Essays on Dynamic Asset Allocation and Performance Measures

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Declaration

Chapter 3 is joint work with Prof. Peter Spencer, University of York, UK. The rest chapters are sole authored. A version of Chapter 4 has been presented at the doctoral sessions of the Financial Management Association (European Meeting) in Barcelona, May 2007, and the European Finance Association in Ljubljana, August 2007, as well as at the C.R.E.T.E. in Naxos, July 2007. It has also been accepted for presentation at the Conference on Asset Management and International Capital Markets in Frankfurt, May 2008, and at the conference of the Financial Management Association (European Meeting) in Prague, June 2008. A previous version of this chapter has appeared in the working paper series of the Department of Economics and Related Studies, University of York (Discussion Paper No. 07/07).

Abstract

The present thesis examines two central issues in financial theory, optimal portfolio choice and investment performance evaluation, when the restrictive assumptions of the traditional static, mean-variance framework of analysis are relaxed.

Chapter 2 presents a series of model specifications for the risky asset's returns and the underlying risk factor and derives the corresponding optimal portfolio choices. It shows how important the modelling assumptions are for the implementation of dynamic asset allocation in practice and it contributes to the literature by examining the impact of horizon effects on portfolio choice in the presence of both predictability and stochastic volatility in asset returns. Moreover, this chapter shows how important is the introduction of an asset that completes the market and allows investors to hedge against the shocks that affect their opportunity set.

Chapter 3 examines the bond portfolio choice of a long-term investor, making use of a macro-finance term structure model that allows for time-varying risk premia. This chapter shows how important is the failure of the expectations hypothesis for both myopic and long-term investors, since the time-variation in the bond premia dictates a market timing behaviour for investment as well as for hedging purposes. Incorporating macroeconomic information, that plays a significant role in bond pricing, we examine how this can be used for the formation of optimal portfolios by long-term investors. Furthermore, this chapter serves as an evaluation of the very

recent term structure models from an asset allocation perspective, drawing the attention to the correlation and the covariance structure of the bond returns.

Chapter 4 employs the Harvey-Siddique asset pricing model and evaluates a sample of UK equity unit trusts, proposing the intercept of this model, that is termed as the *Harvey-Siddique alpha*, as a new performance measure. This asset pricing model adds to the CAPM the returns of a negative coskewness strategy as an extra risk factor. Constructing this factor for the UK stock market, it is shown that negative coskewness bears a high risk premium. This framework allows us to examine how the adoption of specific performance measures generates incentives in fund management. In particular, we provide evidence that fund managers, who are evaluated by mean-variance performance measures, are incentivized to load negative coskewness risk to their portfolios in order to reap the corresponding premium and present it as outperformance.

Chapter 5 overviews the contributions of this thesis, discusses the numerous issues that arise from the present results and outlines the following steps in our research agenda.

Chapter 1

Introduction

1.1 Motivation and structure of the thesis

Most of the theory of finance has been developed within a static, mean-variance framework. The main examples of this framework are the portfolio choice theory of Markowitz (1952), the fund-separation theorem of Tobin (1958) as well as the Sharpe (1964)- Lintner (1965)- Mossin (1966) Capital Asset Pricing Model (CAPM). These pathbreaking contributions served as an excellent first attempt to form optimal portfolios, to price risky assets as well as to evaluate fund managers and investment strategies. Nevertheless, the limitations of this framework are numerous and crucial. Providing motivation for this thesis, a significant portion of investors do not have myopic horizons and asset returns are not normally distributed, hence their higher moments should matter too. This thesis attempts to examine how the formation of optimal portfolios and the evaluation of investment strategies may be modified if we abandon the static, mean-variance framework.

More specifically, we firstly examine how the optimal asset allocation of an investor is modified when his horizon exceeds the next period. Chapters 2 and 3 analyze a series of multiperiod portfolio choice problems, showing how a long-term investor should optimally behave as well as how his optimal decisions differ from the case of a myopic investor. In particular, Chapter 2 presents a series of model specifications for the risky asset's returns and the underlying risk factor and derives the corresponding optimal portfolio choices. This exercise shows how important the modelling assumptions are for the implementation of dynamic asset allocation in practice. It contributes to the literature by examining the impact of horizon effects on portfolio choice in the presence of both predictability and stochastic volatility in asset returns. Moreover, it shows how important is the introduction of an asset that completes the market and allows investors to hedge against the shocks that affect their opportunity set.

Chapter 3 examines the bond portfolio choice of a long-term investor, making use of a macro-finance term structure model that allows for time-varying risk premia. This chapter shows how important is the failure of the expectations hypothesis for both myopic and long-term investors, since the time-variation in the bond premia dictates a market timing behaviour for investment as well as for hedging purposes. Incorporating macroeconomic information, that plays a significant role in bond pricing, we examine how this can be used for the formation of an optimal portfolio by a long-term investor. Furthermore, Chapter 3 serves as an evaluation of the very recent term structure models from an asset allocation perspective, drawing the attention to the correlation and the covariance structure of the bond returns too, apart from the level of their expected returns and their predictability. Finally, in contrast to Chapter 2 that uses the dynamic programming approach, Chapter 3 employs the martingale methodology to solve the optimal portfolio choice problem in both complete and incomplete markets, presenting the plethora of the recently developed techniques for the solution of these problems.

The second issue that this thesis considers is how investment strategies should be evaluated if negative coskewness is priced in financial markets. In particular, Chapter 4 uses the Harvey and Siddique (2000) asset pricing model and evaluates a sample of UK equity unit trusts for the period January 1991- December 2005, proposing the intercept of this model, that is termed as the *Harvey-Siddique alpha*, as a new performance measure. This asset pricing model adds to the CAPM the returns of a negative coskewness strategy as an extra risk factor. Constructing this factor for the UK stock market, it is shown that negative coskewness bears a high risk premium, underlining its importance.

Using the proposed performance measure, the majority of the equity unit trusts significantly underperform their benchmark. Moreover, the results are compared with the performance measures that are commonly used in the literature. This framework also allows us to examine how the adoption of specific performance measures generates incentives in fund management. This is an important issue related to the problem of the ex post state verification of the investment performance. In particular, we provide evidence that fund managers, who are evaluated by mean-variance performance measures, are incentivized to load negative coskewness risk to their portfolios in order to reap the corresponding premium and present it as outperformance. Nevertheless, this is not genuine outperformance for an investor who is averse to negative skewness, hence we document an unfortunate misalignment between managerial incentives and investors' preferences.

Chapter 5 overviews the contributions of this thesis, discusses the numerous issues that arise from the present results and outlines the following steps in our research agenda. The rest of the present chapter introduces the reader to the general intertemporal portfolio choice problem and provides a classification of the fast-growing literature on this topic. It also discusses why negative skewness is priced in financial markets and how this feature is related to the intertemporal risks and the value, size and momentum "anomalies".

1.2 The intertemporal portfolio-consumption choice problem

One of the central issues in financial economics is the determination of the optimal portfolio choice. The classical approach to the problem, suggested by Markowitz (1952), is to maximize a mean-variance objective function. This approach, however, is valid only if the investor has quadratic preferences or the risky asset returns are drawn from an elliptical distribution (see Chamberlain, 1983). Most importantly, this setup explicitly assumes that the investor has a one-period ahead horizon.

Extensive research in the time series behaviour of the stock returns showed that these exhibit a series of stylized facts, such as time-varying volatility[1], negative skewness[2] and predictability.[3] This behaviour led researchers to consider stock returns within a dynamic setting and motivated significant attempts to examine how shocks to a series of macroeconomic and financial risk factors affect the risky assets' returns. These stylized facts can be modelled by assuming that there is a set of underlying stochastic risk factors affecting the risky asset dynamics. Furthermore, if the investment horizon of the investor exceeds the next period, then the optimal portfolio choice problem is significantly modified. In general, as Merton (1971) showed, the optimal portfolio choice of a long-horizon investor will incorporate an intertemporal hedging component, apart from the myopic one. This hedging demand arises due to the incentive of a risk-averse, long-term investor to minimize the volatility of his wealth path, which arises due to shocks in the underlying opportunity set.

[1]See for example the review of Bollerslev et al. (1992).
[2]See for example Andersen et al. (2002).
[3]See the seminal paper of Campbell and Shiller (1988).

The pathbreaking study of Merton (1973) provided a characterization of the solution to this intertemporal problem. Nevertheless, the absence of an analytical solution to the resulting non-linear PDE discouraged any further research. This strand of the literature was revitalized with the increase in computing power and the provision of analytical solutions for special cases of the general problem. The studies of Kim and Omberg (1996), Brennan et al. (1997), Campbell and Viceira (1999) inter alia, examined the impact of returns' predictability by a series of financial variables on the optimal portfolio choice in an incomplete market setting. Chacko and Viceira (2005) dealt with the case of stochastic volatility. On the other hand, Cox and Huang (1989), Wachter (2002) and Detemple et al. (2003) inter alia, working in a complete market setting, made use of the martingale approach to provide an analytical or numerical solution to the general portfolio-consumption choice problem.

1.2.1 The optimization problem

The agent faces the problem of deciding what portion of his wealth he should consume and what to invest as well as how to allocate the portion of the wealth he invests among $n-1$ risky assets and an instantaneously riskless asset. The time-horizon is defined to be finite T. Uncertainty in this problem is described by the probability space $(\Omega, \mathcal{F}, P)$, with Ω being the sample space, P the Wiener measure, and $\mathcal{F}$ the σ-algebra generated by the Brownian motions B_t and B_t^X defined on this space.

The returns of the risky assets are assumed to follow a Stochastic Differential Equation (SDE) and, in particular, the moments of the risky assets are assumed to depend on a set of stochastic factors, symbolized by X. This set of stochastic factors characterizes the investment opportunity set that the agent faces.

Introducing the appropriate notation, the riskless asset follows the Ordinary Differential Equation (ODE):

$$\frac{dP_0}{P_0} = r^f(X)dt \tag{1.1}$$

and the $n-1$ risky assets' returns follow the SDE:

$$\frac{dP_i}{P_i} = \mu_i(X)dt + \Sigma_i(X)dB \tag{1.2}$$

with $i = 1, .., n-1$ where B is a standard $(n-1)$-dimensional Brownian motion on the probability space that we previously defined and X is a k-dimensional vector collecting the stochastic factors.

This state factor follows a Markovian diffusion process given by:

$$dX = \mu^X dt + \Sigma^X dB^X \tag{1.3}$$

where μ^X is a kx1 vector function of X, $\Sigma^X \Sigma^{XT}$ is a kxk matrix function of X, while B^X is a k-dimensional standard Brownian motion on the same probability space.

It is assumed that Σ is almost surely invertible. This is true when the variance-covariance matrix is nonsingular or, in other words, there are no redundant assets included in the asset space. The vectors dB and dB^X are assumed to have a $(n-1)$xk correlation matrix, denoted by ρ.

Denoting by ϕ the $(n-1)$-dimensional vector of the portions of the wealth invested in the corresponding risky assets, the portion of the wealth invested in the riskless asset is $1 - i^T\phi$, where i is a vector of ones.

Therefore, the wealth process (W) is given, in the presence of consumption, by the following diffusion process:

$$dW = (W(\phi^T \frac{dP}{P} + (1 - i^T \phi)\frac{dP_0}{P_0}) - Cdt =$$

$$= W(\phi^T \mu dt + \phi^T \Sigma dB + (1 - i^T \phi) r^f dt) - Cdt =$$

$$= (W(\phi^T (\mu - i^T r^f) + r^f) - C)dt + W\phi^T \Sigma dB \tag{1.4}$$

The investor derives utility from his consumption as well as his terminal wealth. Therefore, he needs to find the optimal level of consumption C^* and the optimal investment ϕ^* in the risky assets, in order to maximize:

$$\max_{C(t),\phi(t)} E_{t_0}[\int_{t_0}^{T} ae^{-\beta(t-t_0)} U(C(t))dt + (1-a)e^{-\beta(T-t_0)} B[W(T),T]] \tag{1.5}$$

subject to the wealth dynamics given by (1.4) and the nonnegativity constraints $C(t) \geq 0$, $W(t) \geq 0$ for all $t \in [t_0, T]$ and $W_{t_0} > 0$.

The first constraint implies that consumption cannot be negative, while the second acts as a natural no-default condition. In other words, zero consumption and zero wealth can be regarded as absorbing barriers in the corresponding processes.

The weight a controls the trade off between utility derived by interim consumption and utility derived by final wealth. If $a = 0$, then utility is derived only by final wealth and there is no interim consumption. If $a = 1$, the agent derives utility only by interim consumption and there is no bequest function. The presented framework is general enough to accommodate for different time rates of preference. The parameter β represents the discounting rate of future streams of consumption and wealth.

The utility functions $U(.)$ and $B(.)$ used above are assumed to be strictly differentiable, strictly increasing, $U'(C) > 0$ and $B'(W) > 0$, and strictly concave, $U''(C) < 0$ and $B''(W) < 0$. Therefore, these utility functions satisfy the local non-satiation property for finite consumption and wealth levels and they refer to a risk-averse investor.

Furthermore, we assume that the standard Inada conditions are satisfied. In particular, $\lim_{c \to 0} U'(C) = \lim_{W \to 0} B'(W) = \infty$ and $\lim_{c \to \infty} U'(C) = \lim_{W \to \infty} B'(W) = 0$. The first Inada condition guarantees that the optimal consumption is nonnegative. Since consumption is financed by wealth, the optimal wealth level should be nonnegative too. In other words, since wealth can be interpreted as the present value of the future stream of consumption, the nonnegativity of consumption implies the nonnegativity of wealth. The second Inada condition essentially guarantees that the solutions are finite. Consequently, using a utility function that satisfies the Inada conditions, we can neglect the wealth and consumption constraints in the maximization procedure. The portfolio-consumption problem collapses to the maximization of equation (1.5) subject to (1.4).

1.2.2 Classification of the Literature

It is generally acknowledged that the literature of dynamic asset allocation begins with the seminal paper of Merton (1971), employing the budget constraint of a multiperiod investor derived in Merton (1969).[4] Merton (1973) provided a supplementary treatment of the issue, deriving an intertemporal capital asset pricing model. However, the lack of an analytical solution for the Partial Differential Equation (PDE) that describes the optimal portfolio-consumption problem for an agent with multiperiod horizon, as well as the lack of the necessary computing power for numerical solutions, prohibited any applications of the Merton setup for more than two decades. These obstacles obliged researchers to follow various approaches of solving this problem. During the last decade, there have been significant contributions, presenting tractable and insightful solutions. This section reviews and classifies this literature.

[4]Samuelson (1969) derived the budget constraint of a multiperiod investor in discrete time.

Complete versus Incomplete markets

The most important classification is among the studies that assume complete markets and those working in an incomplete market setting. Under the assumption of complete markets, the "Martingale approach" can be employed. This methodology builds on the work of Harrison and Kreps (1979) on equivalent martingale measures. The assumption of complete markets implies that every source of risk can be perfectly hedged. No arbitrage and market completeness guarantee the existence and the uniqueness of an equivalent martingale measure.

Using the notation presented in the previous subsection, the matrix of the correlation coefficients between the vector of the Brownian motions of the risky assets (B_t) and the vector of Brownian motions of the risk factors (B_t^X) should be the identity matrix ($\rho = I$). The breakthrough of this methodology is that, as shown in Karatzas et al. (1987) and Cox and Huang (1989) and proved in Cox and Huang (1991), the multiperiod problem can be equivalently written as a static problem.

Since most of the studies employ *ad hoc* specifications for the stock returns dynamics, which are not derived by no-arbitrage conditions, the complete market setting is assumed to be quite restrictive. Therefore, most of the studies (see *inter alia* Brennan et al., 1997), that empirically estimate the parameters of the risky asset dynamics and the risk factors, do not impose such a restriction. On the other hand, Wachter (2002) provides an analytical treatment of the problem when the investor has to choose between a risky asset with mean-reverting returns and the markets are complete. She argues that the assumption of complete markets is realistic in her setup, since the shocks to stock returns have almost perfect negative correlation with the shocks to the Sharpe ratio that serves as a stochastic factor.

Numerical solutions versus explicit solutions

The non-linear second-order PDE derived by Merton cannot be analytically solved as yet. As a result, the first attempts in the literature involved numerical solutions of the PDE using finite difference schemes. Brennan and Schwartz (1996), Brennan et al. (1997), Brennan (1998) and Xia (2001) *inter alia* employ model specifications for which analytical solutions do not exist and, therefore, use numerical solutions. The main disadvantage of these studies is the curse of dimensionality. Despite the increase in computing power, numerical solutions of PDEs become almost impossible as the number of risky assets and stochastic factors increases. Furthermore, the accuracy of the solution becomes very poor as the approximation errors propagate. As Detemple et al. (2005) note, the truncation of the state space and the imposition of boundaries for the value function, as used in finite difference schemes, may lead to a solution with unstable behaviour. Barberis (2000) solves numerically the dynamic programming problem, working in discrete time.

Consequently, recent studies focus on deriving an analytical solution to this problem. Liu (2007) provides a complete treatment of the conditions under which Merton's PDE collapses to a system of ODEs. The necessary assumption is that the risky assets and the risk factors' dynamics are characterized by processes which are "affine-quadratic" in the risk factors. In a complete market setting, an analytical solution can be provided even for the case of intermediate consumption, while under incomplete markets, an analytical solution is given only for the terminal wealth case. The framework of Liu (2007) nests almost all of the studies providing analytical solutions. For example, Kim and Omberg (1996) assume an Ornstein-Uhlenbeck process for the Sharpe ratio of the single risky asset, which follows a Geometric Brownian Motion (GBM) in their model.

Intermediate Consumption versus terminal wealth maximization

As it has already been mentioned, the model specification and the assumption of market completeness are influential features for the type of the problem that is addressed. This choice is closely related to the possibility of deriving analytical solutions as well as the feasibility of numerical solutions. Most of the studies employing numerical solutions (see the previous subsection), as well as Kim and Omberg (1996) and Barberis (2000), assume that the multi-period investor maximizes his value function over terminal wealth, ignoring intermediate consumption. Under complete markets and "affine-quadratic" models for the risky asset dynamics, we can also deal with intermediate consumption using both the Dynamic Programming (see Liu, 2007) and the Martingale approach (see Wachter, 2002). The recent contributions of Detemple et al. (2003, 2005) employ an alternative methodology under complete markets that makes use of Malliavin derivatives. This solution methodology requires Monte Carlo simulations, but it deals with more general risky asset dynamics (e.g. non-affine) and accommodates for the case of intermediate consumption too.

Which moments of the assets' returns are time-varying?

Asset returns exhibit a series of stylized facts such as time-varying risk-premia, stochastic volatility and deviations from normality. As a result, a GBM specification for the risky assets' returns dynamics is a very poor approximation. The literature on dynamic asset allocation has attempted to incorporate these facts in order to examine their impact on portfolio choice.

Chacko and Viceira (2005) examine the case of stochastic volatility for a constant risk premium. Their model specification accommodates for volatility persistence and mean-reversion. It can also yield negative skewness due to the negative correlation between the shocks to the volatility process and the shocks to the risky asset process.

Such a model implies that the multiperiod investor faces a "volatility risk" that he has to hedge away.

Liu et al. (2003) employ the double-jump model of Duffie et al. (2000), which allows for discontinuous movements both in asset prices and their volatility. Such a specification increases the sources of uncertainty for the risky asset and, interestingly, inhibits an "illiquidity" risk, which reflects the probability of a large drop in the risky asset price. An immediate consequence of this possibility is that a risk-averse investor will reduce his risky asset holdings. Das and Uppal (2004) employ a model which also accommodates skewness by assuming a jump process for the asset returns. Furthermore, their study refers to multiple assets, representing various international market indices. As a result, their model captures the fact that the price jumps occur simultaneously across these markets, diminishing the value of international diversification.

Another strand of the literature allows for uncertainty in the moments of the assets' returns. This framework reflects the fact that forecasting is a very difficult task and the out-of-sample portfolio performance using point estimates of the moments is very poor (see e.g. De Miguel et al., 2008). Hence, Brennan (1998) allows the possibility of "estimation risk" for the conditional expectation of the risk premium. This implies that the investor should update his beliefs for the risk premium through time and this "estimation risk" becomes another source of uncertainty, against which the multiperiod investor has to hedge. Practically, this uncertainty reduces the risky asset holdings because the investor does not have exact knowledge about the positivity and the magnitude of the risk premium, i.e. the motivation for investing in the risky asset. In a similar spirit, Xia (2001) assumes that returns are predictable, but the predictability relationship is uncertain. As a result, the multiperiod investor is able to update his beliefs through time, but he also needs to hedge against shocks to this relationship.

Discrete time versus continuous time

Following the work of Merton, most of the studies examine this portfolio problem within a continuous time setting. The similarity with the techniques employed in dynamic asset pricing is one of the reasons why continuous time is preferred. Most importantly, as Merton (1975) argues, continuous time techniques allow us to solve in closed form portfolio choice problems which are analytically intractable in discrete time. This is true for the dynamic programming as well as for the martingale approach.

On the other hand, there are a series of studies that deal with this problem in discrete time. Campbell and Viceira (1999) and Campbell et al. (2003) deal with the case of predictable returns and constant variances. Instead of working with Hamilton-Jacobi-Bellman (HJB) equations, as in continuous time, the studies using discrete time solve Euler equations. In particular, these studies employ loglinearizations of the Euler equations in order to derive the optimal portfolio and consumption expressions. Barberis (2000) examines a buy-and-hold investing strategy when returns are predictable and solves numerically the discrete time version of the dynamic programming problem. Another strand of the literature (see Brandt, 1999, 2007 *inter alia*) uses conditional Euler equations and estimates nonparametrically the optimal portfolio and consumption choices for the various states of the underlying stochastic factors, employing conditional methods of moments.

The main advantage of the studies working with conditional Euler equations is that they do not assume a specific model for the risky asset dynamics but they employ, instead, semiparametric estimations. On the other hand, they bear the disadvantage of resorting to approximations to get explicit solutions. Campbell et al. (2004) demonstrate the correspondence between the discrete and the continuous time case, assuming predictability of stock and bond returns. This study mentions the pitfalls of mapping the discrete time solutions to a continuous time framework.

Choice of utility functions

Another important criterion, according to which studies on dynamic asset allocation can be classified, is the choice of the utility function that the multiperiod investor seeks to maximize. The results crucially depend on this choice, due to the different properties of each utility function. Merton (1971) presented the solution to the problem using an exponential utility function (exhibiting constant absolute risk aversion- CARA) as well as the more general framework of utility functions exhibiting hyperbolic absolute risk aversion (HARA). Most of the recent studies employ time-separable utility functions and more specifically, the power utility function. Such a choice is more realistic in comparison to the exponential one, since the power utility function exhibits constant relative, rather than absolute, risk aversion (CRRA). Furthermore, the separability of this utility function is very convenient, since it reduces the dimension of the value function optimization problem.

On the other hand, there are some papers that examine different utility function specifications. The recursive utility function of Epstein and Zin (1989) has been employed *inter alia* by Campbell and Viceira (1999) and Campbell et al. (2003), while the continuous time version of this utility function, as developed by Duffie and Epstein (1992), has been employed in Campbell et al. (2004).

The use of general conditional Euler equations in Brandt (1999) enables the use of various utility frameworks. Furthermore, the use of Monte Carlo simulations in Detemple et al. (2005) allows for more realistic and complex utility function specifications. In general, the literature faces an important trade off between employing a utility function that replicates realistic features of investors' behaviour and deriving analytical expressions for the portfolio-consumption problem that enable a detailed analysis of their properties.

1.2.3 The Dynamic Programming Approach

We define the value function $I(W(X,t_0),X,t_0)$:

$$I(W(X,t_0),X,t_0) \equiv \max_{C(t),\phi(t)} E_{t_0}[\int_{t_0}^{T} ae^{-\beta(s-t_0)}U(C(s))ds+(1-a)e^{-\beta(T-t_0)}B[W(T),T]] \tag{1.6}$$

This value function represents the maximized expected utility over interim consumption and terminal wealth. It depends on the agent's wealth, time and the stochastic opportunity set.

Defining $t \equiv t_0 + h$, the previous expression can be re-written as:

$$\begin{aligned} I(W(X,t_0),X,t_0) &= \max E_{t_0}[\int_{t_0}^{t_0+h} ae^{-\beta(s-t_0)}U(C(s))ds+ \\ &+\int_{t_0+h}^{T} ae^{-\beta(s-(t_0+h))}U(C(s))ds + (1-a)e^{-\beta(T-t_0)}B[W(T),T] \end{aligned} \tag{1.7}$$

where by definition:

$$\max E_t \int_t^T ae^{-\beta(s-t)}U(C(s))ds + (1-a)e^{-\beta(T-t)}B[W(T),T] = I(W(X,t),X,t) \tag{1.8}$$

Therefore, for h small, we have:

$$I(W(X,t_0),X,t_0) = \max E_{t_0}[\int_{t_0}^{t} ae^{-\beta(s-t_0)}U(C(s))ds + I(W(X,t),X,t)] \quad (1.9)$$

We take a Taylor series expansion of $I(W(X,t),X,t)$ around t_0, and, after applying Ito's lemma due to the stochastic nature of the wealth process, we get:

$$\begin{aligned} I(W(X,t),X,t) &= I(W(t_0),X,t_0) + \frac{\partial I}{\partial W}(W(t) - W(t_0)) + \frac{\partial I}{\partial t}(t - t_0) + \\ &+ \frac{\partial I}{\partial X}(X(t) - X(t_0)) + \frac{\partial^2 I}{\partial W \partial X}(W(t) - W(t_0))(X(t) - X(t_0)) + \\ &+ \frac{1}{2}\frac{\partial^2 I}{\partial X \partial X^T}(X(t) - X(t_0))(X(t) - X(t_0))^T + \frac{1}{2}\frac{\partial^2 I}{\partial W^2}(W(t) - W(t_0))^2 + O(h^2) \end{aligned} \quad (1.10)$$

Substituting expression (1.10) into equation (1.9), we have:

$$\begin{aligned} I(W(X,t_0),X,t_0) &= \max E_{t_0}[\int_{t_0}^{t} ae^{-\beta(s-t_0)}U(C(s))ds + I(W(X,t_0),X,t_0) + \\ &+ \frac{\partial I}{\partial W}(W(t) - W(t_0)) + \frac{\partial I}{\partial t}(t - t_0) + \frac{\partial I}{\partial X}(X(t) - X(t_0)) + \\ &+ \frac{1}{2}\frac{\partial^2 I}{\partial X \partial X^T}(X(t) - X(t_0))(X(t) - X(t_0))^T + \frac{1}{2}\frac{\partial^2 I}{\partial W^2}(W(t) - W(t_0))^2 \\ &+ \frac{\partial^2 I}{\partial W \partial X}(W(t) - W(t_0))(X(t) - X(t_0)) + O(h^2)] \end{aligned} \quad (1.11)$$

or

$$\max E_{t_0}[\int_{t_0}^{t} ae^{-\beta(s-t_0)}U(C(s))ds + \frac{\partial I}{\partial W}(W(t) - W(t_0)) + \frac{\partial I}{\partial t}(t - t_0) +$$
$$+\frac{\partial^2 I}{\partial W \partial X}(W(t) - W(t_0))(X(t) - X(t_0)) + \frac{\partial I}{\partial X}(X(t) - X(t_0)) +$$
$$+\frac{1}{2}\frac{\partial^2 I}{\partial X \partial X^T}(X(t) - X(t_0))(X(t) - X(t_0))^T + \frac{1}{2}\frac{\partial^2 I}{\partial W^2}(W(t) - W(t_0))^2 + O(h^2)] = 0 \tag{1.12}$$

Dividing both sides of the previous expression through h, passing h inside the expectation and taking the limit as $h \to 0$ we get:

$$\max E[ae^{-\beta t}U(C(t)) + \frac{\partial I}{\partial W}dW + \frac{\partial I}{\partial t}dt + \frac{\partial I}{\partial X}dX +$$
$$+\frac{1}{2}\frac{\partial^2 I}{\partial X \partial X^T}(dX)(dX)^T + \frac{\partial^2 I}{\partial W \partial X}dWdX + \frac{1}{2}\frac{\partial^2 I}{\partial W^2}(dW)^2] = 0 \tag{1.13}$$

Defining then:

$$J(W(X,t), \phi(t), X, t) \equiv ae^{-\beta t}U(C(t)) + \frac{\partial I}{\partial W}dW + \frac{\partial I}{\partial t}dt +$$
$$+\frac{\partial I}{\partial X}dX + \frac{\partial^2 I}{\partial W \partial X}dWdX + \frac{1}{2}\frac{\partial^2 I}{\partial X \partial X^T}(dX)(dX)^T + \frac{1}{2}\frac{\partial^2 I}{\partial W^2}(dW)^2 \tag{1.14}$$

we get the Hamilton-Jacobi-Bellman (HJB) principle of optimality:

$$\max E[J(W(X,t), \phi(t), X, t)] = 0 \tag{1.15}$$

Replacing the SDEs for dW and dX, as defined in (1.4) and (1.3) correspondingly, we get the following HJB equation for the dynamic problem that we examine:

$$\max_{C,\phi}\ [ae^{-\beta t}U(C(t))+\frac{\partial I}{\partial t}+\frac{1}{2}W^2\phi^T\Sigma\Sigma^T\phi\frac{\partial^2 I}{\partial W^2}+W(\phi^T(\mu-i^Tr^f)+r^f)\frac{\partial I}{\partial W}+$$
$$+\mu^{XT}\frac{\partial I}{\partial X}-C\frac{\partial I}{\partial W}+\frac{1}{2}Tr(\Sigma^X\Sigma^{XT}\frac{\partial^2 I}{\partial X\partial X^T})+W\phi^T\Sigma\rho^T\Sigma^{XT}\frac{\partial^2 I}{\partial W\partial X}]=0 \quad (1.16)$$

with boundary condition for the value function:

$$I(W(X,T),X,T)=(1-a)B(W(T),T) \quad (1.17)$$

This optimality equation can help us derive the First Order Conditions (FOCs) with respect to the control variables, namely the consumption choice and the vector of the risky assets' portfolio weights. The FOC with respect to the optimal portfolio choice yields:

$$\frac{\partial(.)}{\partial\phi}=0\Rightarrow W^2\phi\Sigma\Sigma^T\frac{\partial^2 I}{\partial W^2}+W(\mu-i^Tr^f)\frac{\partial I}{\partial W}+W\Sigma\rho^T\Sigma^{XT}\frac{\partial^2 I}{\partial W\partial X}=0\Rightarrow$$
$$\Rightarrow\phi^*=-\frac{\frac{\partial I}{\partial W}}{\frac{\partial^2 I}{\partial W^2}W}(\Sigma\Sigma^T)^{-1}(\mu-i^Tr^f)-\frac{1}{\frac{\partial^2 I}{\partial W^2}W}(\Sigma\Sigma^T)^{-1}\Sigma\rho^T\Sigma^{XT}\frac{\partial^2 I}{\partial W\partial X} \quad (1.18)$$

The first term in equation (1.18) is the myopic demand component *à la* Markowitz. It takes into account the risk premium of the risky assets' returns adjusted by their covariance matrix. This expression is multiplied by $-\frac{\frac{\partial I}{\partial W}}{\frac{\partial^2 I}{\partial W^2}W}$, which is a measure of risk aversion (i.e. the concavity of the value function). The second term is the intertemporal hedging demand component. Using this expression, the conditions under which the portfolio choice for the multiperiod investor is identical to the static one can be derived. More specifically:

i) If shifts in the underlying opportunity set do not affect the value function of the investor, i.e. $\frac{\partial^2 I}{\partial W\partial X}=0$. A trivial example is when the value function does not depend on the stochastic factor, while the most commonly referred case is when the investor has a logarithmic value function, i.e. $I(W, X, t) = \log W + \log X$.

ii) If the underlying opportunity set is not stochastic, i.e. $\Sigma^X = 0$. It is important to underline that the intertemporal hedging demand component arises due to the stochastic nature of the underlying opportunity set, not simply because of its time-variation.

iii) If the innovations of the underlying stochastic factors are not correlated with the innovations in the risky asset returns, i.e. $\rho = 0$. Therefore, the need to hedge against the stochastic evolution of the risk factors arises only when this is affecting the returns of the risky assets.

The FOC with respect to consumption yields:

$$\frac{\partial(.)}{\partial C} = 0 \Rightarrow ae^{-\beta t}U'(C(t)) - \frac{\partial I}{\partial W} = 0 \Rightarrow U'(C^*(t)) = \frac{1}{a}\frac{\partial I}{\partial W}e^{\beta t} \tag{1.19}$$

This envelope condition exhibits the trade off between current and future consumption. The optimal consumption level C^* at each time step t equalizes the marginal utility of consumption with the marginal value of saving to finance future consumption, appropriately adjusted for the time value of money and the preferences of the agent. If $U'(C^*(t)) > \frac{1}{a}\frac{\partial I}{\partial W}e^{\beta t}$, then the investor would be better off by increasing current consumption, while if $U'(C^*(t)) < \frac{1}{a}\frac{\partial I}{\partial W}e^{\beta t}$, the agent should reduce current consumption in order to accumulate wealth.

Assuming invertibility of the utility function, the FOC with respect to consumption gives the optimal consumption level:

$$C^* = U_C^{-1}(\frac{1}{a}\frac{\partial I}{\partial W}e^{\beta t}) \tag{1.20}$$

Replacing these FOCs back to the HJB equation, we end up with a second-order non-linear PDE with respect to the value function $I(W,X,t)$:

$$ae^{-\beta t}U(C^*(t)) + \frac{\partial I}{\partial t} - \frac{1}{2}\frac{(\frac{\partial I}{\partial W})^2}{\frac{\partial^2 I}{\partial W^2}}(\mu - i^T r^f)^T(\Sigma\Sigma^T)^{-1}(\mu - i^T r^f) +$$
$$+\frac{\partial I}{\partial W}Wr^f - \frac{\frac{\partial^2 I}{\partial W \partial X}\frac{\partial I}{\partial W}}{\frac{\partial^2 I}{\partial W^2}}[\Sigma^X \rho \Sigma^{-1}(\mu - i^T r^f)]^T - \frac{1}{2}\frac{(\frac{\partial^2 I}{\partial W \partial X})(\frac{\partial^2 I}{\partial W \partial X^T})}{\frac{\partial^2 I}{\partial W^2}}(\Sigma^X \rho\rho^T \Sigma^{XT}) -$$
$$-U_C^{-1}(\frac{1}{a}\frac{\partial I}{\partial W}e^{\beta t})\frac{\partial I}{\partial W} + \frac{1}{2}Tr(\Sigma^X \Sigma^{XT}\frac{\partial^2 I}{\partial X \partial X^T}) + \mu^{XT}\frac{\partial I}{\partial X} = 0 \tag{1.21}$$

As it has been mentioned, there does not exist an analytic solution for this general PDE. Hence, we should either resort to numerical methods or conjecture a particular functional form for the value function. If the investor's preferences with respect to wealth are characterized by a power utility function, then the conjectured form is:

$$I(W,X,t) = e^{-\beta t}\frac{W^{1-\gamma}}{1-\gamma}[f(X,t)]^\gamma \tag{1.22}$$

Using this functional form, the optimal portfolio choice is given by:

$$\phi^* = \frac{1}{\gamma}(\Sigma\Sigma^T)^{-1}(\mu - i^T r^f) + (\Sigma\Sigma^T)^{-1}\Sigma\rho^T\Sigma^{XT}\frac{\partial \ln f}{\partial X} \tag{1.23}$$

and the optimal consumption choice by:

$$C^* = a^{\frac{1}{\gamma}}Wf^{-1} \tag{1.24}$$

This essentially means that the optimal consumption-wealth ratio is given by $\frac{C^*}{W} = a^{\frac{1}{\gamma}}f^{-1}$. Hence, this depends on the weight assigned by the investor to the utility derived by interim consumption as well as on the value of the function f.

Using this conjectured form for the value function, the HJB equation (1.21) becomes:

$$\frac{\partial f}{\partial t} + \frac{1}{2}Tr(\Sigma^X \Sigma^{XT} \frac{\partial^2 f}{\partial X \partial X^T}) + [\mu^X + \frac{1-\gamma}{\gamma}\Sigma^X \rho \Sigma^{-1}(\mu - i^T r^f)]^T \frac{\partial f}{\partial X}$$
$$+\frac{1}{2f}(\gamma - 1)\frac{\partial f}{\partial X^T}(\Sigma^X \Sigma^{XT} - \Sigma^X \rho \rho^T \Sigma^{XT})\frac{\partial f}{\partial X} + a^{\frac{1}{\gamma}}$$
$$+[\frac{1}{2}\frac{1-\gamma}{\gamma^2}(\mu - i^T r^f)^T (\Sigma \Sigma^T)^{-1}(\mu - i^T r^f) + \frac{1-\gamma}{\gamma} r^f - \frac{\beta}{\gamma}]f = 0 \qquad (1.25)$$

with the boundary condition $f(T, X) = (1-a)^{\frac{1}{\gamma}}$.

Liu (2007) derives the general conditions under which this non-linear PDE collapses to a system of solveable ODEs. The requirement is that the drift, μ^X, and the volatility terms, $\Sigma^X \Sigma^{XT}$, of the underlying risk factors are affine-quadratic as well as the terms $(\mu - i^T r^f)^T (\Sigma \Sigma^T)^{-1}(\mu - i^T r^f)$, $\Sigma^X \rho \Sigma^{-1}(\mu - i^T r^f)$, $\Sigma^X \Sigma^{XT} - \Sigma^X \rho \rho^T \Sigma^{XT}$, that appear in the PDE, and the return on the instantaneously riskless asset, r^f, are at most quadratic functions of the vector of risk factors.

1.3 Why is skewness important?

As we previously mentioned, extensive research in the time series behaviour of the stock returns showed that these exhibit a series of stylized facts, such as time-varying volatility, predictability, negative skewness and excess kurtosis. Hence, returns clearly violate the assumption of being normally and identically distributed over time. Equally importantly, the CAPM is documented to fail empirically. In a series of papers, Fama and French (1993, 1995) showed that value and size strategies generate returns that cannot be explained by beta-risk loading. The outperformance of the momentum strategy, documented by Jegadeesh and Titman (1993), is another "anomaly".

Severe criticism to the CAPM assumptions comes from utility theory too. The assumption of quadratic preferences is clearly rejected since it implies increasing absolute risk aversion (IARA). A desirable property for a utility function is that agents are averse to negative skewness and have a preference for payoffs exhibiting positive skewness. This behaviour is termed prudence (see Kimball, 1990). Interestingly, experimental evidence (see Kahneman and Tversky, 1979) showed that there is an asymmetrically higher impact on utility by losses as related to gains, leading to utility frameworks such as Prospect Theory or Disappointment Aversion. These functions imply that agents are even more averse to negative skewness. Hence, aversion to negative skewness is a crucial feature that has been relatively neglected in asset pricing. Harvey and Siddique (2000) provide evidence that the negative coskewness risk is priced in financial markets.

Another very important limitation of the CAPM is its static nature. The recent asset pricing literature has attempted to resolve the documented "anomalies" within an intertemporal framework. The studies of Vassalou (2003), Campbell and Vuolteenaho (2004) and Petkova (2006) provide characteristic examples. This approach has its origins in the Intertemporal CAPM of Merton (1973). The most important observation is that there is a set of underlying risk factors which evolve stochastically through time and affect the dynamics of the assets' returns. It is interesting to observe that the impact of these risk factors on asset returns can be represented by means of higher moments. In other words, the intertemporal risks can be interpreted as higher moments risks.

The next two subsections discuss the interplay of skewness and preferences as well as how the outperformance of the size, value and momentum strategies could be explained as risk premia due to negative skewness.

1.3.1 Skewness and preferences

This subsection examines how skewness in the distribution of an agent's wealth affects his utility. In particular, we deal with utility functions that share the following two properties:

i) Monotonicity, i.e. $U'(W_t) > 0$ and

ii) Concavity, i.e. the individuals exhibit second-order risk aversion and $U''(W_t) < 0$.

We also require that the utility function exhibits Decreasing Absolute Risk Aversion (DARA). This property implies that the wealthier an investor is, the less risk averse he becomes over a given level of investment. Since the Arrow-Pratt coefficient of Absolute Risk Aversion (ARA) is given by:

$$ARA = -\frac{U''(W)}{U'(W)} \tag{1.26}$$

for this to be decreasing in wealth, it should hold that:

$$\frac{\partial(ARA)}{\partial W} < 0 \Rightarrow -\frac{U'''(W)}{U'(W)} + \left(\frac{U''(W)}{U'(W)}\right)^2 < 0 \Rightarrow -\frac{U'''(W)}{U'(W)} < 0 \Rightarrow U'''(W) > 0 \tag{1.27}$$

due to the properties previously stated. The positivity of the third derivative of the utility function along with its concavity describes a prudential behaviour. Leland (1968) and Sandmo (1970) argue that this behaviour is associated with the motive for precautionary savings in the presence of future income/consumption uncertainty. In general, prudence could be characterized as the sensitivity of the optimal choice for a decision variable with respect to its variability (see Kimball, 1990 for an analytical treatment). It is important to note that this behaviour is distinct from risk aversion. In particular, in the case of quadratic utility, $U(W) = aW + bW^2$, the individual has a zero degree of prudence, since $U'''(W) = 0$, even though he is risk averse when $b < 0$.

Extending the previous analysis, it is very informative to examine the interplay between an asset's returns distribution and preferences. In order to examine more formally the impact of skewness on expected utility, it proves useful to perform a Taylor series expansion around the mean level of wealth $\overline{W}$. Then, the expected utility at time t over wealth at time $t+1$ is given by:

$$E_t[U(W_{t+1})] = E_t[\sum_{k=0}^{\infty} \frac{U^{(k)}(\overline{W}_{t+1})(W_{t+1} - \overline{W}_{t+1})^k}{k!}] \tag{1.28}$$

Equation (1.28) can be re-written, under mild assumptions, as:[5]

$$E_t[U(W_{t+1})] = \sum_{k=0}^{\infty} \frac{U^{(k)}(\overline{W}_{t+1})}{k!} E_t[(W_{t+1} - \overline{W}_{t+1})^k] \tag{1.29}$$

In a portfolio choice context, truncating the Taylor series expansion at $k = 2$ corresponds to the familiar mean-variance analysis. If we truncate the series at $k = 3$, the third moment of the wealth distribution is also taken into account, i.e.:

$$E_t[U(W_{t+1})] \simeq U(\overline{W}_{t+1}) +$$
$$+ \frac{U''(\overline{W}_{t+1})}{2!} E_t[(W_{t+1} - \overline{W}_{t+1})^2] + \frac{U'''(\overline{W}_{t+1})}{3!} E_t[(W_{t+1} - \overline{W}_{t+1})^3] \tag{1.30}$$

The expansion of the expected utility shows the importance of asymmetries in the wealth distribution. If we have a symmetric distribution, then the last term vanishes

[5] Lhabitant (1998) argues that if the Taylor series is uniformly convergent, then we also have pointwise convergence and the infinite series ($\sum_{k=0}^{\infty} \frac{U^{(k)}(\overline{W})}{k!}$) can be integrated out of the expectation term-by-term. Lhabitant's assumptions depend on the choice of the utility function. In the case of a power utility function, convergence occurs for wealth levels in the range of $[0, 2\overline{W}]$, which is not such a restrictive range.

for globally differentiable utility functions, even if the agent is prudent. Nevertheless, since most of the assets' returns are characterized by negative skewness, it becomes evident that the mean-variance *Weltanschauung* is only a restrictive case of the general problem.

The impact of skewness is often examined using the power utility function. The main characteristic of this function is that it treats symmetrically utility gains and losses caused by a wealth change of the same magnitude. Actually, this is also a property of the mean-variance analysis. Despite this assumption, there is significant experimental evidence that agents are mainly averse to losses, not just to volatility. The Prospect Theory of Kahnemann and Tversky (1979) as well as the Disappointment Aversion framework of Gul (1991) imply that investors maintain an asymmetric attitude towards losses as compared to gains. In general, this class of value functions captures the feature of first-order risk aversion, analyzed by Segal and Spivak (1990). This feature implies that investors are even more averse to negative skewness in comparison to power utility agents. Hence, in the case of kinked utility functions, one can observe from equation (1.30) that, even if the wealth distribution is symmetric, the third central moment will still affect expected utility due to its asymmetric impact.

1.3.2 Skewness, market "anomalies" and intertemporal risks

In this subsection, it is argued that the documented asset pricing "anomalies", which are currently being explained as premia for intertemporal risks, could be linked to negative skewness. Among the most often cited failures of the CAPM is the outperformance of the size and value strategies. Fama and French (1993, p. 55) questioned whether "...specific fundamentals [can] be identified as state variables that lead to common variation in returns that is independent of the market and carries a different premium than general market risk". This conjecture motivated a significant research effort to associate the returns of the Fama-French portfolios with specific economic and financial variables. For example, Vassalou (2003) creates a mimicking portfolio that proxies news to future GDP growth and argues that this can explain the cross-sectional behaviour of the Fama-French portfolios.

On the other hand, Brennan et al. (2004) use the real interest rate and the Sharpe ratio as intertemporal risk factors. Petkova (2006) devises an asset pricing model using the financial variables that have been employed to predict future stock returns (dividend yield, term spread, default spread and the short term rate). Campbell and Vuolteenaho (2004) propose a two-beta model, explicitly specifying the cash flows and the discount rate as risk factors in an intertemporal asset pricing model. Summarizing, all these studies essentially assume the existence of a set of risk factors that vary stochastically and their innovations are correlated with the innovations to the stock returns. Such a setup generates intertemporal risk prèmia, as shown in Merton (1973).

We argue here that these intertemporal risks could be statically represented in terms of higher moments. To show that formally, we provide a specific example. Let us fix a probability space $(\Omega, \mathcal{F}, P)$ and the information filtration $(\mathcal{F}_t) = \{\mathcal{F}_t : t \geq 0\}$. Let S_t be the stock price and X_t the underlying risk factor at time t. The processes (S_t, X_t) form jointly a Markov process in the state space $D \in R^2$ and they obey to the following SDEs:

$$\frac{dS_t}{S_t} = \mu dt + \sigma_s \sqrt{X} dB_{st} \tag{1.31}$$

$$dX_t = k(\theta - X)dt + \sigma_x \sqrt{X} dB_{xt} \tag{1.32}$$

where $dB = (dB_s, dB_x)'$ is a vector of standard Brownian motions adapted to $(\mathcal{F}_t)$

and the two Brownian motions have a correlation coefficient given by ρ_{sx}. An important observation is that, even though the processes (S_t, X_t) are jointly Markovian, the process of the risky asset price (S_t) is not necessarily Markovian.

Given this setup, it is shown in the Appendix how we can derive the unconditional moments of the stock returns defined by $\Delta s_{t+1} = s_{t+1} - s_t$, where $s_t \equiv \ln S_t$. In particular, we get:

$$E(\Delta s) = \mu \tag{1.33}$$

$$E(\Delta s - E(\Delta s))^2 = \sigma_s^2 \theta \tag{1.34}$$

$$E(\Delta s - E(\Delta s))^3 = \frac{3}{k^2}(e^{-k} + k - 1)\theta \sigma_s \sigma_x \rho_{sx} \tag{1.35}$$

As it is evident from equation (1.35), the skewness of the returns' distribution crucially depends on the sign of the correlation of the shocks. Trivially, if there is no correlation, then the returns' distribution is symmetric. If the correlation is negative, then the returns' distribution is negatively skewed. The point made here is that a myopic investor, who neglects the dynamics of the underlying risk factor X, will erroneously interpret the risky asset's returns as normally distributed. On the other hand, the long-term investor, who takes into account the stochastic evolution of the underlying risk factor will be able to extract the correct, non-normal distribution.

Interestingly, a series of studies in the dynamic asset allocation literature report a negative correlation between the shocks to the risky asset's returns and the shocks to the underlying risk factor. We refer *inter alia* to Brennan et al. (1997) who use the dividend yield, the short and the long rate, Campbell and Viceira (1999) and Barberis (2000), who use the dividend yield, and Wachter (2002), who uses the

Sharpe ratio. This negative correlation gives rise to a hedging demand component in the optimal portfolio choice of a long-term, power utility investor.

Since intertemporal risks can be statically reflected by means of higher moments, it is argued that the Fama-French factors could proxy negative skewness. The study of Harvey and Siddique (2000) establishes such a link. They show that the abnormal returns generated by the value, size and momentum portfolios can be partly explained by the fact that they are negatively coskewed with the market returns. Consequently, the implementation of these strategies is essentially adding negative skewness to the overall portfolio returns. As Harvey and Siddique (2000, p. 1283) comment: "HML (value) and SMB (size) portfolios, to some extent, capture information similar to that captured by skewness". A direct consequence of the previous analysis is that the two approaches to asset pricing -intertemporal risk premia and higher moments risk premia- could be regarded as equivalent.

Examining further this link, stocks with high book-to-market value ratios and small size are dominated by cash-flows (earnings) risk, as argued in Campbell and Vuolteenaho (2004). An earnings' shock has an irreversible effect through time, while an interest rate shock has a partly reversible effect, because interest rates determine the rates of expected returns apart from discounting future cash flows. If investors are forward looking and price these intertemporal risks, a shock to the cash flows will have a higher impact on prices today, while an interest rate shock is expected to have a lower impact due to its reversibility through time. This is sufficient to argue that assets with high earnings risk are expected to incorporate a higher risk premium.

In terms of moments, this argument essentially implies that small size and value stocks are expected to have higher cokurtosis with the overall market returns, because a shock to earnings will have a larger impact on their prices, in comparison to big size and growth stocks, which are mainly characterized by interest rate risk. However, bearing in mind (see Conrad et al., 2002) that negative shocks (bad news) typically have a stronger impact than positive shocks (good news), this feature leads us to the conclusion that the returns of small size and value stocks are negatively coskewed with the market returns.

Providing further reasoning for the size anomaly, it is claimed that small size stocks' returns are expected to be more negatively coskewed in comparison to the big size stocks because they also have lower survival probability rates. Small size stocks are thought to be more vulnerable to negative economic and corporate shocks because they are mainly new companies with lower capital capacity and they have a higher probability of going bankrupt.

Regarding the ouperformance of the momentum strategy, there has not been suggested yet any successful intertemporal framework to provide an explanation of its outperformance. As Campbell and Vuolteenaho (2004, p. 1270) remark: "We are pessimistic about the two-beta model's ability to explain average returns on portfolios formed on past one-year stock returns". The explanation we put forward is due to the limited liability property of the assets and the concept of mean reversion. Within this framework, a poor performance in the previous year reduces the downside risk over the next period. On the other hand, past year's winners are expected to exhibit negatively coskewed returns due to their feature of long-term mean reversion.

1.4 Appendix

In order to find the moments of the stock returns, $\Delta s_{t+1} = s_{t+1} - s_t$, given the presence of the underlying stochastic factor X, we derive the joint conditional characteristic function of the processes (S_t, X_t).

The joint conditional characteristic function is given by:

$$\varphi(u_1, u_2; S_{t+\tau}, X_{t+\tau}|S_t, X_t) = E[\exp(iu_1 S_{t+\tau} + iu_2 X_{t+\tau})|S_t, X_t] \tag{1.36}$$

and satisfies the following Kolmogorov equation:

$$E[\frac{\partial \varphi}{\partial S}dS+\frac{\partial \varphi}{\partial X}dX+\frac{1}{2}\frac{\partial^2 \varphi}{\partial S^2}(dS)^2+\frac{1}{2}\frac{\partial^2 \varphi}{\partial X^2}(dX)^2+\frac{\partial^2 \varphi}{\partial X \partial S}(dS)(dX)+\frac{\partial \varphi}{\partial t}dt] = 0 \tag{1.37}$$

Due to the affine structure of the processes (see Duffie et al., 2000), we conjecture that the functional form of the characteristic function is given by:

$$\varphi(\mathbf{u}; S_{t+\tau}, X_{t+\tau}|S_t, X_t) = \exp[C(\tau;\mathbf{u}) + D_1(\tau;\mathbf{u})S_t + D_2(\tau;\mathbf{u})X_t] \tag{1.38}$$

where $\mathbf{u} = (u_1, u_2)$ along with the terminal conditions $C(0;\mathbf{u}) = 0$, $D_1(0;\mathbf{u}) = iu_1$, $D_2(0;\mathbf{u}) = iu_2$.

Replacing this trial form into the Kolmogorov equation and simplifying we get:

$$\begin{aligned} D_1\mu + D_2 k(\theta - X) + \frac{1}{2}(D_1)^2\sigma_s^2 X + \frac{1}{2}(D_2)^2\sigma_x^2 X + D_1 D_2 \sigma_s \sigma_x \rho_{sx} X &= \\ = \frac{\partial C}{\partial \tau} + \frac{\partial D_1}{\partial \tau}S + \frac{\partial D_2}{\partial \tau}X \end{aligned} \tag{1.39}$$

The method of undetermined coefficients yields the following system of ODEs:

$$\frac{\partial D_1}{\partial \tau} = 0 \Rightarrow D_1 = iu_1 \tag{1.40}$$

$$\frac{\partial C}{\partial \tau} = (D_1)\mu + (D_2)k\theta = iu_1\mu + (D_2)k\theta \tag{1.41}$$

$$\begin{aligned}\frac{\partial D_2}{\partial \tau} &= \frac{1}{2}(D_1)^2\sigma_s^2 + \frac{1}{2}(D_2)^2\sigma_x^2 - D_2 k + D_1 D_2 \sigma_s \sigma_x \rho_{sx} = \\ &= -\frac{1}{2}u_1^2\sigma_s^2 + \frac{1}{2}(D_2)^2\sigma_x^2 + D_2(iu_1\sigma_s\sigma_x\rho_{sx} - k)\end{aligned} \tag{1.42}$$

along with the corresponding terminal conditions.

The ODE with respect to $D_2(\tau)$ is of Riccati type and can be solved analytically. In particular, defining $\tilde{a} \equiv iu_1\sigma_s\sigma_x\rho_{sx} - k$, $\tilde{b} \equiv \frac{1}{2}\sigma_x^2$, $\tilde{c} \equiv -\frac{1}{2}u_1^2\sigma_s^2$, $\tilde{\psi} = \sqrt{\tilde{a}^2 - 4\tilde{b}\tilde{c}}$, the solution is given by:

$$D_2(\tau) = -\frac{2(\exp(\tilde{\psi}\tau) - 1)}{(\tilde{a} + \tilde{\psi})(\exp(\tilde{\psi}\tau) - 1) + 2\tilde{\psi}}\tilde{c} + iu_2\tau \tag{1.43}$$

for $\tilde{\psi} \geq 0$.

Consequently, $C(\tau)$ is given by:

$$C(\tau) = iu_1\mu\tau + iu_2k\theta\tau -$$
$$-2\tilde{c}k\theta[\frac{2\ln[\tilde{a}(\exp(\tilde{\psi}\tau) - 1) + \tilde{\psi}(\exp(\tilde{\psi}\tau) + 1)] - 2\ln(2\tilde{\psi}) - (\tilde{a} + \tilde{\psi})\tau]}{(\tilde{\psi} + \tilde{a})(\tilde{\psi} - \tilde{a})}]$$

Using Lemmas 2 and 3 of Jiang and Knight (2002, p. 201-202), given the joint conditional characteristic function, we can derive the unconditional characteristic function of Δs_{t+1} through:

$$\varphi(u_1, u_2; \Delta s_{t+1}) = \exp[C(1; u_1, 0)]\psi(D_2(1; u_1, 0); X_0) \tag{1.44}$$

where $\psi(D_2(1;u_1,0);X_0)$ is the characteristic function of X_0. This follows a Gamma distribution with mean θ and variance $\frac{\theta\sigma_x^2}{2k}$. Given the solutions to the ODEs and since X_t follows a Gamma distribution, the characteristic function of the stock returns is given by:

$$
\begin{aligned}
&\varphi(u_1,u_2;\Delta s_{t+1}) = \\
&= \exp\{iu_1\mu - 2\tilde{c}k\theta[\frac{2\ln[\tilde{a}(\exp(\tilde{\psi})-1)+\tilde{\psi}(\exp(\tilde{\psi})+1)] - 2\ln(2\tilde{\psi}) - (\tilde{a}+\tilde{\psi})}{(\tilde{\psi}+\tilde{a})(\tilde{\psi}-\tilde{a})}]\} \\
&(1+\frac{2(\exp(\tilde{\psi})-1)}{(\tilde{a}+\tilde{\psi})(\exp(\tilde{\psi})-1)+2\tilde{\psi}}\tilde{c}\frac{\sigma_x^2}{2k})^{-\frac{2k\theta}{\sigma_x^2}} \qquad (1.45)
\end{aligned}
$$

Using this characteristic function, the moments given in equations (1.33), (1.34) and (1.35) are derived.

Chapter 2

Dynamic Asset Allocation: The impact of modelling assumptions and market completion

2.1 Introduction

This chapter initially examines the intertemporal portfolio choice problem employing a series of model specifications for the risky asset's returns dynamics and the underlying risk factor. Using these different specifications, the impact of predictability and stochastic volatility on the optimal portfolio choice is examined, while discussing the necessary assumptions to derive explicit solutions. This is a partial equilibrium approach, where the agent under consideration derives utility from his terminal period wealth in an incomplete market setting. The optimal solutions are derived using the dynamic programming approach, firstly employed by Merton (1971, 1973). The qualitative results for the case of interim consumption are analogous to the terminal wealth case, since, as Wachter (2002) demonstrated, the main impact of interim consumption is to reduce the effective horizon of the terminal wealth investor.

The final specification of the risky asset's returns dynamics used in this paper is a contribution to the literature, since it allows for both returns' predictability and time-varying volatility through the underlying stochastic factor. Until now, these two issues have been separately examined (e.g. see Brennan et al., 1997 and Barberis, 2000, for the case of predictability and Chacko and Viceira, 2005, for the case of stochastic volatility). This model provides the most general framework, since it allows all three possible channels of transmission of a shock affecting the underlying risk factor to the risky asset's returns dynamics, i.e. through the correlation of the shocks, through the drift as well as through the diffusion term of the returns' process. The specification of these dynamics is assumed to be the same as the physical production process used in Cox, Ingersoll and Ross (1985a), drawing a link with the stochastic general equilibrium literature.

An innovative feature of the study is to investigate how the parameter values of the underlying risk factor dynamics affect the optimal solution. In particular, it is examined how the optimal portfolio choice is affected when the speed of mean-reversion and the diffusion coefficient of the underlying stochastic factor, as well as the correlation coefficient of the shocks are modified. In general, in the presence of fast mean-reversion in the risk factor process, the agent is less incentivized to hedge against a shock affecting it, since it is perceived as a transitory shock rather than a permanent one. Regarding the diffusion coefficient, as this increases, the hedging motive grows, since the magnitude of the shocks increases. The impact of the correlation coefficient is also important. If this coefficient is small in absolute value, the hedging ability of the risky asset is weak, hence the corresponding hedging demand is low.

This framework is further exploited in order to introduce a contingent claim on the sole underlying risk factor, to which the investor is assumed to be exposed. In other words, the market becomes now complete from the point of view of this investor. This contingent claim, a zero-coupon bond, is priced as in Cox, Ingersoll and Ross (1985b). Their pricing approach assumes that the market is dominated by logarithmic investors. Hence, the asset space is now expanded to include two risky assets. The aim of this exercise is two-fold: It firstly shows that even when the market is dominated by logarithmic investors, who behave myopically, a long-term investor who acts as a price-taker in this market, should still optimize intertemporally. As a result, his optimal portfolio choice should still incorporate an intertemporal hedging demand component. Secondly, it formally shows the intuitive result that, if an agent is sensitive to one risk factor, only the asset which has perfect correlation with this factor is used for intertemporal hedging, while the other asset could be still useful for its risk premium and the covariance structure of its returns.

Using this expanded asset space, the optimal portfolio choice is determined and the impact of the parameter values of the risk factor process is re-examined. The hedging demand for the zero-coupon bond, that perfectly hedges away the interest rate shocks, increases with the investment horizon. It can be large enough to dominate the portfolio choice of the investor, especially when the diffusion coefficient of the interest rate process is large, the speed of its mean-reversion is low and the correlation between the shocks to the equity and the interest rate process is low in absolute value.

The introduction of the bond provides also the opportunity to examine the impact of market completion on the welfare of the investor. In particular, we calculate the cost of not having access to this contingent claim for a long-term investor. This deprives him of the possibility to diversify his portfolio as well as to hedge intertemporally. The corresponding utility cost is quite high and grows with the investment horizon. To disentangle the diversification from the hedging effect, we also calculate the cost of acting myopically within the complete market setting. The results show that both effects are very significant, exhibiting the double role of this contingent claim. Consequently, the intertemporal dimension of the assets' returns should not be neglected by an investor who is not myopic. This argument highlights the limitation of the use of minimum variance portfolios for long-term investors. In particular, we show that their optimal portfolios would be erroneously regarded as largely inefficient by myopic observers, who take into account only one period-ahead expected returns and standard deviations.

With respect to the modelling assumptions, the main conclusion is that the parameter values of the risky asset's returns and the risk factor dynamics are crucial inputs for the behaviour of the multiperiod agent. Under different assumptions, a plethora of optimal solutions can be derived. This may prove to be a serious problem, if one attempts to practically implement dynamic asset allocation. Therefore, more attention should be paid in the stage of modelling the risky asset returns and the underlying risk factor dynamics.

The rest of this chapter is organized as follows: Section 2.2 presents the case of the simple correlation of shocks, while Section 2.3 examines the impact of returns' predictability and Section 2.4 examines the impact of stochastic volatility. Section 2.5 analyzes the optimal portfolio choices under predictability and stochastic volatility in a complete and an incomplete market setting, while Section 2.6 concludes.

