-axis, right?

113S: Yes, it should be so

114H: OK, look at the graph… they [the two rays in the graph of $f(|x|)$] are symmetric in the y-axis anyway…

115S: Yeah, I know but I think something is wrong with this graph [of $f(|x|)$]…

The goal identified in this episode is for the students to check the accuracy of the graph of $f(|x|)$. This goal was not given or set by the scaffolder but, rather, emerged as a result of S's assessment of the accuracy of the graph of $f(|x|)$. Since H did not see the problem involved in the graph, S had to convince her. This situation brought about a lengthy argumentation which continued throughout this episode. S tried to explain what the problem was by considering the graph of $f(|x|)$ (e.g. 98, 102 and 104) and tried to justify her claims by recognising the surface features of the earlier graphs of $f(|x|)$ (e.g. 98, 109 and 111). H evaluated S's arguments (e.g., 101 and 105) and developed counter-arguments by examining S's explanations and justifications (110, 112 and 114). In the course of these examinations, H built her arguments with recognising the elements involved in S's explanations and justifications. The students, during this argumentation period, were able to connect recognising with building-with actions. However, their arguments were not developed on the basis of the deep mathematical structures involved in the graph of $f(|x|)$ (e.g. 107H and 111S). For example, in 111, S tried to convince H about the inaccuracy of the graph by saying that earlier graphs were symmetric in the y-axis and so should this graph be despite the fact that the graph was symmetric in the y-axis as argued by H in 114. This suggested to me that S intuitively believed the graph was wrong but was unable to give any (mathematically) convincing reason for this. As a result, the students were not able to assemble knowledge artefacts to produce a novel structure. Thus, in this episode, the epistemic actions are identified as recognising and building-with but no constructing action is detected; see the right hand side of the Figure 4.1. provided below.

In terms of interaction, the students attended to one another's utterance and contributed almost equally to the argument. So the failure of students' inability to produce convincing arguments cannot be attributed to their ignorance of, or indifference to, one another's ideas. Although each student held different (probably opposite) views about the accuracy of the graph, this episode was produced jointly without the other being dominant. In this sense, I think that they collaborated rather well but were unable to realise the deficiency in their 'reflecting' method or justify the accuracy (or inaccuracy) of the graph. This I believe suggests that the students' joint efforts were not enough to produce the intended construction of a new method in this episode. In other words, the intended structure of $f(|x|)$ was, at this stage in their development, beyond the students' unassisted joint efforts in this episode.

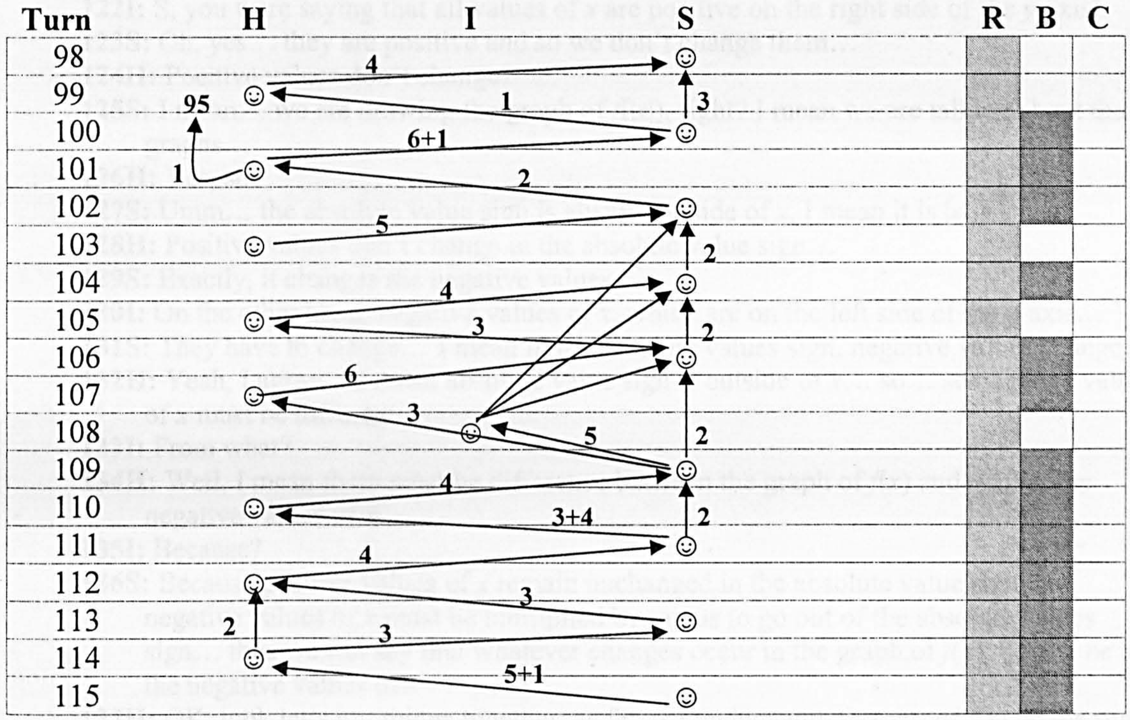


Figure 4.1. Interaction flowchart for episode 3 in task 2.

The legend: **1**: proposing; **2**: elaborating; **3**: explaining; **4**: quest; **5**: agreement; **6**: disagreement.

In this episode, the scaffolder intervened on only one occasion (108) where he recapitulated S's three utterances (see the arrow from 108 to 102, 104 and 106). Why did the scaffolder not intervene and assist the students despite the fact that they clearly needed some assistance in this episode? There appear two essential reasons. First is that the scaffolder was devoted to giving as much autonomy to the students as possible in order to get their active involvement in the activity, which is rather essential for scaffolding (Mercer, 1995, p.74). Secondly he was certainly aware of the difficulties that H&S were experiencing; however, due to lack of a common ground between the scaffolder's perspective and students', which was obvious in the second episode, he possibly wanted the students to come to terms with what he was trying to point out earlier in the second episode.

Episode 4

116I: I'll suggest you something… I would like you to look at and examine the graphs that you have drawn so far… and discuss which part or parts always change and which doesn't! For example, look into the very first graph, which part remained the same and which part changed?

117S: [They return to the graph of $f(|x|)$ obtained for the first question] Look, in this graph… the graph of $f(x)$ until the y-axis didn't change and after y [-axis], it has changed…

118I: Do you mean that the part of $f(x)$ at the positive x, which is always on the right of the y-axis, doesn't change?

119S: Yes that's what I mean… and also on the right side of the y-axis, all of the values of x are positive…

120H: Did it remain unchanged in all of the graphs?

121S: Yes, let's have a look once again every graph that we drew. [They look into other graphs].

122I: S, you were saying that all values of x are positive on the right side of the y-axis!

123S: Oh, yes… they are positive and so we don't change them…

124H: Positive values don't change?

125S: I mean… we are drawing the graph of $f(|x|)$, right? I mean we are talking about these graphs…

126H: Yes, so?

127S: Umm… the absolute value sign is always outside of x, I mean it is $|x|$

128H: Positive values don't change in the absolute value sign…

129S: Exactly, it changes the negative values

130I: On the other hand, negative values of x, which are on the left side of the y-axis…

131S: They have to change… I mean in the absolute values sign, negative values change…

132H: Yeah, I agree… I mean absolute value sign is outside of x… so… so negative values of x must be different

133I: From what?

134H: Well, I mean there must be difference between the graph of $f(x)$ and $f(|x|)$ at the negative values of x

135I: Because?

136S: Because positive values of x remain unchanged in the absolute value sign, but negative values of x must be multiplied by minus to go out of the absolute values sign… thus we can say that whatever changes occur in the graph of $f(x)$, it must be at the negative values of x

137H: OK, look let's put things together: as far as I understand, the part of $f(x)$ on the right hand side (of the y-axis) should remain the same, no matter what. And the part on the left hand side (of the y-axis) should change... do you agree, S?

138S: Yes I think so… I was trying to say that…

139H: Then we made a mistake in the previous graph… I think we should re-draw the graph… [They are once again drawing the graph of $f(|x|)$ for the given graph in C, see Table 4.2. – E].

It is clear from the previous episode that the students' difficulty stemmed from the deficiency of their initial 'reflecting' method with regard to which segment of $f(x)$ is reflected in the line through the y-intercept. The scaffolder was well aware of this problem and, in 116, he instructed H&S and called for a specific action (Tharp and Gallimore, 1988), that is, to investigate all of the graphs of $f(|x|)$ and decide which segment always remains the same and which one changes. In doing so, he explicitly set a goal to and pushed the students into a well-defined pathway. My following descriptions make use of the interaction flowchart generated for this episode as presented in Figure 4.2.

In response to the scaffolder's specific instruction, S started examining the graph obtained for the first question and proposed that the graph of $f(x)$ until the y-axis remains the same which was what she has already said in 109 (category 1 arrow from 117 to 109). The scaffolder, on the basis of his monitoring in the third episode, was aware that this proposal was not enough to eliminate the students' problems. As a result of this analysis, he immediately intervened in 118 by explicitly referring to, and thus hinting at (Bliss et al., 1996), the unchanged segment and also described the critical features (Stone, 1993) of this segment. By this intervention, his intention was to push the students to develop explanations with mathematical reasoning which was of paramount importance for the achievement of the intended construction. This intervention indeed helped the students (starting from S's statement in 119) to clarify the vague language dominant so far in their explanations. S's immediate agreement (category 5 arrow from 119 to 118) and elaboration of her proposal by referring to the specific features of this segment (119) suggests that the scaffolder's intention was accomplished. However, while S was proceeding with the elaboration of her proposal by using the features of this segment, i.e. "all of the values of *x* are positive" (119), H interferes in S's elaboration (120) to check if this segment remained unchanged in all of the graphs of *f*(|*x*|). Although, this interference was possibly necessary to satisfy the goal set by the scaffolder, it put S off her elaboration and they start looking into all other graphs (121). This situation changed the focus of S's elaboration and this had potential to prevent them from developing necessary mathematical ground for the 'reflecting' method. This can be seen in the interaction flowchart (see Figure 4.2.) where category 1 arrow linking 121S to 120H from which category 4 arrows emanates and takes them back to the S's initial proposal in 117.

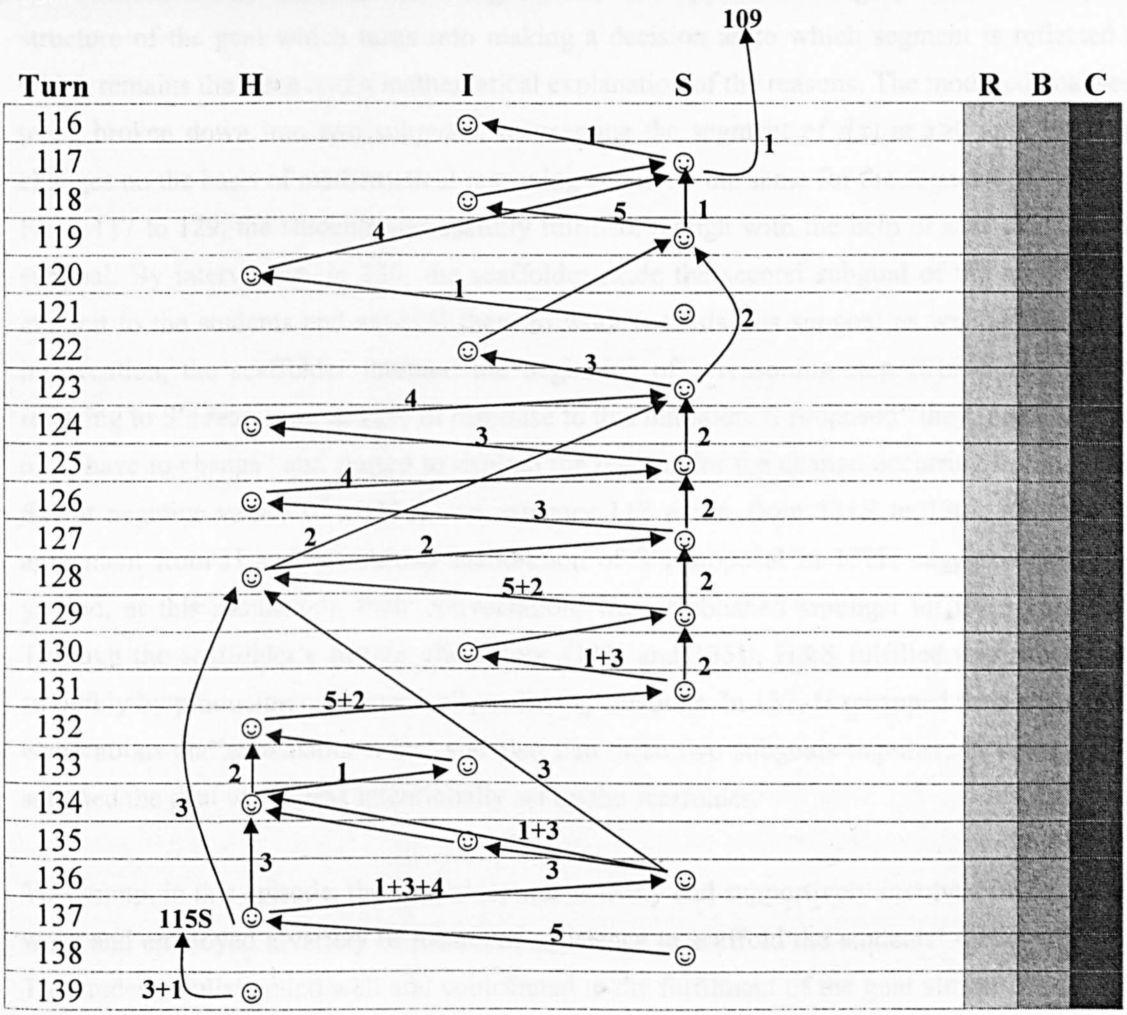


Figure 4.2. Interaction flowchart for episode 4 in task 2.

The legend: **1**: proposing; **2**: elaborating; **3**: explaining; **4**: quest; **5**: agreement; **6**: disagreement.

However, the scaffolder did not want to lose S's developing line of reasoning and intervened in 122 to get the elaboration, which was started in 119S, resumed. This attempt was also successful and took S back to her line of reasoning. In response to the scaffolder's elicitation, S gave explanation to the scaffolder (category 3 arrow from 123 to 122) and also the elaboration resumed (category 2 arrow to 119). These elaboration types of utterances continued for quite a while (see vertical category 2 arrows in the utterances of 131➔129➔127➔125➔123). While these elaborations took place, S also developed explanations to H's quests (see category 3 arrows between 125➔124 and 127➔126). Therefore H played an important role for S's developing elaborations. In fact, these elaboration types of utterances proceeded with the help of H who questioned S (e.g. category 4 arrows emanating from 124 and 126) and provided inputs and feedbacks (two category 2 arrows from 128 to 123 and to 127; category 5+2 arrow from 132 to 131). Although S seemed more active between the utterances of 123-129 in giving explanations and doing elaborations, H soon got attuned to S's developing insights (e.g., 128H and 132H).

When the scaffolder first set the goal to the students in 116, he did not ask for mathematical reasoning for the changes occurring in the graph of $f(x)$. However, through the scaffolder's interventions and the students' following actions, there appears a change (or modification) in the structure of the goal which turns into making a decision as to which segment is reflected and which remains the same *and* a mathematical explanation of the reasons. The modified goal seems to be broken down into two subgoals: to examine the segment of $f(x)$ at $x>0$ and decide if it changes on the basis of mathematical reasoning and to do the same for the segment of $f(x)$ at $x<0$. From 117 to 129, the students successfully fulfilled, though with the help of scaffolder, the first subgoal. By intervening, in 130, the scaffolder made the second subgoal of the modified goal explicit to the students and assisted them to work towards this subgoal as well. In fact by this intervention, the scaffolder initiated the beginning of a reasoning step (Chi et al., 2001) by referring to S's reasoning in 129. In response to this initiation, S proposed "they [negative values of x] have to change" and started to explain the reasons for the change occurring in the graph of $f(x)$ at negative values of x (131) – a category 1+3 arrow from 131S to 130I. An immediate agreement from H and her further elaboration of S's proposal in 132H suggest that common ground, at this moment in their conversation, was established amongst all three participants. Through the scaffolder's further elicitations (133I and 135I), H&S fulfilled this subgoal quite smoothly by producing mathematically valid explanations. In 137, H recapped the results of their elaborations and explanations; and she also tied these two subgoals together. In doing so H&S satisfied the goal which was intentionally set by the scaffolder.

To sum up, in this episode, the scaffolder was actively and supportively involved in the students' work and employed a variety of means of assistance to scaffold the students' works and efforts. The students collaborated well and contributed to the fulfilment of the goal almost equally (eight utterances from H and nine from S). Therefore it can be said that this episode was a product of all the participants' joint efforts and contributions.

In terms of epistemic actions, while H&S worked towards the fulfilment of the goal, they amended their 'reflecting' method and overcame the deficiency involved in this method. Hence they have achieved the formation of the 'reflecting' method which was used to draw the graph of $f(|x|)$ by using the graph of $f(x)$ (see 137H – 139H). This method was undoubtedly new to the students and constructed through the utilisation of existing knowledge artefacts that H&S have at their disposal. These artefacts involve features of absolute value (e.g., 128H and 136S), of linear functions (e.g., 119S), of symmetries (which was referred to as "changes" in 136S and 137H) and of Cartesian grids in terms of x- and y-intervals (134H). In order to construct this new method, throughout this episode, the students pieced these artefacts together and established some links amongst them (e.g., 134 and 136). Both of the students functioned at the same epistemic level and undertook constructing actions; building-with actions in the previous episode have been transformed into constructing actions in this episode. From 117S onwards the students refined their vague arguments and reasoning of the third episode and consequently they developed a mathematical basis for their 'reflecting' method in this episode. While they were not enriched with a new structure in the previous episode, they were now in this episode with a complex structural knowledge that is the 'reflecting' method. The complexity of the 'reflecting' method is related to the fact that it involves some other knowledge pieces as parts such as features of absolute value, symmetry, linear functions and Cartesian grids.

This episode has a unique characteristic in that all of the utterances, including the ones from the scaffolder, are classified as involving constructing actions as can be seen in the right hand side of the flowchart. This classification should not be taken to mean that the scaffolder has also constructed the 'reflecting' method. Then an inevitable question is how the scaffolder's utterances are considered to involve constructing actions despite the fact that construction of the new method occurs on the part of the students? In the later chapters (Chapter 5 section 3.2. and Chapter 7 section 2.) I will attend to this issue and discuss the rationale behind my considerations in great detail. However, I feel it useful to summarise my argument here that construction is a continuous process and involves increasing clarification and progressive evolution of the initial form of a to-be-constructed structure. Construction is achieved through constructing actions which are not discrete and dispersed but rather strongly related to each other. I believe that in this episode even when the scaffolder was talking and contributing and the students were attending to him, the students' construction still continued and thus the scaffolder's utterances represented constructing actions on the part of students. I will argue later that an otherwise position is bound to view constructing actions as discrete and dispersed; and as only limited to a student's specific utterances in isolation from the others. The danger involved in this view is that construction is not considered as a continuously developing process and is reduced to the individual's isolated verbalisations; and this leads us to draw a major distinction between the social and cognitive processes (see Chapter 5, section 3.2. and 4. more on this).

Of course that is not to say that construction continues under any circumstances when the students are not participating. If one is not actively involved in the activity, if one's contribution is not at the same level with others, then we can assume some discrepancies in the epistemic actions, in which case the person's epistemic actions are probably not aligned with the other's or not at the same level with the others. This is exemplified in the following episodes. Given that the students were actively involved in the activity, shared the same goal and the same perspectives within a common mathematical activity in this episode, it is fair to claim that the students were constructing at the moments when they were attending to the scaffolder's utterances even if they were not talking. In the following episodes if the scaffolder's utterance is labelled as constructing action, it should be viewed in the light of this argument.

Episode 5

140H: Look S, this graph is also symmetric in the y-axis...
141S: We took the symmetry of II in the line of y=2
142H: I mean look at the graph of $f(|x|)$ itself... you said that every graph of $f(|x|)$ should be symmetric in the y-axis
143S: Yes all of the graphs were symmetric in the y-axis...and this graph is also symmetric in the y-axis.
144H: So, I think this shows that our graph is correct!
145I: Can we say that this graph is accurate just because there is symmetry in the y-axis?
146S: No I don't think so!
147H: I think we can say... well, at least I can say so
148I: How can you be so sure H?
149H: When we were first drawing the graphs of $f(|x|)$, we said that y takes the same value for x=1 and -1 and likewise x=2 and -2 and so on...
150S: Yes I remember...
151H: Look, if the others are correct, then this should be so...
152I: Why?
153H: Because... look, for example, in this graph [of $f(|x|)$] we can see that for the different values of x, y took the same value because... umm because x is in the absolute value sign
154S: Yeah! I agree with H.
155I: Hang on! I got a problem here! What do you mean 'for the different values of x, y take the same value'?
156H: I mean if for example, x is -1 and 1... y take the same value as x is in the absolute value sign, negative or positive won't make any difference!
157S: Yeah, it is the same
158I: Is it the reason that you think the graph of $f(|x|)$ is symmetric in the y-axis?
159H: Yes, that's why these graphs must be symmetric in the y-axis...
160S: This symmetry exists in all of the graphs that we have drawn!
161H: For example in this graph [she refers to the graph that they obtained in question 4-B] this and that are symmetric
162I: This and that? OK let's name the rays on the graph as I, II, III so that I can follow you [see Table 4.2. – C].
163H: In this graph, I and II are symmetric in the line of y=2; besides II and III are also symmetric in the y-axis...
164S: Yes, true... Symmetric
165H: Yes, they are

Episode 4 ended when the students redrew the graph of $f(|x|)$ that they had obtained inaccurately in episode 3. Episode 5 starts with H's realisation that this graph was also symmetric in the y-axis (140H). Later in an attempt to further elaborate what she realised, she related this 'y-symmetry' aspect of the particular graph of $f(|x|)$ to the other graphs of $f(|x|)$ and proposed this as an indication of the accuracy of the redrawn graph of $f(|x|)$ (see the interaction flowchart presented in Figure 4.3. where category 1 arrow links 144 to 142 and thus, indirectly, to 140). When all three utterances of H in 140, 142 and 144 are taken together, it can be appreciated that her emphasis on the 'y-symmetry' aspect of the graph is not a discrete recognition of an 'obvious' feature of the graph but, rather, she appears to have considered the 'y-symmetry' aspect as a defining property of the graphs of $f(|x|)$. The scaffolder realised the importance of H's realisation and probed into her understanding rather than simply neglecting her proposals and elaboration as can be inferred from 145I and 148I.

One may argue that I am reading too much from the scaffolder's utterances. My answer to this argument is a certain 'no'. In order to provide the rationale behind this answer, I have to go into some details with regard to the graphs of $f(|x|)$. A graph of $f(|x|)$ can be obtained from the linear graph of $f(x)$ by means of two interrelated methods: (1) reflecting $f(x)$ at $x<0$ in the horizontal line crossing the y-intercept; or (2) reflecting $f(x)$ at $x>0$ in the y-axis and also deleting $f(x)$ at $x<0$. The students had already constructed the first method at the end of the fourth episode. The scaffolder was undoubtedly aware of these two methods and his realisation of the importance of H's utterances stemmed from the fact that H started talking about the second method, though not necessarily being really aware of doing so. The scaffolder monitored this based on H's explanations and proposal (140-144), and intended to get her to develop this second method. That is why he was not simply ignoring H's verbalisations. As a result, he set a goal in 145 and 148, that is, to justify the 'y-symmetry' as an indication of the accuracy of $f(|x|)$. He probably expected that in the course of this justification that H would construct the second method since in the fourth episode the students constructed the first method through mathematical justifications. Indeed the scaffolder was proved right in his expectation as H constructed the second method by the end of this episode.

Throughout this episode H was certainly far more actively involved into the activity than S was. This is clear from the interaction flowchart where S's contributions are minor usually in the form of agreement (see category 5 arrows emanating from S) and shallow explanations (e.g., 143 and 160). At the same time, H, in the course of their interaction, was barely referring to her partner but, rather, to her previous utterances, to the scaffolder's or to both (see the vertical arrows). H was constructing the second method while trying to satisfy the goal set by the scaffolder. H's constructing actions between 148 and 154 were scaffolded through deep reasoning questions (Graesser and Person, 1994) (see 148, 152 and 155). These questions seemed to bring about an intellectual push on the part of H; and in order to provide explanations, she utilised earlier utterances which were indicated in the flowchart with long category 3 arrows from 149 to 12H and 13S[3]; from 153 to 127 and from 156 to 149. In this sense, H seemed to be compelled to bring the insights that she gained in the earlier episodes into this episode and these insights were the essence of her explanation type of utterances. Between 146 and 157, there was no explicit reference to the 'y-symmetry' aspect of the graphs of $f(|x|)$ although H was explaining the reasons for this. In 158, the scaffolder referred to this and tried to tie all of the explanations and

[3] **12H**: … if we take x=1, $f(|x|)$ will be –3 again.
13S: Yes, for x=-1, $f(|-1|)$ will be -3 too.

elaborations together with the 'y-symmetry' aspect (a long arrow from 158 to 140). In 159, H's general statement indicated that she tied these explanations to the 'y-symmetry'.

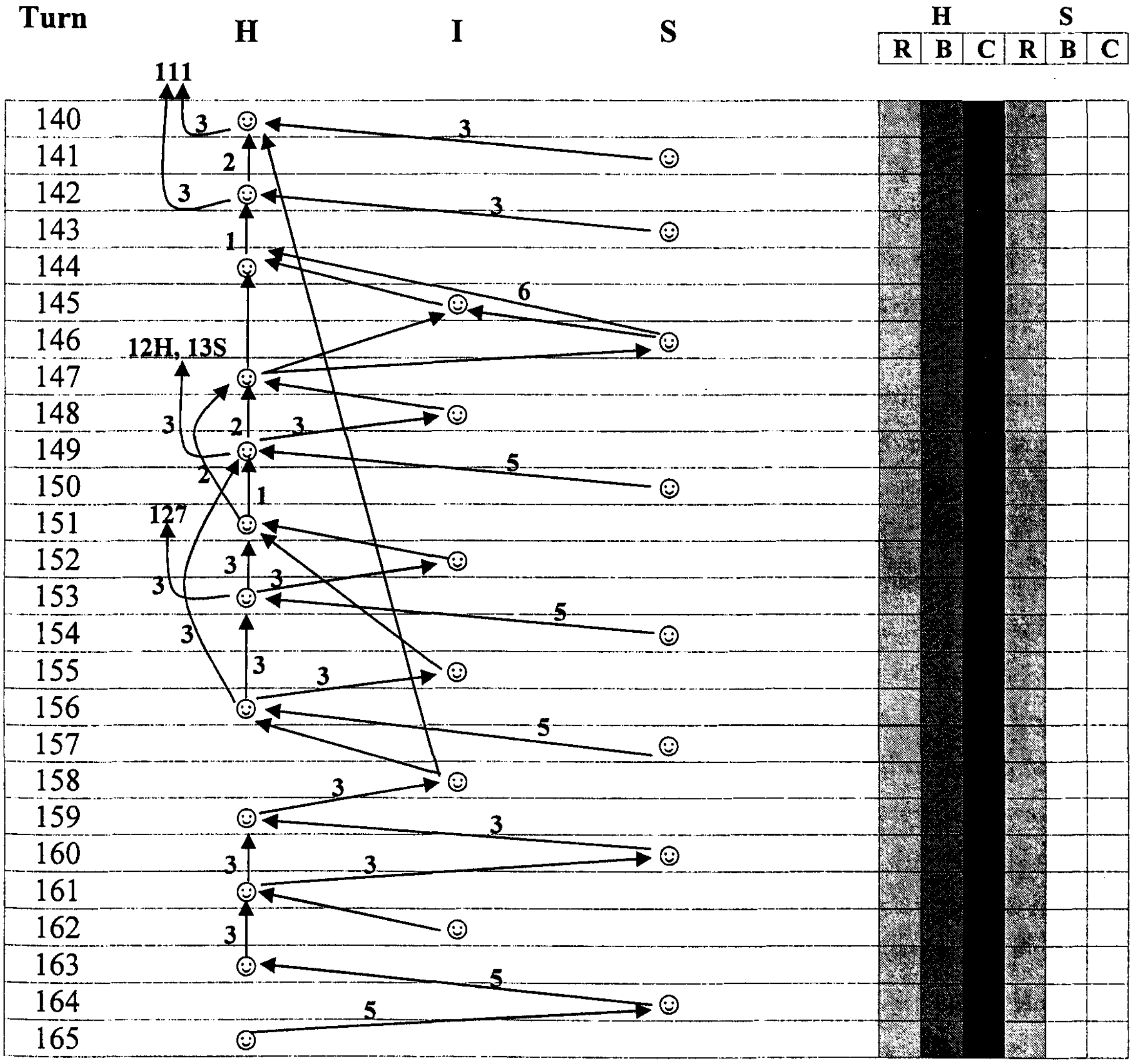


Figure 4.3. Interaction flowchart for episode 5 in task 2.

The legend: **1**: proposing; **2**: elaborating; **3**: explaining; **4**: quest; **5**: agreement; **6**: disagreement.

In terms of epistemic actions, this episode provides a very interesting picture in that two students acted at different epistemic levels, that is, H was at constructing while S at recognising, as depicted in the right hand side of the interaction flowchart. There are four reasons to justify this differentiation. First of all, while H actively participated in the dialogue and produced the essence of this episode, S was rather passive. Some may suggest that learning may occur not only through participation in a dialogue but also through observing or overhearing others participating in it (Clark, 1996; Schober and Clark, 1989). However, there is no evidence that S also constructed the second method. In fact as can be seen in the beginning of the next episode, when H suggested drawing the graph by reflecting in the y-axis (172), S asked for clarification (173S). This suggests to me that while H was enriched with a new knowledge structure, S could not have constructed the second method in this episode. However, that does not mean that S did not gain anything from this episode. What she heard in this episode, in my opinion, prepared her, and created a basis, for her construction of the second method in the next episode.

Second, only H attended to the scaffolder's deep reasoning questions and provided convincing mathematical arguments and explanations (see how often category 3 arrows in the flowchart emanate from H's utterances). However, in the fourth episode, in which both students were considered to be constructing, S (e.g., 136) attended to the scaffolder's deep reasoning question (135I) even though it was directed to H (134). In addition, S's contributions in this episode were usually agreement type of utterances and some shallow explanations which were mere approval of H's reasoning and explanations. S's agreement statements did not involve any further elaborations or explanations. Compare this with H's agreement type of utterances in the fourth episode. For example, H's agreement (132) to S's elaboration and proposal followed a further elaboration of S's proposal. In this sense S's contributions were in this episode not at the same level with H's.

Third, epistemic actions are subjective; while one student can see a deep relation in a structure, another may focus on the surface features. This is the case in this episode. For example, at the outset of this episode, H recognised the 'y-symmetry' aspect of the graph of $f(|x|)$ as a defining property and made use of it as a starting point to develop the second method, but S considered this as a result of their earlier actions (141S) which was common to all other drawn graphs (143S). Whereas H's recognition was part of her constructing action, S's recognition was a result of her realisation that this graph also had the same feature with the other drawn graphs. In this sense there is a differing perspective between the students. This difference can be observed in the way that recognised structures were utilised. For example, when H articulated that y takes the same value for ± 1 and related it to the justification of her proposal (149), S's agreement in 150 and her comment "I remember" suggested to me that she only identified this without relating it to H's proposal.

Finally, the divergence of epistemic actions becomes clearer when H shifted from using the pronoun 'we' to 'I' (147H). From this utterance onwards, H explored her understandings and gatherings (see the vertical arrows) rooted in the earlier parts of the activity (see the long arrows emanating from H pointing to the earlier utterances e.g., 140, 142, 149 and 153). In doing so, she achieved a new construction, the genesis of which can be traced back to the third episode and specifically to the utterances of 111, 112, 113 and 114. It is clear that H in this episode constructed a new method to view the graphs of $f(|x|)$. Despite the fact that there was not a complete articulation of this method, her successful attempt to draw the graph of $f(|x|)$ for question 4-D by reflecting $f(x)$ at $x>0$ according to the y-axis in the next episode corroborates this argument.

Episode 6

166I: OK, let's move on to the next graph given in [question 4-] D.
167S: The graph doesn't change for the positive values of x
168H: Yes
169S: But for the negative values of x
170H: We take the symmetry
171S: We will take symmetry according to the line of y=-2
172H: Oh, no need! I mean we can take the symmetry directly in the y-axis rather than drawing the line [of y=-2] or doing anything else... I mean let's take the symmetry of this part [of $f(x)$ on the right side of the y-axis] according to y-axis
173S: How do you mean?
174H: Look, I mean this is the y-axis right?
175S: Yes
176H: I am saying that we can take the symmetry of the ray on the right [side of the y-axis] and this symmetry is in the y-axis... OK, OK... Let me draw it and show you... [She draws the graph, see Table 4.2. – F]... Isn't this graph correct?
177S: Look, in all of the graphs, the ray on the right [side of the y-axis] remains the same... I mean it doesn't change! The ray on the left [side of the y-axis], which is the part corresponding to the negative values of x, is taken symmetry...
178H: Look I am saying the same thing... but this symmetry is like...
179S: OK that's what I am saying as well...
180H: Then why not taking its symmetry in the y-axis?
181S: Umm...[she pauses and draws the line of y=-2 onto the graph of $f(x)$; see Table 4.2. – F] yes...it doesn't make any difference...
182H: Yes this is the same thing
183S: The first part... I mean the part that doesn't change... if we take its symmetry in the y-axis, we also obtain the same graph [of $f(|x|)$]. Hmm... [She pauses] yeah...
184H: OK, eventually we take the symmetry in the y-axis, huh?
185S: Yes, actually we are saying the same thing!

The goal identified for this episode is for the students to draw the graph of $f(|x|)$ for question 4-D. From the interaction perspective, as can be seen from the flowchart presented in Figure 4.4. below, there appears collaboration between the two students; see the flowchart where the arrows emanating from each student and pointing to one another's previous utterance. Both students attended to each other's explanations and proposals, which are the dominant categories in their interaction (see the arrows carrying the labels of 1 and/or 3). However, the scaffolder did not intervene at all. Even though he was still monitoring the students' actions and was analysing their performance, he did not feel a need to intervene simply because H&S were able to regulate themselves and create their own direction of interaction. In spite of the collaboration between the students, H seems to be on the 'driving seat' in the sense that she produced most of the proposals (six category 1 arrows from H but only two from S), invited her partner to attend to these proposals (176 and 180) and thus gave direction to their interaction. As a result of their interaction, S, towards the end of this episode, constructed the second method, 'y-symmetry', to view the graphs of $f(|x|)$.

In terms of epistemic actions these two students function once again at different epistemic levels. At the beginning of this episode (166-171), both students' actions are identified as recognising and building-with. However, these actions are qualitatively different and therefore in the interaction flowchart these epistemic actions are represented for each of the individuals in different columns. In her initial three utterances of 167, 169, and 171, S recognised the aspects of 'reflecting' method constructed in episode 4 (see a long arrow from 167 to 136) and built the intended graph for question 4-D on the basis of these recognitions (see vertical arrows of 171➔169➔167 and thus to 136). However, although H also recognised the same aspects, these recognising actions, as it were, occurred in relation to her '*y*-symmetry' method constructed in episode 5 as can be seen in 172 (a long arrow from 172 to 159). She used these recognitions not only to articulate her '*y*-symmetry' method (172) but also to build-with them the intended graph successfully (176). This provides evidence that she has already constructed this '*y*-symmetry' method in episode 5 and she was able to recognise and use it. However, the story for S is different between 172 and 179. Her epistemic actions in this interval seemed to me as being 'discrete' recognitions. 'Discrete' here is used in the sense that S only recognised knowledge structures involved in H's explanations in their own right without being connected to any building-with or constructing actions. It seems to me, for example, S recognised the knowledge structures which were mentioned in H's explanations, such as "symmetry … in the *y*-axis" (172); and "this is the *y*-axis" (174). However, asking 'how do you mean' (173) and stating the 'reflecting' method once again in 177 suggest that S's recognitions did not co-join to make sense of H's construction (i.e. '*y*-symmetry') as a whole precisely because H did not construct this method yet.

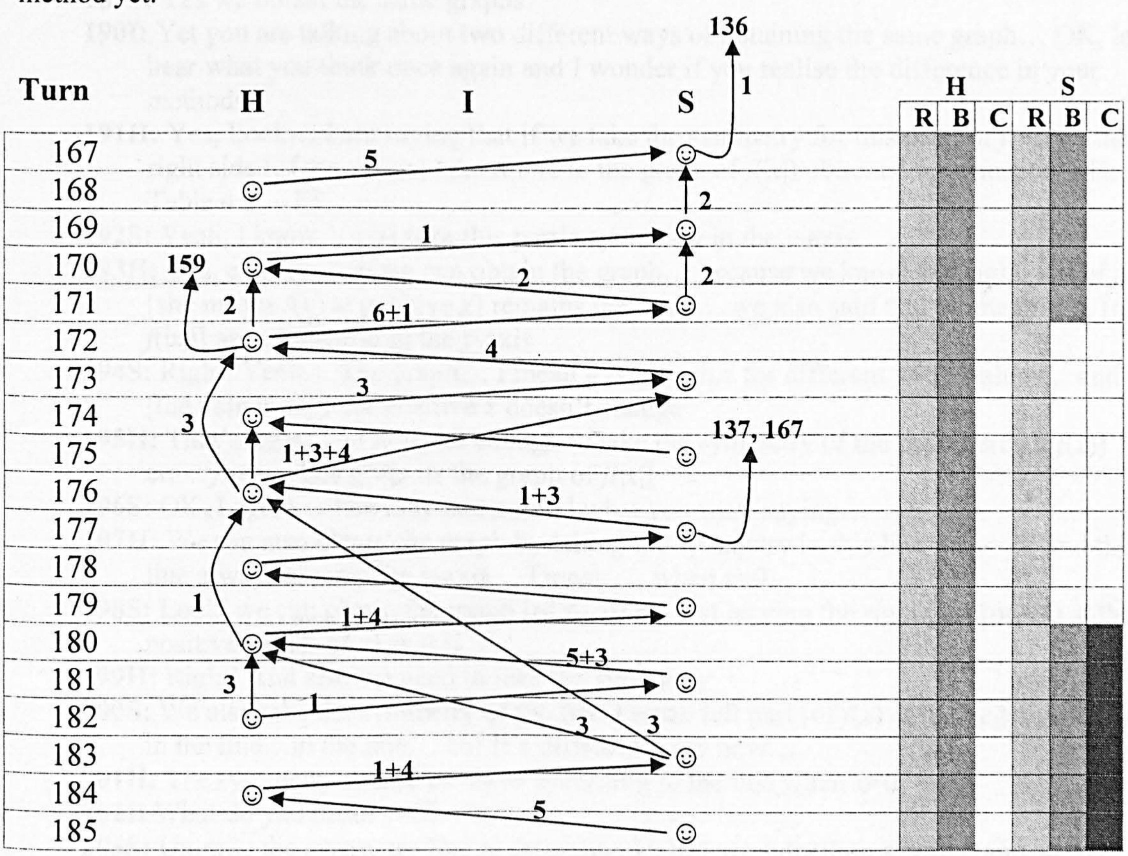


Figure 4.4. Interaction flowchart for episode 6 in task 2.

The legend: **1**: proposing; **2**: elaborating; **3**: explaining; **4**: quest; **5**: agreement; **6**: disagreement.

In the interval 180-185, S seemed to be constructing the '*y*-symmetry' method, which could be traced especially through her utterances 181 and 183 where two long pauses were noted. It is likely that during these two pauses she might have reflected on H's explanations, on H's proposed graph of $f(|x|)$ and on her method (note that S drew the line of y=-2 onto the graph of $f(x)$ in 181). I believe that during these reflections S reorganised her existing knowledge artefacts of symmetry in the y-axis and in the line of y=-2; and she also related them to each other and to the graphs of $f(|x|)$. Although S's construction of the 'y-symmetry' method was not clearly observable in her verbalisations (recall that she paused for quite a while in 181 and 183), her construction of this second method is evidenced in the next episode where she explained how to draw the graphs of $f(|x|)$ by using the 'y-symmetry' method in collaboration with H. It seems that during this episode, H started consolidating her two constructions by means of recognising and building-with actions coming about through her explanations. H's consideration of the two apparently different methods by relying on the product graph as if they were the same suggest that her new constructions were fragile and needed to be consolidated. The issue of consolidation will be explored in Chapter 6.

Episode 7

186I: Actually you are not saying the same things...
187S: But we reach the same graph at the end...
188I: Look, you obtain the same graph of $f(|x|)$
189S: Yes we obtain the same graphs
190I: Yet you are talking about two different ways of obtaining the same graph... OK, let's hear what you think once again and I wonder if you realise the difference in your methods.
191H: Yes, Look... I am saying that if we take the symmetry for this part [of $f(x)$] on the right side [of the y-axis] [she refers to the graph of $f(|x|)$ obtained for question 4-D; see Table 4.2. – F]
192S: Yeah, I know... you take this part's symmetry in the y-axis...
193H: Yes, exactly... so we can obtain the graph... because we know that right part of $f(x)$ [she means $f(x)$ at positive x] remains the same... we also said that all the graphs [of $f(|x|)$] are symmetric in the y-axis
194S: Right! Yeah... The graph... I mean y is the same for different two x values... and [the values of] y for positive x doesn't change
195H: That's right... so it is just enough to take the symmetry of the right part [of $f(x)$] err...yeah... this gives us the graph of $f(|x|)$
196S: OK, I agree... I already understood what you were saying...
197H: We can also obtain the graph by taking the symmetry in this line [of y=-2] and this line always crosses the y-axis... I mean ... when x=0.
198S: Look, we can obtain the graph [of $f(|x|)$] by first leaving the right part [of $f(x)$ at the positive values of x] as it is
199H: Right! And also we need to take the symmetry
200S: We also take the symmetry of the part I mean left part [of $f(x)$ at the negative values] in the line...in the line... oh! It's difficult to say now...
201H: The symmetry should be taken according to the line when x=0
202I: What do you mean x=0?
203S: Umm... the symmetry line is determined when we substitute x with 0 and so we obtain the value of y
204H: Look, we can say that the line always crosses the y-axis and the crossing point is determined by substituting x with 0 in the equation...

By the end of the previous episode, although H&S constructed two different methods to view the graphs of $f(|x|)$, they tended to regard these two as if they were the same 'thing' (182 and 185).

This suggests that they were not able to draw the distinction between these two methods. Although the scaffolder did not intervene in the previous episode, he was still following the students and monitoring their actions based on their interactions. He analysed their performances and felt a need to draw students' attention to the differences involved in these two methods. As a result, at the very beginning of this episode, he intervened and directly stated that they were "not saying the same things" (186). S's response, in 187, to this intervention and H's silence imply that they were too occupied with the accuracy of the sketched graphs rather than the methods themselves. In 190, the scaffolder explicitly set a goal to the students, that is, to state these two methods and point out the differences. Both of the students in collaboration with each other fulfilled this goal by giving an account of first the '*y*-symmetry' method in the interval 191-196, and second the 'reflecting' method between 197-204.

At this point it is interesting to observe that the students relied on the graph of $f(|x|)$ sketched for question 4-D when verbalising their constructed methods. Their articulations, although they included some general elements, focused on this specific graph and referred to the specific 'parts' and 'lines' observed within the graph itself, e.g., "symmetry for *this* part" (191); "*this* part's symmetry" (192); and "taking the symmetry in *this* line" (197). In addition, their articulations of the newly constructed methods were not general mathematical statements. Although they were able to use these new constructions to sketch the intended graphs (especially H), they were experiencing difficulties to articulate their thoughts and seemed to be dependent on the specific graph. In this sense, recognition and use of newly constructed structures seem to precede the development of a language to express these constructions. This is quite an important issue and will be attended to in Chapter 6. In spite of the fact that these articulations were not mathematically general and were not well expressed, they at least provided enough evidence that these two students achieved the intended constructions and thus fulfilled the main goal of the activity.

An important feature of this and the previous episode is that scaffolder gradually handed the responsibility over to the students. The scaffolder's interventions in this and the previous episode occurred infrequently. When these interventions are compared with that of the fourth and fifth episodes, it may be appreciated that he withdrew his assistance to a great extent. This withdrawal did not happen suddenly as he, based on his monitoring and analysing, still provided some minor assistance by, for example, pointing out that the two methods were different. However, his assistance did not go beyond that and he did not provide any direct input because the students were able to discern the difference and to give an account for each method. Therefore, the students can be said to become relatively self-regulated after one or both of them are acquainted with the new constructions.

With regard to epistemic actions, the students acted at the same epistemic level displaying recognising and building-with actions. In their explanations, they repeated and summarised what they have already told earlier and articulated their constructions of the two methods, though not in a 'perfect' mathematical language. In this episode, both of the students were consolidating their constructions by recognising already constructed structures throughout the fourth, fifth and sixth episodes and building explanations with these recognised structures.

2.3. H&S's verbal protocols on the fourth task

In this section, I present and analyse H&S's verbal protocols on the fourth task which is given in Appendix 7. H&S' performance on this task can be viewed as having two distinct stages: (i) from the outset of their work until 145H the point at which H&S were clearly having problems, were 'sticking to substituting' and were not developing a 'better' method to sketch/draw the graphs of $|f(|x|)|$; (ii) from 146I, the point from which the scaffolder regulated H&S actions and foci of attention towards developing a new method. In the first stage, the scaffolder gave H&S almost complete freedom by intentionally limiting his assistance in order to support the students' autonomy and in order to observe how far the students could go without his assistance. In the excerpts below, I present two episodes (episode 1 and 2) from the first stage to give the reader an idea of what H&S were able to do without the scaffolder's assistance. However, having observed the students apparent difficulty in developing a new method to draw the graphs of $|f(|x|)|$, the scaffolder suggested returning back to the first question of the fourth task. Following this suggestion, there occurred five more episodes; episodes 3 to 7. In episode 3, the scaffolder pointed out features of $|f(x)|$ and $f(|x|)$ that H&S constructed in the first two tasks and suggested that they keep these in mind in this task. He then invited H&S, in episode 4, to develop a method by using these earlier constructions. In episodes 5 and 6, H&S constructed a new method by using their constructions of $|f(x)|$ and $f(|x|)$. The new method was used to draw the graphs presented in the fourth question of the task which is detailed in episode 7. Please note that I provided those graphs, in Table 4.3. below, which H&S sketched during their work on the fourth task and which are immediately relevant to the following episodes.

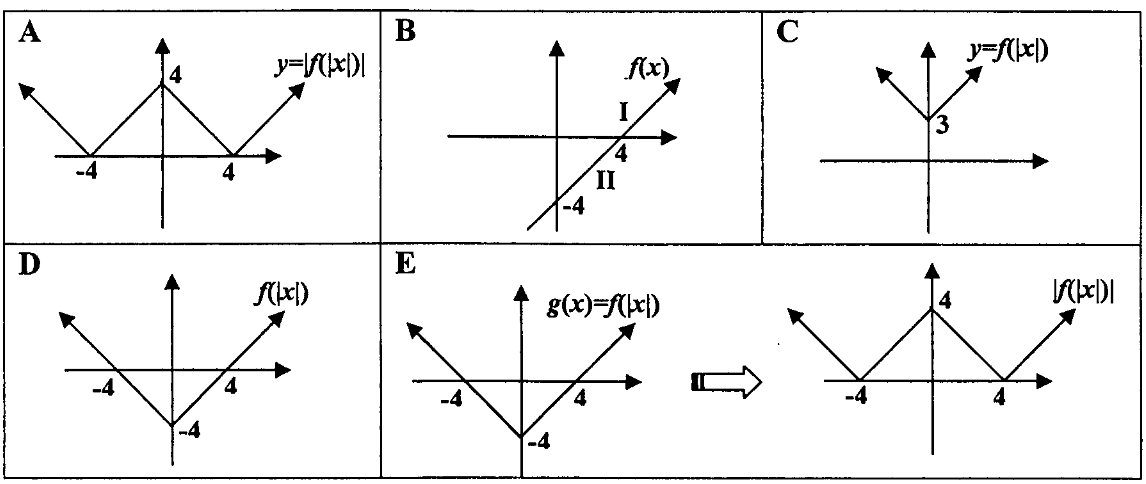


Table 4.3. The graphs that H&S obtained in the fourth task.

Episode 1

The students decided to draw the graph of $|f(|x|)|$ by substituting, but they encountered a problem with regard to whether negative values of x should be substituted to obtain the intended graph.

After a lengthy discussion they agreed to include negative values of x into the graph of $|f(|x|)|$. The students draw the intended graph asked in the first question correctly (see Table 4.3. – A). They then moved on to the second question and searched for some relationships between the graph of $f(x)$ (see Table 4.3. – B) and of $|f(|x|)|$ (see Table 4.3. – A) as asked in the second question. I pick up their conversation at the moment when H started to make comparisons between the graph of $f(x)$ and $|f(|x|)|$.

96H: Well, in the $f(x)$, y takes negative values but in the graph of $|f(|x|)|$, values of y are always positive, they don't take any negative values...
97S: If we say I and II...[see Table 4.3. – B] then part I... I mean positive [values of] y remain the same... but y takes negative values in part II so it is taken symmetrically in the x-axis...
98H: Also when x=0...
99S: We obtain the part of graph in the first quadrant. If we take the symmetry of the parts in the first quadrant according to y-axis... and then take symmetry once again for the negative values...
100I: H, did you understand S?
101H: I understood her, but it seems too long, we will first take the symmetry in the x-axis
102S: Then in the y-axis... negative [values of] y should be taken symmetrically in the x-axis, as there cannot be any negative [value of] y...

......

107S: I think part II is taken symmetrically in the x-axis...
108H: Yes, right.
109S: And then... the whole graph in here [first quadrant] is symmetric in the y-axis... after that we take the symmetry of the negative y according to x-axis...

......

124H: I didn't understand how this part of graph [the line segment in the graph of $|f(|x|)|$ on the second quadrant] came here
125I: You didn't understand how the graph in the second quadrant obtained?
126H: No, I don't see any relation between these two [$f(x)$ and $f(|x|)$] in the second quadrant!
127S: We may understand in the other example... let's go on to the next question, huh?
128H: Yeah, we'd better continue, the next question may help us if we look at them together.

This episode started after the students read the second question whose goal was to find a pattern between the specific graphs of $f(x)$=x-4 and y=$|(|x|\text{-}4)|$. The students seemed to be aligned with the predetermined goal of this question and they searched for a relationship between these two graphs. Their effort to find a relationship continued for 38 turns (91S and 128H, not all shown) throughout this episode. As can be seen from the presented utterances, the students reported the similarities and differences between the graphs and strived to make sense of the changes occurring in the graph of $|f(|x|)|$ in relation to the graph of $f(x)$. While doing this they constantly recognised ostensible features of the graph of $|f(|x|)|$ and tried to use them to obtain the target graph from the graph of $f(x)$. The same kind relationships were recognised by some other students who took part in this study and who have been able to construct a method based on these recognitions. However, H&S were unable to suggest even an initial method on the basis of their observations between these two graphs in this episode. It is probably because the construction of the intended method was beyond these two girls collaborative efforts (see episode 2 below). At the end they decided to move on to the next question (question 3) with the hope that drawing another graph of $|f(|x|)|$ and considering these two graphs of $|f(|x|)|$ together may help them understand the relation (126, 127 and 128). Therefore the students adopted a new goal, that is, to draw the next graph of $|f(|x|)|$ and turn back to this question in order to make sense of the relationship between $f(x)$ and $|f(|x|)|$.

The scaffolder's interventions all through this episode were rather slight (seven interventions, not all shown) and aim to monitor (e.g. 125I) or to encourage the students to share their understandings with each other (e.g. 100I and 95I: "Tell each other whatever you see"). The reason for the slight interventions on the part of the scaffolder was because he wanted to give some autonomy to the students and to observe how far they could go without his assistance.

Episode 2

The students obtained the graph of $|f(|x|)|$ accurately for question 3; see Table 4.3. – C. Then they started talking about this graph by also considering the graph obtained for the question 1 (see Table 4.3. – A).

