Vector Autoregressive Analysis of Macroeconomic Policy

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Abstract

This thesis focuses on the use of Vector Autoregressive (VAR) models in macroeconomic policy analysis. The first chapter discusses a novel approach (PVAR approach) to identify from the VAR a structural model of the economy suitable for the solution of dynamic programming problems. As well dealing dynamic optimisation problems involving a single decision maker, the chapter also shows how to assess optimal policy from either the Markov perfect or the cooperative or the Stackelberg solutions to VAR dynamic games models in which several decision makers compete over the control of the economy. The second chapter compares the empirical performances of the PVAR approach against those of the standard identification methodology by assessing optimal interest rate rules using US data for the period 1960-2003. The empirical results show that feedback rules predicted under the PVAR approach are smoother than those calculated under the standard approach and welfare losses are considerably overstated by the standard approach, regardless of the specification of the objective function. The final chapter proposes an index of the fiscal stance based on the comparison of the targeted debt-GDP ratio with the short run forecast of the debt-GDP ratio of a VAR model formed from the government budget constraint. In contrast to the backward-looking assessments of the literature on fiscal sustainability, the new index is entirely forward-looking and can be used to construct time series of the fiscal stance for the evaluation of fiscal policy over time. The index is computed empirically to assess the fiscal stance of the US, the UK and Germany over the last 25 years.

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Preface

This thesis collects three papers on the use of VAR models for macroeconomic policy analysis. The first paper, entitled *Optimal control of vector autoregressive models*, assesses the use of Vector Autoregressive (VAR) models for the computation of optimal macroeconomic policy rules. I first focus on the problem of identifying from an unrestricted VAR model a stochastic dynamic system in which the endogenous state vector is conditioned upon an exogenous control vector. I describe an identification methodology recently proposed by Wickens (2003) which yields a dynamic system where the state vector equations are conditioned upon both current and lagged values of the exogenous control variables. This technique is evaluated against the conventional identification approach, which conditions changes in the state vector equations only upon the lagged value of the control variables. The new approach is compared with the standard one to evaluate its implications for both the assessment of optimal feedback rules and welfare analysis. In the second part of the paper, I look at the issue of identification within the context of VAR models involving more than one decision maker. Recent works on the computation of optimal policy rules based on VAR models - Sack, (2000), Martin and Salmon (1999), Monti (2003) - focus upon single decision makers control problems. However, macroeconomic models are often characterised by several decision makers simultaneously competing over the control of the economy. Hence, I describe how VAR models can be solved to compute optimal policy rules from either the Markov perfect or the cooperative or the Stackelberg solutions to vector autoregressive dynamic games models.

The second paper, entitled *Assessing optimal monetary policy through VAR models*, analyses both theoretically and empirically a new approach - PVAR method - to formulate optimal policy based on a quadratic intertemporal welfare function and a dynamic constraint extracted from a VAR model of the economy. The paper argues that the VAR under control should not be derived simply by replacing the VAR equations for the policy instruments by an optimal control rule because this alters the stochastic structure of the state vector equations of the VAR and gives a state space representation of the dynamic constraint in which state variables can only respond to lagged values of the control. Instead, one should first transform the VAR in order to condition the non-policy variables on the current value of the policy instruments, then using the resulting sub-system as the dynamic constraint, and finally construct the VAR under control by combining this sub-system with the resulting optimal policy rule. In this way the original stochastic structure of the state vector equations of the VAR is retained. In addition, under the PVAR approach the non-policy variables in the dynamic constraint are conditioned upon both current and lagged values of the control, hence giving a representation of the macroeconomic framework more suitable for policy analysis. In comparing the two approaches, the paper explains the theoretical advantages of the PVAR over the standard method and illustrates its empirical outcomes by examining the formulation of optimal monetary policy rules using US data for the period 1960-2003. I find that feedback rules predicted under the PVAR approach are smoother than those calculated under the standard approach and welfare losses are considerably overstated by the standard approach, regardless of the specification of the objective function.

The final paper, entitled *Measuring the fiscal stance*, proposes an index of the fiscal stance suitable for practical use in short-term policy making. The index is based on a comparison of a target level of the debt-GDP ratio for a given finite horizon with a forecast of the debt-GDP ratio based on a VAR formed from the government budget constraint. This approach to measuring the fiscal stance is different from the literature on fiscal sustainability. We emphasise the importance of having a forward-looking measure of the fiscal stance for the immediate future rather than a test for fiscal sustainability that is backward-looking, or based just on past behaviour which may not be closely related to the current fiscal position. We also describe a bootstrapping methodology that can be easily implemented to attach confidence bands to the index in order to evaluate the statistical significance of the policy prescriptions arising from the empirical computation of the index. We use our methodology to construct a time series of the indices of the fiscal stances of the US, the UK and Germany over the last 25 or more years. We find that both the US and UK fiscal stances have deteriorated considerably since 2000 and Germany's has been steadily deteriorating since unification in 1989, and worsened again on joining EMU. Out-of-sample projections of the index also show that the fiscal stance is expected to improve in the United States and the United Kingdom, while further worsening in Germany.

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Author's declaration

The paper in chapter 3 has been co-written with Michael Wickens and I was mainly in charge of the empirical part. Both chapters 2 and 3 are currently under revision for publication. Errors are mine only.

Chapter 1
Optimal control of vector autoregressive models

Abstract

This paper assesses the use of Vector Autoregressive (VAR) models for the computation of optimal macroeconomic policy rules. I first focus on the problem of identifying from an unrestricted VAR model a stochastic dynamic system in which the endogenous state vector is conditioned upon an exogenous control vector. I describe an identification methodology recently proposed by Wickens (2003) which yields a dynamic system where the state vector equations are conditioned upon both current and lagged values of the exogenous control variables. This technique is evaluated against the conventional identification approach, which conditions changes in the state vector equations only upon the lagged value of the control variables. In the second part of the paper, I look at the issue of identification within the context of VAR models involving more than one decision maker. Recent works on the computation of optimal policy rules based on VAR models - Sack (2000), Martin and Salmon (1999), Monti (2003) - focus upon single decision makers control problems. However, macroeconomic models are often characterised by several decision makers simultaneously competing over the control of the economy. I describe how VAR models can be solved to compute optimal policy rules from either the Markov perfect or the cooperative or the Stackelberg solutions to vector autoregressive dynamic games models.

1.1 Introduction

Vector Autoregressive (VAR) models have been extensively used in macroeconomic policy analysis to assess how the future dynamic of key macroeconomic variables, such as inflation, output and unemployment, is likely to be affected by either an unanticipated policy intervention (policy shock) or a change in the systematic component of the policy (change in the policy rule).

A reduced form VAR model cannot be used as it is to carry out any of these two types of policy analysis, since a set of minimum restrictions has to be imposed on the parameters of the reduced form system to identify the underlying structural model of the economy. Full identification of the structural VAR model is necessary to simultaneously assess policy shocks and changes in the policy rule. However, this paper is only interested in the second type of policy analysis: the change in the VAR policy rule. In particular, the paper deals with the issue of identifying from an unrestricted VAR model a stochastic dynamic linear model suitable for the computation and evaluation of macroeconomic policy rules. In this case full identification is unnecessary, since it is sufficient to identify a so-called semi-structural VAR model, through a partial identification scheme (see, Sack (2000)).

The standard (partial) identification approach is based upon a block Cholesky decomposition of the VAR residuals so that policy changes can affect the state variables only with a lag. In practice, this implies that the dynamic constraint faced by policy makers corresponds with the state vector equations estimated from the reduced form VAR. These equations are then used in isolation from the rest of the VAR to measure feedback rules and carry out welfare analysis. The paper describes an alternative identification approach proposed by Michael Wickens in his 2003 lectures on VAR modelling at the IMF. In contrast to the standard methodology, the new Policy VAR (PVAR) identification approach is based upon a block Cholesky transformation of the reduced form residuals which yields a dynamic system of state variables conditioned upon both current and lagged values of the policy instruments, rather than only the lagged values as in the standard approach. The paper shows that the PVAR approach offers a state space representation of the state vector equations which encompasses the representation computed under the standard approach. Although it does not overcome the Lucas critique (1976) as it applies to VAR models in the analysis of policy changes, the new identification approach does not compound the problem by imposing unnecessary timing restrictions - in the state vector equations - on the interaction between policy and non-policy variables.

The application of the PVAR approach is presented in the context of reduced form VAR models including instrument variables of a single decision maker, in order to outline the implications of the new methodology for the computation of optimal policy rules and welfare analysis. This framework is consistent with conventional macroeconomic policy analyses, which are based upon the computation of decision rules from the solution to stochastic optimal linear regulator problems within VAR models including the policy instruments of a single decision maker, without taking into account other policy makers' reaction functions. I argue that this omission is bound to yield both misspecified policy rules and misleading welfare measurements, whenever there are several decision makers competing over the control of the economy. This is because the certainty equivalence principle, which applies to this type of dynamic optimisation problems, implies that the optimal rule can only minimise the volatility of the deterministic part of the state vector equations, leaving unchanged that of the stochastic part. Since the computation of optimal policy rule depends upon the deterministic part of the stochastic constraint, the omission of variables affecting the deterministic component of the state vector leads to misspecified feedback rules and mismeasurement of the social welfare loss.

The second part of the paper extends then the application of the PVAR identification approach to compute optimal policy rules in the presence of more than one decision maker. In this case, the solution depends upon the number of policy makers and type of strategic interaction among them. I show how optimal policy rules can be computed from VAR dynamic game models with either Markov perfect or cooperative or Stackelberg solutions.

The discussion is articulated in four sections, following this introduction. Section 2 considers the computation of policy rules under the method of dynamic programming when optimisation is carried out with respect to a dynamic linear constraint relating over time state variables, policy instruments and uncorrelated stochastic disturbances. In principle, the constraint can have two different types of timing structures, as policy actions can affect the state vector either instantaneously or with some delay. I show that the choice of the constraint affects (i) the timing structure and the coefficients of the optimal policy rule and, (ii) the measurement of the welfare outcome arising from the system under control.

Section 3 discusses the identification problem in the context of reduced form VAR models, and compares the standard and the PVAR identification approaches. I show how the dynamic constraint obtained from both approaches can be used to compute optimal policy rules and discuss the welfare implications of each approach.

Section 4 extends the computation of optimal policy rules to dynamic optimisation problems involving more than one policy maker. The solution to these problems requires, first of all, the reduced form VAR model to include equations for objective and instrument variables of all decision makers competing over the control of the economy. The identification of the stochastic dynamic constraint then depends upon the type of competition among decision makers. The paper considers three alternative solutions. The Nash solution occurs when a specific objective function is entirely delegated to a single policy maker which optimises with respect to its own policy instruments by taking other decision makers' reaction functions as given. The cooperative solution is obtained when all policy makers pool together their policy instruments to minimise a common objective function. Finally, the Stackelberg solution is examined to compute optimal feedback rules when decision makers compete over the control of the economy as in a leader-followers framework. Section 5 concludes by summarising the main results and highlights the future potential of this research.

1.2 Linear quadratic dynamic programming

1.2.1 Computing optimal policy rules

The computation of an optimal feedback rule involves the solution of an optimisation problem in which a decision maker's loss function is minimised with respect to the available policy instruments and a dynamic constraint relating intertemporally instruments to objective variables. If the decision maker's utility function is approximated with a quadratic form and the intertemporal constraint with a stochastic linear function, the solution to the dynamic optimisation problem can be easily computed by employing the method of dynamic programming.[1] When solved by this method, optimal linear regulator problems are normally referred to as linear quadratic dynamic programming problems, since dynamic programming is used to optimise a quadratic return function, subject to a system of stochastic linear difference equations.[2]

Without loss of generality, the intertemporal quadratic cost function of the policy maker can be written as:

$$L = E_t \sum_{s=0}^{\infty} \beta^{t+s} \left[(\mathbf{y}_{t+s} - \bar{\mathbf{y}})' \, \mathbf{W} \, (\mathbf{y}_{t+s} - \bar{\mathbf{y}}) \right], \tag{1.1}$$

where E_t denotes mathematical expectations conditioned on time t information, $\mathbf{y}'_{t+s} = \left[\begin{array}{cccccc} \mathbf{z}_{1t+s} & \cdots & \mathbf{z}_{1t+s-p} & \mathbf{z}_{2t+s} & \cdots & \mathbf{z}_{2t+s-p} \end{array} \right]'$, $\mathbf{z}_1$ is a vector of endogenous variables, $\mathbf{z}_2$ is a vector of policy instruments, $\bar{\mathbf{y}}$ is a target vector and $\mathbf{W}$ is a symmetric positive semidefinite matrix of policy weights.[3]

The value function $V(\mathbf{y}_t)$, i.e. the minimum value at time t of the welfare loss under the infinite sequence of controls $\{\mathbf{z}_{2t+s}\}_{s=0}^{\infty}$, is given by

$$V(\mathbf{y}_t) = \min_{\{\mathbf{z}_{2t+s}\}_{s=0}^{\infty}} E_t \sum_{s=0}^{\infty} \beta^{t+s} \left[(\mathbf{y}_{t+s} - \bar{\mathbf{y}})' \, \mathbf{W} \, (\mathbf{y}_{t+s} - \bar{\mathbf{y}}) \right].$$

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[1] For detailed discussions about the use of dynamic programming in dynamic optimisation problems, see Bertsekas (2000) and Chow (1976). For applications of dynamic programming to economic problems, in particular to optimal linear regulator problems, see Ljungqvist and Sargent (2004).

[2] The Lagrange technique can also be employed as a method of dynamic optimisation as illustrated in section 5.4. For a description of dynamic economics optimisation by the Lagrange method, see Chow (1997), and, for comparison of dynamic programming and Lagrange method in the assessment of optimal policy, see Chow (1976).

[3] The vector $\mathbf{y}_{t+s}$ can include current and lagged values of both state and instrument variables. The representation of equation (1.1) is sufficiently general to eventually include first differences of the objective function's arguments by imposing ad hoc identities in the off-diagonal elements of the matrix $\mathbf{W}$.

Since L is a quadratic form in $\mathbf{y}_{t+s}-\bar{\mathbf{y}}$, the general structure of the value function can be guessed to be a linear combination of a quadratic, a linear and a constant term, which can be represented as:

$$V(\mathbf{y}_t) = \mathbf{y}_t'\mathbf{P}\mathbf{y}_t - 2\mathbf{y}_t'\mathbf{p} + \mathbf{d}, \tag{1.2}$$

where $\mathbf{P}$ is a positive semidefinite symmetric matrix of coefficients having the same order of $\mathbf{W}$, whereas $\mathbf{p}$ and $\mathbf{d}$ are vectors of coefficients compatible with $\mathbf{y}_{t+s}$.

The Bellman (1957) principle can then be applied to write the value function in the recursive form

$$V(\mathbf{y}_t) = \min_{\{\mathbf{z}_{2t+s}\}_{s=0}^{\infty}} (\mathbf{y}_t-\bar{\mathbf{y}})'\,\mathbf{W}\,(\mathbf{y}_t-\bar{\mathbf{y}}) + \beta E_t\left[V(\mathbf{y}_{t+1})\right] \tag{1.3}$$

and substitution of (1.2) into the value function of equation (1.3) gives the following recursive Bellman (1957) equation:

$$\mathbf{y}_t'\mathbf{P}\mathbf{y}_t - 2\mathbf{y}_t'\mathbf{p} + \mathbf{d} = \min_{\{\mathbf{z}_{2t+s}\}_{s=0}^{\infty}} (\mathbf{y}_t-\bar{\mathbf{y}})'\,\mathbf{W}\,(\mathbf{y}_t-\bar{\mathbf{y}}) + \beta E_t\left[\mathbf{y}_{t+1}'\mathbf{P}\mathbf{y}_{t+1} - 2\mathbf{y}_{t+1}'\mathbf{p} + \mathbf{d}\right]. \tag{1.4}$$

The above dynamic optimisation problem is entirely recursive in the vector $\mathbf{y}_t$ and determination of its solution requires the computation of $\mathbf{y}_{t+1}$ from a dynamic system that relates $\mathbf{y}_{t+1}$ to both the previous period value $\mathbf{y}_t$ and the policy instrument $\mathbf{z}_2$. The constraint can include either $\mathbf{z}_{2t+1}$ or $\mathbf{z}_{2t}$, according to whether the control vector affects the state vector instantaneously or with a lag. Therefore, the stochastic dynamic linear constraint can be written as either

$$\mathbf{y}_{t+1} = \mathbf{a} + \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{z}_{2t+1} + \mathbf{u}_{t+1} \tag{1.5}$$

or

$$\mathbf{y}_{t+1} = \mathbf{c} + \mathbf{C}\mathbf{y}_t + \mathbf{D}\mathbf{z}_{2t} + \mathbf{v}_{t+1}, \tag{1.6}$$

where the vectors $\mathbf{a}$ and $\mathbf{c}$, and the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ includes fixed coefficients, while $\mathbf{u}_t$ and $\mathbf{v}_t$ are vectors of stochastic terms with $E_t[\mathbf{u}_{t+1}] = E_t[\mathbf{v}_{t+1}] = 0$, $E_t[\mathbf{u}_{t+1}\mathbf{u}'_{t+1}] = \mathbf{U}_{t+1}$ and $E_t[\mathbf{v}_{t+1}\mathbf{v}'_{t+1}] = \mathbf{V}_{t+1}$. Moreover, the disturbances in both equations (1.5) and (1.6) are uncorrelated with the instrument vector, that is $E_t[\mathbf{u}_{t+1}\mathbf{z}'_{2t+1}] = \mathbf{0}$ and $E_t[\mathbf{v}_{t+1}\mathbf{z}'_{2t}] = \mathbf{0}$.

The next two sub-sections describe the computation of optimal feedback rules under the two specifications of the dynamic constraint.

First case: $\mathbf{y}_{t+1} = \mathbf{a} + \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{z}_{2t+1} + \mathbf{u}_{t+1}$

The dynamic constraint in equation (1.5) can be employed to take forecasts of $\mathbf{y}_{t+1}$, which can then be substituted into the recursive equation (1.4) so that, after taking expectations, the value function can be written as:

$$V(\mathbf{y}_t) = \left[\begin{array}{c} \mathbf{y}'_t\mathbf{W}\mathbf{y}_t + \beta(\mathbf{a} + \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{z}_{2t+1})'\mathbf{P}(\mathbf{a} + \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{z}_{2t+1}) \\ -2\beta(\mathbf{a} + \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{z}_{2t+1})'\mathbf{p} + \beta E(\mathbf{u}'_{t+1}\mathbf{P}\mathbf{u}_{t+1}) + \beta\mathbf{d} \end{array} \right]. \tag{1.7}$$

Differentiation of $V(\mathbf{y}_t)$ with respect to the control variable $\mathbf{z}_{2t+1}$ gives the first order condition:

$$\frac{\partial V(\mathbf{y}_t)}{\partial \mathbf{z}_{2t+1}} = \beta\mathbf{B}'\mathbf{P}\mathbf{a} + \beta\mathbf{B}'\mathbf{P}\mathbf{A}\mathbf{y}_t + \beta\mathbf{B}'\mathbf{P}\mathbf{B}\mathbf{z}_{2t+1} - \beta\mathbf{B}'\mathbf{p}.$$

Setting the first order condition equal to zero and solving for $\mathbf{z}_{2t+1}$ yields the optimal feedback rule

$$\mathbf{z}_{2t+1} = -\left(\beta\mathbf{B}'\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}'\left[\beta\mathbf{P}\mathbf{a}-\beta\mathbf{p}\right] - \left(\beta\mathbf{B}'\mathbf{P}\mathbf{B}\right)^{-1}\beta\mathbf{B}'\mathbf{P}\mathbf{A}\mathbf{y}_t,$$

which can be written in the compact form:

$$\begin{aligned} \mathbf{z}_{2t} &= \mathbf{f} + \mathbf{F}\mathbf{y}_{t-1}, \\ \mathbf{f} &= -\left(\mathbf{B}'\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}'\left(\mathbf{P}\mathbf{a}-\mathbf{p}\right), \\ \mathbf{F} &= -\left(\mathbf{B}'\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{P}\mathbf{A}. \end{aligned} \tag{1.8}$$

Equation (1.8) shows that under the dynamic constraint (1.5) the optimal feedback rule is a linear function relating the current value of the policy instrument to a constant term and the previous period value of the state vector. A noticeable feature of the solution in equation (1.8) is the absence of stochastic disturbances in the feedback rule. This result is known as certainty equivalence principle, which implies that the solution to a stochastic discounted linear optimal regulator problem is the same deterministic linear feedback rule which would be obtained from the solution to the corresponding deterministic problem.[4]

Computation of the coefficients in the optimal feedback rules requires evaluation of the matrix $\mathbf{P}$ and the vector $\mathbf{p}$. To this end, after substituting the optimal feedback rule into the value function (1.7), the minimum expected welfare cost $V(\mathbf{y}_t)$ can be written as:

$$\mathbf{y}_t'\mathbf{P}\mathbf{y}_t - 2\mathbf{y}_t'\mathbf{p} + \mathbf{d} = \begin{bmatrix} (\mathbf{y}_t-\bar{\mathbf{y}})'\mathbf{W}(\mathbf{y}_t-\bar{\mathbf{y}}) + \\ +\beta\left[\mathbf{a}+\mathbf{A}\mathbf{y}_t+\mathbf{B}(\mathbf{f}+\mathbf{F}\mathbf{y}_t)\right]'\mathbf{P}\left[\mathbf{a}+\mathbf{A}\mathbf{y}_t+\mathbf{B}(\mathbf{f}+\mathbf{F}\mathbf{y}_t)\right] \\ -2\beta\left[\mathbf{a}+\mathbf{A}\mathbf{y}_t+\mathbf{B}(\mathbf{f}+\mathbf{F}\mathbf{y}_t)\right]'\mathbf{p} + \beta E\left(\mathbf{u}_{t+1}'\mathbf{P}\mathbf{u}_{t+1}\right) + \beta\mathbf{d} \end{bmatrix}.$$

After multiplying through and rearranging, the above expression becomes:

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[4] See Brainard (1967) and Ljungqvist and Sargent (2004).

$$\mathbf{y}_t'\mathbf{P}\mathbf{y}_t - 2\mathbf{y}_t'\mathbf{p} + \mathbf{d} = \begin{bmatrix} \mathbf{y}_t'\mathbf{W}\mathbf{y}_t + \beta\mathbf{y}_t'(\mathbf{A}+\mathbf{BF})\mathbf{P}(\mathbf{A}+\mathbf{BF})\mathbf{y}_t + \\ -2\mathbf{y}_t'\mathbf{W}\bar{\mathbf{y}} + 2\beta\mathbf{y}_t'(\mathbf{A}+\mathbf{BF})'(\mathbf{Pa}+\mathbf{PBf}-\mathbf{p}) + \\ +\bar{\mathbf{y}}'\mathbf{W}\bar{\mathbf{y}} + \beta(\mathbf{a}+\mathbf{Bf})'\mathbf{P}(\mathbf{a}+\mathbf{Bf}) + \\ -2\beta(\mathbf{a}+\mathbf{Bf})'\mathbf{p} + \beta E\left(\mathbf{u}_{t+1}'\mathbf{P}\mathbf{u}_{t+1}\right) + \beta\mathbf{d} \end{bmatrix}.$$

Since the term

$$\begin{aligned} (\mathbf{A}+\mathbf{BF})'\mathbf{PBf} &= \mathbf{A}'\mathbf{PBf} + \mathbf{F}'\mathbf{B}'\mathbf{PBf} \\ &= \mathbf{A}'\mathbf{PBf} - \mathbf{A}'\mathbf{PB}(\mathbf{B}'\mathbf{PB})^{-1}\mathbf{B}'\mathbf{PBf} = 0, \end{aligned}$$

the minimum value function becomes:

$$\mathbf{y}_t'\mathbf{P}\mathbf{y}_t - 2\mathbf{y}_t'\mathbf{p} + \mathbf{d} = \begin{bmatrix} \mathbf{y}_t'\mathbf{W}\mathbf{y}_t + \beta\mathbf{y}_t'(\mathbf{A}+\mathbf{BF})\mathbf{P}(\mathbf{A}+\mathbf{BF})\mathbf{y}_t + \\ -2\mathbf{y}_t'\mathbf{W}\bar{\mathbf{y}} + 2\beta\mathbf{y}_t'(\mathbf{A}+\mathbf{BF})'(\mathbf{Pa}-\mathbf{p}) + \\ +\bar{\mathbf{y}}'\mathbf{W}\bar{\mathbf{y}} + \beta(\mathbf{a}+\mathbf{Bf})'\mathbf{P}(\mathbf{a}+\mathbf{Bf}) + \\ -2\beta(\mathbf{a}+\mathbf{Bf})'\mathbf{p} + \beta E\left(\mathbf{u}_{t+1}'\mathbf{P}\mathbf{u}_{t+1}\right) + \beta\mathbf{d} \end{bmatrix}. \quad (1.9)$$

The matrix $\mathbf{P}$ can be calculated by equating the coefficients corresponding to the quadratic terms on both sides of equation (1.9) as:

$$\mathbf{P} = \mathbf{W} + \beta\mathbf{A}'\mathbf{PA} + 2\beta\mathbf{A}'\mathbf{PBF} + \beta\mathbf{F}'\mathbf{B}'\mathbf{PBF}.$$

Substitution of $\mathbf{F} = -(\mathbf{B}'\mathbf{PB})^{-1}\mathbf{B}'\mathbf{PA}$ into the above gives

$$\mathbf{P} = \mathbf{W} + \beta\mathbf{A}'\mathbf{PA} - \beta\mathbf{A}'\mathbf{PB}(\mathbf{B}'\mathbf{PB})^{-1}\mathbf{B}'\mathbf{PA}, \quad (1.10)$$

which is the algebraic matrix Riccati equation for $\mathbf{P}$. Equation (1.10) is highly nonlinear and can be computed numerically by writing:

$$\mathbf{P}_{t+1} = \mathbf{W} + \beta\mathbf{A}'\mathbf{P}_t\mathbf{A} - \beta\mathbf{A}'\mathbf{P}_t\mathbf{B}(\mathbf{B}'\mathbf{P}_t\mathbf{B})^{-1}\mathbf{B}'\mathbf{P}_t\mathbf{A},$$

setting $\mathbf{P}_t \simeq \mathbf{0}$ as initial value and then solving for $\mathbf{P}_{t+1}$. The process is iterated until convergence to the stable value $\mathbf{P}_t = \mathbf{P}_{t+1} = \mathbf{P}$.

The vector $\mathbf{p}$ can be calculated by equating the coefficients corresponding to the linear terms on both sides of equation (1.9), that is:

$$-2\mathbf{p} = -\,2\mathbf{W}\bar{\mathbf{y}} + 2\beta\left(\mathbf{A}+\mathbf{B}\mathbf{F}\right)'\left(\mathbf{P}\mathbf{a}-\mathbf{p}\right),$$

and then solving for $\mathbf{p}$ as:

$$\mathbf{p}=\left[\mathbf{I}-\beta\left(\mathbf{A}+\mathbf{B}\mathbf{F}\right)'\right]^{-1}\left[\mathbf{W}\bar{\mathbf{y}}-\beta\left(\mathbf{A}+\mathbf{B}\mathbf{F}\right)'\mathbf{P}\mathbf{a}\right],$$

where $\mathbf{P}$ is the previously computed stable solution to the algebraic matrix Riccati equation.

Second case: $\mathbf{y}_{t+1} = \mathbf{c} + \mathbf{C}\mathbf{y}_t + \mathbf{D}\mathbf{z}_{2t} + \mathbf{v}_{t+1}$

The computation of optimal policy rules under the dynamic constraint in equation (1.6) requires rewriting the value function as:

$$\tilde{V}\left(\mathbf{y}_t\right) = \mathbf{y}_t'\tilde{\mathbf{P}}\mathbf{y}_t - 2\mathbf{y}_t'\tilde{\mathbf{p}} + \mathbf{d}$$

and the recursive Bellman equation as:

$$\tilde{V}\left(\mathbf{y}_t\right) = \min_{\{\mathbf{z}_{2t+s}\}_{s=0}^{\infty}} \left(\mathbf{y}_t-\bar{\mathbf{y}}\right)'\mathbf{W}\left(\mathbf{y}_t-\bar{\mathbf{y}}\right) + \beta E_t\left[\mathbf{y}_{t+1}'\tilde{\mathbf{P}}\mathbf{y}_{t+1} - 2\mathbf{y}_{t+1}'\tilde{\mathbf{p}} + \mathbf{d}\right]. \tag{1.11}$$

After substituting forecasts of $\mathbf{y}_{t+1}$ from equation (1.6) in the Bellman equation (1.11) and taking expectations, the value function can be written as:

$$\tilde{V}\left(\mathbf{y}_t\right) = \left[\begin{array}{c} \mathbf{y}_t'\mathbf{W}\mathbf{y}_t + \beta\left(\mathbf{c}+\mathbf{C}\mathbf{y}_t+\mathbf{D}\mathbf{z}_{2t}\right)'\tilde{\mathbf{P}}\left(\mathbf{c}+\mathbf{C}\mathbf{y}_t+\mathbf{D}\mathbf{z}_{2t}\right) \\ -2\beta\left(\mathbf{c}+\mathbf{C}\mathbf{y}_t+\mathbf{D}\mathbf{z}_{2t}\right)'\tilde{\mathbf{p}} + \beta E\left(\mathbf{v}_{t+1}'\tilde{\mathbf{P}}\mathbf{v}_{t+1}\right) + \beta\mathbf{d} \end{array}\right].$$

Differentiation of $\tilde{V}\left(\mathbf{y}_t\right)$ with respect to $\mathbf{z}_{2t}$ gives:

$$\frac{\partial\tilde{V}\left(\mathbf{y}_t\right)}{\partial\mathbf{z}_{2t}} = \beta\mathbf{D}'\tilde{\mathbf{P}}\mathbf{c} + \beta\mathbf{D}'\tilde{\mathbf{P}}\mathbf{C}\mathbf{y}_t + \beta\mathbf{D}'\tilde{\mathbf{P}}\mathbf{c} + \beta\mathbf{D}'\tilde{\mathbf{P}}\mathbf{C}\mathbf{y}_t + 2\beta\mathbf{D}'\tilde{\mathbf{P}}\mathbf{D}\mathbf{z}_{2t+1} - 2\beta\mathbf{D}'\tilde{\mathbf{p}}$$

Setting the first order condition equal to zero and solving for $\mathbf{z}_{2t}$, the optimal feedback rule is written as:

$$\mathbf{z}_{2t} = -\left(\mathbf{D}'\tilde{\mathbf{P}}\mathbf{D}\right)^{-1}\mathbf{D}'\left[\tilde{\mathbf{P}}\mathbf{c} - \tilde{\mathbf{p}}\right] - \left(\mathbf{D}'\tilde{\mathbf{P}}\mathbf{D}\right)^{-1}\mathbf{D}'\tilde{\mathbf{P}}\mathbf{C}\mathbf{y}_t,$$

which can be expressed in the compact form:

$$\begin{aligned} \mathbf{z}_{2t} &= \tilde{\mathbf{f}} + \tilde{\mathbf{F}}\mathbf{y}_t, \\ \tilde{\mathbf{f}} &= -\left(\mathbf{D}'\tilde{\mathbf{P}}\mathbf{D}\right)^{-1}\mathbf{D}'\left(\tilde{\mathbf{P}}\mathbf{c} - \tilde{\mathbf{p}}\right), \\ \tilde{\mathbf{F}} &= -\left(\mathbf{D}'\tilde{\mathbf{P}}\mathbf{D}\right)^{-1}\mathbf{D}'\tilde{\mathbf{P}}\mathbf{C}. \end{aligned} \tag{1.12}$$

As for the previous case, the matrix $\tilde{\mathbf{P}}$ is calculated from the stable solution to the recursive algebraic matrix Riccati equation:

$$\tilde{\mathbf{P}}_{t+1}= \mathbf{W}+\beta\mathbf{C}'\tilde{\mathbf{P}}_t\mathbf{C}-\beta\mathbf{C}'\tilde{\mathbf{P}}_t\mathbf{D}\left(\mathbf{D}'\tilde{\mathbf{P}}_t\mathbf{D}\right)^{-1}\mathbf{D}'\tilde{\mathbf{P}}_t\mathbf{C},$$

while the vector $\tilde{\mathbf{p}}$ is computed from:

$$\tilde{\mathbf{p}}= \left[\mathbf{I} - \beta\left(\mathbf{C} + \mathbf{D}\tilde{\mathbf{F}}\right)'\right]^{-1}\left[\mathbf{W}\bar{\mathbf{y}} - \beta\left(\mathbf{C} + \mathbf{D}\tilde{\mathbf{F}}\right)'\tilde{\mathbf{P}}\mathbf{c}\right].$$

Equation (1.12) shows that under the dynamic constraint (1.6), the optimal feedback rule is a deterministic linear function relating the current value of the policy instrument to a constant term and the current value of the state vector. Comparison of the feedback rules in equations (1.8) and (1.12) shows that the choice of the timing structure of the constraint affects both the magnitude of the response coefficients and the timing structure of the feedback rule.

1.2.2 Welfare analysis

The intertemporal loss function in equation (1.1) involves minimisation of the expected volatility of the state vector $\mathbf{y}_{t+s}$ around the vector of targets $\bar{\mathbf{y}}$. To assess the minimum welfare cost associated with policy rules in equations (1.8) and (1.12), it is convenient to decompose the loss function in equation (1.1) as:

$$
\begin{aligned}
L &= L^D + L^S \\
L^D &= E_t \sum_{s=0}^{\infty} \beta^{t+s} \left[(\mathbf{y} - \bar{\mathbf{y}})' \mathbf{W} (\mathbf{y} - \bar{\mathbf{y}}) \right], \\
L^S &= E_t \sum_{s=0}^{\infty} \beta^{t+s} \left[(\mathbf{y}_{t+s} - \mathbf{y})' \mathbf{W} (\mathbf{y}_{t+s} - \mathbf{y}) \right],
\end{aligned}
$$

where $E\left[\mathbf{y}_t\right] = \mathbf{y}$ is the unconditional expectation of the state vector. The term L^D is the deterministic component of the welfare loss, measuring the social cost due to deviation of the private sector's rational expectation equilibrium from the medium term policy targets. The term L^S is the stochastic component of the welfare loss and takes into account the social cost arising from the presence of random disturbances in the dynamic constraint.

Basic matrix algebra can be employed to write L^S as:

$$
\begin{aligned}
L^S &= E_t \sum_{s=0}^{\infty} \beta^{t+s} tr \left[(\mathbf{y}_{t+s} - \mathbf{y})' \mathbf{W} (\mathbf{y}_{t+s} - \mathbf{y}) \right], \\
&= E_t \sum_{s=0}^{\infty} \beta^{t+s} tr \left[\mathbf{W} (\mathbf{y}_{t+s} - \mathbf{y}) (\mathbf{y}_{t+s} - \mathbf{y})' \right], \\
&= \sum_{s=0}^{\infty} \beta^{t+s} tr \mathbf{W} E_t \left[(\mathbf{y}_{t+s} - \mathbf{y}) (\mathbf{y}_{t+s} - \mathbf{y})' \right],
\end{aligned}
$$

which shows that the stochastic component of the welfare cost is a weighted linear combination of the variance of the variables included in the state vector $\mathbf{y}_{t+s}$. Therefore, the computation of L^S requires the assessment in every period $t + s$ of the covariance matrix of the state vector, which can be written as:

$$\mathbf{\Gamma}_{t+s} = E_t\left[(\mathbf{y}_{t+s}-\mathbf{y})(\mathbf{y}_{t+s}-\mathbf{y})'\right]. \tag{1.13}$$

The value $\mathbf{\Gamma}_{t+s}$ under control – and therefore of the minimum stochastic social welfare cost – depends upon the specification of the dynamic equation for $\mathbf{y}_t$ and its computation is discussed in the next two subsections.[5]

First case: $\mathbf{y}_{t+1} = \mathbf{a} + \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{z}_{2t+1} + \mathbf{u}_{t+1}$

Substitution of the optimal rule (1.8) into the dynamic constraint in equation (1.5) gives:

$$\begin{aligned}\mathbf{y}_{t+1} &= \mathbf{a} + \mathbf{A}\mathbf{y}_t + \mathbf{B}(\mathbf{f} + \mathbf{F}\mathbf{y}_t) + \mathbf{u}_{t+1} \\ &= \mathbf{a} + \mathbf{B}\mathbf{f} + (\mathbf{A} + \mathbf{B}\mathbf{F}) + \mathbf{u}_{t+1}\end{aligned}$$

Setting $\mathbf{r} = \mathbf{a} + \mathbf{B}\mathbf{f}$ and $\mathbf{R} = \mathbf{A} + \mathbf{B}\mathbf{F}$, the system under control can be written as:

$$\mathbf{y}_t = \mathbf{r} + \mathbf{R}\mathbf{y}_{t-1} + \mathbf{u}_t. \tag{1.14}$$

Computation of the covariance matrix of the state vector is carried out by first post-multiplying equation (1.14) by $\mathbf{y}'_t$ and taking expectations, which gives:

$$\begin{aligned}E_t\left[\mathbf{y}_t\mathbf{y}'_t\right] &= \mathbf{\Gamma}_t = E_t\left[\mathbf{r}\mathbf{y}'_t\right] + E_t\left[\mathbf{R}\mathbf{y}_{t-1}\mathbf{y}'_t\right] + E_t\left[\mathbf{u}_t\mathbf{y}'_t\right], \\ &= \mathbf{R}E_t\left[\mathbf{y}_{t-1}\mathbf{y}'_t\right] + E_t\left[\mathbf{u}_t\mathbf{u}'_t\right].\end{aligned}$$

[5] For a detailed description of the solution techniques employed in the next two subsections, see Hamilton (1994).

Moreover, transposing equation (1.14), pre-multiplying by $\mathbf{y}_{t-1}$ and taking expectations yields:

$$\begin{aligned} E_t\left[\mathbf{y}_{t-1}\mathbf{y}_t'\right] &= E_t\left[\mathbf{y}_{t-1}\mathbf{r}'\right] + E_t\left[\mathbf{y}_{t-1}\mathbf{y}_{t-1}'\mathbf{R}'\right] + E_t\left[\mathbf{y}_{t-1}\mathbf{u}_t'\right], \\ &= E_t\left[\mathbf{y}_{t-1}\mathbf{y}_{t-1}'\right]\mathbf{R}'. \end{aligned}$$

When $s = 0$, substitution of the above result in equation (1.13) determines the covariance matrix of the state vector under control as:

$$\mathbf{\Gamma}_t = \mathbf{R}\mathbf{\Gamma}_{t-1}\mathbf{R}' + \mathbf{U}_t. \tag{1.15}$$

The above expression is nonlinear and can be solved numerically by setting an initial value $\mathbf{\Gamma}_0 \simeq \mathbf{0}$ and computing the next period value $\mathbf{\Gamma}_1$, which can then be used to compute $\mathbf{\Gamma}_2$ and so on. This recursive procedure can be iterated until convergence to the stable solution $\mathbf{\Gamma}_t = \mathbf{\Gamma}_{t+1} = \mathbf{\Gamma}$, which is used to assess the variance of $\mathbf{y}_t$ in equation (1.13).

Alternatively, an exact solution can be computed by assuming $\mathbf{y}_t$ to be a covariance stationary process, i.e. $\mathbf{\Gamma}_t = \mathbf{\Gamma}_{t-1} = \mathbf{\Gamma}$, so that equation (1.15) becomes:

$$\mathbf{\Gamma} = \mathbf{R}\mathbf{\Gamma}\mathbf{R}' + \mathbf{U}.$$

The vec operator can then be employed to rewrite the above expression as:

$$\begin{aligned} vec\mathbf{\Gamma} &= vec\mathbf{R}\mathbf{\Gamma}\mathbf{R}' + vec\mathbf{U}, \\ &= (\mathbf{R}\otimes\mathbf{R})\, vec\mathbf{\Gamma} + vec\mathbf{U}, \end{aligned}$$

where $\otimes$ indicates the Kronecker product. Solving for $vec\mathbf{\Gamma}$ gives:

$$vec\mathbf{\Gamma} = \left[\mathbf{I} - (\mathbf{R}\otimes\mathbf{R})\right]^{-1} vec\mathbf{U}, \tag{1.16}$$

which is the vec representation of the covariance matrix of the state vector under the optimal policy rule.

Both equations (1.15) and (1.16) are equivalent formulas for the assessment of the variance of the vector $\mathbf{y}_t$ under the optimal rule. Since the variance of the vector $\mathbf{y}_t$ before implementation of the optimal rule can be calculated from the the dynamic constraint (1.5) as

$$\mathbf{\Gamma}_t^* = \mathbf{A}\mathbf{\Gamma}_{t-1}\mathbf{A}' + \mathbf{U}_t,$$

the minimum variance in equation (1.15) can then be subtracted from $\mathbf{\Gamma}_t^*$ to assess the welfare gain from the implementation of the optimal rule:

$$\mathbf{\Gamma}_t^* - \mathbf{\Gamma}_t = \mathbf{A}\mathbf{\Gamma}_{t-1}\mathbf{A}' - \mathbf{R}\mathbf{\Gamma}_{t-1}\mathbf{R}'. \tag{1.17}$$

The result in equation (1.17) shows that the welfare gain is independent from the volatility of the disturbances in equation (1.5). This is a natural implications of the certainty equivalence principle, which states that the implementation of the deterministic optimal rule in the stochastic linear dynamic constraint minimises the volatility of the deterministic part of the state vector, leaving unchanged the variance of the disturbances.

Under the stable solution in equation (1.15), the stochastic component of the welfare loss function can be computed as:

$$L^S = \sum_{s=0}^{\infty} \beta^{t+s} tr\mathbf{W}\mathbf{\Gamma} = \frac{1}{1-\beta} tr\mathbf{W}\mathbf{\Gamma}.$$

If the covariance matrix of the state vector is computed in the vec form as in equation (1.16), – by employing the properties $tr\,(\mathbf{AB}) = tr\,(\mathbf{A}'\mathbf{B}) = (vec\mathbf{A})\,(vec\mathbf{B})$ – L^S can be measured as:

$$L^S = \frac{1}{1-\beta}(vec\mathbf{W})'\left[\mathbf{I} - (\mathbf{R}\otimes\mathbf{R})\right]^{-1} vec\mathbf{U}.$$

Second case: $\mathbf{y}_{t+1} = \mathbf{c} + \mathbf{C}\mathbf{y}_t + \mathbf{D}\mathbf{z}_{2t} + \mathbf{v}_{t+1}$

The welfare analysis can be repeated under the constraint in equation (1.6). Substitution of the optimal rule (1.12) into the constraint gives:

$$\mathbf{y}_{t+1} = \mathbf{c} + \mathbf{D}\tilde{\mathbf{f}} + \left(\mathbf{C} + \mathbf{D}\tilde{\mathbf{F}}\right)\mathbf{y}_t + \mathbf{v}_{t+1}.$$

Setting $\mathbf{q} = \mathbf{c} + \mathbf{D}\tilde{\mathbf{f}}$ and $\mathbf{Q} = \mathbf{C} + \mathbf{D}\tilde{\mathbf{F}}$, the system under control can be written as:

$$\mathbf{y}_{t+1} = \mathbf{q} + \mathbf{Q}\mathbf{y}_t + \mathbf{v}_{t+1}. \tag{1.18}$$

As for the previous case, the covariance matrix of $\mathbf{y}_t$ can be computed as:

$$\tilde{\mathbf{\Gamma}}_t = \mathbf{Q}\mathbf{\Gamma}_{t-1}\mathbf{Q}' + \mathbf{V}_t.$$

The stable solution $\tilde{\mathbf{\Gamma}}_t = \tilde{\mathbf{\Gamma}}_{t+1} = \tilde{\mathbf{\Gamma}}$ to the above equation can be either calculated numerically or in *vec* form as:

$$vec\tilde{\mathbf{\Gamma}} = \left[\mathbf{I} - (\mathbf{Q}\otimes\mathbf{Q})\right]^{-1} vec\mathbf{V}.$$

Moreover, the welfare gain from the implementation of the optimal rule (1.12) into the constraint (1.6) is given by:

$$\tilde{\mathbf{\Gamma}}_t^* - \tilde{\mathbf{\Gamma}}_t = \mathbf{C}\tilde{\mathbf{\Gamma}}_{t-1}\mathbf{C}' - \mathbf{Q}\tilde{\mathbf{\Gamma}}_{t-1}\mathbf{Q}'. \tag{1.19}$$

As for the previous case, the certainty equivalence principle implies that substitution of the optimal rule into the stochastic economy constraint minimises the volatility of the deterministic part of the dynamic equation for the state vector, leaving unchanged the variance of the disturbances. Comparison of the welfare gains in equations (1.17) and (1.19) shows that the choice of the timing structure of the constraint affects not only the dynamic structure of the optimal feedback rule, but also the implied measure of minimum welfare cost.

1.2.3 Assessment

We have shown that the interaction between state and control variables in the dynamic constraint used for the solution to linear quadratic dynamic programming problems affects the timing of the optimal policy rule and the magnitude of the optimal response coefficients. In addition, the dynamic specification of the state vector equations also alters the measurement of the welfare cost under the optimal policy.

The previous analysis discloses at least two issues related to the assessment of optimal policy rules from VAR models, which will be discussed in the rest of the paper.

The first issue concerns the identification of the structural model of the economy from the reduced form VAR. An important feature of optimal linear regulator problems described in the previous two sections is that, after replacing the instrument vector in the constraint with the optimal feedback rule, the dynamic equation for the state vector under control includes only endogenous variables. In fact, equations (1.14) and (1.18) give two VAR representations of the state vector, the first arising from the substitution of the feedback rule (1.8) in the constraint in (1.5) and the second from the substitution of the policy rule (1.12) into (1.6). Researchers can observe and estimate the reduced form VAR coefficients, but may not have a priori knowledge about the decision maker's utility function and the model used to predict future values of the state vector. On the other hand, any reduced form VAR model can be thought as the outcome of a dynamic linear quadratic problem. In this respect, equations (1.14) and (1.18) disclose a fundamental identification problem occurring when estimating a VAR model which includes state and instrument vectors, as the optimal policy rule followed by the decision maker cannot be inferred from the observation of the VAR model alone. In turn, this implies that an identification technique is required to transform the system of endogenous variables into a dynamic constraint suitable for the computation of optimal policy rules, which takes the form of either (1.5) or (1.6).