2.2 Case I: Correlation of shocks

We consider a long-term, risk-averse investor who seeks to maximize his bequest function defined over his terminal wealth, $B[W(T)]$, allocating a portion ϕ of his current wealth to a risky asset with price S and the rest, $1-\phi$, to an instantaneously riskless asset with price P_0. In particular, his value function depends on the level of his wealth W, on a risk factor, which in this section is assumed to be the long-term interest rate r and on time t. The preferences of the investor over his wealth are represented by a power utility function, exhibiting Constant Relative Risk Aversion (CRRA). Hence, the value function of the investor can be essentially written as:

$$I(W(r,t),r,t) = \max_{\phi(t)} E_t[B[W(T),T]] = \frac{W^{1-\gamma}}{1-\gamma}J(r,t) \tag{2.1}$$

with terminal condition $J(r,T) = 1$. Furthermore, γ is the CRRA coefficient with $\gamma \neq 1$.

The first case presented in this section assumes that the underlying risk factor affects the risky asset returns only due to the correlation of the shocks in their processes. The dynamics of the risky asset's returns are given by the SDE:

$$\frac{dS}{S} = \mu dt + \sigma_s dB_s \tag{2.2}$$

where B_s is a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$ and the underlying risk factor, the long-term interest rate r, follows a diffusion process à la Vasicek (1977):

$$dr = \kappa(\theta - r)dt + \sigma_r dB_r \tag{2.3}$$

where B_r is another standard Brownian motion defined on the same probability space.

The innovations to the Brownian motions are allowed to be correlated, with their correlation coefficient given by ρ_{sr}. The model for the interest rate accommodates mean reversion but not time-varying volatility. This model specification satisfies the conditions of Liu (2007) to derive an analytical solution and it is similar in structure to the one examined in Kim and Omberg (1996) and Wachter (2002), with the difference that in their setup the stochastic factor was the market price of risk.

The return of the instantaneously riskless asset is given by the ODE:

$$\frac{dP_0}{P_0} = r^f dt \tag{2.4}$$

The instananeously riskless rate, r^f, is allowed in this section to be distinctly different from the long-term interest rate, r, which serves as a stochastic factor in this case.

Given the description of the problem, the wealth of the agent evolves through time according to the following SDE:

$$dW = \phi W\frac{dS}{S} + (1-\phi)W\frac{dP_0}{P_0} = W[\phi(\mu - r^f) + r^f]dt + \phi W\sigma_s dB_s \tag{2.5}$$

Using the Bellman principle of optimality, presented in Chapter 1, we have the following Proposition for the optimal portfolio choice:

Proposition 2.1

The optimal portfolio choice of the long-term, risk-averse investor who has the value function of equation (2.1), faces the budget constraint of equation (2.5) and the risk factor dynamics given in equation (2.3) is:

$$\phi^* = \frac{1}{\gamma}\frac{(\mu - r^f)}{\sigma_s^2} \tag{2.6}$$

(See Appendix for the proof).

The optimal portfolio choice expression in equation (2.6) contains only the standard Markowitzian component. This first trivial case formalizes the intuitive result, that when the underlying stochastic factor does not influence the investment opportunity set (neither through the drift and the volatility of the risky asset dynamics nor through the risk-free rate), the long-term investor behaves myopically. This result holds because the investor facing a constant equity premium, $\mu - r^f$, and market price of risk, $SR = \frac{\mu - r^f}{\sigma_s}$, does not have any incentive to hedge against the shocks to the underlying risk factor.

2.3 Case II: The impact of predictability

A fundamental issue in financial theory is whether returns are predictable using publicly available information. Despite the extensive research on the topic, the debate is still open. A series of papers in the late 1980s (see *inter alia* Fama and French, 1988 and Campbell and Shiller, 1988), using standard least squares regressions, cast doubt on the assumption that asset prices follow a random walk. In particular, they find that a series of financial variables have significant predictive ability. Fama (1991) interpreted these results as evidence of time-varying risk premia, rather than evidence against market efficiency. In an asset allocation framework, Kandel and Stambaugh (1996) argued that these predictive regressions, though having very low explanatory power, can be of significant economic importance.

This section investigates the impact of predictability on the optimal portfolio choice of a multiperiod investor. The dynamics of the risky asset returns are assumed to be given by the following SDE:

$$\frac{dS}{S} = (\alpha_0 + \alpha_1 r)dt + \sigma_s dB_s \tag{2.7}$$

This specification implies that the long-term interest rate, r, predicts future returns in a linear way; returns are also assumed to exhibit constant volatility. Even though these assumptions may seem quite restrictive, they are similar to most of the studies examining the impact of predictability on dynamic asset allocation. A similar setup has been employed, *inter alia*, in Brennan et al. (1997), Campbell and Viceira (1999) and Barberis (2000).

The stochastic factor, the long-term interest rate r, is assumed to follow the SDE:

$$dr = \kappa(\theta - r)dt + \sigma_r dB_r \tag{2.8}$$

The SDE for the long-term rate implies that it follows a mean-reverting process. This specification is in line with most of the studies applying standard regression analysis to examine the predictability of asset returns through stationary regressors. Nevertheless, the existence of a unit root in the interest rate process is still an open question. It is important to note that the inference for predictability is significantly modified if the interest rate process has a local-to-unity root (see for example Torous et al., 2004). Furthermore, the assumption of constant volatility in the interest rate process is required in order to derive an analytical solution to the optimal portfolio choice problem. The risk factor is allowed to be different from the risk free rate r^f. Hence, the return of the instantaneously riskless asset is assumed to follow the ODE:

$$\frac{dP_0}{P_0} = r^f dt \tag{2.9}$$

Under these assumptions, the risk premium, $\mu - r^f = \alpha_0 + \alpha_1 r - r^f$, and the market price of risk, $SR = \frac{\mu - r^f}{\sigma_s} = \frac{\alpha_0 + \alpha_1 r - r^f}{\sigma_s}$, become time-varying and stochastic through the underlying risk factor r.

The wealth dynamics of the agent are given by:

$$dW = \phi W \frac{dS}{S} + (1-\phi) W \frac{dP_0}{P_0} = W[\phi(\alpha_0 + \alpha_1 r - r^f) + r^f]dt + \phi W \sigma_s dB_s \quad (2.10)$$

Proposition 2.2

The optimal portfolio choice of the long-term, risk-averse investor who has the value function of equation (2.1), faces the budget constraint of equation (2.10) and the risk factor dynamics given in equation (2.8) is

$$\phi^* = \frac{1}{\gamma} \frac{(\alpha_0 + \alpha_1 r - r^f)}{\sigma_s^2} + [d(t) + e(t)r] \frac{\sigma_r}{\sigma_s} \rho_{sr} \quad (2.11)$$

where $d(t)$ and $e(t)$ are the solutions to the of ODEs (2.54) and (2.55) in the Appendix (See Appendix for the proof).

The first term of the optimal portfolio choice in (2.11) represents the static mean-variance component *à la* Markowitz (1952). It depends on the coefficient of relative risk aversion as well as on the risk premium and the variance of the risky asset. The second term represents the intertemporal hedging demand. It depends on the sensitivity of the investor's value function to the interest rate movements, captured by equations $d(t)$ and $e(t)$, as well as on the magnitude of the diffusion coefficients of the risky asset's returns (σ_s) and the interest rate (σ_r) SDEs. It is straightforward to observe that if the long-term rate is not stochastic, i.e. $\sigma_r = 0$, or if the shocks to the interest rate process do not affect the risky asset dynamics, i.e. $\rho_{sr} = 0$, then the intertemporal hedging demand is zero and the optimal portfolio choice collapses to its myopic component. Furthermore, inspecting the ODEs (2.54) and (2.55), in the limit case of $\gamma \rightarrow 1$, which would be equivalent to the case of a logarithmic investor, then $d(t) = e(t) \rightarrow 0$, so the hedging demand would disappear again.

This optimal portfolio choice also demonstrates that, within this framework, the multiperiod agent should be engaged in market timing. In other words, both his myopic and hedging demand components depend on the level of the stochastic factor, apart from his preferences and his time horizon. Hence, it is evident that the existence of predictability crucially modifies the optimal portfolio choice of the long-term investor, as compared to the case of constant investment opportunities. This finding underlines the argument that the portfolio policy of a multiperiod investor can be both quantitatively and qualitatively modified due to different modelling assumptions.

2.4 Case III: The impact of stochastic volatility

This section presents a model, according to which the risky asset returns exhibit stochastic volatility, while the drift of the process is constant. This framework is quite similar to the one employed in Chacko and Viceira (2005). The main difference is that instead of using a latent factor, which is the volatility itself in their framework, we use an observable variable, the dividend y, to induce stochastic, time-varying volatility. In particular, the risky asset returns' dynamics are described by the following SDE:

$$\frac{dS}{S} = \mu dt + \sigma_s \sqrt{\frac{1}{y}} dB_s \tag{2.12}$$

The dividend enters the volatility of the risky asset in an inverse way. Hence, the instantaneous volatility of the risk asset's returns implied by this model is given by $Var(\frac{dS}{S}) = \sigma_s^2 \frac{1}{y}$. In other words, as the risky asset pays a higher dividend, its returns exhibit lower volatility (see Pastor and Veronesi, 2003, for empirical evidence). With respect to the drift of the process, this specification essentially assumes that the expected return is constant. Chacko and Viceira (2005) use as a stochastic factor the precision of asset returns, which is defined as the inverse of the volatility. In our setup, the dividend is assumed to follow the square root process:

$$dy = \kappa(\theta - y)dt + \sigma_y\sqrt{y}dB_y \tag{2.13}$$

where B_y is a standard Brownian motion defined on the same probability space $(\Omega, \mathcal{F}, P)$ and the Brownian motions B_s and B_y are allowed to be correlated with a correlation coefficient given by ρ_{sy}. Apart from mean-reversion, this specification also implies that the volatility of the dividend increases as its level increases too.

The return of the instantaneously riskless asset is given by the ODE:

$$\frac{dP_0}{P_0} = r^f dt \tag{2.14}$$

Under these assumptions, the risk premium, $(\mu - r^f)$, is constant, but the market price of risk, $SR = \frac{(\mu - r^f)\sqrt{y}}{\sigma_s}$, is stochastically time-varying.

Given this setup, it is assumed that the value function of the investor now depends on the level of the dividend as well as on the level of his wealth and his time horizon. Hence, the value function of the investor can be written as:

$$I(W(t), y, t) = \max_{\phi(t)} E_t[B[W(T), T]] = \frac{W^{1-\gamma}}{1-\gamma}J(y,t) \tag{2.15}$$

with terminal condition $J(y,T) = 1$.

Following the same steps as in the previous sections, the dynamics of the agent's wealth process are described by the following SDE:

$$dW = \phi W\frac{dS}{S} + (1-\phi)W\frac{dP_0}{P_0} = W[\phi(\mu - r^f) + r^f]dt + \phi W\sigma_s\sqrt{\frac{1}{y}}dB_s \tag{2.16}$$

Proposition 2.3

The optimal portfolio choice of the long-term, risk-averse investor, who has the value function of equation (2.15), faces the budget constraint of equation (2.16) and the risk factor dynamics given in equation (2.13) is

$$\phi^{*}=\frac{1}{\gamma}\frac{(\mu-r^{f})y}{\sigma_{s}^{2}}+d(t)\frac{\sigma_{y}}{\sigma_{s}}\rho_{sy}y \tag{2.17}$$

with $d(t)=-\frac{2(\exp(\psi(T-t))-1)}{(a+\psi)(\exp(\psi(T-t))-1)+2\psi}\hat{c}$, where $a=(\kappa-\frac{1-\gamma}{\gamma}(\mu-r^{f})\frac{\sigma_{r}}{\sigma_{s}}\rho_{sr})$, $\hat{c}=-\frac{1-\gamma}{2\gamma^{2}}\frac{(\mu-r^{f})^{2}}{\sigma_{s}^{2}}$, $b=-\frac{\gamma}{2}\sigma_{r}^{2}-\frac{1-\gamma}{2}\sigma_{r}^{2}\rho_{sr}^{2}$ and $\psi=\sqrt{a^{2}-4b\hat{c}}$ (See Appendix for the proof).

As in the previous subsection, the optimal portfolio choice implies that the multiperiod agent should be engaged in market timing. In particular, both his myopic and his intertemporal hedging demand components depend on the level of the stochastic factor in a linear way. With respect to the myopic demand component, this is true because the level of the stochastic factor determines the volatility of the risky asset. Regarding the hedging demand component, this effect occurs because the long-term investor seeks to hedge against the shocks to the dividend that would generate higher volatility and, consequently, deterioration of his investment opportunities in the subsequent periods.

The optimal portfolio choice in equation (2.17) becomes identical to the static case, if the shocks to the dividend process are not correlated with the shocks to the risky asset returns ($\rho_{sy}=0$) as well as when the dividend is not stochastic ($\sigma_{y}=0$). In the case of logarithmic preferences, which is equivalent to the limit $\gamma\rightarrow 1$, we would get $d(t)=0$, because $\hat{c}=0$, and the portfolio choice of the multiperiod investor would again be the same as in the static case.

2.5 Case IV: Predictability and stochastic volatility

This section presents a framework in which there are two risky assets with prices S and η correspondingly, and a single underlying state variable, the risk-free rate r. In particular, the dynamics of these processes are correspondingly governed by the following SDEs:

$$\frac{dS}{S} = \mu r dt + \sigma_s \sqrt{r} dB_s \tag{2.18}$$

and

$$\frac{d\eta}{\eta} = \alpha r dt + \omega \sqrt{r} dB_r \tag{2.19}$$

where, as before, B_s and B_r are standard Brownian motions defined on the probability space $(\Omega, \mathcal{F}, P)$. These shocks are not perfectly correlated, i.e. $\rho_{sr} \in (-1, 1)$.

Given this specification, the returns of these processes are predictable through the level of the stochastic factor. Moreover, they exhibit time-varying volatility that depends on the level of the interest rate in a linear way. These models are of the same form as in Cox, Ingersoll and Ross (1985a) and they essentially imply stochastic constant returns to scale, since the distribution of the rate of return on an investment in each of the processes is independent of the scale of the investment. The evolution of the underlying stochastic factor, which in this case is the risk-free rate, denoted by r, is determined by a mean-reverting, square root process:

$$dr = \kappa(\theta - r)dt + \sigma_r \sqrt{r} dB_r \tag{2.20}$$

In this section we examine the optimal portfolio choice of a multiperiod investor who has access only to the first process with price S, but not to the second one. Consequently, in principle, this agent cannot perfectly hedge the risk arising due to the innovations of the Brownian motion B_r. However, he may have the possibility of investing in a contingent claim that is priced on this source of risk, if this becomes available. This contingent claim would eventually allow him to hedge away the interest rate risk. The following two subsections compare the optimal portfolio choice of this agent in two cases:

i) The case where this contingent claim is not available versus

ii) The case where this claim becomes available.

It is important to stress that the key assumption made here is that the asset with value S is priced on a source of risk, captured by the innovations dB_s that does not directly affect the utility of the long-term investor under consideration. His utility, represented by the value function (2.1), is assumed to be affected solely by the risk-free rate r, apart from the level of his terminal wealth and the time horizon of his investment.

2.5.1 Incomplete markets

In the first case, the asset space of the long-term investor contains only the risky asset with price S, which we henceforth call *equity*, and the returns' dynamics of which are given by equation (2.18), as well as an instantaneously riskless asset with price P_0 that obeys the ODE:[1]

$$\frac{dP_0}{P_0} = rdt \tag{2.21}$$

In this case, the risk-premium is given by $(\mu - 1)r$ and the market price of risk is

[1] It should be noted that in order to derive an analytical solution for the optimal portfolio choice in this case, we need to assume that the stochastic factor is the same as the instantaneously risk-free rate. A similar assumption is made in Cox et al. (1985a).

given by $SR = \frac{(\mu-1)\sqrt{r}}{\sigma_s}$. Hence, they are both stochastically time-varying. Denoting by ϕ_e the portion of the wealth invested in the equity, the agent's wealth dynamics are given by:

$$dW = \phi_e W\frac{dS}{S} + (1-\phi_e)W\frac{dP_0}{P_0} = W[\phi_e(\mu-1)r + r]dt + \phi_e W\sigma_s\sqrt{r}dB_s \quad (2.22)$$

Proposition 2.4

The optimal portfolio choice of the long-term, risk-averse investor who has the value function of equation (2.1), faces the budget constraint given in equation (2.22) and the risk factor dynamics given in equation (2.20) is:

$$\phi_e^* = \frac{1}{\gamma}\frac{(\mu-1)}{\sigma_s^2} + d(t)\frac{\sigma_r}{\sigma_s}\rho_{sr} \quad (2.23)$$

with $d(t) = -\frac{2(\exp(\psi(T-t))-1)}{(a+\psi)(\exp(\psi(T-t))-1)+2\psi}\hat{c}$, where $\hat{c} = -(\frac{1-\gamma}{2\gamma^2}\frac{(\mu-1)^2}{\sigma_s^2}+\frac{1-\gamma}{\gamma})$, $b = -\frac{1-\gamma}{2}\sigma_r^2\rho_{sr}^2 - \frac{\gamma}{2}\sigma_r^2$, $a = k$ and $\psi = \sqrt{a^2-4b\hat{c}}$ (See Appendix for the proof).

The optimal portfolio choice expression is composed of two terms. The first term represents the myopic mean-variance component *à la* Markowitz. It depends on the coefficient of relative risk aversion as well as on the risk premium and the variance of the risky asset's returns. The second term represents the intertemporal hedging demand. It depends on the sensitivity of the investor's value function to the interest rate movements as well as on the diffusion coefficients of the risky asset and the interest rate dynamics.

The long-term investor attempts to hedge potential shocks to the interest rate through the second component of his asset demand. Since the correlation between the risky asset and interest rate shocks is not perfect, the investor cannot completely hedge away these shocks. Trivially, if the interest rate is not stochastic, i.e. $\sigma_r = 0$, the optimal portfolio choice is identical to the one in the myopic case. An interesting feature of this model specification is that the investor is not engaged in market timing, neither through his myopic demand nor through his hedging demand.

Optimal portfolio choice, risk aversion and investment horizon

In the present and the following subsections, we examine the optimal portfolio choice of the previously described long-term investor. In particular, we examine how this choice is influenced by the level of his risk aversion and his investment horizon. Employing initially the following set of parameter values: $\{\mu = 1.6,\ \sigma_r = 0.034,\ \sigma_s = 0.9,\ \kappa = 0.1,\ \rho_{sr} = -0.6,\ \theta = 0.058$ and $r = 0.05\}$, it is subsequently examined how the modification of these values affects the behaviour of the investor.

It should be noted that modifying the benchmark parameter values would yield different optimal portfolio choices. This is a standard property of the mean-variance portfolio choice analysis and it carries through in the case of dynamic asset allocation. The various cases we examine for the parameter values show that both the hedging and the myopic demand components are very sensitive to this choice. In general, the expected returns of both the equity and the bond depend on the level of the risk-free rate r. Additionally, the myopic equity demand increases with the value of the parameter μ, while the myopic bond demand increases as the parameter λ increases in absolute value.

Figure 2.1 depicts the portion of wealth invested in the risky asset for three different levels of relative risk aversion, as the investment horizon increases up to 15 years. As the horizon increases, the hedging demand for the risky asset, measured by the difference between the total and the myopic demand, increases regardless of the level of risk aversion. However, the more risk averse the investor is, the greater the hedging motive becomes. In particular, for $\gamma = 4$, the hedging demand component is of the same magnitude as the myopic one if the investment horizon exceeds 10 years, while for $\gamma = 10$ the hedging demand dominates the total demand even for short investment horizons. Moreover, the hedging demand grows fast in the first years, but levels off from a specific horizon onwards. The reason is that the investor has fully exploited by that period all possible hedging opportunities provided by the risky asset.

The impact of the correlation coefficient

A very important determinant of the hedging demand is the extent to which the shocks of the risky asset's returns are correlated with the shocks of the interest rate process. The case of no correlation is trivial, resulting in no hedging demand. The Left panel of Figure 2.2 shows the hedging demand as the correlation between the shocks increases in absolute value for an investor with 15 years of investment horizon and various levels of relative risk aversion. As it is obvious, the greater the correlation is, the higher the hedging demand becomes. The reason is that as the magnitude of the correlation grows in absolute value, the risky asset can be more effectively used to hedge away these shocks. In other words, this risky asset becomes more important as a hedging device against the shocks to the interest rate. In the limit case of perfect negative correlation, the investor can perfectly hedge away these shocks.

It is worth noting that the importance of the risky asset as a hedging device is greater as risk aversion increases. The Right panel of Figure 2.2 demonstrates this point. The investor with $\gamma = 2$ uses the risky asset mainly for investment purposes and his hedging demand is not higher than the half of his myopic demand (ratio <0.5), even in the case of perfect negative correlation. On the other hand, a more risk averse investor, with $\gamma = 10$, will use the risky asset mainly for hedging rather than investment purposes. In particular, his hedging demand will exceed the myopic one even for moderate values of the correlation coefficient ($|\rho_{sr}| > 0.3$).

The impact of the risk factor diffusion coefficient

Figure 2.3 demonstrates how important the diffusion coefficient of the underlying risk factor process is for the optimal hedging demand. Intuitively, the more volatile the interest rate is, the greater the incentive is for a risk averse investor to hedge these fluctuations. The reason is that the same shock is magnified as this coefficient increases and affects more significantly the investor's utility. The Left panel of Figure 2.3 verifies this intuitive argument. As this diffusion coefficient increases, the hedging demand increases, regardless of the level of relative risk aversion.

The Right panel of Figure 2.3 shows that it is the more risk averse investor who will mainly be affected by modifications in the values of the diffusion coefficient. The explanation for this finding is that a highly risk averse investor is the most sensitive one to interest rate shocks that generate fluctuations in his wealth path. As these fluctuations become wider due to the increase in the magnitude of the diffusion coefficient, the more risk-averse investor has a greater incentive to hedge them away.

It is methodologically more appropriate to examine the relative importance of the hedging demand through its ratio with respect to the myopic demand, as in the Right panel of Figure 2.3, instead of simply using its absolute level. The reason is that by using the absolute level, a more risk averse investor may appear to have a lower hedging demand in comparison to a less risk averse investor (see the Left panel of Figure 2.3), because he altogether reduces his exposure to the risky asset, shifting his wealth to the riskless asset and not because he has a minor hedging motive.

The impact of the speed of mean-reversion coefficient

An unexplored issue in the dynamic asset allocation literature is how the speed of mean-reversion of the underlying stochastic factor affects the investor's hedging demand. The Left panel of Figure 2.4 shows the level of the hedging demand for an investment horizon of 15 years and various levels of risk aversion, as the coefficient determining the speed of mean-reversion (κ) of the interest rate process increases. It is evident that the hedging demand decreases as this coefficient increases. As mentioned above, a more clear picture for the impact on the hedging incentives is provided through the ratios of the hedging to myopic demand. As the Right panel of Figure 2.4 demonstrates, the hedging incentive is extremely high when the mean-reversion is extremely slow (a commonly reported characteristic of the interest rate process), especially for highly risk averse investors. As the speed of mean-reversion increases, the hedging motive fades out and almost disappears for moderately risk averse investors ($\gamma = 2$ or $\gamma = 4$).[2]

Though relatively neglected in the literature, this is an important finding that has a sound economic reasoning. When the speed of mean-reversion is slow, then a shock to the interest rate process is persistent and affects the investment opportunity set for a quite long period. In other words, such a persistent shock generates significant fluctuations in the wealth path of the long-term investor, who is highly incentivized to hedge them away. On the other hand, as the speed of mean-reversion increases, this shock is regarded as a transitory one, since it is expected to be absorbed by the process quite quickly. As a result, when the process quickly reverts to its long-run mean, the investor is not significantly affected by such transitory shocks and he is not motivated to hedge them away.

[2] A similar argument has been made by Brennan and Xia (2002) for the speed of mean-reversion of the real interest rate process in their model. The different speed of the mean-reversion in their setup arises due to the different data frequency used for the estimation of the model parameters.

2.5.2 Completing the market

In this subsection an extra asset is introduced, expanding the asset space. This extra asset is a contingent claim written on the non-available risky asset with price η. In order to clarify the terminology, we call this a "complete market case" from the point of view of the investor under examination. This definition implies that there exists an asset, the returns of which are perfectly correlated with the underlying risk factor r, that affects the utility of this long-term investor. The aim of this section is two-fold: We firstly show that even if the market is dominated by logarithmic investors, who behave myopically, a long-term investor, who acts as a price-taker, should still optimize intertemporally. As a result, his optimal portfolio choice should still incorporate an intertemporal hedging demand component. Secondly, we formally show the intuitive result that in a setup with one underlying risk factor, only the asset the returns of which are perfectly correlated with this factor is employed for intertemporal hedging.

The dynamics of the non-available risky asset are given by equation (2.19) and the stochastic factor, as in the previous subsection, is assumed to follow the mean-reverting, square-root process of equation (2.20). We determine now the price of a zero-coupon bond, P^b, which is a contingent claim on the non-available asset. In order to price this contingent claim, we assume that we are in an economy *à la* CIR (1985a), populated by a representative agent, who has a value function $Z(W, r, \tilde{t})$.[3]

The fundamental differential equation that P^b should follow in equilibrium, as adapted by equation 6, p. 388, CIR(1985b), is given by:

[3] Note that $\tilde{t} = \tilde{T} - \tilde{\tau}$ stands for the time horizon of the contingent claim and it is distinct from t, the investment horizon of the power utility agent with value function $I(W, r, t)$.

$$\frac{1}{2}\text{var}(W)\frac{\partial^2 P^b}{\partial W^2}+Cov(W,r)\frac{\partial^2 P^b}{\partial W\partial r}+\frac{1}{2}\sigma_r^2 r\frac{\partial^2 P^b}{\partial r^2}+[rW-C^*]\frac{\partial P^b}{\partial W}+$$
$$\frac{\partial P^b}{\partial r}[\kappa(\theta-r)+\frac{\frac{\partial^2 Z}{\partial W^2}}{\frac{\partial Z}{\partial W}}Cov(W,r)+\frac{\frac{\partial^2 Z}{\partial W\partial r}}{\frac{\partial Z}{\partial W}}\sigma_r^2 r]+\frac{\partial P^b}{\partial \tilde{t}}-rP^b=0 \tag{2.24}$$

where W is the representative investor's wealth and C^* is his optimal consumption level. The equilibrium condition for market-clearing is that the representative agent invests all his wealth in the production process, i.e. $\pi^*=1$, where π^* is his optimal portfolio choice (see the definition in CIR, 1985a, p. 371). Therefore, $Cov(W,r)=\pi^*\omega\sigma_r Wr=\omega\sigma_r Wr$. In particular, the representative agent is assumed to have logarithmic preferences. This implies that $\frac{\partial^2 Z}{\partial W\partial r}=0$ and $\frac{\frac{\partial^2 Z}{\partial W^2}}{\frac{\partial Z}{\partial W}}W=-1$. Since the zero-coupon bond we are pricing is a security, the contractual terms of which do not to depend explicitly on the agent's wealth, then $\frac{\partial P^b}{\partial W}=\frac{\partial^2 P^b}{\partial W^2}=\frac{\partial^2 P^b}{\partial W\partial r}=0$. Consequently, the price of the zero-coupon bond follows the PDE:

$$\frac{1}{2}\sigma_r^2 r\frac{\partial^2 P^b}{\partial r^2}+\frac{\partial P^b}{\partial r}[\kappa(\theta-r)-\lambda r]+\frac{\partial P^b}{\partial \tilde{t}}-rP^b=0 \tag{2.25}$$

with the terminal condition $P^b(r,\tilde{T})=1$, where we have defined $\omega\sigma_r\equiv\lambda$.

Using the assumption that the zero-coupon bond price is an exponential-affine function of the interest rate, its price function can be written as:

$$P^b(r,\tilde{t})=\exp(A(\tilde{t})+B(\tilde{t})r) \tag{2.26}$$

with terminal conditions $A(\tilde{T})=B(\tilde{T})=0$.

Replacing this trial form into the previous PDE we get:

$$\frac{1}{2}B^2(\tilde{t})\sigma_r^2 r+B(\tilde{t})\kappa(\theta-r)+\dot{A}(\tilde{t})+\dot{B}(\tilde{t})r=r+B(\tilde{t})\lambda r \tag{2.27}$$

where $\dot{A}(\tilde{t})$ and $\dot{B}(\tilde{t})$ stand for the corresponding derivatives with respect to time $\tilde{t}$.

Since this expression is affine in r, the following equations are sufficient to hold:

$$\dot{B}(\tilde{t}) = 1 + B(\tilde{t})(\kappa + \lambda) - \frac{1}{2}\sigma_r^2 B^2(\tilde{t}) \tag{2.28}$$

and

$$\dot{A}(\tilde{t}) = -B(\tilde{t})\kappa\theta \tag{2.29}$$

Solving the system of these two ODEs, along with the corresponding terminal conditions, we have:

$B(\tilde{t}) = -\frac{2(\exp(\widetilde{\psi}\tilde{\tau})-1)}{(\tilde{a}+\widetilde{\psi})(\exp(\widetilde{\psi}\tilde{\tau})-1)+2\widetilde{\psi}}\widetilde{c}$ and $A(\tilde{t}) = \frac{2\kappa\theta}{\sigma_r^2}\ln[\frac{2\widetilde{\psi}\exp(\frac{(\tilde{a}+\widetilde{\psi})\tilde{\tau}}{2})}{(\tilde{a}+\widetilde{\psi})(\exp(\widetilde{\psi}\tilde{\tau})-1)+2\widetilde{\psi}}]$, where $\widetilde{b} = -\frac{1}{2}\sigma_r^2$, $\widetilde{a} = \kappa + \lambda$, $\widetilde{c} = 1$, $\widetilde{\psi} = \sqrt{\widetilde{a}^2 - 4\widetilde{b}\widetilde{c}}$. Replacing these functions in equation (2.26), an explicit formula for the zero-coupon bond is derived.

Employing the trial solution (2.26) and the relationship (2.27), the dynamics of the zero-coupon bond returns, $\frac{dP^b}{P^b}$, in a market dominated by logarithmic investors are given by the following SDE:

$$\frac{dP^b}{P^b} = (r + B(\tilde{t})\lambda r)dt + B(\tilde{t})\sigma_r\sqrt{r}dB_r \tag{2.30}$$

Having priced the zero-coupon bond, we now examine the optimal portfolio choice of a risk-averse, long-term investor with a value function as in equation (2.1), who is a price-taker and cannot affect the price formation of this contingent claim. In a vector form, the dynamics of the two available risky assets are given by:

$$\frac{dP}{P} = \mu^P dt + \Sigma dB \tag{2.31}$$

where $\mu^P = \binom{\mu r}{r+B(\tilde{t})\lambda r}$, $\Sigma = \begin{bmatrix} \sigma_s\sqrt{r} & 0 \\ 0 & B(\tilde{t})\sigma_r\sqrt{r} \end{bmatrix}$ and $dB = (dB_s, dB_r)^T$. Correspondingly, the risk factor dynamics can be written as

$$dr = \mu^r dt + \Sigma^r dB_r \tag{2.32}$$

where $\mu^r = \kappa(\theta - r)$ and $\Sigma^r = \sigma_r\sqrt{r}$.

Allocating a portion $\phi = (\phi_e, \phi_b)^T$ of his wealth to the risky assets, *equity* (e) and *bond* (b), and a portion $(1 - i^T\phi)$ to the instantaneously riskless asset, the wealth dynamics of the long-term investor are given by:[4]

$$dW = W(\phi^T \frac{dP}{P} + (1 - i^T\phi)\frac{dP_0}{P_0}) = W(\phi^T(\mu^P - ir) + r)dt + W\phi^T\Sigma dB \tag{2.33}$$

Proposition 2.5

The optimal portfolio choice of the long-term, risk-averse investor who has the value function of equation (2.1), faces the budget constraint given in equation (2.33) and the risk factor dynamics given in equation (2.32) is:

$$\phi^* = \frac{1}{\gamma}(\Sigma\Sigma^T)^{-1}(\mu^P - ir) + d(t)(\Sigma\Sigma^T)^{-1}\Sigma\rho^T\Sigma^{rT} \tag{2.34}$$

where

$$d(t) = -\frac{2(\exp(\psi(T-t)) - 1)}{(a+\psi)(\exp(\psi(T-t)) - 1) + 2\psi}\hat{c} \tag{2.35}$$

with $\hat{c} = \frac{\gamma-1}{2\gamma^2}[\frac{(\mu-1)^2}{\sigma_s^2(1-\rho_{sr}^2)} - \frac{2(\mu-1)\rho_{sr}\lambda}{\sigma_s\sigma_r(1-\rho_{sr}^2)} + \frac{\lambda^2}{\sigma_r^2(1-\rho_{sr}^2)}] + \frac{\gamma-1}{2}(\sigma_r^2\rho_{sr}^2) + \frac{\gamma-1}{\gamma}$, $a = [(1-\gamma)(\frac{\rho_{sr}\sigma_r(\mu-1)}{\sigma_s} + \lambda) + \kappa]$, $b = -\frac{1-\gamma}{2}\sigma_r^2$ and $\psi = \sqrt{a^2 - 4b\hat{c}}$ (See Appendix for the proof).

[4]Note that i is a 2x1 vector of ones.

Corollary 2.1

The intertemporal hedging demand of the long-term, risk-averse investor contains only the zero-coupon bond, the shocks of which are perfectly correlated with the shocks affecting the underlying risk factor. The equity is still useful for its risk-premium and the covariance structure of its returns, but does not have any hedging value.

Proof

From the optimal portfolio choice of equation (2.34), writing the intertemporal hedging demand term in an explicit form, we get:

$$d(t)(\Sigma\Sigma^T)^{-1}\Sigma\rho^T\Sigma^{rT} = \begin{pmatrix} 0 \\ d(t)/B(\tilde{t}) \end{pmatrix} \tag{2.36}$$

Moreover, the total demand in explicit form is given by:

$$\phi^* = \begin{pmatrix} \phi_e \\ \phi_b \end{pmatrix} = \begin{bmatrix} \frac{1}{\gamma}\left[\frac{(\mu-1)}{\sigma_s^2(1-\rho_{sr}^2)} - \frac{\rho_{sr}\lambda}{\sigma_s\sigma_r(1-\rho_{sr}^2)}\right] \\ \frac{1}{\gamma}\left[\frac{\lambda}{\sigma_r^2(1-\rho_{sr}^2)B(\tilde{t})} - \frac{(\mu-1)\rho_{sr}}{\sigma_s\sigma_r(1-\rho_{sr}^2)B(\tilde{t})}\right] + \frac{d(t)}{B(\tilde{t})} \end{bmatrix} \tag{2.37}$$

In the special case that $\rho_{sr} = 0$, we have a full separation of these two demands and the optimal portfolio choice for the equity collapses to the standard one-risky-asset myopic portfolio choice, $\phi_e = \frac{1}{\gamma}\frac{(\mu-1)}{\sigma_s^2}$. In the case we had $\rho_{sr} = \pm 1$, the optimal portfolio choice for the equity would not be determined, as one would expect, since the covariance matrix of the asset returns would not be invertible. This is true because we would then have two risky assets written on the same unique risk factor, making one of these assets redundant.

Optimal portfolio choice and investment horizon

Having solved analytically the portfolio choice problem of the long-term investor in the complete market case, we use specific parameter values to examine its various aspects. In particular, we initially employ the following set of parameter values: $\{\mu = 1.6$, $\sigma_r = 0.034$, $\sigma_s = 0.9$, $\kappa = 0.1$, $\rho_{sr} = -0.6$, $\theta = 0.058$, $\lambda = -0.01$, $r = 0.05$ and $\tilde{\tau} = 20$ years$\}$. This subsection examines how the portfolio choice is affected by the investment horizon, while the following subsections investigate the impact of the magnitude of the correlation coefficient among the shocks (ρ_{sr}), the diffusion coefficient of the interest rate process (σ_r) as well as the coefficient of the speed of mean-reversion (κ).

The Left panel of Figure 2.5 exhibits the equity and bond demand for an investor with $\gamma = 4$, as his investment horizon increases. A myopic investor sells short the bond and has a positive demand for the equity. This holds due to the much lower risk premium the bond is assumed to offer relative to the equity. As equation (2.37) shows, the bond has some diversification merit even for the myopic investor, since the correlation coefficient of the shocks affecting these two assets is negative. The long-term investor, though, finds some extra value in the bond due to its ability to perfectly hedge away the interest rate shocks.

Consequently, as the investment horizon increases, the total demand for the bond increases, since the motive for hedging becomes more significant. As the Right panel of Figure 2.5 shows, the hedging demand dominates the total bond demand as the horizon increases. Moreover, the risk-averse investor under examination may have a total bond demand almost equal to the equity demand, if he has a quite long horizon. This finding shows how an asset with relatively low risk premium may play a significant role in the portfolio of a long-term investor. In particular, its hedging value can be so high as to alter the sign of its demand.

The impact of the correlation coefficient

The hedging role of the bond depends on the correlation of the shocks affecting the equity returns' process with the shocks affecting the interest rate process. The Left panel of Figure 2.6 shows that, as the correlation coefficient decreases in absolute value, the total bond demand significantly increases in comparison to the equity demand. The absolute value of this ratio increases as the investor becomes more risk averse. The Right panel of Figure 2.6 shows that the increase in the bond demand is due to a significant increase in the hedging demand. In particular, for a low magnitude of correlation (in absolute value), the hedging demand dominates the myopic demand. However, if these shocks are highly correlated, the hedging demand for the bond is close to zero, regardless of the level of risk aversion.

To interpret these results, one should take into account the multiple channels of transmission of an interest rate shock to the risky assets' returns processes. For example, an increase in the interest rate implies a negative shock to the equity returns due to the negative correlation. At the same time, this shock increases the expected returns of the equity and the bond, improving the available investment opportunities. However, this shock also increases the equity and bond returns' volatility causing a deterioration of the investment opportunities. Hence, our model specification provides an interesting case where the sign of the hedging demand is not obvious *a priori.*

For the specific parameter values we employed in this subsection, if the correlation coefficient is high in absolute value, then there is little scope for hedging because the negative effects (through the negative correlation and the increase in volatility) due to a positive shock to the interest rate are inherently almost offset by the corresponding positive effects it generates (through the increase of the expected returns). On the other hand, if the correlation is low in absolute value, the first channel of shock transmission is neutralized and the investor becomes more exposed to the interest rate risk. As a result, he is incentivized to hedge away the interest rate shocks.

It is important to note that at the same time, the different magnitudes of correlation have an impact on the diversification role of the bond too. As the correlation decreases in absolute value, the myopic demand for the bond increases. Nevertheless, the hedging demand shift due to the lower correlation (in absolute value) is so high that it dominates any shift in the myopic demand component.

The impact of the risk factor diffusion coefficient

As it has already been discussed, the magnitude of the diffusion coefficient of the interest rate process is a very important input for the hedging demand. The Left panel of Figure 2.7 shows the total bond-to-equity demand ratio for three different values of this coefficient, for an investment horizon of 15 years. We firstly observe that, as the coefficient of relative risk aversion increases, the bond-to-equity demand ratio increases, regardless of the value of the diffusion coefficient. However, a highly risk averse investor, ($\gamma = 10$), will hold less of the bond relative to the equity if the diffusion coefficient is $\sigma_r = 0.06$. *Prima facie*, this result seems to contradict the intuitive argument that, as the shocks to the interest rate get magnified, the relative importance of the hedging asset (i.e. the bond) should increase.

In order to clarify the underlying mechanism, we resort to the Right panel of Figure 2.7, that depicts the ratio (in absolute value) of hedging to myopic demand for the bond. This graph verifies the previous argument, that the hedging component of the bond demand becomes more important as the diffusion coefficient increases. The explanation for the finding of the Left panel is due to the myopic bond demand component. As the diffusion coefficient increases, the bond becomes riskier, hence the myopic bond demand is significantly decreased. This decrease has a greater impact than the increase in the demand due to the hedging motive, so the total bond demand decreases in this case, as the diffusion coefficient increases.

The impact of the speed of mean-reversion

The last feature affecting the bond/equity mix that we examine, is the speed of mean-reversion in the interest rate process. In the incomplete market case, the hedging motive was greater when the speed of mean reversion was slower because a shock affecting the risk factor is more persistent. The Left panel of Figure 2.8 shows that this effect is true in the complete market case too. The hedging demand dominates the total bond demand as the speed of mean reversion is slower. It is also interesting to note that the hedging incentive grows with the agent's risk aversion.

Regarding the bond-to-equity demand ratio, this effect is much more complicated. In particular, we observe that for highly risk averse investors, the total bond demand decreases in comparison to the equity demand, as the speed of mean reversion decreases. Having clarified that the hedging bond demand actually increases in this case, the explanation for this finding is again due to the myopic component. In particular, as the speed of mean-reversion becomes slower, the bond becomes more risky, since the term $B(\tilde{t})$ in the bond returns' dynamics increases in absolute value. Hence, the decrease in the myopic bond demand component of this risk averse investor more than offsets any increase in the hedging demand component, reducing the relative total portion of wealth invested in the bond.

The mean-variance "inefficiency" of the long-term portfolio choice

The fundamental result of dynamic asset allocation is that the portfolio choice of a long-term investor exhibits an additional component due to hedging purposes, apart from the standard myopic component. This result contradicts the mean-variance analysis of Markowitz (1952). Hence, the construction of minimum variance portfolios is of no use for a long-term investor, since the concept of variance has a static dimension in the standard analysis. Nevertheless, the long-term investor seeks to minimize the volatility of his wealth path through time, not simply one-step ahead.

In order to illustrate this concept, we construct in Figure 2.9 the myopic efficient frontier using the properties of the equity and the bond returns, along with the tangency line starting from the risk-free rate $r = 0.05$ in the expected returns' axis. This figure also presents the pairs of expected return and standard deviation arising from the portfolio formed by an agent with 15 years of investment horizon and coefficients of relative risk aversion of $\gamma = 2$, $\gamma = 4$ and $\gamma = 8$ correspondingly. The figure shows that the optimal portfolios of this long-term investor are in the "inefficient region".

Hence, an observer with a static point of view would conjecture that this investor is behaving suboptimally, not exploiting the available investment opportunities. Such a conjecture would be wrong, because the observer does not take into account the intertemporal dimension of the investor's behaviour. In other words, the long-term investor is willing to "sacrifice" a significant level of one-step ahead expected returns by shifting his portfolio mix so as to minimize intertemporal volatility, not just the next period one. This analysis demonstrates how important the intertemporal dimension is for long-term investors and how this can be misinterpreted if viewed from a static perspective.

2.5.3 The impact of market completion

The main concept underlying the process of "completing the market", through the introduction of the zero-coupond bond, is that the long-term investor under examination will be able to perfectly hedge the shocks affecting the underlying risk factor, the only source of intertemporal risk. When the market was incomplete this hedge was only partial. Hence, we expect a significant utility gain for the long-term investor from the process of market completion. Defining as $I_{INC}(W,r,t) = \frac{W^{1-\gamma}}{1-\gamma}J_{INC}(r,t)$, the value function of the investor in the incomplete market case and as $I_{COM}(W,r,t) = \frac{W^{1-\gamma}}{1-\gamma}J_{COM}(r,t)$, the value function in the complete market case, we expect that $I_{INC}(W,r,t) < I_{COM}(W,r,t)$ for $\gamma > 1$. We seek then to find the portion of wealth, CER, which, if deducted from the wealth of the investor in the complete market, would equalize his utility with the utility derived in the incomplete market case:

$$\frac{((1-CER)W)^{1-\gamma}}{1-\gamma}J_{COM}(r,t) = \frac{W^{1-\gamma}}{1-\gamma}J_{INC}(r,t) \tag{2.38}$$

where $J(r,t) = \exp(\gamma(c(t)+d(t)r))$ and the functions $c(t)$, $d(t)$ are correspondingly known for each case (see the proofs of Propositions 2.4 and 2.5 in the Appendix).

In other words, CER can be also interpreted as the portion of wealth that the investor in the incomplete market would pay to have access to the bond that completes the market. This certainty equivalent ratio of wealth (CER), for $\gamma > 1$, is given by:

$$CER = 1 - [\frac{J_{INC}(r,t)}{J_{COM}(r,t)}]^{\frac{1}{1-\gamma}} \tag{2.39}$$

Figure 2.10 shows the CER for three levels of relative risk aversion, as the investment horizon increases. We observe that, regardless of the level of risk aversion, as the horizon increases, the portion of wealth that the long-term investor in the incomplete market would give up in order to have access to the bond is increasing. The cost of not being able to make use of the bond can be as high as 35% of the wealth of an investor with $\gamma = 10$ and 15 years of investment horizon.

Two remarks are useful for the interpretation of this graph: Firstly, the cost does not increase monotonically with the level of risk aversion. This is an expected result since the total demand for the bond has a nonmonotonic behaviour with respect to level of the risk aversion. Secondly, by completing the market, the investor is better off not only due to his enhanced hedging ability, but also due to the use of this bond for diversification purposes.

We disentangle the hedging from the diversification effect by calculating the CER of optimizing intertemporally in comparison to acting myopically within the complete market case. Using the previous steps, the cost of acting myopically is given by:

$$CER = 1 - [J_{INT}(r,t)]^{\frac{1}{\gamma-1}} \tag{2.40}$$

where $I_{INT}(W,r,t) = \frac{W_t^{1-\gamma}}{1-\gamma} J_{INT}(r,t)$ stands for the Intertemporal value function of the long-term investor. The previous expression holds since the value function in the myopic case is given by $I_{MYOP}(W,r,t) = \frac{W_t^{1-\gamma}}{1-\gamma}$.

Figure 2.11 exhibits this certainty equivalent wealth ratio (CER), as the investment horizon increases, showing that the cost of not hedging against the intertemporal risk is very significant and, as expected, it increases with the length of the horizon, since the hedging motive increases. The utility gains are not directly comparable for different levels of relative risk aversion because of the different exposures to the interest rate risk. However, for all these levels of risk aversion, the inability to hedge the interest rate shocks implies significant utility costs.

2.6 Conclusion

This chapter examined the impact of modelling assumptions for the risky asset returns and the risk factor dynamics on the optimal portfolio choice of a long-term investor. It is an attempt to show that the modelling assumptions play a significant role in the solution of this problem and acts as a warning for the practical implementation of dynamic asset allocation, that it is not a straightforward task. Actually, quite the opposite is true.

Firstly, there is a trade off between obtaining closed-form solutions and modelling returns in a flexible way. The "affine-quadratic" dynamics, as characterized in Liu (2007) and employed in this study, allow us to derive explicit solutions, but they cannot accommodate for non-linear models of returns' predictability, non-linear mean reversion or richer stochastic volatility models. Secondly, the issue of whether returns are predictable, as well as how their stochastic volatility is generated, crucially affects the optimal portfolio choice. For example, even though in most of the presented models the long-term investor will be involved in market timing, when both predictability and time-varying volatility is employed as in CIR (1985a), there is no need for market timing.

A very important issue that we examined in this study and that it is not fully exploited in the literature, is the impact of completing the market. In particular, expanding the asset space to include a contingent claim, the returns of which are perfectly correlated with the shocks to the underlying risk factor, significantly affects the optimal portfolio choice. The equity is still important for investment purposes but does not serve for intertemporal hedging any more. The long-term investor manages to perfectly hedge away the underlying risk factor shocks using this contingent claim.

Examining the impact of the market completion on the welfare of the long-term investor, we find that this is significantly improved due to the achieveable perfect hedge. This is a quite important result, underlining the practical purpose of marketing assets that perfectly hedge away known risk factors. Further research is needed to shed light on the welfare implications of hedging instruments, such as volatility and currency options as well as longevity and catastrophe bonds. Identifying the risks that a long-term investor is exposed to, he will be able to immunize his portfolio from unexpected shocks, appropriately employing the corresponding hedging instruments.

A final conclusion of this study is that long-term investors should not make use of the standard mean-variance efficient frontiers. These frontiers take into account only one-period-ahead expected returns and standard deviations, ignoring the intertemporal volatility of the investor's wealth path. It was shown here that the optimal portfolio of a long-term investor would be erroneously regarded as "inefficient" by a myopic observer. Hence, as with the mean-variance portfolio rules, mean-variance efficient frontiers provoke an unfortunate mismatch between preferences, objectives and tools in the asset allocation of long-horizon agents.

2.7 Appendix

Proof of Proposition 2.1

From the Bellman principle of optimality, we derive the following Hamilton-Jacobi-Bellman (HJB) equation:

$$E(dI) = 0 \Rightarrow \frac{\partial I}{\partial W} W[\phi(\mu - r^f) + r^f]dt + \frac{\partial I}{\partial r}\kappa(\theta - r)dt +$$
$$+ \frac{1}{2}\frac{\partial^2 I}{\partial W^2} W^2 \phi^2 \sigma_s^2 dt + \frac{1}{2}\frac{\partial^2 I}{\partial r^2}\sigma_r^2 dt + \frac{\partial I}{\partial t}dt + \frac{\partial^2 I}{\partial W \partial r}\phi W \sigma_s \sigma_r \rho_{sr} dt = 0 \qquad (2.41)$$

The first order condition with respect to the optimal portfolio weight yields:

$$\frac{\partial(.)}{\partial \phi} = 0 \Rightarrow \frac{\partial I}{\partial W} W(\mu - r^f) + \frac{\partial^2 I}{\partial W^2} W^2 \phi \sigma_s^2 + \frac{\partial^2 I}{\partial W \partial r} W \sigma_r \sigma_s \rho_{sr} = 0 \Rightarrow$$
$$\phi^* = -\frac{\frac{\partial I}{\partial W}}{\frac{\partial^2 I}{\partial W^2} W}\frac{(\mu - r^f)}{\sigma_s^2} - \frac{\frac{\partial^2 I}{\partial W \partial r}}{\frac{\partial^2 I}{\partial W^2} W}\frac{\sigma_r}{\sigma_s}\rho_{sr} \qquad (2.42)$$

In order to solve this problem, a trial form for the value function is conjectured:

$$I(W, r, t) = \frac{W^{1-\gamma}}{1-\gamma}\exp(\gamma(c(t) + d(t)r)) \qquad (2.43)$$

with terminal conditions $c(T) = d(T) = 0$, so that $I(W, r, T) = \frac{W_T^{1-\gamma}}{1-\gamma}$.

Substituting the trial solution and the optimal portfolio choice into the HJB equation, the following expression is derived:

$$\frac{1}{2\gamma}\frac{(\mu-r^f)^2}{\sigma_s^2}+r^f+(\mu-r^f)d(t)\frac{\sigma_r}{\sigma_s}\rho_{sr}+\frac{\gamma}{2}d^2(t)\sigma_r^2\rho_{sr}^2+$$
$$\frac{\gamma}{1-\gamma}d(t)\kappa(\theta-r)+\frac{\gamma^2}{2(1-\gamma)}d^2(t)\sigma_r^2+\frac{\gamma}{1-\gamma}(\dot{c}(t)+\dot{d}(t)r)=0 \qquad (2.44)$$

where $\dot{c}$ and $\dot{d}$ denote the corresponding partial derivatives with respect to time t.

This is an affine expression in r. Using the method of undetermined coefficients, we get the following ODEs in $c(t)$ and $d(t)$:

$$\dot{d}(t)=d(t)\kappa \qquad (2.45)$$

and

$$\dot{c}(t)+\frac{1-\gamma}{2\gamma^2}\frac{(\mu-r^f)^2}{\sigma_s^2}+\frac{1-\gamma}{\gamma}r^f+\frac{1-\gamma}{\gamma}(\mu-r^f)d(t)\frac{\sigma_r}{\sigma_s}\rho_{sr}+$$
$$+\frac{1-\gamma}{2}d^2(t)\sigma_r^2\rho_{sr}^2+d(t)\kappa\theta+\frac{\gamma}{2}d^2(t)\sigma_r^2=0 \qquad (2.46)$$

Solving the ODE with respect to $d(t)$ with the help of the terminal condition $d(T)=0$, we get

$$d(t)=0 \qquad (2.47)$$

Substituting the partial derivatives of the value function $I(W,r,t)$ into (2.42) and using that $d(t)=0$, we get the optimal portfolio choice in (2.6).

Proof of Proposition 2.2

The HJB equation of this problem is given by:

$$E(dI)=0\Rightarrow\frac{\partial I}{\partial W}W[\phi(\alpha_0+\alpha_1 r-r^f)+r^f]dt+\frac{\partial I}{\partial r}\kappa(\theta-r)dt+$$
$$+\frac{1}{2}\frac{\partial^2 I}{\partial W^2}W^2\phi^2\sigma_s^2dt+\frac{1}{2}\frac{\partial^2 I}{\partial r^2}\sigma_r^2dt+\frac{\partial I}{\partial t}dt+\frac{\partial^2 I}{\partial W\partial r}\phi W\sigma_s\sigma_r\rho_{sr}dt=0 \tag{2.48}$$

Taking the FOC with respect to ϕ, we find the optimal portfolio choice:

$$\frac{\partial(.)}{\partial\phi}=0\Rightarrow\frac{\partial I}{\partial W}W(\alpha_0+\alpha_1 r-r^f)+\frac{\partial^2 I}{\partial W^2}W^2\phi\sigma_s^2+\frac{\partial^2 I}{\partial W\partial r}W\sigma_r\sigma_s\rho_{sr}=0\Rightarrow$$
$$\phi^*=-\frac{\frac{\partial I}{\partial W}}{\frac{\partial^2 I}{\partial W^2}W}\frac{(\alpha_0+\alpha_1 r-r^f)}{\sigma_s^2}-\frac{\frac{\partial^2 I}{\partial W\partial r}}{\frac{\partial^2 I}{\partial W^2}W}\frac{\sigma_r}{\sigma_s}\rho_{sr} \tag{2.49}$$

In this case, the conjectured functional form for the value function is:

$$I(W,r,t)=\frac{W^{1-\gamma}}{1-\gamma}\exp(\gamma(c(t)+d(t)r+\frac{1}{2}e(t)r^2)) \tag{2.50}$$

Using this trial form, the optimal portfolio choice is given by:

$$\phi^*=\frac{1}{\gamma}\frac{(\alpha_0+\alpha_1 r-r^f)}{\sigma_s^2}+[d(t)+e(t)r]\frac{\sigma_r}{\sigma_s}\rho_{sr} \tag{2.51}$$

Substituting the trial solution and the optimal portfolio choice into the HJB equation (2.48), we get after some simplifications:

$$\frac{1}{2\gamma}\frac{(\alpha_0 - r^f)^2}{\sigma_s^2} + \frac{1}{2\gamma}\frac{\alpha_1{}^2}{\sigma_s^2}r^2 + \frac{1}{\gamma}\frac{(\alpha_0 - r^f)\alpha_1}{\sigma_s^2}r + r^f + \frac{\gamma}{2}(d(t) + e(t)r)^2\sigma_r^2\rho_{sr}^2 +$$
$$+\frac{\gamma}{1-\gamma}e(t)\kappa(\theta - r)r + \frac{\gamma^2}{2(1-\gamma)}d^2(t)\sigma_r^2 + \frac{\gamma^2}{1-\gamma}d(t)e(t)\sigma_r^2 r + \frac{\gamma^2}{2(1-\gamma)}e^2(t)\sigma_r^2 r^2 +$$
$$+\frac{\gamma}{2(1-\gamma)}e(t)\sigma_r^2 + d(t)\frac{\sigma_r}{\sigma_s}\rho_{sr}(\alpha_0 + \alpha_1 r - r^f) + e(t)\frac{\sigma_r}{\sigma_s}\rho_{sr}(\alpha_0 + \alpha_1 r - r^f)r +$$
$$+\frac{\gamma}{1-\gamma}d(t)\kappa(\theta - r) + \frac{\gamma}{1-\gamma}\dot{c}(t) + \frac{\gamma}{1-\gamma}\dot{d}(t)r + \frac{1}{2}\frac{\gamma}{1-\gamma}\dot{e}(t)r^2 = 0 \qquad (2.52)$$

where $\dot{c}$, $\dot{d}$ and $\dot{e}$ denote the corresponding partial derivatives with respect to time t.

This equation is a quadratic expression in r. For this equation to hold, it is sufficient that the following equations hold:

$$\frac{1}{2\gamma}\frac{(\alpha_0 - r^f)^2}{\sigma_s^2} + r^f + \frac{\gamma}{1-\gamma}d(t)\kappa\theta + \frac{\gamma^2}{2(1-\gamma)}d^2(t)\sigma_r^2 + \frac{\gamma}{2}\sigma_r^2\rho_{sr}^2 d^2(t) +$$
$$\frac{\gamma}{2(1-\gamma)}e(t)\sigma_r^2 + d(t)\frac{\sigma_r}{\sigma_s}\rho_{sr}(\alpha_0 - r^f) + \frac{\gamma}{1-\gamma}\dot{c}(t) = 0 \qquad (2.53)$$

$$\frac{1}{\gamma}\frac{(\alpha_0 - r^f)\alpha_1}{\sigma_s^2} - \frac{\gamma}{1-\gamma}d(t)\kappa + \frac{\gamma}{1-\gamma}e(t)\kappa\theta + \frac{\gamma^2}{1-\gamma}d(t)e(t)\sigma_r^2 +$$
$$\gamma e(t)d(t)\sigma_r^2\rho_{sr}^2 + d(t)\frac{\sigma_r}{\sigma_s}\rho_{sr}\alpha_1 + e(t)\frac{\sigma_r}{\sigma_s}\rho_{sr}(\alpha_0 - r^f) + \frac{\gamma}{1-\gamma}\dot{d}(t) = 0 \Rightarrow$$
$$\dot{d}(t) + d(t)\{\frac{1-\gamma}{\gamma}\frac{\sigma_r}{\sigma_s}\rho_{sr}\alpha_1 - \kappa + e(t)[\gamma\sigma_r^2 + (1-\gamma)\sigma_r^2\rho_{sr}^2]\} +$$
$$+e(t)[\frac{\sigma_r}{\sigma_s}\rho_{sr}(\alpha_0 - r^f)\frac{1-\gamma}{\gamma} + \kappa\theta] + \frac{1-\gamma}{\gamma^2}\frac{(\alpha_0 - r^f)\alpha_1}{\sigma_s^2} = 0 \qquad (2.54)$$

and

$$\frac{1}{2\gamma}\frac{\alpha_1{}^2}{\sigma_s^2} - \frac{\gamma}{1-\gamma}e(t)\kappa + \frac{\gamma^2}{2(1-\gamma)}e^2(t)\sigma_r^2 + e(t)\frac{\sigma_r}{\sigma_s}\rho_{sr}\alpha_1 + \frac{\gamma}{2}e^2(t)\sigma_r^2\rho_{sr}^2 + \frac{1}{2}\frac{\gamma}{1-\gamma}\dot{e}(t) = 0 \Rightarrow$$

$$\dot{e}(t) + e(t)\frac{2(1-\gamma)}{\gamma}\left(\frac{\sigma_r}{\sigma_s}\rho_{sr}\alpha_1 - \frac{\gamma}{1-\gamma}\kappa\right) + e^2(t)[\gamma\sigma_r^2 + (1-\gamma)\sigma_r^2\rho_{sr}^2] + \frac{1-\gamma}{\gamma^2}\frac{\alpha_1{}^2}{\sigma_s^2} = 0 \tag{2.55}$$

along with the corresponding terminal conditions.

Solving the system of ODEs in $e(t)$ and $d(t)$, we can replace the solutions into the optimal portfolio choice expression (2.51) in order to get an explicit solution.

The solution for $e(t)$ is given by:

$$e(t) = -\frac{2(\exp(\psi(T-t))-1)}{(a+\psi)(\exp(\psi(T-t))-1)+2\psi}\hat{c} \tag{2.56}$$

for $\psi \geq 0$, where $\psi = \sqrt{a^2 - 4b\hat{c}}$, $a = (2\kappa - \frac{2(1-\gamma)}{\gamma}\frac{\sigma_r}{\sigma_s}\rho_{sr}\alpha_1)$, $b = -\gamma\sigma_r^2 - (1-\gamma)\sigma_r^2\rho_{sr}^2$, $\hat{c} = \frac{\gamma-1}{\gamma^2}\frac{\alpha_1^2}{\sigma_s^2}$. The ODE in $d(t)$ can be solved numerically using a Runge-Kutta scheme.