> **133H:** Look I think the first part [of $f(x)$ at $x>0$] always remains the same… oh does it?
> **134S:** Yes
> **135H:** But in the first question there is something else… I mean a line segment
> **136S:** This graph is also symmetric in the y-axis. But I don't know how it helps us!
> **137H:** We know that the part of $f(x)$ over the x-axis remains the same, right?
> **138S:** Yes… it remains the same and also because y never takes negative values, they are taken symmetrically in the x-axis
> **139H:** But wait! We said this part [of $f(x)$ over the x-axis] didn't change but in this graph [see Table 4.3. – C], it doesn't obey this rule… look at the graph… the part of $f(x)$ in the first quadrant remains the same
> **140S:** Yeah I know, there was a line segment in the first graph [see Table 4.3. – A]
> **141H:** I don't think we can ever understand how to use $f(x)$ to draw the graph of $|f(|x|)|$
> **142S:** The first graph was something like W-shaped… but this graph is V-shaped
> **143H:** They are totally different! How can we speak in a general way? Even this question made things worse rather than helping us.
> **144S:** I think we better stick to substituting; we can answer the next questions by substituting
> **145H:** Yeah after all it is definitely working to draw $|f(|x|)|$

In this episode, the students' goal was the same as the predetermined goal of the third question, that is, to clarify and further develop the initial relationships observed/discovered in the first two questions. After drawing the graph of $|f(|x|)|$ for the third question, the students considered two graphs of $|f(|x|)|$ together. They were certainly aware of the main goal of this task, which was to develop a (general) method to draw the graph of $|f(|x|)|$ by using the graph of $f(x)$ (see 141-143). This awareness was, I believe, a result of their work on the first and second tasks, which had the same organisational structure as the fourth task. In this sense, the students' goals and efforts towards the fulfilment of these goals were not only influenced by the present task on which they were working but also by the previous tasks. For example, in the second task, after drawing the intended graph for the third question, their comments and/or reflections on the relationship between graphs of $f(|x|)$ and of $f(x)$ were limited to only three turns and later they immediately passed on to the fourth question. In addition, they were not, in the second task, concerned much with finding a general method to draw the graphs presented in the fourth question until they started working on the graph given in question 4-B. Nevertheless, in this task, H&S started reflecting on the graphs of $|f(|x|)|$ and on the relations to $f(x)$ right from the second question until the end of this episode, continuing for 55 turns. They clearly strived to develop a general method, which was relatively absent in the corresponding parts of the second task. This is because they knew the organisational structure of this task and were aware that they needed to develop a method to draw the graphs presented in the fourth question by using a 'short-cut' method rather than substituting. This suggests that the students' past experiences on the earlier tasks had a profound impact on the way in which they approached to the present task.

In this, and also the previous, episode, the students collaborated to fulfil their goals in the sense that they attended to each other's proposals and ideas. In spite of their collaboration and painstaking efforts between 91-145, they failed (see 144 and 145) to find a general method and this brought about an understandable frustration which could be realised in their comments such as "I don't think we can ever understand" (141H). It is true that they recognised some important properties of the graphs of $|f(|x|)|$ (e.g. symmetry in the y-axis), but failure to make sense of these recognitions only contributed to their frustration ("I don't know how it helps us"; 136). Their frustration was so strong that they even did not mention about developing a new method after doing some work on the fourth question, which was what they did in the second task, but decided to draw them all by substituting.

It is important to note that the students' adopted goals were the same goals as the predetermined goals of the first three questions of the fourth task. They only managed to satisfy the first goal by successfully drawing the intended graph of $|f(|x|)|$ which was relatively easy to achieve. However, satisfaction of the other goals was strictly related to construction of a new structure. Therefore, consideration of the first and second episodes together demonstrates that construction of the intended structure of $|f(|x|)|$ through the specified path by the task was, at this point in their development, beyond H&S's collaborative efforts.

Episode 3

146I: OK, let's return to the first question once again and talk about it together, OK?
147H: I knew you would suggest that
148S: We are useless!
149I: No, the task is really hard, but you should realise a subtle point, right? In fact, even if you didn't formulate a general rule, your discussions would help you understand the main idea. OK, what is the graph that we are expected to draw? It is $|f(|x|)|$, isn't it?
150H: Yes
151I: You know how to obtain the graph of, say, $|f(x)|$, $|g(x)|$ or $|h(x)|$, don't you?
152H: Yes, we know that
153I: What were you doing? The part over the x-axis
154H&S: Remains the same...

155I: And the part of $f(x)$ under the x-axis…
156H&S: is taken the symmetry
157I: So you know how to get to the graph of a function given in the absolute value
158H: Yes
159I: OK, you also know how to obtain the graph of $f(|x|)$, don't you?
160H: $f(|x|)$? Yes we know…
161S: If we draw the graph of $f(|x|)$ with the help of $f(x)$… it was symmetry in the y-axis as same negative and positive values of x are matched with one value of y…
162H: So the part of $f(x)$ at the positive values of x-remains the same
163S: Yes it remains the same and its symmetry is taken in the y-axis
164H: We know all these things!

This episode began with the scaffolder's suggestion to return to the first question and working this task out together (146). H&S's frustration can be seen in their responses to this suggestion (147&148). The scaffolder was conscious of and tried to control their frustration (Wood et al., 1976) and in fact frustration control can be seen in his feedbacks and in his encouraging and constructive comments right from the beginning of this episode (149) and throughout the rest of this protocol (e.g. 176).

There appear several reasons for the scaffolder's suggestion to return back to the first question and to work together. First of all, he aimed to establish common ground on which they could communicate. It was obvious from H&S's work on the second task (see section 2.2. above, episode 2) that unless all participants share a similar perspective it is unlikely that the scaffolder assists the students in the way that he intends. Secondly, he wanted to formulate and try a new path other than the one specified by the predetermined goals of the task so that H&S could make progress towards the intended construction through this new path. As already mentioned in episode 2, H&S have, so far in this protocol, worked in alignment with the predetermined goals and tried very hard but they could not achieve them. This indicates a necessity of another way to approach to the graphs of $|f(|x|)|$. The scaffolder, as a result of his monitoring and analysing, was aware of the necessity of a new path. The new path that the scaffolder had in mind was to graph $|f(|x|)|$ through the successive application of $|f(x)|$ and $f(|x|)$ to the graph of $f(x)$. In re-emphasising their target as to draw $|f(|x|)|$ and later pointing out the structures of $f(|x|)$ and $|f(x)|$, he was actually hinting his method to the students. It is interesting to note here that, as can be seen in the succeeding episodes, the students developed a new method to draw the graph of $|f(|x|)|$ through the successive application of $f(|x|)$ and $|f(x)|$ rather than of $|f(x)|$ and $f(|x|)$ which was what the scaffolded had in mind. This is particularly important to show that the scaffolder followed up the students' perspectives rather than imposing the method that he had in mind.

After returning to the first question, the scaffolder got the students recollecting what they knew about the graphs of $|f(x)|$ and $f(|x|)$. He was sure that H&S already knew these two structures (see 151 and 159). However, his intention was to remind the students of these structures and thus prepare them for the new path that he had in mind. In doing so, he was also laying the 'building blocks' necessary for the achievement of the intended construction through the new path. The scaffolder was shaping the students' approach and thinking by bringing the necessary and essential structures (for his newly formulated path) to their attention and thus getting them to focus on these structures. As can be seen in the interaction flowchart of this episode as presented in Figure 4.5. below, the scaffolder was the regulative agent as most of the arrows pointed to his utterances in one way or another (see the 'I' column and the arrows pointing to the utterances in there). The students' utterances were basically explaining type of utterances (see category 3 arrows) and there was not even a single proposal from H&S to regulate their actions. Nevertheless the scaffolder tried to get the students' active involvement by 'fill-in-the-blank' kinds of requests (Graesser et al., 1995); see for example, 153 and 155.

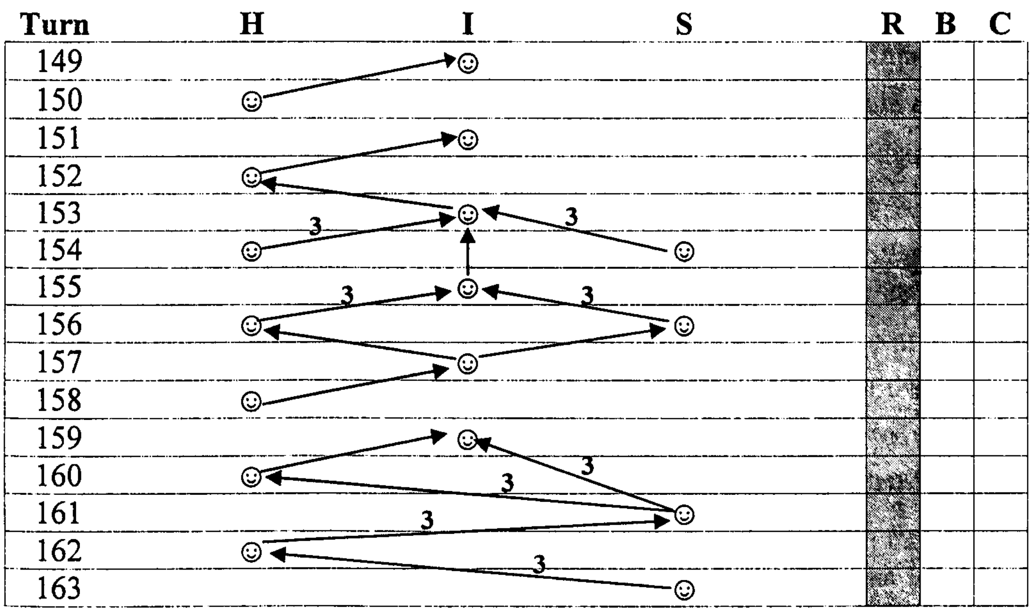


Figure 4.5. Interaction flowchart for episode 3 in task 4.

The legend: **1**: proposing; **2**: elaborating; **3**: explaining; **4**: quest; **5**: agreement; **6**: disagreement.

In terms of epistemic actions, the students were functioning at the level of recognising, as is indicated in Figure 4.5. This is not surprising given that the scaffolder elicited their earlier constructions which were already available to the students.

Episode 4

165I: OK, if you pay a closer attention to the equation… I mean look at the expression itself, [$|f(|x|)|$], it is a combination of these two [$|f(x)|$ and $f(|x|)$]. Do you see that?

166H: Yes, that's right. We already mentioned about this at the beginning…

167S: Yeah, this [$|f(|x|)|$] is a combination of $f(|x|)$ and $|f(x)|$… for example $|f(x)|$ never goes under the x-axis…

168I: OK, let's think about it and consider what you know. How can we use our knowledge to obtain this graph?

169S: Look, I think it makes sense… I mean $|f(x)|$ doesn't pass under the x-axis as y is always positive… and also the graph of $f(|x|)$ is symmetric in the y-axis… that is why the graph of $|f(|x|)|$ doesn't take negative value and is symmetric in the y-axis…

170H: Yeah, it makes sense now… look, if $|f(|x|)|$ is a combination of $f(|x|)$ and $|f(x)|$, can we think about it like a computation with parentheses?

171I: Computation with parentheses?

172H: I mean for example when we are doing computations with some parentheses like… let's say for example, (7-(4+2)), then we follow a certain order…

173S: Right, I understood what you mean… we need to first deal with the parenthesis inside of the expression, is that what you mean?
174H: Yeah, I think it is somehow similar in here, I can sense it but I am unable to clarify…
175S: I know what you mean but how could we determine the parenthesis in here?
176I: You both made an excellent point. OK, let's think about it together! In the expression of $|f(|x|)|$, can we think about the absolute value sign at the very outside of the whole expression as a larger parenthesis, which includes another one just inside

In the previous episode the scaffolder brought the structures of $|f(x)|$ and $f(|x|)$ to the students' attention as they were essential 'building blocks' for his newly formulated method in his mind. These two structures were, in episode 3, introduced to the students discretely without establishing any relation within the context of this task. However, at the very beginning of this episode, the scaffolder tied up these two structures together in relation to the structure of $|f(|x|)|$ (see the long arrow from 165 pointing at 149, 151 and 159 in the interaction flowchart in Figure 4.6.). He explicitly described the structure of $|f(|x|)|$ as a combination of these two structures in order to orient students to the important features (McArthur, et al., 1990) and to the possible relations amongst these three structures. In 168, he invited the students to think about how to obtain the graph of $|f(|x|)|$ by making use of what they already knew. In doing so, he in fact set a goal for the students by prompting them to develop a strategy about how to obtain the graph of $|f(|x|)|$ with regard to the structures of $|f(x)|$ and $f(|x|)$. These interventions prompted S to recognise some properties of the graphs of $|f(x)|$ and $f(|x|)$ in relation to the graphs of $|f(|x|)|$ that they drew earlier (167 and 169). However, the scaffolder neglected S's explanation and elaboration as her recognitions were rooted within the 'failed' path; see the long arrow from 169S to 102&138. His negligence was quite apparent in the flowchart where category 3 arrows pointing to 165I and 168I, but there was not even a single arrow from I to 167S nor to 169S. This negligence could be construed as an indication of his endeavour to get the students adopting a new path different from the 'already-tried-and-failed' one.

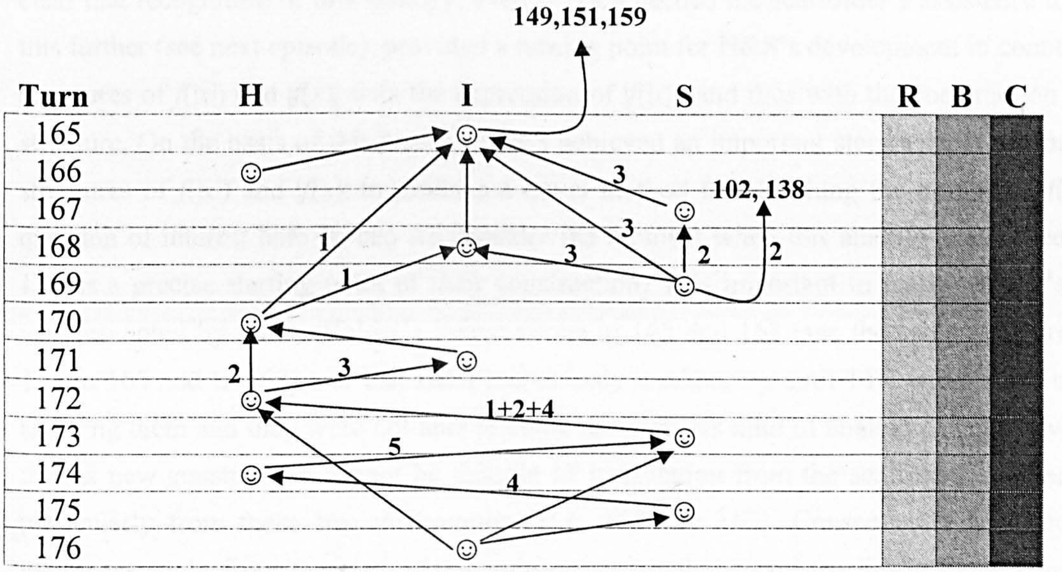


Figure 4.6. Interaction flowchart for episode 4 in task 4.

The legend: **1**: proposing; **2**: elaborating; **3**: explaining; **4**: quest; **5**: agreement; **6**: disagreement.

170H was a turning point for the achievement of the intended construction of the structure of $|f(|x|)|$. In this utterance, H recognised the order of the operational priority of computations including parentheses and proposed that $|f(|x|)|$ might be treated in the same way. This recognition was also shared by S, as she was further elaborating this proposal later in 173 (see arrows through 173➔172➔170). In 171, the scaffolder probed H to understand her intention clearly. Based on H's response and on her interaction with S, he found an opportunity to monitor the students' performance, which was concerned with developing a strategy to obtain the graph of $|f(|x|)|$. As can be seen from the students' interaction between 172H and 175S, the strategy that H&S were developing was appropriate but they were not sure if this approach was reasonable or how they could determine the 'parentheses' in the expression of $|f(|x|)|$. Based on his analysis of the students' encountered difficulties, the scaffolder, in order to encourage the students, gave a positive feedback ("you both made an excellent point") implying that their approach was reasonable. He furthermore accentuated how the absolute value signs in the expression of $|f(|x|)|$ might be used in the similar way to parentheses (176). This assisting intervention in 176 made reference to all the students' turns between 172-175 (see the arrows emanating from 176 pointing directly to 172, 173 and 175; and indirectly to 174 through 175). This suggests that the scaffolder analysed the students' interaction quite carefully to tailor the appropriate type of assistance. Instead of telling the students what to do, he made a proposal about the consideration of the absolute value signs in a similar way to parentheses and presented this in a question format in order to maintain the students' active involvement.

In terms of epistemic actions, I observe constructing actions in this episode. In 170, H recognised a computational priority rule that she knew from her past experience as mathematics student. This recognition was certainly shared by S (see 173) and they attempted to build a strategy by invoking an analogy between the computational priority rule and the expression of $|f(|x|)|$. It is clear that recognition of this analogy, even if H&S needed the scaffolder's assistance to develop this further (see next episode), provided a turning point for H&S's development in connecting the structures of $f(|x|)$ and $|f(x)|$ with the expression of $|f(|x|)|$ and thus with the construction of a new structure. On the basis of this analogy, H&S achieved an important step towards assembling the structures of $f(|x|)$ and $|f(x)|$ to produce a novel method for sketching the graphs of $|f(|x|)|$. The question of interest here is: can we consider the moment when this analogy was stated by H in 170 as a precise starting point of their construction? It is important to realise that H's analogy was prompted by the scaffolder's interventions in 165 and 168 (see the category 1 arrow from 170 to 165 and to 168) and that H&S had already worked, up until 145, without the scaffolder assisting them and they were not able to come up with this kind of analogy. Hence development of this new construction cannot be thought of in isolation from the scaffolder's assistance and particularly from these two interventions (i.e. 165 and 168). Consequently I considered the utterances, right from the start of this episode, as involving constructing actions. However, by labelling these utterances as involving constructing actions, my intention is not to attribute a precise starting point to a new construction. In fact in a later chapter I will attempt to develop the argument that the starting and end point of a construction(s) cannot be precisely demarcated with certain boundaries (see Chapter 7 section 2).

Episode 5

> **177H:** Aha, I got it... I know what we will do.
> **178I:** Could you please tell us?
> **179H:** We can consider $f(|x|)$ as if it was the smaller parenthesis!
> **180I:** Smaller parenthesis?
> **181H:** I mean it should be the first thing that we need to deal with
> **182S:** Yeah, I agree... I think we should begin with the graph of $f(|x|)$ and first draw it
> **183H:** But what next?
> **184S:** Then we can use the absolute value at the outside... in the similar way of doing computations.
> **185H:** But we will be drawing graphs! Can we really do this?
> **186S:** I am not too sure if we can... but it sounds plausible...
> **187I:** What you are doing here is not computation of course... but you are making an analogy, I mean you are making some certain logical assumptions based on your earlier experiences... and I see no problem with that... let's draw the graph by considering what we've just talked about and then decide if it will work or not, huh?

The goal identified in this episode is for the students to devise a plan for the execution of the strategy which was initially suggested in the previous episode. This goal, although not stated in any utterance in this episode, can be discerned from the students' overall interactions which were concerned with how to apply the analogical relation of the computational priority rule when drawing the graphs of $|f(|x|)|$ (see especially 181-186). This goal was not set by the scaffolder but evolved through the interaction among the participants. However, the scaffolder's assistance in 176 and his probing questions contributed to this evolution to a great extent (178&180). In essence the aim of these probing questions was to gain insights into what H has realised (177) and into her proposal (179) in order to monitor how the given assistance in 176 was taken up. The scaffolder was silent between 181 and 186 and he observed the students and analysed their performance on the basis of their interaction. In doing so he made judgements about if and what kind of assistance the students might need. The students put forward that they could first draw the graph of $f(|x|)$ and then consider the absolute value sign at the outside of the expression of $|f(|x|)|$. Although this idea originally proposed by H herself (177&179) and developed through the input of the both students, she later seemed to question her proposal (185) and consulted to S about the aptness of this approach (category 4 arrow from 185 to 184 in the flowchart presented in Figure 4.7. below). S was also not able to say for sure if this approach could work (186). I believe that the students expected, not necessarily consciously, the scaffolder's active involvement; that's one of the reasons why, I think, they stated their uncertainty about the aptness of this approach discussed in this episode. Having monitored the students' difficulties and analysed their performance, in 187, the scaffolder once again intervened to assist them. He initially provided the students with a positive feedback and gave an explanation about why their approach was reasonable. He also assured them that he did not see any problem with their approach. After that he set a new goal to the students and asked them to draw the graph of $|f(|x|)|$, by considering what they have just discussed. In doing so he regulated the students towards the main goal of the activity.

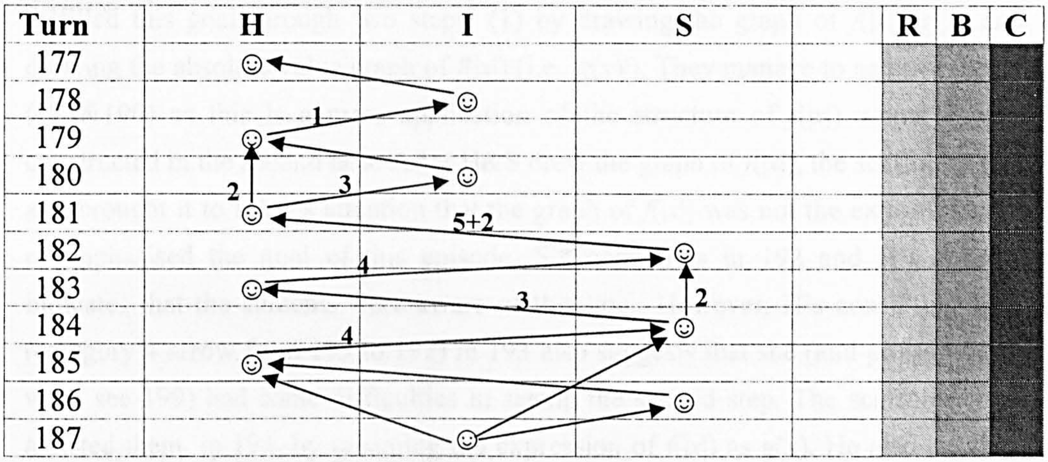


Figure 4.7. Interaction flowchart for episode 5 in task 4.

The legend: **1**: proposing; **2**: elaborating; **3**: explaining; **4**: quest; **5**: agreement; **6**: disagreement.

With regard to epistemic actions, the students in this episode continued with their construction of the new method to draw the graph of $|f(|x|)|$. The students were merging their recognitions of the computational priority rule and of constructions of $|f(x)|$ and $f(|x|)$ together and related them to the graphs of $|f(|x|)|$ to build a plan for execution. This plan was by no means trivial; quite the contrary, it provided the students with a kind of 'blueprint' for their subsequent actions in the next episode and was of fundamental importance to the development of a new method.

Episode 6

188H: What are we doing now?
189S: We will draw first the graph of $f(|x|)$.
190H: OK let's draw the graph now… [They draw the graph of $f(|x|)$, see Table 4.3. – D]
191I: Alright, you drew the graph of $f(|x|)$. But this is not we are expected to find, is it?
192S: No… we will now draw $|f(|x|)|$
193H: Do you know how? Well, the next step is not too clear to me!
194I: OK, just to make your job a bit easier, let's rename $f(|x|)$ as $g(x)$. So what you need to find turns into…
195S: $|g(x)|$
196H: Aha! I can see it now…
197I: What is it?
198H: That means we will draw the absolute value of this graph… I mean we need to take the absolute value of this graph… oh it is so clear now, do you understand S?
199S: Of course, but renaming the expression helped me see it clearly now
200I: OK, let's think about it now, how can we apply absolute value to this graph?
201S: $|g(x)|$ never takes negative values… I mean it never passes under the x-axis
202H: We will be taking the symmetry of the rays [she refers to line segments] under the x-axis
203S: Yes
204H: OK then, let's draw it now. We are now drawing the graph of $|f(|x|)|$
205S: We were taking the symmetry of this part [the line segment into the fourth quadrant] and then we should take the symmetry of that part as well [the line segment into the third quadrant, see Table 4.3. – E].

The goal for this episode is identified as to execute the strategy devised in the previous episode in order to draw the graph of $|f(|x|)|$. This goal was set by the scaffolder at the end of the previous episode in 187. During this episode, H&S worked towards the fulfilment of this goal. They fulfilled this goal through two steps: (1) by drawing the graph of $f(|x|)=g(x)$ and then (2) by drawing the absolute value graph of $f(|x|)$ (i.e. $|g(x)|$). They manage to achieve the first step easily (189&190) as this is a mere application of the structure of $f(|x|)$ which H&S have already constructed in the second task. After H&S drew the graph of $f(|x|)$, the scaffolder (191) intervened and brought it to H&S's attention that the graph of $f(|x|)$ was not the expected outcome and thus re-emphasised the goal of this episode. S's comments in 192 and H's consultation in 193 indicates that the students were aware of this goal. However, H's consulting type of utterance (category 4 arrow from 193 to 192) in 193 also suggests that she (and presumably her partner as well; see 199) had some difficulties in seeing the second step. The scaffolder realised this and assisted them, in 194, by renaming the expression of $f(|x|)$ as $g(x)$. He also implied that drawing the intended graph of $|f(|x|)|$ was indeed an application of the structure of $|f(x)|$ to the graph of $f(|x|)$ (194 and 195). The scaffolder by this intervention in fact specified the second step to direct the students' attention. The scaffolder (200) then invited the students to discuss how to use their knowledge of $|f(x)|$ in drawing the graph of $|g(x)|$. Consequently, a new subgoal emerged for the students, that is, to draw the absolute value graph of $g(x)$. In this episode, it is clear that the scaffolder was an important regulative force in the given context. The scaffolder's regulative function can be seen clearly in the interaction flowchart in Figure 4.8. (especially from 191 onwards), where arrows emanating from the students' utterances directly (192, 195, 198 and 201) or indirectly (through the aforementioned turns) point to the scaffolder's interventions.

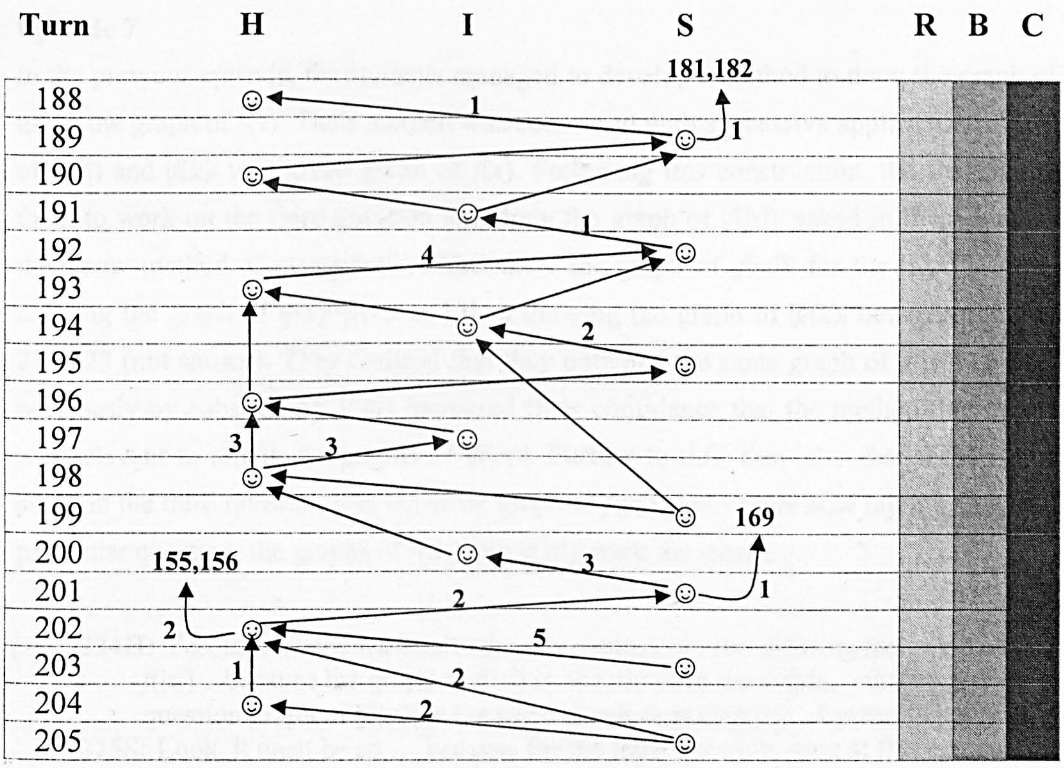


Figure 4.8. Interaction flowchart for episode 6 in task 4.

The legend: **1**: proposing; **2**: elaborating; **3**: explaining; **4**: quest; **5**: agreement; **6**: disagreement.

In terms of epistemic actions, the students utterances display constructing actions which ultimately led H&S to the achievement of the construction of a novel method to draw the graphs of $|f(|x|)|$. This episode provides an illustration of how earlier constructions could become recognising and building-with actions for a further construction. When H&S achieved the first step, they recognised and used the structure of $f(|x|)$. However, there is more to the second step than just a mere recognition and application of the structure of $|f(x)|$ constructed in the first task. The students constructed this structure in the context of the linear graphs. However, applying the structure of $|f(x)|$ to the graph of $f(|x|)$, which are always V-shaped and thus not a linear graph, was completely new to the students. They thus had to reorganise and extend their structure of $|f(x)|$ in order to apply it to a V-shaped graph. It seems to me that this reorganisation comes about through the consideration of the each arm of the graphs of $f(|x|)$ as if it were a linear graph in itself. This may be seen in 205 where S (together with H) first focused on the right arm and reflected the line segment into the fourth quadrant in the x-axis and then did the same for the left arm into the third quadrant. However, this is not the whole story of their construction. Throughout their work in episode 4, 5 and 6, H&S managed to merge the two structures, $|f(x)|$ and $f(|x|)$, together in relation to the structure of $|f(|x|)|$. While doing this they appealed to the known features (to H&S) of these structures and used these to build-with the intended graph of $|f(|x|)|$. They recognised, and used, not only earlier constructions of the previous tasks but also relationships that they discovered at the beginning of their work in this task (see the long arrow emanating from 201S to 169 and thus indirectly to 102 and 138). Thus the construction of this structure was rooted within their earlier constructions as well as within their work in this task.

Episode 7

In the previous episode, the students managed to develop a method to draw the graph of $|f(|x|)|$ by using the graph of $f(x)$. Their method was concerned with successive application of the structures of $f(|x|)$ and $|f(x)|$ to a given graph of $f(x)$. Following this construction, the scaffolder prompted them to work on the third question and draw the graph of $|f(|x|)|$ asked in this question by using their new method. Consequently, H&S drew the graph of $|f(|x|)|$ for the third question by first drawing the graph of $g(x)=|x|+3$ and then drawing the graph of $|g(x)|$ between the utterances of 213-223 (not shown). They realised that they obtained the same graph of $|f(|x|)|$ as they obtained previously by substituting. This increased their confidence that the method that they developed was relevant to obtain the graphs of $|f(|x|)|$. Further to this, they also realised that the graph of $|f(|x|)|$ in the third question was the same graph as $f(|x|)$. They were now talking about why, in this particular question, the graphs of $f(|x|)$ and $|f(|x|)|$ were the same.

> **224H:** Yes, that means we don't take any symmetry after drawing the graph of $f(|x|)$...because the graph of $f(|x|)$ is already over the x-axis... that means for the third question graph of $|f(|x|)|$ is the same graph as $g(x)=f(|x|)$...I mean this is V-shaped.
> **225S:** Look, it must be so ... because for the third question, look at this equation $[|f(|x|)|=|(|x|+3)|]$ even if the absolute value at the outside of the whole expression is removed, we still obtain the same values of y... I mean every value of y is positive for

$f(|x|)$ and so the absolute value sign outside the whole expression doesn't make any difference... so these two graphs should be the same anyway...

As can be seen the students were not uncomfortable with the fact that this graph was V-shaped, which was a hindrance at the beginning of their work (see episode 2). In fact this graph now provides a means by which they deepened their construction. The students later concluded that their method was working. Later on H&S moved on to the question 4 in which four linear graphs were presented to the students who were expected to draw the graph of $|f(|x|)|$ for each one of these. I pick up the conversation at the moment when they started working on this question.

228I: Right let's go on to the 4th question...what will you do in this question?
229H: We will draw the graphs with the same method again.
230S: Yes
231H: [They are talking about the first given graph of f(x) in question 4-A) OK... now, first of all...
232S: The graph of $f(x)$ at the positive values of x will remain the same
233H: First we obtain the graph of $f(|x|)$...
234S: Yes [they are drawing the graph of $f(|x|)$]
235H: Now we will draw its absolute value graph
236S: That means we will take the symmetry in the x-axis
237H: All of the parts over the x-axis remain as they are
238S: The parts under the x-axis... here... these will be cancelled and the symmetries will be taken in the x-axis

As can be seen from the students' interaction, they were able to regulate themselves, proceeded without any help from the scaffolder and obtained the accurate graph of $|f(|x|)|$. The scaffolder was still monitoring the students' performance and analysing their present level and consequently did not feel the need to intervene. Thus in this episode the scaffolder handed the responsibility over to the students. After the students achieved the intended construction, they became relatively self-regulated and in fact in this episode they were practicing the newly constructed structure of $|f(|x|)|$ by recognising and using it in a similar context. Having accurately drawn all of the graphs for the fourth question, the students went on to the fifth question where they were asked to explain how to obtain the graph of $|f(|x|)|$ by using any given graph of $f(x)$. The students' account is reproduced below:

243S: First of all, by making use of the graph of $f(x)$ we obtain the graph of $f(|x|)$ and then obtain $|f(|x|)|$...
244H: To do this, first when drawing $f(|x|)$, part of $f(x)$ at the positive [values of] x remains unchanged... umm then this part is taken symmetrically in the y-axis and err and also part of $f(x)$ at the negative [values of] x is cancelled. After that, we apply absolute value to this graph, and for this... umm... negative values of y are taken symmetry in the x-axis and thus we obtain the graph of $|f(|x|)|$.
245I: Positive values of y
246S: Remains the same, they don't change...

As these verbalisations suggest H&S achieved the construction of a new method, other than substitution, to view the graphs of $|f(|x|)|$.

CHAPTER 5: FORMATION OF CONSTRUCTIONS THROUGH SCAFFOLDING

In this chapter I attend to the first research question of this study, that is, how are the new mathematical constructions formed through scaffolding? Please recall that here the term 'construction' is used in the sense of RBC theory and involves emergence of a new structure through three epistemic actions: recognising, building-with and constructing. Please note that in the rest of my writings, along with the term 'construction', I use two other expressions interchangeably when referring to this process, namely, formation of a new (mathematical) construction and construction of a new (mathematical) structure. When used in connection with H&S's work, these expressions more specifically refer to the methods developed to draw/sketch the absolute value of linear functions (see section 1 below).

In order to answer this research question within the theoretical framework of this study, I will discuss H&S's constructions within their zones of proximal development (ZPD) at stage 1 and 2 (see Chapter 2, section 3.3. and 3.4.). As already discussed in the theoretical framework of this study, it is the stage 1 where students construct new mathematical structures through the regulation and assistance of more capable others e.g., teachers and peers (Tharp and Gallimore, 1988). In this stage, the emphasis is laid essentially on the assistance and regulation, by more capable others, provided to the learners in order to develop new skills. But this consideration raises many questions: What kind of assistance and regulation are we talking about? How do they extend the capability of students and thus result in an achievement of new constructions? Does providing assistance and regulation guarantee the construction of new knowledge? If not, under which circumstances can the learner benefit? Is this assistance and regulation provided only by the physically present participants (peers, scaffolder, or both?)? Or are there some 'others' involved but not physically present in the activity? If so, who are they and how do they affect the formation of a new construction? How can we characterise the social interaction amongst the participants considering the differences and similarities in their perspectives and understandings?

Research carried out in relation to the ZPD (e.g., Wertsch and Hickman, 1987; McLane, 1987; Goldstein, 1999; Meira and Lerman, 2001) points out the legitimacy of asking these questions. For example, McLane (1987) writes that

> ZPD is not simply in the child waiting to be triggered or activated by a more competent member of the culture, but, rather, must be negotiated by the child and the more capable person in a particular context. Whether – and how – it is negotiated depends on *the nature of the specific activity, on the mode of social interaction and the kinds of communicative processes utilised, - and on the particular context that is created* (p.268, emphasis added).

Indeed, with some variation in emphasis and terminology, similar comments have been repeatedly made by many others (e.g. Daniels, 2001; Wells, 1999). All of these questions raised above are important and point to the issues on which I focus my attention. It is through attending to these questions that I will attempt to answer the first research question.

In line with the above-raised questions, this chapter is divided into eight main sections. In the first section, I provide a brief overview of H&S's work on the second and fourth tasks in order to remind the reader of what the students achieved in their work. The second section attempts to characterise different forms of assistance within the ZPD by focusing on two broad categories of assistance: instructional and pedagogical. In the third section I explore why and how the scaffolder's interventions are linked to the students' achievement of new constructions. With regard to the 'why' question, I suggests some causative relationships between the scaffolder's interventions and the achievement of new constructions. Regarding the 'how' question, I propose the idea of 'human mediation' and determine the conditions for this kind of mediation to occur. In the course of my elaboration of human mediation, I argue that cognitive and social developments are closely linked to one another and point out the difficulty of drawing a major distinction between social and cognitive processes. This argument is further developed in the fourth section by drawing on Bakhtinian notions of 'utterance', 'voice' and 'dialogicality'. I utilise these notions to clarify my intention with the term 'social', discuss the mutual influences between the scaffolder's and students' voices, and focus on the influence of the voices from 'absent others' on the formation of new constructions. In the fifth section, I further characterise the interaction between the scaffolder and students by describing two opposing tendencies which exist in any interaction at varying degrees and with relative importance: intersubjectivity and alterity. Common to all these five sections are three indispensable components of any scaffolded discourse: the scaffolder, students and tasks. The sixth section presents a model which is centred on 'emergent goals' and which is concerned with the interconnections amongst these components within the theoretical framework of this study. The seventh section focuses on the 'handover' (of responsibility) principle which is intrinsic to any scaffolding process. The final section concludes this chapter with further research notes in relation to scaffolding and the new constructions.

1. A brief overview of H&S's work

There are two points clear from the analyses of H&S's work: first they constructed structures completely new to them and second constructions of these structures were scaffolded. Regarding the first point, H&S constructed two methods to sketch the graphs of $f(|x|)$ in the course of their work on the second task; and one method to sketch the graphs of $|f(|x|)|$ on the fourth task (apart from substitution which seemed to be available to them before working on these tasks). H&S's constructions in the second task were: drawing the graphs of $f(|x|)$ by the 'reflecting' method, i.e. reflecting $f(x)$ at $x<0$ in the horizontal line crossing the y-intercept of $f(x)$ (see 2: episode 4[1]) and by the 'y-symmetry' method, i.e. reflecting $f(x)$ at $x>0$ in the y-axis (see 2: episode 5 and 6). H&S's construction in the fourth task was drawing the graph of $|f(|x|)|$ through a successive application of the structures of $f(|x|)$ and $|f(x)|$ to the given graph of $f(x)$, which will be referred to as the 'two-step' method.

[1] "**2**: episode **4**" refers to the analysis of H&S's work on the **second** task for the **fourth** episode.

Considering the second point, as already stated in Chapter 2, I view scaffolding as an asymmetric collaboration between a ('more knowledgeable') tutor and ('novice') student(s) *towards* a successful completion of a task within the student's ZPD for which the tutor provides assistance which is augmented or reduced depending on the student's progress. On the basis of this definition, I argue that the constructions of the three methods, stated just above, were scaffolded. To begin with in H&S's work on the second and fourth task, there was an asymmetric collaboration between the scaffolder and the students. The interaction between the scaffolder and H&S was asymmetric (at least in terms of subject matter knowledge) precisely because the scaffolder, from the very start, knew the methods to draw the intended graphs, which H&S was yet to construct. The interaction was also collaborative. The scaffolder was actively involved in the students' construction process by regulating and directing the students' attentions and efforts (see section 3.1. and 3.2. below). However, he was also concerned with the students' active involvement in completing the tasks and the students' active involvement was all too apparent throughout their work (see section 4.2.1. and 4.3.2. below).

Further to this, the successful completion of the tasks was within the H&S's ZPD and the scaffolder provided assistance to the students *towards* this direction. The methods that H&S have constructed were not available to the students before they worked on these tasks. The evidence for this comes from H&S's performance on the diagnostic test (see Chapter 3, section 6.2.) which clearly showed that H&S had already acquired the prerequisite knowledge structures to achieve the intended constructions but were not acquainted with these constructions. Also, it is clear that the scaffolder assisted H&S in completing the tasks successfully and hence in achieving the intended constructions (see section 2 and 3 below). But does that mean the constructions were beyond H&S's unassisted efforts? There are several reasons to believe that the intended constructions were beyond H&S unassisted collective efforts. These reasons will be scrutinised by considering H&S's work on the second and fourth tasks separately.

Focusing on the second task, when working on question 4, H&S suggested the 'reflecting' method which was imprecise and, indeed, ambiguous (2: episode 1). Although the scaffolder attempted to draw H&S's attention to the imprecision and ambiguity of this method (2: episode 2), they were not able to see the problem. The scaffolder then gave them some autonomy to see if they could realise the problem. Although S realised an inconsistency in the graphs obtained by the 'reflecting' method and brought this to H's attention, they had a lengthy but unfruitful discussion which led them nowhere (2: episode 3). However, the scaffolder's interventions (2: episode 4) assisted in bringing about the construction of the 'reflecting' method. In addition to this, the scaffolder considerably assisted the construction of the '*y*-symmetry' method which was first achieved by H and then by S (see 2: episode 5 and 6).

With regard to the fourth task, H&S were given almost full control over their work by the scaffolder who suspended his help almost completely until 145H (see 4: episode 2). At this point, these students were clearly having problems and even gave up developing a method after making so much effort. They were frustrated and seemed to expect assistance (see 4: episode 3; 147H and 148S). H&S's inability and failure to develop a method and their frustration indicated that the structure was very likely to be beyond their unassisted efforts.

However, a truly sceptical mind can question these considerations and may regard them as insufficient to prove that the intended constructions were beyond the H&S's unassisted efforts, arguing that if more freedom was given (perhaps until the end of the tasks!) they might have constructed these methods even without the scaffolder's involvement. In response to this argument, it should be noted that none of the non-scaffolded pairs selected for this study achieved all of the four intended constructions (see Chapter 4, section 1 for a summary of the participating students' performances). Yet H&S achieved all of the constructions. Although the non-scaffolded pairs' failure might be informative, they do not necessarily show that H&S would have also failed if they were not scaffolded. Therefore, it may not be completely clear whether or not H&S would have achieved these constructions without the scaffolder's involvement and assistance. What is clear, however, is that the scaffolder played a crucial role in and contributed to H&S's developing constructions.

Surely seeking a clear-cut answer on the necessity of scaffolder's involvement for the achievement of the constructions eventually leads one into a cul-de-sac. A way out of this can be found when we realise that scaffolded learning does not assume the necessity of students' inability to form new constructions unaided in order to assist their performance. Therefore, the matter of concern here is not so much with questioning the possibility of H&S's achievement of the constructions without the scaffolder but rather with how these constructions were assisted, what role the scaffolder played and how these are related to the mode of interaction amongst the participants. Investigation of these issues necessarily compels one to consider the activity as a whole and to undertake an analysis of a complex set of social, cultural, historical and semiotic dynamics amongst the participants taking part in that activity. In what follows, as part of this complexity, I first focus my attention on the nature and occurrence of the scaffolder's assistance.

2. The scaffolder's assistance

In this section I focus on the type of assistance provided by the scaffolder within the students' potential development level. The extant literature on types of assistance in relation to scaffolding is usually concerned with the scaffolder's spontaneous interventions within the immediate interaction taking place during the course of the activity. Scott (1998, p.70) considers this type of assistance as 'pedagogical means' which consists of "the discursive interventions made by the teacher in spontaneous response to the student's performance". However, there are other means of assistance which comes even before interacting with the students. For example, Rogoff (1990, p.86) refers to these types of assistance as 'structuring situations.' She employs this term in the context of adult-child learning activities, referring to choice of puzzles, and selection of appropriate tools and materials. In a similar vein Scott (1998, p.70) employs the term 'instructional means' when he refers to "teaching activities which are planned ahead of the instruction." Following Scott, I consider assistance within two broad categories: instructional and pedagogical means. I will discuss the nature of these two general types of assistance with regard to H&S's verbal protocols.

2.1. Instructional assistance

Assistance provided through instructional means is often overlooked in the scaffolding literature and researchers do not usually focus on the impact of this kind of assistance on scaffolded learning process. However, the verbal protocols generated for this study point to the importance of instructional assistance. The idea of instructional assistance that I have in mind is broader than the one expressed by Scott and Rogoff and is essentially related to the activity itself. Although some may equate activity to the task (see Cole, 1985, p.158), reading such authors as Wertsch (1998, especially chapter 1), Tharp and Gallimore (1988, especially chapter, 4) and Leont'ev (1981) convinced me of the necessity of considering activity in a broader sense to preserve its unity and essence. Thus my consideration of the activity is more comprehensive and includes, more specifically, the selection and organisation of the students, instructions and expectations conveyed to the students as to how to work (together) on the tasks, utilised tools and materials, design of the learning environment and the tasks. I now elaborate on each one of these aspects by referring to H&S's work.

First of all, H&S were selected by a diagnostic test (see Chapter 3, section 6.2.) in order to make sure that they had the necessary prerequisite knowledge structures (e.g., the notion of symmetry, of linear functions and of absolute value; see Appendix 4) to carry out the tasks. This kind of selection was important to ensure the students' 'cognitive readiness' (Wertsch and Hickmann, 1987, p.262) in relation to the task difficulty. Considering H&S's performance on the diagnostic test which showed that they had already acquired the knowledge structures required to complete the tasks successfully, it can be concluded that H&S were equipped to achieve the target constructions of the tasks. Further to this, H&S were organised to work together and this organisation certainly appeared to assist their progress. This is a subtle means of assistance whose importance lies in the comfort and 'affection' that this organisation provided to H&S. These two girls were used to working together and also expressed their wishes to be paired for this study. Their eagerness to work together, I believe, influenced their evolving interactions and was reflected in the way that they, without restraint, shared their insights (e.g., 2: episode 6), confusions, concerns and difficulties (2: episode 3; 4: episode 1) with one another.

These observations echo Forman's (1989) research findings about the role of peer interaction in the construction of mathematical knowledge. For her study she employed two girls who "declared themselves to be best friends" (ibid, p.60) and found that the evolving nature of the relationship between the girls corresponded to the degree of progress made by the pair. She even observed that the students were not as motivated when they were working individually on the post-test as they were when working collaboratively. I cannot say if H&S would have been less (or more) motivated if they were working individually as they worked together on all four tasks.

However, had the scaffolder forced the girls to work with someone that they did not want to work with or did not feel comfortable with, then it is likely that a very different pattern of interaction would have resulted: the girls' evolving work would likely have been adversely affected by the nature and quality of their personal relationships (Goldstein, 1999). Therefore, organising the students in this way was certainly helpful for H&S's communication.

H&S's eagerness to share their insights and concerns with one another might also be connected to the instruction given by the scaffolder. This instruction (see, Chapter 3, section 5.3.) aimed to encourage H&S to work together in a collaborative manner. They were also expected to talk and discuss issues related to their work and, if necessary, to challenge each other about the appropriateness of their actions, decisions and resulting work. Although such instructions and expectations do not necessarily guarantee perfect collaboration (see Ozmantar, in press), H&S seemed to keep to these conveyed instructions and achieved a 'good' level of collaboration in most of their interactions.

Thirdly, the tools and materials that H&S had at their disposal assisted their developing constructions. The scaffolder provided the students with different coloured pens, rulers, and Cartesian grids printed on papers. The students were asked to draw the graphs with the coloured pens by using rulers on the Cartesian grids printed on the papers. Use of coloured pens was intended to help students realise the changes occurring in the graph of $f(x)$ when transformed into the object graphs. I cannot provide hard evidence that H&S benefited from using coloured pens but the rulers and Cartesian grids did help H&S to sketch much more precise graphs than the ones that might have been sketched by 'hands' on blank paper. Research suggests that precision is an important ingredient in the development of certain mathematical concepts. For instance, Chassapis (1999) points out the importance of precision in circle work as the formal mathematical concept is mediated by the use of a compass. In the case of H&S precision not only assisted them to recognise some properties of the object graphs and to build some patterns in relation to the graphs of $f(x)$ but also assisted them to construct the 'y-symmetry' method in the second task (see 2: episode 5). After sketching the graph of $f(|x|)$ on the Cartesian grid provided, H observed that "for the different values of x, y took the same value" (2: episode 5, 153H). This observation was a crucial step towards the construction of the 'y-symmetry' method and was certainly assisted by the precision occurring as a result of sketching the graph of $f(|x|)$ by using a ruler on the given Cartesian grids where one can see, for example, that "y takes the same value for $x=\pm1$" (2: episode 5, 156H).

Fourthly, the design of the learning environment was also important in assisting students to work without interruption. The tasks were applied in a spare room in H&S's school. Tremendous attention was paid to ensure that H&S worked without any disruption from noise or from any other student in the school. They sat together at a table which was large enough to place all of the learning materials and tools. The direct influence of such an environmental design on students' work cannot be demonstrated precisely but preventing unwanted interferences surely contributed to the activity being carried out smoothly and pleasantly.

My final point is concerned with the tasks. It is obvious that the interaction amongst the participants of scaffolded learning does not take place in a void, but, rather, in the context of a specific task. The task is an indispensable constitutive part of any activity where students' work is scaffolded and it creates the context in which the scaffolder's pedagogical assistance is provided. Although relatively less research attention has been directed to the effect of a task on scaffolded interaction, available research findings suggest that this is an issue of major importance (Gonzalez, 1996). In this respect, closer inspection of the research on scaffolding reveals the existence of different modes of interaction amongst the participants depending on the content of the task, e.g., highly verbal (e.g., a reading task) (Chi et al., 2001) or highly physical (e.g., completion of a puzzle) (Wertsch and Hickman, 1987); how structured a task is and the level of its difficulty (Rogoff, Ellis, and Gardner, 1984). These studies simply suggest that the scaffolder's pedagogic assistance and the way in which it is delivered are greatly influenced by the specificities of the task. Further to this, it is observed in this study that the task has an important regulative effect on the unfolding interaction amongst the participants, the issue to which I will return on several occasions in my following discussions.

The means of instructional assistance elaborated in this section in one way or another affects the scaffolder's pedagogical assistance occurring during the interaction. Although instructional assistance precedes pedagogical assistance, they should not be viewed as functioning separately but rather complementarily. I will now turn to pedagogical assistance and discuss its forms and occurrences.

2.2. Pedagogical assistance

As observed in the analyses of H&S's protocol data, the scaffolder employed several means of pedagogical assistance which was composed of discursive interventions occurring spontaneously in the context of immediate interaction. For instance, he: instructed the students and called for specific actions; described critical features; initiated mathematical reasoning (see 2: episode 4 for the example interventions of each respective type 116, 118, 130). These interventions were, I believe, crucial in aiding H&S's constructions. However, before linking the scaffolder's interventions to the achievement of H&S's constructions I will first consider how, and on which bases, the scaffolder made decisions for these interventions. My consideration will focus on three elements involved in the scaffolder's decision-making process for his interventions: monitoring, analysing and assisting.

As discussed in Chapter 2 (section 2.3.), Scott (1997, 1998) operationalises occurrences of the scaffolder's intervention, suggesting three elements involved in this process: monitoring – scaffolder monitors the present performance of the learner; analysing – scaffolder analyses the nature of any difference between the present performance and the goal level of (or target) performance; assisting – scaffolder responds with an appropriate intervention to support the learner's progression from the present to the goal level of performance. This is what he calls 'action cycle model' (Scott, 1998, p.70), indicating that these three elements occur in a cyclic manner in the scaffolder's interventions.

In this study, the investigation of the data generated in scaffolded situations in general, and H&S data in particular, largely supports Scott's action cycle model. However, the data also suggest that interaction amongst the three elements of this cycle is a more complicated process than just this cycle. The complication stems from the fact that the scaffolder had to simultaneously keep track of students' present performance (monitoring), to relate it to the target performance (analysing), and then to decide and provide 'appropriate' kind and amount of assistance in his spontaneous interventions (assisting). I will now turn to H&S's protocol data to demonstrate the three elements of the action cycle model and discuss the nature and occurrence of them. In doing so, I hope to further develop these three elements by providing detailed accounts. In passing, however, I have to note that the terms 'present performance' and 'target or goal level of performance' used by Scott might be translated into the context of this study, respectively, as 'developing constructions' and 'target constructions'. These terms will at times be used interchangeably in what follows.

In order to monitor the students' present performance, there appear several sources to which the scaffolder had recourse. The first was the students' articulations as expressed through their interactions in relation to what they were doing and going to do. Indeed the students' articulations were the main source to understand their developing constructions and the possible difficulties that they were experiencing or were likely to experience. For example, when H&S proposed the 'reflecting' method for the first time in the second task (see 2: episode 1), it was through their articulation and interaction (see 58S-65S) that the scaffolder (see 2: episode 2, 76I) came to realise the vagueness and imprecision involved in this method.

Monitoring based on the students' articulations required the scaffolder to follow H&S closely and to understand them. Yet, it was not always possible for the scaffolder to understand the students' articulations and/or intentions, which created some obstacles to his monitoring. In that case the scaffolder had recourse to some diagnostic interventions. Two examples of this sort (155I from 2: episode 5 and 171I from 4: episode 4) are provided in the excerpts below.

> **153H:** Look, (…) for the different values of x, y took the same value (…)
> **155I:** What do you mean 'for the different values of x, y take the same value'?
>
> **170H:** (…) Can we think about it [$|f(|x|)|$] like a computation with parentheses?
> **171I:** Computation with parentheses?

Diagnostic interventions were used to obtain clarification on student proposals, elaborations and/or explanations. The scaffolder, in most of the cases, did not intend to assist the students but rather to understand their intentions or to make sure of the accuracy of his understanding of the students intentions (intervention of this sort may assist the students even if not intended to do so).