The second issue related to the assessment of optimal policy rules from VAR models concerns the specifications of the deterministic part of the structural model. The certainty equivalence principle implies that the optimal policy rule is deterministic and can only minimise the volatility of the deterministic part of the state vector in either (1.14) or (1.18). Therefore, welfare measures are sensitive to the specification of the deterministic part of the VAR model, as this in turn defines the structure of the optimal policy rule. Stochastic disturbances in the reduced form VAR model capture the effect on the state vector of any variable other than those explicitly specified in the model, but – as shown in (1.17) and (1.19) – the optimal policy rule does not change the volatility of the stochastic disturbances. Therefore, omission of any variable relevant for the assessment of the optimal policy from VAR models is bound to result in misspecified policy rules and overstated welfare measures. Since traditional VAR analyses of optimal policy rules are based upon reduced form VAR models that include objective and instrument variables of a single decision maker, the main source of misspecification arises from omitting the responses to policy actions of any other decision maker competing over the control of the economy.[6]

The next section focuses on the issue of identification and compares two alternative approaches. Section 5 completes the discussion on the use of VAR model in optimal policy analysis by describing how optimal policy rules can be computed within a dynamic game VAR framework, which includes several decision makers engaging in strategic interaction over the control of the economy.

1.3 Vector Autoregressive models and optimal control

Since the seminal work of Sims (1980) VAR models have been widely used in macroeconomic policy analysis. A natural starting point of an empirical work consists in specifying and estimating a VAR model in which the state vector includes non-policy macroeconomic variables, some of which may be directly targeted by the policy maker, whereas the control vector includes policy variables, which measure the current policy stance.

In the unrestricted VAR both the state and the control vectors are treated as endogenous, whereas policy analysis requires conditioning the state vector equations upon an exogenous control vector which can be freely changed by the policy maker. The policy analysis may focus on assessing the effect of either unanticipated (policy shocks) or anticipated policy interventions (changes in the policy rule) on the future dynamic of the

[6] In particular, optimal monetary policy analysis focuses on the assessment of interest rate rules from VAR models that include objective variables targeted by the central bank, either directly or indirectly, as well as the short term policy interest rate. These models fail to include the effect of policy actions of other decision makers, such as for example fiscal authorities. See Sack (2000), Martin and Salmon (1999) and Goodhart (1999).

variables included in the state vector. In the former case, the estimated effect of policy changes on the future values of the state variables is assessed through the impulse response functions. In the latter case, the purpose of the analysis is to look at the likely pattern of the non-policy variables under rules different from that estimated from the unrestricted VAR and then assess the corresponding welfare implications.

Neither of the two types of analysis can be conducted directly on the reduced form VAR, and a specific set of restrictions is required to identify policy shocks. Full identification of the structural VAR allows computing both impulse response functions and assessing policy changes. This is however unnecessary if the aim of the analysis is only to look at changes in the systematic part of the policy reaction function estimated from the unrestricted VAR model, as only a partial identification scheme, which yields so-called semi-structural representations of the VAR model, is sufficient in this instance.

To illustrate these ideas analytically and set the notation, consider the reduced form VAR model:

$$\mathbf{z}_t = \mathbf{a} + \mathbf{A}(L)\,\mathbf{z}_t + \mathbf{e}_t, \tag{1.20}$$

where $\mathbf{z}_t$ is a set of endogenous macroeconomic variables, $\mathbf{a}$ is a vector of constant terms, L is the lag operator, $\mathbf{A}(L) = \sum_{i=1}^{p} \mathbf{A}_i L^i$ is the matrix describing the systematic adjustment of $\mathbf{z}_t$ in response to its own lags and $\mathbf{e}_t$ is a vector of serially independent reduced form disturbances, with $E[\mathbf{e}_t] = \mathbf{0}$ and $E[\mathbf{e}_t\mathbf{e}_t'] = \Sigma_{\mathbf{e}}$. After ordering state variables before control variables, the endogenous vector $\mathbf{z}_t$ is partitioned as:

$$\mathbf{z}_t' = \begin{bmatrix} \mathbf{z}_{1t} & \mathbf{z}_{2t} \end{bmatrix}',$$

where $\mathbf{z}_{1t}$ is the state vector and $\mathbf{z}_{2t}$ is a control vector. Therefore, the VAR model in equation (1.20) can be partitioned as:

$$\begin{bmatrix} \mathbf{z}_{1,t} \\ \mathbf{z}_{2,t} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{10} \\ \mathbf{a}_{20} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) \\ \mathbf{A}_{21}(L) & \mathbf{A}_{22}(L) \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1,t-1} \\ \mathbf{z}_{2,t-1} \end{bmatrix} + \begin{bmatrix} \mathbf{e}_{1t} \\ \mathbf{e}_{2t} \end{bmatrix}, \tag{1.21}$$

and the covariance matrix of the reduced form disturbances is accordingly partitioned as:

$$\mathbf{\Sigma}_{\mathbf{e}} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{bmatrix},$$

where $\mathbf{\Sigma}_{11}$ is covariance matrix of the disturbances of the state variables, $\mathbf{\Sigma}_{22}$ is the covariance matrix of the disturbances of the control variables, and $\mathbf{\Sigma}_{12}$ is the matrix collecting the covariances between the disturbances of state and control variables.[7]

If the policy maker has full control over the instruments included into the control vector, the policy equations estimated from the unrestricted VAR capture the reaction function of the policy maker over the sample period. In particular, the deterministic part of the control vector $\mathbf{z}_{2t}$ is the systematic policy response to changes in the lagged values of the variables included in the VAR model, while the innovations measure unanticipated changes in policy or policy shocks. Contemporaneously, the equations in the state vector $\mathbf{z}_{1t}$ captures the dynamic of the non-policy variables under the current policy stance.

The reduced form innovations can be thought as a linear combination of the uncorrelated structural shocks ϵ_t, with $E[\epsilon_t] = 0$ and $E[\epsilon_t \epsilon_t'] = \mathbf{\Sigma}_\epsilon$. Following the partition employed for the reduced form VAR model, the relationship between structural and reduced form innovations is written as:

[7] Since $\mathbf{\Sigma}_{\mathbf{e}}$ is symmetric, note that $\mathbf{\Sigma}_{12} = \mathbf{\Sigma}_{12}'$.

$$\begin{bmatrix} \mathbf{e}_{1t} \\ \mathbf{e}_{2t} \end{bmatrix} = \mathbf{B} \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}. \tag{1.22}$$

A well known issue in VAR models is that the structural system is not identifiable given the estimates of the reduced form model, unless an exact number of restrictions is imposed on the structural model. Several approaches have been proposed in the literature to fully identify a structural model from the reduced form VAR.

A first identification approach imposes a recursive structure on contemporaneous interactions between reduced form and structural innovations. Examples of this approach include Sims (1972), Bernanke (1986), Bernanke and Blinder (1992) and Bernanke and Mihov (1998). A well known shortcoming of this identification approach is that the shape of the impulse response function obtained from the structural model critically depends upon the type of triangular structure imposed on the matrix $\mathbf{B}$, namely whether policy variables are ordered before or after non-policy variables. An alternative identification approach exploits information coming from the economic theory to impose restrictions on the long run effects of the disturbances of the reduced form VAR. This approach has been used by Blanchard and Watson (1986), Blanchard and Quah (1989) and Gali (1992) among the many others. As pointed out by Blanchard and Quah (1989) one can retrieve at most only as many types of distinct shocks as there are variables. As a result, identification under this approach is difficult to achieve when the economy is likely to be affected by a number of shocks that is greater than the variables included in $\mathbf{z}_t$. More sophisticated identifications approaches focus on imposing sign restrictions on either the impulse response functions, as in Uhlig (1999), or on the cross-correlation between the impulse responses, as in Canova and De Nicolo' (2002).

Once the VAR has been fully identified by any of the approaches mentioned above, the structural model can be used not only to assess the response of non-policy variables to unanticipated changes in policy, but also the likely implications of changes in the policy rule estimated from the reduced form VAR.

As previously stated, full identification schemes are not necessary to assess the effects of changes in the systematic component of policy. In this case it is sufficient to employ only a partial identification scheme based upon a block Cholesky decomposition of the reduced form disturbances of state and control variables. The next two sub-sections describe two alternative identification approaches and the implications of each of them for the computation of the policy rule and welfare analysis. The first one is based upon the assumption that policy changes can affect the state vector only with a lag, as in equation (1.6). Examples of this approach in monetary policy analysis based on VAR models are in Sack (2000) and Monti (2003), among the others. I will refer to this as standard approach. An alternative approach is to assume the state vector responds to both current and lagged changes in the control vector, as in equation (1.5). This approach has been recently proposed by Wickens (2003) and, to the best of my knowledge, has no been applied in any empirical work so far.

1.3.1 Standard approach

The standard identification approach consists in estimating the unrestricted VAR model in equation (1.21) and then using the estimated equation for $\mathbf{z}_{1t}$ in isolation from the rest of the system to represent the dynamic constraint linking state and control variables. Given the objective function of the policy maker, the constraint can be used to derive an optimal policy rule under dynamic programming which would have a dynamic structure consistent with that in equation (1.12). More generally, the policy rule takes the form:

$$\mathbf{\Lambda z}_{1t} + \mathbf{z}_{2t} = \mathbf{a}_{20}^{*} + \mathbf{A}_{21}^{*}(L)\mathbf{z}_{1,t-1} + \mathbf{A}_{22}^{*}(L)\mathbf{z}_{2t-1}. \tag{1.23}$$

Welfare analysis is conducted under the standard approach by combining the optimal policy rule in (1.23) with the state vector equations in (1.21). This yields the VAR under control

$$\begin{aligned}\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{\Lambda} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{z}_{2t} \end{bmatrix} &= \begin{bmatrix} \mathbf{a}_{10} \\ \widetilde{\mathbf{a}}_{20} \end{bmatrix} + \\ &+ \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) \\ \widetilde{\mathbf{A}}_{21}(L) & \widetilde{\mathbf{A}}_{22}(L) \end{bmatrix}\begin{bmatrix} \mathbf{z}_{1,t-1} \\ \mathbf{z}_{2,t-1} \end{bmatrix} + \begin{bmatrix} \mathbf{e}_{1t} \\ 0 \end{bmatrix}\end{aligned} \tag{1.24}$$

or, re-writing this as a VAR by pre-multiplying by the inverse of the matrix of coefficients of $\mathbf{z}_t$,

$$\begin{aligned}\begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{z}_{2t} \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) \\ -\mathbf{\Lambda A}_{11}(L) + \widetilde{\mathbf{A}}_{21}(L) & -\mathbf{\Lambda A}_{12}(L) + \widetilde{\mathbf{A}}_{22}(L) \end{bmatrix}\begin{bmatrix} \mathbf{z}_{1,t-1} \\ \mathbf{z}_{2,t-1} \end{bmatrix} + \\ &+ \begin{bmatrix} \mathbf{a}_{10} \\ -\mathbf{\Lambda a}_{10} + \widetilde{\mathbf{a}}_{20} \end{bmatrix} + \begin{bmatrix} \mathbf{e}_{1t} \\ -\mathbf{\Lambda e}_{1t} \end{bmatrix}\end{aligned} \tag{1.25}$$

The VAR under control in the standard approach has the following characteristics. First the equations for the non-policy variables are unaffected. Second, the equations for the policy instruments in the VAR under control have a disturbance term that is perfectly correlated with disturbances in the policy equations.

1.3.2 PVAR approach

The PVAR approach suggests the use of a block Cholesky decomposition to the vector $\mathbf{e}_t$ in order to make the reduced form disturbances of the state vector a linear function of the corresponding disturbances of the control vector:

$$\mathbf{e}_{1,t} = \boldsymbol{\epsilon}_{1t} + \mathbf{G}\mathbf{e}_{2,t},$$

where $\boldsymbol{\epsilon}_{1t}$ is the component of $\mathbf{e}_{1,t}$ which is uncorrelated with $\mathbf{e}_{2,t}$. Therefore, the vector $\mathbf{e}_{,t}$ has the following state space representation:

$$\mathbf{e}_t = \begin{bmatrix} \mathbf{e}_{1,t} \\ \mathbf{e}_{2,t} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\epsilon}_{1t} \\ \mathbf{e}_{2,t} \end{bmatrix}.$$

The matrix $\boldsymbol{\Sigma}$ can be used to construct the matrix $\mathbf{G}$ as follows:

$$\mathbf{G} = \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1},$$

which gives the transformation matrix

$$\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix}.$$

The matrix $\mathbf{H}^{-1}$ can be used to map the original system onto a new one in which the disturbances associated with the state variables are uncorrelated with the disturbances associated with the controls:

$$\mathbf{H}^{-1}\mathbf{z}_t = \mathbf{H}^{-1}\mathbf{a} + \mathbf{H}^{-1}\mathbf{A}\,(L)\,\mathbf{z}_{t-1} + \mathbf{H}^{-1}\mathbf{e}_t,$$

which in turn yields the following system of linear equations:

$$\begin{aligned}
\mathbf{z}_{1,t} - \mathbf{G}\mathbf{z}_{2,t} &= [\mathbf{a}_{10} - \mathbf{G}\mathbf{a}_{20}] + [\mathbf{A}_{11}(L) - \mathbf{G}\mathbf{A}_{21}(L)]\,\mathbf{z}_{1,t-1} + \\
&\quad + [\mathbf{A}_{12}(L) - \mathbf{G}\mathbf{A}_{22}(L)]\,\mathbf{z}_{2,t-1} + \mathbf{e}_{1,t} - \mathbf{G}\mathbf{e}_{2,t} \\
\mathbf{z}_{2,t} &= \mathbf{a}_{20} + \mathbf{A}_{21}(L)\,\mathbf{z}_{1,t-1} + \mathbf{A}_{22}(L)\,\mathbf{z}_{2,t-1} + \mathbf{e}_{2,t}.
\end{aligned}$$

Since $\mathbf{e}_{1,t} - \mathbf{G}\mathbf{e}_{2,t} = \epsilon_{1t}$ – and $\mathbf{e}_{2,t}$ and ϵ_{1t} are uncorrelated – the above system is equivalent to

$$\begin{aligned}
\mathbf{z}_{1,t} &= \mathbf{a}_{10} + \mathbf{A}_{11}(L)\,\mathbf{z}_{1,t-1} + \mathbf{A}_{12}(L)\,\mathbf{z}_{2,t-1} + \epsilon_{1t} && (1.26) \\
&\quad - \mathbf{G}\left[\mathbf{a}_{20} + \mathbf{A}_{21}(L)\,\mathbf{z}_{1,t-1} - \mathbf{z}_{2,t} + \mathbf{A}_{22}(L)\,\mathbf{z}_{2,t-1}\right] \\
\mathbf{z}_{2,t} &= \mathbf{a}_{20} + \mathbf{A}_{21}(L)\,\mathbf{z}_{1,t-1} + \mathbf{A}_{22}(L)\,\mathbf{z}_{2,t-1} + \mathbf{e}_{2,t}, && (1.27)
\end{aligned}$$

Equation (1.26) can be separated from the VAR rule in equation (1.27) and employed as dynamic constraint for the evaluation of policy rules. The state vector $\mathbf{z}_{1,t}$ is an autoregressive function of both state and control variables and depends upon the current value of the control vector. The matrix $\mathbf{G}$ and the other autoregressive response coefficients in (1.26) can be computed from either the least squares or maximum likelihood estimates of the reduced form VAR model in equation (1.20).

If we write the optimal control rule as:

$$\mathbf{z}_{2t} = \mathbf{a}_{20}^{*} + \mathbf{A}_{21}^{*}(L)\mathbf{z}_{1,t-1} + \mathbf{A}_{22}^{*}(L)\mathbf{z}_{2t-1},$$

the VAR under control becomes

$$\begin{aligned}
\begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{z}_{2t} \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & \mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11}(L) - \mathbf{G}\mathbf{A}_{21}(L) & \mathbf{A}_{12}(L) - \mathbf{G}\mathbf{A}_{22}(L) \\ \mathbf{A}_{21}^{*}(L) & \mathbf{A}_{22}^{*}(L) \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1,t-1} \\ \mathbf{z}_{2,t-1} \end{bmatrix} + \\
&\quad + \begin{bmatrix} \mathbf{I} & \mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a}_{10} - \mathbf{G}\mathbf{a}_{20} \\ \mathbf{a}_{20}^{*} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & \mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}, \qquad (1.28)
\end{aligned}$$

or

$$\begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{z}_{2t} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}(L) - \mathbf{G}[\mathbf{A}_{21}(L) - \mathbf{A}^*_{21}(L)] & \mathbf{A}_{12}(L) - \mathbf{G}[\mathbf{A}_{22}(L) - \mathbf{A}^*_{22}(L)] \\ \mathbf{A}^*_{21}(L) & \mathbf{A}^*_{22}(L) \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1,t-1} \\ \mathbf{z}_{2,t-1} \end{bmatrix} + \\ + \begin{bmatrix} \mathbf{a}_{10} - \mathbf{G}(\mathbf{a}_{20} - \mathbf{a}^*_{20}) \\ \mathbf{a}^*_{20} \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}. \tag{1.29}$$

Thus, in a VAR under control based on the PVAR method, the new equation for the non-policy variables differs from the original VAR if the new policy rule differs from the original VAR equations for the policy variables. In addition, the VAR under control has a singular error covariance matrix because the new VAR equations for the policy variables are deterministic.

More generally, the vector $\mathbf{z}_{2t}$ in equation (1.27) can be replaced by any deterministic policy rule of the form

$$\Lambda \mathbf{z}_{1t} + \mathbf{z}_{2t} = \mathbf{a}^*_{20} + \mathbf{A}^*_{21}(L)\mathbf{z}_{1,t-1} + \mathbf{A}^*_{22}(L)\mathbf{z}_{2t-1}, \tag{1.30}$$

which gives a new complete model:

$$\begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ \Lambda & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{z}_{2t} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}(L) - \mathbf{G}\mathbf{A}_{21}(L) & \mathbf{A}_{12}(L) - \mathbf{G}\mathbf{A}^*_{22}(L) \\ \mathbf{A}^*_{21}(L) & \mathbf{A}^*_{22}(L) \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1,t-1} \\ \mathbf{z}_{2,t-1} \end{bmatrix} + \\ + \begin{bmatrix} \mathbf{a}_{10} - \mathbf{G}\mathbf{a}_{20} \\ \mathbf{a}^*_{20} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \mathbf{0} \end{bmatrix}$$

and the VAR representation:

$$\begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{z}_{2t} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ \Lambda & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{A}_{11}(L) - \mathbf{G}\mathbf{A}_{21}(L) & \mathbf{A}_{12}(L) - \mathbf{G}\mathbf{A}^*_{22}(L) \\ \mathbf{A}^*_{21}(L) & \mathbf{A}^*_{22}(L) \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1,t-1} \\ \mathbf{z}_{2,t-1} \end{bmatrix} + \\ + \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ \Lambda & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{10} - \mathbf{G}\mathbf{a}_{20} \\ \mathbf{a}^*_{20} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ \Lambda & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \epsilon_{1t} \\ \mathbf{0} \end{bmatrix},$$

As a result, if the policy rule incudes contemporaneous responses to changes in the state variables, then the PVAR approach yields a VAR model under control with a singular variance-covariance matrix because the new VAR equations for the policy variables are deterministic.

1.3.3 Assessment

The PVAR and the standard approach represent two alternative methodologies to identify from an unrestricted VAR model a stochastic dynamic constraint that can be employed for the solution to dynamic linear quadratic problems. From a theoretical point of view the standard approach is equivalent to assume $\mathbf{B}_{12} = \mathbf{0}$ in equation (1.22), i.e. that policy shocks affect state variables with a lag. Consequently, if optimisation is carried out through dynamic programming, the dynamic structure of the optimal policy rule is consistent with that in equation (1.12) which states that the control responds responds to current change in the state vector.

The plausibility of this assumption critically depends upon two factors: the type of macroeconomic variables included in the state vector and the unit of observation of the data. Since the empirical evidence suggests that output and inflation respond with delay to changes in the interest rate, the standard identification approach works well in the context of monetary policy analysis based on VAR models, which include observations of output and inflation in the state vector and of the interest rate in the control. However, the standard approach is inappropriate if the state vector includes other variables which are likely to quickly react to policy changes, such as financial market variables. In addition, it is unlikely that macroeconomic variables respond with delay to policy changes when data are available on a low frequency basis. For example, if data are available on a quarterly basis, then output is likely to respond to changes in policy within the period of observation. As a result, forcing output to respond with a lag, as under the standard approach, would yield a misspecified representation of the dynamic links between state and control variables, hence the wrong solution when computing the optimal policy. Stock and Watson (2001) argue against the use of the standard identification approach even with monthly data. Because most of macroeconomic data are available at least on a quarterly basis, this implies that the standard identification approach is of little use in macroeconomic analysis.

The PVAR identification approach is based upon a block Cholesky decomposition which effectively is equivalent to assume $\mathbf{B}_{21} = \mathbf{0}$ in equation (1.22). This has the advantage of leaving the linear dynamic structure of the state vector equations entirely unrestricted. In fact, under the PVAR approach the state vector equations are conditioned upon both current and lagged value of the control, as well as their own lags. In this respect, the PVAR approach allow the inclusion of a wider range of variables in the state vector equations and it is suitable for the analysis of both high and low frequency data. Indeed, the structural VAR model derived under the standard approach is a special case of that resulting from the PVAR approach, obtained by setting $\mathbf{G} = \mathbf{0}$ in equation (1.26). If optimisation is carried out through dynamic programming, then the dynamic structure of the policy rule obtained under the PVAR approach is consistent with that in equation (1.8). This implies that policy makers respond with a lag to changes in the state vector, which reflects the presence of information lags in the policy response to changes in the non-policy variables. However, the structural VAR model obtained under the PVAR approach can be combined with any unrestricted rule, such as in equation (1.30). In this instance optimisation can be carried out by grid searching through the parameters of (1.30), given the policy maker objective function and the constraint in (1.26). This entirely unrestricted linear framework cannot be achieved under the standard approach.

1.4 Dynamic games

1.4.1 Related Literature

Dynamic optimisation often involves problems which include more than one decision maker. The presence of more than one decision maker give raise to dynamic games models which have been extensively studied in several areas of both microeconomics and macroeconomics. Examples include industrial organisation and price determination - Beggs and Klemperer (1992) Fershtman and Kamien (1987), Karp and Perloff (1989), Reynolds (1991) - exhaustible and renewable resources - Hansen et al. (1985), Lindsey (1989) -, policy credibility - Kydland and Prescott (1977), Barro and Gordon (1986) - international policy coordination - Cohen and Michel (1988), Currie et al. (1989), Miller and Salmon (1985a, b, 1990) - and monetary policy games - Obstfeld (1991), Lockwood and Phillipopoulos (1994).

A branch of the literature has focused on the interactions between monetary and fiscal policy and the alternative welfare implications of different types of interactions between the central bank and the government, for example Pindyk (1977), Tabellini (1988), Benigno and Woodford (2003) and Dixit and Lambertini (2003). A recent empirical assessment of the predictions arising from this literature is provided by Kirsanova et al. (2005). The authors add to the conventional IS curve-Phillips curve-Taylor rule model of Svensson (1997) and Rudebusch and Svensson (1999) two further equations which take into account the conduct of fiscal policy and the dynamic accumulation of public debt. The model is calibrated to study stabilisation policy under alternative types of competitions between monetary and fiscal authorities.

On the other hand, VAR models have been extensively used to assess separately the effects of either monetary policy shocks, for example Bernanke and Blinder (1992), Bernanke and Mihow (1998) and Christiano, Eichenbaum and Evans (1999), or fiscal shocks, see for example Blanchard and Perotti (1999) and Fatas and Mihov (2000). More recently, simultaneous assessment of monetary and fiscal shocks has been proposed by Canzoneri, Cumby and Diba (2002). VAR models have also been employed for the computation of optimal monetary policy rules in the United States, see Sack (2000), and in the United Kingdom, see Martin and Salmon (1998). However, to the best of my knowledge, VAR models have not been exploited so far to evaluate how optimal monetary policy is affected by the presence of the fiscal sector, under alternative types of competitions between monetary and fiscal authorities. Consequently the next subsections discuss the computation of optimal policy rules under alternative strategies of the policy makes when the dynamic structure of the economy is initially observed from the perspective of an unrestricted VAR model.

1.4.2 Setting up the model

A dynamic game model is usually designed by first specifying the objective function, the policy instruments and the stylized forecasting model employed by each decision maker, and then assessing the welfare outcome of policy decisions by taking into account alternative types of interactions among policy makers. So far, little attention has been given to the assessment of dynamic games models when the macroeconomic framework is observed from a reduced form VAR model. In principle, the VAR model has to be specified in terms of objective and instrument variables of all decision makers competing over the control of the economy. The identification problem is more complex than the individual policy maker case, as it requires knowledge of all policy makers' preferences as well as assumptions about the interaction of each policy instrument with both state variables and other policy instruments.

More importantly, the identification strategy and the solution to the dynamic game model depends upon the type of interaction among decision makers. A Markov perfect solution occurs when each decision maker solves his own optimisation problem by taking other players' best strategies as given. A special case of the Markov perfect solution arises when the optimiser knows that others decision makers are committed to an ad hoc policy rule. A cooperative solution results when all decision makers pool together their policy instruments to jointly optimise a common utility function. Finally, a Stackelberg solution occurs when policy makers are ranked in terms of their decision power, as either leader or follower, and act strategically by taking into account other players' optimal responses.

A preliminary issue when using VAR systems to analyse dynamic games models is to appropriately allocate the variables included in the endogenous vector $\mathbf{y}_t$ between the state vector $\mathbf{z}_{1t}$ and the control vector $\mathbf{z}_{2t}$. There are two sets of variables that can be embodied into $\mathbf{y}_t$. The first is a vector $\mathbf{x}_t' = \begin{bmatrix} x_1 & x_2 & \ldots & x_q \end{bmatrix}$ of the so called natural state variables, which refers to macroeconomic variables directly targeted by decision makers, as well as any other variable that indirectly affects the expected value of targeted variables.[8] The second set of variables, when there are $n > 1$ decision makers, includes each policy makers' instrument vector $\boldsymbol{v}_j$, with $j = 1, ..., n$.

While natural state variables are always included into $\mathbf{z}_{1t}$, the location of the instrument vectors $\boldsymbol{v}_j'$s depends upon the type of interaction among policy makers. Under the two Nash solutions, the optimiser treats other policy makers' reaction functions as given and includes their policy instruments into the state vector $\mathbf{z}_{1t}$. Under the cooperative solution, only natural state variables are included into the vector $\mathbf{z}_{1t}$, as all policy instruments are jointly employed in the optimisation. This implies that under Markov perfect and cooperative solutions, the dynamic optimisation problem preserves its recursive structure and can be still solved using the dynamic programming as in the single decision maker case.

Computation of optimal policy rules is more complicated when decision makers interact strategically as in a leader-follower scenario, since standard recursive techniques cannot be applied to compute optimal policy rules. This is because other decision makers' reaction functions cannot be treated as recursive state variables when a decision maker chooses his optimal policy in a rational expectation framework where several decision makers com-

[8] See Ljungqvist and Sargent (2004).

pete over the control of the economy. In this case, the optimal policy is time inconsistent, in the sense that other decision makers respond to the optimal policy and therefore change its expected outcomes. However, Kydland and Prescott (1980) argued that these types of optimal control problems can still be solved by recursive methods by treating other policy makers' reaction functions as recursive bounding constraints. The next four subsections examine the application of the PVAR approach to compute policy rules in VAR based dynamic games having either a Nash or the cooperative or the leader-follower solution.

1.4.3 Linear Markov perfect equilibria

In a dynamic game model with two players, $j = 1, 2$, a linear Markov strategy occurs when at any time t player j's action is restricted to be linearly dependent on the past history of play through the state vector. In this framework a Markov perfect equilibrium is a pair of linear Markov strategies giving the mutual best responses at any possible state of the world. In a dynamic macroeconomic model, the utility function of the decision maker j can be written as:

$$L_{jt} = E_t \sum_{s=0}^{\infty} \beta^{t+s} \left[\left(\mathbf{y}_{t+s} - \bar{\mathbf{y}}_j\right)' \mathbf{W}_j \left(\mathbf{y}_{t+s} - \bar{\mathbf{y}}_j\right) \right],$$

where $\bar{\mathbf{y}}_j$ and $\mathbf{W}_j$ indicate the $j - th$ decision maker's preferences over the policy targets and the arguments of the objective function respectively. The vector $\mathbf{y}_{t+s}$ is specified by treating as state variables all policy instruments different from those of the $j - th$ decision maker, that is:

$$\mathbf{z}'_{1t+s} = \begin{bmatrix} \mathbf{x}_{t+s} & v_{1t+s} & v_{2t+s} & \cdots & v_{j-1t+s} & v_{j+1t+s} & \cdots & v_{nt+s} \end{bmatrix}'$$

and

$$\mathbf{z}_{2t} = \boldsymbol{v}_{jt+s}.$$

The PVAR approach can be employed to transform the VAR model into a dynamic constraint of the form of (1.5). Since this dynamic optimisation problem is entirely recursive in the state vector, the optimal policy rule can then be computed through the dynamic programming method, as for the single policy maker case described in section 2. Under the Markov perfect equilibrium, the $j-th$ decision maker's optimal rule is a linear deterministic function relating the current value of the instrument vector $\boldsymbol{v}_{jt}$ to a constant, lagged values of natural state variables and other policy makers' instruments, that is:

$$\boldsymbol{v}_{jt} = \mathbf{f}_n + \mathbf{F}_n \begin{bmatrix} \mathbf{z}_{1t-1} \\ \cdots \\ \mathbf{z}_{1t-p} \\ \boldsymbol{v}_{jt-1} \\ \cdots \\ \boldsymbol{v}_{jt-p} \end{bmatrix}, \tag{1.31}$$

where the coefficients $\mathbf{f}_n$ and $\mathbf{F}_n$ are computed as in equation (1.8).

An important special case occurs when the optimiser knows that all other policy makers are bounded by a commitment technology, such as an explicit policy rule. The solution to this problem is not a Markov perfect equilibrium because the optimiser takes into account the ad hoc rule of the other policy makers, rather than their best reaction function. To simplify, suppose there are only two decision makers so that the instrument vector is partitioned as

$$\boldsymbol{v}_t' = \begin{bmatrix} \boldsymbol{v}_{1t} & \boldsymbol{v}_{2t} \end{bmatrix}',$$

where v_{1t} includes instruments of decision makers committed to the policy rule, while v_{2t} is the vector of instruments of the optimiser. The reduced form VAR can be partitioned accordingly as:

$$\begin{bmatrix} \mathbf{x}_t \\ v_{1t} \\ v_{2t} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{10} \\ \mathbf{a}_{20} \\ \mathbf{a}_{30} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) & \mathbf{A}_{13}(L) \\ \mathbf{A}_{21}(L) & \mathbf{A}_{22}(L) & \mathbf{A}_{23}(L) \\ \mathbf{A}_{31}(L) & \mathbf{A}_{32}(L) & \mathbf{A}_{33}(L) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t-1} \\ v_{1t-1} \\ v_{2t-1} \end{bmatrix} + \begin{bmatrix} \mathbf{e}_{1t} \\ \mathbf{e}_{2t} \\ \mathbf{e}_{3t} \end{bmatrix}, \quad (1.32)$$

so that natural state variables are separated from the two instrument vectors. Under the PVAR approach, the transformation matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

can be computed to make the instrument vectors of both policy makers exogenous and determine dynamic constraint

$$\begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ v_{Lt} \\ v_{Ft} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a}_{10} \\ \mathbf{a}_{20} \\ \mathbf{a}_{30} \end{bmatrix} +$$

$$+ \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) & \mathbf{A}_{13}(L) \\ \mathbf{A}_{21}(L) & \mathbf{A}_{22}(L) & \mathbf{A}_{23}(L) \\ \mathbf{A}_{31}(L) & \mathbf{A}_{32}(L) & \mathbf{A}_{33}(L) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t-1} \\ v_{Lt-1} \\ v_{Ft-1} \end{bmatrix} +$$

$$+ \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{1t} \\ \mathbf{e}_{2t} \\ \mathbf{e}_{3t} \end{bmatrix}.$$

At this stage, the coefficients corresponding to the instrument vector v_{1t} can be replaced with those of the ad hoc policy rule

$$\mathbf{Q}\mathbf{x}_t + v_{1t} + \mathbf{M}v_{2t} = \mathbf{a}_{20}^* + \mathbf{A}_{21}^*(L)\mathbf{x}_{t-1} + \mathbf{A}_{22}^*(L)v_{1t-1} + \mathbf{A}_{23}^*(L)v_{2t-1} + \mathbf{e}_{2t}^*,$$

to obtain the new model

$$\begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{Q} & \mathbf{I} & \mathbf{M} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \boldsymbol{v}_{1t} \\ \boldsymbol{v}_{2t} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a}_{10} \\ \mathbf{a}^*_{20} \\ \mathbf{a}_{30} \end{bmatrix} +$$
$$+ \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) & \mathbf{A}_{13}(L) \\ \mathbf{A}^*_{21}(L) & \mathbf{A}^*_{22}(L) & \mathbf{A}^*_{23}(L) \\ \mathbf{A}_{31}(L) & \mathbf{A}_{32}(L) & \mathbf{A}_{33}(L) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t-1} \\ \boldsymbol{v}_{1t-1} \\ \boldsymbol{v}_{2t-1} \end{bmatrix} +$$
$$+ \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 & -\mathbf{G}_2 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{1t} \\ \mathbf{e}^*_{2t} \\ \mathbf{e}_{3t} \end{bmatrix}.$$

After erasing the coefficients corresponding to the control vector $\boldsymbol{v}_{2t}$ and solving for the state vector $\begin{bmatrix} \mathbf{x}_t \\ \boldsymbol{v}_{1t} \end{bmatrix}$, the constraint of the economy can be written as:

$$\begin{bmatrix} \mathbf{x}_t \\ \boldsymbol{v}_{1t} \end{bmatrix} =$$
$$= \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 \\ \mathbf{Q} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a}_{10} \\ \mathbf{a}^*_{20} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 \\ \mathbf{Q} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{G}_2 \\ \mathbf{0} \end{bmatrix} \mathbf{a}_{30} +$$
$$+ \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 \\ \mathbf{Q} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & -\mathbf{G}_2 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) & \mathbf{A}_{13}(L) \\ \mathbf{A}^*_{21}(L) & \mathbf{A}^*_{22}(L) & \mathbf{A}^*_{23}(L) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{t-1} \\ \boldsymbol{v}_{1t-1} \end{bmatrix} +$$
$$+ \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 \\ \mathbf{Q} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{G}_1 \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{31}(L) & \mathbf{A}_{32}(L) & \mathbf{A}_{33}(L) \end{bmatrix} \boldsymbol{v}_{2t-1}$$
$$- \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 \\ \mathbf{Q} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{G}_2 \\ \mathbf{M} \end{bmatrix} \boldsymbol{v}_{2t} + \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 \\ \mathbf{Q} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{1t} \\ \mathbf{e}^*_{2t} \end{bmatrix} +$$
$$\begin{bmatrix} \mathbf{I} & -\mathbf{G}_1 \\ \mathbf{Q} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{G}_2 \\ \mathbf{0} \end{bmatrix} \mathbf{e}_{3t},$$

which is compatible with the dynamic system in equation (1.5), with $\mathbf{y}'_t = \begin{bmatrix} \mathbf{z}_{1t} & \ldots & \mathbf{z}_{1t-p} & \mathbf{z}_{2t} & \ldots \end{bmatrix}$

$\mathbf{z}'_{1t} = \begin{bmatrix} \mathbf{x}_t & \boldsymbol{v}_{1t} \end{bmatrix}$ and $\mathbf{z}_{2t} = \boldsymbol{v}_{2t}$. Therefore, under this special case of the Nash solution, the optimal policy rule

$$\boldsymbol{v}_{2t} = \mathbf{f}_r + \mathbf{F}_r \begin{bmatrix} \mathbf{z}_{1t-1} \\ \ldots \\ \mathbf{z}_{1t-p} \\ \boldsymbol{v}_{2t-1} \\ \ldots \\ \boldsymbol{v}_{2t-p} \end{bmatrix}$$

is a linear function embodying optimal responses to changes in the objective variables targeted by the optimiser and rule-based changes of other decision makers' instruments.

1.4.4 Cooperative solution

A cooperative solution occurs when all decision makers coordinate their policy rules in order to achieve common objectives. This scenario can be described by assuming decision makers simultaneously employing all policy instruments to minimise a common objective function, as in equation (1.1), under a common forecasting model of the form of either (1.5) or (1.6). This implies that the vector $\mathbf{z}_{1t+s}$ includes only natural state variables, while the control vector $\mathbf{z}_{2t+s}$ embodies all available policy instruments, that is:

$$\mathbf{z}_{1t+s} = \mathbf{x}_{t+s}$$

and

$$\mathbf{z}'_{2t+s} = \begin{bmatrix} v_{1t+s} & v_{2t+s} & \cdots & v_{nt+s} \end{bmatrix}'.$$

The PVAR approach can be employed to compute a dynamic constraint in which the current value of natural state variables is conditioned upon the current value of the exogenous control vector $\mathbf{z}_2$. The dynamic optimisation problem can then be solved by minimising the cost function (1.1) with respect to $\mathbf{z}_2$ by employing the recursive dynamic programming method as in the single decision maker case. The solution to this dynamic problem implies that each policy maker has to commit to a specific optimal policy rule which relates his own policy instruments to natural state variables, but also takes into account the optimal behaviour of other policy makers. Therefore, the optimal policy rule can be written as:

$$\mathbf{z}_{2t} = \mathbf{f}_c + \mathbf{F}_c \begin{bmatrix} \mathbf{x}_{t-1} \\ \cdots \\ \mathbf{x}_{t-p} \\ \mathbf{z}_{2t-1} \\ \cdots \\ \mathbf{z}_{2t-p} \end{bmatrix},$$

and each equation of $\mathbf{z}_{2t}$ refers to the optimal policy rule of a decision maker under cooperation.

1.4.5 Stackelberg solution

The assessment of either a Nash or the cooperative solution to a dynamic game from a VAR model specified in terms of all objective and instrument variables mainly requires the appropriate allocation of available policy instruments between state and control vectors. Once this task is accomplished, the dynamic game problem is solved by employing the same recursive technique used for the single decision maker case. This is because, the underlying nature of the dynamic optimisation problem under both Nash and cooperative scenarios remains recursive.

When policy makers act strategically the solution to a dynamic optimisation problem is more complex as each decision maker sets the optimal policy given his prediction of other policy makers' reaction functions. In a leader-follower scenario, for instance, the follower's decisions in each period t is influenced by the forecast of the dominant player's next period action. In principle, the dominant player can either confirm or invalidate the follower's prediction. In the latter case the solution to the dynamic problem would exhibit time inconsistency and cannot be computed using a recursive technique. However, if the leader is bound to validate the follower's expectations, the reaction function of the follower can be interpreted as a constraint for the dominant player and a value can be attached to this constraint in terms of the cost of confirming past follower's expectations about the current behaviour of the leader. In this case, the Stackelberg problem is entirely recursive in both the natural state variables and the bounding constraint and can be solved for the dominant player by using standard recursive dynamic programming methods.[9]

Consider the vector of policy instruments partitioned as

$$v_t' = \begin{bmatrix} v_{Lt} & v_{Ft} \end{bmatrix}',$$

where v_{Lt} is the vector of leader's instruments and v_{Ft} is the vector of follower's instruments. The economy constraint is computed from a VAR model of the economy which can be partitioned as:

$$\begin{bmatrix} \mathbf{z}_{1t} \\ v_{Lt} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{10} \\ \mathbf{a}_{20} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) \\ \mathbf{A}_{21}(L) & \mathbf{A}_{22}(L) \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1t-1} \\ v_{Lt-1} \end{bmatrix} + \begin{bmatrix} \mathbf{e}_{1t} \\ \mathbf{e}_{2t} \end{bmatrix}, \tag{1.33}$$

where the state vector $\mathbf{z}_{1t} = \begin{bmatrix} \mathbf{x}_t & v_{Ft} \end{bmatrix}'$ embodies natural state variables and the follower's reaction function, the latter being interpreted as an intertemporal constraint reflecting the leader commitment to confirm in each period the follower's forecast of his actions.[10] Under the PVAR approach, the transformation matrix $\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix}$ can be employed to make the control vector v_{Lt} exogenous as:

$$\begin{aligned} \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1t} \\ v_{Lt} \end{bmatrix} = & \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{a}_{10} \\ \mathbf{a}_{20} \end{bmatrix} + \\ & + \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) \\ \mathbf{A}_{21}(L) & \mathbf{A}_{22}(L) \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1t-1} \\ v_{Lt-1} \end{bmatrix} + \\ & + \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e}_{1t} \\ \mathbf{e}_{2t} \end{bmatrix}. \end{aligned}$$

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9 For a rigorous discussion of the solution to dynamic games models under strategic competition, see Kydland and Prescott (1980) and Ljungqvist and Sargent (2004).

10 The VAR model in (1.33) is a compact system obtained by replacing v_{1t} for v_{Ft} and v_{2t} for v_{Lt} in equation (1.32).

In the single decision maker case, the matrix $\mathbf{G}$ is equal to the product between the covariance matrix of the disturbances in the state and instrument vectors and the variance of the instrument vector's disturbances. In the Stackelberg problem, $\mathbf{G}$ includes also a second term corresponding to the product between the covariance matrix of the disturbances in the bounding constraint and instrument vectors, and the variance of the instrument vector's disturbances.

Under the certainty equivalence principle, the above dynamic constraint for the leader is compatible with the deterministic system:

$$\mathbf{y}_t = \mathbf{A}\mathbf{y}_{1t-1} + \mathbf{B}\boldsymbol{v}_{Lt}, \tag{1.34}$$

where $\mathbf{y}_t' = \begin{bmatrix} \mathbf{z}_{1t} & \dots & \mathbf{z}_{1t-p} & \boldsymbol{v}_{Lt} & \dots & \boldsymbol{v}_{Lt-p} \end{bmatrix}'$ and the vector of state variables $\mathbf{z}_{1t}$ is augmented with $\boldsymbol{v}_{Ft} \quad \dots \quad \boldsymbol{v}_{Ft-p}$ to take into account the cost for the leader accruing from confirming past periods' predictions of the follower about his current behaviour.

If employing a quadratic objective function as in equation (1.1), the optimal policy rule for the leader takes the form:

$$\boldsymbol{v}_{Lt} = \mathbf{f}_o + \mathbf{F}_o\mathbf{y}_{t-1}. \tag{1.35}$$

Given the interpretation of the variables included into $\mathbf{y}_t$, the optimal decision rule of the leader is dependent not only upon the forecasting model employed to predict natural state variables, but also on the follower's best response to the sequence of actions undertaken by the leader, as embedded in the vectors $\boldsymbol{v}_{Ft} \quad \dots \quad \boldsymbol{v}_{Ft-p}$. The response coefficients $\mathbf{f}_o$ and $\mathbf{F}_o$ are calculated as in (1.8), which requires the computation of the stable solution to a matrix Riccati equation as in (1.10). In a Stackelberg optimisation problem, it is convenient to determine the solution to the matrix Riccati equation by employing the Lagrangian method, since this allows a reinterpretation of the coefficients in the stable solution of the matrix Riccati equation $\mathbf{P}$, which can then be exploited to solve the model through recursive techniques. Before forming the Lagrangian, the one-period utility function is decomposed as:

$$(\mathbf{y}_t - \bar{\mathbf{y}})' \mathbf{W} (\mathbf{y}_t - \bar{\mathbf{y}}) = (\mathbf{y}_t - \bar{\mathbf{y}})' \mathbf{R} (\mathbf{y}_t - \bar{\mathbf{y}}) + \boldsymbol{v}'_{Lt} \mathbf{Q} \boldsymbol{v}_{Lt},$$

where $\mathbf{R} + \mathbf{Q} = \mathbf{W}$ and the matrix $\mathbf{R}$ assigns zero values to the diagonal coefficients corresponding to the vector $\boldsymbol{v}_{Lt}$ in $\mathbf{y}_t$. This implies that the Stackelberg problem can be written in the following Lagrangian form:

$$\mathcal{L} = E_t \sum_{s=0}^{\infty} \beta^t \left[\begin{array}{c} (\mathbf{y}_{t+s} - \bar{\mathbf{y}})' \mathbf{R} (\mathbf{y}_{t+s} - \bar{\mathbf{y}}) + \boldsymbol{v}'_{Lt+s} \mathbf{Q} \boldsymbol{v}_{Lt+s} + \\ +2\beta \boldsymbol{\mu}'_{t+s} \left(\mathbf{a} + \mathbf{A} \mathbf{y}_{t+s} + \mathbf{B} \mathbf{u}_{Lt+s+1} - \mathbf{y}_{t+s+1} \right) \end{array} \right],$$

where $\boldsymbol{\mu}_{t+s}$ is a vector of shadow prices associated with the sequence of dynamic constraints in the Lagrangian. Given the definition of $\mathbf{y}_{t+s}$, the vector of shadow prices $\boldsymbol{\mu}_{t+s}$ can be partitioned conformably as:

$$\boldsymbol{\mu}'_{t+s} = \left[\begin{array}{cccccccc} \boldsymbol{\mu}_{\mathbf{x}_{t+s}} & \boldsymbol{\mu}_{\boldsymbol{v}_{Ft+s}} & \cdots & \boldsymbol{\mu}_{\mathbf{x}_{t+s-p}} & \boldsymbol{\mu}_{\boldsymbol{v}_{Ft+s-p}} & \boldsymbol{\mu}_{\boldsymbol{v}_{Lt+s}} & \cdots & \boldsymbol{\mu}_{\boldsymbol{v}_{Lt+s-p}} \end{array} \right]',$$

where the sequence of multipliers $\boldsymbol{\mu}_{\boldsymbol{v}_{Ft+s}} \cdots \boldsymbol{\mu}_{\boldsymbol{v}_{Ft+s-p}}$ represent the set of shadow prices attached to the bounding constraints of the leader reflecting the cost of honoring current and past follower's expectations about future leader's policy choices, as captured by the reaction functions corresponding to $\boldsymbol{v}_{Ft+s},\ldots,\boldsymbol{v}_{Ft+s-p}$.