Proof of Proposition 2.3

The HJB equation of this problem is:

$$E(dI) = 0 \Rightarrow \frac{\partial I}{\partial W}W[\phi(\mu - r^f) + r^f]dt + \frac{\partial I}{\partial y}\kappa(\theta - y)dt +$$
$$+\frac{1}{2}\frac{\partial^2 I}{\partial W^2}W^2\phi^2\sigma_s^2\frac{1}{y}dt + \frac{1}{2}\frac{\partial^2 I}{\partial y^2}\sigma_y^2 y dt + \frac{\partial I}{\partial t}dt + \frac{\partial^2 I}{\partial W \partial y}\phi W\sigma_s\sigma_y\rho_{sy}dt = 0 \tag{2.57}$$

Taking the FOC with respect to ϕ, the optimal portfolio choice is derived:

$$\frac{\partial(.)}{\partial\phi} = 0 \Rightarrow \frac{\partial I}{\partial W}W(\mu - r^f) + \frac{\partial^2 I}{\partial W^2}W^2\phi\sigma_s^2\frac{1}{y} + \frac{\partial^2 I}{\partial W \partial y}W\sigma_y\sigma_s\rho_{sy} = 0 \Rightarrow$$

$$\phi^* = -\frac{\frac{\partial I}{\partial W}}{\frac{\partial^2 I}{\partial W^2}W}\frac{(\mu - r^f)y}{\sigma_s^2} - \frac{\frac{\partial^2 I}{\partial W \partial y}}{\frac{\partial^2 I}{\partial W^2}W}\frac{\sigma_y}{\sigma_s}\rho_{sy}y \tag{2.58}$$

Conjecturing the following functional form for the value function, $I(W, y, t) = \frac{W^{1-\gamma}}{1-\gamma}\exp(\gamma(c(t) + d(t)y))$, and substituting the optimal portfolio choice in the HJB equation, we get the equation:

$$\frac{1}{2\gamma}\frac{(\mu - r^f)^2}{\sigma_s^2}y + (\mu - r^f)d(t)\frac{\sigma_y}{\sigma_s}\rho_{sy}y + \frac{\gamma}{2}d^2(t)\sigma_y^2\rho_{sy}^2 y +$$
$$+\frac{\gamma}{1-\gamma}d(t)\kappa(\theta - y) + \frac{\gamma^2}{2(1-\gamma)}d^2(t)\sigma_y^2 y + \frac{\gamma}{1-\gamma}(\dot{c}(t) + \dot{d}(t)y) = 0 \tag{2.59}$$

This is an affine expression in y. Hence, it is sufficient that the following two equations hold:

$$\dot{d}(t) = d(t)\kappa - \frac{1-\gamma}{2\gamma^2}\frac{(\mu - r^f)^2}{\sigma_s^2} - \frac{1-\gamma}{\gamma}(\mu - r^f)d(t)\frac{\sigma_y}{\sigma_s}\rho_{sy} - \frac{\gamma}{2}d^2(t)\sigma_y^2 - \frac{1-\gamma}{2}d^2(t)\sigma_y^2\rho_{sy}^2 \tag{2.60}$$

and

$$\dot{c}(t) = -\frac{1-\gamma}{\gamma}r^f - d(t)\kappa\theta \tag{2.61}$$

Solving the Riccati ODE with respect to $d(t)$ using the terminal condition $d(T) = 0$, we get:

$$d(t) = -\frac{2(\exp(\psi(T-t)) - 1)}{(a + \psi)(\exp(\psi(T-t)) - 1) + 2\psi}\hat{c}$$

for $\psi \geq 0$, where $a = (\kappa - \frac{1-\gamma}{\gamma}(\mu - r^f)\frac{\sigma_y}{\sigma_s}\rho_{sy})$, $\hat{c} = -\frac{1-\gamma}{2\gamma^2}\frac{(\mu-r^f)^2}{\sigma_s^2}$, $b = -\frac{\gamma}{2}\sigma_y^2 - \frac{1-\gamma}{2}\sigma_y^2\rho_{sy}^2$ and $\psi = \sqrt{a^2 - 4b\hat{c}}$.

Proof of Proposition 2.4

The HJB equation of this problem is given by:

$$
\begin{aligned}
E(dI) = 0 \Rightarrow \frac{\partial I}{\partial W} W[\phi(\mu - 1)r + r]dt + \frac{\partial I}{\partial r}\kappa(\theta - r)dt + \\
+ \frac{1}{2}\frac{\partial^2 I}{\partial W^2} W^2 \phi^2 \sigma_s^2 r dt + \frac{1}{2}\frac{\partial^2 I}{\partial r^2}\sigma_r^2 r dt + \frac{\partial I}{\partial t}dt + \frac{\partial^2 I}{\partial W \partial r}\phi W \sigma_s \sigma_r \rho_{sr} r dt = 0
\end{aligned}
\tag{2.62}
$$

The FOC with respect to the optimal portfolio weight yields:

$$
\begin{aligned}
\frac{\partial(.)}{\partial \phi} = 0 \Rightarrow \frac{\partial I}{\partial W} W(\mu - 1)r + \frac{\partial^2 I}{\partial W^2} W^2 \phi \sigma_s^2 r + \frac{\partial^2 I}{\partial W \partial r} W \sigma_r \sigma_s \rho_{sr} r = 0 \Rightarrow \\
\phi^* = -\frac{\frac{\partial I}{\partial W}}{\frac{\partial^2 I}{\partial W^2} W}\frac{(\mu - 1)}{\sigma_s^2} - \frac{\frac{\partial^2 I}{\partial W \partial r}}{\frac{\partial^2 I}{\partial W^2} W}\frac{\sigma_r}{\sigma_s}\rho_{sr}
\end{aligned}
\tag{2.63}
$$

The following functional form is conjectured for the value function:

$$
I(W, r, t) = \frac{W^{1-\gamma}}{1-\gamma}\exp(\gamma(c(t) + d(t)r))
\tag{2.64}
$$

Using this trial function and replacing the optimal portfolio choice back in the HJB equation, we get:

$$
\begin{aligned}
\frac{1}{2\gamma}\frac{(\mu - 1)^2}{\sigma_s^2} r + r + \frac{\gamma}{1-\gamma} d(t)\kappa(\theta - r) + \frac{\gamma}{2} d^2(t)\sigma_r^2 \rho_{sr}^2 r + \\
+ \frac{1}{2}\frac{\gamma^2}{1-\gamma} d^2(t)\sigma_r^2 r + \frac{\gamma}{1-\gamma}(\dot{c}(t) + \dot{d}(t)r) = 0
\end{aligned}
\tag{2.65}
$$

The method of undetermined coefficients implies that it is sufficient to solve the

ODE in $c(t)$:

$$\frac{\gamma}{1-\gamma}\dot{c}(t) + \frac{\gamma}{1-\gamma}d(t)\kappa\theta = 0 \Rightarrow \dot{c}(t) = -d(t)\kappa\theta \tag{2.66}$$

and the ODE in $d(t)$:

$$\dot{d}(t) + \frac{1-\gamma}{2\gamma^2}\frac{(\mu-1)^2}{\sigma_s^2} + \frac{1-\gamma}{\gamma} - d(t)\kappa + \frac{1-\gamma}{2}d^2(t)\sigma_r^2\rho_{sr}^2 + \frac{\gamma}{2}d^2(t)\sigma_r^2 = 0 \tag{2.67}$$

with the terminal conditions $c(T) = d(T) = 0$. Solving the ODE (2.67) with respect to $d(t)$, we get:

$$d(t) = -\frac{2(\exp(\psi(T-t)) - 1)}{(a+\psi)(\exp(\psi(T-t)) - 1) + 2\psi}\hat{c} \tag{2.68}$$

for $\psi \geq 0$, where $\hat{c} = -(\frac{1-\gamma}{2\gamma^2}\frac{(\mu-1)^2}{\sigma_s^2} + \frac{1-\gamma}{\gamma})$, $b = -\frac{1-\gamma}{2}\sigma_r^2\rho_{sr}^2 - \frac{\gamma}{2}\sigma_r^2$, $a = \kappa$ and $\psi = \sqrt{a^2 - 4b\hat{c}}$.

Moreover, the solution for $c(t)$ is given by:

$$c(t) = -\frac{\kappa\theta}{b}\ln[\frac{2\psi\exp(\frac{(a+\psi)(T-t)}{2})}{(a+\psi)(\exp(\psi(T-t) - 1)) + 2\psi}] \tag{2.69}$$

Proposition 2.5

The corresponding HJB equation for this problem is given by:

$$\begin{aligned} &\frac{\partial I}{\partial t} + \frac{1}{2}W^2\phi\Sigma\Sigma^T\phi^T\frac{\partial^2 I}{\partial W^2} + W(\phi^T(\mu^P - ir) + r)\frac{\partial I}{\partial W} + \\ &+ W\phi^T\Sigma\rho^T\Sigma^{rT}\frac{\partial^2 I}{\partial W \partial r} + \frac{1}{2}Tr(\Sigma^r\Sigma^{rT})\frac{\partial^2 I}{\partial r^2} + \mu^{rT}\frac{\partial I}{\partial r} = 0 \end{aligned} \tag{2.70}$$

Taking the FOC in order to find the optimal portfolio choice, we have:

$$\frac{\partial(.)}{\partial\phi} = 0 \Rightarrow W^2\phi\Sigma\Sigma^T\frac{\partial^2 I}{\partial W^2} + W(\mu^P - ir)\frac{\partial I}{\partial W} + W\Sigma\rho^T\Sigma^{rT}\frac{\partial^2 I}{\partial W\partial r} = 0 \Rightarrow$$

$$\Rightarrow \phi^* = -\frac{\frac{\partial I}{\partial W}}{\frac{\partial^2 I}{\partial W^2}W}(\Sigma\Sigma^T)^{-1}(\mu^P - ir) - \frac{1}{\frac{\partial^2 I}{\partial W^2}W}(\Sigma\Sigma^T)^{-1}\Sigma\rho^T\Sigma^{rT}\frac{\partial^2 I}{\partial W\partial r} \tag{2.71}$$

and substituting this solution into the HJB equation, we have:

$$\frac{\partial I}{\partial t} + \frac{1}{2}\frac{(\frac{\partial I}{\partial W})^2}{\frac{\partial^2 I}{\partial W^2}}(\mu^P - ir)^T(\Sigma\Sigma^T)^{-1}(\mu^P - ir) + \frac{\partial I}{\partial W}Wr + \frac{\frac{\partial^2 I}{\partial W\partial r}\frac{\partial I}{\partial W}}{\frac{\partial^2 I}{\partial W^2}}\Sigma^r\rho\Sigma^{-1}(\mu^P - ir)$$
$$+\frac{1}{2}\frac{(\frac{\partial^2 I}{\partial W\partial r})^2}{\frac{\partial^2 I}{\partial W^2}}(\Sigma^r\Sigma^{rT} - \Sigma^r\rho\rho^T\Sigma^{rT}) + \frac{1}{2}Tr(\Sigma^r\Sigma^{rT})\frac{\partial^2 I}{\partial r^2} + \mu^{rT}\frac{\partial I}{\partial r} = 0 \tag{2.72}$$

Using the trial solution, $I(W,r,t) = \frac{W^{1-\gamma}}{1-\gamma}\exp(\gamma(c(t) + d(t)r))$, the optimal solution becomes:

$$\phi^* = \frac{1}{\gamma}(\Sigma\Sigma^T)^{-1}(\mu^P - ir) + d(t)(\Sigma\Sigma^T)^{-1}\Sigma\rho^T\Sigma^{rT} \tag{2.73}$$

and the HJB equation becomes:

$$\frac{\gamma}{1-\gamma}\dot{c}(t) + \frac{\gamma}{1-\gamma}\dot{d}(t)r + \frac{1}{2\gamma}(\mu^P - ir)^T(\Sigma\Sigma^T)^{-1}(\mu^P - ir) - \gamma d(t)\Sigma^r\rho\Sigma^{-1}(\mu^P - ir)$$
$$+r - \frac{\gamma}{2}(\Sigma^r\Sigma^{rT} - \Sigma^r\rho\rho^T\Sigma^{rT}) + \frac{\gamma}{2}d^2(t)Tr(\Sigma^r\Sigma^{rT}) + \frac{\gamma}{1-\gamma}d(t)\mu^{rT} = 0 \tag{2.74}$$

Using the method of undetermined coefficients, we can equivalently solve the following ODEs:

$$\frac{\gamma}{1-\gamma}\dot{c}(t) + \frac{\gamma}{1-\gamma}d(t)\kappa\theta = 0 \Rightarrow \dot{c}(t) = -d(t)\kappa\theta \tag{2.75}$$

and

$$\frac{\gamma}{1-\gamma}\dot{d}(t)+\frac{\gamma}{2}\sigma_r^2 d^2(t)-[\gamma(\frac{\rho_{sr}\sigma_r(\mu-1)}{\sigma_s}+\lambda)+\frac{\gamma}{1-\gamma}\kappa]d(t)+$$
$$+1+\frac{1}{2\gamma}[\frac{(\mu-1)^2}{\sigma_s^2(1-\rho_{sr}^2)}-\frac{2(\mu-1)\rho_{sr}\lambda}{\sigma_s\sigma_r(1-\rho_{sr}^2)}+\frac{\lambda^2}{\sigma_r^2(1-\rho_{sr}^2)}]+\frac{\gamma}{2}(\sigma_r^2\rho_{sr}^2)=0\Rightarrow$$
$$\dot{d}(t)+\frac{1-\gamma}{2}\sigma_r^2 d^2(t)-[(1-\gamma)(\frac{\rho_{sr}\sigma_r(\mu-1)}{\sigma_s}+\lambda)+\kappa]d(t)+\frac{1-\gamma}{\gamma}+$$
$$+\frac{1-\gamma}{2\gamma^2}[\frac{(\mu-1)^2}{\sigma_s^2(1-\rho_{sr}^2)}-\frac{2(\mu-1)\rho_{sr}\lambda}{\sigma_s\sigma_r(1-\rho_{sr}^2)}+\frac{\lambda^2}{\sigma_r^2(1-\rho_{sr}^2)}]+\frac{1-\gamma}{2}(\sigma_r^2\rho_{sr}^2)=0 \quad (2.76)$$

Solving the Riccati ODE with respect to $d(t)$ using the terminal condition $d(T)=0$, we get:

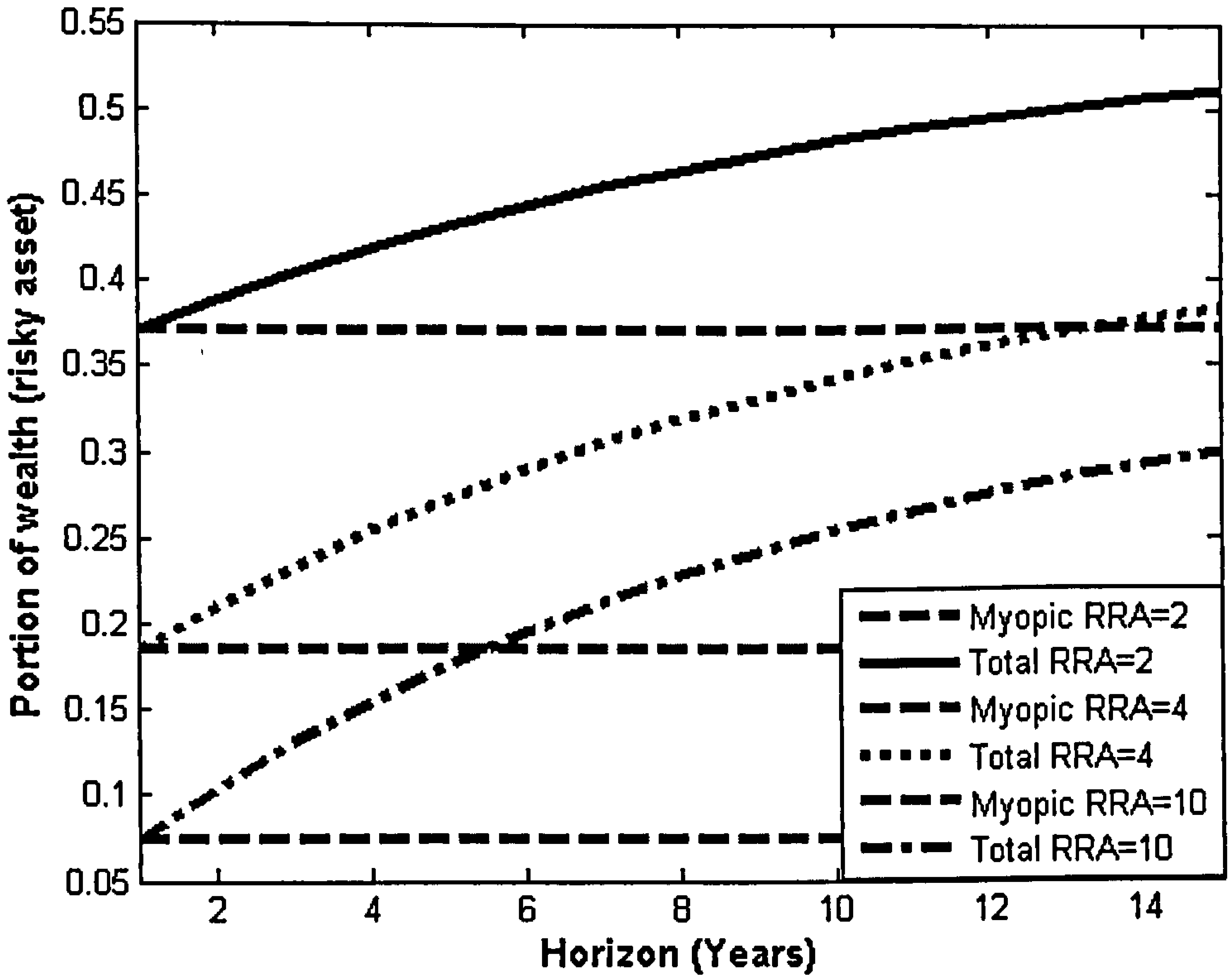
$$d(t)=-\frac{2(\exp(\psi(T-t))-1)}{(a+\psi)(\exp(\psi(T-t))-1)+2\psi}\hat{c} \quad (2.77)$$

for $\psi\geq 0$, where $\hat{c}=\frac{\gamma-1}{2\gamma^2}[\frac{(\mu-1)^2}{\sigma_s^2(1-\rho_{sr}^2)}-\frac{2(\mu-1)\rho_{sr}\lambda}{\sigma_s\sigma_r(1-\rho_{sr}^2)}+\frac{\lambda^2}{\sigma_r^2(1-\rho_{sr}^2)}]+\frac{\gamma-1}{2}(\sigma_r^2\rho_{sr}^2)+\frac{\gamma-1}{\gamma}$, $a=[(1-\gamma)(\frac{\rho_{sr}\sigma_r(\mu-1)}{\sigma_s}+\lambda)+\kappa]$, $b=-\frac{1-\gamma}{2}\sigma_r^2$ and $\psi=\sqrt{a^2-4b\hat{c}}$

Consequently, the solution for $c(t)$ is given by:

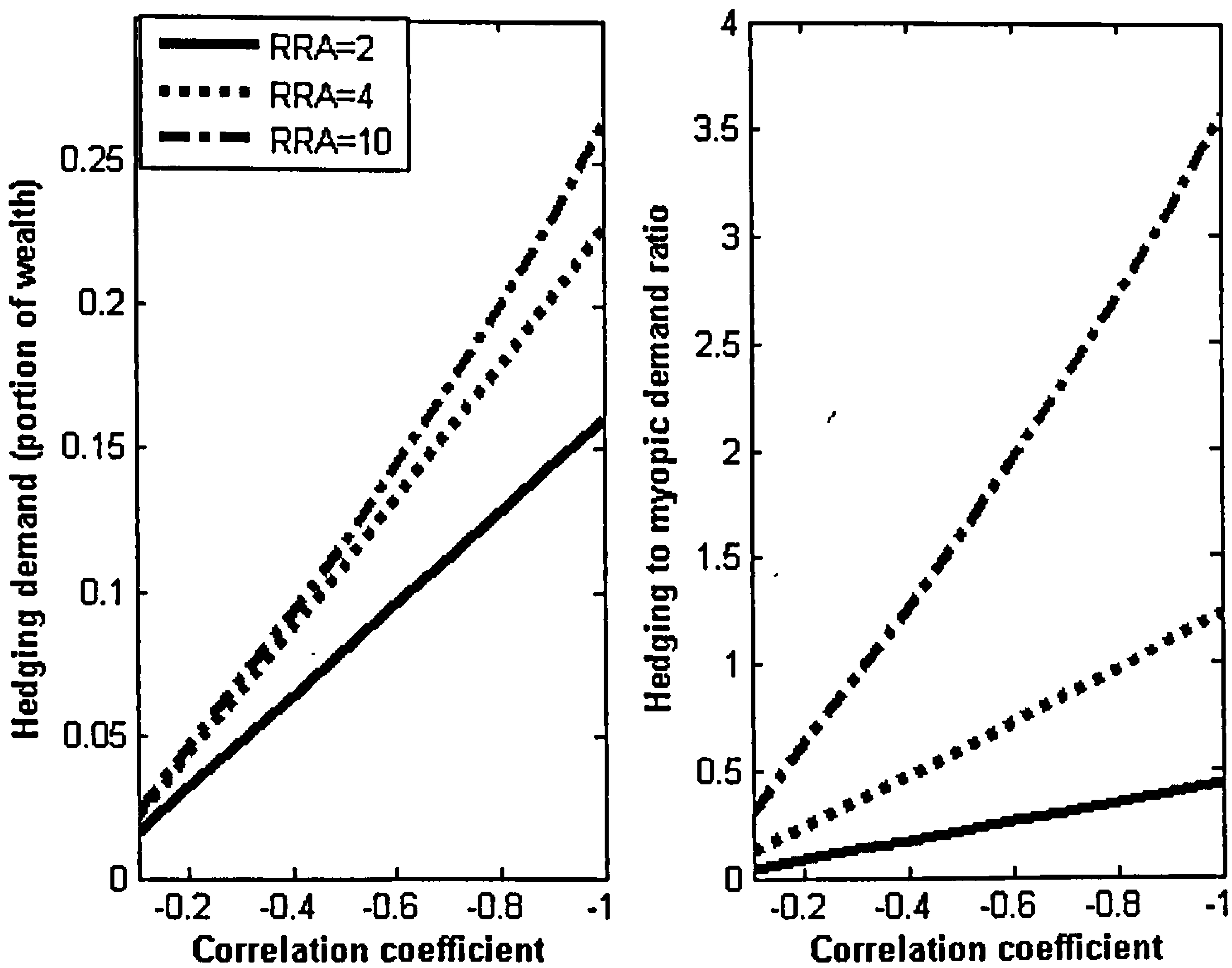
$$c(t)=-\frac{\kappa\theta}{b}\ln[\frac{2\psi\exp(\frac{(a+\psi)(T-t)}{2})}{(a+\psi)(\exp(\psi(T-t)-1))+2\psi}] \quad (2.78)$$

Figure 2.1: Myopic and Total equity demand



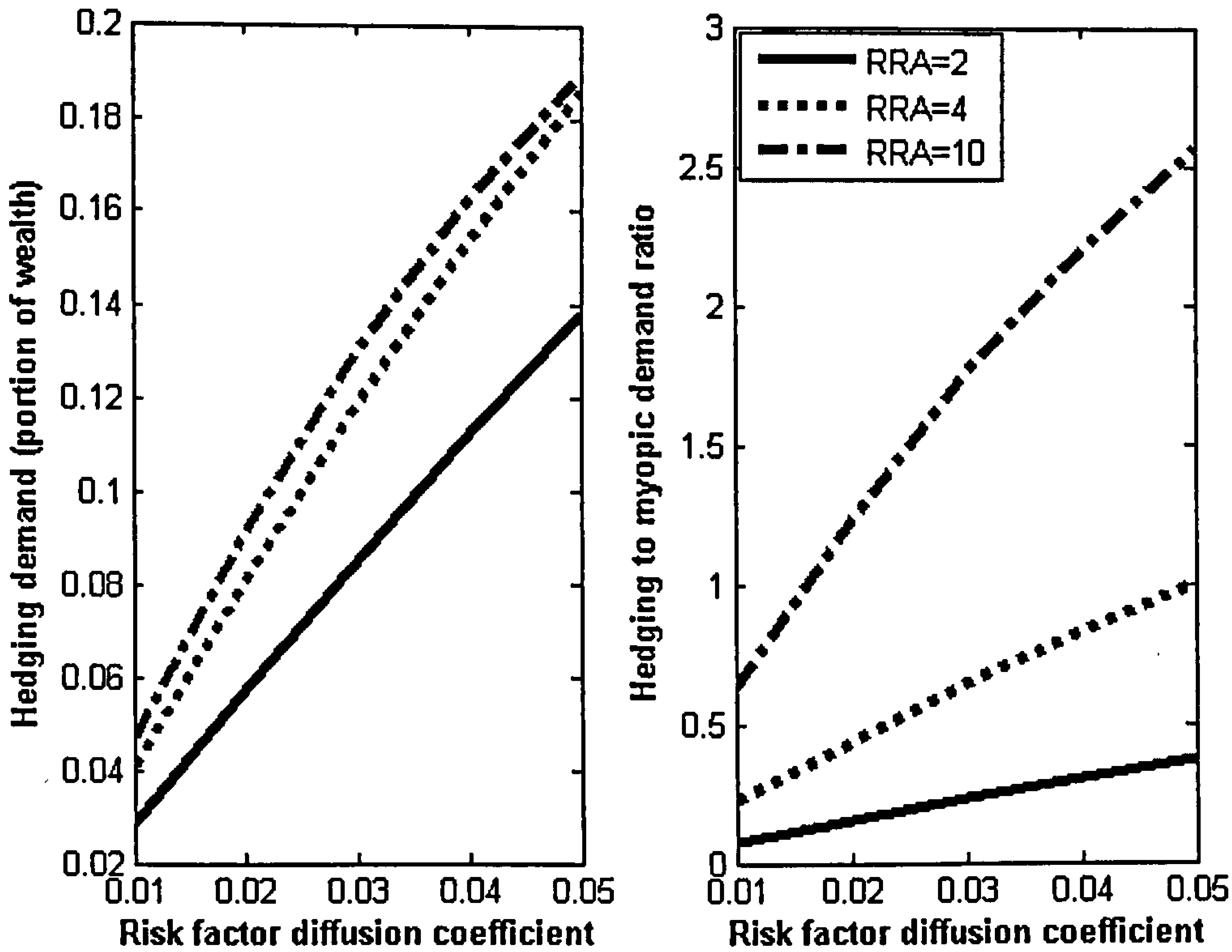
This Figure shows the the myopic and the total equity demand as a portion of an investor's wealth for three degrees of relative risk aversion ($\gamma = 2$, dashed line for the myopic and solid line for the total demand, $\gamma = 4$, dashed line for the myopic and dotted line for the total demand, $\gamma = 10$, dashed line for the myopic and dash-dotted line for the total demand), as the investment horizon increases from 1 to 15 years. The employed parameter values are: $\mu = 1.6$, $\sigma_r = 0.034$, $\sigma_s = 0.9$, $\kappa = 0.1$, $\rho_{sr} = -0.6$, $\theta = 0.058$.

Figure 2.2: The impact of the correlation coefficient on equity demand



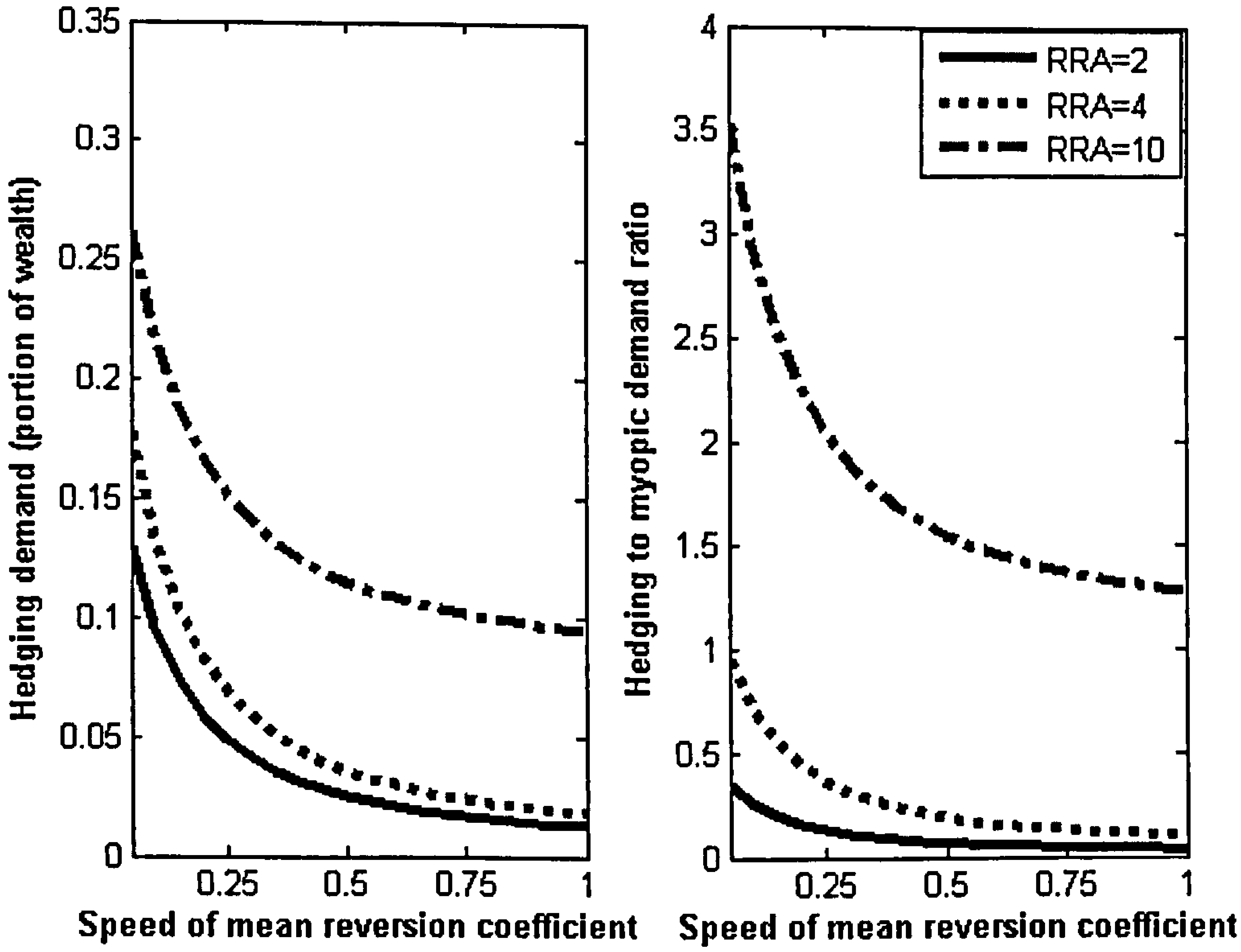
This Figure exhibits the hedging equity demand (left panel) as a portion of an investor's wealth with a horizon of 15 years for different degrees of relative risk aversion ($\gamma = 2$- solid line, $\gamma = 4$- dotted line, $\gamma = 10$- dash-dotted line), as the correlation coefficient varies from $\rho_{sr} = -0.1$ to $\rho_{sr} = -1$. The right panel presents the ratio of the hedging to the myopic equity demand for the same investor as this correlation coefficient varies.

Figure 2.3: The impact of the risk factor's diffusion coefficient on equity demand



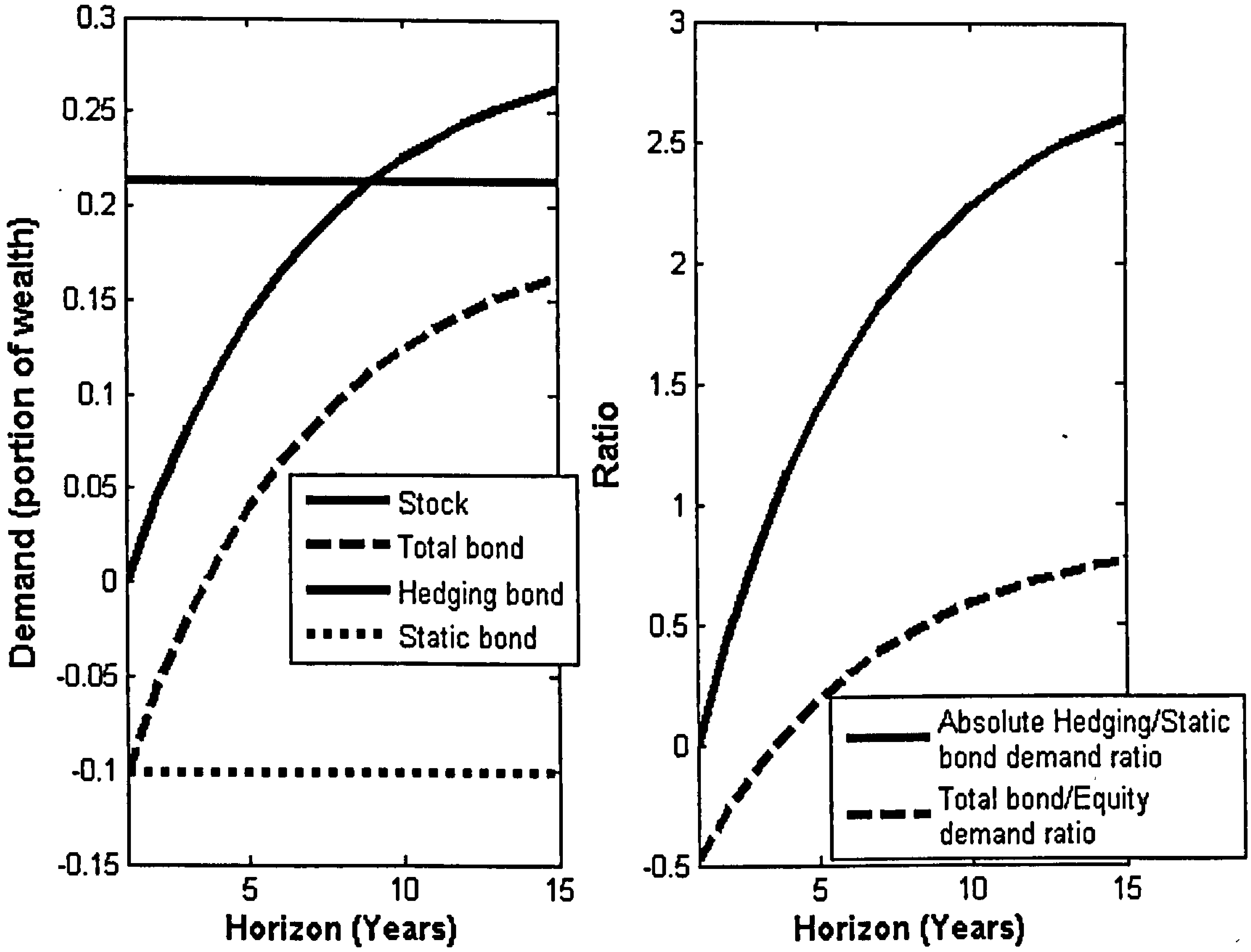
This Figure exhibits the hedging equity demand (left panel) as a portion of an investor's wealth with a horizon of 15 years for different degrees of relative risk aversion ($\gamma = 2$- solid line, $\gamma = 4$- dotted line, $\gamma = 10$- dash-dotted line), as the risk factor's diffusion coefficient increases from $\sigma_r = 0.01$ to $\sigma_r = 0.05$. The right panel presents the ratio of the hedging to the myopic equity demand for the same investor as this diffusion coefficient increases.

Figure 2.4: The impact of the speed of mean-reversion on equity demand



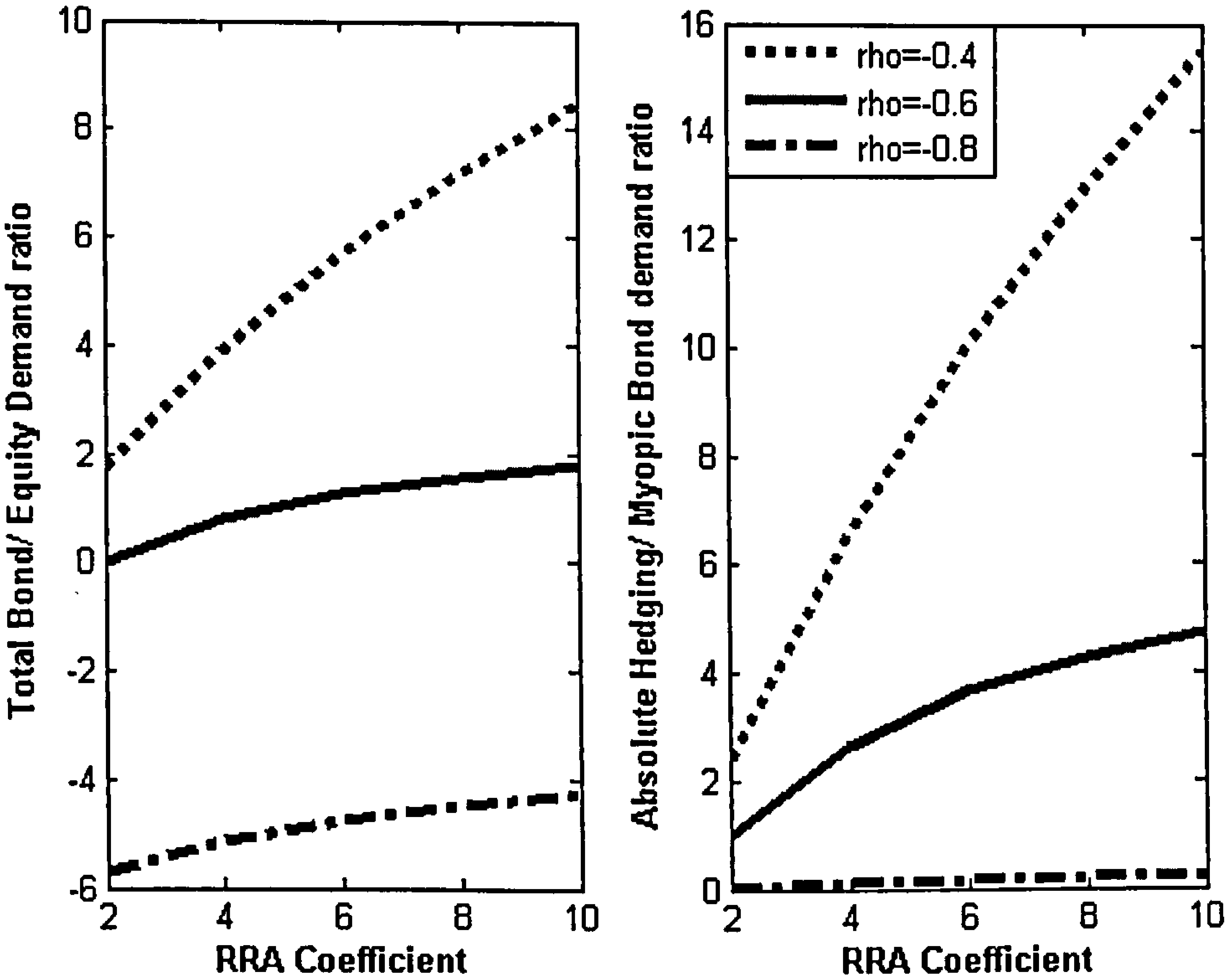
This Figure exhibits the hedging equity demand (left panel) as a portion of an investor's wealth with a horizon of 15 years for different degrees of relative risk aversion ($\gamma = 2$- solid line, $\gamma = 4$- dotted line, $\gamma = 10$- dash-dotted line), as the coefficient of the speed of mean-reversion increases from $\kappa = 0.05$ to $\kappa = 1$. The right panel presents the ratio of the hedging to the myopic equity demand for the same investor as this coefficient increases.

Figure 2.5: Bond and equity demands



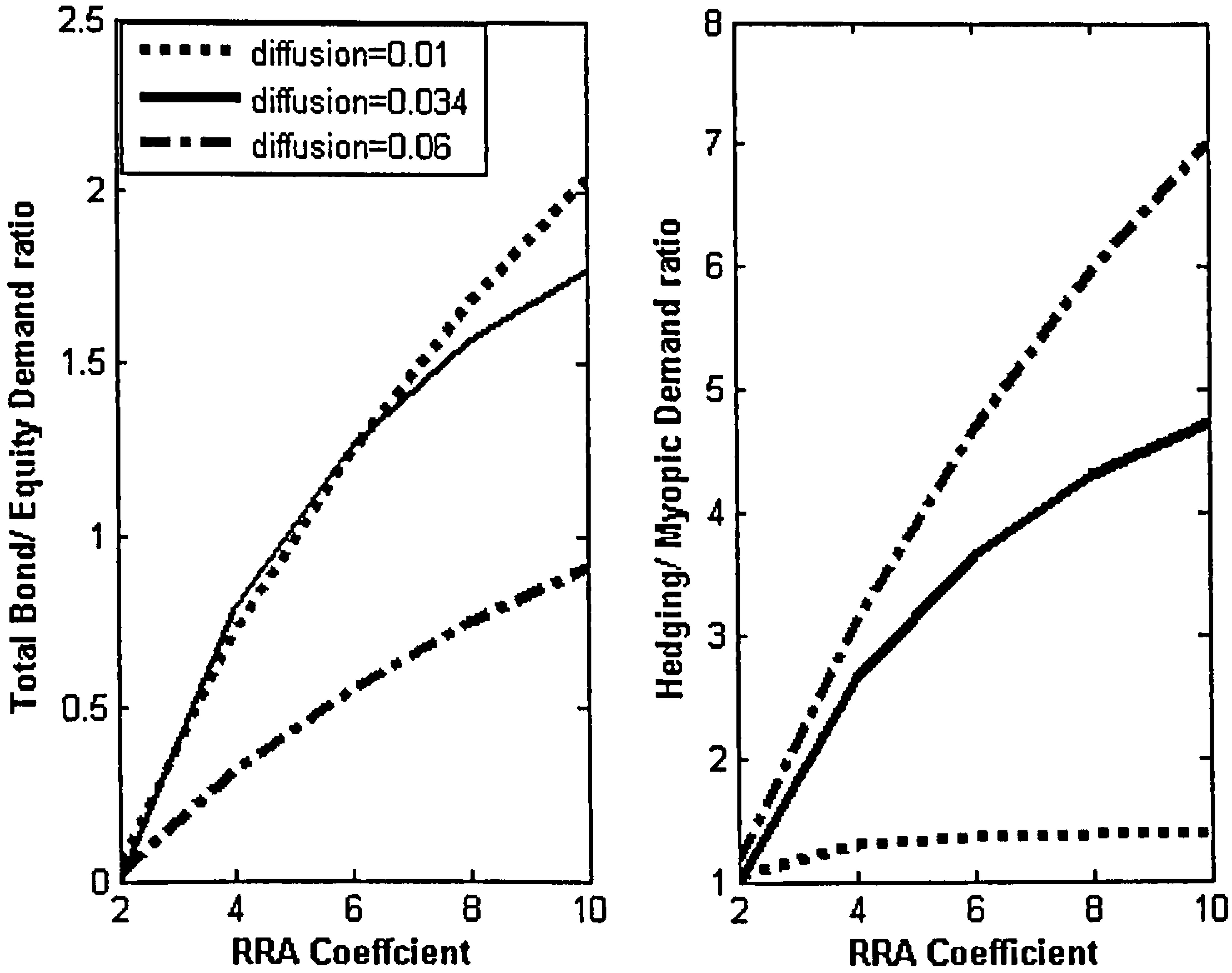
The left panel shows the total demand, as a portion of wealth, for the equity (horizontal solid line), the total demand for the bond (dashed line), the hedging demand for the bond (solid line) and the myopic demand for the bond (horizontal dotted line) for an investor with coefficient of relative risk aversion $\gamma = 4$, as his investment horizon increases from 1 to 15 years. The right panel exhibits the ratio of the total bond to equity demand (dashed line) and the ratio of the hedging to the myopic bond demand in absolute value (solid line) for an investor with relative risk aversion $\gamma = 4$, as his horizon increases. The parameter values used for both panels are: $\mu = 1.6$, $\sigma_r = 0.034$, $\sigma_s = 0.9$, $\kappa = 0.1$, $\rho_{sr} = -0.6$, $\theta = 0.058$, $\lambda = -0.01$ and $r = 0.05$.

Figure 2.6: The impact of the correlation coefficient on bond and equity demands



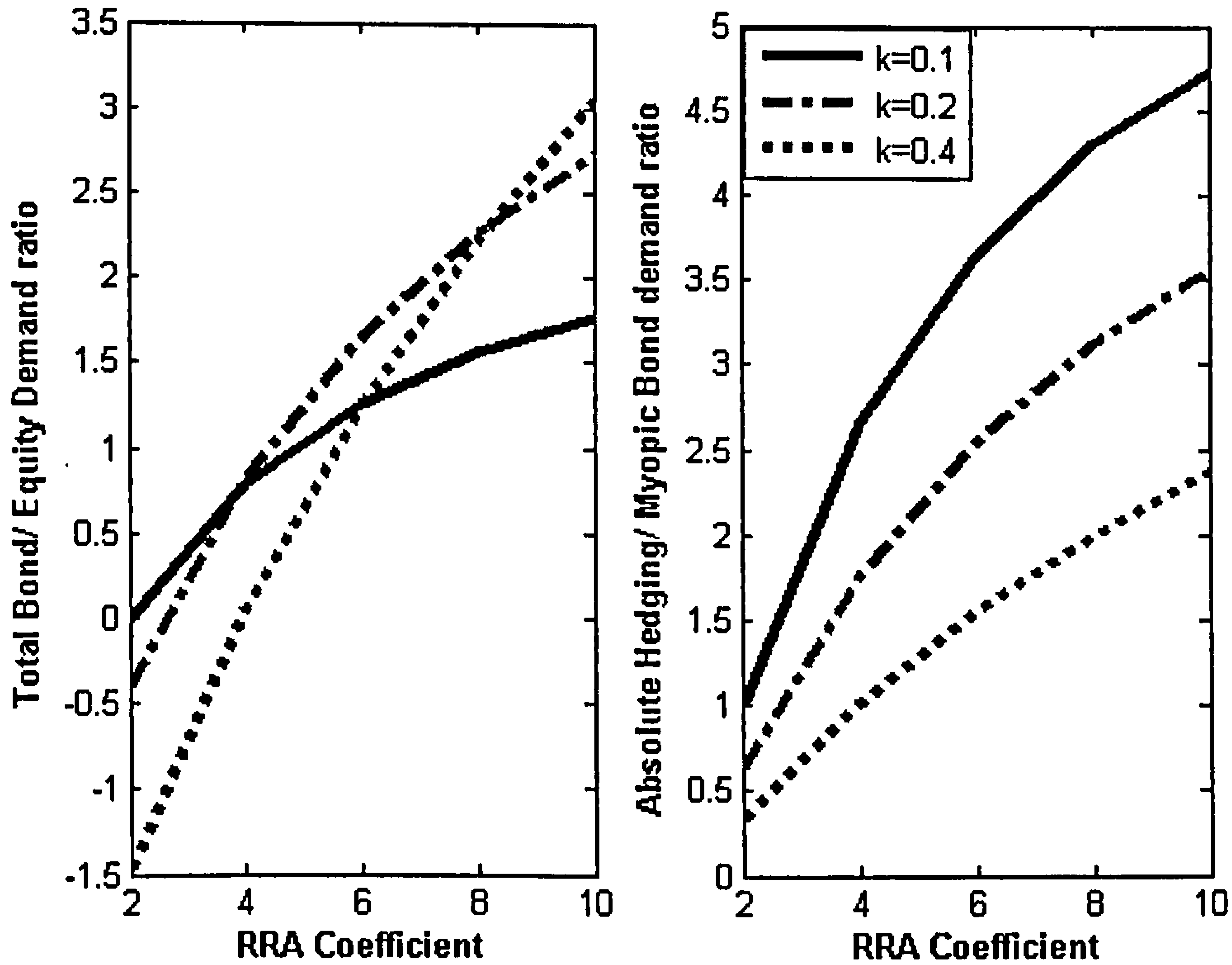
The left panel shows the total bond to equity demand ratio for an investor with 15 years of horizon, as his relative risk aversion increases, for three different values of the correlation coefficient between the shocks affecting the interest rate process and the equity return process. The right panel shows the absolute value of the hedging to myopic bond demand ratio for these values of the correlation coefficient, as the investor's relative risk aversion increases.

Figure 2.7: The impact of the risk factor's diffusion coefficient on bond and equity demands



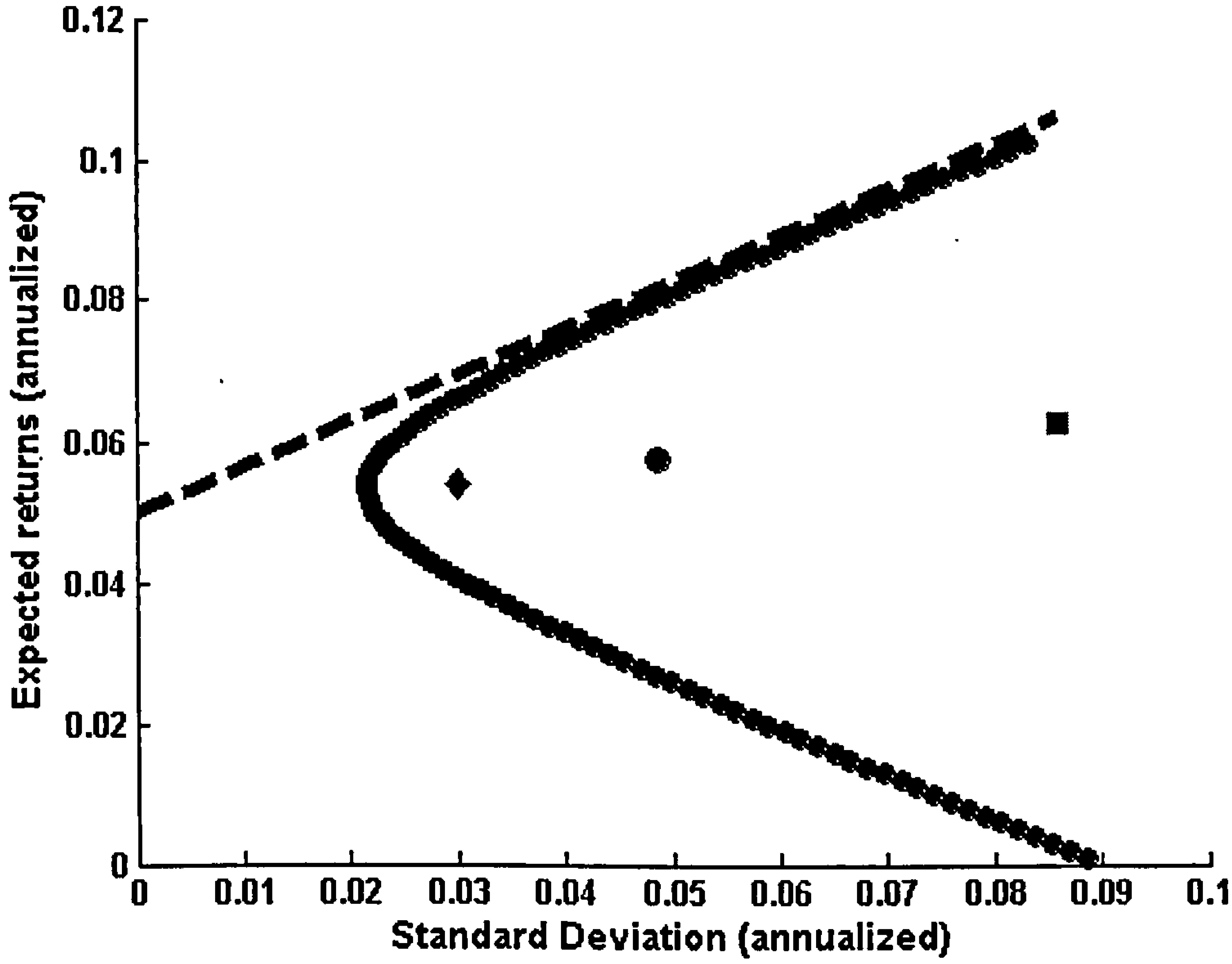
The left panel exhibits the total bond to equity demand ratio for an investor with 15 years horizon, as his relative risk aversion increases, for three different levels of the diffusion coefficient of the interest rate process ($\sigma_r = 0.01$- dotted line, $\sigma_r = 0.034$- solid line, $\sigma_r = 0.06$- dash-dotted line). The right panel shows the absolute hedging to myopic bond demand ratio for these values of the diffusion coefficient, as the investor's relative risk aversion increases.

Figure 2.8: The impact of the speed of mean-reversion on bond and equity demands



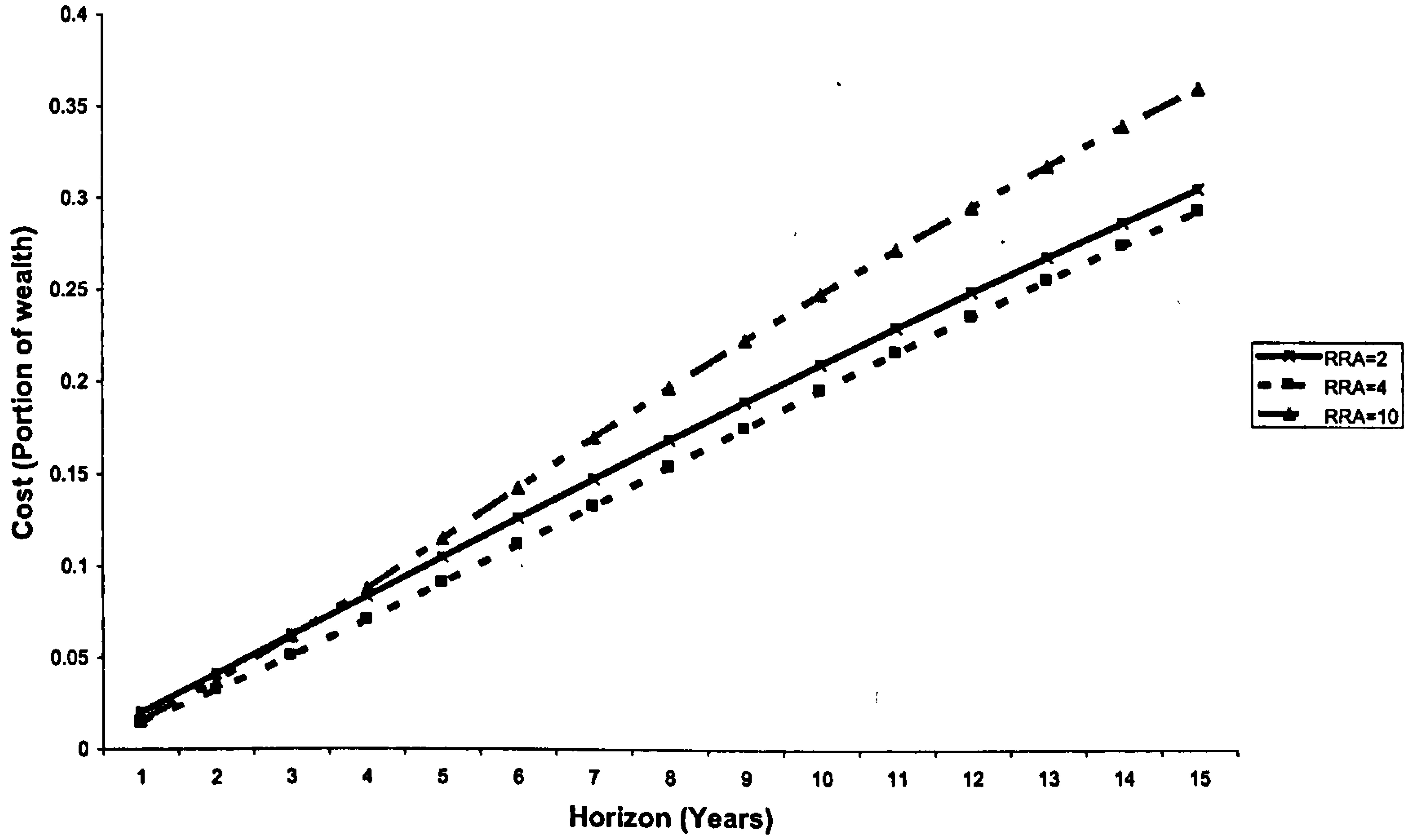
The left panel exhibits the total bond to equity demand ratio for an investor with a horizon of 15 years, as his relative risk aversion increases, for three different values of the coefficient of the speed of mean-reversion ($\kappa = 0.1$- solid line, $\kappa = 0.2$- dash-dotted line, $\kappa = 0.4$- dotted line). The right panel shows the absolute hedging to myopic demand ratio for these three values of the coefficient of the speed of mean-reversion, as the investor's relative risk aversion increases.

Figure 2.9: Myopic efficient frontier and long-term portfolio choices



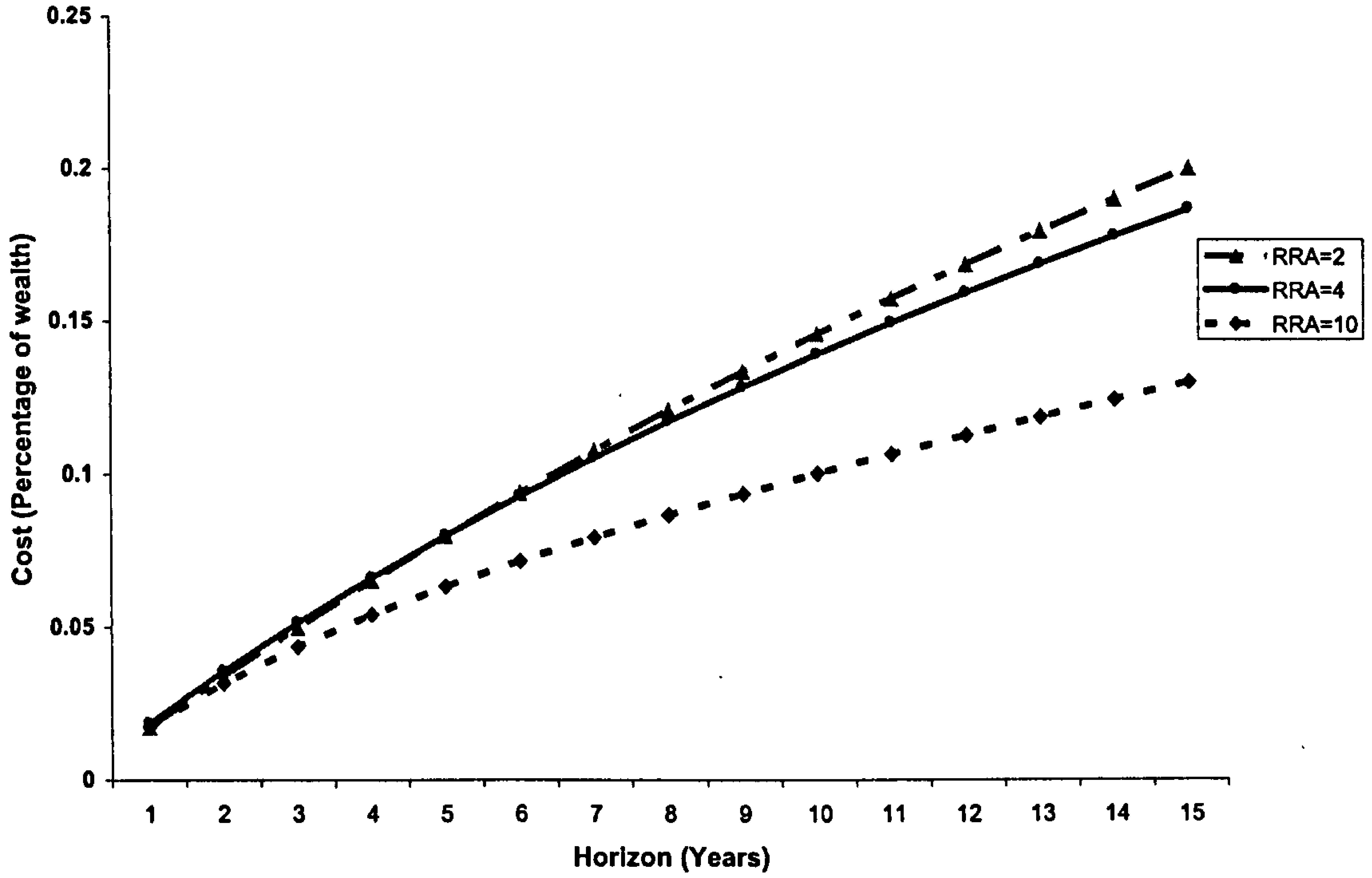
This Figure shows the efficient frontier constructed using the equity and the bond returns. The parameter values used to generate this frontier are: $\mu = 1.6$, $\sigma_r = 0.034$, $\sigma_s = 0.9$, $\kappa = 0.1$, $\rho_{sr} = -0.6$, $\theta = 0.058$, $\lambda = -0.01$ and $r = 0.05$. The dashed line is the tangency line starting from the risk-free rate $r = 0.05$ in the expected returns axis. The square, the circle and the diamond, to the right of the frontier, show the pairs of expected return and standard deviation, derived from the optimal portfolios formed by a long-term investor with 15 years of horizon and coefficients of relative risk aversion equal to $\gamma = 2$ (square), $\gamma = 4$ (circle), $\gamma = 8$ (diamond).

Figure 2.10: Utility gains from market completeness



This Figure shows the portion of the wealth, which, if deducted from the investor acting in the complete market, would yield the same utility as the utility derived by the investor acting in the incomplete market. This portion of wealth (cost of incomplete market) is depicted, as the horizon increases from 1 to 15 years, for three degrees of relative risk aversion ($\gamma = 2$- solid line, $\gamma = 4$- dotted line, $\gamma = 10$- dash-dotted line).

Figure 2.11: Utility gains from optimizing intertemporally



This Figure shows the portion of the wealth, which, if deducted from the investor optimizing intertemporally within the complete market setting, would yield him the same utility as the utility derived by the investor acting myopically. This portion of wealth (cost of acting myopically) is depicted, as the horizon increases from 1 to 15 years, for three degrees of relative risk aversion ($\gamma = 2$- dash-dotted line, $\gamma = 4$- solid line, $\gamma = 10$- dotted line).

Chapter 3

Dynamic Bond Portfolio Choice with Macroeconomic Information

3.1 Introduction

This chapter examines the optimal portfolio choice of a long-term bond investor, who faces a set of macroeconomic risk factors, both observable (inflation and output gap) and latent ones (real interest rate, inflation central tendency and real interest rate central tendency). It makes use of the essentially affine macro-finance term structure model of Dewachter et al. (2006) that allows for time-varying risk premia and consistently estimates the central tendencies simultaneously with the term structure. Employing this setup, the investment as well as the hedging opportunities provided by consistently priced zero-coupon bonds for a power utility agent are examined.