The third source through which the scaffolder monitored the students' developing constructions was through their written responses e.g., sketched graphs and computations. The sketched graphs were, undoubtedly, an important and immediate indication of whether the students were able to apply their knowledge when obtaining the graphs. The scaffolder had the opportunity to observe the students when they were building the graphs and was thus able to monitor their performance. Further to this, the students referred to these obtained graphs in their explanations, elaborations and proposals. This provided the scaffolder with an opportunity to connect their explanations with the graphs so that he could have a better appreciation of their developing insights and the way that the students approached issues (see 2: episode 1 and 3).

Finally, in the course of his monitoring, the scaffolder also made use of the students' non-verbal actions and reactions such as their on- and off-task actions, the time that they spent on certain bits of the task and their frustration resulting from the students' inability to overcome an obstacle or find a solution to their problem(s). Again the utilisation of this source required the scaffolder to make careful observations. For example, in the second task when H&S obtained an erroneous graph, S's realisation of an error and the resulting argumentation with H created a forum for the scaffolder to monitor their insights and developed understandings (2: episode 3). The scaffolder's monitoring took place on the basis of H&S's articulations as well as the time they spent on this particular aspect. The argumentation between H&S lasted 18 turns (2: episode 3), took them a considerable amount of time and they paused on several occasions to understand each other's perspectives and claims (see for example 112H). The noted pauses and amount of time spent in this episode were also informative for the scaffolder who felt a need to intervene after giving so much time to the students.

In addition, the students' frustration might be considered as a non-verbal source for monitoring. For example, H&S's frustration resulting from their inability to come up with a method after spending so much time in the first half of their work on the fourth task (up until 145H) was also instructive for the scaffolder to observe the limits of H&S's collective efforts. The scaffolder might have given more time to the students but he realised that H&S were very frustrated, they even gave up developing a new method (4: episode 2, 144 and 145). On the basis of this monitoring the scaffolder decided to intervene and assist to the students (see 4: episode 2).

As might be inferred from the account given so far the monitoring process in its simplest form consists in the scaffolder's observations of the students' actions and reactions taking place in the course of their work. Generally speaking, on the basis of these observations, the scaffolder made judgements as to whether to assist the students and, if so, judgements on the kind and amount of this assistance. It is this process of judgements that characterises the analysis. In other words, the scaffolder's analysis involves the judgements which are informed by his observations and are directly connected with his interpretation of how the exhibited performance is related to the goal level of performance.

Exemplifying the scaffolder's analysis process with reference to the protocol data presents a formidable challenge: it is a subjective interpretative process which is almost inaccessible to an outsider. After all, in the course of interaction, the scaffolder did not articulate what judgements and observations he made of the students' work. Without those articulations, which are the only way to access to his thoughts, how could one talk about what the scaffolder has analysed? This objection is certainly valid; however, one may draw some inferences on the basis of the given assistance vis-à-vis what analysis might have been done. After all, assisting interventions follow the scaffolder's observations and reflect his judgements, albeit not necessarily fully. Further to this, I have an advantage of being the scaffolder and the researcher who draws the inferences from the scaffolder's assisting intervention as to what analysis was done.

I will now try to exemplify the form of analysis by focusing on episode 5 occurring in the fourth task (4: episode 5; see p.107), in which, I believe, it is relatively easy to trace the scaffolder's analysing process through his assisting intervention. It is useful to remind the reader that the dialogue in this episode occurred after the scaffolder drew H&S's attention to their earlier constructions of $|f(x)|$ and $f(|x|)$ and hinted at the importance of these structures (4: episode 4; 165I). This prompted the students to approach the expression of $|f(|x|)|$ (as a combination of $|f(x)|$ and $f(|x|)$; 170H) through an analogy to the precedence of arithmetic operations involving parentheses. In episode 5, H&S were trying to devise a plan as to how to apply this analogy to the expression of $|f(|x|)|$ when drawing its graph (see 4: episode 4 and 5 for more details).

The scaffolder's intervention in 178 and 180, in episode 5, did not aim to assist but, rather, to monitor what insights H gathered. Based on H&S's interaction, between 181 and 186, I believe, the scaffolder was monitoring as well as analysing the students' performance in relation to target level of performance, which was to draw the graph of $|f(|x|)|$ through successive application of $f(|x|)$ and $|f(x)|$. When we look at and *analyse* the students' performance at any given time as is disclosed through their interactions and articulations, it can be seen that their approach was quite apposite, that they were not sure of the aptness of their analogy and that they were indecisive and seemed to need assurance if they should attempt it. The scaffolder's assisting intervention in 187 addresses all three aspects: he shared H&S's concern that what they were doing was not computation but also assured them that their analogy was fine ("I see no problem with that") and he also pushed the students towards the target level of performance for which he urged H&S to draw the graph of $|f(|x|)|$ on the basis of what they talked about (187I).

The brief examination provided above also reveals that monitoring and analysing actions on the part of the scaffolder were taking place virtually simultaneously. Thus, I posit, that analysing actions are nested in the scaffolder's monitoring actions which may also be used to analyse the differences between the present and target performance level. This suggests that although it is possible to argue the existence of analysing actions in theory, it is, in practice, rather difficult to distinguish between monitoring and analysing actions at any given time in an examination of verbal protocols.

So far it has been argued that the scaffolder's monitoring and analysing actions inform his subsequent assisting interventions. As has been reported in the course of examination of H&S's verbal data, the scaffolder employed a variety of different types of assistance which differed in the degree of specificity, such as giving explanations, asking deep reasoning questions and hinting. A certain type of assistance employed within any specific situation is not necessarily the only appropriate one but just the one selected amongst many others available to the scaffolder. However, it is important to note that the scaffolder's decision as to the type of assistance that he employs also reflects his/her judgements, which need not necessarily be appropriate, about the amount of assistance that the students need in that particular situation. For example, compare the explaining intervention of 187I (4: episode 5) with that of 168I that occurred in the fourth task (4: episode 4). In the 187 intervention, the scaffolder specifically asked the students to draw the graph of $|f(|x|)|$ and gave quite a long explanation but in the 168 intervention he provided them with a "general verbal hint" (Pratt and Savoty-Levine 1998, p.291).

Therefore any assisting intervention reflects the scaffolder's decisions in relation to the type, specificity and amount of assistance deemed to be suitable within the moment. Undoubtedly the decision-making process involved in his act of assisting, based on the observations and resulting judgements, are subjective and may change from scaffolder to scaffolder. For example, consider again 4: episode 5. I have presented my monitoring and resulting analysis of H&S's interactions and noted that H&S were indecisive and needed some assurance to continue. However, someone else may think that H&S's indecisiveness is not a barrier and may not see any need to intervene and assist to the students. However, obviously my perception at the moment of intervention and my consideration afterwards assumed the necessity of this intervention.

An important question arises here: are assisting interventions, informed by monitoring and analysing, bound to lead to success and guarantee progression on the part of the students? Regardless of however well-informed an assisting intervention might be, it is not guaranteed to be beneficial for the students at all the times. As I will argue later (see section 5 below) 'intersubjectivity' or shared common ground amongst the participants is a prerequisite for an intervention of this sort to be fruitful. Further to this the scaffolder may not be able to clearly articulate his intention or may fail to reflect his insights gathered through monitoring and analysis into his assisting interventions. An example of failed intervention as a result of lack of intersubjectivity can be shown from H&S's work in 2: episode 2. In this episode, the scaffolder realised, based on his monitoring and analysing, the vagueness of H&S's initial 'reflecting' method but the students could not understand the scaffolder's intention though he tried thrice (see 2: episode 2, 76I, 80I and 86I; see also comments provided in the analysis of this episode).

I have hitherto attempted to exemplify and characterise the three elements of Scott's action cycle model: monitoring, analysing and assisting which are ultimately concerned with the scaffolder's perception and interpretation of the students' current performance in relation to the goal level of (or target) performance. However, there is a vagueness involved in the consideration of 'the goal level of performance' in Scott's model. At the first glance Scott seems to consider 'the goal' as the main goal of the activity and performance at this level as the target competence which the students are expected to achieve in the course of activity. He also explicitly mentions the recursive nature of his action cycle model in that, after assistance, the scaffolder monitors and analyses the students' new performance and this continues until the main goal of the activity is achieved (Scott, 1998, p.70). I believe that this recursion implicitly assumes the occurrence of monitoring-analysing-assisting actions in relation to intermediate goals emerging in the activity. In fact the verbal data collected for this study invariably suggested that in the course of activity there emerge many subgoals which can be distinguished from, but are subordinate to, the main goal of the activity, achievement of which presupposes attainment of a series of these subgoals[2]. In this connection, I posit that the process of monitoring-analysing-assisting takes place locally in relation to subgoal level and globally in relation to the main goal level (which, in the case of this study, is to develop a method(s) to draw the target graphs). However, the scaffolder's utilisation of these three actions within the immediate context of interaction occurs usually through the subgoal level of performance. After all, the students' immediate interaction and performance at any given time is mainly concerned with the attainment of a subgoal. Nevertheless performance at the subgoal level needs to be constantly monitored and analysed in relation to main goal level of performance so that the scaffolder could provide appropriate regulation towards the target competence through spontaneous interventions.

I will now turn to H&S's verbal data and try to concretise this argument on the basis of 4: episode 5; 177H-187I. The emergent subgoal of this episode was to devise a plan as to how to apply the analogy of precedence of arithmetic operations involving parentheses to the expression of $|f(|x|)|$ when drawing its graph. The scaffolder's monitoring and analysing actions occurred in the immediate context of the interaction but were also concerned simultaneously with the subgoal level and the main goal level (i.e. to construct a method to draw the graph of $|f(|x|)|$). In relation to the subgoal level of performance the students planned, being compatible with their analogy, to draw the graph for the expression of $|f(|x|)|$ starting from the inside, i.e. first to draw the graph of $f(|x|)$ (181H and 182S) and second applying the absolute value at the outside to this graph (184S). When the scaffolder intervened in 187, he first gave feedback assuring H&S of the appropriateness of this plan ("I see no problem with that") which was concerned with the subgoal level of performance and then asked the students to execute this plan to draw the graph of $|f(|x|)|$ precisely because he presupposed and foresaw the importance of the execution of this plan in the students' achievement of the main goal of the activity.

[2] This observation conforms to activity theory (Leont'ev, 1981) and will be discussed in section 6 below.

An important implication of this observation is that one of the requirements of successful scaffolding is profound subject matter knowledge as the scaffolder needs to observe the students' subgoal level of performance and simultaneously relate it to the expected level of performance. This requires the scaffolder to achieve a delicate regulation of the students' efforts at the subgoal level and map this level of performance onto main goal level of performance. In this regard, Tharp and Gallimore (1988) eloquently put forth the importance of subject matter knowledge when they write that

> a profound knowledge of subject matter is required of teachers who seek to assist performance. Without such knowledge, teachers cannot be ready to promptly assist performance, because they cannot quickly reformulate the goals of the interaction; they cannot map the child's conception of the task goal onto the superordinate knowledge structures of the academic discipline that is being transmitted (*sic*) (p.35).

Although this section centred on pedagogical assistance, the reader may be surprised that there was no explicit reference to the discussion of pedagogy itself. Throughout this section I preferred a relational use of the term pedagogy with regard to discursive and spontaneous interventions aimed to assist the achievement of a construction. I intentionally refrain from a detailed discussion of this term due to the space limitations and because it is beyond my aims in this section. However, I recognise the importance of pedagogy in any scaffolded discourse for it is a fundamental aspect of one's approach to teaching and learning. In this regard, for example, Davies (1994, p.26) argues that "pedagogy involves a vision of (theory, set of beliefs) about society, human nature, knowledge and production, in relation to educational ends, with terms and rules inserted as to the practical and mundane means of their realisation." Surely the scaffolder in this study held his own views on "society, human nature, knowledge and production" and these views inescapably influenced his pedagogic practice during his scaffolding. In connection with pedagogic practice, drawing on the (neo)Vygotskian perspectives, Daniels (2001) goes as far as to consider it "as a fundamental social context through which cultural production-reproduction takes place (p.6)." The pedagogic practice of individual scaffolders and its direct relation to their observations and decisions are important issues and I believe that further research with regard to these issues would be of value to further understand the dynamics of the subjectivity of this process and the individual motives involved in pedagogic practices of the scaffolders.

3. Linking interventions to the achievement of constructions

The discussion so far has focused on the occurrence and characterisation of instructional and pedagogical assistance provided via the scaffolder's interventions. However, the reader may realise that the 'assisting potential' of these interventions is merely assumed. It is one thing to assume that scaffolder's interventions 'assist' the students and quite another to demonstrate that they *actually* 'assist'. It is yet quite another thing to establish the links between these interventions and the students' achievement of intended constructions by asking 'how' and 'why' questions. With so much written about the scaffolded learning for almost 30 years, the scaffolding literature, ironically, does not provide a framework for a full understanding of how and why scaffolding is conducive to new learning or the formation of new constructions. This deficiency is realised by many such as Stone (1998, p.350) who, in an important review of scaffolding literature, points out this fact:

> [a] concern from existing discussions of scaffolding that I would like to stress is ... that the mechanisms by which new learning takes place during such adult-child interactions need greater specification. ... the actual details of such a process have not been clarified in subsequent discussions. More important, perhaps, are concerns that the discussions of possible mechanisms have underemphasized potentially key components.

As this study is concerned with the role of scaffolding in the formation of a new construction, it is inevitably necessary to offer my view on why and how scaffolding leads students to the formation of a construction(s). In this section, I propose some answers, though not necessarily definitive, to these two questions by focusing on H&S's verbal data. In relation to the 'why' question, I attempt to establish some causative relationships between the scaffolder's interventions and the students' developing constructions. I then focus on the 'how' question, to which my answer is, the 'mediation' of construction(s) through the scaffolder's interventions.

3.1. Causative relationships between interventions and the achievement of constructions

Five major factors can be observed from the overall analysis of the scaffolder's interventions which had a direct impact on, and hence can be connected to, the students' developing constructions: (1) regulating the students by setting subgoals; (2) reducing the students' uncertainty; (3) directing the students' attentions and efforts; (4) elicitation of deep explanatory reasoning; and (5) remediation of the students' errors. The impact of interventions cannot be strictly divided across these factors; an intervention may serve more than one purpose e.g. setting a subgoal to reduce uncertainty. Thus it is somewhat artificial to isolate them, but I shall do so for the clarity of communication but I will, at times, cross-reference one to another.

With regard to the first factor, the scaffolder was engaged in deliberate regulation of the students. The scaffolder's efforts in this respect were surely deliberate precisely because his role in the activity was to help the students achieve the intended constructions. In order to regulate the students he broke down the main goal of the activity into subgoals and further supported the students to attain these subgoals. The importance of setting subgoals to regulate students' work is implicit in the writings of many who focus on tutor-student(s) dyads (Fleer, 1992; Moschkovich, 2004). For example, Fleer (1992), using examples from the lesson transcripts of three very different teachers, attempts to identify teacher-child interaction which scaffolds scientific thinking. She notes that one of the important characteristics of science instruction that fosters conceptual understanding is the establishment of goal-oriented context between students and teachers. In another study, Moschkovich (2004), in an empirical investigation of appropriating mathematical practices, concludes that the student in her study appropriated new ways of acting by setting new goals. Her observation is that it is the act of determining goals for self-regulation that the student has appropriated. These studies, amongst others, implicitly assume the importance of regulation through subgoals; however, little research has been directed to the investigation of regulation through setting new subgoals.

Regulation through subgoals is a rather intricate issue and requires a simultaneous consideration of mutual understanding, realisation of joint attention and communication of intentions (real or perceived) amongst the participants of an activity. However, I will not attend to these intricacies for the time being but, rather, attend to them in following sections. I will here merely focus on how the scaffolder regulated the students through new subgoals. In order to understand the scaffolder's efforts to break down the main goal of the activity into subgoals, it is important to realise that the scaffolder in this study not only had a vision of the main goal of the activity but also the target construction and the possible paths that could, potentially, take the students to the achievement of that construction. These visions helped the scaffolder regulate the students through setting subgoals. The subgoals set by the scaffolder can be distinguished from the main goal of the activity but are a constitutive part of it. An important assumption that connects the regulation to the achievement of a construction is that the attainment of the main goal of the activity, which is to form the target constructions, presupposes the fulfilment of a series of these subgoals. These subgoals will later be designated as 'emergent goals' in section 6 below where I elaborate this assumption fully but here it suffices to mention this.

The most common way the scaffolder set a subgoal was via direct requests or instructing the students to perform certain actions. For example, in the second task (episode 3), when the students found it difficult to decide which part of $f(x)$ was to be reflected, the scaffolder, rather than telling them the right answer, set a subgoal to get them discover the changes in the graph of $f(x)$ when transformed into the graph of $f(|x|)$ by asking directly to examine the earlier graphs as is reproduced below (2: episode 4)

> **116I:** ... I would like you to look at and examine the graphs that you have drawn so far... and discuss which part or parts always change, and which doesn't! For example, look into the very first graph, which part remained the same and which part changed?

However, direct instruction was not the only method the scaffolder employed to set a new subgoal. He, at times, created situations which led to new subgoals. For example, in the second task (episode 5) the scaffolder challenged H when she affirmed the accuracy of the graph of $f(|x|)$ on the basis of the occurrence of symmetry in the y-axis. As a result a subgoal emerged from the scaffolder's interrogation, i.e. to justify her proposal, as is shown below.

> **144H:** So, I think this [the symmetry in the y-axis] shows that our graph is correct!
> **145I:** Can we say this graph is accurate just because there is symmetry in the y-axis?
> **147H:** ... at least I can say so!
> **148I:** How can you be so sure H?

In general, the purpose of the scaffolder regulating the students through setting a new subgoal was to put them on a path through which they could supposedly make progress and thus move closer to the achievement of the intended construction. This was, in a sense, an attempt to simplify the role of the students rather than the task in order to ensure that the demands placed on them were not too complex (Wood, 1991) because the demands of the target construction was divided into 'manageable chunks' over these subgoals. In doing so the scaffolder also reduced the students' degree of uncertainty because the subgoals gave the students certain purposes to fulfil.

The second factor was the scaffolder's interventions to reduce and handle the students' uncertainty. H&S's uncertainty about the appropriateness of their proposals, elaborations and explanations certainly appeared and reappeared throughout their developing constructions. As an example consider once again 4: episode 5. This episode occurred after the scaffolder hinted that the students should consider the absolute value signs in the expression of $|f(|x|)|$ (4: episode 4; 176I). Following this hint, H exclaimed "Aha! I got it[3]" (4: episode 5, 177H); a reaction which is likely to happen when one overcomes a mental blockage which potentially results from an uncertainty. Further to this, H&S's uncertainty with regard to their explanations and elaborations (see 181H to 184S) within this episode was quite apparent when they said, for example, "Can we *really* do this?" and "I am *not too sure* if we can!" (185H and 186S). The students in this episode were considered as acting at the constructing level (see the right hand side of the interaction flowchart in Figure 4.7. on p.108) and needed to forge new connections amongst the available structures of $f(|x|)$, $|f(x)|$ and precedence of computational priority. Surely the students' uncertainty stems from the fact that they were confronting such an issue for the first time and were thus acting in an *unfamiliar* situation and also lacked a vision of the target construction.

In fact, in the course of construction of a new structure, uncertainty seems to be inevitable. This is so because, in order to achieve a construction, one needs not only to recognise and use but also to reorganise the available knowledge structures, put them together, relate them to the new situation and forge new connections amongst them. Further to this, all these actions need to be carried out in an 'unfamiliar situation'. Indeed construction is the process through which students become acquainted and familiar with the new structure, which presupposes students' unfamiliarity with the to-be-constructed structure prior to construction. Considering that students have no clear picture or vision of the construction to be formed, for otherwise they can be said to have already constructed, they necessarily confront uncertainty, albeit at varying degrees, when striving to construct something unavailable and thus unfamiliar to them. Wood (1991) quite eloquently puts forth the relationship between unfamiliarity and uncertainty:

> When we find ourselves needing to act in a very unfamiliar situation, uncertainty is high and our capacity to attend to and remember objects, features and events within the situation is limited. ... Children, being novices of life in general, are potentially confronted with more uncertainty than the more mature and, hence, their abilities to select, remember and plan are limited in proportion. Without help in organising their attention and activity, children may be overwhelmed by uncertainty. The more knowledgeable can assist them in

[3] The original expression in Turkish was "Ha! Tamam yaa."

> organising their activities, by reducing uncertainty, breaking down a complex task into more manageable steps or stages (pp.105-106).

In the above quotation, Wood's argument, in its simplest form, suggests that uncertainty stems from unfamiliarity and that learners for whom uncertainty is very high need some support in order to reduce this uncertainty. What constitutes very high uncertainty might be debatable; however, it is quite conceivable that the scaffolder in the episode was likely to be triggered to intervene (see 4: episode 5; 187I) and set a subgoal following H&S's statements indicating their uncertainty. When we look into his 187 intervention, it can be seen that he not only gave feedback to reduce H&S's uncertainty but also specified a target ("let's draw the graph"), i.e. set a subgoal to the students.

As Wood (ibid.) hinted, one way to help students deal with the uncertainty is to organise their attention, for uncertainty is likely to distract the attention, which may result in disorganised efforts in the use of the mathematical structures at the students' disposal. Therefore, specifying a target or setting a subgoal, especially at the moments of uncertainty, is a crucial regulative strategy to help the students not only overcome their uncertainty but also to give a direction to their efforts and attention.

This brings us to the third factor of the scaffolder's intervention, to direct the students' attentions and efforts. We can surely see that the scaffolder was trying to give direction to the students' efforts by focusing their attention on particular features of the mathematical structures. For example, as H&S were working to reach the target of drawing the graph of $|f(|x|)|$ by two steps (4: episode 6), their uncertainty reappeared (193H) about how to draw $|f(|x|)|$. In response, the scaffolder intervened and labelled $f(|x|)$ as $g(x)$ (194I) implying that drawing the graph of $|f(|x|)|$ was the same as drawing the graph of $|g(x)|$, and this was well appreciated by the students (198H and 199S). The scaffolder further intervened (200I) to direct H&S's attentions and efforts to apply their knowledge of $|f(x)|$ to the already-drawn graph of $f(|x|)$. The scaffolder's position in this episode, and allegedly throughout their work, might be best portrayed by Bruner's term as serving the students as a "vicarious form of consciousness" (Bruner, 1985, p.24) due to the fact that the scaffolder knew what H&S needed to attend to owing to his vision of the target construction which allowed him to bring subtle aspects to the students' attention, which might otherwise have been blocked or been indiscernible.

It was in this episode that H&S constructed a new method to draw the graph of $|f(|x|)|$, for which the scaffolder played an important role in the management of their attentions and efforts. The management of attention in collaborative learning environments is indeed considered as critically important to achieve new learning (Barron, 2003; van Oers, 2001). Mason (1989) has written about the importance of shifts in one's attention to which he attributed a focal role in doing and learning mathematics. In order for a shift of attention to come about he, with others, draws attention to the vitality of students working with an agent who, in a situation, has an awareness of what to attend to and what to ignore. In this connection Mason and Spence (1999) argue that

> coming to know is essentially a matter of shifts in the structure of attention, in what is attended to, in what is stressed and what consequently ignored with what connections ... Knowing is not a simple matter of accumulation. It is rather a state of awareness, of preparedness to see in the moment. That is why it is so vital for students to have the opportunity to *be in the presence of someone who is aware of the awarenesses that constitute their mathematical 'seeing'*(p.151, emphasis added)."

It is not perfectly clear what 'mathematical seeing' might be; however, from their writings it might be inferred that 'seeing' is realising what is known and/or knowing to use it. In the case of construction, this involves successful use of available structures (building-with) when required within the situation. However, as Mason and Spence (ibid., p.135) propose, use of available knowledge "depends on the structure of attention in the moment, depends on what one is aware of" because "no-one can act if they are unaware of a possibility to act; no-one can act unless they have an act to perform." By implication, this proposition suggests that if students are not aware of the importance and necessity of their knowledge for a new construction, they are unable to make use of them as they (or their attention) are 'blocked'. In fact this was the case for H&S when they were working on the fourth task. They failed to recognise the connection between their knowledge of $|f(x)|$ and $f(|x|)$, and the expected construction of $|f(|x|)|$. Unless one is aware of the potential of these structures, one cannot use them in a new construction. It is in this connection that the scaffolder's efforts to direct H&S attentions and efforts become clearer. In the fourth task (episode 3), the scaffolder first brought $|f(x)|$ and $f(|x|)$ to H&S's attention and got them to recollect what they know about these structures. Later, at the outset of the fourth episode, he drew the students attention to the expression of $|f(|x|)|$ and suggested viewing this as a combination of $|f(x)|$ and $f(|x|)$ (165I). Only then, only after ensuring that these two structures were now in the students' focus of attention, did he set a new subgoal by inviting them to work out an idea as to how to use $|f(x)|$ and $f(|x|)$ in connection with $|f(|x|)|$ (168I).

The fourth factor identified in the scaffolder's interventions was the elicitation of 'deep explanatory reasoning'. I intentionally chose the expression 'deep explanatory reasoning' over 'explanation' in order to emphasise that the scaffolder was not 'just' concerned to elicit any explanation but rather explanations rooted in mathematical reasoning. But what exactly is the difference between these two? Deep explanatory reasoning is a specific form of explanation and the most elementary feature that distinguishes these two, as I see it, is the 'depth' of reasoning on the basis of interrelationships amongst mathematical structures involved in an explanation and required by the situation. In order to elucidate this rather obscure-looking differentiation, let us consider two examples from the second task, both were articulated by S and each could be counted as an explanation but only the second one, in my opinion, involves deep explanatory reasoning.

109S (episode 3): Yes, I mean shouldn't we take the symmetry of this part? Look at this graph; the part of $f(x)$ until the y-axis is unchanged and the remaining part is reflected. I mean if I name the rays as I and II ... shouldn't II be reflected in [the line of] y=2?

136S (episode 4): Because positive values of x remains unchanged in the absolute value sign, but negative values of x must be multiplied by minus to go out of the absolute value... thus we can say that whatever changes occur in the graph of $f(x)$, it must be at the negative values of x.

In both of these utterances, S was arguing that the segment of $f(x)$ at x<0 should be reflected or 'changed' when it is transformed into the graph of $f(|x|)$. As can be clearly seen, these two 'explanation' type of utterances are rather different in that the first essentially relies on the ostensible features of the graph, lacking depth of mathematical reasoning, whereas the second capitalises on the interrelationships amongst the involved mathematical structures. At this juncture it should be noted that I am not arguing that the first type of explanation utterances are devoid of benefit; quite the contrary they might be of crucial importance in leading to deeper level of understandings. However, strength and significance of the second type of utterances are undeniable particularly in the course of construction which requires assembling knowledge artefacts to produce new ones through forging new connections and this is quite likely through mathematical reasoning.

In this study the scaffolder seemed to be more concerned with the elicitation of deep explanatory reasoning. To do so he usually asked deep reasoning questions. But just which ones are those? Graesser and his colleagues have done a considerable amount of work on this matter and they define these questions as the ones that manifest logical reasoning, causal reasoning, or goal-oriented reasoning, and questions starting with 'why', 'how', 'why not', 'what if' (see Graesser and Person, 1994; Graesser, Person and Magliano, 1995 for further details). The scaffolder's use of these questions was not inadvertent; he intentionally employed them to spark off an 'intellectual push' on the part of the students. For instance in the second task, the construction of the 'y-symmetry' method was basically scaffolded by his demand on H to provide answers to deep reasoning questions (see 2: episode 5; 148I, 152I and 155I). His desire to elicit deep explanatory reasoning can further be seen in his efforts to initiate the beginning of mathematical reasoning. As an example, consider the utterances below extracted from 2: episode 4.

117S: (...) the graph of $f(x)$ until the y-axis didn't change, and after y [axis], it has changed.
118I: Do you mean that the part of $f(x)$ at the positive x, which is always on the right of the y-axis, doesn't change?

S's explanation in 117 has come about in response to the scaffolder's explicit instruction to decide which segment of $f(x)$ (at x<0 or x>0) changes in the graph of $f(|x|)$. However, her account was simply vague ("$f(x)$ until the y-axis") and devoid of structural features necessary to develop deep explanatory reasoning. By intervening in 118, the scaffolder, suggesting a 'recap' of S's explanation (as if it were what she meant!), hinted at which segment did not change and purposefully brought in some critical features of this segment into the students' focus of attention

("$f(x)$ at *positive x*", which S did not mention) to initiate a reasoning step to justify why it does not change. Indeed in the rest of this episode, the students produced explanations involving deep reasoning by focusing on the positive values of x and relating it to the features of absolute value and hence formed the expected construction, though with the help of the scaffolder.

The fifth factor was with regard to the remediation of errors. It is quite clear that the scaffolder easily detected and attempted to remedy the errors and deficiencies in H&S's explanations, elaborations and proposals. In fact it is a common finding of studies into human tutoring that if students make an error and are unable to detect it immediately themselves, the tutors almost always point it out in order to ensure that solution does not stray very far from a correct solution path (VanLehn, Siler, Murray, Yamauchi, & Bagget, 2003). The scaffolder in this study tried not to tell the students directly what the error or deficiency was, though he at times explicitly pointed out the existence of a problem. Rather than supplying the students with the correct information he tried to get them to correct their errors by themselves. For example, by the end of episode 6 in the second task, although the students constructed two methods, i.e. 'reflecting' and '*y*-symmetry', they were not able to draw the distinction between these two and they tended to regard them as if they were the same "thing" (182H and 185S). Having realised this deficiency, the scaffolder explicitly stated the existence of two different methods and asked them to distinguish between the two (2: episode 7, 186I and 190I). As a result, H&S not only articulated these two methods but also managed to draw the distinction, thus remedying their deficiency. However, the scaffolder was not always successful in his efforts to remedy the students' deficiencies. As mentioned before, he failed, owing to the lack of intersubjectivity, to get H&S to realise the vagueness of their 'reflecting' method when they first proposed it.

I have hitherto explored five factors of scaffolder's interventions in connection with their impact on the students' developing constructions. Having observed their positive effects on the construction of new structures not only in H&S's work but also in the other scaffolded students' work, I here attempted to relate my observations to the relevant literatures. I believe that these factors could help us develop some important scaffolding strategies. These strategies could target setting subgoals, reducing uncertainties, directing attentions, eliciting deep explanatory reasoning and remediation of student errors. These strategies could even provide us with a framework to analyse and evaluate the scaffolders' effectiveness. However, further empirical research is clearly needed to investigate the effect of these observed factors in some other studies before obtaining an analytical framework on the basis of the arguments provided here.

In this section, my consideration focused on the establishment of some causative relationships between the scaffolder's interventions and the students' developing constructions. In the next section, I will scrutinise how the scaffolder's interventions lead the students to the formation of a new construction. In this respect I will go into a detailed elaboration of the interventions in relation to epistemic actions with a particular emphasis on constructing actions and attempt to develop the idea of scaffolder's mediating role.

3.2. Mediation of constructions through the scaffolder's interventions

In the fields of (socio)cultural psychology and education, the idea of mediated action owes a great deal to the works of Vygotsky. He proposes that higher mental processes and human actions in general are mediated by tools (Vygotsky, 1978, 1986). Vygotsky distinguishes two general classes of tools which can mediate the actions: technical and psychological tools (1981a, p.137). Technical tools, e.g., a knife or a hammer, are directed at the objects and processes in nature whereas psychological tools mediate human's own mental functioning:

> By being included in the process of behaviour, the psychological tool alters the entire flow and structure of mental functions. It does this by determining the structure of a new instrumental act just as a technical tool alters the process of a natural adaptation by determining the form of labour operations (1981a, p.137).

But what are these psychological tools? Vygotsky provides a suggestive list, rather than a definition, which includes "language; various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams, maps, and mechanical drawings; all sort of conventional signs; etc. (ibid., p.137)." Yet the features of the mediating tools (e.g., material or ideal or both? tools or artefacts?) are a matter of debate amongst different parties involved in the studies of mediated action (Daniels, 2001; Wells, 1999; Kozulin, 1998; Wertsch, 1998; Cole, 1996). A detailed consideration of these debates is not the interest of this study; however what is clear from those debates is that most of human mental actions and psychological functions of perception, memory, attention and so forth are mediated by psychological tools[4] which are "inherently situated culturally, institutionally, and historically" (Wertsch, 1998, p.24).

However, in Vygotsky's account mediators of human actions and psychological functions are not limited to technical and psychological tools; he also envisioned a third class of mediators, that is, other human beings (Kozulin, 1998). It is in this connection that I wish to pursue the idea of scaffolder acting as a mediator for the students' epistemic actions and hence constructions. In this section my intention is to establish the conditions for such mediation to come about and discuss its implications (and importance) for the students' developing constructions. However, I first need to consider the idea of human mediator in more detail.

The idea of mediation through another individual (often an adult) is implicitly and explicitly mentioned throughout Vygotsky's writings. An implicit indication of this idea can be seen in his well-known statement of general genetic law of cultural development (1978, 1981b), that is "every function in the child's cultural development appears twice: first, on the social level, and

[4] Wertsch (1998) prefers using 'mediational means' as a generic term which subsumes technical as well as psychological tools.

later, on individual level; first *between* people [intermental] and then *inside* the child [intramental] (1978, p.57)." The implicit mediating role of an adult assumed in this statement can be seen in his description of the emergence of meaning for what he calls "indicatory gesture" (1981b, p.160). According to Vygotsky, when initially a child tries to grasp an object which is beyond his/her reach, his/her hands stop and hover in mid-air. "Here we have a child's movements", writes Vygotsky (ibid, p.161), that "do nothing more than objectively indicate an object". However, when the mother comprehends the importance of the movement as an indicator, an essential change occurs in the situation.

> The indicatory gesture becomes a gesture for others. In response to the child's unsuccessful grasping movement, a response emerges not on the part of the object but on the part of *another human*. Thus *other people* introduce the primary sense into this unsuccessful grasping movement. And only afterward, owing to the fact [that] they have already connected the unsuccessful grasping movement with the whole objective situation, do children themselves begin to use movement as an indication (ibid, 161; emphasis added).

Perhaps the most explicit reference to the mediating role of other humans within a situation can be seen in a citation provided by Ivic (1989, p.429):

> In 1932 Vygotsky wrote that: It is through the *mediation of others*, through the *mediation of the adult* that the child undertakes activities. Absolutely everything in the behaviour of the child is merged and rooted in social relations. Thus, the child's relations with reality are from the start social relations, so that the newborn baby could be said to be in the highest degrees a social being (emphasis added).

Surely this general position is again closely linked to Vygotsky's general genetic law but here Ivic explicitly spells out the mediating role of the adult for the children to undertake an activity. Further to this, as can be seen from these two quotations, adult (or another individual) is given a focal position as a mediator of 'meaning' – "the child's relations with reality are from the start social relations". Vygotsky's interest in human mediation in this sense is partly motivated by his desire to account for how external speech experienced first in interaction with *others* on the intermental plane is gradually 'internalised' and becomes inner speech for self-directed mental activity in the intramental plane. Vygotsky finds the connection between these two planes in the mediating function of signs and particularly of speech; and the adult's role in this connection is conceived as "a carrier of signs, symbols, and meanings" (Kozulin, 1998, p.64). Thus he focuses on language and other sign systems (i.e. psychological tools) in terms of how they are part of and mediate human actions (Wertsch, 1991).

Thus Vygotsky's theory of human mediation, in its simplest form, suggests that an individual acts as a mediator of meaning and thus functions as a vehicle of symbolic tools in the activities of and in communication with other people in sociocultural settings via use of language. But does an individual act as a mediator of meaning (e.g., of indicatory gesture) in any activity or under any circumstances? Vygotsky hints that human mediation is not always possible and depends on some conditions. For example in his example of indicatory gesture, he suggests that emergence of meaning of this gesture requires the mother to *comprehend* the child's movement as an indicator and only then "the situation changes in an essential way" (Vygotsky, 1981b, p.161). Unfortunately Vygotsky does not elaborate the activities of human mediators beyond their function as a vehicle of symbolic tools and thus he does not clarify the conditions under which this mediation can occur.

This lacuna has also been recognised by Kozulin (1998) who attempts to take Vygotsky's ideas of human mediation further and suggests some conditions necessary for such a mediation to come about. To do so, he heavily draws on the works of Reuven Feuerstein who, with others, developed the theory of mediated learning experience (MLE) (see Feuerstein, Rand and Rynders, 1988). MLE was originally designed to aid the learning process of retarded children who had not learned through traditional cultural mediation of their parents. The theory is particularly interested in the causes of the individual differences in cognitive development. Although the theory accepts the effects of organismic and environmental factors in the performance of retarded children, it also argues that the effect of these factors can be minimised through mediation of an adult, resulting in significant improvements in cognitive-developmental outcomes. According to MLE this mediation and the resulting positive effects can only be possible if the mediated interaction meet four "universal" criteria: intentionality, reciprocity, meaning and transcendence (Feuerstein et al., 1988, p.62).

Intentionality refers to the adult's efforts to infuse a learning situation with a sense of purpose by "constantly focusing on the child's state of attention, problem-solving strategies, mistakes, and insights" and is considered as necessary to transform the "interactive situation from accidental into purposeful" (Kozulin, 1998, p.66). Hence mediation requires intentional acts and behaviours on the part of the adult within the given learning situation. However, the adult is not the only actor of this situation and the learner's response is also crucial in the mediated interaction. This is related to the second criterion of reciprocity which can be said to exist if the learner responds to the adult's intentional acts and behaviours in some way, whether it be verbal, non-verbal or even visual (Collins, 2001). Reciprocity highlights the active involvement of the learner in the learning situation and on the adult's efforts to ensure this. However, in order to respond to the adult's intentional acts, the learner needs, at least, to have a sense of intention and see the relevance. This constitutes the third criterion, that of meaning which emphasises conducting a learning event in a meaningful way for otherwise it might become "a mere sequence of strange, behavioural acts – devoid of purpose and affective investment, the situations loses its mediational potential" (Kozulin, 1998, p.67). The final, and most important, criterion is transcendence. A mediated interaction is transcendent in nature if the adult is concerned with going beyond the content and context of the interaction and brings about some changes on the part of learners (e.g., enriching their repertoire of knowledge) beyond their immediate needs (Feuerstein et al., 1988).

I have so far briefly sketched out two theories of human mediation in the schools of Vygotsky and of Feuerstein. I recognise that this issue has not been treated in the same way in those schools which employed the term with different purposes and within different explanatory frameworks as briefly mentioned above. However, I found strong complementary features in both of those theories which assisted me in developing insights with regard to mediation of the students' developing constructions through the scaffolder's interventions in scaffolded discourse.

In order to clarify the complementarity that I have in mind, I need to be more precise about what is meant by mediation and for this purpose I return to Vygotsky (1981a). In his explanation of mediated action through psychological tools, Vygotsky gives the examples of a knot in a handkerchief, a string on one's finger and a mnemonic scheme (p.138) in an act of, say, remembering. In order to elucidate the mediation, let us take a concrete example where a string is used to mediate remembering. Imagine someone going shopping who wants to remember to buy, say, bread and attaches a string to his/her finger to ensure he/she remembers this. In the market, if this person remembers to buy the bread when he/she sees the piece of string, then this act of remembering is mediated through this string. If this person remembers to buy the bread even without using the string, then the act of remembering is not mediated by the string, though it is still attached to his/her finger. By analogy, albeit like any analogy it has its limits, my position is that if the students in scaffolded discourse undertake an action through the scaffolder's intervention, whether it be as a response to a question, following an explanation, prompted and so on, then this action is, I argue, mediated by the scaffolder's intervention. To give a concrete example, consider the utterances below taken from 4: episode 4:

> **165I:** Ok, if you pay a closer attention to the equation... I mean look at the expression itself, $|f(|x|)|$, it is a combination of these two [of $|f(x)|$ and $f(|x|)$]. Do you see that?
>
> ......
>
> **170H:** Look I think it makes sense! if $|f(|x|)|$ is a combination of $f(|x|)$ and $|f(x)|$, can we think about it like a computation with parentheses?

H's act of recognition that "$|f(|x|)|$ is a combination of $|f(x)|$ and $f(|x|)$" is mediated because it follows the scaffolder's prompt in 165 after which she exclaims "it makes sense!". Please note that I do not see H's utterance as a simple copy or repetition of the scaffolder's utterance because she is now using this recognition in connecting the expression of $|f(|x|)|$ to the 'computational priority' (building-with) and even gave an example (see 172H). Thus, in H's utterance not only is the act of recognising but also the resulting building-with is mediated through the scaffolder's utterance in 165.

Although I am talking about human mediation, I am focusing on the discourse and language employed through speech. The question arises here: is this not a reduction of human mediation to the use of language? This is a perplexing issue and some reflections are necessary. I recognise that human mediation is not limited to the use of language. This point is convincingly argued in the studies of Feuerstein, Wertsch, Rogoff and others. For instance Rogoff (1990) demonstrates the typicality and importance of nonverbal forms of communication and context manipulation for children's 'guided participation' in the cultural practices of some non-western countries.

Moreover, in any interactive situation, a significant amount of information is also communicated nonverbally such as gaze directions, gestures and facial expressions (see Fox, 1993). These clearly indicate the possibility and existence of nonverbal mediation and force us to reject a reductionist view of human mediation to the use of language. However, focusing on language to understand human mediation does not distort the phenomenon itself nor does it invalidate the insights gathered. This is so first because, as Vygotsky suggests, language functions as a mediator of the social activity by enabling participants to plan, coordinate and review their actions through external speech (see Wells, 1999, pp.6-7). Second because it is impossible to separate the speaker from the speech which does not come into being by itself but is uttered with certain intentions. Third because my intention here is not to provide a fine-grained account of human mediation but to develop some insights as to how the achievement of new constructions are mediated by a human mediator (i.e. scaffolder) in a discursive activity. Fourth because when we analyse human mediators' functions in discursive activities, we heavily rely on the language employed in this course, whether spoken or written, since this is one of the most elementary ways that an understanding of their function can be achieved. This is especially true in the case of this study which only has the spoken words of the participants. Thus in what follows I will focus on the scaffolder's interventions through his utterances in explaining his function of mediation, with an awareness that there is more to human mediation than the use of language.

Considering my focus on the scaffolder as a mediator whose discursive interventions are under scrutiny, some authors such as Wertsch (1998) would, for example, argue that any action is undertaken with a cultural tool ("individual(s)-acting-with-mediational-means", 1991, p.12) and might also argue that focusing merely on the scaffolder as a mediator may not reflect the whole picture. It is certainly true that other cultural tools which are semiotic in nature (such as the equations, expressions such as $f(|x|)$ or $|f(|x|)|$ and absolute value sign), are employed by the participants in the course of their interaction and communication. It is also true that a sole focus on the scaffolder as a mediator may not reflect the whole picture, but the whole picture is too complex to capture 'at one shot'. My effort to focus on the scaffolder as a mediator is not to be construed as a rejection of the semiotic tools employed by the individuals in undertaking an action, but, rather, as privileging a certain aspect of the phenomena under investigation to gain some insights into the mediating role of the scaffolder, and doing this with an awareness of the existence and importance of other mediational means.

So far I tried to state my understanding of what mediation is (and is not), exemplified some incidents where the scaffolder mediated the recognising and building-with actions, established the rationale behind my interest in the scaffolder's interventions to appreciate his mediating role and the rationale behind my ignorance of the other mediational means in my discussion. In doing so I hope to have laid down the building blocks for my proceeding discussion. I now return to my main concern in this section: what roles does the scaffolder's mediation play in the students' developing constructions? An obvious answer is that mediation facilitated the students' constructions. This is easily justified when we consider the previous subsection where it was argued that the scaffolder reduced the students uncertainty, directed their attention and so on. These do have some facilitating effects on the students' achievement of constructions. However, repeated readings of the scaffolded students' verbal data suggested to me that the essence of the mediation does not lie in this facilitating effect and I realised that the scaffolder's mediation through his interventions led to remarkable changes in the students' performance which otherwise did not seem to be likely. To understand this, Vygotsky's discussion of the effect of psychological tools was useful and he, in this regard, writes that "psychological tool alters the entire flow and structure of mental functions" (Vygotsky, 1981a, p.137). Reading his arguments and considering the students' data made me realise that the significance of the scaffolder's mediation does not stem from that the interventions simply facilitate the students' performance that could have occurred without them; instead from the fact that it brings about some essential transformations in the students' ways of seeing, talking and acting.

But does any intervention mediate students' actions and thus result in these essential transformations? On closer inspection of the data, I realised that the scaffolder at times succeeded and at other times failed to act as a mediator. In an explanation of the scaffolder's failure or success to mediate the students' constructions, I found Feuerstein's theory of mediation useful and it is in this connection that Vygotsky's and Feuerstein's theories are complementary in relation to my data. That is, I realised that the transformations that I mentioned are likely to occur when the scaffolder's mediation meet the four criteria put forward by Feuerstein and his colleagues: intentionality, reciprocity, meaning and transcendence.

I will now return to the protocol data and discuss how the scaffolder enacted his mediating role; the resulting changes in the students' way of seeing, talking and acting; and the necessity of the four conditions for a construction to be mediated. In order to understand the mediating role of the scaffolder, I focused on his pedagogical interventions. The extent of the resulting transformations can be best understood when we compare the episodes where the scaffolder intervened with the ones where he did not intervene. For this reason I will make some comparative comments between episodes 3 and 4 in the second task. In the former episode the students were acting mainly at the building-with level; and the scaffolder intervened only once (108I). The latter episode was marked with students acting at the constructing level; achieved the construction of the 'reflecting method'; and the scaffolder's interventions and the effects were all too apparent.

In episode 3, H&S were talking about if the graph obtained for question 4-C was correct or not, about which segment of $f(x)$ (at $x<0$ or $x>0$) should be reflected and about the reasons for these. After a lengthy argumentation, however, they could not convince each other about their views on these issues. The students' difficulty stemmed from the deficiency of their 'reflecting' method with regard to which segment of $f(x)$ was reflected in the horizontal line intercepting the y-axis. Having realised this, at the very beginning of episode 4, the scaffolder intervened and asked the students to look at the earlier graphs and decide which segment of $f(x)$ had changed and which had not (116I). This intervention was surely intentional and intention certainly transcended the immediate needs of the students in that the scaffolder was not just concerned to get H&S to report on some previous graphs so that they could see who was correct but his intention was to get the students to recognise changes occurring in the graphs and thus amend their ambiguous method. S's response "the graph of $f(x)$ until the y-axis didn't change" (117S) was rather ambiguous and did not meet the transcending intention of the scaffolder. The scaffolder once again intervened in 118 and 'rephrased' S's response by clearly indicating the important properties of this segment of $f(x)$ ("part of $f(x)$ at the *positive* x, which is on the *right*..."), recognition of which was essential for the students to fulfil the transcendent intention of the scaffolder, that is, constructing the 'reflecting method'. Immediately following this, S in 119 reproduced the scaffolder's utterance almost completely but this was not a simple copy of what has been said to her as she did this with her own recognition (because she was now able to use it, see 123, 125 and 127), which is mediated through the scaffolder's intervention.

Following the initial three interventions (116I, 118I and 122I) an essential transformation in S's way of seeing is apparent (see 123, 125 and 127). She was now looking at a graph of $f(|x|)$, and seeing in it a particular line segment ("on the right side of the y-axis"; 119S) with some particular qualities (positive xs; 119S); seeing in it a relation of absolute value sign (127S); seeing a 'link' in it with the expression of $f(|x|)$ (125S); and seeing a reason in it for being unchanged (123). The extent of this transformation becomes clearer when one looks into S' way of seeing these graphs in episode 3 where she merely observed and used their ostensible features such as some "parts" and "symmetries" (e.g., 107 and 109). Soon H also got attuned to S and adopted S's perspective (128H). Throughout this episode, particularly from 127 onwards, there was a dramatic change in the way that these students talked about the graphs of $f(|x|)$. To better appreciate this change, first consider episode 3 in which H&S's talk was chiefly based on their beliefs and expectations (see how frequent the auxiliary verb 'should' was used in their talk); and their reasoning, if ever stated, remained merely related to the similarities and differences of the graphs (e.g., 107 and 109). Nevertheless in episode 4, their talk involved explanations with deep mathematical reasoning rather than their beliefs and expressed almost always with certainty (see how often "must" was used; e.g., 132H and 136S). During this episode the scaffolder sustained H&S's ways of talking and seeing by initiating reasoning steps or asking deep reasoning questions (131I, 133I and 135I) and thus mediating their constructions.

Finally, these differences in their ways of seeing and talking were also reflected in their actions. In the third episode H&S were acting at the recognising and/or the building-with level as they were usually concerned with justifying their statements without drawing on mathematical features of the involved structures (e.g. 109S). However, in episode 4, their actions were transformed into constructing actions as they were now seeing and talking about deep structural relationships by using essential knowledge artefacts (e.g. 136S). Two aspects of episode 4 are noted: first that interaction amongst the participants were reciprocal as they all participated in it and made significant contributions and second that the students appreciated the scaffolder's interventions and were able to draw on what has been brought to their attention (see 118-119 and 130-131-132) and in this sense 'meaning' was successfully conveyed to the students.

Exactly similar transformations as a result of the scaffolder's mediation can be observed in the fourth task. In episode 3, the scaffolder mediated the students' recognising actions by bringing the structures of $|f(x)|$ and $f(|x|)$ to the focus of their attention. In episode 4, after his suggestion of considering $|f(|x|)|$ as a combination of $|f(x)|$ and $f(|x|)$, a transformation started in the students' seeing (seeing "precedence of operations" in the expression $|f(|x|)|$; see episode 4), talking (talking about the graphs of $|f(x)|$ and $f(|x|)$ *in* $|f(|x|)|$; see episode 5) and acting (merging the graphs of $|f(x)|$ and $f(|x|)$ into a single graph; see episode 6). These transformations were maintained through the scaffolder's interventions (see especially 176I, 187I, 194I and 201I). The importance of these transformations resulting from the scaffolder's mediation can be better appreciated when we compare H&S's work in episode 4, 5 and 6 with the first half of their work (until 145H) where the scaffolder intentionally limited his assistance and consequently H&S stated their intention to give up developing a method other than substitution (see 4: episode 2).

An important aspect of the mediation is that the interventions which meditate the students' constructing actions themselves can be considered as constituting constructing actions *for* the students. In other words, at the moments of mediation, even when the scaffolder intervenes and does the talking, and the students are listening and attending to him, the scaffolder's utterances, I argue, can be counted as constructing on the part of students. This was the case in 2: episode 4 discussed just above. This episode has a unique characteristic in that all of the utterances, whether be it from the scaffolder or the students, were classified as involving constructing actions, as can be seen in the right hand side of the interaction flowchart created for this episode (see Figure 4.2. on p.88).

The inevitable question is: how can the scaffolder's utterances be considered to constitute constructing actions *for* the students despite the fact that these actions are subjective and occur on the part of the students? A satisfactory answer to this question is not straightforward and it is intimately related to one's understanding and interpretation of constructing actions. I believe that construction is a continuous process and involves increasing clarification and progressive evolution of the initial form of a to-be-constructed structure. Construction is achieved through constructing actions which are not discrete and dispersed but, rather, are strongly related to one another (see Chapter 7, section 2 for more on this). Thus, I posit, during the mediating interventions, as is the case in episode 4, the students' construction still continues even if they are not talking but simply listening to the scaffolder and attending to his interventions which represent constructing actions on the part of the students.

What evidence can I give to justify this? To begin with, if one does not consider the scaffolder's utterances in this episode as constructing actions on the part of the students, then one necessarily assumes that construction only occurs when the students are talking and thus producing utterances. This necessarily implies that if and when the scaffolder intervenes and contributes to the activity, the students' construction process stops and it later resumes when they start talking. This view forces us to regard the constructing actions as discrete and dispersed and only limited to the students' specific utterances in an isolated manner. The danger involved in this view is that construction is not considered as a continuously developing process and reduces it to the individual's verbalisations. In addition, this view renders the essence of mediation meaningless in that it inevitably draws a major distinction between cognitive and social processes. If we follow this line of reasoning and accept an existence of such a distinction, then where does the cognition reside: 'in the individual's minds' separated from its social origins? Like Rogoff (1995), I believe, and indeed the analysis of the data made me believe, that cognitive development is a social development i.e. specific processes that the participants communicate and interact with are the substance of cognitive development. I will return to this issue in section 4 below, but as an illustration of the kind of phenomenon that I have in mind, for now, let us reconsider episode 4 and examine it closer. For example, consider the utterances 130-131-132; if we put them altogether, we obtain:

> "**(130I)** On the other hand, negative values of x, which are on the left side of the y-axis... **(131S)** They have to change... I mean in the absolute values sign, negative values change... **(132H)** [Yeah, I agree]... I mean absolute value sign is outside of x... so... so negative values of x must be different."

In 130 the scaffolder initiates the beginning of a reasoning step; in 131 S takes this further with the same line of reasoning; and 132 H completes the reasoning step. Thus if the bit in the square bracket is omitted, these three utterances from three participants become seamless with the same line of reasoning, with the same focus and with the same goal; resulting in a realisation that "whatever changes occur in the graph of $f(x)$, it must be at the negative values of x" (136S; see also 137H). This, I believe, suggests that even when the scaffolder is talking or intervening, or put it another way, when the students do not utter anything but attend to the scaffolder's intervention, construction does not stop; quite the contrary it continues to evolve. This simply displays the difficulty of drawing a major distinction between social and cognitive processes.

