THE UNIVERSITY OF YORK

NAVIER-STOKES EQUATIONS AND VECTOR ADVECTION

By

Mikhail Neklyudov

(M.Sc. Moscow State University, Russia)

being a Thesis submitted for the Degree of

Doctor of Philosophy

in the University of York

Department of Mathematics

DECEMBER 2006

To my mother.

Abstract

In this thesis, certain systems of linear parabolic equations called vector advection equations will be considered. These equations are of great current scientific interest because they appear in magnetohydrodynamics and also it models certain properties of three dimensional Navier-Stokes equations which does not appear in the model of scalar advection.

The thesis consists of six chapters. The first chapter is a review of existing relevant literature. The second chapter contains preliminary material necessary for further chapters.

In the third chapter it is shown that solution of vector advection equations is self-dual in a certain sense described in the thesis. It is established that the so called regularity space of vector transport operator changes with time reversal of velocity v. Also the classical result of Serrin-Prodi-Ladyzhenkaya on the existence of strong solution of Navier-Stokes equations is reproved.

In the fourth chapter the Feynman-Kac type formulas for the vector advection operator have been proved. Another way to prove Feynman-Kac type formula can be found in Busnello, Flandoli, Romito (2005). Our approach permits us to find other non classical Feynman-Kac formulae for vector transport operator.

In the fifth chapter we study the asymptotic behaviour for certain class of parabolic stochastic partial differential equations. First we prove a backward uniqueness result and the existence of the spectral limit for abstract SPDEs and then show how it can be applied for some linear and nonlinear SPDEs. Our results generalize the results proved in Ghidaglia (1986) for non stochastic PDE.

In the last chapter we prove existence of a global solution for the random vortex filament equation. This equation appear in fluid dynamics in the theory of three dimensional Euler equations. Existence of a global solution for smooth initial conditions has been shown in a preprint work of Berselli, Gubinelli. We work in the framework of rough space theory, see e.g. Gubinelli (2004) and assume that initial condition is a closed curve of Hölder class with exponent $\nu > \frac{1}{3}$. In particular, this result covers the case of Brownian loop.

Table of Contents

Acknowledgements

First of all I would like to thank Professor Zdzisław Brzeźniak, my supervisor. He has shown me how to conduct coherent and rigorous research and introduced me to the area of SPDEs. Furthermore, He was very patient with me and gave me a lot of his time. I am very grateful for many suggestions and insights he has given me in the time of our collaboration. He also provided support to me in different non-academical ways. I have learned a lot from him and hope to learn more in the future.

I wish to thank Professor Tomasz Zastawniak, Professor Ben Goldys and Professor Oleg Smolyanov for their interest in my work and some helpful suggestions.

I would like to thank Professor David Elworthy, Dr. Xue-Mei Li, Professor Zdzisław Brzeźniak, Professor Huai Zhong Zhao, the organizers of East Midlands Stochastic analysis seminar (2004, 2005, 2006), organizers of the IMA PI Summer Program for Graduate Students "Stochastic Partial Differential Equations and Environmental and Geophysical Modeling" (Wyoming, 2005), Professor Smolyanov and Professor Shavgulidze, the organizers of the seminar "Infinite dimensional analysis and mathematical physics"(Moscow, 2004, 2005, 2006) for the wonderfully prepared meetings.

I am grateful to the Overseas Research Students Awards Scheme, the University of Hull and University of York for financial support towards a research studentship.

I thank My Mum, Brother and Grandmother. It would be impossible to write this work without their support and love.

Misha Neklyudov

Hull, United Kingdom
December, 2006

Chapter 1

Introduction

The main subject of this thesis is investigation of certain properties of vector transport equations and corresponding properties of Navier-Stokes equations. The motivation of this research comes from the need to understand the motion of fluid particles in turbulent fluid flows.

Theory of models describing turbulent fluid flows is a very active research field both in mathematics and physics. It is believed that some physical properties of turbulence such as anomalous scaling can be modelled by passive scalar advection equations, see [47], [31]. Mathematical study of the passive scalar equations has been started only recently in the works of Le Jan, Raimond [51] and Lototsky, Rozovskii [55], [56]. In the case of irregular velocity vector field which appears in turbulence the standard framework of PDEs is not sufficient to study the scalar advection equation. Le Jan and Raimond have introduced a new concept of generalized solutions of the scalar advection model through the Wiener chaos decomposition. They observed that this solution does not necessarily corresponds to the flow of trajectories of the velocity vector field. They found that the solutions have two regimes, see also work of Gawędzki and Vergassola [41] for motivation and more informal approach:

- Coalescent flow of maps i.e. solution corresponds to the flow of particles that coalesce at some moment of time.

- Diffusive regime i.e. solution correspond to branching of trajectories.

In connection with these works it is natural to ask whether similar properties hold for solutions of the vector advection equations.

In this thesis I have considered vector advection equations and studied some of its properties. It turns out that these properties do not have analogues in the scalar advection case. In particular, I show that these equations are self-dual in a certain sense. As a corollary, I find estimates for the vorticity of solution of the Navier-Stokes equations (NSEs for brief) which could be used to reprove the classical result of Ladyzhenskaya [49], Serrin [69] and Prodi [64] about existence of the strong solution of NSEs. Also the self-duality property allows me to establish the "optimal" space (in a sense described in Chapter 3) for the domain of the vector advection operator. Another interesting consequence of the self-duality property is that existence of solutions imply their uniqueness and vice versa. I hope that in the future I will be able to find more applications of my theory.

Another property of the vector advection equations studied in the thesis is a non classical form of the Feynman-Kac type formula. I prove such a formula for both the two dimensional vector advection equations and for the two dimensional NSEs. Thus we show that there exist two different "path integral" representations of the fluid flow.

The question of finding probabilistic representation of solution to NSEs have drawn attention of many mathematicians. Different Feynman-Kac type representations of solution of Navier-Stokes equations were considered in works by Busnello [17], Busnello, Flandoli and Romito [18], Constantin and Iyer [23] (see also work of Constantin [22] for Euler equations), Rapoport [66], Albeverio and Belopolskaya [2], Le Jan and Sznitman [52], see also the bibliographic survey in [18].

Constantin in [22] studied the Lagrangian formulation of the Euler equations. He reformulated the incompressible Euler equations through the inverse of the Lagrangian map and proved the local existence of a solution to the resulting equation, which he called the active vector formulation of Euler equations. He also noticed that the resulting equation is a generalization of the Clebsch variables representation. He also discussed a blow up issue in terms of different geometrical criteria connected with the behaviour of the gradient and the inverse gradient of the Lagrangian map.

In the article [17] by Busnello, see also [18], the Lagrangian formulation of the two and three dimensional Navier-Stokes equations in conjunction with probabilistic treatment of the relationship between velocity and vorticity, i.e. the Biot-Savar law in three dimensional case, were used to prove global existence and uniqueness of solutions to the two dimensional NSEs and local existence and uniqueness to three dimensional NSEs. In both papers the Bismut-Elworthy-Li integration by parts formula, see [30] and references therein, is used to give a probabilistic representation of velocity.

Constantin and Iyer in [23] suggested a different Lagrangian formulation of the NSEs. They used the active vector formulation developed in Constantin [22] and they replaced the Lagrangian trajectories by a stochastic flow. This interpretation of the Lagrangian formulation allowed the authors to get results similar to those by Busnello, Flandoli and Romito. Furthermore, they were able to get a similar representation for other hydrodynamic models, including the viscous Burgers equation and LANS-alpha models.

Our starting point for the Feynman-Kac type representation of solution to the vector advection equations is a generalization of the classical Kelvin Theorem about conservation of circulation of velocity along with the flow of the inviscid fluid. I establish an analog of the Kelvin Theorem for the vector advection equations which states that the circulation of the velocity v along the flow (X_t^s), $0 \leq s \leq t \leq T$ defined by the following stochastic differential equation

$$\begin{aligned} dX_t^s(x) &= v(t, X_t^s(x))dt + \sqrt{2\nu}dW_t \\ X_s^s(x) &= x \end{aligned}$$

is a martingale. By taking the mathematical expectation I immediately get the Feynman-Kac formula. The stochastic flow (X_t^s), $0 \leq s \leq t \leq T$ has been used by Busnello [17], Constantin and Iyer [23], Albeverio and Belopolskaya [2]. The main difficulty in obtaining new a'priori estimates for the solutions of the vector advection equations is the presence in the Feynman-Kac formula of the gradient with respect to the initial data. Similar difficulty appears in the study of the three dimensional Euler and Navier-Stokes equations, see works by Flandoli, Constantin and others listed earlier. In connection with this problem it is natural to ask whether there exists other non trivial flows for which generalization of the Kelvin Theorem and the Feynman-Kac type representation holds true. In the thesis I answer positively in the two dimensional case. I show that the flow generated by the following SDE can be used:

$$\begin{aligned} dX_t^s(x) &= \sqrt{2\nu}\sigma_1(X_t^s(x))dW_t, 0 \leq s \leq t \leq T \\ X_s^s(x) &= x \end{aligned}$$

where

$$\sigma_1(x) = \begin{pmatrix} \cos\frac{\phi(x)}{\nu} & -\sin\frac{\phi(x)}{\nu} \\ \sin\frac{\phi(x)}{\nu} & \cos\frac{\phi(x)}{\nu} \end{pmatrix}, x \in \mathbb{R}^2.$$

ϕ is a stream function defined by $v = \nabla^{\perp}\phi$. As a consequence the standard Feynman-Kac type formula (4.1.7) is simplified as in (4.2.6).

In the three dimensional case the question of possibility to model flow in this way remains open, see though Question 4.3.6 for the present state of the knowledge about the problem.

The second topic investigated in the thesis is the asymptotic behaviour at time goes to infinity of solutions to certain linear and nonlinear stochastic PDEs. I show the backward uniqueness and the existence of the spectral limit for a quite general class of linear parabolic SPDEs and, under certain regularity assumptions, for certain nonlinear SPDEs. In particular, my results are applicable to linear parabolic SPDEs with gradient noise and Navier-Stokes equations with multiplicative noise. Similar questions for deterministic PDEs were studied in the works of Foias, Saut [34] for Navier-Stokes equations and Ghidaglia [42] for general deterministic parabolic PDEs. Moreover, Foias, Saut [34], [35] were able to show existence and smoothness of corresponding spectral manifolds. It would be interesting to extend these results for stochastic case.

In the last chapter of the thesis I prove existence of global solutions to the random filament equation. This equation appears as an approximate model for time evolution of an incompressible inviscid fluid under the assumption that the vorticity vanishes outside some neighborhood of a certain time-dependent closed curve, see [32], [33], [8], [68] and the book [20]. Some numerical approximations, see [5], [75], imply that the regions of "big" vorticity have a form of a "filament" and therefore, this model can be considered as mathematical idealization of the motion of the fluid. Berselli and Gubinelli [7] have shown global existence of solution in the case of initial condition belongs to the Sobolev space $H^{1,2}$. In this work I establish existence and uniqueness of a global solution for a larger class of initial data, including Hölder functions with exponent $\nu \in (\frac{1}{3}, 1]$. This is of utmost importance because it includes, for instance, the Brownian loops, see [9],p.1849. I use framework of rough path theory, see [58], [43] and references therein. Local existence in this class has been proved in [9]. A generalization of this result to manifold valued loops along the lines suggested by Brzeźniak and Léandré in [14] is one of open problems discussed at the end of Chapter 6.

Chapter 2

Preliminary Material

In this chapter I shall present general settings of the thesis. I shall first briefly recall some basic notation and standard theorems from vector analysis. Then I shall present some abstract tools i.e. Lebesgue integration theory, functional analysis and theory of evolution equations. In particular, I shall be concerned with embedding theorems, Gagliardo-Nirenberg inequalities, existence and uniqueness theorems for abstract evolution equations. I will conclude the chapter with few results from stochastic analysis which will be used throughout the thesis.

2.1 Basic notations

Here I will present some standard theorems of Analysis.

I begin by introducing some standard notation of vector calculus. Suppose that $a = (a^1, a^2, a^3), b = (b^1, b^2, b^3) \in \mathbb{R}^3$ then $a \times b$ is a cross product of two vectors defined by

$$(a \times b)^1 = a^3b^2 - a^2b^3, (a \times b)^2 = a^1b^3 - a^3b^1, (a \times b)^3 = a^2b^1 - a^1b^2.$$

The cross product has following properties:

$$(a \times b, c)_{\mathbb{R}^3} = (a, b \times c)_{\mathbb{R}^3}, a, b, c \in \mathbb{R}^3 \tag{2.1.1}$$

$$|a \times b|_{\mathbb{R}^3} \leq |a|_{\mathbb{R}^3}|b|_{\mathbb{R}^3}, a, b \in \mathbb{R}^3. \tag{2.1.2}$$

Operators curl : $C^\infty(\mathbb{R}^3, \mathbb{R}^3) \to C^\infty(\mathbb{R}^3, \mathbb{R}^3)$ and div : $C^\infty(\mathbb{R}^3, \mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$ are defined by identities

$$\operatorname{curl} u = \left(\frac{\partial u^3}{\partial x_2} - \frac{\partial u^2}{\partial x_3}, \frac{\partial u^1}{\partial x_3} - \frac{\partial u^3}{\partial x_1}, \frac{\partial u^2}{\partial x_1} - \frac{\partial u^1}{\partial x_2}\right),$$

$$\operatorname{div} u = \sum_{i=1}^{3} \frac{\partial u^i}{\partial x_i}.$$

We have following well known theorems (see for instance, [70], p.135):

Theorem 2.1.1 (The divergence or Green's Theorem). *Let $S \subset \mathbb{R}^3$ be a C^1-class closed surface which is a boundary of a bounded domain $D \subset \mathbb{R}^3$. Choose the outward normal vector field $\vec{n}$ to the surface. Then if $u : D \to \mathbb{R}^3$ is of C^1 class,*

$$\iiint_D \operatorname{div} u \, dx = \iint_S u \cdot \vec{n} \, d\sigma,$$

where $d\sigma$ is the surface measure.

Theorem 2.1.2 (Stokes's Theorem). *Let $S \subset \mathbb{R}^3$ be a bounded and open two-sided surface bounded by a closed non-intersecting curve $\Gamma \subset \mathbb{R}^3$ (simple closed curve), $\vec{n}$ be an outward normal vector field to the surface S. Assume that $D \subset \mathbb{R}^3$ is a domain such that $S \subset D$ and $u : D \to \mathbb{R}^3$ is of C^1 class. Then*

$$\int_\Gamma u \cdot dx = \int_S (\operatorname{curl} u) \cdot \vec{n} \, d\sigma.$$

In particular, if curl u *is equal to zero on D, then $\int_\Gamma u \, dx = 0$.*

2.2 Abstract Tools

Here I present some standard results from functional analysis and integration theory.

2.2.1 Lebesgue integration theory

I will present here the classical theorems on Lebesgue Integration theory, see for instance [37], and some fundamental inequalities. Let $(X, \mathcal{F}, \mu)$ be a measure space i.e. X is a space, $\mathcal{F}$ is a σ-algebra of subsets of X and μ is a measure with domain $\mathcal{F}$. The sets of $\mathcal{F}$ are measurable sets.

Let f be a real-valued function defined on a measurable set X_0 of X. We say that f is a measurable function if the inverse image of any open set in $\mathbb{R}$ is a measurable set. We say that f is an extended real valued function if we allow it to have values of $+\infty$ or $-\infty$. In this case we add in the definition of measurability the requirement that the sets $f^{-1}(+\infty), f^{-1}(-\infty)$ be measurable.

A function f is called a simple function if there is a finite number of mutually disjoint measurable sets $E_1, \dots, E_m$ and real numbers $\alpha_1, \dots, \alpha_m$ such that

$$f = \sum_{i=1}^{m} \alpha_i \chi_{E_i}(x),$$

where χ_E is an indicator function of set E i.e.

$$\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E. \end{cases}$$

A simple function $f = \sum_{i=1}^{m} \alpha_i \chi_{E_i}$ on a measure space $(X, \mathcal{F}, \mu)$ is said to be integrable if $\mu(E_i) < \infty$ for all the indices i for which $\alpha_i \neq 0$. The integral of f is the number $\sum_{i=1}^{m} \alpha_i \mu(E_i)$. We denote this sum also $\int f(x) d\mu(x)$ or $\int f \, d\mu$.

Definition 2.2.1. *An extended real-valued, measurable function f, on a measure space $(X, \mathcal{F}, \mu)$ is said to be integrable if there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of integrable simple functions having the following properties:*

(a) $\int_X |f_k(x) - f_m(x)| d\mu \to 0$ *as* $k, m \to \infty$.

(b) $\lim_{k\to\infty} f_k(x) = f(x)$ *a.e.*

Theorem 2.2.2 (Lebesgue Dominated Convergence Theorem (see [37], p.54)). *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of integrable functions defined on* X *that almost everywhere converge to a measurable function* f. *If there exists an integrable function* g *defined in* X, *such that almost everywhere for all* n, $|f_n(x)| \leq g(x)$, *then* f *is integrable,* $\lim_{n\to\infty} \int_X |f_n(x) - f(x)| d\mu = 0$, *and*

$$\lim_{n\to\infty} \int_X f_n(x) d\mu = \int_X f(x) d\mu.$$

Theorem 2.2.3 (Lebesgue monotone Convergence Theorem (see [37], p.58)). *Let* $\{f_n\}_{n=1}^{\infty}$ *be a monotone increasing sequence of integrable functions and their integrals are bounded from above i.e. there exists a constant* $K < \infty$ *such that for all* $n \in \mathbb{N}$*:*

$$\int_X f_n(x) d\mu \leq K$$

Then, f_n *is convergent a.e. on* X *and if we denote* $f(x) = \lim_{n\to\infty} f_n(x)$ *(exists a.e. on* X*),* f *is integrable and*

$$\lim_{n\to\infty} \int_X f_n(x) d\mu = \int_X f(x) d\mu.$$

Theorem 2.2.4 (Fatou's lemma (see [37], p.58)). *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of measurable nonnegative functions which converges a.e. on* X *to* f *such that*

$$\int_X f_n(x) d\mu \leq K, n \in \mathbb{N}$$

then f *is integrable on* X *and*

$$\int_X f(x) d\mu \leq K.$$

Theorem 2.2.5 (Absolute continuity of Lebesgue Integral(see [37], p.53)). *Let* f *be integrable function on* X. *Then for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $|\int_A f(x) d\mu| < \varepsilon$ *provided* A *is a measurable set such that* $\mu(A) < \delta$.

At the end of this section I present the following well known inequalities (see [37], p.96):

Theorem 2.2.6 (Hölder's inequality). *If* $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, u, v *are measurable functions defined on* X *such that* $\int_X |u|^p \, d\mu < \infty$ *and* $\int_X |v|^q \, d\mu < \infty$, *then* $\int_X |uv| \, d\mu < \infty$ *and* $\int_X |uv| \, d\mu \le (\int_X |u|^p \, d\mu)^{1/p} (\int_X |v|^q \, d\mu)^{1/q}$.

Theorem 2.2.7 (Minkowski's inequality). *If* $1 \le p \le \infty$, u, v *are measurable functions defined on* X *such that* $\int_X |u|^p \, d\mu < \infty$ *and* $\int_X |v|^p \, d\mu < \infty$, *then* $(\int_X |u+v|^p \, d\mu)^{1/p} \le (\int_X |u|^p \, d\mu)^{1/p} + (\int_X |v|^p \, d\mu)^{1/p}$.

Theorem 2.2.8 (Young inequality). *Suppose* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0$.*Then* $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.

2.2.2 Functional analysis results

The fixed point theory is concerned with the conditions which guarantee that a map $F : X \to X$ of a topological space X into itself admits one or more fixed points. The Banach fixed point theorem is the simplest yet the most important result in this respect. Now I present the Banach Fixed Point Theorem, for more details I refer to Dugundji and Granas [29] (p.10) and Theorem 3.8.2, p.119 in Friedman [37].

Theorem 2.2.9 (Banach Fixed Point Theorem). *Let* (Y, d) *be a complete metric space and let* $F : Y \to Y$ *be a contractive map, i.e. there exists* $M < 1$ *such that for any* $x, y \in Y$, $d(F(x), F(y)) \le M d(x, y)$. *Then* F *has a unique fixed point* $x_0 \in Y$, *i.e. there exists a unique* $x_0 \in Y$ *such that* $F(x_0) = x_0$.

Banach-Alaoglu Theorem together with compact embedding theorems, are the basic tools used in proofs of existence of solutions of PDE. Now I present Banach-Alaoglu Theorem, for more details I refer to Friedman [37] (Theorem 4.12.2, p.169).

Theorem 2.2.10 (Banach-Alaoglu Theorem). *Let X be a Banach space and let X^* be its dual. Then the closed unit ball of X^* is compact in the weak-* topology.*

Remark 2.2.11. Friedman ([37]) have used different terminology. The notion of weak topology he introduced is the same as the notion of weak-* topology in modern terminology.

Next I present one abstract compact embedding theorem in Banach spaces, for proof I refer to [72] (Theorem 2.1, p.184).

Theorem 2.2.12. *Let X_0, X, X_1 be three Banach spaces such that*

$$X_0 \subset X \subset X_1, \tag{2.2.1}$$

where injections are continuous and

$$X_i \text{ is reflexive, } i = 0, 1, \tag{2.2.2}$$

$$\text{the injection } X_0 \subset X \text{ is compact.} \tag{2.2.3}$$

Let $T > 0$, $\alpha_0 > 1$, $\alpha_1 > 1$ be fixed finite numbers and

$$Y = \{v \in L^{\alpha_0}(0,T;X_0), v' = \frac{dv}{dt} \in L^{\alpha_1}(0,T;X_1)\} \tag{2.2.4}$$

be a Banach space equipped with the norm

$$||v||_Y = ||v||_{L^{\alpha_0}(0,T;X_0)} + ||v'||_{L^{\alpha_1}(0,T;X_1)}. \tag{2.2.5}$$

Then injection of Y into $L^{\alpha_0}(0,T;X_0)$ is compact.

2.2.3 Approximation theorem

Here I will recall a concept of mollifier and present standard approximation theorem.

Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a C_0^∞ function defined by

$$\rho(x) = \begin{cases} 0, & |x| \geq 1; \\ ce^{-\frac{1}{1-|x|^2}}, & |x| < 1, \end{cases}$$

where c is a constant such that $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$.

Definition 2.2.13. *A function $u : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$ is a Lebesgue measurable set, is locally integrable if and only if u is measurable and for any compact subset $K \subset X$, $\int_K |u(x)|\,dx < \infty$. We denote space of locally integrable functions on X by $L^1_{loc}(X)$*

Suppose that function $u \in L^1_{loc}(X)$. We call

$$J_\varepsilon(u) = \frac{1}{\varepsilon^n} \int_X \rho(\frac{x-y}{\varepsilon}) u(y) dy$$

a mollifier of u. Properties of mollifier are summarized in the following theorem for which we refer to Friedman [36] Theorem 6.2 (p.12) and Agmon [1] Theorem 1.5, 1.6, 1.7, 1.8 for more details.

Theorem 2.2.14. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable function that vanishes outside a measurable set $X \subset \mathbb{R}^n$.*

1. *If u is locally integrable in X, then $J_\varepsilon(u) \in C^\infty(\mathbb{R}^n, \mathbb{R})$.*

   If in addition, the support of u is contained in a compact subset K of X and if $\varepsilon < \operatorname{dist}(K, \partial X)$, then $J_\varepsilon(u) \in L^p(X, \mathbb{R})$ and $J_\varepsilon(u) \in C_0^\infty(X, \mathbb{R})$;

2. *If for $1 \leq p \leq \infty$, $u \in L^p(X, \mathbb{R})$, then $|J_\varepsilon(u)|_{L^p} \leq |u|_{L^p}$;*

3. *If for $1 \leq p \leq \infty$, $u \in L^p(X, \mathbb{R})$, then $J_\varepsilon(u) \to u$ in $L^p(X, \mathbb{R})$ as $\varepsilon \to 0$.*

2.3 Functional spaces and embedding theorems

In this section I will recall some basic notations and properties of Sobolev spaces.

Let $D \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be either $\mathbb{R}^n$ or open bounded domain with smooth boundary $\Gamma = \partial D$. Assume that $d \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu > 0$. I will use following notation.

1. If $\nu \in \mathbb{N}$, then $C^\nu(D, \mathbb{R}^d)$ is the space of ν times continuously differentiable functions from D to $\mathbb{R}^d$. By $C_b^\nu(D, \mathbb{R}^d)$ we denote the set of all $u \in C^\nu(D, \mathbb{R}^d)$ that are bounded together with their derivatives up to (inclusive) order ν.

2. If $\nu \notin \mathbb{N}$, then $C^\nu(D, \mathbb{R}^d)$ is the space of those functions u belonging to $C^{[\nu]}(D, \mathbb{R}^d)$ whose $[\nu]$-order partial derivatives are Hölder continuous function with exponent $\nu - [\nu]$. The subspace $C_b^\nu(D, \mathbb{R}^d)$ of $C^\nu(D, \mathbb{R}^d)$ consisting of those functions u satisfying
$$|u|_{\nu,D} = \sum_{|\alpha|\le[\nu]} \sup_{x\in D} |D^\alpha u(x)| + \sum_{|\alpha|=[\nu]} \sup_{x\ne y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{\nu-[\nu]}} < \infty$$
is a Banach space (with norm $|\cdot|_{\nu,D}$).

3. $C^\infty(D, \mathbb{R}^d) = \bigcap_{m>0} C^m(D, \mathbb{R}^d)$, $C_b^\infty(D, \mathbb{R}^d) = \bigcap_{m>0} C_b^m(D, \mathbb{R}^d)$

4. $C_0^\infty(D, \mathbb{R}^d) = \{f \in C^\infty(D, \mathbb{R}^d) | \operatorname{supp} f$ – compact subset of $D\}$

5. $\mathcal{D}(D, \mathbb{R}^d) = \{f \in C_0^\infty(D, \mathbb{R}^d) | \operatorname{div} f = 0\}$

6. $H_0^{k,p}(D, \mathbb{R}^d)$- completion of $C_0^\infty(D, \mathbb{R}^d)$ with respect to norm
$$(\sum_{l=1}^{k} \sum_{|\alpha|\le l} \int_D |\nabla^\alpha f|_{\mathbb{R}^d}^p \, dx)^{1/p}.$$
$L_0^p(D, \mathbb{R}^d) = H_0^{0,p}(D, \mathbb{R}^d)$;

7. $H^{k,p}(D,\mathbb{R}^d)$– completion of $C^\infty(D,\mathbb{R}^d)$ w.r.t. the same norm, $L^p(D,\mathbb{R}^d)$=$H^{0,p}(D,\mathbb{R}^d)$;

8. In the case of $d = 1$ I will often omit second argument in notations of spaces above and write for example $C^m(D)$ instead of $C^m(D,\mathbb{R})$. Similarly, in the case when $d = n$ I will use bold fonts, for instance, I will write $\mathbf{H}^{k,p}(D)$ instead of $H^{k,p}(D,\mathbb{R}^n)$ and $\mathbf{L}^p(D)$ instead of $L^p(D,\mathbb{R}^n)$;

9. In the case of divergence free functions I will use the `mathbb` font and subscript $_{sol}$. For example, I will denote $\mathbb{H}^{k,p}_{sol}(D) = \{f \in \mathbf{H}^{k,p}(D) | \operatorname{div} f = 0\}$ and similar notation will be used for other spaces.

In the following I will often consider the case of $p = 2$ and $D = \mathbb{R}^n$. In this case there exists another equivalent definition of Sobolev space $H^{s,2}(\mathbb{R}^n)$, $s \in \mathbb{R}$ by fourier transform, see Lions and Magenes [53], Temam [72] and Bensoussan [10].

Definition 2.3.1. *If $u \in L^2(\mathbb{R}^n) \bigcap L^1(\mathbb{R}^n)$, the Fourier transform $\hat{u}$ is defined by*

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} u(x)\, dx. \tag{2.3.1}$$

One can show that $|\hat{u}|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^{n/2}} |u|_{L^2(\mathbb{R}^n)}$. Therefore, the map $u \mapsto \hat{u}$ has a unique extension to a bounded linear map $\mathcal{F}$ from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

I set $\hat{u} = \mathcal{F}(u)$ and

$$u = \bar{\mathcal{F}}\hat{u} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ixy} \hat{u}(y)\, dy.$$

One can also show that the above extension is in fact onto and $\mathcal{F}\bar{\mathcal{F}} = \bar{\mathcal{F}}\mathcal{F} = \mathrm{id}$. In particular, $\mathcal{F}$ is a linear isomorphism of $L^2(\mathbb{R}^n)$. We have the following result, see Lions and Magenes [53] Theorem 1.1.2 for a proof.

Theorem 2.3.2. *If $m \in \mathbb{N}$, then*

$$H^{m,2}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : (1+|\xi|^2)^{m/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n)\}. \tag{2.3.2}$$

and the norm

$$\|u\|_{H^{m,2}(\mathbb{R}^n)} = |(1+|\xi|^2)^{m/2}\hat{u}|_{L^2(\mathbb{R}^n)} \tag{2.3.3}$$

is equivalent to the standard norm $|\cdot|_{m,2}$.

Taking into account Theorem 2.3.2 we can define $H^{s,2}(\mathbb{R}^n)$, $s \in \mathbb{R}$ as follows:

$$H^{s,2}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |(1+|\xi|^2)^{s/2}\hat{u}(\xi)|^2\, d\xi < \infty\}.$$

Now I will recall Sobolev embedding Theorem and version of Gagliardo-Nirenberg inequality (see Friedman [36], Theorem 1.9.3 (p.24) and Theorem 1.10.1 (p.27)):

Theorem 2.3.3 (Sobolev embedding Theorem). *Suppose $D \subset \mathbb{R}^n$ is open bounded set with smooth boundary or $\mathbb{R}^n$. Then we have*

$$H^{k,p}(D) \subset L^q(D) \text{ if } \frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{p} - \frac{k}{n} > 0,$$

$$H^{k,p}(D) \subset C(D) \text{ if } kp > n,$$

$$H^{k,p}(D) \subset C^\nu(D) \text{ if } 0 \leq \nu < k - \frac{n}{p}$$

and the embeddings are continuous. In the second case every function $f \in H^{k,p}(D)$ has a continuos version $\hat{f}$ such that $\hat{f} = f$ almost everywhere in D.

Theorem 2.3.4 (Gagliardo-Nirenberg inequality). *Let $1 \leq q, r \leq \infty$ and $j, m \in \mathbb{N}$ satisfying $0 \leq j < m$. Then for any $u \in C_0^m(\mathbb{R}^n)$,*

$$|D^j u|_{L^p(\mathbb{R}^n)} \leq C|D^m u|^a_{L^r(\mathbb{R}^n)}|u|^{1-a}_{L^q(\mathbb{R}^n)}, \tag{2.3.4}$$

where $\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}$ for all $\frac{j}{m} \leq a \leq 1$ and C is a constant depending only on n, m, j, q, r, a with the following exception:

If $m - j - \frac{n}{r}$ is a nonnegative integer, then inequality (2.3.4) holds only for $\frac{j}{m} \leq a < 1$.

2.4 Theory of abstract parabolic equations

The definitions and results of this section are standard and I refer the reader to [53], [72] and [71].

Let V and H be two separable Hilbert spaces, $V \subset H$ and embedding is continuous, V dense in H; let $|\cdot|_H$ and $||\cdot||_V$ denote the norms in H and V, $(\cdot,\cdot)_H$ scalar product in H. Identifying H with its dual H' we have $V \subset H \cong H' \subset V'$. Duality relation between V and V' I denote $< \cdot,\cdot >_{V',V}$. I will call triple $V \subset H \cong H' \subset V'$ a Gelfand Triple. In the following when it is clear from the context which norm (or duality relation) I am using I will often omit indexes i.e. write $|\cdot|$ instead of $|\cdot|_H$ (or $< \cdot,\cdot >$ instead of $< \cdot,\cdot >_{V',V}$).

Possibility of identification H and H' follows from Riesz Theorem. Here I will present more general Lax-Milgram Theorem (Lemma 2.2.1 in [71], p.26).

Theorem 2.4.1 (Lax-Milgram Theorem). *Assume that $B : H \times H \to \mathbb{R}$ is a quadratic form and there exist positive constants c and C such that*

(continuity) $|B(u,v)| \leq C|u||v|$

(coercivity) $|B(u,u)| \geq c|u|^2$

for all $u, v \in H$. Under these conditions, if $F \in H'$ then there exists an element $u \in H$ such that $F(v) = B(u,v)$ for all $v \in H$. Furthermore, u is uniquely determined by F.

Now if we consider the case $B(\cdot,\cdot) = (\cdot,\cdot)_H$ we immediately get Riesz Theorem and identification of H and H'.

Definition 2.4.2 (Coercive form). *Let $a : [0,T] \times V \times V \to \mathbb{R}^1$ be a continuous bilinear form i.e. we suppose that*

$$|a(t,u,v)| \leq c||u||\,||v||, t \in [0,T], u, v \in V$$

and

$$\textit{the function } t \to a(t,u,v) \textit{ is measurable for any fixed } u,v \in V.$$

Then we say that the form a is coercive if there exists $\lambda \in \mathbb{R}$, $\alpha > 0$ *such that*

$$|a(t,v,v)| + \lambda|v|^2 \geq \alpha||v||^2, v \in V. \tag{2.4.1}$$

Since, for fixed $t \in [0,t]$, *the form* $v \to a(t,u,v)$ *is continuous on* V, *we have:*

$$a(t,u,v) =< A(t)u, v >, A(t)u \in V'. \tag{2.4.2}$$

Thus, we have operator

$$A \in L^{\infty}([0,T], \mathcal{L}(V,V')) \tag{2.4.3}$$

(defined by (2.4.2)*) which corresponds to the coercive form a. Condition of coercivity* (2.4.1) *of the form a can be reformulated in terms of operator* A *as follows:*

there exists $\alpha > 0, \lambda \in \mathbb{R}$ *such that*

$$< A(t)u, u >_{V',V} \geq \alpha|u|_V^2 + \lambda|u|_H^2, u \in V. \tag{2.4.4}$$

I will call operator satisfying conditions (2.4.3), (2.4.4) *coercive.*

We have following correspondence between properties of form and operator:

Theorem 2.4.3 (Theorem 2.2.3 in [71], p.29). *Let a be coercive symmetric form satisfying condition* (2.4.1) *with* $\lambda = 0$, *then for each* $t \in [0,T]$ $A(t)$ *is positive definite and self-adjoint,* $D(A(t)^{1/2}) = V$, *and*

$$a(t,u,v) = (A(t)^{1/2}u, A(t)^{1/2}v), \; t \in [0,T], u,v \in V.$$

Remark 2.4.4 (remark 2.2.1 in [71], p.29). For $\lambda > 0$, replace $a(u,v)$ by $a(u,v) + \lambda(u,v)$ and A by $A + \lambda$ then the conclusion of Theorem 2.4.3 still holds.

For parabolic equations with coercive operators we have following existence and uniqueness theorem:

Theorem 2.4.5 (Chapter 3, Theorem 4.1 and remark 4.3 in [53], p.238). *If $A \in L^\infty([0,T], \mathcal{L}(V,V'))$ is coercive operator, $u_0 \in H$, $f \in L^2(0,T;V')$ then equation*

$$\begin{aligned} \frac{du}{dt} + Au &= f \\ u(0) &= u_0 \end{aligned}$$

has unique solution $u \in L^2(0,T;V)$, $u' \in L^2(0,T;V')$. Moreover, u satisfies an estimate

$$|u|_H^2(t) + \alpha \int_0^t |u(s)|_V^2 ds \leq (1+2\lambda t)e^{2\lambda t}(|u_0|_H^2 + \frac{1}{4\alpha}\int_0^t |f|_{V'}^2 ds) \tag{2.4.5}$$

and $u \in C(0,T;H)$.

The following lemma is of independent interest (see chapter 3, Lemma 1.2 in [72], p.260) and will be used in the proof of Theorem 2.4.5.