There are a series of ways that this chapter contributes to the literature. Firstly, it incorporates macroeconomic information in an asset allocation context and shows how this can be of significant use for both a myopic and a long-term investor. Despite the conclusion of Cochrane (2007, p. 242) that finance has a lot to say about macroeconomics, macroeconomic information has been relatively neglected in the dynamic asset allocation literature. The stochastic factors affecting the investment opportunity set are commonly assumed to be financial variables, such as the dividend yield (see Barberis, 2000) and the Sharpe ratio (see Wachter, 2002) that exhibit only a modest degree of predictive ability. Nevertheless, macroeconomic variables have been shown to be significant predictors of future bond returns (see Ang and Piazzesi, 2003).[1]

Secondly, it focuses on bond portfolio choice that is relatively unexplored in the literature, since the majority of the studies on dynamic asset allocation examine stock-only portfolios. The notable exceptions are the studies of Campbell and Viceira (2001), Brennan and Xia (2002) and Sangvinatsos and Wachter (2005). The degree of the bond yields' predictability as well as the bonds' no-arbitrage pricing, based on the underlying stochastic factors, imply that a bond portfolio setting provides an even more robust framework to examine, in contrast to most of the literature that uses ad hoc assumptions for the asset returns' dynamics (see Brennan et al., 1997 for an early example). Furthermore, bonds-only portfolios are extremely important for the fund management industry and for central banks.

The related work on dynamic bond portfolio choice includes the studies of Brennan and Xia (2000) and Sorensen (1999), who assume that interest rates are as in Vasicek (1977). A similar framework has been employed in Xia (2002), who examines the impact of limited access to nominal bonds on an investor's welfare. Munk et al. (2004) examine the stock-bond mix of a long-term power utility investor in the presence of mean-reverting returns, stochastic interest rates à la Vasicek and inflation uncertainty. This framework has been further exploited for the derivation of the optimal portfolio-consumption policy of a long-term investor by Munk and Sorensen (2004). In this study, apart from a Vasicek interest rate model, a three-factor,

[1]Flavin and Wickens (2003) argue that macroeconomic variables contain significant information for the covariance structure of the assets' returns, showing that this property can be of great importance for mean-variance investors.

non-Markovian, Heath- Jarrow-Morton term structure model is also employed. The results show that the hedging demand is very sensitive to the choice of the term structure model. Theoretical approaches to bond-only portfolio selection problems have been considered by Schroder and Skiadas (1999), Kargin (2003), Tehranchi and Ringer (2004) as well as Liu (2007).

Our study also allows for time-varying bond premia, in contrast to Campbell and Viceira (2001) and Brennan and Xia (2002) who assume them to be constant. Hence, we can capture the failure of the expectations hypothesis. Moreover, we use a macro-finance term structure model rather than the purely latent factor term structure model that has been commonly used in the finance literature and has been employed by Sangvinatsos and Wachter (2005). Unlike the latent factor framework that lacks a clear economic interpretation, the macro-finance model of Dewachter et al. (2006) allows us to reach more robust conclusions for bond portfolio choice. It additionally allows the central tendencies of inflation and the real interest rate to affect bond premia and hence the optimal portfolio choice, a concept not considered before to the best of our knowledge. Furthermore, the existence of five underlying risk factors enables us to examine the implications of portfolio selection among multiple bonds.

In addition to the previous issues, this study also serves as an evaluation of the employed term structure model from an asset allocation perspective. The term structure literature has mainly focused on fitting past and predicting future yields, while the asset allocation perspective allows us to examine a series of more general issues. In particular, the covariance and correlation structure of the bonds' returns is extremely important for the formation of optimal portfolios. Equally important are the implied market prices of risk that affect the sensitivities of an investor's wealth to the underlying macroeconomic factors. In other words, the implied risk premia contain significant information for the degree of risk aversion and the horizon of market participants.

Finally, we can derive conclusions with respect to what a long-term investor regards as a riskless asset. This very important issue has been repeatedly considered in the literature (see Modigliani and Sutch (1965), Stiglitz (1970) and Fischer (1975) for early discussions), but it was only recently that Wachter (2003) provided a formal theoretical treatment. In particular, we are able to introduce real bonds in the asset menu of the investor, that helps us investigate how the definition of the riskless asset is modified when the investor's horizon changes. The introduction of real bonds allows us also to examine their role for investment and hedging purposes.

The rest of this chapter is organized as follows: Section 3.2 outlines the employed term structure model, while Section 3.3 discusses the data issues and the implications of the estimated model. Section 3.4 derives the optimal portfolio choice for the long-term investor in complete markets and Section 3.5 derives the corresponding portfolio choice in an incomplete market setting. Section 3.6 discusses the results for the portfolio choice in both complete and incomplete bond markets, while Section 3.7 concludes.

3.2 The term structure model

3.2.1 Risk factors

We employ the setup of Dewachter et al. (2006). There are five stochastically time-varying risk factors: the output gap y, the inflation rate π, the real interest rate ρ, the inflation central tendency π^* and the central tendency of the real interest rate ρ^*.[2]

[2]For precision, the real instantaneously risk-free rate prevailing in an economy with stochastic price level as derived in Section 3.4.2 is slightly different from the definition of Dewachter et al. (2006), but we keep the same notation for ease of reference.

The dynamics of the risk factors, $(y, \pi, \rho, \pi^*, \rho^*)$, are characterized by the following SDEs:

$$dy = [\kappa_{yy}y + \kappa_{y\pi}(\pi - \pi^*) + \kappa_{y\rho}(\rho - \rho^*)]dt + \sigma_y dw_y \tag{3.1}$$

$$d\pi = [\kappa_{\pi y}y + \kappa_{\pi\pi}(\pi - \pi^*) + \kappa_{\pi\rho}(\rho - \rho^*)]dt + \sigma_\pi dw_\pi \tag{3.2}$$

$$d\rho = [\kappa_{\rho y}y + \kappa_{\rho\pi}(\pi - \pi^*) + \kappa_{\rho\rho}(\rho - \rho^*)]dt + \sigma_\rho dw_\rho \tag{3.3}$$

$$d\pi^* = \kappa_{\pi^*\pi^*}(\pi^* - \theta_{\pi^*})dt + \sigma_{\pi^*} dw_{\pi^*} \tag{3.4}$$

$$d\rho^* = \kappa_{\rho^*\rho^*}(\rho^* - \theta_{\rho^*})dt + \sigma_{\rho^*} dw_{\rho^*} \tag{3.5}$$

where $w_j(t), j = \{y, \pi, \rho, \pi^*, \rho^*\}$ denote independent standard Brownian motions defined on the probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F}$ and time set $[0, T]$, $0 \leq T < \infty$..

The assumed dynamics of the output gap (y), inflation (π) and the real interest rate (ρ) imply that these are affected by the short-run deviations from their central tendencies (the central tendency of the output gap is zero). Moreover, this structure implies that the central bank follows a feedback rule for the real interest rate. The central tendencies of the inflation (π^*) and the real interest rate (ρ^*) are allowed to be mean-reverting processes, capturing possible inertia in the adjustment process.

Vector X contains these risk factors:

$$X = (y,\ \pi,\ \rho,\ \pi^*,\ \rho^*)^T \tag{3.6}$$

Re-writing the dynamics of the risk factors in a vector form, we have:

$$dX = [\bar{\psi} + KX]dt + Sdw \tag{3.7}$$

where $\bar{\psi} = (0,\ 0,\ 0,\ -\kappa_{\pi^*\pi^*}\theta_{\pi^*},\ -\kappa_{\rho^*\rho^*}\theta_{\rho^*})^T$ is a 5x1 vector, dw is a 5x1 vector

$$dw = (dw_y,\ dw_\pi,\ dw_\rho,\ dw_{\pi^*}, dw_{\rho^*})^T \tag{3.8}$$

K is a 5x5 matrix with elements

$$K = \begin{bmatrix} \kappa_{yy} & \kappa_{y\pi} & \kappa_{y\rho} & -\kappa_{y\pi} & -\kappa_{y\rho} \\ \kappa_{\pi y} & \kappa_{\pi\pi} & \kappa_{\pi\rho} & -\kappa_{\pi\pi} & -\kappa_{\pi\rho} \\ \kappa_{\rho y} & \kappa_{\rho\pi} & \kappa_{\rho\rho} & -\kappa_{\rho\pi} & -\kappa_{\rho\rho} \\ 0 & 0 & 0 & \kappa_{\pi^*\pi^*} & 0 \\ 0 & 0 & 0 & 0 & \kappa_{\rho^*\rho^*} \end{bmatrix} \tag{3.9}$$

and $S = \text{diag}(\sigma_y, \sigma_\pi, \sigma_\rho, \sigma_{\pi^*}, \sigma_{\rho^*})$ is a diagonal 5x5 matrix.

The last assumption essentially implies that there is no interrelationship in the volatility structure of the macro factors. Following Duffee (2002), the market price of risk is assumed to be time-varying and affine in the risk factors, given by $\xi = S\Lambda + S^{-1}\Xi X$, where Λ is a 5x1 vector and Ξ is a 5x5 matrix, the elements of which are estimated from the joint model. The matrix Ξ contains the sensitivities of the prices of risk to the macroeconomic factors. In particular, to avoid identification problems, only a restricted set of the market prices of risk is estimated (see Dai and Singleton, 2000 for a discussion). This term structure model can be classified as $A_0(5)$ within the family of the essentially affine models.

3.2.2 Bond returns

The price of a zero-coupon default-free nominal bond at time t maturing at time $t+\tau \equiv T$ is assumed to be given by $P(X,t) = \exp(-a(\tau) - b(\tau)^T X)$, where $a(\tau)$ is a scalar and $b(\tau)$ is a 5x1 vector, with initial conditions $a(0) = 0$ and $b(0) = 0_{5x1}$. The no-arbitrage condition under the risk neutral measure Q requires that the expected excess returns of this bond are zero, since the market price of risk is zero under Q.

Given the market price of risk ξ, it holds that

$$d\tilde{w} = dw + \xi dt \tag{3.10}$$

where $\tilde{w}(t) = (\tilde{w}_y, \tilde{w}_\pi, \tilde{w}_\rho, \tilde{w}_{\pi^*}, \tilde{w}_{\rho^*})^T$ is a vector of independent standard Brownian motions under the risk neutral measure.

The dynamics of X under Q are given by:

$$dX = [\bar{\psi} - S^2\Lambda + (K - \Xi)X]dt + Sd\tilde{w} \tag{3.11}$$

Therefore, the no-arbitrage condition for the bond price under Q implies that the following PDE should hold:

$$(\bar{\psi} - S^2\Lambda + (K-\Xi)X)^T \frac{\partial P}{\partial X} + \frac{1}{2}\frac{\partial^2 P}{\partial X \partial X^T} S^2 + \frac{\partial P}{\partial t} = rP \tag{3.12}$$

where r is the instantaneously nominal risk-free rate, given by $r \equiv \pi + \rho = \delta_1^T X$, where $\delta_1 = (0\ 1\ 1\ 0\ 0)^T$.

Substituting the partial derivatives of P into the PDE and using the method of undetermined coefficients, we end up with a system of ODEs in $a(\tau)$ and $b(\tau)$:

$$\frac{\partial a}{\partial \tau} = \bar{\psi}^T b - \frac{1}{2} Tr(b^T S^2 b) - \Lambda^T S^2 b \tag{3.13}$$

and

$$\frac{\partial b}{\partial \tau} = \delta_1 + (K - \Xi)^T b \tag{3.14}$$

along with the corresponding initial conditions. These ODEs are solved numerically using a Runge-Kutta scheme.[3]

Under the risk-neutral measure Q, the absence of arbitrage opportunities dictates that the returns' dynamics of the zero-coupon nominal bond i in this setup are given by:

$$\frac{dP_i}{P_i} = rdt - b(\tau)^T S d\tilde{w} \tag{3.15}$$

Switching from the risk-neutral measure Q to the physical measure P, the bond returns' dynamics under the physical measure are given by the SDE:

$$\begin{aligned} \frac{dP_i}{P_i} &= (r - b(\tau)^T S\xi)dt - b(\tau)^T S dw \\ &= (r - b(\tau)^T S^2 \Lambda - b(\tau)^T \Xi X)dt - b(\tau)^T S dw \end{aligned} \tag{3.16}$$

3.3 Data, Estimation and implications

3.3.1 Data

In order to estimate the term structure model, Dewachter et al. (2006) make use of data for the yields on zero-coupon US Treasury bonds with maturities of 3 and 6 months and 1, 2, 5 and 10 years.[4] A quarterly frequency is adopted for the period 1964:Q1 to 1998:Q4. Inflation was constructed by taking the annual percentage change in the CPI index provided by the IMF and the output gap series were con-

[3]The ODEs in this chapter were numerically solved in MATLAB 7.0 using the function **ode45** and in Mathematica 6 using the function **ndsolve**.

[4]The data are available on Gregory Duffee's website.

structed based on data provided by the Congressional Budget Office. The filtered time series for the five factors are depicted in Figure 3.1 and Figure 3.2.[5] It is important to observe that there is significant variation in the underlying macroeconomic variables, hence we expect that the investment opportunity set significantly varies through time.

3.3.2 Estimation

In order to estimate in an efficient way both the dynamics of the underlying risk factors and the prices of risk, Dewachter et al. (2006) employ a Kalman filter algorithm.[6] We refer to Section 3 of their study for the proper definition of the measurement and the transition equation of the model as well as for its implications for the macroeconomic dynamics. The estimates of the parameters and their standard errors are given in Table 3.1. The results show that the market prices of risk are time-varying indeed.[7] Moreover, the volatilities of the risk factors are estimated to be relatively low, an important observation for the rest of the analysis.

3.3.3 Implications for the bonds' returns

Nominal Bonds' sensitivities to the risk factors

Given the estimates of the term structure model and the market prices of risk, we firstly examine the sensitivities of the bond returns to the underlying risk factors. This will help us understand how the expected returns and the volatilities of the

[5] We would like to thank Marco Lyrio for the provision of the data.

[6] The presence of unobservable factors dictates the use of a filtering procedure. Duffee and Stanton (2008) argue that for Gaussian models the Kalman filter yields more efficient estimates in comparison to the Efficient Method of Moments when the time series are highly persistent.

[7] It is important to note that, following the dynamic asset allocation literature, we use the point estimates of the coefficients. Even though the unrestricted coefficients of the matrix Ξ are significantly different from zero, this is not true for the constant component of the inflation and the output gap risk premia, i.e. Λ_y and Λ_π. Setting either parameter to zero could slightly affect the analysis.

bonds for various maturities are formed. In particular, we solve the ODEs in $b(\tau)$ given by equation (3.14) in order to derive the factor loadings. Figure 3.3 presents these loadings adjusted for the corresponding bond maturity, $(b(\tau)/\tau)$.

The inflation and the real rate affect only the very short maturities. Practically, their impact on the nominal bonds with maturity beyond one year is negligible. Negligible is also the impact of the output gap, regardless of the bonds' maturity. On the other hand, the central tendency of inflation has a dominant impact on the prices of bonds with maturities longer than one year. The impact of the real rate central tendency is also significant for similar maturities.

It should be noted that Figure 3.3 presents the factor loadings adjusted for the corresponding maturity $(b(\tau)/\tau)$, while for the bond returns and their volatilities, it is the plain values of $b(\tau)$ that matter. Hence, one can observe that it is mainly the inflation central tendency π^* that affects the nominal bond returns and volatilities for maturities longer than one year.

Expected excess bond returns

An extremely important input for any portfolio choice problem is the expected excess returns offered by the available assets. Hence, it is interesting to examine the implications of this model for the expected excess returns of the nominal bonds for various maturities. In particular, given (3.16), these are given by $-b(\tau)^T S^2 \Lambda - b(\tau)^T \Xi X$. Figure 3.4 plots the expected excess returns of the nominal zero-coupon bonds with maturities 1, 3, 5, 7 and 10 years for the whole sample period.

There are a series of observations to make: Firstly, there is significant time-variation in the expected excess returns. This finding underlines the importance of the returns' predictability through a set of underlying risk factors that generates horizon effects for an investor's optimal portfolio choice. The expectations hypothesis that assumes constant term premia and has been used by Campbell and Viceira

(2001) and Brennan and Xia (2002) for dynamic bond portfolio choice is clearly rejected, given the statistical significance of the time-varying risk coefficients in Ξ.

Secondly, this term structure model implies much more "reasonable" risk premia in comparison to a series of previously used models.[8] Thirdly, another attractive feature of this essentially affine term structure model is that the variation in the expected excess returns is significantly higher than their average level. As Duffee (2002) notes, this is a necessary qualification for a model to be consistent with the variety of yield curve shapes observed in the data. The commonly used completely affine models fail to meet this requirement and, as a result, they are less fit for the selection of optimal portfolios.

Furthermore, it should be noted that the excess returns tend to strongly comove through time, despite the quite diverse economic conditions encountered in the sample period. We conjecture that this is true because the expected excess returns are mainly driven by the inflation central tendency.[9] For example, variations in the output gap have almost no effect on the bonds' returns. As a final comment, it should be stressed that for most of the periods, the 1-year zero-coupon bond has a relatively high expected excess return (higher than 0.5%). This fact may significantly affect the formation of optimal portfolios.

The Covariance and Correlation structure of bonds' returns

The term structure literature usually neglects the models' implications for the covariance and correlation structure of the bonds' returns. However, this is an extremely important input for portfolio selection that is traditionally built within a mean-

[8] See for example the extreme premia reported in Sangvinatsos and Wachter (2005) as well as the ones implied by the model of Dewachter and Lyrio (2006).

[9] This observation is analogous to the result of Cochrane and Piazzesi (2005) that a single factor can explain much of the time variation in expected excess bond returns. It is also related to the known result of Litterman and Scheinkmann (1991) that the parallel shifts of the yield curve can be captured by a single factor.

variance framework. Given the coefficient estimates and the factor loadings $b(\tau)$, we report in Table 3.2 the covariances and correlations for the 1, 2, 3, 5, 7 and 10-year nominal zero-coupon bonds' returns.

The volatilities of the returns are extremely low, especially for short maturities. An explanation for this result is that the factor that has a dominant role for bond pricing, the inflation central tendency, is estimated to have an extremely low volatility (see Table 3.1). Furthermore, this $A_0(5)$ term structure model does not allow for time-varying bond returns' volatilities and it ignores possible covariances between the various risk factors. According to Duffee (2002), the limited ability of the essentially affine models to capture the time variation in conditional variances is the price to pay for their superior forecasting power. This trade off is significantly taken into account when we report our results.

With respect to the correlations of the returns, these are found to be extremely high. In particular, for bonds with similar maturities the correlation is higher than 0.95. Again, this characteristic significantly affects the formation of optimal portfolios, since including bonds of similar maturities in the asset menu will lead to a nearly singular covariance matrix, leading to extreme results.

The maximal Sharpe ratio

Given the market price of risk $\xi = S\Lambda + S^{-1}\Xi X$, an interesting way to examine the implications of the present term structure model for an investor is to derive the Maximal Sharpe Ratio (MSR), given by the norm of ξ, i.e. $MSR = \sqrt{\xi^T \xi}$. Figure 3.5 depicts the MSR for the sample period. The MSR is always positive, because it is assumed that the investor can take both long and short positions in any asset. There are two very significant observations to make: Firstly, there are a series of periods where the (perceived) MSR is extremely high. As we discussed in the previous subsections, this finding is due to the combination of the relatively high expected excess returns and the very low volatilities implied by the estimated term structure model. This finding is essentially equivalent to a "bond premium puzzle", drawing the analogy to the equity premium puzzle of Mehra and Prescott (1985), since a high reward for bearing risk in the bond market implies highly volatile marginal utilities for the investors. Secondly, there is significant time-variation in the investment opportunities. This fact underlines the significance of horizon effects and market timing in portfolio selection.

3.4 Portfolio choice in complete markets

3.4.1 Real wealth and consumption and access to nominal bonds

This section examines the optimal portfolio choice in a complete nominal bond market for a long-term investor, who takes into account the stochastic evolution of the underlying risk factors. Formally, the investor has to allocate his wealth W at time t among 5 zero-coupon default-free bonds with different maturities (T_1, T_2,..., T_5), the returns of which are given in equation (3.16), and cash yielding the nominal instantaneously risk-free rate, $\frac{dP_0}{P_0} = rdt$. The portions of his wealth allocated to each of the zero-coupon bonds are collected in the 5x1 vector $\phi = (\phi_1, \phi_2, ..., \phi_5)^T$, while the portion of the wealth invested in the instantaneously riskless asset is $\phi_0 = 1 - i^T\phi$, where i is a vector of ones.

Utility over terminal real wealth

We firstly examine the case where the investor's utility is defined over terminal real wealth. Given the bond returns' dynamics, the wealth dynamics of the investor are given by the following SDE:

$$dW = W(\phi^T \frac{dP}{P} + (1 - i^T \phi) \frac{dP_0}{P_0}) =$$

$$= W(\phi^T [-B(\tau)^T S^2 \Lambda - B(\tau)^T \Xi X)] + r) dt - W \phi^T B(\tau)^T S dw \tag{3.17}$$

where now $\frac{dP}{P}$ is the vector of the 5 zero-coupon bond returns. Hence, $B(\tau)$ is now a matrix 5x5, since this stacks the column vectors $b(\tau)$ for each of the 5 bonds in a matrix form. In other words, $B(\tau) = (b(\tau_1), b(\tau_2), .., b(\tau_5))$, where $(\tau_1, \tau_2, ...,$ $\tau_5)$ are the durations of the 5 zero-coupon bonds.

Since the vector of the shocks affecting the returns of the risky assets dw is the same as the vector of shocks affecting the underlying risk factor dynamics, we can use the Martingale approach of Cox and Huang (1989) and Karatzas et al. (1987) to solve the optimal portfolio choice problem. The main observation is that there exists a unique pricing kernel m. This pricing kernel is a Stochastic Discount Factor (SDF) that converts the risky asset dynamics into a martingale process and it can be interpreted as a system of Arrow-Debreu prices. Given that ξ is the market price of risk in the complete nominal market, the dynamics of the nominal SDF are given by:

$$\frac{dm}{m} = -r dt - \xi^T dw \tag{3.18}$$

with initial value $m_{t_0} = 1$. The crucial observation for the martingale methodology is that the process $m_t W_t$ is a martingale too. As a result, $E_{t_0}[m_T W_T] = W_{t_0}$.

The long-term investor seeks to maximize

$$\max E_{t_0} \left\{ \frac{(\frac{W_T}{\Pi_T})^{1-\gamma}}{1-\gamma} \right\} \tag{3.19}$$

subject to the constraint that

$$E_{t_0}[m_T W_T] = W_{t_0} \tag{3.20}$$

In order to examine this problem, we need to specify the dynamics of the price level Π_t. Following Brennan and Xia (2002), the price level dynamics are given by the SDE:

$$\frac{d\Pi}{\Pi} = \pi dt + \sigma_\Pi^T dw \tag{3.21}$$

where σ_Π is a 5x1 vector. In general, the shocks affecting the stochastic price level need not be perfectly correlated with the shocks to the underlying risk factors in X.[10] In our setup, however, we get this perfect correlation since one of the underlying risk factors is the inflation rate π. In other words, we can specify the vector σ_Π as $\sigma_\Pi = (0\ \sigma_\pi\ 0\ 0\ 0)^T$, where σ_π is the diffusion coefficient of the inflation rate process specified in Section 3.2. As a result, we are in the "complete markets case" as termed by Brennan and Xia (2002)[11], in the sense that the shocks to the bonds' returns are perfectly correlated with the shocks affecting the underlying risk factors including the price level. Hence, we can use the martingale methodology for this case too.

Consequently, the dynamic portfolio choice problem becomes a static one. Forming the Lagrangian function:

$$L(W_T, l) = E_{t_0}\left\{\frac{(\frac{W_T}{\Pi_T})^{1-\gamma}}{1-\gamma}\right\} - l[E_{t_0}[m_T W_T] - W_{t_0}] \tag{3.22}$$

and taking the First Order Condition (FOC) with respect to the terminal wealth

[10]This case, which is equivalent to an incomplete market case, is examined by Sangvinatsos and Wachter (2005), where the underlying risk factors are latent factors with no clear macroeconomic interpretation and hence their shocks cannot be perfectly correlated with the shocks to the price level dynamics.

[11]As Brennan and Xia (2002, p. 1206) note, this is the case when "the expected rate of inflation π is not observable but must be inferred from the observation of the price level itself". This is exactly true in the Dewachter et al. (2006) term structure model that we are using, since the rate of inflation is calculated as the annual percentage change in the CPI.

we have:

$$\frac{\partial(.)}{\partial W_T} = 0 \Rightarrow W_T = (\Pi_T^{1-\gamma} l m_T)^{-\frac{1}{\gamma}} \Rightarrow \frac{W_T}{\Pi_T} = (l m_T \Pi_T)^{-\frac{1}{\gamma}} \tag{3.23}$$

where l is the Lagrange multiplier. Defining the variable $Z_t \equiv (l m_t \Pi_t)^{-1}$, real wealth at time t can be written as:

$$\frac{W_t}{\Pi_t} = (l m_t \Pi_t)^{-\frac{1}{\gamma}} = Z_t E_t[(Z_T)^{\frac{1}{\gamma}-1}] \tag{3.24}$$

As a result, wealth W is a function of Z, Π, X and t. Consequently, we can write it as $G(Z, \Pi, X, t) \equiv F(X, t) \Pi Z^{\frac{1}{\gamma}} = W(t)$.

Applying Ito's lemma, the dynamics of the variable Z are given by:

$$\frac{dZ}{Z} = (r - \pi + \sigma_\Pi^T \sigma_\Pi + \xi^T \xi - \sigma_\Pi^T \xi) dt + (\xi - \sigma_\Pi)^T dw \tag{3.25}$$

while under the risk neutral measure Q they are given by:

$$\frac{dZ}{Z} = (r - \pi + \sigma_\Pi^T \sigma_\Pi) dt + (\xi - \sigma_\Pi)^T d\tilde{w} \tag{3.26}$$

It should be also noted that the price level dynamics under the risk neutral measure Q are given by:

$$\frac{d\Pi}{\Pi} = (\pi - \sigma_\Pi^T \xi) dt + \sigma_\Pi^T d\tilde{w} \tag{3.27}$$

and the dynamics of X under Q are given by (3.11).

As Wachter (2002) argues, we can interpret wealth as a zero-coupon bond that pays a final amount W_T at time T. The no-arbitrage condition for a zero-coupon bond $G(Z, \Pi, X, t)$ implies that under the physical measure P, its instantaneous expected excess returns should be equal to the market price of risk multiplied by the diffusion of the wealth process G. It actually proves easier to work under the risk-neutral measure Q. In this case, the instantaneous expected excess returns should be equal to zero, since the market price of risk is zero under Q.

This argument implies that the following PDE should hold:

$$\frac{\partial G}{\partial Z}Z(r-\pi+\sigma_\Pi^T\sigma_\Pi)+[\bar{\psi}-S^2\Lambda+(K-\Xi)X]^T\frac{\partial G}{\partial X}+\frac{\partial G}{\partial \Pi}\Pi(\pi-\sigma_\Pi^T\xi)+$$
$$+\frac{\partial G}{\partial t}+\frac{1}{2}Tr(SS^T\frac{\partial^2 G}{\partial X\partial X^T})+\frac{1}{2}\frac{\partial^2 G}{\partial Z^2}Z^2(\xi-\sigma_\Pi)^T(\xi-\sigma_\Pi)+\frac{1}{2}\frac{\partial^2 G}{\partial \Pi^2}\Pi^2\sigma_\Pi^T\sigma_\Pi+$$
$$+(S\sigma_\Pi)^T\Pi\frac{\partial^2 G}{\partial \Pi\partial X}+\frac{\partial^2 G}{\partial Z\partial \Pi}\Pi Z(\xi-\sigma_\Pi)^T\sigma_\Pi+Z(\xi-\sigma_\Pi)^TS\frac{\partial^2 G}{\partial Z\partial X}=rG \quad (3.28)$$

along with the terminal condition $G(Z_T,X_T,\Pi_T,T)=\Pi_T Z_T^{\frac{1}{\gamma}}$.

It is also crucial to note that the optimally invested wealth should have a diffusion term identical to the diffusion term of the wealth process G. Therefore, for the optimal portfolio choice ϕ, it should hold that:

$$G(\phi^T(-B^TS)=\frac{\partial G}{\partial Z}Z(\xi-\sigma_\Pi)^T+\frac{\partial G}{\partial \Pi}\Pi\sigma_\Pi^T+(S\frac{\partial G}{\partial X})^T \quad (3.29)$$

Proposition 3.1

Let us conjecture the following form for the function $G(Z,X,\Pi,t)$:

$$G(Z,X,\Pi,t)=\Pi_t Z_t^{\frac{1}{\gamma}}F(X,t)=\Pi_t Z_t^{\frac{1}{\gamma}}\exp[\frac{1}{\gamma}(\frac{1}{2}X^TQ(t)X+d(t)^TX+c(t))] \quad (3.30)$$

where $Q(t)$, $d(t)$, $c(t)$ are 5x5 matrix, 5x1 vector and scalar functions of time correspondingly with terminal conditions $Q(T)=0_{5x5}$, $d(T)=0_{5x1}$ and $c(T)=0$.

For an investor who has a power utility over real terminal wealth with coefficient of relative risk aversion $\gamma\neq 1$ and has access to nominal bonds, his optimal portfolio choice is given by:

$$\phi_t = \frac{1}{\gamma}(B^T S^2 B)^{-1}(-B^T S^2 \Lambda - B^T \Xi X_t) + (1 - \frac{1}{\gamma})(B^T S^2 B)^{-1}(-B^T S)\sigma_\Pi$$
$$+\frac{1}{\gamma}(B^T S^2 B)^{-1}(-B^T S)S[d(t) + \frac{1}{2}(Q(t) + Q(t)^T)X_t] \tag{3.31}$$

with the functions $d(t)$ and $Q(t)$ satisfying the system of ODEs (3.99) - (3.100) given in the Appendix. The remainder $\phi_0 = 1 - i^T\phi$ is invested in the nominal instantaneously riskless asset (See Appendix for the Proof).

The first two terms of the optimal portfolio choice expression (3.31) compose the myopic term *à la* Markowitz (1952). In particular, the second term arises because the investor seeks to maximize his utility over real wealth having access to nominal bonds that are priced under the nominal SDF. The third term provides the hedging demand component *à la* Merton (1973). There are two interesting observations to make:

i) The long-term power utility investor has a hedging demand apart from the standard mean-variance myopic one. This depends on the diffusion of the bond returns dynamics as well as on the sensitivity of the investor's wealth to the risk factors, represented here by the functions $d(t)$ and $Q(t)$. It is straightforward to observe that, if the investor is not sensitive to any shifts in the risk factors, then there is no intertemporal hedging demand component, since $\frac{\partial G}{\partial X} = \mathbf{0}$.

ii) Both the myopic and the hedging bond demand components are characterized by market timing, i.e. the optimal portfolio choice depends on the current level of the risk factors. The reason for this behaviour is that bond returns are predictable through the risk factors in X, hence the investor needs to take this information into account for investment as well as for hedging purposes.

Utility over interim real consumption

The previous setup can be generalized to the case where the agent seeks to maximize his utility over real interim consumption. In particular, the long-term investor seeks to maximize:

$$\max_{C(t),\phi(t)} E_{t_0} \int_{t_0}^{T} e^{-\delta(t-t_0)} \frac{(C_t/\Pi_t)^{1-\gamma}}{1-\gamma} dt \tag{3.32}$$

where δ is the investor's time discount rate. The wealth dynamics are now slightly different since we allow for consumption:

$$dW = W(\phi^T \frac{dP}{P} + (1 - i^T\phi)\frac{dP_0}{P_0}) - Cdt =$$

$$= W(\phi^T[-B(\tau)^T S^2 \Lambda - B(\tau)^T \Xi X)] + r)dt - Cdt - W\phi^T B(\tau)^T S dw \tag{3.33}$$

Given the nominal SDF m, the discounted consumption stream of the agent must be financed by his initial wealth. As a result, the static budget constraint of the problem is given by:

$$E_{t_0} \int_{t_0}^{T} m_t C_t dt = W_{t_0} \tag{3.34}$$

This can be re-written as:

$$W_{t_0} m_{t_0} = E_{t_0}[\int_{t_0}^{T} C_t m_t dt] \Rightarrow W_{t_0} = m_{t_0}^{-1} E_{t_0}[\int_{t_0}^{T} C_t m_t dt] \tag{3.35}$$

This relationship holds for any time point $t \in [t_0, T]$. Forming the Lagrangian of the static problem and taking the FOC with the respect to consumption, we get:

$$\frac{\partial(.)}{\partial C_t} = 0 \Rightarrow C_t^* = e^{-\frac{\delta}{\gamma}(t-t_0)} (\Pi_t^{1-\gamma} l m_t)^{-\frac{1}{\gamma}} \Rightarrow \frac{C_t^*}{\Pi_t} = e^{-\frac{\delta}{\gamma}(t-t_0)} (l m_t \Pi_t)^{-\frac{1}{\gamma}} \tag{3.36}$$

Employing the variable $Z_t \equiv (l m_t \Pi_t)^{-1}$ and substituting in the budget constraint the optimal solution for consumption, real wealth at t can be written as:

$$\frac{W_t}{\Pi_t} = m_t^{-1} E_t[\int_t^T e^{-\frac{\delta}{\gamma}(s-t)} (l m_s \Pi_s)^{-\frac{1}{\gamma}} m_s \Pi_s ds] = Z_t E_t[\int_t^T e^{-\frac{\delta}{\gamma}(s-t)} Z_s^{\frac{1}{\gamma}-1} ds] \quad (3.37)$$

Hence, nominal wealth W_t is a function of Z, Π, X and t and we can define the function $\hat{G}(Z, X, \Pi, t)$:

$$\hat{G}(Z, X, \Pi, t) \equiv W_t = \Pi_t Z_t E_t[\int_t^T e^{-\frac{\delta}{\gamma}(s-t)} Z_s^{\frac{1}{\gamma}-1} ds] \quad (3.38)$$

with the boundary condition:

$$\hat{G}(Z, X, \Pi, T) = 0 \quad (3.39)$$

In this case, $\hat{G}(Z, X, \Pi, t)$ can be re-written as:

$$\hat{G}(Z, X, \Pi, t) = \int_t^T e^{-\frac{\delta}{\gamma}(s-t)} G(Z, X, \Pi, s) ds \quad (3.40)$$

where $G(Z, X, \Pi, s)$ was previously defined.

Hence, in the case of interim consumption, wealth is interpreted as a bond that pays a stream of consumption coupons. Given this interpretation, this "bond" should satisfy the corresponding no-arbitrage condition for a coupon bond. As Wachter (2002) shows, instead of solving the implied PDE for the coupon bond, we can equivalently solve the PDE derived in the terminal wealth case, given in (3.28).[12]

Corollary 3.1

Given the relationship between the functions $\hat{G}(Z, X, \Pi, t)$ and $G(Z, X, \Pi, t)$ as

[12] This is essentially an application of Lemma 2 in Liu (2007, p. 35). This Lemma states that, if for a function $f(t, X)$ it holds that $\frac{\partial f}{\partial t} + \mathcal{L}f = 0$ with terminal condition $f(T, X) = 1$, where $\mathcal{L}$ is the Dynkin operator, then the function $\hat{f} = \int_t^T f(s, X) ds$ satisfies the equations $\frac{\partial \hat{f}}{\partial t} + \mathcal{L}\hat{f} + 1 = 0$ and $\hat{f}(T, X) = 0$.

well as the conjectured functional form for $G(Z, X, \Pi, t)$ given in (3.30), the optimal consumption-wealth ratio of an investor who seeks to maximize his power utility function over interim real consumption and has access to nominal bonds is given by:

$$\frac{C_t^*}{W_t^*} = \frac{e^{-\frac{\delta}{\gamma}(t-t_0)}}{\int_t^T e^{-\frac{\delta}{\gamma}(s-t)} F(X,s) ds} \tag{3.41}$$

and his optimal portfolio choice ϕ_t is given by:

$$\phi_t = \frac{1}{\gamma}(B^T S^2 B)^{-1}(-B^T S^2 \Lambda - B^T \Xi X_t) + (1-\frac{1}{\gamma})(B^T S^2 B)^{-1}(-B^T S)\sigma_\Pi +$$
$$+\frac{1}{\gamma}(B^T S^2 B)^{-1}(-B^T S) S \frac{\int_t^T [\, e^{-\frac{\delta}{\gamma}(s-t)} F(X,s)(d(s)^T + \frac{1}{2}(Q(s)+Q(s)^T)X)] ds}{\int_t^T e^{-\frac{\delta}{\gamma}(s-t)} F(X,s)\, ds} \tag{3.42}$$

where the functions $c(t)$, $d(t)$ and $Q(t)$ satisfy the system of ODEs in (3.98) - (3.100). The remainder $\phi_0 = 1 - i^T \phi$ is invested in the nominal instantaneously riskless asset (See Appendix for the Proof).

The previous expressions show that the long-term investor with utility over interim consumption behaves quite differently in comparison to a myopic investor. His consumption-wealth ratio is time-varying and it depends on the underlying macroeconomic conditions. Consequently, the long-term investor attempts to "time the market" in his consumption decisions as well as in his optimal portfolio choice. Moreover, the hedging demand given in (3.42) can be interpreted as a weighted average of a series of terminal wealth hedging demands, where the weights depend on the present discounted value of consumption.

The expression (3.41) outlines the mechanism, according to which the investor trades off consumption and savings. This trade off depends on his horizon as well as his degree of relative risk aversion. As Wachter (2002) notes, for $\gamma > 1$ the consumption-wealth ratio tends to rise when investment opportunities are high and it tends to fall when investment opportunities become poorer.

The inverse of (3.41) presents the ratio of wealth to consumption, bearing a resemblance to a dividend discount model. Consequently, the present value of wealth depends on the current macroeconomic conditions as well as on the preferences and the horizon of the investor. For $\gamma > 1$, an increase in the time discount rate, δ, decreases the wealth-consumption ratio. It should be also noted that when $T \to \infty$, as in Campbell and Viceira (2001), this expression is similar to a Laplace transform of $F(X, s)$ with $\frac{\delta}{\gamma}$ being the corresponding discount parameter.

3.4.2 Introducing real bonds

If the investor is interested in real wealth and consumption, then there is the scope of adding real bonds in the asset space. The term structure model we have employed allows us to introduce and price real bonds in a convenient way. In this subsection we price real zero-coupon bonds using the real SDF. Starting from the dynamics of the nominal SDF m:

$$\frac{dm}{m} = -rdt - \xi^T dw \tag{3.43}$$

and applying Ito's lemma to the function $M = m\Pi$, we can find the dynamics of the real SDF M. In particular, these are given by the SDE:

$$\frac{dM}{M} = \frac{d(m\Pi)}{m\Pi} = -(r - \pi + \sigma_\Pi^T \xi)dt - (\xi - \sigma_\Pi)^T dw \tag{3.44}$$

There are two observations to make, given the dynamics of the real SDF. Firstly, the real instantaneously risk free rate is given by $r - \pi + \sigma_\Pi^T \xi$. This is distinct from the conventionally characterized as real rate $\rho = r - \pi$. The difference is the extra term, $\sigma_\Pi^T \xi$, arising due to the stochastic evolution of the price level. Secondly, the market price of risk under the real SDF is also modified and it is given by $\xi - \sigma_\Pi$.

It is assumed that the price of a real zero-coupon bond, P^R, is an affine function of the underlying risk factors in X and time t. In particular, the price of the real bond is given by:

$$P^R(t, X) = \exp(-a^R(\tau) - b^R(\tau)^T X) \tag{3.45}$$

where $a^R(\tau)$ is a scalar and $b^R(\tau)$ is a 5x1 vector, with initial conditions $a^R(0) = 0$ and $b^R(0) = 0_{5x1}$.

Under the risk neutral measure Q, the no-arbitrage condition for the real zero-coupon bonds states that their expected excess returns over the real instantaneously risk free rate, $r - \pi + \sigma_\Pi^T \xi$, should be equal to zero. It should be noted that under the real SDF, switching from the risk-neutral measure Q to the physical measure P, it holds that $d\tilde{w} = dw + (\xi - \sigma_\Pi)dt$. Therefore, the dynamics of X under Q are now given by:

$$dX = [\bar{\psi} - S^2\Lambda + S\sigma_\Pi + (K - \Xi)X]dt + Sd\tilde{w} \tag{3.46}$$

Hence, the no-arbitrage condition for the real bond prices under Q implies the following equation:

$$(\bar{\psi} - S^2\Lambda + S\sigma_\Pi + (K - \Xi)X)^T \frac{\partial P^R}{\partial X} + \frac{1}{2}\frac{\partial^2 P^R}{\partial X \partial X^T}S^2 + \frac{\partial P^R}{\partial t} = (r - \pi + \sigma_\Pi^T \xi)P^R \tag{3.47}$$

Substituting the partial derivatives of P^R in the PDE and using the method of undetermined coefficients, we get a system of ODEs in $a^R(\tau)$ and $b^R(\tau)$:

$$\frac{\partial a^R}{\partial \tau} = \bar{\psi}^T b^R - \frac{1}{2}Tr(b^{RT} S^2 b^R) - \Lambda^T S^2 b^R + \sigma_\Pi^T S\Lambda + \sigma_\Pi^T S^T b^R \tag{3.48}$$

and

$$\frac{\partial b^R}{\partial \tau} = (\delta_1 - \delta_2) + (K - \Xi)^T b^R + \Xi^T S^{-1} \sigma_\Pi \tag{3.49}$$

where $\delta_2 = (0\ 1\ 0\ 0\ 0)^T$, along with the corresponding initial conditions.

Under the risk-neutral measure Q, the absence of arbitrage opportunities dictates that the returns' dynamics of the real zero-coupon bond i in this setup are given by:

$$\frac{dP_i^R}{P_i^R} = (r - \pi + \sigma_\Pi^T \xi)dt - b^R(\tau)^T S d\tilde{w} \tag{3.50}$$

Consequently, under the physical measure P, the bond returns dynamics are given by:

$$\frac{dP_i^R}{P_i^R} = (r - \pi + \sigma_\Pi^T \xi - b^R(\tau)^T S \xi + b^R(\tau)^T S \sigma_\Pi)dt - b^R(\tau)^T S dw \tag{3.51}$$

Figure 3.6 shows the factor loadings $b^R(\tau)$, adjusted for the corresponding maturity τ, i.e. $b^R(\tau)/\tau$. In comparison to the corresponding factor loadings of the nominal bonds, the main difference is that both the inflation and the output gap have an important effect on the real bond prices and returns for short maturities. The real rate ρ affects only the very short maturities. As in the case of nominal bonds, the inflation central tendency is the dominant factor for the real bond prices. Nevertheless, its impact is now less pronounced for longer maturities that are also slightly more affected by the real rate central tendency.

Figure 3.7 shows the expected excess returns of the real bonds over the real risk free rate, given by $-b^R(\tau)^T S^2 \Lambda - b^R(\tau)^T \Xi X + b^R(\tau)^T S \sigma_\Pi$. It is interesting to observe that the real bonds with long maturities exhibit negative expected excess returns for a significant portion of our sample period. In general, for a large portion of our sample period, the 1-year and the 3-year real zero-coupon bonds are characterized by higher expected excess returns in comparison to the 7-year and the 10-year bonds. Table 3.3 presents the covariance and the correlation structure of the real bonds' returns. The results document the very low volatilities and the very high correlations of the returns.

Some of the cases that we examine in the subsequent empirical results include both nominal and real bonds in the asset menu of the long-term investor. For consistency, both types of bonds should be expressed under the same SDF (either nominal or real). When we deal with this case, we choose to express the real bonds' returns dynamics under the nominal SDF. To this end, we need to apply Ito's lemma to the function $(P^R\Pi)$. It should be reminded that the real bonds' returns dynamics under the real SDF are given by equation (3.51). Applying Ito's lemma to the function $(P^R\Pi)$, we find the dynamics of the real bonds' returns under the nominal SDF. In particular, these are given by:

$$\frac{d(P_i^R\Pi)}{(P_i^R\Pi)} = (r + \sigma_\Pi^T\xi - b^R(\tau)^T S\xi)dt - (b^R(\tau)^T S - \sigma_\Pi^T)dw \tag{3.52}$$

Consequently, when we examine the optimal portfolio choice of the long-term investor in the presence of both real and nominal zero-coupon bonds, the previous expression for the real bond returns' dynamics under the nominal SDF is being used. For example, the diffusion term of a real bond's returns under the nominal SDF, given by $-(b^R(\tau)^T S - \sigma_\Pi^T)$, has been employed in the expression of the optimal portfolio choice given in equation (3.31) in the place of the corresponding diffusion term of the nominal bond's returns dynamics, given by $-b(\tau)^T S$.

3.4.3 Real wealth and consumption and access to real bonds

Utility over terminal real wealth

In this subsection, we examine the case of an investor who maximizes utility over terminal real wealth, when he has access to a complete real bond market. The real wealth dynamics of this investor are given by:

$$d(\frac{W}{\Pi}) = \frac{W}{\Pi}(\phi^{RT}[-B^R(\tau)^T S(\xi-\sigma_\Pi)]+r-\pi+\sigma_\Pi^T\xi)dt+\frac{W}{\Pi}\phi^{RT}(-B^R(\tau)^T S)dw \quad (3.53)$$

where ϕ^R is the vector of the portions of his real wealth $\frac{W}{\Pi}$ allocated to each of the 5 available real zero-coupon bonds and $B^R(\tau) = (b^R(\tau_1), b^R(\tau_2), .., b^R(\tau_5))$.

The crucial observation is that using the real pricing kernel M, the process $M_t\frac{W_t}{\Pi_t}$ is a martingale too. As a result, we get the static budget constraint $E_{t_0}[M_T\frac{W_T}{\Pi_T}] = \frac{W_{t_0}}{\Pi_{t_0}}$. The investor seeks to maximize (3.19) subject to this constraint. Forming the corresponding Lagrangian function and taking the FOC with respect to the terminal real wealth we have:

$$\frac{\partial(.)}{\partial(W_T/\Pi_T)} = 0 \Rightarrow \frac{W_T}{\Pi_T} = (lM_T)^{-\frac{1}{\gamma}} \quad (3.54)$$

where l is the corresponding Lagrange multiplier.

We define the variable $Z_t^R \equiv (lM_t)^{-1}$, hence real wealth at time t can be written as:

$$\frac{W_t}{\Pi_t} = (lM_t)^{-\frac{1}{\gamma}} = Z_t^R E_t[(Z_T^R)^{\frac{1}{\gamma}-1}] \quad (3.55)$$

Given the definition of the real SDF, $M = m\Pi$, one may note that $Z_t^R = Z_t$. Hence, the dynamics of Z_t^R are given by (3.25) under the physical measure P. However, since the market price of risk under the real SDF is $(\xi - \sigma_\Pi)$, the dynamics of

Z_t^R under the risk neutral measure Q are now given by:

$$\frac{dZ^R}{Z^R} = (r - \pi + \sigma_\Pi^T \xi)dt + (\xi - \sigma_\Pi)^T d\tilde{w} \tag{3.56}$$

and the dynamics of X under Q are given by (3.46).

Real wealth $\frac{W}{\Pi}$ is a function of Z^R, X and t. Consequently, we can define $G^R(Z^R, X, t) \equiv F^R(X,t)(Z^R)^{\frac{1}{\gamma}} = \frac{W_t}{\Pi_t}$, with the terminal condition $G^R(Z_T^R, X_T, T) = (Z_T^R)^{\frac{1}{\gamma}}$. Consequently, real wealth in this case can be interpreted as a real zero-coupon bond paying a final amount at time T. This means that under the risk neutral measure, its expected excess returns (over the real risk free rate) should be equal to zero. In other words, G^R should satisfy under Q:

$$\begin{aligned}
&\frac{\partial G^R}{\partial Z^R} Z^R (r - \pi + \sigma_\Pi^T \xi) + [\bar{\psi} - S^2 \Lambda + S\sigma_\Pi + (K - \Xi)X]^T \frac{\partial G^R}{\partial X} + \\
&+ \frac{\partial G^R}{\partial t} + \frac{1}{2} Tr(SS^T \frac{\partial^2 G^R}{\partial X \partial X^T}) + \frac{1}{2} \frac{\partial^2 G^R}{(\partial Z^R)^2} (Z^R)^2 (\xi - \sigma_\Pi)^T (\xi - \sigma_\Pi) + \\
&+ Z^R (\xi - \sigma_\Pi)^T S \frac{\partial^2 G^R}{\partial Z^R \partial X} = (r - \pi + \sigma_\Pi^T \xi) G^R
\end{aligned} \tag{3.57}$$

Moreover, the optimally invested real wealth should have a diffusion term identical to the diffusion term of the real wealth process G^R. Therefore, the optimal portfolio choice ϕ^R should satisfy:

$$G^R \phi^{RT} (-B^{RT} S) = \frac{\partial G^R}{\partial Z^R} Z^R (\xi - \sigma_\Pi)^T + (S \frac{\partial G^R}{\partial X})^T \tag{3.58}$$

Proposition 3.2

Let us conjecture the following form for the function $G^R(Z^R, X, t)$:

$$G^R(Z^R,X,t) = (Z_t^R)^{\frac{1}{\gamma}} F^R(X,t) = (Z_t^R)^{\frac{1}{\gamma}} \exp[\frac{1}{\gamma}(\frac{1}{2}X^T Q^R(t) X + d^R(t)^T X + c^R(t))] \tag{3.59}$$

where $Q^R(t)$, $d^R(t)$, $c^R(t)$ are 5x5 matrix, 5x1 vector and scalar functions of time correspondingly with terminal conditions $Q^R(T) = \mathbf{0}_{5x5}$, $d^R(T) = \mathbf{0}_{5x1}$ and $c^R(T) = 0$.

For an investor who has a power utility over real terminal wealth with coefficient of relative risk aversion $\gamma \neq 1$ and has access to real bonds, his optimal portfolio choice is given by:

$$\begin{aligned}\phi_t^R &= \frac{1}{\gamma}(B^{RT}S^2B^R)^{-1}(-B^{RT}S^2\Lambda - B^{RT}\Xi X_t + B^{RT}S\sigma_\Pi) \\ &+ \frac{1}{\gamma}(B^{RT}S^2B^R)^{-1}(-B^{RT}S)S[d^R(t) + \frac{1}{2}(Q^R(t) + Q^R(t)^T)X_t]\end{aligned} \tag{3.60}$$

with the functions $d^R(t)$ and $Q^R(t)$ satisfying the system of ODEs (3.99) - (3.100). The remainder $\phi_0^R = 1 - i^T\phi^R$ is invested in the instantaneously real riskless asset (See Appendix for the Proof).

Utility over interim real consumption

This subsection examines the case where the investor maximizes his utility over interim real consumption. Using the real SDF M, his initial real wealth should finance the discounted real consumption stream:

$$E_{t_0}[\int_{t_0}^T M_t \frac{C_t}{\Pi_t} dt = \frac{W_{t_0}}{\Pi_{t_0}} \tag{3.61}$$

The investor seeks to maximize (3.32) subject to the previous constraint. Forming the Lagrangian and taking the FOC with respect to real consumption, we get:

$$\frac{\partial(.)}{\partial(C_t/\Pi_t)} = 0 \Rightarrow \frac{C_t^*}{\Pi_t} = e^{-\frac{\delta}{\gamma}(t-t_0)}(lM_t)^{-\frac{1}{\gamma}} \tag{3.62}$$

Using the definition of $Z_t^R \equiv (lM_t)^{-1}$and the optimal real consumption choice, real wealth at time t can be written as:

$$\frac{W_t}{\Pi_t} = M_t^{-1}E_t[\int_t^T e^{-\frac{\delta}{\gamma}(s-t)}(lM_s)^{-\frac{1}{\gamma}}M_s ds] = Z_t^R E_t[\int_t^T e^{-\frac{\delta}{\gamma}(s-t)}(Z_s^R)^{\frac{1}{\gamma}-1}ds] \tag{3.63}$$

If we define

$$\hat{G}^R(Z^R, X, t) \equiv \frac{W_t}{\Pi_t} = Z_t^R E_t[\int_t^T e^{-\frac{\delta}{\gamma}(s-t)}(Z_s^R)^{\frac{1}{\gamma}-1}ds] \tag{3.64}$$

with the boundary condition:

$$\hat{G}^R(Z^R, X, T) = 0 \tag{3.65}$$

then, $\hat{G}^R(Z^R, X, t)$ can be re-written as:

$$\hat{G}^R(Z^R, X, t) = \int_t^T e^{-\frac{\delta}{\gamma}(s-t)}G^R(Z^R, X, s)ds \tag{3.66}$$

where $G^R(Z^R, X, t)$ was previously defined.

In this case, real wealth can be interpreted as a real coupon bond that pays a stream of discounted real consumption coupons. Following this observation, the dynamics of this bond should obey the corresponding no-arbitrage condition. Nevertheless, as we have already mentioned, we do not need to solve a different PDE than the one derived for the terminal wealth case, given in (3.57).

Corollary 3.2

Given the relationship between the functions $\hat{G}^R$ and G^R and the conjectured functional form for G^R given in (3.59), the optimal real consumption-wealth ratio of an investor who seeks to maximize his power utility function over interim real consumption and has access to real bonds is given by:

$$\frac{C_t^*/\Pi_t}{W_t^*/\Pi_t} = \frac{e^{-\frac{\delta}{\gamma}(t-t_0)}}{\int_t^T e^{-\frac{\delta}{\gamma}(s-t)} F^R(X,s)ds} \tag{3.67}$$

Furthermore, his optimal portfolio choice is given by:

$$\phi_t^R = \frac{1}{\gamma}(B^{RT}S^2B^R)^{-1}(-B^{RT}S^2\Lambda - B^{RT}\Xi X_t + B^{RT}S\sigma_\Pi) +$$
$$+\frac{1}{\gamma}(B^{RT}S^2B^R)^{-1}(-B^{RT}S)S\frac{\int_t^T [\, e^{-\frac{\delta}{\gamma}(s-t)} F^R(X,s)(d^R(s)^T + \frac{1}{2}(Q^R(s) + Q^R(s)^T)X)]ds}{\int_t^T e^{-\frac{\delta}{\gamma}(s-t)} F^R(X,s)\; ds} \tag{3.68}$$

with the functions $c^R(t)$, $d^R(t)$ and $Q^R(t)$ satisfying the system of ODEs in (3.98) - (3.100). The remainder, $\phi_0^R = 1 - i^T\phi^R$, is invested in the real instantaneously riskless asset.

3.5 Portfolio choice in incomplete markets

In this section, we examine the optimal portfolio choice of the long-term investor with a power utility function over terminal wealth in an incomplete bond market. The market incompleteness arises because the investor has access to fewer bonds in comparison to the number of the underlying risk factors. Therefore, in an incomplete nominal bond market, $B(\tau)$ is a 5xn matrix with $n < 5$ and in an incomplete real bond market, $B^R(\tau)$ is a 5xn matrix with $n < 5$.

3.5.1 Access to nominal bonds

In an incomplete nominal bond market, any price of risk $\bar{\xi}$ can be written as:

$$\bar{\xi} = (-B^TS)^T(B^TS^2B)^{-1}(-B^TS)\bar{\xi} + (\bar{\xi} - (-B^TS)^T(B^TS^2B)^{-1}(-B^TS)\bar{\xi}) \quad (3.69)$$

The first term of this expression, $(-B^TS)^T(B^TS^2B)^{-1}(-B^TS)\bar{\xi} = \xi$ is the unique price of risk that both prices and it is spanned by the available assets, i.e. it is the price of risk in the complete market case that we examined in the previous section. The second term $\bar{\xi} - (-B^TS)^T(B^TS^2B)^{-1}(-B^TS)\bar{\xi} \equiv v$ is the residual of the projection of $\bar{\xi}$ onto the available assets and lies in the null space of $(-B^TS)$, hence it holds that $(-B^TS)v = \mathbf{0}$. Note that in a complete market $v = \mathbf{0}$.

Consequently, in the incomplete market case, the nominal SDF m^v is associated with the price of risk $\bar{\xi} = \xi + v$. The dynamics of m^v are given by:

$$\frac{dm^v}{m^v} = -rdt - (\xi + v)^T dw \quad (3.70)$$

In general, the budget constraint $E_{t_0}[m_T^v W_T] = W_{t_0}$ should be satisfied by any pricing kernel m^v. However, we cannot optimize with respect to this budget constraint because it is not possible to replicate the resulting process for wealth by trading in the underlying assets, since we are in an incomplete market. He and Pearson (1991) argue that it is sufficient to verify this budget constraint for a single pricing kernel m^{v^*}. This is termed as the minimax pricing kernel because it is the kernel that minimizes the agent's maximized utility. In other words, we can solve the portfolio choice problem in the incomplete market case as in the complete market case by "adding" the assets that are necessary to complete the market and at the same time modifying their returns' process in a way that their optimal portfolio weights are zero. This is accomplished by the specific minimax kernel m^{v^*}, because it guarantees that the "added" assets have such properties that the investor does not want to trade them.

Hence, the pricing kernel we use follows the SDE:

$$\frac{dm^{v^*}}{m^{v^*}} = -rdt - (\xi + v^*)^T dw \tag{3.71}$$

The solution of the optimal portfolio choice problem follows the same steps as in the previous section. The agent seeks to maximize:

$$\max E_{t_0} \left\{ \frac{(\frac{W_T}{\Pi_T})^{1-\gamma}}{1-\gamma} \right\} \tag{3.72}$$

subject to the constraint

$$E_{t_0}[m_T^{v^*} W_T] = W_{t_0} \tag{3.73}$$

The nominal wealth dynamics are the same as in (3.17). The FOC of the problem is:

$$\frac{\partial(.)}{\partial W_T} = 0 \Rightarrow W_T = (\Pi_T^{1-\gamma} l m_T^{v^*})^{-\frac{1}{\gamma}} \Rightarrow \frac{W_T}{\Pi_T} = (l m_T^{v^*} \Pi_T)^{-\frac{1}{\gamma}} \tag{3.74}$$

where l is the corresponding Lagrange multiplier.

Defining the variable $Z_t^{v^*} \equiv (l m_t^{v^*} \Pi_t)^{-1}$, real wealth at time t can be written as:

$$\frac{W_t}{\Pi_t} = (l m_t^{v^*} \Pi_t)^{-\frac{1}{\gamma}} = Z_t^{v^*} E_t[(Z_T^{v^*})^{\frac{1}{\gamma}-1}] \tag{3.75}$$

As a result, wealth W_t is a function of $Z_t^{v^*}$, Π, X and t. Consequently, we can define the function $G^I(Z^{v^*}, \Pi, X, t) \equiv F^I(X,t)\Pi(Z^{v^*})^{\frac{1}{\gamma}} = W(t)$.

Applying Ito's lemma, the dynamics of the variable Z^{v^*} are given by:

$$\frac{dZ^{v^*}}{Z^{v^*}} = (r-\pi+\sigma_\Pi^T\sigma_\Pi+(\xi+v^*)^T(\xi+v^*)-\sigma_\Pi^T(\xi+v^*))dt+(\xi+v^*-\sigma_\Pi)^T dw \quad (3.76)$$

while, under the risk neutral measure Q, they are given by:

$$\frac{dZ^{v^*}}{Z^{v^*}} = (r-\pi+\sigma_\Pi^T\sigma_\Pi)dt+(\xi+v^*-\sigma_\Pi)^T d\tilde{w} \quad (3.77)$$

It should be also noted that the price level dynamics under the risk neutral measure Q are given by:

$$\frac{d\Pi}{\Pi} = (\pi-\sigma_\Pi^T(\xi+v^*))dt+\sigma_\Pi^T d\tilde{w} \quad (3.78)$$

while the corresponding dynamics of X are given by:

$$dX = [\bar{\psi}+KX-S(\xi+v^*)]dt+Sd\tilde{w} \quad (3.79)$$

We can interpret again nominal wealth as a zero-coupon bond that pays a final amount W_T at time T. The no-arbitrage condition for a zero-coupon bond $G^I(Z^{v^*}, X, \Pi, t)$ under the risk-neutral measure implies that its instantaneous expected excess returns should be equal to zero. Following this observation, we get the PDE:

$$\begin{aligned}
&\frac{\partial G^I}{\partial Z^{v^*}}Z^{v^*}(r-\pi+\sigma_\Pi^T\sigma_\Pi)+[\bar{\psi}+KX-S(\xi+v^*)]^T\frac{\partial G^I}{\partial X}\\
&+\frac{\partial G^I}{\partial \Pi}\Pi(\pi-\sigma_\Pi^T(\xi+v^*))+\frac{\partial G^I}{\partial t}+\frac{1}{2}Tr(SS^T\frac{\partial^2 G^I}{\partial X\partial X^T})+\frac{1}{2}\frac{\partial^2 G^I}{\partial \Pi^2}\Pi^2\sigma_\Pi^T\sigma_\Pi\\
&+\frac{1}{2}\frac{\partial^2 G^I}{(\partial Z^{v^*})^2}(Z^{v^*})^2(\xi+v^*-\sigma_\Pi)^T(\xi+v^*-\sigma_\Pi)+(S\sigma_\Pi)^T\Pi\frac{\partial^2 G^I}{\partial \Pi\partial X}\\
&+\frac{\partial^2 G^I}{\partial Z^{v^*}\partial \Pi}\Pi Z^{v^*}(\xi+v^*-\sigma_\Pi)^T\sigma_\Pi+Z^{v^*}(\xi+v^*-\sigma_\Pi)^T S\frac{\partial^2 G^I}{\partial Z^{v^*}\partial X}=rG^I
\end{aligned} \quad (3.80)$$

with the terminal condition $G^I(Z_T^{v^*}, X_T, \Pi_T, T) = \Pi_T(Z_T^{v^*})^{\frac{1}{\gamma}}$.