However, as already mentioned, in order for a mediation and the resulting effects, as discussed so far, on the students to occur, the scaffolder's interventions need to meet four conditions: intentionality, reciprocity, meaning and transcendence. With regard to the first condition, in every scaffolded situation, with no exception, the scaffolder's assisting interventions are intentional (whether they fulfil their intention or not is another matter). Although the scaffolding literature rarely makes explicit reference to the intentionality of the scaffolder, thinking of the scaffolding situation without intentionality renders the scaffolder's role in working with students random. The very essence of scaffolding process relies on giving assistance to the students and adjusting the amount and type of it depending on the students' current level of performance; and surely these are not possible without having an intention to do so.

The conditions of reciprocity and meaning are basically concerned with how an assisting intervention is responded to by the learners, which determines if and when a mediation takes place. More specifically, and simply, reciprocity refers to students' active involvement in the activity and their responsiveness to the interventions. Unless one has taken an action in response to an intervention, mediation is non-existent. Meaning refers to the relevance of the scaffolder's intervention to the students' current efforts *and* how and if the students perceive the relevance of the intervention. A closer inspection of the verbal protocols reveals that in the episodes where the students achieve constructions, the scaffolder's interventions met these two criteria (see the discussion of episode 4 given above). However, when he failed to meet the criteria of reciprocity and/or meaning, his interventions also failed to mediate constructions. Thus the data suggest that key to a mediation in a scaffolded situation is an achievement of reciprocity and meaning. In order to illustrate this argument, I consider episode 2 and 5 in the second task.

In the beginning of episode 2, the students successfully sketched the graph of $f(|x|)$ for the given graph in question 4-B by using their reflecting method despite all its ambiguity. They then decided to check the accuracy of this graph (75H) and at this very moment the scaffolder intervened and asked which segment of $f(x)$ was reflected (76I). His purpose was to draw H&S's attention to the ambiguity of their 'reflecting' method. However, this intervention was not immediately relevant to what H&S were doing at that moment and the scaffolder did not clarify the relevance either. As a result the students' answer was completely irrelevant to the intervention; rather than explaining which segment was reflected they talked about the symmetry line (77H and 78S). The scaffolder tried twice more by rephrasing his question (82I and 86I) but again the students did not see the purpose (simply because it was not conveyed to them) and their response remained extraneous to the scaffolder's intention. If the scaffolder explained his concern, made the problematic point in H&S's 'reflecting' method clear or got them to realise his intention, his interventions might have been successful to mediate the students' constructions. Yet he simply failed as his interventions were not meaningful to the students.

In episode 5, the story was rather different. H constructed the '*y*-symmetry' method through the scaffolder's interventions but not S. The reason seems to be that the interaction amongst the participants was not reciprocal in that H was actively involved and produced the essence of this episode but S could not actively contribute to the interaction. Further to this, H was producing convincing mathematical explanations in attending to the scaffolder's interventions whereas S was making slight contributions (usually in the form of agreement and some shallow explanations, see Figure 4.3., p.93). I also speculate that the whole experience in this episode was not quite meaningful to S. This is because when H suggested using the '*y*-symmetry' method in episode 6, S did not understand what H meant or how to draw the graph by using this method despite H's demonstration (see 2: episode 6) and despite the fact that throughout episode 5 S constantly agreed with H's explanations (except in one incident, 146S). This suggests the existence of some strong differences in their perspectives as S could not see the relevance between H's explanations and H's construction of the '*y*-symmetry' method. Thus when the scaffolder's interventions do not achieve reciprocity, then mediation did not appear to happen. The issues of reciprocity and meaning have clear links with that of intersubjectivity to which I will attend in section 5 below.

The final condition is that of transcendence which is basically concerned with whether the learner(s) is enriched with new knowledge. In order to clarify this, let us consider a hypothetical situation in which a student is working with the scaffolder on the second task of this study. This student can complete the task by drawing all the graphs of $f(|x|)$ without developing a general method to draw the graphs of these kinds. Then the condition of transcendence suggests that mediation did not take place in this case because the learning experience remained limited only to the immediate need that elicited it, i.e. to complete the task by drawing the graphs. Thus when this condition is translated into the context of this study, it simply suggests that in order to say a construction is mediated, one needs to show that a new construction is achieved. When we return to H&S' verbal data, there were two constructions in the second task (the 'reflecting' and 'y-symmetry' methods) and one in the fourth task (the 'two-step' method). The students' accounts and applications surely showed that these constructions were achieved. In this sense the scaffolder's interventions met the condition of transcendence in the episodes in which these constructions were achieved.

On the basis of these considerations, an ideal characterisation of the interventions in mediating constructions might be formulated as follows: these interventions should be intentional; meaningfully conveyed to the students who are to be actively involved and mutually engaged in the activity; and result in an enrichment on the part of the students with a new construction(s).

4. The nature of the 'social' in the formation of constructions

In the preceding subsection cognitive processes have been viewed as closely linked to social processes and the difficulty of drawing a major distinction between the two was also emphasised. But some important issues were left unattended. For example, in the previous subsection, the social processes were elaborated on the basis of the scaffolder's interventions and of the resulting changes in the students' constructions. Then the inevitable question arises: is the link between social and cognitive processes limited to the 'more knowledgeable' scaffolder's interventions and hence influences? If not, what influences do the students have in social processes? Further to this, does an emphasis on the inseparability of social and cognitive processes suggest that a mathematical construction is not achievable without 'social interaction'? Some show situations in which a 'lone learner' is working on a task, 'thinking hard', and consequently able to achieve new constructions; see Kidron and Dreyfus (2004) for a description of such a learner. So, if the construction is a 'social' process, how can one explain these cases? Or if the construction does not necessarily involve 'social interactions', how can one talk about the inseparability of social and cognitive processes?

Certainly all these questions are legitimate and warrant further considerations. Key to these considerations is the notion of the 'social': what do we mean by the social and social interaction? The term 'social' in the context of this study is employed in two senses. One sense of the social is concerned with the participation of more than one individual in the immediate context of the activity. I will use the term 'social interaction' when I refer to the verbal communication amongst the immediate participants of an activity. The second sense of the 'social' is concerned with the involvement of 'others' who are not physically present within the immediate context of the activity but whose 'voices' speak through and interact with the voices of physically present participants. The first sense of the social is fairly clear and reflects a common use of the term; and indeed most of the research in the sociocultural tradition has long dealt with the term in this sense (see, for example, Rogoff, 1990; Mercer, 1995). However, although it resonates with the writings of many (e.g., Wertsch, 1998), the second sense is not as clear as the first one and is in need of explication. My understanding of the social in these two senses, as will be discussed throughout, reflects a broader and more cultural and historical view. My interpretation of the social in these two senses was strongly influenced by Bakhtin (1981, 1986) and his writings on such notions as 'utterance', 'voice' and 'dialogicality'. Therefore I will first briefly detail some relevant Bakhtinian concepts and subsequently attend to the questions posed above and hence further develop and ponder on the influences of the social forces operating in the formation of mathematical constructions.

4.1. Some central Bakhtinian concepts

Bakhtin (1984) was interested in creative aspects of human language and found this creativity, to a great extent, in the plurality of human experience, dialogicality of human speech and of human life. In Bakhtin's view, the dialogue, in general terms, is a constant interaction between meanings which potentially affect others ('other' here as opposed to 'self') (Emerson and Holquist, 1981, p.426). His account of dialogue is very general, not limited to face-to-face verbal interaction but concerns verbal communication of any type whatsoever. In this respect even a book (i.e. "verbal performance in print") is dialogic as it engages in an ideological colloquy on a large scale: "it responds to something, objects to something, affirms something, anticipates possible responses and objections, seek support and so on" (Voloshinov, 1973, p.95).

Bakhtin's approach to analysis of dialogicality in verbal communications focused on the utterance, a speech act, "the real unit of speech communication" (1986, p.71). In Bakhtin's account the utterance is an appropriate unit for an analysis of dialogicality because it is "filled with *dialogic overtones*" (1986, p.92) in that it involves not only active participation of the speaker (writer) but also the hearer (reader) and both of these are inseparable components in defining an utterance (Matejka and Titunik, 1986, p.2). Hence a true understanding of the role of an utterance in a dialogue needs to take into account the speaking personality who produces the utterance and this is what Bakhtin called 'voice'.

According to Bakhtin, voice is the speaking personality, speaking consciousness and has always a will or a desire behind it (Emerson and Holquist, 1981, p.434). By voice, or the speaking consciousness, Bakhtin puts the emphasis on broader issues of a speaking subject's perspective, conceptual horizon, intention and world-view (Wertsch, 1991, p.51). Central to Bakhtin's account of voices is that they co-exist in the production of an utterance, that is, voices always come into contact and operate in a social milieu for otherwise "the utterance does not and cannot exist" (1986, p.99). In other words, an utterance, whether spoken or written, is "always expressed from a point of view" (Clark and Holquist, 1984, p.10 cited in Wertsch, 1991), from a voice, and is always directed to at least an addressee, a second voice, who varies from being an immediate participant of the dialogue to an "indefinite, unconcretised *other*" (see Bakhtin 1986, p.95 for more on addressees). This dialogic quality of utterances is what Bakhtin called addressivity, "the quality of turning to someone else" (1986, p.99). Bakhtin's concern with addressivity in the utterance includes both a concern with who is doing the speaking and a concern with who is being addressed (Wertsch, 1991, p.53).

In Bakhtin's view the intimate connection between the speaker and addressee in producing utterances is intrinsically linked to the idea of responsivity. In this regard he noted that every utterance must be regarded primarily as a response to a preceding utterance(s) (ibid., 1986, p.91). Thus responsivity applies not only in listening (to the addressee) but also in speaking (to the speaking consciousness). For the addressee, to perceive and understand another's utterance is "to take an active, responsive attitude toward it. ... sooner or later what is heard and actively understood will find its response in the subsequent speech or behaviour of the listener" (ibid, 1986, pp.68-69). The speaker's utterances are themselves responses to some previous utterances, either from within the immediate conversation or from a previous occasion, and are shaped by an anticipation of a response. In this sense any utterance, regardless of its seemingly monologic or univocal external presentation, is internally dialogic or social. Therefore, the answer to the question "who is doing the speaking" from the Bakhtin's perspective is necessarily "at least two voices" (Wertsch, 1991, p.63).

By invoking voice, addressivity and responsivity, Bakhtin's argument is that an utterance does not come about as an isolated act but is always situated or contextualised within an activity of verbal communication by the utterances that both precede and follow. While explaining the source of personal voice in the production of a particular utterance, Bakhtin writes:

> The unique speech experience of each individual is shaped and developed in continuous and constant interaction with others' individual utterances. This experience can be characterised to some degree as the process of assimilation – more or less creative – of others' words (and not the words of language). Our speech, that is, all our utterances (including creative works) is filled with other's words, varying degrees of otherness or varying degrees of 'our-own-ness,' varying degrees of awareness and detachment. These words of others carry with them their own expression, their own evaluative tone, which we assimilate, rework and accentuate (1986, p.89).

Here Bakhtin suggests that voices, in the course of production of an utterance within a verbal communication, take on and reproduce other people's voices either directly through speaking their words as if they were their own ("our-own-ness"), or through the use of reported speech ("otherness") (see also Voloshinov, 1973, part III; Bakhtin, 1981, pp.293-294). This process of producing unique utterances by invoking other's voices involves a specific kind of dialogicality or multivoicedness that Bakhtin termed "ventriloquation", "the process whereby one voice speaks through another voice or voice type in a social language" (Wertsch, 1991, p.59). Perhaps with this in mind "our utterances" write Cheyne and Tarulli (1999, p.12) " are inhabited by the voices of others." However, words and voices of others do not simply enter into one's own voice, they are selectively assimilated by the speaking consciousness. On account of the value-laden nature of language, this selective assimilation of the words and voices of others, Bakhtin suggests, is part of "the ideological becoming of a human being" (1981, p.341). Hence a voice and its particular utterances inescapably convey commitments and enact particular value judgements (Cazden, 1993).

Two particular aspects of Bakhtin's view on dialogicality, utterances and voices stand out in importance with regard to the notion of the social in relation to my purpose. The first is that the social could involve, though not necessarily be confined to, a face-to-face verbal interaction which evolves around and entails highly complicated negotiations amongst the immediate physical participants of a discursive activity. In the context of my study, Bakhtin's writings provide insight into understanding of the role of the students in shaping the scaffolder's utterances (interventions). The second is that the social may transcend the boundaries of the immediate context of the activity and involve the interaction with the voices of others who are distant in time and space. This is particularly important in that it questions, and indeed compels us to reconsider, the veracity of our assumption of a 'lone' leaner, a learner in 'solitude'. This aspect of the social provides us with a good starting point to deal with the questions posed at the outset of this section in terms of the integrity of social and cognitive processes. At this juncture it should be noted that these two interpretations of the social that I provided within the Bakhtinian framework are not necessarily the 'only' two; there are others (see, for example, Wertsch, 1998, Chap. 4; Wells, 1999). However, these two are particularly relevant to my following discussions and are, to me, directly observable in the verbal data collected for this study.

When taken together, in the context of this study, these two interpretations of the social indicate that the formation of mathematical constructions always takes place in a social milieu, under the influences of social forces. It is my intention in this section to delve into and elaborate on this. To do so, I will look into H&S's verbal data by considering the fundamental Bakhtinian question: who is talking (what voices are present) in the scaffolded discourse? In connection with this question, I will first focus on the voices of the immediate participants (i.e. the scaffolder and students). Secondly, I will discuss the 'other voices' which ventriloquate through the immediate participants' voices and which reside in the tasks. Finally I will provide a discussion of how these voices interact in the formation of the new constructions. On several occasions in my discussion I will also consider how the voices and its utterances enacted some particular value judgements and are negotiated through the utterances of the immediate participants.

4.2. Interaction of the scaffolder's and students' voices

Under this heading I concentrate on two interrelated issues. The first is to discuss the influence of the students' voices on the scaffolder's interventions. Following this, I will attempt to further elaborate if and how constructions take place in a social milieu. In this respect I will argue that constructions are socially produced irrespective of whether or not they take place in overtly social contexts. Viewing the notion of the social from Bakhtin's perspective led me to realise the historical forces operating during the formation of the constructions. Hence, the second issue to which I will attend is a consideration of the social and historical aspects of the constructions.

4.2.1. Influence of the students' voices on the scaffolder's interventions

When the fundamental Bakhtinian question is posed 'what voices are present in the scaffolded discourse' the immediate answer is 'the voices of the scaffolder and the students'. In section 3 above, I focused on the voice and utterances (interventions) of the scaffolder in the discourse which produced the constructions. But what role do the students' voices play in regulating this discourse? In fact this question has long been the concern of those involved in research related to scaffolded learning. Some argue that scaffolded learning promotes a conception of (adult) tutor-student interaction as one-sided in nature, in favour of the adult and thus neglecting the importance of the perspective of the students in the course of interaction (e.g., Harste, Woodward and Burke, 1984; and Searle, 1984). Thus viewed, scaffolding might be argued to lead to a 'transmission' view of learning. Given that the initial discussions of scaffolding were focused on describing role of an adult in a process of assisting the students in working out the task (see Stone, 1993), these interpretations might be considered valid.

However, recent interpretations of scaffolded learning make it fairly clear that this conception does not depict a learning situation in which the students simply follow the instructions of the scaffolder towards the completion of the task through step-by-step demonstration and/or explanations and thus act as a mere recipient of the scaffolder's 'wisdom'. In this connection, many (e.g., Wood and Wood, 1996; Mercer, 1995; Tharp and Gallimore, 1988) noted the necessity of the students' active involvement in the scaffolded task completion and repeatedly emphasised the 'contingency principle' (Wood, 1991) as an essential quality of scaffolding in all settings whether it be parent-child or teacher-student assemblies. The contingency principle in its simplest form refers to provision of the assistance which is augmented or withdrawn depending on the students' developing competence and could be achieved by carefully monitoring and analysing the students' current progress in relation to the expected task achievement.

Nonetheless, the issue, in practice, is more complex than noting the active involvement of the students and the contingency principle. It is important to recognise that scaffolding involves an asymmetry between the parties involved. There are asymmetries of, at least, knowledge and power, which could result in a productive interaction as well as in imposition with regard to the students. But again the question is: can we consider those practices, in which an adult imposes his/her understanding and expectations on the students, as scaffolding? Wood (1991) and Mercer (1995) refuse to consider these situations as involving scaffolding; however, an imposition may occur without adult being aware of it. That does not mean that in such a situation new learning cannot occur; of course new learning may occur even in such a situation. But then the issue of the learner's influence becomes a dilemma which lies in the differentiation between theoretical considerations and practical implementations. There does not appear an easy way out of this dilemma; yet it would be certainly presumptive to assume that scaffolding always proceeds from the scaffolder to the student whose main objective is to follow the given instructions as "a diligent apprentice to an all-powerful master" (Daniels, 2001, p.65).

Through the analysis of the scaffolded students' data in this study, I came to realise a complex set of social and semiotic dynamics involved in a scaffolding practice which informed me about the nature of the students' influences on the scaffolder's interventions. This complexity became clear to me when I especially viewed the verbal data from Bakhtin's perspective. In the case of H&S, I am also convinced of the subtle yet strong influences of the students' voices in shaping the scaffolder's interventions. In this part of my discussion, I shall undertake an analysis of H&S's verbal data from a Bakhtinian perspective in order to demonstrate these subtle influences. However this undertaking will not be exhaustive in this subsection but will continue throughout the following sections as well (see section 5 and 6 below).

In the verbal data, generally speaking, throughout the scaffolded discourse, the students' voices actively shaped the voice of the scaffolder who was active in shaping them. In order to exemplify and clarify this observation, let us return to H&S's verbal data in 4: episode 4 and to the interaction flowchart generated for this episode (Figure 4.6., p.105). In this episode I concentrate on the scaffolder's voice and initially on his two utterances expressed in 165I and 168I which are strongly connected and clearly qualify as assisting interventions. The effect of the students' voices in shaping these interventions (or utterances) can be seen in that these utterances relied on the students' utterances produced previously in episode 3, presupposed them to be known and somehow took them into account (Bakhtin, 1986, p.91). These two utterances (165 directly and 168 indirectly through 165, see Figure 4.6.) have direct referential links with the scaffolder's utterances of 149, 151 and 159 in episode 3 and thus to the students' responses or reactions to these three utterances in episode 3 (Figure 4.5., p.104, see the arrows from either H or S pointing to the scaffolder's utterances in the 'I' column). Thus these two utterances (i.e. 165I and 168I) relied on the students' responses in episode 3 in that they drew on what the students uttered about the graphs of $|f(x)|$ and $f(|x|)$. Furthermore, 165 and 168 were not self-sufficient in and of themselves in that their semantic existence presupposed the students' previous utterances: in 165I – "it is a combination of these two [but what are these two?]" and in 168I – "how can we use our knowledge [what knowledge?]". Within the given context, it is undoubtedly clear what these utterances refer to; yet, this shows that the scaffolder was aware of and took into account the students' responses and utterances while producing his own utterances. Thus the influence occurs bi-directionally between the scaffolder and the students since the scaffolder's utterances can themselves only be understood as responses to the students'.

In the above paragraph I tried to demonstrate that the scaffolder's utterances were shaped by the students' preceding utterances. However, Bakhtin argues that utterances are not only shaped by the preceding but also by the subsequent links in the chain of speech communication. But how can what is not yet uttered shape what is being uttered? This is a delicate matter and closely related to the notion of addressivity. As already mentioned, in Bakhtin's view, an utterance is always directed to an addressee whose role in a speech communication is not that of a passive listener but of an active participant. From the very beginning, an utterance is created in anticipation of possible responsive reactions, for whose sake, the utterance is constructed (Bakhtin, 1986, p.94). With this in mind, when we consider the two utterances, 165 and 168, once again, it might be realised that these two were shaped by the scaffolder's expectation of some responses from the students. He was, in 165, for example, expecting the students to see that "expression, $|f(|x|)|$, is a combination of $|f(x)|$ and $f(|x|)$." Surely this utterance was constructed for the sake of encountering this response, i.e. "do you see that?" – 165; and indeed it generated the intended response from the students (166H and 167S). It does not matter whether the response is the same as the expected one(s) but what matters is the fact that the scaffolder expected answers, explanations or contributions which may not be always predicted. In this sense these utterances, 165 and 168, were also shaped by an expectation of the students' responsive reactions which constituted the subsequent links in the speech communication. Of course these links did not exist at the moment of producing these utterances, after all there is no arrow in the interaction flowcharts pointing at a proceeding utterance. But after the response was elicited a clear link/arrow forward can be shown; for example, an arrow from 165I to 170 is imaginable on the basis that these two refer to $|f(|x|)|$ as a combination of $|f(x)|$ and $f(|x|)$.

The above argument simply suggests that the students, as participants in the scaffolded discourse and as addressees, shape the scaffolder's utterances. Nevertheless it is important to note that the extent of the students' influence in shaping the scaffolder's utterances in this sense are somehow related to how they are being perceived by the scaffolder as the addressees of his utterances. In this connection Bakhtin (1986, p.95) writes, "both the composition and, particularly, the style [expressive and emotionally evaluative aspect] of the utterance depend on those to whom the utterance is addressed, how the speaker (or writer) senses and imagines his addressees, and the force of their effect on the utterance." Bakhtin here suggests that perception of the addressee is reflected even in the construction and expression of the utterances. This very powerful observation points to the existence of social negotiations even in the very construction of the participants' utterances. This negotiation works in two ways: on the one hand the scaffolder's utterances in their composition and style depend on his perception of the students as the addressees of his utterances; on the other, the students' responding utterances, being also directed to the scaffolder, reflect how they sense and perceive the scaffolder as well. Therefore the extent to which the students are influential on the voice of the scaffolder may vary from student to student depending on how the scaffolder perceives the students and how the students perceive the scaffolder and his/her role.

However, with this argument, my intention is not to rule out the possibility of some moments in a scaffolded discourse when the students comply with the authority of the scaffolder who may act as a determining and directive tutor. In fact, at times, this might even be necessary to establish common ground (see section 5 below on 'intersubjectivity and alterity' for more on this). This was the case, for example, in 4: episode 3 where the scaffolder aimed to bring about a shift in the students' approach to the graphs of $|f(|x|)|$. Until this episode, the students tried to develop a method by observing the similarities and differences in the graphs of $|f(|x|)|$ that they had sketched. However, at the end of episode 2, they clearly failed to develop a method on this basis. In episode 3 the scaffolder shifted the students' attention from the graphs themselves to the expression of $|f(|x|)|$ and in this connection he brought the structures of $|f(x)|$ and $f(|x|)$ to the students' focus of attention. In doing so he established common ground on which his following interventions would be provided. In episode 3 he was determining and directing the discourse by first introducing the graphs of $|f(x)|$ (151I) and then the graphs of $f(|x|)$ (159I). The students' compliance with the authority of the scaffolder was all too apparent in this episode. However, if we consider the scaffolded discourse holistically, it becomes difficult to characterise the entire discourse by attributing a directive and determining tutor role to the scaffolder.

In this respect, in the following episodes, one can see essential changes in the position of the scaffolder in that he was no longer acting as a determining tutor but was negotiating with the students. For example, in episode 4 when the students recognised the computational priority rule in connection with the expression of $|f(|x|)|$ (4: episode 4, 170H), the scaffolder was no longer acting as a sole determiner of the discourse. He was now negotiating with the students' views and suggestions which grew on the grounds of mathematically valid analogies. If we look at the scaffolder's intervention in 176I, we can realise that his intervention was taking into account the students' proposals, explanations and elaborations (see the interaction flowchart in Figure 4.6., p.105, where arrows from 176 are referring to all H and S's utterances either directly or indirectly) and thus negotiating with them. Following the scaffolder's 176 intervention, a new episode started (4: episode 5) whose goal was to devise a plan as to how to use the graphs of $|f(x)|$ and $f(|x|)$ in sketching the graph of $|f(|x|)|$. This goal was not set by the scaffolder but emerged as a result of H's (177H and 179H) interpretation of 176I intervention; and therefore it was negotiated with, not imposed on, the students.

I recognise that the scaffolder's intervention in 176I was (and arguably later interventions as well, e.g., 187I) leading the students; but I believe that the scaffolder's leading role was also part of the negotiations between the students and the scaffolder. This is more akin to the comment on the issue of addressee, discussed above, of noting the influence of the mutual perception of the participants. This is also closely related to what Ozmantar and Monaghan (2005) call 'pedagogic resonance' which concerns the scaffolder and students' mutual understanding of the social context of cultural reproduction-production. They argue that our histories as learners/teachers instil us with expectations regarding learning/teaching. If a scaffolder has, say, a particularly 'open' approach to teaching, e.g. tries to avoid 'leading' the student, and an adolescent student has been taught in a 'didactic' manner from early childhood, then it is possible that the students will find the scaffolding experience frustrating and/or unproductive. They call this situation one of 'low pedagogic resonance'.

In this study, all parties directly involved were educated in Turkish state schools. They shared a common pedagogic basis within the social context of the Turkish education culture. My use of the term 'culture' here simply is concerned with common practices of particular communities (Rogoff, 1990, p.110) and this involves the Turkish schools where the common practice, at least in most of mathematics courses, is frequently that 'teachers lead and students follow'. I am not concerned here with the effectiveness of this practice, but my point is that as they were educated in Turkish state schools, I believe, the students expected the scaffolder to lead and the scaffolder acted accordingly and this was a tacit negotiation. This may be more clearly seen in the fourth task in which students have once failed (episode 2) and afterwards the scaffolder explicitly declared his intention to help them (episode 3; 146I). Thus H&S knew that the scaffolder was ready to help and they were, more often than not, uncertain as to how to proceed. Whenever H&S felt uncertainty, they clearly indicated this; and in so doing their aim, I believe, was not only to share their concerns with each other but also to get the scaffolder's contribution and his lead. To clarify this, let us return to one of those moments occurring in 4: episode 5 where H&S were talking about drawing the graph of $|f(|x|)|$ by first drawing $f(|x|)$ and then applying the structure of $|f(x)|$. H (185) asked "Can we really do this?" and S (186) replied "I am not too sure if we can". The addressees of these utterances were not only H and S but also the scaffolder. Thus these two utterances were also directed to the scaffolder and, even in producing these utterances, H&S were also considering the scaffolder as the addressee who already declared his intention to assist. By these utterances, I believe, H&S was tacitly asking the scaffolder to lead. Similar comments can also be made, for example, for 4: episode 4, 174-175-176; and 4: episode 6, 193-194.

4.2.2. Social and historical aspects of the constructions

In my previous discussion I tried to give an account of a complex set of social and semiotic dynamics involved in scaffolded discourse and to draw attention to the mutual influences in shaping the participants' voices which were thus often in need of negotiation. My consideration also pointed out the existence of some cultural dynamics in the scaffolded discourse. In the light of these observations, I will now consider the social and historical forces operating during the formation of the constructions. In this connection, my general position can be formulated as follows: constructions are socially produced, individually appropriated[5] and historically situated.

[5] The term 'appropriated' may suggest some links with the issues of 'appropriation' and 'mastery' which are used as alternative conceptions to 'internalisation' (see Wertsch, 1998; Wells, 1999; Nuthal, 1999; Rogoff, 1990). Whilst there might be some links here I do not use this term with these issues in mind.

My use of the term 'social' here involves both the interaction between the immediate participants of the activity and involvement of 'other's voices' (see section 4.3. below) but, for the time being, I am merely concerned with the interaction between the immediate participants.

In order to illustrate the kind of phenomenon that I have in mind let us return to H&S's verbal data and more specifically 2: episode 5. It was in this episode that H constructed the 'y-symmetry' method by attending to the scaffolder's deep reasoning questions and developing mathematically valid arguments to the scaffolder's challenges. In order to illuminate my general position on the basis of this episode, a brief review of H&S's progress in the second task is required. In episode 3, H&S sketched the graph of $f(|x|)$ for question 4-C incorrectly and, as a result of S's assessment of the accuracy of this graph, the students talked about whether the graph was accurate or not. In episode 4, they, with the help of the scaffolder, constructed the 'reflecting' method and re-drew the graph of $f(|x|)$ and they did so accurately at the end of this episode. At the outset of episode 5, H recognised that this graph was "symmetric in the y-axis" (140) and proposed this as a sign of the accuracy of this graph (144). These two recognitions were of critical importance to the construction of the 'y-symmetry' method. My argument is that the construction of this method was produced in social interaction, was individually appropriated and was historically situated; and this is true for any construction (but recall that the term 'social' here is not limited to the face-to-face verbal communication).

The 'y-symmetry' method was produced in social interaction. Following H's recognitions in 140, 142 and 144, the scaffolder immediately intervened by asking probing questions and through the scaffolder's mediation H constructed this method. This was so because while H initially saw the symmetric feature as a sign of the accuracy of this sort of graphs, by attending to the scaffolder's questions she elaborated and refined her proposal, forged new connections to explain occurrence of the symmetry (149, 153 and 156) and came up with the idea that these sort of graphs must be symmetric in the y-axis (159). Her use of this method to draw the graph for question 4-D in episode 6 clearly shows that H formed this construction.

However, some objections may arise against my argument that H constructed the 'y-symmetry' method through the mediation of the scaffolder and therefore it was produced in social interaction. One may argue, for example, that the scaffolder did not appear to have any direct influence on H's recognitions at the moments of the creation of the utterances (2: episode 5, 140-142-144). In other words, H's recognition, especially in 140, came almost 'out of nowhere' and that H did these recognitions by herself. On the basis of this observation a truly sceptical mind may argue that perhaps at the moments of these recognitions H has already constructed the 'y-symmetry' method and H may have recognised what she has already constructed. This line of reasoning may question and ask if it does any justice to consider the constructions as social products. It is true that H was the recognising subject, but did this insight (140) completely belong to her? When we return to the interaction flowchart constituted for this episode (see Figure 4.3., p.93), it can be seen that her recognition in 140 is referentially linked to 111S in which S suggested that a graph of $f(|x|)$ should be symmetric in the y-axis and this suggestion emerged from S's observation of the graphs that H&S had drawn *collaboratively* throughout their earlier works in the utterances of 22H, 38H, 53S and 74S (not all shown in the protocols). Without those graphs, which were jointly drawn, would S be making this suggestion? Without S's suggestion would H be proposing the idea of 'y-symmetry' in 140? Perhaps more importantly why, at the moment when S suggested the idea of y-symmetry in 111, was H not able to see the importance of this suggestion (see 2: episode 3; 112H and 114H)? Why did H appreciate the importance of the idea of 'y-symmetry' only after the construction of the 'reflecting' method?

When viewed from Bakhtin's perspective, H can be said to have developed her utterance in 140 through interaction with others' (the scaffolder's and S's) individual utterances. Further to this, H's utterance in 140 was filled with the words of S as if they were H's own. In other words S's voice was now ventriloquating through the voice of H. In 142, however, H said "you told that every graph of $f(|x|)$ should be symmetric in the y-axis". In so doing H clearly accepted that the idea did not originally belong to her but initially suggested by her partner, S. But of course there was a categorical difference in the way that utterance 140 was created: H was creating this utterance with her own intention, with her own expression and with her own evaluative tone (Bakhtin, 1986, p.89). Through ventriloquation, H took over S ideas but transformed them to suit her own needs. Thus viewed it can be said that H's construction of the 'y-symmetry' method from the start to end was produced in social interaction.

This analysis shows that what might be seen as purely cognitive (i.e. act of recognising 'y-symmetry') is indeed profoundly rooted in social practices. This brings us back to my argument, stated in section 3.2. above, that the specific processes that the participants communicate and interact with are the substance of cognitive development. Nonetheless this does not mean that participation in a social activity is what constitutes the essence of cognitive development. If it were so, how would we accommodate the individual differences or where would the fine line lie between social practices and individual differences in cognitive development? However we view social, it cannot be conceived of without individuals themselves and individual differences. That is why the research literature on social interaction in small groups is full of reports discussing why and how some individuals manage to learn something out of participation in a learning task while others could not (e.g., Barron, 2000 and 2003). Hence, regardless of whatever reasons might be given for the differences, sharing the same social milieu does not necessarily mean the same level of development, the same level of understanding and the same level of acquisition. My point is that constructions are individually appropriated, meaning that a construction is neither a copy of what has become available on the social plane nor a copy of what was socially invented and that it is a process of meaning-making, sense-making and making 'something' one's own and using this 'something' for one's own purposes.

In order to illustrate my argument, I once again return to 2: episode 5. In this episode, S was sharing the same social milieu with H, hearing the words of H and the scaffolder, and was even in constant agreement (with one exception, 146) with H's explanations, elaborations and proposals (see how often category 5 arrows emanated from S in Figure 4.3., p.93). Yet in episode 6 in which H&S were expected to sketch the graph of $f(|x|)$ for question 4-D, S did not even realise the possibility of using the '*y*-symmetry' method despite the fact that H explained (172H) and illustrated it (176H) to S. It may seem quite odd that she was agreeing with her partner in episode 5 virtually at all times but not able to use these insights, which were made available in the social plane, when required in episode 6. One may posit that S was not 'sincere' in her agreement. I cannot tell how sincere S was in her agreements, though I believe she was; but I feel that it is not a matter of sincerity but a matter of seeing and hearing the same things but understanding quite different things (see the comments on the analysis of episode 5) and therefore acting at different epistemic levels (see the right hand side of the interaction flowchart in Figure 4.3.). In a nutshell S could not (or perhaps 'did not') appropriate H's insights in this episode but did so later in episode 6. In contrast to S, H appropriated S's suggestion in 111 (episode 3) and used this for her own purposes and it sparked off her construction in episode 5 as discussed above. On the basis of this consideration, it can be clearly stated that epistemic actions are subjective and sensitive to individual differences.

Finally, constructions are historically situated in small-time scales in the completion of tasks in overtly social situations; and constructions are also situated in wider time scales in relation to the individuals' personal histories. Regarding the small-time scales, formation of constructions does not take place in isolated moments of produced utterances but rather relies on what has already been uttered (what has already been recognised and built-with). This is akin to Bakhtin's (1986) claim that any utterance is a link in a very complexly organised and continuous chain of other utterances. In this regard, let us consider 2: episode 5 one more time. In this episode H's utterances were not only linked to one another (see the horizontal arrows in Figure 4.3.) but also linked to the utterances from the earlier episodes; see the long arrows in the flowchart (Figure 4.3.) emanating, for example, from 153, 149, 142 and 140. Therefore H's constructing actions, and thus the construction of the '*y*-symmetry' method, in this episode were rooted in some of the earlier utterances across different episodes. Most of those referents did not represent constructing actions, some were building-with and some were recognising. Therefore, what has already been recognised and built-with was also important for the achievement of constructions. Moreover, constructions are also situated within the students' personal histories, meaning that utterances are linked to some previous occasions in the students' personal experiences in mathematics. Throughout their work, H&S's utterances involved recognition and use of, for example, the features of absolute value and linear functions, and of symmetries and reflections. All of these features were available to the students before working on the tasks and were related to their earlier experiences (i.e. personal histories) as mathematics students.

4.3. Other voices in the scaffolded discourse

In the previous section I attended to the Bakhtinian question "what voices are present" by considering the voices of the immediate participants. In this section, however, I will attempt to establish that the scaffolded discourse was "inhabited by the voices of others" (Cheyne and Tarulli, 1999, p.12) who were not physically present within the immediate context of the activity. To do so, I will analyse the scaffolded discourse through three essential elements common to any such discourse: the student(s), the scaffolder and the task(s). I will also consider particular value judgements that other voices conveyed through the utterances of the participants and of the tasks. My general aim here is to lay the foundation of the idea that mathematical constructions are inherently social whether or not they occur in an overtly social context. Elaboration of this idea will be undertaken in section 4.4. below where I amplify the interaction of the voices in the formation of constructions.

4.3.1. Voices within the students' utterances

I focus on two instances of voices present in H&S's utterances: substituting (2: episode 1; 4: episode 2) and the computational priority rule (4: episode 4). Like many other students who participated in this study, at the outset of their work on the first, second and fourth tasks, H&S immediately began to draw the graphs of $f(x)$, $f(|x|)$ and $|f(|x|)|$ by substitution. A common practice in Turkish mathematics classroom is to draw graphs by substitution. This is usually first introduced in Grade 7 (13-14 years old students) and although other methods are used later, e.g. using the gradient and the y-intercept in the case of linear functions, teachers tend to instruct students to draw the graphs by substitution. H&S's immediate use of, and apparent preference for, substitution over another method to draw the graphs of, say, $|f(|x|)|$, e.g. breaking the equation into x-axis interval cases, is certainly related to their earlier experiences as mathematics students. Given this, H&S's teachers' voices can be said to ventriloquate through H&S's utterances and in their actions (using substitution to sketch the graphs). H&S's use of substitution also enacts particular values. For example, in the fourth task, when they were drawing the graphs of $|f(|x|)|$, they considered substitution as a 'definitely working' method (4: episode 2, 145H) and did not question the validity of this method although it gave them two very different V- and W-shaped graphs of $|f(|x|)|$ (see Table 4.3. – A and – C, p.99). When they failed to construct another method at the end of episode 2, they both decided to stick to substitution (4: episode 2, 144S and 145H); this certainly suggests that H&S made value judgements about substitution as a method.

Teachers' voices were also, I believe, present when the computational priority rule was invoked in 4: episode 4. The order of precedence of arithmetic operations and brackets is the focus for a great deal of student work in Turkish junior high school mathematics classrooms and is tested in the high-stakes university entrance exam. Students are taught that if a computation involves nested parentheses, then they should work 'from the inside to the outside'. When the scaffolder suggested considering $|f(|x|)|$ as a combination of $|f(x)|$ and $f(|x|)$ (4: episode 4, 165I), H recognised the computational priority rule (170H). Later S also recognised and further elaborated on how to deal with the computational priority (173S). Although there were no parentheses (other than the ones enclosing $|x|)$ in $|f(|x|)|$ and the scaffolder did not have parentheses in mind in his 165I utterance (see below), H 'heard' parentheses; it was, I posit, again a ventriloquated teacher's voice that she heard and repeated.

4.3.2. Voices within the scaffolder's utterances and actions

A host of voices are present in the scaffolder's utterances and actions. The main ones are the voices of academics: authors he read and tutors whom he had discussions with as he started his PhD studies. I focus on authors to exemplify the other voices involved in the scaffolder's utterances and actions. Since the scaffolder is the same person as the writer of these lines, I will use the pronoun 'I' at times to refer to the scaffolder in what follows.

Extensive reading is an important and formative activity for many novice researchers. In my case, reading significantly contributed to my developing understanding of what scaffolding is and how to provide it. I studied the scaffolding literature on human tutoring (e.g., Graesser and Person, 1994) and developmental research (Rogoff, 1990) and learnt of the potential hazards involved in tutoring. Leseman and Sijsling (1996), for example, argue that tutoring may cast students into an essentially passive role and tutors may ignore the learner's perspective. In addition strong value judgements that students should be actively involved in the learning task (e.g., Mercer, 1995) were implicit, and often explicit (Wood, 1991). I was also persuaded by the research findings that students' active responses play a crucial role in enhancing learning (e.g., Chi et al., 2001). Further to this, many research studies on human tutoring (e.g., Chi et al. ibid.) state that tutors tend to give unnecessarily extensive explanations to the students even when they do not need it and Leinhardt (2001) argues that good explanations are those which are given to the students when they signal that they do not understand and which are targeted at the students' confusion, lack of understanding and misunderstanding. Influence of these voices, and many unmentioned others, was apparent in my scaffolding practice: to support students' autonomy; to obtain the active involvement of students in the tasks; to give 'good' explanations; to avoid unnecessary explanations and interventions; and to take the students' perspectives into account in the course of scaffolding their work.

I now return to the protocol data on the fourth task and provide examples that reveal the voices of other researchers in my scaffolding practice with H&S. In order to support H&S's autonomy, I gave them almost complete freedom until the end of episode 2 (144S and 145H), the point at which H&S were clearly having problems, were 'sticking to substituting' and were not developing a 'better' method. After this point the frequency of my regulative interventions increased considerably. I tried, however, to support H&S's autonomy and active involvement in the task by inviting them to develop their own insights rather than telling them how to work out things. For example, in the crucial 165I intervention (episode 4) I prompted H&S to see that $|f(|x|)|$ is a combination of $|f(x)|$ and $f(|x|)$ but left it to the students to take it further. In addition, I delayed giving further assistance until I felt such assistance necessary, e.g.: 176I comes after H "I am unable to clarify" and S "how could we determine"; 187I comes after H "Can we really do this?" and S "I'm not too sure". Finally, I tried to tailor the type and extent of assistance that I gave to support H&S's developing insights and perspectives. For example, when H suggested thinking of the expression of $|f(|x|)|$ "like a computation with parentheses" (170H), I was aware that there were no parentheses in the expression (other than the ones enclosing $|x|$) and actually I did not have the parentheses in mind. When I intervened in episode 3 and 4, the idea that I had in mind was to get the students to draw the graphs of $|f(|x|)|$ by using the features of the graphs of $|f(x)|$ and $f(|x|)$ (i.e. $|f(|x|)|$ is combination of $|f(x)|$ and $f(|x|)$: so there can be no negative values of y then apply $|f(x)|$; and because of $f(|x|)$, the graph of $|f(x)|$ at positive x should be reflected). But I followed up the students' perspective rather than forcing them to follow the path that I had in mind and I feel that the students' approach was more efficient than what I was thinking of.

As can be seen, the scaffolder's utterances are saturated with others' voices which ventriloquated through his actions and decisions. All these voices were expressed from a particular point of view, conveyed certain value judgements: what a tutor should or should not do while working with the students. Through ventriloquation, the scaffolder took over others' ideas and values, tried them on and transformed them to suit his own needs and purposes (Wells, 1999), that is, he has appropriated the voices of others, value judgements that they conveyed and the particular perspectives. This was the process through which his understanding of scaffolding and approach to how to provide it was shaped. This was the part of his ideological development; it was the part of his process of "ideological becoming" (Bakhtin, 1981, p.341).

4.3.3. Voices in the tasks

The most obvious voice residing in any task is that of its author who, in the case of this study, was the scaffolder. However, at least in the case of this study, there were many other voices ventriloquated through the utterances of the tasks but I focus here on the voices of the authors of the some resources utilised to create the tasks, of the piloting students and of the academics.

The idea of absolute value of linear functions did not simply belong to the author of the tasks. It has been 'borrowed' from and inspired by the work and writings of others. These functions have long attracted the attention of many professionals (e.g., Arcidiacono, 1983; Horak, 1994) and explored in many websites such as the one called 'exploremath' (www.exploremath.com). In the course of preparation and design of the tasks, the author had recourse to these resources and utilised the available materials. Therefore, voices of the authors of the texts and the designers of the websites were existent in the tasks.

Further to this, voices of the piloting students resided in the tasks as well. Subsequent to preparation of the first drafts, the tasks were, for example, piloted on some fellow PhD students (see Chapter 3, section 6.3. for more on piloting). On the basis of their advice, some questions were modified. For instance, the initial form of question 1 in the second task, was "Draw the graph of $f(|x|)=|x|-4$ and comment on any pattern." Piloted students suggested that if the students' attention was focused on some sort of symmetrical pattern in this question, then it might give them a specific starting point to search for a symmetrical pattern which is rather crucial to develop the target methods. Therefore this question was modified by adding "comment on any patterns or symmetries." My intention here is neither to provide the full list of the modifications nor to make a case in favour or against the benefit of these modifications; but rather to emphasise the fact that the piloting students' voices ventriloquated through the tasks.

There were also the voices of the academics in the tasks. In preparing the tasks, there were several issues needed dealing with including the structure of the questions and the time interval between the applications of the successive tasks. In terms of the former, there were several alternatives e.g., open-ended vs. closed-ended. With this matter the author of the tasks had discussions with his supervisors who suggested that it would be more fruitful to use open-ended questions which were more likely to be engaging and to lend themselves to verbalisations and thus attest to epistemic actions. So it was decided to use open-ended questions in the tasks (that is not to argue that closed-ended questions would not allow students to achieve constructions). Therefore, the voices of the supervisors were also present in the tasks. With regard to the time interval, the author had a meeting with Peter Tomlinson, an important figure in educational psychology, who drew attention to the issue of consolidation and the importance of some sort of practice in the tasks. As a result of this meeting, question 4 was added to the tasks in order to give the students an opportunity to do some practice . Thus Peter's voice also ventriloquated through the tasks.

My consideration so far points out clearly that the tasks were, far from being 'a set of printed questions', filled with the voices of others, or as Bakhtin (1986, p.92) puts it, "with the dialogic overtones." Further to this, when working on the tasks, the students were not only interacting with the author's voice but also with that of the abovementioned others'.

My final comment on this matter concerns the particular value judgements that those voices enacted. Each of those voices had a desire behind it, had an anticipation and had a particular "perspective, conceptual horizon, intention and world view" (Wertsch, 1991, p.51): anticipation of the benefit of hints; a perspective on the students' engagement in open-ended questions (e.g. more verbalisations); intention to observe epistemic actions; and a view on the knowledge acquisition (e.g., importance of practice). These value judgements were somehow implicit or explicit in the author's appropriation of those voices as he reflected them in the design of the tasks. These value judgements, however, do not inexorably reflect the 'absolute truth', if such a thing exists at all, but a particular 'world view'.

4.4. Interaction of the voices in the formation of constructions

Up until now, I tried to establish that in the course of the formation of new constructions the voices, from physically present participants and from the others who were not present in the activity but whose voices ventriloquated through the utterances of the participants and of the tasks, can be detected. The effect of the others' voices on the unfolding interaction was all too apparent. For example, in the fourth task, teachers' voices of substitution and computational priority were utilised in the activity and a great deal of interaction evolved in and around the use of these ideas. In a similar vein, the scaffolder's actions were, to a certain extent, determined by the voices from the relevant literature which influenced his decisions as to how to intervene in the students' work. Further to this, the participants were interacting with the voices residing in the tasks by virtue of which ideas to be discussed, explained and elaborated were often determined. In a nutshell, many different voices other than the immediate participants were existent and interacting in the formation of the new constructions.

My consideration thus far simply suggests the existence of a 'virtual collaboration', a 'virtual' interaction, a form of interaction which does not necessarily require the physical presence of the person(s) interacting in the immediate context of activity. Similar arguments are, in fact, implicit (Tharp and Gallimore, 1988, p.57) and explicit in the writings of many (Wertsch, 1991; Vygotsky, 1987). In this connection Vygotsky (1987, p.216) writes:

> When the school child solves a problem at home on the basis of a model that he has been shown in class, he continues to act in collaboration, though at the moment the teacher is not standing near him. From a psychological perspective, the solution of the second problem is similar to this solution of a problem at home. It is a solution accomplished with the teacher's help. This help – this aspect of collaboration – is invisibly present. It is contained in what looks from the outside like the child's independent solution of the problem.

The kind of phenomenon that Vygotsky had in mind might be exemplified with my earlier consideration of the students invoking substitution to draw the graphs. Even though, at the moment of using this method, H&S's teacher(s) was not standing near them, the teacher(s) was invisibly present in virtue of his/her voice. Therefore these students were not only interacting with each other (or the scaffolder) but also with their teacher(s). However, that is not to say that other voices would always be useful and/or helpful for the students to achieve a construction. There may be several voices involved in the interaction which are not necessarily in agreement but rather conflicting. In that case the students' constructions may have even been hampered by those voices. Nor is it to say that interacting with the voices of absent others in the course of a construction is more (or less) important/effective than with that of physically present persons.

However, what I am saying is that even if a student is deemed to be working 'alone' – true, a student may be *physically* 'alone' during his/her work – on a task and achieves a construction, this student can by no means be conceived of having achieved this construction, without the involvement of other voices, in total isolation. The student is, at least, interacting with the voices of his/her teacher and/or of the author of the tasks. With this in mind, I reconsider Kidron and Dreyfus' (2004) study, cited at the beginning of section 4, in which a 'lone' learner's construction about the bifurcation of dynamic processes is elaborated. The authors note that the learner in their study 'learned alone' with recourse only to books, some websites and a graphic calculator. I have no problem to accept that this learner was 'alone' in the sense that she was not interacting with any individual who was *physically* present during the formation of her construction. However, the learner was obviously interacting with the voices of many others, at least the authors of the books and the designers of the websites. She was getting virtual support and assistance from the others; the construction was achieved by the involvement of the voices of others. This is my point when I argue that mathematical constructions are socially produced; they are inherently social whether or not they occur in an overtly social context. They are shaped and developed in continuous and constant interaction with others who could be physically present at the moments of these constructions and/or could be distant in space and time. As Maybin (1993, p.132) states, "We have no alternative but to use the words of others, but we do have some choice over whose voices we appropriate, and how we reconstruct the voices of others within our own speech." That is why constructions involve the voices of others in one way or another. That is why it is impossible to draw a major theoretical line between the social and cognitive processes. My intention here is not to trivialise the role of individuals; of course there are differences in capabilities, of course qualities such as perseverance, motivation and determination make differences in the achievement of new constructions. But my point here is simply that regardless of however persevering, motivated and determined one might be, there is no way, none, in which one can achieve a construction in complete isolation from the voices of others.

5. Intersubjectivity and alterity

In the previous section I attempted to provide an account of the social aspects of the constructions by drawing on the voices of the immediate participants and of the others in the scaffolded discourse. As part of this attempt, I focused on the nature of the social interactions, tried to specify the mutual influences amongst the immediate participants and noted the complexity of this process. In this section I shall try to further specify the nature of social interactions amongst the immediate participants by outlining two opposing tendencies which may be seen as characterising any social interaction: 'intersubjectivity' and 'alterity'. The importance of these two tendencies may vary depending on the specific conditions of the interactions, yet both are always at work in any given discourse. In what follows I shall first elaborate on these two tendencies and then provide illustrations from H&S's verbal data in order to demonstrate the interplay between the two. However, I shall make no attempt to provide a comprehensive review of these two tendencies nor shall I deal with the myriad of issues being taken up to extend these two tendencies to, for example, the emergence of self and self-world, self-culture relationships (on these issues see Coelho and Figueiredo, 2003; Simao, 2003; Hermans, 2001). Instead I limit my considerations to outlining what I see as the basics of these two tendencies.

5.1. What is meant by intersubjectivity and alterity?

The problem of intersubjectivity in essence concerns the conditions under which participants in a dialogue (dialogue here in the broadest sense) achieve a coherent and viable interaction. This is, according to Rommetveit (1985), a question of how two persons who engage in a conversation can transcend their 'private worlds' and understand each other as, he contends, what any person sees going on in a situation is private. He argues that people have no direct access into one another's beliefs and thoughts; however, they can be talked about and may become a temporarily shared reality. Thus

> the solitary observer may try to transform his 'private' outlook on the situation into a social reality simply by telling some other person about it. Once the other person accepts the invitation to listen and engage in a dialogue … the two of them are jointly committed to a temporarily shared social world, established and continually modified by acts of communication (Rommetveit, 1979, p.10 cited in Wertsch, 1998, p.112).

Rommetveit here lays the emphasis on the mutual commitment to the talked-about reality in order to make sense of what the other's private world is, in order to achieve a 'state' of intersubjectivity. This perspective is compatible with Uhlenbeck's (1978) 'make sense' principle of ordinary speech. He describes it with the following terms (ibid, p.190):

> It says that the hearer always takes the view that what the speaker is saying somehow makes sense. It is this certitude which makes him try to infer – on the basis of lingual and extra lingual evidence available to him – what the speaker actually is conveying to him.

Although it is not explicitly stated, this formulation implies the necessity of mutual commitment so as to strive to make sense of the talked-about reality. Two particular aspects of Uhlenbeck's formulation are noteworthy with regard to intersubjectivity. The first is that it points out the importance of what has already been talked about ("on the basis of lingual and extra lingual evidence available to him") so as to understand the speaker's intention and meaning. This points to the necessity of, at least to some degree, shared understanding of what has gone before. Schegoff (1991) views shared understanding as a crucial ingredient for the coherence and viability of an interaction. In this regard he writes (ibid, p.157)

> The very coherence and viability of the course of [direct] interaction [between persons], jointly produced by the participants through a series of moves in a series of moments that are each built in some coherent fashion with respect to what went before, depends on some considerable degree of shared understanding of what has gone before, both proximately and distally, and what alternative courses of action lie ahead.

The second aspect of Uhlenbeck's formulation is that it implies that the hearer may not be able to make sense of the talked-about reality or may make wrong inferences. This situation can bring about divergences in understandings, that, according to Schegoff (1991, p.158), embody 'breakdowns' of intersubjectivity, that is, 'trouble' "in socially shared cognition of the talk and conduct in the interaction." In order to prevent these breakdowns and achieve the coherence and viability of interaction, authors, such as Clark (1996), insist on the establishment of a common ground on which participants of a dialogue can interact successfully, can share their understandings with one another. In Clark's view, establishment of a common ground requires common reference points and depends on the listener's assumptions about the speaker's referents and the speaker's presuppositions about the listener's background information. If the speaker presupposes incorrectly too much background information for the listener, then the message does not go through, in which case a common ground has failed to be established, and intersubjectivity is broken down.

In the light of my considerations so far, a state of intersubjectivity may be said to be concerned with, broadly speaking, the extent to which different aspects of an activity is shared amongst the participants and/or held in common: perspective, attention, understanding, communicational agreement, assumptions and presuppositions, and mutual commitments (see also Matusov, 2001; Wertsch, 1998).