Lemma 2.4.6. *If $u \in L^2(0,T;V)$, $u' \in L^2(0,T;V')$ then u is almost everywhere equal to a function continuous from $[0,T]$ into H and we have following equality, which holds in the scalar distribution sense on $(0,T)$:*

$$\frac{d}{dt}|u|^2 = 2 < u', u > . \tag{2.4.6}$$

As a consequence we have:

Corollary 2.4.7. *If $f, g \in L^2(0,T;V)$, $f', g' \in L^2(0,T;V')$ then*

$$\frac{d}{dt}(f,g)_H = < f', g >_{V',V} + < f, g' >_{V',V} . \tag{2.4.7}$$

In the case when our operator is time independent we have following important property:

Theorem 2.4.8 (Theorem 3.6.1 of [71],p.76). *Let $A \in \mathcal{L}(V, V')$ be coercive operator(i.e. it satisfies* (2.4.4)*). Then operator $-A$ generates analytic semigroup in H and V'.*

Theorem 2.4.9 (Theorem 3.2 p.22 of [54]). *Suppose that A-closed unbounded operator in H, such that*

$A + p$ is an isomorphism of $D(A) \to H$ for $p = \zeta + i\eta$, $\zeta > \zeta_0$, $\eta \in \mathbb{R}$, such that

$$||(A+p)^{-1}||_{\mathcal{L}(H,H)} \leq \frac{c}{1+|p|}, c - constant.$$

Then for $f \in L^2(0,T;H)$, there exists unique function $u \in L^2(0,T;D(A))$ satisfying

$$u' + Au = f, u(0) = u_0 \in [D(A), H]_{\frac{1}{2}}$$

As a corollary we have following proposition:

Proposition 2.4.10. *Let $A \in \mathcal{L}(V, V')$ be coercive operator. Then for given $f \in L^2(0,T;H)$, $u_0 \in V$ there exists a unique solution $u \in L^2(0,T;D(A)) \cap C(0,T;V)$ of the problem:*

$$\begin{aligned} \frac{du}{dt} + \nu Au &= f \qquad (2.4.8)\\ u(0) &= u_0 \end{aligned}$$

and it satisfies $u' \in L^2(0,T;H)$. Moreover, for a constant $C = C(\lambda, T, \nu)$ independent of u_0 and f,

$$|u'|^2_{L^2(0,T;\mathcal{H})} + \nu^2|u|^2_{L^2(0,T;D(A))} \leq C(|f|^2_{L^2(0,T;\mathcal{H})} + |u_0|^2_V). \qquad (2.4.9)$$

Proof of Proposition 2.4.10. It follows from Theorem 3.6.1 p.76 of [71] that $-A$ generates analytic semigroup in $\mathcal{H}$. Therefore, existence and uniqueness of solution u follows from Theorem 3.2 p.22 of [54]. It remains to show (2.4.9). Define $X = \{u \in L^2(0,T;D(A)) : u' \in L^2(0,T;H)\}$, $Q \in \mathcal{L}(X, V \times L^2(0,T;H))$, $Qu = (u(0), u' + Au)$. Then Q is one-to-one and onto operator and, according to open mapping theorem, there exists continuous inverse operator $Q^{-1} \in \mathcal{L}(V \times L^2(0,T;H), X)$ and (2.4.9) follows. □

Now I consider important example of Stokes operator:

Example 2.4.11. Suppose that $D \subset \mathbb{R}^n$– open bounded domain or $\mathbb{R}^n$,

$$\mathcal{V} := \{u \in C_0^\infty(D, \mathbb{R}^n) : \nabla \cdot u = 0\};$$

$$H := \text{closure of } \mathcal{V} \text{ in } L^2(D, \mathbb{R}^n);$$

$$V := \text{closure of } \mathcal{V} \text{ in } H_0^1(D, \mathbb{R}^n).$$

and $\tilde{a} : V \times V \to \mathbb{R}^1$ is defined by

$$\tilde{a}(u, v) = \sum_{i,j=1}^{n} \int_D \nabla_i u^j \nabla_i v^j dx$$

Form $\tilde{a} : V \times V \to \mathbb{R}^1$ is coercive, bilinear, continuous and symmetric. Therefore, from Lax-Milgram theorem follows that for any $f \in V'$ there exists unique $u \in V$ such that

$$\tilde{a}(u, v) + \lambda(u, v) =< f, v >_{V',V}, \forall v \in V \tag{2.4.10}$$

Define $A \in \mathcal{L}(V, V')$ by identities $\tilde{a}(u, v) =< Au, v >_{V,V'}$. Then operator $A \in \mathcal{L}(V, V')$ is self-adjoint and coercive by Theorem 2.4.3 and definition 2.4.2. We notice that this operator is a Stokes operator, see section 2.6 below.

Remark 2.4.12. Notation of this example shall be used later (unless otherwise stated).

2.5 Helmholtz Decomposition

In this section I shall recall one characterization of space H, which appear in the study of Navier-Stokes equations. I omit proof which can be found in Galdi [39], Temam [72] or references therein.

The proof of the following Theorem 2.5.1 can be found in Galdi [39], Theorem III.1.1m p.107. This result states that the Hilbert space $\mathbb{L}^2(D)$ can be decomposed into a sum of two orthogonal subspaces H and G. The first of these spaces contains the set of all smooth divergence free vectors of compact support in D as a dense subset. The second subspace contains the gradients of all single-valued functions defined in D. And the decomposition theorem implies the existence of unique orthogonal (Leray or Helmholtz) projection P from $\mathbb{L}^2$ into H. It follows that $P : \mathbb{L}^2 \to H$ is linear, bounded, idempotent ($P^2 = P$), $\mathrm{Range}(P) = H$ and $\ker(P) = G$.

Theorem 2.5.1. *Let $D \subset \mathbb{R}^n$ be an open bounded set with a sooth boundary. Then H and*

$$G = \{u \in \mathbb{L}^2(D) : u = \nabla p, \text{ for some } p \in W^{1,2}_{loc}(D)\}$$

are orthogonal subspaces of $\mathbb{L}^2(D)$.

Moreover $\mathbb{L}^2(D) = H \oplus G$.

2.6 Stokes operator

In this section we define Stokes operator and present some of its properties, see [21] for details.

We assume that domain D has boundary of C^2 class.

Definition 2.6.1. *The Stokes operator $A : D(A) \to H$ is defined by*

$$D(A) = \mathbb{H}^{2,2}_{sol}(D) \cap V, A := -P\Delta.$$

We have the following proposition (theorems 4.3,4.4 of [21]).

Proposition 2.6.2. *(1) A is self-adjoint and positive definite operator.*

(2) A is isomorphism from V to V'.

(3) A has inverse operator $A^{-1} : H \to H$ and A^{-1} is a compact operator.

(4) There exists orthonormal basis $\{e_j\}_{j=1}^{\infty}$ *of H and sequence* $\{\lambda_j\}_{j=1}^{\infty}$ *such that*

$$Ae_j = \lambda_j e_j, j = 1, \ldots \tag{2.6.1}$$

$$0 < \lambda_1 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots \tag{2.6.2}$$

$$\lim_{j \to \infty} \lambda_j = +\infty \tag{2.6.3}$$

and $\{e_j\}_{j=1}^{\infty} \subset D(A)$.

Remark 2.6.3 (proposition 4.5 of [21]). If boundary of D is of C^{l+2} class, then $e_j \in \mathbb{H}_{sol}^{l+2}(D)$.

Due to the proposition 2.6.2 we can define $D(A^{\alpha})$ as follows:

Definition 2.6.4. *Let* $\alpha \in \mathbb{R}$. *Define*

$$D(A^{\alpha}) = \{u \in H : u = \sum_{j=1}^{\infty} u_j e_j, \sum_{j=1}^{\infty} \lambda_j^{2\alpha} |u_j|^2 < \infty\}.$$

and

$$A^{\alpha} u := \sum_{j=1}^{\infty} \lambda_j^{\alpha} u_j e_j \textit{ for } u = \sum_{j=1}^{\infty} u_j e_j.$$

Remark 2.6.5. We notice that $D(A^{\alpha})$ is Hilbert space with scalar product

$$(u, v)_{D(A^{\alpha})} = \sum_{j=1}^{\infty} u_j v_j \lambda_j^{2\alpha}.$$

Remark 2.6.6. From definition of $A^{\alpha}, \alpha \in \mathbb{R}$ it immediately follows that $A^{\alpha} : D(A^{\alpha+\rho}) \to D(A^{\rho})$ is isomorphism for each $\rho \in \mathbb{R}$.

Lemma 2.6.7. *We have for* $\alpha > \beta$ *that* $D(A^{\alpha}) \subset D(A^{\beta})$ *and the embedding is compact.*

Proof of Lemma 2.6.7. We will identify $u \in D(A^{\alpha})$ with sequence $\{u_j\}_{j=1}^{\infty}$ such that $u = \sum_{j=1}^{\infty} u_j e_j$. We will also identify $D(A^{\alpha})$ with the set of sequences $\{u_j\}_{j=1}^{\infty}$ with finite norm

$$\sum_{j=1}^{\infty} \lambda_j^{2\alpha} |u_j|^2.$$

Let B be a unit ball in $D(A^\alpha)$. It is enough to show that any sequence $\mathbf{u} = \{u^k\}_{k=1}^\infty \subset B$ has finite ε cover for each $\varepsilon > 0$ i.e. for each $\varepsilon > 0$ there exists finite set $Q_\varepsilon = \{q_1, \dots, q_l\}$ such that $||\mathbf{u} - Q_\varepsilon||_{D(A^\beta)} < \varepsilon$. Let us show first that each element of $\mathbf{u}$ can be uniformly approximated in $D(A^\beta)$ by its finite dimensional projection. We have that $\lim\limits_{j\to\infty} \lambda_j = \infty$. Therefore, there exists $d \in \mathbb{N}$ such that $\lambda_{d+1}^{2(\beta-\alpha)} < \varepsilon/2$. Then

$$||u^k - \sum_{l=1}^{d} u_j^k e_j||^2_{D(A^\beta)} = \sum_{j=d+1}^{\infty} \lambda_j^{2\beta} |u_j^k|^2 \le$$
$$\lambda_{d+1}^{2(\beta-\alpha)} \sum_{j=d+1}^{\infty} \lambda_j^{2\alpha} |u_j^k|^2 \le \lambda_{d+1}^{2(\beta-\alpha)} < \varepsilon/2. \tag{2.6.4}$$

Denote $\mathbf{u}_d$ set of finite dimensional projections of elements of $\mathbf{u}$. We have that the set $\mathbf{u}_d$ is bounded, countable and lies in finite dimensional space $X_d = \{u \in H : u = \sum\limits_{j=1}^{d} u_j e_j\} \cong \mathbb{R}^d$. Therefore, it is compact and there exist $Q_\varepsilon \subset X_d$ such that

$$||\mathbf{u}_d - Q_\varepsilon||^2_{D(A^\beta)} < \varepsilon/2. \tag{2.6.5}$$

Thus, it follows from (2.6.4) and (2.6.5) that

$$||\mathbf{u} - Q_\varepsilon||^2_{D(A^\beta)} < \varepsilon.$$

□

2.7 Interpolation theory and positive definite self-adjoint operators

In this section we present some facts from interpolation theory we will need in the thesis. We will consider only the simplest set up, see [74], [6] and [61] for much deeper explanation of the theory, and describe only real interpolation K-method.

Suppose X_0, X_1 are two banach spaces continuously embedded in vector space $\mathcal{A}$. We say that $X = (X_0, X_1)$ is compatible pair in this case. We equip spaces $X_0 \cap X_1$ and $X_0 + X_1$ with the norms

$$||u||_{X_0 \cap X_1} = (||u||_{X_0}^2 + ||u||_{X_1}^2)^{\frac{1}{2}}$$

and

$$||u||_{X_0+X_1} = \inf_{\substack{u=u_0+u_1 \\ u_0 \in X_0, u_1 \in X_1}} \{(||u||_{X_0}^2 + ||u||_{X_1}^2)^{\frac{1}{2}}\}.$$

Define K-functional as follows

$$K(t,u,X) = \inf_{\substack{u=u_0+u_1 \\ u_0 \in X_0, u_1 \in X_1}} \{(||u||_{X_0}^2 + t^2||u||_{X_1}^2)^{\frac{1}{2}}\}, t > 0, u \in X_0 + X_1.$$

Next we define weighted L_q-norm,

$$||f||_{\theta,q} = \left(\int_0^\infty |t^{-\theta} f(t)|^q \frac{dt}{t} \right)^{\frac{1}{q}}, 0 < \theta < 1, 1 \leq \theta < \infty.$$

Definition 2.7.1. *We define*

$$(X_0, X_1)_{\theta,q} = \{u \in X_0 + X_1 : ||K(\cdot, u, X)||_{\theta,q} < \infty\}$$

and put

$$||u||_{(X_0,X_1)_{\theta,q}} = ||K(\cdot, u, X)||_{\theta,q}.$$

Lemma 2.7.2 ([61],p.320)*(i) If $u \in X_0 \cap X_1$, then $u \in (X_0, X_1)_{\theta,q}$ and there exist constant $c = c(\theta, q)$ such that*

$$||u||_{(X_0,X_1)_{\theta,q}} \leq c(\theta,q)||u||_{X_0}^{1-\theta}||u||_{X_1}^{\theta} \leq c(\theta,q)||u||_{X_0 \cap X_1}.$$

(ii) $(X_0, X_1)_{\theta,q} \subset X_0 + X_1$

(iii) If $X = (X_0, X_1), Y = (X_0, Y)$ *are compatible pairs and* $Y \subset X_1$ *is a continuous embedding, then*

$$(X_0, X_1)_{\theta,q} \subset (X_0, Y)_{\theta,q}.$$

Proof. Parts i and ii are proved in Lemma B.4 [61],p.320. Part iii follows from definition of interpolation pair. □

We will need also following two theorems:

Theorem 2.7.3 (see [74], p.167). *Assume* X *be a Hilbert space,* $\Lambda : X \to X$ *be a positive definite self-adjoint operator. Then*

$$(D(\Lambda^\alpha), D(\Lambda^\beta))_{\theta,2} = D(\Lambda^{\alpha\theta+\beta(1-\theta)}), \alpha, \beta \geq 0, \theta \in (0,1) \tag{2.7.1}$$

Theorem 2.7.4 (see [61], p.330). *If* $s_0, s_1 \in \mathbb{R}$ *and* Ω *is an open non-empty subset of* $\mathbb{R}^n$*,then*

$$(H^{s_0,2}(\Omega), H^{s_1,2}(\Omega))_{\theta,2} = H^{s,2}(\Omega) \text{for } s = (1-\theta)s_0 + \theta s_1, \theta \in (0,1).$$

2.8 Stochastic analysis

I assume basic knowledge of stochastic analysis in Hilbert spaces, see [24], and present here certain definitions and results for completeness. In this thesis I will use only very particular case, which can be easily deduced from finite dimensional results. The following notation will be used throughout. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with increasing right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0} \subset \mathcal{F}$, W_t-standard $\mathbb{R}^n$ valued wiener process, X be a Hilbert space, $\mathcal{L}^1(X)$-Banach space of trace-class operators with $Tr\phi$ denote the trace of $\phi \in \mathcal{L}^1(X)$, $M^k(0,T;X)$ denote the space of X-valued processes ξ_t such that

1. ξ_t is $\mathcal{F}_t$-measurable for $t \in [0,T]$.

2. $\mathbb{E}\int_0^T |\xi_s|_X^k ds < \infty$.

Suppose also that $V \subset H \cong H' \subset V'$ is a Gelfand triple. Then for any $\xi \in M^2(0,T;H^n)$ we can define stochastic integral $M_t = \sum_k \int_0^T \xi^k(s)dw_s^k$ as an H-valued random variable given by equality

$$(h, \sum_k \int_0^T \xi^k(s)dw_s^k)_H = \sum_k \int_0^T (h, \xi^k(s))dw_s^k \forall h \in H.$$

M_t- is a continuous H-valued martingale on the segment $[0,T]$ and one can prove following Ito formula (Theorem 1.2 p.135 from Pardoux [63]).

Theorem 2.8.1 (Ito formula). *Suppose:*

$$u \in M^2(0,T;V)$$

$$u_0 \in H$$

$$v \in M^2(0,T;V')$$

$$\phi \in M^2(0,T;H^n), \text{ with:}$$

$$u(t) = u_0 + \int_0^t v(s)ds + \sum_k \int_0^t \phi^k(s)dw_s^k.$$

Let $\psi : H \to \mathbb{R}$ be a twice differentiable functional, which satisfies following assumptions:

(i) ψ, ψ' and ψ'' are locally bounded.

(ii) ψ and ψ' are continuous on H

(iii) $\forall Q \in \mathcal{L}^1(H)$, $Tr(Q \circ \psi'')$ is a continuous functional on H.

(iv) If $u \in V$, $\psi'(u) \in V$; $u \to \psi'(u)$ is continuous from V (with the strong topology) into V endowed with the weak topology.

(v) $\exists k$ *s.t.* $||\psi'(u)||_V \leq k(1 + ||u||_V), \forall u \in V$.

Then:

$$\psi(u(t)) = \psi(u_0) + \int_0^t < v, \psi'(u) > ds + \sum_k \int_0^t (\psi'(u), \phi^k) dw_s^k + \frac{1}{2} \sum_i \int_0^t (\psi''(u)\phi^i, \phi^i) ds. \tag{2.8.1}$$

2.8.1 Comparison theorems

Here I present Gronwall lemma and its analog for stochastic analysis i.e. Comparison theorem for one dimensional diffusions (chapter vi, Theorem 1.1 in [44],p.352):

Theorem 2.8.2 (Comparison theorem). *Suppose we have:*

$\rho : [0, \infty) \to \mathbb{R}$ *– strictly increasing function such that* $\rho(0) = 0$, $\int_0^\infty \frac{ds}{\rho(s)^2} = \infty$.

$\sigma \in C([0, \infty) \times \mathbb{R}, \mathbb{R})$ *such that*

$$|\sigma(t, x) - \sigma(t, y)| \leq \rho(|x - y|), t \geq 0, x, y \in \mathbb{R}.$$

$b_1, b_2 \in C([0, \infty) \times \mathbb{R}, \mathbb{R})$ *such that* $b_1(t, x) < b_2(t, x)$, $t \geq 0, x \in \mathbb{R}$.

two $(\mathcal{F}_t)$*-adapted processes* $x_1(t, \cdot)$, $x_2(t, \cdot)$.

$B(t, \cdot)$*-one dimensional brownian motion such that* $B(0) = 0$ *a.s..*

two real $(\mathcal{F}_t)$*-adapted processes* $\beta_1(t, \cdot)$, $\beta_2(t, \cdot)$.

Assume that the following conditions are satisfied with probability one:

$$x_i(t) - x_i(0) = \int_0^t \sigma(s, x_i(s)) dB_s + \int_0^t \beta_i(s) ds, \ i = 1, 2, \tag{2.8.2}$$

$$\beta_1(t) \leq b_1(t, x_1(t)), t \geq 0. \tag{2.8.3}$$

$$\beta_2(t) \geq b_2(t, x_2(t)), t \geq 0, \tag{2.8.4}$$

$$x_1(0) \leq x_2(0). \tag{2.8.5}$$

Then, with probability one, we have

$$x_1(t) \leq x_2(t), t \geq 0. \tag{2.8.6}$$

Furthermore, if the pathwise uniqueness of solutions holds for at least one of the following SDEs

$$X(t) - X(0) = \int_0^t \sigma(s, X(s))dB_s + \int_0^t b_i(s, X(s))ds, i = 1, 2$$

then the same conclusion holds under the weakened condition

$$b_1(t, x) \leq b_2(t, x), t \geq 0, x \in \mathbb{R}.$$

Lemma 2.8.3 (Gronwall inequality). *Let $T > 0$ and $c \geq 0$. Let u be a Borel measurable bounded nonnegative function on $[0, T]$, and let v be a nonnegative integrable function on $[0, T]$. Assume that $u \cdot v$ is integrable on $[0, T]$ and*

$$u(t) \leq c + \int_0^t v(s)u(s)\, ds, \quad t \in [0, T]. \tag{2.8.7}$$

Then

$$u(t) \leq ce^{\int_0^t v(s)\, ds}, \quad t \in [0, T]. \tag{2.8.8}$$

Chapter 3

Selfduality of vector transport equation

In this chapter I study the vector advection equations of the following form

$$\begin{aligned}\frac{\partial G}{\partial t} &= -\nu AG + \operatorname{curl}(v(t) \times G), t \in [0, T],\\ G(0,\cdot) &= G_0.\end{aligned}$$

Here we assume that H is the Hilbert space defined in Chapter 2, Example 2.4.11. Moreover, A is the Stokes operator defined also in that chapter in section 2.6. Function $v \in L^r(0, T; \mathbb{L}^s(\mathbb{R}^3))$, where $\frac{2}{r} + \frac{3}{s} = 1$, $r, s \geq 0$, and the initial data $G_0 \in H$ are assumed to be known. A precise definition of a solution is given below in definition 3.1.6. I will prove that the solution of the above problem is self-dual in a sense described below. Self-duality allows us to show certain properties of the vector transport operator. In particular, I will show that the $\mathcal{L}(\mathbb{H}^{k,2}_{sol}, \mathbb{H}^{k,2}_{sol})$ and $\mathcal{L}(\mathbb{H}^{1-k,2}_{sol}, \mathbb{H}^{1-k,2}_{sol})$ norms of the operator of vector transport operator $\mathcal{T}_t^v$ defined in formula (3.3.2), where $\mathbb{H}^{k,2}_{sol}$ is defined in section 2.3, are equal, see Corollary 3.3.5. Moreover, I prove that the space $\mathcal{L}(\mathbb{H}^{\frac{1}{2},2}_{sol}, \mathbb{H}^{\frac{1}{2},2}_{sol})$ is in certain sense optimal for vector transport operator, see Corollary 3.3.7. This duality can be understood as generalization of invariance of helicity for Navier-Stokes equations, see Corollary 3.3.8. I also reprove, see again Corollary 3.3.8, the classical result of Serrin [69], Prodi [64], Ladyzhenskaya [49] on the existence of a strong solution to the Navier-Stokes equations if velocity v satisfies an additional assumption that

$$v \in L^r(0,T;\mathbb{L}^s(\mathbb{R}^3)),$$

for some $r > 2, s > 3$ such that $\frac{2}{r} + \frac{3}{s} = 1$.

3.1 General Setting

In this section I will introduce the equations I am interested in and I will state the existence and uniqueness results. These results will be proved in the next section. I will use notation introduced in Example 2.4.11.

3.1.1 Definition of Nonlinear term

Definition 3.1.1. *Let us define a trilinear form $\tilde{b} : C_0^\infty(D) \times \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ by*

$$\tilde{b}(v,f,\phi) = < P(v \times \operatorname{curl} f), \phi >_{V',V}, v \in C_0^\infty(D), f \in \mathcal{D}, \phi \in \mathcal{D}. \tag{3.1.1}$$

Lemma 3.1.2. *For any $\delta, \varepsilon > 0$, $v \in C_0^\infty(D)$, $f \in \mathcal{D}$, $\phi \in \mathcal{D}$ there exists $C_\delta > 0$ such that*

$$|\tilde{b}(v,f,\phi)|^2 \le |f|_V^2 |\phi|_V^2 (\varepsilon^{1+\delta/3} + \frac{C_\delta}{\varepsilon^{1+3/\delta}} |v(t)|^{2+\frac{6}{\delta}}_{\mathbb{L}^{3+\delta}(D)}), \tag{3.1.2}$$

$$|\tilde{b}(v,f,\phi)| \le \frac{1}{2}||f||_V^2 + \frac{1}{2}(\varepsilon^{1+\delta/3}||\phi||_V^2 + \frac{C_\delta}{\varepsilon^{1+3/\delta}} |v(t)|^{2+\frac{6}{\delta}}_{\mathbb{L}^{3+\delta}(D)} |\phi|_H^2). \tag{3.1.3}$$

$$|\tilde{b}(v,f,\phi)|^2 \le |\phi|_H^2 (\varepsilon^{1+\delta/3}||f||^2_{D(A)} + \frac{C_\delta}{\varepsilon^{1+3/\delta}} |v|^{2+\frac{6}{\delta}}_{\mathbb{L}^{3+\delta}(D)} |f|_V^2) \tag{3.1.4}$$

To prove Lemma 3.1.2 I will need following result.

Lemma 3.1.3. *For any $\delta, \varepsilon > 0$ there exists $C_\delta > 0$ such that*

$$||f \times g||^2_{\mathbb{L}^2(D)} \le \varepsilon^{1+\delta/3}||f||^2_V + \frac{C_\delta}{\varepsilon^{1+3/\delta}}|g|^{2+\frac{6}{\delta}}_{\mathbb{L}^{3+\delta}(D)}|f|^2_H, \; f \in V, g \in H. \tag{3.1.5}$$

Proof of Lemma 3.1.3. Let $p = 3 - \frac{2\delta}{1+\delta}$, $q = \frac{3+\delta}{2}$, $\theta = \frac{3}{3+\delta}$. Then $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\begin{aligned} ||f \times g||^2_{\mathbb{L}^2(D)} &\le \int_D |f|^2|g|^2 dx \le \\ &|f|^2_{\mathbb{L}^{2p}(D)}|g|^2_{\mathbb{L}^{2q}(D)} \le \\ &(||f||^\theta_V |f|^{1-\theta}_H)^2 |g|^2_{\mathbb{L}^{2q}(D)} \le \\ \varepsilon^{1+\delta/3}||f||^2_V &+ \frac{C_\delta}{\varepsilon^{1+3/\delta}}|g|^{2+\frac{6}{\delta}}_{\mathbb{L}^{3+\delta}(D)}|f|^2_H. \end{aligned}$$

where the first inequality follows from inequality (2.1.2), the second follows from Hölder inequality, the third follows from Gagliardo-Nirenberg inequality (Theorem 2.3.4) and fourth inequality follows from Young inequality. □

Proof of Lemma 3.1.2. We have for any $(v, f, \phi) \in C_0^\infty(D) \times \mathcal{D} \times \mathcal{D}$ that

$$|\tilde{b}(v, f, \phi)|^2 = | < v(t) \times \phi, \mathrm{curl}\, f >_{V',V} |^2 \le \tag{3.1.6}$$

$$\begin{aligned} |\,\mathrm{curl}\, f|^2_H |v(t) \times \phi|^2_H \le ||f||^2_V (\varepsilon^{1+\delta_0/3}||\phi||^2_V + \frac{C_{\delta_0}}{\varepsilon^{1+3/\delta_0}}|v(t)|^{2+\frac{6}{\delta_0}}_{\mathbb{L}^{3+\delta_0}(D)}|\phi|^2_H) \le \\ |f|^2_V |\phi|^2_V (\varepsilon^{1+\delta_0/3} + \frac{C_{\delta_0}}{\varepsilon^{1+3/\delta_0}}|v(t)|^{2+\frac{6}{\delta_0}}_{\mathbb{L}^{3+\delta_0}(D)}) \end{aligned}$$

where equality (3.1.6) follows from (2.1.1), second inequality follows from Lemma 3.1.3.

Similarly,

$$|\tilde{b}(v, f, \phi)| = | < v(t) \times \phi, \mathrm{curl}\, f >_{V',V} | \le \tag{3.1.7}$$

$$|\,\mathrm{curl}\, f|_H |v(t) \times \phi|_H \le \frac{1}{2}||f||^2_V + \frac{1}{2}|v(t) \times \phi|^2_H \le$$

$$\frac{1}{2}||f||^2_V + \frac{1}{2}(\varepsilon^{1+\delta_0/3}||\phi||^2_V + \frac{C_{\delta_0}}{\varepsilon^{1+3/\delta_0}}|v(t)|^{2+\frac{6}{\delta_0}}_{\mathbb{L}^{3+\delta_0}(D)}|\phi|^2_H), \tag{3.1.8}$$

and

$$|\tilde{b}(v, f, \phi)|^2 = | < v(t) \times \operatorname{curl} f, \phi >_{V',V} |^2 \leq$$
$$|\phi|_H^2 |v(t) \times \operatorname{curl} f|_H^2 \leq |\phi|_H^2 (\varepsilon^{1+\delta/3} |f|_{D(A)}^2$$
$$+ \frac{C_\delta}{\varepsilon^{1+3/\delta}} |v|_{\mathbb{L}^{3+\delta}(D)}^{2+\frac{6}{\delta}} |f|_V^2)$$

□

Fix $\delta_0 > 0$. It follows from inequality (3.1.2) that for any $\delta_0 > 0$ trilinear form $\tilde{b}$ is continuous with respect to the $L^{3+\delta_0}(D) \times V \times V$ topology. Therefore, for any $\delta_0 > 0$ there exist continuous trilinear form $b : L^{3+\delta_0}(D) \times V \times V \to \mathbb{R}$ such that

$$b(\cdot, \cdot, \cdot)|_{C_0^\infty(D) \times \mathcal{D} \times \mathcal{D}} = \tilde{b}.$$

Moreover,

$$b(v, f, \phi) = -(v \times \phi, \operatorname{curl} f)_H. \tag{3.1.9}$$

Indeed, the form on the left hand side of equality (3.1.9) is equal to the form on the right hand side of equality (3.1.9) for $(v, f, \phi) \in C_0^\infty(D) \times \mathcal{D} \times \mathcal{D}$ and both forms are continuous in $L^{3+\delta_0}(D) \times V \times V$. Hence, for each $(v, f) \in L^{3+\delta_0}(D) \times V$ $b(v, f, \cdot) \in V'$ and therefore the following definition is well posed.

Definition 3.1.4. *Let us define a bilinear operator* $B : L^{3+\delta_0}(D) \times V \to V'$ *by*

$$< B(v, f), \phi >_{V',V} = b(v, f, \phi), v \in L^{3+\delta_0}(D), f \in V, \phi \in V.$$

Corollary 3.1.5. *There exists a constant* $C_{\delta_0} > 0$ *such that for any* $(v, f) \in L^{3+\delta_0}(D) \times V$,

$$|B(v, f)|_{V'}^2 \leq ||f||_V^2 (\varepsilon^{1+\delta_0/3} + \frac{C_{\delta_0}}{\varepsilon^{1+3/\delta_0}} |v(t)|_{\mathbb{L}^{3+\delta_0}(D)}^{2+\frac{6}{\delta_0}}). \tag{3.1.10}$$

Moreover, if $(v, f) \in L^{3+\delta_0}(D) \times D(A)$ *then* $B(v, f) \in H$ *and*

$$|B(v, f)|_H^2 \leq (\varepsilon^{1+\delta_0/3} ||f||_{D(A)}^2 + \frac{C_{\delta_0}}{\varepsilon^{1+3/\delta_0}} |v|_{\mathbb{L}^{3+\delta_0}(D)}^{2+\frac{6}{\delta}} |f|_V^2) \tag{3.1.11}$$

Proof of Corollary 3.1.5. Proof immediately follows from Lemma 3.1.2. □

3.1.2 Equations

Assume that $F_0 \in H, f \in L^2(0,T;V')$. I consider the following two problems:

$$\frac{\partial F}{\partial t} = -\nu AF - B(v(t), F) + f, F(0) = F_0 \tag{3.1.12}$$

$$\frac{\partial G}{\partial t} = -\nu AG - \operatorname{curl}(v(t) \times G) + f, G(0) = F_0 \tag{3.1.13}$$

Definition 3.1.6. *I call element F (respectively G) of $L^2(0,T;V) \cap L^\infty(0,T;H)$ a solution of equation* (3.1.12)*(resp.* (3.1.13)*) if F (resp. G) satisfies* (3.1.12) *(resp.* (3.1.13)*) in the distribution sense.*

In next two Propositions I state existence and regularity results for solutions of (3.1.12) and (3.1.13).

Proposition 3.1.7. *Suppose that* $(F_0, f) \in H \times L^2(0,T;V')$ *and*

$$v \in L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D)) \text{ for some } \delta_0 > 0. \tag{3.1.14}$$

Then

(i) there exists the unique solution F of problem (3.1.12) *and*

$$|F(t)|_H^2 + \nu \int_0^t ||F(s)||_V^2 ds \le K_1\Big(\int_0^t |v(s)|_{\mathbb{L}^{3+\delta_0}}^{2+6/\delta_0} ds, \nu\Big) \times \Big(|F_0|_H^2 + \frac{C}{\nu}\int_0^t |f(s)|_{V'}^2 ds\Big). \tag{3.1.15}$$

Moreover, if the following stronger version of (3.1.14) *is satisfied*

$$v \in L^\infty(0,T;\mathbb{L}^{3+\delta_0}(D)) \text{ for some } \delta_0 > 0. \tag{3.1.16}$$

then $F' \in L^2(0,T;V')$.

(ii) If in addition $(F_0, f) \in V \times L^2(0,T;H)$, *condition* (3.1.16) *is satisfied, then* $F \in C([0,T], V) \cap L^2(0,T;D(A))$.

(iii) If $(F^{(k)}(0), f^{(k)}) \in V \times L^2(0,T;H)$, $v^{(k)}$ *satisfies* (3.1.16), $k = 1,\ldots,n$ *then* $F^{(k)} \in C([0,T],V), k = 1,\ldots,n$.

Remark 3.1.8. I have used the class of functions $L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))$, $\delta_0 > 0$ for the parameter function v because it is possible to prove energy inequality (3.1.15) for solutions of (3.1.12) if $v \in L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))$, $\delta_0 > 0$. I would like to mention that the energy inequality (3.1.15) does not automatically follow from the type of equation (3.1.12) as in the case of scalar advection. Indeed, in the case of scalar advection we have for $F \in L^\infty(0,T;L^2(D)) \cap L^2(0,T;H_0^{1,2}(D))$ following equation

$$\frac{\partial F}{\partial t} = \nu\triangle F + (v\nabla)F, F(0) = F_0 \in L_0^2(D) \tag{3.1.17}$$

and formally speaking, under the following condition of incompressibility

$$\operatorname{div} v = 0$$

one can take scalar product of equation (3.1.17) with F in $L^2(D)$, integrate the result w.r.t. time and from incompressibility condition and integration by parts it follows standard a'priori estimate of F in space $L^\infty(0,T;L^2(D)) \cap L^2(0,T;H_0^{1,2}(D))$. For vector advection to get similar a'priori estimate we need some additional integrability condition on v.

Remark 3.1.9. I notice that on the one hand, our class $L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))$ is a Serrin regularity class (If $r = 2 + \frac{6}{\delta_0}$, $s = 3 + \delta_0$ then $\frac{2}{r} + \frac{3}{s} = 1$) and therefore, any weak solution of Navier-Stokes equations belonging to this class is strong solution. On the other hand, I have been unable to prove that under assumption (3.1.14) solution F of (3.1.12) satisfies condition $F' \in L^2(0,T,V')$. Problem which appears here is similar to the problem with weak solution u of Navier-Stokes equations, see [72], for which it is also not proved that $u' \in L^2(0,T;V')$.

For the second equation we have:

Proposition 3.1.10. *Suppose* $(F_0, f) \in H \times L^2(0,T;V')$ *and condition* (3.1.14) *is satisfied. Then*

(i) there exists unique solution G *of equation* (3.1.13) *such that* $G' \in L^2(0,T;V')$ *and we have an estimate*

$$|G(t)|_H^2 + \nu \int_0^t ||G(s)||_V^2 ds \le K_1\Big(\int_0^t |v(s)|_{\mathbb{L}^{3+\delta_0}}^{2+6/\delta_0} ds, \nu\Big) \times$$
$$\Big(|G_0|_H^2 + \frac{C}{\nu}\int_0^t |f(s)|_{V'}^2 ds\Big). \qquad (3.1.18)$$

(ii) If $(F_0, f) \in V \times L^2(0,T;H)$, v *satisfies* (3.1.14) *and* $v \in L^2(0,T,V)$, *then* $G \in C([0,T],V) \cap L^2(0,T;D(A))$.

(iii) If $(G^{(k)}(0), f^{(k)}) \in V \times L^2(0,T;H)$, $v^{(k)}$ *satisfies* (3.1.14) *and* $v^{(k)} \in L^2(0,T,V)$, $k = 1,\dots,n$ *then* $G^{(k)} \in C([0,T],V), k = 1,\dots,n$.

Corollary 3.1.11. *Assume that* $F_0 \in H$,

$$f^{(k)} \in L^2(0,T;H), k \in \mathbb{N}.$$

Assume also that $v^{(k)}$ *satisfies condition* (3.1.14) *for any* $k \in \mathbb{N}$. *Then solution of equation* (3.1.12) *is in* $C^\infty((0,T] \times D)$.

Similarly for equation (3.1.13) we have

Corollary 3.1.12. *Assume that* $F_0 \in H$,

$$f^{(k)} \in L^2(0,T;H), k \in \mathbb{N}$$

and $v^{(k)}$ *satisfies condition* (3.1.16) *for any* $k \in \mathbb{N}$. *Then solution of equation* (3.1.13) *is in* $C^{\infty}((0,T] \times D)$.