Moreover, the optimally invested wealth should have a diffusion term equal to the diffusion term of the wealth process G^I. Therefore, the optimal portfolio choice ϕ^I should satisfy:

$$G^I(\phi^I)^T(-B^TS) = \frac{\partial G^I}{\partial Z^{v^*}} Z^{v^*}(\xi + v^* - \sigma_\Pi)^T + \frac{\partial G^I}{\partial \Pi}\Pi\sigma_\Pi^T + (S\frac{\partial G^I}{\partial X})^T \tag{3.81}$$

Proposition 3.3

Let us conjecture the following form for the function $G^I(Z^{v^*}, X, \Pi, t)$:

$$\begin{aligned} G^I(Z^{v^*}, X,\ \Pi,\ t) &= \Pi_t(Z_t^{v^*})^{\frac{1}{\gamma}} F^I(X,t) = \\ &= \Pi_t(Z_t^{v^*})^{\frac{1}{\gamma}} \exp[\frac{1}{\gamma}(\frac{1}{2}X^TQ^I(t)X + d^I(t)^TX + c^I(t))] \end{aligned} \tag{3.82}$$

where $Q^I(t)$, $d^I(t)$, $c^I(t)$ are 5x5 matrix, 5x1 vector and scalar functions of time correspondingly, with terminal conditions $Q^I(T) = 0_{5x5}$, $d^I(T) = 0_{5x1}$ and $c^I(T) = 0$.

For an investor who has a power utility over real terminal wealth with coefficient of relative risk aversion $\gamma \neq 1$ and has access to an incomplete nominal bond market, the minimax pricing kernel is given by:

$$v^* = (1-\gamma)[(\sigma_\Pi^T)^\perp]^T - (S^\perp)^T[d^I + \frac{1}{2}(Q^I + (Q^I)^T)X] \tag{3.83}$$

where $(\sigma_\Pi^T)^\perp$ is the residual of the projection of σ_Π^T onto the traded assets, $(\sigma_\Pi^T)^\perp = \sigma_\Pi^T - \sigma_\Pi^T(-B^TS)^T(B^TS^2B)^{-1}(-B^TS)$ and $(S^\perp)$ is the residual of the projection of S onto the traded assets, $S^\perp = S - S(-B^TS)^T(B^TS^2B)^{-1}(-B^TS)$.

Moreover, his optimal portfolio choice is given by:

$$
\begin{aligned}
\phi_t^I &= \frac{1}{\gamma}(B^T S^2 B)^{-1}(-B^T S^2 \Lambda - B^T \Xi X_t) + (1-\frac{1}{\gamma})(B^T S^2 B)^{-1}(-B^T S)\sigma_\Pi \\
&+\frac{1}{\gamma}(B^T S^2 B)^{-1}(-B^T S)S[d^I(t) + \frac{1}{2}(Q^I(t) + Q^I(t)^T)X_t] \qquad (3.84)
\end{aligned}
$$

with the functions $d^I(t)$ and $Q^I(t)$ satisfying the system of ODEs in (3.110)-(3.111) given in the Appendix. The remainder $\phi_0^I = 1 - i^T\phi^I$ is invested in the nominal instantaneously riskless asset (See Appendix for the Proof).[13]

3.5.2 Access to real bonds

The derivation of the optimal asset allocation for a long-term investor with power utility over terminal wealth in an incomplete real bond market follows the same steps. Any price of risk $\overline{\xi - \sigma_\Pi}$ can be written as:

$$
\begin{aligned}
&\overline{\xi - \sigma_\Pi} = (-B^{RT}S)^T(B^{RT}S^2B^R)^{-1}(-B^{RT}S)(\overline{\xi - \sigma_\Pi}) + \\
&+((\overline{\xi - \sigma_\Pi}) - (-B^{RT}S)^T(B^{RT}S^2B^R)^{-1}(-B^{RT}S)(\overline{\xi - \sigma_\Pi})) \qquad (3.85)
\end{aligned}
$$

The first term of this expression, $(-B^{RT}S)^T(B^{RT}S^2B^R)^{-1}(-B^{RT}S)(\overline{\xi - \sigma_\Pi}) = \xi - \sigma_\Pi$ is the unique price of risk that both prices and it is spanned by the available real assets, i.e. it is the price of risk in the complete real market case we examined in the previous section. The second term

$$
(\overline{\xi - \sigma_\Pi}) - (-B^{RT}S)^T(B^{RT}S^2B^R)^{-1}(-B^{RT}S)(\overline{\xi - \sigma_\Pi}) \equiv V
$$

is the residual of the projection of the price of risk onto the available real bonds and lies in the null space of $(-B^{RT}S)$, hence it holds that $(-B^{RT}S)V = \mathbf{0}$.

[13]Note that the analogy of Wachter (2002) between the case of utility over terminal wealth and the case of utility over interim consumption does not carry through in the incomplete market case.

Consequently, in the incomplete market case, the real SDF, M^V, is associated with the price of risk $(\overline{\xi - \sigma_\Pi}) = \xi - \sigma_\Pi + V$. Following the same argument of He and Pearson (1991), we seek to find the minimax pricing kernel M^{V^*}, which guarantees that the real bonds "added" to complete the market have such properties that the investor does not want to trade them.

Hence, the pricing kernel we use follows the SDE:

$$\frac{dM^{V^*}}{M^{V^*}} = -(r - \pi + \sigma_\Pi^T(\xi + V^*))dt - (\xi + V^* - \sigma_\Pi)^T dw \tag{3.86}$$

The agent seeks to maximize:

$$\max E_{t_0} \left\{ \frac{(\frac{W_T}{\Pi_T})^{1-\gamma}}{1-\gamma} \right\} \tag{3.87}$$

subject to the constraint $E_{t_0}[M_T^{V^*} \frac{W_T}{\Pi_T}] = \frac{W_{t_0}}{\Pi_{t_0}}$. Forming the corresponding Lagrangian function and taking the FOC with respect to terminal real wealth, we have:

$$\frac{\partial(.)}{\partial(W_T/\Pi_T)} = 0 \Rightarrow \frac{W_T}{\Pi_T} = (lM_T^{V^*})^{-\frac{1}{\gamma}} \tag{3.88}$$

where l is the corresponding Lagrange multiplier. We define the variable $Z_t^{V^*} \equiv (lM_t^{V^*})^{-1}$, hence real wealth at time t can be written as:

$$\frac{W_t}{\Pi_t} = (lM_t^{V^*})^{-\frac{1}{\gamma}} = Z_t^{V^*} E_t[(Z_T^{V^*})^{\frac{1}{\gamma}-1}] \tag{3.89}$$

The dynamics of $Z_t^{V^*}$ are given by (3.76) under the physical measure P. However, since the market price of risk under the real SDF is $(\xi + V^* - \sigma_\Pi)$, the dynamics of $Z_t^{V^*}$ under the risk neutral measure Q are now given by:

$$\frac{dZ_t^{V^*}}{Z_t^{V^*}} = (r - \pi + \sigma_\Pi^T(\xi + V^*))dt + (\xi + V^* - \sigma_\Pi)^T d\tilde{w} \tag{3.90}$$

and the dynamics of X under Q are given by:

$$dX = [\bar{\psi} + KX - S(\xi + V^* - \sigma_\Pi)]dt + Sd\tilde{w} \tag{3.91}$$

Real wealth $\frac{W}{\Pi}$ is a function of $Z_t^{V^*}$, X and t. Consequently, we can define

$$G^{IR}(Z^{V^*}, X, t) \equiv F^{IR}(X,t)(Z^{V^*})^{\frac{1}{\gamma}} = \tfrac{W_t}{\Pi_t},$$

with the terminal condition $G^{IR}(Z_T^{V^*}, X_T, T) = (Z_T^{V^*})^{\frac{1}{\gamma}}$. Consequently, real wealth in this case can be interpreted as a real zero-coupon bond paying a final amount at time T. This interpretation implies that under the risk neutral measure, its expected excess returns (over the real risk free rate) should be equal to zero. In other words, G^{IR} should satisfy under Q:

$$\begin{aligned}
&\frac{\partial G^{IR}}{\partial Z^{V^*}} Z^{V^*}(r - \pi + \sigma_\Pi^T(\xi + V^*)) + [\bar{\psi} + KX - S(\xi + V^* - \sigma_\Pi)]^T \frac{\partial G^{IR}}{\partial X} + \\
&+ \frac{\partial G^{IR}}{\partial t} + \frac{1}{2} Tr(SS^T \frac{\partial^2 G^{IR}}{\partial X \partial X^T}) + \frac{1}{2} \frac{\partial^2 G^{IR}}{(\partial Z^{V^*})^2} (Z^{V^*})^2 (\xi + V^* - \sigma_\Pi)^T (\xi + V^* - \sigma_\Pi) \\
&+ Z^{V^*}(\xi + V^* - \sigma_\Pi)^T S \frac{\partial^2 G^{IR}}{\partial Z^{V^*} \partial X} = (r - \pi + \sigma_\Pi^T(\xi + V^*)) G^{IR}
\end{aligned} \tag{3.92}$$

Moreover, the optimally invested wealth should have a diffusion term equal to the diffusion term of the real wealth process G^{IR}. Therefore, the optimal portfolio choice ϕ^{IR} should satisfy:

$$G^{IR}(\phi^{IR})^T(-B^{RT}S) = \frac{\partial G^{IR}}{\partial Z^{V^*}} Z^{V^*}(\xi + V^* - \sigma_\Pi)^T + (S\frac{\partial G^{IR}}{\partial X})^T \tag{3.93}$$

Proposition 3.4

Let us conjecture the following form for the function $G^{IR}(Z^{V^*}, X, t)$:

$$G^{IR}(Z^{V^*}, X, t) = (Z^{V^*})^{\frac{1}{\gamma}} F^{IR}(X, t) =$$
$$= (Z^{V^*})^{\frac{1}{\gamma}} \exp[\frac{1}{\gamma}(\frac{1}{2}X^T Q^{IR}(t) X + d^{IR}(t)^T X + c^{IR}(t))] \tag{3.94}$$

where $Q^{IR}(t)$, $d^{IR}(t)$, $c^{IR}(t)$ are 5x5 matrix, 5x1 vector and scalar functions of time correspondingly, with terminal conditions $Q^{IR}(T) = \mathbf{0}_{5x5}$, $d^{IR}(T) = \mathbf{0}_{5x1}$ and $c^{IR}(T) = 0$.

For an investor who has a power utility over real terminal wealth with coefficient of relative risk aversion $\gamma \neq 1$ and has access to an incomplete real bond market, the minimax pricing kernel is given by:

$$V^* = -(S^{\perp})^T [d^{IR} + \frac{1}{2}(Q^{IR} + (Q^{IR})^T) X] \tag{3.95}$$

where $(S^{\perp})$ is the residual of the projection of S onto the traded real bonds, $S^{\perp} = S - S(-B^{RT}S)^T (B^{RT} S^2 B^R)^{-1} (-B^{RT} S)$.

Moreover, his optimal portfolio choice is given by:

$$\phi_t^{IR} = \frac{1}{\gamma}(B^{RT} S^2 B^R)^{-1}(-B^{RT} S^2 \Lambda - B^{RT} \Xi X_t + B^{RT} S \sigma_{\Pi})$$
$$+ \frac{1}{\gamma}(B^{RT} S^2 B^R)^{-1}(-B^{RT} S) S [d^{IR}(t) + \frac{1}{2}(Q^{IR}(t) + Q^{IR}(t)^T) X_t] \tag{3.96}$$

with the functions $d^{IR}(t)$ and $Q^{IR}(t)$ satisfying the system of ODEs (3.116)-(3.117). The remainder, $\phi_0^{IR} = 1 - i^T \phi^{IR}$, is invested in the instantaneously real riskless asset (See Appendix for the Proof).

3.6 Results

This section reports the optimal portfolio choices of an investor who maximizes his utility over real terminal wealth for various degrees of relative risk aversion and investment horizons. The first subsection refers to the case of a complete bond market while the second subsection refers to the case of an incomplete bond market, where the incompleteness arises because the available bonds are fewer than the number of the underlying risk factors.

Following Sangvinatsos and Wachter (2005), the results that we report refer to a specific combination of the underlying risk factors. Nevertheless, instead of setting ad hoc values for the risk factors, we employ the values of the macroeconomic variables that were actually experienced in 1975:Q1.[14] This helps us derive an economic interpretation for our results. This specific choice is made by taking into account the implications of the term structure model for the bonds' expected excess returns and their covariances, as discussed in Section 3.3.3. In particular, we have chosen this date because it yields reasonable expected excess returns for the bonds relative to their volatilities. Given the bond premium puzzle implication of this model, other periods of our sample would yield extreme results, obscuring the analysis.

3.6.1 Portfolio choice in complete markets

In a complete market, an investor may form a portfolio with multiple bonds. In particular, the existence of five risk factors in the term structure model of Dewachter et al. (2006) allows us to examine the investor's bond portfolio choice among five nominal bonds. Panel A of Table 3.5 presents these choices for $\gamma = 4$ and $\gamma = 10$, as well as when the investment horizon increases up to $T = 10$ years. The results show that, in case five bonds are available, the investor should take extreme long

[14]These refer to an output gap of $y = -5.89\%$, inflation $\pi = 10.43\%$, real interest rate $\rho = -5.21\%$, inflation central tendency $\pi^* = 4.55\%$ and real rate central tendency $\rho^* = 0.67\%$.

and short positions. The reason for this behaviour is the extremely high correlation of the bonds' returns, mentioned in Section 3.3.3. Even very small differences in the risk-return trade off of the various bonds, motivate the investor to hold highly leveraged positions.

This may be a quite puzzling observation, but it is a direct consequence of the estimated term structure model. There are various ways that these extreme positions could be much smoother in reality. Firstly, a series of institutional investors, such as pension and bond funds, are not allowed to hold short positions. Such a constraint would lead to zero positions in the bonds that theoretically appear to be sold short. Secondly, transactions costs could actually make the perceived differences in the risk-return trade off disappear, neutralizing the incentive of holding so extreme positions.

Furthermore, one should be very cautious when interpreting these results, because they refer to ex post estimates of the risk factors' and bond returns' volatilities. In particular, the volatility of the inflation central tendency, that mainly affects bond returns, was estimated to be extremely low over the sample period. However, the perceived volatility in the long-run inflation expectations may be much higher ex ante, so an investor may regard the bond returns' volatilities to be much higher than what this model implies. Consequently, if the level of risk is increased, the ex ante maximal Sharpe ratio would be much lower and the small differences in the bonds' premia would not lead to major demand shifts. In other words, if parameter uncertainty is taken into account as in Barberis (2000), Xia (2001) and Garlappi et al. (2007), bond demands may not be as extreme as the results in Table 3.5 suggest.

The next issue to examine is how real bonds can be mixed with nominal bonds in a portfolio. This exercise is motivated by the significantly lower correlations of the returns of these two different types of bonds, as Table 3.4 shows. It should be noted that in this case the expected excess returns as well as the volatilities and correlations of the real bonds are derived using their dynamics under the nominal SDF. We have shown how to derive the real bonds' returns dynamics in (3.52). Therefore, when we derive the optimal portfolio choice of the long-term investor from equation (3.31) in the presence of both real and nominal bonds, the dynamics of the real bonds' returns given by the SDE in (3.52) are appropriately used.

In particular, Panel B of Table 3.5 reports the results of bond portfolio selection among five bonds, i.e. nominal 1, 5 and 10-year bonds and real 3 and 7-year bonds, when the investor derives utility from his terminal wealth. The long and short positions are relatively large for low degrees of risk aversion in the myopic case ($T = 0$), but they are not as extreme as in the case where only nominal or only real bonds are available. Small differences in the risk-return trade offs are not extremely magnified in this case, because each bond also plays a significant diversification role in the formation of the optimal portfolios, since the correlations of the bonds' returns are now considerably less than perfect.

With respect to the hedging demands, their magnitude depends on the investor's horizon as well as on his degree of relative risk aversion. In particular, hedging demands tend to increase in absolute value as the horizon increases and tend to decrease as the degree of risk aversion increases. It is interesting to observe that the investor tries to match the duration of the bonds with his horizon. In other words, he tries to optimally combine the bonds so as to create a synthetic bond portfolio that has a maturity corresponding to his horizon. These results verify the argument of Brennan and Xia (2002).

An interesting issue to examine is how the definition of the riskless asset is modified, as the horizon of the investor increases. It is well known that an infinitely risk-averse myopic investor ($\gamma \rightarrow \infty$, $T = 0$) with utility over nominal wealth would assign a zero portfolio weight to the available risky bonds, investing his wealth only in the instantaneously nominal riskless asset that yields the nominal risk-free rate r. On the other hand, for the myopic investor with utility over real terminal wealth, the portion of wealth invested in the instantaneously nominal riskless asset is not exactly equal to one. This can be seen by setting $\gamma \to \infty$ and $T = 0$ in (3.31). This substitution yields $\phi_0 = 1 - i^T(B^T S^2 B)^{-1}(-B^T S)\sigma_\Pi$, which is equal to 1.05 in the particular case we examine, as reported in Panel A of Table 3.6. The explanation for this result is that the available assets are nominal, while the myopic investor we examine seeks to form a mean-variance efficient portfolio in real terms. Therefore, he should take into account the inflation risk loadings of the various bonds. In particular, while the total demand for the nominal bonds is $i^T\phi = -0.05$, the internal allocation exhibits significant long and short positions, exploiting the differences in the inflation risk loadings.

Examining the optimal portfolio choice of a long-term investor ($T > 0$), we set $\gamma = 100,000$ to approximate infinite relative risk aversion. Panel A of Table 3.6 reports the portfolio choices among five nominal bonds as the investor's horizon increases up to $T = 10$ years. These results show that the instantaneously nominal riskless asset is not risk-free for a long-term investor, due to the reinvestment risk that it generates, as explained in Stiglitz (1970).

Moreover, the results show that none of the nominal zero-coupon bonds can play the role of the riskless asset, even when the maturity of the bond is identical to the horizon of the investor. This finding can be explained by the fact that the investor cares about his real terminal wealth but has access only to nominal bonds and it reflects the argument that a nominal bond cannot be a perfect hedging instrument for shocks in the real wealth process. This is true even when the inflation risk is relatively low, because the infinitely risk averse long-term investor would not like to be exposed to any unhedgeable shock affecting his real wealth. Consequently, the investor attempts to proxy the non-existing riskless asset by taking positions in the available nominal bonds and the optimal mix depends on his horizon as well as on the available bond maturities.

The introduction of a real bond market allows us to examine how this infinitely risk-averse investor would behave, if he could allocate his wealth among real zero-coupon bonds, priced under the real SDF. For the myopic case ($T = 0$), the risk-free asset is the instantaneously real riskless asset yielding $r - \pi + \sigma_{\Pi}^{T}\xi$. Setting $\gamma \rightarrow \infty$ and $T = 0$ in (3.60), we get $\phi^R = \mathbf{0}$ and $\phi_0^R = 1$. Panel B of Table 3.6 reports this case as well as the optimal bond portfolio choice for non-myopic investment horizons. The striking result is that the infinitely risk averse long-term investor, who cares about his real wealth at a terminal date, should allocate his wealth to a single zero-coupon bond that has a duration equal to his horizon.

The previous result has been proved theoretically by Wachter (2003) and it is also stated in Liu (2007), but it has not been shown empirically in the literature, to the best of our knowledge.[15] Moreover, this finding underlines the importance of introducing real bonds in an economy (see Viard, 1993 inter alia for a discussion). It should be noted that this result offers an explanation for the existence of assets with negative expected excess returns, as the real bonds exhibited in a significant portion of our sample (see Figure 3.7). In particular, these bonds are held by long-term investors who are attracted by the significant hedging value that these assets incorporate. The traditional myopic framework could not have justified their existence.

The framework we have been using allows us also to examine the sensitivities of the investor's wealth with respect to shifts in the underlying macroeconomic factors. In particular, these sensitivities are affected by the agent's degree of risk aversion as well as his investment horizon. In the case of an investor who derives utility from his real terminal wealth and faces a complete nominal bond market, his wealth elasticities are given by $\frac{\partial G}{\partial X}\frac{1}{G}$, where G is defined in (3.30). For the current analysis, it proves

[15]This is because Campbell and Viceira (2001) consider utility over consumption and access to zero-coupon bonds, while Sangvinatsos and Wachter (2005) consider utility over real terminal wealth but access only to nominal zero-coupon bonds.

more informative to report the norm of these elasticities, given by $\sqrt{(\frac{\partial G}{\partial X}\frac{1}{G})^T(\frac{\partial G}{\partial X}\frac{1}{G})}$.

The top panel in Figure 3.8 reports this norm as the horizon increases for various levels of relative risk aversion. The results show that, for low levels of relative risk aversion ($\gamma < 10$) and long investment horizons, these elasticities are extremely high. Only a combination of short horizons and very high levels of risk aversion would yield a low norm of elasticities. This effect explains the significant shifts in the hedging demands that we report in our portfolio choice results when the investment horizon or the degree of relative risk aversion is modified. Unreported results show that the wealth elasticity with respect to the inflation central tendency is the dominant one.

For a significant portion of investors, it is reasonable to assume that they seek to maximize utility over interim consumption, not just over wealth at a terminal date. This is because an investor may have set a series of intermediate consumption goals (mortgage payments, fixed liabilities etc.). This behaviour significantly modifies the hedging demand of the investor as this can be seen by comparing (3.42) with (3.31). As Wachter (2002) notes, the introduction of consumption significantly reduces the effective horizon of the investor.

Consequently, it is interesting to examine the impact of consumption on the wealth sensitivities with respect to the underlying risk factors. We calculate the wealth elasticities in the case of an agent with utility over real interim consumption, given by $\frac{\partial \hat{G}}{\partial X}\frac{1}{\hat{G}}$, where $\hat{G}$ is defined in (3.40). The bottom panel in Figure 3.8 presents the norm of these elasticities, given by $\sqrt{(\frac{\partial \hat{G}}{\partial X}\frac{1}{\hat{G}})^T(\frac{\partial \hat{G}}{\partial X}\frac{1}{\hat{G}})}$, for various levels of relative risk aversion as the horizon increases, setting $\delta = 0.06$. In comparison to the terminal wealth case, these elasticities are much smaller now, confirming the reduction in the effective investment horizon when interim consumption is introduced. Nevertheless, for low degrees of relative risk aversion as well as for long horizons, the wealth sensitivities remain large. These large sensitivities, arising due to the high market prices of risk estimated by the term structure model, are difficult to be reconciled with the smoothness characterizing actual consumption data, hence this finding is equivalent to a premium puzzle in the bond markets. We need a very high degree of risk aversion and a very short horizon, even in the presence of consumption, in order to get low levels of wealth elasticities.

3.6.2 Portfolio choice in incomplete markets

The multi-factor term structure model of Dewachter et al. (2006) allowed us to examine the formation of portfolios with five zero-coupon bonds, within a complete market setting. Nevertheless, there are a series of reasons why an investor may actually allocate his wealth to a restricted set of bonds. In particular, this section examines the portfolio choice among two or three zero-coupon bonds and the corresponding riskless asset. Within our setup, this leads to an incomplete market setting, which was analyzed in Section 3.5.

More specifically, it is common for institutional investors to face short selling constraints, disabling them to fully exploit the differences in the bonds' risk-return profiles by holding leveraged positions. Most importantly, transaction costs may actually make these risk-returns differences disappear for bonds of similar maturities. As Figure 3.4 and Figure 3.7 show, the expected excess returns are very similar for bonds with close maturities and they tend to strongly co-move through time. For example, the expected excess return of the 5-year nominal zero-coupon bond was, on average, only 0.34% higher than the expected excess return offered by the 3-year nominal zero-coupon bond for our sample period. Hence, transaction costs of the order of 0.2% could make bonds of very close maturities practically redundant.

Liquidity considerations would offer an additional reason why an investor may be willing to hold positions in a restricted set of bonds. While our portfolio choice exercise implicitly assumes that the zero-coupon bonds with prespecfied maturities are always available, this may not be true in practice. In particular, bonds of specific maturities may not be liquid enough, hence an investor may not be able to fully capture the corresponding perceived expected returns that are implied by the estimated term structure model.[16] Consequently, this investor may prefer to hold a restricted set of highly liquid bonds and avoid loading illiquidity risk to his portfolio. Finally, it should be reminded that most of the dynamic bond portfolio studies make use of two or three risk factors to price bonds (see Campbell and Viceira, 2001 and Sangvinatsos and Wachter, 2005 correspondingly). The low dimension of these models simplifies the asset allocation problem, indicating that investors should form portfolios that are composed of only two or three bonds.

We firstly examine the portfolio choice among a 3-year and a 10-year nominal zero-coupon bond and the nominal instantaneously riskless asset. Panel A of Table 3.7 presents the optimal allocations for $\gamma = 4$ and $\gamma = 10$, as the horizon increases from $T = 0$ to $T = 10$ years. For the specific macroeconomic conditions that we have selected, the corresponding excess returns and their covariances imply that the myopic demand for the 10-year bond is much higher than the demand for the 3-year bond. Furthermore, the demand for the nominal instantaneously riskless asset is quite high. Nevertheless, long-term investors have significant hedging bond demands that dominate the corresponding total demands as the horizon increases. Moreover, the magnitude of the hedging demand is greater for low levels of relative risk aversion and the investor should actually borrow at the instantaneously riskless rate.

The next issue to examine in this subsection is the impact of macroeconomic shifts on optimal portfolio choice. This is an attractive feature of our study with respect to the rest of the literature, since we employ an essentially affine term structure model with a clear macroeconomic interpretation. Hence, unlike Campbell and Viceira (2001) and Brennan and Xia (2002) who assume constant term premia, both myopic

[16] See Amihud and Mendelson (1991) for the impact of transaction costs and liquidity on bond yields.

and hedging demands in our model are affected by the level of the underlying risk factors. In other words, the long-term investor is involved in market timing both in his myopic and his hedging demand, as it is evident from (3.31). Moreover, the modifications of the underlying risk factors are due to specific macroeconomic effects, in contrast to the latent factor model used by Sangvinatsos and Wachter (2005).

As it was mentioned in Section 3.3.3, the central tendency of inflation, π^*, has a dominant effect on bond returns. Consequently, we examine the impact on portfolio choice when this tendency increases or decreases by one standard deviation, keeping the rest of the factors constant. The portfolio choices reported in Panel B of Table 3.7 show that the increase in this central tendency implies a significant increase in the excess returns of the 3-year and the 10-year bond. As a result, the risk-return trade off of these bonds is modified, making the 3-year bond much more attractive now in comparison to the benchmark case. Not only the myopic demand for this bond is higher, but this is also true for the corresponding hedging demands. The total demand for the 10-year bond is lower for short horizons and it significantly increases only when the investor's horizon approaches the ten years. The total hedging demand still decreases as the degree of relative risk aversion increases, but its magnitude is now larger in comparison to the benchmark case.

On the other hand, a decrease in the central tendency of inflation has the opposite effect, as the results in panel C of Table 3.7 show. The reduction in the bonds' premia modifies their risk-return trade off in such a way that the myopic investor sells short the 3-year bond. There is some hedging role for this bond, but this has become very limited and only for investment horizons close to the bond's maturity. For long horizons, the investor has a significantly negative hedging demand for this bond, since he mainly makes use of the 10-year bond as a hedging instrument. The magnitude of the total bond demand is now quite a lot smaller in comparison to the benchmark macroeconomic conditions, prevailing in 1975:Q1.

Pricing real bonds within our setup, we concluded that they exhibit a different behaviour in comparison to nominal bonds. This observation motivates us to examine the optimal allocation of real wealth when a 3-year and a 10-year real bond are available, apart from the instantaneously real risk free rate. Panel A of Table 3.8 shows the myopic as well as the total bond demands, as the investor's horizon increases for the macroeconomic conditions prevailing in 1975:Q1. Even though both bonds have negative expected excess returns, the risk-return trade off that they exhibit motivates the myopic investor to hold a significant long position in the 3-year bond by selling short the 10-year real bond. As the horizon increases, the hedging demand for the 10-year bond becomes increasingly positive. In general, the investor attempts to combine the bonds' maturities so as to match his horizon.

Since the factor loadings of the macroeconomic risks are modified in the case of real bonds, it is interesting to examine the impact of macroeconomic change in this case too. Panel B of Table 3.8 shows the optimal portfolio choice when the inflation central tendency is increased by one standard deviation, while Panel C of Table 3.8 presents the case of the corresponding reduction in the inflation central tendency. In the first case, the increase in the premium of the 3-year real bond significantly increases both the myopic and the hedging demand for this bond for short horizons. On the other hand, the reduction in the central tendency of inflation reduces the reward for holding both the 3-year and the 10-year real bond. Consequently, the magnitude of both the myopic and the hedging bond demands becomes lower in comparison to the benchmark case.

Expanding the asset space to include a third nominal bond leads to larger long and short positions. Panel A of Table 3.9 shows the allocation of wealth to the nominal zero-coupon bonds with 1, 5 and 10-year maturities for various investment horizons when the investor maximizes utility over real terminal wealth. The magnitude of the hedging demands depends crucially on the horizon and it tends to decrease as the investor becomes more risk averse. The inclusion of another bond gives the investor more flexibility in his attempt to create a hedging bond portfolio with duration that matches his horizon. The 10-year nominal bond plays a significant hedging role mainly for investors with long horizons.

Panel B of Table 3.9 reports the corresponding optimal portfolio choices when an investor can allocate his wealth among real zero-coupon bonds with 1, 5 and 10-year maturities. The 1-year bond offers a significantly higher premium in comparison to the negative premia offered by the 5-year and 10-year bonds. Consequently, the myopic investor should take a significant long position in the 1-year bond, selling short the 5-year bond. On the other hand, the 10-year bond incorporates significant hedging value for a long-term investor, especially when his horizon is longer than 5 years. These results also show that access to multiple real bonds may offer the required flexibility to long-term investors who seek to hedge away shocks to their real wealth process in an optimal way.

Subsequently, we examine the optimal asset allocation to one real and two nominal bonds. This is an interesting combination to examine because the investor can actually improve the diversification of his portfolio, extract the premia from nominal bonds and make use of the real bond's hedging value. It should be reminded that when deriving the optimal portfolio choice for this case, the real bond's SDE under the nominal SDF, given by (3.52), is appropriately used in equation (3.84). Panel C of Table 3.9 shows the portfolio choice results. Bond demands are much lower in magnitude and less sensitive to shifts in the investment horizon and the degree of relative risk aversion. This is due to the much lower correlation in the returns of the available bonds. The 5-year real bond plays a hedging role, especially for an investor with a horizon close to five years. The 1-year bond is used mainly for its attractive risk-return profile, while the 10-year nominal bond plays a very significant hedging role for horizons longer than five years.

3.6.3 Sensitivity analysis

The last issue to examine is how sensitive are the portfolio choices that we previously presented to the choice of the benchmark date 1975:Q1. The extreme swings in the maximal Sharpe ratio, illustrated in Figure 3.5, indicate that both the myopic and the hedging demands will be subject to extreme swings too. This is due to the fact that under time-varying risk premia, the investor should be involved in market timing for investment as well as for hedging purposes. The large shifts in the macroeconomic factors as well as the very high wealth elasticities of the power utility investor with respect to these factors predict these extreme swings in portfolio choices.

We plot in Figure 3.9 the total myopic bond demand for a power utility investor with $\gamma = 10$, who has access to a 3-year and a 10-year nominal zero-coupon bond for the period 1964:Q1 to 1998:Q4 (solid line). The vector of the myopic bond demands is given by the first two terms in equation (3.84). The total myopic bond demand exhibits significant shifts through time, verifying the argument that the time-variation in the bond premia is a very important issue that even a myopic investor should take into account. The evolution of the total myopic demand resembles the evolution of the maximal Sharpe ratio that is illustrated in Figure 3.5.

More impressive are the extreme shifts in the total hedging bond demand. Figure 3.9 illustrates the evolution of the sum of the hedging demands for these two nominal bonds. The vector of the hedging demands is given by the third term in equation (3.84) for a power utility investor with $\gamma = 10$, who has a horizon of $T = 3$ and $T = 10$ years correspondingly. For both cases, the shifts are extreme for even small changes in the macroeconomic factors. Furthermore, the magnitude of the total hedging demand is very high and overwhelmingly dominates the corresponding total myopic bond demand for every single period. It is also very interesting to observe that the total hedging bond demand is almost the same for both investment horizons, to the extent that the two lines are indistinguishable. As we have previously analyzed, the investment horizon plays a crucial role for the internal composition of the hedging bond portfolio, not for its total magnitude. The magnitude as well as the extreme shifts in the total hedging bond demand are consequences of the huge wealth elasticities of the long-term power utility investor, illustrated in Figure 3.8, as well as the considerable variation in the investment opportunity set. Therefore, these results underline our previous conclusions, showing that the time-variation in bond premia is an extremely important issue that a long-term investor should not neglect in his attempt to hedge away undesirable shocks.

We repeat the previous sensitivity analysis for the case of a power utility investor with $\gamma = 10$, who has access to a 3-year and a 10-year real zero-coupon bond for the period 1964:Q1 to 1998:Q4. Figure 3.10 illustrates the total myopic bond demand (solid line), the total hedging bond demand for an investor with a horizon of $T = 3$ years (triangle-marked line) as well as the corresponding total hedging bond demand when the investment horizon is $T = 10$ years (dashed line). The vector of the myopic bond demands is given by the first term in equation (3.96), while the vector of the hedging bond demands is given by the second term in (3.96).

The results show that the shifts in the total myopic bond demand as well as in the total hedging bond demand are of great magnitude for the case of real bonds too. A visual inspection of Figure 3.10 shows that the evolution of the total myopic and total hedging bond demands follow closely the evolution of the maximal Sharpe ratio. This finding confirms again the argument that the power utility investor should rebalance his bond portfolio according to the shifts in the macroeconomy. Moreover, the magnitude of the total hedging bond demand of the power utility investor with $\gamma = 10$ is greater than the corresponding total myopic bond demand for every period in our sample. Nevertheless, the difference in their magnitudes is less pronounced as compared to the case of nominal zero-coupon bonds. Given this evidence, it should be finally noted that the period we employed as our benchmark in this study, i.e. 1975:Q1, was selected on the basis that it yielded a low magnitude of myopic and hedging demands, facilitating our analysis.

3.7 Conclusion

This chapter examined the dynamic bond portfolio choice of a long-term, power utility investor. Using the macro-finance term structure model of Dewachter et al. (2006), we were able to provide a clear macroeconomic interpretation for the formation of bond premia and to examine how shifts in the macroeconomy affect the formation of portfolios. We have also documented how the concept of the riskless asset is defined in the case of utility over terminal real wealth, when the investor has access either to nominal or real bonds. Until now, this issue has been explored in the literature only separately for each case.

Our results can be of significant importance for institutional investors, such as pension funds. Matching the duration of a nominal bond with the investment horizon is a legitimate practice for infinitely risk averse agents who have utility over nominal wealth at a terminal date. However, if the investors are interested in real wealth, this practice does not provide a safe strategy, because the nominal bond is risky in real terms. In this case, a real bond with the appropriate duration becomes the riskless asset. Furthermore, in the presence of consumption or liabilities and payments in the case of pension funds, the effective investment horizon is considerably reduced.

This framework has also allowed us to provide reconciliation for the existence of assets with low or even negative expected excess returns, such as long-term real bonds. Traditional mean-variance theory cannot provide an explanation why an investor should hold these bonds. Our results verify the popular opinion that these bonds are mainly held for hedging purposes, especially when investors care for real wealth and consumption. Moreover, the real bonds could be included in a broader portfolio because their returns exhibit much lower correlation with the returns of nominal bonds, hence they may be useful instruments for diversification.

With respect to the term structure literature, we provide an evaluation of the essentially affine models from an asset allocation perspective. While the focus of the literature is on fitting past and predicting future yields, the covariance and correlation structure that estimated models imply for the bond returns is relatively neglected. It is shown that this is a major concern if one wishes to implement these models for portfolio choice, because the estimated volatilities are extremely low and the correlations of the returns are extremely high. A potential solution is to use a term structure model that allows for time-varying conditional volatilities, as in Spencer (2007), or to adopt a Bayesian approach for portfolio choice, as in Garlappi et al. (2007), assuming parameter uncertainty that effectively increases the returns' volatilities.

Last, the high market prices of risk yield extreme wealth sensitivities to movements in the underlying risk factors, generating hedging demands of large magnitude. This effect becomes moderate only if we assume very short horizons and very high degrees of relative risk aversion. It is hard to reconcile these large sensitivities within the commonly used power utility framework in the presence of horizon effects. Consequently, we document a premium puzzle in the bond market too. Future research could examine whether the myopic loss aversion framework of Benartzi and Thaler (1995) or the framework of loss aversion with narrow framing, as recently examined in Barberis et al. (2006), may yield more realistic conclusions.

3.8 Appendix

Proof of Proposition 3.1

Employing the conjectured functional form for G, we can substitute the corresponding terms into the PDE (3.28) and recalling the definition of the price of risk ξ as well as the fact that the nominal risk-free rate is given by $r = \delta_1^T X$ and $r - \pi = (\delta_1 - \delta_2)^T X$, where $\delta_1 = (0\ 1\ 1\ 0\ 0)^T$ and $\delta_2 = (0\ 1\ 0\ 0\ 0)^T$, this PDE can be written as:

$$\begin{aligned}
&(\frac{1}{\gamma} - 1)(\delta_1 - \delta_2)^T X + \frac{1}{2\gamma}\bar{\psi}^T(Q + Q^T)X + \frac{1}{\gamma}\bar{\psi}^T d + \frac{1}{2\gamma}X^T K^T(Q + Q^T)X + \\
&\quad + \frac{1}{\gamma}d^T KX + \frac{1-\gamma}{2\gamma^2}\Lambda^T S^T S\Lambda + \frac{1-\gamma}{\gamma^2}\Lambda^T \Xi X + \frac{1-\gamma}{2\gamma^2}X^T \Xi^T S^{-2}\Xi X - \\
&\quad - \frac{1-\gamma}{\gamma^2}\sigma_\Pi^T S\Lambda - \frac{1-\gamma}{\gamma^2}\sigma_\Pi^T S^{-1}\Xi X + (\frac{1}{\gamma} - 1)\sigma_\Pi^T S\Lambda + (\frac{1}{\gamma} - 1)\sigma_\Pi^T S^{-1}\Xi X + \\
&\quad + \frac{1}{\gamma}(1 - \frac{1}{\gamma})\sigma_\Pi^T S[\frac{1}{2}(Q + Q^T)X + d] + \frac{1-\gamma}{2\gamma^2}\sigma_\Pi^T\sigma_\Pi + \frac{1}{2\gamma^2}d^T SS^T d + \\
&\quad + \frac{1}{8\gamma^2}X^T(Q + Q^T)SS^T(Q + Q^T)X + \frac{1}{2\gamma^2}d^T SS^T(Q + Q^T)X + \\
&+ \frac{1}{4\gamma}Tr(SS^T(Q + Q^T)) + \frac{1-\gamma}{2\gamma^2}\Lambda^T S^T S(Q + Q^T)X + \frac{1-\gamma}{2\gamma^2}X^T\Xi^T(Q + Q^T)X + \\
&\quad + \frac{1-\gamma}{\gamma^2}\Lambda^T S^T Sd + \frac{1-\gamma}{\gamma^2}d^T\Xi X + \frac{1}{2\gamma}X^T\dot{Q}X + \frac{1}{\gamma}\dot{d}^T X + \frac{1}{\gamma}\dot{c} = 0 \qquad (3.97)
\end{aligned}$$

where $\dot{c}$, $\dot{d}$, $\dot{Q}$ stand for the derivatives of c, d and Q with respect to time t.

From the previous expression, gathering terms in $X^T[.]X$, X and the scalar term, we get the following system of ODEs:

$$\dot{c}+\bar{\psi}^T d+\frac{1-\gamma}{2\gamma}\Lambda^T S^T S\Lambda+\frac{1}{2\gamma}d^T SS^T d+\frac{1}{4}Tr(SS^T(Q+Q^T))+$$
$$+\frac{1-\gamma}{\gamma}\Lambda^T S^T Sd-\frac{1-\gamma}{\gamma}\sigma_\Pi^T S\Lambda+(1-\gamma)\sigma_\Pi^T S\Lambda+(1-\frac{1}{\gamma})\sigma_\Pi^T Sd+\frac{1-\gamma}{2\gamma}\sigma_\Pi^T\sigma_\Pi=0 \tag{3.98}$$

$$\dot{d}^T+(1-\gamma)(\delta_1-\delta_2)^T+\frac{1}{2}\bar{\psi}^T(Q+Q^T)+d^T K+\frac{1-\gamma}{\gamma}\Lambda^T\Xi-\frac{1-\gamma}{\gamma}\sigma_\Pi^T S^{-1}\Xi+$$
$$+(1-\gamma)\sigma_\Pi^T S^{-1}\Xi+\frac{1}{2\gamma}d^T SS^T(Q+Q^T)+\frac{1-\gamma}{2\gamma}\Lambda^T S^T S(Q+Q^T)+$$
$$+\frac{1}{2}(1-\frac{1}{\gamma})\sigma_\Pi^T S(Q+Q^T)+\frac{1-\gamma}{\gamma}d^T\Xi=0 \tag{3.99}$$

$$\dot{Q}+K^T(Q+Q^T)+\frac{1-\gamma}{\gamma}\Xi^T S^{-2}\Xi+\frac{1}{4\gamma}(Q+Q^T)SS^T(Q+Q^T)+\frac{1-\gamma}{\gamma}\Xi^T(Q+Q^T)=0 \tag{3.100}$$

along with the corresponding terminal conditions. This is a system of 31 ODEs. Furthermore, substituting the functional form of G into (3.29), we get the optimal portfolio choice in (3.31).

Proof of Corollary 3.1

The expression for the optimal consumption-wealth ratio is derived by substituting the definitions of $\hat{G}(Z,X,\Pi,t)$ and $G(Z,X,\Pi,t)$ into equations (3.36) and (3.38). For the optimal portfolio choice, we have that the diffusion component of the dynamics of $\hat{G}(Z,X,t)$ is given by $\frac{\partial\hat{G}}{\partial Z}Z(\xi-\sigma_\Pi)+\frac{\partial\hat{G}}{\partial\Pi}\Pi\sigma_\Pi+S\frac{\partial\hat{G}}{\partial X}$. Consequently, the optimal portfolio choice, ϕ, should satisfy:

$$\hat{G}\phi^T(-B^T S)=\frac{\partial\hat{G}}{\partial Z}Z(\xi-\sigma_\Pi)^T+\frac{\partial\hat{G}}{\partial\Pi}\Pi\sigma_\Pi^T+(S\frac{\partial\hat{G}}{\partial X})^T \tag{3.101}$$

Substituting into this equation the definitions of $\hat{G}$ and G, we get the optimal portfolio choice in (3.42). The ODEs are the same as in the terminal wealth case, because we can equivalently solve PDE (3.28) involving G instead of the corresponding PDE involving $\hat{G}$.

Proof of Proposition 3.2

Substituting the conjectured functional form for G^R into the PDE (3.57), we end up with exactly the same equation as in (3.97) with respect to $Q^R(t)$, $d^R(t)$ and $c^R(t)$. As a result, the ODEs that $Q^R(t)$, $d^R(t)$ and $c^R(t)$ should satisfy are of the same form as in (3.98) - (3.100).

Substituting the conjectured form of the function $G^R(Z^R, X, t)$ into (3.58), we get the optimal portfolio choice in (3.60).

Proof of Corollary 3.2

The expression for the optimal consumption-wealth ratio is derived by substituting the definitions of $\hat{G}^R(Z^R, X, t)$ and $G^R(Z^R, X, t)$ into equations (3.64) and (3.62). Solving the equivalent PDE (3.57), we end up with equation (3.97). As a result, the system of ODEs for $Q^R(t)$, $d^R(t)$ and $c^R(t)$ are of the same form as in (3.98) - (3.100).

Since the investor has access to real bonds, his real wealth dynamics are given by:

$$\begin{aligned} d(\frac{W}{\Pi}) &= \frac{W}{\Pi}(\phi^{RT}[-B^R(\tau)^T S(\xi - \sigma_\Pi)] + r - \pi + \sigma_\Pi^T \xi)dt - Cdt \\ &\quad + \frac{W}{\Pi}\phi^{RT}(-B^R(\tau)^T S)dw \end{aligned} \tag{3.102}$$

Consequently, given the diffusion component of the dynamics of $\hat{G}^R(Z^R, X, t)$, the optimal portfolio choice should satisfy

$$\hat{G}^R \phi^{RT}(-B^{RT} S) = \frac{\partial \hat{G}^R}{\partial Z^R} Z^R(\xi - \sigma_\Pi)^T + (S\frac{\partial \hat{G}^R}{\partial X})^T \tag{3.103}$$

Substituting the conjectured form of the function $\hat{G}^R$ into this equation, we get the optimal portfolio choice in (3.68).

Proof of Proposition 3.3

Substituting the conjectured form for G^I into (3.81), the optimal portfolio choice should satisfy:

$$(\phi^I)^T(-B^TS) = \frac{1}{\gamma}\xi^T + \frac{1}{\gamma}(v^*)^T + (1-\frac{1}{\gamma})\sigma_\Pi^T + \frac{1}{\gamma}(\frac{1}{2}X^T(Q^I + Q^{IT}) + d^{IT})S \quad (3.104)$$

According to the argument of He and Pearson (1991), v^* should guarantee that the unhedgeable parts of σ_Π^T and S should drop out. Note that σ_Π^T can be written as:

$$\sigma_\Pi^T = \sigma_\Pi^T(-B^TS)^T(B^TS^2B)^{-1}(-B^TS) + [\sigma_\Pi^T - \sigma_\Pi^T(-B^TS)^T(B^TS^2B)^{-1}(-B^TS)] \quad (3.105)$$

where the first component is the projection of σ_Π^T onto the available assets and the second component is the residual of the projection, $(\sigma_\Pi^T)^\perp$. Similarly, S can be written as:

$$S = S(-B^TS)^T(B^TS^2B)^{-1}(-B^TS) + [S - S(-B^TS)^T(B^TS^2B)^{-1}(-B^TS)] \quad (3.106)$$

where $S^\perp = S - S(-B^TS)^T(B^TS^2B)^{-1}(-B^TS)$ is the residual of the projection of S onto the available assets. It should be noted that under complete markets, $(\sigma_\Pi^T)^\perp = \mathbf{0}$ and $S^\perp = \mathbf{0}$.

So, v^* should satisfy the following condition:

$$v^* = (1-\gamma)[(\sigma_\Pi^T)^\perp]^T - (S^\perp)^T[(\frac{1}{2}(Q^I + Q^{IT})X + d^I)] \quad (3.107)$$

Substituting this expression into the optimal portfolio choice (3.104), we get:

$$(\phi^I)^T(-B^TS) = \frac{1}{\gamma}\xi^T + (1-\frac{1}{\gamma})(\sigma_\Pi^T - (\sigma_\Pi^T)^\perp) + \frac{1}{\gamma}(\frac{1}{2}X^T(Q^I+Q^{IT}) + d^{IT})(S - S^\perp) \Rightarrow$$

$$(\phi^I)^T(-B^TS) = \frac{1}{\gamma}\xi^T + (1-\frac{1}{\gamma})\sigma_\Pi^T(-B^TS)^T(B^TS^2B)^{-1}(-B^TS) +$$
$$+\frac{1}{\gamma}(\frac{1}{2}X^T(Q^I+Q^{IT}) + d^{IT})S(-B^TS)^T(B^TS^2B)^{-1}(-B^TS) \qquad (3.108)$$

Multiplying this expression by $(-B^TS)^T(B^TS^2B)^{-1}$ and taking the transpose, we derive the optimal portfolio choice expression in (3.84).

Moreover, substituting the conjectured form for G^I and the expression for v^* into the PDE (3.80), we can derive an equation from which, if we collect the terms in $X^T[.]X$, X and the scalar correspondingly, we get the following system of ODEs:

$$\dot{c}^I + \bar{\psi}^T d^I + \frac{1-\gamma}{2\gamma}\Lambda^T S^T S\Lambda + \frac{1}{2\gamma}d^{IT}SS^Td^I + \frac{1}{4}Tr(SS^T(Q^I+Q^{IT}))+$$
$$+\frac{1-\gamma}{\gamma}\Lambda^TS^TSd^I - \frac{1-\gamma}{\gamma}\sigma_\Pi^TS\Lambda + (1-\gamma)\sigma_\Pi^TS\Lambda + (1-\frac{1}{\gamma})\sigma_\Pi^TSd^I + \frac{1-\gamma}{2\gamma}\sigma_\Pi^T\sigma_\Pi +$$
$$+\frac{(1-\gamma)^2}{\gamma}(\sigma_\Pi^T)^\perp Sd^I - \frac{1-\gamma}{2\gamma}d^{IT}S^\perp S^Td^I + (1-\gamma)^2(1-\frac{1}{\gamma})(\sigma_\Pi^T)^\perp\sigma_\Pi +$$
$$+(1-\gamma)(\frac{1}{\gamma}-1)d^{IT}S^\perp\sigma_\Pi + \frac{(1-\gamma)^3}{2\gamma}(\sigma_\Pi^T)^\perp[(\sigma_\Pi^T)^\perp]^T -$$
$$-\frac{(1-\gamma)^2}{\gamma}(\sigma_\Pi^T)(S^\perp)^Td^I + \frac{1-\gamma}{2\gamma}d^{IT}S^\perp(S^\perp)^Td^I = 0 \qquad (3.109)$$

$$\dot{d}^{IT} + (1-\gamma)(\delta_1 - \delta_2)^T + \frac{1}{2}\bar{\psi}^T(Q^I + Q^{IT}) + d^{IT}K + \frac{1-\gamma}{\gamma}\Lambda^T\Xi - \frac{1-\gamma}{\gamma}\sigma_\Pi^T S^{-1}\Xi +$$
$$+(1-\gamma)\sigma_\Pi^T S^{-1}\Xi + \frac{1}{2\gamma}d^{IT}SS^T(Q^I + Q^{IT}) + \frac{1-\gamma}{2\gamma}\Lambda^T S^T S(Q^I + Q^{IT}) +$$
$$+\frac{1}{2}(1 - \frac{1}{\gamma})\sigma_\Pi^T S(Q^I + Q^{IT}) + \frac{1-\gamma}{\gamma}d^{IT}\Xi + \frac{(1-\gamma)^2}{2\gamma}\sigma_\Pi^T S(Q^I + Q^{IT}) -$$
$$-\frac{1-\gamma}{\gamma}d^{IT}S^\perp S(Q^I + Q^{IT}) + \frac{1-\gamma}{2}(\frac{1}{\gamma} - 1)\sigma_\Pi^T(S^\perp)^T(Q^I + Q^{IT}) -$$
$$-\frac{1-\gamma}{2\gamma}(\sigma_\Pi^T)^\perp(S^\perp)^T(Q^I + Q^{IT}) + \frac{1-\gamma}{2\gamma}d^{IT}S^\perp(S^\perp)^T(Q^I + Q^{IT}) = \dot{0} \qquad (3.110)$$

$$\dot{Q}^I + K^T(Q^I + Q^{IT}) + \frac{1-\gamma}{\gamma}\Xi^T S^{-2}\Xi +$$
$$+\frac{1}{4\gamma}(Q^I + Q^{IT})SS^T(Q^I + Q^{IT}) + \frac{1-\gamma}{\gamma}\Xi^T(Q^I + Q^{IT}) +$$
$$+\frac{1-\gamma}{4\gamma}(Q^I + Q^{IT})S^\perp(S^\perp)^T(Q^I + Q^{IT}) - \frac{1-\gamma}{2\gamma}(Q^I + Q^{IT})S^\perp S(Q^I + Q^{IT}) = 0$$
$$(3.111)$$

along with the corresponding terminal conditions.

Proof of Proposition 3.4

Substituting the conjectured form for G^{IR} into (3.93), the optimal portfolio choice satisfies:

$$(\phi^{IR})^T(-B^{RT}S) = \frac{1}{\gamma}(\xi - \sigma_\Pi)^T + \frac{1}{\gamma}(V^*)^T + \frac{1}{\gamma}(\frac{1}{2}X^T(Q^{IR} + Q^{IRT}) + d^{IRT})S \quad (3.112)$$

Again, V^* should guarantee that the unhedgeable part of S should drop out. Hence, V^* should satisfy the following condition:

$$V^* = -(S^{\perp})^T[(\frac{1}{2}(Q^{IR} + Q^{IRT})X + d^{IR})] \tag{3.113}$$

where $S^{\perp} = S - S(-B^{RT}S)^T(B^{RT}S^2B^R)^{-1}(-B^{RT}S)$ is the residual of the projection of S on the available real bonds.

Substituting the expression for V^* into (3.112), we get:

$$\begin{aligned}(\phi^{IR})^T(-B^{RT}S) &= \frac{1}{\gamma}(\xi - \sigma_\Pi)^T + \frac{1}{\gamma}(\frac{1}{2}X^T(Q^{IR} + Q^{IRT}) + d^{IRT})(S - S^{\perp}) = \\ &= \frac{1}{\gamma}(\xi - \sigma_\Pi)^T + \frac{1}{\gamma}(\frac{1}{2}X^T(Q^{IR} + Q^{IRT}) + d^{IRT})S(-B^{RT}S)^T(B^{RT}S^2B^R)^{-1}(-B^{RT}S)\end{aligned} \tag{3.114}$$

Multiplying this expression by $(-B^{RT}S)^T(B^{RT}S^2B^R)^{-1}$ and taking the transpose, we derive the optimal portfolio choice expression in (3.96).

Substituting the conjectured form for G^{IR} and the expression for V^* into the PDE (3.92), we can derive an expression from which, if we gather the terms in $X^T[.]X$, X and the scalar, we get the following system of ODEs:

$$\begin{aligned}&\dot{c}^{IR} + \bar{\psi}^T d^{IR} + \frac{1-\gamma}{2\gamma}\Lambda^T S^T S\Lambda + \frac{1}{2\gamma}d^{IRT}SS^Td^{IR} + \frac{1}{4}Tr(SS^T(Q^{IR} + Q^{IRT})) + \\ &+\frac{1-\gamma}{\gamma}\Lambda^T S^T S d^{IR} - \frac{1-\gamma}{\gamma}\sigma_\Pi^T S\Lambda + (1-\gamma)\sigma_\Pi^T S\Lambda + (1-\frac{1}{\gamma})\sigma_\Pi^T S d^{IR} + \frac{1-\gamma}{2\gamma}\sigma_\Pi^T\sigma_\Pi \\ &+\frac{\gamma-1}{\gamma}d^{IRT}S^{\perp}Sd^{IR} + (1-\gamma)(\frac{1}{\gamma} - 1)d^{IRT}S^{\perp}\sigma_\Pi + \frac{1-\gamma}{2\gamma}d^{IRT}S^{\perp}(S^{\perp})^T d^{IR} = 0\end{aligned} \tag{3.115}$$

$$\dot{d}^{IRT} + (1-\gamma)(\delta_1 - \delta_2)^T + \frac{1}{2}\bar{\psi}^T(Q^{IR} + Q^{IRT}) + d^{IRT}K + \frac{1-\gamma}{\gamma}\Lambda^T\Xi - \frac{1-\gamma}{\gamma}\sigma_\Pi^T S^{-1}\Xi +$$
$$+(1-\gamma)\sigma_\Pi^T S^{-1}\Xi + \frac{1}{2\gamma}d^{IRT}SS^T(Q^{IR} + Q^{IRT}) + \frac{1-\gamma}{2\gamma}\Lambda^T S^T S(Q^{IR} + Q^{IRT}) +$$
$$+\frac{1}{2}(1-\frac{1}{\gamma})\sigma_\Pi^T S(Q^{IR} + Q^{IRT}) + \frac{1-\gamma}{\gamma}d^{IRT}\Xi + \frac{1-\gamma}{2\gamma}d^{IRT}S^\perp(S^\perp)^T(Q^{IR} + Q^{IRT})$$
$$+\frac{\gamma-1}{\gamma}d^{IRT}S^\perp S(Q^{IR} + Q^{IRT}) + \frac{1-\gamma}{2}(\frac{1}{\gamma} - 1)\sigma_\Pi^T(S^\perp)^T(Q^{IR} + Q^{IRT}) = 0 \quad (3.116)$$

$$\dot{Q}^{IR} + K^T(Q^{IR} + Q^{IRT}) + \frac{1-\gamma}{\gamma}\Xi^T S^{-2}\Xi +$$
$$+\frac{1}{4\gamma}(Q^{IR} + Q^{IRT})SS^T(Q^{IR} + Q^{IRT}) + \frac{1-\gamma}{4\gamma}(Q^{IR} + Q^{IRT})S^\perp(S^\perp)^T(Q^{IR} + Q^{IRT})$$
$$+\frac{1-\gamma}{\gamma}\Xi^T(Q^{IR} + Q^{IRT}) - \frac{1-\gamma}{2\gamma}(Q^{IR} + Q^{IRT})S^\perp S(Q^{IR} + Q^{IRT}) = 0 \qquad (3.117)$$

along with the corresponding terminal conditions.

Table 3.1: Estimated coefficients

	y	π	ρ	π^*	ρ^*
$\kappa_{y,\cdot}$	-0.3146 (0.0638)**	-1.0748 (0.1765)**	-0.4555 (0.1711)**		
$\kappa_{\pi,\cdot}$	0.3854 (0.064)**	-0.2452 (0.0446)**	-0.1319 (0.0249)**		
$\kappa_{\rho,\cdot}$	-0.0685 (0.0197)**	-5.1575 (0.4009)**	-5.3035 (0.3718)**		
$\kappa_{\pi^*,\cdot}$				-0.0036 (0.0012)**	
$\kappa_{\rho^*,\cdot}$					-0.4849 (0.0347)**
$\theta_{\cdot}$				0.0224 (0.0367)	0.0137 (0.0057)**
$\Lambda_{\cdot}$	-63.9951 (65.1136)	34.8309 (63.3615)	32.5392 (14.2244)**	-21.9563 (11.0275)**	
$\Xi_{\rho,\cdot}$	0.0082 (0.003)**	-0.3672 (0.1068)**	-0.9849 (0.2204)**	-1.1884 (0.3716)**	-1.5318 (0.4773)**
$\sigma^2_{\cdot}$	0.000279 (0.000043)**	0.000146 (0.000019)**	0.001545 (0.000042)**	0.000067 (0.000010)**	0.000253 (0.000043)**

Notes: This table shows the estimated parameters for the dynamics in X and the market price of risk ξ as reported in Table II of Dewachter et al. (2006). Robust standard errors are given in the parentheses. ** and * indicate statistical significance at the 5% and the 10% level correspondingly.

Table 3.2: Covariances and Correlations of Nominal bonds

Panel A: Covariance Matrix						
	1-yr	2-yr	3-yr	5-yr	7-yr	10-yr
1-yr	0.0004					
2-yr	0.0007	0.0014				
3-yr	0.0009	0.0019	0.0026			
5-yr	0.0013	0.0026	0.0037	0.0054		
7-yr	0.0016	0.0032	0.0046	0.0069	0.0088	
10-yr	0.0019	0.0041	0.0058	0.0088	0.0115	0.0153
Panel B: Correlation Matrix						
	1-yr	2-yr	3-yr	5-yr	7-yr	10-yr
1-yr	1					
2-yr	0.974	1				
3-yr	0.947	0.994	1			
5-yr	0.897	0.964	0.987	1		
7-yr	0.848	0.927	0.961	0.993	1	
10-yr	0.782	0.871	0.916	0.968	0.991	1

Notes: Panel A shows the covariance matrix of the nominal zero-coupon bond returns for various maturities. Panel B shows the corresponding correlation matrix.

Table 3.3: Covariances and Correlations of Real bonds (real SDF)

Panel A: Covariance Matrix						
	1-yr	2-yr	3-yr	5-yr	7-yr	10-yr
1-yr	0.0005					
2-yr	0.0009	0.0018				
3-yr	0.0013	0.0025	0.0035			
5-yr	0.0017	0.0034	0.0048	0.0066		
7-yr	0.0019	0.0038	0.0066	0.0075	0.0087	
10-yr	0.0021	0.0041	0.0075	0.0081	0.0094	0.0104
Panel B: Correlation Matrix						
	1-yr	2-yr	3-yr	5-yr	7-yr	10-yr
1-yr	1					
2-yr	0.982	1				
3-yr	0.964	0.995	1			
5-yr	0.933	0.976	0.991	1		
7-yr	0.91	0.956	0.975	0.995	1	
10-yr	0.885	0.932	0.951	0.975	0.99	1

Notes: Panel A shows the covariance matrix of the real zero-coupon bond returns for various maturities. Panel B shows the corresponding correlation matrix.

Table 3.4: Mixture of nominal and real bonds (nominal SDF)

Panel A: Covariance Matrix					
	1-yr (N)	3-yr (R)	5-yr (N)	7-yr (R)	10-yr (N)
1-yr (N)	0.0004				
3-yr (R)	0.0009	0.0042			
5-yr (N)	0.0013	0.0037	0.0054		
7-yr (R)	0.0016	0.0061	0.0061	0.0093	
10-yr (N)	0.0019	0.0054	0.0088	0.0091	0.0153
Panel B: Correlation Matrix					
	1-yr (N)	3-yr (R)	5-yr (N)	7-yr (R)	10-yr (N)
1-yr (N)	1				
3-yr (R)	0.741	1			
5-yr (N)	0.897	0.773	1		
7-yr (R)	0.819	0.965	0.859	1	
10-yr (N)	0.782	0.673	0.968	0.766	1

Notes: Panel A shows the covariance matrix of a mixture of nominal and real zero-coupon bond returns for various maturities. For consistency, we have used the dynamics of the real bonds' returns under the nominal SDF, as they are given by the SDE (3.52). (N) indicates a nominal bond while (R) indicates a real bond. Panel B shows the corresponding correlation matrix.