After taking expectations, the first order conditions with respect to $\mathbf{y}_t$ and $\boldsymbol{v}_{Lt+1}$ are calculated as:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{y}_t} = -2\beta^t \boldsymbol{\mu}_t + 2\beta^t \mathbf{R}\mathbf{y}_t + 2\beta^{t+1}\mathbf{A}'\boldsymbol{\mu}_{t+1}, \tag{1.36}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{v}_{Lt+1}} = 2\beta^{t+1}\mathbf{B}'\boldsymbol{\mu}_{t+1} + 2\beta^{t+1}\mathbf{Q}\boldsymbol{v}_{Lt+1}. \tag{1.37}$$

Solving the first order condition (1.37) for $\boldsymbol{v}_{Lt+1}$ and substituting the solution in equation (1.34), the dynamic constraint can be written as:

$$\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{Q}^{-1}\mathbf{B}'\boldsymbol{\mu}_{t+1} \tag{1.38}$$

and the system of first order conditions obtained by combining (1.36) and (1.38) has the following state space representation:

$$\begin{bmatrix} \mathbf{I} & \mathbf{B}\mathbf{Q}^{-1}\mathbf{B}' \\ \mathbf{0} & \beta\mathbf{A}' \end{bmatrix} \begin{bmatrix} \mathbf{y}_{t+1} \\ \boldsymbol{\mu}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{R} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{y}_t \\ \boldsymbol{\mu}_t \end{bmatrix}, \tag{1.39}$$

with stable solution given by:

$$\boldsymbol{\mu}_t = \mathbf{P}\mathbf{y}_t. \tag{1.40}$$

To compute $\mathbf{P}$, substitution of the solution (1.40) into the system in equation (1.39) yields:

$$\left(\mathbf{I} + \mathbf{B}\mathbf{Q}^{-1}\mathbf{B}'\mathbf{P}\right)\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t \tag{1.41}$$

$$\beta\mathbf{A}'\mathbf{P}\mathbf{y}_{t+1} = (-\mathbf{R} + \mathbf{P})\,\mathbf{y}_t. \tag{1.42}$$

Since

$$\left(\mathbf{I} + \mathbf{B}\mathbf{Q}^{-1}\mathbf{B}'\mathbf{P}\right)^{-1} = \mathbf{I} - \beta\mathbf{B}\left(\mathbf{Q} + \beta\mathbf{B}'\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{P},$$

equation (1.41) can be solved for $\mathbf{y}_{t+1}$ to obtain:

$$\mathbf{y}_{t+1} = \left[\mathbf{A} - \beta\mathbf{B}\left(\mathbf{Q} + \beta\mathbf{B}'\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{P}\mathbf{A}\right]\mathbf{y}_t.$$

After setting $\mathbf{Q} = \mathbf{0}$ and substituting $\mathbf{R}$ for $\mathbf{W}$, the term $\beta\left(\mathbf{Q} + \beta\mathbf{B}'\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{P}\mathbf{A}$ becomes equivalent to the matrix $\mathbf{F}$ from equation (1.8) and $\mathbf{y}_{t+1}$ is written as:

$$\mathbf{y}_{t+1} = (\mathbf{A} - \mathbf{B}\mathbf{F})\,\mathbf{y}_t. \tag{1.43}$$

Premultiplying both sides in the above expression by the term $\beta\mathbf{A}'\mathbf{P}$ gives:

$$\beta\mathbf{A}'\mathbf{P}\mathbf{y}_{t+1} = \beta\left(\mathbf{A}'\mathbf{P}\mathbf{A} - \mathbf{A}'\mathbf{P}\mathbf{B}\mathbf{F}\right)\mathbf{y}_t,$$

which can be equated to the right hand side of (1.42) and solved for $\mathbf{P}$ to obtain:

$$\mathbf{P} = \mathbf{W} + \beta\mathbf{A}'\mathbf{P}\mathbf{A} - \beta\mathbf{A}'\mathbf{P}\mathbf{B}\left(\mathbf{B}'\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{P}\mathbf{A},$$

which shows that the stable solution $\mathbf{P}$ to the dynamic system (1.39) is the same algebraic matrix Riccati required to compute the response coefficients in the optimal feedback rule (1.35).

Therefore, the matrix $\mathbf{P}$ can be partitioned in order to decode all information associated with the Lagrangian multipliers of the constraint in equation (1.40) as:

$$\begin{bmatrix} \boldsymbol{\mu}_{\kappa t} \\ \boldsymbol{\mu}_{\upsilon_F t} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\kappa}_t \\ \boldsymbol{\upsilon}_{Ft} \end{bmatrix},$$

where $\boldsymbol{\mu}_{\kappa t}$ is the vector of shadow prices associated with

$$\boldsymbol{\kappa}_t' = \begin{bmatrix} \mathbf{x}_t & \mathbf{x}_{t-1} & \boldsymbol{\upsilon}_{Ft-1} & \cdots & \mathbf{x}_{t-p} & \boldsymbol{\upsilon}_{Ft-p} & \boldsymbol{\upsilon}_{Lt} & \cdots & \boldsymbol{\upsilon}_{Lt-p} \end{bmatrix}'.$$

The solution to $\boldsymbol{\mu}_{\upsilon_F t}$ is calculated as

$$\boldsymbol{\mu}_{\upsilon_F t} = \mathbf{P}_{21}\boldsymbol{\kappa}_t + \mathbf{P}_{22}\boldsymbol{\upsilon}_{Ft},$$

which leads to the following formulation of the vector $\mathbf{u}_{Ft}$:

$$\upsilon_{Ft} = \mathbf{P}_{22}^{-1}\mu_{\upsilon_F t} - \mathbf{P}_{22}^{-1}\mathbf{P}_{21}\kappa_t.$$

The state vector and the vector of shadow prices can now be written, respectively, as:

$$\mathbf{y}_t = \begin{bmatrix} \kappa_t \\ \upsilon_{Ft} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{P}_{22}^{-1}\mathbf{P}_{21} & \mathbf{P}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \kappa_t \\ \mu_{\upsilon_F t} \end{bmatrix} \tag{1.44}$$

and

$$\mu_{\upsilon_F t} = \begin{bmatrix} \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \kappa_t \\ \upsilon_{Ft} \end{bmatrix}.$$

The optimal leader decision rule in equation (1.35) is thus computed as:

$$\upsilon_{Lt} = -\mathbf{f}_o - \mathbf{F}_o \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{P}_{22}^{-1}\mathbf{P}_{21} & \mathbf{P}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \kappa_t \\ \mu_{\upsilon_F t} \end{bmatrix}. \tag{1.45}$$

Finally, both equations (1.44) and (1.45) can be embodied in the dynamic constraint (1.43) to write the state vector under control and optimal feedback rule of the follower respectively as:

$$\begin{aligned} \begin{bmatrix} \kappa_{t+1} \\ \mu_{\upsilon_F t+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{F}_o) \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{P}_{22}^{-1}\mathbf{P}_{21} & \mathbf{P}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \kappa_t \\ \mu_{\upsilon_F t} \end{bmatrix} \\ \upsilon_{Ft} &= \begin{bmatrix} -\mathbf{P}_{22}^{-1}\mathbf{P}_{21} & \mathbf{P}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \kappa_t \\ \mu_{\upsilon_F t} \end{bmatrix}. \end{aligned}$$

1.5 Conclusion

This paper addresses the issue of measuring policy rules when the economy is represented by a VAR model. I describe a new partial-identification approach which has the advantage, over the existing methodologies, of leaving the dynamic structure of the state vector equations in the semi-structural VAR model entirely unrestricted, since it conditions the dynamic of the state variables to both current and lagged values of the control vector, rather than the lagged values alone as under the standard approach. The problem with the standard identification approach is well summarised by Stock and Watson in their 2001 review paper on VAR models: "... the timing conventions in VARs do not necessarily reflect real-time data availability, and this undercuts the common method of identifying restrictions based on timing assumptions. For example, a common assumption made in structural VARs is that variables like output and inflation are sticky and do not respond "within the period" to monetary policy shocks. This seems plausible over the period of a single day, but becomes less plausible over a month or quarter..." [Stock, and Watson, (2001), pp.112].

In this respect, I believe, the new identification approach represents an improvement over the existing methodologies. The approach can be implemented for the computation of optimal policy rules within conventional output-inflation-interest rate VAR models used for monetary policy analysis to evaluate to what extent the choice of the identification approach affects the response coefficients in the interest rate under the optimal rule and the welfare measurements. A similar analysis can then be extended to the evaluation of optimal fiscal rules. Furthermore, the PVAR approach is preferable to the standard one when the state vector equations includes financial market variables such as stock market prices or term structure which are bound to react immediately to policy changes. In this respect, the PVAR identification approach can be used for assessing optimal macroeconomic policy rules within macro-finance VAR models of the economy. Finally, as already argued in section 4, there is a growing interest in the macroeconomic literature on the simultaneous assessment of optimal monetary and fiscal policy. This literature points out that optimal policy and welfare measures depends upon the type of interactions between monetary and fiscal authorities. So far, and to the best of my knowledge, a preliminary assessment of these finding has been carried out by using stylised - backward looking - models of the economy. I describe how VAR models can be employed to compute optimal policy rules under multiple policy makers competing over the control of the economy. This provides an alternative framework against which one can compare the traditional findings of the literature on the interactions between monetary and fiscal authorities. The empirical implementation of this framework is part of my future research agenda.

Chapter 2
Assessing optimal monetary policy through VAR models

Abstract

This paper assesses both theoretically and empirically a new approach - PVAR method - to formulate optimal policy based on a quadratic intertemporal welfare function where the dynamic constraint of the economy is derived from a VAR model. The paper argues that the VAR under control should not be derived simply by replacing the VAR equation for the policy instruments by an optimal control rule because this alters the stochastic structure of the state vector equations, and gives a state space representation of the dynamic constraint in which state variables can only respond to lagged values of the control. Instead, one should first transform the VAR in order to condition the non-policy variables on the policy instruments, then use the resulting sub-system as the dynamic constraint, and finally construct the VAR under control by combining this sub-system with the resulting optimal policy rule. In this way the original stochastic structure of the state vector equations of the VAR is retained. In addition, under the PVAR approach the state variables in the dynamic constraint are conditioned upon both current and lagged values of the control, hence giving a representation of the macroeconomic framework more suitable for policy analysis. In comparing the two approaches, the paper explains the theoretical advantages of the PVAR over the standard method and applies both methods by examining the formulation of optimal monetary policy rules using US data for the period 1960-2003. The empirical findings

show that feedback rules predicted under the PVAR approach are smoother than those calculated under the standard approach and welfare losses are considerably overstated by the standard approach, regardless of the specification of the objective function. We suggest that since the whole process is easily automated, the PVAR method may provide a useful benchmark for use in real time against which to compare other, probably far more labour intensive, policy choices.

2.1 Introduction

This paper studies optimal monetary policy rules obtained from the maximisation of an intertemporal quadratic objective function based on a trade-off between inflation and output subject to a dynamic constraint that is derived from a vector autoregressive (VAR) model of the economy. The choice of a quadratic objective function in inflation and output (or the output gap) reflects common practice in the control and inflation targeting literatures, for example, Rudebusch and Svensson (1999) and Sack (2000). A more formal justification was provided by Rotemberg and Woodford (1998) - see also Woodford (2003) - who showed that such a quadratic function can be derived as an approximation to a micro founded macro model with standard preferences in terms of consumption.

As far as the choice of the intertemporal constraint relating targeted variables to policy instruments is concerned, pioneering works of Kydland and Prescott (1977), Barro and Gordon (1983) and Rogoff (1985) studied optimal monetary policy within rational expectations models of output and inflation. In contrast, Rudebusch and Svensson (1999) used a dynamic constraint obtained from an entirely backward looking model of output and inflation. Clarida, Gali and Gertler (1999) and Rotemberg and Woodford (1998) looked at monetary policy from the perspective of New Keynesian models, thus assessing optimal rules by using a dynamic constraint with forward looking components in both the aggregate demand and supply equations. Optimal policy will clearly be affected by this choice. A more agnostic approach, that seeks to avoid imposing a constraint not supported by the data, is to use instead a data-based VAR. Since it is a theory free model, the assessment of policy rules from a VAR model is not biased by the choice of the structural model of the economy employed by the decision maker. However, using VAR models to study optimal monetary policy presents two major issues. The first is related to the fact that a change of policy rule alters the VAR model. As a result, there must be a concern that any VAR is vulnerable to structural change. In principle, therefore, the Lucas Critique applies here. In practice, however, like Rudebusch (2002), we find that structural change to a VAR as a result of changing policy appears not to be much in evidence. Without wishing to claim that basing policy on VAR is a first-best approach compared with using a correctly-specified structural model, given the difficulty of agreeing on what that structural model should be, using a VAR may still provide a helpful benchmark against which to compare a first-best policy and any other policies such as one based on a Taylor rule or a policy of discretion.

The second issue arising when assessing optimal policy rules from VAR models is that in a VAR all variables are endogenous, whereas computation of an optimal policy rule requires a dynamic constraint in which state variables are conditioned upon exogenous policy instruments. For this reason an identification technique is required to extract from the VAR a state space representation of the economy in which state variables are conditioned on the exogenous vector of policy instruments. The standard identification approach results in a dynamic constraint in which policy actions can only have a delayed effect on the state vector. Sack (2000) shows how this may be accomplished using a VAR in which the disturbances in the equations for the non-policy and policy variables are assumed to be uncorrelated. Martin and Salmon (1999) also use a VAR but with a different set of identifying restrictions from Sack. Having obtained the optimal policy rule, forecasts of the non-policy variables are derived from the VAR under control by replacing the original VAR equations for the policy instruments by the optimal policy rule. We refer to this methodology as the standard approach. Stock and Watson (2001) have made a related suggestion, namely to replace the interest rate equation in a VAR with a Taylor rule. This has the added drawback that the Taylor rule may not be an optimal choice.

Rather than make any assumptions about the correlation structure of the disturbances in the state vector equations of the VAR, we estimate the VAR unrestrictedly and then derive the dynamic constraint relating the non-policy variables to the policy instruments by transforming the VAR so that the non-policy variables are conditioned on the policy instruments. Having derived the optimal policy rule, we construct the VAR under control by combining the sub-system of equations for the non-policy variables that make up the dynamic constraint with the optimal rule. We refer to this approach as the policy VAR (PVAR) method.

The paper examines in detail the theoretical and empirical implications for optimal monetary policy of the PVAR approach and compares its outcomes with those arising from the standard methodology. From a theoretical perspective, the paper demonstrates that the choice of the identification approach has profound effects on both the dynamic structure of the policy rule and the magnitude of its response coefficients. Under the standard approach substitution of the optimal policy rule into the economy constraint minimises the volatility of the deterministic part of the state vector equation alone. The paper proves that under the PVAR approach the same operation minimises the volatility of the stochastic component of the state vector, as well as its deterministic part.

Empirical comparison of the PVAR and the standard approach is carried out by assessing optimal interest rate rules under alternative specifications of the objective function using US data for the period 1960-2003. The results show that optimal policy rules predicted under the PVAR approach deliver smoother feedback rates than those predicted under the standard approach. In line with Rudebush (2001), this finding corroborates the view that a plausible explanation of the gap between the response coefficients of the optimal and the VAR interest rate rule lies in the misspecification of the forecasting model employed by the policy maker, rather than in the uncertainty surrounding the precise values of the model's parameters.[11] Moreover, welfare losses computed under the PVAR approach are lower than those calculated under the standard approach, regardless of the specification of the objective function. This outcome occurs because, after substitution of the optimal rule into the dynamic constraint, output and inflation forecasts obtained from the PVAR approach are systematically smoother and faster converging toward their targets than under the standard approach.

[11] For a survey on the effects of uncertainty on optimal policy rules, see Sack and Wieland (2000).

The discussion is articulated in five sections following this introduction. The next section describes the PVAR approach and compares it with the standard methodology. Section 3 shows how dynamic programming is used under the two methodologies to compute optimal policy rules. Section 4 describes the data and the empirical specification of the dynamic optimisation problem under both approaches. Section 5 comments on the main empirical findings and Section 6 concludes. State space representations of the dynamic constraint under the PVAR and the standard approach, as well as all tables and figures are reported in appendices at the end of the paper.

2.2 Formulating the dynamic constraint from a VAR

VAR models have been widely employed for the assessment of optimal macroeconomic policy rules (Sack (2000), Martin and Salmon (1999) Monti (2003)). This is accomplished in several steps. The first step consists in specifying and estimating a reduced form VAR model including state variables targeted by the policy maker and control variables, which represent the instruments used by the policy maker in the conduct of policy. The state vector may also include other variables which are not directly targeted by the policy maker, but may be involved in the transmission mechanism from the policy instruments to the policy targets.

The second step consists in the identification of the dynamic constraint of the economy from the reduced form VAR. For the purpose of computing and evaluating optimal policy rules, this second task is accomplished by imposing a block Cholesky decomposition between the disturbances of the state and control variables. The resulting semi-structural

VAR gives a sufficient state space representation of the dynamic of the economy for the computation of optimal policy rules. In particular, the traditional identification approach is based upon the assumption that policy actions have no immediate effect on state variables. Consequently, the dynamic constraint of the economy coincides with the VAR equations for the non-policy variables. This constraint is combined with a welfare function of the policy maker - commonly chosen to be a quadratic function of the targets around their desired values - in order to compute optimal policy rules. If optimisation is carried out through dynamic programming, then the optimal policy rule relates the instruments to the current and lagged values of the non-policy variables, as well as lagged values of the control vector. Welfare analysis is carried out by replacing the original VAR equations for the policy instruments with the optimal policy rule to form a new VAR in the state variables, the VAR under control. The main drawback of this methodology is that it is only valid if, in the reduced form VAR, the disturbances of the non-policy and policy variables are uncorrelated. In addition, Stock and Watson (2001) argue that the standard identification approach is of limited use in macroeconomic policy analysis as it implausibly imposes a delay in the response of output and prices to changes in policy.

In response to these criticisms, an alternative identification approach, hereafter PVAR approach, based upon an identification methodology which is valid when the disturbances of the non-policy and policy variables are correlated in the reduced form VAR. The PVAR approach yields a semi-structural VAR model in which changes in the state variables are related to current and lagged values of the control vector, as well as lags of the state vector. Therefore, the PVAR approach has the advantage, over the existing methodology, of not imposing any timing restriction in the state space representation of the dynamic constraint of the economy. Under the PVAR method, we estimate the VAR unrestrictedly and then derive the dynamic constraint relating the non-policy variables to the policy instruments by transforming the VAR so that the non-policy variables are conditioned on the current values of the policy instruments. Having derived the optimal policy rule, the VAR under control is constructed by combining the sub-system of equations for the non-policy variables that make up the dynamic constraint with the optimal rule.

To illustrate analytically the PVAR approach, consider a generic reduced form VAR (p) model:

$$\mathbf{z}_t = \mathbf{a} + \mathbf{A}(L)\,\mathbf{z}_t + \mathbf{e}_t,$$

where $\mathbf{z}_t$ is a $q \times 1$ vector of endogenous variables, $\mathbf{a}$ is a vector of constant terms, L is the lag operator, $\mathbf{A}(L) = \sum_{i=1}^{p} \mathbf{A}_i L^i$, with $\mathbf{A}_i$ indicating a $q \times q$ matrix of lag i coefficients, $\mathbf{e}_t$ is a vector of stochastic disturbances such that $E[\mathbf{e}_t] = 0$ and $E[\mathbf{e}_t \mathbf{e}_t'] = \Sigma$. To order state variables before control variables, the vector $\mathbf{z}_t$ is partitioned as $\mathbf{z}_t' = \begin{bmatrix} \mathbf{z}_{1,t}' & \mathbf{z}_{2,t}' \end{bmatrix}$, where $\mathbf{z}_{1,t}$ is a $s \times 1$ vector of states and $\mathbf{z}_{2,t}$ is a $c \times 1$ vector of controls. The reduced form VAR can be partitioned accordingly as

$$\begin{bmatrix} \mathbf{z}_{1,t} \\ \mathbf{z}_{2,t} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{10} \\ \mathbf{a}_{20} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) \\ \mathbf{A}_{21}(L) & \mathbf{A}_{22}(L) \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1,t} \\ \mathbf{z}_{2,t} \end{bmatrix} + \begin{bmatrix} \mathbf{e}_{1t} \\ \mathbf{e}_{2t} \end{bmatrix}, \tag{2.46}$$

where $\mathbf{A}_{11}(L) = \sum_{i=1}^{p} \mathbf{A}_{11i} L^i$, $\mathbf{A}_{12}(L) = \sum_{i=1}^{p} \mathbf{A}_{12i} L^i$, $\mathbf{A}_{21}(L) = \sum_{i=1}^{p} \mathbf{A}_{21i} L^i$, $\mathbf{A}_{22}(L) = \sum_{i=1}^{p} \mathbf{A}_{22i} L^i$, whereas $\mathbf{e}_{1t}$ and $\mathbf{e}_{2t}$ are vectors of reduced form disturbances corresponding to state and control variables respectively. The covariance matrix of the disturbances is also partitioned as:

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},$$

where Σ_{11} is the $s \times s$ covariance matrix of the disturbances of the state vector, Σ_{22} is the $c \times c$ covariance matrix of the disturbances of the control vector, and Σ_{12} is $c \times s$ the matrix collecting the covariances between the disturbances of state and control vectors.[12]

Bernanke and Blinder (1992), whose purpose was to identify the shocks rather than perform optimal control analysis, started with a structural VAR:

$$\mathbf{B}\mathbf{z}_t = \mathbf{a} + \mathbf{A}(L)\,\mathbf{z}_t + \mathbf{u}_t,$$

where the disturbances $\mathbf{u}_t$ of the policy and non-policy instruments are assumed to be uncorrelated. They then consider two possible identification schemes to separate state and control variables: partitioning $\mathbf{B}$, they set either $\mathbf{B}_{12}$ or $\mathbf{B}_{21}$ equal to zero. If the variables in $\mathbf{z}_t$ are ordered non-policy and policy as before then this implies, respectively, that either the policy variables affect the non-policy variables with a lag or vice-versa. They then impose further restrictions on either $\mathbf{B}_{12}$ or $\mathbf{B}_{21}$ to compute impulse response functions. Bernanke and Mihov (1998) argue in favour of the restriction $\mathbf{B}_{12} = \mathbf{0}$. They also point out that this restriction may not be suitable if the data period is so long that the non-policy variables have time to react to the policy instruments within the period of observation. The restrictions imposed by Sack, Bernanke and Blinder, and Bernanke and Mihov are sometimes known as partial identification of a VAR, or using a semi-structural VAR. Martin and Salmon (1999), who also consider optimal policy with a VAR, argue that

12 Since Σ is symmetric, note that $\Sigma_{12} = \Sigma_{12}'$.

identifying a VAR through the sort of recursive restrictions used by Sack is unsatisfactory. Instead they use selective contemporaneous non-recursive restrictions to the disturbances.

Under the PVAR method, a block Cholesky decomposition is applied to the partitioned vector $\mathbf{e}_t$ in order to make the reduced form disturbances in the state vector a linear function of the corresponding disturbances in the control vector, that is:

$$\mathbf{e}_{1,t} = \epsilon_{1t} + \mathbf{G}\mathbf{e}_{2,t},$$

where ϵ_{1t} is the component of $\mathbf{e}_{1,t}$ which is uncorrelated with $\mathbf{e}_{2,t}$. Therefore, the vector $\mathbf{e}_{,t}$ has the following state space representation:

$$\mathbf{e}_t = \begin{bmatrix} \mathbf{e}_{1,t} \\ \mathbf{e}_{2,t} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \epsilon_{1t} \\ \mathbf{e}_{2,t} \end{bmatrix}.$$

The covariance matrix Σ can be used to construct the matrix G as follows:

$$\mathbf{G} = \Sigma_{12}\Sigma_{22}^{-1}, \tag{2.47}$$

which in turn gives the transformation matrix:

$$\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ 0 & \mathbf{I} \end{bmatrix}.$$

The matrix $\mathbf{H}^{-1}$ can be employed to map the original system onto a new one in which the disturbances associated with the state variables are uncorrelated with the disturbances associated with the controls:

$$\mathbf{H}^{-1}\mathbf{z}_t = \mathbf{H}^{-1}\mathbf{a} + \mathbf{H}^{-1}\mathbf{A}(L)\,\mathbf{z}_{t-1} + \mathbf{H}^{-1}\mathbf{e}_t,$$

which in turn yields the following system of linear equations:

$$\begin{aligned} \mathbf{z}_{1,t} - \mathbf{G}\mathbf{z}_{2,t} &= [\mathbf{a}_{10} - \mathbf{G}\mathbf{a}_{20}] + [\mathbf{A}_{11}(L) - \mathbf{G}\mathbf{A}_{21}(L)]\,\mathbf{z}_{1,t-1} + \\ &\quad + [\mathbf{A}_{12}(L) - \mathbf{G}\mathbf{A}_{22}(L)]\,\mathbf{z}_{2,t-1} + \mathbf{e}_{1,t} - \mathbf{G}\mathbf{e}_{2,t} \\ \mathbf{z}_{2,t} &= \mathbf{a}_{20} + \mathbf{A}_{21}(L)\,\mathbf{z}_{1,t-1} + \mathbf{A}_{22}(L)\,\mathbf{z}_{2,t-1} + \mathbf{e}_{2,t}. \end{aligned}$$

Since $\mathbf{e}_{1,t} - \mathbf{G}\mathbf{e}_{2,t} = \epsilon_{1t}$ – and $\mathbf{e}_{2,t}$ and ϵ_{1t} are uncorrelated – the above system is equivalent to

$$\begin{aligned} \mathbf{z}_{1,t} &= \mathbf{a}_{10} + \mathbf{A}_{11}(L)\,\mathbf{z}_{1,t-1} + \mathbf{A}_{12}(L)\,\mathbf{z}_{2,t-1} + \epsilon_{1t} \qquad (2.48) \\ &\quad -\mathbf{G}\left[\mathbf{a}_{20} + \mathbf{A}_{21}(L)\,\mathbf{z}_{1,t-1} - \mathbf{z}_{2,t} + \mathbf{A}_{22}(L)\,\mathbf{z}_{2,t-1}\right] \\ \mathbf{z}_{2,t} &= \mathbf{a}_{20} + \mathbf{A}_{21}(L)\,\mathbf{z}_{1,t-1} + \mathbf{A}_{22}(L)\,\mathbf{z}_{2,t-1} + \mathbf{e}_{2,t}, \qquad (2.49) \end{aligned}$$

with $\mathbf{z}_{2,t}$ uncorrelated with ϵ_{1t}. Hence, under the PVAR approach the state vector $\mathbf{z}_{1,t}$ is conditioned on the exogenous control vector $\mathbf{z}_{2,t}$. As a result, equation (2.48) can be used in isolation from the rest of the system to represent the law of motion of the state vector. Equation (2.49) can be replaced by any other policy rule and combined with equation (2.48) to form a complete new model of the economy under control. If the policy rule takes the general form:

$$\mathbf{\Lambda}\mathbf{z}_{1t} + \mathbf{z}_{2t} = \mathbf{a}_{20}^{*} + \mathbf{A}_{21}^{*}(L)\mathbf{z}_{1,t-1} + \mathbf{A}_{22}^{*}(L)\mathbf{z}_{2t-1} + \mathbf{e}_{2t}^{*},$$

then the complete model of the economy is given by:

$$\begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ \mathbf{\Lambda} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{z}_{2t} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{10} \\ \mathbf{a}_{20}^{*} \end{bmatrix} + \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) \\ \mathbf{A}_{21}^{*}(L) & \mathbf{A}_{22}^{*}(L) \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1,t-1} \\ \mathbf{z}_{2,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \mathbf{e}_{2t}^{*} \end{bmatrix}.$$

The above model can be written as the VAR system:

$$
\begin{aligned}
\begin{bmatrix} \mathbf{z}_{1t} \\ \mathbf{z}_{2t} \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ \boldsymbol{\Lambda} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{a}_{10} \\ \mathbf{a}_{20}^{*} \end{bmatrix} + \\
&+ \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ \boldsymbol{\Lambda} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11}(L) & \mathbf{A}_{12}(L) \\ \mathbf{A}_{21}^{*}(L) & \mathbf{A}_{22}^{*}(L) \end{bmatrix} \begin{bmatrix} \mathbf{z}_{1,t-1} \\ \mathbf{z}_{2,t-1} \end{bmatrix} \\
&+ \begin{bmatrix} \mathbf{I} & -\mathbf{G} \\ \boldsymbol{\Lambda} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\epsilon}_{1t} \\ \mathbf{e}_{2t}^{*} \end{bmatrix}.
\end{aligned}
$$

The standard approach is as a special case of the PVAR approach, obtained by setting $\mathbf{G} = \mathbf{0}$ in equation (2.48). This is because $\mathbf{G} = \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}$ and the standard identification assumes that $\boldsymbol{\Sigma}_{12} = \mathbf{0}$. Thus, the two identification approaches lead to quite different state space representation of the dynamic constraint. Under the PVAR approach the response of the state vector to a current change in the policy instrument is embodied in the matrix $\mathbf{G}$, while the coefficients of the vector $\mathbf{z}_{2t-1}$ are treated as part of the state vector. Under the standard approach, the state space representation of the structural model ultimately coincides with the state equations of the reduced form VAR.[13]

It is important to assess the implications of the two approaches for the stochastic properties of the VAR. This is because the stochastic properties of the state vector after substitution of any policy rule, either optimal or sub-optimal, are crucially different under the two identification approaches. The expected value of $\mathbf{z}_{1,t}$ from equation (2.48) is given by:

$$E\left[\mathbf{z}_{1,t}\right] = \mathbf{a}_{10} + \mathbf{A}_{11}\left(L\right)\mathbf{z}_{1,t-1} + \mathbf{A}_{12}\left(L\right)\mathbf{z}_{2,t-1},$$

13 A detailed description of the two state space representations is provided in appendix A.

because the term in square brackets in equation (2.48) is equal to $\mathbf{e}_{2t}$, which has zero expected value by definition. Since $E[\mathbf{z}_{1,t}]$ is equal to the deterministic part of the state vector, the same expected value for $\mathbf{z}_{1,t}$ is obtained from the stochastic dynamic equation derived under the standard approach and the original VAR. Therefore, the PVAR transformation does not alter the stochastic properties of the state vector. However, when the vector $\mathbf{z}_{2,t-1}$ is replaced with another policy rule, the expected values of $\mathbf{z}_{1,t}$ obtained from equations (2.48) depends upon the value of G. Under the PVAR approach, the forecast error of the new system under control, obtained after substitution of a policy rule in equation (2.48), is given by ϵ_{1t} since the expected value of the term in square brackets is nonzero under control. In contrast, when the matrix G is forced to be equal to zero, as under the standard approach, the substitution of the same policy rule in equation (2.48) implies that the forecast error is given by $\mathbf{e}_{1t}$. Therefore, substitution of the optimal rule into the dynamic constraint computed under the standard approach reduces the volatility of the deterministic part of the state vector, leaving unaffected the volatility of the stochastic term. In contrast, under the PVAR approach, substitution of the optimal rule into the dynamic constraint has the effect of reducing the volatility of both the deterministic and the stochastic component of the state vector.

2.3 Optimal policy with a VAR

The optimal control of a time-separable inter-temporal quadratic objective function constrained by a stochastic linear dynamic system is well known. The solution may be obtained either by using the method of dynamic programming or the method of Lagrange multipliers; both techniques lead to the same solution.[14] When dynamic programming is used the problem is commonly referred to as linear quadratic dynamic programming. We wish to compare the optimal solution based on the PVAR method with that based on the standard method using a VAR. We therefore derive the solutions for the policy variables $\mathbf{z}_{2t}$ for the case where the dynamic constraint determining the non-policy variables $\mathbf{z}_{1t}$ is the conditional VAR in equation (2.48), and for the case where the equations for $\mathbf{z}_{1t}$ are those in the reduced form VAR, equation (2.46).

In general, the quadratic loss function of the policy maker can be written as:

$$L_t = E_t \sum_{s=0}^{\infty} \beta^s \left[(\mathbf{y}_{t+s} - \bar{\mathbf{y}})' \mathbf{W} (\mathbf{y}_{t+s} - \bar{\mathbf{y}}) \right], \tag{2.50}$$

where E_t denotes mathematical expectations conditioned on time t information, $\mathbf{y}'_{t+s} = \begin{bmatrix} \mathbf{z}_{1t+s} & \cdots & \mathbf{z}_{1t+s-p} & \mathbf{z}_{2t+s} & \cdots & \mathbf{z}_{2t+s-p} \end{bmatrix}'$, $\mathbf{z}_1$ is a vector of endogenous variables, $\mathbf{z}_2$ is a vector of policy instruments, $\bar{\mathbf{y}}$ is a target vector and $\mathbf{W}$ is a symmetric positive semidefinite matrix of policy weights.[15]

The value function $V(\mathbf{y}_t)$, i.e. the minimum value at time t of the welfare loss under the infinite sequence of controls $\{\mathbf{z}_{2t+s}\}_{s=0}^{\infty}$, is given by:

$$V(\mathbf{y}_t) = \min_{\{\mathbf{z}_{2t+s}\}_{s=0}^{\infty}} E_t \sum_{s=0}^{\infty} \beta^t \left[(\mathbf{y}_{t+s} - \bar{\mathbf{y}})' \mathbf{W} (\mathbf{y}_{t+s} - \bar{\mathbf{y}}) \right].$$

Since L is a quadratic form in $\mathbf{y}_{t+s} - \bar{\mathbf{y}}$, the general structure of the value function can be guessed to be a linear combination of a quadratic, a linear and a constant term and represented as:

[14] An accurate comparison of the use of dynamic programming and Lagrange multipliers tecniques for the assessment of optimal policy rules can be found in Chow (1976).

[15] The vector $\mathbf{y}_t$ can include current and lagged values of both state and instrument variables. The representation of equation (2.50) is sufficiently general to eventually include first differences of the objective function's arguments by imposing ad hoc identities in the off-diagonal elements of the matrix $\mathbf{W}$.

$$V(\mathbf{y}_t) = \mathbf{y}_t'\mathbf{P}\mathbf{y}_t - 2\mathbf{y}_t'\mathbf{p} + \mathbf{d}, \tag{2.51}$$

where $\mathbf{P}$ is a positive semidefinite symmetric matrix of coefficients having the same order of $\mathbf{W}$, whereas $\mathbf{p}$ and $\mathbf{d}$ are vectors of coefficients compatible with $\mathbf{y}_t$. The Bellman (1957) principle can then be applied to write the value function in the recursive form

$$V(\mathbf{y}_t) = \min_{\{\mathbf{z}_{2t+s}\}_{s=0}^{\infty}} (\mathbf{y}_t - \bar{\mathbf{y}})' \mathbf{W} (\mathbf{y}_t - \bar{\mathbf{y}}) + \beta E_t [V(\mathbf{y}_{t+1})] \tag{2.52}$$

and substitution of the expression in (2.51) into the value function of equation (2.52) gives the following recursive Bellman (1957) equation:

$$\mathbf{y}_t'\mathbf{P}\mathbf{y}_t - 2\mathbf{y}_t'\mathbf{p} + \mathbf{d} = \min_{\{\mathbf{z}_{2t+s}\}_{s=0}^{\infty}} (\mathbf{y}_t - \bar{\mathbf{y}})' \mathbf{W} (\mathbf{y}_t - \bar{\mathbf{y}}) + \beta E_t \left[\mathbf{y}_{t+1}'\mathbf{P}\mathbf{y}_{t+1} - 2\mathbf{y}_{t+1}'\mathbf{p} + \mathbf{d}\right]. \tag{2.53}$$

2.3.1 Standard approach

The dynamic constraint in the standard approach is based on the sub-system of equations for $\mathbf{z}_{1t}$ in the original VAR, equation (2.46). In state-space (companion) form it can be re-written as:

$$\mathbf{y}_{t+1} = \mathbf{c} + \mathbf{C}\mathbf{y}_t + \mathbf{D}\mathbf{z}_{2t} + \mathbf{v}_{t+1}, \tag{2.54}$$

where $\mathbf{v}_{t+1} = \mathbf{e}_{t+1}$ and $E_t[\mathbf{v}_{t+1}\mathbf{z}_{2t}'] = \mathbf{0}$. Maximising the value function (2.53) subject to (2.54) gives the optimal rule, see Sack(2000) and Ljungqvist and Sargent (2004). The optimal solution is given by:

$$\begin{aligned} \mathbf{z}_{2t} &= \tilde{\mathbf{f}} + \tilde{\mathbf{F}}\mathbf{y}_t, \\ \tilde{\mathbf{f}} &= -\left(\mathbf{D}'\tilde{\mathbf{P}}\mathbf{D}\right)^{-1}\mathbf{D}'\left(\tilde{\mathbf{P}}\mathbf{c} - \tilde{\mathbf{p}}\right), \\ \tilde{\mathbf{F}} &= -\left(\mathbf{D}'\tilde{\mathbf{P}}\mathbf{D}\right)^{-1}\mathbf{D}'\tilde{\mathbf{P}}\mathbf{C}. \end{aligned} \tag{2.55}$$

where $\tilde{\mathbf{P}}$ is calculated from the stable solution to the recursive algebraic matrix Riccati equation:

$$\tilde{\mathbf{P}}_{t+1} = \mathbf{W} + \beta\mathbf{C}'\tilde{\mathbf{P}}_t\mathbf{C} - \beta\mathbf{C}'\tilde{\mathbf{P}}_t\mathbf{D}\left(\mathbf{D}'\tilde{\mathbf{P}}_t\mathbf{D}\right)^{-1}\mathbf{D}'\tilde{\mathbf{P}}_t\mathbf{C},$$

while the vector $\tilde{\mathbf{p}}$ is computed from:

$$\tilde{\mathbf{p}} = \left[\mathbf{I} - \beta\left(\mathbf{C} + \mathbf{D}\tilde{\mathbf{F}}\right)'\right]^{-1}\left[\mathbf{W}\bar{\mathbf{y}} - \beta\left(\mathbf{C} + \mathbf{D}\tilde{\mathbf{F}}\right)'\tilde{\mathbf{P}}\mathbf{c}\right].$$ [16]

Hence, in the standard method, equation (2.55), the policy instrument responds contemporaneously to $\mathbf{y}_t$. The matrix Riccati equation is non-linear but satisfies a fixed-point theorem, the solution for $\tilde{\mathbf{P}}$ must be therefore be obtained through numerical iteration.

Substituting (2.55) into (2.54) gives the state-space representation

$$\mathbf{y}_{t+1} = \mathbf{q} + \mathbf{Q}\mathbf{y}_t + \mathbf{v}_{t+1}, \tag{2.56}$$

where $\mathbf{q} = \mathbf{c} + \mathbf{D}\tilde{\mathbf{f}}$ and $\mathbf{Q} = \mathbf{C} + \mathbf{D}\tilde{\mathbf{F}}$. Equations (2.56) represents the VAR under control, which gives the behaviour of the state vector on the implicit assumption that the policy instruments have always been generated by the above policy rule, and there has been no switch of policy.

The expected loss under the standard approach may be evaluated as:

[16] See, for example, Ljungqvist and Sargent (2000), p. 1012-1014.

$$L_t = \sum_{s=0}^{\infty} \beta^s \left[\left(E_t \mathbf{y}_{t+s} - \bar{\mathbf{y}} \right)' \mathbf{W} \left(E_t \mathbf{y}_{t+s} - \bar{\mathbf{y}} \right) \right] + \sum_{s=0}^{\infty} \beta^s tr \mathbf{W} \tilde{\mathbf{\Gamma}}_s, \tag{2.57}$$

where

$$\begin{aligned}
\tilde{\mathbf{\Gamma}}_s &= E_t \left[\left(\mathbf{y}_{t+s} - E_t \mathbf{y}_{t+s} \right) \left(\mathbf{y}_{t+s} - E_t \mathbf{y}_{t+s} \right)' \right] \\
&= \left(\mathbf{I} + \mathbf{Q} + \ldots + \mathbf{Q}^{s-1} \right) \tilde{\Omega} \left(\mathbf{I} + \mathbf{Q} + \ldots + \mathbf{Q}^{s-1} \right)' \\
&= \mathbf{Q} \tilde{\mathbf{\Gamma}}_{s-1} \mathbf{Q}' + \tilde{\Omega}
\end{aligned}$$

where $\tilde{\mathbf{\Gamma}}_0 \simeq 0$ and $E\mathbf{v}_t\mathbf{v}_t' = \tilde{\Omega}$. Thus $\tilde{\mathbf{\Gamma}}_s$ is is the conditional covariance matrix of $\mathbf{y}_{t+s}$ given information at time t. Equation (2.57) shows that the expected welfare cost can be decomposed into two parts. The first term on the right hand side of equation (2.57) is the deterministic component of the welfare cost and measures the cost due to the unconditional expectation of the vector $\mathbf{y}_t$ being different from the long run target. The second term is the stochastic component of the welfare cost, which depends upon the volatility of the vector $\mathbf{y}_{t+s}$. In particular, $\tilde{\mathbf{\Gamma}}_s$ measures the volatility of the forecast error due to the presence of the disturbances $\mathbf{e}_{1t+s}$, which cause deviations of $\mathbf{y}_{t+s}$ from its expected path.

2.3.2 PVAR method

In the PVAR approach the dynamic constraint is based on equation (2.48). In state-space form it can be re-written as:

$$\mathbf{y}_{t+1} = \mathbf{a} + \mathbf{A}\mathbf{y}_t + \mathbf{B}\mathbf{z}_{2t+1} + \mathbf{u}_{t+1} \tag{2.58}$$

where $\mathbf{u}_{t+1} = \epsilon_{t+1}$ and $E_t \left[\mathbf{u}_{t+1} \mathbf{z}_{2t+1}' \right] = \mathbf{0}$. Maximisation of (2.53) subject to (2.58) gives the optimal rule:

$$\begin{aligned} \mathbf{z}_{2t} &= \mathbf{f} + \mathbf{F}\mathbf{y}_{t-1}, \\ \mathbf{f} &= -\left(\mathbf{B}'\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}'\left(\mathbf{P}\mathbf{a} - \mathbf{p}\right), \\ \mathbf{F} &= -\left(\mathbf{B}'\mathbf{P}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{P}\mathbf{A}. \end{aligned} \tag{2.59}$$

[17]

where $\mathbf{P}$ is the solution to the time-invariant Riccati equation:

$$\mathbf{P}_{t+1} = \mathbf{W} + \beta\mathbf{A}'\mathbf{P}_t\mathbf{A} - \beta^2\mathbf{A}'\mathbf{P}_t\mathbf{B}\left(\mathbf{B}'\mathbf{P}_t\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{P}_t\mathbf{A}$$

and the vector $\mathbf{p}$ can be calculated from:

$$\mathbf{p} = \left[\mathbf{I} - \beta\left(\mathbf{A} + \mathbf{B}\mathbf{F}\right)'\right]^{-1}\left[\mathbf{W}\bar{\mathbf{y}} - \beta\left(\mathbf{A} + \mathbf{B}\mathbf{F}\right)'\mathbf{P}\mathbf{a}\right].$$

This solution differs from that of Chow (1976), pp. 156-160 and 176-178 due to the presence of the discount factor. The loss function, equation (2.50), differs from Chow's which replaces $\beta^s\mathbf{W}$ by the more general $\mathbf{W}_s$. As for the previous case, the Riccati equation is highly nonlinear and it can only be solved numerically, for example, by setting initial values $\mathbf{P}_t \simeq \mathbf{0}$ and iterating until convergence into the stable value $\mathbf{P}_t = \mathbf{P}_{t+1} = \mathbf{P}$.

The behaviour of the state vector under control is usually expressed in state-space form, which is obtained by substituting the optimal rule (2.59) into equation (2.58) to give:

$$\mathbf{y}_{t+1} = \mathbf{r} + \mathbf{R}\mathbf{y}_t + \mathbf{u}_{t+1}, \tag{2.60}$$

where $\mathbf{r}_t = \mathbf{a} + \mathbf{B}\mathbf{f}$ and $\mathbf{R} = \mathbf{A} + \mathbf{B}\mathbf{F}$.

The loss function, equation (2.50), may be evaluated under control by re-writing it as:

$$L_t = \sum_{s=0}^{\infty} \beta^s \left[(E_t \mathbf{y}_{t+s} - \bar{\mathbf{y}})' \mathbf{W} (E_t \mathbf{y}_{t+s} - \bar{\mathbf{y}}) \right] + \sum_{s=0}^{\infty} \beta^s tr \mathbf{W} \mathbf{\Gamma}_s,$$

where $\mathbf{\Gamma}_s$ now measures the volatility of the forecast error due to the presence of random disturbances ϵ_{t+s} which cause deviations of $\mathbf{y}_{t+s}$ from its expected path. $\mathbf{\Gamma}_s$ may be obtained from equation (1). Denoting $E\mathbf{u}_t\mathbf{u}_t' = \mathbf{\Omega}$ then, as $\mathbf{u}_{t+s}$ has a constant variance,

$$\begin{aligned} \mathbf{\Gamma}_s &= E_t \left[(\mathbf{y}_{t+s} - E_t \mathbf{y}_{t+s}) (\mathbf{y}_{t+s} - E_t \mathbf{y}_{t+s})' \right] \\ &= \left(\mathbf{I} + \mathbf{R} + \ldots + \mathbf{R}^{s-1} \right) \Omega \left(\mathbf{I} + \mathbf{R} + \ldots + \mathbf{R}^{s-1} \right)' \\ &= \mathbf{R} \mathbf{\Gamma}_{s-1} \mathbf{R}' + \Omega \end{aligned}$$

where $\mathbf{\Gamma}_0 = 0$.

To summarise, if the original VAR disturbances are correlated, then the PVAR method should be followed instead of the standard approach which assumes that they are uncorrelated, and hence that $\mathbf{G} = \mathbf{0}$. The PVAR approach has the further advantage of yielding a semi-strucural VAR in which state variables respond to current and lagged values of the control, rather than only lagged values of the control as in the standard methodology. In this respect, the PVAR approach does not impose any timing restriction on the dynamic of state and control variables in the state space representation of the economy. One could argue that the drawback of the PVAR approach is that it yields optimal policy rules in which the control vector responds with a lag to changes in the state variables, equation (2.55), whereas the standard identification approach yields a policy rule in which the control vector responds to current changes in the state variables, equation (2.59). However, this result is not related to the identification approach but rather to the technique adopted for the computation of the optimal rule, i.e. dynamic programming. In fact, one could specify - under both identification approaches - a policy rule with a dynamic structure entirely unrestricted:

$$\mathbf{\Lambda z}_{1t} + \mathbf{z}_{2t} = \mathbf{a}^*_{20} + \mathbf{A}^*_{21}(L)\mathbf{z}_{1,t-1} + \mathbf{A}^*_{22}(L)\mathbf{z}_{2t-1},$$

and then grid search through the parameters to maximise the objective function in equation (2.50). Within this optimisation framework, the optimal policy rule computed under the PVAR approach includes feedback to both current and lagged values of the state vector.