3.2 Proof of propositions 3.1.7 and 3.1.10

Proof of Proposition 3.1.7.(i) I will divide proof in three steps a), b), c):

a) Let us consider special case when $v \in L^{\infty}(0,T;\mathbb{L}^{3+\delta_0}(D))$. We will use Theorem 2.4.5 with Gelfand triple $V \subset H \cong H' \subset V'$. Denote $A(t) = \nu A + B(v(t),\cdot)$. We need to check whether conditions (2.4.3) and (2.4.4) are satisfied. We have for $f \in V$

$$< A(t)f, f >_{V',V} = \nu\tilde{a}(f,f) + < B(v(t),f), f >_{V',V} . \tag{3.2.1}$$

Last term in the expression (3.2.1) can be estimated as follows:

$$\begin{aligned} | < B(v(t),f), f >_{V',V} | \leq \frac{1}{2}||f||_V^2 + \frac{1}{2}(\varepsilon^{1+\delta_0/3}||f||_V^2 \\ + \frac{C_{\delta_0}}{\varepsilon^{1+3/\delta_0}}|v(t)|_{\mathbb{L}^{3+\delta_0}(D)}^{2+\frac{6}{\delta_0}}|f|_H^2) \end{aligned} \tag{3.2.2}$$

where inequality (3.2.2) follows from estimate (3.1.3). Thus from estimate (3.2.2) and continuity of form $\tilde{a}$ follows that,

$$||A(t)||_{\mathcal{L}(V,V')} \leq C\nu + C_2|v(t)|_{\mathbb{L}^{3+\delta_0}(D)}.$$

Coercivity condition also follows from estimate (3.1.3). We have for $f \in V$, $t \in [0,T]$,

$$\begin{aligned} | < A(t)f, f >_{V',V} | = |\nu\tilde{a}(f,f) + < B(v(t),f), f >_{V',V} | \geq \\ \frac{\nu}{2}||f||_V^2 - \frac{C}{\nu}(\varepsilon^{1+\delta_0/3}||f||_V^2 + \frac{C_{\delta_0}}{\varepsilon^{1+3/\delta}}|v(t)|_{\mathbb{L}^{3+\delta_0}(D)}^{2+\frac{6}{\delta_0}}|f|_H^2). \end{aligned}$$

Let us choose $\varepsilon > 0$ such that $\frac{\nu}{2} - \frac{C}{\nu}\varepsilon^{1+\delta_0/3} > 0$. Then we get coercivity estimate (2.4.4). Thus, by Theorem 2.4.5, first statement of the Proposition follows in our special case.

b) To prove Proposition in the general case I will show the energy inequality for solutions of equation (3.1.12) when $v \in L^\infty(0,T;\mathbb{L}^{3+\delta_0}(D))$. From step (a) we know that solution $F \in L^2(0,T;V)$ such that $F' \in L^2(0,T;V')$ exists and unique. Then, from Lemma 2.4.6 and equality (3.1.9) follows that

$$\frac{1}{2}\frac{d}{dt}|F|_H^2 = -\nu||F||_V^2 + < f, F >_{V',V} - < B(v,F), F >_{V',V} =$$
$$= -\nu||F||_V^2 + < f, F >_{V',V} + (\operatorname{curl} F, v \times F)_H.$$

Therefore, by applying the Young inequality, I infer that

$$|F(t)|_H^2 + 2\nu\int_0^t |F(s)|_V^2 ds - \int_0^t (\operatorname{curl} F(s), v(s) \times F(s))_H ds = |F(0)|_H^2 +$$
$$\int_0^t < f(s), F(s) >_{V',V} ds \le |F(0)|_H^2 + \frac{\nu}{2}\int_0^t |F(s)|_V^2 ds + \frac{C}{\nu}\int_0^t |f(s)|_{V'}^2 ds.$$

The term $\int_0^t (\operatorname{curl} F(s), v(s) \times F(s))_H ds$ can be estimated as follows:

$$|\int_0^t (\operatorname{curl} F(s), v(s) \times F(s))_H ds| \le \frac{\nu}{4}\int_0^t |\operatorname{curl} F|_H^2 ds +$$
$$\frac{C}{\nu}\int_0^t |v(s) \times F(s)|_H^2 ds \le \frac{\nu}{4}\int_0^t |\operatorname{curl} F|_H^2 ds +$$
$$\frac{C}{\nu}\int_0^t (\varepsilon^{1+\delta_0/3}|F(s)|_V^2 + \frac{C_{\delta_0}}{\varepsilon^{1+\delta_0/3}}|v(s)|_{\mathbb{L}^{3+\delta_0}}^{2+6/\delta_0}|F(s)|_H^2) ds \le$$
$$\le (\frac{\nu}{4} + \frac{C}{\nu}\varepsilon^{1+\delta_0/3})\int_0^t |F(s)|_V^2 ds + \frac{C_{\delta_0}}{\nu\varepsilon^{1+\delta_0/3}}\int_0^t |v(s)|_{\mathbb{L}^{3+\delta_0}}^{2+6/\delta_0}|F(s)|_H^2 ds$$

Let $\varepsilon > 0$ such that $\frac{\nu}{4} + \frac{C}{\nu}\varepsilon^{1+\delta_0/3} = \frac{\nu}{2}$ then

$$|F(t)|_H^2 + \nu\int_0^t ||F(s)||_V^2 ds \le |F(0)|_H^2 + \frac{C}{\nu}\int_0^t |f(s)|_{V'}^2 ds +$$

$$+\frac{C_{\delta_0}}{\nu\varepsilon^{1+\delta_0/3}}\int_0^t |v(s)|^{2+6/\delta_0}_{\mathbb{L}^{3+\delta_0}}|F(s)|^2_H ds$$

and, from Gronwall lemma follows that

$$|F(t)|^2_H \leq (|F(0)|^2_H + \frac{C}{\nu}\int_0^t |f(s)|^2_{V'}ds)e^{C(\delta_0,\nu)\int_0^t |v(s)|^{2+6/\delta_0}_{\mathbb{L}^{3+\delta_0}}ds}.$$

Thus

$$|F(t)|^2_H + \nu\int_0^t ||F(s)||^2_V ds \leq K_1(|F(0)|^2_H + \frac{C}{\nu}\int_0^t |f(s)|^2_{V'}ds)$$
$$(1+\int_0^t |v(s)|^{2+6/\delta_0}_{\mathbb{L}^{3+\delta_0}}ds)e^{C(\delta_0,\nu)\int_0^t |v(s)|^{2+6/\delta_0}_{\mathbb{L}^{3+\delta_0}}ds}. \quad (3.2.3)$$

(c) The general case. Let $v \in L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))$ then by Theorem 2.2.14 there exists a sequence of functions $\{v_n\}_{n=1}^{\infty}$ such that $v_n \in L^{\infty}(0,T;\mathbb{L}^{3+\delta_0}(D))$ and $v_n \to v$ in $L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))$. Let F_n be a corresponding sequence of solutions of equation (3.1.12) with v replaced by v_n. Then from inequality (3.2.3) it follows that the sequence $\{F_n\}_{n=1}^{\infty}$ lies in a bounded set of $L^{\infty}(0,T;H)\cap L_2(0,T;V)$. Therefore, by the Banach-Alaoglu Theorem there exists subsequence $\{F_{n'}\}$ and $F^* \in L^{\infty}(0,T;H)$ such that for any $q \in L^1(0,T;H)$

$$\int_0^T (F_{n'} - F^*, q(s))_H ds \to 0 \quad (3.2.4)$$

Similarly, from the Banach-Alaoglu Theorem follows that one can find subsequence $\{F_{n''}\}$ of $\{F_{n'}\}$ which converges to $F^{**} \in L^2(0,T;V)$ in weak topology of $L^2(0,T;V)$ i.e. for any $q \in L^2(0,T;V')$

$$\int_0^T < F_{n''} - F^{**}, q(s) >_{V',V} ds \to 0. \quad (3.2.5)$$

In particular, (3.2.4) and (3.2.5) are satisfied for $q \in L^2(0,T;H)$. Therefore $F^* = F^{**} \in L^\infty(0,T;H) \cap L_2(0,T;V)$. Put $F = F^*$. Now it remains to show that F satisfies equation (3.1.12) in the weak sense. Let $\psi \in C^\infty([0,T],\mathbb{R})$, $\psi(1) = 0$, $h \in V$. Then by part (a) of the proof I have

$$-\int_0^T (F_n(s),h)_H\psi'(s)ds + \int_0^T < B(v_n,F_n),h >_{V',V} \psi(s)ds + \nu\int_0^T \tilde{a}(F_n(s),h)\psi(s)ds = (F_0,h)_H\psi(0) + \int_0^T < f(s),h >_{V',V} \psi(s)ds.. \tag{3.2.6}$$

From (3.2.4), responsibly (3.2.5), immediately follows convergence of the first term, responsibly third term, on the left hand side of this equality. For the second term I have

$$|\int_0^T < B(v_n,F_n) - B(v,F),h >_{V',V} \psi(s)ds| \le |\int_0^T < B(v_n - v,F_n),h >_{V',V} \psi(s)ds| + |\int_0^T < B(v,F_n - F),h >_{V',V} \psi(s)ds| = I_n + II_n.$$

Let $\varepsilon > 0$ be fixed. For any $\varepsilon_2, \varepsilon_3 > 0$ I have by inequality (3.1.3)

$$I_n \le \varepsilon_3 \int_0^T |\,\mathrm{curl}\, F_n|_H^2 ds +$$

$$\frac{C}{\varepsilon_3}\int_0^T (\varepsilon_2|h|_V^2 + \frac{C}{\varepsilon_2}|v_n - v|_{\mathbf{L}^{3+\delta_0}(D)}^{2+\frac{6}{\delta_0}}|h|_H^2)|\psi|^2 ds =$$

$$\varepsilon_3||F_n||^2_{L_2(0,T;V)} + \frac{C\varepsilon_2}{\varepsilon_3}|h|^2_V\int\limits_0^T|v|^2ds +$$

$$\frac{C|h|^2_H}{\varepsilon_3\varepsilon_2}\int\limits_0^T|v_n - v|^{2+\frac{6}{\delta_0}}_{\mathbb{L}^{3+\delta_0}(D)}|v|^2ds.$$

Taking into account that F_n is bounded in $L_2(0,T;V)$ and v_n converges to v in $L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))$ I can choose $\varepsilon_2, \varepsilon_3$ and $N = N(\varepsilon)$ in such way that $I_n \le \frac{\varepsilon}{2}$ $n \ge N$.

For II_n I have $II_n = |\int\limits_0^T < F_n - F, \text{curl}(v \times h) >_{V',V} \psi(s)ds|$. From (3.1.5) follows that $v \times h \in L_2(0,T;H)$ and, therefore, $\text{curl}(v\times h) \in L_2(0,T;V')$ and convergence of II_n to 0 follows from (3.2.5). Uniqueness of F follows from energy inequality. Thus, first statement of the Proposition 3.1.7 is proved.

(ii) To prove [ii] I follow an idea from [15], see also [11].

Lemma 3.2.1. *Let $g : [0,T] \to \mathbb{R}$ be measurable function such that $\int\limits_0^T |g(s)|ds < \infty$. Then for any $\delta > 0$ there exists partition $\{T_i\}_{i=1}^n$ of interval $[0,T]$ such that $\int\limits_{T_i}^{T_{i+1}} |g(s)|ds < \delta$, $i = 1,\ldots,n$.*

Proof. Immediately follows from absolute continuity of Lebesgue integral. (Theorem 5 p.301,[46]). □

Local existence of solution. Let $X_T = \{F : [0,T] \to D(A) | |F|^2_{X_T} = \nu^2|F|^2_{L^2(0,T;D(A))} + |F'|^2_{L^2(0,T;H)} < \infty\}$. Define a map $\Phi_T : X_T \to X_T$ by $\Phi_T(z) = G$, where G is a solution of the problem

$$G' + \nu AG = f - B(v(t), z), G(0) = F_0 \quad (3.2.7)$$

Lemma 3.2.2. *If v satisfies assumption* (3.1.14), *$f \in L^2(0,T;H)$, $F_0 \in V$ then $B(v(t), z) \in L^2(0,T;H)$ and map Φ_T is well defined.*

Proof. It is enough to prove that $B(v(t), z) \in L^2(0, T; H)$. Then correctness of definition of Φ_T will follow from Proposition 2.4.10. I have from inequality (3.1.11):

$$||B(v(t), z)||^2_{L^2(0,T;H)} \leq C_1(\varepsilon, \delta_0)||z||^2_{L^2(0,T;\mathbb{H}^2_{sol}(D))}$$
$$+C_2(\varepsilon, \delta_0)|z|^2_{C(0,T;V)}|v|_{L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))}$$

and the result follows from Lemma 2.4.6. □

Let us show that there exists such $T_1 \leq T$ that Φ_{T_1} is contractive map. I have by Proposition 2.4.10 and inequality (3.1.11)

$$||\Phi_t(z_1) - \Phi_t(z_2)||^2_{X_t} \leq C_1||B(v, z_1 - z_2)||^2_{L^2(0,t;H)} \leq$$
$$C_1\varepsilon^{1+\delta_0/3}|z_1 - z_2|^2_{L^2(0,t;D(A))} +$$
$$C_1\frac{C_\delta}{\varepsilon^{1+3/\delta}}|z_1 - z_2|^2_{C(0,t;V)}|v|_{L^{2+6/\delta_0}(0,T;\mathbb{L}^{3+\delta_0}(D))} \leq$$
$$C_1\varepsilon^{1+\delta_0/3}|z_1 - z_2|^2_{X_t} + C_1\frac{C_\delta}{\varepsilon^{1+3/\delta}}|z_1 - z_2|^2_{X_t}|v|_{L^{2+6/\delta_0}(0,t;\mathbb{L}^{3+\delta_0}(D))})$$

Now let us choose $\varepsilon > 0$ that $C_1\varepsilon^{1+\delta_0/3} = 1/2$ and denote $K = C_1\frac{C_\delta}{\varepsilon^{1+3/\delta}}$. I have

$$||\Phi_t(z_1) - \Phi_t(z_2)||^2_{X_t} \leq (1/2 + K|v|_{L^{2+6/\delta_0}(0,t;\mathbb{L}^{3+\delta_0}(D))})|z_1 - z_2|^2_{X_t} \tag{3.2.8}$$

Choose $t = T_1$ such that $|v|_{L^{2+6/\delta_0}(0,T_1;\mathbb{L}^{3+\delta_0}(D))} \leq d = \frac{1}{3K}$ then Φ_{T_1} is an affine contraction map and by Banach fixed point theorem there exists fixed point $F \in X_{T_1}$ of Φ_{T_1}. Then F is a solution of equation (3.1.12) on interval $[0, T_1]$.

Global existence of solution. From Lemma 3.2.1 and assumption (3.1.14) it follows that I can find partition $0 = T_0 < T_1 < \ldots < T_{k-1} < T_k = T$ of interval $[0, T]$ such that $|v|_{L^{2+6/\delta_0}(T_i,T_{i+1};\mathbb{L}^{3+\delta_0}(D))} < 1/3K$, $i = 0, \ldots, k-1$. Therefore, I can use estimate (3.2.8) and Banach fixed point theorem iteratively to define global solution.

(iii) To deduce [iii] I will use a method suggested by R.Temam in [73]. I will consider only the case $k = 1$. General case follows by induction. Let us recall that

$$A(t) = \nu A + B(v(t), \cdot).$$

I differentiate equation (3.1.12) w.r.t. t (in weak sense) and get equation for F':

$$\frac{dF'}{dt} = -A(t)F' + B(v'(t), F) + f'$$

Now from assumptions of statement [ii] follows that it is enough to prove that $B(v'(t), F) \in L^2(0, T; H)$ and use part [i]. I have from inequality (3.1.11)

$$\int_0^T |B(v'(t), F)|_H^2 dt \leq$$

$$\varepsilon^{1+\delta_0/3} \int_0^T || \operatorname{curl} F||_V^2 dt + \frac{C_{\delta_0}}{\varepsilon^{1+3/\delta_0}} \times$$

$$\int_0^T |v'(t)|^2_{\mathbb{L}^{3+\delta_0}(D)} | \operatorname{curl} F|_H^2 dt \leq \varepsilon^{1+\delta_0/3} |F|_{L^2(0,T;D(A))} +$$

$$+ \frac{C_{\delta_0}}{\varepsilon^{1+3/\delta_0}} ||F||^2_{C(0,T;V)} \int_0^T |v'(t)|^{2+\frac{6}{\delta_0}}_{\mathbb{L}^{3+\delta_0}(D)} dt < \infty$$

Where $F \in C(0, T; V)$ by Lemma 2.4.6.

□

Proof of Proposition 3.1.10. The proof is very similar to the proof of previous proposition.

a) Let us consider special case when $v \in L^\infty(0, T; \mathbb{L}^{3+\delta_0}(D))$. I will use Theorem 2.4.5 with Gelfand triple $V \subset H \cong H' \subset V'$. Denote $B(t) = \nu A + \operatorname{curl}(v(t) \times \cdot)$. I need to check whether conditions (2.4.3) and (2.4.4) are satisfied. I have for $f \in V$

$$< B(t)f, f >_{V',V} = \nu \tilde{a}(f, f) + < \operatorname{curl}(v(t) \times f), f >_{V',V} =$$

$$= \nu \tilde{a}(f, f) + < v(t) \times f, \operatorname{curl} f >_{V',V} . \qquad (3.2.9)$$

Now I can use estimate (3.2.2) and continuity of form $\tilde{a}$ to get

$$||B(t)||_{\mathcal{L}(V,V')} \leq C\nu + C_2|v(t)|_{\mathbb{L}^{3-\delta_0}(D)}.$$

Coercivity estimate can be proved in the same way as in the proof of Proposition 3.1.7. Therefore, by Theorem 2.4.5 first statement of the Proposition is proved for our special case.

b) To prove Proposition in general case I will show energy inequality for solutions of (3.1.13) when $v \in L^\infty(0,T;\mathbb{L}^{3+\delta_0}(D))$. From step (a) I know that solution $F \in L^2(0,T;V)$ such that $F' \in L^2(0,T;V')$ exists and unique. Then, from Lemma 2.4.6 follows that

$$\frac{1}{2}\frac{d}{dt}|G|_H^2 = -\nu||G||_V^2 + < f, G >_{V',V} - < v \times G, \operatorname{curl} G >_{V',V} =$$
$$= -\nu||G||_V^2 + < f, G >_{V',V} + (\operatorname{curl} G, v \times G)_H$$

Therefore,

$$|G(t)|_H^2 + 2\nu \int_0^t |G(s)|_V^2 ds - \int_0^t (\operatorname{curl} G(s), v(s) \times G(s))_H ds = |G(0)|_H^2 +$$
$$\int_0^t < f(s), G(s) >_{V',V} ds \leq |G(0)|_H^2 + \frac{\nu}{2}\int_0^t |G(s)|_V^2 ds + \frac{C}{\nu}\int_0^t |f(s)|_{V'}^2 ds$$

where inequality follows from Young inequality. Term $\int_0^t (\operatorname{curl} G(s), v(s) \times G(s))_H ds$ can be estimated in the same way as in Proposition 3.1.7. Thus

$$|G(t)|_H^2 + \nu \int_0^t ||G(s)||_V^2 ds \leq K_1(|G(0)|_H^2 + \frac{C}{\nu}\int_0^t |f(s)|_{V'}^2 ds)$$
$$(1 + \int_0^t |v(s)|_{\mathbb{L}^{3+\delta_0}}^{2+6/\delta_0} ds) e^{C(\delta_0,\nu)\int_0^t |v(s)|_{\mathbb{L}^{3+\delta_0}}^{2+6/\delta_0} ds}. \quad (3.2.10)$$

c) General case. Now, let $v_n \in L^\infty(0,T;\mathbb{L}^{3+\delta_0}(D))$ is a sequence of functions such that $v_n \to v \in L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D)), n \to \infty$ in topology of $L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))$; G_n is a corresponding sequence of solutions of (3.1.13). Then from (3.2.10) follows that sequence $\{G_n\}_{n=1}^\infty$ lie in a bounded set of $L^\infty(0,T;H)\cap L_2(0,T;V)$. Using the same argumentation as in Proposition 3.1.7 I can find subsequence $\{G_{n'}\}_{n'=1}^\infty$ weakly converging to $G \in L^\infty(0,T;H)\cap L_2(0,T;V)$ which satisfies (3.1.13) in a weak sense. Uniqueness of G follows from energy inequality. The only difference with previous Proposition is that I can prove that $G' \in L^2(0,T,V')$. I have

$$||G'||^2_{L^2(0,T,V')} = ||B(\cdot)G||^2_{L^2(0,T,V')} \le \int_0^T |\nu AG + \operatorname{curl}(v(t)\times G(t))|^2_{V'}dt \le$$

$$\nu^2||G||^2_{L^2(0,T,V)} + \int_0^T |v(t)\times G(t)|^2_H dt \le \nu^2||G||^2_{L^2(0,T,V)} +$$

$$\int_0^T (C_1|G(t)|^2_V + C_2|v(t)|^{2+6/\delta_0}_{\mathbb{L}^{3+\delta_0}}|G(t)|^2_H)dt \le$$

$$C_3||G||^2_{L^2(0,T,V)} + C_2||G||^2_{L^\infty(0,T,H)}|v|_{L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))} < \infty$$

Thus, first statement of the Proposition is proved. [i] and [ii] are proved in the same way as in Proposition 3.1.7:

(i) Local existence of solution. Let $X_T = \{F||F|^2_{X_T} = \nu^2|F|^2_{L^2(0,T;D(A))} + |F'|^2_{L^2(0,T;H)} < \infty\}$. Define a map $\Phi_T : X_T \to X_T$ by $\Phi_T(z) = G$, where G is a solution of the problem

$$G' + \nu AG = f - \operatorname{curl}(v(t)\times z), G(0) = F_0 \tag{3.2.11}$$

Lemma 3.2.3. *If v satisfies assumption* (3.1.14), $v \in L^2(0,T;V)$, $f \in L^2(0,T;H)$, $F_0 \in V$ *then* $\operatorname{curl}(v(t)\times z) \in L^2(0,T;H)$ *and map Φ_T is well defined.*

Proof. It is enough to prove that $\operatorname{curl}(v(t)\times z) \in L^2(0,T;H)$.Then correctness of definition of Φ_T will follow from Proposition 2.4.10. I have:

$$||\operatorname{curl}(v(t) \times z)||^2_{L^2(0,T;H)} \leq C(||z\nabla v||^2_{L^2(0,T;H)} + ||v\nabla z||^2_{L^2(0,T;H)}) \leq$$
$$C|z|_{C(0,T;V)}|v|_{L^2(0,T;V)} \qquad (3.2.12)$$

and the result follows from Lemma 2.4.6. □

Let us show that there exists such $T_1 \leq T$ that Φ_{T_1} is contractive map. I have by Proposition 2.4.10 and Lemma 3.1.3

$$||\Phi_t(z_1) - \Phi_t(z_2)||^2_{X_t} \leq C_1||\operatorname{curl}(v(t) \times (z_1 - z_2))||^2_{L^2(0,t;H)} \leq$$
$$C|z_1 - z_2|^2_{C(0,t;V)}|v|^2_{L^2(0,t;V)} \leq C|z_1 - z_2|^2_{X_t}|v|^2_{L^2(0,t;V)}$$

Now let us choose $t = T_1$ such that $|v|_{L^2(0,t;V)} < 1/2$ then Φ_{T_1} is an affine contraction map and by Banach fixed point theorem there exists fixed point $F \in X_{T_1}$ of Φ_{T_1}. Then F is a solution of equation (3.1.12) on interval $[0, T_1]$.

Global Existence of solution. From Lemma 3.2.1 and assumption (3.1.14) it follows that I can find partition $0 = T_0 < T_1 < \ldots < T_{k-1} < T_k = T$ of interval $[0, T]$ such that $|v|_{L^2(T_i,T_{i+1};V)} < 1/2$, $i = 0, \ldots, k-1$. Therefore, I can use estimate (3.2.8) and Banach fixed point theorem iteratively to define global solution.

(ii) I will consider only the case $k = 1$. General case follows by induction. I differentiate equation (3.1.12) w.r.t. t (in weak sense) and get equation for F':

$$\frac{dF'}{dt} = -A(t)F' + \operatorname{curl}(F \times v'(t)) + f'$$

Now from assumptions of statement [ii] it follows that it is enough to prove that $\operatorname{curl}(F \times v'(t)) \in L^2(0, T; H)$ and then use part [i]. By estimate (3.2.12) and Lemma 2.4.6 I get the result.

□

Proof of Corollaries 3.1.11 and 3.1.12. Here I follow remark 3.2, p.90 in [73].

I have a weak solution F of equation (3.1.12)(corr. (3.1.13)) in $L^\infty(0,T;H)\cap L^2(0,T;V)$ by proposition 3.1.7(corr. 3.1.10).

Choose arbitrarily small $t_0 > 0$ such that $F(t_0) \in V$. By proposition 3.1.7 (corr. 3.1.10) I have that $F \in L^2(t_0,T;D(A)) \cap C(t_0,T;V)$.

I choose $t_1 > t_0$, t_1 arbitrarily close to t_0 such that $F(t_1) \in D(A)$ then (3.1.12) (corr. (3.1.13)) shows that $F'(t_1) \in H$. I conclude by proposition 3.1.7 (corr. 3.1.10) that $F' \in L^\infty(t_1,T;H) \cap L^2(t_1,T;V)$.

I choose $t_2 > t_1$, $t_2 - t_1$ arbitrarily small such that $F'(t_2) \in V$ and conclude that $F' \in L^2(t_2,T;D(A)) \cap C(t_2,T;V)$, etc.

Finally I get that $F^{(k)} \in C([t_l,T],V), k = 1,\dots,n$ for t_l arbitrarily close from 0, n arbitrarily large. The regularity follows by Sobolev embedding theorem. □

3.3 Duality

Here I state main theorem of the chapter and deduce some corollaries.

Theorem 3.3.1. *Suppose that $F_0 \in H$, $G_0 \in H$, $v \in L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))$. Let F and G be solutions of problems*

$$\begin{aligned} \frac{\partial F}{\partial t} &= -\nu AF - B(v(t),F), t \in [0,T], \qquad (3.3.1)\\ F(0,\cdot) &= F_0,\\ \frac{\partial G}{\partial t} &= -\nu AG + \operatorname{curl}(v(T-t)\times G), t \in [0,T], \qquad (3.3.2)\\ G(0,\cdot) &= G_0. \end{aligned}$$

Then the following identity holds

$$(F(t), G(T-t))_H = (F(0), G(T))_H, t \in [0,T]. \tag{3.3.3}$$

Proof of Theorem 3.3.1. First step. Let us prove theorem in the case of smooth initial data and smooth v. I can find $F_0^\varepsilon \in C^\infty(\overline{D}) \cap H$, $G_0^\varepsilon \in C^\infty(\overline{D}) \cap H$, $v^\varepsilon \in C_b^\infty([0,T] \times \overline{D}) \cap L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))$ such that $F_0^\varepsilon \to F_0$, $\varepsilon \to 0$ in H, $G_0^\varepsilon \to G_0$, $\varepsilon \to 0$ in H and $v^\varepsilon \to v$, $\varepsilon \to 0$ in $L^\infty(0,T;\mathbb{L}^{3+\delta_0}(D))$. It follows from Corollaries 3.1.11 and 3.1.12 that there exists solutions $F^\varepsilon \in C(0,T;H) \cap C^\infty((0,T] \times \overline{D})$, $G^\varepsilon \in C(0,T;H) \cap C^\infty((0,T] \times \overline{D}))$ of equations

$$\begin{aligned} \frac{\partial F^\varepsilon}{\partial t} &= -\nu A F^\varepsilon - P(v^\varepsilon(t) \times \operatorname{curl} F^\varepsilon) \\ F^\varepsilon(0,\cdot) &= F_0^\varepsilon \\ \frac{\partial G^\varepsilon}{\partial t} &= -\nu A G^\varepsilon + \operatorname{curl}(v^\varepsilon(T-t) \times G^\varepsilon) \\ G^\varepsilon(0,\cdot) &= G_0^\varepsilon \end{aligned}$$

Therefore, for $t \in (0,T]$ I have:

$$\begin{aligned} &\frac{d}{dt}(F^\varepsilon(t), G^\varepsilon(T-t))_{\mathbb{L}^2(D)} = < \frac{d}{dt}F^\varepsilon(t), G^\varepsilon(T-t) >_{\mathbb{L}^2(D)} - \\ &(F^\varepsilon(t), \frac{d}{dt}G^\varepsilon(T-t))_{\mathbb{L}^2(D)} = \nu(P(\triangle F^\varepsilon(t)), G^\varepsilon(T-t))_{\mathbb{L}^2(D)} \\ -(P(v(t) \times \operatorname{curl} F^\varepsilon(t)), &G^\varepsilon(T-t))_{\mathbb{L}^2(D)} - \nu(F^\varepsilon(t), P(\triangle G^\varepsilon(T-t)))_{\mathbb{L}^2(D)} \\ &-(F^\varepsilon(t), \operatorname{curl}(v(t) \times G^\varepsilon(T-t)))_{\mathbb{L}^2(D)} = (i) - (ii) - (iii) - (iv) \end{aligned}$$

It follows from the fact that $\operatorname{div} F^\varepsilon = \operatorname{div} G^\varepsilon = 0$, $F^\varepsilon|_{\partial D} = G^\varepsilon|_{\partial D} = 0$ and formula of integration by parts that $(F^\varepsilon, \nabla\psi)_{\mathbb{L}^2(D)} = (G^\varepsilon, \nabla\psi)_{\mathbb{L}^2(D)} = 0$ for any $\psi \in C^\infty(\overline{D})$. Thus, I have

$$(i) = (P(\triangle F^\varepsilon(t)), G^\varepsilon(T-t))_{\mathbb{L}^2(D)} = (\triangle F^\varepsilon(t), G^\varepsilon(T-t))_{\mathbb{L}^2(D)},$$

$$(ii) = (P(v(t) \times \operatorname{curl} F^\varepsilon(t)), G^\varepsilon(T-t))_{\mathbb{L}^2(D)} =$$

$$(v(t) \times \operatorname{curl} F^\varepsilon(t), G^\varepsilon(T-t))_{\mathbb{L}^2(D)} \tag{3.3.4}$$

and

$$(iii) = (F^\varepsilon(t), P(\triangle G^\varepsilon(T-t)))_{\mathbb{L}^2(D)} = (F^\varepsilon(t), \triangle G^\varepsilon(T-t))_{\mathbb{L}^2(D)}$$

Therefore, I get $(i) - (iii) = 0$ by Green formula. From (2.1.1), (3.3.4) and formula

$$\int_D u \operatorname{curl} v dx - \int_D v \operatorname{curl} u dx = \int_{\partial D} (u \times v, \vec{n}) d\sigma$$

follows that

$$(ii) = (v(t) \times \operatorname{curl} F^\varepsilon(t), G^\varepsilon(T-t))_{\mathbb{L}^2(D)} =$$

$$-(\operatorname{curl} F^\varepsilon(t) \times v(t), G^\varepsilon(T-t))_{\mathbb{L}^2(D)} =$$

$$-(\operatorname{curl} F^\varepsilon(t), v(t) \times G^\varepsilon(T-t))_{\mathbb{L}^2(D)} = -(iv).$$

Thus, I have $\frac{d}{dt}(F^\varepsilon(t), G^\varepsilon(T-t))_{\mathbb{L}^2(D)} = 0, t \in (0,T]$. Also, from regularity of F^ε, G^ε follows that $(F^\varepsilon(t), G^\varepsilon(T-t))_{\mathbb{L}^2(D)} \in C^\infty((0,T]) \cap C([0,T])$. As a result I get (3.3.3).

Second step. Let us show that $F_\varepsilon(t) \to F(t)$ in weak topology of H and $G_\varepsilon \to G$ in $C([0,T], H)$ topology. Then I have

$$|(F(t), G(T-t)) - (F^\varepsilon(t), G^\varepsilon(T-t))| =$$

$$|(F - F^\varepsilon(t), G(T-t)) + (F^\varepsilon(t), G - G^\varepsilon(T-t))| \leq$$

$$|(F - F^\varepsilon(t), G(T-t))| + |F^\varepsilon(t)|_H |G - G^\varepsilon(T-t)|_H \leq$$

$$|(F - F^\varepsilon(t), G(T-t))| + |F_0^\varepsilon|_H \sup_{t \in [0,T]} |G - G^\varepsilon(t)|_H \to 0, \varepsilon \to 0$$

i.e. $(F(t), G(T-t))_H = \lim_{\varepsilon \to 0} (F^\varepsilon(t), G^\varepsilon(T-t))_H$ and the result follows from first step. To show weak convergence of $F_\varepsilon(t)$ to $F(t)$ I notice first that F_ε converges to F in weak topology of $L^\infty(0,T;H)$ by Banach-Alaoglu theorem (proof is exactly the same as proof of convergence of F_n to F in Proposition 3.1.7). Also, I have from Banach-Alaoglu theorem that $F^\varepsilon(t)$ weakly converges to some $\Psi(t) \in H$. Let us show that $\Psi(t) = F(t)$. I have $\int_0^T (F^\varepsilon(s) - F(s), q(s))_H ds \to 0$ for any $q \in L^1(0,T;H)$. Put $q(s) = \xi \frac{1}{\sqrt{2\pi\tau}} e^{-|t-s|^2/2\tau} \in L^1(0,T;H)$, $\xi \in H$, $K(\varepsilon,\tau) = \int_0^T (F^\varepsilon(s),\xi)_H \frac{1}{\sqrt{2\pi\tau}} e^{-|t-s|^2/2\tau} ds$. I have

$$\lim_{\varepsilon\to 0}\lim_{\tau\to 0} K(\varepsilon,\tau) = (\Psi(t),\xi), \lim_{\tau\to 0}\lim_{\varepsilon\to 0} K(\varepsilon,\tau) = (F(t),\xi)$$

and $0 \le K(\varepsilon,\tau) \le |\xi|_H |F_0|_H$. Therefore, I get $F(t) = \Psi(t)$. Thus, it remains to show that $G_\varepsilon \to G$ in $C([0,T],H)$ topology. Denote $R^\varepsilon = G^\varepsilon - G$. Then I have

$$\begin{aligned} \frac{\partial R^\varepsilon}{\partial t} &= -\nu A R^\varepsilon + \operatorname{curl}(v^\varepsilon(T-t) \times R^\varepsilon) + \operatorname{curl}((v^\varepsilon - v) \times G) \\ R^\varepsilon(0,\cdot) &= G_0^\varepsilon - G_0. \end{aligned}$$

I have from energy inequality (3.2.10) that

$$\begin{aligned} |R^\varepsilon|^2_{C(0,T;H)} &\le C(|v^\varepsilon|_{L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))}) \times \\ &(|G_0^\varepsilon - G_0|_H^2 + |\operatorname{curl}((v - v^\varepsilon) \times G)|^2_{L^2(0,T;V')}) \le \\ C(|v^\varepsilon|_{L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))}) &(|G_0^\varepsilon - G_0|_H^2 + |(v - v^\varepsilon) \times G|^2_{L^2(0,T;H)}) \le \\ C(|v|_{L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))}) &(|G_0^\varepsilon - G_0|_H^2 + \tau^{1+\delta_0/3} \int_0^T |G(s)|_V^2 ds + \\ &\frac{C_{\delta_0}}{\tau^{1+3/\delta_0}} |G|^2_{C(0,T;H)} |v^\varepsilon - v|_{L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))}) \end{aligned} \tag{3.3.5}$$

where I have used Lemma 3.1.3 in last inequality and $\tau-$ arbitrary positive number. Now, from convergence v^ε to v in $L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(D))$, G_0^ε to G_0 in H and (3.3.5) I have the result.

□

From now on I will consider case $D = \mathbb{R}^3$. I notice that in this case if F is a solution of Problem 3.1.12 with parameters (F_0, f, v), then $\operatorname{curl} F$ is a solution of Problem 3.1.13 (with parameters $(\operatorname{curl} F_0, \operatorname{curl} f, v)$).

Definition 3.3.2. *Let $\mathcal{T}_T^v : H \to H$ be the vector transport operator defined by $\mathcal{T}_T^v(F_0) = F(T)$, where F is the unique solution of equation* (3.3.1).

Define also the operator of time reversal by $S^T(v)(t) = -v(T-t)$. Then from Theorem 3.3.1 I infer that

Corollary 3.3.3. *Assume that $F_0 \in V$, $G_0 \in H$, $v \in L^{2+\frac{6}{\delta_0}}(0, T; \mathbb{L}^{3+\delta_0}(\mathbb{R}^3))$. Then I have following duality relation:*

$$(\operatorname{curl} F_0, \mathcal{T}_T^{S^T(v)} G_0)_H = (\operatorname{curl} \mathcal{T}_T^v F_0, G_0)_H. \tag{3.3.6}$$

Remark 3.3.4. This Corollary can be used to define operator $\mathcal{T}_T^v$ on the functions from Sobolev spaces with negative index.

Corollary 3.3.5. *Assume that either v satisfies condition* (3.1.14) *or there exists unique solution $F \in L^\infty(0, T; H)$ of equation* (3.3.1) *with initial condition $F_0 \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ such that duality relation* (3.3.6) *holds. Then*

$$||\mathcal{T}_T^v||_{\mathcal{L}(\mathbb{H}^{k,2}_{sol}, \mathbb{H}^{k,2}_{sol})} = ||\mathcal{T}_T^{S^T(v)}||_{\mathcal{L}(\mathbb{H}^{1-k,2}_{sol}, \mathbb{H}^{1-k,2}_{sol})}, \; k \in [0, 1]. \tag{3.3.7}$$

Proof of Corollary 3.3.5. Indeed, by (3.3.6) I have

$$||\mathcal{T}_T^v||_{\mathcal{L}(\mathbb{H}^{k,2}_{sol}, \mathbb{H}^{k,2}_{sol})} = \sup_{\phi, \psi \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)} \frac{| < \mathcal{T}_T^v \phi, \psi > |}{||\phi||_{\mathbb{H}^{k,2}_{sol}} ||\psi||_{\mathbb{H}^{-k,2}_{sol}}} =$$

$$\sup_{\phi, \psi \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)} \frac{| < \operatorname{curl} \mathcal{T}_T^v \phi, \operatorname{curl}^{-1} \psi > |}{||\phi||_{\mathbb{H}^{k,2}_{sol}} ||\psi||_{\mathbb{H}^{-k,2}_{sol}}} =$$

$$\sup_{\phi,\psi\in\mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)} \frac{|<\operatorname{curl}\phi, \mathcal{T}_T^{S^T(v)}\operatorname{curl}^{-1}\psi>|}{||\phi||_{\mathbb{H}^{k,2}_{sol}}||\psi||_{\mathbb{H}^{-k,2}_{sol}}} =$$

$$\sup_{\phi,\psi\in\mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)} \frac{|<\phi, \mathcal{T}_T^{S^T(v)}\psi>|}{||\phi||_{\mathbb{H}^{k-1,2}_{sol}}||\psi||_{\mathbb{H}^{1-k,2}_{sol}}} =$$

$$||\mathcal{T}_T^{S^T(v)}||_{\mathcal{L}(\mathbb{H}^{1-k,2}_{sol},\mathbb{H}^{1-k,2}_{sol})}$$

□

Definition 3.3.6. *By X_α I denote the class of all functions $u : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ such that*

(i)

$$u \in L^\infty(0,T;H).$$

(ii) There exists unique solution of equation (3.3.1) *with parameters u and $S^t(u|_{[0,t]})$, $t\in[0,T]$.*

(iii)

$$|\mathcal{T}_t^u|_{\mathcal{L}(\mathbb{H}^{\alpha,2}_{sol},\mathbb{H}^{\alpha,2}_{sol})} < \infty, \quad t\in[0,T]$$

Then the following result follows from Corollary 3.3.5

Corollary 3.3.7. *Assume that $\alpha\in[0,1]$ then $X_\alpha = X_{1-\alpha}\subset X_{\frac{1}{2}}$. Space $X_{\frac{1}{2}}$ is invariant with respect to scaling $(\Psi_\lambda u)(t,x) = \lambda u(\lambda^2 t, \lambda x)$, $\lambda\le 1$, $t\in[0,T]$, $x\in\mathbb{R}^3$.*

Proof of Corollary 3.3.7. Property $X_\alpha = X_{1-\alpha}$ immediately follows from Corollary 3.3.5 and definition of X_α. Let us show that $X_\alpha\subset X_{\frac{1}{2}}$. Let $u\in X_\alpha$. Then $\forall t\in[0,T]$ I have

$$\mathcal{T}_t^u \in \mathcal{L}(\mathbb{H}^{\alpha,2}_{sol},\mathbb{H}^{\alpha,2}_{sol}), \mathcal{T}_t^u \in \mathcal{L}(\mathbb{H}^{1-\alpha,2}_{sol},\mathbb{H}^{1-\alpha,2}_{sol}).$$

Indeed, it follows by definition of X_α that $S^t(u|_{[0,t]})\in X_\alpha$ and by Corollary 3.3.5 I have that

$$|\mathcal{T}_t^u|_{\mathcal{L}(\mathbb{H}^{1-\alpha,2}_{sol},\mathbb{H}^{1-\alpha,2}_{sol})} = |\mathcal{T}_t^{S^t(u|_{[0,t]})}|_{\mathcal{L}(\mathbb{H}^{\alpha,2}_{sol},\mathbb{H}^{\alpha,2}_{sol})}.$$

Therefore, by interpolation theorem ([74]) I have that

$$\mathcal{T}_t^u \in \mathcal{L}([\mathbb{H}^{\alpha,2}_{sol}, \mathbb{H}^{1-\alpha,2}_{sol}]_{1/2}, [\mathbb{H}^{\alpha,2}_{sol}, \mathbb{H}^{1-\alpha,2}_{sol}]_{1/2}),$$

i.e.

$$\mathcal{T}_t^u \in \mathcal{L}(\mathbb{H}^{\frac{1}{2},2}_{sol}, \mathbb{H}^{\frac{1}{2},2}_{sol}).$$

Third property follows from identity

$$\mathcal{T}_t^{\Psi_\lambda(u)}\Psi_\lambda(F_0) = \Psi_\lambda(\mathcal{T}_t^u F_0)$$

and boundedness of scaling operators Ψ_λ and $\Psi_\lambda^{-1} = \Psi_{\frac{1}{\lambda}}$ in $\mathbb{H}^{\frac{1}{2},2}_{sol}$. □

Consider family of spaces $(\mathbb{H}^{\alpha,2}_{sol}, X_\alpha), \alpha \in [0,1]$. Then Corollary 3.3.7 shows that the space $(\mathbb{H}^{\frac{1}{2},2}_{sol}, X_{\frac{1}{2}})$ is optimal for definition of vector transport operator $\mathcal{T}_t^u$ in the following sense: The set $u \in X_\alpha$ for which I have

$$\mathcal{T}_t^u \in \mathcal{L}(\mathbb{H}^{\alpha,2}_{sol}, \mathbb{H}^{\alpha,2}_{sol}), \alpha \in [0,1].$$

is the largest one for $\alpha = \frac{1}{2}$. Now I will get classical existence result of Serrin-Prodi- Ladyzhenskaya ([69], [64], [49]) which in our context means that $\bigcup_{\delta_0>0} L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(\mathbb{R}^3)) \subset X_1 = X_0$.