Table 3.5: Portfolio choice among five bonds

Panel A: 5 Nominal bonds									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=1	T=5	T=10	T=0	T=1	T=5	T=10
1-yr (N)	0.20%	20.88	24.76	30.34	21.93	10.57	15.10	21.21	13.09
3-yr (N)	0.65%	-218.49	-243.74	-281.20	-231.35	-102.24	-127.45	-166.48	-118.55
5-yr (N)	0.91%	757.66	810.83	820.84	729.94	334.49	385.80	410.69	317.84
7-yr (N)	1.17%	-788.08	-825.91	-779.61	-723.58	-339.46	-376.39	-350.25	-287.31
10-yr (N)	1.71%	236.62	243.95	219.23	212.55	100.15	107.57	89.40	79.37
Panel B: 5 Nominal and real bonds under the nominal SDF									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=1	T=5	T=10	T=0	T=1	T=5	T=10
1-yr (N)	0.20%	-1.47	-1.85	-5.81	-5.22	-0.81	-0.56	-3.69	-3.44
3-yr (R)	0.28%	17.32	17.81	15.63	15.19	7.27	7.74	6.09	5.42
5-yr (N)	0.91%	11.59	13.72	15.65	13.21	4.64	5.64	7.52	5.72
7-yr (R)	-1.39%	-16.57	-16.75	-13.67	-13.93	-6.79	-6.95	-4.46	-4.41
10-yr (N)	1.71%	-2.45	-3.23	-2.42	0.06	-0.97	-1.34	-1.01	1.09

Notes: This Table shows the optimal portfolio choice of an investor who has utility over real terminal wealth and access to five bonds. Panel A shows the portfolio choice among five nominal bonds as derived in (3.31) for the macroeconomic conditions prevailing in 1975:Q1 ($y = -5.89\%$, $\pi = 10.43\%$, $\rho = -5.21\%$, $\pi^* = 4.55\%$ and $\rho^* = 0.67\%$). Panel B shows the corresponding portfolio choice among 3 nominal and 2 real bonds. The dynamics for the returns of these 2 real bonds that have been employed in Panel B are given by the SDE in (3.52) and they are stated under the nominal SDF. The hedging demand for horizon T is given by the difference between the total demand for horizon T and the demand for $T = 0$ (myopic demand). The allocation to the nominal instantaneously riskless asset is equal to one minus the sum of the total bond demands in each case.

Table 3.6: Bond portfolio choice for an infinitely risk-averse investor

Panel A: 5 Nominal bonds						
	T=0	T=1	T=3	T=5	T=7	T=10
1-yr (N)	3.69	4.90	9.44	10.41	8.69	6.09
3-yr (N)	-24.74	-23.43	-50.18	-52.05	-40.38	-26.79
5-yr (N)	52.21	45.98	88.61	78.96	49.93	25.56
7-yr (N)	-40.38	-33.41	-55.75	-38.44	-12.52	3.57
10-yr (N)	9.17	6.99	8.97	2.23	-4.64	-7.38
Instantaneously risk-free asset	1.05	-0.03	-0.09	-0.10	-0.08	-0.05
Panel B: 5 Real bonds						
	T=0	T=1	T=3	T=5	T=7	T=10
1-yr (R)	0.00	1.00	0.00	0.00	0.00	0.00
3-yr (R)	0.00	0.00	1.00	0.00	0.00	0.00
5-yr (R)	0.00	0.00	0.00	1.00	0.00	0.00
7-yr (R)	0.00	0.00	0.00	0.00	1.00	0.00
10-yr (R)	0.00	0.00	0.00	0.00	0.00	1.00
Instantaneously risk-free asset	1.00	0.00	0.00	0.00	0.00	0.00

Notes: This Table shows the optimal portfolio choice of an infinitely risk averse investor ($\gamma = 100,000$), who has utility over real terminal wealth and access to five bonds. Panel A shows the portfolio choice among five nominal bonds for the macroeconomic conditions prevailing in 1975:Q1. Panel B shows the corresponding portfolio choice when the investor has access to five real bonds. The hedging demand for horizon T is given by the difference between the total demand for horizon T and the demand when $T = 0$ (myopic demand). The allocation to the corresponding instantaneously riskless asset is equal to one minus the sum of the total bond demands in each case, provided in the last line of each panel.

Table 3.7: Portfolio choice among two nominal bonds

Panel A: Benchmark case 1975:Q1									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=3	T=5	T=10	T=0	T=3	T=5	T=10
3-yr	0.65%	0.05	5.17	4.66	1.94	0.03	3.79	3.88	2.01
10-yr	1.71%	0.26	0.29	1.28	3.12	0.10	0.02	0.63	2.12
Panel B: Increase in the inflation central tendency									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=3	T=5	T=10	T=0	T=3	T=5	T=10
3-yr	0.88%	0.84	9.39	8.77	5.47	0.35	6.07	6.13	3.82
10-yr	1.96%	0.002	-0.24	0.94	3.15	-0.004	-0.23	0.52	2.32
Panel C: Decrease in the inflation central tendency									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=3	T=5	T=10	T=0	T=3	T=5	T=10
3-yr	0.42%	-0.74	0.94	0.56	-1.59	-0.28	1.52	1.64	0.19
10-yr	1.46%	0.52	0.82	1.62	3.09	0.20	0.26	0.74	1.91

Notes: This Table shows the optimal portfolio choice of an investor who has utility over real terminal wealth and access to two nominal bonds, given by (3.84). Panel A shows the portfolio choice for the macroeconomic conditions prevailing in 1975:Q1 ($y = -5.89\%$, $\pi = 10.43\%$, $\rho = -5.21\%$, $\pi^* = 4.55\%$ and $\rho^* = 0.67\%$). Panel B shows the corresponding portfolio choice when the inflation central tendency π^* is increased by one standard deviation ($\sigma_{\pi^*} = 0.81\%$), while Panel C shows the corresponding choice when the inflation central tendency π^* is decreased by one standard deviation. The hedging demand for horizon T is given by the difference between the total demand for horizon T and the demand when $T = 0$ (myopic demand). The allocation to the nominal instantaneously riskless asset is equal to one minus the sum of the total bond demands in each case.

Table 3.8: Portfolio choice among two real bonds

Panel A: Benchmark case 1975:Q1									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=3	T=5	T=10	T=0	T=3	T=5	T=10
3-yr	-0.25%	4.76	5.17	3.13	0.03	1.90	2.63	1.38	-0.82
10-yr	-1.59%	-3.01	-0.85	1.16	3.33	-1.20	-0.02	1.41	3.04
Panel B: Increase in the inflation central tendency									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=3	T=5	T=10	T=0	T=3	T=5	T=10
3-yr	0.12%	5.65	6.32	3.89	0.19	2.26	3.12	1.59	-1.05
10-yr	-1.14%	-3.39	-0.02	2.37	4.96	-1.36	0.52	2.24	4.21
Panel C: Decrease in the inflation central tendency									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=3	T=5	T=10	T=0	T=3	T=5	T=10
3-yr	-0.62%	3.87	4.01	2.37	-0.14	1.55	2.14	1.16	-0.59
10-yr	-2.06%	-2.63	-1.68	-0.05	1.69	-1.05	-0.56	0.58	1.87

Notes: This Table shows the optimal portfolio choice of an investor who has utility over real terminal wealth and access to real bonds, given by (3.96). Panel A shows the portfolio choice for the macroeconomic conditions prevailing in 1975:Q1 ($y = -5.89\%$, $\pi = 10.43\%$, $\rho = -5.21\%$, $\pi^* = 4.55\%$ and $\rho^* = 0.67\%$). Panel B shows the corresponding portfolio choice when the inflation central tendency π^* is increased by one standard deviation ($\sigma_{\pi^*} = 0.81\%$), while Panel C shows the corresponding choice when the inflation central tendency π^* is decreased by one standard deviation. The hedging demand for horizon T is given by the difference between the total demand for horizon T and the demand when $T = 0$ (myopic demand). The allocation to the real instantaneously riskless asset is equal to one minus the sum of the total bond demands in each case.

Table 3.9: Portfolio choice among three bonds

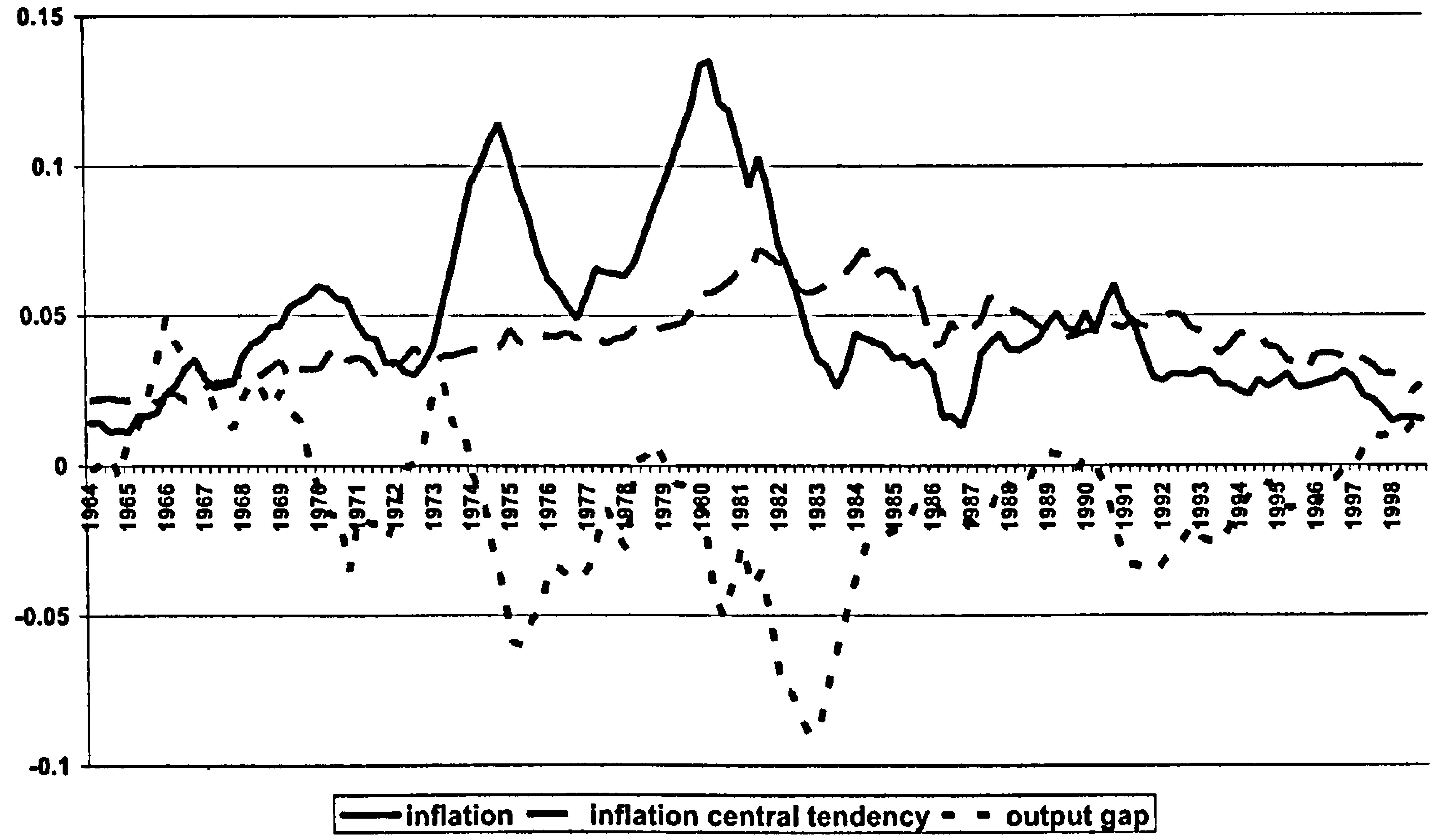
Panel A: 3 Nominal bonds (nominal SDF)									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=1	T=5	T=10	T=0	T=1	T=5	T=10
1-yr (N)	0.20%	1.74	0.73	-5.96	-3.89	0.14	-0.25	-5.77	-4.31
5-yr (N)	0.91%	-1.61	1.27	8.66	4.09	-0.27	1.53	8.08	4.74
10-yr (N)	1.71%	0.99	-0.07	-1.07	2.13	0.25	-0.42	-1.66	0.91

Panel B: 3 Real bonds (real SDF)									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=1	T=5	T=10	T=0	T=1	T=5	T=10
1-yr (R)	0.34%	29.69	31.68	30.51	31.71	11.87	13.42	12.52	13.40
5-yr (R)	-1.37%	-12.96	-13.45	-15.90	-20.92	-5.18	-5.44	-6.64	-10.67
10-yr (R)	-1.59%	3.83	4.47	9.46	13.76	1.53	1.87	5.13	8.78

Panel C: 3 Nominal and real bonds under the nominal SDF									
		$\gamma = 4$				$\gamma = 10$			
	Premia	T=0	T=1	T=5	T=10	T=0	T=1	T=5	T=10
1-yr (N)	0.20%	2.98	4.71	4.81	2.28	0.87	2.00	1.99	1.17
5-yr (R)	-0.84%	-1.42	-0.99	0.83	-0.37	-0.46	-0.14	1.08	0.63
10-yr (N)	1.71%	0.60	0.63	3.15	3.74	0.23	0.23	1.30	2.43

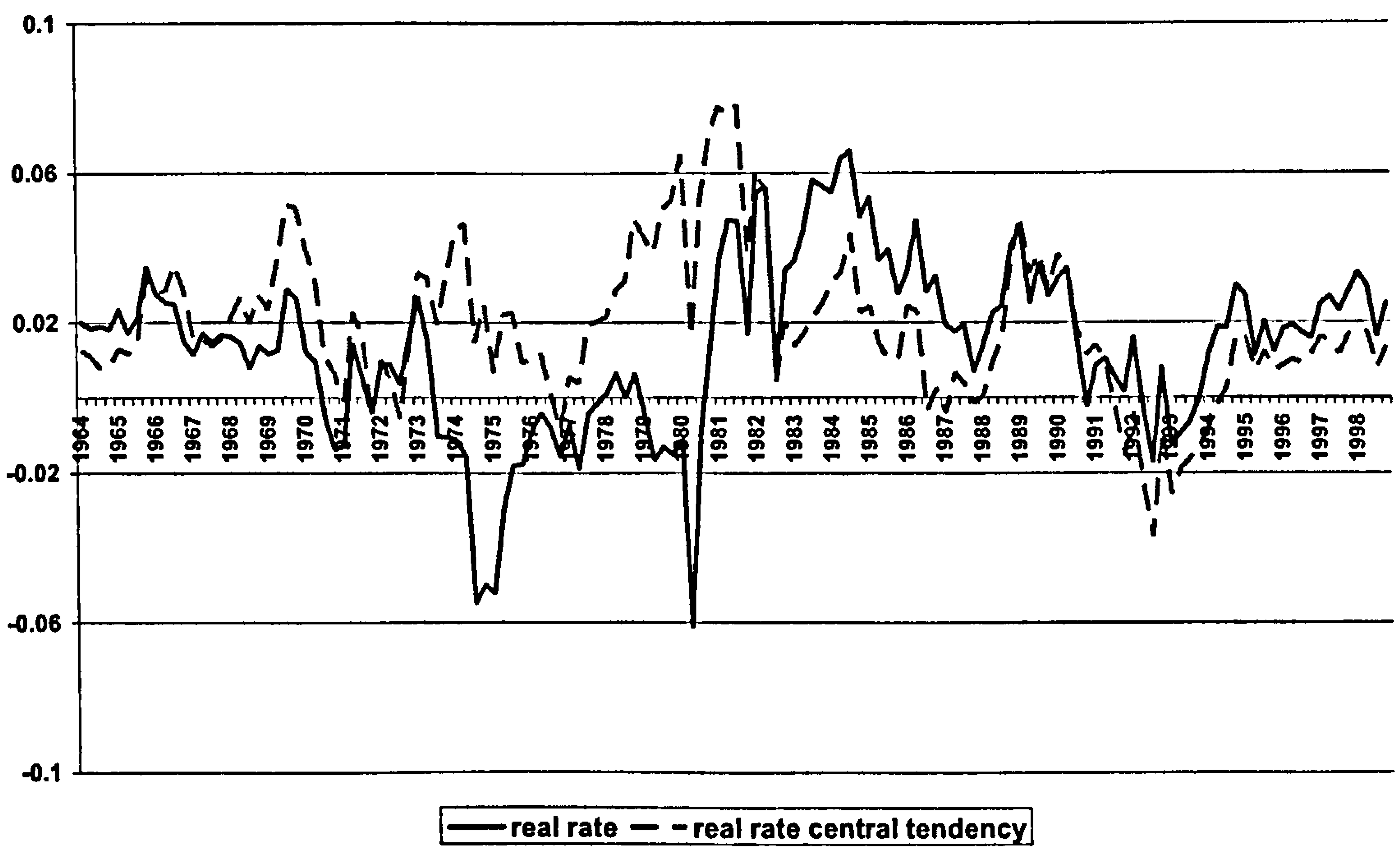
Notes: This Table shows the optimal portfolio choice of an investor who has utility over real terminal wealth and access to three bonds for the macroeconomic conditions prevailing in 1975:Q1. Panel A shows the allocation among three nominal bonds, given by (3.84), while Panel B shows the allocation among three real bonds, given by (3.96). Panel C shows the corresponding allocation among two nominal and one real bond. The dynamics for the returns of the real bond are given by the SDE in (3.52) and they are stated under the nominal SDF. The hedging demand for horizon T is given by the difference between the total demand for horizon T and the demand when $T = 0$ (myopic demand). The allocation to the instantaneously riskless asset is equal to one minus the sum of the total bond demands in each case.

Figure 3.1: Output gap, inflation and inflation central tendency



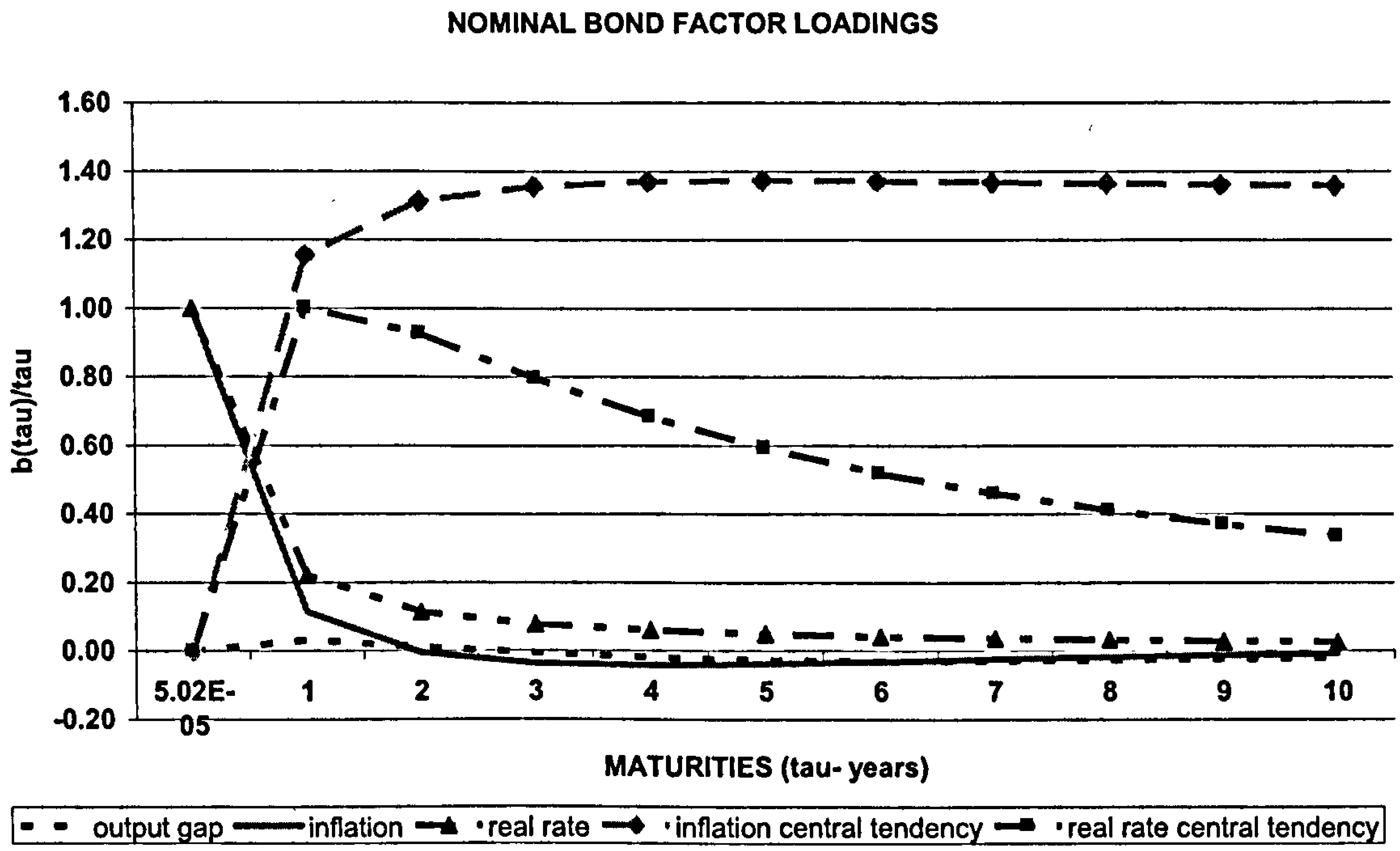
This figure plots the output gap y (dotted line) and the inflation series π (solid line), along with the filtered series for the central tendency of inflation π^* (dashed line) during the sample period 1964:Q1 to 1998:Q4.

Figure 3.2: Real interest rate and real interest rate central tendency



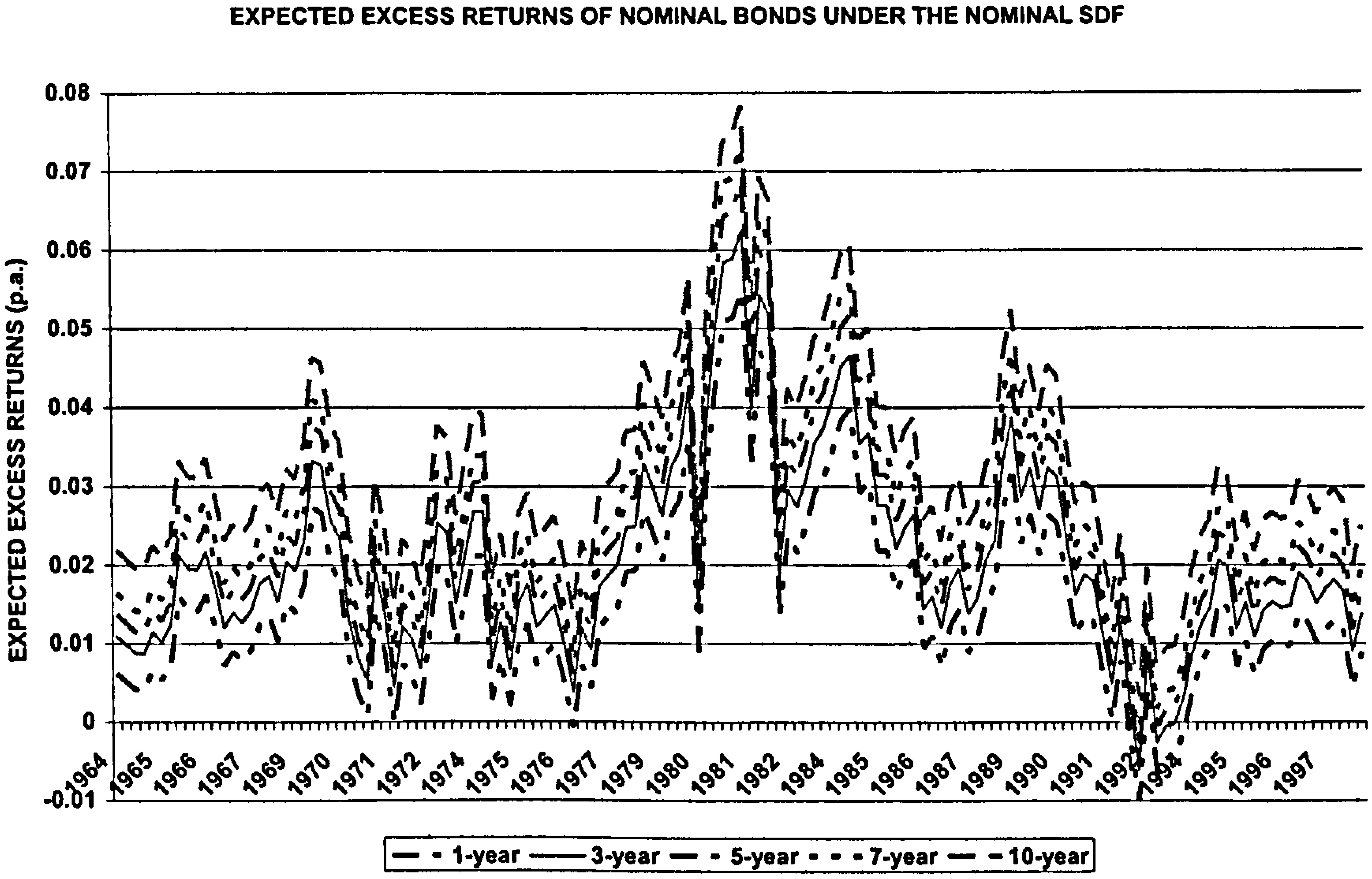
This figure plots the filtered series of the real interest rate ρ (solid line) and the central tendency of the real interest rate ρ^* (dashed line), during the sample period 1964:Q1 to 1998:Q4.

Figure 3.3: Factor Loadings of nominal zero-coupon bonds



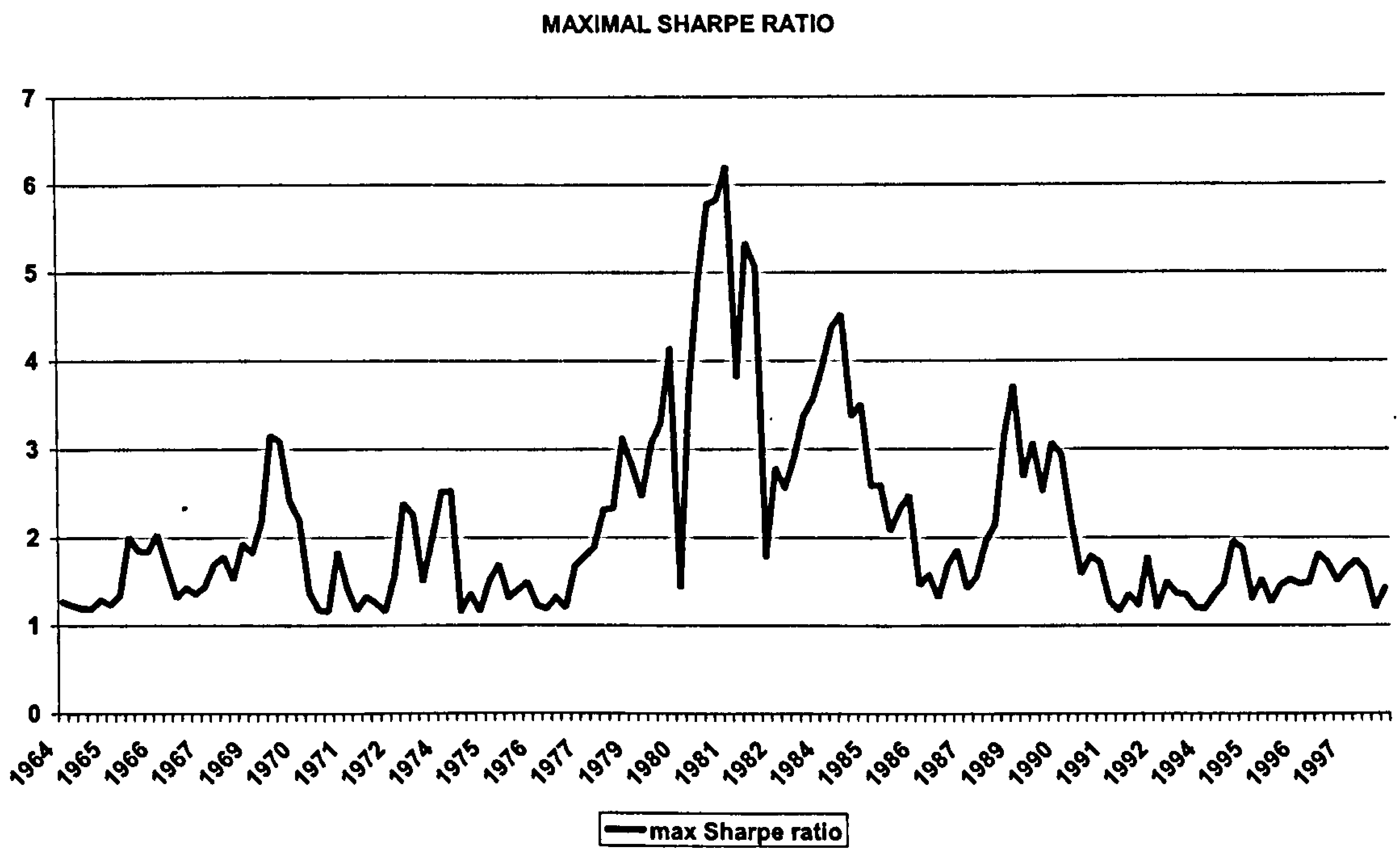
This figure plots the loadings of the output gap y (dotted line), inflation π (solid line), real rate ρ (double dot-dashed line), inflation central tendency π^* (dashed line) and real rate central tendency ρ^* (dot-dashed line) on the nominal zero-coupon bonds, adjusted for the corresponding maturities, i.e. $b(\tau)/\tau$.

Figure 3.4: Expected excess returns of nominal zero-coupon bonds



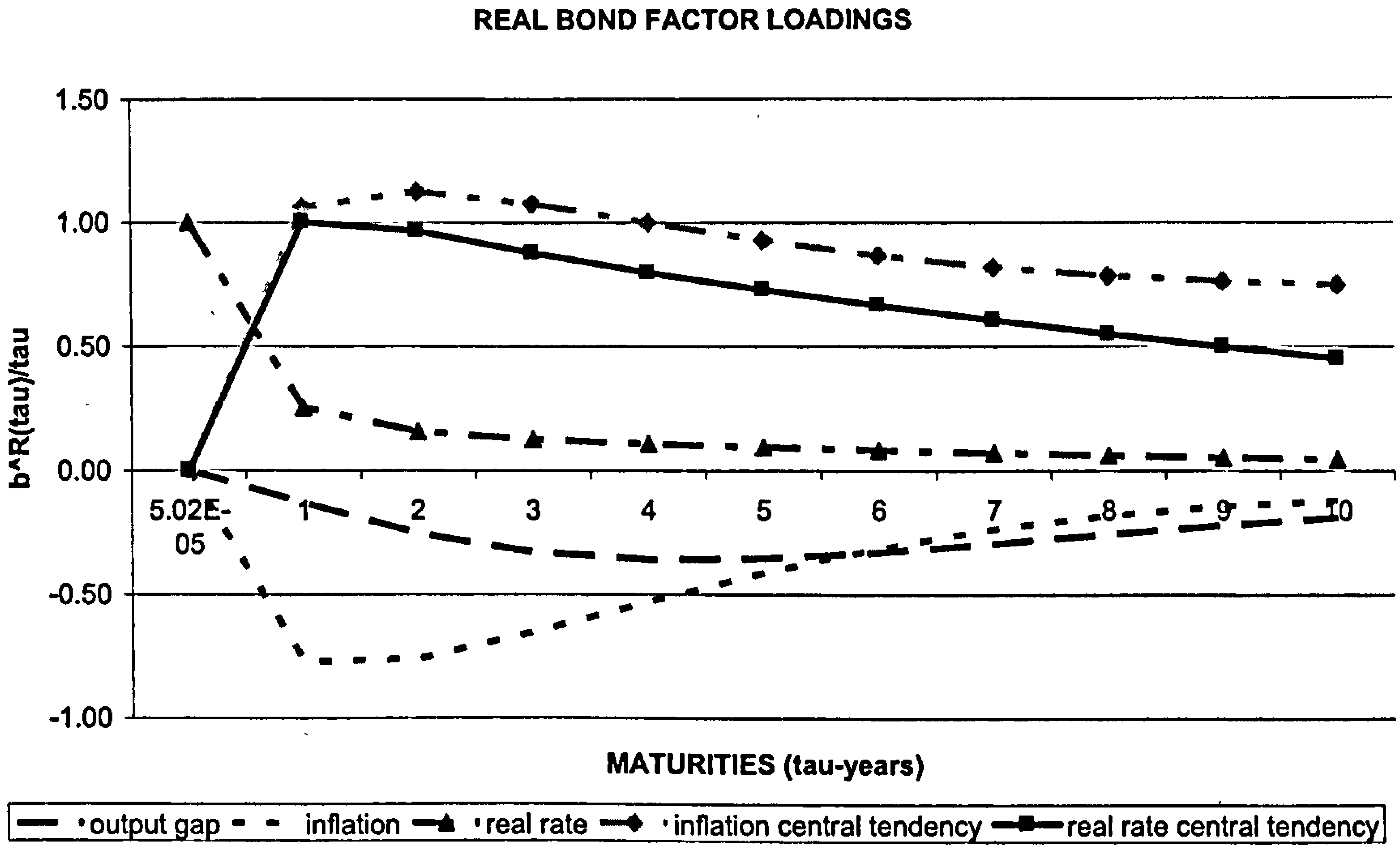
This figure shows the expected excess returns of the nominal zero-coupon bonds with maturities of 1, 3, 5, 7 and 10 years over the sample period 1964:Q1 to 1998:Q4

Figure 3.5: Maximal Sharpe Ratio



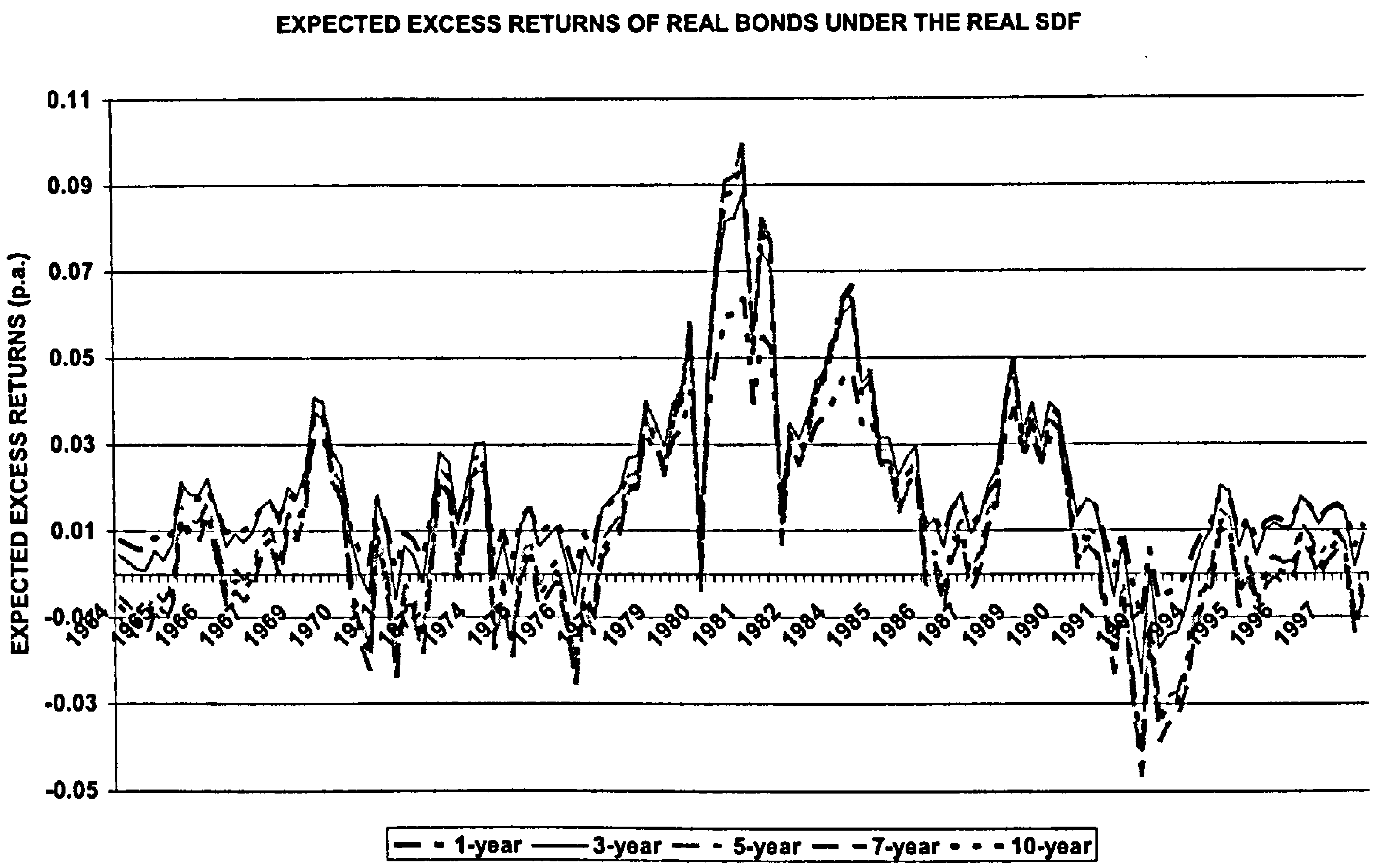
This figure plots the Maximal Sharpe ratio, given by the norm $\sqrt{\xi^T \xi}$, over the sample period 1964:Q1 to 1998:Q4.

Figure 3.6: Factor Loadings of real zero-coupon bonds



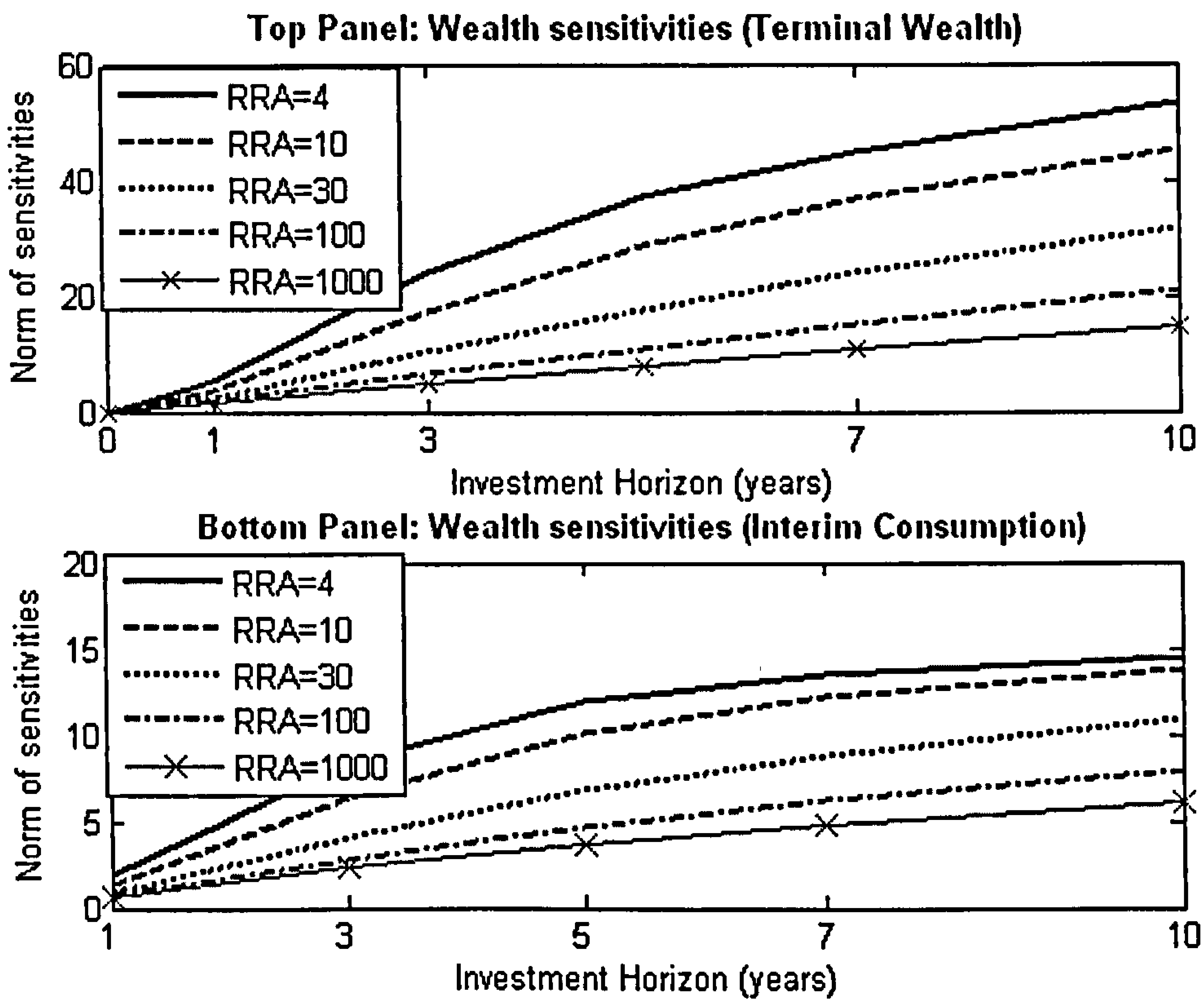
This figure plots the loadings of the output gap y (dashed line), inflation π (dotted line), real rate ρ (dot-dashed line), inflation central tendency π^* (double dot-dashed line) and real rate central tendency ρ^* (solid line) on the real zero-coupon bonds, adjusted for the corresponding maturities, i.e. $b^R(\tau)/\tau$.

Figure 3.7: Expected excess returns of real zero-coupon bonds



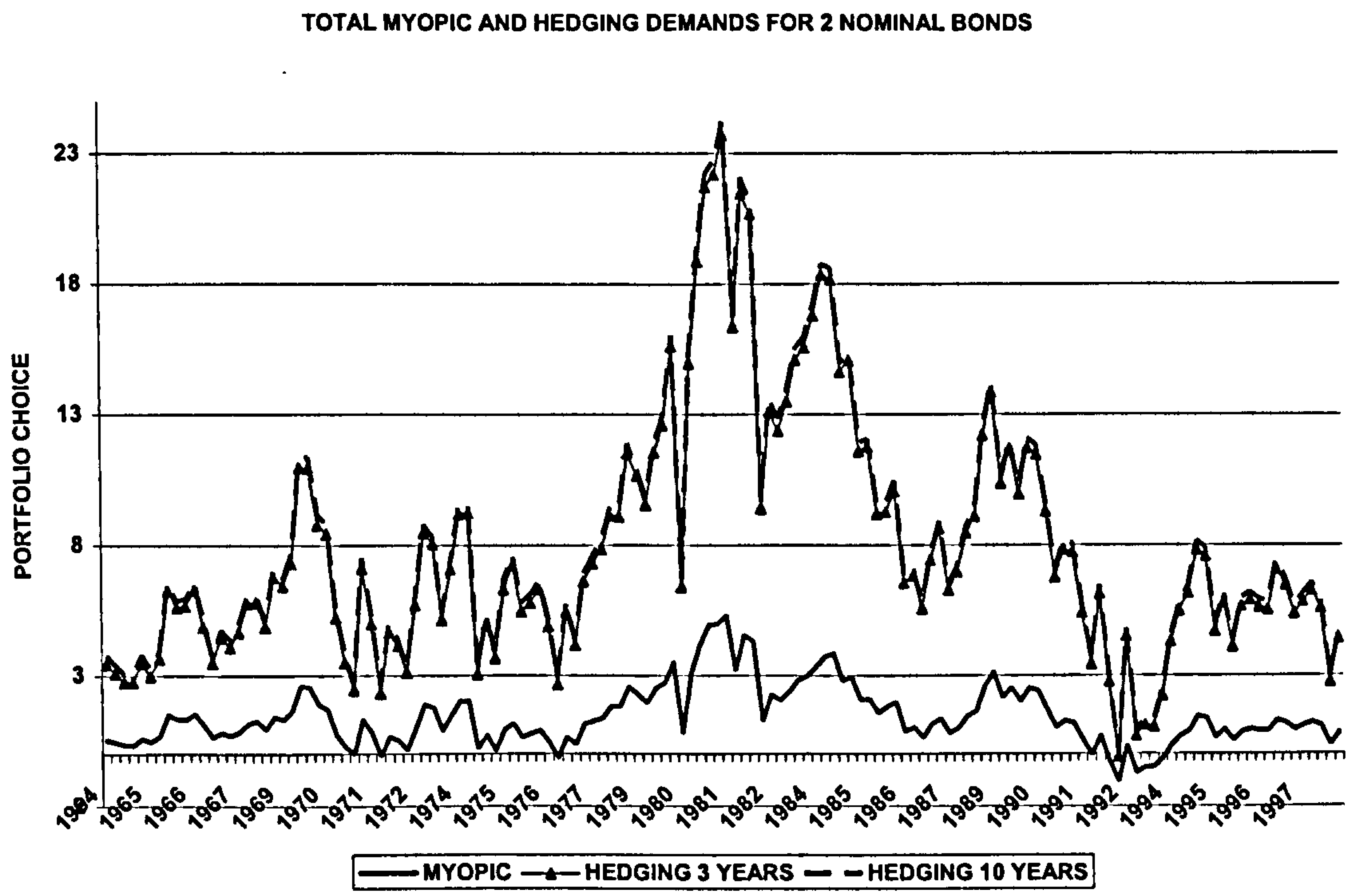
This figure shows the expected excess returns of the real zero-coupon bonds under the real SDF with maturities of 1, 3, 5, 7 and 10 years, over the sample period 1964:Q1 to 1998:Q4

Figure 3.8: Wealth sensitivities with respect to the risk factors



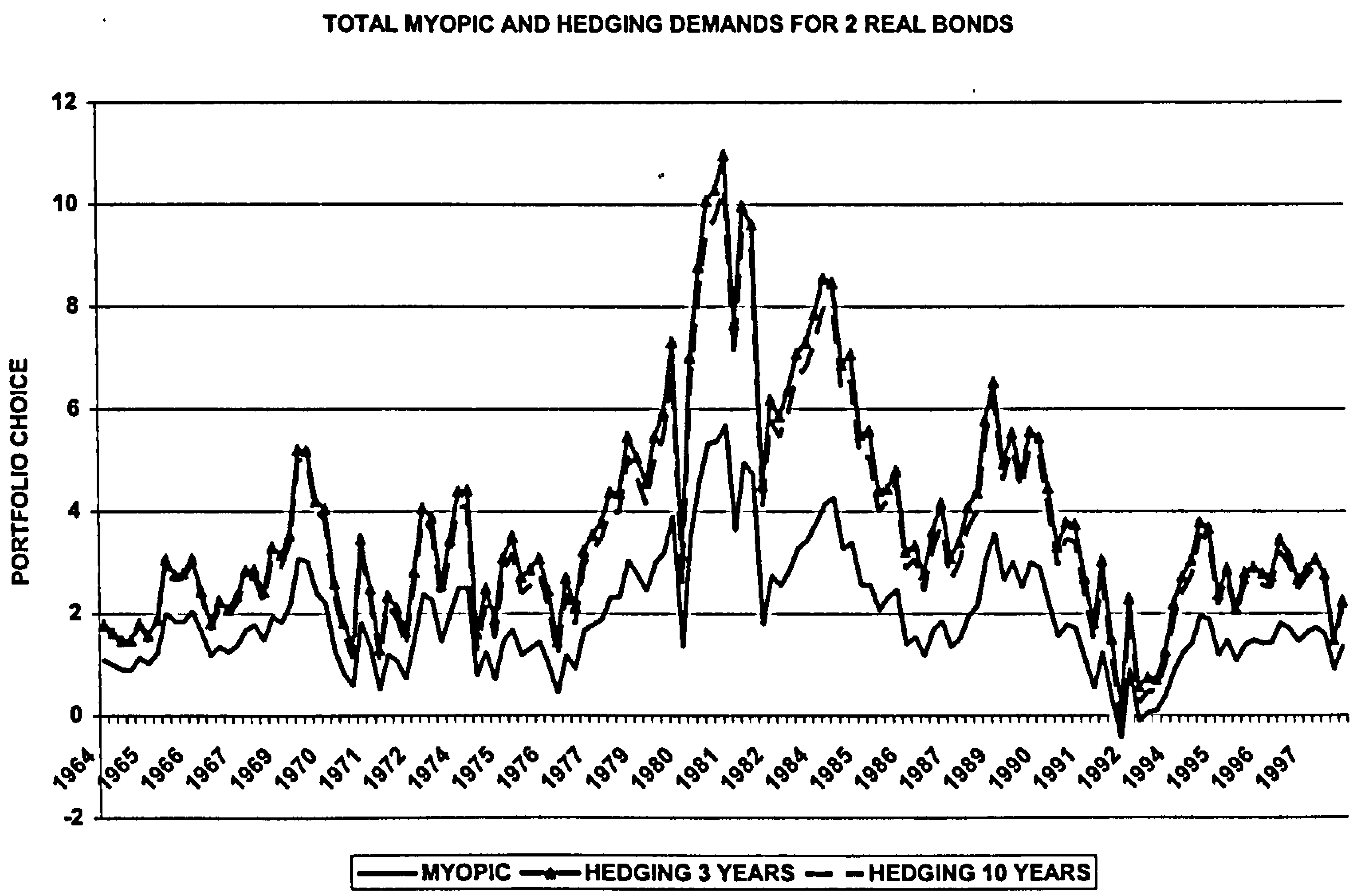
This figure shows the norm of the wealth sensitivities to the underlying risk factors for various degrees of Relative Risk Aversion (RRA), as the investment horizon increases. The top panel shows these norms for the case of an investor with utility over real terminal wealth, given by $\sqrt{(\frac{\partial G}{\partial X}\frac{1}{G})^T(\frac{\partial G}{\partial X}\frac{1}{G})}$. The bottom panel shows the corresponding norms for the case of an investor with utility over interim real consumption with $\delta = 0.06$, given by $\sqrt{(\frac{\partial \hat{G}}{\partial X}\frac{1}{\hat{G}})^T(\frac{\partial \hat{G}}{\partial X}\frac{1}{\hat{G}})}$.

Figure 3.9: Total myopic and hedging portfolio choices for 2 nominal bonds



This figure shows the total myopic bond portfolio choice (solid line) of a power utility investor with coefficient of relative risk aversion $\gamma = 10$, who has access to a 3-year and a 10-year nominal zero-coupon bonds for the period 1964:Q1 to 1998:Q4. This sum is given by $i^T\phi^{myopic}$ where ϕ^{myopic} consists of the first two terms in equation (3.84). The triangle-marked line shows the total hedging bond demand for the power utility investor with $\gamma = 10$, who has a horizon of $T = 3$ years and access to the same nominal bonds. The dashed line shows the corresponding total hedging bond demand for the investor with a horizon of $T = 10$ years. For each case, this sum is given by $i^T\phi^{hedging}$, where $\phi^{hedging}$ is given by the third term in equation (3.84).

Figure 3.10: Total myopic and hedging portfolio choices for 2 real bonds



This figure shows the total myopic bond portfolio choice (solid line) of a power utility investor with coefficient of relative risk aversion $\gamma = 10$, who has access to a 3-year and a 10-year real zero-coupon bonds for the period 1964:Q1 to 1998:Q4. This sum is given by $i^T \phi^{myopic}$ where ϕ^{myopic} consists of the first term in equation (3.96). The triangle-marked line shows the total hedging bond demand for the power utility investor with $\gamma = 10$, who has a horizon of $T = 3$ years and access to the same real zero-coupon bonds. The dashed line shows the corresponding total hedging bond demand for the investor with a horizon of $T = 10$ years. For each case, this sum is given by $i^T \phi^{hedging}$, where $\phi^{hedging}$ is given by the second term in equation (3.96).

Chapter 4

Performance Measures and Incentives: Loading negative coskewness to outperform the CAPM

4.1 Introduction

The most important development in financial markets during the last decades is their domination by institutional investors. The Office of National Statistics (ONS) reports that the individual ownership of the companies listed in the London Stock Exchange (LSE) has decreased from 54% in 1963 to just 12.8% in 2006. One of the most successful investment vehicles is the unit trust.[1] According to the Investment Company Institute (ICI), the 1,903 unit trusts with a British domicile were managing $786 bil. in December 2006.[2]

[1] The term "unit trust" corresponds to the most commonly used term "open-end mutual fund". Henceforth, the two terms will be interchangeably used.

[2] See the 2007 ICI Fact Book for further details.

The outstanding success of mutual funds is due to the fact that they provide access to professional management and a highly diversified portfolio even for investors with low initial capital. In a world of perfect competition and symmetric information, investing in actively managed mutual funds would be an optimal solution. However, the delegated nature of fund management creates a series of problems related to asymmetric information.

The informational asymmetries between the mutual fund shareholder (principal) and the manager (agent) may cause problems in three levels (see Spencer, 2000, for an analytical discussion). Firstly, there is the case of *adverse selection.* The most capable managers are more likely to get employed in different types of investment vehicles, where the compensation structure is directly linked to performance. Closed-end funds, hedge funds and private banking services could attract the best managers. Secondly, there is the *moral hazard* problem. Once the investment has taken place, the manager could attempt to expropriate wealth from the investors, most commonly by charging a high expense ratio. The third issue refers to the *ex post state verification,* i.e. the problem of fairly evaluating the investment outcome. To resolve the last problem, it is necessary to have an objective way of measuring managerial ability.

Measuring investment performance has attracted a lot of attention in the literature and various measures have been suggested, according to which managers should be classified and rewarded. The most commonly used measures are the Sharpe ratio, the Treynor ratio and Jensen's alpha, the intercept of the CAPM. The literature on US mutual funds using these measures is voluminous, dating back to Treynor (1965) and Sharpe (1966). For UK funds, these measures have been employed for performance evaluation purposes inter alia by Leger (1997), Blake and Timmermann (1998), Quigley and Sinquefield (2000) as well as Thomas and Tonks (2001).

All these measures have their roots in the static mean-variance world of Markowitz

(1952). Consequently, there are a series of inherent problems in these measures, since they neglect the intertemporal risk premia priced in capital markets (see the seminal work of Merton, 1973, and the recent study of Campbell and Vuolteenaho, 2004) as well as the risk premia arising due to the deviation of asset returns from normality (see Kraus and Litzenberger, 1976 and Harvey and Siddique, 2000).

Severe criticism to the CAPM assumptions comes from utility theory too. The assumption of quadratic preferences is clearly rejected since it implies increasing absolute risk aversion. A desirable property of a utility function is that agents are averse to negative skewness and have a preference for payoffs exhibiting positive skewness. This behaviour is termed prudence (see Kimball, 1990). Furthermore, experimental evidence indicates that there is an asymmetrically higher impact on utility by losses in comparison to gains, leading to utility frameworks such as Prospect Theory (Kahneman and Tversky, 1979) or Disappointment Aversion (Gul, 1991). These value functions imply that agents are even more averse to negative skewness.

Nevertheless, aversion to negative skewness is a crucial feature that has been relatively neglected in asset pricing. Notable exceptions are the studies of Harvey and Siddique (2000), Dittmar (2002) and Smith (2007) for the US market as well as Hung et al. (2004) for the UK market. The latter study exploits the Huang-Litzenberger Stochastic Discount Factor (SDF) approach to examine whether a coskewness or a cokurtosis risk factor is priced in the London Stock Exchange. In particular, the squared and the cubic excess market returns have been added as additional regressors to a Fama-French asset pricing model. The results of this study do not lead to unambiguous results with respect to whether higher co-moments risk premia are priced.

The recent performance evaluation literature uses Carhart's alpha, the intercept of the Carhart (1997) regression that extends the CAPM by adding the size, value and momentum strategies as risk factors. The literature on US mutual funds is vast. The studies of Bollen and Busse (2004), Chen et al. (2004) and Kosowski et al. (2006) provide recent examples. Cuthbertson et al. (2007) provide an excellent survey of the literature. With respect to the UK, multi-factor models have been used by Tonks (2005) for the performance evaluation of pension funds as well as by Otten and Bams (2002), Fletcher and Forbes (2002) and Cuthbertson et al. (2006) for the evaluation of equity trusts. These studies provide overwhelming evidence that the majority of the funds exhibit negative managerial ability.

Nevertheless, there are two major problems with this measure: Firstly, there is no robust, universally accepted theoretical reason why these strategies should be considered as risk factors. As a result, we run into the problem of characterizing randomly fluctuating returns as risk factors. Secondly, this measure attributes returns to specific strategies, not their fundamental source of risk. Consequently, there will always be the incentive for fund managers to construct alternative strategies that outperform this measure too. For instance, there are a series of studies suggesting that portfolios of stocks sorted on the basis of their liquidity may yield positive Carhart alphas (see Acharya and Pedersen, 2005), giving rise to an illiquidity risk factor (see Pastor and Stambaugh, 2003).

This chapter has two main aims: The first is to review the assumptions on which the most commonly used performance measures are based and to examine the incentives that these measures generate. Fund managers try to distinguish themselves from their peers on the basis of these measures and they respond to this incentive by adopting investment strategies that generate excess returns and help them outperform. The second aim is to propose an appropriate performance measure for risk-averse and prudent investors that is based on sound economic theory and to evaluate the performance of UK equity unit trusts according to this measure.

In particular, the Harvey and Siddique (2000) asset pricing model, which adds a zero-cost negative coskewness strategy as an extra factor to the CAPM, is proposed to be the most appropriate one for a prudent investor. This model takes into account the risk premia formed in capital markets due to the participants' aversion to negative skewness.[3] The intercept of this model, which we term as the *Harvey-Siddique alpha*, is employed as a measure to evaluate the performance of UK equity unit trusts during the period January 1991- December 2005. To the best of our knowledge, the only study that has explicitly used the Harvey-Siddique model for performance evaluation purposes is Moreno and Rodriguez (2005) for the case of Spanish funds. A series of other performance evaluation studies have incorporated the concept of skewness into their analysis. For instance, Fletcher and Forbes (2004) use the Huang-Litzenberger SDF approach for the evaluation of UK equity unit trusts, while Ding and Shawky (2007) employ this framework for the evaluation of hedge funds. Skewness has been also taken into account by studies utilizing the Data Envelopment Analysis framework (see Joro and Na, 2006).

In order to perform our analysis, the returns of a zero-cost coskewness spread portfolio have been calculated for the UK, showing that the average monthly return of this strategy was 2.09% p.a. over the period January 1991-December 2005. Previewing our results, the median unit trust investing in the FTSE All Share universe had a Harvey-Siddique alpha of −2.36% p.a., while the corresponding median Jensen alpha was −1.77% p.a. and the median Carhart alpha was −2.32% p.a. With respect to the managers' incentives, it is shown that almost all of the examined trusts had a positive loading on the negative coskewness spread strategy. Most interestingly, the trusts with the highest Jensen alphas were those with the highest loadings on the negative coskewness factor. We also provide evidence that the nonnormality of the trusts' performance distribution can be partly attributed to heterogeneous risk-taking, with the coskewness strategy being a main source of this heterogeneity.

[3] Adcock (2007) stresses the importance of taking skewness into account when measuring portfolio performance.

The main message is that prudent investors should employ the *Harvey-Siddique alpha* in order to neutralize the incentive of trust managers to load coskewness risk. For these investors, most of the UK unit trusts exhibited a significantly negative managerial ability. However, managers were very successful in reaping the negative coskewness premium priced in the market, boosting their returns and correctly responding to their incentives, since they have been evaluated according to mean-variance measures that regard this premium to be a "free lunch".

The rest of this chapter is organized as follows: Section 4.2 discusses the relationship between skewness, preferences and asset pricing. Section 4.3 reviews the most commonly used performance measures, discussing the incentives they generate and Section 4.4 provides the details of the data and the related methodological issues. Section 4.5 discusses the results of the unit trusts' performance evaluation and Section 4.6 presents some further results, while Section 4.7 concludes.

4.2 Why is negative coskewness risk priced in the markets?

4.2.1 Skewness and preferences

Given the discussion in Chapter 1, it is expected that a risk-averse and prudent investor has a preference over a positively skewed payoff distribution and an aversion towards a negatively skewed one. There is significant evidence in the markets supporting this argument. The popular portfolio insurance products are protecting investors against downside risk. Moreover, modern risk management mainly deals with the avoidance of extreme negative returns. The most characteristic example is the measurement of Value-at-Risk (VaR).

Furthermore, option-implied distributions, especially after the October 1987 crash, are typically negatively skewed. In particular, deep out-of-the-money puts, which are popular instruments for portfolio insurance, have quite high prices relative to the ones implied by the Black and Scholes (1973) model. As a result, the implied volatility-strike price graph exhibits a "smirk", in contrast to the constant volatility assumption of Black and Scholes. This feature of option prices has been termed *crashophobia* (see Jackwerth, 2004). On the other hand, the preference for positive skewness is evident in lotteries. Agents are willing to participate in lotteries with positively skewed payoffs (see for example, Golec and Tamarkin, 1998), even though these have negative expected values, i.e. they are unfair games. It is worth mentioning that the participation in such unfair games increases as positive skewness increases (e.g. jackpots in lotteries). A similar explanation has been offered by Tufano (2008) for the success of UK premium bonds.

The impact of skewness is often examined using the power utility function, that treats symmetrically utility gains and losses caused by a wealth change of the same magnitude. Actually, this is also a property of the mean-variance analysis. Despite this assumption, there is significant experimental evidence that agents are mainly averse to losses, not just to volatility. The Prospect Theory of Kahnemann and Tversky (1979) as well as the Disappointment Aversion framework of Gul (1991) imply that investors maintain an asymmetric attitude towards losses as compared to gains. In general, this class of value functions captures the feature of first-order risk aversion (see Segal and Spivak, 1990) and implies that investors are even more averse to negative skewness in comparison to power utility agents.