2.4 Computing optimal monetary policy rules

Inflation targeting policies carried out in western countries over the last 15 years consist in changing the monetary policy instruments, mainly the short term interest rate, in order to achieve either explicit or implicit medium term inflation targets. As a consequence, the literature on the computation of optimal monetary policy rules usually describes central banks' preferences through ad hoc discounted quadratic loss functions embodying control over a linear weighted combination of inflation, output and changes in the short term interest rate, that is:

$$L = E_t \sum_{s=0}^{\infty} \beta^{t+s} \left[\lambda_\pi \left(\pi_{t+s} - \pi\right)^2 + \lambda_y y^2_{t+s} + \lambda_{\Delta rs} \Delta rs^2_{t+s}\right], \qquad (2.61)$$

where π_t is the rate of inflation, y_t is the output gap, rs_t is the nominal short-term interest rate controlled by the central bank, $\lambda_\pi = 1$ is the policy weight attached to inflation, while λ_y and $\lambda_{\Delta rs}$ measure the policy weights attached to y and Δrs relatively to inflation. The term Δrs_t^2, which measures the first-difference in the instrument rate, reflects the cost attached to changes in policy and it is referred to as interest rate smoothing term.[18]

The loss function is minimised with respect to the intertemporal sequence of interest rates $\{rs_{t+s}\}_{s=0}^{\infty}$ and a stochastic linear model representing the dynamic structure of the economy. Although the central bank targets only output and inflation, the optimal feedback rule has to embody responses to actions of any other decision maker that can contribute to the determination of output and inflation, either directly or indirectly. In macroeconomic analyses, the two main candidates are the fiscal authorities and the private sector. Fiscal authorities play a key role in smoothing the path of the nominal economy, mainly by setting the automatic stabilisers and, in principle, the response of the central bank to short-run fluctuations of real output and prices should be negatively correlated with the strength of the automatic stabilisers.[19] The private sector reacts to monetary policy actions by changing short and long term spending decisions, as captured by the term structure of the interest rates. The transmission mechanism is completed by the link between the fiscal and the private sector, as reflected by the correlation between the budget deficit and the long-run real interest rate.[20] Therefore, the optimal feedback rule of an inflation targeting central bank has to include feedback not only to changes in output and inflation, but also to adjustments in government's spending and financing decisions and in the term structure of the interest rates.

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18 For a critical review of the optimal monetary policy literature, see Svensson (2003).

19 For a detailed discussion, see Taylor (1995, 2000).

20 Canzoneri, Cumby and Diba (2002) discuss theoretical arguments in support of this transmission channel and provide empirical evidence of a positive and statistically significant effect of the budget surplus on the interest rate.

In light of this discussion, optimal monetary policy rules are computed by specifying a VAR model of the U.S. economy in which the endogenous vector $\mathbf{z}_t$ includes:

$$\mathbf{z}_t' = \left[\begin{array}{cccccccc} \frac{b_t}{y_t} & \frac{g_t}{y_t} & \frac{v_t}{y_t} & R_t & y_t & \pi_t & rl_t & rs_t \end{array} \right],$$

where $\frac{b_t}{y_t}$ is the net liability-to-GDP ratio, $\frac{g_t}{y_t}$ is the government net expenditure-to-GDP ratio, $\frac{v_t}{y_t}$ is the government revenue-to-GDP ratio, R_t is the effective interest rate on net government liabilities, y_t is the output gap, π_t is the rate of inflation, rl_t is the nominal long-term interest rate and rs_t is the nominal short-term interest rate.[21]

Given the specification of the objective function in equation (2.61), the first seven variables in $\mathbf{z}_t'$ are non-policy variables, whereas the only policy instrument is rs_t. Note that y_t and π_t are directly targeted by the central bank, whereas the other non-policy variables contribute to the specification of the transmission mechanism. In fact, the first four variables in $\mathbf{z}_t$ capture the response of the fiscal sector to monetary and non-monetary policy actions, while rl_t addresses adjustments in the term structure of the interest rates following monetary policy actions.

The empirical assessment employs annual data for the period 1960 to 2005, which are plotted in Figure 2.1.[22] The VAR model is estimated by selecting a lag dimension of 2, as this is the minimum number of lags required to produce serially uncorrelated residuals in all equations of the system. To check the structural stability of the VAR I computed re-

[21] Specifically, b refers to the consolidated gross financial liabilities of the government sector net of short term financial assets, such as cash, bank deposits, loans to the private sector, etc.; rs refers to interest rates on the three-month deposits; and rl refers to the ten-year government bond yield. The measures of $\frac{b}{y}$, $\frac{g}{y}$, $\frac{v}{y}$ and R are consistent with the government budget constraint and π is calculated as the rate of change in the GDP deflator.

[22] The OECD Economic Outlook is the source of all data except for the output gap series, taken from the Federal Reserve Bank of St. Louis' database.

cursive Chow tests, but with one marginal exception none were significant. I also examined recursive estimates of the VAR coefficients. These showed little variation beyond the initial start-up observations, hence supporting Rudebusch's (2002) conclusion that a monetary policy VAR for the US does not display much evidence of the sort of structural instability predicted by the Lucas Critique.

Following Rudebusch and Svensson (1999), optimal interest rate rules are computed under five alternative specifications of the objective function, obtained by varying the relative weights λ_y and $\lambda_{\Delta rs}$ and summarised in table 2.1. The first objective function sets the benchmark case, which assumes output having the same weight of inflation and the interest rate smoothing term having a weight of 0.5. Objective functions 2 and 3 look at the consequences of having a policy weight attached to output which is 5 times respectively lower or higher than that of inflation. Specifications 4 and 5 consider the effect of varying the policy weight attached to Δrs, by setting $\lambda_\pi = \lambda_y = 1$ and $\lambda_{\Delta rs}$ either equal to one or to one-tenth of λ_π.

Optimal policy rules are assessed under both the PVAR and the standard approach, and the state space representations of the dynamic constraint under the two identification approaches are reported in appendix A. Finally, the target vector $\bar{\mathbf{y}}$ is assumed to correspond with the expected value of the variables included in $\mathbf{y}$, measured from the average sample value of the state vector. Therefore, the computation of the welfare cost function in this paper takes only into account the stochastic component of equation (**??**).

2.5 Empirical results

The purpose of the empirical analysis is threefold. First, the predicted volatility of policy instruments is assessed under the PVAR and standard approach for alternative specifications of the objective function. Next, the performances of policy rules computed from both approaches are evaluated by comparing their implied welfare costs. Finally, sensitiveness analysis is carried out to appraise the impact of alternative choices of policy weights on the results, by examining the welfare effects of gradually switching the policy focus from inflation to output stabilisation. All results are reported in appendix B.

Figure 2.2 plots optimal interest rates computed from both the PVAR and the standard approach under the five specifications of the objective function described in table 2.1. Each panel in the figure includes the optimal policy instrument computed under both approaches for a specific set of policy weights, as well as the interest rate predicted from the reaction function estimated in the original VAR model. For all specifications of the objective function, the optimal policy instrument computed under the standard approach is closer to the VAR interest rate than that computed under the PVAR approach. In the latter case, evidence of a much larger gap is observed in the first half of the 1960s and from the second half of the 1970s to the first half of the 1980s. In addition, feedback rules computed from the PVAR approach seem to deliver smoother interest rates than those measured from the standard approach.

Table 2.2 presents summary statistics of the series in figure 2.2, namely the standard deviation (SD) of each interest rate and the average absolute distance (AAD) of each optimal rate from the VAR interest rate. The former statistics measure the overall volatility of the policy instrument under a specific feedback rule, whereas the latter compute the difference between the optimal and the VAR reaction functions. The statistics are calculated over the whole sample period (1960-2006), as well as the first (1960-1983) and second half (1984-2006). Two main patterns emerge from the table. First, the PVAR approach tends to deliver policy rules less reactive to changes in the state variables than predicted by the standard approach. Standard deviations of optimal rates predicted from the PVAR approach are consistently smaller than those predicted from the standard approach under all specifications of the objective function.[23] Second, the statistics confirm that optimal interest rate rules measured under the standard approach are closer to the VAR reaction function than those derived from the PVAR approach. The AADs for the whole sample and the first-half sub-sample are lower when calculated from the PVAR than the standard approach under all specifications of the objective function. This pattern is reversed during the second half of the sample when the AADs of rules computed from the PVAR approach are lower than the corresponding statistics from the standard approach. However, comparison of the average AADs for the five objective functions shows that the statistic is fairly stable between the two sub-samples under the standard approach, while the reversion of the pattern in the second half of the sample is caused by the reduction of the AAD under the PVAR approach in that period.

Figure 2.3 plots 20-periods ahead forecasts of output and inflation calculated after substitution of each optimal rule into the corresponding dynamic constraint. Panel 3.a includes forecasts for rules computed under objective functions 1, 2 and 3, whereas panel

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[23] The only exception is for optimal policy rates computed under objective function 3 in the second half of the sample, which display similar volatility under both approaches.

3.b displays forecasts obtained under objective functions 4 and 5. Each panel also includes output and inflation forecasts obtained from the original VAR and the policy targets. The figure shows that output and inflation stabilisation occurs more rapidly under the PVAR than the standard approach. Output forecasts under the PVAR approach are smoother and converge faster towards the target, while under the standard approach they have in the first quarter of the forecasting horizon a hump-shape for all specifications of the objective function. Inflation forecasts also appear to be less volatile when computed from the PVAR approach. As for the case of output, inflation forecasts under the standard approach are particularly volatile during the first 5 periods, before converging towards the target at a smoother pace.

The above predictions, together with those of the interest rate smoothing term, are employed to measure welfare losses as in equation (2.61) for the five specifications of the objective function. Tables 2.3 and 2.4 report the calculated total welfare costs, as well as – to appraise the contribution of each component of the welfare function – standard deviations of forecasts of y, π and Δrs. Each table also displays standard deviations of the forecasts of the three policy variables obtained from the original VAR model, and both unweighted and weighted welfare costs measured under the PVAR and the standard approach. In particular, table 2.3 shows welfare costs computed by using undiscounted forecasts, i.e. $\beta = 1$, whereas table 2.4 repeats the analysis by using $\beta = 1/(1-\rho)$, where ρ is equal to the corresponding sample average of 5.42 per cent. As observed for figure 3, output and inflation forecasts obtained from the PVAR approach are less volatile than the corresponding predictions from the standard approach. Under the PVAR approach output volatility is reduced, on average, by about 50 per cent and inflation volatility by about 40 per cent. Under the standard approach the average reduction in output and inflation volatility is of about 20 and 10 per cent, respectively. Consequently, the overall loss under the PVAR approach is lower than that calculated under the standard approach, for all specifications of the objective function. Discounting has the effect of smoothing the expected standard deviations over the horizon period, leaving unchanged the relative differences between the outcomes of the two identification techniques and the alternative specifications of the objective function.

The final part of the empirical assessment looked at the sensitiveness of welfare measurement to alternative policy actions. In particular, the sensitiveness analysis is carried out by solving the dynamic optimisation problem and computing the welfare cost iteratively 990 times, for λ_y ranging from 0.1 to 10. Since $\lambda_\pi = 1/\lambda_y$, any increase in the policy weight attached to the standard deviation of output corresponds to a proportional decrease in the policy weight attached to the standard deviation of inflation. The first step of the iteration, $\lambda_y = 0.1$ and $\lambda_\pi = 10$, approximates a strict inflation targeting policy, whereas the last step, $\lambda_y = 10$ and $\lambda_\pi = 0.1$, simulates a strict output targeting regime. Therefore, the key purpose of the analysis is to unfold the change in the welfare cost due to the gradual switch of the policy focus from inflation to output stabilisation.

To evaluate the contribution of each component of the welfare cost function as the policy weight attached to output volatility relative to inflation volatility increases, figure 2.4 plots changes in the standard deviation of y, π and Δrs under the PVAR approach while λ_y ranges from 0.1 to 10. The figure includes three panels as the computation is carried out by fixing $\lambda_{\Delta rs}$ respectively at 0.1, 0.5 and 1. At the beginning of the iteration marginal increases of λ_y augment the volatility of both y and π, while reducing the volatility of Δrs. These patterns, more evident in the panel corresponding to $\Delta rs = 1$, quickly fade away and output volatility begins to decline, while Δrs volatility starts to rise, as λ_y increases. Inflation volatility decreases considerably in each panel up until λ_y reaches 3.89, 3.57 and 4.44 – for $\lambda_{\Delta rs}$ respectively equal to 0.1, 0.5 and 1 – before slowly rising until the end of the iteration.[24] Therefore, when λ_y is low, increasing the weight attached to output relative to inflation reduces both output and inflation volatility. However, there is a critical combination of policy weights after which a further increase in λ_y reduces the volatility of output at the cost of a higher volatility in inflation.

Figure 2.5 repeats the previous analysis for the standard approach. At each level of $\lambda_{\Delta rs}$, output volatility monotonously decreases while λ_y rises. Inflation volatility declines to reach minimum values when λ_y is equal to 2.60 for $\lambda_{\Delta rs} = 0.1$, to 4.16 for $\lambda_{\Delta rs} = 0.5$ and to 5.24 for $\lambda_{\Delta rs} = 1$.[25] Similarly to inflation, the volatility of Δrs decreases at the beginning of the iteration and begins to increase as soon as λ_y reaches, respectively in each panel, 2.21, 2.85 and 3.21.[26] The declining pattern in output volatility is similar across the three panels, while the volatility of inflation is more heterogeneous, indicating that under the standard approach inflation stabilisation is more sensitive to the choice of $\lambda_{\Delta rs}$ than output stabilisation. Comparison of figures 2.4 and 2.5 clearly shows that both levels

[24] The volatility of inflation reaches the minimum values of respectively 0.46, 0.41 and 0.39. The corresponding values of λ_π are respectively 0.26, 0.28 and 0.23.

[25] The minimum value of inflation volatility at each of the three levels of $\lambda_{\Delta rs}$ is 0.90, 0.78 and 0.72. The corresponding λ_π's are 0.38, 0.24 and 0.19 respectively.

[26] The minimum standard deviation of the interest rate smoothing term, at each level of $\lambda_{\Delta rs}$, is 0.36, 0.31 and 0.29, reached at λ_π respectively equal 0.45, 0.35 and 0.31.

and changes in the volatility of output, inflation and the interest rate smoothing term, with respect to λ_y, are lower under the PVAR than the standard approach.

Figure 2.6 plots the change in the total welfare loss for alternative choices of λ_y under the PVAR and standard approach to appraise the sensitiveness of the total welfare cost to different values of the policy weights. The figure essentially confirms the evidence arising from the previous two graphs, since the total welfare loss computed under the PVAR approach is lower than that computed under the standard approach, at any combination of the policy weights. The figure shows that in general the total welfare cost declines during the first quarter of the iteration and increases afterwards. In the first quarter of the iteration, the patterns observed for the PVAR approach suggest that the reduction in output and inflation volatility dominates the increase in volatility coming from Δrs. This pattern is reversed after the first quarter because the decrease in output volatility is more than compensated by the increase in the volatility of the other two variables. Under the standard approach the initial reduction in the total welfare loss is due to the simultaneous reduction in the volatility of all targeted variables during the first quarter of the iteration. This pattern is reversed afterwards because, as under the PVAR approach, the fall in output volatility is more than compensated by the increase in volatility of the other two variables. Figure 2.6 also shows that the total loss has a unique local minimum value which varies between the two identification approaches and considerably depends upon the choice of λ_y and $\lambda_{\Delta rs}$. In particular, visual inspection suggests that the larger $\lambda_{\Delta rs}$, the higher the value of λ_y corresponding to the minimum welfare cost.

Table 2.5 presents summary statistics of the total welfare series plotted in figure 2.6. The table reports the minimum and the maximum values of the welfare cost, the corresponding policy weights, the average and the standard deviation of each series. Under the PVAR approach, the minimum value of the welfare cost decreases as $\lambda_{\Delta rs}$ increases, ranging from 1.86 per cent when $\lambda_{\Delta rs} = 0.1$ to 1.58 per cent when $\lambda_{\Delta rs} = 1$. In general, the total welfare cost is fairly stable across different values of $\lambda_{\Delta rs}$, being on average about 1.80 per cent with standard deviation of about 5.5 to 6 per cent. Also under the standard approach the minimum value of the welfare cost decreases as $\lambda_{\Delta rs}$ increase, even though minimum values are higher than those computed for the PVAR approach, ranging from 2.15 per cent when $\lambda_{\Delta rs} = 0.1$ to 1.75 per cent when $\lambda_{\Delta rs} = 1$. In addition, the average loss for the three values of $\lambda_{\Delta rs}$ is larger – about 2.20 per cent – and considerably more sensitive to variations in the policy weights – standard deviation of about 30 per cent – than that computed under the PVAR approach.

Figure 2.7 plots the efficiency frontier for the feasible combinations of standard deviations in output and inflation achievable by the optimal policy rules under the PVAR and the standard approach, while increasing λ_y from 0.1 to 10 and for values of $\lambda_{\Delta rs}$ equal to either 0.1 or 0.5 or 1. The shape of all efficiency frontiers plotted in the three panels ultimately reflects the patterns outlined in figures 4 and 5 for output and inflation volatility. The span of the frontier is smaller under the PVAR than the standard approach, regardless of the policy weight attached to the interest smoothing term. Each frontier shows that when the volatility of output is high, changes in the policy rules resulting from an increase in λ_y have the effect of reducing inflation, as well as output, volatility. Output volatility reaches a critical value, around 0.5 per cent under the PVAR approach and 0.8 per cent under the standard approach, after which any further increase in λ_y requires inflation to be more volatile.

Table 2.6 presents summary statistics of the efficiency frontiers plotted in figure 2.7, focusing, in particular, on the portions of the frontiers displaying the trade-off between output and inflation. The table shows that the length and the shape of the trade-off area varies considerably between the two identification approaches. Under the PVAR approach the trade-off area on the efficiency frontier covers almost 60 per cent of the iterations and it is considerably stable across the three values of $\lambda_{\Delta rs}$. On average, the trade-off begins when the policy weight attached to inflation volatility declines to about a quarter of that attached to output volatility. The net welfare gain (NWG) from increasing λ_y, calculated as the difference between the welfare gain from the reduction in output volatility and the welfare cost from the increase in inflation volatility, is rather negligible, as it is almost zero when $\lambda_{\Delta rs} = 0.1$, and 0.01 or 0.02 per cent when $\lambda_{\Delta rs}$ equals either 0.5 or 1. Under the standard approach, the length of the trade-off area on the efficiency frontier is more sensitive to the values $\lambda_{\Delta rs}$, covering about either 75 or 60 or 50 per cent of the frontier when $\lambda_{\Delta rs}$ is equal to respectively 0.1, 0.5 and 1.[27] Finally, the net welfare change is larger than that predicted under the PVAR approach and negative for all three values of $\lambda_{\Delta rs}$.

[27] The trade-off starts when λ_π declines respectively to about 0.4, 0.25 and 0.2.

2.6 Conclusion

This paper suggests a way of formulating optimal policy based on a VAR that avoids many of the problems found in the standard approach. For example, and perhaps the most important advantage, the PVAR method yields a dynamic structural representation of the economy in which state variables are conditioned upon current and lagged values of the policy instruments rather than lagged values alone, as under the standard method. This state space representation is more suitable in macroeconomic policy analysis since output and prices respond within the periods of observation to a change in policy even when observed in a monthly basis.

Since the whole process is easily automated, the PVAR method may provide a useful benchmark for use in real time against which to compare other, probably far more labour intensive, policy choices. Although basing optimal policy on a VAR has the merit of simplicity, it is not without its drawbacks. The paper shows that as a result of implementing optimal policy, the VAR under control is different from the original VAR. This is not necessarily a problem in itself, but it does draw attention to the fact that any previous changes of policy are likely to have caused structural change in the original VAR. This shows the vulnerability - at least in theory - of any VAR to structural change. The problem is further exacerbated because the VAR is just a particular time series representation of a structural model. If the parameters of the structural model alter as a result of policy changes, then we would expect the VAR coefficients to change too. In practice, like Rudebusch (2002), we find little evidence of structural change in the dynamics of a VAR suitable for analysing monetary policy for the US.

Another drawback of using a VAR is that it is not suitable for handling the effects on non-policy variables of anticipated policy changes. One cannot avoid using a structural rational expectations model if one wishes to analyse this problem. To avoid any misapprehensions, therefore, we emphasise that in arguing the merits of adopting the PVAR method for formulating policy based on a VAR, we are not suggesting that using a VAR is necessarily preferable to using a well specified structural model.

The PVAR method is compared empirically with the standard methodology by analysing monetary policy for the US since 1960 under different specifications of the welfare function. The results suggest that the path of the interest rate obtained using the PVAR method would have been smoother than that obtained under the standard approach. This suggests that the excess of volatility typically found in studies on optimal US monetary policy under the standard approach (Sack (2000)) can be also imputed to the misspecification of the forecasting model employed by the policy maker, as well as the uncertainty surrounding the precise values of the model's parameters.

Chapter 3
Measuring the fiscal stance

In this paper we propose an index of the fiscal stance suitable for practical use in short-term policy making. The index is based on a comparison of a target level of the debt-GDP ratio for a given finite horizon with a forecast of the debt-GDP ratio based on a VAR formed from the government budget constraint. This approach to measuring the fiscal stance is different from the literature on fiscal sustainability. We emphasise the importance of having a forward-looking measure of the fiscal stance for the immediate future rather than a test for fiscal sustainability that is backward-looking, or based just on past behaviour which may not be closely related to the current fiscal position. We also describe a bootstrapping methodology that can be easily implemented to attach confidence bands to the index in order to evaluate the statistical significance of the policy prescriptions arising from the empirical computation of the index. We use our methodology to construct a time series of the indices of the fiscal stances of the US, the UK and Germany over the last 25 or more years. We find that both the US and UK fiscal stances have deteriorated considerably since 2000 and Germany's has been steadily deteriorating since unification in 1989, and worsened again on joining EMU. Out-of-sample projections of the index also show that the fiscal stance is expected to improve in the United States and the United Kingdom, while further worsening in Germany.

3.1 Introduction

Recent concerns in 2004 and 2005 about the fiscal stances of the US, France and Germany and of possible reforms to the EU's Stability and Growth Pact (largely due to the errant fiscal positions of France and Germany) have renewed interest in the issue of how to measure the fiscal stance. In this paper we propose an index of the fiscal stance suitable for practical use in short-term policy making. We take a very different approach from the literature on fiscal sustainability even though, like this literature, it is based on the government intertemporal budget constraint. We emphasise the importance of having a forward-looking measure of the fiscal stance that focuses on the implications of the current fiscal stance for the immediate future. We argue against focusing on formal tests of the stationarity of debts and deficits as they are backward-looking and not necessarily a good guide to the current stance of fiscal policy. The index is based on a comparison of a target level of the debt-GDP ratio for a given finite horizon with a forecast of the debt-GDP ratio based on a VAR formed from the government budget constraint. By using a VAR forecasting model we avoid basing the index on a particular theoretical model of the economy, and the index is simple to compute and readily automated. We use our methodology to examine the fiscal stances of the US, the UK and Germany over the last 25 or more years. We find that both the US and UK fiscal stances have deteriorated considerably since 2000 and Germany's has been steadily deteriorating since unification in 1989 and worsened again on joining EMU. The index can also be employed to forecast the fiscal stance over the short run. Our result show that the fiscal stance is expected to improve in the US and the UK, but not in Germany. The emphasis on the fiscal stance, as opposed to fiscal sustainability, is a key feature of this paper. Determining whether the current fiscal stance is sustainable has proved difficult and controversial, and has limited applicability in evaluating fiscal policy in the short run. Typically, tests for fiscal sustainability focus on the dynamic properties of past debts and deficits and assume that these processes will continue into the infinite future with a view to establishing whether the present value of future primary surpluses are sufficient to meet current government debt obligations. There are obvious problems with this approach. First, a failure to satisfy a test for fiscal sustainability does not necessarily have any implications for the current fiscal stance. A government could argue that fiscal sustainability can be achieved by changing future fiscal policy so that sufficient surpluses would be generated. Or, it may be that rejection of fiscal sustainability was due to past fiscal policy and that subsequent changes had removed the problem. In both cases, the time series properties of past debts and deficits would no longer be relevant for current policy. Second, a failure to satisfy a test for fiscal sustainability has little immediate relevance if financial markets are still willing to hold government debt, perhaps in the belief that governments will make the appropriate changes to fiscal policy in the future. Third, a test statistic is not a user-friendly way of representing fiscal policy. Something more transparent is required such as an index series that can capture changes in the fiscal stance over time. Fourth, in the related literature on inter-temporal current account sustainability, the outcome of the test for sustainability depends on whether consumption is modelled correctly. We seek a measure of the fiscal stance based on the government constraint that is theory free. Although the outcome of tests for fiscal sustainability have not played much of a role in discussions on fiscal policy, a measure of the current fiscal stance would still be helpful. Such a measure should be easy to represent and compute and not depend on a particular theoretical model of the economy. Governments need to know the likely consequences of their current fiscal stance for their debt obligations and the costs of borrowing and of servicing the debt. Markets need to know the risks associated with the fiscal stance in order to price government debt. The Maastricht Treaty was an attempt to ensure that fiscal policy was set appropriately in the run-up to EMU so that the temptation to inflate away debts was avoided. Its successor, the Stability and Growth Pact, seeks to avoid fiscal spillovers from one country to another which might affect monetary policy or euro-debt obligations. It is increasingly recognised, however, that such fiscal rules are neither necessary nor sufficient. Whatever the fiscal framework, a crucial ingredient is an appropriate measure of the current fiscal stance. The index we propose is concerned with forecasting whether the debt-GDP ratio is likely to exceed or fall below a pre-specified target over a pre-specified time horizon. Given the time horizon and the target level of the debt-GDP ratio at the end of that horizon, the index is based on a comparison of the desired change in the debt-GDP ratio and a forecast of the present value of the current level of the debt-GDP ratio over the horizon derived from a simple VAR forecasting model of the economy. If the index exceeds unity then the current fiscal stance is said to be inconsistent with the debt objective over the horizon in the sense that debt is forecast to rise above target; if the index is less than unity then the fiscal stance is said to be consistent with the debt objective. The choice of a VAR model is to avoid taking a particular view of the economy and to permit the method to be easily automated. The VAR is based on a log-linear approximation to the government's inter-temporal budget constraint in order that interest rates, inflation and growth are allowed to be time varying. This approach is in contrast to much of the literature on fiscal sustainability where interest rates, inflation and growth are held constant over the forecast horizon in order to eliminate the non-linearities that their time variation would introduce into the intertemporal budget constraint. The paper is set out as follows. In Section 2 we examine a number of different ways of writing the government budget constraint and establish our notation. In Section 3 we present an analysis of fiscal sustainability with a view to showing its limitations in providing a useful measure of the current fiscal stance. We provide an intuitive rationale for the various tests for fiscal sustainability that have been proposed in the literature and discuss the technical problems in implementing these tests. We also comment on the implications of this analysis of fiscal sustainability for the debt and deficit limits of the EU's Stability and Growth Pact. In section 4 we describe how, by using a log-linear approximation to the government budget constraint we can derive our proposed fiscal index and show how it can implemented using VAR analysis. In Section 5 we calculate the index for the US, the UK and Germany over the period from the 1970's to 2005. Section 6 shows further implications of the index for policy analysis. In particular, we describe a bootstrapping methodology that can be easily implemented to construct confidence bands for the index of the fiscal stance. We also suggest that projections of the index over the future provide useful insights about the evolution and implication of fiscal policy in the medium run. Our findings are summarized in Section 7.

3.2 The government budget constraint

We begin by considering the nominal government budget constraint (GBC), the sustainability of fiscal policy and the implications of various fiscal rules, such as the EU's Stability and Growth Pact.[28] The nominal GBC can be written

$$P_t g_t + (1 + R_t)B_{t-1} = B_t + \Delta M_t + P_t T_t \tag{3.62}$$

where g_t is real government expenditure including real transfers to households, T_t is total real taxes and M_t is the stock of outside nominal, non-interest bearing money in circulation that is supplied by the government (the central bank) at the start of period t, B_t is the nominal value of government bonds issued at the end of period t, R_t is the average interest rate on bonds issued at the end of period $t-1$ and $R_t B_{t-1}$ is total interest payments made in period t.[29] Thus the left-hand side of equation (3.62) is total nominal expenditures in period t and the right-hand side is total revenues plus additions to government current financial resources.

The equivalent real GBC can be derived from the nominal GBC by dividing through the nominal GBC by the general price level P_t. This gives

$$g_t + (1 + R_t)\frac{P_{t-1}}{P_t}\frac{B_{t-1}}{P_{t-1}} = T_t + \frac{B_t}{P_t} + \frac{M_t}{P_t} - \frac{P_{t-1}}{P_t}\frac{M_{t-1}}{P_{t-1}}$$

[28] There is a substantial literature on these issues. Most of it goes back some way in time. See, for example, Hamilton and Flavin (1986), Trehan and Walsh (1988, 1991), Kremers (1989), Wilcox (1989), Blanchard et al (1990), Bohn (1992, 1995, 1998, 2005), Hakkio and Rush (1991), Buiter et al (1993), Ahmed and Rogers (1995) and Wickens and Uctum (2000). There is also a related literature on current account sustainability, see Wickens and Uctum (1993).

[29] In practice governments issue bonds at a discount and redeem them at par. Thus if all bonds were for one period, then $B_t = P_t^B B_t^G$ where B_t^G is the number of bonds issued in period t each with price $P_t^B = \frac{1}{1+R_{t+1}}$ and $B_{t-1}^G = (1 + R_t)B_{t-1}$.

or

$$g_t + (1 + r_t)b_{t-1} = T_t + b_t + m_t - \frac{1}{1 + \pi_t} m_{t-1} \tag{3.63}$$

where $\pi_t = \frac{\Delta P_t}{P_{t-1}}$ is the rate of inflation, b_t is the real stock of government debt, m_t is the real stock of money and r_t is the real rate of interest defined by

$$1 + r_t = \frac{1 + R_t}{1 + \pi_t}$$

and implying that approximately $r_t \simeq R_t - \pi_t$.

The GBC can also be expressed in terms of proportions of nominal or real GDP by dividing through the nominal GBC by nominal GDP $P_t y_t$, where y_t is real GDP. We obtain

$$\frac{g_t}{y_t} + \frac{1 + R_t}{(1 + \pi_t)(1 + \gamma_t)} \frac{b_{t-1}}{y_{t-1}} = \frac{T_t}{y_t} + \frac{b_t}{y_t} + \frac{m_t}{y_t} - \frac{1}{(1 + \pi_t)(1 + \gamma_t)} \frac{m_{t-1}}{y_{t-1}} \tag{3.64}$$

where γ_t is the rate of growth of GDP and $\frac{T_t}{y_t}$ is the average tax rate.

The total nominal government deficit (or public sector borrowing requirement, PSBR) is defined as

$$P_t D_t = P_t g_t + R_t B_{t-1} - P_t T_t - \Delta M_t$$

hence $\frac{D_t}{y_t}$, the real government deficit as a proportion of GDP is

$$\begin{aligned}\frac{D_t}{y_t} &= \frac{g_t}{y_t} + \frac{R_t}{(1 + \pi_t)(1 + \gamma_t)} \frac{b_{t-1}}{y_{t-1}} - \frac{T_t}{y_t} - \frac{m_t}{y_t} + \frac{1}{(1 + \pi_t)(1 + \gamma_t)} \frac{m_{t-1}}{y_{t-1}} \\ &= \frac{b_t}{y_t} - \frac{1}{(1 + \pi_t)(1 + \gamma_t)} \frac{b_{t-1}}{y_{t-1}}\end{aligned}$$

The right-hand side shows the net borrowing required to fund the deficit expressed as a proportion of GDP.

We also define the nominal primary deficit $P_t d_t$ (the total deficit less debt interest payments) as

$$P_t d_t = P_t D_t - R_t B_{t-1}$$

which implies that

$$\frac{d_t}{y_t} = \frac{D_t}{y_t} - \frac{R_t}{(1+\pi_t)(1+\gamma_t)} \frac{b_{t-1}}{y_{t-1}}$$

Hence the ratio of the primary deficit to GDP is

$$\begin{aligned} \frac{d_t}{y_t} &= \frac{g_t}{y_t} - \frac{T_t}{y_t} - \frac{m_t}{y_t} + \frac{1}{(1+\pi_t)(1+\gamma_t)} \frac{m_{t-1}}{y_{t-1}} \\ &= \frac{b_t}{y_t} - \frac{1+R_t}{(1+\pi_t)(1+\gamma_t)} \frac{b_{t-1}}{y_{t-1}} \end{aligned} \tag{3.65}$$

This is a non-linear difference equation in $\frac{b_t}{y_t}$. If we define

$$1 + \rho_t = \frac{1+R_t}{(1+\pi_t)(1+\gamma_t)}$$

where approximately, $\rho_t = R_t - \pi_t - \gamma_t = r_t - \gamma_t$, the real interest rate adjusted for economic growth, then equation (3.65) can be written as

$$\frac{b_t}{y_t} = (1+\rho_t)\frac{b_{t-1}}{y_{t-1}} + \frac{d_t}{y_t} \tag{3.66}$$

This is the key equation for determining the sustainability of fiscal policy. We note that the evolution of $\frac{b_t}{y_t}$ can also be written in terms of the total deficit since

$$\frac{b_t}{y_t} = \frac{1}{(1+\pi_t)(1+\gamma_t)} \frac{b_{t-1}}{y_{t-1}} + \frac{D_t}{y_t} \tag{3.67}$$

For positive inflation and growth this is a stable difference equation.

3.3 Fiscal sustainability

Fiscal sustainability concerns the evolution of $\frac{b_t}{y_t}$ and whether it remains finite or explodes. The fiscal stance is said to be sustainable if $\frac{b_t}{y_t}$ is finite and if financial markets are willing to hold the level of debt that emerges. Before describing our proposed new procedure for determining whether the fiscal stance is sustainable we review the principal methods available in the literature. All of these methods take equation (3.66) as their starting point. In discussing sustainability it is convenient to distinguish between two cases: where the discount rate ρ_t (and hence R_t,π_t and γ_t) is assumed to be constant and where it is allowed to be time varying.[30]

3.3.1 Constant discount rate

If ρ_t is assumed to be constant then from equation (3.66) $\frac{b_t}{y_t}$ evolves according to the difference equation

$$\frac{b_t}{y_t} = (1+\rho)\frac{b_{t-1}}{y_{t-1}} + \frac{d_t}{y_t} \tag{3.68}$$

where $1+\rho = \frac{1+R}{(1+\pi)(1+\gamma)}$ or, approximately, $\rho = R - \pi - \gamma$. The solution for $\frac{b_t}{y_t}$ depends on whether the equation (3.68) is stable or unstable. We consider both cases.

Case1: $\rho < 0$ (stable case)

In this case $\frac{1+\mathrm{R}}{(1+\pi)(1+\gamma)} < 1$ and equation (3.68) is a stable difference equation, and hence can be solved *backwards* by successive substitution. The expected value of the debt-GDP ratio in n period's time conditional on information at time t is

[30] Ahmed and Rogers (1995) and Bohn (1995, 2005) argue that the appropriate discount rate to use for discounting future primary surpluses is the inter-temporal marginal rate of substitution. In a complete markets full general equilibrium model this would be the real rate of return used here.

$$E_t(\frac{b_{t+n}}{y_{t+n}}) = (1+\rho)^n \frac{b_t}{y_t} + \sum_{s=0}^{n-1} (1+\rho)^{n-s} E_t(\frac{d_{t+s}}{y_{t+s}}) \tag{3.69}$$

Taking the limit as $n \longrightarrow \infty$ gives the transversality condition

$$\lim_{n\longrightarrow\infty} (1+\rho)^n \frac{b_t}{y_t} = 0 \tag{3.70}$$

If this holds then we obtain

$$\lim_{n\longrightarrow\infty} E_t(\frac{b_{t+n}}{y_{t+n}}) = \lim_{n\longrightarrow\infty} \sum_{s=1}^{n} (1+\rho)^{n-s} E_t(\frac{d_{t+s}}{y_{t+s}}) \tag{3.71}$$

The evolution of the debt-GDP ratio depends on that of $\frac{d_t}{y_t}$. Suppose that $\frac{d_t}{y_t}$ may be stochastic but is expected to grow at the rate λ, then

$$E_t(\frac{d_{t+s}}{y_{t+s}}) = (1+\lambda)^s \frac{d_t}{y_t} \tag{3.72}$$

It follows that

$$\begin{aligned}
\lim_{n\longrightarrow\infty} E_t(\frac{b_{t+n}}{y_{t+n}}) &= \lim_{n\longrightarrow\infty} \sum_{s=1}^{n} (1+\rho)^{n-s} (1+\lambda)^s \frac{d_t}{y_t} \\
&= \lim_{n\longrightarrow\infty} (1+\lambda) \left(\frac{(1+\lambda)^n - (1+\rho)^n}{\lambda - \rho} \right) \frac{d_t}{y_t} \\
&= -\frac{1}{\rho} \frac{d_t}{y_t} \quad if\ \lambda = 0
\end{aligned} \tag{3.73}$$

If $\rho, \lambda < 0$ then $\lim_{n\longrightarrow\infty} E_t(\frac{b_{t+n}}{y_{t+n}}) = 0$ and it will explode if $\lambda > 0$. Thus, the debt-GDP ratio will remain finite and positive if the ratio of the primary surplus to GDP $(-\frac{d_t}{y_t})$ does not explode. We note that if $\lambda < 0$ then $\frac{d_t}{y_t}$ is a stationary I(0) process and the expected, or long-run value of the debt-GDP ratio is zero. And if $\lambda = 0$ then $\frac{d_t}{y_t}$ is a non-stationary I(1) process, and hence $\frac{b_t}{y_t}$ will also be I(1). Moreover, $\frac{b_t}{y_t}$ and $\frac{d_t}{y_t}$ will be cointegrated with cointegrating vector $(1, \frac{1}{\rho})$. Fiscal policy is therefore sustainable provided $\frac{b_t}{y_t}$ does not grow over time.

Case 2: $\rho > 0$ **(unstable case)**

In this case $0 < \frac{(1+\pi)(1+\gamma)}{1+R} < 1$ and equation (3.68) is an unstable difference equation and hence must be solved *forwards*, not backwards as follows:

$$\begin{aligned}\frac{b_t}{y_t} &= \frac{1}{1+\rho}E_t(\frac{b_{t+1}}{y_{t+1}} - \frac{d_{t+1}}{y_{t+1}}) \\ &= (1+\rho)^{-n} E_t(\frac{b_{t+n}}{y_{t+n}}) - \sum_{s=1}^{n}(1+\rho)^{-s} E_t(\frac{d_{t+s}}{y_{t+s}}) \qquad (3.74)\end{aligned}$$

Taking limits as $n \to \infty$ gives the transversality condition

$$\lim_{n\to\infty}(1+\rho)^{-n} E_t(\frac{b_{t+n}}{y_{t+n}}) = 0 \qquad (3.75)$$

which implies that

$$\frac{b_t}{y_t} = \sum_{s=1}^{\infty}(1+\rho)^{-s} E_t(\frac{-d_{t+s}}{y_{t+s}}) \qquad (3.76)$$

We note that the right-hand side of equation (3.76) is the expected present value of current and future primary surpluses expressed as a proportion of GDP. This condition implies that current and future surpluses will be sufficient to pay-off current debt.

Suppose once more that $\frac{d_t}{y_t}$ is expected to evolve according to equation (3.72) then

$$\begin{aligned}\frac{b_t}{y_t} &= \sum_{s=1}^{\infty}(1+\rho)^{-s}(1+\lambda)^s(\frac{-d_t}{y_t}) \qquad (3.77) \\ &= \frac{1+\lambda}{\rho-\lambda}(\frac{-d_t}{y_t}) \quad if\ -1<\lambda<\rho,\ \rho>0\end{aligned}$$

Thus, provided that the current level of the debt-GDP ratio does not exceed the right-hand side, fiscal policy is sustainable and the debt-GDP ratio will grow at the rate λ, the same rate as $\frac{-d_t}{y_t}$.

If $\frac{-d_t}{y_t}$ is stationary then $-1 < \lambda < 0$ and $\frac{b_t}{y_t}$ will also be stationary. If $\lambda = 0$, so that $\frac{-d_t}{y_t}$ is I(1) then we obtain the same condition as equation (3.73)

$$\frac{b_t}{y_t} = \frac{1}{\rho}(\frac{-d_t}{y_t}) \tag{3.78}$$

implying that $\frac{b_t}{y_t}$ will be I(1) and cointegrated with $\frac{-d_t}{y_t}$.

These results can be compared with a number of well-known empirical tests for fiscal sustainability and provide some insight into the rationale behind the tests. The test of Hamilton and Flavin (1986) is based on the following version of equation (3.74)

$$\frac{b_t}{y_t} = A_0\,(1+\rho)^{-t} - \sum_{s=1}^{\infty}(1+\rho)^{-s}\,E_t(\frac{d_{t+s}}{y_{t+s}})$$

except that real debt and the real primary deficit is used rather than $\frac{b_t}{y_t}$ and $\frac{d_t}{y_t}$. On the null hypothesis that the transversality condition holds $A_0 = 0$.

Trehan and Walsh (1988) propose a cointegration test for fiscal sustainability. They measure debt and the primary deficit in real terms rather than as proportions of GDP, but Hakkio and Rush (1991) employ the test expressing the variables as proportions of GDP. If the variables have unit roots and are cointegrated with cointegrating vector $(\rho, 1)$ then fiscal policy is sustainable. (Or, if government expenditures and revenues are I(1), then the cointegrating vector with debt must be $(\rho, 1, -1)$.) This result follows immediately from equations (3.73) and (3.78). Alternatively, if the cointegrating relation between debt and the primary deficit is

$$\frac{d_t}{y_t} + \alpha \frac{b_t}{y_t} = u_t$$

where u_t is I(0), then from equation 3.68),

$$(1+\alpha)\frac{b_t}{y_t} = (1+\rho)\frac{b_{t-1}}{y_{t-1}} + u_t$$

It follows that $\frac{b_t}{y_t}$ has a unit root if $\alpha = \rho$.

3.3.2 Time-varying discount rate

In practice, ρ_t will be time-varying, not constant and so these tests will in general be invalid. We therefore revert to the original budget constraint, equation (3.66). This may be solved forwards to obtain

$$\frac{b_t}{y_t} = E_t[(\Pi_{s=1}^{n}\frac{1}{1+\rho_{t+s}})\frac{b_{t+n}}{y_{t+n}}] - E_t[\sum_{s=1}^{n}(\Pi_{i=1}^{s}\frac{1}{1+\rho_{t+i}})\frac{d_{t+s}}{y_{t+s}}] \tag{3.79}$$

if

$$\delta_{t,s} = \Pi_{i=1}^{s}\frac{1}{1+\rho_{t+i}} \leq 1 \quad for\ all\ s \geq 1$$

Hence fiscal solvency depends on the transversality condition

$$\lim_{n\to\infty} E_t[(\Pi_{s=1}^{n}\frac{1}{1+\rho_{t+s}})\frac{b_{t+n}}{y_{t+n}}] = 0 \tag{3.80}$$

which implies that

$$\frac{b_t}{y_t} = E_t[\sum_{s=1}^{\infty}(\Pi_{i=1}^{s}\frac{1}{1+\rho_{t+i}})(\frac{-d_{t+s}}{y_{t+s}})] \tag{3.81}$$

Like equation (3.76), equation (3.81) says that the present value of current and future primary surpluses must be sufficient to offset current debt liabilities. The difference is that the discount rate is compounded from time-varying rates.

In order to analyse sustainability we define the variables

$$\begin{aligned} x_t &= \delta_{t,n}\frac{b_t}{y_t} \\ z_t &= \delta_{t,n}\frac{d_t}{y_t} \end{aligned}$$

We may now write equation (3.66) as

$$\Delta x_t = z_t$$

Fiscal sustainability now requires the transversality condition

$$\lim_{n\to\infty} E_t(x_{t+n}) = 0$$

and implies that

$$x_t = -\lim_{n\to\infty} E_t[\sum_{s=1}^{n} z_{t+s}]$$

Wilcox (1989) shows that fiscal sustainability is satisfied if x_t is a zero-mean stationary process. Wickens and Uctum (2000) prove a more general result that does not require x_t to be stationary. They show that fiscal sustainability is satisfied if z_t is a zero-mean stationary process. It then follows that x_t will be an I(1) process. Trehan and Walsh (1991) argue that fiscal policy is sustainable with a variable discount rate if the *total* deficit is stationary. This result follows directly from equation (3.67). As it is a stable difference equation if nominal growth is positive, $\frac{b_t}{y_t}$ is finite (and stationary) if $\frac{D_{t+s}}{y_{t+s}}$ is stationary.

3.3.3 Stability and Growth Pact (SGP)

This was based on the original Maastricht conditions that $\frac{b_t}{y_t}$ must be less than 0.6 and $\frac{D_t}{y_t}$ must be less than 0.03. It can be shown that these conditions are neither necessary, nor sufficient for fiscal sustainability. Much has already written on the issue of necessity. To show insufficiency, consider equation (3.67) assuming that inflation and growth are constant and $\frac{b_t}{y_t}$ and $\frac{D_t}{y_t}$ are constant at these maximum values. Hence we obtain the condition

$$\begin{aligned}\frac{b}{y} &= \frac{(1+\pi)(1+\gamma)}{(1+\pi)(1+\gamma)-1}\frac{D}{y} \\ &\simeq \frac{1}{\pi+\gamma}\frac{D}{y}\end{aligned}$$

It follows, therefore, that

$$\pi+\gamma \simeq \frac{\frac{D}{y}}{\frac{b}{y}}$$

Thus, given the limits on debt and deficits specified under the SGP, the nominal rate of growth must not be less than $\frac{0.03}{0.6} \equiv 5\%$. If nominal growth were less than this then debt would rise above 60% even if the deficit limit were satisfied. Conversely, even if the deficit or debt limits were exceeded, the appropriate rate of nominal growth would still be consistent with fiscal sustainability. For example, if the deficit exceeds 3% it is still possible for the debt-GDP ratio to satisfy the 60% limit if nominal growth exceeds 5%. This shows that, in general, the SGP is neither necessary nor sufficient for fiscal sustainability.