Corollary 3.3.8. *Assume that u is a weak solution of the Navier-Stokes equations without force satisfying Serrin condition* $u \in L^{2+\frac{6}{\delta_0}}(0,T;\mathbb{L}^{3+\delta_0}(\mathbb{R}^3))$ *for some* $\delta_0 > 0$. *If* $G_0 \in H$, $u(0) \in V$ *then*

$$(\operatorname{curl} u(0), \mathcal{T}_T^{S^T(u)}G_0)_H = (\operatorname{curl} u(T), G_0)_H, \tag{3.3.8}$$

$$||\operatorname{curl} u(T)||_H \le ||\mathcal{T}_T^{S^T(u)}||_{\mathcal{L}(H,H)}||\operatorname{curl} u(0)||_H \tag{3.3.9}$$

and u is a strong solution of Navier-Stokes equations i.e. $u \in L^\infty(0,T;V)$.

Proof of Corollary 3.3.8. By Proposition 3.1.7 there exist unique solution $F \in L^2(0,T;V) \cap L^\infty(0,T;H)$ of equation (3.1.12) with initial condition $F_0 = u(0)$ and $v = u$. I notice that u is also solution of (3.1.12) by Navier-Stokes equations. Thus, $F = u$ and I have (3.3.8) by Theorem 3.3.1. Therefore, I have

$$||\operatorname{curl} u(t)||_H \le ||\mathcal{T}_T^{ST(u)}||_{\mathcal{L}(H,H)}||\operatorname{curl} u(0)||_H$$

and by boundedness of operator $\mathcal{T}_T^{ST(u)}$ (Proposition 3.1.7) I get the result. □

Remark 3.3.9. I notice that relation (3.3.8) is a generalization of helicity invariant

$$\int_{\mathbb{R}^3} (u, \operatorname{curl} u)_{\mathbb{R}^3} dx,$$

see e.g. [60], of Euler equations for Navier-Stokes equations. Indeed, if I formally consider transport operator $\mathcal{T}_T$ for the case $\nu = 0$ and put $G(0) = u(T)$ in the right hand side of (3.3.8) then, under assumption that Euler equations has a unique solution, I get that $\mathcal{T}_T^{ST(u)} u(T) = u(0)$.

In the next section we will consider some modification of Navier-Stokes equations for which space $\mathbb{H}_{sol}^{\frac{1}{2},2}$ is a natural space for the solution.

3.4 Modified Navier-Stokes equations

In this section I will prove existence of the unique global strong solution for certain modification of the three dimensional Navier-Stokes equations. We consider the following system of equations:

$$\begin{cases} \frac{\partial \mathbf{F}}{\partial t} + \mathbf{F} \times A^{\frac{1}{2}}\mathbf{F} &= -\nu\Delta\mathbf{F} + M(s) + \nabla p \\ \operatorname{div} \mathbf{F} &= 0, F|_{\partial D} = 0 \\ \mathbf{F}(0) &= \mathbf{F}_0. \end{cases}$$

Here, as usual $F = F(t, x)$ is the unknown velocity field, F_0 is the initial velocity field (known data) and M is the external force. Moreover, A is the Stokes operator introduced in section 2.6. Please note, that on the right handside we have the usual laplacian and not the Stokes operator. These equations are a modification of NSEs in the sense that the term $\operatorname{curl} F$ is replaced by $A^{1/2}F$. The latter is in some sense a pseudodifferential operator of order 1 and the former a differential operator of order 1. One can notice that $\operatorname{curl}^2 = (A^{1/2})^2 = A$.Thus, our modification is in some sense a natural one.

As a result of our modification helicity balance for three dimensional N.-S. equations will be transformed into additional a priori estimate for equation (3.4).

3.4.1 Nonlinearity and its properties

Definition 3.4.1. *Let us define trilinear form $\tilde{l} : \mathcal{D} \times \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ by*

$$\tilde{l}(f, g, h) = (f \times A^{\frac{1}{2}}g, h)_H, \ f, g, h \in \mathcal{D}. \tag{3.4.1}$$

We will collect properties of $\tilde{l}$ in the following lemma

Lemma 3.4.2. *There exist constants $C_1(D), C_2(D) > 0$ such that for any $f, g, h \in \mathcal{D}$ we have*

$$\tilde{l}(f, g, f) = 0 \tag{3.4.2}$$

$$\tilde{l}(f, g, A^{\frac{1}{2}}g) = 0 \tag{3.4.3}$$

$$|\tilde{l}(f, g, h)| \leq C_1|h|_H|f|_V|A^{\frac{3}{4}}g|_H \tag{3.4.4}$$

$$|\tilde{l}(f, g, h)| \leq C_2|g|_V|f|_V|h|_{D(A^{\frac{1}{4}})} \tag{3.4.5}$$

Proof of Lemma 3.4.2. Properties 3.4.2 and 3.4.3 immediately follows from the definition of the vector product. Let us choose and fix $f, g, h \in \mathcal{D}$. All the constants below are independent from f,g and h.

Now we shall prove property (3.4.4). From the Schwarz and the Hölder inequality we infer that

$$|\tilde{l}(f,g,h)| \le |h|_H |f \times A^{\frac{1}{2}} g|_H \le |h|_H |f|_{\mathbf{L}^6(D)} |A^{\frac{1}{2}} g|_{\mathbf{L}^3(D)}. \tag{3.4.6}$$

Moreover, we have that

$$V \subset \mathbf{H}^{1,2} \subset \mathbf{L}^6(D), \tag{3.4.7}$$

and

$$D(A^{\frac{1}{4}}) \subset \mathbb{H}^{\frac{1}{2},2}_{sol} \subset \mathbf{L}^3(D). \tag{3.4.8}$$

Indeed, embeddings (3.4.7) and $\mathbb{H}^{\frac{1}{2},2}_{sol} \subset \mathbf{L}^3(D)$ follow from the Sobolev embedding theorem 2.3.3. Thus it remains to show that $D(A^{\frac{1}{4}}) \subset \mathbb{H}^{\frac{1}{2},2}_{sol}$. From Theorem 2.7.3, definition of Stokes operator and part iii of Lemma 2.7.2 we infer that

$$D(A^{\frac{1}{4}}) = (H, D(A))_{\frac{1}{4},2} = (H, \mathbb{H}^{2,2}_{sol}(D) \cap V)_{\frac{1}{4},2} \subset (H, \mathbb{H}^{2,2}_{sol}(D))_{\frac{1}{4},2} \cap (H, V)_{\frac{1}{4},2}$$

Now by Theorem 2.7.4 we have that

$$(H, \mathbb{H}^{2,2}_{sol}(D))_{\frac{1}{4},2} \subset \mathbb{H}^{\frac{1}{2},2}_{sol}(D).$$

Taking into account that $(H, V)_{\frac{1}{4},2} \subset H$ we get that $D(A^{\frac{1}{4}}) \subset \mathbb{H}^{\frac{1}{2},2}_{sol}(D) \cap H = \mathbb{H}^{\frac{1}{2},2}_{sol}$.

Therefore, it follows from (3.4.7) and (3.4.8) that there exist constants $C_3, C_4 > 0$ such that

$$|f|_{\mathbf{L}^6(D)} \le C_3 |f|_V, \ |f|_{\mathbf{L}^3(D)} \le C_4 |f|_{D(A^{\frac{1}{4}})}.$$

Combining these inequalities with inequality (3.4.6) we get property 3.4.4.

Similarly, it follows from inequality (2.1.1), Hölder inequality and the Sobolev embedding theorem that

$$|\tilde{l}(f,g,h)| = |(A^{\frac{1}{2}} g, f \times h)_H| \le |A^{\frac{1}{2}} g|_H |f \times h|_H \le$$

$$\le |g|_V |f|_{\mathbf{L}^6(D)} |h|_{\mathbf{L}^3(D)} \le C_2 |g|_V |f|_V |h|_{D(A^{\frac{1}{4}})}.$$

□

From inequality (3.4.5) we infer that the form $\tilde{l}$ is continuous with respect to the $V \times V \times V$ topology. Since $\mathcal{D}$ is dense in V there exists continuous trilinear form $l : V \times V \times V \to \mathbb{R}$ such that

$$l(\cdot, \cdot, \cdot)|_{\mathcal{D}\times\mathcal{D}\times\mathcal{D}} = \tilde{l}.$$

Moreover, the norm of the extension l is the same as of $\tilde{l}$. Hence, for each $(f, g) \in V \times V$ $l(f, g, \cdot) \in V'$ and therefore the following definition is well posed.

Definition 3.4.3. *Let us define a bilinear operator $L : V \times V \to V'$ by*

$$< L(f, g), \phi >_{V',V} = l(f, g, \phi), f, g \in V, \phi \in V.$$

Corollary 3.4.4. *There exists constant $C_1 > 0$ such that for any $(f, g, h) \in V \times V \times V$,*

$$< L(f, g), f >_{V',V} = 0 \tag{3.4.9}$$

$$| < L(f, g), h >_{V',V} | \leq C_1 |f|_V |g|_V |h|_{D(A^{\frac{1}{4}})}. \tag{3.4.10}$$

Furthermore, if $f \in V, g \in D(A^{\frac{3}{4}})$ then

$$< L(f, g), A^{\frac{1}{2}} g >_{V',V} = 0. \tag{3.4.11}$$

If in addition, $(f, g, h) \in V \times D(A^{\frac{3}{4}}) \times H$ then

$$| < L(f, g), h >_{V',V} | \leq |f|_V |g|_{D(A^{\frac{3}{4}})} |h|_H. \tag{3.4.12}$$

3.4.2 Existence and uniqueness theorem

I consider the following problem:

$$\frac{\partial \mathbf{F}}{\partial t} = -\nu A\mathbf{F} + L(\mathbf{F}, \mathbf{F}) + M, \tag{3.4.13}$$

$$F(0) = F_0. \tag{3.4.14}$$

Definition 3.4.5. *Assume that* $F_0 \in D(A^{\frac{1}{4}})$, *external force* $M \in L^2(0,T; D(A^{-\frac{1}{4}}))$. *Then a function* $F \in C([0,T], D(A^{\frac{1}{4}})) \cap L^2(0,T; D(A^{\frac{3}{4}}))$ *is a solution of equation* (3.4.13) *if* F *satisfies* (3.4.13) *in the distribution sense and* F *satisfies* (3.4.14).

Theorem 3.4.6. *Suppose that* $F_0 \in D(A^{\frac{1}{4}})$, $M \in L^2(0,T; D(A^{-\frac{1}{4}}))$. *Then*

(i) There exists a unique function $F \in C([0,T], D(A^{\frac{1}{4}})) \cap L^2(0,T; D(A^{\frac{3}{4}}))$, $F' \in L^2(0,T; D(A^{-\frac{1}{4}}))$ *that is a solution of problem* (3.4.13)-(3.4.14).

(ii) Furthermore, if $F_0 \in V$ *and the external force* $M \in L^2(0,T; H)$ *then* $F \in C([0,T], V) \cap L^2(0,T; D(A))$.

Proof of Theorem 3.4.6. I will use standard technique described, for instance in Temam[72].

(i) Existence of solution. The proof of existence will consist of four steps a), b), c), d):

a) Galerkin approximation of solution. Recall that By proposition 2.6.2 A is self adjoint, positive definite and there exist a sequence $\{e_j\}_{j=1}^{\infty}$ of eigenvectors of A, corresponding to the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ (enumerated in increasing order). We have that for any $j \in \mathbb{N}$ $e_j \in D(A)$ and $\{e_j\}_{j=1}^{\infty}$ forms orthonormal basis of H. For each $n \in \mathbb{N}$ one can define an approximate solution as follows:

$$F_n(t) = \sum_{k=1}^{n} g_{kn}(t) e_k \tag{3.4.15}$$

and

$$(F_n'(t), e_k) + \nu(AF_n(t), e_k)_H + l(F_n(t), F_n(t), e_k) = (M(t), e_k), \tag{3.4.16}$$

$$t \in [0,T], k = 1, \dots, n,$$

$$F_n(0) = F_{0n}. \tag{3.4.17}$$

where F_{0n} is an orthogonal projection of F_0 onto the space X_n spanned by $\{e_j\}_{j=1}^{n}$.We notice that system (3.4.16)-(3.4.17) is just projection of problem (3.4.13)-(3.4.14) to the linear span of first n vectors of basis $\{e_j\}_{j=1}^{\infty}$. Denote $M_k = (M, e_k)$. Then $M_k \in L^2(0, T)$, $k \in \mathbb{N}$. Combining (3.4.15) with (3.4.16) and (3.4.17) we get nonlinear system of differential equations for functions $g_{kn}(t), t \in [0, T], k = 1, \dots, n$:

$$g'_{kn}(t) + \nu\lambda_k^2 g_{kn}(t) + \sum_{i,j=1}^{n} l(e_i, e_j, e_k) g_{in}(t) g_{jn}(t) = M_k(t). \tag{3.4.18}$$

$$g_{kn}(0) = (F_{0n}, e_k), t \in [0, T], k = 1, \dots, n. \tag{3.4.19}$$

The nonlinear system (3.4.18)-(3.4.19) has a maximal solution defined on some interval $[0, T_m]$, see [19], where $T_m \leq T$ such that if $T_m < T$, then $\lim_{t \to T_m} |F_n(t)|_H = \infty$. We shall prove later that this is not the case and therefore $T_m = T$.

b) A priori estimates. Let $F_n(t), t \in [0, T_n)$ be the maximal solution of the problem (3.4.18)-(3.4.19) existence of which was established in part (a). First a priori estimate (called also energy estimate) is the same as for Navier-Stokes equations. Multiplying equation (3.4.18) by g_{kn} and adding these equations for $k = 1, \dots, n$ and taking into account (3.4.9), we get

$$(F'_n(t), F_n(t))_H + \nu |F_n(t)|_V^2 = (M(t), F_n), t \in [0, T_n). \tag{3.4.20}$$

Let us fix $t \in [0, T_n)$. Integrating equality (3.4.20) from 0 to t and using Young inequality we have

$$|F_n(t)|_H^2 + \nu \int_0^t |F_n(s)|_V^2 \, ds \leq |F_{0n}|_H^2 + \frac{1}{\nu} \int_0^T |M(s)|_{V'}^2 ds$$

$$\leq |F_0|_H^2 + \frac{1}{\nu} \int_0^T |M(s)|_{V'}^2 ds, \; t \in [0, T_n). \tag{3.4.21}$$

It readily follows from inequality (3.4.21) that the sequence $\{F_n\}_{n=1}^{\infty}$ is bounded in $L^{\infty}(0, T_n; H) \cap L^2(0, T_n; V)$. Furthermore, inequality (3.4.21) implies in particular, that $\limsup_{t \nearrow T_m} |F_n(t)| < \infty$ provided $T_n < T$. Hence, we infer that $T_n = T$.

Next we will deduce the second a'priori bound for the sequence F_n. This bound is intrinsic property of our problem and it does not hold for the proper Navier-Stokes equations. We multiply (3.4.18) on $\lambda_k^{\frac{1}{2}} g_{kn}$ and add these equations for $k = 1, \ldots, n$. Taking into account (3.4.11), we get

$$(F_n'(t), A^{\frac{1}{2}} F_n(t))_H + \nu(AF_n(t), A^{\frac{1}{2}} F_n(t))_H = (M(s), A^{\frac{1}{2}} F_n(t)), t \in [0, T]. \quad (3.4.22)$$

Fix $t \in [0, T]$. Integrating equality (3.4.22) from 0 to t and using Young inequality we have

$$|F_n(t)|^2_{D(A^{\frac{1}{4}})} + \nu \int_0^t |F_n(s)|^2_{D(A^{\frac{3}{4}})} ds \leq |F_{0n}|^2_{D(A^{\frac{1}{4}})} + \frac{1}{\nu} \int_0^T |M(s)|^2_{D(A^{-\frac{1}{4}})} ds$$
$$\leq |F_0|^2_{D(A^{\frac{1}{4}})} + \frac{1}{\nu} \int_0^T |M(s)|^2_{D(A^{-\frac{1}{4}})} ds, t \in [0, T]. \quad (3.4.23)$$

Hence sequence $\{F_n\}_{n=1}^{\infty}$ is bounded in $L^{\infty}(0, T; D(A^{\frac{1}{4}})) \cap L^2(0, T; D(A^{\frac{3}{4}}))$.

Now we shall show the following corollary of the last two estimates. We deduce that the sequence $\{F_n\}_{n=1}^{\infty}$ is bounded in $L^4(0, T; V)$. We have by Lemma 2.7.2 and $V = D(A^{\frac{1}{2}})$ (see Theorem 2.4.3) that

$$|F_n|_V^2 \leq |F_n|^2_{D(A^{\frac{1}{2}})} \leq |F_n|_{D(A^{\frac{1}{4}})} |F_n|_{D(A^{\frac{3}{4}})}.$$

Therefore

$$|F_n|^4_{L^4(0,T;V)} = \int_0^T |F_n|_V^4 ds \leq \int_0^T |F_n|^2_{D(A^{\frac{1}{4}})} |F_n|^2_{D(A^{\frac{3}{4}})} ds$$

$$\leq |F_n|^2_{L^{\infty}(0,T;D(A^{\frac{1}{4}}))} |F_n|^2_{L^2(0,T;D(A^{\frac{3}{4}}))} \leq \frac{1}{\nu} |F_{0n}|^4_{D(A^{\frac{1}{4}})} + \frac{|F_{0n}|^2_{D(A^{\frac{1}{4}})}}{\nu^2} \int_0^T |M(s)|^2_{D(A^{-\frac{1}{4}})} ds$$

$$\leq \frac{1}{\nu} |F_0|^4_{D(A^{\frac{1}{4}})} + \frac{|F_0|^2_{D(A^{\frac{1}{4}})}}{\nu^2} \int_0^T |M(s)|^2_{D(A^{-\frac{1}{4}})} ds \quad (3.4.24)$$

We will need one more estimate. Let us show that $\{F_n'\}$ is bounded in $L^2(0,T;D(A^{-\frac{1}{4}}))$. Define $P_{X_n}: V' \to X_n$ as an extension of the usual orthogonal projection from H onto X_n. We notice that system of equations (3.4.16) can be rewritten as

$$F_n' = -\nu AF_n - P_{X_n}L(F_n) + P_{X_n}M.$$

Therefore, in order to show that $\{F_n'\}$ is bounded in $L^2(0,T;D(A^{-\frac{1}{4}}))$, it is enough to show that sequences $\{AF_n\}_{n=1}^{\infty}$ and $\{L(F_n)\}_{n=1}^{\infty}$ are bounded in $L^2(0,T;D(A^{-\frac{1}{4}}))$. Boundedness of sequence $\{AF_n\}_{n=1}^{\infty}$ in $L^2(0,T;D(A^{-\frac{1}{4}}))$ follows from the a'priori estimate (3.4.23) and continuity of A as operator from $D(A^{\frac{3}{4}})$ to $D(A^{-\frac{1}{4}})$.

Next let us observe that in view of inequalities (3.4.10) and (3.4.24) we have that

$$|L(F_n)|^2_{L^2(0,T;D(A^{-\frac{1}{4}}))} = \int_0^T |L(F_n)|^2_{V'}ds \leq C\int_0^T |F_n|^4_V ds$$
$$\leq C(\nu, |F_0|_{D(A^{\frac{1}{4}})}, |M|_{L^2(0,T;D(A^{-\frac{1}{4}}))}). \quad (3.4.25)$$

Therefore, we infer that there exists a constant $C = C(\nu, |F_0|_{D(A^{\frac{1}{4}})}, |M|_{L^2(0,T;D(A^{-\frac{1}{4}}))})$ such that for all $n \in \mathbb{N}$

$$|F_n'|^2_{L^2(0,T;D(A^{-\frac{1}{4}}))} \leq C(|F_n|^2_{L^2(0,T;D(A^{\frac{3}{4}}))} + |L(F_n)|^2_{L^2(0,T;D(A^{-\frac{1}{4}}))})$$
$$+|M|_{L^2(0,T;D(A^{-\frac{1}{4}}))} \leq C. \quad (3.4.26)$$

c) Topology of convergence of Galerkin approximation. From the part (b) we infer that the sequence $\{F_n\}_{n=1}^{\infty}$ is bounded in the Banach space $L^\infty(0,T;H) \cap L_2(0,T;V)$. Therefore, by the Banach-Alaoglu Theorem there exists subsequence $\{F_{n'}\}$ and $F^* \in L^\infty(0,T;H)$ such that $F_n \to F$ weakly-*, which means that for any $q \in L^1(0,T;H)$

$$\int_0^T (F_{n'} - F^*, q(s))_H ds \to 0. \quad (3.4.27)$$

Similarly, from the Banach-Alaoglu Theorem it follows that one can find a subsequence $\{F_{n''}\}$ of $\{F_{n'}\}$ which converges to $F^{**} \in L^2(0,T;V)$ in the weak topology of $L^2(0,T;V)$ i.e. for any $q \in L^2(0,T;V')$

$$\int_0^T < F_{n''} - F^{**}, q(s) >_{V',V} ds \to 0. \qquad (3.4.28)$$

In particular, Note that (3.4.27) and (3.4.28) are satisfied for any $q \in L^2(0,T;H)$. Therefore, $\int_0^T < F* - F**, q >= 0$ for all $q \in L^2(0,T;H)$ and so $F^* = F^{**} \in L^\infty(0,T;H) \cap L^2(0,T;V)$. Put $F = F^*$.

Now we will show that there exists subsequence $\{F_{n'''}\}$ of $\{F_{n''}\}$ which converges strongly in $L^2(0,T;V)$. It is enough to show that the sequence $\{F_n\}$ is precompact in $L^2(0,T;V)$. We have following chain of continuous embeddings

$$D(A^{\frac{3}{4}}) \subset V = D(A^{\frac{1}{2}}) \subset V'$$

Furthermore, embedding $D(A^{\frac{3}{4}}) \subset V$ is compact (see Lemma 2.6.7 and recall that $V = D(A^{\frac{1}{2}})$). It follows from estimates (3.4.23) and (3.4.26) that the set $\{F_n\}$ is bounded in

$$Y = \{\psi | \psi \in L^2(0,T;D(A^{\frac{3}{4}})); \psi' \in L^2(0,T;V')\}.$$

Now we are in the position to use Theorem 2.2.12 with the data $X_0 = D(A^{\frac{3}{4}})$, $X = V$, $X_1 = V'$, $\alpha_0 = \alpha_1 = 2$. We have by the Theorem 2.2.12 that the injection of Y into $L^2(0,T;V)$ is compact and therefore, the sequence $\{F_n\}$ is precompact in $L^2(0,T;V)$. Below we will assume without loss of generality that the sequence $\{F_n\}_{n=1}^{\infty}$ converges to F in strong topology of $L^2(0,T;V)$ and satisfies (3.4.27). Notice that because $F_n \to F$ strongly in $L^2(0,T;V)$ also (3.4.28) is satisfied.

(d) Convergence of Galerkin approximation. The convergence result of step c) enable us to prove that F is a solution of problem (3.4.13)-(3.4.14). Now I will proceed essentially as in the proof of part i) step c) of Proposition 3.1.7.

Let $\psi \in C^\infty([0,T],\mathbb{R})$ such that $\psi(1) = 0$. Then by part (a) of the proof I have

$$-\int_0^T (F_n(s), h)_H \psi'(s)ds + \int_0^T < L(F_n, F_n), h >_{V',V} \psi(s)ds +$$
$$\nu \int_0^T \tilde{a}(F_n(s), h)\psi(s)ds = (F_{0n}, h)_H \psi(0) + \int_0^T < M(s), h >_{V',V} \psi(s)ds. \quad (3.4.29)$$

for $h = e_j, j = 1,\ldots,n$. Let us observe that from (3.4.27), resp. (3.4.28), it follows that 1st term, resp. 3rd term, in (3.4.29) converge to

$$-\int_0^T (F(s), h)_H \psi'(s)ds,$$

resp.

$$\nu \int_0^T \tilde{a}(F(s), h)\psi(s)ds.$$

For the second term we have following inequality

$$|\int_0^T < L(F_n, F_n) - L(F, F), e_j >_{V',V} \psi(s)ds| \leq$$
$$|\int_0^T < L(F_n - F, F_n), h >_{V',V} \psi(s)ds| +$$
$$|\int_0^T < L(F, F_n - F), h >_{V',V} \psi(s)ds| = I_n + II_n.$$

It follows from inequality (3.4.10) and convergence $F_n \to F$ in $L^2(0,T;V)$ that

$$I_n \leq C\int_0^T |F_n - F|_V |F_n|_V |\psi(s)| |h|_V ds \leq |F_n - F|_{L^2(0,T;V)} |F_n|_{L^2(0,T;V)} |h|_V |\psi|_{L^\infty} \xrightarrow[n\to\infty]{} 0.$$

and similarly $II_n \to 0, n \to \infty$. Thus, we get

$$-\int_0^T (F(s),h)_H \psi'(s)ds + \int_0^T < L(F,F),h >_{V',V} \psi(s)ds +$$

$$\nu \int_0^T \tilde{a}(F(s),h)\psi(s)ds = (F_0,h)_H \psi(0). \qquad (3.4.30)$$

for $h = e_1, \ldots, e_n, \ldots$. Since both sides of (3.4.30) are linear and continuous in V, we have that (3.4.30) holds for any $h \in V$.

As the result we have shown that $F \in L^\infty(0,T;H) \cap L^2(0,T;V)$ is a solution of equation (3.4.13) in distribution sense. Furthermore, it immediately follows from a'priori estimate (3.4.23) that $F \in L^\infty(0,T;D(A^{\frac{1}{4}})) \cap L^2(0,T;D(A^{\frac{3}{4}}))$. Moreover, it follows from (3.4.26) that $F' \in L^2(0,T;D(A^{-\frac{1}{4}}))$ and, consequently, that $F \in C(0,T;D(A^{\frac{1}{4}}))$ by Lemma 2.4.6. Indeed, if we identify $D(A^{\frac{1}{4}})$ with its adjoint then we have Gelfand triple $D(A^{\frac{3}{4}}) \subset D(A^{\frac{1}{4}}) \cong D(A^{\frac{1}{4}})^* \subset D(A^{-\frac{1}{4}})$.

Uniqueness of solution. Assume that there exists two solutions F_1 and F_2 of equation (3.4.13). Denote $F = F_1 - F_2$. Then we have the following equation for F

$$\begin{aligned} \frac{\partial F}{\partial t} &= -\nu AF + L(F_1) - L(F_2) \\ F(0) &= 0 \end{aligned}$$

By part i of the proof we have that $F \in L^2(0,T;D(A^{\frac{3}{4}}))$ and $F' \in L^2(0,T;D(A^{-\frac{1}{4}}))$. Hence, by Lemma 2.4.6 we have that

$$|F(t)|^2_{D(A^{\frac{1}{4}})} + 2\nu \int_0^t |F(s)|^2_{D(A^{\frac{3}{4}})} ds \le \left| \int_0^t l(F_1,F_1,A^{\frac{1}{2}}F) - l(F_2,F_2,A^{\frac{1}{2}}F)ds \right| =$$

$$\left| \int_0^t l(F,F_1,A^{\frac{1}{2}}F) + l(F_2,F,A^{\frac{1}{2}}F)ds \right| = \left| \int_0^t l(F,F_1,A^{\frac{1}{2}}F)ds \right| = I. \qquad (3.4.31)$$

where I have used property 3.4.11 in the last equality. Consequently, we have by inequality (3.4.12) that

$$I \leq \int_0^t |F|_V^2 |F_1|_{D(A^{\frac{3}{4}})} ds \leq \int_0^t |F|_{D(A^{\frac{1}{4}})} |F|_{D(A^{\frac{3}{4}})} |F_1|_{D(A^{\frac{3}{4}})} ds \leq$$

$$\nu \int_0^t |F(s)|^2_{D(A^{\frac{3}{4}})} ds + \frac{C}{\nu} \int_0^t |F|^2_{D(A^{\frac{1}{4}})} |F_1|^2_{D(A^{\frac{3}{4}})} ds. \tag{3.4.32}$$

Combining estimates (3.4.31) and (3.4.32) we get that

$$|F(t)|^2_{D(A^{\frac{1}{4}})} + \nu \int_0^t |F(s)|^2_{D(A^{\frac{3}{4}})} ds \leq \frac{C}{\nu} \int_0^t |F|^2_{D(A^{\frac{1}{4}})} |F_1|^2_{D(A^{\frac{3}{4}})} ds. \tag{3.4.33}$$

Since $F_1 \in L^2(0, T; D(A^{\frac{3}{4}}))$ it follows by Gronwall lemma that

$$|F(t)|^2_{D(A^{\frac{1}{4}})} \leq 0, t \in [0, T],$$

i.e. $F_1 = F_2$ in $C(0, T; D(A^{\frac{1}{4}}))$.

(ii) Let us prove that $F \in L^\infty(0, T; V) \cap L^2(0, T; D(A))$. It is enough to show that Galerkin approximation sequence $\{F_n\}_{n=1}^\infty$ is bounded in $L^\infty(0, T; V) \cap L^2(0, T; D(A))$. I multiply equation (3.4.16) on $\lambda_j g_{jk}$ for each $j = 1, \ldots, k$ and add these equations. We have

$$\frac{d}{dt}|F_n|_V^2 + 2\nu |AF_n|_H^2 + l(F_n, F_n, AF_n) = 0. \tag{3.4.34}$$

Consequently,

$$|F_n(t)|_V^2 + 2\nu \int_0^t |AF_n(s)|_H^2 ds \leq |F_{0n}|_V^2 + \int_0^T |l(F_n, F_n, AF_n)| ds. \tag{3.4.35}$$

Second term of right part of (3.4.35) can be estimated as follows

$$\int_0^T |l(F_n, F_n, AF_n)| ds \leq \int_0^T |AF_n|_H |F_n|_V |F_n|_{D(A^{\frac{3}{4}})} ds \leq \nu \int_0^T |AF_n(s)|_H^2 ds$$

$$+\frac{C}{\nu}\int_0^T |F_n|_V^2 |F_n|^2_{D(A^{\frac{3}{4}})} ds. \tag{3.4.36}$$

where first inequality follows from (3.4.12) and the second inequality follows from Young inequality. Combining (3.4.35) and (3.4.36) we get

$$|F_n(t)|_V^2 + \nu\int_0^t |AF_n(s)|_H^2 ds \leq |F_{0n}|_V^2 + \frac{C}{\nu}\int_0^T |F_n|_V^2 |F_n|^2_{D(A^{\frac{3}{4}})} ds. \tag{3.4.37}$$

Therefore, it follows by Gronwall inequality and a'priori estimate (3.4.23) that

$$|F_n(t)|_V^2 \leq |F_{0n}|_V^2 e^{\frac{C}{\nu}\int_0^T |F_n|^2_{D(A^{\frac{3}{4}})} ds} \leq |F_0|_V^2 e^{\frac{C}{\nu^2}|F_0|^2_{D(A^{\frac{1}{4}})}} < \infty, \tag{3.4.38}$$

i.e. sequence $\{F_n\}_{n=1}^{\infty}$ is bounded in $L^{\infty}(0,T;V)$. Moreover, combining (3.4.37) and (3.4.38) we get boundedness of sequence $\{F_n\}_{n=1}^{\infty}$ in $L^{\infty}(0,T;V)\cap L^2(0,T;D(A))$. Thus, $F \in L^{\infty}(0,T;V)\cap L^2(0,T;D(A))$. To show that $F \in C([0,T],V)$ we need to prove that $F' \in L^2(0,T;H)$ and the result will follow from Lemma 2.4.6 and the fact that $D(A) \subset V \cong V' \subset H$ is a Gelfand triple. From system of equations (3.4.13), assumption on M and the fact that $F \in L^{\infty}(0,T;V)\cap L^2(0,T;D(A))$(proved above) we infer that it is enough to show that $L(F,F) \in L^2(0,T;H)$. By inequality (3.4.12) we have

$$|L(F,F)|^2_{L^2(0,T;H)} \leq \int_0^T |F|_V^2 |F|^2_{D(A^{3/4})} ds \leq |F|^2_{L^{\infty}(0,T;V)} |F|^2_{L^2(0,T;D(A^{3/4}))} < \infty.$$

□

Chapter 4

Feynman-Kac Formula for vector transport equation

The aim of this chapter is to prove the Feynman-Kac type formulae for solutions to the vector advection equations, see Propositions 4.1.3 and 4.4.1. A different approach to this problem can be found in works [18] and [23]. Our approach permits us to find other non-classical Feynman-Kac formulae for the vector transport operator, see Propositions 4.2.1, 4.2.4 for the 2D case and Proposition 4.3.5 for the 3D case. One can notice that Proposition 4.3.5 is, in certain sense, a generalization of Propositions 4.2.1 to 3D case. It would be interesting to find the generalization of Proposition 4.2.4 to 3D case, see discussion in question 4.3.6.

4.1 Formulae of Feynman-Kac Type.

In this section I will suggest a physical meaning to the operator $\mathcal{T}_T^{S^T(\cdot)}$ defined in Definition 3.3.2 and in the same time I will deduce a Feynman-Kac type formula for the solutions of the vector advection equations. From now on I suppose that $v \in C_0^\infty([0,T] \times \mathbb{R}^n)$ and $D = \mathbb{R}^n$. We also assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete filtered probability space. Let $(W_t)_{t\geq 0}$ be an $\mathbb{R}^m$-valued Wiener process defined on this space.

We begin with the following preliminary but basic result.