The reader may realise that I prefer using the expression of 'a state of intersubjectivity' over 'intersubjectivity'. The term 'state' here is intentional and implies the existence of intersubjectivity in a relative, instead of in a 'pure', form. This is because although in theory it is possible to talk about 'pure' intersubjectivity, in practice it rarely happens. Pure intersubjectivity ultimately presupposes a communication which can be described in terms of an 'information-transmission' model. The transmission model of communication that I have in mind, simply, and perhaps oversimplified, applies to both written and spoken language by virtue of which speaker (writer) sends an encoded message with a single meaning to the hearer (reader) who passively receives the message and either decodes or fails to decode it (see Wertsch, 1991 for more on this). The essential problem with this view is that it ignores the listener's (or addressee's) influence on the speaker, treats the message as conveying an unaltered meaning which can be understood outside the flow of speech communication, and attributes a passive role to the listener ignoring the differences in, for example, his/her interpretations, perceptions and perspectives on the talked-about reality. That is not to deny the possibility of the transmission of some sort of information but rather that what is 'transmitted' is, more often than not, altered.

This brings us to the issue of alterity. The term 'alterity' is derived from the writings of Bakhtin and has, relatively recently, entered into the educational and psychological discourse through studies concerned with intersubjectivity (e.g., Wertsch, 1998; Coelho and Figueiredo, 2003). Before going into the Bakhtin's notion of alterity, it should be noted that he does not deny the importance of the understanding of what is being conveyed in the course of a dialogue. In this respect he writes that

> To understand another person's utterance means to orient oneself with respect to it, to find the proper place for it in the corresponding context. For each word of the utterance that we are in process of understanding, we, as it were, lay down a set of our own answering words. The greater their number and weight, the deeper and more substantial our understanding will be ... (Voloshinov, 1973, p.102).

Here in this quotation Bakhtin is clearly concerned with the individuals' understanding of one another in the course of a dialogue and the importance that he attaches to this is quite evident throughout his writings. However, here we have a rejection of transmission of meaning through language. Because, according to Bakhtin, as discussed above, any utterance involves at least two voices, the speaker and addressee, and the meaning and understanding is produced through the voices coming into contact and inter-animating one another. Thus it is not only the speaker producing the utterances but also the hearer or addressee is indispensable in the creation of meaning. In other words, the meaning is not transmitted but created through inter-animation of the voices. This clearly shows that Bakhtin attributes to the 'other' (hearer or addressee) a focal position in the creation of meaning of an event, of the talked-about reality. Further to this, in his view, diversity of the other's outlook and the differences that the other brings into a situation are all necessary to achieve new understandings and creation of new meanings. This becomes more evident when Bakhtin (1990, p.87) writes

> In what way would it enrich the event if I merged with the other, and instead of *two* there would be now only *one*? And what would I myself gain by the other's merging with me? If he did, he would see and know no more than what I see and know myself; he would merely repeat in himself that want of any issue out of itself which characterises my own life. Let him rather remain outside of me, for in that position he can see and know what I myself do not see and do not know from my own place, and he can essentially enrich the event of my own life.

Here Bakhtin explicitly spells out the importance of other voices, which do not need to (and perhaps should not) be in total agreement with the one doing the speaking, in order to achieve a development. He clearly values the 'otherness' of the other who speaks from a different perspective, conceptual horizon and world view. Thus he puts value judgements on differences and conflicting voices, rather than on consensus, and perceives them as an essential component of development and enrichment. As Cheyne and Tarulli (1999, p.14) noted in Bakhtin's view "the distance and difference of others is not only always retained but deemed essential. It is from differences in understanding that dialogue and thought are productive". This line of argument suggests that the meaning of an event is always created by virtue of the co-existence or co-being with the different. For Bakhtin differentiating becomes the basic condition for living: "existence is a dialogue that has the settlement of diversity as its basic condition for emergence" (Simao, 2003, p.455). Thus in Bakhtin's view, alterity should be accepted and defined positively rather than associated with some kind of insufficiency.

In sum a state of alterity is concerned, broadly speaking, not only with the differences in perspectives, conceptual horizons and world views but also with the differences and the changes occurring in perceptions, understandings and interpretations in the course of a dialogue. The general point to be made about intersubjectivity and alterity is that in the course of a communication both of these tendencies are often co-existent at different degrees and with relative importance. Therefore, a communication cannot be best understood in terms of one or the other in isolation but rather through a consideration of both of these tendencies. Thus, the challenge is, as Wertsch (1998, p.111) notes, to recognise "how these forces are part of an integrated, dynamic picture." The challenge becomes even greater when it comes to relating these two forces to the achievement of mathematical constructions by considering the scaffolded discourse which requires paying simultaneous attention to the communication between the scaffolder and the students and between the students themselves, and doing this in connection with the tasks on which the participants' conversation is being shaped. In the rest of this section I offer my view on how intersubjectivity and alterity constitutes the dynamics of scaffolded discourse towards the achievement of constructions by focusing on the immediate participants. I will further attend to this issue in section 6 below, on 'emergent goals', this time by considering the tasks as well. In passing it should be noted that the issues of intersubjectivity and alterity as described here have some clear links with and implications for Piagetian and Vygotskian theories of cognitive development; yet, I will not go into details due to the space limitations and because these are not central to my following arguments. The reader can find more on these issues in Wertsch (1998), Rogoff (1990) and Cheyne and Tarulli (1999).

5.2. An illustration of the interplay between intersubjectivity and alterity

The question of interest in this section is how intersubjectivity and alterity constitute the dynamics of social interaction and propels the scaffolded discourse towards the achievement of new constructions. Although I believe that my following observations about these two forces hold for any scaffolded discourse, I do not intend to generalise them beyond the confines of the verbal data generated for this study.

When we focus on the scaffolder's and the students' utterances it becomes evident that they have no way of direct access to their individual thoughts, individual interpretations of the situation and individual meanings gathered from the situation, i.e. individual 'private worlds'. However these 'private worlds' becomes relatively accessible, not necessarily completely and truly as they are, to the others when they are talked about, transformed into speech. The participants during their speech assume that what is being said is somehow making sense to the listener. They assume some aspects of the talked-about reality, e.g., graphs, features of absolute values and of linear functions, are shared and held in common amongst themselves. Therefore they take *a priori* some aspects of the activity as being shared. They formulate their understandings, observations and interpretations, in sum all their communicative acts, with an anticipation that they will be understood as intended. It is through this anticipation and endeavour to communicate their 'private worlds', they trade on one another's 'truth' until such a time that discrepancies in the individual understandings, perspectives and interpretations come to the surface. Subsequent to a realisation of the differences, what is being taken as shared and thus unquestioned before becomes the focus of attention and constitutes the basis on which the participants need to come to terms with one another's 'private world'. In this course, the participants may need to negotiate their perspectives and to modify their interpretations of the talked-about reality. To formulate in one sentence: intersubjectivity is assumed until such a time that alterity comes clearly to the surface which is negotiated to attain a new state of intersubjectivity. Thus a state of intersubjectivity is required to communicate successfully but alterity is required to generate new understandings; and both are necessary for the achievement of constructions. Intersubjectivity and alterity should not be viewed as rigid constructs as if they were demarcated with certain boundaries. As mentioned earlier, they are often co-existent, mutually constituted and the boundaries, if they exist at all, are blurred. In order to concretise this formulation I now return to H&S's verbal data in the second task and attempt to illustrate my understanding of the interplay between alterity and intersubjectivity as hitherto briefly sketched out.

H&S in their work on the second task initially proposed the 'reflecting' method (2: episode 1, 58S - 65S) which was not sufficiently well specified and was even ambiguous. The scaffolder was aware of this deficiency. After H&S drew the graph of $f(|x|)$ for question 4-B, in order to draw the students' attention to the lack of specificity of this method, he intervened in 76I (2: episode 2) and asked H&S which part of $f(x)$ was reflected. This question was not immediately relevant to the context and the scaffolder presupposed, incorrectly, too much background information of the students and thus did not clearly state his intention behind this intervention by making, for example, lucid to the students the deficiency of their reflecting method. Surely H&S did not have direct access to the scaffolder's 'private world' and were not able to understand his purpose. Further to this, the shared context at that point in the conversation was not yet sufficient to allow the scaffolder's intention and meaning to get across the students. As a result, H&S responses remained extraneous to the scaffolder's intervention and they talked about the symmetry line rather than the reflection. This surely suggests that a state of intersubjectivity is necessary to achieve a successful communication.

A closer inspection of the rest of this episode also reveals that the scaffolder and the students did not share the same perception of the talked-about reality. That is, the scaffolder was concerned with the deficiency and general applicability of the 'reflecting' method (see 2: episode 2, 80I, 81H and 86I) whereas the students were concerned with the specific graph at hand (83S and 85S) rather than with their 'reflecting' method. It was even questionable that the students made any connection between the scaffolder's interventions and the 'reflecting' method. Thus it could be argued that the students were not able to see the issue from the scaffolder's perspective. In fact similar observations were made by many studies concerned with learning through adult guidance (see Parret-Clermont, Perret and Bell, 1991). This observation is not surprising given that the scaffolder, from the start, already had a vision of what the students were to construct. It is observation such as this that has led many to view the increasing intersubjectivity as one dimension along which cognitive development occurs (see Wertsch, 1998, Chap. 4). This view, when the cognitive development is thought of in a narrow sense as achievement of a construction, can also be corroborated in H&S's verbal data. For example, in the course of H&S's construction of the 'reflecting' method, one can easily see increasing intersubjectivity amongst the participants (see 2: episode 4; especially 130-139). However, a single-minded focus on intersubjectivity may blind us to appreciate alterity, which is also important for the achievement of the constructions.

In this respect it is useful to turn our attention to 2: episode 3 in which H&S were talking about the accuracy of the graph of $f(|x|)$ obtained for question 4-C by their ill-developed 'reflecting' method. They were both looking at the same graph yet interpreting its accuracy differently: while S thought that the graph was wrong H was not convinced of this. In this sense there was an apparent alterity in their perspectives. As a result they had a long argument. In my previous comments I emphasised that their arguments in this episode were not quite fruitful in that they were not able to produce mathematically convincing arguments and find a resolution. However, these arguments were not useless either. Throughout their arguments, they at least realised that there were some problems not only with the graph obtained for question 4-C but also with the application of their 'reflecting' method. Moreover, what was taken as shared (i.e. the application of the 'reflecting' method) and thus unquestioned before (i.e. which part of $f(x)$ is reflected) became the focus of the students' attention. In episode 2, it was the scaffolder who tried to bring this to the students' attention but this was not successful. However, in episode 3 it was thanks to the differences in H&S's assessments of the accuracy of the graph that the ambiguity of their method came to the surface (2: episode 3, 109S-115H).

These differences in interpretations and understandings created a need for the students to negotiate their perspectives. The term 'negotiation' here is employed to indicate the exchange of (usually opposing or at least differing) ideas to reveal the 'private worlds' unknown to the others towards the establishment of a consensus or agreement, which might or might not happen. In this sense, negotiation can be seen in that each of the students stated their own (differing) positions in relation to the accuracy of the obtained graph and the required changes in the graph of $f(x)$ when transformed into the graph of $f(|x|)$ and also the students talked about one another's claims. What is important here is to realise that the need for negotiation occurred when the differences in perspectives clearly came to the surface in the course of H&S's work. However, they could not bring a resolution to this until the scaffolder intervened in episode 4 and took part in the negotiation process. In episode 4, we can see some modifications in the perspectives and approaches of the each student. S, for example, no longer tried to justify her claims on the basis of ostensible features of the earlier drawn graphs, which was the case in episode 3, but rather drew on the features of the mathematical structures (e.g., features of absolute value and of linear functions). H, on the other hand, changed her opinion that the graph of $f(|x|)$ obtained for question 4-C in episode 3 was correct and even gave reasons for the inaccuracy of the graph (episode 4, 137H and 139H). In this sense both of the students changed, or at least modified, their interpretations of the talked-about reality.

At this juncture it is essential to note that despite the obvious existence of alterity in 2: episode 3, a state of intersubjectivity did exist too. There was surely common ground (discussing the accuracy of the graph) and mutual commitment (which can be seen in their desire to understand one another, see, for example, 104-105-106 and 111-112) between the students who shared the common reference points (e.g., "graph", "the parts", "symmetry" and "lines") through which they have communicated and understood each other's intention, though not necessarily perfectly. Existence of a state of intersubjectivity can also be clearly seen in the interaction flowchart (Figure 4.1., p.86) where all of the arrows from one student to another are referentially linked. Thus intersubjectivity and alterity co-existed in this episode.

The students' arguments in episode 3 were also helpful for the scaffolder to adjust the kind and amount of his assistance on the basis of his monitoring and analysing actions. In episode 4, when the scaffolder asked H&S to look into the earlier graphs and decide which part was unchanged

(116I), the shared context at that point in the conversation was sufficient for the students to appreciate the intention of this intervention due to their earlier discussions in episode 3. This is clear from the students' immediate reaction to return to the earlier graphs to examine them with this purpose (2: episode 4, 117S). Thus existing alterity in episode 3 led to the attainment of a new state of intersubjectivity in episode 4. Actually in episode 4, the existence of intersubjectivity is pretty clear in that the participants shared common ground on which, while creating utterances, they drew heavily on the previous utterances as if they were uttered by the speaker (see, for example, from 127 to 132). This shows that the listeners made appropriate assumptions about the speaker's referents and the speaker correctly presupposed the listener's background information. However, in this episode it was again not only intersubjectivity but also alterity at work. This is a subtle point and in order to make this point clearly I consider the scaffolder's interventions in this episode.

Recall the mediating role of the scaffolder as discussed before. In this respect, it was argued that the construction of the 'reflecting' method was mediated through the scaffolder's interventions. While it is indisputably true that some information is conveyed from the scaffolder to the students and vice versa, it is apparent that the scaffolder's interventions served as a mechanism to generate new understandings on the part of the students. This would be unlikely to happen unless there was alterity in the information conveyed to the students. To elucidate this point, let us consider the communication between 116-119 and especially 118-119 as reproduced below.

> **118I:** Do you mean that the part of $f(x)$ at the positive x, which is always on the right of the y-axis, doesn't change?
> **119S:** Yes that's what I mean... and also on the right side of the y-axis, all of the values of x are positive...

As already stated, following this exchange there was a categorical change in the way that S saw the graph of $f(|x|)$. She was seeing in it a particular line segment ("on the right side of the y-axis"; 119S) with some particular qualities (positive xs; 119S); seeing in it a relation of absolute value sign (127S); seeing a 'link' in it with the expression of $f(|x|)$ (125S); and seeing a reason in it for being unchanged (123). But how could these dramatic changes have occurred if there was no alterity? After all, 119 is almost a restatement of 118. If the information in 119 was transmitted to S without any alterity, how would the resulting changes in S's seeing be explained? After all, neither 118 nor the preceding utterances contained information or messages which can be directly linked to S's new understandings following 118 as indicated above. When viewed from a Bakhtinian standpoint, it can be argued that these new understandings were generated through the inter-animation of different voices each of which was speaking from a different perspective and conceptual horizon; each of which was interpreting and perceiving the talked-about reality in accordance with different 'private worlds'. In this sense the very 'otherness' of the other involved in the communication was responsible for the generation of new understandings, emergence of new constructions (recall also H&S's arguments in episode 3). That is, none of these voices in isolation, but all of these voices in conjunction were generating the new constructions.

My final words in this section are on the thorny issue of mutual influences between the students and scaffolder. Is the scaffolder the mere actor propelling the interaction towards the achievement of constructions? Do the students just follow the scaffolder's instructions and react to his interventions and consequently achieve constructions? Or does scaffolding proceed from the scaffolder to the students? None of these questions can be answered with a simple 'yes' or 'no'. The issues of intersubjectivity and alterity hitherto discussed once again show the complexity of the scaffolding process and the intricacy of mutual influences amongst the parties involved. On the one hand, attainment of a state of intersubjectivity to establish common ground is a prerequisite to communicate successfully and this cannot be done disregarding the students and their background knowledge. On the other hand, the issue of alterity shows the importance of the differences in perspectives and interpretations to generate new meanings. When the necessity and importance of these two opposing tendencies in scaffolded discourse are realised, it becomes rather difficult to put either the scaffolder or the students at a focal position, or to attribute an essential role to one over the other. After all the scaffolder's utterances (or interventions) can themselves only be understood as responses to the students whose understandings and interpretations of those utterances are what makes the difference in the progress towards constructions.

Thus far I tried to demonstrate how the intersubjectivity and alterity are constitutive part of a dynamic picture in connection with the formation of a new construction(s) by considering the scaffolded discourse. What is missing in my discussion in relation to these two forces is the role of the tasks which create a context for the involvement of the participants and shape the interaction to a considerable extent. This will be one of the foci in the next section where the issue of emergent goals is discussed.

6. Emergent goals

Up until this point in my discussion, I have elaborated on the scaffolder's interventions, offered my views on how these interventions are linked to the achievement of new mathematical constructions, delved into the nature of social interaction in scaffolded discourse by focusing on the immediate participants and the voices of absent others and tried to characterise social interaction in terms of intersubjectivity and alterity. All these are but different dimensions of the general phenomenon under investigation i.e. 'scaffolded mathematical constructions'. Central to my investigation of these different dimensions were three indispensable elements of the activity in which the constructions were achieved: the scaffolder, students and tasks. Although implicit in my writings, I did not make any particular attempt to explicitly demonstrate how these three elements are interconnected with one another within the theoretical framework of this study (see Chapter 2, section 3.4.). In other words, how are the scaffolder, students and tasks related to one another in the construction of a new mathematical structure within the students' ZPD at stage 1? It is my aim in this section to attend to this question regarding H&S's work. Although there might be some other ways, through the repeated readings of H&S's verbal data, I find the idea of 'emergent goals' useful to demonstrate the interconnections amongst these three elements. In what follows I will first clarify my meaning of 'emergent goals' and then suggest a model which will be illustrated on the basis of H&S's verbal data.

6.1. Towards a model: goals and emergent goals

In this part of my study, I will briefly consider the notion of 'goal' within two well-known frameworks: Leont'ev's (1981) activity theory and Saxe's (1991) 'emergent goal' model. Both of these frameworks were formative to my developing understanding of the 'emergent goals' and their roles in explaining how the scaffolder, students and tasks fit together into a complex whole in the formation of a new construction.

Although Leont'ev's activity theory has undergone critical examinations (see Kozulin, 1998) and been further developed since it had been first proposed (see the volume edited by Engestrom, Miettinen and Punamaki, 1999), I shall make no attempt to go into these details. Instead I shall limit my consideration to some of the essentials of this theory insofar as they are relevant to my following discussions. Leont'ev (1981) conceived of human activity as the basic unit of analysis for the investigation of cognitive development. According to him, activity, in its simplest form, is composed of a subject(s) (an individual or group of individuals) and an object, mediated by a tool. Being the main distinguishing feature of an activity, an object is held by the subject, motivates the activity and gives it a specific direction. The motive of the activity is realised in virtue of actions which are carried out through operations that are dependent on specific conditions. In his account, action is the most important component of human activity and always subordinated to a conscious goal. Leont'ev (1981, p.61) writes,

> any kind of well-developed activity presupposes the attainment of a series of concrete goals, some of which are rigidly ordered. In other words, an activity is usually carried out by some aggregate of actions subordinated to partial goals, which can be distinguished from the overall goal. In this process ... the overall goal functions to realise a conscious motive, which is converted into a motive-goal precisely because it is conscious.

The argument presented in this quotation may be better understood by invoking Leont'ev's (1978) well-known illustration of a tribe's hunting-gathering. Members of this tribe execute different actions with different goals. Some make hunting equipment, others frighten the prey towards the other members who kill the game. The activity here is hunting and realised by the aggregate of actions subordinated to partial goals (e.g., to frighten the prey, to use hunting equipment and to kill the prey). These partial goals are clearly conscious and distinguishable from the overall goal, i.e. to obtain food and/or clothing. The overall goal here serves to realise a conscious motive of the hunting activity, i.e. to stay alive.

Two aspects of Leont'ev's argument are particularly noteworthy here: first the overall or main goal of an activity is realised by the attainment of a series of partial goals and second these partial goals are conscious. In the context of this study, the first aspect provides a good deal of insight into analysis of the activity, in which scaffolded constructions were achieved, by decomposing the activity into a cycle of goal-directed actions. However, I found the second aspect problematic. Are the partial goals subordinated to the main goal of the activity always conscious? In the above illustration of hunting-gathering, consciousness of the partial goals are evident. This is precisely because the tribe has a plan how to conduct the hunting activity by assigning certain roles to the individuals which are relatively stable, i.e. some frighten and others kill. Nevertheless it is not hard to imagine some moments when the conduct of activity does not proceed as planned. Under these circumstances the individuals may need to reformulate their actions in relation to some (partial) goals which *emerge* as a result of the demands of the situation and which need not necessarily be conscious. As an illustration of this argument consider H&S's work in 2: episode 6. Following the scaffolder's prompt in 166I, H&S set about sketching the graph of $f(|x|)$ for question 4-D. However, there occurred an 'unexpected' problem here: while S proposed to sketch the graph by the 'reflecting' method (169S and 171S), H attempted to draw the graph by the 'y-symmetry' method (172H). This apparent difference in H&S's approach to sketch the target graph stemmed from the fact that H, but not S, constructed the 'y-symmetry' method in episode 5; and this led to the emergence of some new (partial) goals for the students. For example, H had to explain to S the 'y-symmetry' method (172H) and illustrate it (176H). S, on the other hand, tried to understand H's explanation (173S) and explained her own understanding as to how to draw the graphs of this sort (177S). However, it is questionable that the students were conscious that they adopted some new goals other than just sketching the graph of $f(|x|)$ for question 4-D. Therefore, consciousness of a goal reflects an idealisation, not a rule, and it seems presumptive to assume that every partial goal is conscious.

Another framework in which the notion of 'goal' plays a key role is that of Saxe's (1991) 'emergent goal' model. Although Saxe has been considerably influenced by Leont'ev's activity theory (see Cobb, Jaworski and Presmeg, 1996), his treatment of goals is rather different. In Saxe's account goals are practice-linked and, as Monaghan (2004) suggests, can be conceived of as 'little' and often unconscious goals which come into being and fade away. The basic assumption Saxe holds is that individuals construct new understandings in their goal-directed cultural activities; and the central problem that he addresses is that of explaining how individuals' personal goals become interwoven with the socially organised activities in which they participate. The answer to this problem lies in Saxe's observation that goals are "emergent phenomena shifting and taking new forms as individuals use their knowledge and skills alone and in interaction with others to organise their immediate contexts" (1991, p.17). Thus goals are not fixed or static constructions but rather emerge in the course of a cultural activity, that is, they take form and shift as the individuals participate in practice (Saxe and Bermudez, 1996). Saxe provides a model of emergent goals by specifying four parameters (see Figure 5.1.): activity structures, prior understandings, conventions and artefacts, and social interactions.

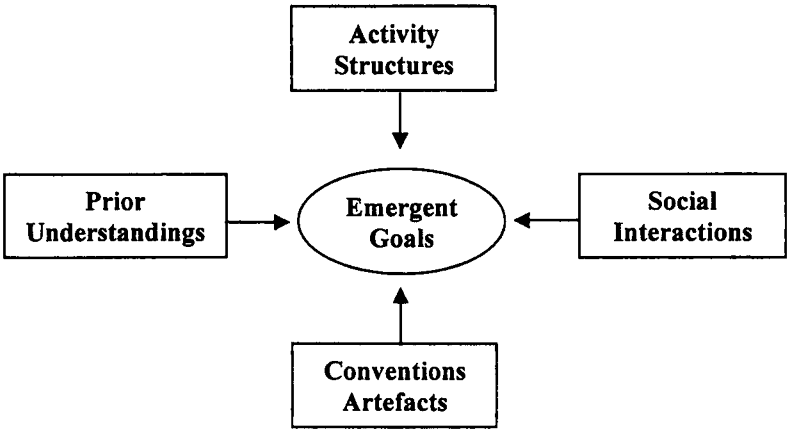


Figure 5.1. Saxe's four-parameter Emergent Goals model

I will now briefly explain each of these parameters and provide illustrations on the basis of H&S's verbal data. The first parameter, activity structures, refers to "general tasks that must be accomplished in the practice- and task-linked motives" (Saxe, 1991, p.17). In the context of this study, an obvious candidate for this component is the tasks on which the students worked. In order to complete the tasks H&S had to attend to the questions each of which brought about a new goal such as to draw the graph of, say, $f(x)$ and $f(|x|)$. To achieve this goal, they had recourse to some known (to H&S) procedures such as substitution and commented on some patterns of symmetries. This is related to their prior understandings, which is the second parameter. H&S' prior understandings both "constrain and enable the goals they construct in practices" (ibid, p.18). For example, question 1 in the, say, fourth task, did not specify how to draw the graph of $|f(|x|)|$ but H&S preferred substituting different values of x into the given equation and using the substitution was an emergent goal enabled by H&S's prior understandings of drawing the graphs of linear functions. In the use of substitution as a method to draw the graphs, H&S used Arabic numeric system and while commenting on the patterns of symmetries they were using the Euclidian metric, albeit being oblivious of doing so. This brings us to the third parameter: conventions and artefacts that consist of "the cultural forms that have emerged over the course of social history" (ibid, p.18). Finally the social interactions parameter is concerned with relationships and the resulting influences amongst participants, which in the case of this study was the scaffolder, students and some absent others, i.e. this was an instructional setting.

6.2. A model of emergent goals within the ZPD

I have so far briefly considered the notion of goal within Leont'ev's activity theory and Saxe's emergent goal model. The insights gathered from the idea of emergent goals in Saxe's framework helped me understand and develop interconnections amongst the three elements: the scaffolder, students and tasks. The insights that Leont'ev's activity theory provided were also useful for me to see how the emergent goals build upon one another in the course of formation of a construction. Against the background of these two frameworks I came up with a model in an attempt to display interconnections, as they were, amongst these three elements around emergent goals within the theoretical framework of my study as presented in Figure 5.2. below.

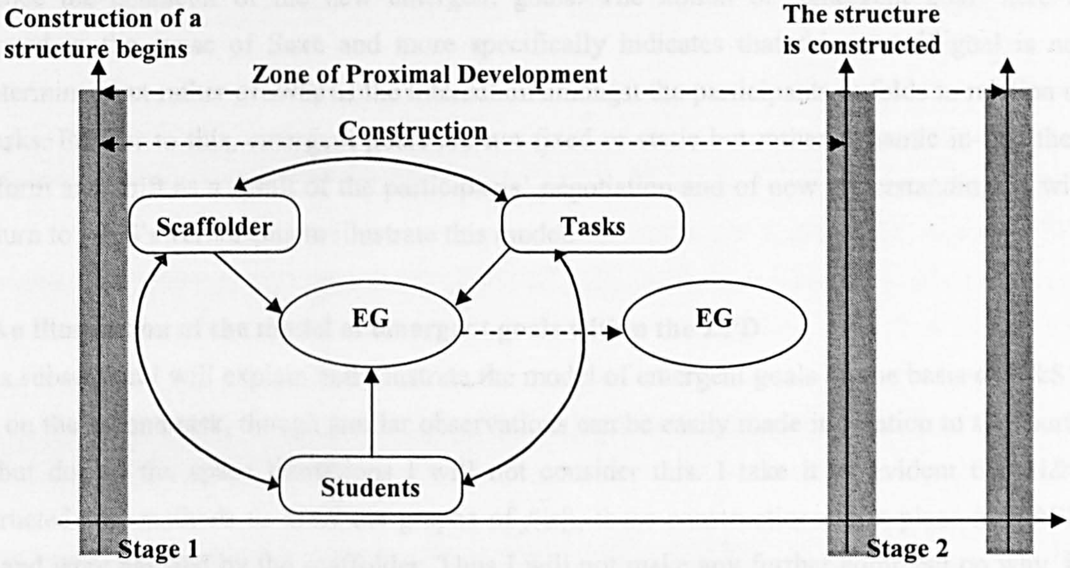


Figure 5.2. A model of emergent goals (EG) within the ZPD

I will briefly describe this model here but return to it in the next subsection where I elaborate the model and illustrate its functioning on the basis of H&S's verbal data. This model is akin to Saxe's but there are some differences. The parameter 'activity structures' is represented by 'tasks' in my model. The 'social interactions' parameter in Saxe model is here represented with the bi-directional arrows between the elements: the scaffolder, students and tasks. The parameter 'prior understandings' is intrinsic to the 'scaffolder' and 'students' in my model. More specifically, students' prior understandings involve their earlier knowledge about, for example, the notion of absolute values, of symmetries and reflections, and of linear functions as well as their understandings of social practices, e.g., their understanding of collaboration. The scaffolder's prior understandings involve mathematical content knowledge about the absolute value linear functions as well as pedagogical knowledge, e.g., how to scaffold the students' work. In my illustration I will not make further comment on this to avoid unnecessary repetitions as I have already mentioned the participants' prior understandings throughout my previous writings. The parameter 'conventions and artefacts' is not represented in my model. This is because an analysis of this parameter requires some further theoretical discussion as to what constitutes conventions and the nature of artefacts in mathematical discourse; and I feel I can say enough to make my point in this section without recourse to this parameter.

Central to this model are two assumptions. The first is, in line with Leont'ev's activity theory, that the main goal of the activity is attained through fulfilment of a series of emergent goals which are distinguishable from, yet subordinated to, the main goal itself. In the case of this study, the main goal of the activities depends on the tasks and can be considered as constructing a method(s) to draw the graphs of $|f(x)|$, $f(|x|)$ and $|f(|x|)|$ by using the original graph of $f(x)$ and consolidating the constructions of $|f(x)|$ and $f(|x|)$. The second assumption is related to the nature of the emergent goals, that is, they are built upon each other in the sense that achievement or failure of a prior emergent goal gives rise to new emergent goals. Thus, prior emergent goals influence the character of the new emergent goals. The notion of 'emergent goal' here is employed in the sense of Saxe and more specifically indicates that this sort of goal is not predetermined but rather evolves as the interaction amongst the participants unfolds in relation to the tasks. Further to this, emergent goals are not fixed or static but rather dynamic in that they take form and shift as a result of the participants' negotiation and of new understandings. I will now turn to H&S's verbal data to illustrate this model.

6.3. An illustration of the model of emergent goals within the ZPD

In this subsection I will explain and illustrate the model of emergent goals on the basis of H&S's work on the second task, though similar observations can be easily made in relation to the fourth task but due to the space limitations I will not consider this. I take it as evident that H&S constructed two methods to draw the graphs of $f(|x|)$, these constructions took place in H&S's ZPD and were assisted by the scaffolder. Thus I will not make any further comment on why, in the proposed model, the construction is considered as taking place within the students' ZPD (see Chapter 2, section 3.3. and 3.4.) Instead I focus on the emergent goals in relation to the second task, scaffolder and students. It is important to note here that in the proposed model the horizontal lines, indicating the beginning and end of a construction(s), are shaded precisely because it is virtually impossible to tell the exact moments when a construction starts and ends.

In H&S's verbal data on the second task, three sorts of goals can be identified: the main goal of the activity, predetermined goals of the questions presented in the second task (see Chapter 3, section 6.6. for these goals) and emergent goals. The main goal of the activity was the construction of the structure of $f(|x|)$ which was deliberately assigned by the scaffolder. Achievement of this goal was dependent on a successful completion of the second task. This task consists of purposeful questions each of which has their own goal(s) determined by the scaffolder. In this sense, the completion of task can be considered as a pathway, among many other possible ones, constituted by predetermined goals which supposedly and potentially take the students to a formation of the intended (by the scaffolder) construction(s). However, the predetermined goal structure of the questions exist only in the 'head' of the scaffolder, it does not have to be necessarily seen and interpreted by the students in the same way as the scaffolder does (cf. 'intersubjectivity and alterity' above). Thus, interpretation and perception of the questions and their predetermined goals result in the emergence of a new goal. For example, in question 4, four graphs of $f(x)$ without equations were presented and the predetermined goal was to urge the students to develop a (general) method to draw the graphs of $f(|x|)$ by using the given graphs of $f(x)$ rather than the equations. Despite the fact that the scaffolder tried to call attention to this (2: episode 1, 44I), the students interpreted this goal differently (45H) and decided to find the equation first. Consequently, a new goal different from the predetermined one emerged, that is, to draw the graph of $f(|x|)$ by substitution and then try to develop a general method.

Along with predetermined goals, the scaffolder also influences the emergence of a new goal. This issue has already been discussed in section 3.1. above; however, I briefly comment on this issue to clarify the scaffolder's influence on the emergence of a new goal. As the scaffolder had a clear vision of the target competence level and main goal of the activity, he regulated the students towards the achievement of the target construction by setting new goals. To do this, he monitored and analysed the students' performance on the basis of their actions and interactions and provided assistance which, at times, led to a new emergent goal. For example, when the students realised that they could draw a graph of $f(|x|)$ by reflecting a segment of $f(x)$ (episode 1), they were not aware which segment of $f(x)$ was being reflected. The scaffolder monitored this and, as a consequence of his analysis, intervened to assist the students (episode 2, 76I). The goal of the intervention was to have the students realise which segment of $f(x)$ was reflected. However, the students misinterpreted the intervention and talked about the symmetry line (episode 2, 77H). As a result, the students failed to fulfil this goal because their explanations remained specific to the graph at hand. The scaffolder may also set an explicit goal, as a form of assistance, for the students to move forward. For example, when the students got the graph of $f(|x|)$ wrong in question 4-C in episode 3 and could not see why it was so (see episode 3), the scaffolder, in 116I (episode 4), set an explicit goal, that is, to examine the graphs of $f(|x|)$ and decide which part has changed and which part remained the same. Hence emergence and fulfilment of a goal, to a certain extent, depends on how the students perceive and interpret the given assistance and relate it to the context of the activity.

Emergent goals are also dependent on how the students interact with and influence each other and on how they perceive and assess their resulting work. For example, when they got the graph of $f(|x|)$ wrong in question 4-C, S's assessment about the graph's accuracy (episode 3) and H's perception of the resulting graph led to emergence of a new goal, that is, to evaluate and justify if the graph was right or wrong. The students were unable to produce mathematically valid arguments and counter-arguments, and this contributed to their failure to fulfil this emergent goal. In fact it was as a result of this failure that the scaffolder set an explicit goal in 116I.

My consideration hitherto simply suggests that emergent goals are neither in the 'head' of the scaffolder nor that of the students, they are negotiated in the interaction itself (see also Saxe, Gearhart and Guberman, 1984). It is also important to realise that the emergent goals are not static but take form and shift as a result of this negotiation and of new understandings. This argument can be vividly illustrated by the emergent goals in episode 6. At the beginning of this episode, H&S's goal was to draw the graph of $f(|x|)$ by using the graph of $f(x)$ given in question 4-D. It is obvious that both H&S shared this goal (see 166I-170S) which was in line with the predetermined goal of question 4. Whilst S suggested drawing this graph by the 'reflecting' method (171S) H insisted on using 'y-symmetry' (172H and 176H). The idea of 'y-symmetry' was a new understanding for H which had just been developed in episode 5. The initial goal of drawing the graph of $f(|x|)$ took a new form through H&S's interaction, that is, to decide which method to use in drawing the graph. Moreover, the initial goal for H shifted to explain and illustrate how the graph of $f(|x|)$ can be obtained by the 'y-symmetry' method (172, 174 and 176). S's goal also shifted to understand how to use the 'y-symmetry' method (181S, 183 and 185).

At this juncture I need to comment on a possible objection to my interpretation of the emergent goal structure that is assumed to be held by the participants. I have so far simply stated what goals emerged for the participants and I have just noted these goal structures as I see them. For example, in the above paragraph, I noted that in episode 6, H's goal was 'to explain and illustrate her '*y*-symmetry' method to her partner' and S's goal was 'to understand how to use the '*y*-symmetry' method'; recall that I consider emergent goals as often unconscious and I do not contend H&S were conscious of these goals. Authors, such as Yackel (1996), argue that these interpretations of goals are in the mind of observer, who in my case is the researcher writing these lines, and need not necessarily correspond to the students' actual goals within the given moment. I certainly agree with this comment but need to note that some of the emergent goals are clearly stated by the students. For example, in episode 1, H in 47 said "we should find the equation for the first graph" and then later stated "let's find the equation" (47H). In this case the emergent goal of finding the equation is indisputably clear. Whether this goal is shared by S is another matter and related to 'intersubjectivity and alterity' and I think I said enough about this in section 5 of this chapter. However, in some other cases, the goals were not explicitly stated as is the case in my previous example of noting H's and S's goals in episode 6. When the goals were not clearly stated, I made inferences about the participants' goals on the basis of their verbalisations and actions, and referred to the appropriate utterances. Although I paid tremendous attention to make appropriate inferences, I accept that my inferences may not reflect the actual goal structures in the 'mind' of the participants but this neither invalidates my theoretical considerations that goals exist, emerge through interaction, are negotiated and shaped in the light of new understandings nor creates an essential challenge to the proposed model.

In relation to the two assumptions of the proposed model (section 6.2. above), it is clear from H&S's verbal data that attainment or failure of initial emergent goals impacts on subsequent goals. For example, H&S in episode 2 failed to satisfy the emergent goal of making a general mathematical statement as to which segment of $f(x)$ was to be reflected. As a result of this failure, they also failed to obtain an accurate graph of $f(|x|)$ for question 4-C, which brought about the emergence of a new goal, that is, to look into the graph to check its accuracy (episode 3). Subsequent to these emergent goals, a new goal set by the scaffolder in 116I of episode 4 emerged. The students fulfilled this emergent goal, though with the help of the scaffolder, in episode 4, and consequently constructed the 'reflecting' method. This simply shows that goals continuously and successively emerge and that fulfilment of a series of these emergent goals, though not necessarily all of them, may finally lead students to the formation of the intended construction(s).

A closer inspection of H&S's verbal data indicates some links between emergent goals and epistemic actions. It appears that the students need to undertake some epistemic actions in order to realise emergent goals. For example, having read question 4, H&S decided to draw the graph of $f(|x|)$ for question 4-A by substitution. In order to realise this goal, they recognised that the equation of the graph presented in question 4-A can be obtained by "using intersection points" (episode 1, 47H) and then found out the equation of this graph. So attainment of this goal required H&S to recognise some properties of the graph of $f(x)$ and then built the equation with these recognitions. However, when their goal was to justify the changes occurring in the graph of $f(x)$ when transformed into the graph of $f(|x|)$, they had to forge new connections amongst their existing knowledge about absolute values, linear functions and symmetries (see episode 4). In other words, in order to attain this goal H&S undertook constructing actions. Hence I posit that emergent goals determine the nature of and initiate a series of epistemic actions.

My final comments are on the mutual influences between the scaffolder and students. The description of the proposed model points out dynamic and dialectical interrelationships amongst the scaffolder and students whose interaction is, to a great extent, regulated by the demands of the task. The critical point is that the participants continuously influence each other and the emergence of new goals. That is, how the scaffolder perceives and interprets (monitoring and analysing) the students' work in relation to the task and then intervenes (assisting); and how the intervention is perceived and interpreted by the students, and how the scaffolder interprets the new situation and so on. This continues all the way through the formation of a construction and all these interactions take place with regard to the tasks which create the context for involvement of the participants.

The proposed model and my previous writings were mainly concerned with how the constructions were scaffolded within the stage 1 of the students' ZPDs. However, an important aspect of the scaffolding metaphor is yet to be discussed, that is, handover of the responsibility which suggests a transition from the stage 1 to stage 2 within the ZPD. This will be the next issue to deal with.

7. Handover of the responsibility

Handover of the responsibility (Bruner, 1983) from the scaffolder to student(s) in task completion is a distinctive and intrinsic feature of any scaffolding practice. The idea of handover suggests that the scaffolder should be sensitive to the developing competence of the students and gradually reduce the amount of assistance given to support the students. This is, in fact, the ultimate aim of Wood's (1991) contingency principle, Scott's (1998) action cycle model and Collins, Brown and Newman's (1989) 'fading' principle which more specifically suggests that

> Once the learner has a grasp of the target skill, the master reduces (or fades) his participation, providing only limited hints, refinements, and feedback to the learner, who practices successively approximating smooth execution of the whole skill (p.456).

Handover (or fading) depends on whether the students are able to progress without the scaffolder's assistance. It requires the scaffolder to monitor and analyse the students' developing competence, which in the case of this study refers to the students' developing construction(s), and adjust the amount of assistance and gradually reduce it to the level of none. A complete handover of responsibility indicates a transfer of control and regulation from the scaffolder to the students. In the scaffolding literature some associate this kind of transfer with 'internalisation' (Langer and Applebee, 1986). Here I first briefly reflect on the notion of 'internalisation' and then return to the issue of handover.

The notion of 'internalisation' is a key conception to Vygotsky's ZPD and the general genetic law of cultural development. He uses this term to describe the process through which an intermental functioning becomes intramental functioning (see Wertsch and Stone, 1985 for more on this). Reader may be surprised that despite the central role of this notion in the ZPD, I have not employed this term in my previous discussions. I purposefully avoided using this term in my writings for several reasons. First of all, although Vygotsky draws heavily on this notion in his writings, he, unfortunately, does not provide, in my opinion, a sufficiently well specified account of it. As a result, on the basis of his writings, this conception has been interpreted by many in quite different ways (see Rogoff, 1990; Nuthall, 1999; Wells, 1999); yet there does not appear any adequate account of the internalisation process (see also Rogoff, Matusov and White, 1996). Secondly, this notion runs through the writings of some other influential figures such as that of Piaget's (1978); and I found it difficult to see how exactly they can be reconciled or distinguished on the basis of empirical data. Thirdly, the notion of internalisation seems to suggest a differentiation between the 'external' and 'internal'. But what are the 'external' and 'internal'? On the basis of what could an 'entity' be considered as external and/or internal? Perhaps more importantly how can the 'external' and 'internal' be conceptualised? Wertsch (1998, p.48) also realises the difficulty involved in the conception of internalisation and writes

> it [internalisation] encourages us to engage in the search for internal concepts, rules, and other such mental entities that are quite suspect in the eyes of philosophers ... and cognitive scientists ... The construct of internalisation also entails a kind of opposition, between external and internal processes, that all too easily leads to the kind of mind-body dualism that has plagued philosophy and psychology for centuries.

Finally I do not feel that I can add something new to the existing debates in relation to this notion. Therefore, I avoided using this notion and in fact, I believe, I was and will be able to make my points without recourse to the process of internalisation. I now return to my main theme of handover of responsibility.

In the theoretical framework of this study (Chapter 2, section 3.4.), handover of the responsibility corresponds to a transition from stage 1 to stage 2 of the ZPD. That is to say, the students become relatively less dependent on the scaffolder's assistance and become relatively more able to regulate themselves. When we return to H&S's verbal data, we can observe the occurrence of handover in their works but only after they achieved the intended constructions. For example, in the fourth task, H&S's dependence on the scaffolder's assistance in order to construct the 'two-step' method to draw the graphs of $|f(|x|)|$ was all too apparent throughout episode 3, 4, 5 and 6. Following this construction, they moved on to question 4 where they were expected to sketch the graphs of $|f(|x|)|$ for four given graphs of $f(x)$ (see episode 7). They sketched the expected graphs without any need of support from the scaffolder. They were able to regulate themselves and the scaffolder, having realised their competency in sketching the graphs, did not feel any need to intervene and therefore handed the responsibility of completing the remainder of the task over to the students. In a similar vein, in the second task, the scaffolder was greatly involved in the students' formation of two constructions of the 'reflecting' (episode 4) and '*y*-symmetry' (episode 5) methods. Having achieved these two constructions, H did not need the scaffolder's assistance in episode 6 while sketching the graph of $f(|x|)$ for question 4-D and she was also able to illustrate her understandings to her partner, S. Once again, the scaffolder did not feel any need to intervene in episode 6. However, having realised the students' confusion over the two methods, he intervened to get H&S to clarify the differences between the methods (episode 7, 186I, 188I and 190I). But even then his intervention was limited to noting that these two methods were different and did not go beyond a request for an explanation of the two methods to distinguish them. In the rest of episode 7, he did not intervene except for one occasion in 202I and thus again handed the responsibility over to the students. H&S were able to account for the differences between these two methods and were able to regulate themselves.

On the basis of these observations, I posit that the complete handover of the responsibility is likely to indicate that the students have become acquainted with a new structure and hence formed a construction(s). However, it is important to note that despite the apparent self-regulation, the students' constructions were not fully developed in that they were not able to make general mathematical statements to describe their methods. For example, while describing their 'reflecting' method in episode 7, despite a few general statements (e.g., 198S), H&S drew heavily on the specific features of the graphs at hand such as "in *this* line … when x=0" and had some difficulties to express themselves (e.g., 200S). Similar observations can be made for their account of '*y*-symmetry' method (episode 7). So some important questions arise here: what is the initial state of the newly constructed structures? Can the new mathematical structures (i.e. new methods) be used in further activities immediately following their constructions? if so, under which circumstances? If not, why? All of these questions are related to the issue of consolidation that I attend to in the next chapter. However, before doing this, I will conclude this chapter with some further research notes.

8. Final remarks and further research notes

In this chapter I have investigated the formation of mathematical constructions through scaffolding within the framework of RBC theory and the ZPD. This investigation initially focused on the scaffolder's interventions and I attempted to link these interventions to the students' developing constructions. I suggested some causative relationships and proposed a mechanism centred on the idea of human mediation. Furthermore, drawing on Bakhtinian notions of utterance, voice and dialogicality, I also offered my views on the mutual influences between the scaffolder and students. Analysing the verbal data from Bakhtin's perspective, I addressed some general issues related to the nature of social interaction in the formation of mathematical constructions. This analysis involved the interaction between the immediate participants of the activity, the scaffolder and students, as well as some absent others who were present in virtue of their voices in the interaction. In an attempt to further characterise the social interaction in scaffolded discourse, I reflected on two opposing tendencies: alterity and intersubjectivity. I also provided an emergent goal model within the general theoretical framework of this study so as to make explicit the interconnections amongst three elements common to any scaffolded discourse: the scaffolder, students and tasks. Finally I elaborated on the handover principle.

In this chapter, my observations and resulting arguments were developed on the basis of a small number of cases. I tried to justify and exemplify my observations and arguments by referring to the relevant literature and to the verbal protocols of two 17-year-old students' work on two tasks. That the arguments relied on a small number of cases and were exemplified on the basis of two students' work is a limitation to be noted. However, although this limitation may raise some questions as to the substantiation of my arguments and observations, the data gathered from the small number of cases allowed me to carry out fairly well detailed analysis of the cases and develop new insights. Yet I need to note that my arguments and observations need further investigations.

Another limitation of this study is that the researcher and scaffolder were the same person. Although this situation was quite helpful in the course of analysis of his interventions, it was nonetheless a bias to be noted. This is a bias first because he, being the researcher, was rather enthusiastic and highly motivated to provide 'good scaffolding' with the students in their work. Second because he was, from the very start, aware that his interventions would be scrutinised and this probably influenced his scaffolding practice. This bias thus raises a question as to the typicality of the observed behaviours of the scaffolder.

Yet it is important to note that the scaffolder was, I believe, a successful tutor in his scaffolding and this kind of successful tutoring is not easily found in the relevant literature (see VanLehn et al., 2003; Chi et al., 2001). There appear several reasons for this successful tutoring. First of all, the scaffolder gained considerable experience in scaffolding students on these particular tasks. He scaffolded pre-pilot, pilot and actual study students over the same four tasks. Secondly, before actual data collection, he analysed some of the piloted students' verbal protocols and found a chance to see shortcomings in his scaffolding. This was important to make self-reflections on the way he intervened. Thirdly, he had discussion with his supervisors on the piloted students' verbal protocols and they warned him against hasty interventions of which he was oblivious at the time. Finally his subject matter knowledge related to the absolute value of linear functions helped him recognise the relevance of the students' different approaches and perspectives.

The very reason of the above-mentioned bias (i.e. motivation and enthusiasm), and experience gained through self-reflections, discussions with some more experienced people in the relevant field and content knowledge seem to be key ingredients of successful scaffolding. This brings us to the a general concern that under which circumstances can one provide successful scaffolding? What kind of training and practice can be useful towards successful scaffolding? Of course scaffolding cannot be conceived of separately from the students and tasks but I believe it is important to undertake further studies to determine the conditions on the part of a scaffolder to act successfully in a given situation.

In relation to the successful scaffolding, an important observation, in my opinion, was, as discussed before, the transformations in the students' way of seeing, talking and acting that sensitive and supportive interventions could bring about in the formation of new constructions. But how can we use these insights to design and organise classroom activities in which the students could be supported towards the formation of constructions? This is indeed a true concern for many researchers interested in scaffolding. For example, a recent volume of 'The Journal of the Learning Sciences' (2004) is devoted to this issue. The papers published in this special volume attempt to utilise the insights gathered over the years from the studies of scaffolding and propose some ways to use it in complex teaching and learning settings, ones involving, for example, computerised tools in classrooms. However there are certain difficulties in applying scaffolding in classrooms such as the large number of students that the teachers need to deal with, the different levels of students' understandings and some management issues. Under these circumstances it is not hard to imagine the difficulty of analysing and monitoring the students current level of understandings and providing appropriate assistance. Yet, given the potential gains of scaffolding, I believe that it would be valuable to carry out studies to develop some teaching and learning activities inspired by this notion.

Having noted the scaffolder's success in this study, I need to acknowledge that this success was, to a certain extent, related to the students, H&S, who demonstrated a harmonious interaction. However, this kind of harmony in students' interactions is an exception rather than a rule (see Sfard, 2001, Barron, 2003). This harmony might be attributed to the instructions given to the students on how to work together (see Chapter 3, section 5.3.). Nevertheless these instructions did not work out with some students taking part in this study as reported elsewhere (see Ozmantar, in press). The reason for this harmony, I believe, lies in the fact that these students were used to working together, respected each other's needs and expectations and expressed themselves comfortably without any fear of humiliation or resentment. Thus it can be said that H&S cared about each other's understandings. Further to this the scaffolder cared about the students and their understandings. Therefore the success of scaffolding certainly involves caring relationships of the participants, involves affective issues.

The question here is: what role does affect play in cognitive development? I believe that affect constitutes an important dimension in the formation of constructions through the ZPD. By drawing on Noddings' (1984) notion of 'caring', Goldstein (1999) for example, convincingly argues that the caring relationships of the participants is one of the dimensions, an important one, along which learning through interaction is achieved within the ZPD. In a similar vein Wells

(1999, p.331) also notes the importance of affect by referring to the learner's feelings: "learning in the ZPD involves all aspects of the learner – acting, thinking and feeling; it not only changes the possibilities for participation but also transforms the learner's identity." This is not the place to go into detailed discussion of the affective dimension of cognitive development within the ZPD. I believe, however, this is an important issue which merits further research. In fact some research initiation in this direction has already begun in relation to RBC theory (Williams, 2002). But this is a new area of research and a true understanding of learning within the ZPD requires further initiations in this direction than already undertaken.

CHAPTER 6: CONSOLIDATION OF THE NEWLY CONSTRUCTED STRUCTURES

As already discussed in the theoretical framework of this study, RBC theory posits that the genesis of an abstraction passes through three stages: (a) a need for a new structure; (b) the construction of a new structure through recognising and building-with actions and (c) the consolidation of the new structure. The initial work of Hershkowitz et al (2001) and a companion paper by the same authors (Dreyfus et al., 2001) deal with the stages (a) and (b) but merely mention the importance of consolidation of the newly constructed structures. It is through consolidation, these authors *a priori* assume, that the new structures become more familiar to the students who consequently will progressively be able to recognise and use these structures with increased ease. They, however, call for further research regarding the consolidation stage (c) to gain a better appreciation of this process on the basis of empirical data.

This study was designed to investigate the construction of a new structure as well as its consolidation. It was mainly for this reason that four consecutive tasks were prepared (see Chapter 3, section 6). Investigating the students' verbal protocols over four tasks provided crucial insights in relation to the second research question: what is the nature of consolidation? In this chapter, in relation to this research question, I will, more specifically, seek answers to the questions: What is the initial state of new constructions? What changes may occur during the consolidation? If and how consolidated constructions are used in further activities?

In answering these questions, this chapter is organised into five sections. In the first section I provide a brief review of literature related to the issue of consolidation. Secondly some background information is given on the protocol data utilised to answer the questions posed above. In the third section, substantial verbal data of one student's work will be presented in seven episodes with some commentaries. Following the presentation of these episodes, I discuss five particular issues: the initial state of new constructions, changes coming about in the course of consolidation, a re-consideration of the consolidation model proposed by Dreyfus and Tsamir (2004), task design in consolidation and some reflections on language development, use of examples and establishment of interconnections. The final section will focus on consolidation in relation to the ZPD on the basis of the theoretical framework of this study and conclude with some further research notes.

1. Consolidation: a literature review

The issue of consolidation is often studied in the field of psychology and neuroscience (e.g. McGaugh, 2000, Spear, 1978). Researchers in these fields often focus on the consolidation of memory – retrograde amnesia – and its function in relation to 'forgetting' and 'retention of information'. In this connection, the literature hypothesises that memory of newly learned information is relatively impermanent and disrupted by the learning of other information immediately after the original learning and that processes underlying new memories initially continue in a fragile state and take some time to become 'fixed' (see for example McGaugh, 2000 and Cantania, 1998). McGaugh (ibid.) comments that this hypothesis still guides research investigating time-dependent involvement of neural systems and cellular processes enabling lasting memory.