3.4 An index of the fiscal stance

All of these tests of fiscal sustainability are of limited practicality. The main problem is that the tests are based on the past behaviour of debts and deficits whereas the sustainability of current fiscal stance is related to their future behaviour. The test outcome could be dominated by an influential, but anomalous, period in the distant past yet the current fiscal stance may still be sustainable. Even if the current fiscal stance is not sustainable, governments could claim that a policy change planned for the future would make it sustainable. As a result, the tests provide an ineffective constraint on fiscal policy, especially in the near future. This suggests that we need a more forward-looking approach that focuses on the short-term implications of the current fiscal stance. As the fiscal position varies over time, it would be helpful to have a measure that reflects this and enables historical comparisons to be made. We therefore propose constructing an index number series of the current fiscal stance.

The index is based on the inter-temporal government budget constraint. The index measures the ratio of the desired change in the discounted debt-GDP ratio over a given time horizon relative to the forecast change. The target debt-GDP ratio at the end of the horizon could be, for example, a particular number such as the 60% SGP limit, a percentage reduction or the maintenance of the current level of debt. The forecast change in the debt-GDP ratio is, in effect, the present value of current and future primary surpluses. Future primary surpluses and discount rates are forecast using a VAR based on the variables in the governmnent budget constraint. Any other forecasting model could be used instead, including a structural model of the whole economy. The reasons for choosing a such a VAR are its simplicity and its ease of replication and automation for any economy. We also wish to try to avoid taking a particular view on macroeconomic theory and on the structural of the economy. Since time variation in the future discount rate may be of importance, we base the VAR on our log-linear approximation to the government budegt constraint. The use of an index of sustainability was initially proposed by Blanchard et al. (1990) and Buiter et al. (1993). Their indices are based on a comparison of the current debt-GDP ratio and that n periods ahead with given fixed values of the deficit and discount rate. The main shortcoming of these indices of fiscal sustainability is that the future dynamic of all macroeconomic variables employed to forecast fiscal policy is set in advance through ad-hoc assumptions rather than being determined endogenously from a model of the economy (see, Cuddington 1997). In contrast, the index of the fiscal stance proposed in this paper overcomes these issues and generalises the existing indices by allowing the deficit and discount rate to be time-varying and endogenous.

3.4.1 Log-linearising the GBC

The index of the fiscal stance proposed in this paper is based upon a log-linear approximation to the government budget constraint. In principle, the log-linear approximation can be taken about several points.

One option is the steady-state solution of the GBC, assuming it exists. The problem with the steady-state solution is that, even when it exists, it may be difficult to detect in small samples, not least because, even in small samples, government control over the level and the way of financing public spending implies that long run fiscal targets are unstable and bounded to change under different fiscal regimes. One could also argue that the existence of the steady state already implies long run fiscal sustainability, hence there no need of further testing it or constructing an index of the fiscal stance. This is true to the extent that one is interested solely about the long run implications of fiscal policy. The index of the fiscal stance proposed in this paper aims at assessing the short and medium run effects of the current fiscal stance, given its long run position. In addition, it would be preferable to have a measure of the fiscal stance which can always be computed, regardless of whether or not the steady state can be detected empirically.

As a result, a second option would be to specify an intertemporal objective function for the government and then computing the optimal level of debt-GDP given the constraint of the economy. In this case, the GBC can be log-linearised about the average optimal level of debt computed from the solution to this optimisation problem.

In this paper we follow a much simpler approach and log-linearise the GBC about a balanced-budget position, in which the constant debt-GDP ratio is chosen to be consistent with a specific level set by the government. In particular, we choose the sample average of the debt-GDP ratio. Alternatively, when the index is computed for European countries the obvious candidate is the 60 per cent ratio established in the SGP. We then take the sample average of both the government revenue-GDP ratio and the interest rate on government debt. Finally, the level of government spending is determined so that it is consistent with the balanced budget position and the average measures of debt, revenue and interest rate described above.

In computing the index, as the primary deficit can take negative values, it is necessary to write the GBC in terms of total expenditures g_t and total revenues v_t both of which are strictly positive. We therefore re-write the GBC, equation (3.64), as

$$\frac{b_t}{y_t} = \frac{g_t}{y_t} - \frac{v_t}{y_t} + (1 + \rho_t)b_{t-1}$$

where

$$\frac{v_t}{y_t} = \frac{T_t}{y_t} + \frac{m_t}{y_t} - \frac{1}{(1 + \pi_t)(1 + \gamma_t)} \frac{m_{t-1}}{y_{t-1}}$$

Next we approximate the GBC about the balanced-budget solution described above. As described above, we assume $\frac{b_t}{y_t}$, $\frac{v_t}{y_t}$ and ρ_t to be constant and equal to their sample average, which we denote with $\frac{b}{y}$, $\frac{v}{y}$ and ρ respectively. The corresponding balanced budget level of $\frac{g_t}{y_t}$, denoted as $\frac{g}{y}$, is determined from to the GBC therefore satisfies

$$\frac{g}{y} = \frac{v}{y} - \rho\frac{b}{y}.$$

To log-linearise, the GBC may be re-written as

$$f(x_t) = \exp\,[\ln\frac{b_t}{y_t}] - \exp\,[\ln\frac{g_t}{y_t}] + \exp\,[\ln\frac{v_t}{y_t}] - \exp\,[\ln\,(1+\rho_t) + \ln\frac{b_{t-1}}{y_{t-1}}] = 0.$$

Noting that a first-order Taylor series approximation to $h(x_t) = \exp[\ln x_t]$ about $\ln x$ is

$$h(x_t) = x[1 + (\ln x_t - \ln x)]$$

a log-linear approximation to the GBC is given by

$$\begin{aligned} \ln\frac{b_t}{y_t} &\simeq c + \frac{g}{b}\ln\frac{g_t}{y_t} - \frac{v}{b}\ln\frac{v_t}{y_t} + (1+\rho)\ln(1+\rho_t) + (1+\rho)\ln\frac{b_{t-1}}{y_{t-1}} \qquad (3.82) \\ c &= -\rho\ln\frac{b}{y} - \frac{g}{b}\ln\frac{g}{y} + \frac{v}{b}\ln\frac{v}{y} - (1+\rho)\ln(1+\rho). \end{aligned}$$

As $\ln(1+\rho_t) \simeq \rho_t$, in effect the discount rate is an additional variable in the equation. Thus, by employing a log-linear transformation of the GBC, we may analyse fiscal position when the deficit and discount rate are time-varying using, once more, a constant coefficient linear difference equation.

Whether the difference equation is a stable or unstable depends on the sign of ρ. Assuming that $\rho > 0$, we solve the equation forwards to obtain

$$\ln \frac{b_t}{y_t} = (1+\rho)^{-n} E_t(\ln \frac{b_{t+n}}{y_{t+n}}) - \sum_{s=1}^{n} (1+\rho)^{-s} E_t(k_{t+s}) \tag{3.83}$$

$$k_t = c + \frac{g}{b} \ln \frac{g_t}{y_t} - \frac{v}{b} \ln \frac{v_t}{y_t} + (1+\rho) \ln(1+\rho_t) \tag{3.84}$$

where k_t is, in effect, the logarithmic equivalent of the primary deficit. The transversality condition is therefore

$$\lim_{n \to \infty} (1+\rho)^{-n} E_t(\ln \frac{b_{t+n}}{y_{t+n}}) = 0 \tag{3.85}$$

which implies that

$$\ln \frac{b_t}{y_t} = -\sum_{s=1}^{\infty} (1+\rho)^{-s} E_t(k_{t+s}). \tag{3.86}$$

If k_t is stationary then $\ln \frac{b_t}{y_t}$, and hence $\frac{b_t}{y_t}$, remains finite and stationary. This may occur due to the individual terms of k_t being stationary, or due to some being I(1) but being cointegrated with the appropriate cointegrating vector. If k_t and each component of k_t are I(1) then, if they also cointegrated with cointegrating vector given by the coefficients in the definition of c, then fiscal sustainability is still satisfied. From equation (3.82) the cointegrating vector is:

$$\ln \frac{b_t}{y_t} \simeq -\frac{c}{\rho} - \frac{g}{\rho b} \ln \frac{g_t}{y_t} + \frac{v}{\rho b} \ln \frac{v_t}{y_t} + \frac{1+\rho}{\rho} \ln(1+\rho_t)$$

3.4.2 Constructing the index

The basis of our proposed index is the inter-temporal log-linearized budget constraint equation (3.82). This can be re-written as:

$$(1+\rho)^{-n} E_t(\ln \frac{b_{t+n}}{y_{t+n}}) - \ln \frac{b_t}{y_t} = \sum_{s=1}^{n} (1+\rho)^{-s} E_t(k_{t+s})$$

which can be interpreted as determining in logarithmic terms the present value of primary deficits required to achieve an expected change in discounted debt. If we replace $E_t[\ln \frac{b_{t+n}}{y_{t+n}}]$ by a target level $\ln(\frac{b_{t+n}}{y_{t+n}})^*$ then we can determine whether future values of k_t are consistent with satisfying a particular target change in discounted debt given by:

$$(1+\rho)^{-n} \ln(\frac{b_{t+n}}{y_{t+n}})^* - \ln \frac{b_t}{y_t} = \sum_{s=1}^{n} (1+\rho)^{-s} E_t(k_{t+s}) \tag{3.87}$$

The left-hand side of equation (3.87) can be interpreted as the desired change in discounted debt between periods t and $t+n$. The right-hand side is the logarithmic equivalent of the present value of the primary surpluses required to achieve this desired change in discounted debt. We replace $E_t(k_{t+s})$ by forecasts of the future values of kt based on the information available at time t, including the current fiscal stance.

A measure of whether the current fiscal stance is likely to achieve the debt objective is obtained by comparing the two sides of equation (3.87). If, for example, the aim is to decrease discounted debt then the left-hand side will be negative and the right-hand side gives the present value of the primary surplus required to achieve this reduction in debt. We therefore base our measure of the consistency of the current fiscal stance with the $n-$period debt objective on the gap between the objective and the forecast outcome:

$$FS(t,n) = [(1+\rho)^{-n} \ln(\frac{b_{t+n}}{y_{t+n}})^* - \ln \frac{b_t}{y_t}] - \sum_{s=1}^{n} (1+\rho)^{-s} E_t(k_{t+s}) \gtreqless 0.$$

Our index is:

$$\begin{aligned} FSI(t,n) &= \exp[FS(t,n)] \\ &= \frac{K_{t,n}}{b_t/y_t} \\ \ln K_{t,n} &= (1+\rho)^{-n} \ln(\frac{b_{t+n}}{y_{t+n}})^* - \sum_{s=1}^{n} (1+\rho)^{-s} E_t(k_{t+s}) \\ k_t &= c + \frac{g}{b} \ln \frac{g_t}{y_t} - \frac{v}{b} \ln \frac{v_t}{y_t} + (1+\rho)\ln(1+\rho_t) \\ c &= -\rho \ln \frac{b}{y} - \frac{g}{b} \ln \frac{g}{y} + \frac{v}{b} \ln \frac{v}{y} - (1+\rho)\ln(1+\rho). \end{aligned}$$

As $n \rightarrow \infty$ the first term in $\ln k_{t,n}$ tends to zero and the index can be interpreted as comparing the existing level of the debt-GDP ratio, with the resources to pay it off. The index may be interpreted as follows:

(i) $FSI(n) = 1$ the debt-GDP ratio in period $t+n$ is forecast to be on target

(ii) $FSI(n) > 1$ the debt-GDP ratio in period $t+n$ is forecast to be below target

(ii) $FSI(n) < 1$ the debt-GDP ratio in period $t+n$ is forecast to be above target

Only in case (iii) is the forecasted present value of the primary surplus insufficient to achieve the desired change in the debt-GDP ratio. In this sense, the current fiscal stance would not be sustainable.

In practice, the special case considered by Buiter and Blanchard of maintaining a constant debt-GDP ratio over the planning horizon will usually be of most interest. In this case

$$FS(t,n) = [(1+\rho)^{-n} - 1]\ln\frac{b_t}{y_t} - \sum_{s=1}^{n}(1+\rho)^{-s}E_t(k_{t+s}) \gtreqless 0$$

The index then becomes

$$\begin{aligned} FSI(t,n) &= \exp[FS(t,n)] \\ &= \frac{K_{t,n}}{b_t/y_t} \\ \ln K_{t,n} &= (1+\rho)^{-n}\ln\frac{b_t}{y_t} - \sum_{s=1}^{n}(1+\rho)^{-s}E_t(k_{t+s}) \end{aligned} \tag{3.88}$$

Since in this case:

$$\begin{aligned} \ln\frac{b_t}{y_t} &= (1+\rho)^{-n}\ln\frac{b_t}{y_t} - \sum_{s=1}^{n}(1+\rho)^{-s}E_t(k_{t+s}) \\ &= -\frac{1}{1-(1+\rho)^{-n}}\sum_{s=1}^{n}(1+\rho)^{-s}E_t(k_{t+s}) \\ &\simeq -\frac{1}{n\rho}\sum_{s=1}^{n}(1+\rho)^{-s}E_t(k_{t+s}) \end{aligned}$$

the index could also be calculated as

$$\begin{aligned} FSI(t,n) &= \frac{K^*_{t,n}}{b_t/y_t} \\ \ln K^*_{t,n} &= -\frac{1}{1-(1+\rho)^{-n}}\sum_{s=1}^{n}(1+\rho)^{-s}E_t(k_{t+s}) \end{aligned} \tag{3.89}$$

where the numerator is now proportional to the present value of primary surpluses. We consider this case in our empirical examples below.

3.4.3 Forecasting the fiscal variables

In order to compute the index, we require forecasts of the variables of the following vector $\mathbf{z}_t$:

$$\mathbf{z}_t = \left(\frac{b_t}{y_t}, \frac{g_t}{y_t}, \frac{v_t}{y_t}, y_t, \pi_t, IRL_t, IRS_t\right)$$

where $\frac{b_t}{y_t}, \frac{g_t}{y_t}$ and $\frac{v_t}{y_t}$ denote government debt, revenue and net spending, respectively, all measured as a proportion to GDP; y_t is the output gap measured as deviation of real GDP from a quadratic trend; π_t is the inflation rate computed from the growth rate of the GDP deflator; IRL_t is the long-term interest rate; and IRS_t is the short term interest rate.

For the reasons given above, we use a VAR(p) to obtain these forecasts. This is a simple forecasting scheme that is easily implemented and is theory free. We denote the VAR by:

$$\mathbf{z}_t = \mathbf{A}_0 + \sum_{i=1}^{p} \mathbf{A}_i \mathbf{z}_{t-i} + \mathbf{e}_t, \tag{3.90}$$

where $\mathbf{e}_t \sim i.i.d.[\mathbf{0}, \Sigma]$. The vector of variables $\mathbf{z}_t$ may be I(0) or I(1). For forecasting purposes it is unnecessary to take account any non-stationarity or cointegration among the variables. Equally, if cointegration exists, a cointegrated VAR could be estimated instead of a levels VAR and the cointegrated VAR could then be written in levels to obtain (3.90). We also note that to improve the forecasts $\mathbf{z}_t$ may contain additional variables to those that appear in the budget constraint.

$n-$period ahead forecasts may be obtained using the companion form

$$\mathbf{Z}_t = \mathbf{B}_0 + \mathbf{B}\mathbf{Z}_{t-1} + \mathbf{u}_t.$$

where $\mathbf{Z}_t'=[\mathbf{z}_t', \mathbf{z}_{t-1}', ..., \mathbf{z}_{t-p+1}]$, $\mathbf{u}_t'=[\mathbf{e}_t', \mathbf{0}, ..., \mathbf{0}]$, $\mathbf{B}_0' = [\mathbf{A}_0', 0, ..., 0]$ and

$$\mathbf{B} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & . & . & \mathbf{A}_{p-1} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & . & . \\ \mathbf{0} & . & \mathbf{I} & \mathbf{0} & . \\ . & . & . & . & . \\ . & . & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}$$

The forecast of $\mathbf{Z}_{t+n}$ is

$$E_t[\mathbf{Z}_{t+s}] = \sum_{i=0}^{s-1} \mathbf{B}^i\mathbf{B}_0 + \mathbf{B}^s\mathbf{Z}_t$$

Expressing k_t as the following linear function of $\mathbf{z}_t$

$$k_t = a + \boldsymbol{\beta}'\mathbf{z}_t$$

and defining the selection matrix $\mathbf{S} = [\mathbf{I}, \mathbf{0}, \mathbf{0}, .., \mathbf{0}]$ such that

$$\mathbf{z}_t = \mathbf{S}\mathbf{Z}_t$$

we obtain

$$\begin{aligned} FS(t,n) &= \ln K^*_{t,n} - \ln \frac{b_t}{y_t} \\ &= -\frac{1}{1-(1+\rho)^{-n}} \sum_{s=1}^{n} \{(1+\rho)^{-s} \left[a + \boldsymbol{\beta}'\mathbf{S}(\sum_{i=0}^{s-1} \mathbf{B}^i\mathbf{B}_0 + \mathbf{B}^s\mathbf{Z}_t)\right]\} - \ln \frac{b_t}{y_t} \end{aligned}$$

As the last term $\ln \frac{b_t}{y_t}$ is also a linear function of $\mathbf{Z}_t$, $FS(t,n)$ could just be written as

$$FS(t,n) = a_n + \mathbf{b}_n'\mathbf{Z}_t$$

where a_n is a scalar dependent on the time horizon and $\mathbf{b}_n$ is a vector. This emphasizes that $FS(t,n)$ is based on information available at time t,and in particular the current fiscal stance. Increasing the forecast horizon alters a_n and $\mathbf{b}_n$, but not $\mathbf{Z}_t$.

To implement this in practice it will be necessary to estimate a_n and $\mathbf{b}_n$ from the VAR estimates. The choice of ρ and c could be based, for example, on the average values in the sample, their time t values or their average values over the forecast period. A time series for $FS(t, n)$ could be calculated from the sample either using all of the sample observations to estimate the VAR, or recursively using only observations up to period t.

Note that for the empirical computation of the index, no measure of ρ_t is included in the VAR. This is because we take only the n-periods ahead forecast of $\frac{b_t}{y_t}$, $\frac{g_t}{y_t}$ and $\frac{v_t}{y_t}$ and then compute the corresponding value for ρ_t in such a way that the one-period GBC is satisfied in its forecasts. This implies that one can construct the index of the fiscal stance also by comparing the debt-GDP ratio forecasted $n-$periods ahead with the corresponding targeted level, rather than constructing the right-hand side of the GBC as described above. We do not follow this alternative approach because in this case the index cannot be decomposed, hence it would not be possible to understand the reasons of the misalignment between the forecasted debt and the its targeted level.

3.5 Indices of the fiscal stance of the US, the UK and Germany

We now construct a time series of the index of the fiscal stance for the US, the UK and Germany. For the US we consider three horizons: one-year, two-years and five-years ahead. For the UK and Germany we use just a one-year horizon. We assume that the aim in each period is to maintain the current level of the debt-GDP ratio. Hence, we use the version of the index given by equation (3.89). The data are annual and range from 1960 to 2005 for the US, from 1970 to 2005 for the UK, and from 1977 to 2009 for Germany. The data sources and the construction of the variables are described in the Appendix. There are minor differences in definitions for the different countries. For example, the debt data for the US are measured as net liabilities. This is different from the Maastricht definition of debt but, given the definitions of the other variables, is consistent with the government budget constraint. Table 3.1 gives the average values for v, b , g, b and ρ in Germany, the UK and the US.

Table 3.1: Balanced-budget GBC (in percentage)

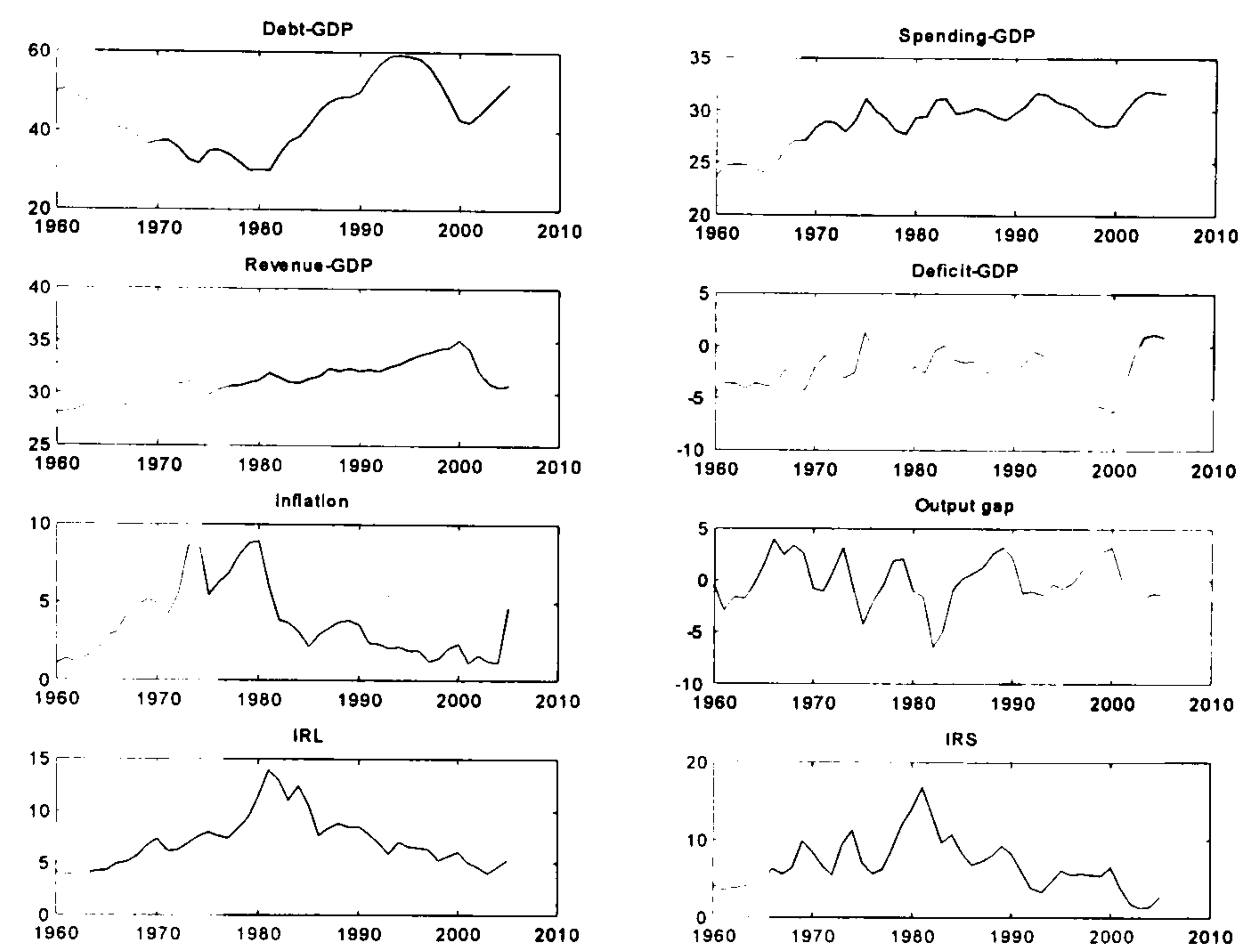
	b	g	v	ρ
Germany	48.89	43.17	44.13	1.96
United Kingdom	35.22	40.74	41.64	2.57
United States	43.81	28.86	31.24	5.44

Note: b, v and ρ are sample averages, g is constructed from balanced budget equation $v = g + \rho b$

3.5.1 The United States

Figure 3.1 gives a plot of eight key variables: the debt-GDP ratio (Debt-GDP), the government spending in goods and services as a proportion of GDP (Spending-GDP), the government revenue as a proportion of GDP (Revenue-GDP), the implied primary deficit-GDP ratio (Deficit-GDP), the inflation rate computed from the change in the GDP deflator (Inflation), the deviation of real GDP from a quadratic trend (output gap), the long term interest rate (IRL) and the short term interest rate (IRS).

Figure 3.1: US data plot



Augmented Dickey-Fuller tests for these variables using up to 6 lags suggest that we cannot reject a unit root for any variable other than the output gap. As we are using the VAR only for forecasting we estimate a VAR in levels of the variables and ignore any possible cointegration arising from the variables that have unit roots. For space reasons we do not report the results from the Augmented Dickey-Fuller and the VAR estimates, but we note that a lag of 1 produces serially uncorrelated residuals.

We examine fiscal sustainability based a constant target debt-GDP ratio for three horizons: one-year, two-years and five-years ahead. For each horizon we present four figures. Figures 3.2.1, 3.2.2 and 3.2.5 are plots of $FSI(n)$, the index of the fiscal stance. We recall that $FSI(n) < 1$ implies that the debt-GDP ratio is forecast to be above target. The forecasts are based on estimates of the VAR for the whole sample.

Figures 3.3-3.5 give various breakdowns of the index into its component parts. Thus, Figures 3.3.1, 3.3.2 and 3.3.3 are plots of $ln\frac{b_t}{y_t}$ and the forecast logarithm of the present value of current and future primary surpluses, $\ln K_{t,n}$, which we denote in the graph by $EPVGBC(n)$. There are three components to $FS(t,n)$: the desired change in discounted debt $PVdb(n)$, the present value of the primary surplus $PVs(n)$ and the term for the discount factor, $PVrho(n)$. These are plotted in Figures 3.4.1, 3.4.2 and 3.4.5. Finally, in Figures 3.5.1, 3.5.2 and 3.5.5 we plot the two components of $PVs(n)$. These are the present value of revenues $PVv(n)$ and of expenditures $PVg(n)$.

(i) One-year horizon

Figure 3.2.1

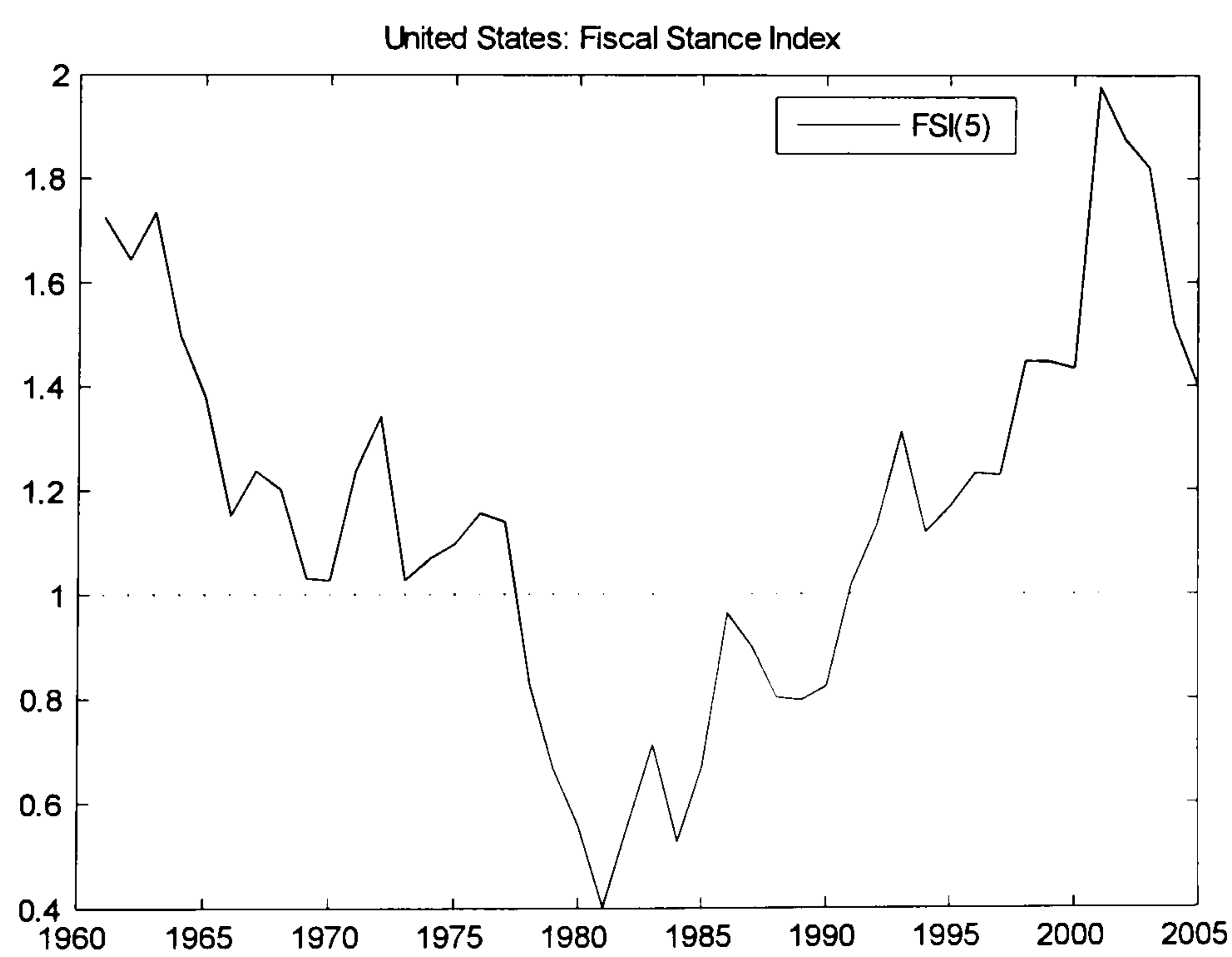


Figure 3.3.1

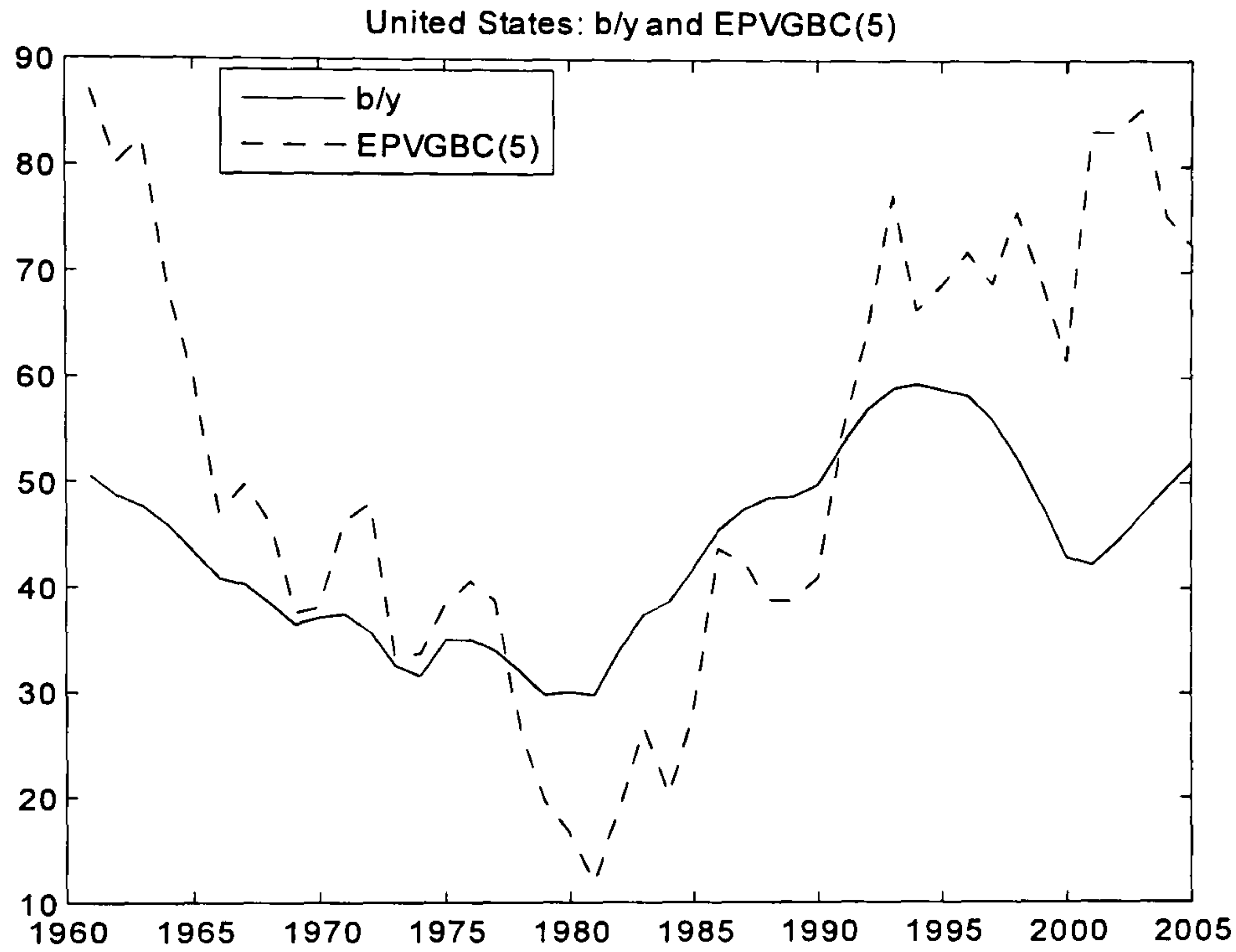


Figure 3.4.1

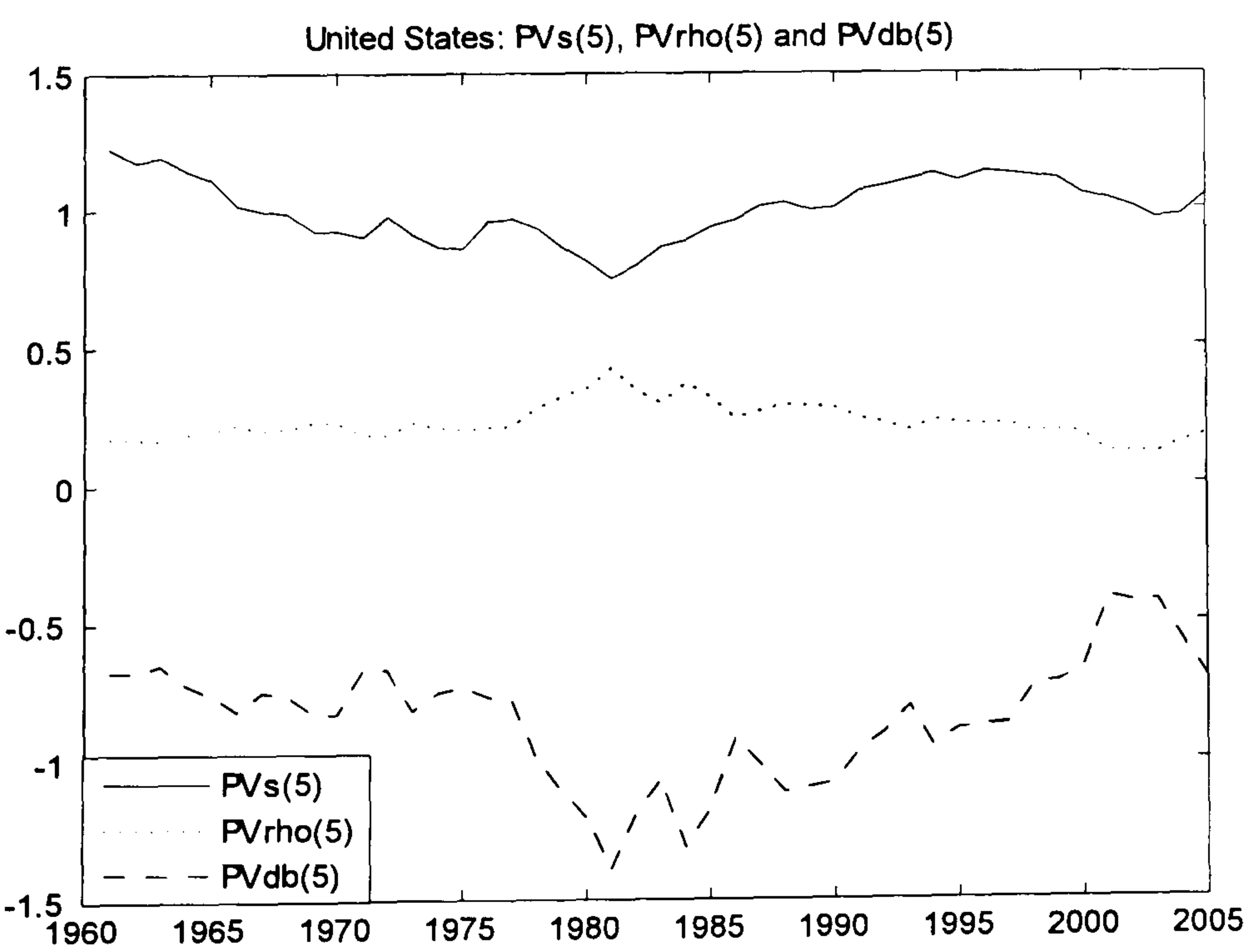
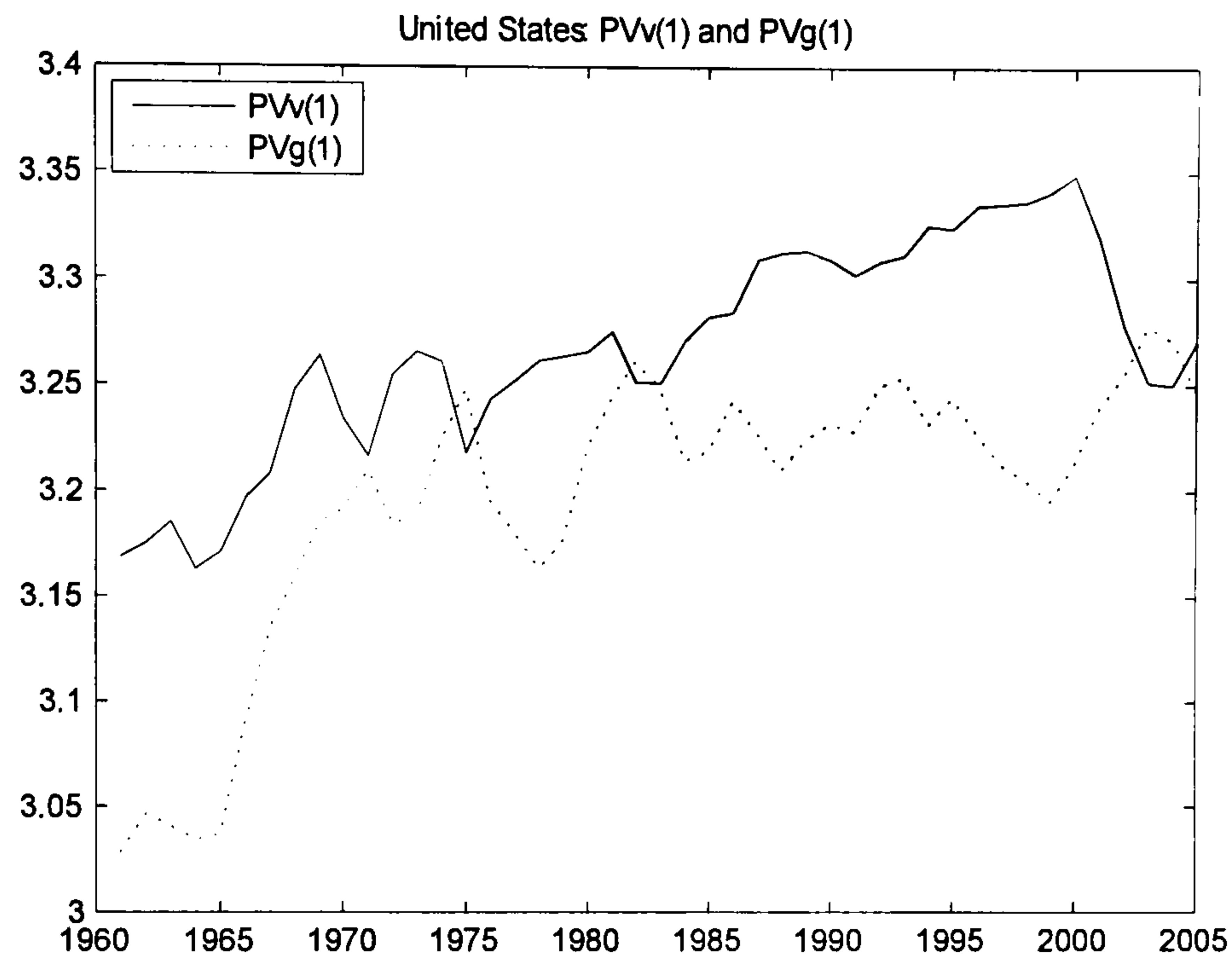