Proposition 4.1.1. *Let $\sigma(t,\cdot) \in C_b^{2,\alpha}(\mathbb{R}^n, \mathbb{R}^n \otimes \mathbb{R}^m)$, $a(t,\cdot) \in C_b^{1,\alpha}(\mathbb{R}^n, \mathbb{R}^n)$, $t \in [0,T]$. Assume that Γ is a closed loop of C^1 class in $\mathbb{R}^n$. Let $F \in C^{1,2}([0,T] \times \mathbb{R}^n, \mathbb{R}^n)$, $X = X_t(x,\omega) : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$-be defined by:*

$$dX_t(x) = a(t, X_t(x))dt + \sigma(t, X_t(x))dW_t$$
$$X_0(x) = x.$$

Then

$$\int\limits_{X_t(\Gamma)} \sum_{k=1}^n F^k(t,x)dx_k = \int\limits_{\Gamma} \sum_{k=1}^n F^k(0,x)dx_k + \int\limits_0^t \int\limits_{X_s(\Gamma)} \sum_{k=1}^n \left(\frac{\partial F^k}{\partial t} + \sum_{j=1}^n a^j \left(\frac{\partial F^k}{\partial x_j} - \frac{\partial F^j}{\partial x_k}\right) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 F^k}{\partial x_i \partial x_j} \sum_{m=1}^n \sigma^{im}\sigma^{jm} \right) dx_k ds + \tag{4.1.1}$$
$$+\frac{1}{2} \int\limits_0^t \int\limits_{X_s(\Gamma)} \sum_{k=1}^n \left(\sum_{j,l} \frac{\partial F^j}{\partial x_l} \sum_m \sigma^{lm} \frac{\partial \sigma^{jm}}{\partial x_k} \right) dx_k ds + \int\limits_0^t \int\limits_{X_s(\Gamma)} \sum_{k,j=1}^n F^j(s,x) \frac{\partial \sigma^{jl}}{\partial x_k} dx_k dw_s^l + \int\limits_0^t \int\limits_{X_s(\Gamma)} \sum_{k=1}^n \left(\sum_{i,l=1} \frac{\partial F^k}{\partial x_i} \sigma^{il} \right) dx_k dW_s^l.$$

Proof of Proposition 4.1.1. I will denote division of contour Γ as follows $\pi = \{z_j\}_{j=0}^k$, where $d(\pi) = \max_{j=0,\dots,k-1} |z_{j+1} - z_j|$-diameter of division. Then I have:

$$\int\limits_{X_t(\Gamma)} \sum_{k=1}^n F^k(t,x)dx_k = \lim_{d(\pi)\to 0} \sum_{k,j=1}^n F^k(t, X_t(z_j))(X_t^k(z_{j+1}) - X_t^k(z_j)) =$$
$$\lim_{d(\pi)\to 0} \sum_{k,j=1}^n \left(F^k(0, z_j)(z_{j+1} - z_j) + \int\limits_0^t (X_s^k(z_{j+1}) - X_s^k(z_j)) \right. \tag{4.1.2}$$

$$d(F^k(s,X_s(z_j))) + \int_0^t F^k(s.X_s(z_j))d(X^k_s(z_{j+1}) - X^k_s(z_j)) +$$
$$+\tfrac{1}{2}\langle X^k_\cdot(z_{j+1}) - X^k_\cdot(z_j), F^k(\cdot, X_\cdot(z_j))\rangle_t) = (i) + (ii) + (iii) + (iv)$$

where I have used formula $X_tY_t = X_0Y_0 + \int_0^t X_sdY_s \int_0^t Y_sdX_s + \frac{1}{2}\langle X, Y\rangle_t$([44]). Below I will consider 4 terms of equality (4.1.2) separately: $(i) = \int_\Gamma \sum_{k=1}^n F^k(0,x)dx_k$

$$(ii) = \lim_{d(\pi)\to 0} \sum_{k,j=1}^n \int_0^t (X^k_s(z_{j+1}) - X^k_s(z_j))d(F^k(s,X_s(z_j))) = \lim_{d(\pi)\to 0} \sum_{k,j=1}^n$$
$$\int_0^t (X^k_s(z_{j+1}) - X^k_s(z_j))(\frac{\partial F^k}{\partial t} + \sum_{i=1}^n \frac{\partial F^k}{\partial x_i}a^i + \frac{1}{2}\sum_{l,m=1}^n \frac{\partial^2 F^k}{\partial x_l \partial x_m}\sum_{p=1}^n \sigma^{lp}\sigma^{mp})$$
$$(s,X_s(z_j))ds + \int_0^t (X^k_s(z_{j+1}) - X^k_s(z_j))(\sum_{l,i=1}^n \frac{\partial F^k}{\partial x_i}\sigma^{il}(s,X_s(z_j))dw^l_s) =$$
$$\sum_{k=1}^n \int_0^t \int_{X_s(\Gamma)} (\frac{\partial F^k}{\partial t} + \sum_{i=1}^n \frac{\partial F^k}{\partial x_i}a^i + \frac{1}{2}\sum_{l,m=1}^n \frac{\partial^2 F^k}{\partial x_l \partial x_m}\sum_{p=1}^n \sigma^{lp}\sigma^{mp})(s,x)dx_k ds +$$
$$\sum_{k=1}^n \int_0^t \int_{X_s(\Gamma)} (\sum_{l,i=1}^n \frac{\partial F^k}{\partial x_i}\sigma^{il}(s,x)dx_k dw^l_s)$$

$$(iii) = \lim_{d(\pi)\to 0} \sum_{k,j=1}^n \int_0^t F^k(s,X_s(z_j))((a^k(s,X^k_s(z_{j+1})) - a^k(s,X^k_s(z_j)))ds +$$
$$(\sigma^{kl}(s,X^k_s(z_{j+1})) - \sigma^{kl}(s,X^k_s(z_j)))dw^l_s) =$$
$$= \sum_{k=1}^n \int_0^t \int_{X_s(\Gamma)} F^k(s,x)d_xa^k(s,x)ds + \sum_{k=1}^n \int_0^t \int_{X_s(\Gamma)} F^k(s,x)d_x\sigma^{kl}(s,x)dw^l_s$$

$$(iv) = \lim_{d(\pi)\to 0} \sum_{k,j=1}^n \frac{1}{2}\langle X^k_\cdot(z_{j+1}) - X^k_\cdot(z_j), F^k(\cdot, X_\cdot(z_j))\rangle_t = \frac{1}{2}\lim_{d(\pi)\to 0} \sum_{k,j=1}^n$$

$$\langle \int_0^{\cdot} (\sigma^{kl}(s, X_s^k(z_{j+1})) - \sigma^{kl}(s, X_s^k(z_j)))dw_s^l, \int_0^{\cdot} \sum_{i,m=1}^{n} \frac{\partial F^k}{\partial x_i} \sigma^{im}(s, X_s(z_j))dw_s^m \rangle_t =$$

$$= \frac{1}{2} \lim_{d(\pi)\to 0} \sum_{k,j=1}^{n} \int_0^t \sum_{i,l=1}^{n} \frac{\partial F^k}{\partial x_i} \sigma^{il}(s, X_s(z_j))(\sigma^{kl}(s, X_s^k(z_{j+1})) - \sigma^{kl}(s, X_s^k(z_j)))ds =$$

$$= \sum_{k=1}^{n} \int_0^t \int_{X_s(\Gamma)} \sum_{i,l=1}^{n} \frac{\partial F^k}{\partial x_i} \sigma^{il}(s, x) d_x \sigma^{kl}(s, x) ds.$$

where I have used boundedness of corresponding expressions on arbitrary closed contour $\Gamma \subset \mathbb{R}^n$ to exchange integral and limit signs. □

Corollary 4.1.2. *Define flow (X_t^s), $0 \le s \le t \le T$ by equality*

$$\begin{aligned} dX_t^s(x) &= v(t, X_t^s(x))dt + \sqrt{2\nu}dW_t \quad (4.1.3)\\ X_s^s(x) &= x \end{aligned}$$

Let F be a solution of the following linear equation [1,2]

$$\frac{\partial F}{\partial t} = -\nu A_0 F + P((v(T-t)\nabla)F - \nabla F v(T-t)) \quad (4.1.4)$$

$$F(0, \cdot) = F_0 \quad (4.1.5)$$

where $F_0 \in C_0^\infty(\mathbb{R}^n)$, $v \in C_0^\infty([0,T] \times \mathbb{R}^n)$. Then $M_t = \int_{X_t^{T-s}(\Gamma)} \sum_{k=1}^{n} F^k(T-t)dx_k$, $t \in [T-s, T]$ is a local martingale w.r.t. t.

Proof of Corollary 4.1.2. It follows immediately from Proposition 4.1.1. □

This fact is generalization of Kelvin Theorem see e.g. [59], p.26. Indeed, in the case $\nu = 0$ local martingale M_t is a constant and $X_t^s(x)$ is a position of a particle at time t

[1] *which coincides with equation 3.3.1 in the case $n = 3$*

[2] *In this chapter we say that F is a solution of PDE if $F \in C_{t,x}^{1,2}$ and it satisfies PDE in the classical sense*

starting from point x at time s. Now, I have following formula of Feynman-Kac type for solution of equation (4.1.4).

Proposition 4.1.3. *Assume that* $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ *is a solution of equation* (4.1.4), *flow* (X_t^s), $0 \le s \le t \le T$ *is given by* (4.1.3) *and conditions of corollary* (4.1.2) *are satisfied. Assume also that for some* $\beta > 0$ *and any smooth closed loop* Γ

$$\mathbb{E}| \int\limits_{X_t^{T-s}(\Gamma)} F^k(T-t,x)dx_k|^{1+\beta} < \infty. \tag{4.1.6}$$

Then

$$F(s,x) = P(\mathbb{E}(F_0(X_T^{T-s}(x))\nabla X_T^{T-s}(x))) \tag{4.1.7}$$

Proof of Proposition 4.1.3. Define

$$M_t^s = \int\limits_{X_t^{T-s}(\Gamma)} \sum_{k=1}^{3} F^k(T-t)dx_k. \tag{4.1.8}$$

The process $(M_t^s), t \in [T-s,T]$ is a local martingale by the Corollary 4.1.2. Therefore, by the uniform integrability condition (4.1.6) I infer that M_t^s is martingale. Then I can take mathematical expectation and get $\mathbb{E}M_T^s = \mathbb{E}M_s^s$ and the result follows. □

Remark 4.1.4. Condition (4.1.6) is satisfied if, for instance, $F \in L^\infty([0,T] \times \mathbb{R}^n)$ and

$$\int\limits_0^T |\nabla v|_{L^\infty}(s)ds < \infty.$$

Remark 4.1.5. Another method of proving formula (4.1.7) is presented in the article of [18], see also literature therein. Their approach based upon extension of standard Feynman-Kac formula for parabolic equation on more general system of linear parabolic equations with potential term (system 3.2,p.306 of [18]). Extension is carried on by means of the method of new variables introduced by Krylov [48]. Moreover, in [18] formula (4.1.7) is used to prove local existence and uniqueness result for Navier-Stokes equations. The idea of generalization of Kelvin Theorem is taken from [62], see also [23].

In connection with formula (4.1.7) I can pose the following question. Is the flow (X_t^s), $0 \leq s \leq t \leq T$ given by (4.1.3) the only possible flow which give us a solution of equation (4.1.4) by means of formula (4.1.7)? The answer on this question is negative. In next two paragraphs I will consider separately examples for dimensions $n = 2$ and $n = 3$.

4.2 The case n=2

Proposition 4.2.1. *Suppose that* $v \in C_0^\infty([0,T] \times \mathbb{R}^2, \mathbb{R}^2)$, $\psi, \psi^{-1} \in C^1(\mathbb{R}, \mathbb{R})$, $\phi = \psi(\operatorname{rot} v)$, $F_0 \in C_0^\infty(\mathbb{R}^2)$. *Let* (X_t^s), $0 \leq s \leq t \leq T$ *be the stochastic flow corresponding to*

$$\begin{aligned} dX_t^s(x) &= v(t, X_t^s(x))dt + \sqrt{2\nu}\sigma_1(X_t^s(x))dW_t, \qquad (4.2.1) \\ X_s^s(x) &= x, x \in \mathbb{R}^2, \end{aligned}$$

where

$$\sigma_1(x) = \begin{pmatrix} \cos\phi(x) & -\sin\phi(x) \\ \sin\phi(x) & \cos\phi(x) \end{pmatrix}, x \in \mathbb{R}^2.$$

Assume that $F \in C^{1,2}([0,T] \times \mathbb{R}^n, \mathbb{R}^n)$ *is a solution of equation* (4.1.4) *such that for some* $\beta > 0$ *and any smooth closed loop* Γ *condition* (4.1.6) *is satisfied. Then formula* (4.1.7) *is satisfied.*

Proof of Proposition 4.2.1. Suppose that condition (4.1.6) is fulfilled. Then, it is enough to show that process $(M_t^s), t \in [T-s, T]$ defined by formula (4.1.8) (with the flow $(X_t^s), 0 \leq s \leq t \leq T$ generated by (4.2.1)) is a local martingale. We have, for $t \in [0, T]$

$$\int\limits_{X_t^{T-s}(\Gamma)} \sum_{k=1}^n F^k(T-t, x)dx_k = \int\limits_\Gamma \sum_{k=1}^n F^k(s,x)dx_k + \int\limits_{T-s}^t \int\limits_{X_\tau^{T-s}(\Gamma)} \sum_{k=1}^n \left(\frac{\partial F^k}{\partial t} + \right.$$

$$\sum_{j=1}^{n} v^j (\frac{\partial F^k}{\partial x_j} - \frac{\partial F^j}{\partial x_k}) + \nu \sum_{i,j=1}^{n} \frac{\partial^2 F^k}{\partial x_i \partial x_j} \sum_{m=1}^{n} \sigma_1^{im} \sigma_1^{jm} \Bigg) dx_k d\tau +$$

$$+\nu \int_{T-s}^{t} \int_{X_\tau^{T-s}(\Gamma)} \sum_{k=1}^{n} \left(\sum_{j,l} \frac{\partial F^j}{\partial x_l} \sum_{m} \sigma_1^{lm} \frac{\partial \sigma_1^{jm}}{\partial x_k} \right) dx_k d\tau + \sqrt{2\nu} \int_{T-s}^{t} \int_{X_\tau^{T-s}(\Gamma)}$$

$$\sum_{k,j=1}^{n} F^j (T-\tau, x) \frac{\partial \sigma_1^{jl}}{\partial x_k} dx_k dw_\tau^l + \sqrt{2\nu} \int_{T-s}^{t} \int_{X_\tau^{T-s}(\Gamma)} \sum_{k=1}^{n} \left(\sum_{i,l=1} \frac{\partial F^k}{\partial x_i} \sigma_1^{il} \right) dx_k dW_\tau^l.$$

Hence, because σ_1 is an orthogonal matrix and F satisfies equation (4.1.4) I have that

$$\frac{\partial F^k}{\partial t} + \sum_{j=1}^{n} v^j (\frac{\partial F^k}{\partial x_j} - \frac{\partial F^j}{\partial x_k}) + \nu \sum_{i,j=1}^{n} \frac{\partial^2 F^k}{\partial x_i \partial x_j} \sum_{m=1}^{n} \sigma_1^{im} \sigma_1^{jm} =$$

$$\frac{\partial F^k}{\partial t} + \sum_{j=1}^{n} v^j (\frac{\partial F^k}{\partial x_j} - \frac{\partial F^j}{\partial x_k}) + \nu \triangle F^k = \frac{\partial p}{\partial x_k}.$$

Therefore, it is enough to show that

$$\int_{T-s}^{t} \int_{X_\tau^{T-s}(\Gamma)} \sum_{k=1}^{n} \left(\sum_{j,l} \frac{\partial F^j}{\partial x_l} \sum_{m} \sigma_1^{lm} \frac{\partial \sigma_1^{jm}}{\partial x_k} \right) dx_k d\tau = 0.$$

Since σ_1 is orthogonal matrix we have that

$$\sum_{m} \sigma_1^{lm}(x) \sigma_1^{jm}(x) = \delta_{lj}. \tag{4.2.2}$$

One can differentiate (4.2.2) w.r.t. x_k and get

$$\sum_{m} \sigma_1^{lm} \frac{\partial \sigma_1^{jm}}{\partial x_k} = - \sum_{m} \sigma_1^{jm} \frac{\partial \sigma_1^{lm}}{\partial x_k}. \tag{4.2.3}$$

Thus, as $n = 2$, it means that it is enough to calculate

$$\sum_{m} \sigma_1^{1m} \frac{\partial \sigma_1^{2m}}{\partial x_k}.$$

We have

$$\sum_m \sigma_1^{1m} \frac{\partial \sigma_1^{2m}}{\partial x_k} =$$

$$\cos\phi \frac{\partial}{\partial x_k}(\sin\phi) - \sin\phi \frac{\partial}{\partial x_k}(\cos\phi) = \frac{\partial \phi}{\partial x_k}$$

and, therefore,

$$\int_{T-s}^{t} \int_{X_\tau^{T-s}(\Gamma)} \sum_{k=1}^{n} \left(\sum_{j,l} \frac{\partial F^j}{\partial x_l} \sum_m \sigma_1^{lm} \frac{\partial \sigma_1^{jm}}{\partial x_k} \right) dx_k d\tau =$$

$$\int_{T-s}^{t} \int_{X_\tau^{T-s}(\Gamma)} (\frac{\partial F^1}{\partial x_2} - \frac{\partial F^2}{\partial x_1}) d\phi d\tau = \int_{T-s}^{t} \int_{X_\tau^{T-s}(\Gamma)} \psi^{-1}(\phi) d\phi d\tau = 0.$$

□

Remark 4.2.2. The construction of Proposition 4.2.1 can be easily generalized to the case $n = 3$ in the following way. Suppose

$$\begin{aligned} dX_t^s(x) &= v(t, X_t^s(x))dt + \sqrt{2\nu}\sigma_1(X_t^s(x))dW_t, 0 \le s \le t \le T \qquad (4.2.4) \\ X_s^s(x) &= x, x \in \mathbb{R}^3 \end{aligned}$$

where

$$\sigma_1(x) = \begin{pmatrix} \cos\phi(x) & -\sin\phi(x) & 0 \\ \sin\phi(x) & \cos\phi(x) & 0 \\ 0 & 0 & 1 \end{pmatrix}, x \in \mathbb{R}^3,$$

and $\phi = \psi((\text{curl } v)^1)$, ψ^{-1}, $\psi \in C^1(\mathbb{R}, \mathbb{R})$ (also similar construction can be made for other components of the curl v). Truly three dimensional rotations σ_1 will be considered in next paragraph.

Remark 4.2.3. In this example I notice that the law of the flow given by (4.2.1) and the law of canonical flow (4.1.3) are the same. Next example shows that it is possible to find a flow which have got a law of wiener process.

Proposition 4.2.4. *Suppose that v is of C_0^∞ class and* $\operatorname{div} v = 0$. *Define for each $x \in \mathbb{R}^2$, $s \in [0,T]$ the stochastic flow $X_t^s(x), t \in [s,T]$ corresponding to the solution of equation*

$$\begin{aligned} dX_t^s(x) &= \sqrt{2\nu}\sigma_1(X_t^s(x))dW_t, 0 \le s \le t \le T \\ X_s^s(x) &= x, x \in \mathbb{R}^2 \end{aligned} \tag{4.2.5}$$

where

$$\sigma_1(x) = \begin{pmatrix} \cos\frac{\phi(x)}{\nu} & -\sin\frac{\phi(x)}{\nu} \\ \sin\frac{\phi(x)}{\nu} & \cos\frac{\phi(x)}{\nu} \end{pmatrix}, x \in \mathbb{R}^2,$$

ϕ defined by $v = \nabla^\perp\phi$(such ϕ exists because of $\operatorname{div} v = 0$*). Assume also that $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a solution of equation* (4.1.4) *(with $F_0 \in C_0^\infty(\mathbb{R}^2)$) such that for some $\beta > 0$ and any smooth closed loop Γ condition* (4.1.6) *is satisfied. Then formula* (4.1.7) *is satisfied.*

Proof of Proposition 4.2.4. Similarly to Proposition 4.2.1 I get that $\int\limits_{X_t^{T-s}(\Gamma)} \sum\limits_{k=1}^{2} F^k(T-t)dx_k, t \in [T-s,T]$ is a local martingale. Indeed, correction term in (4.1.1) due to rotation of Brownian Motion is equal to $\int\limits_{T-s}^{t} \int\limits_{X_\tau^{T-s}(\Gamma)} (\frac{\partial F^1}{\partial x_2} - \frac{\partial F^2}{\partial x_1})d\phi ds$, see previous Proposition, and if $v = \nabla^\perp\phi$ this is exactly first order term of two dimensional equation (4.1.4). □

Remark 4.2.5. Under assumptions of Proposition 4.2.4 formula (4.1.7) can be simplified as follows

$$F(s,x) = P(\mathbb{E}(F_0(x + \sqrt{2\nu}(W_T - W_{T-s}))\nabla X_T^{T-s}(x))) \tag{4.2.6}$$

where $X_t^s(x))$ is defined by (4.2.5).

Remark 4.2.6. Proposition 4.2.4 and formula (4.1.1) show difference between passive scalar advection and vector advection equations. In case of scalar advection no gradient of flow appears in Feynman-Kac type formula and, correspondingly, solution of scalar advection is completely defined by the law of flow itself. Since rotation of brownian motion cannot influence the law of the flow I cannot do the trick for scalar advection.

Question 4.2.7. In connection to the Proposition 4.2.4 I can pose following question: Prove directly (not through formula (4.1.1)) that the limit $\nu \to 0$ in the formula (4.2.6) exists and converges to the solution of 2D Euler equations?

4.3 The case n=3

I will need following definitions. Let $\hat{\cdot}$ be a linear isomorphism defined by

$$\hat{\cdot} : \mathbb{R}^3 \ni \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \in \mathfrak{so}(3).$$

It is called the hat-map isomorphism. Let also $\exp : \mathfrak{so}(3) \to SO(3)$ be standard exponential map. Define a map $BCH : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathfrak{so}(3)$ by

$$\exp(BCH(\hat{u}, \hat{v})) = \exp(\hat{u}) \exp(\hat{v}), \hat{u}, \hat{v} \in \mathfrak{so}(3).$$

I will now deduce exact type of "correction" which appear in formula (4.1.1).

Proposition 4.3.1. *Define* $\sigma : [0,T] \times \mathbb{R}^3 \to SO(3)$ *by* $\sigma(t,x) = \exp(\widehat{a(t,x)}), t \in [0,T], x \in \mathbb{R}^3$, $a \in C^1([0,T] \times \mathbb{R}^3, \mathbb{R}^3)$. *Then*

$$\sum_m \sigma^{\cdot m} \frac{\partial \sigma^{\cdot m}}{\partial x_k} = (1 - \cos|a|) \widehat{b \times \frac{\partial b}{\partial x_k}} + \sin|a| \widehat{\frac{\partial b}{\partial x_k}} + \hat{b} \frac{\partial |a|}{\partial x_k} \tag{4.3.1}$$

where $\vec{b} = \frac{\vec{a}}{|a|}$.

Proof of Proposition 4.3.1. I will use the following Baker-Campbell-Hausdorff formula in $\mathfrak{so}(3)$, see e.g. [45], p. 630.

Proposition 4.3.2. *If $u, v \in \mathbb{R}^3$ then*

$$BCH(\hat{u}, \hat{v}) = \alpha \hat{u} + \beta \hat{v} + \gamma [\hat{u}, \hat{v}]$$

where $[\hat{u}, \hat{v}]$ denotes the commutator of $\hat{u}$ and $\hat{v}$, and α, β, and γ are real constants defined by

$$\alpha = \frac{\sin^{-1}(d)}{d}\frac{a_1}{\theta}, \beta = \frac{\sin^{-1}(d)}{d}\frac{b_1}{\phi}, \gamma = \frac{\sin^{-1}(d)}{d}\frac{c_1}{\theta\phi},$$

where a_1, b_1, c_1 and d are defined as

$$\begin{aligned} a_1 &= \sin\theta\cos^2(\phi/2) - \sin\phi\sin^2(\theta/2)\cos\angle(u,v), \\ b_1 &= \sin\phi\cos^2(\theta/2) - \sin\theta\sin^2(\phi/2)\cos\angle(u,v), \\ c_1 &= \frac{1}{2}\sin(\theta)\sin(\phi) - 2\sin^2(\theta/2)\sin^2(\phi/2)\cos\angle(u,v), \\ d &= \sqrt{a_1^2 + b_1^2 + 2a_1 b_1\cos\angle(u,v) + c_1^2\sin^2\angle(u,v)}. \end{aligned}$$

In the above formulae $\theta = |u|$, $\phi = |v|$, and $\angle(u,v)$ is the angle between the two vectors u and v.

I have

$$\begin{aligned} \sum_m \sigma^{\cdot m}\frac{\partial \sigma^{\cdot m}}{\partial x_k} = \exp(-\hat{a})\frac{\partial}{\partial x_k}\exp(\hat{a}) = \exp(-\hat{a}) \times \\ \lim_{\delta\to 0}\frac{1}{\delta}(\exp(\hat{a}(x+\delta e_k)) - \exp(\hat{a}(x))) = \\ \lim_{\delta\to 0}\frac{1}{\delta}(\exp(-\hat{a})\exp(\hat{a}(x+\delta e_k)) - \mathrm{id}) = \\ \lim_{\delta\to 0}\frac{1}{\delta}(\exp(BCH(-\hat{a}, \hat{a}(x+\delta e_k))) - \mathrm{id}) = \lim_{\delta\to 0}\frac{BCH(-\hat{a}, \hat{a}(x+\delta e_k))}{\delta} = \\ \lim_{\delta\to 0}\frac{\alpha(\delta)(-\hat{a}(x)) + \beta(\delta)\hat{a}(x+\delta e_k) + \gamma(\delta)[-\hat{a}(x), \hat{a}(x+\delta e_k)]}{\delta} = (*) \end{aligned}$$

where in the last equality I have used Proposition 4.3.1 with $u = -\hat{a}(x)$, $v = \hat{a}(x+\delta e_k)$.

Therefore,

$$(*) = \lim_{\delta\to 0}\beta(\delta)\frac{\hat{a}(x+\delta e_k)-\hat{a}}{\delta} + \hat{a}(x)\lim_{\delta\to 0}\frac{\beta(\delta)-\alpha(\delta)}{\delta} -$$

$$\lim_{\delta\to 0}\gamma(\delta)[\hat{a}(x), \frac{\hat{a}(x+\delta e_k)-\hat{a}(x)}{\delta}] = \frac{\partial\hat{a}}{\partial x_k}\lim_{\delta\to 0}\beta(\delta) + \hat{a}\lim_{\delta\to 0}\frac{\beta(\delta)-\alpha(\delta)}{\delta} -$$

$$\widehat{(a\times\frac{\partial a}{\partial x_k})}\lim_{\delta\to 0}\gamma(\delta)$$

So, I need to calculate

$$(i) = \lim_{\delta\to 0}\beta(\delta), (ii) = \lim_{\delta\to 0}\frac{\beta(\delta)-\alpha(\delta)}{\delta}, (iii) = \lim_{\delta\to 0}\gamma(\delta).$$

From (4.3.2) follows that I need to calculate asymptotics of $a_1(\delta), b_1(\delta), c_1(\delta), d(\delta), \delta\to 0$.

I have

$$\theta = |a|(x), \phi = |a|(x+\delta e_k) = |a|(x) + \delta\frac{\partial}{\partial x_k}|a| + \underline{o}(\delta),$$

$$\cos(\angle(u,v)) = \frac{(-a(x), a(x+\delta e_k))}{|a|(x)|a|(x+\delta e_k)} = -1 + \bar{o}(\delta^2)$$

$$a_1 = \sin|a|(\frac{1+\cos|a|(x+\delta e_k)}{2}) - \sin|a|(x+\delta e_k)\times$$
$$(\frac{1-\cos|a|(x)}{2})(-1+\bar{o}(\delta^2)) =$$
$$\sin|a|(\frac{1+\cos(|a|+\delta\frac{\partial}{\partial x_k}|a|)}{2}) + (\frac{1-\cos|a|}{2})\times$$
$$\sin(|a|+\delta\frac{\partial}{\partial x_k}|a|) + \bar{o}(\delta^2) =$$
$$\frac{\sin|a|}{2}(1+\cos|a| - \sin|a|\frac{\partial}{\partial x_k}|a|\delta) +$$
$$(\frac{1-\cos|a|}{2})(\sin|a| + \cos|a|\frac{\partial}{\partial x_k}|a|\delta) + \bar{o}(\delta^2) =$$
$$= \sin|a|(x) - \frac{1}{2}(1-\cos|a|)\frac{\partial}{\partial x_k}|a|\delta + \bar{o}(\delta^2) \qquad (4.3.2)$$

Similarly,

$$\begin{aligned} b_1 = \sin|a|(x+\delta e_k)(\frac{1+\cos|a|}{2}) - \sin|a| \times \\ (\frac{1-\cos|a|(x+\delta e_k)}{2})(-1+\bar{o}(\delta^2)) = \\ \sin(|a|+\delta\frac{\partial}{\partial x_k}|a|)(\frac{1+\cos|a|}{2}) + \\ \sin|a|(\frac{1-\cos(|a|+\delta\frac{\partial}{\partial x_k}|a|)}{2}) + \bar{o}(\delta^2) = \\ (\sin|a| + \cos|a|\frac{\partial}{\partial x_k}|a|\delta)(\frac{1+\cos|a|}{2}) + \\ \frac{1}{2}\sin|a|(\cos|a| - \delta\sin|a|\frac{\partial}{\partial x_k}|a|) + \bar{o}(\delta^2) = \\ \sin|a| + \frac{1}{2}(1+\cos|a|)\frac{\partial}{\partial x_k}|a|\delta + \bar{o}(\delta^2) \end{aligned} \tag{4.3.3}$$

$$c_1 = 1 - \cos|a| + \bar{o}(\delta) \tag{4.3.4}$$

$$d = \bar{o}(\delta) \tag{4.3.5}$$

From (4.3.2), (4.3.3), (4.3.4) and (4.3.5) I get

$$(iii) = \lim_{\delta\to 0}\frac{\sin^{-1}(d)}{d}\frac{c_1}{|a|(x)|a|(x+\delta e_k)} = \frac{1-\cos|a|}{|a|^2}$$

$$(i) = \lim_{\delta\to 0}\frac{\sin^{-1}(d)}{d}\frac{b_1}{|a|(x+\delta e_k)} = \frac{\sin|a|}{|a|}$$

$$\begin{aligned} (ii) = \lim_{\delta\to 0}\frac{\sin^{-1}(d)}{d}\frac{1}{\delta}(\frac{\sin|a| + \frac{1}{2}(1+\cos|a|)\frac{\partial}{\partial x_k}|a|\delta + \bar{o}(\delta^2)}{|a|(x+\delta e_k)} - \\ \frac{\sin|a|(x) - \frac{1}{2}(1-\cos|a|)\frac{\partial}{\partial x_k}|a|\delta + \bar{o}(\delta^2)}{|a|}) = \frac{|a|-\sin|a|}{|a|^2}\frac{\partial}{\partial x_k}|a| \end{aligned}$$

Thus, I get

$$\exp(-\hat{a})\frac{\partial}{\partial x_k}\exp(\hat{a}) = \frac{\sin|a|}{|a|}\frac{\partial\hat{a}}{\partial x_k} + \frac{|a|-\sin|a|}{|a|^2}\frac{\partial}{\partial x_k}|a|\hat{a} + \tag{4.3.6}$$

$$\frac{\cos|a|-1}{|a|^2}\widehat{a \times \frac{\partial a}{\partial x_k}} \tag{4.3.7}$$

If I put $b = \frac{a}{|a|}$ and insert it in (4.3.7) I get (4.3.1). □

Corollary 4.3.3. *Let* (X_t^s), $0 \leq s \leq t \leq T$ *be the stochastic flow corresponding to*

$$\begin{aligned} dX_t^s(x) &= v(t, X_t^s(x))dt + \sqrt{2\nu}\sigma_1(t, X_t^s(x))dW_t, \\ X_s^s(x) &= x, x \in \mathbb{R}^3 \end{aligned} \tag{4.3.8}$$

where $\sigma_1(t,x) = \exp(\hat{a})(t,x)$, $b = \frac{a}{|a|} \in S(2)$. *Then I have*

$$\begin{aligned} \int\limits_{X_t^{T-s}(\Gamma)} \sum_{k=1}^{n} F^k(T-t,x)dx_k = \int\limits_{\Gamma} \sum_{k=1}^{n} F^k(s,x)dx_k + \int\limits_{T-s}^{t} \int\limits_{X_\tau^{T-s}(\Gamma)} \sum_{k=1}^{n} \Bigg(\frac{\partial F^k}{\partial t} + \\ \sum_{j=1}^{n} v^j (\frac{\partial F^k}{\partial x_j} - \frac{\partial F^j}{\partial x_k}) + \nu \triangle F^k \Bigg) dx_k d\tau + \\ +\nu \int\limits_{T-s}^{t} \int\limits_{X_\tau^{T-s}(\Gamma)} (\operatorname{curl} F, (1-\cos|a|) b \times \frac{\partial b}{\partial x_k} + \sin|a| \frac{\partial b}{\partial x_k} + b \frac{\partial |a|}{\partial x_k}) dx_k d\tau + \\ \sqrt{2\nu} \int\limits_{T-s}^{t} \int\limits_{X_\tau^{T-s}(\Gamma)} \sum_{k=1}^{n} \Bigg(\sum_{i,l=1} (\frac{\partial F^k}{\partial x_i} - \frac{\partial F^i}{\partial x_k}) \sigma_1^{il} \Bigg) dx_k dW_\tau^l. \end{aligned} \tag{4.3.9}$$

Proof of Corollary 4.3.3. Immediately follows from Proposition 4.3.1 and identity

$$\sum_{i,j} \frac{\partial F^i}{\partial x_j} (\hat{a})^{ij} = (\operatorname{curl} F, a).$$

□

Remark 4.3.4. Vector b has physical meaning of an axis of rotation σ and $\phi = |a|$ is an angle of rotation.

Now, I will give three dimensional analog of two dimensional Proposition (4.2.1).

Proposition 4.3.5. *Let* (X_t^s), $0 \leq s \leq t \leq T$ *be the stochastic flow corresponding to*

$$\begin{aligned} dX_t^s(x) &= v(t, X_t^s(x))dt + \sqrt{2\nu}\sigma_1(t, X_t^s(x))dW_t, \\ X_s^s(x) &= x \end{aligned} \tag{4.3.10}$$

where $\sigma_1(t,x) = \exp(\hat{a})(t,x)$, $a = \operatorname{curl} F$. *Assume also that* $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ *is a solution of equation* (4.1.4) *(with* $F_0 \in C_0^\infty(\mathbb{R}^n)$*) such that for some* $\beta > 0$ *and any smooth closed loop* Γ *condition* (4.1.6) *is satisfied. Then the formula* (4.1.7) *holds true.*

Proof of Proposition 4.3.5. I have $b = \frac{\operatorname{curl} F}{|\operatorname{curl} F|}$, $|b| = 1$, $|a| = |\operatorname{curl} F|$ and therefore

$$(\operatorname{curl} F, \frac{\partial b}{\partial x_k}) = |\operatorname{curl} F|(b, \frac{\partial b}{\partial x_k}) = 0.$$

Similarly,

$$(\operatorname{curl} F, b \times \frac{\partial b}{\partial x_k}) = |\operatorname{curl} F|(b, b \times \frac{\partial b}{\partial x_k}) = 0,$$

and

$$(\operatorname{curl} F, b)\frac{\partial |\operatorname{curl} F|}{\partial x_k} = \frac{1}{2}\frac{\partial |\operatorname{curl} F|^2}{\partial x_k}$$

□

Question 4.3.6. It would be interesting to generalize of Proposition 4.2.4 to the three dimensional case. In view of Corollary 4.3.3 in order to find such generalization it is enough to prove that there exists a triple

$$(b, \phi, \psi) \in (L^\infty([0,T], C^\infty(\mathbb{R}^3, S^2)), L^\infty([0,T], C^\infty(\mathbb{R}^3, S^1)), L^\infty([0,T], C^\infty(\mathbb{R}^3, \mathbb{R})))$$

such that

$$(\cos\phi - 1)(\operatorname{curl} F, b \times \frac{\partial b}{\partial x_k}) + \sin\phi(\operatorname{curl} F, \frac{\partial b}{\partial x_k}) + (\operatorname{curl} F, b)\frac{\partial \phi}{\partial x_k} + \\ + \frac{\partial \psi}{\partial x_k} = \frac{(v \times \operatorname{curl} F)^k}{\nu}, k = 1, 2, 3, \qquad (4.3.11)$$

where F is a solution of equation (4.1.4), ν -corresponding parameter (here I suppose that $v \in C^\infty$). I notice that system (4.3.11) is time independent and, therefore it is enough to consider the system for every fixed time $t \in [0,T]$. If v is two dimensional (i.e. $v_3 = 0$,

v_1, v_2 does not depend upon x_3) and $\operatorname{div} v = 0$ than $b = (0,0,1)$, $\phi = \phi_1/\nu$, where ϕ_1 is a stream function for v, $\psi = 0$ (see Proposition 4.2.4). In three dimensional case the problem is completely open. One of the possibilities to narrow the problem is to consider the case when $F = u$ corresponding to the case of Navier-Stokes equations.

Question 4.3.7. Here is another question connected with system (4.3.11). How variables b, ϕ, ψ depend upon ν? Can I tend ν to 0 in representation (4.3.11)? In two dimensional case and under condition of incompressibility $\operatorname{div} v = 0$ representation (4.3.11) holds also in the limit of $\nu = 0$. Indeed, stream function for v (which exists because $\operatorname{div} v = 0$) in two dimensional case is independent of F and ν.

Remark 4.3.8. I can reformulate system (4.3.11) in the following way.

Let (α, β) be the following parametrization of vector $b \in S^2$

$$b = (\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha), \quad \alpha \in S^1 \cong (0, 2\pi], \beta \in S^1 \cong (0, 2\pi]$$

and denote

$$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (\alpha, \beta, \phi), \quad \alpha, \beta, \phi \in S^1 \cong (0, 2\pi].$$

Then system (4.3.11) can be written as follows:

$$\sum_i \Psi_i(\vec{\alpha}, \operatorname{curl} F)\frac{\partial \alpha_i}{\partial x_k} + \frac{\partial \psi}{\partial x_k} = \frac{(v \times \operatorname{curl} F)^k}{\nu}, k = 1, 2, 3 \tag{4.3.12}$$

where

$$\Psi_1(\vec{\alpha}, \omega) = (\cos\alpha_3 - 1)(\omega_2\cos\alpha_2 - \omega_1\sin\alpha_2) - \sin\alpha_3\sin\alpha_1 \times (\omega_1\cos\alpha_2 + \omega_2\sin\alpha_2) + \omega_3\cos\alpha_1\sin\alpha_3, \tag{4.3.13}$$

$$\Psi_2(\vec{\alpha}, \omega) = (\cos\alpha_3 - 1)(\frac{\sin 2\alpha_1}{2}(\omega_2\sin\alpha_2 - \omega_1\cos\alpha_2) - \omega_3\cos^2\alpha_1) - \sin\alpha_3\cos\alpha_1(\omega_1\sin\alpha_2 + \omega_2\cos\alpha_2), \tag{4.3.14}$$

$$\Psi_3(\vec{\alpha}, \omega) = \omega_1\cos\alpha_1\cos\alpha_2 + \omega_2\cos\alpha_1\sin\alpha_2 + \omega_3\sin\alpha_1. \tag{4.3.15}$$

Thus, roughly speaking, locally (in the region of point x_0 where $\operatorname{curl} F(x) \cong \operatorname{curl} F(x_0)$) our system (4.3.11) reduces to existence of change of coordinates $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\sum_i \Psi_i(\vec{\alpha}, \operatorname{curl} F(x))d\alpha_i = \sum_k \frac{(v \times \operatorname{curl} F)^k(x)dx_k}{\nu} + dv$$

i.e. two forms $\sum_i \Psi_i(y, \operatorname{curl} F(x))dy_i$ and $\sum_k \frac{(v\times \operatorname{curl} F)^k(x)dx_k}{\nu}$ are locally equal up to change of coordinates and adding exact form.

4.4 Another Feynman-Kac type formula

Another application of Proposition 4.1.1 is Feynman-Kac type formula for solutions of equation:

$$\begin{cases} \dfrac{\partial F}{\partial t} = -\nu A_0 F + (v(T-\cdot)\cdot\nabla)F - (F\cdot\nabla)v(T-\cdot), \\ F(0) = F_0, t>0, x\in\mathbb{R}^n, \end{cases} \tag{4.4.1}$$

where A_0 is a Stokes operator, $F_0 \in H$, v satisfies condition (3.1.14). For simplicity I formulate the result for $n = 3$.