In the field of education, relatively less research attention has been directed towards the issue of consolidation. There are some studies concerned with consolidation in connection with 'mastery' and skill acquisition, and maintenance of these skills in the development of self-regulatory learning behaviour (e.g., Meichenbaum and Biemiller, 1998). However, there is little research concerned with consolidation and abstraction. There appear to be only three papers considering the issue of consolidation within the framework of RBC theory of abstraction. Tabach, Hershkowitz and Schwarz (2001) and Tabach and Hershkowitz (2002) examine the construction of knowledge and its consolidation. They touch on the importance and necessity of the consolidation after the construction of new knowledge structures and they appear to point to the fragility of new structures but they do not analyse the process of consolidation.

A major contribution, I believe, to understanding the process of consolidation comes from Dreyfus and Tsamir (2004). They analyse the protocol data of one student working on the comparison of infinite sets and conclude that consolidation is a long-term process in which an abstraction becomes so familiar that it is available to the student in a flexible manner. They identify three modes of thinking that take place in the course of consolidation: building-with; reflecting on the building-with; and reflection on "a wide range of mathematical and psychological issues" (ibid., p.297). They claim that building-with actions are the most direct and elementary means of consolidation. They characterise the consolidation of an abstraction with the psychological and/or cognitive constructs: immediacy, self-evidence, confidence, flexibility and awareness. I will not employ Dreyfus and Tsamir's constructs in this study to avoid a narrow line of enquiry in a new area of research; yet I comment on their work later in the discussion section.

2. Background for the protocol data

In this chapter, I focus on a student's work on the third task which was prepared to consolidate the constructions of the first and second tasks (i.e. $|f(x)|$ and $f(|x|)$). Although I have already discussed how this task was prepared (see Chapter 3, section 6), I feel it useful to remind the reader briefly of that process again. Four sequential tasks were initially designed (see Appendix 3). The mathematical focus of the first, second and third tasks was for students to draw/sketch the graphs of the linear absolute value functions $|f(x)|$, $f(|x|)$ and $|f(|x|)|$ respectively. The fourth task was designed to consolidate all of the constructions in the first, second and third tasks. Following the piloting of these initial tasks (see Chapter 3, section 6.3.), it was realised that the fourth task did not provide consolidation opportunities for the students (see Chapter 3, section 6.4. for more on this). Consequently the organisation of the initial four tasks was rearranged such that the consolidation task, the initial fourth task, became the third task, amended to consolidate only the constructions $|f(x)|$ and $f(|x|)$ and that the initial third task becomes the fourth task. So in the actual data collection, all of the students worked first on the graph of $|f(x)|$; second on $f(|x|)$;

third on a task for the consolidation of these two; and fourth on the graph of $|f(|x|)|$ (see Appendix 7).

The amended third task is presented in Appendix 7 and is composed of five questions. Question 1 presents a linear function and asks students to draw the graph of $|f(x)|$ and $f(|x|)$. Students are free to choose a solution strategy. The aim is to examine the state of the new structures and observe how students obtain the graphs. Question 2 asks students to state how they would obtain the graphs of $|f(x)|$ and $f(|x|)$ for $f(x)=ax+b$. The aim of this question is to lead students into discussion. Question 3 and 4 present claims from three imaginary students about how to obtain the graphs of $|f(x)|$ and $f(|x|)$, respectively, from a given graph of $f(x)$. All of the claims in these two questions are different and incorrect (though they are designed to be 'intelligently incorrect'!). The aim is to engage students in discussion and for students to justify and clarify their ideas. Question 5 aims to focus on the difference between $|f(x)|$ and $f(|x|)$ from graphic considerations alone.

In what follows some excerpts from one student's verbal protocol on the amended third task are presented. The details as to how the verbal protocols have been prepared for the analysis were discussed in Chapter 3 (section 8). In this connection, I suffice to say that the student's verbal protocols were audio-recorded, transcribed and then translated from Turkish into English. Principles guiding translation were: (1) that the English should be clear; and (2) faithfulness to the original intent, e.g. wording as close as possible to the original Turkish.

I select a single student rather than a pair of students for this helps me focus on consolidation without getting involved in considerations of issues of social interaction, which increases the complexity of the analysis and are already discussed in the previous chapter. I call the student Tugay (not his real name but a common Turkish name). He was one of three scaffolded individuals; all three consolidated the constructions, made in the first and second tasks, in the third task (see Chapter 4, section 1 for an overview of all participating students' performances on the tasks). Although I will make some comments on the interviewer/scaffolder's assistance in the final section, this issue will not be the main concern. This is not because the issue of assistance and scaffolding is irrelevant to consolidation but because this issue was my explicit focus regarding the first research question which was attended to in Chapter 5 and discussions provided in this respect will shed light on the role of assistance and scaffolding in relation to consolidation. As I am not concerned with scaffolding, I prefer to use the term 'interviewer', in my considerations below, to 'scaffolder' which was dominantly used to describe the function of the interviewer in the previous chapters. I have chosen Tugay to investigate consolidation because he was at ease and expressed himself clearly during his work with the interviewer.

3. Episodes in the verbal protocols

I do not present the complete protocol of Tugay's work in the third task; this would be very long and would detract from analysis. In selecting protocol excerpts I attempt to present an overview of the work and include all excerpts referred to in the discussion section. The main focus in the protocol excerpts will be on $f(|x|)$, not on $|f(x)|$. This is for reasons of space and because matters relating to consolidation are more apparent in Tugay's work with $f(|x|)$ as he found this more difficult than $|f(x)|$.

Before going into Tugay's work on the third task, it is useful to know what he achieved in the first two tasks. At the end of the first and second tasks Tugay was asked to briefly explain how to obtain the graph of, respectively, $|f(x)|$ and $f(|x|)$ given the graph of $f(x)$. With regard to $|f(x)|$ he explained:

> ... when a graph of $f(x)$ is given, the first thing to do is to take the symmetry of the segment of $f(x)$ under the x-axis so that it becomes symmetric in the x-axis by flipping it up over the x-axis... and when we flip it up over the x-axis, we obtain a graph where y is constantly positive and x is negative or positive... when we look at the examples that we did, we draw them initially by finding points but later we practically draw the graph of the function by folding the area under the x-axis over the x-axis... After flipping up and pasting this over the x-axis we can obtain the graph of $|f(x)|$.

Regarding the graph of $f(|x|)$ he explained:

> Even if we are not given any equation we can obtain the graph of $f(|x|)$... To do so err... intersection point of the function of $f(x)$ and y err for x=0, through the point of intersection of y, I mean we first find the value of y for x=0... then we draw the line of ...I mean drawing the line passing through this value... I mean say $y=a$... we draw the line of $y=a$... the segment [of $f(x)$] at the negative values of x, up or down...I mean we take a symmetry in this line by flipping this part [of $f(x)$ at the negative values of x] up or down and the graph is usually V-shaped... umm always V-shaped and this gives us the graph of the function of $f(|x|)$.

These explanations suggest that Tugay at the end of the first and second tasks constructed a method to sketch the graphs of, respectively, $|f(x)|$ and $f(|x|)$ from the graph of $f(x)$.

The protocol excerpts presented below are divided into seven episodes. The first two episodes deal with Tugay's work on questions 1 and 2. I omit the section of the protocol where Tugay works on question 3 as this deals with $|f(x)|$. Episodes 3, 4, 5 and 6 deal with Tugay's work on question 4. I divide the work on this question into four parts because Tugay's verbalisations display a distinct development in his work on this question. Episode 7 deals with Tugay's work on question 5. After the episodes I briefly present what Tugay did in the fourth task. This is interesting in itself, in that it shows how he went on to use his abstractions (i.e. consolidated constructions) in building a further construction, $|f(|x|)|$, and this informs theoretic discussion in this chapter and the next. In the following 'I' refers to the interviewer and 'T' refers to Tugay. Each complete utterance is given a new line number. I provide comments after each episode.

Episode 1

> **2T:** [Tugay reads question 1.] Umm how can we do this?
> **3I:** How are you planning to obtain these graphs?

4T: As far as I remember from the earlier tasks, I can either first draw the graph of $f(x)$ and then take the symmetries accordingly or I can substitute different values of x and then draw the graphs.

5I: OK then, let's talk about them before you proceed. What do you remember about the graphs of $|f(x)|$?

6T: If I remember correctly we take the symmetry of the negative values of y in the x-axis.

7I: Negative values of y?

8T: As far as I remember we flip up the part of $f(x)$ at negative values of y over the x-axis to obtain the graph of $|f(x)|$.

9I: What about $f(|x|)$?

10T: For the graph of $f(|x|)$, we draw a line parallel to y-axis through the intersection point of $f(x)$ and y-axis. Then we take the symmetry of a ray by flipping up and down accordingly. But I am not too sure how! Maybe according to the given graph, I guess.

11I: OK. You told me what you remember about these graphs. I just wonder how you could draw these graphs now?

12T: Umm, I think it'd better if I substitute different values of x and then draw the graphs because I feel more secure in that way. Maybe afterwards I can use what I developed before...

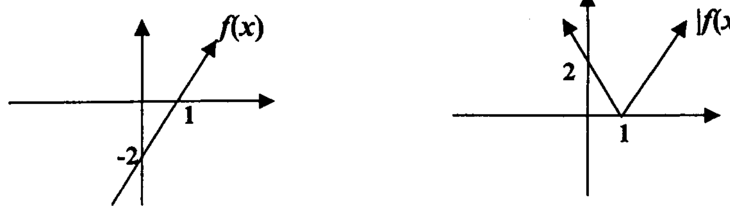
13I: OK.

Recall that Tugay constructed valid structures for $|f(x)|$ and $f(|x|)$ in the first two tasks. When he was asked to state how to obtain the graphs of $|f(x)|$ and $f(|x|)$ using $f(x)$, he came up with two ways of doing this: first drawing the graph of $f(x)$ and then 'taking symmetries' or substituting. Although substituting was a valid method to obtain the intended graphs, the aim of the task was for students to draw the graphs from the graph of $f(x)$. Although Tugay remembered elements of how to draw the graphs such as taking symmetries, his language was not precise and he also stated that he was not sure about how the symmetries were taken. This uncertainty was apparent in the linguistic forms he used, e.g. "as far as I remember" (4T), "if I remember correctly" (6T) and "I am not too sure... I guess" (10T). These statements indicate that the knowledge structures he constructed in the first and second tasks are rather fragile and need to be consolidated. He did not use the constructions and resorted to substituting because, he said, he felt "more secure in that way" (12T).

Episode 2

Tugay completes question 1 by substituting (14T – 42T, not shown) and moves on to question 2.

44T: I think I use the first question for this. Let's see... but I need to draw the graph of $f(x)$ first so that I can see what happens...[He draws the graph of $f(x)$ and $|f(x)|$ (see below)].



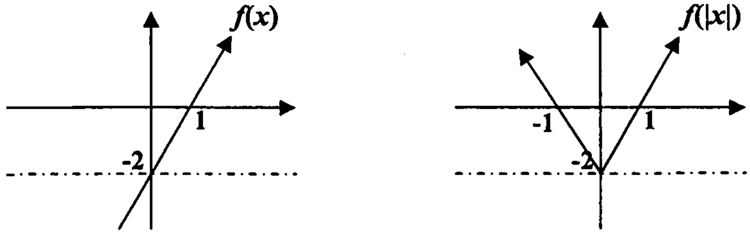
48T: Umm, as can be seen, one part of $f(x)$ remains the same, which is over the x-axis. And the ray which belongs to $f(x)$ under the x-axis becomes ... umm I mean lower part is flipped up over the x-axis. Because instead of –2 the ray passes through +2 and also when we take this part symmetrically it goes through (-1,4) which is (-1, -4) in the graph of $f(x)$. So yeah, it is symmetric in the lower part but no change in the positive y values.

49I: In order to draw the graph of $|f(x)|$ ...

50T: Umm, to draw the graph of $|f(x)|$, one needs to take the symmetry of the negative values of y in the x-axis.

51I: What about $f(|x|)$?

52T: [He looks at the graphs of $f(x)$ and $f(|x|)$, see below].



... for the graph of $f(|x|)$, umm after x becomes negative, I mean... umm first we find the intersection point of $f(x)$ and y-axis. Then we draw a line which passes through this point and which is also parallel to the y-axis. I think I better explain with the above example. We draw the line of y=-2 because it is the intersection of $f(x)$ and y-axis. Then the part of $f(x)$ under this line will be flipped up to obtain the graph of $f(|x|)$.

53I: You mean you reflect in the line of y=-2?

54T: Hmm yes, yes symmetry but the line... I mean the symmetry line changes according to $f(x)$. [Work on question 2 ends here.]

A question of interest in this episode is, to what extent was Tugay's use of a specific function necessary to his elaboration? Question 2 asks for a general description explained with an example. Tugay used the example from question 1 but his explanation appeared to be tied to this example, though he did recognise (54T) that what he calls a 'symmetry line' may change if another function is used. One cannot state for certain but the use of a specific function appears to be intrinsic (if not necessary) to his thoughts at this stage in his development. For example, in 52T he began to articulate how to obtain the graph of $f(|x|)$ in general terms but then returned to his specific graph saying "I think I better explain with the above example". Indeed all of his statements in this episode made reference to specific graphs (even 54T). Maybe he could have proceeded without them but he certainly appeared to be dependent on them. During this episode, he did not express his thoughts without recourse to citing specific examples; in this sense these structures may be said to be 'rigid' at this point in his development.

Episode 3

Between utterances 56T and 107I Tugay worked on question 3, which is concerned with $|f(x)|$, and I omit this part of the protocol. I pick up the discussion as Tugay starts question 4.

108T: [He reads Aylin's statement] umm, let's see...

109I: You may better understand what Aylin says if you illustrate it on a grid.

110T: OK, but I wonder if a graph of $f(|x|)$ can take negative values, I mean under the x-axis!...Well, yes, of course it could be...I mean...let's draw a random graph [he draws a linear graph]. What was I asked? In order to draw the graph of $f(|x|)$, we should draw a line passing through the intersection of the y-axis and $f(x)$... and then we take the symmetry in that line.

111I: You took the symmetry of the part of $f(x)$ corresponding to the negative values of x?

112T: Yeah, I took the symmetry at the negative values of x and then flipped it up. So the graph of $f(|x|)$ does not have to be over the x-axis all the time.
113I: It does not?
114T: No, it may have some part under the x-axis as well.
115I: You showed that Aylin's reasoning is wrong, but how about her method? She may propose a correct method with the wrong reasoning?
116T: Negative values of $f(x)$
117I: Should take symmetry in the x-axis, she says...
118T: No... negative values... but wait a minute... absolute value... [He re-reads Aylin's statement]
119I: What she says is, briefly, part of $f(x)$ under the x-axis should be taken symmetry in the x-axis.
120T: Yeah I know but it's wrong. Because when the symmetry is taken in the x-axis we obtain a graph like this...
121I: So?
122T: So, it will be different from the graph that I drew. Thus it cannot be so... I mean I don't agree with Aylin.

In this episode Tugay convinced himself that the graph of $f(|x|)$ need not necessarily be above the x-axis. Piloting alerted me to a misconception that if a function includes an absolute value sign, then the graph must be above the x-axis (since absolute value signs makes negative values positive) and an aim of this question was to allow students to confront this misconception (which Tugay successfully did). However, although he convinced himself he did not, in my opinion, offer a convincing argument or justification that he was right and Aylin was wrong other than note that Aylin's method was incompatible with his in that they produced different graphs.

Episode 4

123I: OK what do you think about the Cem's theory?
124T: [He reads Cem's statement.] Are both graphs the same for the positive x values? ... [He looks at the graphs of $f(x)$ and $f(|x|)$ in the first question]... both graphs appear to be the same... for the positive x values... [He re-reads Cem's statement] 'There is no difference'... yes... there is no difference...
125I: Do you think he is right?
126T: There seems no difference for now...but I mean we have to consider the whole theory to come to a decision... [He re-reads Cem's statement and reads aloud the second part] "but we cannot say anything about the difference for the negative x values, which depends on the equation of $f(x)$"... yes...umm... negative x values for the graph of $f(|x|)$... [He reads the second part once again]. I think we can say something about the graph of $f(|x|)$ for the negative x values...because only $f(x)$... I mean when the x values are negative, then we take this part symmetrically in a line which parallels to the y-axis... so we already obtain the difference between these two graphs. That means that we can say that only a symmetry would be the difference...[He looks at the two graphs obtained for the first question]... but the difference between $f(x)$ and $f(|x|)$?... [A long pause]... Well, of course, the difference is evident... I mean while $f(x)$ is linear, the graph of $f(|x|)$ is something like the shape of V...but one arm of the V is symmetric at the negative x values...
127I: I am not too sure if I understood you right. So let's go through what Cem says together; he says that $f(x)$ does not change for the positive values of x... which is this part [the ray on the right of the y-axis]
128T: Hmm yes that's right
129I: But for the part of $f(x)$ corresponded to the negative x values we cannot say anything, it depends on the equation of $f(x)$; I think he means that one needs to substitute some values of x

130T: Yeah?
131I: Only then we can find the other part of the graph of $f(|x|)$, he says.
132T: But we did find the $f(|x|)$ without substituting x values. I mean in the last task and for this task we found the same graphs when we substituted and when we took the symmetry. So I think we can obtain the graph without substituting but only taking the appropriate symmetry.
133I: Well, how can we convince Cem that you're right? I mean why Cem isn't right but you are? Is it simply because what you were saying was right for a couple of examples that you solved and you have seen the pattern only for those examples? Maybe Cem also found this theory while working on several examples which contradicted what you are saying? Obviously we do not have the right to say 'ok my rule is right simply because it worked on several examples' while talking about mathematics!
134T: Yes, actually... yes you're right.... Well, first of all, I remember that for the positive x values the graph of $f(x)$ remains absolutely the same, well I don't know if I can say 'absolutely'.
135I: OK.
136T: But for the negative values of x, it was enough to take the symmetry. In fact we made use of analytic geometry for the solutions so...but I am not sure if what I am saying is definite...I am confused...

Three aspects of Tugay's work in this episode are particularly noteworthy: he was correct; he was not confident when his statements were challenged mathematically; and he did not appear to see how the various mathematical features he has introduced interrelate. Tugay explains his 'V' construction well in 126T. 127I-131I re-examined Cem's statement and Tugay (132T) explained that substitution was not necessary. The interviewer then (133I) questioned the generality of both Cem's and Tugay's statements in mathematical terms. Tugay's responses when challenged (134T and 136T) were hesitant (not "absolutely", "not ... definite", "I am confused"). This hesitancy even applied to his claims about $f(|x|)$ for positive values of x which he had previously drawn without any hesitancy as well as his claim that the graph of $f(|x|)$ for negative values of x could be found from the graph of $f(x)$ alone without need of an equation. This hesitancy I think is tied up with the fact that he did not relate his V rule to mathematical structures known to him: features of absolute value, of symmetry and of linear graphs. Although his V rule arose from these known structures, there is no evidence here that he has developed interrelationships amongst these concepts and the structure of $f(|x|)$, i.e. he has not developed a 'web of meaning' (Noss and Hoyles, 1996, chap. 5) and that without this he was not confident when challenged mathematically.

Episode 5

Between 137I – 150T (not shown) the interviewer led Tugay to look at the graph of $f(|x|)$ from a different perspective. He returned to the first graph and examined $f(|x|)$ for ±3 and ±1 and they noted that $f(|x|)$ has the same values each times.

151I: What would be the reason for this?
152T: I think this is because of the absolute value sign, I mean it is outside of the x.
153I: That means...
154T: That means... regardless of the sign of the values of x, they will be matching the same value of y.
155I: What does this tell us about the symmetry?

156T: So it tells us perhaps that all of the graphs of $f(|x|)$ are symmetric in the y-axis. Actually I remember that I told something about it on the second task but I did not realise today though.

157I: Perhaps? When you say perhaps I feel [Tugay interrupts].

158T: Well yes I mean I need to look at once again! [He examines the graphs]... yes, all of the graphs must definitely be symmetric in the y-axis because different values of x with different signs must have the same vale of y, which is why it must be symmetric in the y-axis.

159I: So if we are talking about the symmetry in the *y-axis*, then what does it tell us about the position of the graph of $f(x)$ in relation to the graph of $f(|x|)$?

160T: Yes, umm... now I am quite sure that we can say everything clearly about the position of the $f(|x|)$ in relation to graph of $f(x)$... I mean...

161I: If we return to Cem's arguments, what would you tell him?

162T: Now it is evident that we can say that there is no difference between the graphs of $f(x)$ and $f(|x|)$ at the positive x values. At the same time, we can say surely that the part corresponding to the negative x values must be the symmetry of the ray which is on the right side of the y-axis. So Cem is wrong. I mean we can say how to obtain the graph even without needing an equation.

I return to two of the issues discussed after the previous episode: language and interconnections. There is a noticeable change in the confidence of Tugay's language in this episode compared to the previous one. The language of the previous episode was marked by hesitancy and uncertainty. In this episode, however, especially in 158T, 160T and 162T Tugay used expressions such as "must definitely", "quite sure" and "surely" all of which signify self-confidence. The interviewer's focus on $\pm n$ values did not provide Tugay with new knowledge but enabled him to focus his attention, for the first time in this task, on this symmetric aspect of $f(|x|)$ and this focus allowed him to connect his knowledge about absolute values and symmetry with regard to $f(|x|)$. This interconnection or web of meaning, I believe, prompted his increased mathematical certainty.

Episode 6

Tugay moves on to Arzu's argument. He reads this and immediately responds:

164T: No, it is not so... I mean when we take the symmetry of the graph [of $f(x)$] at positive values of x in the y-axis we obtain the $f(|x|)$.

The interviewer thrice challenges Tugay but Tugay is sure about his statement.

171I: Well, why does the part of $f(x)$ at positive x values remain unchanged? How do you justify this mathematically?

172T: Why does positive x values not change? Because... I mean every value will be positive in the absolute value...it does not matter for positive values whether they are in the absolute value sign or not because it is positive anyway so it does not change. But the negative values differ if they are in the absolute value. I mean when they are in the absolute value sign, then they change, they alter into positive...and thus result changes...so when one substitutes, for example -2 for x in the $f(x)$, then one would obtain a different result from the result of $f(|x|)$ when one substitutes –2... because |-2| is a positive value and this is –2 in the $f(x)$. So they are totally different.

173I: So, for positive values of $f(|x|)$ [Tugay interrupts him]

174T: OK, let me put it another way, in the $f(|x|)$ when we substitute positive values we obtain a result which is the same result as of $f(x)$. But if we substitute negative values in the equation of $f(|x|)$ we get different result from the result of $f(x)$ when the same negative values are substituted in the $f(x)$.

175I: Which shows that?

176T: Yeah, that proves that the graph of $f(x)$ at the positive x-values is exactly the same graph as the graph of $f(|x|)$. On the other hand, as the negative x values change in the $f(|x|)$ so does the graph of $f(x)$ when transformed into the graph of $f(|x|)$... I think I made my point, right?

177I: Well, yes, I mean I think so, but I wish to hear from you how we can obtain the graph of $f(|x|)$ from the graph of $f(x)$ once again because I am a little bit confused after so much discussion.

178T: Well actually we can obtain the graph of $f(|x|)$ in two different ways. The first one is that umm...we can draw a parallel line to the y-axis through the intersection point of $f(x)$ and y-axis. And then for the negative x values we can take the symmetry of that part of graph in this line. Secondly, well... I think this is easier... that is we can take... umm... I mean the graph of $f(x)$ at the positive x values...remains the same; I mean we can take the symmetry of this part in the y-axis and cancel the part of $f(x)$ at the negative x-values... and so this is $f(|x|)$...yeah... yes definitely so.

Tugay quickly and assuredly evaluates Arzu's claim after one reading. He stated his disagreement with Arzu eloquently and precisely and described a correct way to obtain the graph of $f(|x|)$. His quick and assured articulations throughout this episode contrasts strongly with his hesitancy at the beginning of the protocol. Further to this he made general statements about the graphs of $f(|x|)$ without relying on specific examples (see 178T). Indeed there is a shift in the relationship between specific examples and general statements. Early in the task, e.g. 52T, he explained through examples but in this episode he used examples to illustrate his already made general statements, e.g. 172T. His structure of $f(|x|)$ was also flexible in the sense that he was able to talk about it in general terms, emphasised different aspects and even defined two different ways to obtain the graph of $f(|x|)$ from the graph of $f(x)$ (174T, 176T and 178T). He was self-confident in his claims and able to defend them when challenged (172T). The substantial difference in the quality of Tugay's arguments and explanations in this episode could be considered a result of his increased familiarity, by virtue of time spent on the task, with the new structure. What is more, I believe he established interconnections as resulting in a qualitative jump in understanding. In episode 3 Tugay realised that the graph of $f(|x|)$ can take negative values (can be under the x-axis); in episode 5 he understood that the graph of $f(x)$ for positive values of x remains unchanged, that this can be reflected in the y-axis to obtain the graph of $f(|x|)$ for negative values of x and that the segment of $f(x)$ at the positive values of x remains the same in the graph of $f(|x|)$. Each of these aspects has undergone a great deal of examination but the product of this work, I believe, evidenced in this protocol is greater than the sum of the parts.

Episode 7

Tugay proceeds to question 5. He quickly and correctly commented on each of the six graphs. For brevity I only provide protocol excerpts from a and d (see Appendix 7, task 3).

186T: OK... if I first talk about (a), then it could be the graph of $|f(x)|$ because there is no negative values of y.

187I: Right!

188T: Besides, the part that corresponded to positive values of x is taken symmetrically in the y-axis, which means that it could be the graph of $f(|x|)$ as well. Yeah this graph could belong to both $|f(x)|$ and $f(|x|)$.

210T: [He is talking about the graph d.] Well let me summarise, this is not the graph of $|f(x)|$ because ... yes it does not have any negative values [of y] but the symmetry does not start from the x-axis. And it is the graph of $f(|x|)$... obviously symmetry is on the y-axis, the y values for the positive xs do not change, and what is more, x=-2 and x=+2 takes the same value of y. So yes clearly it is the graph of $f(|x|)$.

Tugay's explanations were precise and assured mathematical statements. His competent identification of each of these graphs as a graph of $|f(x)|$ and/or $f(|x|)$ indicates that he has consolidated his constructions of $|f(x)|$ and $f(|x|)$.

Tugay's use of his consolidations in the fourth task

In this section some excerpts are presented from the Tugay's work in the fourth task where he used the abstractions of $f(|x|)$ and $|f(x)|$ to construct a method to sketch the graphs of $|f(|x|)|$.

114T: If I understood you right, you suggest that I draw the graph in two steps.

115I: Two steps?

116T: I mean, first is to find the $f(|x|)$ which we label it as $g(x)$, and then to obtain $|g(x)|$?

117I: Yeah that is what we talked, right?

118T: OK

119I: Look, we have the graph of $f(x)$ for the first question, in fact you drew it anyway but we have it...so if you use this graph to obtain $g(x)$

120T: Oh yeah, I can leave the ray of $f(x)$ at the positive values of x, and take it symmetrically in the y-axis... it comes from here and goes through... and symmetry.... [He draws the graph of $f(|x|)$].

121I: So now you get the graph of $g(x)$, which is...

122T: Yes, this is actually the graph of $f(|x|)$ and now we will find the graph of $|g(x)|$ and so we will draw $|f(|x|)|$... [He draws the $|g(x)|$ by taking the symmetries of the line segments under the x-axis and thus obtain the intended graph of $|f(|x|)|$] ... so there are two line segments and I need them positive so I will take their symmetries in the x-axis.

In 114T – 122T Tugay achieved the intended construction of the graph of $|f(|x|)|$ by using two steps[1]: (1) by drawing the graph of $f(|x|)$ and then (2) by drawing the absolute value graph of $f(|x|)$. This new construction relied on Tugay's earlier constructions of $|f(x)|$ and $f(|x|)$. Tugay later moved on to the question 5 in which he was asked to explain how to obtain the graph of $|f(|x|)|$ by using the graph of $f(x)$.

148T: OK by making use of $f(x)$... umm first of all we take the symmetry of the $f(x)$ at the positive values of x in the y-axis, in so doing we obtain the graph of $f(|x|)$. Then... after that if there is any negative values of y in this graph, then we take also the symmetry of these negative values of y... like as if drawing the graph of $|f(x)|$ and flip it up over

[1] At this point I should comment that the interviewer assisted Tugay to draw the graph of $|f(|x|)|$ in two steps, which was more or less the same way that H&S came up with in their work on the fourth task. Tugay's work took place after H&S's and the interviewer was apparently influenced by H&S's construction of $|f(|x|)|$.

the x-axis. We have to do so because there cannot be any negative values of y in the graph of $|f(|x|)|$. And so we can obtain the graph of $|f(|x|)|$.

149I: How can you make sure that your formulation is correct?

150T: Well, OK… first of all according to this method there is no negative values of y in the graph of $|f(|x|)|$ because the absolute value sign which is outside of the whole expression makes it impossible to exist any negative values of y, that is the first evidence. In addition, umm… we first find $g(x)$ which is $f(|x|)$, and we know that the graph of $f(|x|)$ is symmetric in the y-axis so our rule is correct once again.

Tugay competently applied his two constructions, $|f(x)|$ and $f(|x|)$, in a new context and used them to construct the graph of $|f(|x|)|$. He did not appear to have any difficulty in using these constructions or in expressing himself. His recognition and use of these two constructions in this task suggest that the constructions of the first and second tasks have become abstractions (i.e. consolidated constructions; see Chapter 7, section 3.2. for more on this) which can be used in further constructions.

4. Discussion

I address five particular aspects of consolidation in relation to new constructions. In what follows, I consider both $f(|x|)$ and $|f(x)|$ but my main focus is $f(|x|)$. I begin with a discussion of the initial state of the two absolute function constructions and changes that were observed during the consolidation process. Then, in relation to Tugay's verbal data, I comment on the model proposed by Dreyfus and Tsamir (2004). The fourth subsection outlines issues in task design. Finally I will make some speculative comments on the interrelations in language development, use of examples and developing new connections.

4.1. The initial state of the new constructions

As Tugay's explanations at the end of the first and second task suggests he has constructed methods to obtain the graphs of $|f(x)|$ and $f(|x|)$. However, when he started the third task, he was not confident in the validity of these constructions. He was, for example, able (10T) to describe how to obtain the graphs of $f(|x|)$ from the graph of $f(x)$, but his comments "I'm not too sure" suggests that he was not certain about these constructions. He also expressed feelings of insecurity (12T) with regard to these constructions as a means of obtaining the graphs of $f(|x|)$ and $|f(x)|$. His comment "if I substitute" suggests that he is uncertain about the validity of the constructions formed in the first and second tasks. A hesitancy in defending constructed methods for a considerable period after their construction was common in all the protocols of students who made these constructions. In Tugay's case one can see his uncertainty reappearing as the interviewer probes different aspects of the graphs. In 136T, for example, he stated that he was not sure if his symmetry argument for negative values of x was correct and stated "I am confused".

In the early parts of the protocols of the third task students made extensive use of specific examples and these examples were used as a basis for formulating their ideas. Only students who consolidated the constructions in this task went beyond specific examples and then only in the latter parts of the protocols. This is not surprising but it does draw attention to an apparent need to ground the new constructions in concrete examples. In Tugay's case he stated, (44T), "I need to draw the graph of $f(x)$ first so that I can see what happens". He did not talk about the relationship between the graphs of $f(x)$ and $f(|x|)$ until he had drawn them (see episode 2). The points and lines he constructed were prefaced with demonstrative adjectives in his discourse: "this point ... this line" (52T) – he appeared to be unable to formulate his constructions in general mathematical terms free from specific lines and points. This dependence on specific examples in explanations and the uncertainty noted above indicate that the new constructions are fragile and need to be consolidated.

4.2. Changes in the course of consolidation

In the course of his work on the third task, Tugay appeared to consolidate his constructions of $|f(x)|$ and $f(|x|)$. In this respect I focus on: reconstruction of the constructions, increased resistance to challenges, developing a language for the constructions and greater flexibility.

It appears that Tugay reconstructed his constructions of $|f(x)|$ and $f(|x|)$ in the initial stages of the third task. Reconstruction is a process in which constructions are derived as in past constructions, i.e. the constructions are not simply recalled. In between 44T - 54T, Tugay was combining and manipulating various bits of information about absolute values, symmetries and graphs. This process continues throughout the third task. For example, later in the protocol (172T) one can see Tugay's justification that the graph of $f(|x|)$ is the same as the graph of $f(x)$ for positive values of x by combining bits of information and actively reorganising them. My intention here is not to equate reconstruction with consolidation but reconstruction appears to be an important part of consolidation.

It seems that it is the fragility of new constructions that makes students reluctant to use them to counter challenges. In the course of consolidation, however, students begin to resist challenges by establishing interconnections between the new constructions and established mathematical knowledge and by reasoning with these constructions.

Tugay established interconnections between the graph of $f(|x|)$, absolute values, symmetry and linear functions (154T, 158T, 172T and 174T). In 172T, for example, Tugay explained why $f(x)$ and $f(|x|)$ were the same for positive values of x by establishing connections between the graph of $f(|x|)$, absolute values and $f(x)$. Shortly after (176T) a change in the tone of his assertions can be noted, "that proves ... I think I made my point". This change to a confident tone continues from 176T, e.g. compare "yes definitely so" (178T) with "I don't know if I can say absolutely" (134T). This aspect of this construction, that $f(x)$ and $f(|x|)$ are the same for positive values of x, appears to be fully consolidated as he used this to confidently elaborate how to obtain the graph of $f(|x|)$ (174T, 176T and 178T).

I believe that interconnections between Tugay's new constructions and his existing knowledge were not sufficiently well established in the beginning of the third task and that this situation contributed to his insecurity and uncertainty in his claims about his new constructions. I therefore posit that the more connections students make between the new constructions and existing knowledge, the more meaningful and accessible the new constructions become, and students become more confident and resistant to challenges.

Apart from the confidence of Tugay's language, as the constructions of the first and second tasks were consolidated in the third task, there was a qualitative shift in the clarity and precision of his language in the course of the third task. It appears to me that language development (to describe a new construction) has a dialectical relationship with the consolidation of the construction. For example, at the end of the second task (recall Tugay's explanation of how to obtain the graph of $f(|x|)$) and at the beginning of the third task (10T and 52T), Tugay's description of the graph of $f(|x|)$ lacks precision and is slightly ambiguous whereas in 178T his mathematical language is precise and unambiguous. A lack of precision in the initial part of the third task is not surprising, but language development during the task is significant with regard to consolidation in that the language of the new constructions needs time to develop.

Students' use of examples is closely related to this development in their language of the construction. Prior to consolidation students appear to need concrete examples to formulate their thoughts but after consolidation they appear to use examples to demonstrate assertions. Tugay, for example, in the second task and in 52T, articulated his thoughts by referring to specific properties of graphs. In 172T, however, he used examples to convince the interviewer, to clarify and to justify his assertions.

The use of specific examples to articulate thoughts suggests that the new constructions are somewhat inflexible. When Tugay, for example, was asked to give an account of the graphs of $f(|x|)$ (52T) he appeared to begin stating a general rule, "after x becomes negative", but then focused on a specific graph. Later in this protocol (174T), however, he quickly provided an alternative way to view $f(|x|)$ – and did so without recourse to a specific example. The phrase "let me put it another way", along with the confident and precise way he stated this other way, suggest that he has consolidated this construction and was using it flexibly.

4.3. Comments on Dreyfus and Tsamir's consolidation model

I deliberately chose not to employ Dreyfus and Tsamir's (2004) constructs in the analysis of Tugay's protocols, to avoid a narrow line of enquiry in a new area of research. Their paper, however, is an important one and it is useful to make some comparative comments.

Dreyfus and Tsamir isolate three distinct modes of thinking in the consolidation process: building-with, reflecting on the building-with and reflecting. A re-examination of Tugay's protocol with regard to these modes reveals that building-with was the dominant mode of thinking throughout the third task. He occasionally reflected on building-with, for example, when he said in 136T, "it was enough to take the symmetry. In fact we made use of analytic geometry". In Tugay's protocols it was not observed, however, what Dreyfus and Tsamir call 'reflection', "an impressive display of general reflection on a wide range of mathematical and psychological issues" (ibid, p.297). It does not seem that 'reflection' is an essential part of consolidating a construction.

Dreyfus and Tsamir claim (ibid, pp.297-298) that consolidation occurs both in using new constructions and while reflecting on them. Tugay's data supports this (if 'reflecting' is taken to mean 'considering' rather than 'reflection' as described immediately above). In the early stages of the third task Tugay reconstructed his new structures and later developed convincing arguments to defend his claims where he both used and reflected on the new constructions. This helped him to establish interconnections between his established mathematical knowledge and the new constructions.

Dreyfus and Tsamir put forward five psychological and/or cognitive constructs associated with the progressive consolidation of a construction: immediacy, self-evidence, confidence, flexibility and awareness. The data broadly support this. The issues of confidence and flexibility have been already discussed. Regarding 'immediacy', there are clear indications in Tugay's protocol that this develops during consolidation. At the beginning of the third task (12T) Tugay was rather slow in describing how to draw the graph of $f(|x|)$ and somewhat hesitant in evaluating the initial two propositions in the fourth question (108T and 110T; 124T and 126T). However, towards the end of the task he was quickly describing (162T and 178T) ways to obtain the graph of $f(|x|)$ and reacted to the third proposition in question 4 (164T) almost immediately after reading it. Regarding self-evidence (i.e. "the obviousness that the use of a structure has for the student" (ibid, p.298)) and awareness (which "enables the student to reflect on related mathematical and instructional issues" (ibid, p.298)), these appear to be so embedded within the process of consolidation that they may not always be evidenced by particular utterances.

4.4. Task design

Task design is a matter of critical importance in mathematics education research and in instruction. A number of comments were already made on the design of tasks for consolidating a construction earlier in Chapter 3 (section 6.4. and 6.5.). The importance of this area merits an overall consideration of task design issues together with a theoretic framework. It seems obvious to me that a task[2] should be designed to reside in student's zone of proximal development (ZPD, Vygotsky, 1978) for, otherwise, the task is either too simple for the student (and we cannot trace the development of their thinking) or the task is beyond their current capabilities (and we simply record them being unable to progress with the task). Given this it was surprising that the initial third task, to construct a structure for $|f(|x|)|$ from the new constructions $f(|x|)$ and $|f(x)|$, was

[2] This arguably applies to all research and instructional tasks but this statement certainly applies to research tasks designed to probe students' consolidation of a construction.

beyond the students' ZPD. On the basis of considerations in the piloting section (see Chapter 3, section 6.3.) I suggest that recognition of a construction is not sufficient for a construction to be built-with in the ZPD but that recognition of an abstraction (a consolidated construction) is necessary. This I believe is a useful general principle in task design.

Consideration of the problems experienced in the pilot study (see Chapter 3, section 6.3.) in the light of Dreyfus an Tsamir's (2004) work convinced me that discussing the new structure in search of mathematical reasons and justifications would be important to consolidation. This led to the design of questions 3 and 4 in the revised third task and the evidence in Tugay's protocol for question 4 (episodes 3 – 6) certainly supports this belief. But the discussion between Tugay and the interviewer was not just any old discussion, it concerned claims that Tugay had to think deeply about in order to evaluate (was in his ZPD) and Tugay's development was scaffolded (but the details of scaffolding are omitted in this chapter). Further to this, as noted above, increasing precision and clarity of Tugay's language to communicate his new constructions were achieved in the course of his consolidation. Therefore, I posit that creating opportunities for discussion grounded in mathematical reasoning, to develop a language for the new construction, is an important feature in designing tasks to consolidate a new construction.

My final word on the task design for consolidating a new construction is that tasks are designed to optimise students' progression from undeveloped structures to fully developed structures. This was a feature of the third task used in this study (see Appendix 7). Question 1 presented a specific function but question 2 withdrew the support this specific function provided. Tugay's protocol in episode 2 certainly suggest that he was not comfortable with this support being withdrawn but it appears to have forced him to attempt to think beyond specific examples. Questions 3 and 4, as noted immediately above, were designed to facilitate discussion with reasoning and the structural interconnections he made in episodes 3 – 6 were significant. Beyond issues of language their resides, in Aylin, Cem and Arzu's claims, a need to focus on linking different structural features (symmetry, positive/negative x-axis sub-domains, absolute values). Question 5 was similar to questions 3 and 4 in that Tugay had to focus on structural features of his construction, this time expressed graphically instead of written statements. As it happened Tugay had consolidated his constructions by the time he reached question 5 and found this question relatively easy, but if he had not done so, then it is likely that it would have assisted him in making interconnections which is an important aspect of consolidation.

4.5. Language, examples and interconnections

The final focus is on a dialectic between the development of a language to describe a new construction, the use of examples and knowledge interconnections. This subsection will be more speculative than the other subsections but I think the issues are worth commenting on.

I share Vygotsky's (1978, pp.22-24) conviction that speech has an organisational role in the development of knowledge. In the initial two sections above (4.1. and 4.2.), some comments were made on the confidence of Tugay's language but the development of a language to describe Tugay's construction can also be discerned. Immediately following the formation of new constructions (recall Tugay's explanations as to how to obtain the graphs of $|f(x)|$ and $f(|x|)$ at the end of the first and second tasks) and early on in the third task (10T and 52T) Tugay's language in his explanations were not fully developed to communicate his new constructions. It is true that we can see a mathematically developed language to describe intersection points, parallel lines and equations of lines but not for the newly acquired constructions. This is especially clear in Tugay's account of $f(|x|)$ at the end of second task where he often switched *to and fro* between general statements on and specific descriptive features of the graphs, e.g., "intersection point ... for x=0"; "we draw the line of y=a"; "the graph is V-shaped"; (see also 52T). Further to this, one can see some redundant expressions and frequent repetitions, which suggest his language and thoughts related to the new construction were somehow disorganised, e.g., "then we draw a line... I mean drawing the line passing through this value... I mean say y=a... we draw the line of y=a" In these statements he was trying to explain and/or describe the symmetry line.

However, when the new constructions were consolidated (see especially from 178T onwards), the language he employed to communicate his thoughts in relation to the new constructions became rather fluent and precise, and was composed of general mathematical expressions which are stated pretty clearly. It is striking to see, for example, how clearly he expressed the two ways of obtaining the graphs of $f(|x|)$ in 178T. The qualitative jump in the development of Tugay's language in the course of consolidation becomes quite evident when he stated that "we can draw a parallel line to the y-axis through the intersection point of $f(x)$ and y-axis. And then for the negative x values we can take the symmetry of that part of graph in this line." Here we have Tugay's account of $f(|x|)$, the clarity of which is obvious. Interestingly, with regard to language development, Hershkowitz et al. (2001) see this as an indication that a construction has become an abstraction: "Generally if the construction is an abstraction, the learners develop in parallel a language for expressing their new knowledge and using it to explain or justify" (ibid., p.212).

With regard to reasoning there is, as noted above, a developmental change in Tugay's use of examples in the third task. Examples appear essential to Tugay's explanations in the second task (recall his explanation at the end of the second task) and in the early part of the protocol, e.g. 52T; they are a denotational reference for tying meaning to the construction. As the construction is consolidated meaning appears to reside in the structural links established and examples, if used at all, take on the role of illustrating these structural links (e.g. 172T). Tugay's development with regard to examples is parallel to his development with regard to language; and they seem to be linked, as Vygotsky (1978) notes with regard to action and speech:

> Initially speech follows actions, is provoked and dominated by activity. At a later stage, however, when speech is moved to the starting point of an activity, a new relation between word and action emerges. Now speech guides, determines, and dominates the course of action (ibid., p.28).

But, as Noss and Hoyles (1996, p.124) note, with regard to 'webs of meaning' and 'situated abstraction', "Mathematics is more than action in the sense of activity-with-objects. It is activity with relationships..." The relationships amongst different aspects of a construction (which in the case of $f(|x|)$ involve, for example, symmetry line(s), a matching relationship between two different values of x with one value of y in the graph and reflecting the positive values of x in the y-axis) are developed by establishing interconnections between known (to Tugay) and the new structures through recognising and building-with actions. I believe that establishing interconnections between new constructions and existing knowledge structures is a necessary aspect of abstraction and that without this it is impossible to achieve the vertical reorganisation of mathematical knowledge which RBC theory posits. It does not matter (indeed, it is expected) that these interconnections are initially not sufficiently well established. Attainment of well-established interconnections is an aspect of the consolidation of a construction.

5. Final remarks: some reflections on consolidation and the ZPD

In this chapter I have focused on the issue of consolidation of a new construction by drawing on a student's verbal protocols. The analysis of this student's work suggested that new constructions are initially fragile, this fragility manifesting itself by virtue of hesitancy in defending formulated constructions and dependence on the specific examples as a basis for formulating ideas. This fragility suggests a need to consolidate the new constructions. In the course of consolidation, it is observed that earlier constructions are reconstructed, used in a flexible manner and expressed with general mathematical statements. Further to this, the student became increasingly resistant to the challenges by establishing interconnections between the new and already acquired knowledge structures.

In this final section, I will make some brief comments on the issue of consolidation in connection with the ZPD by referring to the theoretical framework of this study (see Chapter 2, section 3.3. and 3.4.) and conclude with some further research notes. In the theoretical framework of this study, consolidation is depicted as commencing towards the end of the first stage and continuing through the second stage towards the third stage of the ZPD (see Figure 2.3., p.34). This depiction has two particular aims. The first is to emphasise that the new constructions are not fully developed (i.e. they do not become abstractions immediately following constructions, see Chapter 7, section 3.2.) and are in need of consolidation. This was true at least in the case of Tugay and other students participating in this study. The second is to indicate that in the course of consolidation the learner is relatively more self-regulated in comparison with his/her performance in the first stage of the ZPD during the formation of a new construction. A closer inspection of Tugay's verbal data suggests that even if his new constructions were fragile and inflexible, he was relatively self-regulated while consolidating his new constructions in that he was able to draw the graphs, remembered some aspects of earlier constructions and commented on three fictitious students' 'intelligently incorrect' claims (see Appendix 7, task 3 – question 4).

However, Tugay needed the interviewer's assistance to consolidate his new constructions (see especially episode 4 and 5). The interviewer in fact, by challenging and pointing out critical features (e.g., a matching relationship of two different values of x with one value of y), helped Tugay develop interconnections which were at the heart of Tugay's consolidation. In Vygotskian terms Tugay's constructions were not 'fossilised' (1978, p.68) in the earlier part of his verbal protocols on the third task. Fossilisation is the feature describing the third stage of the ZPD and indicates the 'fixity' and 'distance' of new constructions from the social forces of change (Tharp and Gallimore, 1988, p.38). On the one hand Tugay's initial hesitancy to counter the interviewer's challenges (see episode 4) simply suggests that his constructions were still under the influence of the social forces of change; on the other hand his subsequent resistance to challenges and his confidence in defending his constructions (see episode 6) indicates that he achieved a transition from the second to third stage representing 'fossilised performance' which in the case of this study involves Tugay's consolidated constructions. I am not here concerned with the exact moments of these transitions but the existence of these transitions inform the relationships between RBC theory of abstraction and the ZPD as has already been discussed in the theoretical framework of this study.

My final words in this chapter concern some further research notes. The arguments and observations presented in this chapter were developed on the basis of a limited number of cases and were basically illustrated on the basis of a single student's verbal data. In this respect further theoretical and empirical research is clearly needed to substantiate/verify these arguments. Furthermore, there are some other more practice-linked questions to which I have not attended. For example, what kind of teaching-learning activities create opportunities for consolidation? What strategies can teachers in classrooms with large number of students employ for consolidation? How can the insights gathered in this and other studies into the process of consolidation inform teachers' practices and design of activities? These questions are by no means trivial and require substantial research. My considerations on the issue of task design (see section 4.4. above) could potentially inform these questions; however, further research is needed to provide greater details and operationalise these observations.

CHAPTER 7: CRITICAL REFLECTIONS ON RBC THEORY

In this chapter I will attend to the third and final research question: To what extent the RBC theory of abstraction is valid beyond the cases presented by the original authors? Subsequent to its first publication in 2001, RBC theory has attracted the attention of many (e.g., Bikner-Ahsbahs, 2004; Kidron & Dreyfus, 2004; Schwarz, Dreyfus, Hadas, & Hershkowitz, 2004; Stehlikova, 2003; Tabach, Hershkowitz and Schwarz, 2001; Tsamir & Dreyfus, 2002; Williams, 2002, 2003, 2004; Wood & McNeal, 2003; Ozmantar and Roper, 2004; Monaghan and Ozmantar, 2004). Constructs of RBC theory have been employed in the analyses of these studies which produced some empirical evidence as to the validity of this theory especially in terms of three epistemic actions. However, none of these studies critically evaluated this theory as a whole.

The necessity of critical evaluation of this theory became obvious to me especially after my participation in PME-28, in 2004. I took part in an informal meeting with many of the above-cited authors and some others interested in this theory and the meeting continued for approximately two hours. This meeting was to me intellectually stimulating and I found a chance to hear others' insightful comments, concerns (and sometimes challenges) and two of the original authors' (Tommy Dreyfus and Baruch Schwarz) reactions and difficulties in clarification of some aspects of RBC theory. Some of the particular issues/questions raised and discussed in this meeting involved: To what extent is the social aspect of the theory intrinsic and indispensable to the arguments? Is it a construction or abstraction being consolidated? What clearly differentiates between constructing actions and construction? What is a 'structure' to be constructed and/or abstracted? Does construction and/or abstraction bring about an awareness on the part of a learner/student? How could this theory inform teaching-learning activities in classroom contexts?

None of these questions are trivial or easy to answer. However, I will offer my view on these questions/issues by evaluating and providing critical reflections on RBC theory in the light of the data generated in this study. To do this, I will focus on three key dimensions which characterise RBC theory: its epistemological and sociocultural principles, epistemic actions and genesis of abstractions as proposed by Hershkowitz, Schwarz and Dreyfus (2001; referred to as HSD hereafter). Throughout my evaluation, I will suggest some modifications to the theory and discuss the rationale behind these modifications. I will also note some issues which require further research to deepen and enrich our understanding of mathematical abstraction.

1. Critical reflections on sociocultural and epistemological principles

HSD's RBC theory of abstraction is a dialectical materialist approach which views the genesis of an abstraction as an undeveloped initial entity which develops through the use of mediational means and social interaction. RBC theory's sociocultural principles rely on the works of Vygotsky (1986) and Leont'ev (1981) and epistemological principles draw heavily on Davydov (1990). In what follows I will first focus on sociocultural and then epistemological principles and exemplify my arguments on the basis of H&S's verbal protocols. However it should be noted that when referring to the original authors' work, I will use the term 'abstraction' in describing their arguments as this term is employed in their studies. Nonetheless I will avoid employing this term in describing my data and prefer to use the term 'construction' until I discuss the relationship between abstraction and construction in section 3.2. where I argue that an abstraction is in fact a consolidated construction.

1.1. Sociocultural principles

HSD embrace Leont'ev's (1981) exposition of activity theory in considering abstraction as an activity so as to underline the importance of context which is according to them "a personal and social construct that includes the student's social and personal histories, conceptions, artefacts and social interaction" (ibid. p.202). I personally found this aspect of RBC theory particularly powerful in that it realises the importance of the social and historical forces crucial to knowledge acquisition in its formulation. This theory also takes into account the variation of understandings across settings, which is apparent in the authors' assertion that abstraction is not an objective universal process but rather a subjective process (Dreyfus, Hershkowitz and Schwarz, 2001). The data presented in this study support the subjectivity of the abstraction process. For example, in their work on the second task in episode 5 (see the interaction flowchart for this episode, Figure 4.3., p.93) H&S were acting at different epistemic levels: H at constructing but S at recognising. They were sharing the same social milieu and part of the same activity but following different developmental trajectories. It was in this episode where H constructed the '*y*-symmetry' method. However, S had to wait until the second half of the episode 6 (see from 179S onwards, p.95) to achieve the construction of this method. This clearly exemplifies the variation of individual understandings and development within the same activity.

It is clear in HSD's writings that they are concerned with the social aspect of RBC theory in that abstraction is characterised as a process taking place in physical and social settings. However, in their original paper, HSD are mainly concerned with epistemic actions and construction of a new structure rather than social aspect of their theory. In order to complement and further develop the social aspect of this theory, the same authors published another paper (Dreyfus et al., 2001), which aim to investigate the process of abstraction in peer interaction groups. In this paper they aim to show that RBC theory applies in an environment that is rich in social interactions and propose some patterns of interactions which favour the formation of an abstraction. However, as I read these two papers, I became increasingly confused about (and uncomfortable with) their treatment of 'social interaction' in relation to their theory. On one hand social interaction is seen as a characteristic feature of RBC theory and thus intrinsic to the formation of an abstraction. On the other hand they assert that construction "might often occur when students sit alone and think hard about mathematics" (HSD, 2001, p.212).

The problem here is that if social interaction is a characteristic feature of (and thus intrinsic to the formation of) an abstraction, then what kind of social interaction exists in an individual student's solitary work? Ironically, my confusion about their treatment of the notion of 'social' even increased when I read their second paper which is supposedly elaborating on the social aspect of their theory. The reason for my confusion is related to the fact that they see socially rich environments as creating a forum in which participants' verbalisations may attest to epistemic actions and thus making them observable. Now the question arises: Do HSD see social interaction as a methodological requirement into the investigation of epistemic actions rather than viewing this as an intrinsic component of abstraction process?