There are a number of regulatory and psychological issues related to loss aversion. Moreover, pension funds and insurance companies usually face legal obligations to pay out fixed or quasi-fixed amounts. The same holds true for households with liabilities over mortgages, loan installments or fees. Habit formation is another example of anchoring one's preferences around a reference point and being reluctant to accept any wealth level below that point. This mixture of obligations and preferences make pension funds, insurance companies and individuals extremely averse to negative movements in asset prices.

4.2.2 Coskewness in asset pricing

The central problem in Asset Pricing is to find a valid SDF, M, for future payoffs. Formally, the SDF is assumed to be positive (see Harrison and Kreps, 1979), it is unique under complete markets and satisfies the following relationship:

$$P_t = E_t[M_{t+s}X_{t+s}] \tag{4.1}$$

where P_t is the price of an asset at time t and X_{t+s} denotes the asset's payoff at time $t+s$.

In a one-period ahead framework, $R^G_{t+1} = \frac{X_{t+1}}{P_t} \equiv 1 + R_{t+1}$ is employed to re-write equation (4.1) as:

$$1 = E_t[M_{t+1}R^G_{t+1}] \tag{4.2}$$

It is straightforward to derive the following relationships (see Smith and Wickens, 2002), which relate the SDF, the gross return of a risky asset, R_{t+1}, as well as that of a risk-free asset, r^f_t :

$$1 = E_t(M_{t+1})(1 + r^f_t) \Rightarrow E_t(M_{t+1}) = \frac{1}{1 + r^f_t} \tag{4.3}$$

and

$$E_t(R^G_{t+1}) = \frac{1 - Cov(M_{t+1}, R^G_{t+1})}{E_t(M_{t+1})} \tag{4.4}$$

Combining these two equations we get a central result in asset pricing theory:

$$E_t(R_{t+1}) - r_t^f = -(1 + r_t^f)Cov(M_{t+1}, R_{t+1}) \tag{4.5}$$

This equation implies that the expected excess return of a risky asset depends on the covariance of the SDF with this risky return.

Harvey and Siddique (2000) use the marginal rate of substitution $\frac{U'(W_{t+1})}{U'(W_t)}$ as a SDF to show the implications of this specification. Taking a first-order Taylor series expansion of $U'(W_{t+1})$ around W_t, we get the standard CAPM analysis. However, there is no particular reason why the truncation of the Taylor series expansion should occur at the first order. If the truncation takes place at the second order, then:

$$\begin{aligned}
U'(W_{t+1}) &\simeq U'(W_t) + U''(W_t)(W_{t+1} - W_t) + \frac{U'''(W_t)(W_{t+1} - W_t)^2}{2!} \Rightarrow \\
\frac{U'(W_{t+1})}{U'(W_t)} &\simeq 1 + \frac{U''(W_t)W_t}{U'(W_t)} R_{m,t+1} + \frac{U'''(W_t)W_t^2}{2U'(W_t)} R_{m,t+1}^2 = \\
&= 1 - \gamma R_{m,t+1} + \frac{U'''(W_t)W_t}{2U''(W_t)} \frac{U''(W_t)W_t}{U'(W_t)} R_{m,t+1}^2 = \\
&= 1 - \gamma R_{m,t+1} + \frac{1}{2}\gamma\eta R_{m,t+1}^2
\end{aligned} \tag{4.6}$$

where we have used the simple budget constraint $W_{t+1} = W_t(1 + R_{m,t+1})$ with R_m being the market return, the definition of the Arrow-Pratt measure of Relative Risk Aversion, $\gamma \equiv -\frac{U''(W_t)W_t}{U'(W_t)}$ and the coefficient of Relative Prudence $\eta \equiv -\frac{U'''(W_t)W_t}{U''(W_t)}$, as defined by Kimball (1990). Furthermore, defining $\tilde{b} \equiv \frac{U''(W_t)W_t}{U'(W_t)} = -\gamma$ and $\tilde{c} \equiv \frac{U'''(W_t)W_t^2}{2U'(W_t)} = \frac{1}{2}\gamma\eta$, the SDF implied by (4.6) can now be written as:

$$M_{t+1} = 1 + \tilde{b}R_{m,t+1} + \tilde{c}R_{m,t+1}^2 \tag{4.7}$$

This is not a linear SDF, since the squared market returns are involved. Recalling the fundamental asset pricing equation (4.5) of the SDF approach, the expected excess return of an asset now depends on the covariance of this asset's returns not only with the market return but also with the squared market return. This is exactly what coskewness measures. In the case of a prudent and risk-averse investor, we have that $\eta > 0 \Rightarrow \tilde{c} > 0$, so the fundamental equation of asset pricing (4.5) can be written as:

$$E_t(R_{t+1}) - r_t^f = -(1 + r_t^f)\tilde{b}Cov(R_{m,t+1,}R_{t+1}) - (1 + r_t^f)\tilde{c}Cov(R_{m,t+1,}^2R_{t+1}) \quad (4.8)$$

Therefore, for a given level of $Cov(R_{m,t+1,}R_{t+1})$, we have two cases with respect to $Cov(R_{m,t+1,}^2R_{t+1})$. On the one hand, if $Cov(R_{m,t+1,}^2R_{t+1}) > 0$, then $E_t(R_{t+1}) - r_t^f$ is now lower in comparison to the case of $\tilde{c} = 0$. This implies that if the risky asset's returns are positively coskewed with the market returns, then this asset will bear a lower risk premium. On the other hand, if $Cov(R_{m,t+1,}^2R_{t+1}) < 0$, then $E_t(R_{t+1}) - r_t^f$ is now higher. In other words, a prudent investor seeks an extra risk premium in order to hold an asset, the returns of which are characterized by negative coskewness. Therefore, if financial markets are populated by prudent investors, expected returns should be higher for assets exhibiting negative coskewness. This is a key result in our analysis, predicting the existence of a negative coskewness premium.

4.3 Performance measures and incentives in fund management

4.3.1 Raw returns

Since the work of Markowitz (1952), it has been understood that there exists a direct positive relationship between risk and returns. However, managers and funds are still often ranked according to their raw returns. The new breed of funds, appearing as "absolute return" seeking funds, reflects the lack of understanding of the link between risk and return. Using raw returns as a performance measure essentially means that the investor is indifferent to risk, i.e. his utility is not decreasing in volatility/risk, and risk premia are thought to be "free lunches". If a manager is evaluated according to raw returns, he will be incentivised to undertake the highest possible risk.

4.3.2 Sharpe ratio

Most of the performance studies have been evaluating investment strategies according to their risk-adjusted returns. One of the most commonly used measures of risk-adjusted performance is the Sharpe ratio due to Sharpe (1966):

$$SR = \frac{E(R_p) - r^f}{\sigma_p} \tag{4.9}$$

where $E(R_p)$ is the average fund's return over a specific period, r^f is the risk-free rate and σ_p is the standard deviation of the fund's returns in the same period. Using this measure, fund managers do not have the incentive to invest in more volatile assets, since higher volatility essentially penalizes their excess returns.

The Sharpe ratio is a purely mean-variance measure, neglecting higher moments. Nevertheless, these higher moments bear risk premia in a market with prudent investors, as previously discussed. Consequently, the rational response of the fund manager is to invest in assets that exhibit negative skewness in order to reap the corresponding risk premium and be classified as a Winner.

There are a series of examples documenting the existence of these strategies. Investing in emerging countries' bonds as well as non-investment grade bonds is a straightforward case. These bonds have a higher probability of default in comparison to investment grade bonds. As a result, their returns are more negatively skewed and they provide higher yields. If a manager matches bonds with the same volatility but with different degrees of skewness, he will achieve a higher Sharpe ratio- until the default occurs.

Goetzmann et al. (2007) analyze methods of maximizing a portfolio's Sharpe ratio using derivatives. Shorting different fractions of out-of-the-money puts and calls creates a negatively skewed distribution of returns and leads to the maximum Sharpe ratio. Their example also shows that hedge funds and other investment vehicles, that use derivative assets, can manipulate their Sharpe ratio. Leland (1999) provides an example of a dynamic strategy of cash and stocks as well as static strategies using options that generate negative skewness and outperform in terms of the Sharpe ratio. Finally, Adcock (2005) shows how to exploit skewness in order to construct a hedge fund that has superior mean-variance performance. Given this evidence, such funds should not be evaluated by mean-variance measures.

The inappropriateness of the Sharpe ratio for skewed returns is also mentioned in Ziemba (2005), who suggests a modification of the Sharpe ratio to emphasize the importance of the downside risk. The Symmetric Downside-Risk Sharpe Ratio (DSR) is given by:

$$DSR = \frac{E(R_p) - r^f}{\sqrt{2(\sigma_-^2)}} \tag{4.10}$$

with

$$\sigma_-^2 = [1/(T-1)] \sum_{t=1}^{T} (R_{p,t} - E(R_p))_-^2 \tag{4.11}$$

and the returns used are only those below the average $E(R_p)$. Essentially, this measure adjusts the excess returns by using the semi-variance instead of variance.

4.3.3 Jensen Alpha and Treynor Ratio

Within the CAPM framework, Jensen (1968) introduced the intercept of the following regression as a measure for the fund manager's ability:

$$r_{p,t} = \alpha_{Jensen} + \beta_p r_{m,t} + \epsilon_t \tag{4.12}$$

where $r_{p,t}$ stands for the excess returns of the trust, $r_{m,t}$ the excess return of a suitable market index and β_p the fund's CAPM beta. The intercept (α_{Jensen}) shows whether the manager has added any value over and above the return justified by the risk he had undertaken. The concept of risk here is summarized in the CAPM beta. Closely related is the measure proposed by Treynor (1965):

$$TR = \frac{E(R_p) - r^f}{\beta_p} \tag{4.13}$$

Following the spirit of the Sharpe ratio, the Treynor ratio adjusts excess returns for the corresponding CAPM beta risk (β_p).

As it has been discussed, the CAPM is a static mean-variance measure, neglecting all other sources of risk, in particular those arising due to the higher moments and the stochastic evolution of the underlying risk factors affecting the asset returns. Consequently, if evaluated according to the CAPM, managers are incentivised to employ strategies which load intertemporal and higher co-moments risks in their portfolios. It is known that fund managers construct strategies that exploit patterns such as the size, value and momentum "anomalies" in order to add value to their portfolios. Temporary success of these strategies generates a positive Jensen alpha classifying the manager as a Winner. These strategies are supposed to have zero CAPM beta risk but they are not necessarily riskless.

4.3.4 Carhart Alpha

The basic doctrine of financial theory is that "free lunches", in the spirit of Harrison and Kreps (1979), should be ruled out.[4] Furthermore, since the Fama-French (1993) and momentum (see Jegadeesh and Titman, 1993) strategies are very simple to construct and implement, their returns cannot be regarded as genuinely added value. Reflecting these arguments, Carhart (1997) suggested the intercept of the four-factor model:

$$r_{p,t} = \alpha_{Carhart} + \beta_p r_{m,t} + \beta_1 SMB_t + \beta_2 HML_t + \beta_3 MOM_t + \epsilon_t \tag{4.14}$$

i.e. the *Carhart alpha,* as a performance measure.

The Carhart regression (4.14) essentially attributes the fund's returns generated by the size (SMB), value (HML) and momentum (MOM) strategies to the corresponding risk factors. The intercept of this regression ($\alpha_{Carhart}$) reveals the value the manager has added to his portfolio above what the beta risk could justify and these known strategies could generate. The most important feature of this measure is that it neutralizes the incentive to adopt these strategies, since they are recognized as risk factors.

Despite the significance of this contribution, the Carhart measure has two main disadvantages. Firstly, managers would still try to find patterns in stock returns in order to outperform on the basis of this measure too. Even though the Carhart model reduces the possible opportunities, there exist other such strategies that generate abnormal returns within this model (see e.g. the illiquidity risk factor in Pastor and Stambaugh, 2003). The need for a more general measure capturing all these

[4] Accepting "free lunches" would be equivalent to discarding asset allocation. If there exist strategies that add value to portfolios without undertaking any further risk, then the optimal portfolio choice collapses to an infinite demand schedule for these strategies.

types of risks is obvious. Secondly, there is no robust theory explaining why the size, value and momentum strategies are characterized as risk factors. Hence, this model may misinterpret randomly fluctuating returns for risk factors, penalizing genuinely added value. Consequently, a measure based on sound economic theory is needed.

4.3.5 Harvey-Siddique alpha

Given the theoretical motivation of Section 4.2, it is argued that a prudent investor should evaluate his investments according to the following model:

$$r_{p,t} = \alpha_{HS} + \beta_p r_{m,t} + c(S^- - S^+)_t + \varepsilon_t \tag{4.15}$$

where $(S^- - S^+)$ stands for the returns of the negative coskewness spread strategy, defined in the next section. The intercept of this model, (a_{HS}), termed as the *Harvey-Siddique alpha*, will give us the value added by the manager over and above the covariance and negative coskewness risks.

As Harvey and Siddique (2000, p. 1276) point out, this asset pricing model has two main advantages over a model that includes the squared market returns as a factor (e.g. as in Kraus and Litzenberger, 1976): Firstly, the measure of standardized coskewness is constructed by residuals, so it is by construction independent of the market return and, secondly, β_p in (4.15) is similar to the standard CAPM beta. Moreover, standardized coskewness is unit free and analogous to a factor loading. Apart from the parsimony in comparison to the Carhart (1997) measure, the Harvey-Siddique alpha is also more general since it captures the excess returns from every possible strategy that loads negative skewness to the portfolio and it is based on an asset pricing model built within a rigorous utility theory framework.

4.4 Data and Methodology

We follow the methodology of Harvey and Siddique (2000) to construct the zero-cost negative coskewness portfolio ($S^- - S^+$). Using 60 monthly excess returns $r_{i,t}$, we regress the market model for each individual stock i

$$r_{i,t} = \alpha_i + \beta_i r_{m,t} + \varepsilon_{i,t} \tag{4.16}$$

extracting the residuals $\varepsilon_{i,t}$, which are by definition orthogonal to the excess market returns $r_{m,t}$. Therefore, these residuals are net of the systematic risk as this is measured by the covariance of the stock returns with the market returns. However, they still incorporate the coskewness risk. Therefore, we can get a measure of the standardized coskewness of each stock's returns with the market returns. This is given by:

$$\beta_{i,SKD} = \frac{E[\varepsilon_{i,t}\varepsilon_{m,t}^2]}{\sqrt{E[\varepsilon_{i,t}^2]E[\varepsilon_{m,t}^2]}} \tag{4.17}$$

where $\varepsilon_{i,t}$ is the residual previously extracted from the market model for stock i at time t and $\varepsilon_{m,t}$ is the deviation of the excess market return at time t from the average excess market return in the examined period.

Ranking the stocks according to this coskewness measure, we form a value-weighted portfolio of the 30% stocks with the most positively coskewed returns (S^+), while the 30% stocks with the most negatively coskewed returns form another portfolio (S^-). The next step is to find the returns of these portfolios on the 61st month. The spread of these two portfolios' returns ($S^- - S^+$) will yield the return generated by the self-financing strategy of buying the stocks with the most negatively coskewed returns and selling short the stocks with the most positively coskewed returns.

To construct the coskewness measure, β_{SKD}, we employ data for monthly returns and market values of all stocks being listed in the FTSE All Share Index during the period 1986-2005 with at least 61 observations. The number of stocks utilized to create the coskewness portfolios varied from 413 (with market value of £339,404 million) in December 1991 to 581 in January 2004 (with market value of £1,045,331 million). The risk-free rate is given by the interbank monthly rate and the market returns are the returns of the FTSE All Share Index. The source for the data and the FTSE All Share Index listings is Thomson Datastream and Worldscope respectively.

The coskewness portfolios' returns are constructed for the period January 1991-December 2005. Table 4.1 presents the average returns of the zero-cost coskewness spread portfolio ($S^{-} - S^{+}$) and the market excess returns for various periods. A striking feature of the zero-cost portfolio is that it yielded, on average, a return of 2.09% p.a., over the period 1991-2005, having a very low standard deviation. Figure 4.1 shows these returns along with the excess market returns. The subperiod analysis showed that the negative coskewness risk was more highly priced in the last subperiod, i.e. January 2001- December 2005.

Following Cuthbertson et al. (2006), we proxy the size strategy returns (SMB) as the difference between the monthly returns of the Hoare Govett Small Cap and the FTSE 100 indices and the value strategy returns (HML) as the spread between the monthly returns of the MSCI UK Growth and the MSCI UK Value indices. The returns of the momentum strategy (MOM) were calculated by ranking all available stocks at time t according to their returns between the months $t-12$ and $t-1$. The top 30% (value-weighted) of these stocks were classified as Winners and the bottom 30% as Losers. The spread of their monthly returns at $t+1$ is taken as the momentum strategy return. The portfolios were rebalanced on a monthly basis.

Table 4.1 indicates that the Size and Value strategies yielded positive returns mainly during the subperiod January 2001- December 2005. Estimating the measures of standardized coskewness of these strategies for the period January 2001- December

2005, these were $\beta_{SKD}^{SMB} = -0.257$ for the Size (SMB) strategy and $\beta_{SKD}^{HML} = -0.107$ for the Value (HML) strategy. The Momentum (MOM) strategy yielded a positive average return throughout the examined period, but this was even higher during the last subperiod. The measure of standardized coskewness had a value of $\beta_{SKD}^{MOM} = -0.08$ for the whole period and $\beta_{SKD}^{MOM} = -0.43$ for the last subperiod. Hence, all these strategies had negatively coskewed returns, especially when they yielded high average returns. As a result, a trust manager who followed these strategies was loading negative skewness to his portfolio, extracting the corresponding premium.

For the UK unit trusts, the Lipper Fund Database is used to acquire Net Asset Values (NAV) on a monthly basis.[5] We select the unit trusts which are marked for sale in the UK and they have domicile either in the UK or overseas. The performance evaluation study refers to unit trusts which have the FTSE All Share Index as a fund manager benchmark in the Lipper Database, justifying the use of the returns of this index as a proxy for market returns. To alleviate the problem of survivorship bias, the database we utilize also includes unit trusts which have ceased operations before 2005. To have a meaningful performance study, only trusts with more than 61 observations of NAVs were employed. This selection leaves us with 273 unit trusts having more than 60 monthly returns for the period January 1991- December 2005. The minimum number of trusts in our final dataset is 150 in 1991 and the maximum number is 273 in 2004. Table 4.2 provides the number of trusts for each subperiod as well as their average excess returns.

[5]It should be noted that the NAV is calculated after the deduction of management fees and other expenses.

4.5 Unit trusts' performance

4.5.1 Average returns and Sharpe ratios

The average excess returns earned by the equity trusts are given in Panel A of Table 4.2. An interesting observation is that the trusts exhibited consistently lower returns in comparison to the excess returns of the FTSE All Share Index in the whole sample period as well as in the three examined subperiods. The high fees they charged is regarded to be the main reason why this underperformance is observed.[6] Non-stock holdings could provide another explanation, but the lack of holdings data does not allow us to test it formally. However, the results of the CAPM analysis showed that the average beta of the trusts was 0.93, providing some evidence for this argument.

Panel A of Table 4.2 also provides the average Sharpe ratios of the trusts for the period January 1991- December 2005. For the calculation of this average, the individual Sharpe ratio of each trust with more than 60 monthly returns was firstly calculated. It should be reminded that during bear market phases, Sharpe ratios can take negative values. The purpose of calculating this ratio is to compare it with the Downside-Risk Sharpe Ratio (DSR) that replaces the variance of the monthly returns with their semi-variance. The average DSRs are also reported in Panel A of Table 4.2. The most important finding is that the average DSR is much lower (in absolute value) in comparison to the average Sharpe ratio regardless of the examined period. The explanation for this finding is that the returns' semi-variance was much higher than their variance; consequently, the excess returns were much more severely penalized under the DSR measure.

The previous result is consistent for all of the individual trusts and shows that the trusts' returns were indeed negatively skewed. This is an interesting finding

[6] Khorana et al. (2008) claim that the equity trusts marked for sale in the UK had an average Total Expense Ratio of 1.42% p.a. in 2002. The corresponding Total Shareholer Cost was 2.48%.

for prudent investors who are averse to negative skewness as well as for loss averse investors who are even more averse to this feature. It also shows that if the DSR replaces the Sharpe ratio as a performance measure, the managers will be incentivized to avoid large negative returns, meeting the preferences of their clients.

4.5.2 Jensen alpha

Focusing now on measures based on asset pricing models, the first measure to employ is Jensen's alpha, given by the intercept of regression (4.12). Over the period January 1991- December 2005, the median trust had a Jensen alpha of -1.77% p.a. Figure 4.2 shows the distribution of the estimated Jensen alphas for all the trusts during this period. It is evident that the majority of the trusts have negative alphas, but their distribution is positively skewed. This finding implies that there are a few trusts who have quite high positive alphas. Having ranked the trusts according to their alphas, Table 4.3 shows the corresponding values for various percentiles of the distribution. The upper 25% of the trusts had a positive alpha, though very few of these estimates were statistically significant. On the other hand, the bottom 45% of the trusts exhibited alphas of less than -2% p.a.. These results are in accordance with the previous literature, as this is reviewed in Cuthbertson et al. (2007).

The common practice of ranking trusts according to their alpha point estimates could be misleading, since the standard error of the estimate is not taken into account. It has been suggested by Kosowski et al. (2006) that ranking trusts according to their t-statistics is more appropriate, since this adjusts the point estimate for its in-sample variability (standard error). Table 4.4 presents such a ranking, using the corresponding t-values.[7] Using a 95% confidence interval for the t-statistic, only 5% of the trusts exhibited significantly positive managerial ability. On the other hand,

[7] In this study, reported t-values use Newey-West (1987) heteroskedasticity and autocorrelation consistent standard errors.

more than 30% of the trusts exhibited significantly negative managerial ability.

An immediate conclusion from the shape of the alphas' distribution is that according to this static, mean-variance measure, managerial ability existed, but only for a very small portion of the trust managers. Furthermore, even the quadratic investors who chose the bottom 30% trusts would have been significantly better off if they had invested in a low-cost index fund. Since we deal with net returns, high expenses and management fees could well be a reason for the significant underperformance of many trusts. The nonnormality of the alphas' distribution[8] can be explained in two ways: Either managers exhibit heterogeneous abilities, with a few of them being highly skilful or these managers adopted heterogeneous risk-taking strategies. The next subsections investigate further these hypotheses.

4.5.3 Carhart alpha

This subsection evaluates the trusts by using the intercept of the Carhart asset pricing model in (4.14) as a performance measure. The rankings of the trusts according to their Carhart alphas, along with the corresponding t-statistics and p-values are given in Table 4.3, while Table 4.4 provides the rankings according to the t-values of the Carhart alpha point estimates. In comparison to Jensen's alpha, the main conclusion is that the achieved performance is now much lower for every percentile of the trusts' distribution. Figure 4.3 plots the distribution of the trusts' Carhart alphas.

In particular, the median trust's Carhart alpha was -2.32% p.a. over the period January 1991- December 2005. Therefore, it is argued that attributing the returns of the size, momentum and value strategies to the corresponding risk factors, as the Carhart regression does, significantly diminishes the performance of the trust managers. Interestingly, there were only very few funds that exhibited a positive and

[8]The Kolmogorov-Smirnov test was employed to formally test the hypothesis of normality for the standardized alphas. The hypothesis of normality was rejected at levels even lower than 1%.

statistically significant alpha over this period. This result points to the argument that, if an investor considers the documented "anomalies" as risk factors, he would be much better off by investing in a low-cost index fund rather than the median UK equity unit trust. Nevertheless, the relative ranking of the funds according to their Carhart alphas is similar to their corresponding ranking according to their Jensen alphas. Formally, the Spearman's correlation coefficient of these two rankings is $\rho = 0.916$.[9]

With respect to managerial incentives, Figure 4.4 depicts the loadings of the trusts' returns on the size, value and momentum factors over the whole sample period. It is interesting to observe that the managers mainly followed size strategies. There is no evidence for a unanimous adoption of value and momentum strategies, since the estimates of the corresponding loadings are evenly distributed around zero for the examined trusts. In particular, while 199 of the trusts had a significantly (at the 5% level) positive loading on the size strategy, only 48 of them had a significantly positive loading on the momentum strategy and just 11 of them on the value strategy. These findings are in line with Cuthbertson et al. (2006) and in contrast to the stylized facts of the US literature (see Carhart, 1997), i.e. that the managers consistently follow value and momentum strategies.

The explanation for this finding lies in the fact that these strategies yielded highly volatile returns which were not consistently positive throughout our sample period. Consequently, we would not expect managers to stick to specific strategies, especially when these are not performing well. This is a fundamental issue demonstrating the main disadvantage of the Carhart model, that is to assume that managers follow specific strategies even though these do not persistently yield positive returns and do

[9]The Spearman's rank correlation coefficient is given by $\rho = 1 - \frac{6\sum d_i^2}{n(n^2-1)}$, with $d_i = x_i - y_i$, where x_i and y_i are correspondingly the rankings of fund i according to the measures under examination and n is the number of funds.

not rigorously represent any specific risk factor. This is also an important issue when comparing asset pricing models on the basis of a selection criterion. In particular, we employ the Schwartz Information Criterion (SIC), that captures the trade off between obtaining the better fit model and using the fewest possible regressors.[10] Our regression analysis shows that on the basis of this criterion, for the 38% of the funds the CAPM should be preferred. The reason is that for these funds, the enhancement of the explanatory power in the Carhart model is not sufficient to justify the use of three additional regressors. Therefore, the parsimony of the asset pricing model is a highly desirable property.

4.5.4 Harvey-Siddique alpha

This subsection presents the results of the unit trusts' evaluation using the Harvey-Siddique asset pricing model given in (4.15). Interestingly, the examined trusts had a median Harvey-Siddique alpha of -2.36% p.a. This is much lower than the median Jensen alpha. Figure 4.2 plots the distribution of the Harvey-Siddique alphas along with the distribution of the Jensen alphas. It is evident that the whole distribution is shifted to the left, as we switch from the Jensen alpha to the Harvey-Siddique alpha. The main explanation for this difference is that trust managers followed coskewness strategies, earning positive returns which were regarded as "abnormal returns" according to the CAPM. If a manager had genuinely added value to his portfolio, without adding negative skewness, then there should be no difference between the results derived by these two measures.

To verify this conjecture, it is interesting to note that 263 out of the total 273 trusts had a positive loading (coefficient $\hat{c}$) on the coskewness factor and this positive

[10] Formally, the Schwartz Information Criterion (SIC) takes the value SIC$= n\ln(\frac{RSS}{n}) + (k+1)\ln(n)$, where n is the number of observations, RSS is the sum of squared residuals and k is the number of explanatory variables in the model. This expression is an increasing function of RSS and k. The model with the lower value is preferred.

loading was statistically significant at the 5% level for 175 funds during the examined period. Figure 4.5 plots the density of the loadings on the coskewness strategy (solid line), showing that more than 90% of the trusts had a positive coefficient estimate and more than the 50% of the trusts had a coefficient point estimate of more than 0.25. This finding confirms that the majority of the funds were employing strategies which essentially loaded negative skewness, without this implying that they consciously followed the specific negative coskewness spread strategy we analyzed in the previous section.[11] The Spearman correlation coefficient for the rankings of the trusts according to their Jensen and Harvey-Siddique alphas is found to be $\rho = 0.969$.

Ranking the trusts according to their Harvey-Siddique alphas, Table 4.3 reports their estimates for various percentiles of this distribution. It is striking to observe that only the 16% of the trusts had positive alphas. On the other hand, the 55% of the funds had an alpha of less than -2% p.a.. Ranking the trusts according to the t-values of these alphas in Table 4.4, the results are equivalent. Only 3 funds had significantly positive alphas at the 5% level, while 41% of the trusts had significantly negative alpha estimates. With respect to the distribution of the alphas, this is now closer to normality,[12] being less positively skewed in comparison to the Jensen alphas' distribution.

Interestingly, the two trusts with the highest Jensen alphas (15.5% and 13.71% p.a. correspondingly), which account for the extreme positive tail of the distribution, are the trusts with the 2nd and 8th (out of 273) highest loadings of the coskewness risk factor (with coefficient point estimates of $\hat{c} = 1.53$ and $\hat{c} = 0.98$ correspondingly). Hence, the conjecture of heterogeneous risk-taking we previously made is supported

[11] Fund management practice shows that managers try to find and exploit patterns in stock returns in order to generate portfolios that beat the measures according to which they are evaluated. Therefore, a negative coskewness strategy does not necessarily mean that the manager consciously picks stocks with this characteristic, but that the strategies he implements actually mimick this statistical characteristic.

[12] The null hypothesis of normality is marginally rejected at the 5% level using the Kolmogorov-Smirnov test.

by the results and part of this heterogeneity is due to the negative coskewness risk. With respect to the explanatory power of the Harvey-Siddique model, we found that this should be preferred to the CAPM on the basis of the SIC for all of the trusts we examined. The enhancement in the explanatory power through the addition of the coskewness risk factor comes at the minimum possible cost, i.e. only one extra regressor, leading to lower values of SIC for each trust. As a result, the parsimony of the Harvey-Siddique model is verified to be a highly attractive characteristic for the sample of trusts we examine.

There are two main conclusions from these results: The first is that prudent investors, who are averse to negative skewness and should use the Harvey-Siddique alpha to evaluate their trust managers, would have been better off by investing in a low-cost index fund, as compared to more than 80% of the available trusts over the period January 1991- December 2005. The second conclusion is that managers were very successful in reaping the negative coskewness premium, presenting it as "added value" and higher Jensen alpha. Figure 4.6 presents in a scatterplot the estimate of the Jensen alpha for each trust versus the estimate of the negative coskewness strategy loading ($\hat{c}$), demonstrating this positive relationship. The Pearson correlation coefficient of these two variables is 0.45. More formally, regressing the Jensen alpha point estimates on the coskewness factor loadings, we get the following result (t-values are given in the parentheses):

$$\alpha_{Jensen} = \underset{(-11.8)}{-0.0318} + \underset{(9.32)}{0.0593}\ \hat{c} \qquad R^2 = 24.3\%$$

The regression results show that a large part of the variation in the Jensen alphas can be explained by the trusts' coskewness factor loadings alone. This relationship is strongly significant. This finding confirms the argument that the trusts with the highest Jensen alphas were on average those that loaded most of the negative coskewness risk. In other words, it can be argued that unit trusts would have been useful investment vehicles for agents with quadratic preferences that regard the coskewness premium as a "free lunch", but not for prudent investors.

Comparing the results derived from the Harvey-Siddique models with corresponding results derived from the Carhart model, we find that for 63% of the examined trusts the Harvey-Siddique model should be preferred on the basis of the SIC. Again, the parsimony of this model is a highly attractive feature. The distribution of the Harvey-Siddique alphas is very similar to the distribution of the Carhart alphas, as this can be seen from Figure 4.3. This result provides support for the argument that the coskewness factor may partly capture the information contained in the Carhart regressors. Finally, the Spearman correlation coefficient between the trusts' rankings according to the Carhart and Harvey-Siddique alphas is $\rho = 0.9362$.

4.6 Further results

4.6.1 Adding the coskewness factor to the Fama-French and Carhart models

In this subsection we examine whether the previous results are modified if we take the Fama-French or the Carhart model as the benchmark and augment it by adding the negative coskewness factor $(S^- - S^+)$. This approach has been taken by Hung et al. (2004) in an asset pricing context and by Ding and Shawky (2007) in a performance evaluation context. In particular, we examine if the addition of the coskewness factor affects the conclusions with respect to trusts' managerial ability, whether the evidence for the loading of the negative coskewness risk is robust to the inclusion of other risk factors as well as whether the augmented models exhibit superior explanatory power.

As it has been previously discussed, the Fama-French model adds to the CAPM a size and a value risk factor. Consequently, the Fama-French alpha (α_{FF}) is the intercept of the following regression model:

$$r_{p,t} = \alpha_{FF} + \beta_p r_{m,t} + \beta_1 SMB_t + \beta_2 HML_t + \epsilon_t \tag{4.18}$$

Employing this asset pricing model for the 273 trusts of our sample during the period January 1991- December 2005, we estimate their Fama-French alphas. Figure 4.7 exhibits the distribution of these trusts' alphas. This is very similar to the distribution of the Carhart alphas illustrated in Figure 4.3. The magnitude of the Fama-French alphas is also similar to the Carhart ones. In particular, the median trust has a Fama-French alpha of -2.19% p.a. It is interesting to note that 220 of these trusts had a negative Fama-French alpha point estimate and for 113 of them this point estimate is significantly negative at the 5% level. The Spearman rank correlation coefficient between the Fama-French and the Carhart measures is extremely high, equal to $\rho = 0.979$.

Adding the negative coskewness factor to the Fama-French model, the augmented regression to be estimated is given by:

$$r_{p,t} = \alpha_{FF+CSK} + \beta_p r_{m,t} + \beta_1 SMB_t + \beta_2 HML_t + c(S^- - S^+)_t + \epsilon_t \tag{4.19}$$

The intercept of this model, α_{FF+CSK}, shows the value that the manager added to the portfolio once the returns generated by the size, value and coskewness strategies have been attributed to the corresponding risk factors. We estimate this augmented model and the distribution of the corresponding alphas is plotted in Figure 4.7. The inclusion of the coskewness factor shifts the whole distribution of the Fama-

French alphas to the left. The median trust had an augmented Fama-French alpha of -2.57% p.a. The estimated alpha was negative for 231 of the trusts, while this estimate was significantly negative at the 5% level for 123 of these trusts. With respect to the trusts' rankings according to these two measures, their Spearman correlation coefficient is equal to $\rho = 0.974$.

The dotted curve in Figure 4.5 illustrates the density of the estimated coskewness factor loadings ($\hat{c}$) derived from regression (4.19). The magnitude of these loadings is slightly reduced relative to the Harvey-Siddique model, but they are still predominantly positive. In particular, 234 out of the total 273 trusts had a positive loading on the coskewness factor and for 101 of these trusts this loading was significantly positive at the 5% level. The explanation we put forward for this finding is that the value and size factors partly capture the negative coskewness risk. Nevertheless, the fact that the estimated coskewness loadings do not evaporate in the presence of these factors leads to the conclusion that the coskewness factor we have employed in this study carries important information for the trusts' returns.

On the basis of the SIC, the augmented Fama-French model should be preferred to its standard version for 32% of the examined trusts. This is an important result if we take into account the fact that the SIC heavily penalizes the inclusion of an additional regressor. To stress the importance of the coskewness factor, it should be noted that comparing the standard Fama-French model with the Harvey-Siddique one on the basis of the SIC, the latter is preferred for 66% of the trusts. The explanatory power of the Harvey-Siddique model coupled with its attractive parsimony make it the preferred one.

We repeat the previous analysis, adding now the negative coskewness factor to the Carhart model which is given in (4.14). Hence, the augmented Carhart model is now given by:

$$r_{p,t} = \alpha_{Carhart+CSK} + \beta_p r_{m,t} + \beta_1 SMB_t + \beta_2 HML_t + \beta_3 MOM_t + c(S^- - S^+)_t + \epsilon_t \tag{4.20}$$

The intercept of this model, $\alpha_{Carhart+CSK}$, shows the value the manager added to the portfolio once the returns generated by the size, value, momentum and coskewness strategies have been attributed to the corresponding risk factors. The estimated alphas are very similar to the ones derived by the augmented Fama-French model, to the extent that the two distributions are not distinguishable. The median trust has an augmented Carhart alpha of -2.52% p.a. For 232 of the trusts, the estimated alpha was negative, while this estimated value was significantly negative for 124 trusts. The ranking of the trusts based on the augmented Carhart model is very similar to the ranking based on its standard version. The corresponding Spearman rank correlation coefficient is equal to $\rho = 0.979$.

The estimated loadings on the coskewness factor ($\hat{c}$) are very similar to the ones derived by the augmented Fama-French model. Their distribution is illustrated by the dashed curve in Figure 4.5. Even though their magnitude is slightly reduced with respect to the Harvey-Siddique model, they are still positive for 229 of the examined trusts. Moreover, for 106 of these trusts the coskewness factor loading was significantly positive at the 5% level. Therefore, the argument that the trusts' managers loaded negative coskewness to their portfolios during the examined period holds true even when we include the commonly used value, size and momentum risk factors.

Comparing the Carhart model with its augmented version on the basis of the SIC, the Carhart model is preferred for the 65% of the trusts. These results provide further support for the argument that the Carhart factors mimick the coskewness factor to some extent. In other words, the addition of the coskewness factor to the

Carhart model is not expected to immensely enhance the explanatory power of the augmented model. Nevertheless, the finding that the inclusion of the coskewness factor leads to a lower SIC value for the rest 35% of the trusts acts as a warning for the fitness of the Carhart model. The information contained in the negative coskewness factor is of great importance for the appropriate performance attribution of a large number of funds.

4.6.2 Subperiod analysis

Due to the high turnover of trust managers as well as the different market phases they face, it is interesting to examine whether the previous findings are robust for shorter time periods. Therefore, the total period is split into three subperiods of 5 years each. Panel A of Table 4.2 provides the average returns the trusts achieved as well as their Sharpe ratios and their DSRs. Panel B of Table 4.2 presents their average Jensen, Carhart and Harvey-Siddique alphas. The average DSRs are consistently lower (in absolute value) in comparison to the average Sharpe ratios for all three subperiods. With respect to their average returns, there has been a significant improvement in the subsequent subperiods as compared to the initial period of January 1991- December 1995, when trusts underperformed the market by more than 400 bps.

There are three possible explanations for this improvement: The first is that there may have been a decrease in the expenses of the industry due to higher competition provoked by the entry of new trusts, so more of the managers' ability was finally captured by the individual shareholder.[13] The second explanation is that the trusts may have been more exposed to beta risk from 1996 onwards. This hypothesis cannot be supported by the data, since the average beta estimate of the trusts remained relatively stable and close to 0.93 for all three subperiods. The third explanation is that managers may have added more of the coskewness premium to their portfolios

[13]This hypothesis is not testable, since no data on UK unit trusts' expenses were available.

during the last two subperiods.

As Table 4.1 presents, the coskewness spread strategy yielded high positive returns after 1995. These returns were as high as 3.39% p.a. during the period January 2001- December 2005. Interestingly, the number of funds having positive loadings of the coskewness risk increased as this premium was increasing. During the period January 1991- December 1995 there were 113 trusts out of the total 150 having a positive loading (and only 26 of them being statistically significant at the 5% confidence level), from January 1996 to December 2000 there were 167 out of 197 trusts with positive loadings (with 46 of them being statistically significant), while during the period January 2001- December 2005, as many as 252 out of the total 265 trusts were loading negative coskewness risk (with 160 of these coefficients being statistically significant). This was actually the period that the value, size and momentum strategies yielded high positive returns, characterized by negative coskewness. Hence, trust managers responded very quickly to the existence of this premium, employing strategies that mimicked negative coskewness. Moreover, they correctly acted according to their incentives, since most of them were evaluated either through their raw returns or through mean-variance measures.

The previous analysis explains the significant improvement in the trusts' Jensen alphas over the subperiods presented in Table 4.5, where the median trust had a Jensen alpha of -4.1% p.a. during the period January 1991- December 1995 and then significantly improved to -2.01% p.a. and -1.4% p.a. in the next subperiods. Hence, these results provide further evidence that the trusts were successful in reaping the negative coskewness premium, actually increasing their exposure to this risk during the period its premium was at its highest levels. While this strategy would have yielded a significant gain for a quadratic investor, it does not do so for a prudent one, because at the same time it loads the negative skewness risk that he is averse to.

With respect to Carhart alphas, Table 4.5 presents the trusts' rankings according to their point estimates and Table 4.6 presents their rankings according to their t-values for these three subperiods. The general conclusion is that alphas are significantly reduced as we switch from the CAPM to the Carhart model. The most significant reduction in the performance, however, takes place in the third subperiod, January 2001- December 2005. As the results show, even the top-ranked trust had a Carhart alpha of only 4.93% p.a. in comparison to a Jensen alpha of 11.11% p.a., while the median trust scored quite badly, achieving a Carhart alpha of −2.81% p.a. Moreover, the investors who selected the bottom 30% of the trusts experienced a significantly negative performance of less than −3.73% p.a.

The explanation for this significant modification of the results is that during the third subperiod of our sample the size, value and momentum strategies yielded very high positive returns. Hence, for the managers who followed these strategies during this subperiod, these returns were translated into a higher Jensen alpha, but not into a high Carhart alpha, since they were attributed as premia to the corresponding factors. This result shows how different conclusions an investor can extract using different asset pricing models for performance evaluation.

In order to examine the evolution of managerial ability for a prudent investor, Table 4.5 presents the Harvey-Siddique alphas and their t-statistics for the three subperiods across various percentiles of the distribution. With respect to their point estimates, in all three periods, less than 30% of the trusts had positive alphas. The median trust severely underperformed during the period 1991-1995 exhibiting a Harvey-Siddique alpha of −4.25% p.a. This performance was significantly improved in the period 1996-2000, but the median trust still had an alpha of −2.43% p.a. Nevertheless, this improvement was not continued in 2001-2005, since the median fund achieved a Harvey-Siddique alpha of −2.68% p.a.

Figure 4.8 plots the distributions of the trusts' Harvey-Siddique alphas for each of the subperiods. While the distribution of alphas in 1996-2000 was shifted to the right in comparison to the previous subperiod, it was then shifted to the left during the period 2001-2005, exhibiting a large concentration of values around the mean. Ranking trusts according to the t-values of their Harvey-Siddique alphas in Table 4.6, it is surprising to see that in the second and the third subperiod, apart from the top two trusts, there was no other trust with a significantly positive alpha. On the other hand, in all three subperiods, more than 30% of the trusts had significantly negative Harvey-Siddique alphas.

4.6.3 Bootstrap analysis

The previous subsections relied on t-statistics to examine the significance of the performance measures' point estimates. This is a valid procedure under the Gauss-Markov assumption of normality for the regressions' residuals. Nevertheless, the nonnormality of the alphas' distribution and the evidence of heterogeneous risk-taking may cast doubts on the validity of the normality assumption, especially for the trusts at the tails of the alphas' distribution. If the residuals are not normally distributed, then the t-statistics may lead to spurious results and the extreme alpha estimates may be due to sampling variability, i.e. luck.

We perform the Jarque-Bera (JB) test for normality in the residuals derived by the various asset pricing models we have employed.[14] For the market model, 177 out of the 273 funds have non-normal residuals (at the 5% level). The number of trusts with non-normal residuals drops to 173 when we employ the Fama-French model and 166 for the Carhart model. Using the Harvey-Siddique model we had 172 trusts with non-normal residuals at the 5% level. Hence, there is sufficient evidence that

[14]The Jarque-Bera test is a goodness-of-fit measure of departure from normality, based on the sample kurtosis and skewness. The test statistic JB is defined as $JB = \frac{n}{6}(S^2 + \frac{(K-3)^2}{4})$, where n is the number of observations, S is the measure of skewness and K is the degree of kurtosis. The JB statistic follows a $\chi^2(2)$ distribution, having as its null hypothesis that the data are drawn from a normal distribution.

the normality assumption is violated for a large portion of these funds. In order to examine the impact of sampling variability on the results discussed in the previous sections, this subsection employs a simple bootstrap methodology (see Hall, 1992, for an introduction) at the individual fund level.[15]

In particular, we employ the following bootstrap procedure, accordingly adjusted for the Harvey-Siddique asset pricing model. Extracting the time series of residuals $\{\hat{\epsilon}_{i,t}, t = T_{i0}, ..., T_{i1}\}$ for each trust i from the regression:

$$r_t = \hat{\alpha}_{HS} + \hat{\beta} r_{m,t} + \hat{c}(S^- - S^+)_t + \hat{\epsilon}_t \tag{4.21}$$

we draw a sample with replacement for each of these trusts and create a pseudo-time series of resampled residuals $\{\hat{\epsilon}^{b^I}_{i,t_\epsilon}, t_\epsilon = s^{b^I}_{T_{i0}}, ..., s^{b^I}_{T_{i1}}\}$, where b^I is an index for the bootstrap number and where each of the time series indices $s^{b^I}_{T_{i0}}, ..., s^{b^I}_{T_{i1}}$ are drawn randomly from $[T_{i0}, ..., T_{i1}]$.

Using this pseudo-time series of resampled residuals, we construct for each trust i a time-series of monthly excess returns $r^{b^I}_{t_\epsilon}$ for each bootstrapped sample b^I, employing the already estimated set of coefficients $\{\hat{\alpha}_{HS}, \hat{\beta}, \hat{c}\}$:

$$r^{b^I}_{t_\epsilon} = \hat{\alpha}_{HS} + \hat{\beta} r_{m,t} + \hat{c}(S^- - S^+)_t + \hat{\epsilon}^{b^I}_{t_\epsilon} \tag{4.22}$$

for $t = T_{i0}, ..., T_{i1}$ and $t_\epsilon = s^{b^I}_{T_{i0}}, ..., s^{b^I}_{T_{i1}}$. We subsequently use these pseudo-returns to re-estimate regression (4.15) for each bootstrap sample b^I and we get an alpha estimate $\{\hat{\alpha}^{b^I}_{HS}\}$. Repeating the previous steps 1,000 times, we have a set of 1,000 alpha estimates for each fund i, having bootstrapped its residuals. This distribution of bootstrapped alphas can help us derive confidence intervals for the point estimate

[15] Kosowski et al. (2006) suggest a novel cross-sectional bootstrap methodology to distinguish "skill" from "luck" in US mutual funds. Cuthbertson et al. (2006) also employ this novel bootstrap methodology to evaluate UK unit trusts for a series of commonly used performance measures. In contrast, our study employs a simpler, fund-by-fund bootstrap approach.

without imposing any parametric assumption on the fund's residuals.

The aim of this subsection is to derive the distribution of the bootstrapped alphas for each of the funds in Table 4.3, i.e. for different percentiles of the Harvey-Siddique alphas' distribution for the entire sample period. This will help us derive more robust conclusions with respect to the statistical significance of the estimated Harvey-Siddique alphas. In particular, the last line of Table 4.3 reports the corresponding p-values derived by the bootstrap methodology. The results show that the conclusions regarding the significance of the estimated Harvey-Siddique alphas do not get modified if we use the bootstrapped residuals.

Figure 4.9 illustrates the bootstrapped alphas' distribution (solid curve) for a series of funds. The vertical line shows the corresponding point estimate of the Harvey-Siddique alpha. We also plot the density of alphas under the conventional regression analysis (dashed line), in order to have a benchmark of comparison. The inference under the bootstrap methodology is not considerably different from the standard approach. In particular, apart from the top trust in Panel A1, the two alternative distributions are almost identical for the rest of the trusts.

Controlling for the impact of sampling variation is more crucial when the residuals' distribution is asymmetric. However, as it can be seen from Figure 4.9, the derived bootstrapped alphas' distributions for a series of funds are relatively symmetric, hence there is no noteworthy difference in terms of statistical inference as compared to the parametric case of the standard regression analysis. This is a quite interesting result, in light of the skewed bootstrapped alphas' distributions derived in Kosowski et al. (2006). The explanation we put forward is that the inclusion of a coskewness factor may considerably contribute to the symmetry of the residuals' distribution, since it attributes highly skewed returns to the corresponding risk factor, unlike other models that would regard them as residuals. The nonnormality in the regressions' residuals under the Harvey-Siddique model is mainly due to their excess kurtosis.

4.6.4 Can the Harvey-Siddique model explain the trusts' returns?

This subsection examines the explanatory power of the Harvey-Siddique asset pricing model with respect to the funds' returns. There are two main issues that are of interest: Firstly, it is examined whether this model captures in a better way the variability of the funds' returns and, secondly, we consider whether the factor loadings of this model can explain the level of the funds' average returns. In particular, we make use of the 148 funds that had 180 monthly observations (no missing value) during the period January 1991- December 2005.

With respect to the variability of these funds' returns, using the CAPM, the average R^2 was as high as 80.4%. This is an expected as well as a necessary result, if the selected benchmark is an appropriate one for the funds we evaluate. Adding the coskewness factor, the average adjusted R^2 for these funds using the Harvey-Siddique model is 81.27%. This is again an anticipated result because the returns of the added coskewness factor ($S^- - S^+$) are not highly variable as Table 4.1 and Figure 4.1 show. On the other hand, the average adjusted R^2 of the funds using the Carhart model is 85.4%. The increase in the explanatory power of the Carhart model is due to the fact that the added size, value and momentum factors are highly variable as Table 4.1 shows. It should be reminded that due to the definition of the least squares methodology, the more volatile the regressors are, the more flexible the model is to fit the dependent variable.

These results show that the CAPM does a very good job in capturing the co-movement of the funds' returns with their benchmark's returns. Consequently, the addition of extra factors cannot considerably increase the R^2 of the multi-factor models, especially when these exhibit low variability, as the coskewness factor does. The motivation for the use of this extra factor is the failure of the CAPM to explain the level of the funds' average returns. This is a common finding in the empirical asset pricing literature, implying that the beta of the fund with respect to its benchmark is not an appropriate measure of risk. If this was true, then there should be a positive relationship between the funds' betas and their average excess returns. Figure 4.10 plots the average excess returns of the 148 funds against their estimated CAPM betas, showing that the positive relationship predicted by the standard financial theory does not hold. More specifically, the slope coefficient of this relationship is negative but statistically insignificant. The average excess returns of the funds are not related to their beta risk exposure.

On the other hand, Figure 4.11 plots the average funds' excess returns against their coskewness factor loadings, ($\hat{c}$), as estimated by the Harvey-Siddique model. It is interesting to document a strong and statistically significant positive relationship between the level of the funds' average excess returns and their exposure to the coskewness risk. This very important result implies that the most appropriate measure of risk is the coskewness factor loading of the fund. It should be noted that the beta estimates for the funds derived by the Harvey-Siddique model are the same with the beta estimates derived by the CAPM, since the coskewness risk factor is by construction orthogonal to the beta risk. Consequently, we can argue that the Harvey-Siddique model dominates the CAPM, because it inherits its high explanatory power for the variability of the funds' returns and it additionally manages to explain their cross-sectional average returns, revealing what is the appropriate concept of risk that is priced in the markets and yields the corresponding premia that the funds reap.

4.7 Conclusion

Higher moments in asset returns is a relatively neglected issue in the investment performance evaluation literature. This issue becomes even more important, if one takes into account the experimental evidence that large negative returns affect utility asymmetrically more than positive returns do. Consequently, a prudent investor should not use mean-variance measures to evaluate his investments, because they neglect his actual preferences and regard the negative coskewness premium as a "free lunch".

In the case of delegated asset management, this issue is very crucial, since the fund manager, if evaluated through mean-variance measures, will falsely interpret that the fund shareholder has no preferences over skewness and he will be incentivized to follow tactical asset allocation strategies that load this type of risk, in order to reap the corresponding premium. Clearly, this situation generates a mismatch between objectives and outcomes, leading to erroneous conclusions with respect to the *ex post state verification* of the investment performance.

The limitations of the static, mean-variance measures motivate the adoption of a performance measure that adjusts for the negative coskewness risk, documented to be priced in the UK stock market. The Harvey-Siddique two-factor asset pricing model is qualified to be appropriate for a prudent investor and it has a sound theoretical basis, unlike the Carhart asset pricing model. The intercept of this model, which we term as the *Harvey-Siddique alpha*, will reveal the genuine outperformance for such an investor, resolving the *ex post verification problem*.

This measure was employed for the evaluation of the UK equity unit trusts that had the FTSE All Share Index as their benchmark for the period January 1991-December 2005. The vast majority of the trusts exhibited a negative Harvey-Siddique alpha, significantly underperforming their benchmark. Actually, the median under performance of the trusts (-2.32%) for prudent investors was of greater magnitude than the current average expense ratio they charge (*circa* 1.5%).

Interestingly, most of the trusts loaded negative coskewness to their portfolios, capturing part of the corresponding premium and correctly responding to their incentives, since they are currently being evaluated through mean-variance measures. This finding shows how a prudent investor would misinterpret this premium for genuinely added value, if he was using such a measure too. Hence, the call for the shift of interest from outperforming to matching investors' preferences and objectives becomes even more important, reflecting the advice of Charles Ellis (2005, p. 115) not to play "the Loser's Game of trying to beat the market- a game that almost every investor will eventually lose".

Table 4.1: Excess Market, Size, Value, Momentum and Coskewness returns

	Excess Market	$(S^{-}-S^{+})$ Strategy	Size strategy	Value strategy	Momentum strategy
Panel A: Total Period 1991-2005					
Aver. Monthly Returns	0.42%	0.174%	0.175%	0.18%	0.169%
St. Deviation	4.01%	0.96%	3.65%	2.81%	1.39%
Panel B: Subperiod 1991-1995					
Aver. Monthly Returns	0.73%	0.021%	-0.08%	0.09%	0.15%
St. Deviation	4.09%	0.62%	3.63%	2.14%	0.74%
Panel C: Subperiod 1996-2000					
Aver. Monthly Returns	0.63%	0.21%	-0.001%	0.001%	0.09%
St. Deviation	3.71%	1.07%	4.01%	3.52%	1.64%
Panel D: Subperiod 2001-2005					
Aver. Monthly Returns	-0.09%	0.28%	0.61%	0.44%	0.26%
St. Deviation	4.21%	1.11%	3.31%	2.61%	1.62%

Notes: This Table presents the average monthly returns and the corresponding standard deviation of the coskewness, the size, the value and the momentum strategy along with the excess market returns. Panel A presents the results for the total period January 1991- December 2005, Panel B the corresponding results for the subperiod January 1991- December 1995, Panel C for the subperiod January 1996- December 2000 and Panel D for the subperiod January 2001- December 2005.

Table 4.2: Unit trusts' descriptive statistics

Panel A: Trusts' returns and Sharpe Ratios				
Periods	Number of trusts ≥ 60 obs.	Trusts' Average Excess Returns (p.a.)	Average Sharpe Ratio	Average Downside Sharpe Ratio
1991-1995	150	4.61%	0.089	0.051
1996-2000	197	5.76%	0.116	0.065
2001-2005	265	-2.01%	-0.04	-0.022
1991-2005	273	2.93%	0.039	0.024

Panel B: Trusts' Jensen, Carhart and Harvey-Siddique alphas			
	Average Jensen alpha (p.a.)	Average Carhart alpha (p.a.)	Average Harvey-Siddique alpha (p.a.)
1991-1995	-3.65%	-3.37%	-3.81%
1996-2000	-1.30%	-1.57%	-1.91%
2001-2005	-0.98%	-2.68%	-2.33%
1991-2005	-1.23%	-1.97%	-2.12%

Notes: Panel A presents the number of trusts with more than 60 observations, their average annualized excess returns as well as the average Sharpe ratios and Downside Sharpe ratios during the whole sample period period, January 1991- December 2005, and the subperiods January 1991- December 1995, Janurary 1996- December 2000 and January 2001- December 2005. Panel B reports the average annualized Jensen, Carhart and Harvey-Siddique alphas of the trusts for the same periods.

Table 4.3: Alpha rankings, Total Period 1991-2005

	Top	1%	5%	10%	30%	median	70%	90%	95%	99%	Bottom
Jensen alpha	15.51%	9.05%	4.57%	2.31%	-0.37%	-1.77%	-2.96%	-4.32%	-4.89%	-6.08%	-15.67%
t-stat	1.97	1.96	2.31	1.35	-0.27	-0.97	-3.59	-3.38	-2.62	-3.89	-3.97
p-value	0.03	0.03	0.01	0.09	0.37	0.25	<0.01	<0.01	0.01	<0.01	<0.01
Carhart alpha	13.68%	4.92%	3.01%	1.08%	-1%	-2.32%	-3.37%	-4.66%	-5.48%	-7.15%	-16.49%
t-stat	1.98	3.2	0.98	0.51	-2.64	-1.56	-3.88	-4.25	-2.42	-2.57	-4.23
p-value	0.03	<0.01	0.25	0.35	0.01	0.13	<0.01	<0.01	0.01	0.01	<0.01
H-S alpha	11.20%	6.45%	2.87%	0.88%	-1.20%	-2.36%	-3.69%	-4.97%	-5.68%	-6.94%	-17.07%
t-stat	1.43	1.84	1.01	0.49	-0.88	-2.34	-3.53	-3.61	-2.37	-4.27	-4.28
p-value	0.14	0.08	0.21	0.35	0.29	0.01	<0.01	<0.01	0.01	<0.01	<0.01
boots. p-value	0.06	0.03	0.14	0.31	0.18	0.01	<0.01	<0.01	<0.01	<0.01	<0.01

Notes: This Table presents the rankings of the trusts over the whole sample period, January 1991- December 2005, based on the estimates of the annualized Jensen, Carhart and Harvey-Siddique (H-S) alphas. In particular, the annualized alpha estimates for various percentiles of the trusts' rankings are reported along with their corresponding t-statistics and p-values. Reported t-values use Newey-West (1987) heteroskedasticity and autocorrelation consistent standard errors. The last line of the Table reports the corresponding p-values for the Harvey-Siddique alpha of each fund, as they are derived from the bootstrap methodology discussed in Section 4.6.3.

Table 4.4: t-statistics rankings, Total period, 1991-2005

	Top	1%	5%	10%	30%	median	70%	90%	95%	99%	Bottom
t-Jensen alpha	2.81	2.61	1.91	0.85	-0.18	-1.14	-2.05	-3.59	-4.18	-4.80	-6.96
Jensen alpha	13.71%	6.99%	6.05%	1.73%	-0.40%	-1.32%	-3.24%	-5.21%	-3.28%	-6.07%	-3.25%
p-value	<0.01	0.01	0.07	0.27	0.39	0.21	0.05	<0.01	<0.01	<0.01	<0.01
t-Carhart alpha	3.20	1.98	1.38	0.64	-0.65	-1.55	-2.47	-4.03	-4.39	-4.97	-7.14
Carhart alpha	4.92%	13.6%	1.45%	1.57%	-1.17%	-3.64%	-1.87%	-4.45%	-3.45%	-5.96%	-3.35%
p-value	<0.01	0.03	0.16	0.33	0.33	0.08	<0.01	<0.01	<0.01	<0.01	<0.01
t-H-S alpha	2.20	1.98	1.31	0.40	-0.71	-1.54	-2.60	-3.90	-4.58	-5.33	-6.70
H-S alpha	5.85%	9.55%	6.12%	1.41%	-1.45%	-1.55%	-4.17%	-4.82%	-4.46%	-4.70%	-3.21%
p-value	0.01	0.03	0.09	0.37	0.25	0.08	0.01	<0.01	<0.01	<0.01	<0.01

Notes: This Table presents the rankings of the trusts over the whole sample period, January 1991- December 2005, based on the t-statistics of the Jensen, Carhart and Harvey-Siddique alphas. In particular, the t-statistics for various percentiles of the trusts' rankings are reported along with their corresponding annualized alphas and p-values. Reported t-values use Newey-West (1987) heteroskedasticity and autocorrelation consistent standard errors.