Proposition 4.4.1. *If* (X_t^s), $0 \le s \le t < \infty$ *is the flow corresponding to problem* (4.1.3) *such that there exists* $\beta > 0$:

$$\mathbb{E}\Big|\int_{X_t^{T-s}(S)} F^1(T-t,x)dx_2dx_3 + F^2(T-t,x)dx_3dx_1 + F^3(T-t,x)dx_1dx_3\Big|^{1+\beta} < \infty \tag{4.4.2}$$

for any smooth surface $S \subset \mathbb{R}^3$ *with smooth boundary* Γ *and all* $0 \le T-s \le t \le T$. *Then the solution of equation* (4.4.1) *with* $F_0 \in C_0^\infty(\mathbb{R}^n)$, $v \in C_0^\infty([0,T]\times\mathbb{R}^n)$ *satisfies*

$$F^1(s,x) = \mathbb{E}[F_0^1(X_T^{T-s}(x))(\frac{\partial X_T^{T-s,2}}{\partial x_2}\frac{\partial X_T^{T-s,3}}{\partial x_3} - \frac{\partial X_T^{T-s,2}}{\partial x_3}\frac{\partial X_T^{T-s,3}}{\partial x_2})$$

$$+F_0^2(X_T^{T-s}(x))(\frac{\partial X_T^{T-s,3}}{\partial x_2}\frac{\partial X_T^{T-s,1}}{\partial x_3} - \frac{\partial X_T^{T-s,3}}{\partial x_3}\frac{\partial X_T^{T-s,1}}{\partial x_2})$$

$$+F_0^3(X_T^{T-s}(x))(\frac{\partial X_T^{T-s,1}}{\partial x_2}\frac{\partial X_T^{T-s,2}}{\partial x_3}-\frac{\partial X_T^{T-s,1}}{\partial x_3}\frac{\partial X_T^{T-s,2}}{\partial x_2}) \tag{4.4.3}$$

$$F^2(s,x)=\mathbb{E}[F_0^1(X_T^{T-s}(x))(\frac{\partial X_T^{T-s,2}}{\partial x_3}\frac{\partial X_T^{T-s,3}}{\partial x_1}-\frac{\partial X_T^{T-s,2}}{\partial x_1}\frac{\partial X_T^{T-s,3}}{\partial x_3})$$

$$+F_0^2(X_T^{T-s}(x))(\frac{\partial X_T^{T-s,3}}{\partial x_3}\frac{\partial X_T^{T-s,1}}{\partial x_1}-\frac{\partial X_T^{T-s,3}}{\partial x_1}\frac{\partial X_T^{T-s,1}}{\partial x_3})$$

$$+F_0^3(X_T^{T-s}(x))(\frac{\partial X_T^{T-s,1}}{\partial x_3}\frac{\partial X_T^{T-s,2}}{\partial x_1}-\frac{\partial X_T^{T-s,1}}{\partial x_1}\frac{\partial X_T^{T-s,2}}{\partial x_3}) \tag{4.4.4}$$

$$F^3(s,x)=\mathbb{E}[F_0^1(X_T^{T-s}(x))(\frac{\partial X_T^{T-s,2}}{\partial x_1}\frac{\partial X_T^{T-s,3}}{\partial x_2}-\frac{\partial X_T^{T-s,2}}{\partial x_2}\frac{\partial X_T^{T-s,3}}{\partial x_1})$$

$$+F_0^2(X_T^{T-s}(x))(\frac{\partial X_T^{T-s,3}}{\partial x_1}\frac{\partial X_T^{T-s,1}}{\partial x_2}-\frac{\partial X_T^{T-s,3}}{\partial x_2}\frac{\partial X_T^{T-s,1}}{\partial x_1})$$

$$+F_0^3(X_T^{T-s}(x))(\frac{\partial X_T^{T-s,1}}{\partial x_1}\frac{\partial X_T^{T-s,2}}{\partial x_2}-\frac{\partial X_T^{T-s,1}}{\partial x_2}\frac{\partial X_T^{T-s,2}}{\partial x_1}) \tag{4.4.5}$$

Proof of Proposition 4.4.1. Proof follows from Proposition 4.1.3. Indeed, if G is a solution of equation (4.1.4) then $F = \operatorname{curl} G$ is a solution of (4.4.1). For solution G of (4.1.4) I have got representation by formula (4.1.7) of Feynman-Kac type. Integrating it w.r.t. closed contour Γ I get

$$\int_\Gamma \sum_k G^k(s,x)dx_k = \mathbb{E}(\int_{X_T^{T-s}(\Gamma)} \sum_k G_0^k(x)dx_k). \tag{4.4.6}$$

Now, result immediately follows from Stokes Theorem. □

Remark 4.4.2. Feynman-Kac type formula (4.4.3)-(4.4.5) in the case of $\nu = 0$ degenerates in equation for characteristics, see e.g. [37], of the following infinite dimensional PDE of first order. Denote Y a set of smooth surfaces $S \subset \mathbb{R}^n$ with smooth boundary Γ, TY set of vector fields on Y and

$$\tilde{F} : Y \ni S \mapsto \int_S (F, \vec{n}) d\sigma \in \mathbb{R}.$$

Assume that $v \in C_0^\infty([0,T] \times \mathbb{R}^n)$, F is a solution of equation (4.4.1) with parameters $v, \nu = 0$. Then $\tilde{F}$ satisfy equation

$$\frac{\partial \tilde{F}}{\partial t} = D_{\tilde{v}} \tilde{F} \tag{4.4.7}$$

where $D_{\tilde{v}}$ is directional derivative along with vector field $\tilde{v} \in TY$ given by

$$Y \ni S \mapsto \bigcup_{x \in S} v(x) \in TY,$$

and equation for characteristics of (4.4.7)(i.e. solution of (4.4.7) is constant along with characteristics) is exactly our Feynman-Kac type formula.

Chapter 5

Backward Uniqueness of SPDEs

The aim of this Chapter is to study the asymptotic behaviour for large times of solutions to a certain class of parabolic stochastic partial differential equations. In particular, I will prove the backward uniqueness result and the existence of the spectral limit for abstract SPDEs and then show how these results can be applied to some concrete linear and nonlinear SPDEs. For example, I will consider linear parabolic SPDEs with gradient noise and stochastic NSEs with multiplicative noise. My results generalize the results proved in [42] for deterministic PDEs.

5.1 Backward Uniqueness and existence of Spectral Limit for abstract parabolic SPDE

Let us recall some notation from previous sections. Assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a filtered complete probability space, $(w_t)_{t\geq 0}$ is an $\mathbb{R}^n$-valued Wiener process and $V \subset H = H' \subset V'$ is a Gelfand triple. We assume that $A(t)$, $t \in [0, \infty)$ is a linear bounded operator from V to V' and $B_k(t)$, $k = 1, \cdots, n$, $t \in [0, \infty)$ is linear bounded operator from V to H and from H to V'.

Definition 5.1.1. *I call progressively measurable stochastic process $u_t, t \geq 0$ with values in H a solution of equation*

$$\begin{cases} du + (A(t)u + F(u))dt + \sum_{k=1}^{n} B_k(t)udw_t^k = 0, \\ u(0) = u_0 \in H \end{cases} \tag{5.1.1}$$

if there exists such $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ that for any $t \in [0, T]$, $\omega \in \Omega'$

$$u(t, \omega) = u_0 - \int_0^t (Au(s, \omega) + F(u))ds - \sum_{k=1}^{n} \int_0^t B^k u(s, \omega)dw_s \tag{5.1.2}$$

and $u \in M^2(0, T; V) \cap L^2(\Omega, C([0, T], H))$.

Theorem 5.1.2. *Suppose that the families of operators $A(t)$ and $B^k(t)$, $k = 1, \ldots, n$, $t \in [0, T]$ satisfy the following additional assumptions.*

There exists $\tilde{A}' \in L^1(0, T; \mathcal{L}(V, V'))$ such that for all $\phi \in V, \psi \in V'$

$$\frac{d}{dt} < \tilde{A}(\cdot)\phi, \psi > \ = \ < \tilde{A}'(\cdot)\phi, \psi > . \tag{5.1.3}$$

There exists $\alpha > 0$ and $\lambda \in \mathbb{R}$ such that for all $u \in V$

$$2 < A(\cdot)u, u > +\lambda|u|^2 \geq \alpha||u||^2 + \sum_{k=1}^{n} |B_k(\cdot)u|^2. \tag{5.1.4}$$

There exists $\phi \in L^2(0, T)$ such that for all $u \in V$

$$\sum_{k=1}^{n} | < u, B_k(\cdot)u > | \leq \phi(\cdot)|u|_H^2. \tag{5.1.5}$$

There exists $K_1 \in L^2(0, T)$, $K_2 \in L^1(0, T)$ such that

$$C = \sum_{k=1}^{n} B_k^*[\tilde{A}, B_k] \leq K_1 \operatorname{id} + K_2 \tilde{A}. \tag{5.1.6}$$

There exists $L_1, L_2 > 0$ *such that for all* $x \in D(\tilde{A})$

$$\sum_k ||B_k(\cdot)x||_V \leq L_1|A(\cdot)x|_H + L_2|x|_H. \tag{5.1.7}$$

There exists $\beta, \gamma > 0$ *such that for all* $x \in V$

$$| < A(\cdot)x, x > | \leq \beta||x||^2 + \gamma|x|_H^2. \tag{5.1.8}$$

Assume that $u_0 \in H$, u *is a solution of* (5.1.1), $\exists\delta_0 > 0$ *such that* $u \in M^{2+\delta_0}(0, T; D(\tilde{A}))$, F *satisfies inequality*

$$|F(u)(t)| \leq n(t)||u||_V \text{ for a.a.} t \in [0, T], \tag{5.1.9}$$

$$\exists\kappa(\delta_0) > 2 + \frac{4}{\delta_0} : \mathbb{E}e^{\kappa(\delta_0)\int_0^T n^2(s)ds} < \infty. \tag{5.1.10}$$

Then from $u(T) = 0, \mathbb{P}a.s.$ *(as an element of H) follows that* $u(t) = 0, t \in [0, T]\mathbb{P}a.s.$.

Remark 5.1.3. From Theorem 1.4, p.140 from [63] follows that there exists unique solution u of equation (5.1.1).

Remark 5.1.4. Assumption 5.1.10 is satisfied if, for instance, $n \in L^2(0, T)$.

Proof of Theorem 5.1.2. I will argue by contradiction. Suppose that there exists an event $R \subset \Omega$, $c > 0$, $t_0 \in [0, T)$ such that $\mathbb{P}(R) > 0$ and $|u(t_0, \omega)|_H \geq c > 0, \omega \in R$. Note that $R \in \mathcal{F}_{t_0}$. Therefore, I can without any loss of generality suppose that $\mathbb{P}(R) = 1$. Otherwise, I could consider everywhere below instead of measure $\mathbb{P}$ the conditional measure $\frac{1}{\mathbb{P}(R)}\chi_R(\cdot)\mathbb{P}$ and denote it by the letter $\mathbb{P}$.

We have $u(\cdot, \omega) \in C([t_0, T], H), \mathbb{P} - a.s.$. Consequently, we have alternative:

[i] $|u(t)|_H > 0 \forall t \in (t_0, T]\mathbb{P} - a.s.$

[ii] If I denote $\tau(\omega) = \inf_t\{t \in (t_0, T], |u(t)|_H = 0\}$ then $t_0 < \tau \leq T$ is correctly defined.

In the first case we have a contradiction. Let us consider the second case. To prove the Theorem it is enough to show that there exist probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that for any $t \in (t_0, \tau]$ $\tilde{\mathbb{E}}|u(t)|_H^2 \geq c > 0$ (where $\tilde{\mathbb{E}}$ is a mathematical expectation w.r.t. measure $\mathbb{Q}$). Indeed, by taking the limit as $t \to \tau$ we get contradiction because $u(\tau) = 0$. Define $d\mathbb{Q}^\varepsilon = M_T^\varepsilon d\mathbb{P}$ where $M_t^\varepsilon = \exp(-2\sum_k \int_0^t \frac{<u, B_k u>}{|u|_H^2+\varepsilon} dw_s^k - 2\int_0^t \sum_k \frac{(<u, B_k u>)^2}{(|u|_H^2+\varepsilon)^2} ds)$ - stochastic exponent. We have $\mathbb{E}M_t^\varepsilon = 1$ because of assumption (5.1.5) and

$$dM_t^\varepsilon = -2M_t^\varepsilon \sum_k \frac{< u, B_k u >}{|u|_H^2 + \varepsilon} dw_s^k.$$

I denote $\tilde{\mathbb{E}}^\varepsilon$ mathematical expectation w.r.t. measure $\mathbb{Q}^\varepsilon$. Let

$$\psi^\varepsilon(t) = -\frac{1}{2} M_t^\varepsilon \log(|u(t)|_H^2 + \varepsilon), t \in [0, T].$$

We have

$$d\psi^\varepsilon(t) = -\frac{1}{2}(\log(|u|_H^2 + \varepsilon) dM_t^\varepsilon + M^\varepsilon d\log(|u|_H^2 + \varepsilon) + d < M^\varepsilon, \log(|u|_H^2 + \varepsilon) >_t)$$

Also we have following equality:

$$-d\frac{1}{2}\log(|u(s)|_H^2 + \varepsilon)) = \left(\frac{< (A - \frac{1}{2}\sum_{k=1}^n B_k^* B_k)u + F(u), u >}{(|u|_H + \varepsilon)^2} + \frac{\sum_{k=1}^n < u, B_k u >^2}{(|u|_H^2 + \varepsilon)^2}\right)(s) ds + \sum_{k=1}^n \frac{< u, B_k u > dw_s^k}{|u|_H^2 + \varepsilon}$$

It follows from the Itô formula (Theorem 2.8.1) and

$$\begin{aligned} F'(x)h_1 &= \frac{2 < x, h_1 >}{|x|_H^2 + \varepsilon}, x, h \in H. \\ F''(x)(h_1, h_2) &= \frac{2}{|x|_H^2 + \varepsilon}(< h_2, h_1 > - \frac{2 < x, h_2 >< x, h_1 >}{|x|_H^2 + \varepsilon}), x, h_1, h_2 \in H. \end{aligned}$$

Indeed, a function F defined by $F(x) = \log(|x|_H^2 + \varepsilon)$, $x \in H$ is of C^∞ class, has locally bounded continuous derivatives, its derivative has got no more than linear growth and assumptions (iii) and (iv) of Theorem 2.8.1 are satisfied. Denote

$$\widetilde{\Lambda}_\varepsilon^F = \frac{< (A - \frac{1}{2}\sum_{k=1}^{n} B_k^* B_k)u + F(u), u >}{|u|_H^2 + \varepsilon} + \frac{\sum_{k=1}^{n} < u, B_k u >^2}{(|u|_H^2 + \varepsilon)^2}.$$

$$\widetilde{\Lambda}_\varepsilon = \frac{< (A - \frac{1}{2}\sum_{k=1}^{n} B_k^* B_k)u, u >}{|u|_H^2 + \varepsilon} + \frac{\sum_{k=1}^{n} < u, B_k u >^2}{(|u|_H^2 + \varepsilon)^2}.$$

Therefore, we get

$$d\psi^\varepsilon(s) = M_s^\varepsilon(\widetilde{\Lambda}_\varepsilon^F ds + \sum_{k=1}^{n} \frac{< u, B_k u > dw_s^k}{|u|_H^2 + \varepsilon} +$$

$$+ \log(|u|_H^2 + \varepsilon) \sum_{k=1}^{n} \frac{< u, B_k u > dw_s^k}{|u|_H^2 + \varepsilon} + \frac{\sum_{k=1}^{n} < u, B_k u >^2}{(|u|_H^2 + \varepsilon)^2} ds). \quad (5.1.11)$$

Now from assumption (5.1.5) and (5.1.11) we get

$$\frac{1}{2}\mathbb{E}M_{t_0}^\varepsilon \log|u(t_0)|_H^2 - \frac{1}{2}\mathbb{E}M_t^\varepsilon \log|u(t)|_H^2 \le C_1 + C_2 \int_{t_0}^{t} \mathbb{E}M_s^\varepsilon \widetilde{\Lambda}_\varepsilon^F(s) ds.$$

Therefore, if I prove that $\mathbb{E}M_t^\varepsilon \widetilde{\Lambda}_\varepsilon(s)^F \le C(s)$ such that $K = \int_{t_0}^{T} |C(s)| ds < \infty$ we will have that

$$\tilde{\mathbb{E}}^\varepsilon \log(|u(t)|_H^2 + \varepsilon) \ge \tilde{\mathbb{E}}^\varepsilon \log(|u(t_0)|_H^2 + \varepsilon) - 2K, t \in [t_0, T]$$

and

$$\tilde{\mathbb{E}}^\varepsilon(|u(t)|_H^2 + \varepsilon) = \tilde{\mathbb{E}}^\varepsilon e^{\log(|u(t)|_H^2 + \varepsilon)} \ge e^{\tilde{\mathbb{E}}^\varepsilon \log(|u(t)|_H^2 + \varepsilon)} \ge e^{\tilde{\mathbb{E}}^\varepsilon \log(|u(t_0)|_H^2 + \varepsilon) - 2K} > 0.$$

Taking the limit $\varepsilon \to 0$ we get the Theorem. It remains to find an estimate for $\int_{t_0}^{t} \tilde{\mathbb{E}}^\varepsilon \widetilde{\Lambda}_\varepsilon^F(s) ds$.

We have an estimate

$$\int_{t_0}^{t} \tilde{\mathbb{E}}^{\varepsilon}\widetilde{\Lambda}_{\varepsilon}^{F}(s)ds \leq \int_{t_0}^{t} \tilde{\mathbb{E}}^{\varepsilon}\widetilde{\Lambda}_{\varepsilon}(s)ds + \tilde{\mathbb{E}}^{\varepsilon}\int_{t_0}^{t} \frac{n(s)|u|_H||u||_V}{|u|_H^2+\varepsilon}ds \leq \int_{t_0}^{t} \tilde{\mathbb{E}}^{\varepsilon}\widetilde{\Lambda}_{\varepsilon}(s)ds$$

$$+ (\int_{t_0}^{t} n^2(s)ds)^{1/2}\tilde{\mathbb{E}}^{\varepsilon}(\int_{t_0}^{t} \frac{|u|_H^2||u||_V^2}{(|u|_H^2+\varepsilon)^2}ds)^{1/2} \leq \int_{t_0}^{t} \tilde{\mathbb{E}}^{\varepsilon}\widetilde{\Lambda}_{\varepsilon}(s)ds + C\tilde{\mathbb{E}}^{\varepsilon}(\int_{t_0}^{t} \frac{||u||_V^2}{|u|_H^2+\varepsilon}ds)^{1/2}$$

$$\leq \int_{t_0}^{t} \tilde{\mathbb{E}}^{\varepsilon}\widetilde{\Lambda}_{\varepsilon}(s)ds + C(\int_{t_0}^{t} \tilde{\mathbb{E}}^{\varepsilon}\widetilde{\Lambda}_{\varepsilon}(s)ds)^{1/2}.$$

Therefore, it is enough to estimate term $\tilde{\mathbb{E}}^{\varepsilon}\widetilde{\Lambda}_{\varepsilon}(s)$. We have an assumption (5.1.5) and hence without loss of generality we can put

$$\widetilde{\Lambda}_{\varepsilon} = \frac{< (A - \frac{1}{2}\sum_{k=1}^{n} B_k^* B_k)u, u >}{|u|_H^2 + \varepsilon}.$$

Thus, $\widetilde{\Lambda}_{\varepsilon} = \frac{<\tilde{A}u,u>}{|u|_H^2+\varepsilon}$. Let now $\{e_i\}_{i\geq 1}$ be an orthonormal basis in H , P_N be a projection on first N elements of the basis, $Q_N = \mathrm{id} - P_N$. Define $\tilde{A}_N = P_N^*\tilde{A}P_N$, $\widetilde{\Lambda}_{\varepsilon}^N = \frac{<\tilde{A}_N u,u>}{|u|_H^2+\varepsilon}$, $F(x) = \frac{<\tilde{A}_N x,x>}{|x|_H^2+\varepsilon}$. Then

$$\begin{aligned} F'(x)h_1 &= \frac{2 < \tilde{A}_N x, h_1 >}{|x|_H^2+\varepsilon} - \frac{2 < \tilde{A}_N x, x >< x, h_1 >}{(|x|_H^2+\varepsilon)^2}, \\ F''(x)(h_1,h_2) &= 2\frac{< \tilde{A}_N h_1, h_2 >}{|x|_H^2+\varepsilon} - 4\frac{< \tilde{A}_N x, h_1 >< x, h_2 >}{(|x|_H^2+\varepsilon)^2} \\ &- 4\frac{< \tilde{A}_N x, h_2 >< x, h_1 >}{(|x|_H^2+\varepsilon)^2} - 2\frac{< \tilde{A}_N x, x >< h_2, h_1 >}{(|x|_H^2+\varepsilon)^2} \\ &+ 8\frac{< \tilde{A}_N x, x >< x, h_1 >< x, h_2 >}{(|x|_H^2+\varepsilon)^3}. \end{aligned}$$

By the Itô formula we have

$$d\widetilde{\Lambda}_{\varepsilon}^N = \frac{< \tilde{A}_N' u, u >}{|u|_H^2+\varepsilon}ds + \frac{2 < \tilde{A}_N u, du >}{|u|_H^2+\varepsilon} - \frac{2 < \tilde{A}_N u, u >< u, du >}{(|u|_H^2+\varepsilon)^2}$$

$$+ \sum_{k=1}^{n} \left(\frac{< \tilde{A}_N B_k u, B_k u >}{|u|_H^2 + \varepsilon} - \frac{4 < \tilde{A}_N u, B_k u >< u, B_k u >}{(|u|_H^2 + \varepsilon)^2} \right.$$

$$\left. - \frac{< \tilde{A}_N u, u > |B_k u|_H^2}{(|u|_H^2 + \varepsilon)^2} + \frac{4 < \tilde{A}_N u, u >< u, B_k u >^2}{(|u|_H^2 + \varepsilon)^3} \right) ds$$

From identity $du + (Au + F(u))dt + \sum_{k=1}^{n} B_k u dw_t^k = 0$ follows that

$$d\widetilde{\Lambda}_\varepsilon^N = -\frac{2 < \tilde{A}_N u, Au + F(u) >}{|u|_H^2 + \varepsilon} ds + \frac{2 < \tilde{A}_N u, u >< u, Au + F(u) >}{(|u|_H^2 + \varepsilon)^2} ds$$
$$+ \frac{< \tilde{A}'_N u, u >}{|u|_H^2 + \varepsilon} ds + \sum_{k=1}^{n} \frac{< \tilde{A}_N B_k u, B_k u >}{|u|_H^2 + \varepsilon} ds - \sum_{k=1}^{n} \frac{< \tilde{A}_N u, u > |B_k u|_H^2}{(|u|_H^2 + \varepsilon)^2} ds$$
$$+ \left(\sum_{k=1}^{n} \frac{2 < \tilde{A}_N u, u >< u, B_k u >}{(|u|_H^2 + \varepsilon)^2} - \sum_{k=1}^{n} \frac{2 < \tilde{A}_N u, B_k u >}{|u|_H^2 + \varepsilon}\right)$$
$$\left(dw_s^k + 2 \frac{< u, B_k u >}{|u|_H^2 + \varepsilon} ds\right) \quad (5.1.12)$$

Therefore, we get

$$d(M_t^\varepsilon \widetilde{\Lambda}_\varepsilon^N) = M_s^\varepsilon \left(-\frac{2 < \tilde{A}_N u, Au + F(u) >}{|u|_H^2 + \varepsilon} ds + \frac{2 < \tilde{A}_N u, u >< u, Au + F(u) >}{(|u|_H^2 + \varepsilon)^2} ds \right.$$
$$+ \frac{< \tilde{A}'_N u, u >}{|u|_H^2 + \varepsilon} ds + \sum_{k=1}^{n} \frac{< \tilde{A}_N B_k u, B_k u >}{|u|_H^2 + \varepsilon} ds - \sum_{k=1}^{n} \frac{< \tilde{A}_N u, u > |B_k u|_H^2}{(|u|_H^2 + \varepsilon)^2} ds$$
$$\left. - \sum_{k=1}^{n} \frac{2 < \tilde{A}_N u, B_k u >}{|u|_H^2 + \varepsilon} dw_s^k \right) \quad (5.1.13)$$

We can rewrite (5.1.13) as

$$d(M_t^\varepsilon \widetilde{\Lambda}_\varepsilon^N) = M_s^\varepsilon \left(\frac{< \tilde{A}_N u, u > (2 < Au, u > - \sum_{k=1}^{n} |B_k u|^2)}{(|u|_H^2 + \varepsilon)^2} \right.$$

$$\left. + \frac{< \tilde{A}'_N u, u >}{|u|_H^2 + \varepsilon} - \frac{2 < \tilde{A}_N u, Au >}{|u|_H^2 + \varepsilon} + \sum_{k=1}^{n} \frac{< \tilde{A}_N B_k u, B_k u >}{|u|_H^2 + \varepsilon} \right) ds$$

$$+ \quad M_s^\varepsilon\Big(\frac{2<\tilde{A}_N u, u><F(u), u>}{(|u|_H^2+\varepsilon)^2} - \frac{2<\tilde{A}_N u, F(u)>}{|u|_H^2+\varepsilon}\Big)ds$$

$$- \quad M_s^\varepsilon \sum_{k=1}^n \frac{2<\tilde{A}_N u, B_k u>}{|u|_H^2+\varepsilon} dw_s^k$$

$$= \quad M_s^\varepsilon \left(\frac{2<\tilde{A}_N u, u><\tilde{A}u, u>}{(|u|_H^2+\varepsilon)^2} - \frac{2<\tilde{A}_N u, Au>}{|u|_H^2+\varepsilon} \right.$$

$$+ \quad \frac{2<\tilde{A}_N u, u><F(u), u>}{(|u|_H^2+\varepsilon)^2} - \frac{2<\tilde{A}_N u, F(u)>}{|u|_H^2+\varepsilon} + \frac{<\tilde{A}'_N u, u>}{|u|_H^2+\varepsilon}$$

$$+ \quad \left. \sum_{k=1}^n \frac{<\tilde{A}_N B_k u, B_k u>}{|u|_H^2+\varepsilon} \right) ds - M_s^\varepsilon \sum_{k=1}^n \frac{2<\tilde{A}_N u, B_k u>}{|u|_H^2+\varepsilon} dw_s^k \tag{5.1.14}$$

Drift term can be written as:

$$\begin{aligned} M^\varepsilon(2\widetilde{\Lambda}_\varepsilon^N \frac{<\tilde{A}u, u>}{|u|_H^2+\varepsilon} - 2\frac{<\tilde{A}_N u, Au>}{|u|_H^2+\varepsilon} + \\ 2\widetilde{\Lambda}_\varepsilon^N \frac{<F(u), u>}{|u|_H^2+\varepsilon} - 2\frac{<\tilde{A}_N u, F(u)>}{|u|_H^2+\varepsilon} + \\ +\frac{<C_N u, u>}{|u|_H^2+\varepsilon}) = M^\varepsilon(-2\frac{<\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u, \tilde{A}u>}{|u|_H^2+\varepsilon} - \\ 2\frac{<\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u, F(u)>}{|u|_H^2+\varepsilon} + \frac{<C_N u, u>}{|u|_H^2+\varepsilon}) = (i) + (ii) + (iii) \end{aligned} \tag{5.1.15}$$

where $C_N = \sum_{k=1}^n B_k^*[\tilde{A}_N, B_k] + \tilde{A}'_N(\cdot)$. We have

$$\begin{aligned} (i) = M^\varepsilon(-2\frac{|\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u|^2}{|u|_H^2+\varepsilon} - 2\frac{<\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u, (\tilde{A} - \tilde{A}_N)u>}{|u|_H^2+\varepsilon} - \\ 2\frac{<\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u, \widetilde{\Lambda}_\varepsilon^N u>}{|u|_H^2+\varepsilon}) = M^\varepsilon(-2\frac{|\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u|^2}{|u|_H^2+\varepsilon} - 2\frac{\varepsilon(\widetilde{\Lambda}_\varepsilon^N)^2}{|u|_H^2+\varepsilon} - \\ 2\frac{<\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u, (\tilde{A} - \tilde{A}_N)u>}{|u|_H^2+\varepsilon}) \end{aligned}$$

Therefore, we have

$$M_t^\varepsilon \widetilde{\Lambda}_\varepsilon^N(t) + 2\int_{t_0}^{t} M_s^\varepsilon \frac{|\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u|^2}{|u|_H^2+\varepsilon} ds + 2\varepsilon \int_{t_0}^{t} M_s^\varepsilon \frac{(\widetilde{\Lambda}_\varepsilon^N)^2}{|u|_H^2+\varepsilon} ds = M_{t_0}^\varepsilon \widetilde{\Lambda}_\varepsilon^N(t_0) -$$

$$2\int_{t_0}^{t} M_s^\varepsilon \frac{< \tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u, (\tilde{A}-\tilde{A}_N)u >}{|u|_H^2+\varepsilon} ds - 2\int_{t_0}^{t} M_s^\varepsilon \frac{< \tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u, F(u) >}{|u|_H^2+\varepsilon} ds +$$

$$\int_{t_0}^{t} M_s^\varepsilon \frac{< C_N u, u >}{|u|_H^2+\varepsilon} ds - \int_{t_0}^{t} M_s^\varepsilon \sum_{k=1}^{n} \frac{2 < \tilde{A}_N u, B_k u >}{|u|_H^2+\varepsilon} dw_s^k =$$

$$= (i) + (ii) + (iii) + (iv) + (v) \quad (5.1.16)$$

Now we can estimate (ii) by Young inequality:

$$\int_{t_0}^{t} M_s^\varepsilon \frac{< \tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u, (\tilde{A}-\tilde{A}_N)u >}{|u|_H^2+\varepsilon} ds \le \varepsilon_1 \int_{t_0}^{t} M_s^\varepsilon \frac{|\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u|^2}{|u|_H^2+\varepsilon} ds +$$

$$+\frac{C}{\varepsilon_1}\int_{t_0}^{t} M_s^\varepsilon \frac{|(\tilde{A}-\tilde{A}_N)u|^2}{|u|_H^2+\varepsilon} ds, \varepsilon_1 < \frac{1}{2}$$

and similarly (iii). As the result, we get

$$M_t^\varepsilon \widetilde{\Lambda}_\varepsilon^N(t) + \int_{t_0}^{t} M_s^\varepsilon \frac{|\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u|^2}{|u|_H^2+\varepsilon} ds + 2\varepsilon \int_{t_0}^{t} M_s^\varepsilon \frac{(\widetilde{\Lambda}_\varepsilon^N)^2}{|u|_H^2+\varepsilon} \le M_{t_0}^\varepsilon \widetilde{\Lambda}_\varepsilon^N(t_0)$$

$$+\frac{C}{\varepsilon_1}\int_{t_0}^{t} M_s^\varepsilon \frac{|(\tilde{A}-\tilde{A}_N)u|^2}{|u|_H^2+\varepsilon} ds + \frac{C}{\varepsilon_2}\int_{t_0}^{t} M_s^\varepsilon \frac{|F(u)|^2}{|u|_H^2+\varepsilon} ds$$

$$+\int_{t_0}^{t} M_s^\varepsilon \frac{< C_N u, u >}{|u|_H^2+\varepsilon} ds - \int_{t_0}^{t} M_s^\varepsilon \sum_{k=1}^{n} \frac{2 < \tilde{A}_N u, B_k u >}{|u|_H^2+\varepsilon} dw_s^k \quad (5.1.17)$$

Let us estimate the term (iv). We have by definition of C_N and C the following chain of inequalities:

$$| < C_N x, x > | \le | < Cx, x > | + |\tilde{A}'_N(\cdot)|_{\mathcal{L}(V,V')} |x|_V^2$$

$$+|\sum_k < (\tilde{A}_N - \tilde{A})B_kx, B_kx > | + | < (\tilde{A}_N - \tilde{A})x, \sum_k B_k^* B_k x > |$$

$$\leq K_1(\cdot)|x|_H^2 + (K_2(\cdot) + |\tilde{A}'(\cdot)|_{\mathcal{L}(V,V')})| < \tilde{A}x, x > |$$

$$+|\sum_k < \tilde{A}Q_N B_k x, Q_N B_k x > | + |(\tilde{A}_N - \tilde{A})x||x|_{D(\tilde{A})}$$

$$\leq K_1(\cdot)|x|_H^2 + (K_2(\cdot) + |\tilde{A}'(\cdot)|_{\mathcal{L}(V,V')})| < \tilde{A}x, x > |$$

$$+|(\tilde{A}_N - \tilde{A})x||x|_{D(\tilde{A})} + \sum_k ||Q_N B_k x||_V^2.$$

where assumption (5.1.6) has been used in second inequality and assumption (5.1.8) in last inequality. Therefore

$$\int_{t_0}^t M_s^\varepsilon \frac{< C_N u, u >}{|u|_H^2 + \varepsilon} ds \leq \int_{t_0}^t K_1(s) M_s^\varepsilon ds + \int_{t_0}^t (K_2(s) + |\tilde{A}'(s)|_{\mathcal{L}(V,V')}) M_s^\varepsilon \widetilde{\Lambda}_\varepsilon^N(s) ds +$$

$$(\int_{t_0}^t M_s^\varepsilon \frac{|(\tilde{A}_N - \tilde{A})u|^2}{|u|_H^2 + \varepsilon} ds)^{1/2} (\int_{t_0}^t M_s^\varepsilon \frac{|\tilde{A}u|^2}{|u|_H^2 + \varepsilon} ds)^{1/2} + \int_{t_0}^t M_s^\varepsilon \frac{\sum_k ||Q_N B_k u||_V^2}{|u|_H^2 + \varepsilon} ds \quad (5.1.18)$$

I will denote $K_3(\varepsilon, N, \omega) = \int_{t_0}^t M_s^\varepsilon \frac{|(\tilde{A}_N - \tilde{A})u|^2}{|u|_H^2 + \varepsilon} ds$, $K_4(\varepsilon, N) = \int_{t_0}^t M_s^\varepsilon \frac{\sum_k ||Q_N B_k u||_V^2}{|u|_H^2 + \varepsilon} ds$, $K_5(\varepsilon) = \int_{t_0}^t M_s^\varepsilon \frac{|\tilde{A}u|^2}{|u|_H^2 + \varepsilon} ds$, $K_6(s) = |\tilde{A}'(s)|_{\mathcal{L}(V,V')}$. Combining (5.1.17), (5.1.18) with (5.1.9) we get

$$M_t^\varepsilon \widetilde{\Lambda}_\varepsilon^N(t) + \int_{t_0}^t M_s^\varepsilon \frac{|\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u|^2}{|u|_H^2 + \varepsilon} ds + 2\varepsilon \int_{t_0}^t M_s^\varepsilon \frac{(\widetilde{\Lambda}_\varepsilon^N)^2}{|u|_H^2 + \varepsilon}$$

$$\leq M_{t_0}^\varepsilon \widetilde{\Lambda}_\varepsilon^N(t_0) + \int_{t_0}^t K_1(s) M_s^\varepsilon ds + \frac{C}{\varepsilon_1} K_3(\varepsilon, N) + (K_3(\varepsilon, N) K_5(\varepsilon))^{1/2} + K_4(\varepsilon, N)$$

$$+ \frac{C}{\varepsilon_2} \int_{t_0}^t (n^2(s) + K_2(s) + K_6(s)) M_s^\varepsilon \widetilde{\Lambda}_\varepsilon^N(s) ds - \int_{t_0}^t M_s^\varepsilon \sum_{k=1}^n \frac{2 < \tilde{A}_N u, B_k u >}{|u|_H^2 + \varepsilon} dw_s^k$$

(5.1.19)

Now we notice that expression $\frac{|(\tilde{A}-\tilde{A}_N)u|^2}{|u|_H^2+\varepsilon}$ can be bounded from above by $\frac{|\tilde{A}u|^2}{2\varepsilon}$. Then we have

$$\tilde{\mathbb{E}}K_3(\varepsilon,N) = \tilde{\mathbb{E}}^\varepsilon \int_{t_0}^{t} \frac{|(\tilde{A}-\tilde{A}_N)u|^2}{|u|_H^2+\varepsilon} ds$$

$$\leq \frac{C}{\varepsilon}(\mathbb{E}((M_t^\varepsilon)^{1+\frac{2}{\delta_0}}))^{\frac{\delta_0}{2+\delta_0}}(\mathbb{E}\int_{t_0}^{t} |\tilde{A}u|^{2+\delta_0} ds)^{\frac{2}{2+\delta_0}} < \infty$$

where last inequality follows from the fact that $u \in M^{2+\delta_0}(0,T;D(\tilde{A}))$. Similarly, we have $\tilde{\mathbb{E}}K_4(\varepsilon,N) < \infty$ uniformly w.r.t. N and $\tilde{\mathbb{E}}K_5(\varepsilon) < \infty$. Therefore, we can find $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ such that $\forall \omega \in \Omega', N \in \mathbb{N}$ $K_3(\varepsilon,N) < \infty$, $K_4(\varepsilon,N) < \infty$, $K_5(\varepsilon) < \infty$, $u \in L^2(0,T;D(\tilde{A}))$. Moreover, we notice that $K_3(\varepsilon,N)$, $K_4(\varepsilon,N)$, $\widetilde{\Lambda}_\varepsilon^N$ are monotone w.r.t. N. Tending N to ∞ in (5.1.19) and noticing that $\lim_{N\to\infty} K_3(\varepsilon,N) = 0$, $\lim_{N\to\infty} K_4(\varepsilon,N) = 0$, $\widetilde{\Lambda}_\varepsilon(t) = \lim_{N\to\infty} \widetilde{\Lambda}_\varepsilon^N(t)$ we get

$$M_t^\varepsilon \widetilde{\Lambda}_\varepsilon(t) + \int_{t_0}^{t} M_s^\varepsilon \frac{|(\tilde{A}-\widetilde{\Lambda}_\varepsilon)u|^2}{|u|_H^2+\varepsilon} ds + 2\varepsilon \int_{t_0}^{t} M_s^\varepsilon \frac{(\widetilde{\Lambda}_\varepsilon)^2}{|u|_H^2+\varepsilon} ds$$

$$\leq M_{t_0}^\varepsilon \widetilde{\Lambda}_\varepsilon(t_0) + \int_{t_0}^{t} K_1(s) M_s^\varepsilon ds + \frac{C}{\varepsilon_2}\int_{t_0}^{t} (n^2(s) + K_2(s) + K_6(s)) M_s^\varepsilon \widetilde{\Lambda}_\varepsilon(s) ds$$

$$- \int_{t_0}^{t} M_s^\varepsilon \sum_{k=1}^{n} \frac{2 < \tilde{A}u, B_k u >}{|u|_H^2+\varepsilon} dw_s^k \quad (5.1.20)$$