In fact the vagueness of HSD's treatment of social interaction in relation to their theory was a matter of debate in the informal meeting we had in PME-28. One of the participants even questioned the necessity of social interaction for the achievement of an abstraction. Another participant asked quite frankly the question: is social interaction emphasised in this theory because it is 'fashionable' in these days to include the notion of 'social' in the new theories (though not exactly in these words)? As a matter of fact, when I first met RBC theory and decided to work on it, I had similar difficulties in appreciating the role of social interaction. It was clear to me that if an activity takes place in overtly social contexts where more than one individual participates in an activity, the influence of the participants and of their interaction surely makes a difference. But in many articles, I came across such statements as "social organisational processes are an inherent characteristic of learning – whether or not it occurs in an overtly social context" (Forman, 1996, p.117). My difficulty was as to how to interpret this sort of statements: where was the social interaction in a 'lone' student's learning by studying, say, a mathematics book? The challenge for me became even greater as I read papers such as that of Bruner's (1985, p.32) who writes that

> aspirant members of a culture learn from their tutors, the vicars of their culture, how to understand the world. That world is symbolic world in the sense that it consists of conceptually organised, rule-bound belief systems about what exists, about how to get to goals, about what is to be valued. There is no way, none, in which a human being could possibly master that world without the aid and assistance of others for, in fact, that world *is* others.

How was I to interpret Bruner here? It was understandable to me that people are born in a world, in a culture which has its own rules, rule-bound belief systems and certain values that members are supposed to learn. But my question was: can a person not learn without a 'tutor'? My answer was then that a person can of course learn, even if with some difficulty, without needing a tutor. Surely there are books, the Internet and other sources that could be used for this purpose; so why was this insistence on the social interaction and the assistance of a tutor? The similar difficulties continued until I read Wertsch's two well-known books: *Mind as Action* (1998) and *Voices of the Mind* (1991); and later the writings of Bakhtin (1981, 1986) and then Wells (1999) and Vygotsky (1986). It is through reading these authors I realised that social interaction is not and cannot be limited to the physically present participants: there is a 'virtual interaction' with the voices of others who are distant in time and space. Therefore, even if an individual is working alone on, say, a book, this person is interacting, at least, with the voice(s) of the author(s) who are acting as tutor(s) and assisting the reader/individual.

My intention here is not to go into the detailed discussion of the issue of 'voices'; I have already done this in Chapter 5 when discussing the interaction of the voices in the formation of a construction. By employing Bakhtinian notion of 'voice', I hope to have clarified that social interaction (virtual or actual) is intrinsic to the formation of a construction and thus to an abstraction (in my sense of the word; see section 3.2. below). My writings in this respect, I believe, provide further clarifications to RBC theory in terms of the nature of 'social', to which I devoted a section in Chapter 5 (section 4).

In relation to the notion of 'social', Wertsch (1998) would argue that all human actions, and epistemic actions are no exception to this, are socially situated by virtue of the 'mediational means' which are employed in the course of activity and which are part of any cultural, historical and institutional setting:

> virtually all human action, be it on the individual or social interactional plane, is socioculturally situated; even when an individual sits in solitude and contemplates something, she is socioculturally situated by virtue of the mediational means she employs (ibid, p.109).

This brings us to the issue of tools/artefacts and mediation. The RBC authors assert that vertical reorganisation of previously constructed mathematics involves the use of artefacts available to a student(s). According to RBC theory, artefacts include material objects, material tools, e.g., computerised ones, and immaterial ones such as language and procedures (Dreyfus et al., 2001). In general the term 'artefact' in this theory is concerned with everything which potentially mediates students' learning and construction of new knowledge structures. Although the nature of artefacts/tools (i.e. material or ideal) is a matter of debate in the relevant literature (see for example, Daniels, 2001; Monaghan, 2003), there appears an agreement in the sociocultural tradition that virtually all human actions, which surely involve epistemic actions, are mediated. In the mathematics education literature, especially that concerned with 'learning' and 'doing' mathematics in technology-rich environments (e.g., use of CAS, Excel, Logo) great deal of attention is paid to the idea of 'tool' mediation (e.g., Noss and Hoyles, 1996; Ruthven, 2002). These studies make a powerful case that learning and doing mathematics changes in dramatic ways with the involvement of tools which themselves become source of cognitive development. One of the most important insights of these studies to me is the observation that tools used in 'doing' mathematics shape and are shaped by the students' actions (Noss and Hoyles, 1996). Especially the French work on 'instrumentation', in the last decade or so, (see Artigue, 2002) has compiled considerable amount of research on this matter (see Monaghan (2003) for a brief overview).

Clearly abstraction and construction of new knowledge is strongly dependent on mediation of tools/artefacts available to the students. However, I did not focus on this aspect of RBC theory in my discussions. I believe further research is required to achieve a more detailed understanding of the mediation of tools in the process of abstraction. I think it will be an exciting research programme to study the tools/artefacts aspects of RBC theory in relation to 'instrumentation'. The 'anthropological approach' that instrumentation adopts is different to RBC theory but has the potential to enrich RBC theory. The issue of 'instrumentation', in particular, may contribute to our understanding of the genesis of knowledge construction with tools/instruments in greater detail. RBC theory on the other hand could provide instrumentation and the anthropological approach with detailed analyses of knowledge construction (or 'cognitive development') in technology rich environments.

Although HSD's theoretical considerations remain limited to tool/artefact mediation, they nevertheless recognise the mediating potential of a human being (i.e. human mediation). In this respect they provide a few instances of the interviewer mediating the student's epistemic actions in their study (HSD, 2001, p.220). However they do not fully elaborate the idea of human mediation, its implications for knowledge construction and the conditions under which such mediation could occur. In my study, on the basis of H&S's verbal data, I attended to the idea of human mediation and considered this in great detail (see Chapter 5 section 3.2.) by focusing on the scaffolder's (discursive) interventions and the students' related utterances. My analyses and observations corroborate HSD's assertion that "constructing is mediated by human interaction" (ibid, p.220). Further to this, my consideration of human mediation complements and adds detail to RBC theory in determining the circumstances for such mediation to come about (recall the four conditions I proposed: intention, reciprocity, meaning and transcendence) and in describing the transformations (in the students' way of seeing, talking and acting) occurring on the part of students in the course of such mediation.

1.2. Epistemological principles

HSD suggest three epistemological principles of abstraction: (1) abstraction requires theoretical thought, in the sense of Davydov; it may also include elements of empirical thought; (2) a process of abstraction leads from initial unrefined abstract entities to a novel structure; (3) the novel structure comes into existence through reorganisation of abstract identities and through establishment of new internal links within the initial entities and external links among them (HSD, 2001, p.202). These principles essentially stem from Davydov's (1990) 'method of ascent' which assumes a dialectical relationship between the abstract and concrete in the course of abstraction. In this section, I will briefly detail the 'method of ascent' and then discuss it in relation to H&S's verbal data. I will also elaborate on some implications of this dialectical view as suggested by the data.

According to Davydov (1990) the genesis of an abstraction starts from a preliminary, immature first form (or, as worded by Davydov, "a simple, undifferentiated, undeveloped form", p.278), which need not be internally nor externally consistent. The development of abstraction progresses from analysis, at the initial stage, to synthesis (p.291). It ends with a consistent and highly structured final form. This process, described by Davydov as the "ascent to the concrete, from undeveloped to the developed" is accomplished "through real interconnections among phenomena" (pp.302-303). These real interconnections can be achieved by means of 'theoretical thought' rather than 'empirical thought'. In Davydov's view, empirical thought is concerned with establishing external links amongst the features of reality. Davydov uses the term 'external' to indicate that these sort of links are required to establish "particular connections and relationships" (p.253) and can be expressed verbally "as the results of sensory observations" (p.255) (e.g., observing similarities and differences between things). However, establishment of real interconnections requires theoretical thought which he describes as "an idealisation of the basic aspect of practical activity involving objects and of the reproduction in that activity of the universal forms of things, their measures, and their laws" (p. 249). For Davydov, theoretical thought is necessary to establish internal links or "essential relationships [which] cannot be observed directly by the senses, since they are not given in available, established, resultative, and disassociated being" (p.255).

The development through abstraction as posited by Davydov and adopted by HSD is relevant to H&S's performance revealed through their work on both task 2 and 4. In order to illustrate this, let us return to the students' work on the second task. When H&S came up with their initial 'reflecting' method (2: episode 1; 60S: "the part of $f(x)$ until the y-axis remains the same, and then the remaining part is taken symmetrically"), they were drawing heavily on the specific features of the graph of $f(|x|)$ at hand (58S and 59H) and were observing the similarities between the graphs of $f(|x|)$ that they had obtained earlier. For example, they focused on specific points from which symmetry starts (e.g., "from the value of y at x=0" – 61H; "the graph takes the value of –2 at x=0" – 62S; "the point of (0, -2) – 64H); they reported certain similarities between the graphs of $f(|x|)$ (e.g., "it [the symmetry] is the same in this graph as well" – 60S).

This 'reflecting' method proposed in 58S and 60S was not applied consistently to draw the graphs of $f(|x|)$ by the students precisely because it was, as Davydov suggests, in its immature first form which had to wait to be developed until episode 4. For example, when H&S applied this method to draw the graph given in question 4-B (see episode 2, p.83), they were successful despite all the vagueness of the method. Yet when they applied it to the graph given in question 4-C, they obtained an erroneous graph (see episode 3, p.84). Even when they applied this method with some success in episode 2, H&S were still concerned with specific features and properties of the graphs; e.g., "the graph of $f(x)$ will be the same until the point at x=0" – 69H. This clearly suggests that in the course of their construction, H&S were concerned with particular instances and employed empirical thought.

Although H&S came up with the initial 'reflecting' method through empirical thought, this was not sufficient for them to achieve the construction of this method which required theoretical thought. This observation could be best illustrated on the basis of H&S's work in episodes 3 and 4. In episode 3, H&S were talking about the accuracy of their erroneous graph of $f(|x|)$ which they sketched by using their initial 'reflecting' method. S asserted that the graph was wrong and tried to convince her partner, H. However, neither of them produced convincing arguments as to the (in)accuracy of the graph which remained unresolved. However, in episode 4 (pp.86-87), with the involvement of the scaffolder, they constructed the 'reflecting' method. The telling difference between the students' arguments in these episodes is related to the mode of reasoning that they employed. In Davydovian terms, H&S can be said to have employed empirical thought in episode 3 but theoretical thought in episode 4. To clarify this, let us take two example explanations from S; one from episode 3 and the other from episode 4. In both of these explanation statements, S was arguing that the segment of $f(x)$ at $x<0$ should be reflected (which S put it as 'changed') when it is transformed into the graph of $f(|x|)$.

> **109S** (episode 3): Yes, I mean shouldn't we take the symmetry of this part [of $f(x)$ at $x<0$]? Look at this graph; the part of $f(x)$ until the y-axis is unchanged and the remaining part is reflected. I mean if I name the rays as I and II ... shouldn't II be reflected in [the line of] $y=2$?

> **136S** (episode 4): Because positive values of x remain unchanged in the absolute value sign, but negative values of x must be multiplied by minus to go out of the absolute value... thus we can say that whatever changes occur in the graph of $f(x)$, it must be at the negative values of x.

S's first statement (109S) was basically concerned with justifying the argument of "taking symmetry" on the basis of her observation of ostensible features of a previously drawn graph ("the part of $f(x)$ until the y-axis is unchanged and the remaining part is reflected"). In other words, she was interconnecting features of 'reality', i.e. suggesting reflection of $f(x)$ at $x<0$ due to the similar features of the earlier graph of $f(|x|)$. However these observations as such did not lead the students to further clarify their 'reflecting' method. In contrast, S's statement in 136 was concerned with highly developed mathematical reasoning. She was reproducing "universal forms of things, their measures, and their laws" (Davydov, 1990, p. 249), e.g., "positive values of x remains unchanged in the absolute value" "negative values of x must be multiplied by minus." In this connection, she was at this point in her development establishing Davydovian "real interconnections" between the structural features of absolute value, the notion of symmetry (she expressed this as "changes"), a linear graph of $f(x)$ and positive/negative x-axis sub-domains. From Davydov's perspective, these interconnections are "internal, essential relationships" which are not immediately available to the "sensory observations" (ibid., p.255). That is not to deny the importance of empirical thought, which clearly was important for H&S's development but rather to emphasise the necessity of theoretical thought required to establish Davydovian real interconnections.

This illustration, in my opinion, clearly corroborates the epistemological principles of RBC theory. Implicit to these principles is the dialectic relationship between the concrete and abstract during the development occurring through the abstraction process. However, HSD do not clearly state their view on the concrete and abstract. Further to this I, personally, feel that this relationship was not made sufficiently explicit in their analyses and discussions. It is my intention in the rest of this section to focus my attention on this dialectical relationship.

As noted in the literature review section (see Chapter 2 section 1.3.2.), my own position on the abstract and concrete, which is I believe compatible with RBC theory, is as follows: the abstract is complex, complexity in the sense of having some "other ideas as parts" (Ohlsson and Lehtinen, (1997, p.42), goes beyond particular instances and is related to theoretical thought in the sense of Davydov. The concrete on the other hand is concerned with particular instances and experiences and is often related to empirical thought in the sense of Davydov. The critical point is that the development occurring through abstraction is not from the concrete to the abstract (as assumed by classical view, see van Oers, 2001 for a critique) but, rather, a dialectical *to and fro* between the concrete and the abstract.

In fact this dialectical relationship was implicitly discussed in my analyses of H&S's work on task 2 presented just above. The 'reflecting' method constructed in episode 4 was an abstract structure. It is clearly more complex than the component knowledge structures: the notion of symmetry, reflecting, features of absolute value and properties of linear functions. This complexity is related to the fact that in their 'reflecting' method, H&S integrated these components into a single structure and needed to employ theoretical thought as discussed above. Further to this, this method was not concerned solely with a particular graph of $f(|x|)$; this is clear from, for example, H's statement that "the part of $f(x)$ on the right hand side (of the y-axis) should remain the same, no matter what. And the part on the left hand side should change" (2: episode 4; 137H). Here in this statement we can see "a grasp and a rational expression, not only of the existence of certain things and their properties, but also of their possibility, as such, with a subsequent determination of the conditions of their manifestation in a certain form" (Davydov, 1990, p.278).

However, this "grasp and rational expression" was achieved through particular instances and (concrete) examples which gave rise to the initial form of 'reflecting' method (58S and 60S) as discussed above. This method was reconsidered when applied to a particular (concrete) graph erroneously in episode 3. This particular error motivated H&S to further clarify their initial 'reflecting' method in episode 4 in which this method was constructed. When the scaffolder asked H&S to explain their 'reflecting' method in episode 7 (p.97), we can see the students returning again to particular instances and properties of the graphs that they had at hand at that moment: e.g. "taking the symmetry *in this line* … when x=0" – 197H. This simply suggests a dialectical *to and fro* between the concrete and abstract in the course of construction. This dialectical relationship is also observed in the course of consolidation which was examined on the basis of Tugay's verbal protocols where it was also clear that Tugay needed to ground his new constructions in concrete examples (see Chapter 6 section 4.1.). Therefore, in the course of abstraction (in my sense) the concrete and abstract are dialectically intertwined.

An important implication of this dialectical relationship is that both the concrete and abstract are linked rather than detached and they both are necessary for individuals to appreciate the other. That is, the abstract and concrete mutually constitute the 'meaning' and both kinds of knowledge act as resources of 'meaning-making' for each other. H&S's work on task 4 provides a nice illustration of this contention. H&S achieved a construction of $|f(|x|)|$ by developing the 'two-step' method through successive application of the structures of $f(|x|)$ and $|f(x)|$ to the graph of $f(x)$. The 'two-step' method constructed to sketch the graphs of $|f(|x|)|$ was an abstract structure as it was not merely concerned with a particular graph of $|f(|x|)|$, was complex and involved theoretical thought. To begin with, the 'two-step' method went beyond a particular graph of $|f(|x|)|$. This is because H&S achieved this construction towards the end of episode 6 where they integrated the structures of $|f(x)|$ and $f(|x|)$ into a single one, $|f(|x|)|$. The level of complexity, in the sense of Ohlsson and Lehtinen (1997), can be seen here clearly: $|f(|x|)|$ is more complex than each of its component structures of $|f(x)|$ and $f(|x|)$ and both of these structures are more complex than their components of reflecting, symmetry, linear function and absolute value. While constructing the structure of $|f(|x|)|$, H&S employed theoretical thought through which they interconnected $|f(x)|$ and $f(|x|)$ by means of the analogy of precedence of arithmetical operations (see 4: episode 4, 5 and 6). In the course of construction of the 'two-step' method, H&S also used concrete examples and particular instances (e.g., "we need to take the absolute value of *this* graph" – 198H; "symmetry of *this part* … symmetry of *that part*" – 205S). Therefore, it is clear that these particular instances helped H&S construct the 'two-step' method which was an abstract structure.

What is interesting here is that following the construction of the 'two-step' method, particular instances became more sensible to H&S. For instance, before the construction of the 'two-step' method, H&S initially sketched the graph of $|f(|x|)|$, asked by question 3, by substitution in episode 2. However, since this graph was V-shaped, H&S at the initial stages of their work on task 4 could not make sense of this and did not know how to interpret this 'unexpected' graph (recall that the first graph H&S drew was W-shaped); and consequently they simply gave up developing a method other than substitution (see 4: episode 2, p.101). Following their construction of the 'two-step' method, H&S once again sketched the graph of $|f(|x|)|$ asked by question 3 but this time by using their new method. Having accurately sketched the graph of $|f(|x|)|$ which was the same graph as the one obtained by substitution, H&S comprehended the reason for the shape of this graph and S commented:

> **225S (episode 7):** Look, it must be so [V-shaped]… because for the third question, look at the equation [$|f(|x|)|=|(|x|+3)|$] even if the absolute value at the outside of the whole expression is removed, we still obtain the same values of y… I mean every value of y is positive for $f(|x|)$ and so the absolute value sign outside the whole expression doesn't

make any difference... so these two graphs [of $f(|x|)$ and $|f(|x|)|$] should be the same anyway...

As can be seen in S's explanation, following the construction of the 'two-step' method to view the graphs of $|f(|x|)|$, this particular instance of $|f(|x|)|$ became understandable to the students and inspired their confidence (e.g., "it must be so") rather than creating confusion. In other words, the students grasped the rationale behind the 'unexpected' shape of this particular graph of $|f(|x|)|$ and were able to explain the reason for this in the light of their new construction of $|f(|x|)|$. These observations also clearly suggest that in the course of abstraction the abstract and concrete are dialectically connected and that each is utilised by the individuals to comprehend the other. In other words, the shape of a particular graph of $|f(|x|)|$ became sensible to H&S when viewed in the light of the insights that they gained from their new abstract 'two-step' method.

2. Critical reflections on the epistemic actions

HSD observe three epistemic actions in the process of abstraction: recognising, building-with and constructing. I believe that these epistemic actions are one of the most important aspects of their theory in that these actions provide an analytical tool for an empirical investigation of the abstraction process. The data in this study largely supports the occurrence of these epistemic actions (see below). However, my analysis of the data suggests some further insights into constructing actions. In what follows I will briefly exemplify these epistemic actions from H&S's verbal data. I also differentiate between the constructing actions and 'construction' on the basis of the data. Finally I consider the nested relationship amongst these actions as posited by HSD.

In the framework of RBC theory, recognising means identifying a mathematical structure inherent in a given mathematical situation. The examples of recognising actions are abundant in H&S's verbal data. H&S's recognising actions varied widely, ranging from a realisation of symbolic notations (e.g., absolute value sign, equations of linear functions) to particular properties and features of their earlier constructions (e.g. negative x-axis sub-domain of the linear graphs, intersection points and features of absolute value) and to analogical relationships (e.g., precedence of arithmetic operations). As HSD rightly state, recognising actions occurred with a purpose that went beyond the act of recognition. Almost all of H&S's recognitions came about when they were explaining, elaborating and/or proposing an idea, answering the questions posed in the tasks, replying (and/or reacting) to one another or to the scaffolder's interventions.

Building-with, in HSD's account, consists of combining existing artefacts in order to satisfy a goal such as solving a problem or justifying a statement. That is, building-with actions are those actions with which the students use their recognitions to solve a problem, explain a situation or reflect on a process by invoking known (to the students) strategies, rules or theorems. In H&S's data, the building-with actions can be easily detected, for example, in their method of substitution to draw the graphs of $f(x)$, $f(|x|)$ and $f(|x|)$; in their strategy to obtain the equation of

$f(x)$ from its graphical presentation by using intersection points (2: episode 1; 45H and 47H); in their explanation of the analogy of computational priority rule (4: episode 4). Building-with actions were undertaken (by H&S) to achieve a goal(s): substitution to draw the graphs; obtaining the equation to be able to use the method of substitution; and analogy of computational priority to propose a possible relationship amongst $|f(x)|$, $f(|x|)$ and $|f(|x|)|$.

Although building-with actions helped H&S to develop new constructions, these actions, as rightly observed by HSD, did not immediately enrich the students with a new structure. For example, when H&S started working on task 2, they obtained the graphs of $f(x)$ and $f(|x|)$ by substitution. They substituted for different values of x into the given equation. Their computations, marking the points on the given grids, uniting them to sketch the graphs are all examples of building-with actions. These graphs sketched through building-with actions did not immediately result in an emergence of the 'reflecting' or 'y-symmetry' method. However, as H&S were developing these methods (see episode 4 – 117S; and episode 5 – 149H) they made use of these graphs. In this sense, building-with actions were necessary and useful but not sufficient to achieve the construction of new methods.

Constructing consists of assembling knowledge artefacts to produce a new structure with which the participants become acquainted. Through constructing actions mathematical elements are combined together, structured, organised and developed into more complex elements. According to HSD thus these actions are necessary to achieve the vertical reorganisation of knowledge artefacts or pieces. As a result of constructing actions, in HSD's view, a new abstract entity/structure, which was initially inaccessible, emerges. The data largely supports this but adds some more details into the nature of this epistemic action.

In my writings in the previous chapters, I was careful in the language I used to attend to a subtle differentiation between the 'construction' and 'constructing actions'. I employed the terms 'construction', 'formation of a construction', 'construction of a new (mathematical) structure' when referring to the emergence of a new structure which requires constructing actions; and the term 'constructing actions' was used as epistemic actions in the sense of HSD. Although HSD do not differentiate between the terms 'constructing actions' and 'construction' (they loosely equate these two), as a result of my analyses of the students' verbal protocols I was convinced of the importance to differentiate between these two.

In H&S's data, constructing actions became explicit when the students were assembling knowledge artefacts into more complex one(s). These constructing actions often occurred as 'little' segments or pieces in the students' utterances. However, these 'little' segments were not discrete or dispersed but rather strongly interrelated to one another. These constructing actions were of critical importance for the emergence or construction of a new structure. But the construction (or emergence) of a new structure was not (and in fact cannot be) limited to those episodes where H&S were observed to undertake constructing actions. The starting and end points of a construction cannot be precisely demarcated with certain boundaries. Further to this, even if the students were not explicitly engaged in constructing actions, changes in the students' understandings and thus construction proceeded in a state of constant flux. This observation is compatible with (and in fact at the heart of) the dialectical view of abstraction. In this connection Davydov (1990, p.253) writes that

> Within the evolving natural whole, all things are constantly changing, passing into other things, vanishing. But each thing, according to dialectics, does not merely change or disappear – it passes into its own other, which, within some broader interaction of things, proceeds as a necessary consequence of the being of the thing that has vanished, retaining everything positive from it (within the limits of all nature this is also a universal connection).

When Davydov's argument is translated into the context of this study, it suggests that construction takes place through constant changes, is in a state of constant flux, and involves increasing clarification and progressive evolution of the initial form of a to-be-constructed structure. In the course of this evolution, some new understandings emerge, some others (e.g. 'preliminary' ones) vanish, though "retaining everything positive" from them. Thus the formation of a new structure is subject to a constant cycle of changes. My argument is that these changes not only occur at the moments when the students overtly undertake constructing actions but also when they undertake building-with and recognising actions even if these changes are not explicitly visible in the students' behaviour.

I will now return to H&S's verbal data to illustrate and clarify my arguments. In this respect, I focus my attention on task 2, though exactly similar observations can be made in H&S's work on task 4. In my writings in the previous chapters I argued that H&S achieved (i.e. completed the formation of) the construction of the 'reflecting' method in episode 4. However, it is not possible to tell *the exact moment* when the formation of this construction was completed. It is true that H's verbalisation in 137 and H&S's application of this method in 139 clearly suggests that H&S formed the construction of the 'reflecting' method *in* this episode. But we cannot be precise about the exact moment when the construction of this method was completed, which, perhaps, was formed before it had been verbalised. The construction of this method was clearly dependent on constructing actions which are observable within episode 4.

A closer inspection of constructing actions manifested through H&S's verbalisations in this episode is consistent with my suggestion that constructing actions occurred as 'little' segments or pieces within the students' utterances: "right side of the y-axis ... values of x are positive" (119S); "so we don't change them" (123S); "the absolute value sign is always outside of x" (127S); "positive values don't change in the absolute value sign" (128H); "in the absolute value sign, negative values change" (131S); "negative values of x must be different" (132H); "there must be difference between the graph of $f(x)$ and $f(|x|)$ at the negative values of x" (134H); "whatever changes occur in the graph of $f(x)$, it must be at the negative value of x" (136). As can be seen, H&S's formation of the 'reflecting' method, as proposed in 137H, was connected with the constructing actions as revealed in the above-quoted utterances through which H&S were assembling the knowledge artefacts (e.g., features of absolute value, positive/negative x-axis sub-domains; the idea of symmetry and reflection which was communicated as 'change') in order to produce a new one (i.e. the 'reflecting' method). However, these 'little' segments/pieces were closely interconnected to one another and the ideas were tied together through these 'little' pieces.

However, it would not be a wise decision to limit the formation of the construction of the 'reflecting' method to episode 4, though we can surely say H&S completed the formation of this method *in* this episode. The genesis of this construction can be traced back to H&S's work in episode 1 in which the initial first form of the 'reflecting' method was proposed in 58S and 60S. In their first proposal, H&S were merely engaging in building-with actions in that they were reflecting on the ostensible features of the graphs (see 58S-65S). These graphs were obtained by substitution which, as discussed above, again indicates building-with actions. Further to this, when H&S applied this initial method erroneously in episode 3, they had a lengthy discussion as to the accuracy of this graph. Their discussions in episode 3 did not go beyond building-with actions. However, we simply cannot overlook the importance and contribution of these building-with actions to H&S's formation of the 'reflecting' method which was completed in episode 4. It is true that we may not be able to see the changes occurring in episodes 1, 2 and 3 as clearly as we see in episode 4. However, the changes in the students' understandings were taking place even when they were not visible in the students' performances or behaviours.

In this respect, H's construction of the 'y-symmetry' method provides a nice illustration to clarify this argument. The genesis of H's construction of the 'y-symmetry' method can be traced back to their discussions of the accuracy of the graph of $f(|x|)$ for the given graph of $f(x)$ in question 4-C (2: episode 3). In episode 3, S argued that the graph was wrong on the basis of her observation that "these two rays [the rays on the left and right side of the y-axis] will be symmetric in the y-axis … all other graphs were symmetric in the y-axis" (2: episode 3, 111S). In fact the graph, although inaccurately sketched, was symmetric in the y-axis (see Table 4.2.–D on p.81). Having realised this, H was not convinced of the inaccuracy of the graph: "they are symmetric in the y-axis anyway" (114H). On the surface, there did not seem a change in H's perception and understanding of the graphs of $f(|x|)$ (i.e. she still believed that the graph was accurate!). However, having developed the 'reflecting' method and redrawn the graph of $f(|x|)$, this time correctly, H recognised that this redrawn graph was symmetric in the y-axis (2: episode 5; 140H and 142H). In episode 4, there was no mention about the y-symmetry aspect of the graphs of $f(|x|)$ at all. However, H's understanding of the graphs of $f(|x|)$ changed since their discussion in episode 3. While H was critical of S's argument of y-symmetry in episode 3, she now, in episode 5, viewed this y-symmetry aspect as an indication of the accuracy of this sort of graphs (see 144H, 145I and 147H). In fact on the basis of this realisation and the scaffolder's mediation, H constructed the 'y-symmetry' method in episode 5. In Davydovian terms, H's understanding and perception of the graphs of $f(|x|)$ were "constantly changing, passing into other things, vanishing" but she was "retaining everything positive" (1990, p.253) from what is vanished (e.g., retaining the relevance of the y-symmetry which was suggested by S and which was initially perceived as unrelated to the accuracy of $f(|x|)$; this perception apparently changed later in episode 5).

This analysis also shows that formation of a construction cannot be limited to the episode(s) where constructing actions were observed. However, we cannot be sure as to the precise moment when a formation of a construction commences either. For example, did the construction of 'y-symmetry' start in 111S where y-symmetry aspect of the graph was first proposed? We cannot be sure; perhaps H had already recognised this aspect when they drew some other graph of $f(|x|)$ by substitution but did not report this.

My final words in this section are concerned with the nested relationship amongst the epistemic actions. The data generated in this study strongly corroborate HSD's argument that these three epistemic actions are dynamically nested: that building-with actions are nested in constructing actions and recognising actions are nested in building-with actions and in constructing actions. This nested relationship was invariably observed in every protocol data generated for this study. I will briefly illustrate this nested relation on the basis of H&S's verbal data. H&S's building-with actions surely involved recognising actions. As an example, I consider the method of substitution on which H&S heavily relied to answer, at least, the first three questions of the tasks. When sketching the intended graphs with this method, H&S substituted different values of x into the given equation and found corresponding values of y. In doing so, they obtained several associated ordered pairs and mapped them onto the Cartesian grid and then combined them to sketch the intended graph. While doing all these, they recognised, for example, algebraic expression of the equations, different signs (e.g. absolute value) and features of the grids (e.g., horizontal line represents the x-axis and vertical the y-axis; and ordered pairs can be mapped onto the grid according to these axes). Constructing actions also involved both building-with and recognising actions. For example when H&S were developing the 'reflecting' method, they forged connections between the graphs of $f(|x|)$ and the relevant structures by recognising and using, for example, the features of absolute value, symmetry and properties of the linear graphs. Finally, the epistemic actions were, as observed by HSD, subjective and sensitive to individual differences as has been seen in H&S's work in episodes 5 and 6 where these students were acting at different epistemic levels. I suffice to note this as I already made a number of comments on this matter in my analysis of H&S's verbal data (see Chapter 4, section 2.2. – episode 5 and 6).

3. Critical reflections on the genesis of abstraction

According to HSD the genesis of an abstraction passes through three stages: (a) a need for a new structure; (b) the construction of a new entity/structure through recognising and building-with actions and (c) the consolidation of the new abstract entity/structure. Regarding stage (a) it appears evident to me that without a need, the students do not attempt to construct a new structure. In this study, this need was created by virtue of the demands of the task, and of the absent and present participants in the activity through interaction. Regarding stage (b) I have already made a number of comments on the construction of new structure throughout this chapter. Stage (c) was the subject of the Chapter 6 and detailed discussions on this matter were provided there. However, in this section, I would like to address two important questions: what is a structure that is being constructed? and is it a new construction or an abstraction that is consolidated?

3.1. What is constructed?

In this section I attend to the question: what is it that the learners/students constructed (or abstracted)? Although HSD are basically concerned with the process aspect of abstraction (occurrence and identification of epistemic actions) their theory accepts the existence of a final product of abstraction – the new structure(s). HSD use the term 'structure' in the sense of Davydov (1990), that the development of an abstraction from an undeveloped initial entity involves establishing an internal structure with links that ultimately results in a differentiated and structured entity. The term 'structure' is further used (see Dreyfus and Tsamir, 2004) as a generic term for abstracted structures, methods, strategies and concepts. However, the things which can be considered as 'structures' can be easily expanded to include, for example, mathematical terms and laws and a formal description within a written mathematical theory such as a definition. So the question of interest here is: what is a structure?

The problem is an old one and not peculiar to RBC theory. For example, researchers concerned with the issue of internalisation kept asking (and are still asking): what gets internalised? Is it speech or "rules of action, in the service of goals, which become activated by symbol systems such as language and diagrams" (Wood and Wood, 1996b) or maybe some other 'things'? Likewise, researchers concerned with the issue of appropriation, which has emerged as an alternative conception to internalisation, asked the same question: what is it that learners appropriate? Having reviewed the literature on the issue of appropriation, Moschkovich (2004, p.54) notes that: "Learners have been described as appropriating a broad spectrum of things ranging from information or skills, to meanings for words, to interpretations of a task, to ways of acting and thinking, or to discourses and social practices." Or if we consider Wertsch's (1998) suggestion that the learners master some skills in using cultural tools (e.g., number systems), then a structure may be equated with some sort of cultural tools.

The term structure is also used in some other theories of abstraction such as that of Noss and Hoyles' (1996) 'situated abstraction' and 'webbing'. In these authors' account, 'situated abstraction' describes "how learners construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed" (ibid., p.122). The idea of 'webbing' conveys "the presence of a structure that learners can draw upon and reconstruct for support" (ibid., 108). A structure in Noss and Hoyles' view vary widely from, for example, an understanding of a mathematical idea (e.g. the idea of tangent, perpendicular lines) to the properties of computational settings (e.g., specific icons, particular experience in using, say, Logo and Cabri, opening the appropriate menu item in a software). In their examples, it is relatively clear what 'structures' are; however, at a theoretical level, this term is used rather loosely and vaguely; and I have similar problems here.

Returning back to H&S's verbal data, the structures that were constructed, I argued, were three methods to sketch/draw the graphs of $f(|x|)$ ('reflecting' and 'y-symmetry' methods) and $|f(|x|)|$ ('two-step' method). However, in the course of H&S's construction of these structures, it can be observed that these students also developed a language to describe their methods. And indeed language development was also an important aspect of Tugay's consolidation. So should the term 'structure' also involve, or indicate, some sort of language development? In this regard Noss and Hoyles (1996, p.124) note that "mathematics requires a language – there must be a preservation of the balance of action and description." But what about the transformations occurring in the students' ways of seeing and acting during construction as discussed in Chapter 5 (section 3.2.)? Are they also part of the structures? Surely these transformations are part of the students' development, evolving through new understandings and different dimensions of the constructed structures.

Apparently answering the question that what is constructed (or abstracted) at a theoretical level or in a general manner is not a simple endeavour. This is precisely because the construction of a structure involves multifaceted development on the part students: in the ways of acting, seeing, talking, thinking, using, recognising and so on. Considering these complexities I do not have much problem with using the term such as 'structure' in a generic way to describe what the students acquire(d) and/or develop(ed). However I believe that it is important for the studies concerned with abstraction (whether in the sense of RBC theory or in relation to any other) to clarify and describe in detail (i.e. to be precise) what structure(s) that the students construct(ed) and do this with a realisation that this development is not a simple but a multifaceted one.

3.2. Consolidating an abstraction or a construction?

When I started working on RBC theory, I was perplexed for a very long time about HSD's differentiation between construction and abstraction. In HSD's view, abstraction is a process leading to an outcome, i.e., a new abstract structure, which they call "abstracted entity" or "abstraction" (HSD, 2001, p.201). In their second paper (Dreyfus et al., 2001), they call the new structure a "construction", thus equating these two. Now the question is: what is the difference between an abstraction and a construction? Are they the same? If they are, why do HSD conceptualise a single process with two names? Can we call a new construction an abstraction? If these two are different, what exactly is this difference(s)? With these questions in mind, reading their papers (and later Dreyfus and Tsamir's (2004) paper in which the issue of consolidation is investigated), I became increasingly uncomfortable (and in fact confused) by their treatment of these terms. For example, in their original paper HSD write that "further stages of the process of abstraction are needed to consolidate ... newly constructed knowledge"

(2001, p.211). So HSD clearly see the new structures ("constructed knowledge" – construction) as part of the abstraction process. But in their second paper (Dreyfus et al., 2001) they talk about consolidation of abstractions and thus appear to view the consolidation of new structures as distinct from the abstraction process. So HSD are inconsistent in their use of the terms construction, consolidation and abstraction (even in their use of the term 'constructing', but I already attended to this and stated my own position in section 2 of this chapter).

In the informal meeting held at PME-28, I brought this inconsistency into the attention of the original authors and the others. The reaction that I got from the original authors was that we were dealing with a new theory whose concepts are still vague and my concern was considered as a pedantic focus on words! But unless these concepts are settled clearly, how could this new theory be used in the analyses of abstraction? It is precisely the realisation of this vagueness that motivated me to suggest clarifications on the basis of the empirical data and of the evaluation of RBC theory.

It appears to me that when the RBC authors refer to the consolidation of an abstraction or structure, they are actually referring to the consolidation of the construction, seen by them as the final stage of the abstraction. But surely the final stage of an abstraction is when the abstraction is sufficiently familiar and established that the structured knowledge can be used (recognised and built-with) in further activities. However, this sufficient familiarity was not observed immediately following the new constructions but was established during the consolidation activity. Hence I claim that what is consolidated is the construction, not the abstraction, and that consolidation is an integral part of abstraction (schematically my view is shown in the theoretical framework of the study; see Figure 2.3. on p.34). Indeed, I have been careful in my use of language all the way through my writings to attend to this subtle but important distinction; but on what basis do I propose this?

I see this distinction as more than a pedantic focus on words but as a matter at the heart of RBC theory of abstraction. RBC theory is based on Davydov's dialectical materialistic theory of thought which essentially proposes a production of theoretical knowledge.

> The theoretical knowledge that arises on the basis of *transformation* of object reflects their internal relationships and connections. In the reproduction of an object in the form of theoretical knowledge, thought *goes beyond* the limits of sensory conceptions (1990, p.300).

And according to Davydov's 'method of ascent to the concrete', on the basis of which RBC theory was proposed, making an abstraction concrete is the ultimate stage which

> requires its conversion into well-developed theory by *deriving* and explaining a system's particular phenomena from its universal base (ibid., p.301).

Theoretical thought in the form of an abstraction transforms how we interpret phenomena, its "internal relationships and connections" or "essential relationships" (ibid., p.255) which are not immediately available to the senses. However, in Tugay's case he could not initially give an account of $f(|x|)$ without sketching some example graphs of $f(x)$ and $f(|x|)$ (see Chapter 6, episode 2, pp.185-186) and without making some observations in terms of similarities and differences between these two. In this sense his initial constructions were not abstractions as they could not go "beyond the limits of sensory conceptions". However, after consolidation we can observe Tugay ascending back to the concrete and evaluating the given graphs in question 5 (of the revised task 3, see Appendix 7) in terms of whether they were the graphs of $|f(x)|$ and/or $f(|x|)$ on the basis of a general consideration of the consolidated constructions (i.e. abstractions) of $|f(x)|$ and $f(|x|)$. In this regard Tugay, for example, explained that

> **210T (episode 7, p.191):** [He is talking about the graph d given in question 5] Well let me summarise, this is not the graph of $|f(x)|$ because ... yes it does not have any negative values [of y] but the symmetry does not start from the x-axis. And it is the graph of $f(|x|)$... obviously symmetry is on the y-axis, the y values for the positive xs do not change, and what is more, x=-2 and x=+2 takes the same value of y. So yes clearly it is the graph of $f(|x|)$.

Here in his statement, Tugay can be said, in Davydovian terms, to be "*deriving* and explaining a system's particular phenomena from its universal base"; the particular phenomena being the given graphs and the universal base being his abstractions of $|f(x)|$ and $f(|x|)$.

Further to this, an unconsolidated construction cannot be used to create new theoretical knowledge (because it has not been consolidated!) and so is not, in the Davydovian sense of the term, an abstraction. This is acknowledged by HSD and by Dreyfus and Tsamir (2004) but they continue to speak of consolidating an abstraction. There is evidence that an unconsolidated construction cannot be used to create new theoretical knowledge in the pilot work reported earlier (see Chapter 3, section 6.3.). In the initial third task, pilot study students did recognise $f(|x|)$ and $|f(x)|$ but they were not able to use them to construct $|f(|x|)|$. In other words, without consolidation, vertical reorganisation did not happen for pilot study students. If new constructions are not used how can the students make vertical reorganisations that RBC theory posits? Unless the new constructions become abstractions (consolidated constructions), previously constructed mathematical knowledge is not reorganised into a new mathematical structure(s). This is evident in Tugay's protocols. At the beginning of the third task the new constructions were fragile; by the end of the third task Tugay was using them flexibly; in the fourth task Tugay used his $f(|x|)$ and $|f(x)|$ structures to construct the graph of $f(|x|)|$ (the fourth task: 114T - 122T; see Chapter 6, section 3, p.191) for a specific $f(x)$ and later (the fourth task: 148T, see pp.191-192) clearly explained how to do this for an arbitrary linear function. Thus Tugay was sufficiently familiar with the new structures such that he could use his abstractions of $|f(x)|$ and $f(|x|)$ to vertically reorganise them into a new structure (i.e. $|f(|x|)|$).

In my consideration of abstraction, this term refers to both a process and a product. As a process, abstraction involves two processes: the process of construction through which a new (abstract) structure emerges and the process of consolidation through which the new construction becomes available to students in a flexible manner and is expressed confidently with general mathematical statements. As a product, an abstraction is, in my view, a consolidated construction which can be recognised and used in further abstractions to construct new knowledge structures.

4. Final remarks and further research notes

In this chapter, on the basis of data generated for this study, I evaluated RBC theory by focusing on three constitutive dimensions of this theory: sociocultural and epistemological principles, epistemic actions and three stages of the genesis of abstraction. Throughout my evaluation, I attempted to clarify some key constructs of this theory which are employed ambiguously by HSD. These include the abstract and concrete, constructing actions, construction, consolidation and abstraction. In this final section, I will suggest some issues which merit further research.

The first issue that requires further research attention is language development. In this study, I observed students developing a language to communicate new constructions. However, immediately following constructions, the language was not fully developed in the sense that students were not able to communicate their new constructions fluently with precise (mathematical) statements. For example, H&S were oscillating between general statements to express the structure of $f(|x|)$ and specific properties of these graphs (see H&S's work on task 2; Chapter 4, section 2.2. – episode 7). Further to this, H&S had considerable difficulties to describe certain features of their new constructions, e.g., "we also take the symmetry of the part I mean left part ... in the line... in the line... oh! It is difficult to say now" (2: episode 7; 200S). In this utterance S was trying to explain the horizontal symmetry line intersecting the y-axis. However, she had difficulties to express this. Language development, however, seems to continue in the course of consolidation stage of an abstraction as was the case in Tugay's work on the third task. But is the development of appropriate language a necessary component of construction? Of consolidation? Of abstraction? In the case of this study, these questions might be answered positively, but even then further research is clearly required to understand the dynamics of the language development in the course of an abstraction.

The second issue is related to awareness. The importance of this issue is also realised by HSD and Dreyfus and Tsamir (2004). Are the students aware that they have constructed something new, something unknown to them before? Or do they become aware of their new constructions in the course of consolidation? If so how does this awareness help them use the new constructions when required? In the case of my study, the students seemed to me to be aware of their new constructions. This is, perhaps, because of the organisation of the tasks whose final questions asked the students to explain the new construction(s). However, the issue of awareness is an important one and requires further research.

Thirdly a theoretical discussion on the issue of 'erroneous' abstraction is necessary. In this study, it was observed that some of the students formed misconstructions. For example one pair working without scaffolded help constructed the structure $f(|x|)$ in relation to the slope of a given function of $f(x)$. They came up with the idea that "if the slope is positive then we take the symmetry of positive part [i.e. $f(x)$ at $x>0$] but if the slope is negative then we take the symmetry of negative part [i.e. $f(x)$ at $x<0$] in the y-axis". Surely this is a misconstruction. Then can (or should) we consider misconstructions, if they are consolidated, as abstractions? In my discussions in this study and other RBC related studies always provide examples of appropriate constructions and abstractions. In fact Davydov also gives an account of abstraction which necessarily progresses in a positive direction. So there does not appear a room for misconstructions in the considerations of abstractions. But it is likely that the students will make misconstructions (and thus erroneous abstractions). The challenge here is whether these misconstructions can be used in further activities to construct new knowledge structures. This is because RBC theory suggests a recursive model of abstraction in that the product of earlier abstraction should be used in further abstractions. If these misconstructions cannot be used in further abstractions, can we still consider them as abstractions? Surely answers to these questions are not straightforward and require empirical and theoretical considerations.

Fourthly, HSD, following Davydov (1990), distinguish between 'everyday' and scientific' concepts and RBC theory is interested in the abstraction of scientific concepts in the domain of mathematics. Scientific concepts are, in Davydov's account, considered as more complex, general and abstract. HSD argue that their model can be used to study abstraction of scientific concepts which are surely not limited to the domain of mathematics. I believe it would be an interesting research programme to apply this theory in other domains such as physics, chemistry and biology. I speculate that similar observations in terms of epistemic actions and the genesis of an abstraction can be made in other domains as well.

The fifth issue is related to the 'concept formation'. On almost every occasion when I talked about my study and RBC theory of abstraction, I was persistently confronted with the same question: what is the difference between abstraction and 'concept formation'? My answer was invariably: what is 'concept formation'? Surely the relationship between abstraction and concept formation lies inescapably in one's understanding and account of 'concept formation'. Do we view a concept, as suggested by traditional logic, as an "abstract universal generality expressed in words" (Davydov, 1990, p.246)? Or do we follow Vygotsky's division between the 'spontaneous' and scientific' concepts? In Vygotsky's account, spontaneous concepts are rich in meaning for children, but they are local and not linked with one another. In contrast 'scientific concepts' are abstract but empty yet they become enriched with meaning in interaction with (and offer generality to) the spontaneous concepts (Saxe, 1991, pp.12-13). Or perhaps we could consider concepts in relation to development of sign systems over the course of a culture's social and historical advancement and concept formation in terms of construction and use of signs to communicate and regulate some psychological functions (Valsiner, 2001). RBC theory, as discussed above, embraces Davydov's division of everyday and scientific concepts; the former develops though empirical thought and the latter through theoretical thought. So, as can be seen, there are many different interpretations of the term 'concept' and apparently there is not a single account of 'concept formation'. However, I believe it would be a valuable study to investigate the issue of abstraction in relation to different accounts of concept formation.

The sixth issue which requires further research concerns 'metacognition'. The idea of metacognition has explicitly been on the agenda of mathematics educators for almost 30 years. Broadly speaking metacognition is concerned with an 'efficient' and 'effective' use of already acquired knowledge and skills. More specifically, it can be described as an awareness of how one learns; awareness of when one does and does not understand; knowledge of how to use available information to achieve a goal; ability to judge the cognitive demands of a particular task; knowledge of what strategies to use for what purposes; and assessment of one's progress both during and after performance (see Flavell, 1979; Schoenfeld, 1985; Gourgey, 1998). Researchers have, by and large, studied metacognition within problem solving activities in which students have already acquired adequate subject matter knowledge but are unable to use them to solve complex problems. In the context of RBC theory, this suggests that metacognition is studied in relation to building-with actions. However, to the best of my knowledge, this issue has not been explicitly investigated in the course of new knowledge construction. In this study, even a superficial inspection of H&S's verbal data could easily verify the importance of metacognitive strategies such as assessing the accuracy and rationality of the results, assessing the understandings, reflections on the earlier solutions (see Goos and Galbraith; 1996). I believe that the epistemic actions that RBC theory suggests constitute a useful analytic tool to carry out elaborate analyses to gain valuable insights into the effect of metacognition on the construction of new knowledge. I believe this is a new area of research which deserves substantial attention.

Finally, further research is needed to utilise the implications of RBC theory within classroom contexts. In this respect, the dialectical view of knowledge construction and the issue of consolidation that RBC theory emphasises are critically important to inform classroom practices. Particularly RBC theory and my observations alike point out the importance of theoretical thought in establishing 'real interconnections' not only in the course of knowledge construction but also in consolidation. Yet the importance of empirical thought should not be underestimated. So the critical point here seems to me to achieve a balance between these two while designing teaching/learning activities, in the course of instructions and devising tasks. However, implementing and utilising these insights requires substantial research effort.

CHAPTER 8: CONCLUSION

This research study was mainly motivated by two aims: to investigate the validity of RBC theory and to investigate the role of scaffolding in the process of abstraction. These two aims gave raise to three particular research questions:

1. How are new mathematical constructions formed through scaffolding?
2. What is the nature of consolidation?
3. To what extent is RBC theory of abstraction proposed by Hershkowitz et al. valid beyond the cases presented by the original authors?

In what follows, by focusing on each of these research questions, I will first briefly sketch out the main findings and observations gathered from the analyses of the verbal data generated by the students who worked on four sequential tasks over four days. Following this I will present the shortcomings of this study as well as the contributions that it makes to the relevant literature.

How are new mathematical constructions formed through scaffolding?

In my answer to this research question, I presented two cases in which a pair of girls worked together with the scaffolder whose assisting interventions were rather crucial to their formation of new constructions. In my discussion of these cases, I initially focused my attention on characterising the scaffolder's assistance which was examined under two broad categories: instructional and pedagogical. Instructional assistance is essentially related to the preparation and organisation of the activity; and pedagogical assistance refers to the scaffolder's assisting interventions coming about spontaneously in the conduct of the activity concerning the immediate interaction amongst the participants (i.e. the students and scaffolder). I exemplified both type of assistance and argued that both type of assistance are complementary to each other.

Following this, I attempted to establish the links between the scaffolder's assisting interventions and the resulting changes occurring on the part of students in their formation of new constructions by asking 'why' and 'how' questions. In response to the former, I suggested five causative relationships: (1) deliberate regulation of the students through subgoals; (2) reducing the students' uncertainty; (3) directing the students attentions and efforts; (4) elicitation of deep explanatory reasoning; and (5) remediation of the students' errors. In response to the 'how' question, I developed the idea that the students' constructions were mediated by the scaffolder's interventions (i.e. the idea of human mediation). My argument was that the effect of the scaffolder's interventions lies in its mediating power which eventually brings about essential transformations in the students' ways of seeing, talking and acting. However, I also argued that in order for such mediation to occur, the scaffolder's interventions should meet four criteria: intentionality, reciprocity, meaning and transcendence.

In the course of my elaboration of human mediation, I argued that cognitive and social developments are closely linked to one another and pointed out the difficulty of drawing a major distinction between social and cognitive processes. In order to further develop this argument, by drawing on the Bakhtinian notions of 'utterance', 'voice' and 'dialogicality', I focused on the nature of ' the social' in connection with the construction of a new mathematical structure. This consideration led me to conclude that new mathematical structures are socially constructed, individually appropriated and historically situated. In this regard, I made it clear that the notion of social not only involves face-to-face verbal interaction of the participants but it also transcends the boundaries of the immediate context of activity and involves interaction with the voices of others who are distant in time and space. I discussed the involvement of others' voices by exemplifying the voices residing in the students' utterances, in the scaffolder's utterances and in the tasks.

These considerations revealed that the verbal interaction in scaffolded discourse entails highly complex negotiations amongst the immediate participants of the activity. This complexity was observed to stem partly from two opposing tendencies which characterise any verbal communication: intersubjectivity and alterity. A state of intersubjectivity is concerned with the extent to which different aspects of an activity is shared amongst the participants and/or held in common. Alterity on the other hand is connected to differences in participants' perspectives, understandings and perceptions. I argued and exemplified that these two tendencies are often co-existent at different degrees and with relative importance.

The issues of intersubjectivity and alterity showed how the participants in verbal communication in scaffolded discourse constantly influence one another. That is, the scaffolder's interventions, which are active in shaping the students' actions, are also actively shaped by the students' actions during their formation of new constructions in relation to the tasks. In order to make this observation explicit, I suggested a model by considering the idea of 'emergent goals'. This model was illustrated with the students' verbal protocols. In this illustration, I demonstrated how emergent goals take form and shift as a result of the participants' negotiation and of new understandings; and constructions are achieved through the fulfilment of a series of emergent goals which are distinguishable from, yet subordinated to, the main goal of the activity. It was also argued that the emergence of subgoals is contingent upon at least four dialectically interrelated parameters: the task, the scaffolder's interventions, students' interpretation of the task and of the scaffolder's interventions and prior emergent goals. Finally I elaborated on the 'handover principle' which suggests a transfer of control and regulation from the scaffolder to students and which was observed to come about following the formation of constructions.

What is the nature of consolidation?

In my answer to this research question, I presented a student's verbal protocols on the third task which was designed to consolidate the constructions of the first and second tasks. This student, working with the scaffolded help, successfully completed and thus achieved the intended constructions of the first and second tasks and also consolidated these constructions during his work on the third task. Further to this, he was able to use his consolidated constructions when required on the fourth task. In my analysis of this student's work, I initially tackled two particular questions: What is the initial state of new constructions? And what changes may occur during the consolidation?

This student, at the initial stages of his work on the third task, was rather dependent on the specific examples as a basis for formulating ideas and was particularly uncertain about the validity of his arguments regarding his new constructions. The mathematical language that he used to express his new constructions was ambiguous and at times incoherent. He was also unable to defend his claims regarding his new constructions when challenged by the scaffolder. All these indicated that the new constructions are initially in a fragile state and in need of consolidation.