Figure 3.5.1



(ii) Two year horizon

Figure 3.2.2

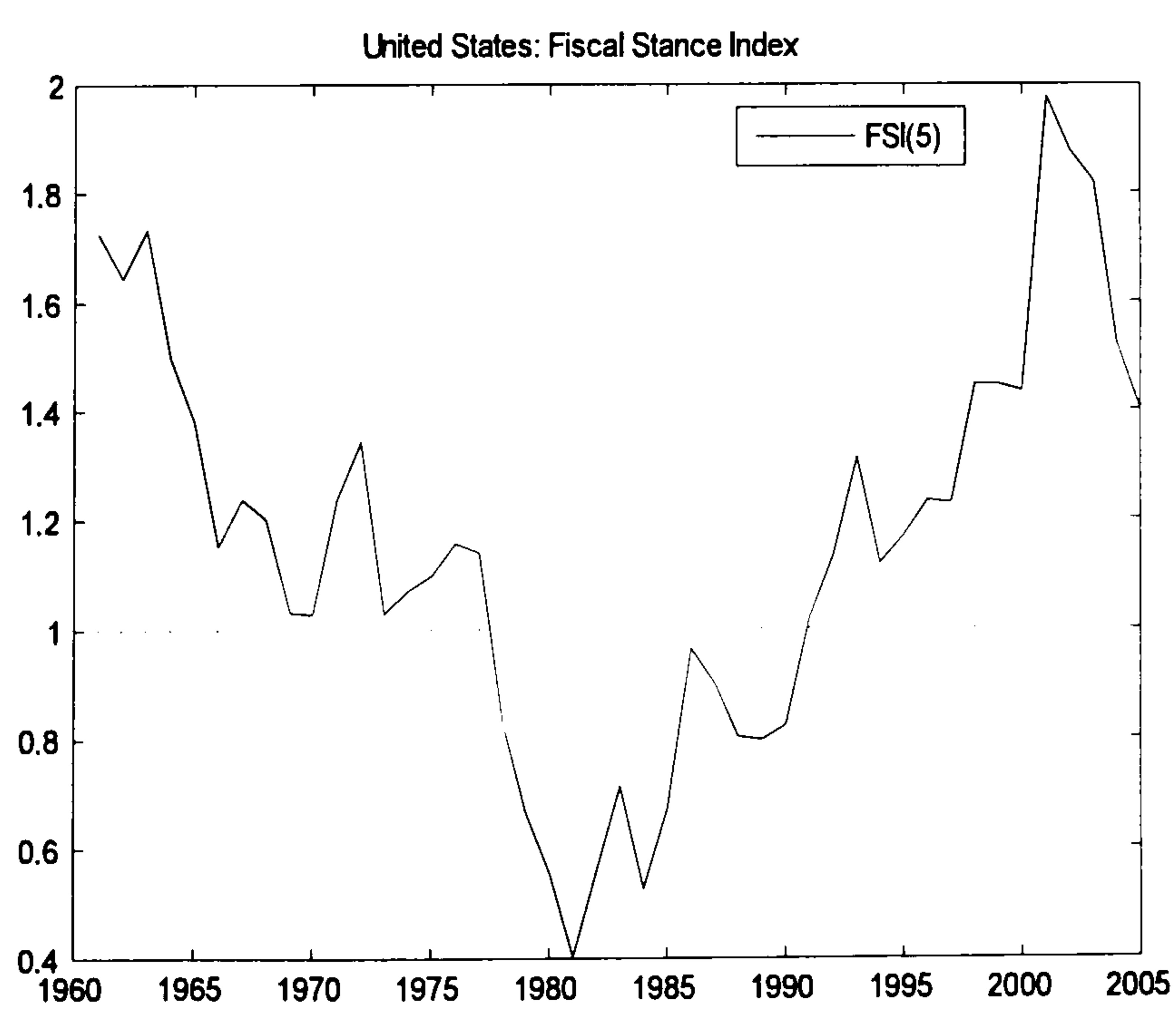


Figure 3.3.2

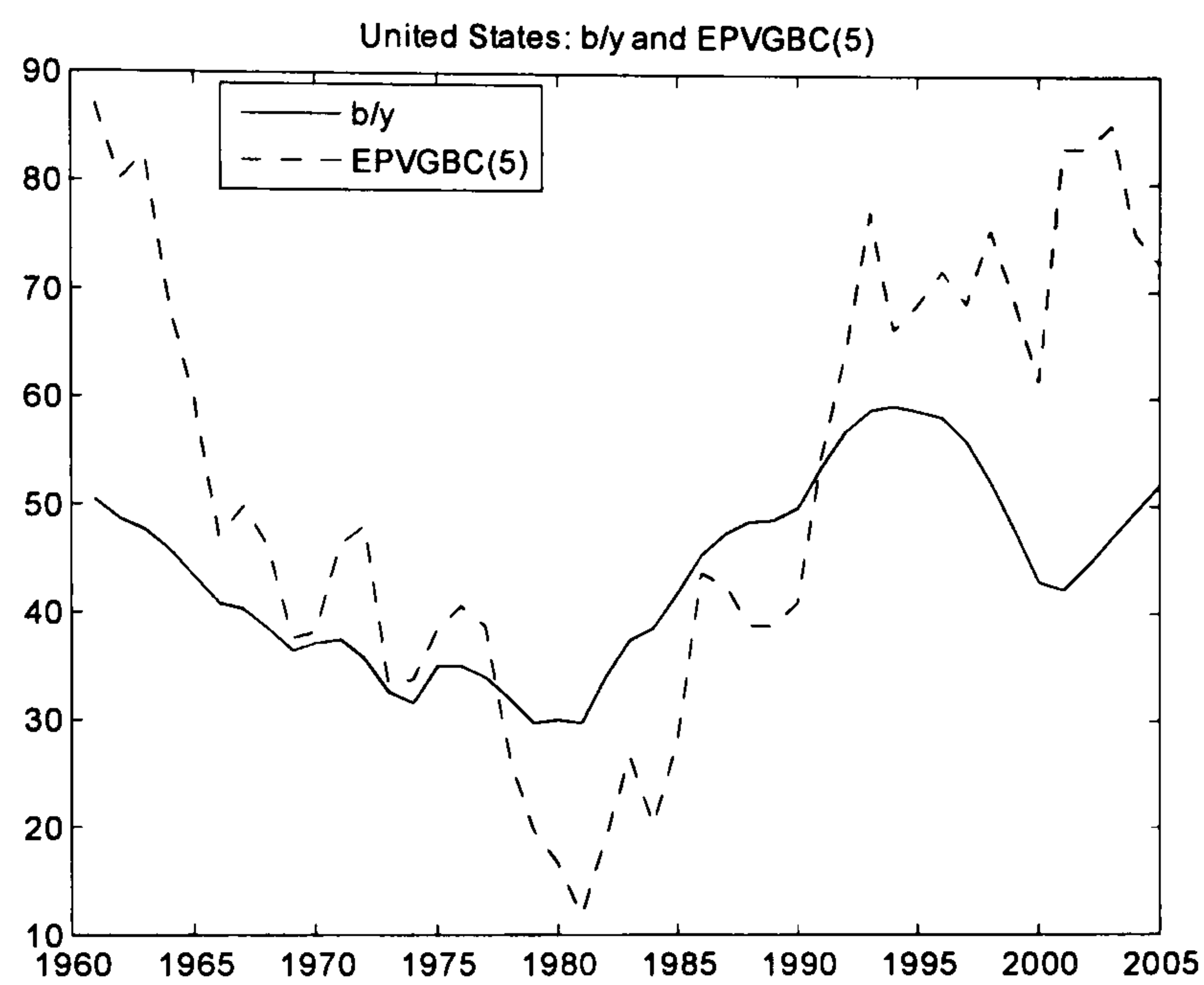


Figure 3.4.2

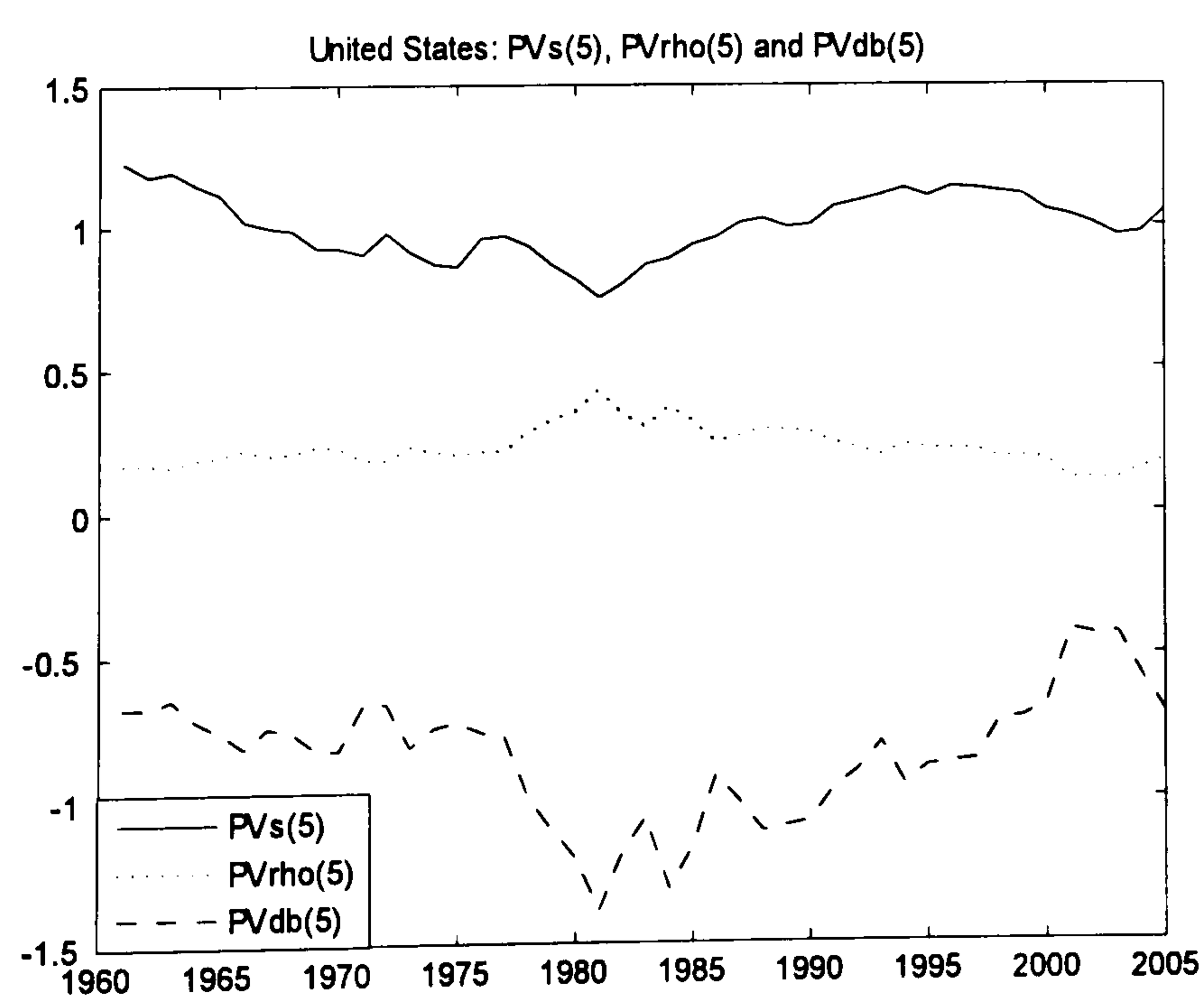
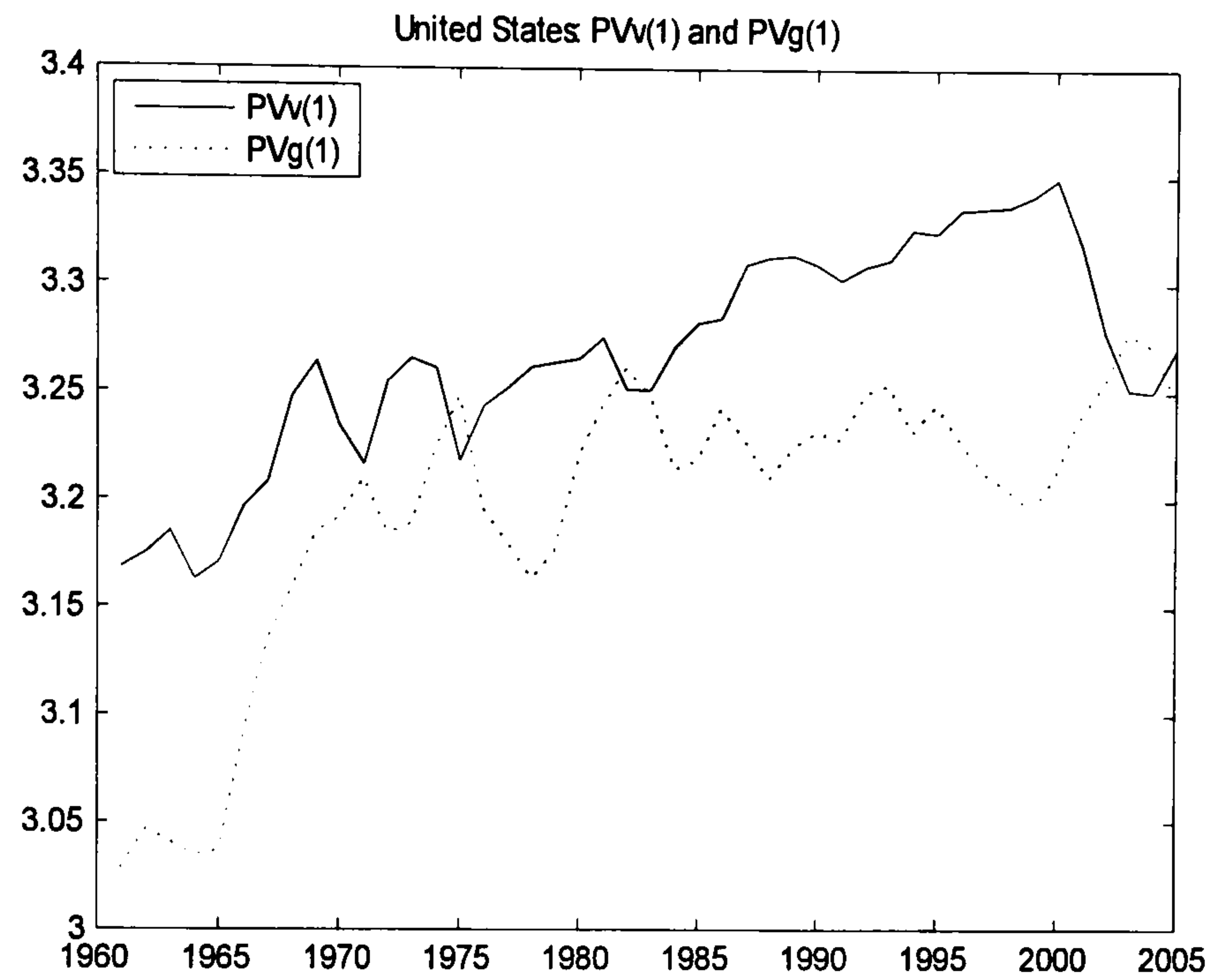


Figure 3.5.2



(iii) Five-year horizon

Figure 3.2.5

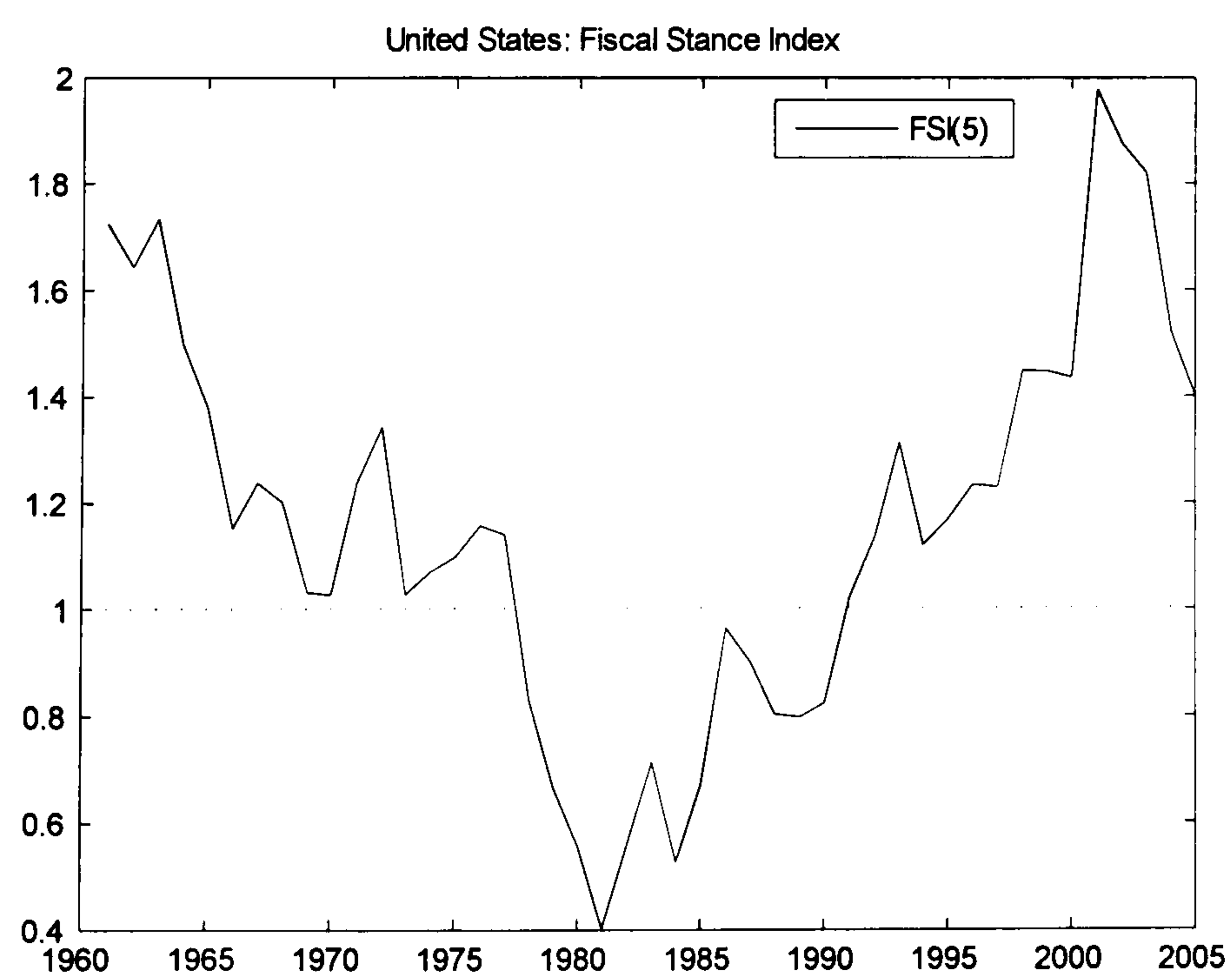


Figure 3.3.5

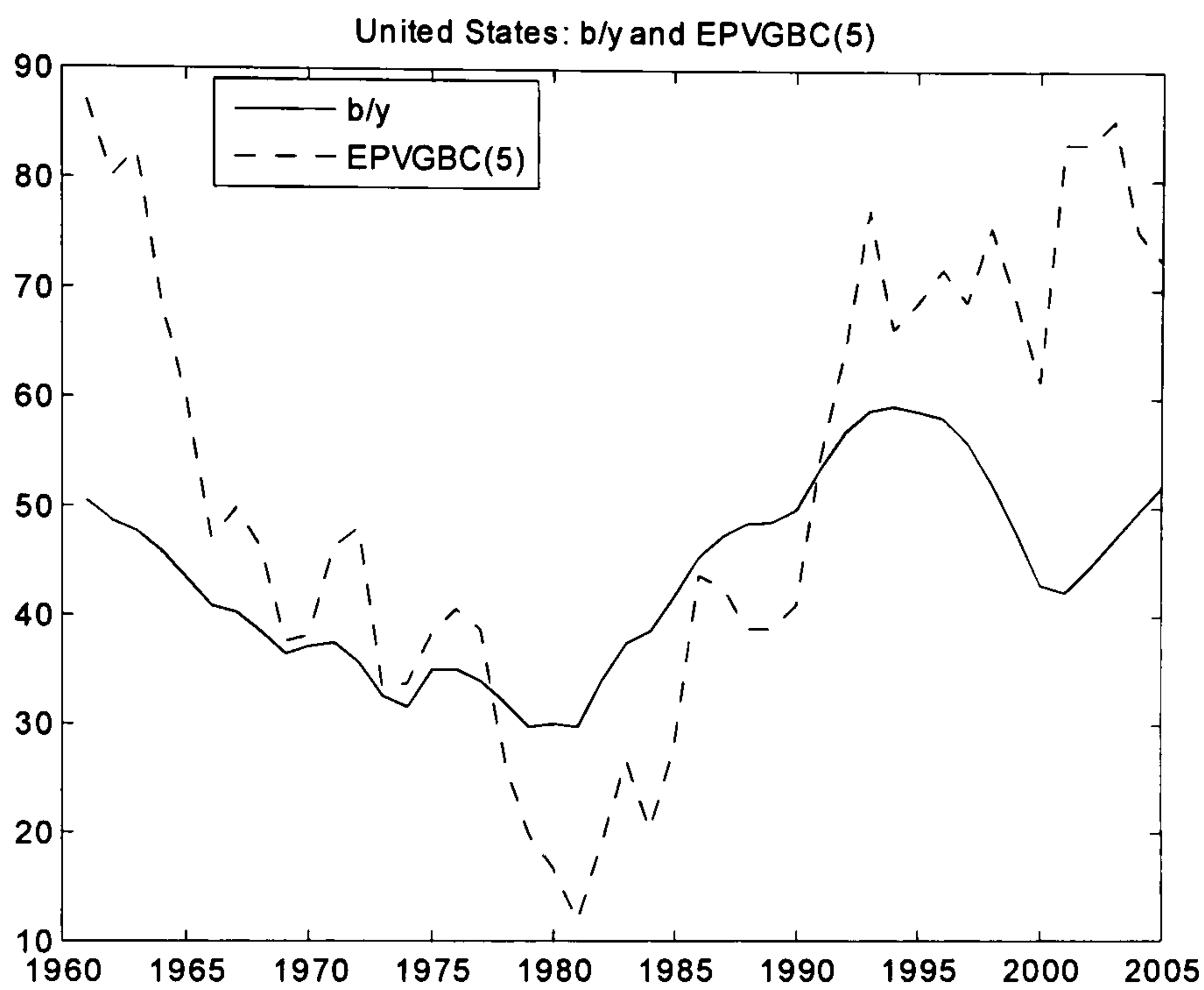


Figure 3.4.5

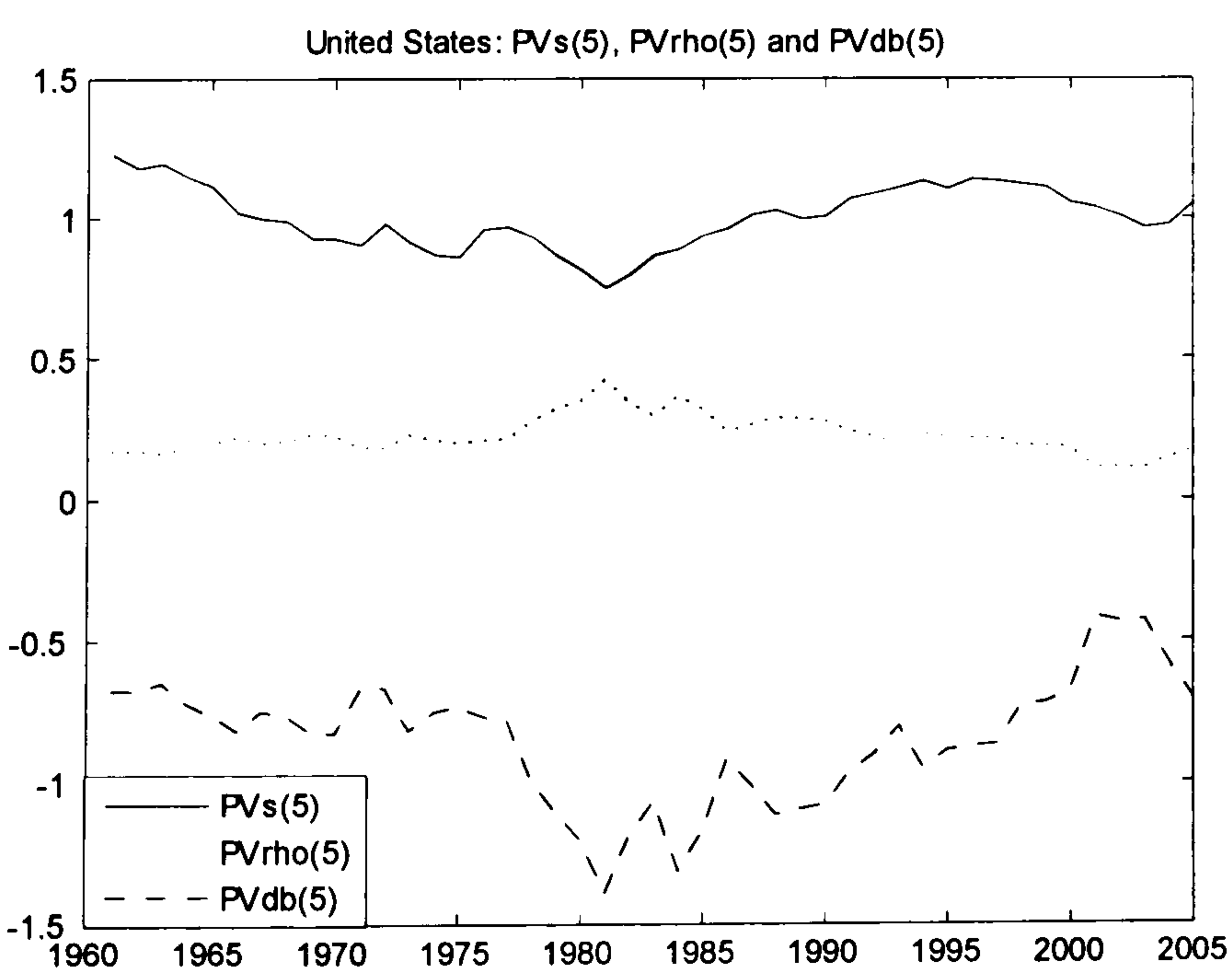
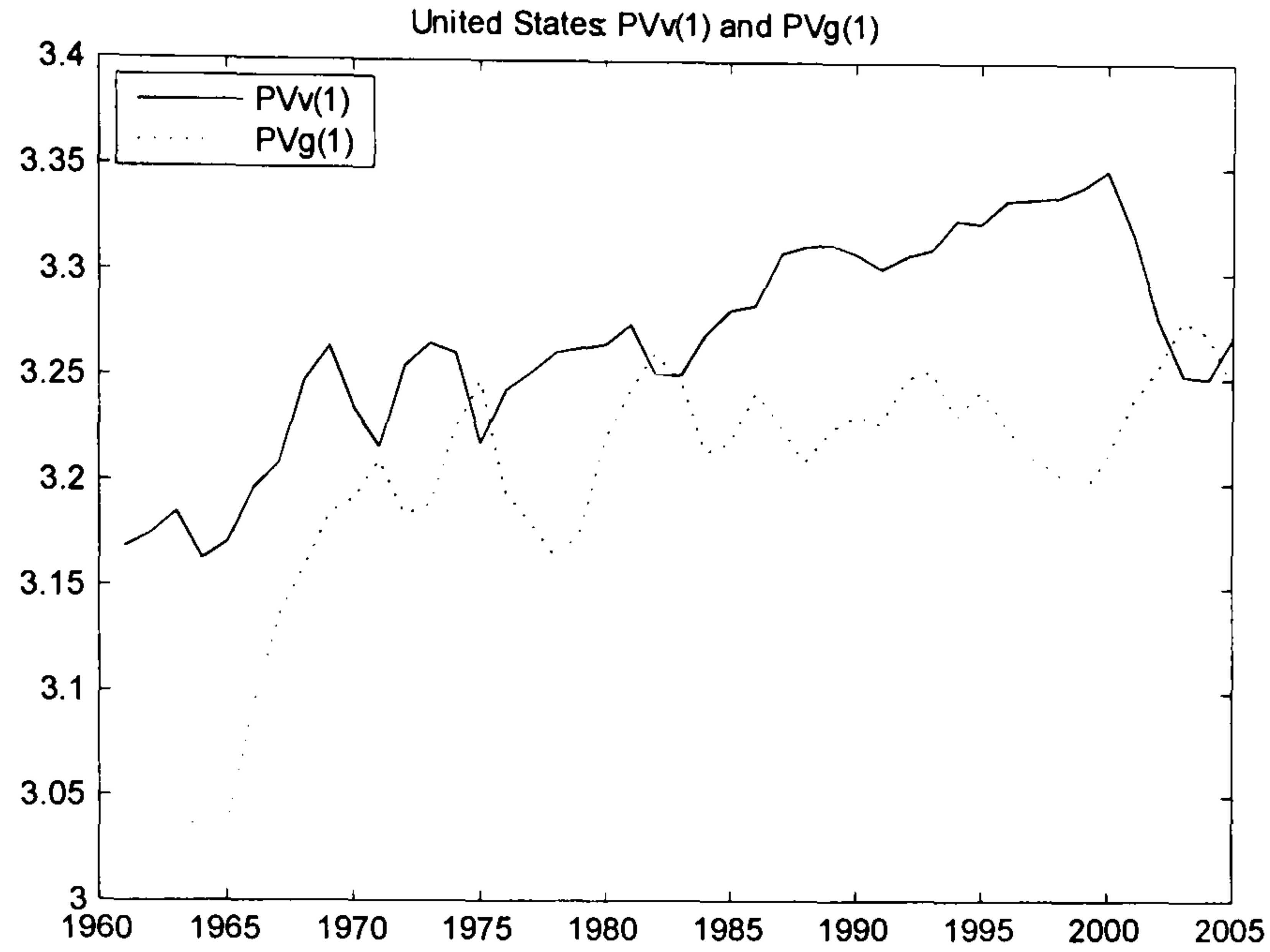


Figure 3.5.5



We observe that FSI(n), the index of the fiscal stance, exceeds unity for any length of time only during the early 1960's and the 1990's. In the other periods it is either roughly equal to unity (implying that the fiscal stance is compatible with a non-rising debt-GDP ratio) or less than unity (implying that the debt-GDP ratio is rising).

From 2001 the FSI strongly indicates a rising level of the debt-GDP ratio at each horizon. The FSI is also less than unity for the period ending in 1989. The start date of this period depends on the time horizon. For one-year and two-year horizons it is similar, consisting of most of the 1980's, but for the five-year horizon it extends back through the 1970's, almost to 1965. Thus the 1990's marked a period of US fiscal recovery which ended in around 2000.

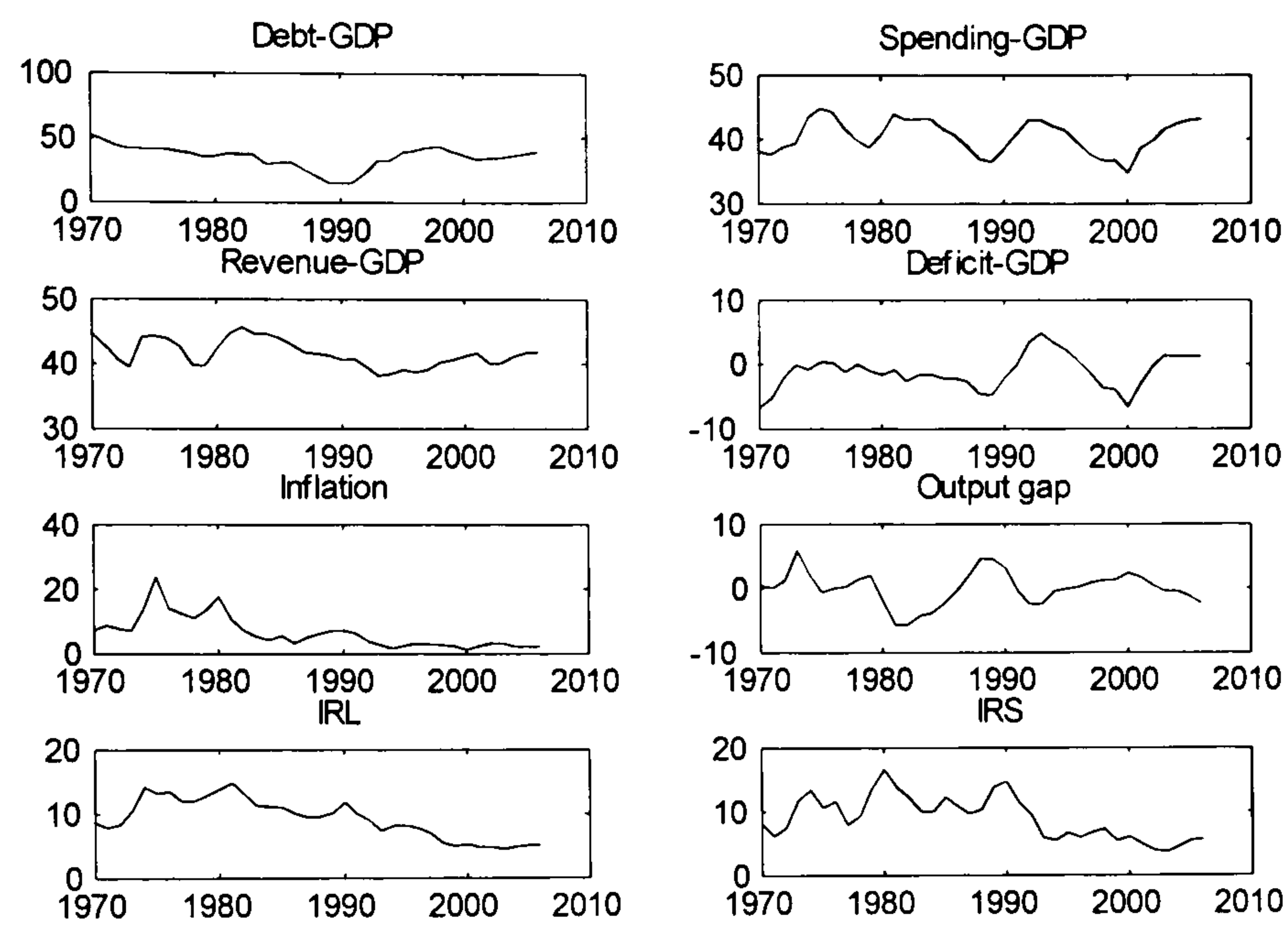
Decomposing the index into its components, we find that FSI<1 for the period 1979-1994 when the debt-GDP ratio rose substantially. We also find that variations in the present value of forecast primary surpluses are the main determinant of fluctuations in the index.

The change in debt target and the discount factor nearly offset each other. This is because we have assumed a constant discounted debt target and so the discount factor is the variable causing the change in discounted debt term to fluctuate. The present values for expenditures and revenues are similar before 1995 but are different thereafter. In the period 1995-2001 the present value of revenues exceed those of expenditures thereby producing a fiscal recovery. After 2001 the present value of expenditures exceed those of revenues. This fiscal deterioration was due to a combination of rising expenditures and sharply falling revenues. Fluctuations in the discount rate make an additional, but not large, contribution. To summarize, there is clear evidence of a break in US fiscal policy from 2001 that has resulted in a rising debt-GDP ratio no matter the horizon over which we look. This fiscal stance would be unsustainable if maintained. The cause is a combination of a rising present value of expenditures and of sharply falling revenues. There have been previous periods when the fiscal stance also led to a rising debt-GDP ratio, most notably from 1979-1994. This was not fully corrected until the period 1995-2000 when the present value of expenditures was reduced and was much lower than that of revenues.

3.5.2 The United Kingdom

The data are annual for the period 1970 to 2005 and are plotted in Figure 3.6.

Figure 3.6: UK data



Based once again on a levels VAR(1), but considering only a one-year horizon, we obtain the measures of the index reported in Figures 3.7-3.10.

Figure 3.7

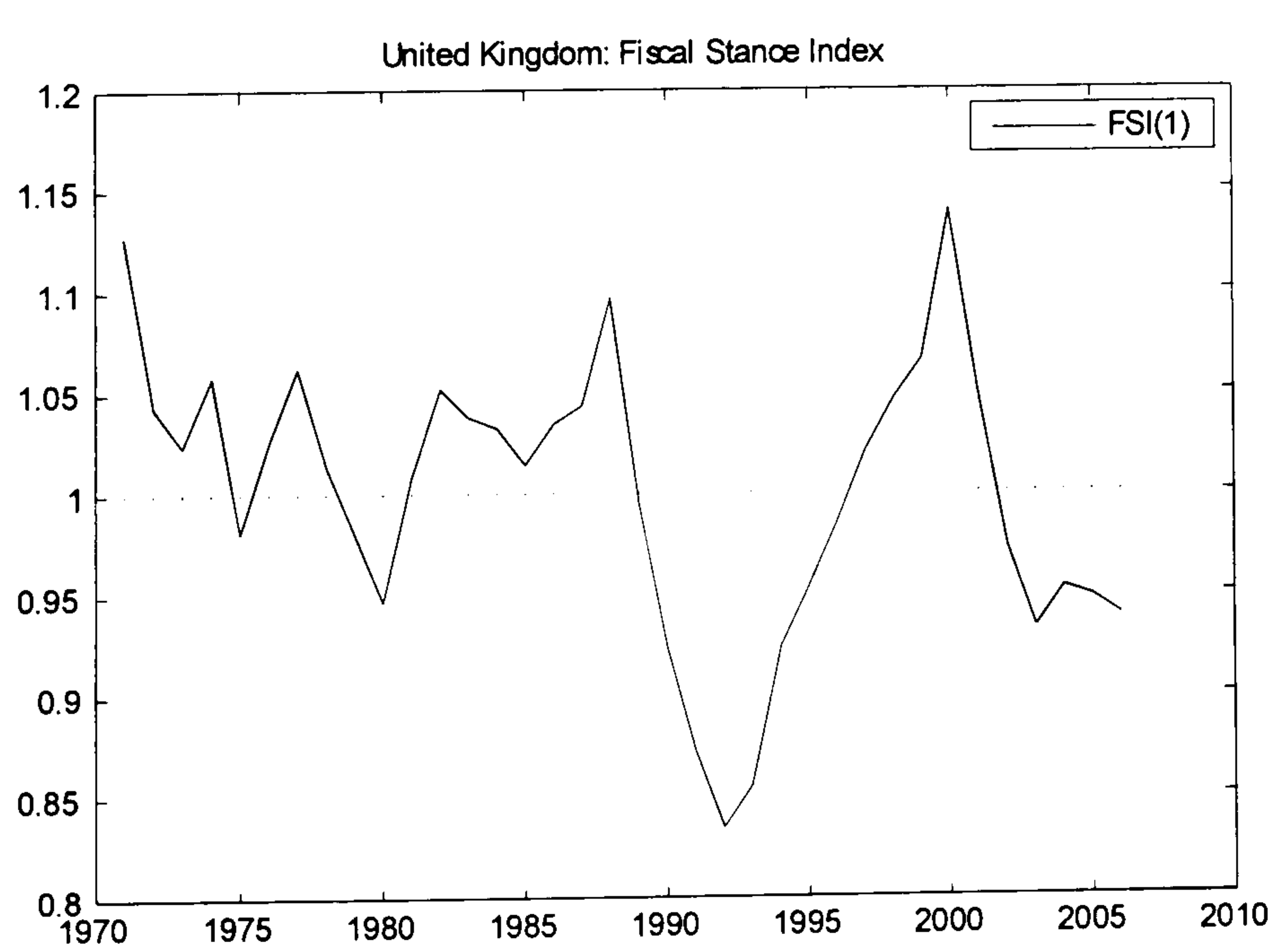


Figure 3.8

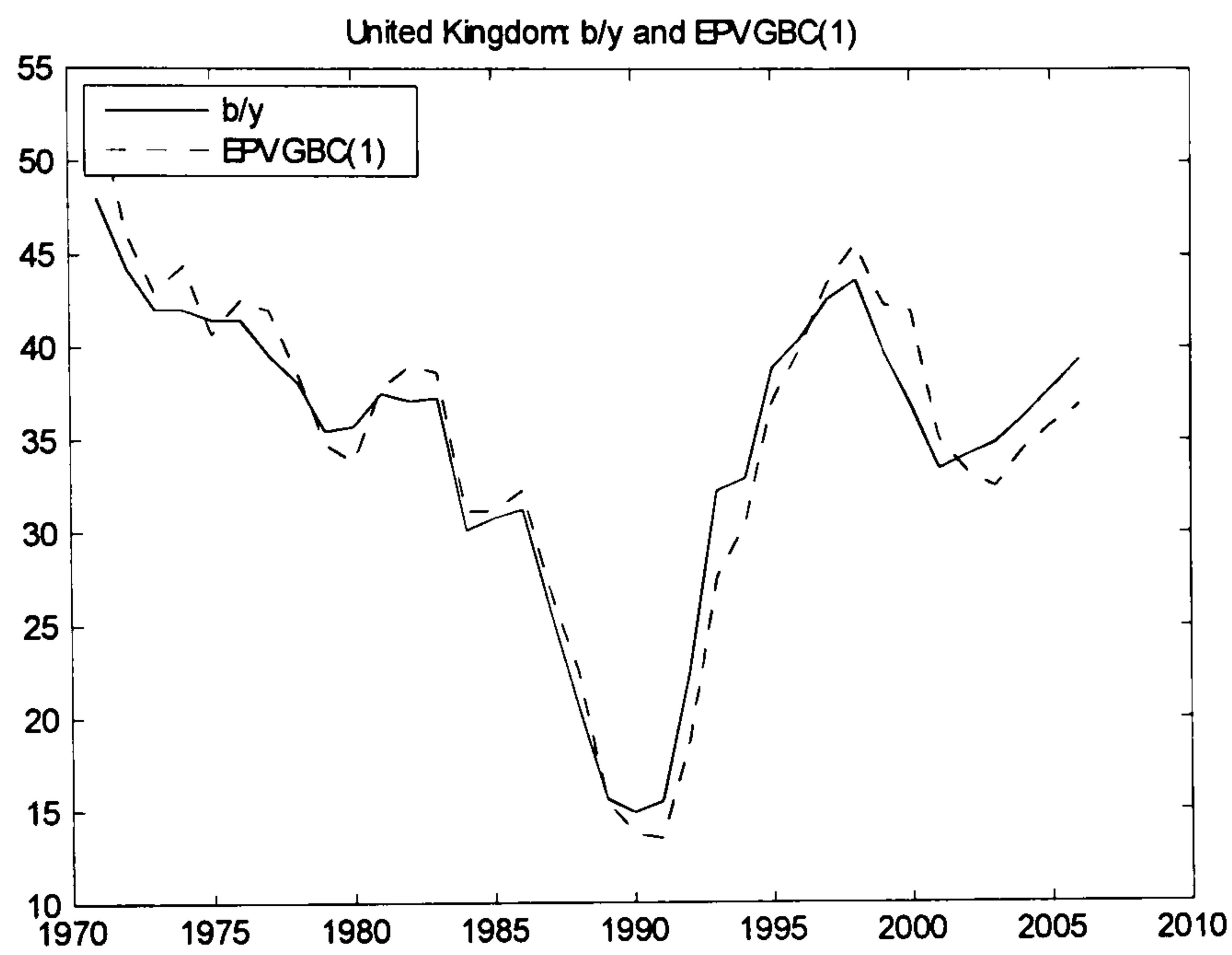


Figure 3.9

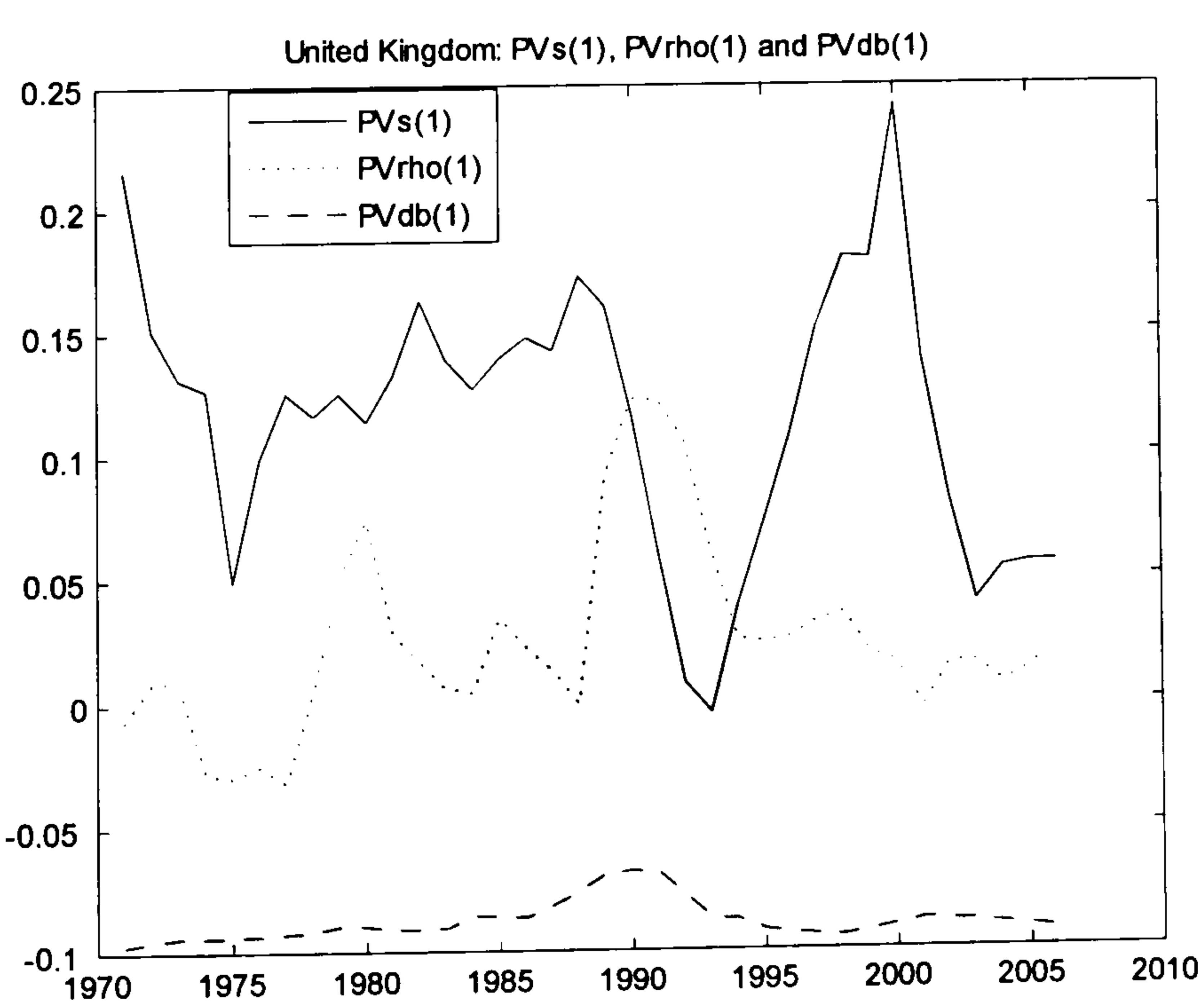
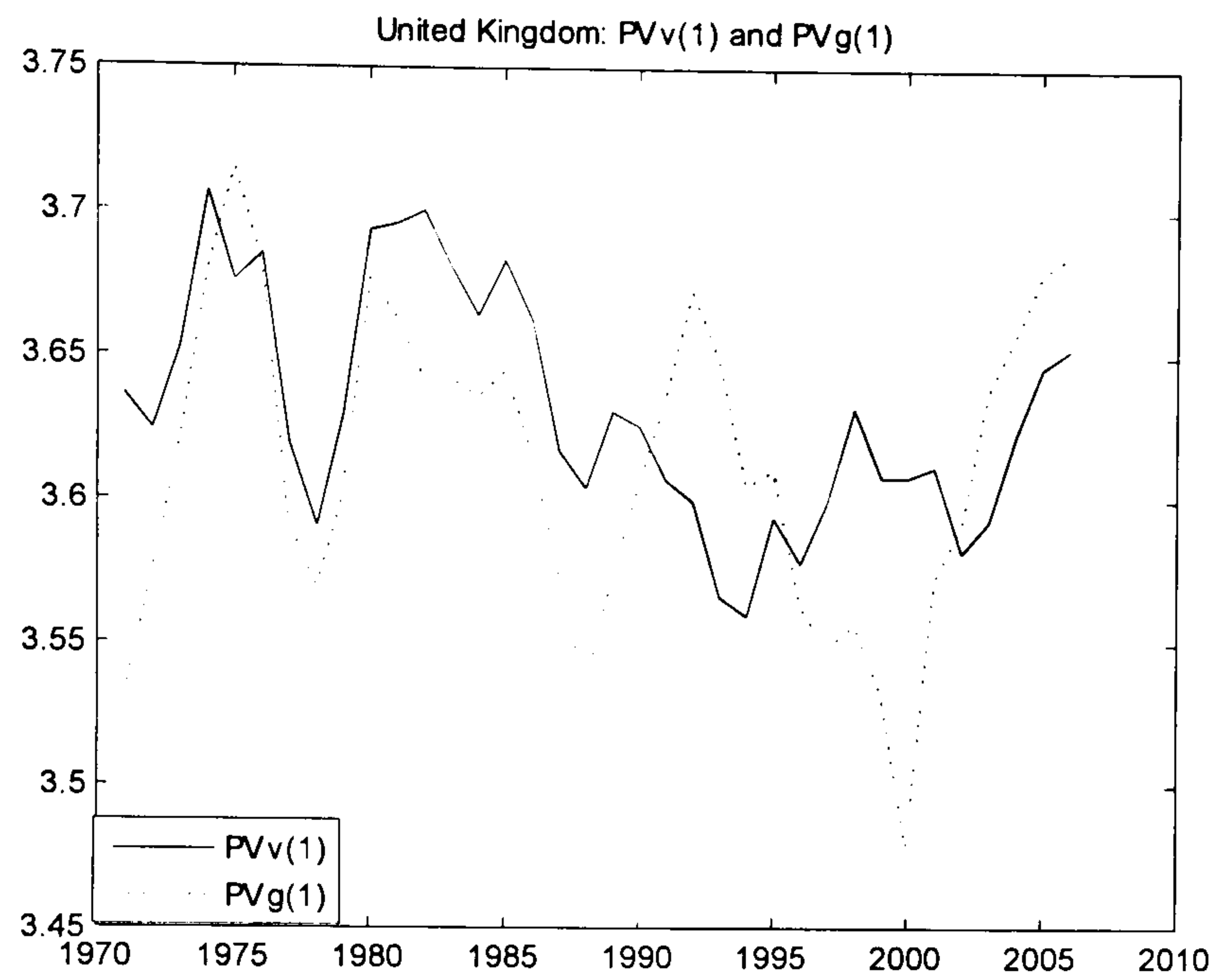


Figure 3.10

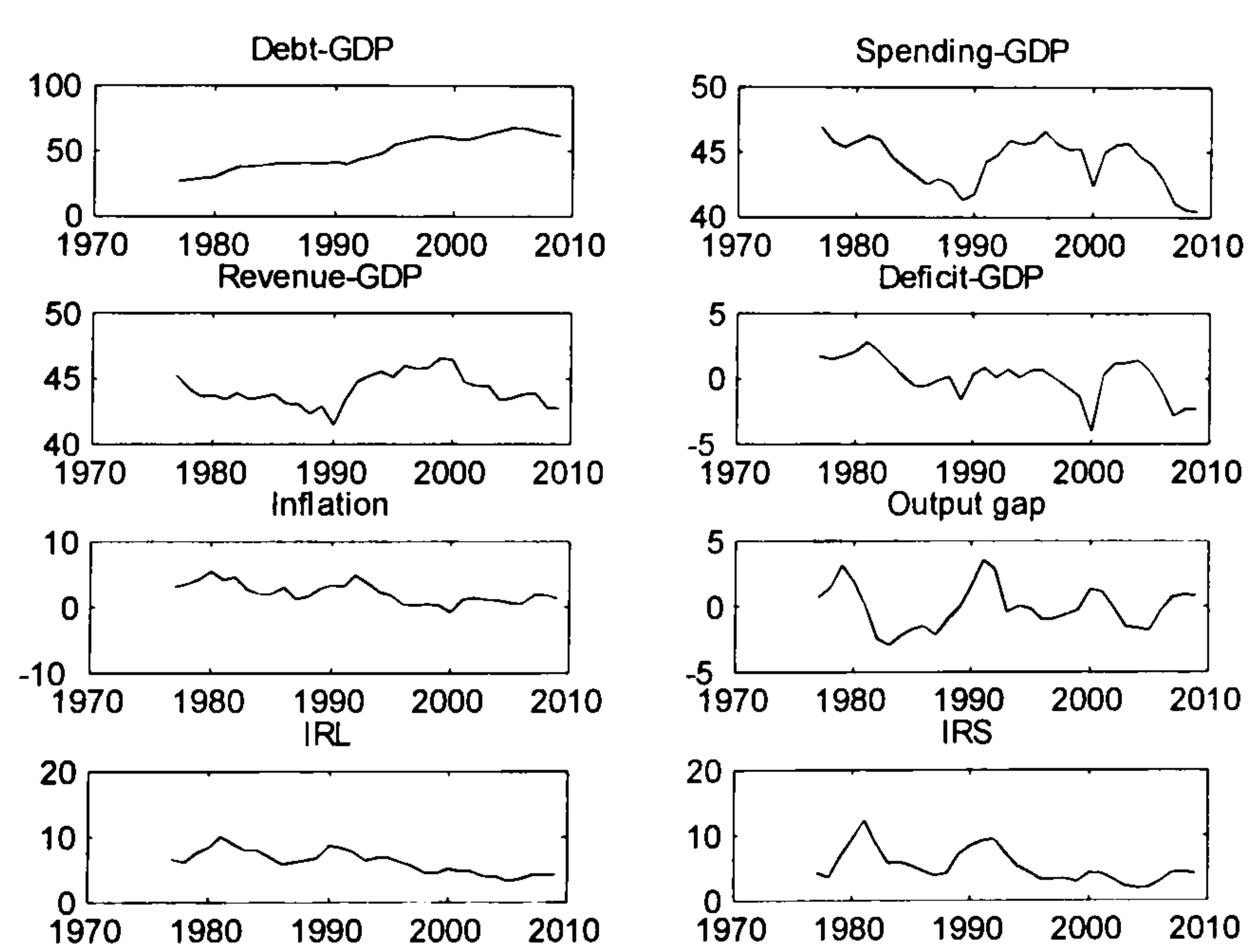


We observe only two brief periods where FSI > 1. These are 1986-1988 and 1997-2000. From 1971-1984 and after 2000 FSI < 1 often by a considerable margin. The period 1984-2005 has four clear episodes. From 1984-1989 there were falls in the debt-GDP ratio and in both revenues and expenditures in present value terms resulting in an improving fiscal position. This was a period where privatization receipts were used to pay off debt, even though the assets were not included in our measure of debt, namely, net government liabilities. From 1989-1992, when sterling left the ERM, the fiscal position deteriorated sharply due to rising expenditures. This may even have been a contributory factor in the speculation against sterling in 1992. After 1992 the debt-GDP rose steadily as it did in the US, but expenditures, after continuing to rise, turned down, which caused an improvement in the fiscal stance. From 1996-2001 there was a marked improvement in the fiscal position mainly due to rising revenues from the upturn in economic activity. From 2001 the fiscal stance deteriorated again due to expenditures (which started to increase in 1998) rising much more than revenues. The Chancellor of the Exchequer has said throughout his tenure that the UK is meeting its fiscal targets, but this evidence indicates that this has not precluded an obvious decline in the sustainability of the UK's fiscal stance.