Let us denote X_t^ε–solution of equation

$$X_t^\varepsilon = M_{t_0}^\varepsilon \widetilde{\Lambda}_\varepsilon(t_0) + \int_{t_0}^{t} K_1(s) M_s^\varepsilon ds + \frac{C}{\varepsilon_2}\int_{t_0}^{t} (n^2(s) + K_2(s) + K_6(s)) X_s^\varepsilon ds$$

$$- \int_{t_0}^{t} M_s^\varepsilon \sum_{k=1}^{n} \frac{2 < \tilde{A}u, B_k u >}{|u|_H^2+\varepsilon} dw_s^k \quad (5.1.21)$$

Then, we have that

$$M_t^\varepsilon \widetilde{\Lambda}_\varepsilon(t) + \int_{t_0}^{t} M_s^\varepsilon \frac{|(\tilde{A} - \widetilde{\Lambda}_\varepsilon)u|^2}{|u|_H^2 + \varepsilon} ds \le X_t^\varepsilon. \quad (5.1.22)$$

Indeed, it is enough to subtract from inequality (5.1.20) identity (5.1.21) and use Gronwall lemma. Now, we can calculate X_t^ε:

$$X_t^\varepsilon = M_{t_0}^\varepsilon \widetilde{\Lambda}_\varepsilon(t_0) e^{\int_{t_0}^{t}(n^2(s)+K_2(s)+K_6(s))ds}$$
$$+ \int_{t_0}^{t} e^{\int_s^t (n^2(\tau)+K_2(\tau)+K_6(\tau))d\tau} (K_1(s) M_s^\varepsilon ds - M_s^\varepsilon \sum_{k=1}^{n} \frac{2 < \tilde{A}u, B_k u >}{|u|_H^2 + \varepsilon} dw_s^k) \quad (5.1.23)$$

Denote

$$L_t^\varepsilon = \int_{t_0}^{t} e^{\int_s^t (n^2(\tau)+K_2(\tau)+K_6(\tau))d\tau} M_s^\varepsilon \sum_{k=1}^{n} \frac{2 < \tilde{A}u, B_k u >}{|u|_H^2 + \varepsilon} dw_s^k.$$

We have that

$$X_t^\varepsilon = M_{t_0}^\varepsilon \widetilde{\Lambda}_\varepsilon(t_0) e^{\int_{t_0}^{t}(n^2(s)+K_2(s)+K_6(s))ds}$$
$$+ \int_{t_0}^{t} e^{\int_s^t (n^2(\tau)+K_2(\tau)+K_6(\tau))d\tau} K_1(s) M_s^\varepsilon ds - L_t^\varepsilon. \quad (5.1.24)$$

By definition L_t^ε is a local martingale. I will show that in our assumptions it is martingale. It is enough to show that there exists $\delta > 0$

$$\mathbb{E}|L_t^\varepsilon|^{1+\delta} < \infty.$$

Let p_i, q_i, $i = 1, 2, 3$ be real numbers such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $p_i > 1$, $i = 1, 2, 3$. By successive application of Hölder inequality we have:

$$\mathbb{E}|L_t^\varepsilon|^{1+\delta} \le C\mathbb{E} \int_{t_0}^{t} e^{(1+\delta)\int_s^t (n^2+K_2+K_6)d\tau} (M_s^\varepsilon)^{1+\delta} \sum_{k=1}^{n} \frac{|\tilde{A}u|^{1+\delta} |B_k u|^{1+\delta}}{(|u|_H^2 + \varepsilon)^{1+\delta}} ds$$

$$\le C(\mathbb{E}e^{(1+\delta)q_1\int\limits_{t_0}^{t}(n^2+K_2+K_6)d\tau})^{\frac{1}{q_1}}\mathbb{E}\int\limits_{t_0}^{t}(M_s^\varepsilon)^{(1+\delta)p_1}\sum_{k=1}^{n}\frac{|\tilde{A}u|^{(1+\delta)p_1}|B_ku|^{(1+\delta)p_1}}{(|u|_H^2+\varepsilon)^{(1+\delta)p_1}}ds$$

$$\le C(\varepsilon)(\mathbb{E}e^{(1+\delta)q_1\int\limits_{t_0}^{t}(n^2+K_2+K_6)d\tau})^{\frac{1}{q_1}}(\mathbb{E}\int\limits_{t_0}^{t}(M_s^\varepsilon)^{(1+\delta)p_1q_2}ds)^{\frac{1}{q_2}}$$

$$\times\int\limits_{t_0}^{t}\sum_k\mathbb{E}|\tilde{A}u|^{(1+\delta)p_1p_2}|B_ku|^{(1+\delta)p_1p_2}ds\le C(\varepsilon)(\mathbb{E}e^{(1+\delta)q_1\int\limits_{t_0}^{t}(n^2+K_2+K_6)d\tau})^{\frac{1}{q_1}}$$

$$\times(\mathbb{E}\int\limits_{t_0}^{t}(M_s^\varepsilon)^{(1+\delta)p_1q_2}ds)^{\frac{1}{q_2}}(\int\limits_{t_0}^{t}\mathbb{E}|\tilde{A}u|^{(1+\delta)p_1p_2q_3}ds)^{\frac{1}{q_3}}$$

$$\times(\int\limits_{t_0}^{t}\mathbb{E}\sum_k|B_ku|^{(1+\delta)p_1p_2p_3}ds)^{\frac{1}{p_3}}=S$$

Choose $\delta>0,p_1,p_2,p_3>1$ in the following way: $(1+\delta)p_1p_2=1+\gamma, p_3=\frac{2}{1+\gamma}, \gamma=\frac{\delta_0}{4+\delta_0}$. Then, we have $(1+\delta)p_1p_2p_3=2$, $(1+\delta)p_1p_2q_3=2+\delta_0$, $(1+\delta)q_1=\kappa(\delta_0)>2+\frac{4}{\delta_0}$ and we get that $S<\infty$ by regularity assumption on u. Thus L_t^ε is martingale, $\mathbb{E}L_t^\varepsilon=0$ and it follow from (5.1.24), (5.1.22) and Hölder inequality

$$\mathbb{E}M_t^\varepsilon\widetilde{\Lambda}_\varepsilon(t)\le\mathbb{E}X_t^\varepsilon\le C(t_0)(\mathbb{E}\widetilde{\Lambda}_\varepsilon(t_0)^{1+\delta_0}+||K_1||_{L^2(0,T)}(t-t_0))\mathbb{E}e^{\kappa(\delta_0)\int\limits_{t_0}^{t}n^2(s)ds} \tag{5.1.25}$$

From Fatou lemma follows that $\tilde{\mathbb{E}}\widetilde{\Lambda}(t)\le\sup\limits_{\varepsilon>0}\tilde{\mathbb{E}}^\varepsilon\widetilde{\Lambda}_\varepsilon(t)$. Therefore, we get our estimate from (5.1.25).

Remark 5.1.5. I notice that in the case of antisymmetric B_k, $k=1,\dots,n$ instead of assumption that $u\in M^{2+\delta_0}(0,T;D(\tilde{A}))$(where $\delta_0>0$) can be used weaker assumption that $u\in M^2(0,T;D(\tilde{A}))$.

Corollary 5.1.6. *Under assumptions of Theorem 5.1.2 either* $u(t) = 0, t \in [0, T]$ $\mathbb{P}$*-a.s. or* $u(t) > 0, t \in [0, T]$ $\mathbb{P}$*-a.s..*

I will use in the following Theorem the same notation as in Theorem 5.1.2.

Theorem 5.1.7. *Suppose that u–solution of equation* (5.1.1)*, u is not identically* 0 $\mathbb{P}$*-a.s., assumptions* (5.1.8), (5.1.7), (5.1.4) *of Theorem 5.1.2 are satisfied, $\tilde{A}$ does not depend upon time, assumption* (5.1.6) *satisfied with parameters $K_1 = 0$, $K_2 \in L^2(T_0, \infty)$, and following assumptions are satisfied*

For all $T > T_0$

$$\mathbb{P}(u \in L^2(T_0, T; D(\tilde{A}))) = 1, \tag{5.1.26}$$

There exist $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ such that for a.a. $t \in [T_0, \infty)$, $\omega \in \Omega'$

$$|F(u)(t)| \leq n(t)||u||_V, n(\cdot, \omega) \in L^2(T_0, \infty). \tag{5.1.27}$$

There exists $\phi \in L^2(T_0, \infty)$ such that

$$\sum_{k=1}^{n} | < u, B_k(\cdot)u > | \leq \phi(\cdot)|u|_H^2, \tag{5.1.28}$$

There exists $C_1^k \in L^2(T_0, \infty), k = 1, \ldots, n$ such that

$$| < \tilde{A}u, B_k(\cdot)u > | \leq C_1^k(\cdot)| < \tilde{A}u, u > | \tag{5.1.29}$$

Then there exists $\tilde{\Lambda}^\infty : \Omega \to \sigma(\tilde{A})$ such that

$$\lim_{t \to \infty} \tilde{\Lambda}(t) = \tilde{\Lambda}^\infty . \mathbb{P} - a.s.$$

Proof of Theorem 5.1.7. Without loss of generality we will suppose that $T_0 = 0$ below. I will prove first existence of the limit $\lim_{t \to \infty} \tilde{\Lambda}(t)$. By the same arguments as in the Theorem

5.1.2 we have inequality (5.1.19) i.e.

$$M_t^\varepsilon \widetilde{\Lambda}_\varepsilon^N(t) + \int_{t_0}^{t} M_s^\varepsilon \frac{|\tilde{A}_N - \widetilde{\Lambda}_\varepsilon^N u|^2}{|u|_H^2 + \varepsilon} ds + 2\varepsilon \int_{t_0}^{t} M_s^\varepsilon \frac{(\widetilde{\Lambda}_\varepsilon^N)^2}{|u|_H^2 + \varepsilon}$$

$$\leq M_{t_0}^\varepsilon \widetilde{\Lambda}_\varepsilon^N(t_0) + \int_{t_0}^{t} K_1(s) M_s^\varepsilon ds + \frac{C}{\varepsilon_1} K_3(\varepsilon, N) + (K_3(\varepsilon, N) K_5(\varepsilon))^{1/2} + K_4(\varepsilon, N)$$

$$+ \frac{C}{\varepsilon_2} \int_{t_0}^{t} (n^2(s) + K_2(s)) M_s^\varepsilon \widetilde{\Lambda}_\varepsilon^N(s) ds - \int_{t_0}^{t} M_s^\varepsilon \sum_{k=1}^{n} \frac{2 < \tilde{A}_N u, B_k u >}{|u|_H^2 + \varepsilon} dw_s^k \quad (5.1.30)$$

for any $t, t_0 \geq 0$. Now one can notice that expression $\frac{|(\tilde{A} - \tilde{A}_N) u|^2}{|u|_H^2 + \varepsilon}$ can be bounded from above by $\frac{|\tilde{A} u|^2}{2\varepsilon}$. Then we have

$$K_3(\varepsilon, N) = \int_{t_0}^{t} M_s^\varepsilon \frac{|(\tilde{A} - \tilde{A}_N) u|^2}{|u|_H^2 + \varepsilon} ds \leq \frac{C}{\varepsilon} \sup_{s \leq t} M_s^\varepsilon \int_{0}^{t} |(\tilde{A} - \tilde{A}_N) u|^2 ds < \infty, \mathbb{P} - \text{a.s.}$$

Indeed, we have

$$\mathbb{P}(\sup_{s \leq t} M_s^\varepsilon > \lambda) \leq \frac{1}{\lambda}$$

by Doob inequality and, therefore, $\sup_{s \leq t} M_s^\varepsilon < \infty, \mathbb{P} - \text{a.s.}$. Moreover,

$$\int_{0}^{t} |(\tilde{A} - \tilde{A}_N) u|^2 ds \leq 2 \int_{0}^{t} |\tilde{A} u|^2 ds < \infty, \mathbb{P} - \text{a.s.}$$

by assumption (5.1.26). Similarly, $K_4(\varepsilon, N) < \infty$ uniformly w.r.t. N, $K_5(\varepsilon) < \infty$ $\mathbb{P}$ – a.s.. Thus, we can find $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$ such that $\forall \omega \in \Omega', N \in \mathbb{N}$ $K_3(\varepsilon, N) < \infty$, $K_4(\varepsilon, N) < \infty$, $K_5(\varepsilon) < \infty$, $u \in L^2(0, T; D(\tilde{A})), \forall T > 0$. Moreover, one can notice that $K_3(\varepsilon, N)$, $K_4(\varepsilon, N)$, $\widetilde{\Lambda}_\varepsilon^N$ are monotone w.r.t. N. Tending N to ∞ in (5.1.19) and noticing that $\lim_{N \to \infty} K_3(\varepsilon, N) = 0$, $\lim_{N \to \infty} K_4(\varepsilon, N) = 0$, $\widetilde{\Lambda}_\varepsilon(t) = \lim_{N \to \infty} \widetilde{\Lambda}_\varepsilon^N(t)$ we get

$$M_t^\varepsilon \widetilde{\Lambda}_\varepsilon(t) + \int_{t_0}^{t} M_s^\varepsilon \frac{|(\tilde{A} - \widetilde{\Lambda}_\varepsilon) u|^2}{|u|_H^2 + \varepsilon} ds + 2\varepsilon \int_{t_0}^{t} M_s^\varepsilon \frac{(\widetilde{\Lambda}_\varepsilon)^2}{|u|_H^2 + \varepsilon} ds \leq$$

$$M^{\varepsilon}_{t_0}\widetilde{\Lambda}_{\varepsilon}(t_0) + \int_{t_0}^{t} K_1(s)M^{\varepsilon}_s ds + \frac{C}{\varepsilon_2}\int_{t_0}^{t}(n^2(s) + K_2(s))M^{\varepsilon}_s\widetilde{\Lambda}_{\varepsilon}(s)ds$$

$$-\int_{t_0}^{t} M^{\varepsilon}_s \sum_{k=1}^{n} \frac{2 < \tilde{A}u, B_k u >}{|u|^2_H + \varepsilon} dw^k_s. \tag{5.1.31}$$

Consequently, it follows (in the same way as in the Theorem 5.1.2) (5.1.22) i.e. we have

$$M^{\varepsilon}_t\widetilde{\Lambda}_{\varepsilon}(t) + \int_{t_0}^{t} M^{\varepsilon}_s \frac{|(\tilde{A} - \widetilde{\Lambda}_{\varepsilon})u|^2}{|u|^2_H + \varepsilon} ds \le M^{\varepsilon}_{t_0}\widetilde{\Lambda}_{\varepsilon}(t_0)e^{\int_{t_0}^{t}(n^2(s)+K_2(s))ds} - L^{\varepsilon}_{t_0,t} \tag{5.1.32}$$

where

$$L^{\varepsilon}_{t_0,t} = \int_{t_0}^{t} e^{\int_{s}^{t}(n^2(\tau)+K_2(\tau))d\tau} M^{\varepsilon}_s \sum_{k=1}^{n} \frac{2 < \tilde{A}u, B_k u >}{|u|^2_H + \varepsilon} dw^k_s. \tag{5.1.33}$$

Multiplying (5.1.32) on $e^{-\int_0^t (n^2(\tau)+K_2(\tau))d\tau}$ and denoting

$$S^{\varepsilon}_t = e^{-\int_0^t (n^2(\tau)+K_2(\tau))d\tau} M^{\varepsilon}_t\widetilde{\Lambda}_{\varepsilon}(t), N^{\varepsilon}_t = e^{-\int_0^t (n^2(\tau)+K_2(\tau))d\tau} \int_{t_0}^{t} M^{\varepsilon}_s \frac{|(\tilde{A} - \widetilde{\Lambda}_{\varepsilon})u|^2}{|u|^2_H + \varepsilon} ds.$$

We have

$$S^{\varepsilon}_t + N^{\varepsilon}_t \le S^{\varepsilon}_{t_0} - 2\int_{t_0}^{t} e^{-\int_0^s (n^2(\tau)+K_2(\tau))d\tau} M^{\varepsilon}_s \sum_{k=1}^{n} \frac{2 < \tilde{A}u, B_k u >}{|u|^2_H + \varepsilon} dw^k_s =$$

$$S^{\varepsilon}_{t_0} - 2\int_{t_0}^{t} S^{\varepsilon}_s \sum_{k=1}^{n} \frac{< \tilde{A}u, B_k u >}{| < \tilde{A}u, u > |} dw^k_s \tag{5.1.34}$$

Putting $\varepsilon = 0$ in inequality (5.1.34) we get

$$S_t + N_t \le S_{t_0} - 2\int_{t_0}^{t} S_s \sum_{k=1}^{n} \frac{< \tilde{A}u, B_k u >}{| < \tilde{A}u, u > |} dw^k_s,$$

where $S_t = S^0_t$, $N_t = N^0_t$. By comparison Theorem for one dimensional diffusions (Theorem 2.8.2) we have

$$S_t + N_t \le S_{t_0} e^{-2\int_{t_0}^{t}\sum_{k=1}^{n} \frac{<\tilde{A}u, B_k u>}{|<\tilde{A}u,u>|} dw^k_s - 2\int_{t_0}^{t}\sum_{k=1}^{n} \frac{|<\tilde{A}u, B_k u>|^2}{|<\tilde{A}u,u>|^2} ds}. \tag{5.1.35}$$

Denote

$$\vartheta_{t_0,t} = e^{-2\int\limits_{t_0}^{t}\sum\limits_{k=1}^{n}\frac{<\tilde{A}u,B_k u>}{|<\tilde{A}u,u>|}dw_s^k - 2\int\limits_{t_0}^{t}\sum\limits_{k=1}^{n}\frac{|<\tilde{A}u,B_k u>|^2}{|<\tilde{A}u,u>|^2}ds}$$

–exponential martingale in the formula (5.1.35). We have from assumption (5.1.29) and martingale convergence Theorem ([65], p.8) that $\mathbb{E}\vartheta_{t_0,t} = 1$. Indeed,

$$\mathbb{E}\vartheta_{t_0,t}^{1+\delta} \leq \mathbb{E}e^{C(\delta)\int\limits_{t_0}^{\infty}|C_1^k(s)|^2 ds} < \infty.$$

Hence, by martingale convergence Theorem there exists $\lim\limits_{t\to\infty}\vartheta_{t_0,t} = \vartheta_{t_0,\infty} < \infty$ $\mathbb{P}$-a.s.. and $\lim\limits_{t_0\to\infty}\vartheta_{t_0,\infty} = 1$ $\mathbb{P}$-a.s.. Thus,

$$\limsup_{t\to\infty} S_t \leq S_{t_0}\vartheta_{t_0,\infty}$$

and, therefore,

$$\limsup_{t\to\infty} S_t \leq \liminf_{t_0\to\infty} S_{t_0}\vartheta_{t_0,\infty} = \liminf_{t_0\to\infty} S_{t_0}\mathbb{P} - \text{a.s.}$$

i.e. there exists

$$\lim_{t\to\infty} S_t = \lim_{t\to\infty} e^{-\int\limits_0^t (n^2(\tau)+K_2(\tau))d\tau} M_t\widetilde{\Lambda}(t)\mathbb{P} - \text{a.s..} \tag{5.1.36}$$

Moreover, it follows from assumption (5.1.27) that there exists

$$\lim_{t\to\infty} e^{-\int\limits_0^t (n^2(\tau)+K_2(\tau))d\tau} = e^{-\int\limits_0^\infty (n^2(\tau)+K_2(\tau))d\tau} \neq 0\mathbb{P} - \text{a.s..} \tag{5.1.37}$$

Furthermore, from assumption (5.1.28) follows that M_t is uniformly integrable exponential martingale and, as a result, we have that there exists

$$\lim_{t\to\infty} M_t = M_\infty \neq 0\mathbb{P} - \text{a.s..} \tag{5.1.38}$$

Combining (5.1.36), (5.1.37) and (5.1.38) we get existence of

$$\lim_{t\to\infty} \widetilde{\Lambda}(t) = \widetilde{\Lambda}^\infty, \mathbb{P} - \text{a.s..} \tag{5.1.39}$$

It remains to show that $\widetilde{\Lambda}^\infty \in \sigma(\tilde{A})$. We infer from (5.1.35) that

$$\lim_{t\to\infty} N_t \leq S_{t_0}\vartheta_{t_0,\infty} < \infty, \mathbb{P}-\text{a.s.}. \tag{5.1.40}$$

i.e.

$$\lim_{t\to\infty} e^{-\int\limits_0^t (n^2(\tau)+K_2(\tau))d\tau} \int\limits_{t_0}^t M_s \frac{|(\tilde{A}-\widetilde{\Lambda})u|^2}{|u|_H^2} ds < \infty, \mathbb{P}-\text{a.s.}. \tag{5.1.41}$$

Therefore,

$$\int\limits_{t_0}^\infty M_s|(\tilde{A}-\widetilde{\Lambda}(s))\frac{u}{|u|_H}|^2 ds < \infty, \mathbb{P}-\text{a.s.}. \tag{5.1.42}$$

Denote $\psi = \frac{u}{|u|_H}$. It follows from (5.1.42) that there exists sequence $t_j \to \infty$, $j \to \infty$ such that $(\tilde{A}-\widetilde{\Lambda}(t_j))\psi(t_j) \to 0$ in H. Therefore, $h_j = (\tilde{A}-\widetilde{\Lambda}^\infty)\psi(t_j) \to 0$ as $j \to \infty$. If $\widetilde{\Lambda}^\infty \notin \sigma(\tilde{A})$ then $(\tilde{A}-\widetilde{\Lambda}^\infty)^{-1} \in \mathcal{L}(V', V)$. Since $h_j \to 0$ in V' I have $\psi(t_j) = (\tilde{A}-\widetilde{\Lambda}^\infty)^{-1}h_j \to 0$ in H. This is contradiction with the fact that $|\psi(t)|_H = 1$.

5.2 Applications

Now I will show how to apply Theorems 5.1.2 and 5.1.7 to certain linear and nonlinear SPDEs.

5.2.1 Backward Uniqueness.

Linear SPDEs.

I will consider following equation:

$$du + (Au + F(u))dt + \sum_{k=1}^n B_k u dw_t^k = f dt + \sum_{k=1}^n g_k dw_t^k. \tag{5.2.1}$$
$$f \in M^2(0,T;V), g \in M^2(0,T;D(A))$$

where operators A, B_k, F $k = 1, \dots, n$ satisfy the same assumptions as in the Theorem 5.1.2. I will suppose that n (from assumption (5.1.9)) is nonrandom and $n \in L^2(0,T)$. Then, I notice that assumption (5.1.9) is satisfied. Applying Theorem 5.1.2 I have the following result:

Theorem 5.2.1. *Let u_1, u_2 be two solutions of (5.2.1), such that there exists $\delta_0 > 0$*

$$u_1, u_2 \in M^{2+\delta_0}(0,T; D(\tilde{A})). \tag{5.2.2}$$

Then if $u_1(T) = u_2(T)$, $\mathbb{P}$-a.s., $u_1(t) = u_2(t), t \in [0,T]$ $\mathbb{P}$-a.s..

Proof. Denote $s = u_1 - u_2$. Applying Theorem 5.1.2 to s I immediately get the result. □

Example 5.2.2. Assume $b_k, c, \sigma_k \in C^{0,1}_{t,x}([0,T] \times \mathbb{R}^n)$, $k = 1, \dots, n$ and following inequalities are satisfied:

$$\sup_{x\in\mathbb{R}^n} \sum_k |\nabla\sigma_k(t,\cdot)|^2 \in L^2(0,T),$$
$$\sup_{x\in\mathbb{R}^n} \sum_k |b_k(t,\cdot)| + |c(t,\cdot)| \in L^2(0,T).$$

Then equation

$$du = (\Delta u + \sum_k b_k(t,\cdot)\frac{\partial u}{\partial x_k} + c(t,\cdot)u + f)dt + \sum_k (\sigma_k(t,\cdot)\frac{\partial u}{\partial x_k} + g_k) \circ dw_t^k$$

(where stochastic integral is in Stratonovich sense) satisfies conditions of the Theorem 5.2.1. Indeed, we have in this case that

$$\tilde{A} = -\Delta, F = -\sum_k b_k(t,\cdot)\frac{\partial}{\partial x_k} - c(t,\cdot), B_k = \sigma_k(t,\cdot)\frac{\partial}{\partial x_k},$$
$$H = L^2(\mathbb{R}^n), V = H^{1,2}_0(\mathbb{R}^n).$$

We need to check only assumption (5.1.6). Other conditions are trivial. We have

$$\sum_k ([\tilde{A}, B_k]u, B_k u)_H = \sum_k \int_{\mathbb{R}^n} \sigma_k \frac{\partial u}{\partial x_k}(\sigma_k \frac{\partial \Delta u}{\partial x_k} - \Delta(\sigma_k \frac{\partial u}{\partial x_k}))dx =$$

$$\sum_k \int_{\mathbb{R}^n} |\nabla\sigma_k|^2 |\frac{\partial u}{\partial x_k}|^2 dx \le \sup_{x\in\mathbb{R}^n} \sum_k |\nabla\sigma_k(t,\cdot)|^2 |u|_V^2.$$

Existence of regular solutions have been established in [63].

Remark 5.2.3. Instead of Laplacian one can consider operator $A(t) = \sum_{ij} a^{ij}(t)\frac{\partial^2}{\partial x_i \partial x_j}$ where matrix $a = (a^{ij}) : [0,T] \to \mathbb{R}^n$ is uniformly (w.r.t. t) positively definite.

Spde with quadratic nonlinearity

In this case we can also apply Theorem 5.1.2. Though, assumptions on regularity of solutions will be very strong (assumption (5.2.6) below).

$$du + (Au + B(u,u) + R(u))dt + \sum_{k=1}^n B_k u dw_t^k = f dt + \sum_{k=1}^n g_k dw_t^k. \tag{5.2.3}$$
$$u(0) = u_0 \in H, f \in L^2(0,T;H), g \in M^2(0,T;H)$$

where operators A, B_k, $k = 1,\dots,n$ satisfy the same assumptions as in the Theorem 5.1.2, $B \in \mathcal{L}(V\times V, V')$, $R \in \mathcal{L}(V,H)$ and

$$|B(u,v)| + |B(v,u)| \le K\|u\|_V|Av|, \forall u \in V, v \in D(\tilde{A}). \tag{5.2.4}$$

Applying Theorem 5.1.2 I have the following result:

Theorem 5.2.4. *Let* u_1, u_2 *be two solutions of* (5.2.3), *such that there exists* $\delta_0 > 0$

$$u_1, u_2 \in M^{2+\delta_0}(0,T; D(\tilde{A})), \tag{5.2.5}$$

$$\exists\kappa(\delta_0) > 2 + \frac{4}{\delta_0} \text{ such that } \mathbb{E}e^{\kappa(\delta_0)\int_{t_0}^t |u_i|^2_{D(\tilde{A})}(s)ds} < \infty, i = 1,2. \tag{5.2.6}$$

Then if $u_1(T) = u_2(T)$, $\mathbb{P}$-*a.s.*, $u_1(t) = u_2(t), t \in [0,T]$ $\mathbb{P}$-*a.s.*.

Proof of Theorem 5.2.4. I denote $s = u_1 - u_2$. Then we have

$$ds + (As + B(u_1,s) + B(s,u_2) + R(s))dt + \sum_{k=1}^n B_k s dw_t^k = 0. \tag{5.2.7}$$

From (5.2.4) follows that

$$|B(u_1, s) + B(s, u_2) + R(s)| \leq [||R||_{\mathcal{L}(V,H)} + K(|Au_1| + |Au_2|)]||s||_V \leq$$
$$C[||R||_{\mathcal{L}(V,H)} + K(|\tilde{A}u_1| + |\tilde{A}u_2|)]||s||_V$$

and according to (5.2.5), (5.2.6) $||R||_{\mathcal{L}(V,H)} + K(|\tilde{A}u_1| + |\tilde{A}u_2|)$ satisfy assumption (5.1.9-5.1.10) and Theorem 5.1.2 applies to s.

Remark 5.2.5. It would be interesting to understand if it is possible to find weaker assumptions under which Theorem 5.1.2 is still valid. One possible option is to try to follow the line of proof of Theorem 5.1.7(Do everything $\mathbb{P}$-a.s.!)

In the framework above fall Navier-Stokes equations with multiplicative noise.

5.2.2 Existence of spectral limit.

Suppose that u is a solution of equation

$$\begin{cases} du + (A(\cdot)u + B(u,u) + R(u))dt + \sum_{k=1}^{n} B_k(\cdot)u\,dw_t^k = 0, \\ u(0) = u_0 \in H, \end{cases} \tag{5.2.8}$$

where operators $A(\cdot)$, $B_k(\cdot)$, $k = 1, \ldots, n$ satisfy the same assumptions as in the Theorem 5.1.7, $B \in \mathcal{L}(V \times V, V')$, $R \in \mathcal{L}(V, H)$ and

$$|B(u,v)| + |B(v,u)| \leq K||u||_V|Av|, \forall u \in V, v \in D(\tilde{A}). \tag{5.2.9}$$

Applying Theorem 5.1.7 I have the following result:

Theorem 5.2.6. *Let u be solution of* (5.2.8), *such that*

$$\exists T_0 > 0 \mathbb{P}(u \in L^2(T_0, \infty; D(\tilde{A}))) = 1, \tag{5.2.10}$$

Then there exists $\tilde{\Lambda}^{\infty} : \Omega \to \sigma(\tilde{A})$ *such that*

$$\lim_{t\to\infty} \tilde{\Lambda}(t) = \tilde{\Lambda}^{\infty}.\mathbb{P} - a.s.$$

Remark 5.2.7. In the case of $B = 0$ instead of condition (5.2.10) I can put weaker condition

$$\exists T_0 > 0 \mathbb{P}(u \in L^2(T_0, T; D(\tilde{A}))) = 1, \forall T > T_0 \tag{5.2.11}$$

Example 5.2.8. 2D Stokes equations with multiplicative noise:

$$du + (\nu Au + B(u))dt + \sigma(t)u \circ dw_t = 0, \tag{5.2.12}$$

$$u(0) = u_0 \in H, u \in L^2(0,T;V) \cap L^{\infty}(0,T;H), \text{ where}$$

$$H = \{u \in L^2(D,\mathbb{R}^n) | \operatorname{div} u = 0, (u \cdot \vec{n})|_{\Gamma} = 0\},$$

$$V = H_0^{1,2}(D,\mathbb{R}^2) \cap H, \ \sigma \in L^2(0,\infty),$$

$A = -P\triangle$-Stokes operator (P-projection on divergence free fields), $B(u, w) = P((u\nabla)w)$, w_t -one dimensional wiener process. It is enough to check condition (5.2.10). I have $N_t = e^{\int_0^t \sigma(s)dw_s} = e^{\int_0^t \sigma(s)dw_s - \frac{1}{2}\int_0^t \sigma^2(s)ds} e^{\frac{1}{2}\int_0^t \sigma^2(s)ds}$ and from condition $\sigma \in L^2(0,\infty)$ follows that

$$M_t = e^{\int_0^t \sigma(s)dw_s - \frac{1}{2}\int_0^t \sigma^2(s)ds}$$

is uniformly integrable martingale. Indeed, I have

$$\mathbb{E}M_t^2 = e^{\int_0^t \sigma^2(s)ds}\mathbb{E}e^{2\int_0^t \sigma(s)dw_s - 2\int_0^t \sigma^2(s)ds} = \tag{5.2.13}$$

$$e^{\int_0^t \sigma^2(s)ds} = e^{||\sigma||^2_{L^2(0,\infty)}} < \infty \tag{5.2.14}$$

and uniform integrability follows, for instance, from Theorem 3.1,chapter 3 of [67],p.68. Hence there exists limit $M_{\infty} = \lim_{t\to\infty} M_t$ and by Burkholder-Davis-Gundy inequality([67]) I get that

$$\mathbb{E}(\sup_{s\le\infty} M_s) \le 1 + C\mathbb{E} < M >_{\infty} < \infty.$$

Therefore, there exists set $\Omega' \subset \Omega$, $P(\Omega') = 1$ such that for any $\omega \in \Omega'$

$$k(\omega) = \sup_{s \leq \infty} N_t(\omega) < \infty. \tag{5.2.15}$$

Fix $\omega \in \Omega'$. Let us consider following equation

$$dv + (\nu Av + N_t(\omega)B(v))dt = 0, v(0) = u_0, v \in L^\infty(0,T;H) \cap L^2(0,T;V) \tag{5.2.16}$$

Then I notice that u given by the formula

$$u = ve^{\int_0^t \sigma(s)dw_s}$$

satisfies equation (5.2.12). Thus it is enough to prove that $v(\omega) \in L^2(0,\infty;D(A))$. Then I will have from condition (5.2.15) that $u = N_.(\omega)v(\omega) \in L^2(0,\infty;D(A))$, $\omega \in \Omega'$ and the result follows. Equation (5.2.16) for fixed $\omega \in \Omega'$ is a system of deterministic Navier-Stokes equations and, therefore existence of solution $v \in L^\infty(0,T;H) \cap L^2(0,T;V) \forall T > 0$ will follow from Theorem 3.1 p.282 in [72]. Let us show that if $v(0) = u(0) \in V$ then $v \in L^2(0,\infty;D(A))$. Firstly, taking scalar product of (5.2.16) with v I get energy estimate:

$$|v|^2_{L^\infty(0,T;H)} + 2\nu|v|^2_{L^2(0,T;V)} \leq |u_0|^2_H.$$

Since the estimate is independent of T I get $v \in L^\infty(0,\infty;H) \cap L^2(0,\infty;V)$. Secondly, taking scalar product of (5.2.16) with Av I have

$$\frac{1}{2}\frac{d}{dt}||v||^2_V + \nu|Av|^2 + N_t(\cdot) < B(v), Av >= 0 \tag{5.2.17}$$

Integrating w.r.t.time and using Young inequality I get

$$||v(t)||^2_V + 2\nu \int_0^t |Av(s)|^2 ds \leq ||v(0)||^2_V + \nu \int_0^t |Av(s)|^2 ds + \frac{k^2}{\nu} \int_0^t |B(v)|^2_H ds \tag{5.2.18}$$

For $B(\cdot)$ I have following estimate (Lemma 3.8 p.313 of [72])

$$|B(u)| \leq c_1|u|^{1/2}||u||_V|Au|^{1/2} \forall v \in V \cap \mathbf{H}^2(D). \tag{5.2.19}$$

Inserting this estimate into (5.2.18) I have

$$\begin{aligned} ||v(t)||_V^2 + \nu \int_0^t |Av(s)|^2 ds &\leq ||v(0)||_V^2 + \frac{k^2}{\nu} \int_0^t |Av||v|||v||_V^2 ds \\ &\leq ||v(0)||_V^2 + \frac{k^2}{\nu}(\nu^2 \int_0^t \frac{|Av|^2}{2k^2} ds + C\frac{k^2}{2\nu^2} \int_0^t |v|^2||v||_V^4 ds) \\ &= ||v(0)||_V^2 + \frac{\nu}{2} \int_0^t |Av(s)|^2 ds + c\frac{k^3}{\nu^3} \int_0^t |v|^2||v||_V^4 ds \end{aligned}$$

where second inequality follows from Young inequality. Thus I have

$$||v(t)||_V^2 + \frac{\nu}{2} \int_0^t |Av(s)|^2 ds \leq ||v(0)||_V^2 + c\frac{k^3}{\nu^3} \int_0^t |v|^2||v||_V^4 ds. \tag{5.2.20}$$

Therefore, from Gronwall lemma follows that

$$\begin{aligned} ||v(t)||_V^2 + \frac{\nu}{2} \int_0^t |Av(s)|^2 ds &\leq ||v(0)||_V^2 e^{c\frac{k^3}{\nu^3} \int_0^t |v|^2||v||_V^2 ds} \\ &\leq ||v(0)||_V^2 e^{c\frac{k^3}{\nu^3} |v|^2_{L^\infty(0,\infty;H)} ||^2_{L^2(0,\infty;V)}} < \infty \end{aligned}$$

and I have that $v \in L^2(0, \infty; D(A))$.

Chapter 6

Global evolution of random vortex filament equation

In this chapter we prove existence of global solution for random vortex filament equation. We consider following equation

$$\frac{d\gamma}{dt} = u^{\gamma(t)}(\gamma(t)), t \in [0, T] \qquad (6.0.1)$$

$$\gamma(0) = \gamma_0. \qquad (6.0.2)$$

Here $\gamma : [0, T] \to D_X \subset \mathcal{C}$ is some trajectory in the subset D_X of $\mathcal{C}$ of continuous closed curves in $\mathbb{R}^3$, u^Y, $Y \in D_X \subset \mathcal{C}$ is a vector field given by

$$u^Y(x) = \int_Y \nabla\phi(x - y) \times dy. \qquad (6.0.3)$$

where $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function which satisfies certain assumptions (see Hypothesis 6.2.1). Exact meaning of the line integral above and set D_X we consider will be explained below. This equation appear in fluid dynamics in the theory of three dimensional Euler equations. Result of Beale, Kato, and Majda [4] suggest that possible singularity of Euler equations appear when vorticity field of fluid blows up. Some numerical simulations of 3D turbulent fluids show that regions where vorticity is big have a form of "filament", see, for instance [5], [75]. As a consequence, see [8] and [68], problem (6.0.1)-(6.0.3) can be deduced from the assumption that vorticity of the liquid is concentrated around some curve. We will work in the framework of rough space theory developed by T. J. Lyons and co-authours, see [57], [58] and references therein, and assume that initial condition is a closed curve of Hölder class with exponent $\nu \in (\frac{1}{3}, 1]$.

6.1 Definition and Properties of Rough Path integrals

In this section we define rough path integral and state some of its properties. We mainly follow [43] and [9].