Table 4.5: Alpha rankings, Subperiod Analysis

	Top	1%	5%	10%	30%	median	70%	90%	95%	99%	Bottom
Panel A: January 1991- December 1995											
Jensen alpha	14.83%	7.13%	3.01%	1.48%	-2.41%	-4.10%	-5.44%	-7.21%	-8.31%	-14.79%	-15.05%
t-stat	2.88	2.18	1.45	0.27	-1.86	-2.86	-3.58	-2.62	-2.77	-3.16	-3.49
p-value	<0.01	0.04	0.14	0.38	0.07	<0.01	<0.01	0.01	<0.01	<0.01	<0.01
Carhart alpha	12.91%	8.45%	3.67%	1.15%	-2.23%	-3.68%	-5.03%	-6.86%	-7.91%	-13.57%	-14.07%
t-stat	3.36	2.43	2.22	0.41	-1.14	-2.12	-2.96	-4.52	-3.80	-4.39	-3.47
p-value	<0.01	0.02	0.03	0.36	0.21	0.04	<0.01	<0.01	<0.01	<0.01	<0.01
H-S alpha	14.23%	6.50%	2.71%	0.97%	-2.58%	-4.25%	-5.49%	-7.33%	-8.70%	-15.12%	-15.25%
t-stat	2.82	2.14	1.35	0.18	-1.50	-1.44	-3.58	-1.25	-2.98	-3.23	-3.51
p-value	<0.01	0.04	0.16	0.39	0.13	0.14	<0.01	0.18	<0.01	<0.01	<0.01
Panel B: January 1996- December 2000											
Jensen alpha	19.97%	13.76%	6.44%	3.05%	-0.51%	-2.01%	-3.11%	-4.98%	-6.02%	-7.65%	-9.03%
t-stat	1.93	1.75	1.66	1.34	-0.12	-1.27	-0.44	-1.94	-1.95	-2.25	-3.48
p-value	0.06	0.09	0.1	0.16	0.39	0.17	0.36	0.06	0.06	0.03	<0.01
Carhart alpha	16.42%	10.64%	4.67%	1.83%	-0.5%	-1.99%	-3.28%	-4.86%	-6.1%	-7.77%	-9.41%
t-stat	2.26	2.22	0.75	0.72	-0.22	-1.08	-2.18	-0.90	-2.61	-3.71	-4.07
p-value	0.03	0.03	0.29	0.28	0.39	0.22	0.04	0.26	0.01	<0.01	<0.01
H-S alpha	14.19%	11.11%	4.36%	1.95%	-0.72%	-2.43%	-3.54%	-5.36%	-6.81%	-8.21%	-10.13%
t-stat	1.48	1.43	1.21	0.41	-0.28	-1.03	-1.94	-2.11	-3.32	-2.44	-4.01
p-value	0.13	0.14	0.19	0.36	0.38	0.23	0.06	0.03	<0.01	0.01	<0.01
Panel C: January 2001- December 2005											
Jensen alpha	11.11%	10.02%	5.32%	2.73%	-0.04%	-1.40%	-2.73%	-3.93%	-5.07%	-11.69%	-20.78%
t-stat	2.24	2.24	1.27	1.03	-0.01	-0.78	-1.47	-2.39	-2.84	-2.54	-2.06
p-value	0.03	0.03	0.17	0.23	0.4	0.29	0.13	0.02	<0.01	0.01	0.05
Carhart alpha	4.93%	4.69%	2.37%	0.39%	-1.38%	-2.81%	-3.73%	-5.37%	-6.85%	-12.5%	-25.7%
t-stat	2.16	1.38	0.98	0.23	-0.62	-1.79	-3.18	-2.41	-3.33	-2.93	-2.53
p-value	0.04	0.15	0.25	0.39	0.32	0.09	<0.01	0.02	<0.01	<0.01	0.01
H-S alpha	8.97%	8.01%	3.13%	0.76%	-1.35%	-2.68%	-3.65%	-4.86%	-5.87%	-11.94%	-26.24%
t-stat	2.01	2.07	0.82	0.21	-0.82	-0.96	-2.61	-1.47	-2.02	-2.50	-2.61
p-value	0.05	0.04	0.28	0.38	0.28	0.25	0.01	0.14	0.05	0.01	<0.01

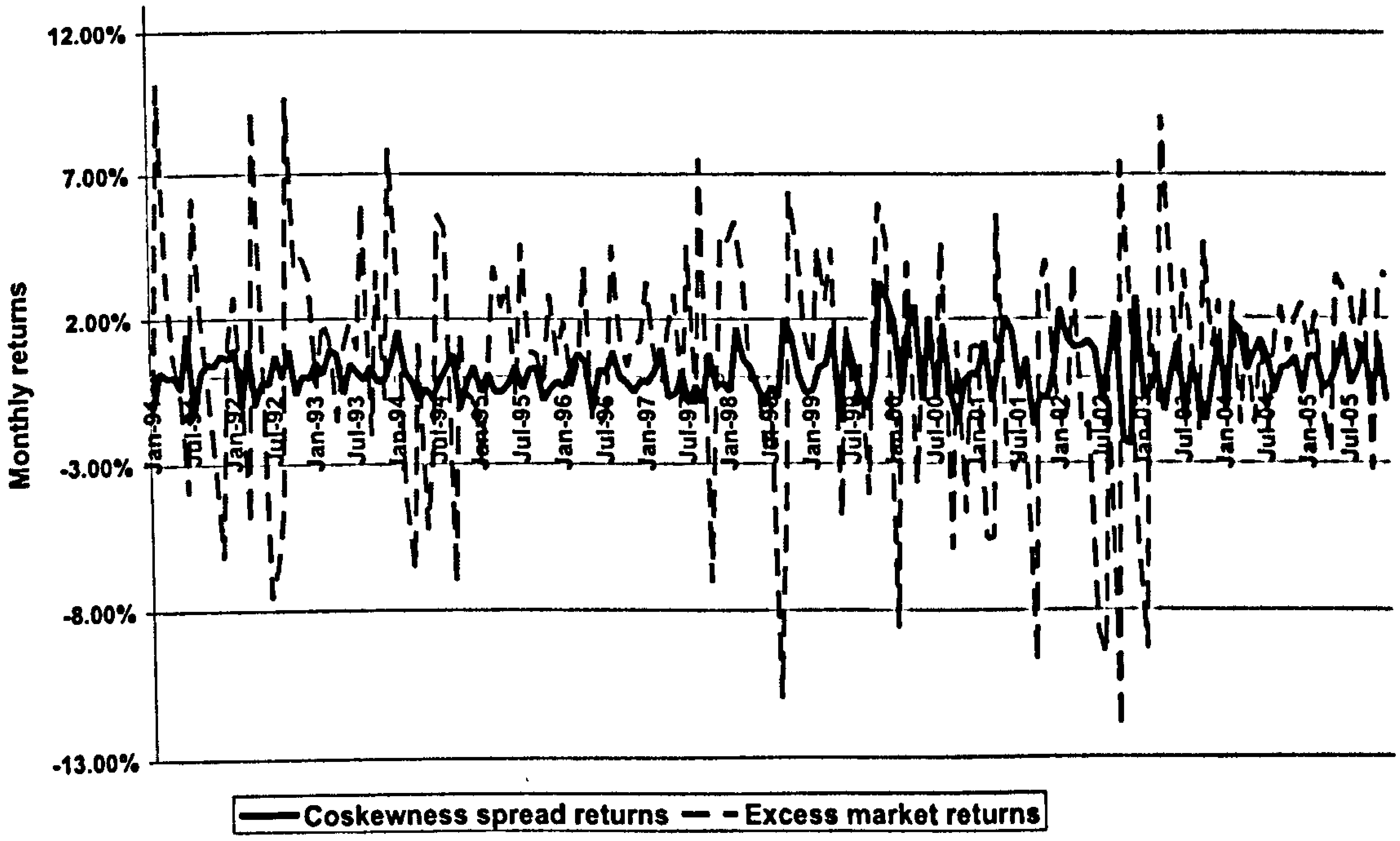
Notes: This Table presents the rankings of the trusts based on the estimates of the annualized Jensen, Carhart and Harvey-Siddique alphas for the three subperiods examined: January 1991- December 1995, January 1996- December 2000 and January 2001- December 2005. In particular, the annualized alpha estimates for various percentiles of the trusts' rankings are reported along with their corresponding t-statistics and p-values. Reported t-values use Newey-West (1987) heteroskedasticity and autocorrelation consistent standard errors.

Table 4.6: t-statistics rankings, Subperiod Analysis

	Top	1%	5%	10%	30%	median	70%	90%	95%	99%	Bottom
Panel A: January 1991- December 1995											
t-Jensen	2.88	2.18	1.14	0.27	-0.98	-1.86	-2.70	-3.18	-3.56	-4.63	-4.74
Jensen alpha	14.83%	7.13%	2.23%	1.48%	-2.06%	-2.41%	-6.29%	-6.92%	-8.76%	-8.89%	-9.84%
p-value	<0.01	0.04	0.21	0.38	0.25	0.07	0.01	<0.01	<0.01	<0.01	<0.01
t-Carhart	3.36	2.57	1.53	0.48	-1.14	-1.90	-2.72	-3.47	-3.84	-4.52	-4.92
Carhart alpha	12.91%	5.55%	5.52%	0.67%	-2.23%	-3.35%	-6.15%	-14.07%	-5.31%	-6.86%	-9.2%
p-value	<0.01	0.01	0.08	0.36	0.21	0.05	<0.01	<0.01	<0.01	<0.01	<0.01
t-H-S	2.81	2.14	1.08	0.18	-1.00	-1.86	-2.71	-3.24	-3.68	-4.69	-4.74
H-S alpha	14.23%	6.50%	2.43%	0.97%	-4.56%	-2.43%	-5.80%	-6.25%	-5.75%	-9.84%	-9.06%
p-value	<0.01	0.04	0.22	0.39	0.24	0.07	0.01	<0.01	<0.01	<0.01	<0.01
Panel B: January 1996-December 2000											
t-Jensen	2.38	1.94	1.35	0.77	-0.20	-0.78	-1.49	-2.27	-2.52	-3.45	-4.85
Jensen alpha	11.78%	19.97%	3.05%	4.56%	-0.73%	-2.46%	-2.58%	-3.09%	-3.99%	-6.93%	-3.07%
p-value	0.02	0.06	0.16	0.29	0.39	0.29	0.13	0.03	0.02	<0.01	<0.01
t-Carhart	2.65	2.63	1.47	0.75	-0.18	-0.97	-1.64	-2.42	-2.68	-3.71	-4.86
Carhart alpha	10.49%	7.64%	2.16%	4.67%	-0.5%	-1.59%	-4.97%	-4.04%	-3.71%	-7.7%	-3.12%
p-value	0.01	0.01	0.14	0.31	0.39	0.25	0.1	0.02	0.01	<0.01	<0.01
t-H-S	2.16	1.65	1.13	0.63	-0.29	-0.96	-1.64	-2.31	-2.77	-3.49	-4.73
H-S alpha	10.83%	7.90%	2.59%	5.94%	-0.70%	-2.44%	-5.36%	-5.02%	-4.87%	-6.19%	-3.06%
p-value	0.04	0.1	0.21	0.32	0.38	0.25	0.1	0.03	<0.01	<0.01	<0.01
Panel C: January 2001-December 2005											
t-Jensen	2.74	2.27	1.70	0.96	-0.02	-0.72	-1.46	-2.29	-2.63	-4.43	-5.30
Jensen alpha	10.97%	9.78%	8.58%	4.39%	-0.03%	-1.53%	-3.24%	-4.93%	-3.58%	-3.21%	-2.95%
p-value	<0.01	0.03	0.09	0.25	0.40	0.31	0.14	0.03	0.01	<0.01	<0.01
t-Carhart	2.17	1.38	0.98	0.18	-0.82	-1.57	-2.11	-2.95	-3.29	-5.01	-6.20
Carhart alpha	4.93%	4.69%	2.37%	0.41%	-1.23%	-3.92%	-4.52%	-7.29%	-5.71%	-3.71%	-4.15%
p-value	0.04	0.16	0.25	0.39	0.27	0.12	0.04	<0.01	<0.01	<0.01	<0.01
t-H-S	2.07	1.77	0.99	0.27	-0.62	-1.27	-2.00	-2.85	-3.02	-4.21	-5.69
H-S alpha	8.01%	4.32%	3.96%	0.89%	-2.88%	-3.30%	-3.97%	-4.81%	-3.60%	-3.17%	-3.80%
p-value	0.05	0.08	0.24	0.38	0.33	0.18	0.05	<0.01	<0.01	<0.01	<0.01

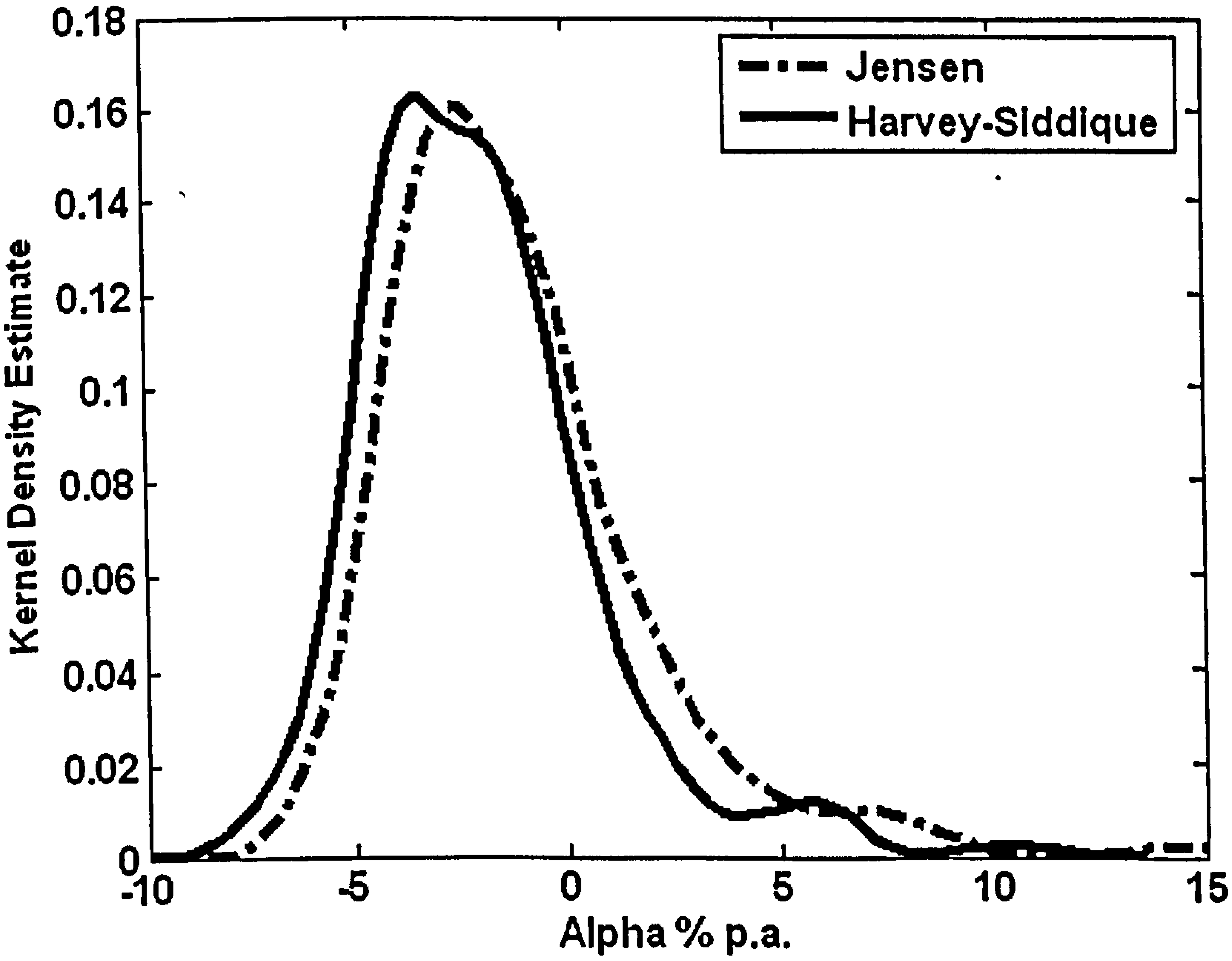
Notes: This Table presents the rankings of the trusts based on the t-statistics of the Jensen, Carhart and Harvey-Siddique alphas for the three subperiods examined: January 1991- December 1995, January 1996- December 2000 and January 2001- December 2005. In particular, the t-statistics for various percentiles of the trusts' rankings are reported along with their corresponding annualized alphas and p-values. Reported t-values use Newey-West (1987) heteroskedasticity and autocorrelation consistent standard errors.

Figure 4.1: Excess market returns and coskewness strategy returns



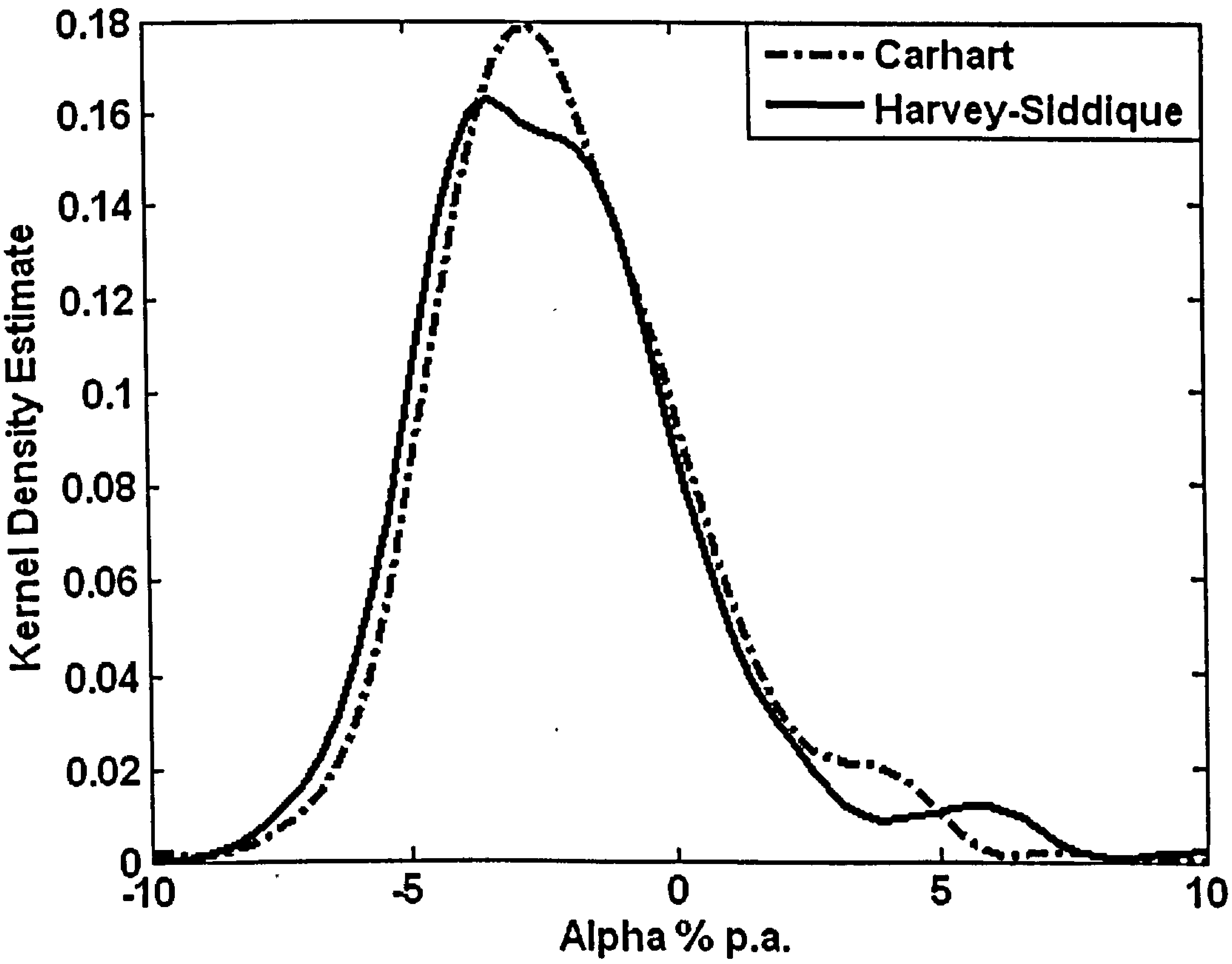
This Figure shows the monthly excess returns of the FTSE All Share Index (dashed line) and the monthly returns of the zero-cost coskewness spread strategy (solid line), as defined in Section 4.4, during the period January 1991-December 2005.

Figure 4.2: Jensen and Harvey-Siddique alphas



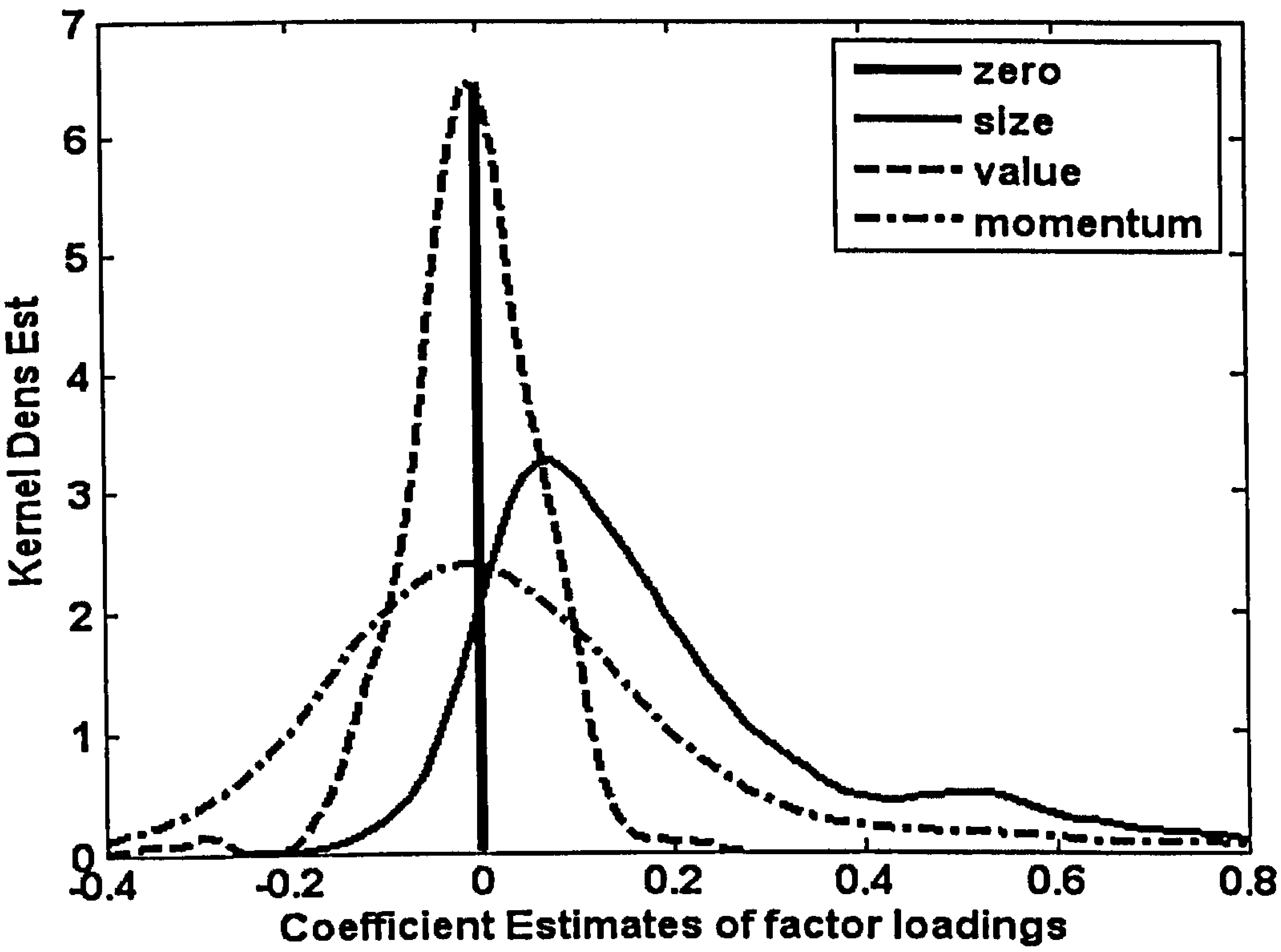
This Figure shows the distribution of the trusts' Jensen alpha estimates (dash-dotted line) and the distribution of the trusts' Harvey-Siddique alpha estimates (solid line) during the period January 1991- December 2005. This distribution as well as the distributions in the following figures were smoothed using a kernel density estimator. We employed a Gaussian kernel function and the corresponding optimal bandwith (see Silverman, 1986).

Figure 4.3: Carhart and Harvey-Siddique alphas



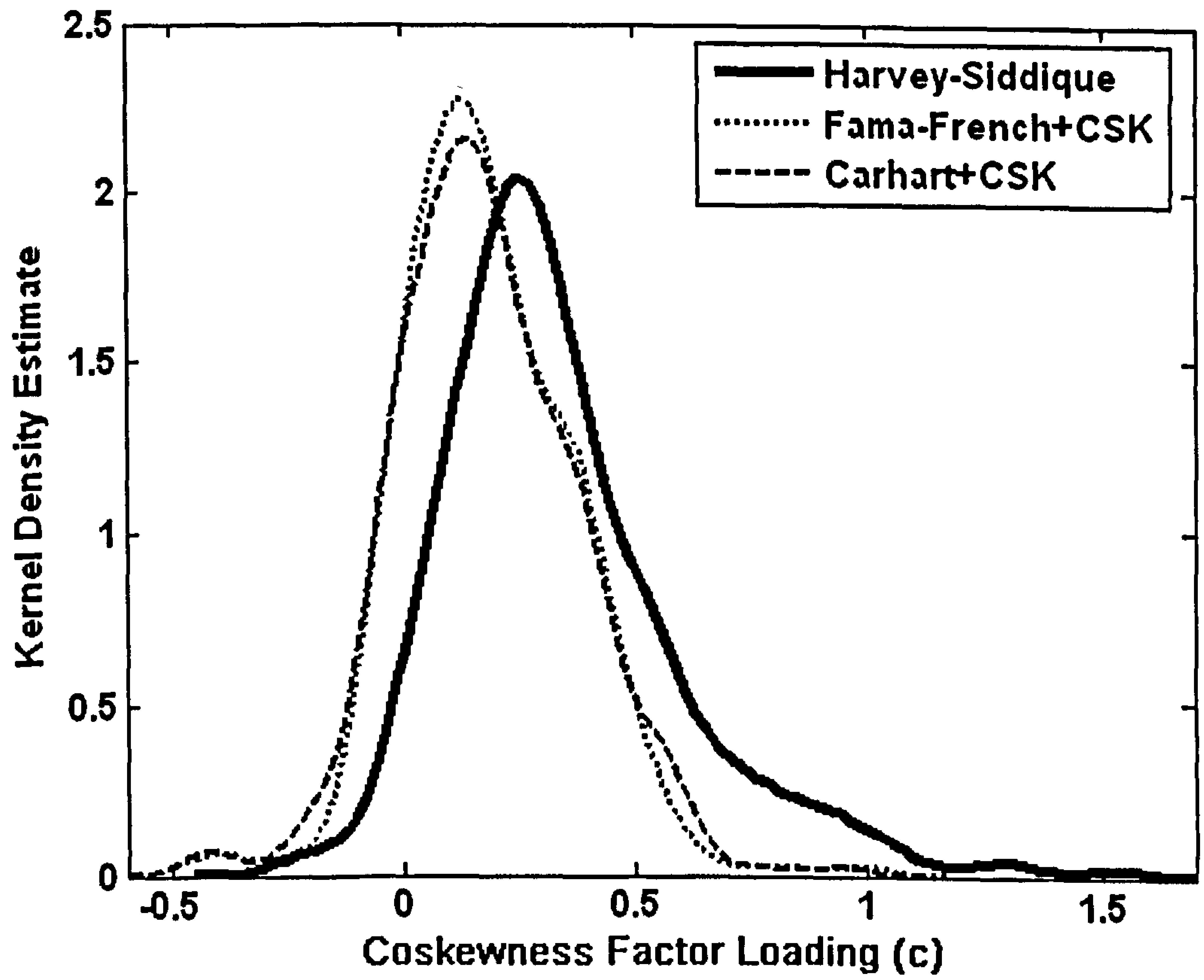
This Figure shows the distribution of the trusts' Carhart alpha estimates (dash-dotted line) and the distribution of the trusts' Harvey-Siddique alpha estimates (solid line) during the period January 1991- December 2005.

Figure 4.4: Size, value and momentum factor loadings



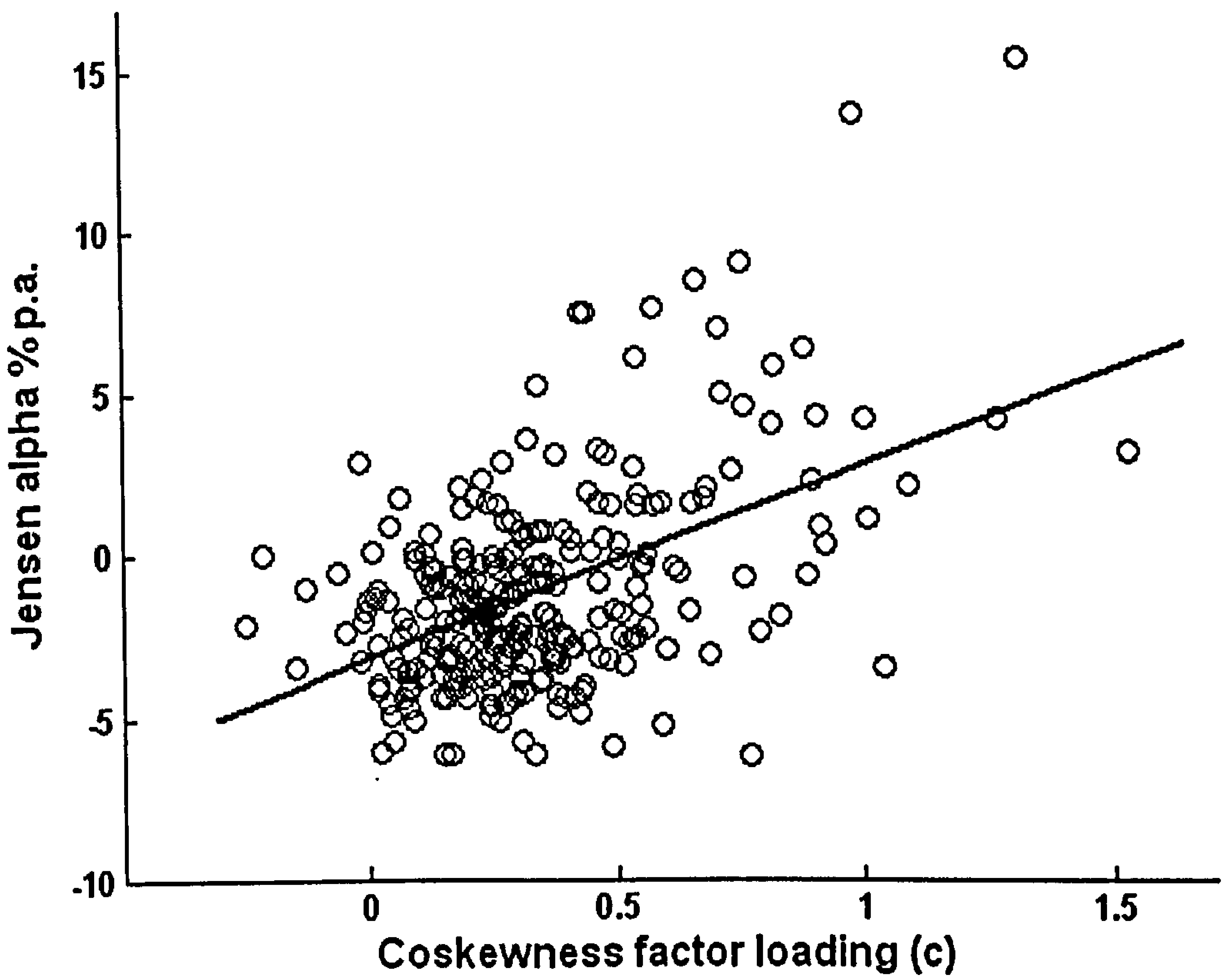
This Figure plots the densities of the estimated loadings on the size ($\widehat{\beta}_1$, solid curve), value ($\widehat{\beta}_2$, dashed curve) and momentum ($\widehat{\beta}_3$, dash-dotted curve) strategies for each trust, derived from the regression (4.14) for the period January 1991-December 2005. The vertical line corresponds to a zero coefficient.

Figure 4.5: Coskewness factor loadings



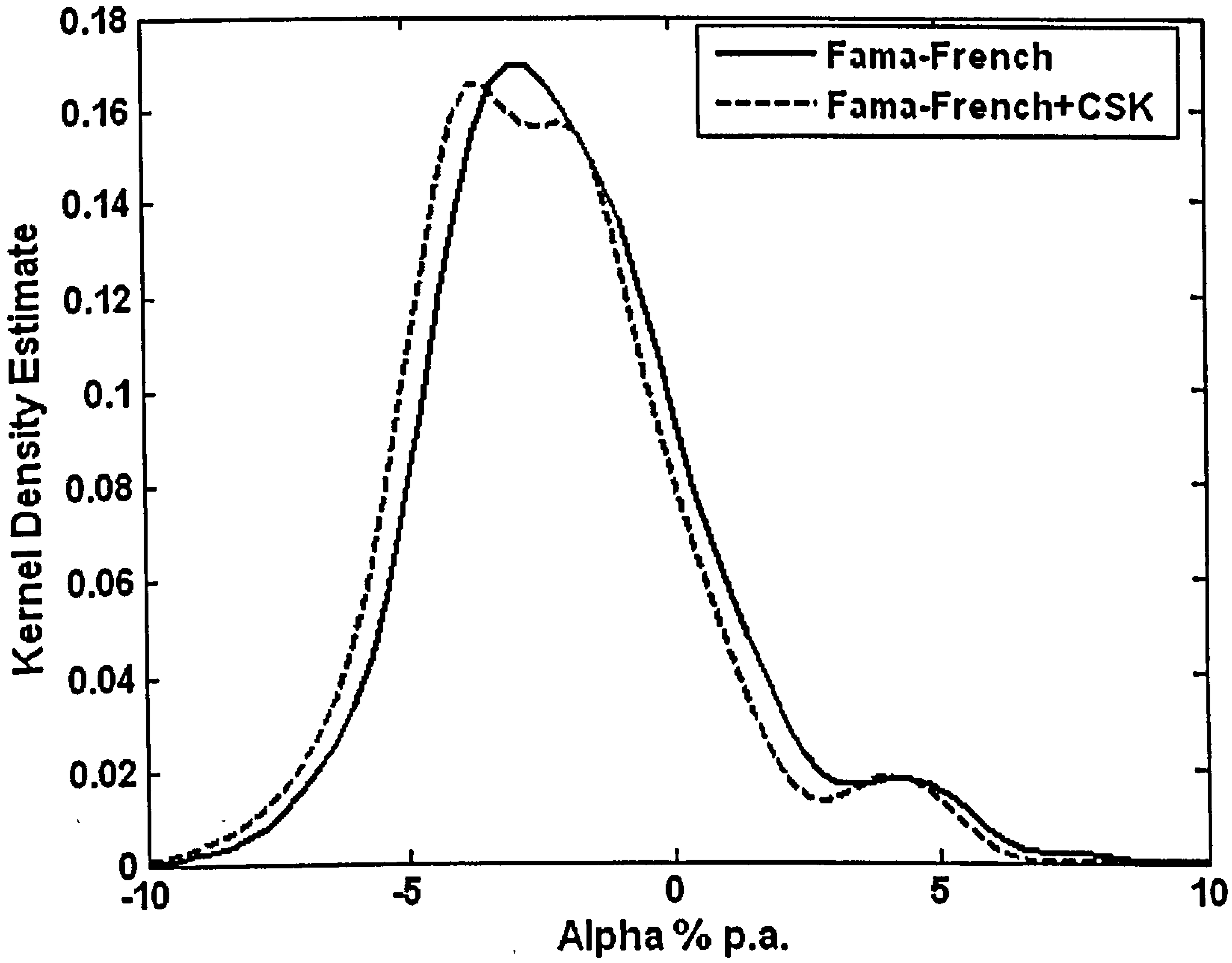
This Figure plots the densities of the estimated factor loadings ($\widehat{c}$) on the coskewness strategy from three asset pricing models. The solid curve shows the density of the estimated factor loadings ($\widehat{c}$) from the Harvey-Siddique regression (4.15) for each trust. The dotted curve shows the corresponding density of the estimated coskewness factor loadings derived from the augmented Fama-French model, given in (4.19). The dashed curve shows the density of the estimated coskewness factor loadings derived from the augmented Carhart model, which is defined in (4.20). The estimates of these loadings refer to the period January 1991-December 2005.

Figure 4.6: Jensen alphas and coskewness factor loadings



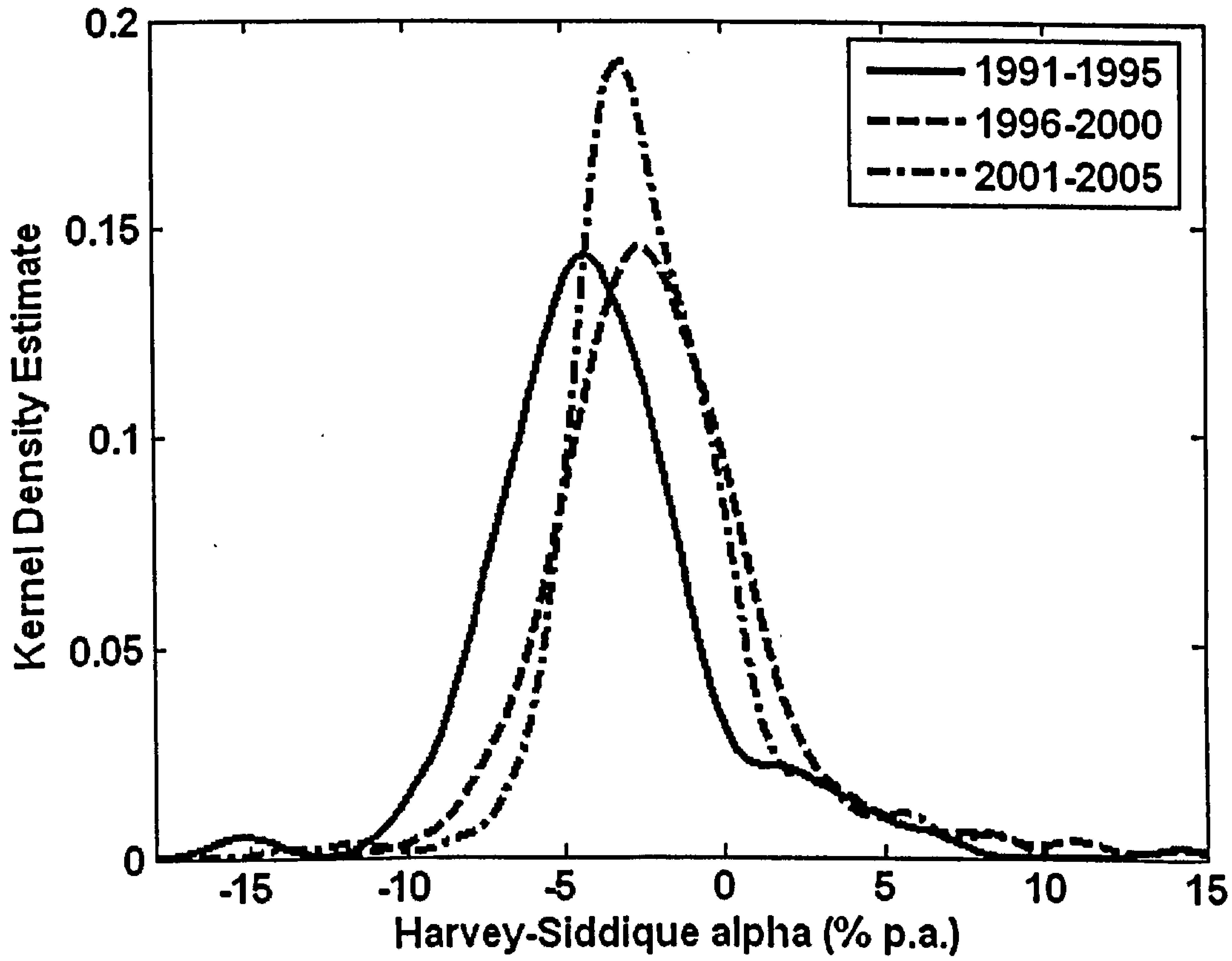
This Figure presents the scatterplot of the Jensen alpha estimates from regression (4.12) versus the coefficient estimates ($\widehat{c}$) on the coskewness strategy from regression (4.15) for each trust for the period January 1991- December 2005. It also plots the fitted values from a standard least squares regression involving these variables.

Figure 4.7: Fama-French and Fama-French augmented alphas



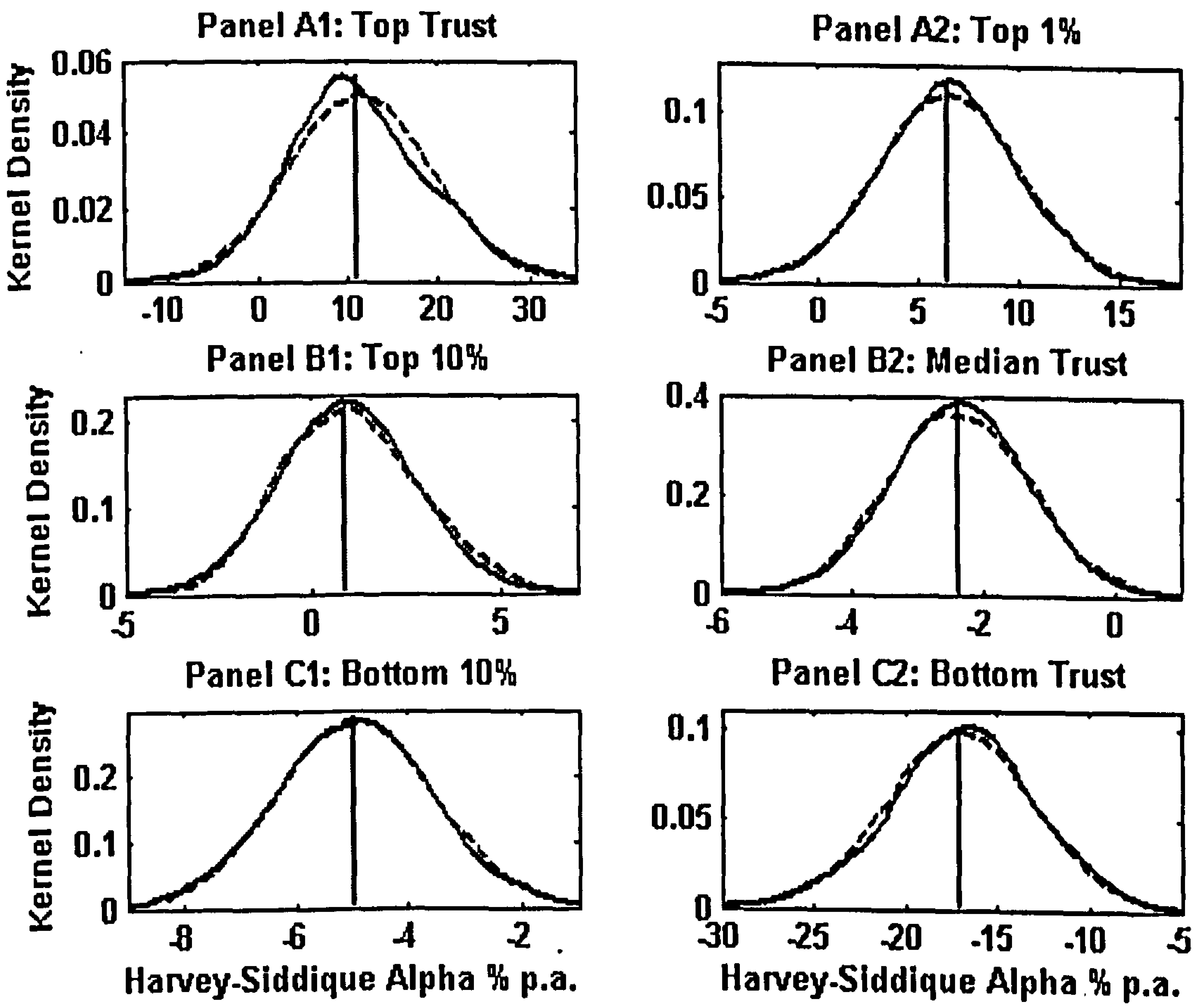
This Figure shows the distribution of the trusts' estimated alphas derived from the Fama-French asset pricing model, which is given in (4.18) (solid curve) as well as the corresponding distribution of the alphas derived from the augmented Fama-French model, which is given in (4.19) (dashed curve). These alpha estimates of the trusts refer to the period January 1991- December 2005.

Figure 4.8: Subperiod Harvey-Siddique alphas



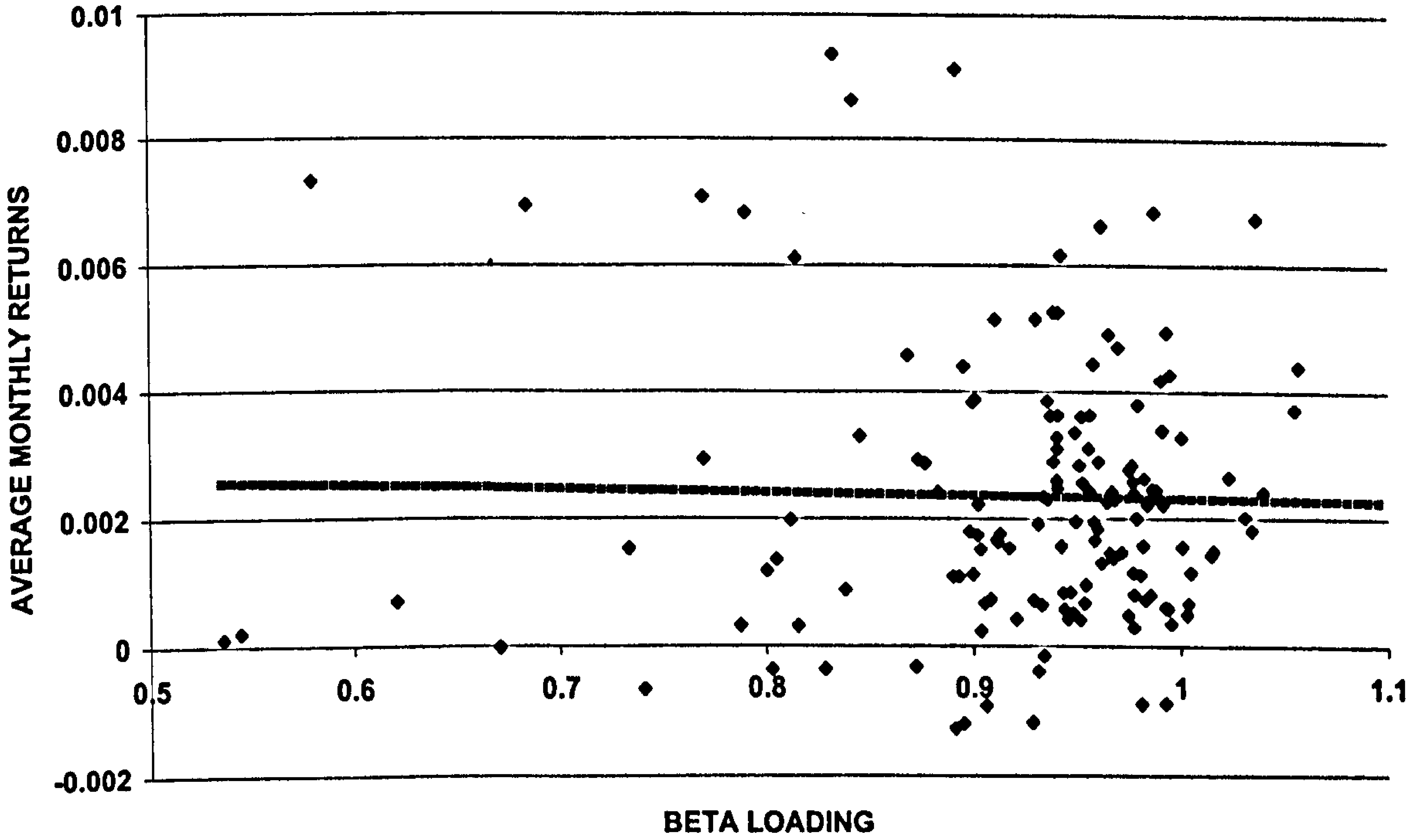
This Figure shows the distribution of the trusts' Harvey-Siddique alphas for the periods: January 1991- December 1995 (solid line), January 1996- December 2000 (dashed line) and January 2001- December 2005 (dash-dotted line).

Figure 4.9: Bootstrapped Harvey-Siddique alphas



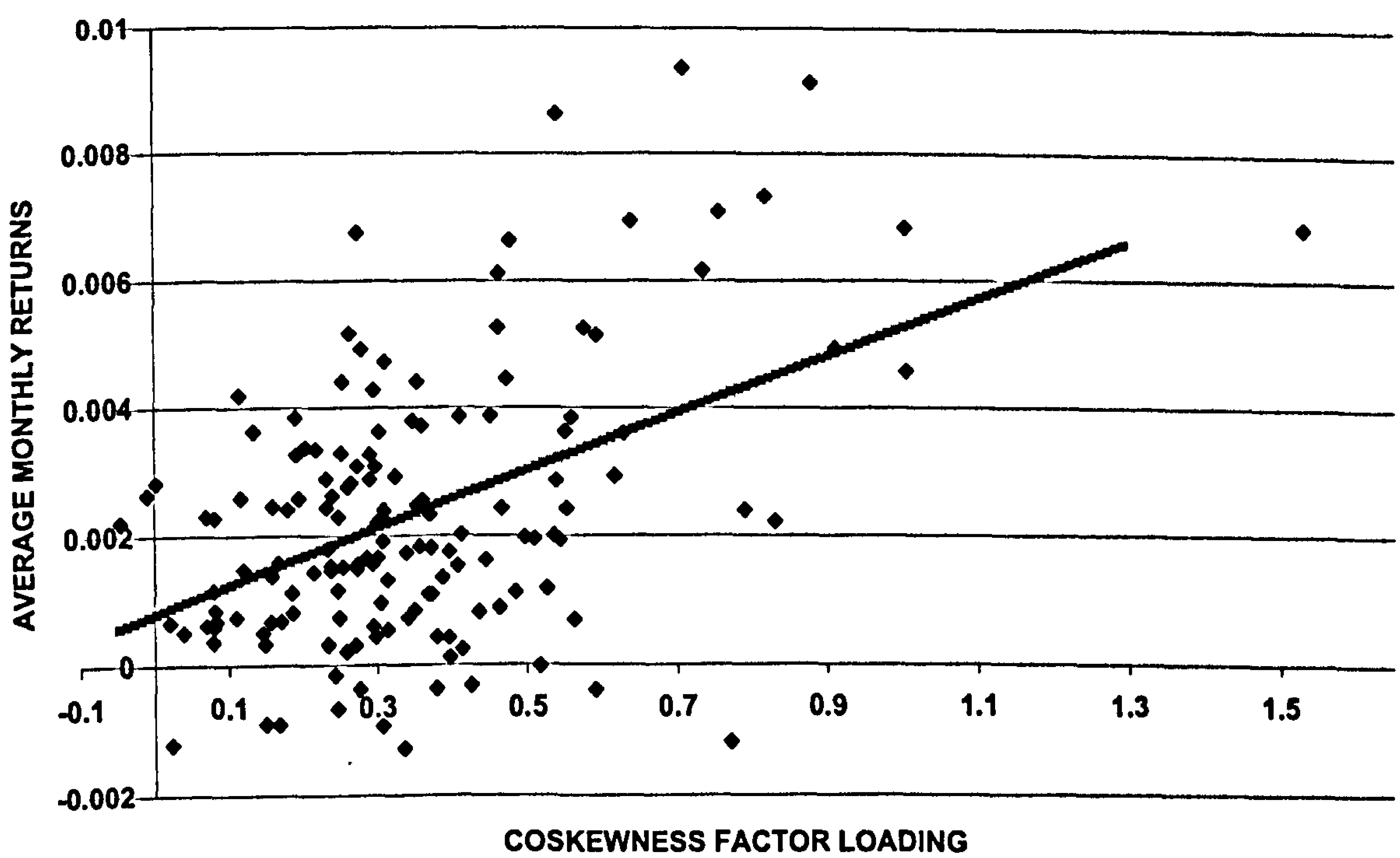
This Figure compares the density of the bootstrapped Harvey-Siddique alphas (solid curve), derived through the methodology discussed in Section 4.6.3, with the corresponding alphas' density under the assumption of normality for the trusts' residuals (dashed curve). These two curves generate alternative confidence intervals around the actual estimate of the Harvey-Siddique alpha that is represented by the vertical line. The Figure plots these confidence intervals for six trusts ranked according to their Harvey-Siddique alpha estimate for the period January 1991- December 2005.

Figure 4.10: The failure of the CAPM



This Figure presents the scatterplot of the average monthly excess returns of 148 funds versus their beta estimates derived by either the CAPM or the Harvey-Siddique model in (4.15), for the period January 1991- December 2005. It also plots the fitted values from a standard least squares regression involving these variables.

Figure 4.11: Average monthly excess returns and coskewness risk loadings



This Figure presents the scatterplot of the average monthly excess returns of 148 funds versus their coefficient estimates ($\widehat{c}$) on the coskewness risk factor from regression (4.15) for the period January 1991- December 2005. It also plots the fitted values from a standard least squares regression involving these variables.

Chapter 5

Conclusions

The present thesis examined two central issues in financial theory, optimal portfolio choice and investment performance evaluation, when some of the restrictive assumptions of the traditional framework of analysis are relaxed. In particular, we firstly derived the optimal portfolio choice for an investor who has a long-term horizon and faces a set of underlying risk factors that vary stochastically through time and affect the investment opportunity set. The predictability of future stock and bond returns as well as the stochastic volatility that these returns exhibit can be modelled using this framework. The main conclusion of our study is that the optimal portfolio choice crucially depends on the horizon of the investor. A significant hedging demand may arise, obliging the investor to deviate from the standard mean-variance portfolio choice, suggested by Markowitz (1952) within a static framework. The hedging motives may be so important that the hedging demand may dominate the total demand for the risky asset. These results show that the risky assets incorporate an extra value apart from their premia and their diversification role; they can be used as hedging instruments by risk-averse, long-term investors, who seek to minimize the volatility of their wealth's path through time.

The previous issues were also re-examined in a bond portfolio setting, that has been relatively neglected in the dynamic asset allocation literature. In particular, the failure of the expectations hypothesis and the current developments in the term structure literature motivate significant timing and horizon effects in bond portfolio management. The time-variation in the bond premia obliges long-term investors to hold significant hedging positions. The use of a macro-finance term structure model enabled us to provide an economic interpretation for the formation of optimal portfolios and to examine how shifts in the macroeconomy should affect the investors' decisions. This framework also allowed us to introduce real bonds in the asset space and to examine their role for portfolio choice, documenting their significant hedging role. Most importantly, we can rigorously confirm the popular advice that an infinitely risk-averse, long-term investor who seeks to maximize his utility over real wealth at a terminal date should hold a real zero-coupon bond with maturity that is equal to his horizon.

The interesting results derived in Chapters 2 and 3, are essentially the first small steps in a wide avenue of future research. Abandoning the restrictive static framework and the unrealistic assumption of i.i.d. returns, researchers are motivated to examine a series of topics related to optimal asset allocation. A crucial issue that needs to be considered is the modelling assumptions used for the risky assets' returns and the dynamics of the underlying risk factors. Chapter 2 showed that this is an absolutely important input for any dynamic asset allocation study. The plethora of cases and assumptions, that yield different and, sometimes, contradicting conclusions create a significant obstacle that we need to overcome. Emprical asset pricing studies should focus on the properties of the asset returns' dynamics and the characteristics that a series of predictive financial and macroeconomic variables actually exhibit. Financial econometrics should also help us by providing the appropriate tools for the estimation of continuous-time models in the presence of latent variables. Even though modelling and estimating jump processes has attracted little attention in the literature, it is an extremely important issue that will play a significant role in the future dynamic asset allocation studies.

An issue related to the estimation of the asset returns' and risk factors' dynamics is parameter uncertainty. The vast majority of the dynamic asset allocation studies make use of the point estimates of the various parameters, neglecting the statistical significance of these estimates. This issue becomes even more important if we take into account the fact that in the presence of near-unit root processes, such as the dividend yield and the interest rate, statistical inference becomes notoriously unreliable. An obviously more realistic and more prudential approach is to incorporate this parameter uncertainty into the optimal portfolio choice problem. Brennan (1998), Barberis (2000), Xia (2001) and Garlappi et al. (2007) provide examples of this approach, showing to us how differently the long-term investor should behave if he adopts a Bayesian framework, forming priors about the parameter values employed in the investment decision and updating appropriately these priors as he collects information. It is expected that future research will focus on this issue.

In the light of the recent developments in experimental studies and utility theory, an important limitation of the framework we adopted is the use of a smooth, differentiable power utility function. As we mentioned in the Introduction and in Chapter 4, there are a series of reasons why investors are expected to exhibit loss aversion. In this case, it is necessary to employ kinked utility functions for the study of optimal portfolio choice. Nevertheless, there are very few academic studies that employ these utility functions. The main reason is that the dynamic portfolio choice problems become notoriously difficult to solve if we use non-differentiable utility functions. Berkelaar et al. (2004) provide significant insights for the horizon effects in portfolio choice under loss aversion, but they need to assume that the market price of risk and the interest rate are not stochastically time-varying, in order to derive analytical solutions. More work is definitely necessary in order to examine the hedging incentives of multiperiod loss-averse investors.

Another important limitation of Chapters 2 and 3 is that they are purely partial equilibrium studies. In other words, the multiperiod investor we examine is a price-taker, who cannot affect asset prices, taking the parameter estimates as exogenously given. Having understood the impact of horizon effects on the individual portfolio choice, the next step is to examine the impact of horizon effects on asset prices, risk premia, volatility and interest rates within a general equilibrium framework. There have been a series of studies examining the interplay between agents with heterogeneous preferences (see Basak and Shapiro, 2001) or heterogeneous beliefs (see Basak, 2005), but again very few studies allow for stochastically time-varying risk factors. A unifying approach that takes into account investors' heterogeneity in the presence of a stochastically time-varying opportunity set seems very appealing and it is expected to provide a more realistic description of the financial markets.

The second issue that this thesis dealt with is how the deviation from the mean-variance world and the assumption of normality for asset returns modifies asset pricing and, as a result, the performance measurement of investments. In particular, Chapter 4 examined whether a negative coskewness risk factor is priced in the UK stock market and employed the Harvey-Siddique asset pricing model to evaluate a sample of UK equity unit trusts. Apart from showing that the negative coskewness risk bears a relatively high premium, this chapter also outlined the incentives that the use of mean-variance measures generates for fund managers.

The main argument is that, if managers are evaluated through mean-variance measures, which do not take into account the negative coskewness risk, then they are motivated to load this type of risk in their portfolios, in order to reap the corresponding premium and present it as genuine outperformance. This issue becomes particularly important if we take into account the fact that the majority of the individual investors exhibit loss aversion that makes them even more averse to this type of risk. Furthermore, this mismatch between the investors' preferences and the managers' incentives becomes even more serious, due to the delegated nature of the fund management procedure. There is essentially a misalignment of interests between the principal (trust shareholder) and his agent (manager), obscuring the ex post state verification of the outcome of the fund management service. Interestingly, we show that the coskewness risk loading can explain the cross-sectional average excess returns of the trusts as well as their Jensen alphas.

The results of Chapter 4 call for a more rigorous study of the empirical asset pricing models. These models have been extensively used in the literature for corporate finance studies, for the performance evaluation of investment strategies as well as for risk management purposes. The use of purely mean-variance models, such as the CAPM, has been clearly rejected, due to the evidence for the outperformance of size, value and momentum strategies. Academic research keeps discovering other factors that outperform the CAPM, while the industry keeps exploiting these "anomalies", presenting them as genuine outperformance. The use of multi-factor models, such as the Fama-French 3-factor model and the Carhart 4-factor model is a positive step forward, since the excess returns offered by these strategies are regarded to be risk premia rather than "free lunches". Nevertheless, these extra factors are strategy-specific and they are not based on any rigorous theoretical argument. Surprisingly, the Harvey-Siddique model, that is based on rigorous utility theory, has not yet attracted the attention of researchers.

The CAPM was very attractive because it managed to synopsize the various types of risk in a single statistical characteristic, the covariance of an asset's returns with the returns of its benchmark. The Harvey-Siddique model makes use of the same idea, trying to capture the risks that the CAPM does not, using another statistical characteristic, namely the coskewness of an asset's returns with the returns of its benchmark. The parsimony and generality of this model are attractive features, making it preferable to the strategy-specific asset pricing models. Most importantly, this model builds on the recent developments in utility theory, experimental studies as well as financial econometrics. The fact that investors are not averse to volatility but mainly to downside risk, exhibiting first-order risk aversion or loss aversion, the argument that negative news have an asymmetric impact on asset returns and the ample evidence that asset returns exhibit negative skewness, deviating from symmetry and normality, guarantee the robust foundations and the superiority of the Harvey-Siddique model. Consequently, this is a very promising model and it should be employed and tested for different markets and different asset classes in order to derive robust conclusions.

On the other hand, it is very important to identify what are the fundamental characteristics that give rise to this negative coskewness risk. Unfortunately, modern financial theory tends to neglect the impact of the companies' fundamentals on stock prices. Abandoning the mean-variance and i.i.d. world, we should explore more systematically the significant information that it is contained in the companies' financial statements, the impact that macroeconomic fluctuations have on asset prices as well as how these are affected by institutional characteristics and market microstructure effects. There are already a series of important studies along these lines, but their results are analyzed from a very narrow perspective. Negative coskewness may be able to provide the unifying concept for all these issues.

More specifically, the degree of leverage and financial flexibility that a company exhibits is a characteristic that is expected to affect its returns in different economic conditions (see Gamba and Triantis, 2008). Highly leveraged and financially constrained companies are the first to be badly hit in a credit "crunch". The dividend yield and the total payout ratio of a company is another issue that we should seriously take into account (see Boudoukh et al., 2007). Companies with high dividend yields and payout ratios are expected to offer "protection" to investors during a crisis. Moreover, the quality of corporate governance is expected to have an impact on asset returns too (see Gompers et al., 2003). Good governance could act as a "shield" in a negative economic environment, while badly run companies would be the first to go bankrupt (see Mitton, 2002).

Furthermore, the extent to which companies have underfunded pension funds is an extemely important issue, because such a deficit is ultimately a liability (see Franzoni and Marin, 2006). Companies with huge deficits in their pension funds are expected to be fiercely hit by negative shocks that further increase these deficits. With respect to microstructure effects, illiquidity may be another source of risk related to the negative coskewness risk. Amihud (2002) and Acharya and Pedersen (2005) show that illiquidity risk is priced in financial markets. Investors tend to avoid assets that are thinly traded, exhibit wide bid-ask spreads and high transaction costs because they are afraid that they will have to sell these assets at a high discount, suffering major losses, especially in periods of "flight to quality" and "flight to safety".

On the aggregate level, the negative coskewness risk premium is expected to vary through time. Therefore, since this is a systematic risk factor that is priced in financial markets, it is important to examine whether it is predictable as well as what are the underlying macroeconomic and financial variables that can help us determine its price. Until now, a series of variables have been used to proxy negative future economic conditions. The most common choices are the default spread between corporate bonds and the term spread between short and long-term bond maturities. It is argued that monetary policy significantly affects the price of the coskewness risk. Most of the company-specific characteristics, that we previously referred to, are affected by the credit conditions and the cost of money, hence monetary variables may determine the coskewness premium in a predictable way. Market paticipants' expectations may also help us to determine the price of this type of risk, hence opinion surveys and information derived by options and futures markets could be employed in a unified approach.

This thesis was an initial step in the complex and fascinating world of investment decisions, asset prices and the fund management industry. It served as an introduction to the concepts and the problems that arise once we abandon the unrealistic static, mean-variance world. Every step beyond the textbook treatment of these topics is like diving in an ocean. At the same time, however, it provides a strong motivation for future research into the topics that we outlined in this last section. Interesting times lie ahead!

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