In the course of consolidating new constructions, however, it was observed that the student became gradually less dependent on the specific examples which were in fact used to exemplify his understandings and convince the scaffolder rather than being a basis in expressing his thoughts. He also became able to express his constructions in general mathematical terms and with considerable precision. Further to this, he, in the later stages of his work on the third task, became increasingly resistant to challenges by establishing interconnections between the new and already acquired knowledge structures, was able to defend his constructions confidently and to express himself quite flexibly in different ways. In sum, in the course of consolidation, it is observed that earlier constructions are reconstructed, used in a flexible manner, expressed confidently with general mathematical statements. These observations suggested to me that there exists a dialectic between the development of a language to describe a new construction, the use of examples and knowledge interconnections. On the basis of this student's overall performance in relation to the task, I argued that in designing a consolidation task it is important to create opportunities for students to discuss the newly constructed structures in search of mathematical reasons and that tasks should be designed to optimise students' progression from undeveloped structures to fully developed structures.

To what extent is RBC theory of abstraction valid beyond the cases presented by the original authors?

In my attempt to answer this research question, I presented some critical reflections and evaluations on the basis of the data generated for this study by focusing on three constitutive dimensions of RBC theory: sociocultural and epistemological principles, epistemic actions and three stages of the genesis of abstraction. My reflections on the sociocultural principles of this theory pointed out some ambiguities. For example, it was not clear whether this theory views social interaction as a methodological requirement into the investigation of the epistemic actions or as an intrinsic component of the abstraction process. In this respect I argued that social interaction (virtual or actual) is intrinsic to the formation of a construction and thus to an abstraction. Moreover, my analyses and observations corroborated Hershkowitz et al.'s assertion that construction is mediated by human interaction. Further to this, my consideration of human mediation complemented and added details to RBC theory in determining the circumstances for such mediation to come about and in describing the transformations occurring on the part of students in the course of such mediation.

Regarding epistemological principles, this theory heavily drew on Davydov's 'method of ascent' and the data generated in this study largely supported the development, as assumed by Davydov, occurring during the formation of a mathematical abstraction. However, my analysis also suggested that the dialectical relationship between the concrete and abstract, as assumed by Davydov's 'method of ascent', is underdeveloped in RBC theory. In this connection, I attempted to clarify the terms 'concrete' and 'abstract', suggested my view on how this dialectic occurs by referring to the students' verbal protocols and focused on the implications of this dialectical relationship.

Following this, I turned my attention to the epistemic actions (i.e. recognising, building-with and constructing) which, Hershkowitz et al. claim, can be identifiable in any abstraction process. The students' verbal protocols largely supported the occurrence and existence of these epistemic actions. Further to this, as suggested by Hershkowitz et al., there was a clear nested relationship amongst these three epistemic actions. Nevertheless, my analyses suggested further insights especially into constructing actions. Although HSD do not differentiate between the terms 'constructing actions' and 'construction' (they loosely equate these two), as a result of my analyses of the students' verbal protocols I was convinced of the importance of a differentiation between these two. Constructing actions often occurred as 'little' segments or pieces in the students' utterances. However, these 'little' segments were not discrete or dispersed but rather strongly interrelated to one another. These constructing actions were of critical importance for the emergence or construction of a new structure. However, I argued that the construction of a new structure takes place in a state of constant 'flux' and continues to evolve even if the students are engaged in building-with and recognising actions.

In my evaluation of the genesis of abstraction, I addressed two particular questions: what is a 'structure' that is being constructed/abstracted? And is it a new construction or abstraction being consolidated? In relation to the first question, I noted that the term 'structure' is used as a generic term for abstracted methods, strategies, concepts, mathematical terms and laws, and a formal description within a written mathematical theory such as a definition. However, I pointed out the difficulty of providing a fine-grained account of what a structure is and involves at a theoretical level, in a general manner. This was precisely because the abstraction of a structure involves a multifaceted development on the part of students: in the ways of acting, seeing, talking, thinking, using, recognising and so on. I hence noted that it is important for studies concerned with abstraction (whether in the sense of RBC theory or in relation to any other) to clarify and describe in detail (i.e. to be precise) what structure(s) that the students abstracted and doing this with realisation that this development is not a simple but a multifaceted one.

Regarding the second question, I stated that the RBC authors are inconsistent in their use of the terms construction, consolidation and abstraction. In my attempt to clarify the use of these terms, I contended that the final stage of an abstraction is when the abstraction is sufficiently familiar and established that the structured knowledge can be used (recognised and built-with) in further activities. However, this sufficient familiarity was not observed immediately following the new constructions but was established during the consolidation activity. On the basis of this, I put forward the case that what is consolidated is the construction, not the abstraction, and that consolidation is an integral part of abstraction.

Up until now, I briefly sketched out my answers to the three research questions which this study was set out to explore and noted the main arguments. However, there were some shortcomings and biases of this research study that should be noted here. The first is related to the language translation. As mentioned before, the data for this study was collected in Turkey and hence the students' original verbalisations (i.e. verbal data) occurred in Turkish, the verbal data later translated into English. Despite the fact that I paid considerable attention to the faithfulness of the original content, through translation, some of the properties of language are inevitably affected. For example, when the English translation of the dialogues is read, it sounds more formal than it originally is. However, I do *not* think that this sort of differences between the translated version and the original version of the verbal protocols invalidate my arguments and observations.

Secondly, in collecting the verbal data, I preferred audio-recording the students' verbalisations rather than video recording their whole behaviours. Thinking of feasibility, this was surely easier and practical. However, in the course of my analyses, I realised that it would not be a wise decision to separate the students' verbal and non-verbal behaviours and that their non-verbal behaviours could also provide valuable insights into our understandings of affective dimensions of the knowledge construction. In future research therefore I believe it may be more profitable to collect visual data whenever possible.

Thirdly, in this study, my observations and resulting arguments relied on a small number of cases. Considering the budget and time constraints, and the efforts required to carry out a large-scale study, I believe, it was the right decision. However, although this limitation may raise some questions regarding the substantiation of my arguments and observations, I was cautious not to make hasty generalisations throughout my writings. Further to this, the data gathered from the small number of cases allowed me to carry out fairly well detailed analysis of the cases and develop new insights. As Schoenfeld (1985) notes, there is always a trade-off between depth and breadth when one performs a small number of detailed analyses like the ones described in this study. From such analyses one begins to obtain a reasonably clear understanding of the phenomenon under investigation. Nonetheless it is difficult to get a sense of the typicality of the observations in relation to that phenomenon. I felt, however, such detailed analyses of a small number of cases were necessary and appropriate for this study as it aimed to investigate a relatively new area of research which demands a concurrent inspection of the participants' behaviours and relationships together with an elaboration of the process of mathematical construction and consolidation. Therefore more theoretical and empirical research is certainly necessary to substantiate/verify my observations and arguments presented in this study.

Finally, as mentioned before, the scaffolder and researcher was the same person in this study. Although this situation was quite helpful in the course of the analysis of the scaffolder's interventions, it was nonetheless a bias to be noted. This is a bias first because he, being the researcher, was rather enthusiastic and highly motivated to provide 'good scaffolding' with the students in their work. Second because he was, from the very start, aware that his interventions would be scrutinised and this probably influenced his scaffolding practice. This bias thus raises a question as to the typicality of the observed behaviours of the scaffolder and clearly suggests the need of further studies in relation to the observed behaviours.

Despite these shortcomings, I believe that this research study contributed to the relevant research literature in many respects. First of all, by evaluating the RBC theory of abstraction, I hoped to have clarified, further developed, complemented and provided amendments to this theory. I personally believe that putting to the test existing models and theories by trying them out in new research studies, identifying aspects which need more elaboration, identifying the weaknesses and strengths and suggesting amendments are all valuable research activities for the community of research in mathematics education. This is particularly important for theories which are quite influential such as RBC theory for otherwise there is a danger of researchers being unreflective, if not ignorant, consumers of educational theories and models. In this sense, I believe that the evaluation of RBC theory is an important contribution to the field.

Secondly, the issue of knowledge consolidation that this research study attempted to develop is another important contribution. The issue of consolidation, especially within the mathematics education community, has long been neglected. This might be due to the strong emergence of constructivist theories which associate the idea of consolidation with a behaviouristic approach to learning in terms of 'drill-and-practice.' However, it is still common practice, at least in the Turkish context, for many teachers to assign homework to the students, spending much classroom time on practising the newly learnt topics in mathematics. My arguments regarding the issue of consolidation suggest that all this time spent on practising newly learnt mathematics is not in vain as long as these practice activities are prepared carefully and purposefully towards a flexible and confident use of new mathematical knowledge by establishing interconnections between what is already known and is just learnt.

Another important contribution of this study in my opinion lies in its efforts to develop links between the scaffolded help and new learning. With so much written about scaffolded learning within the last 30 years, the research community still spend much of their efforts on discussing what is and is not scaffolding. I do not deny the importance of this issue; however, it is now time to turn our attention to the other more important issues such as how new learning comes about through scaffolding. Is it because the students hear the scaffolder and thus achieve a new learning? Surely it is not that simple. In this study, I attempted to shed some light on this matter by suggesting some causative relationships and by developing the idea of human mediation. In my attempt, RBC theory surely provided me with a valuable analytical tool into developing these ideas. I am aware that these ideas are not definitive answers, yet they provide, I think, a good starting point in developing (and perhaps deepening) our understandings about scaffolding and learning.

Fourthly, my considerations regarding scaffolding and new knowledge construction suggest that there is more to scaffolding than it was initially thought. Considering that the initial arguments of scaffolding were centred on describing the role of an adult tutor in assisting some children towards a task completion, this study suggests that scaffolding is a much more subtle and intricate phenomenon, which involves a complex set of social, cultural, historical, contextual and semiotic dynamics. Throughout my discussions, I argued with examples that scaffolded discourse involves many dynamics such as value judgements, individual's personal histories, common cultural practices, voices of absent others and certain patterns of interaction. Further to this, scaffolding is often seen in relation to intersubjectivity (this is probably because of Vygotsky's insistence on the development of intersubjectivity in the ZPD, see Cheyne and Tarulli, 1999). However, in this study I argued and illustrated that the issue of alterity is also part of any scaffolded discourse and important in developing new understandings; and this is a part of the dynamism involved in scaffolded discourse.

Fifthly, in the theoretical framework of this study, I attempted to show that ideas inspired by the RBC theory of abstraction are compatible with (and in fact can be viewed within) the Vygotskian notion of the ZPD. An important implication of this attempt is that development occurring through mathematical abstractions as envisioned by RBC theory can be projected on to the students' potential development level. In this connection, this study suggested that such development is indeed a multifaceted one which reflects itself through the transformations of the students' ways of acting, seeing, talking, thinking, using, recognising and so on. However, I argued that it does not do any justice to consider such development purely in terms of cognitive growth which is indeed intertwined with social processes.

Yet another contribution of this study lies in the realisation that the development coming about through the formation of abstractions can be best described in terms of a dialectical *to and fro* between the concrete and abstract. This view is fundamentally different from empiricist accounts which dominated Western mathematics education writings in the 20th century. This view also challenges the dominant dualistic perspectives seen as intrinsic to mathematics such as informal/formal, process/object, specific/general and local/global (see Andresen, 2005).

There is a tendency in mathematics education literature which assumes an ascending (hierarchical) development from the former to the latter ones. The dialectical view that this study promotes questions the justification in thinking mathematical ideas (and learning mathematics) in terms of a global hierarchy around these dualities, and particularly the arrangement of such a hierarchy along a concrete/abstract dimension. My observations of a dialectic development between the concrete and the abstract make these hierarchical arrangements quite questionable and I believe that it is a poor conceptualisation to view these dualities in terms of bipolar opposites and to assume a development, slowly but surely, from the former to the latter ones (e.g., from informal to formal mathematics; from processes to objects; from specifics to generals). Nevertheless I am aware that this position requires further justification on the basis of further empirical evidence and theoretical discussions.

Finally, the contribution of this study to the relevant literature also lies in the questions/issues raised which warrant further research attention. I have already noted these issues/questions in the relevant chapters but I here briefly mention some of them and conclude this chapter. How important is 'affect' in the formation of new mathematical abstractions? How could the insights gathered from scaffolding be used in designing and organising classroom activities in which the students could be supported towards the formation of abstractions? What kind of training and practice can be useful towards successful scaffolding in supporting the students' knowledge construction? What kind of teaching-learning activities create opportunities for consolidation? What strategies can teachers in classrooms with large number of students employ for consolidation? How can the insights gathered in this and other studies into the process of consolidation inform teachers' practices and design of activities? Is the development of appropriate language a necessary component of construction? Of consolidation? Of abstraction? What dynamics are involved in developing mathematical language to express new constructions? Are the students aware that they constructed something new, something unknown to them before? Or do they become aware of their new constructions in the course of consolidation? If so how does this awareness help them use the new constructions when required? Does RBC theory hold in the domains other than mathematics such as physics, chemistry and biology?

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Appendix 1: Think-aloud exercises and warm-up questions

The students working under think-aloud instructions were given the below set of four questions as warming up exercises. They were explained how to do 'thinking-aloud' while especially working on question 4 of below. These questions were also given to all other students before they started working on the actual tasks. The pairs working with and without scaffolded help were given the instructions as to how to work together. The pairs while working on the question 4 of below were explained how to collaborate together.

Four warm-up questions:

1. What is the absolute value of a number? Define and explain the concept of 'absolute value'.
2. Which real numbers are changed by the absolute value function?
3. Find the value of $\left|-\frac{3}{5}\right|$, $|-5|$, $\left|-\sqrt{3}\right|$, $|2|$.
4. The value of $|x| + x$ equals to 0. Explain for which x values this equation is always true.

Appendix 2: 'Hints' given to the individual students doing thinking-aloud

These hints were given to the individual students working under thinking-aloud instructions to prevent frustration when they got stuck and were unable to proceed at the beginning of their work on task 1 and 2.

The hints for the task 1:

i. Make an x, y chart for the function of $f(x)=x+2$.

ii. Make another chart for the function of $|f(x)|= |x+2|$ for the same x values.

iii. How does the y values of $f(x)$ compare to the y values of $|f(x)|$?

The hints for the task 2:

i. Make an x, y chart for the function of $f(x)=x-4$.

ii. Make another chart for the function of $f(|x|)=|x|-4$ for the same x values.

iii. How does (x, y) for $f(x)$ compare to $(-x, y)$ for $f(|x|)$.

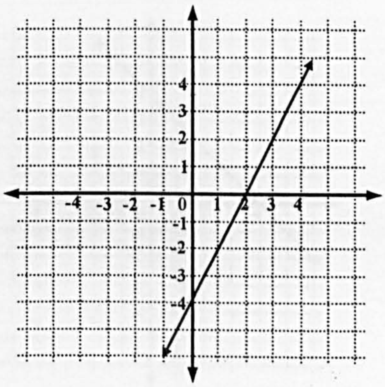
Appendix 3: The initial four tasks

In this appendix, I present the four sequential tasks in their initial form as used for the pilot study. The organisation of the first three tasks was identical apart from the mathematical focus i.e. $|f(x)|, f(|x|)$ and $|f(|x|)|$. The mathematical focus of task 1, 2 and 3 was for students to draw/sketch the graphs of the linear absolute value functions $|f(x)|, f(|x|)$ and $|f(|x|)|$ by making use of the graph of $f(x)$. In task 1 and 2 it was expected that students would draw/sketch the graphs of $|f(x)|$ and $f(|x|)$ from their knowledge of the graph of $f(x)$ and of the absolute values of numbers. It was then expected that students would use this new knowledge to construct the graph of $|f(|x|)|$ in the task 3. Question 4 of task 1, 2 and 3 were designed to provide an opportunity for consolidation. Two other opportunities for consolidation were envisaged: task 3 (on $|f(|x|)|$) was intended to build upon the structures constructed in task 1 and 2 (i.e. $|f(x)|$ and $f(|x|)$) and thus consolidate these structures; task 4, which had six questions, designed to consolidate all of the constructions in tasks 1, 2 and 3.

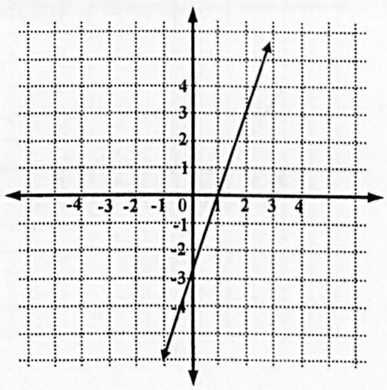
These four tasks have been revised and redesigned subsequent to pilot study which located task design faults and the inefficiency of the consolidation opportunities and particularly of task 4 which was re-developed. The revised form of the tasks which were used in the main study is presented in Appendix 7.

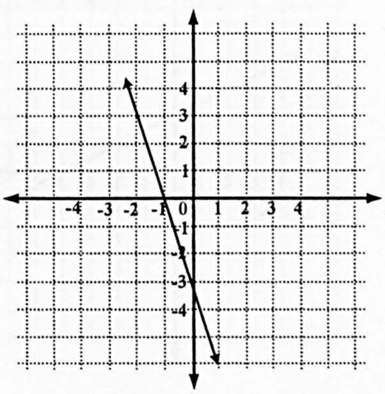
Task 1

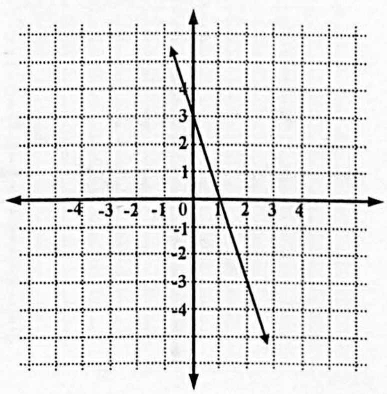
1. A function of f is defined on the set of real numbers as $f(x)=x+2$. Draw the graph of $|f(x)|=|x+2|$ and comment on any patterns or symmetries.
2. Do you see any relationship between the graph of $|f(x)|=|x+2|$ and the graph of $f(x)=x+2$? Explain your answer.
3. The graph of $f(x)=2x–4$ is given below. Can you obtain the graph of $|f(x)|=|2x-4|$ from the graph of $f(x)$? Explain your reasoning.

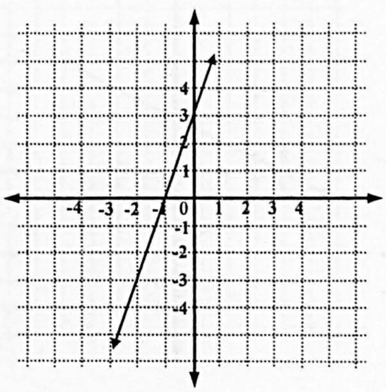


4. There are four different graphs of $f(x)$ given below. Find the graphs of $|f(x)|$ by making use of the graphs of $f(x)$.





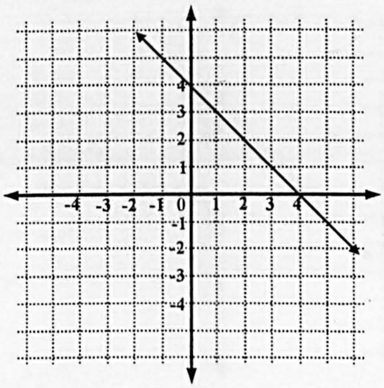




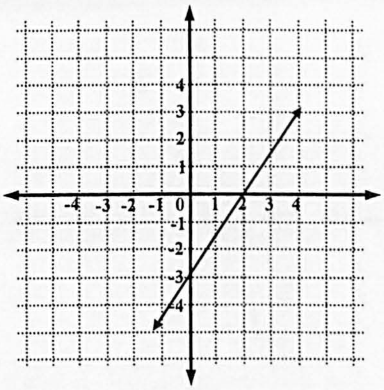
5. How would you explain one of your friends how to draw the graph of $|f(x)|$ by using the graph of $f(x)$? Demonstrate that your explanation correct by using the above-given graphs.

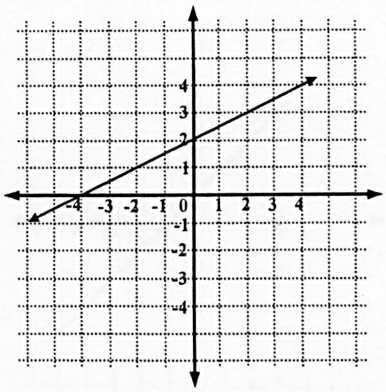
Task 2

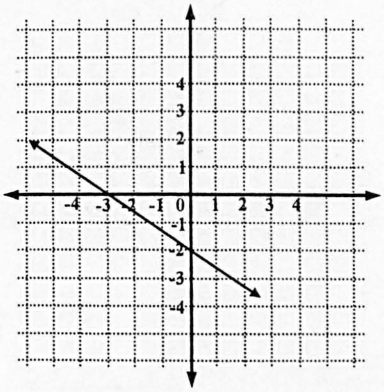
1. A function of f is defined on the set of real numbers as $f(x)=x-4$. Draw the graph of $f(|x|)=|x|-4$ and comment on any patterns or symmetries.
2. Do you see any relationship between the graph of $f(x)=x-4$ and the graph of $f(|x|)=|x|-4$? Explain your answer.
3. The graph of $f(x)= -x+4$ is given below. Can you obtain the graph of $f(|x|)= -|x|+4$ from the graph of $f(x)$? Explain your reasoning.

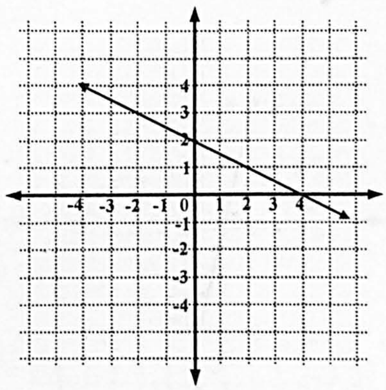


4. There are four different graphs of $f(x)$ given below. Find the graphs of $f(|x|)$ by making use of the graphs of $f(x)$.





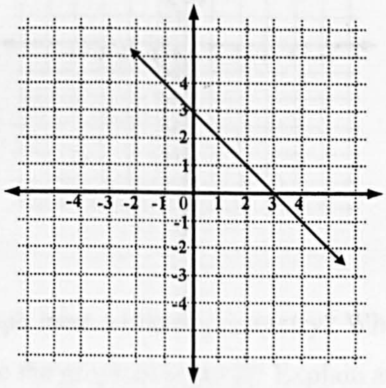




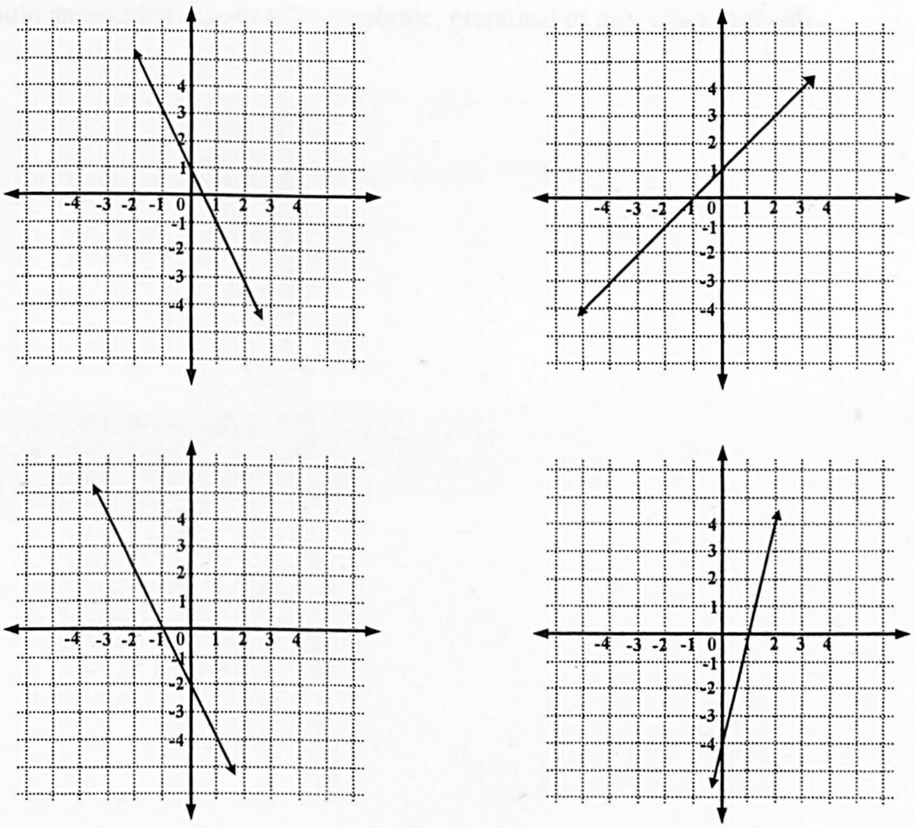
5. How would you explain one of your friends how to draw the graph of $f(|x|)$ by using the graph of $f(x)$? Demonstrate that your explanation correct by using the above-given graphs.

Task 3

1. A function of f is defined on the set of real numbers as $f(x)=2x-6$. Draw the graph of $|f(|x|)|=|(2|x|-6)|$ and comment on any patterns or symmetries.
2. Do you see any relationship between the graph of $f(x)=2x-6$ and the graph of $|f(|x|)|=|(2|x|-6)|$? Explain your answer.
3. The graph of $f(x)= -x+3$ is given below. Can you obtain the graph of $|f(|x|)|=|(-|x|+3)|$ from the graph of $f(x)$? Explain your reasoning.



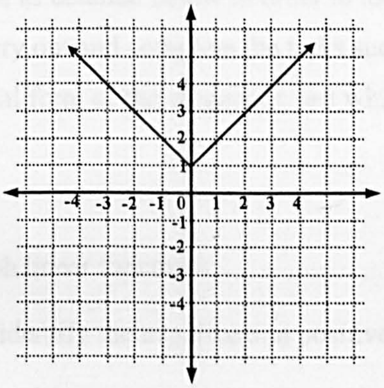
4. There are four different graphs of $f(x)$ given below. Find the graphs of $|f(|x|)|$ by making use of the graphs of $f(x)$.



5. How would you explain one of your friends how to draw the graph of $|f(|x|)|$ by using the graph of $f(x)$? Demonstrate that your explanation correct by using the above-given graphs.

Task 4

1. What is the equation of the line that contains the line segment located in the first quadrant drawn below?



2 Does the graph have a line of symmetry? Where?

3 Could this be the graph of $y=|x+2|$? Explain and justify your answer.

4 Could this be the graph of $y=|x|+2$? Explain and justify your answer

5 Could this be the graph of $y=|(|x|+2)|$? Explain and justify your answer

6 Suppose $y=mx+b$. What values of b make the graphs of $y=|mx+b|$ and $y=|m|x|+b|$ identical? You could answer the question by algebraic, graphical or any other methods.

Appendix 4: Decomposition of the topic of absolute value linear functions

The topic of absolute value linear function was decomposed into three basic topics and some further subtopics as detailed below in order to identify the prerequisite knowledge structures necessary to carry out and complete the tasks successfully. This decomposition was used to prepare the initial form of the diagnostic test which is presented in Appendix 5.

i. To graph linear functions.

- To identify the negative and positive x and y values on the Cartesian grid.
- To make an x, y chart for a linear function.
- To read the graph of a linear function.
- To obtain a linear function from two given points.

ii. To reflect linear functions in a line of symmetry.

- To find the reflection of a line segment in the axes.
- To find the reflection of a line segment in a given line of symmetry.
- To find the co-ordinates of the reflected points.
- To identify the line of symmetry.

iii. To evaluate the features of absolute value.

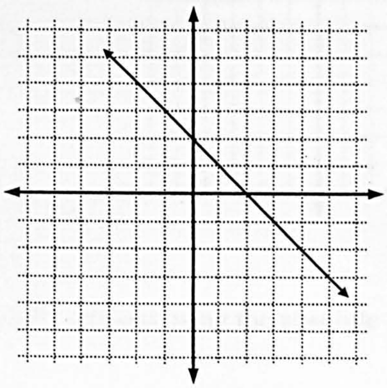
- The effect of absolute value on any real number.

Appendix 5: The initial form of the diagnostic test

In order to select the appropriate students who have already acquired the prerequisite knowledge structures, a diagnostic test was prepared. In order for a student to be selected for this study, one had to answer all of the first eight questions correctly but none of the last two questions correctly. In this appendix I present the initial form of the diagnostic test which was revised following the pilot study. The revised form of the diagnostic test, which was used to select the students for the actual study, is presented in Appendix 8.

1. The graph of $f(x)$ is given below. According to this graph complete the table below.

x	0		-4	
y	2	0		5

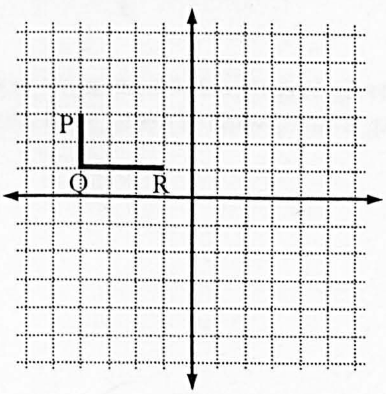


2. Draw the graph of $f(x)= 2x–4$ on the grid given below and show all your working. (*NB: The grid is not shown here*).

3. Answer the following three questions on the grid given below.
   i. Plot the points (-4, 1) and (1, -4) and draw the straight line which passes through these points.
   ii. Write down the co-ordinates of the points where your line crosses the two axes.
   iii. Write down the equation that connects x and y.

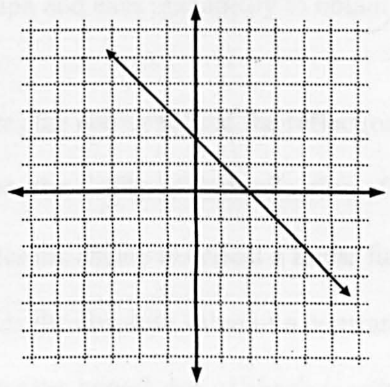
(*NB: The grid is not shown here*).

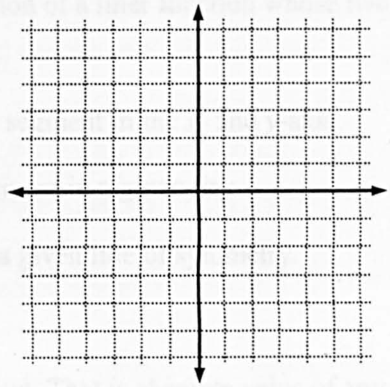
4. There is a shape given on the Cartesian grid below.
   i. Draw the reflection of the PQR in the x-axis and write down the co-ordinates of the reflection in the x-axis.
   ii. Let P\`Q\`R\` be reflection of the PQR in the x-axis. Draw the reflection of P\`Q\`R\` in the y-axis and find the co-ordinates of the reflection in the y-axis.



5. On the Cartesian grid given below, (*NB: The grid is not shown here.*)
   i. Plot the points A (-4, -2) and B (-2, -4) and draw the line segment between these two points.
   ii. Draw the reflection of **AB** in the line of y=1 and find the reflected co-ordinates of **A`** and **B`**.

6. The graph of $f(x)= -x+2$ is given below. Draw the reflection of $f(x)$ in the x-axis on the given Cartesian grid below.

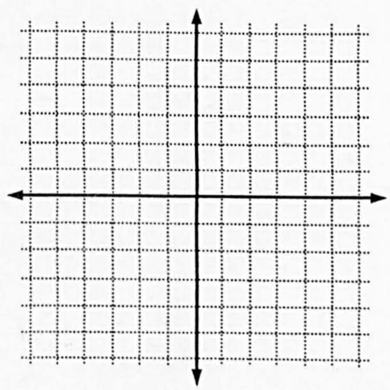




7. Rewrite the expression of |2 + |3 |(-2) + |4-7| | without using the absolute value sign.

8. Find the set of solution for the inequality of $7+|x|< 6$. Explain your answer.

9. Draw the graph of $f(x)=|3x-6|$ on the Cartesian grid below . Show all of your working.



10. Draw the graph of $f(x)=|2x|+4$ on the Cartesian grid given below. Show all of your working. (*NB: The grid is not shown here; it is the same grid as given above*).

Appendix 6: Reasons behind the each question in the diagnostic test

Q1 refers to Question 1 in the diagnostic test presented in Appendix 5.

Q1: Evaluates the ability to read a graph of linear function.

Q2: Evaluates the ability to sketch the graph for a given equation.

Q3: Evaluates the knowledge about the Cartesian co-ordinates, ability to find the intersection points of graph and axes and ability to obtain the equation of a liner function whose two points are given.

Q4: Evaluates the ability to find the reflection of a line segment in the x- and y-axis.

Q5: Evaluates the ability to find reflection of a line segment in a given line.

Q6: Evaluates the ability to reflect a linear function in a given line of symmetry.

Q7: Evaluates the absolute value of a constant.

Q8: Evaluates the knowledge of absolute value definition. That is absolute value of any constant cannot be less than 0 in other words it is always greater than or equals to zero.

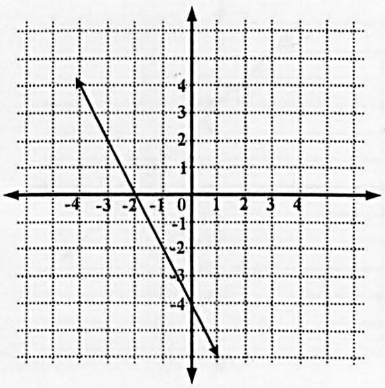
Q9 and **Q10** are straightforward evaluations of the intended constructions of the first and second tasks. These two questions are intended to evaluate whether the students are already acquainted with these constructions.

Appendix 7: Revised version of the tasks as used in the main study data collection

The initial four tasks, presented in Appendix 3, were redesigned subsequent to the pilot study which gave rise to the revised version of the tasks as presented in this appendix. In this revised version of the tasks, the initial task 4, which was prepared to consolidate the constructions of task 1, 2 and 3, was dismissed since it did not work as intended. The initial task 1 and 2 remained the same, though some slight changes took place in the wordings of the questions. An intermediate task 3 was prepared to consolidate the constructions of task 1 and 2. The initial task 3 was given to the students as task 4 in this revised version. So in the actual data collection, all of the students worked first on the graph of $|f(x)|$; second on $f(|x|)$; third on a task for the consolidation of these two; and fourth on the graph of $|f(|x|)|$. The revised four tasks were given to the actual study students in the same order as presented in this appendix.

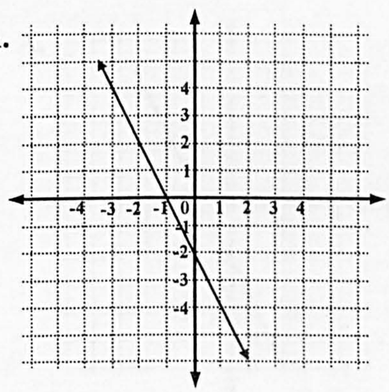
Task 1

1. A function of f is defined on the set of real numbers as $f(x)=x+2$. Draw the graph of $|f(x)|=|x+2|$ and comment on any patterns or symmetries.
2. Do you see any relationship between the graph of $|f(x)|=|x+2|$ and the graph of $f(x)=x+2$? Explain your answer.
3. The graph of $f(x)= -2x-4$ is given below. Can you obtain the graph of $|f(x)|=|-2x-4|$ from the graph of $f(x)$? Explain your answer.

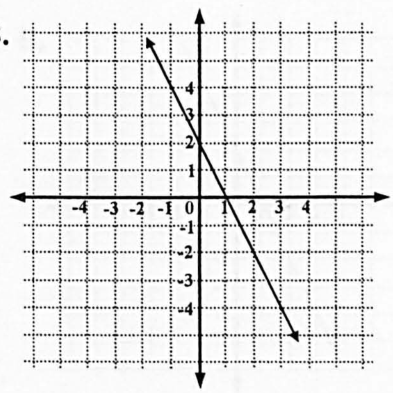


4. There are four different graphs of $f(x)$ given below. Find the graphs of $|f(x)|$ by making use of the graphs of $f(x)$.

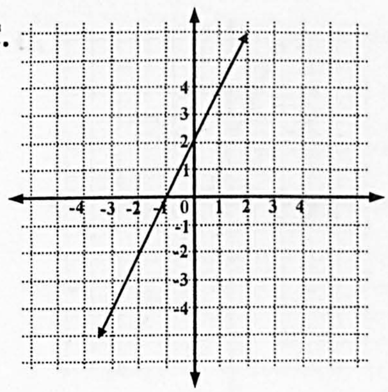
A.



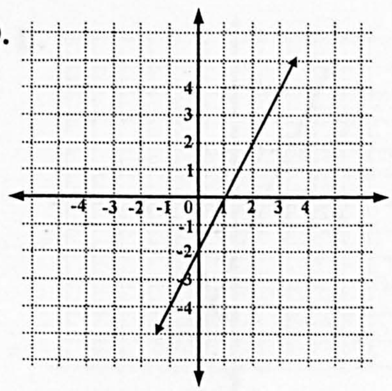
B.



C.



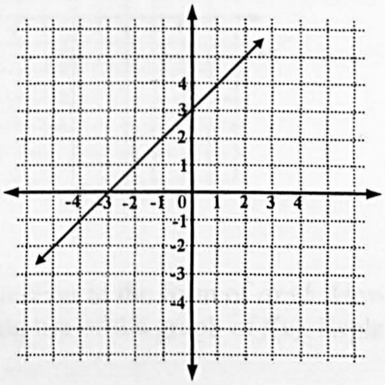
D.



5. How would you explain one of your friends how to draw the graph of $|f(x)|$ by using the graph of $f(x)$? Demonstrate that your explanation correct by using the above-given graphs.

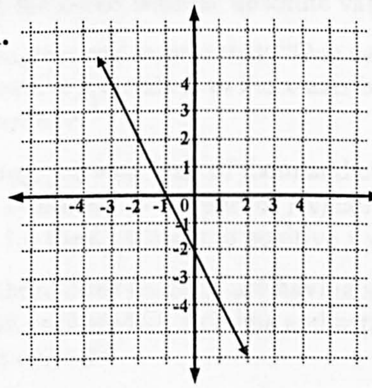
Task 2

1. A function of f is defined on the set of real numbers as $f(x)=x-4$. Draw the graph of $f(|x|)=|x|-4$ and comment on any patterns or symmetries.
2. Do you see any relationship between the graph of $f(x)=x-4$ and the graph of $f(|x|)=|x|-4$? Explain your answer.
3. The graph of $f(x)=x+3$ is given below. Can you obtain the graph of $f(|x|)=|x|+3$ from the graph of $f(x)$? Explain your answer.

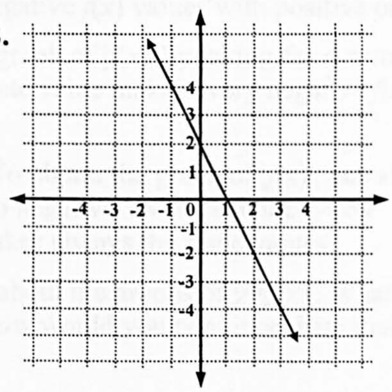


4. There are four different graphs of $f(x)$ given below. Find the graphs of $f(|x|)$ by making use of the graphs of $f(x)$.

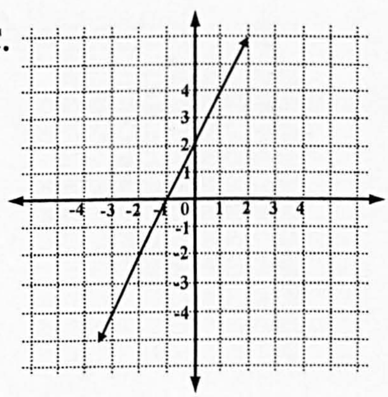
A.



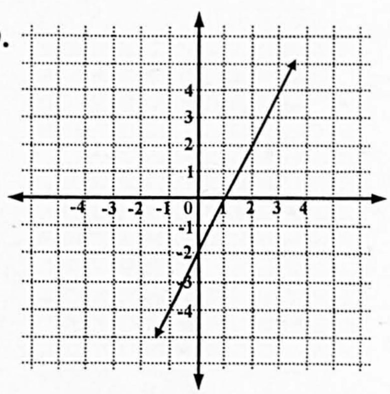
B.



C.



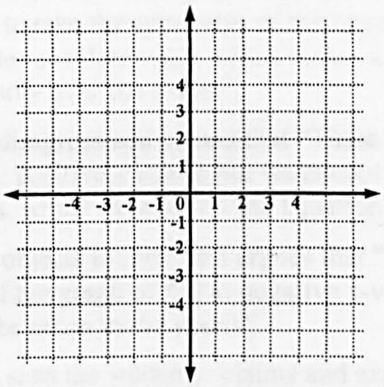
D.

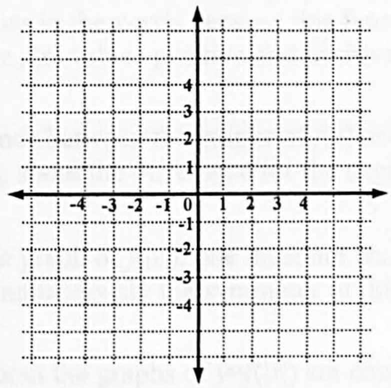


5. How would you explain one of your friends how to draw the graph of $f(|x|)$ by using the graph of $f(x)$? Demonstrate that your explanation correct by using the above-given graphs.

Task 3

1. A linear function $f(x)$ is defined as $f(x)=2x-2$. Draw the graphs of $|f(x)|$ and $f(|x|)$ on the grids given below.





2. $f(x)$ is a linear function in the form of $ax+b$. How would you draw the graph of $y=|f(x)|$ and $y=f(|x|)$ by making use of the graph of $f(x)$. Explain your answer.

3. Three friends are talking about how the graph of $y=|f(x)|$ can be obtained, given the graph of $f(x)$ which is a linear function.

   Aylin says that: "To obtain the graph of $|f(x)|$, one needs to take the symmetry of the graph of $f(x)$ in the x-axis because absolute value replace negative $f(x)$ values with positive ones."

   Cem disagrees and argues that: "One can obtain the graph of $|f(x)|$ by taking the symmetry of the positive $f(x)$ values in the x-axis because absolute value makes every negative $f(x)$ values positive."

   Arzu disagrees with both of them and claims that: "To obtain the graph of $|f(x)|$, one should take the symmetry of the part of $f(x)$ corresponding to negative x-values in the x-axis because for the negative and positive x values $|f(x)|$ takes always the same values.

Apparently these three students are having some trouble about the graphs of $y=|f(x)|$. What can you say about each student's claims and explanations? How would you convince them that your suggestion is correct?

4. The same students are now talking about how they can obtain the graph of $y=f(|x|)$ by using the graph of $f(x)$ which is a linear function.

   Aylin says that "The graph of $f(|x|)$ must be over the x-axis. To obtain the graph of $f(|x|)$, one needs to take the symmetry of the negative $f(x)$ values in the x-axis because this function includes absolute value which makes every negative $f(x)$ values positive and positive values exist only over the x-axis".

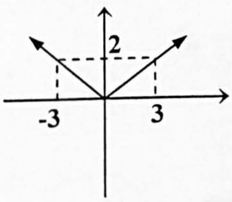
   Cem disagrees and claims that "There is no difference between the graphs of $f(x)$ and $f(|x|)$ for the positive x values but we cannot say anything about the difference for the negative x values, which depends on the equation of $f(x)$".

   Arzu objects to them and argues that "To obtain the graph of $f(|x|)$, one must not change the part of the graph of $f(x)$ at negative x-values and simultaneously the symmetry of this part must be taken in the y-axis".

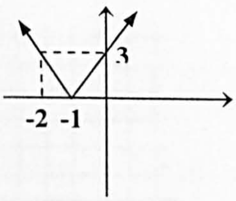
As can be seen the students' claims and explanations about the graphs of $y=f(|x|)$ are once again in conflict with each other. What can you say about each student's claims and explanations? How would you convince them that your suggestion is correct?

5. Which of the given graphs could be considered as a graph of $y=|f(x)|$ and/or $y=f(|x|)$. Explain your reasoning.

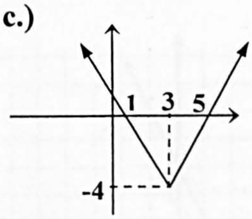
a.)



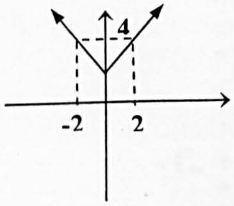
b.)



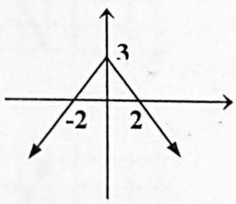
c.)



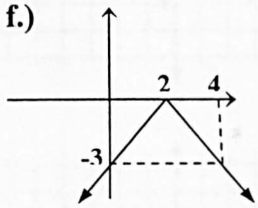
d.)



e.)

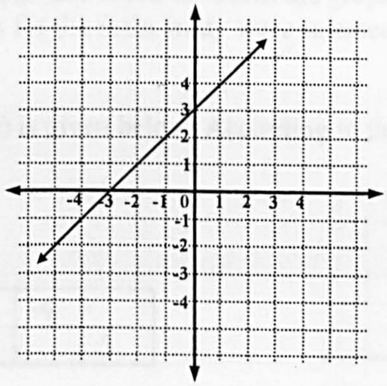


f.)



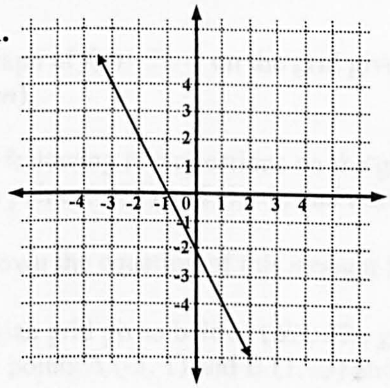
Task 4

1. A function of f is defined on the set of real numbers as $f(x)=x-4$. Draw the graph of $|f(|x|)|=|(|x|-4)|$ and comment on any patterns or symmetries.
2. Do you see any relationship between the graph of $f(x)=x-4$ and the graph of $|f(|x|)|=|(|x|-4)|$? Explain your answer.
3. The graph of $f(x)=x+3$ is given below. Can you obtain the graph of $|f(|x|)|=|(|x|+3)|$ from the graph of $f(x)$? Explain your answer.

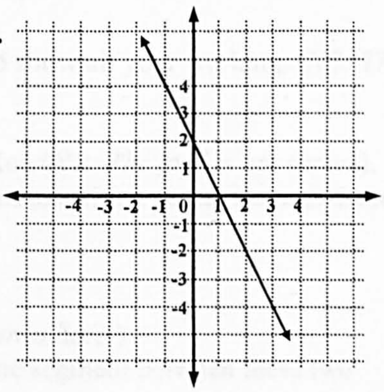


4. There are four different graphs of $f(x)$ given below. Find the graphs of $|f(|x|)|$ by making use of the graphs of $f(x)$.

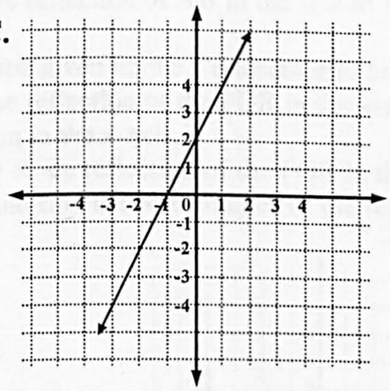
A.



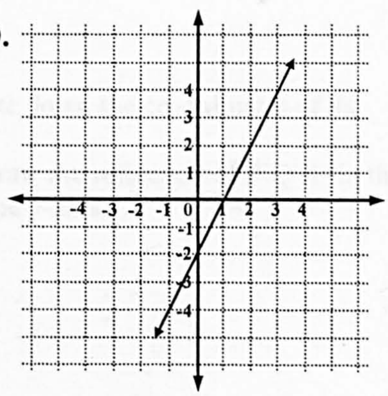
B.



C.



D.



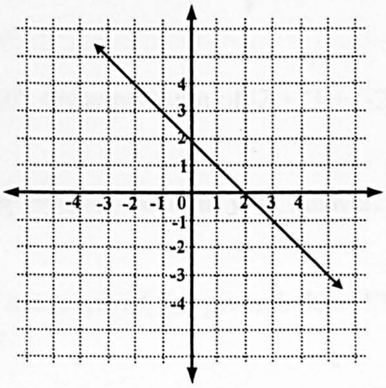
5. How would you explain one of your friends how to draw the graph of $|f(|x|)|$ by using the graph of $f(x)$? Demonstrate that your explanation correct by using the above-given graphs.

Appendix 8: Revised version of the diagnostic test used for the selection of sample for the main study data collection

The initial diagnostic test, presented in Appendix 5, was subject to some changes following the pilot study. The main change took place within the last two questions, 9 and 10, of the initial test. These two questions were used to assess whether the students have already formed the intended abstractions or not. In the initial test, the students were presented two equations of, respectively, $|f(x)|$ and $f(|x|)$ and expected to obtain their graphs. However, in this revised form, presented in this appendix, instead of giving two equations of $|f(x)|$ and $f(|x|)$, the students were given two linear graphs and asked to sketch the graphs of $|f(x)|$ and $f(|x|)$ by using these linear graphs. The students for the main study were selected by means of this test presented here.

1. The graph of $f(x)$ is given below. According to this graph complete the table below.

x	0		-4	
y	2	0		5

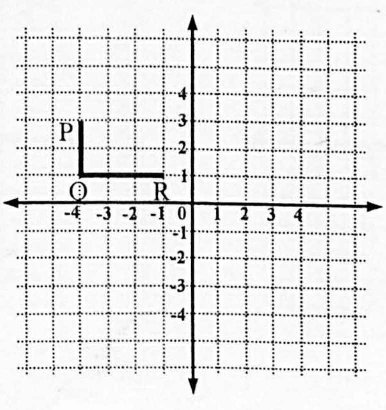


2. Draw the graph of $f(x)= 2x–4$ on the grid given below and show all your working. (*NB: The grid is not shown*).

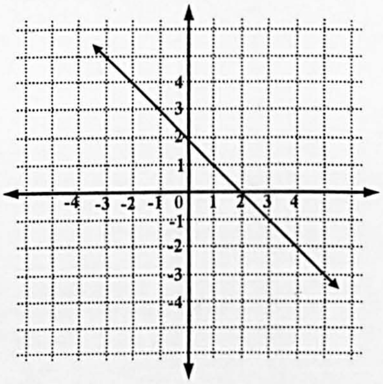
3. Answer the following two questions on the grid given below(*NB: The grid is not shown*).
   i. Plot the points (-4, 1) and (1, -4) and draw the straight line which passes through these points.
   ii. Write down the equation of this straight line.

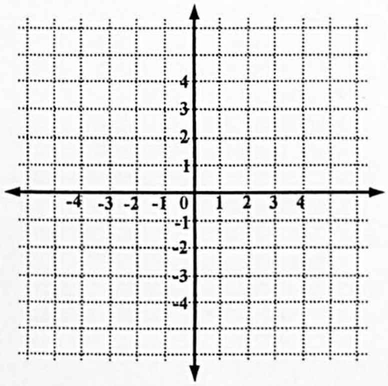
4. On the Cartesian grid given below, (*NB: The grid is not shown here.*)
   i. Plot the points A (-3, 1) and B (1, -3) and draw the line segment between these two points.
   ii. Draw the reflection of **AB** in the line of y=1.

5. There is a shape given on the Cartesian grid below.
   i. Draw the reflection of the PQR in the x-axis and write down the co-ordinates of the reflection in the x-axis.
   ii. Let P\`Q\`R\` be reflection of the PQR in the x-axis. Draw the reflection of P\`Q\`R\` in the y-axis and find the co-ordinates of the reflection in the y-axis.



6. The graph of $f(x) = -x+2$ is given below. Draw the reflection of $f(x)$ in the x-axis on the given Cartesian grid below.

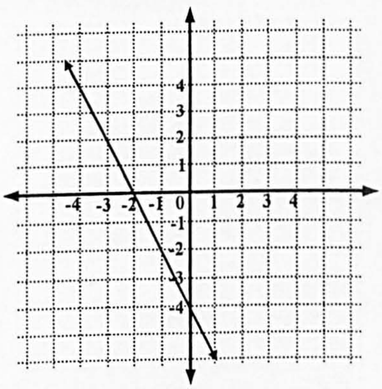


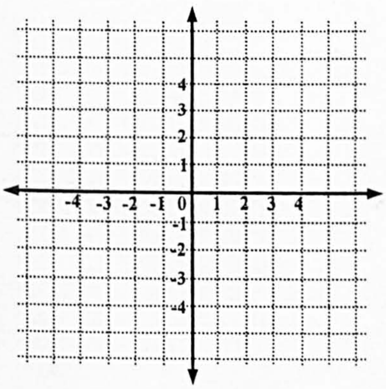


7. Using the features of absolute value, simplify the expression of $|2 + |3 + (-2)| - |4-7||$.

8. Find the set of solution for the inequality of $7+|x| < 6$. Explain your answer.

9. Sketch the graph of $|f(x)|$ by making use of the graph of $f(x)$ given below. Show all your work on the paper and explain your answer.





10. Sketch the graph of $f(|x|)$ by making use of the graph of $f(x)$ given below. Show all your work on the paper and explain your answer.