3.5.3 Germany

The data are annual for the period 1960 to 2005 and are plotted in Figure 3.11.

Figure 3.11: Germany data



The results on fiscal sustainability for the period from 1977 are reported in Figures 3.12-3.15 for a one-year horizon.

Figure 3.12

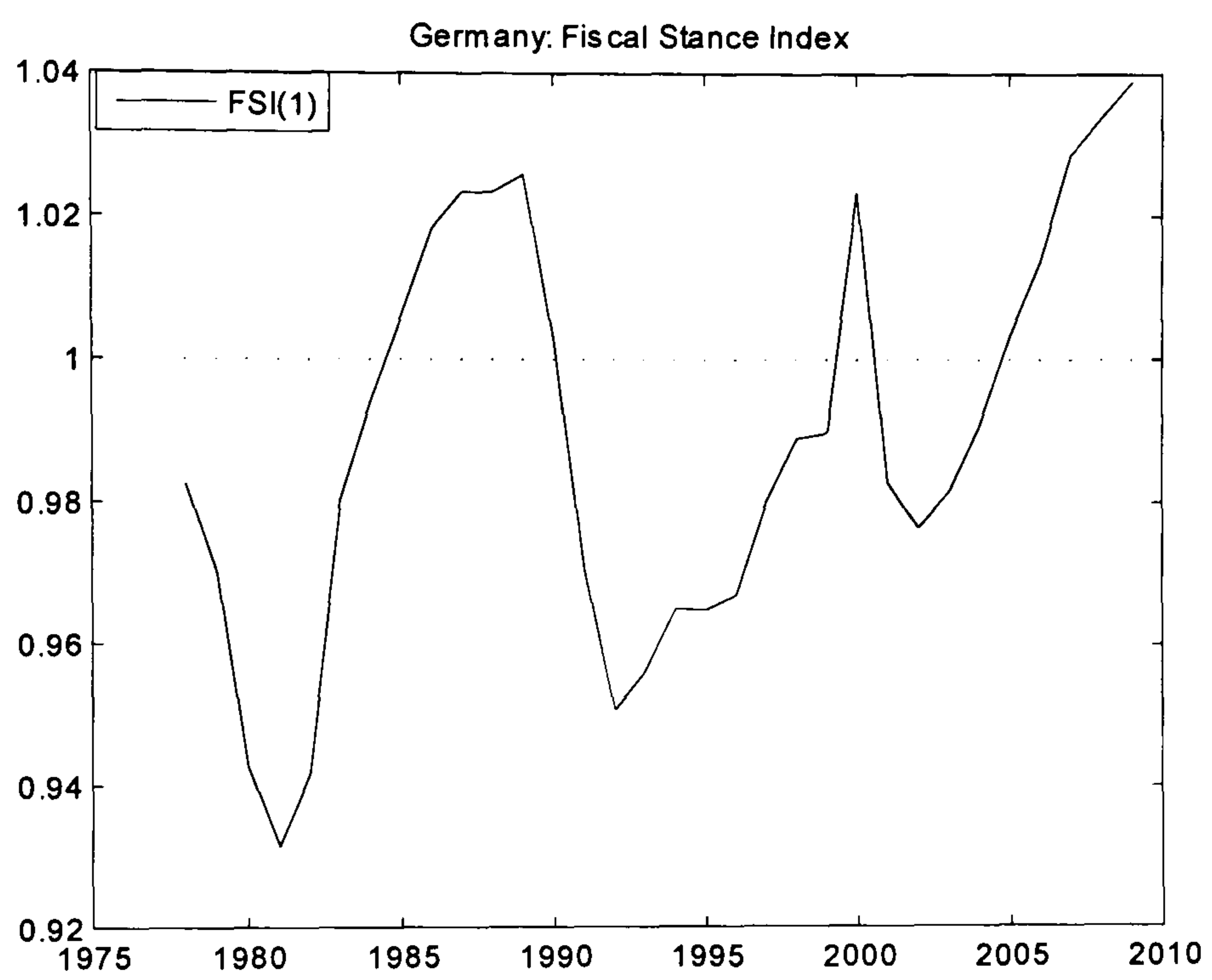


Figure 3.13

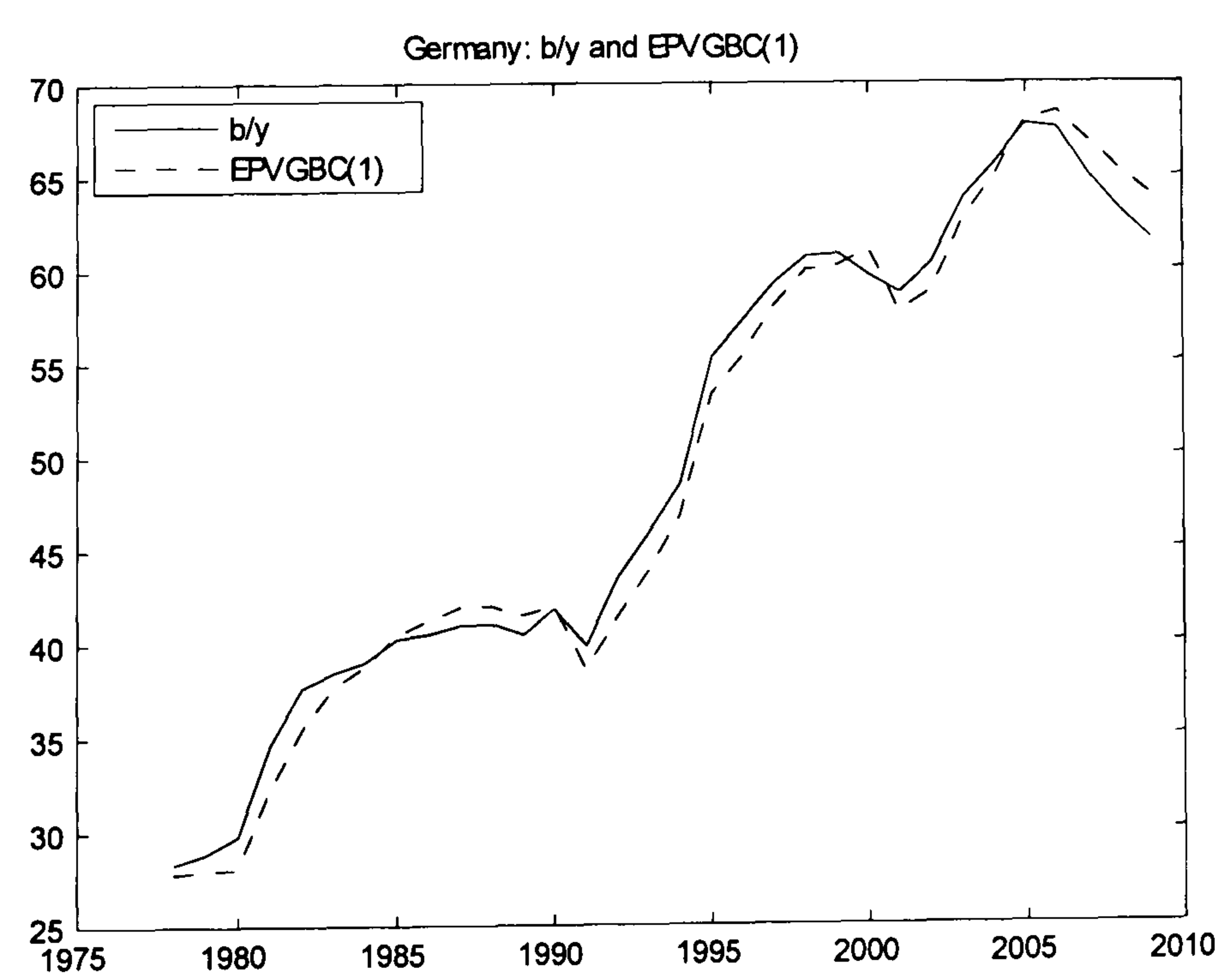


Figure 3.14

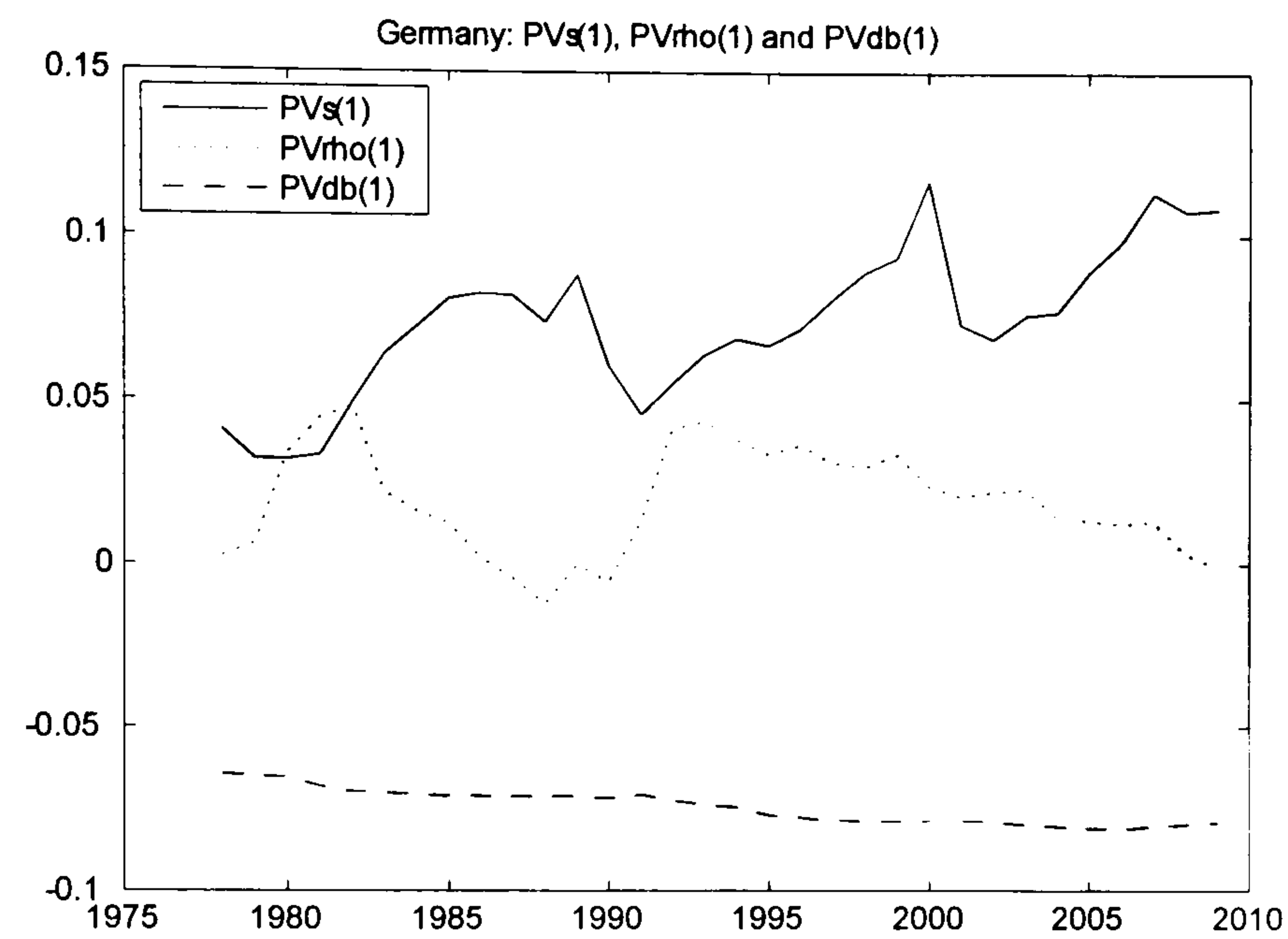
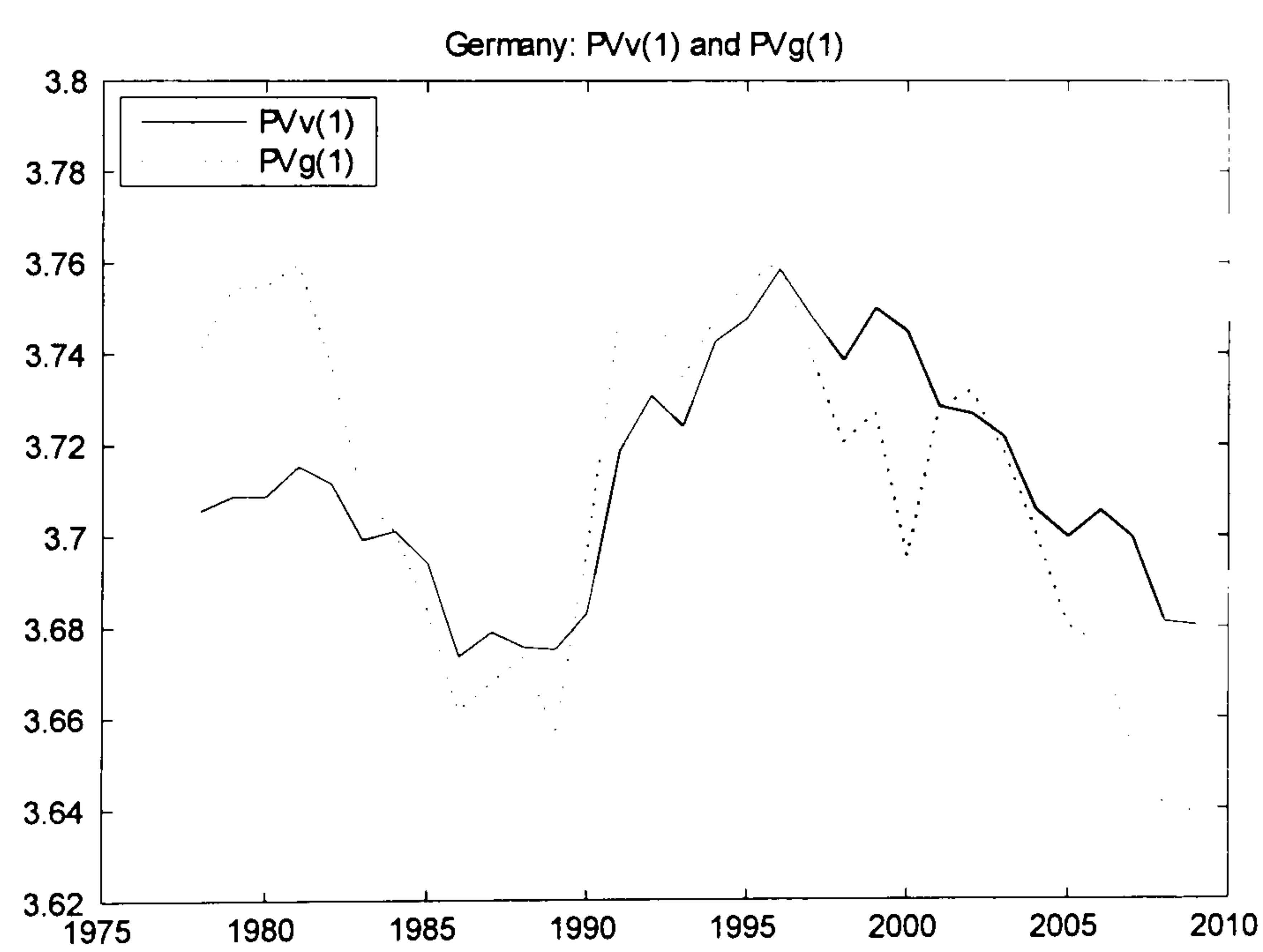


Figure 3.15



There has been a steady deterioration in the F SI over the whole period since 1977. There were two occasions when the index worsened sharply. They are in 1989 on German unification, and again in 1999 shortly after EMU began. Both events seem to have been very harmful to the fiscal stance. Throughout the period the debt-GDP ratio has risen and, with the exception of the period 1992-1999, the fiscal position has gradually deteriorated. The improvement during the period 1992-1999 coincides with improvements in the US and UK and is due to sustained economic growth causing a rise in tax revenues. But since expenditures also increased during this period, the improvement in the German fiscal stance was less marked that for those of the US and UK. Since 1999 the fiscal stance has continued to worsen as expenditures, although falling over the period, have exceeded revenues which have also decreased. The observed secular decline in the German fiscal stance reflects and supports the widespread perception that Germany may need structural reform.

3.6 Using the fiscal stance index for policy analysis

The index of the fiscal stance proposed in this paper gives a useful benchmark against which to compare the short term implications of the current fiscal policy. In this section we use a bootstrapping technique to add confidence bands to the index in order to measure the statistical significance of its policy prescriptions. In principle, since the index is based upon VAR forecasts, one could bootstrap the VAR forecasts and then compute at each stage of the iteration the FSI. There are at least two problems with this approach. First, there is not agreement in the literature on how to bootstrap VAR forecasts, in particular when the VAR may include nonstationary variables (see, for a review, Berkowitz and Kilian (2000)). The second issue is that much of the literature on bootstrapping the predictions arising from AR(p) and VAR(p) models focuses on out-of-sample forecasts, whereas the index is computed by taking forecasts through the whole sample period. This is an issue particularly relevant because the bootstrap of out-of-sample AR(p) and VAR(p) forecasts is generally accomplished by setting the last p observations in each bootstrapping sample equal to the last p observations in the original sample. In this way the bootstrapped forecasts are always constructed by using the same type of information in each bootstrap sample. (See, Thombs and Schucany (1990)). The FSI exploits in-sample forecasts from the VAR, so that predictions are based on an information set that changes with the sample. In this case, a possible solution is to estimate the VAR recursively and then compute and bootstrap the FSI at each point in the sample. This would be however be inconsistent with the way in which we compute the FSI in the paper, as we estimate the VAR only once and over the whole sample. In light of these issues we follow a much straightforward approach. We recognise that the FSI is ultimately a time series which can be approximated with an AR(p) model. Hence, the estimated parameters of any AR(p) model can be bootstrapped following the standard Stine (1987) algorithm. This works as follows:

1. Set up the AR(p) model: $A(L)\, y_t = e_t$
2. Estimate the parameters $\hat{A}(L)$ and compute the vector of residuals $\hat{e}_t$.
3. Compute rescaled residuals $\hat{\epsilon}_t = \hat{e}_t / \sqrt{\frac{T-p}{T-2p}}$
4. Resample with replacement from the rescaled residuals $\hat{\epsilon}_t$ to generate the bootstrap innovations $\hat{\epsilon}_t^*$
5. Generate a bootstrap sample $\{y_t^*\}$ from the original sample $\{y_t\}$ leaving the first $p-$observations in $\{y_t^*\}$ equal to those in $\{y_t\}$.
6. Generate pseudo-data from $\hat{A}(L)\, y_t^* = \epsilon_t^*$

7. Calculate the bootstrap parameter estimates $\hat{A}_p^*(L)$.

8. Compute the predicted series $\{\hat{y}_{t+h}^*\}$ from $\hat{y}_t^* = \hat{A}(L)\, y_t^*$.

9. Repeat 4-8 $B-$times and build the confidence interval.

In particular, we used an AR(6) and constructed 10000 bootstrapped samples. Figures 3.16-3.18 plot the FSI(1) for the United States, the United Kingdom and Germany with the corresponding 10 per cent confidence bands.

Figure 3.16

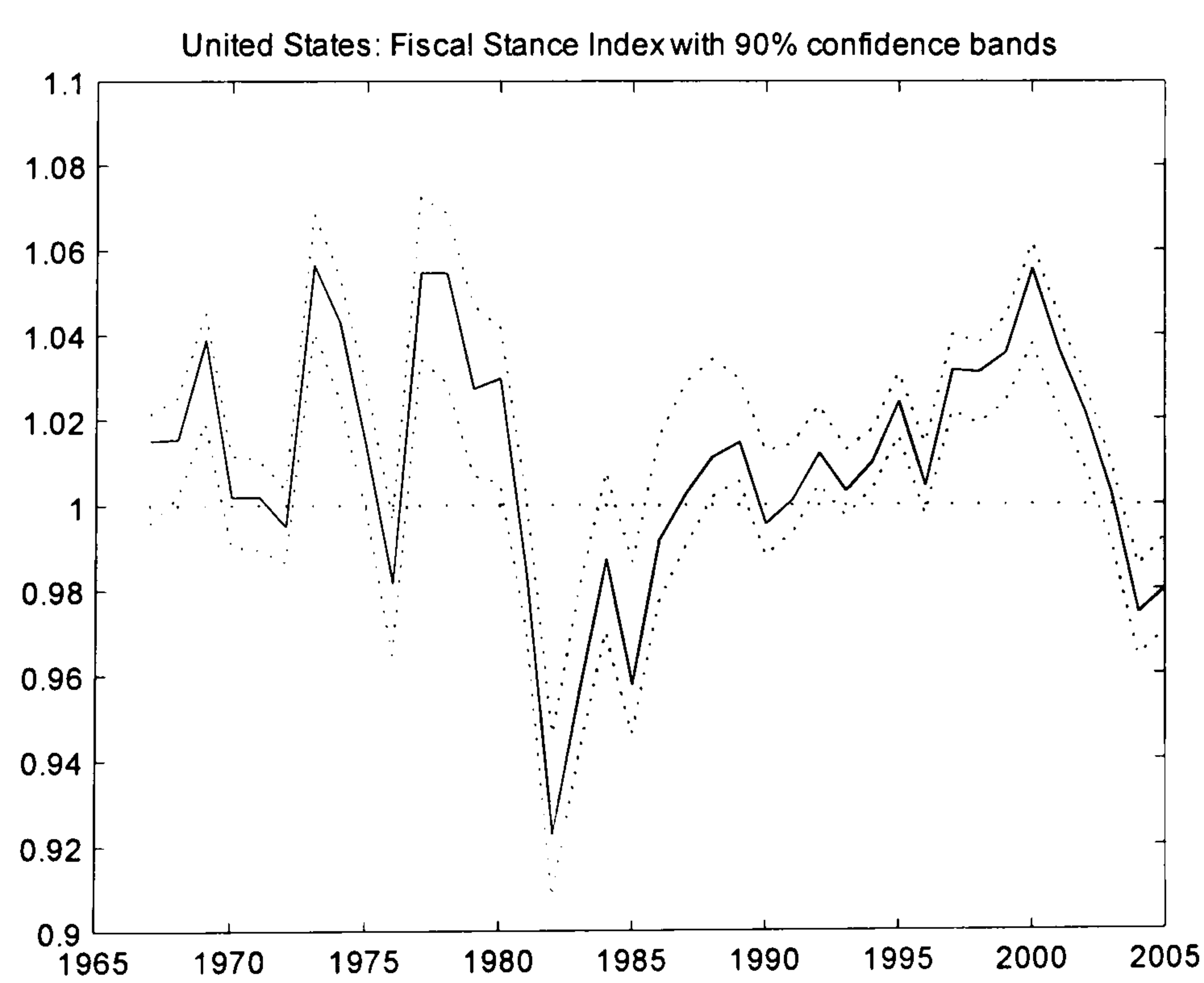


Figure 3.17

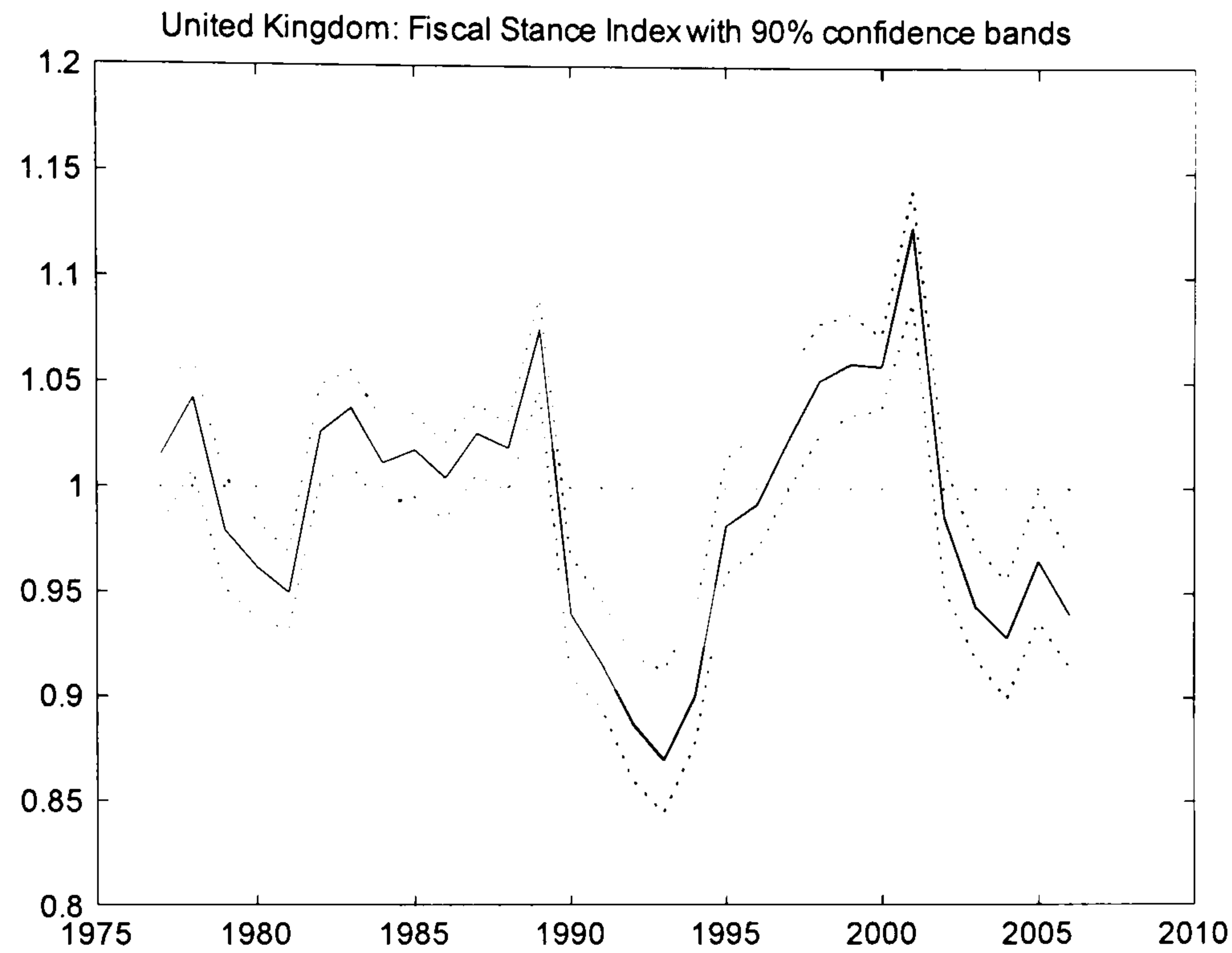
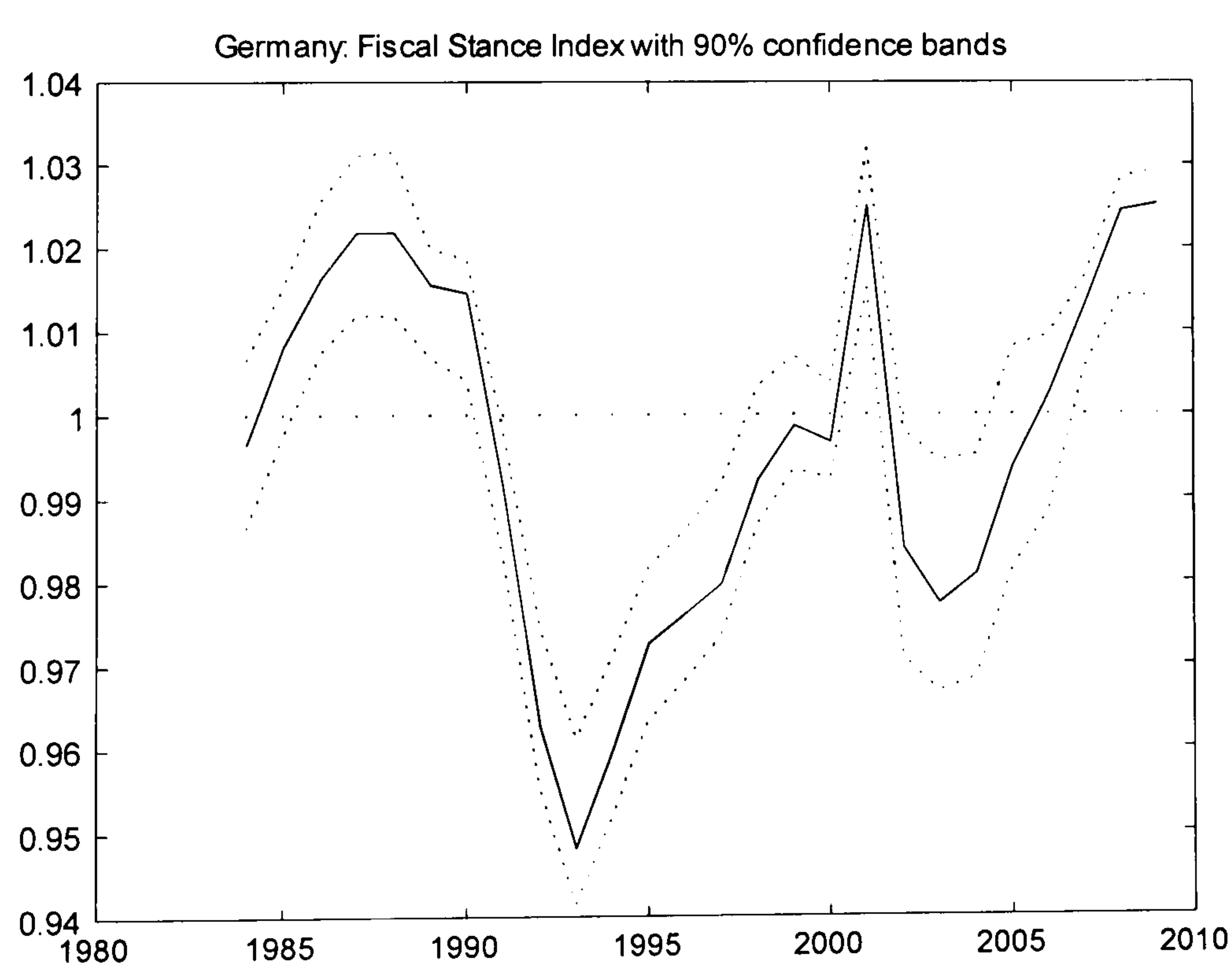


Figure 3.18



The results show that there are several periods in which the FSI index is either above or below 1 and well between the 10 per cent confidence bands. In particular, in the United

States the index is below 1 and within the confidence bands during the first half of the 1980's and the period 2003-2005. The index is above 1 throughout the 1990's and for short periods of time in the late 1960's, as well as mid and late 1970's. In the United Kingdom, the index is considerably greater than one only during the second half of the 1990's and early 2000. The index is significantly below 1 during the first half of the 1990's and after 2001. The results for Germany display a significant deterioration after the 1990 and the year 2001, but also a significant recovery since 2005.

We conclude by showing a further use of the index of the fiscal stance FSI to assess the likely implications that the current fiscal stance has over the future. As proposed in the previous section we fit an AR(p) model on the FSI series computed for each country. Next we take the $n-$periods ahead prediction of the index to evaluate its dynamic over the short run. The forecasts are displayed in figures 3.19, 3.20 and 3.21, for the Unites States, the United Kingdom and Germany respectively. As well as the predicted path, each graph also plots the forecasts standard error, determined by taking into account both the error variance and parameter uncertainty.

Figure 3.19: US forecast FSI(1)

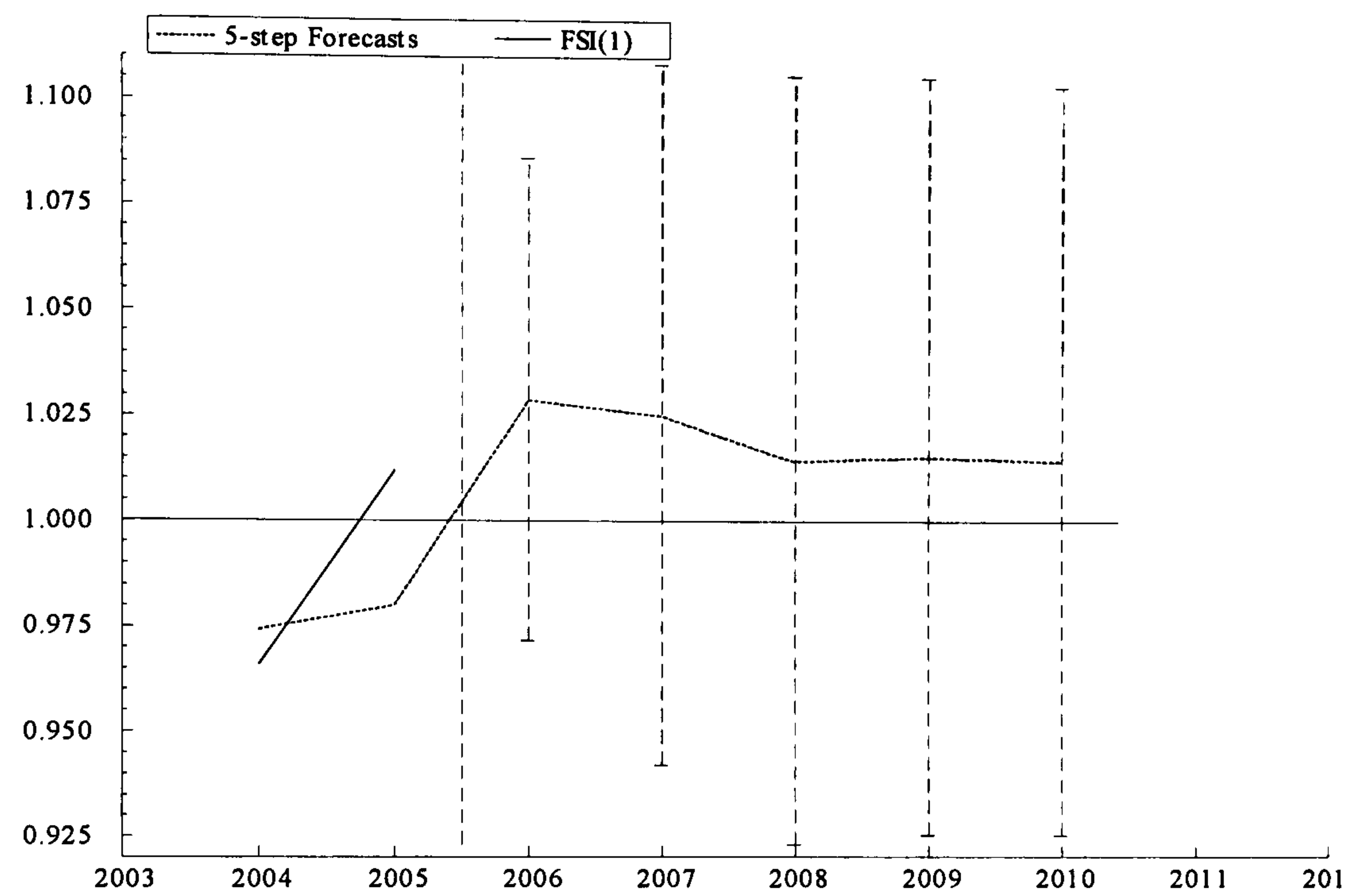


Figure 3.20: UK forecast FSI(1)

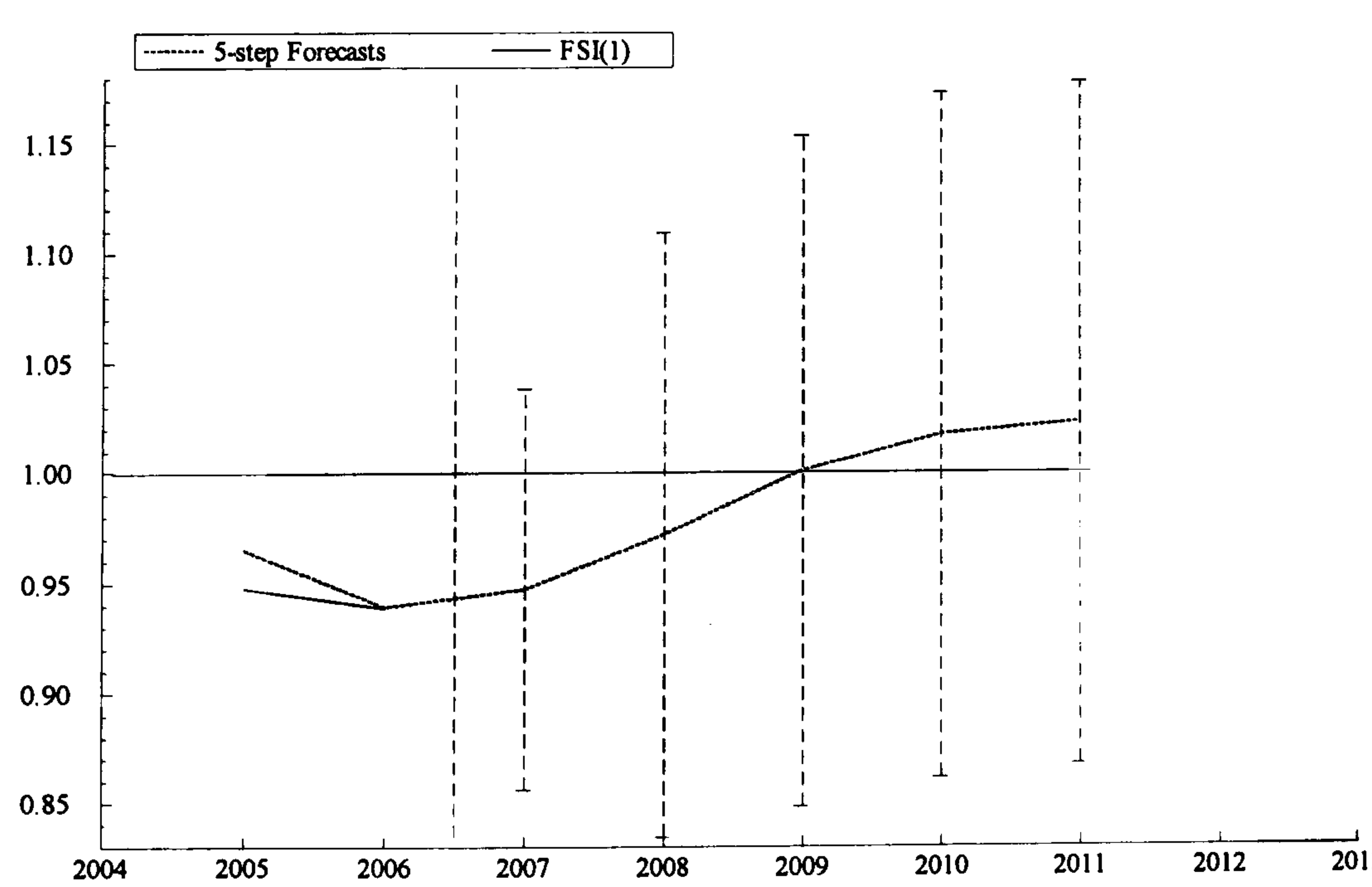
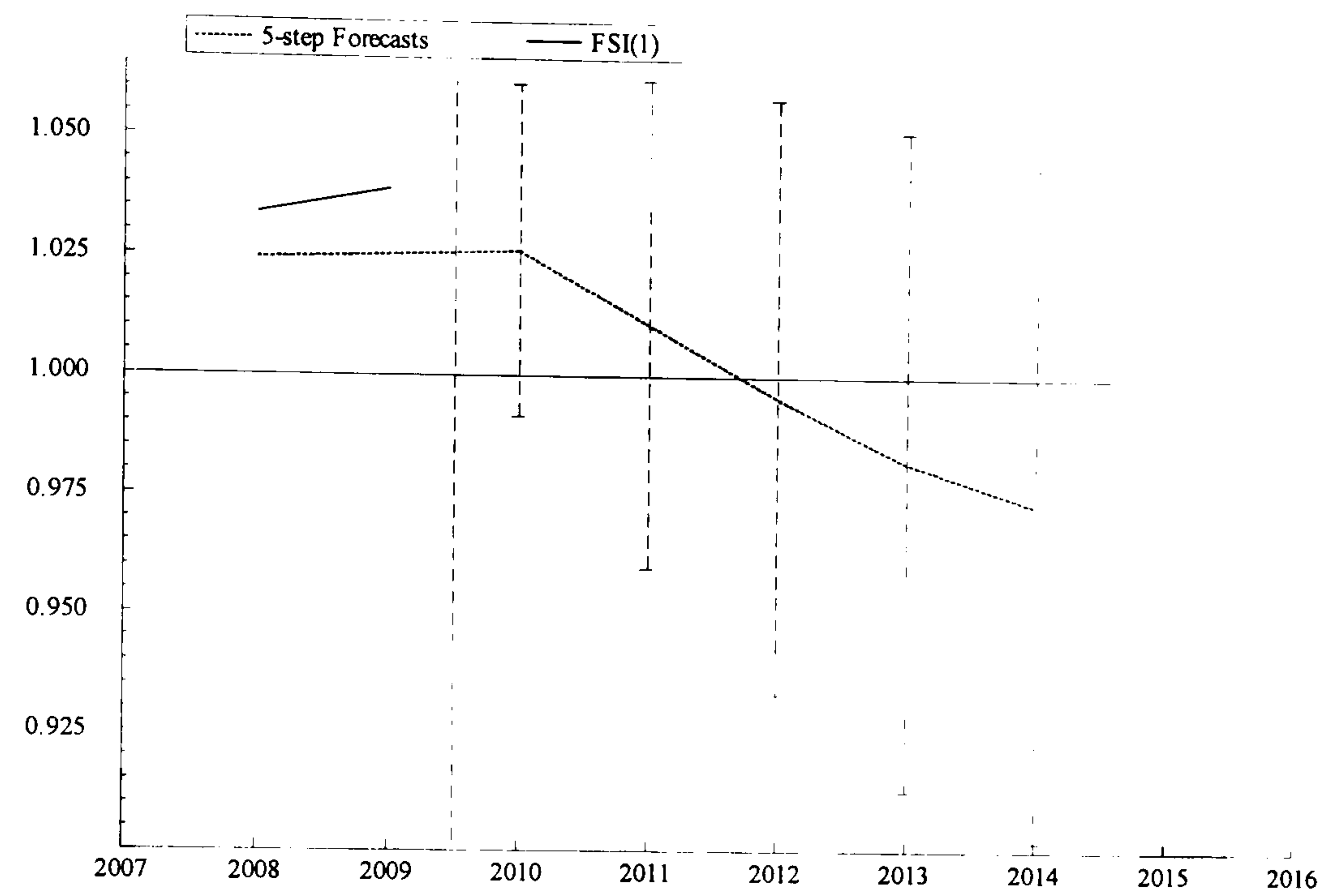


Figure 3.21: Germany forecast FSI(1)



In particular, the results show that the fiscal stance is forecasted to remain above 1 in both the United States and the United Kingdom, and to deteriorate below 1 in Germany.

3.7 Conclusion

In this paper we have proposed the construction of an index to measure the current fiscal stance. We have distinguished this from existing measures of the sustainability of the fiscal stance and argued that such tests, which focus on the past, may not be a helpful guide to the current stance of fiscal policy. Like the tests for fiscal sustainability, this index is based on the government inter-temporal budget constraint. The main differences are that the index is forward looking, it applies to a finite time horizon, and it uses a log-linear approximation to the government budget constraint which enables the inflation, economic growth and interest rates to be time varying rather than constant. In effect, the index is based on a comparison of the forecast and the desired debt-GDP ratio over that horizon where the forecast is constrained to satisfy the government budget constraint. We propose the use of a VAR forecasting model based on the government budget constraint as this is simple to compute and easily automated. We have shown how to identify individual components of the index that may be causing problems for the fiscal stance. We have applied this methodology to three countries: the US, the UK and Germany. In the UK and US the index of fiscal sustainability has fluctuated considerably with periods when the debt-GDP ratio has risen followed by periods when it has fallen. During the period of strong economic growth in the 1990's the fiscal positions of all three countries improved considerably, but in recent years the fiscal stance in all three countries has been steadily deteriorating. Our index indicates that a continuation of the present fiscal stances is leading to a period of marginal fiscal recovery in the US and in the UK, while the German fiscal position is expected to deteriorate over the medium run.

Appendix A
State space representation (Chapter 2)

The dynamic constraint computed under the standard approach corresponds to the state vector of the reduced form VAR in equation (2.46) and has the same dynamic structure of the constraint in equation (2.54). The term $\mathbf{A}_{12}(L)\,\mathbf{z}_{2,t}$ in equation (2.46) can be written as:

$$\mathbf{A}_{12}(L)\,\mathbf{z}_{2t} = \sum_{i=1}^{p} \mathbf{A}_{12i} L^i \mathbf{z}_{2t} = \mathbf{A}_{12.1}\mathbf{z}_{2t-1} + \sum_{i=2}^{p} \mathbf{A}_{12i} L^i \mathbf{z}_{2t-1} = \left[\mathbf{A}_{12.1} + \tilde{\mathbf{A}}_{12}(L)\right] \mathbf{z}_{2t-1},$$

where $\mathbf{A}_{12.1}$ are the coefficients of the first lag of $\mathbf{z}_{2t}$, while $\tilde{\mathbf{A}}_{12}(L)$ includes the coefficients of all others lags.

Therefore, under the standard approach the state space representation of the dynamic constraint is obtained by setting:

$$\mathbf{y}_t = \left[\begin{array}{c} \mathbf{z}_{1,t} \\ \mathbf{z}_{1,t-1} \\ \vdots \\ \mathbf{z}_{1,t-p+1} \\ \hline \mathbf{z}_{2,t} \\ \mathbf{z}_{2,t-1} \\ \vdots \\ \mathbf{z}_{2,t-p+1} \\ \hline \Delta\mathbf{z}_{2,t} \\ \Delta\mathbf{z}_{2,t-1} \\ \vdots \\ \Delta\mathbf{z}_{2,t-p+1} \end{array}\right], \quad \mathbf{a} = \left[\begin{array}{c} \mathbf{a}_{10} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{array}\right], \quad \mathbf{C} = \left[\begin{array}{c} \mathbf{A}_{12.1} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hline \mathbf{I} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{array}\right], \quad \mathbf{v}_t = \left[\begin{array}{c} \mathbf{e}_{1,t} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hline \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{array}\right]$$

and

$$
\mathbf{A}=\left[\begin{array}{ccccc|ccccc|cccc}
 & \mathbf{A}_{11}(L) & & & & 0 & & \tilde{\mathbf{A}}_{12}(L) & & & 0 & 0 & \dots & 0 \\
\mathbf{I} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & \mathbf{I} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\hline
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & \mathbf{I} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \mathbf{I} & 0 & 0 & 0 & \dots & 0 \\
\hline
0 & 0 & \dots & 0 & 0 & -\mathbf{I} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & 0 & \mathbf{I} & -\mathbf{I} & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \dots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -\mathbf{I} & 0 & 0 & 0 & \dots & 0
\end{array}\right].
$$

On the other hand, the dynamic constraint under the PVAR approach in equation (2.48) is compatible with a dynamic constraint of equation (2.58). Thus, the appropriate state space representation of the model under the PVAR approach is given by:

$$
\mathbf{y}_t=\left[\begin{array}{c}
\mathbf{z}_{1,t}\\ \mathbf{z}_{1,t-1}\\ \vdots \\ \mathbf{z}_{1,t-p+1}\\ \hline \mathbf{z}_{2,t}\\ \mathbf{z}_{2,t-1}\\ \vdots\\ \mathbf{z}_{2,t-p+1}\\ \hline \Delta\mathbf{z}_{2,t}\\ \Delta\mathbf{z}_{2,t-1}\\ \vdots\\ \Delta\mathbf{z}_{2,t-p+1}
\end{array}\right],\ \mathbf{a}=\left[\begin{array}{c}
\mathbf{A}_{10}-\mathbf{G}\mathbf{A}_{20}\\ 0\\ \vdots\\ 0\\ \hline 0\\ 0\\ \vdots\\ 0\\ \hline 0\\ 0\\ \vdots\\ 0
\end{array}\right],\ \mathbf{B}=\left[\begin{array}{c}
\mathbf{G}\\ 0\\ \vdots\\ 0\\ \hline \mathbf{I}\\ 0\\ \vdots\\ 0\\ \hline \mathbf{I}\\ 0\\ \vdots\\ 0
\end{array}\right],\ \mathbf{u}_t=\left[\begin{array}{c}
\mathbf{e}_{1,t}-\mathbf{G}\mathbf{e}_{2,t}\\ 0\\ \vdots\\ 0\\ \hline 0\\ 0\\ \vdots\\ 0\\ \hline 0\\ 0\\ \vdots\\ 0
\end{array}\right]
$$

and

$$
\mathbf{A}=\left[\begin{array}{ccccc|ccccc|cccc}
\multicolumn{5}{c|}{\mathbf{A}_{11}(L)-\mathbf{G}\mathbf{A}_{21}(L)} & \multicolumn{5}{c|}{\mathbf{A}_{12}(L)-\mathbf{G}\mathbf{A}_{22}(L)} & 0 & 0 & \cdots & 0 \\
\mathbf{I} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathbf{I} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \mathbf{I} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \mathbf{I} & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & -\mathbf{I} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \mathbf{I} & -\mathbf{I} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \cdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -\mathbf{I} & 0 & 0 & 0 & \cdots & 0
\end{array}\right].
$$

Appendix B
Tables and figures (Chapter 2)

Table 2.1: Alternative specifications of the objective function

	Policy Weights		
OF	λ_{π}	λ_y	$\lambda_{\Delta rs}$
1	1	1	0.5
2	1	0.2	0.5
3	1	5	0.5
4	1	1	1
5	1	1	0.1

Notes: OF=Objective function; the OF is described in equation (2.61).

Table 2.2: Interest rate rules under alternative specifications of the objective function, summary statistics

Sample	1960-2006		1960-1983		1984-2006	
	SD	AAD	SD	AAD	SD	AAD
	Original VAR					
	2.80		2.76		2.30	
OF	PVAR approach					
1	2.22	1.62	1.00	2.31	2.02	0.97
2	2.17	1.33	1.17	1.78	1.78	0.90
3	2.65	2.29	1.80	3.40	2.50	1.24
4	2.18	1.29	1.24	1.77	2.01	0.83
5	2.70	2.43	1.84	3.50	2.18	1.41
Mean	2.38	1.79	1.41	2.55	1.10	1.07
OF	Standard approach					
1	3.03	1.31	2.29	1.44	2.28	1.17
2	3.14	1.25	2.46	1.31	2.19	1.19
3	2.95	1.55	2.10	1.78	2.49	1.31
4	2.89	1.13	2.27	1.21	2.19	1.04
5	3.27	1.59	2.41	1.79	2.44	1.40
Mean	3.06	1.36	2.31	1.51	2.32	1.22

Notes: OFs=Objective functions as described in table 1; SD= Standard Deviation, in percentage; AD= Average Absolute Distance between optimal and VAR rule, in percentage; mean=average value across the five OFs.

Table 2.3: Welfare loss analysis under alternative specifications of the objective function, undiscounted forecasts

	SD							
	y	π	Δrs	Tot	y	π	Δrs	Tot
	Original VAR							
	1.27	1.35	0.42	**3.04**				
OF	PVAR approach							
	unweighted				weighted			
1	0.60	0.52	0.56	**1.69**	0.60	0.52	0.28	**1.41**
2	0.75	0.52	0.56	**1.74**	0.15	0.63	0.18	**0.96**
3	0.46	0.45	0.85	**1.76**	2.31	0.45	0.43	**3.19**
4	0.67	0.54	0.45	**1.66**	0.67	0.54	0.45	**1.66**
5	0.52	0.54	0.80	**1.86**	1.02	0.98	0.24	**2.24**
OF	Standard approach							
	unweighted				weighted			
1	1.02	0.98	0.48	**2.48**	1.02	0.98	0.24	**2.24**
2	1.09	1.05	0.56	**2.71**	0.22	1.05	0.28	**1.55**
3	0.88	0.87	0.34	**2.09**	4.41	0.87	0.17	**5.45**
4	0.97	0.98	0.40	**2.36**	0.97	0.98	0.40	**2.36**
5	1.10	0.98	0.62	**2.70**	1.10	0.98	0.06	**2.14**

Notes: OF=Objective functions as described in table 1;
SD=Standard Deviation, in percentage.

Table 2.4: Welfare loss analysis under alternative specifications of the objective function, discounted forecasts

	SD							
	y	π	Δrs	Tot	y	π	Δrs	Tot
	Original VAR							
	1.01	0.50	0.36	**1.87**				
OF	PVAR approach							
	unweighted				weighted			
1	0.52	0.18	0.44	**1.14**	0.52	0.18	0.22	**0.92**
2	0.62	0.14	0.27	**1.03**	0.12	0.14	0.14	**0.40**
3	0.43	0.29	0.69	**1.41**	2.14	0.29	0.34	**2.77**
4	0.57	0.16	0.34	**1.07**	0.57	0.16	0.34	**1.07**
5	0.46	0.23	0.66	**1.34**	0.46	0.23	0.07	**0.75**
OF	Standard approach							
	unweighted				weighted			
1	0.84	0.43	0.43	**1.70**	0.84	0.43	0.22	**1.49**
2	0.88	0.48	0.50	**1.86**	0.18	0.48	0.25	**0.91**
3	0.76	0.45	0.29	**1.50**	3.82	0.45	0.14	**4.42**
4	0.79	0.41	0.35	**1.55**	0.79	0.41	0.35	**1.55**
5	0.93	0.47	0.56	**1.95**	0.93	0.47	0.06	**1.45**

Notes: OF=Objective functions as described in table 1;
SD=Standard Deviation, in percentage.

Table 2.5: Welfare loss computed from PVAR and standard approach under alternative choices of λ_y and λ_π, summary statistics

	PVAR			Standard approach		
	Loss	λ_y	λ_π	Loss	λ_y	λ_π
			$\lambda_{\Delta rs} = 0.1$			
min	1.86	1.14	0.88	2.15	2.56	0.39
max	2.02	10	0.10	3.58	0.10	10
mean	1.94			2.64		
SD	5.42			34.99		
			$\lambda_{\Delta rs} = 0.5$			
min	1.65	1.98	0.51	1.88	3.87	0.26
max	1.84	10	0.1	3.40	0.1	10
mean	1.74			2.10		
SD	6.05			27.84		
			$\lambda_{\Delta rs} = 1$			
min	1.58	2.57	0.39	1.75	4.92	0.20
max	1.80	0.1	10	3.25	0.1	10
mean	1.65			1.92		
SD	5.26			28.16		

Notes: SD=Standard Deviation, in percentage; λ_y ranges from 0.1 to 10; $\lambda_\pi = 1/\lambda_y$.

Table 2.6: Output/inflation trade-off under PVAR and standard approach, summary statistics

	PVAR		Standard approach	
	From	to	From	to
		$\lambda_{\Delta rs} = 0.1$		
λ_y	10	3.77	10	2.59
λ_π	0.1	0.27	0.1	0.39
SD(y)	0.417	0.425	0.75	0.88
SD(π)	0.471	0.462	1.21	0.90
NWG	-0.001		-0.18	
		$\lambda_{\Delta rs} = 0.5$		
λ_y	10	3.59	10	4.17
λ_π	0.1	0.28	0.1	0.24
SD(y)	0.426	0.460	0.732	0.763
SD(π)	0.435	0.411	0.877	0.782
NWG	0.01		-0.064	
		$\lambda_{\Delta rs} = 1$		
λ_y	10	4.32	10	5.20
λ_π	0.1	0.23	0.1	0.19
SD(y)	0.440	0.483	0.713	0.723
SD(π)	0.410	0.389	0.763	0.718
NWG	0.022		-0.035	

Notes: SD=Standard Deviation, in percentage; $\lambda_\pi = 1/\lambda_y$;
NWG= Net Welfare Gain=ΔSD(y) $-$ ΔSD(π) .

Figure 2.1: Data plot

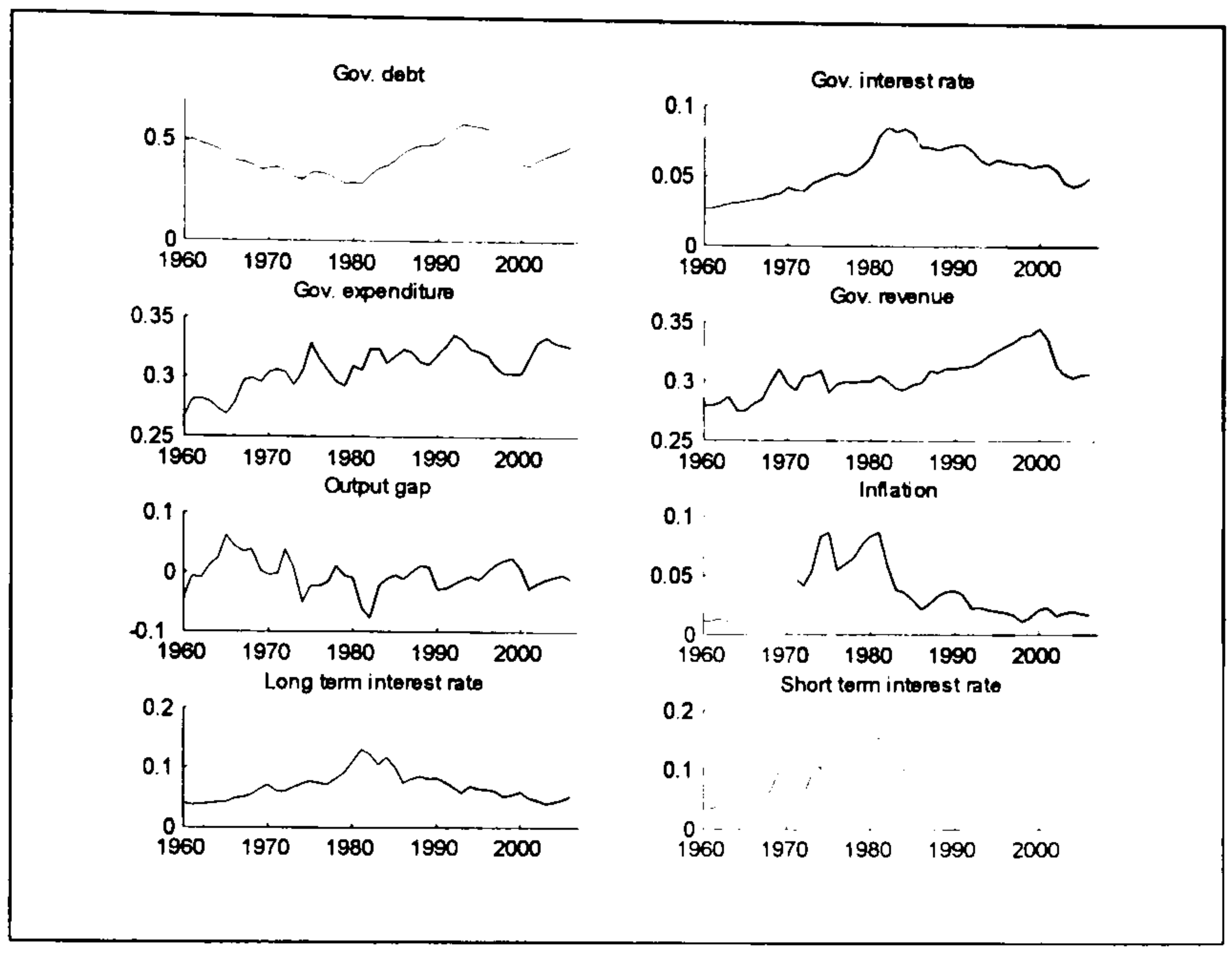


Figure 2.2: Interest rate rules from original VAR, PVAR and standard approach, under alternative specification of the objective function

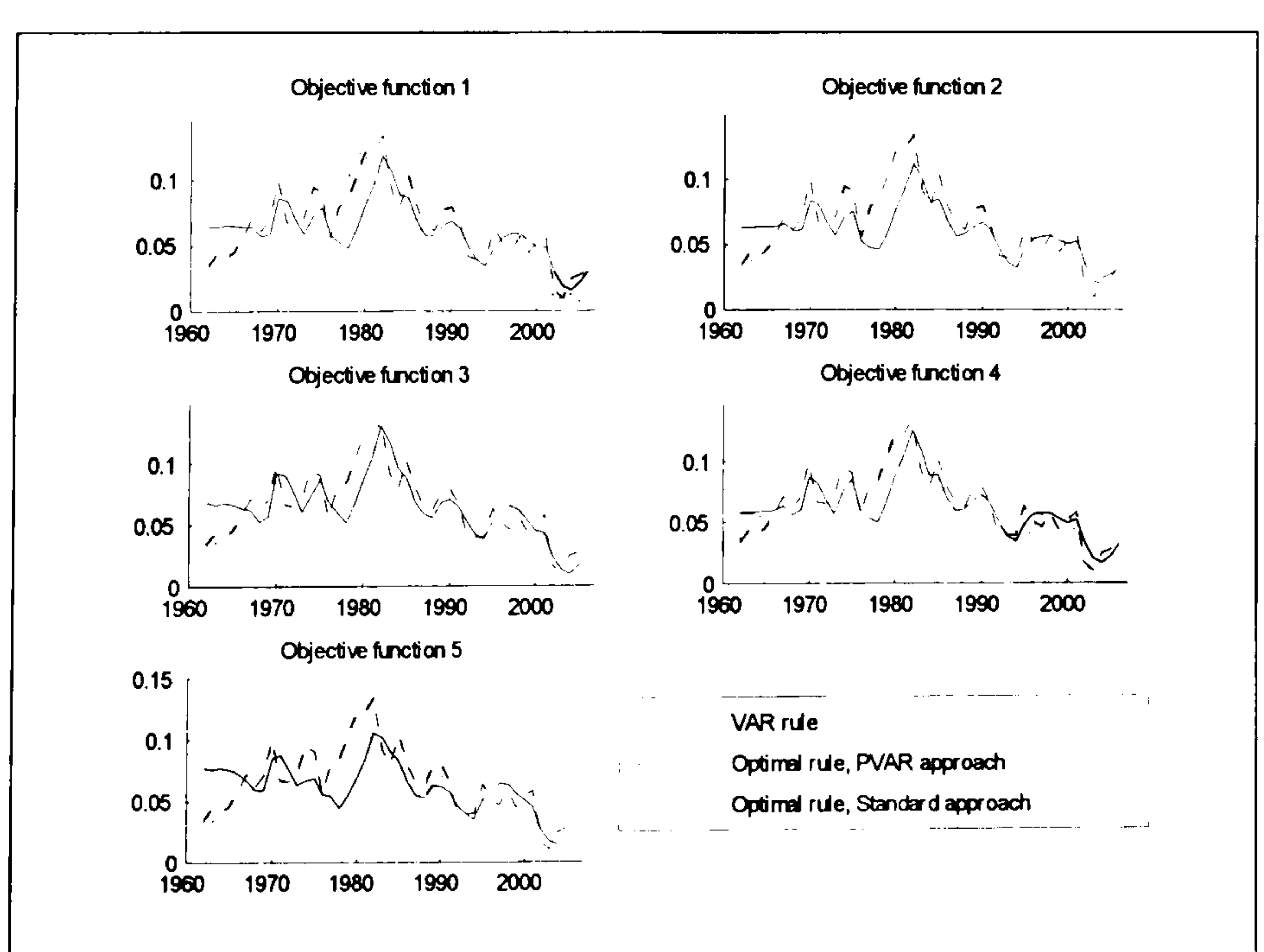


Figure 2.3.a: Output and inflation forecasts from original VAR, PVAR and standard approach, objective function 1, 2 and3

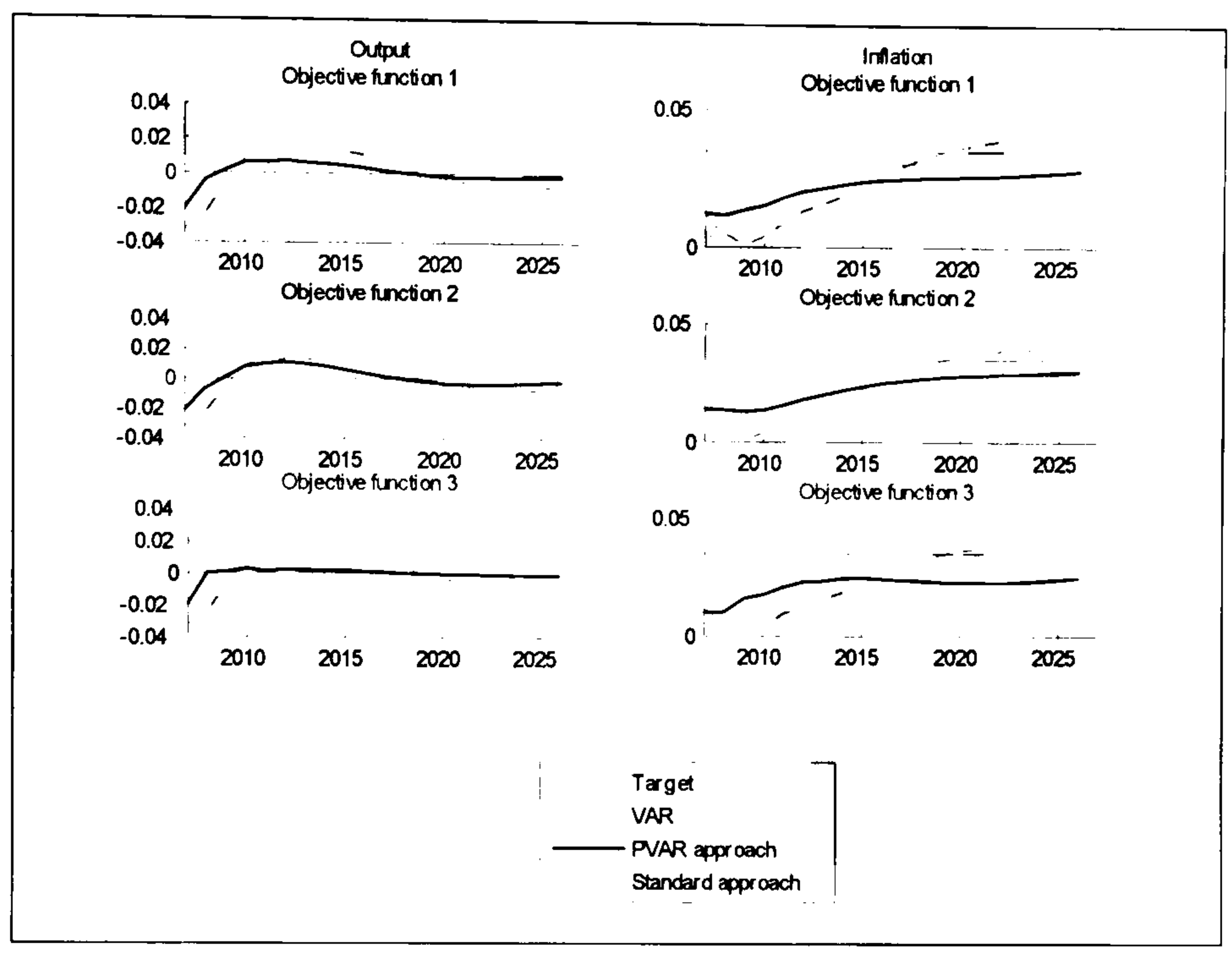


Figure 2.3.b: Output and inflation forecasts from original VAR, PVAR and standard approach, objective function 4 and 5

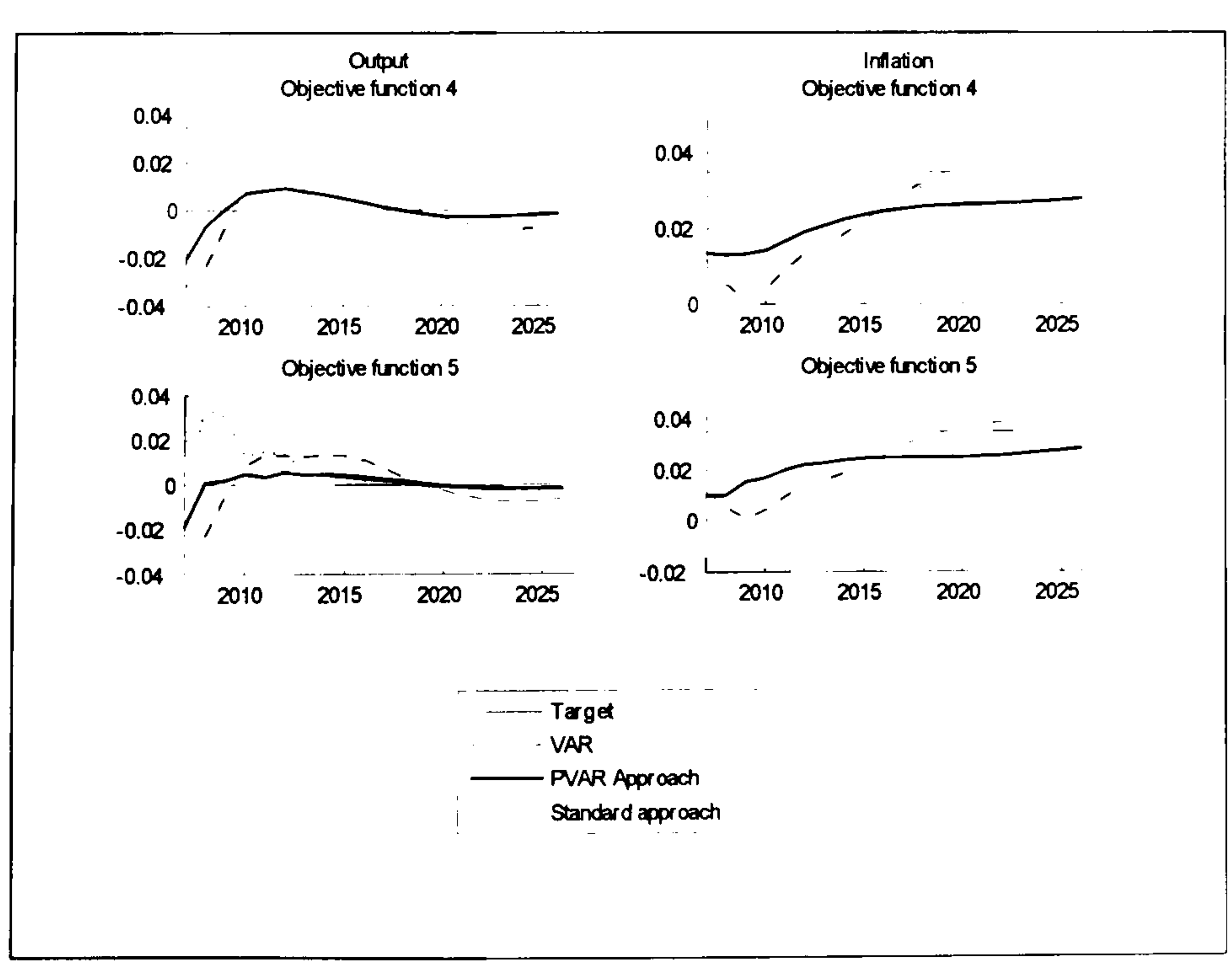


Figure 2.4: Change in standard deviation of welfare loss components under alternative choices of λ_y and λ_π, PVAR approach

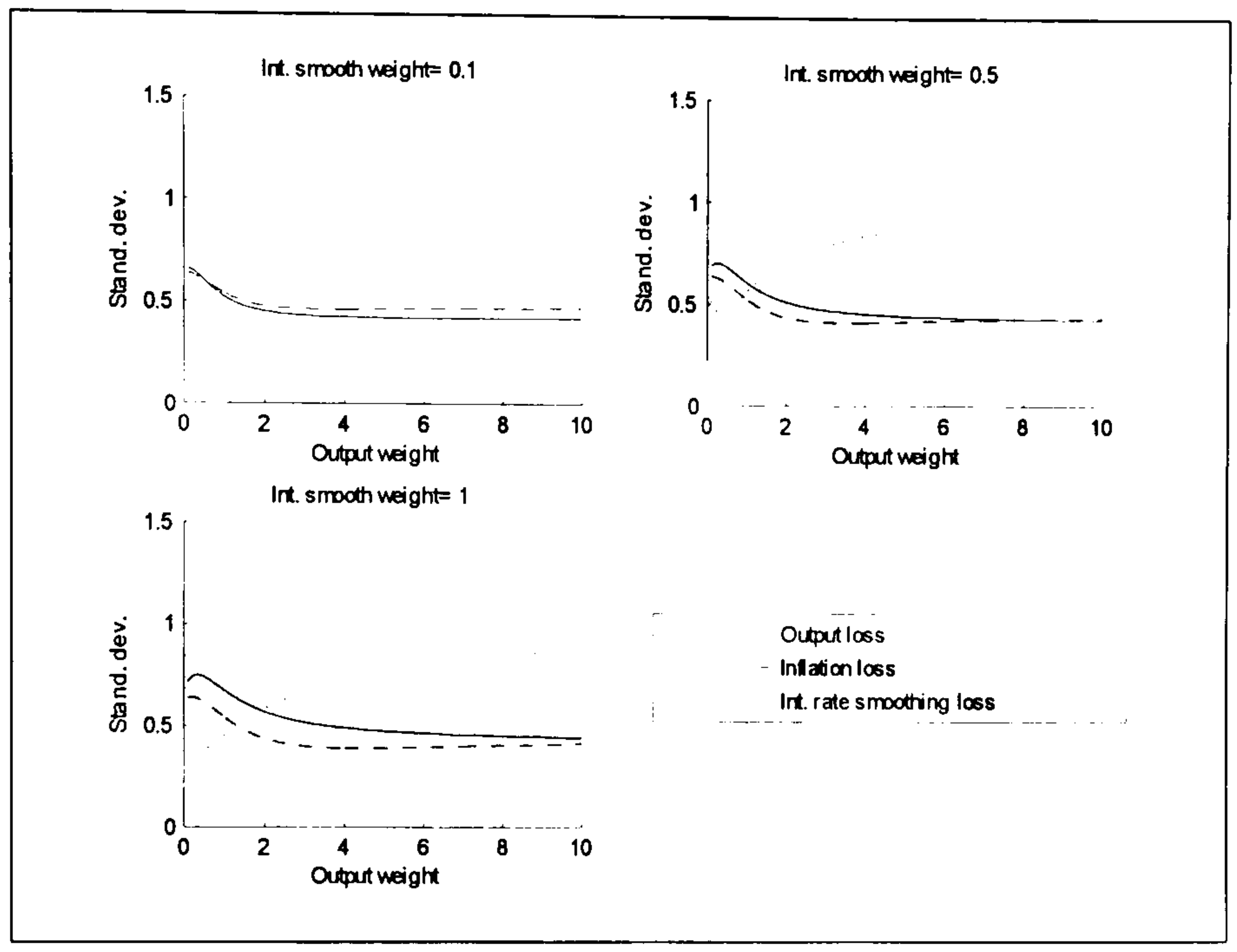


Figure 2.5: Change in standard deviation of welfare loss components under alternative choices of λ_y and λ_π, standard approach

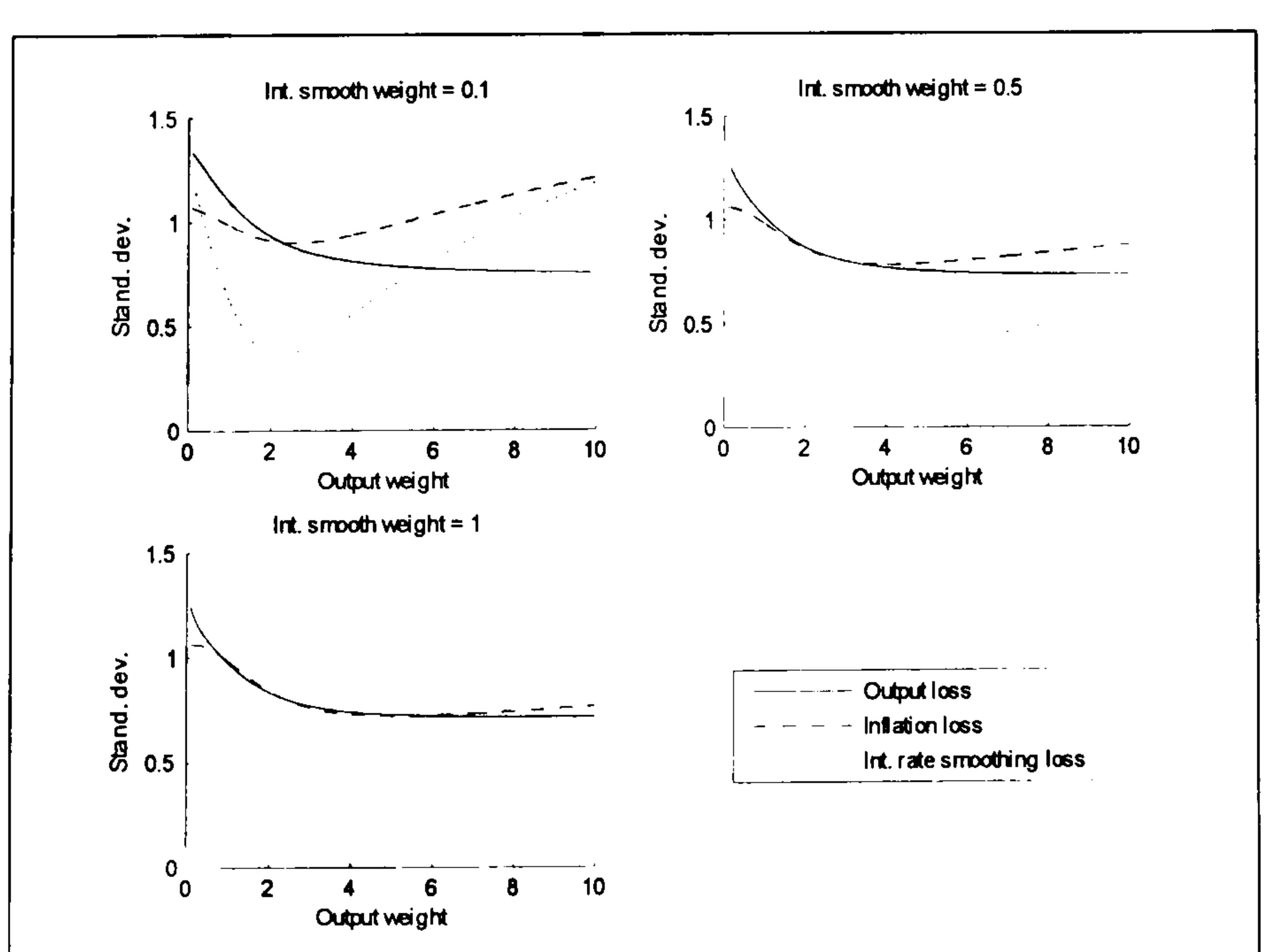


Figure 2.6: Change in welfare loss under alternative choices of λ_y and λ_π, PVAR and standard approach

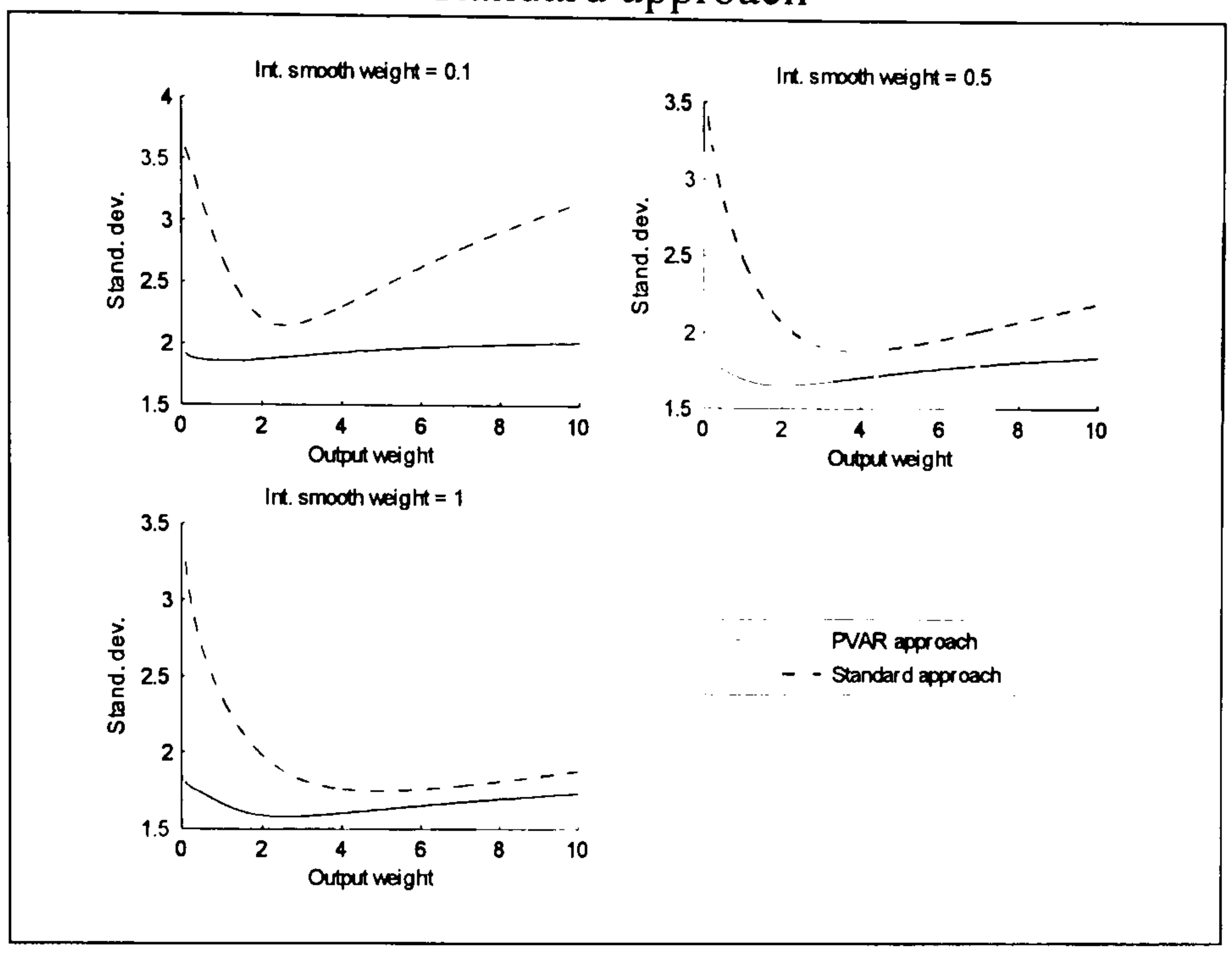
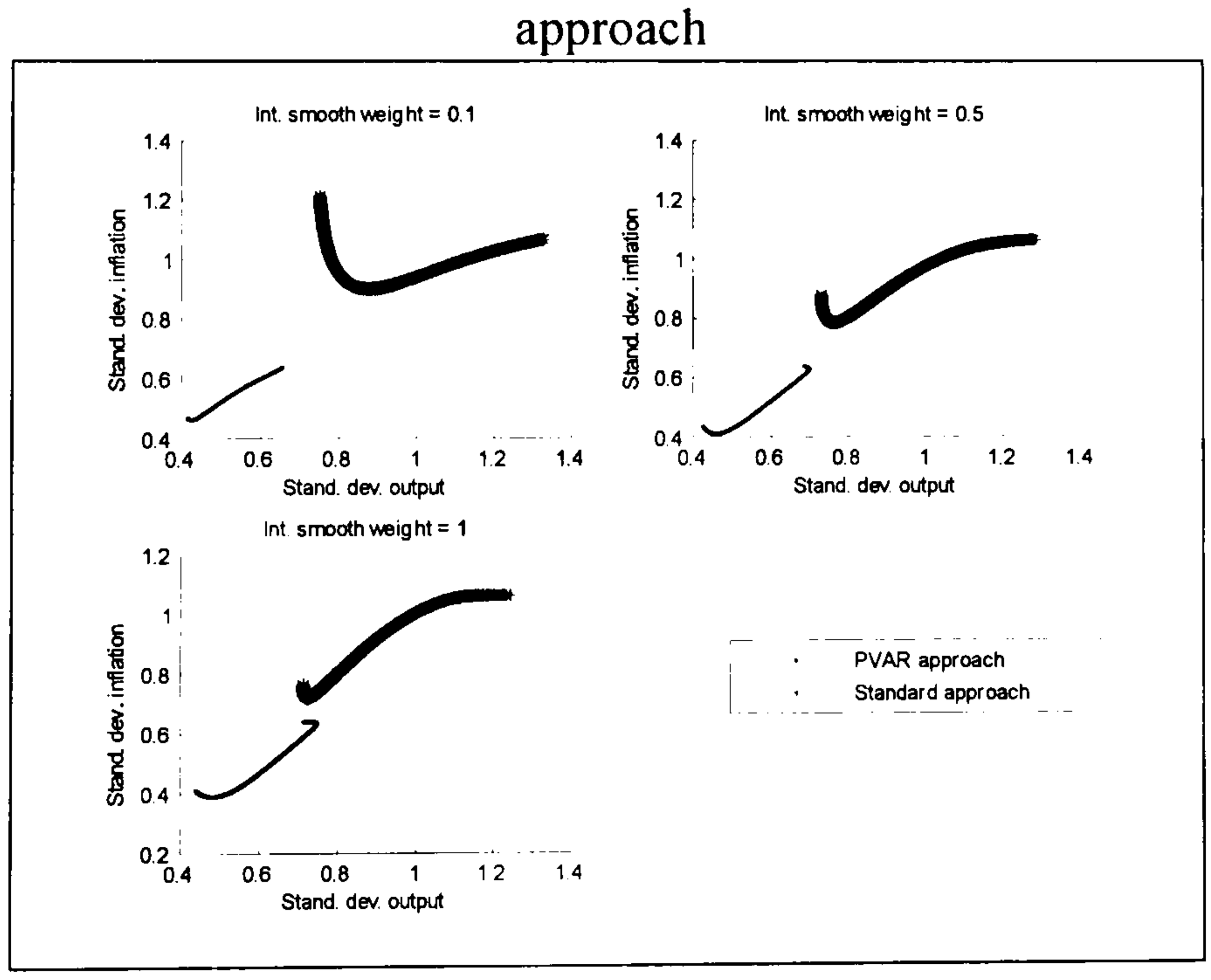


Figure 2.7: Efficiency frontiers of optimal interest rate rules under PVAR and standard approach



Appendix C
Data (Chapter 3)

The US data are annual for the period 1960 to 2005 and are taken from the *OECD* Economic Outlook database and are described in the *OECD* Economic Outlook Database Inventory and on the Annex Tables session of the Sources and Methods.

GDP, Value, at market prices, of gross domestic product;

$GNFL$, Value of government net financial liabilities[31];

$PGDP$, deflator of GDP at market prices;

$GGINTP$, Value of gross government interest payments;

$GGINTR$, Value of gross government interest receipts;

$GNINTP$, Value of net government interest payments[32];

$YPGT$, Value of government total disbursement;

$YRGT$, Value of government total receipts;

IRS, Short-term nominal interest rate (in percentages)[33];

IRL, Long-term interest rate (in percentages)[34].

The variables used in this study are then calculated as follows:

1. $\frac{b_t}{y_t}$ is $GNFL$ in percentage of GDP.

---

31 This variable refers to the consolidated gross financial liabilities of the government sector net of short-term financial assets, such as cash, bank deposits, loans to the private sector etc.

32 $GGINTP = GNINTP - GNINTR$

33 U.S. rates refer to interest rates on United States dollar three-month deposits in London, UK interest rates are 3-month rates on interbank loans, while Germany interest rates refer to the 3-month FIBOR rate.

34 Rates refer to the ten-year government bond yield for the US and the UK, while they refer to the federal bond yield in the case of Germany.

2. $\frac{v_t}{y_t}$ is $YRGT$ in percentage of GDP.

3. $\frac{g_t}{y_t}$ is $YPGT$ minus $GGINTP$ in percentage of GDP.

4. π_t is the annual rate of change in the natural logarithm of $PGDP$.

5. y_t is the output GAP measured as deviation of real GDP from a quadratic trend.

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