Definition 6.1.1. *Assume V_1, V_2 are two Banach spaces and $t \in \mathbb{R}^+$. Define*

$$\tilde{C}^t((V_1)^2, V_2) = \{f \in C((V_1)^2, V_2) | \text{ such that} |f|_{\tilde{C}^t((V_1)^2,V_2)} = \sup_{a \neq b, a, b \in V_1} \frac{|f(a,b)|_{V_2}}{|a-b|^t_{V_1}} < \infty\}.$$

We will often use space $\tilde{C}^t((S^1)^2, \mathbb{R}^3)$ and denote it by $\tilde{C}^t$. The space $\tilde{C}^t$ endowed with the norm $|\cdot|_{\tilde{C}^t}$ is a Banach space.

Definition 6.1.2. *Let us fix $X \in C^\nu(S^1, \mathbb{R}^3)$. We say that path $Y \in C(S^1, \mathbb{R}^3)$ is weakly controlled by X if there exist functions $Z \in C^\nu(S^1, L(\mathbb{R}^3, \mathbb{R}^3))$ and $R \in \tilde{C}^{2\nu}((S^1)^2, \mathbb{R}^3)$ such that*

$$Y(\xi) - Y(\eta) = Z(\eta)(X(\xi) - X(\eta)) + R(\xi, \eta), \xi, \eta \in S^1. \tag{6.1.1}$$

Let $\mathcal{D}_X$ be the set of pairs (Y, Z), where $Y \in C(S^1, \mathbb{R}^3)$ is a path weakly controlled by X, and $Z \in C^\nu(S^1, L(\mathbb{R}^3, \mathbb{R}^3))$ is such that $R \in \tilde{C}^{2\nu}((S^1)^2, \mathbb{R}^3)$, where R is defined by representation (6.1.1). We notice that this is vector space. Define the semi-norm in $\mathcal{D}_X$ as follows

$$||(Y, Z)||_{\mathcal{D}_X} = |Z|_{C^\nu} + ||R||_{\tilde{C}^{2\nu}}, \tag{6.1.2}$$

where

$$R(\xi, \eta) = Y(\xi) - Y(\eta) - Z(\eta)(X(\xi) - X(\eta)), \xi, \eta \in S^1. \tag{6.1.3}$$

Furthermore, define the following norm

$$||(Y, Z)||^*_{\mathcal{D}_X} = ||Y||_{\mathcal{D}_X} + |Y|_{C(S^1, \mathbb{R}^3)}. \tag{6.1.4}$$

Then, one can prove that $(\mathcal{D}_X, ||\cdot||^*_{\mathcal{D}_X})$ is a Banach space. From now on we will denote elements of $\mathcal{D}_X$ by (Y, Y') and the corresponding R will be denoted by R^{Y}. We will often omit to specify Y' when it is clear from the context and write $||Y||_{\mathcal{D}_X}$ instead of $||(Y, Y')||_{\mathcal{D}_X}$.

Definition 6.1.3. *Let* $\Pi : \mathcal{D}_X \ni (Y, Z) \mapsto Y \in C(S^1, \mathbb{R}^3)$ *be the natural projection.*

We will need following properties of $\mathcal{D}_X$, see [43].

Lemma 6.1.4. $\Pi(\mathcal{D}_X) \subset C^\nu(S^1, \mathbb{R}^3)$.

Proof of Lemma 6.1.4. Immediately follows from inequality

$$||Y||_{C^\nu} \leq ||Y||^*_{\mathcal{D}_X}(1 + ||X||_{C^\nu}). \tag{6.1.5}$$

□

Lemma 6.1.5. *Let* $\phi \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ *and* $(Y, Z) \in \mathcal{D}_X$. *Then*

$$(W, W') := (\phi(Y), \phi'(Y)Z) \in \mathcal{D}_X \tag{6.1.6}$$

and the remainder has the following representation

$$R^W(\xi,\eta) = \phi'(Y(\xi))R(\xi,\eta) + (Y(\eta) - Y(\xi))$$

$$+ \int_0^1 [\nabla\phi(Y(\xi) + r(Y(\eta) - Y(\xi))) - \nabla\phi(Y(\xi))]dr, \xi, \eta \in S^1. \tag{6.1.7}$$

where R is the remainder for Y w.r.t. X given by (6.1.3)*. Furthermore, there exists a constant $K \geq 1$ such that*

$$||\phi(Y)||_{\mathcal{D}_X} \leq K||\nabla\phi||_{C^1}||Y||_{\mathcal{D}_X}(1 + ||Y||_{\mathcal{D}_X})(1 + ||X||_{C^\nu})^2. \tag{6.1.8}$$

Moreover, if $(\tilde{Y}, \tilde{Z}) \in \mathcal{D}_{\tilde{X}}$ and

$$(\tilde{W}, \tilde{W}') := (\phi(\tilde{Y}), \phi'(\tilde{Y})\tilde{Z})$$

then

$$|W' - \tilde{W}'|_{C^\nu} + |R^W - R^{\tilde{W}}|_{\tilde{C}^{2\nu}} + |W - \tilde{W}|_{C^\nu} \leq$$

$$C(|X - \tilde{X}|_{C^\nu} + |Y' - \tilde{Y}'|_{C^\nu} + |R^Y - R^{\tilde{Y}}|_{\tilde{C}^{2\nu}} + |Y - \tilde{Y}|_{C^\nu}) \tag{6.1.9}$$

with

$$C = K||\phi||_{C^3}(1 + ||X||_{C^\nu} + ||\tilde{X}||_{C^\nu})^3|(1 + ||Y||_{\mathcal{D}_X} + ||\tilde{Y}||_{\mathcal{D}_{\tilde{X}}})^2. \tag{6.1.10}$$

In the case $X = \tilde{X}$ we have

$$||\phi(Y) - \phi(\tilde{Y})||_{\mathcal{D}_X} \leq K||\nabla\phi||_{C^2}||Y||_{\mathcal{D}_X}$$

$$(1 + ||Y||_{\mathcal{D}_X} + ||\tilde{Y}||_{\mathcal{D}_X})^2(1 + ||X||_{C^\nu})^4||Y - \tilde{Y}||_{\mathcal{D}_X}. \tag{6.1.11}$$

Proof of Lemma 6.1.5. See [43], Proposition 4 for all statements of the Lemma, except (6.1.7) (which is actually also proven, though not stated explicitly). Let us show (6.1.7). Denote $y(r) = Y(\xi) + r(Y(\eta) - Y(\xi)), r \in [0, 1]$. Then

$$\phi(y(1)) - \phi(y(0)) = \int_0^1 \phi'(y(r))y'(r)dr =$$

$$\sum_k (Y^k(\eta) - Y^k(\xi)) \int_0^1 \frac{\partial\phi}{\partial x_k}(y(r))dr = \sum_k \frac{\partial\phi}{\partial x_k}(Y(\xi))(Y^k(\eta) - Y^k(\xi))$$

$$+\sum_k (Y^k(\eta) - Y^k(\xi)) \int_0^1 [\frac{\partial\phi}{\partial x_k}(y(r)) - \frac{\partial\phi}{\partial x_k}(Y(\xi))]dr$$

$$= \sum_{k,l} \frac{\partial\phi}{\partial x_k}(Y(\xi))(Y')^{kl}(X^l(\eta) - X^l(\xi)) + \sum_k \frac{\partial\phi}{\partial x_k}(Y(\xi))(R^Y)^k(\xi, \eta)$$

$$+\sum_k (Y^k(\eta) - Y^k(\xi)) \int_0^1 [\frac{\partial\phi}{\partial x_k}(y(r)) - \frac{\partial\phi}{\partial x_k}(Y(\xi))]dr, \qquad (6.1.12)$$

and the result follows. □

Now we define integral of path Y weakly controlled by X w.r.t. another path Z, weakly controlled by X. We will need one more definition.

Definition 6.1.6. *Let $\nu > \frac{1}{3}$. We say that couple $\mathbb{X} = (X, \mathbb{X}^2)$, $X \in C^\nu(S^1, \mathbb{R}^3)$, $\mathbb{X}^2 \in C^{2\nu}((S^1)^2, L(\mathbb{R}^3, \mathbb{R}^3))$ is a ν-rough path if the following condition is satisfied:*

$$\mathbb{X}^2(\xi, \rho) - \mathbb{X}^2(\xi, \eta) - \mathbb{X}^2(\eta, \rho) = (X(\xi) - X(\eta)) \otimes (X(\eta) - X(\rho)), \xi, \eta, \rho \in S^1 \quad (6.1.13)$$

Remark 6.1.7. If $\nu > 1$ and $\mathbb{X}$ is a ν-rough path, then $\mathbb{X}$ is identically pair of constants $(X(0), 0)$. Indeed, in this case X is Hölder function with exponent more than 1 i.e. constant $X(0)$. Hence, $\mathbb{X}^2 = 0$.

Remark 6.1.8. If $\nu \in (\frac{1}{2}, 1]$ then $\mathbb{X}^2$, the second component of a ν-rough path $\mathbb{X} = (X, \mathbb{X}^2)$, is uniquely determined by its first component. Indeed,

$$\mathbb{X}^{2,ij}(\xi,\eta) = \int_{\xi}^{\eta} (X_\rho^i - X_\eta^i) dX_\rho^j, \xi, \eta \in S^1, i, j = 1, 2, 3 \tag{6.1.14}$$

where the integral is understood in the sense of Young, see [76]. One can show that $\mathbb{X}^2$ defined by formula (6.1.14) satisfies conditions of Definition 6.1.6. Let us show uniqueness of $\mathbb{X}^2$. Assume that there exists another $\mathbb{X}_1^2$ which satisfies definition 6.1.6. Put $G(\xi) = \mathbb{X}^2(\xi, 0) - \mathbb{X}_1^2(\xi, 0)$. Then by condition 6.1.13

$$\mathbb{X}^2(\xi, \rho) - \mathbb{X}_1^2(\xi, \rho) = G(\xi) - G(\rho),$$

and, since $\mathbb{X}^2 \in \tilde{C}^{2\nu}$, G is a Hölder function of order more than 1 i.e. 0. Therefore, $\mathbb{X}_1^2 = \mathbb{X}^2$.

Note that by identity (6.1.13) it follows that $\mathbb{X}^2(\xi, \xi) = 0, \xi \in S^1$.

Assumption 6.1.9. *We say that our ν-rough path $(X, \mathbb{X}^2)$ is an approximable ν-rough path if there exist a sequence $(X_n, \mathbb{X}_n^2)$ such that*

$$X_n \in C^\infty(S^1, \mathbb{R}^3),$$

$$\mathbb{X}_n^{2,ij}(\xi,\eta) = \int_{\xi}^{\eta} (X_n^i(\rho) - X_n^i(\eta)) dX_n^j(\rho), \xi, \eta \in S^1, i, j = 1, 2, 3.$$

and

$$|X_n - X|_{C^\nu} + |\mathbb{X}_n^2 - \mathbb{X}^2|_{\tilde{C}^{2\nu}} \to 0, n \to \infty. \tag{6.1.15}$$

Example 6.1.10. Let $\{B_t\}_{t\in[0,1]}$ be standard three dimensional brownian bridge such that $B_0 = B_1 = x_0$ and let $\mathbb{B}^{2,ij}$ be the area process

$$\mathbb{B}^{2,ij}(\xi,\eta) = \int_{\xi}^{\eta} (B^i_\rho - B^i_\eta) dB^j_\rho, i, j = 1, 2, 3$$

where integral can be understood either in Stratonovich or in Ito sense. Then, this couple $(B, \mathbb{B}^2)$ is a ν-rough path (see [9], p.1849). Moreover, it is approximable ν-rough path. Indeed, it follows from Theorem 3.1 in [38] that we can approximate X with piecewise linear dyadic X'_n in the sense of assumption 6.1.9a.s..

From now on we suppose that approximable ν-rough path $\mathbb{X} = (X, \mathbb{X}^2)$ and corresponding Banach space $\mathcal{D}_X$ are fixed.

Lemma 6.1.11. *Let $\pi = \{\xi_0 = \xi < \xi_1 < \ldots < \xi_n = \eta\}$ be a finite partition of $[\xi, \eta]$ and $d(\pi) = \sup_i |\xi_{i+1} - \xi_i|$ is a mesh of π. If $Y, Z \in \mathcal{D}_X$ then the limit*

$$\lim_{d(\pi)\to 0} \sum_{i=0}^{n-1} [Y(\xi_i)(Z(\xi_{i+1}) - Z(\xi_i)) + Y'(\xi_i)Z'(\xi_i)\mathbb{X}^2(\xi_{i+1}, \xi_i)] \tag{6.1.16}$$

exists and is denoted by definition by

$$\int_{\xi}^{\eta} Y dZ.$$

Proof of Lemma 6.1.11. See [43], Theorem 1. □

Remark 6.1.12. In the case of $\nu > \frac{1}{2}$ line integral defined in the Lemma 6.1.11 is reduced to the Young definition of line integral $\int Y dZ$. Indeed, it is enough to notice that second term in formula (6.1.16) is of the order $O(|\xi_{i+1} - \xi_i|^{2\nu})$, $2\nu > 1$. Obviously, line integral does not depend upon Y', Z' in this case.

Lemma 6.1.13. *Assume* $Y, W \in \mathcal{D}_X$, $\tilde{Y}, \tilde{W} \in \mathcal{D}_{\tilde{X}}$. *Define* $Q, \tilde{Q} : (S^1)^2 \to \mathbb{R}$ *by identities*

$$Q(\eta,\xi) := \int_{\xi}^{\eta} Y dW - Y(\xi)(W(\eta) - W(\xi)) - Y'(\xi)W'(\xi)\mathbb{X}^2(\eta,\xi), \eta, \xi \in S^1, \quad (6.1.17)$$

$$\tilde{Q}(\eta,\xi) := \int_{\xi}^{\eta} \tilde{Y} d\tilde{W} - \tilde{Y}(\xi)(\tilde{W}(\eta) - \tilde{W}(\xi)) - \tilde{Y}'(\xi)\tilde{W}'(\xi)\tilde{\mathbb{X}}^2(\eta,\xi), \eta, \xi \in S^1. \quad (6.1.18)$$

then $Q, \tilde{Q} \in \tilde{C}^{3\nu}$. *Moreover, there exists constant* $C = C(\nu) > 0$ *such that for all* $Y, W \in \mathcal{D}_X$

$$||Q||_{\tilde{C}^{3\nu}} \leq C(1 + ||X||_{C^\nu} + ||\mathbb{X}^2||_{\tilde{C}^{2\nu}})||Y||_{\mathcal{D}_X}||W||_{\mathcal{D}_X}. \quad (6.1.19)$$

Furthermore,

$$||Q - \tilde{Q}||_{\tilde{C}^{3\nu}} \leq C(1 + ||X||_{C^\nu} + ||\mathbb{X}^2||_{\tilde{C}^{2\nu}}) \\ ((||Y||_{\mathcal{D}_X} + ||\tilde{Y}||_{\mathcal{D}_{\tilde{X}}})\varepsilon_W + (||W||_{\mathcal{D}_X} + ||\tilde{W}||_{\mathcal{D}_{\tilde{X}}})\varepsilon_Y + \varepsilon_X). \quad (6.1.20)$$

where

$$\varepsilon_Y = |Y' - \tilde{Y}'|_{C^\nu} + |R^Y - R^{\tilde{Y}}|_{\tilde{C}^{2\nu}} + |Y - \tilde{Y}|_{C^\nu},$$

$$\varepsilon_W = |W' - \tilde{W}'|_{C^\nu} + |R^W - R^{\tilde{W}}|_{\tilde{C}^{2\nu}} + |W - \tilde{W}|_{C^\nu},$$

$$\varepsilon_X = (||Y||_{\mathcal{D}_X} + ||\tilde{Y}||_{\mathcal{D}_{\tilde{X}}})(||W||_{\mathcal{D}_X} + ||\tilde{W}||_{\mathcal{D}_{\tilde{X}}})(|X - \tilde{X}|_{C^\nu} + |\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{\tilde{C}^{2\nu}}).$$

Proof of Lemma 6.1.13. See [43], Theorem 1. For formula (6.1.20) see [43], p.104, formula (27). □

By Lemmas 6.1.11 and 6.1.5 for any $A \in C^2(\mathbb{R}^3, L(\mathbb{R}^3, \mathbb{R}^3)), Y \in \mathcal{D}_X$ we can define rough path integral $V^Y : \mathbb{R}^3 \to \mathbb{R}$ as follows

$$V^Y(x) := \int_{S^1} A(x - Y)dY, x \in \mathbb{R}^3. \quad (6.1.21)$$

We have following bounds on its regularity:

Lemma 6.1.14. *Let $Y \in \mathcal{D}_X$, $\tilde{Y} \in \mathcal{D}_{\tilde{X}}$, then there exists $C_1 = C_1(\nu)$, $C_2 = C_2(\mathbb{X})$ such that for any integer $n \geq 0$,*

$$||\nabla^n V^Y||_{L^\infty} \leq 4C_1C_2^3||\nabla^{n+1}A||_{C^1}||Y||^2_{\mathcal{D}_X}(1+||Y||_{\mathcal{D}_X}) \tag{6.1.22}$$

and

$$||\nabla^n V^Y - \nabla^n V^{\tilde{Y}}||_{L^\infty} \leq C(\nu)|A|_{C^{n+3}}C_X^4(1+||Y||_{\mathcal{D}_X}+||\tilde{Y}||_{\mathcal{D}_{\tilde{X}}})^3$$
$$(|X-\tilde{X}|_{C^\nu}+|\mathbb{X}^2-\tilde{\mathbb{X}}^2|_{\tilde{C}^{2\nu}}+|Y'-\tilde{Y}'|_{C^\nu}+|R^Y-R^{\tilde{Y}}|_{\tilde{C}^{2\nu}}+|Y-\tilde{Y}|_{C^\nu}). \tag{6.1.23}$$

where

$$C_X = 1+|X|_{C^\nu}+|\tilde{X}|_{C^\nu}+|\mathbb{X}^2|_{\tilde{C}^{2\nu}}+|\tilde{\mathbb{X}}^2|_{\tilde{C}^{2\nu}}.$$

In the case of $X = \tilde{X}$ inequality can be written as

$$||\nabla^n V^Y - \nabla^n V^{\tilde{Y}}||_{L^\infty} \leq 16C_1C_2^3||\nabla^{n+1}A||_{C^2}||Y||_{\mathcal{D}_X}(1+||Y||_{\mathcal{D}_X})^2||Y-\tilde{Y}||^*_{\mathcal{D}_X}. \tag{6.1.24}$$

Proof of Lemma 6.1.14. Inequalities (6.1.22) and (6.1.24) were proved in [9], Lemma 7. Now we will show (6.1.23). It is enough to consider the case of $n = 0$. By formulas (6.1.19) and (6.1.18) we have

$$V^Y - V^{\tilde{Y}}(x) = A(x-Y(0))(Y(1)-Y(0)) - A(x-\tilde{Y}(0))(\tilde{Y}(1)-\tilde{Y}(0)) +$$
$$(A(x-Y))'(0)Y'(0)\mathbb{X}^2(0,1) - (A(x-\tilde{Y}))'(0)\tilde{Y}'(0)\tilde{\mathbb{X}}^2(0,1) + Q^x(0,1) - \tilde{Q}^x(0,1)$$

where Q^x and $\tilde{Q}^x$ (given by formulas (6.1.19) and (6.1.18)) satisfy inequality (6.1.20) and we have identified S^1 with $[0,1]$. Therefore, $Y(1) = Y(0)$, $\tilde{Y}(1) = \tilde{Y}(0)$. Hence, we have

$$|V^Y - V^{\tilde{Y}}|_{L^\infty} \leq \sup_x |(A(x-Y))'(0)Y'(0)\mathbb{X}^2(0,1) - (A(x-\tilde{Y}))'(0)\tilde{Y}'(0)\tilde{\mathbb{X}}^2(0,1)|$$
$$+ \sup_x |Q^x(0,1) - \tilde{Q}^x(0,1)|. \tag{6.1.25}$$

For the first term on the r.h.s. we have

$$
\begin{aligned}
&|(A(x-Y))'(0)Y'(0)\mathbb{X}^2(0,1) - (A(x-\tilde{Y}))'(0)\tilde{Y}'(0)\tilde{\mathbb{X}}^2(0,1)| \\
&\qquad \leq |(dA(x-Y(0))Y'(0)Y'(0) - dA(x-\tilde{Y}(0))\tilde{Y}'(0)\tilde{Y}'(0))\mathbb{X}^2(0,1)| \\
&\qquad\qquad + |dA(x-\tilde{Y}(0))\tilde{Y}'(0)\tilde{Y}'(0)(\mathbb{X}^2(0,1) - \tilde{\mathbb{X}}^2(0,1))| \\
&\qquad \leq |\mathbb{X}^2|_{\tilde{C}^{2\nu}} |dA(x-Y(0))Y'(0)Y'(0) - dA(x-\tilde{Y}(0))\tilde{Y}'(0)\tilde{Y}'(0)| \\
&\qquad\qquad + |A|_{C^2}|Y'|^2_{L^\infty}|\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{\tilde{C}^{2\nu}} \\
&\leq |\mathbb{X}^2|_{\tilde{C}^{2\nu}}|Y'|^2_{L^\infty}|A|_{C^2}|Y' - \tilde{Y}'|_{L^\infty} + |\mathbb{X}^2|_{\tilde{C}^{2\nu}}|A|_{C^1}(|Y'|_{L^\infty} + |\tilde{Y}'|_{L^\infty})|Y' - \tilde{Y}'|_{L^\infty} \\
&\qquad\qquad + |A|_{C^2}|Y'|^2_{L^\infty}|\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{\tilde{C}^{2\nu}}. \qquad (6.1.26)
\end{aligned}
$$

By (6.1.20) we can estimate second term as follows

$$
|Q^x - \tilde{Q}^x|_{\tilde{C}^{3\nu}} \leq C((||A(x-Y)||_{\mathcal{D}_X} + ||A(x-\tilde{Y})||_{\mathcal{D}_{\tilde{X}}})\varepsilon_Y + (||Y||_{\mathcal{D}_X} + ||\tilde{Y}||_{\mathcal{D}_{\tilde{X}}})\varepsilon_A + \varepsilon_X). \qquad (6.1.27)
$$

where

$$
\varepsilon_Y = |Y - \tilde{Y}|_{C^\nu} + |Y' - \tilde{Y}'|_{C^\nu} + |R^Y - R^{\tilde{Y}}|_{\tilde{C}^{2\nu}},
$$

$$
\varepsilon_A = |A(x-Y) - A(x-\tilde{Y})|_{C^\nu} + |A(x-Y)' - A(x-\tilde{Y})'|_{C^\nu} + |R^{A(x-Y)} - R^{A(x-\tilde{Y})}|_{\tilde{C}^{2\nu}},
$$

$$
\varepsilon_X = (||A(x-Y)||_{\mathcal{D}_X} + ||A(x-\tilde{Y})||_{\mathcal{D}_{\tilde{X}}})(||Y||_{\mathcal{D}_X} + ||\tilde{Y}||_{\mathcal{D}_{\tilde{X}}})(|X - \tilde{X}|_{C^\nu} + |\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{\tilde{C}^{2\nu}}).
$$

By formula (6.1.9) we can estimate ε_A as follows

$$
\begin{aligned}
|\varepsilon_A| \leq K|A|_{C^3}(1 + |X|_{C^\nu} + |\tilde{X}|_{C^\nu})^3(1 + |Y|_{D_X} + |\tilde{Y}|_{D_{\tilde{X}}})^2 \times \\
(|X - \tilde{X}|_{C^\nu} + |Y - \tilde{Y}|_{C^\nu} + |Y' - \tilde{Y}'|_{C^\nu} + |R^Y - R^{\tilde{Y}}|_{\tilde{C}^{2\nu}}). \qquad (6.1.28)
\end{aligned}
$$

By inequality (6.1.8) we infer that

$$
||A(x-Y)||_{\mathcal{D}_X} \leq K|A|_{C^2}|Y|_{D_X}(1 + |Y|_{D_X})(1 + |X|_{C^\nu})^2, \qquad (6.1.29)
$$

and similarly,

$$||A(x-\tilde{Y})||_{\mathcal{D}_{\tilde{X}}} \leq K|A|_{C^2}|\tilde{Y}|_{D_{\tilde{X}}}(1+|\tilde{Y}|_{D_{\tilde{X}}})(1+|\tilde{X}|_{C^\nu})^2. \quad (6.1.30)$$

Therefore, combining (6.1.27) with (6.1.28), (6.1.29) and (6.1.30) we get

$$|Q^x-\tilde{Q}^x|_{\tilde{C}^{3\nu}} \leq C(\nu)|A|_{C^{n+3}}(1+|X|_{C^\nu}+|\tilde{X}|_{C^\nu})^4(1+||Y||_{\mathcal{D}_X}+||\tilde{Y}||_{\mathcal{D}_{\tilde{X}}})^3$$
$$(|X-\tilde{X}|_{C^\nu}+|\mathbb{X}^2-\tilde{\mathbb{X}}^2|)_{\tilde{C}^{2\nu}}+|Y'-\tilde{Y}'|_{C^\nu}+|R^Y-R^{\tilde{Y}}|_{\tilde{C}^{2\nu}}+|Y-\tilde{Y}|_{C^\nu}). \quad (6.1.31)$$

Hence, the result follows from (6.1.26) and (6.1.31). □

We will denote for any $Y\in\mathcal{D}_X$, $\tilde{Y}\in\mathcal{D}_{\tilde{X}}$

$$|Y-\tilde{Y}|_D=|X-\tilde{X}|_{C^\nu}+|\mathbb{X}^2-\tilde{\mathbb{X}}^2|_{\tilde{C}^{2\nu}}+|Y'-\tilde{Y}'|_{C^\nu}+|R^Y-R^{\tilde{Y}}|_{\tilde{C}^{2\nu}}+|Y-\tilde{Y}|_{C^\nu}.$$

6.2 Random Filaments evolution problem

Let $\mathcal{D}_{\mathbb{X},T}=C([0,T],\mathcal{D}_X)$ be a vector space with the usual supremum norm

$$|F|_{\mathcal{D}_{\mathbb{X},T}}=\sup_{t\in[0,T]}|F(t)|^*_{\mathcal{D}_X}. \quad (6.2.1)$$

Obviously $\mathcal{D}_{\mathbb{X},T}$ is a Banach space. Assume also that the function ϕ appeared in the formula (6.0.3) satisfies following hypothesis.

Hypothesis 6.2.1. *(i) $\phi:\mathbb{R}^3\to\mathbb{R}$ is even function.*

(ii) the Fourier transform of ϕ is real and non-negative function:

$$\hat{\phi}(k)\geq 0, k\in\mathbb{R}^3$$

(iii)

$$\int_{\mathbb{R}^3}(1+|k|^2)^2\hat{\phi}(k)dk<\infty$$

Example 6.2.2. The function $\phi(\cdot) = \frac{\Gamma}{(|\cdot|^2+\mu^2)^{1/2}}, \mu > 0$ is smooth and satisfies hypothesis 6.2.1, see p.6 of [7].

Then the following local existence and uniqueness Theorem for problem (6.0.1)-(6.0.3) has been proved in [9], see Theorem 3,p.1842.

Theorem 6.2.3. *Assume* $\phi \in C^6(\mathbb{R}^3, \mathbb{R})$, $\nu \in (\frac{1}{3}, 1)$, $\mathbb{X} = (X, \mathbb{X}^2)$ *is a* ν*-rough path,* $\gamma_0 \in \mathcal{D}_X$. *Then there exists a time* $T_0 = T_0(\nu, |\phi|_{C^5}, \mathbb{X}) > 0$ *such that the problem* (6.0.1)-(6.0.3) *has unique solution in* $\mathcal{D}_{\mathbb{X},T_0}$.

Our aim is to prove global existence of solution of the problem (6.0.1)-(6.0.3) under assumptions of Theorem 6.2.3 and additional hypothesis 6.2.1 i.e. we shall prove

Theorem 6.2.4. *Assume* $T > 0$, $\gamma_0 \in \mathcal{D}_X$, $\phi \in C^6(\mathbb{R}^3, \mathbb{R})$,$\phi$ *satisfies hypothesis 6.2.1,* $\mathbb{X}$ *is an approximable* ν*-rough path,* $\nu \in (\frac{1}{3}, 1)$. *Then the problem* (6.0.1)-(6.0.3) *has unique solution in* $\mathcal{D}_{\mathbb{X},T}$.

We will need the following definition.

Definition 6.2.5. *Let* $\phi \in C^4(\mathbb{R}^3, \mathbb{R})$, $\gamma \in \mathcal{D}_X$. *Put*

$$\mathcal{H}_X(\gamma) = \frac{1}{2} \int\limits_{S^1} \vec{\psi}^\gamma(\gamma(\xi)) \cdot d\vec{\gamma}(\xi), \tag{6.2.2}$$

where

$$\vec{\psi}^\gamma(x) = \int\limits_{S^1} \phi(x - \gamma(\eta)) d\vec{\gamma}(\eta).$$

$\mathcal{H}_X(\gamma)$ *is called the energy of path* γ.

Remark 6.2.6. Definition (6.2.2)-(6.2.5) is well posed. Indeed, by Lemma 6.1.14 $\vec{\psi}^\gamma \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ and, therefore, it follows by Lemma 6.1.5 that $\vec{\psi}^\gamma(\gamma(\cdot)) \in \mathcal{D}_X$.

Remark 6.2.7. Assume that $\nu > \frac{1}{2}$ and $\gamma \in C^1(S^1, \mathbb{R}^3)$. Then by Remark 6.1.12 the line integrals in the definition of energy are understood in the sense of Young and

$$\mathcal{H}_X(\gamma) = \frac{1}{2}\int_{S^1}\int_{S^1} \phi(\vec{\gamma}(\xi) - \vec{\gamma}(\eta))(\frac{d\vec{\gamma}}{d\xi}(\xi), \frac{d\vec{\gamma}}{d\eta}(\eta))d\xi d\eta. \quad (6.2.3)$$

Lemma 6.2.8. *Assume $\phi \in C^4(\mathbb{R}^3, \mathbb{R})$. Then there exists constant $C = C(\nu, \mathbb{X})$ such that for all $\gamma \in \mathcal{D}_X$*

$$|\mathcal{H}_X(\gamma)| \leq C|\phi|_{C^4}|\gamma|^4_{\mathcal{D}_X}(1 + |\gamma|_{\mathcal{D}_X})^2. \quad (6.2.4)$$

Moreover, the map $\mathcal{H}_X : \mathcal{D}_X \to \mathbb{R}$ is locally Lipshitz i.e. for any $R > 0$ there exists $C = C(R)$ such that for any $\gamma, \tilde{\gamma} \in \mathcal{D}_X$, $|\gamma|_{\mathcal{D}_X} < R$, $|\tilde{\gamma}|_{\mathcal{D}_X} < R$ we have

$$|\mathcal{H}_X(\gamma) - \mathcal{H}_X(\tilde{\gamma})| \leq C(R)|\gamma - \tilde{\gamma}|^*_{\mathcal{D}_X}. \quad (6.2.5)$$

Furthermore, for any $R > 0$ there exists $C = C(R)$ such that for any $\gamma \in \mathcal{D}_X$, $\tilde{\gamma} \in \mathcal{D}_{\tilde{X}}$,

$$|\gamma|_{\mathcal{D}_X} < R, |\tilde{\gamma}|_{\mathcal{D}_{\tilde{X}}} < R,$$

$$C_{X,\tilde{X}} = |X|_{C^\nu} + |\tilde{X}|_{C^\nu} + |\mathbb{X}^2|_{\tilde{C}^{2\nu}} + |\tilde{\mathbb{X}}^2|_{\tilde{C}^{2\nu}} < R$$

we have

$$|\mathcal{H}_X(\gamma)-\mathcal{H}_{\tilde{X}}(\tilde{\gamma})| \leq C(R)(|X-\tilde{X}|_{C^\nu}+|\mathbb{X}^2-\tilde{\mathbb{X}}^2|_{\tilde{C}^{2\nu}}+|\gamma-\tilde{\gamma}|_{C^\nu}+|\gamma'-\tilde{\gamma}'|_{C^\nu}+|R^\gamma-R^{\tilde{\gamma}}|_{\tilde{C}^{2\nu}}).$$

Proof of Lemma 6.2.8. First we will show inequality (6.2.4). By representation (6.1.17) we have

$$\mathcal{H}_X(\gamma) = \frac{1}{2}(\vec{\iota}^\gamma(\gamma(0))(\vec{\gamma}(1) - \vec{\gamma}(0))) +$$
$$\left[d\vec{\iota}^\gamma(\gamma(0))\gamma'(0)\right]\gamma'(0)\mathbb{X}^2(1, 0) + Q(0, 1) = I + II + III, \gamma \in \mathcal{D}_X.$$

Since $\vec{\gamma}(1) = \vec{\gamma}(0)$ we infer that $I = 0$. Concerning the second term by Lemma 6.1.14 we have the following estimate

$$|II| \leq ||\mathbb{X}^2||_{\tilde{C}^{2\nu}} ||\nabla\vec{\psi}||_{L^\infty} |\gamma'|^2_{L^\infty} \leq$$
$$\leq C(\nu, \mathbb{X}) |\phi|_{C^3} |\gamma|^4_{\mathcal{D}_X} (1 + |\gamma|_{\mathcal{D}_X}). \tag{6.2.6}$$

For third term we infer from inequality (6.1.19)

$$|III| \leq |Q||_{\tilde{C}^{3\nu}} \leq C(\nu, \mathbb{X}) |\vec{\psi}(\vec{\gamma})|_{\mathcal{D}_X} |\vec{\gamma}|_{\mathcal{D}_X}. \tag{6.2.7}$$

Then by Lemmas 6.1.5 and 6.1.14 we have

$$|\vec{\psi}^\gamma(\vec{\gamma})|_{\mathcal{D}_X} \leq C(\nu, \mathbb{X}) |\psi^\gamma|_{C^2} |\vec{\gamma}|_{\mathcal{D}_X} (1 + |\vec{\gamma}|_{\mathcal{D}_X}) \leq$$
$$C(\nu, \mathbb{X}) |\phi|_{C^4} |\vec{\gamma}|^3_{\mathcal{D}_X} (1 + |\vec{\gamma}|_{\mathcal{D}_X})^2 \tag{6.2.8}$$

Combining (6.2.6), (6.2.7) and (6.2.8) we get inequality (6.2.4).

Now we will prove inequality (6.2.6). By formula (6.1.18) we have

$$\mathcal{H}_X(\gamma) - \mathcal{H}_{\tilde{X}}(\tilde{\gamma}) = \frac{1}{2} \Big[(\nabla\psi^\gamma(\gamma(0))\gamma'(0)\gamma'(0) - \nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))\tilde{\gamma}'(0)\tilde{\gamma}'(0))\mathbb{X}^2(1,0)$$
$$+\nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))\tilde{\gamma}'(0)\tilde{\gamma}'(0)(\mathbb{X}^2(1,0) - \tilde{\mathbb{X}}^2(1,0)) + Q(0,1) - \tilde{Q}(0,1) \Big] = I + II + III \tag{6.2.9}$$

The first term in (6.2.9) can be represented as follows

$$\begin{aligned} I &= (\nabla\psi^\gamma(\gamma(0))\gamma'(0)\gamma'(0) - \nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))\tilde{\gamma}'(0)\tilde{\gamma}'(0))\mathbb{X}^2(1,0) \\ &= \big[(\nabla\psi^\gamma(\gamma(0)) - \nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0)))\gamma'(0)\gamma'(0) \\ &+ \nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))(\gamma'(0) - \tilde{\gamma}(0))\gamma'(0) \\ &+ \nabla\psi^{\tilde{\gamma}}(\tilde{\gamma}(0))\tilde{\gamma}(0)(\gamma'(0) - \tilde{\gamma}(0)) \big] \mathbb{X}^2(1,0) = A + B + C \end{aligned} \tag{6.2.10}$$

The first term in (6.2.10) can be estimated as follows

$$\begin{aligned}|A| &= |(\nabla\psi^{\gamma}(\gamma(0)) - \nabla\psi^{\tilde\gamma}(\tilde\gamma(0)))\gamma'(0)\gamma'(0)\mathbb{X}^2(1,0)| \\ &\leq ||\mathbb{X}^2||_{\tilde C^{2\nu}}|\gamma|^2_{\mathcal{D}_X}(|\nabla\psi^{\gamma}(\gamma(0)) - \nabla\psi^{\gamma}(\tilde\gamma(0))| + |\nabla\psi^{\gamma}(\tilde\gamma(0)) - \nabla\psi^{\tilde\gamma}(\tilde\gamma(0))|) \\ &\leq ||\mathbb{X}^2||_{\tilde C^{2\nu}}|\gamma|^2_{\mathcal{D}_X}(|\psi^{\gamma}|_{C^2}|\gamma(0) - \tilde\gamma(0)| + C_X^4|\phi|_{C^4}(1 + |\gamma|_{\mathcal{D}_X} + |\tilde\gamma|_{\mathcal{D}_{\tilde X}})^3|\gamma - \tilde\gamma|_D \\ &\leq KC_X^4|\phi|_{C^4}(1 + |\gamma|_{\mathcal{D}_X} + |\tilde\gamma|_{\mathcal{D}_{\tilde X}})^3|\gamma - \tilde\gamma|_D, \quad (6.2.11)\end{aligned}$$

where second inequality follows from inequality (6.1.23) and third one from inequality (6.1.22). For second term we have by inequality (6.1.22)

$$\begin{aligned}|B| &\leq C||\mathbb{X}^2||_{\tilde C^{2\nu}}|\gamma|_{\mathcal{D}_X}|\phi|_{C^3}|\tilde\gamma|^2_{\mathcal{D}_X}(1 + |\tilde\gamma|_{\mathcal{D}_X})|\gamma - \tilde\gamma|_D \leq \\ &\qquad CC_X(1 + |\gamma|_{\mathcal{D}_X} + |\tilde\gamma|_{\mathcal{D}_{\tilde X}})^3|\gamma - \tilde\gamma|_D. \quad (6.2.12)\end{aligned}$$

Similarly, we have for third term

$$|C| \leq C(\nu, \mathbb{X}, |\gamma|_{\mathcal{D}_X}, |\tilde\gamma|_{\mathcal{D}_X})|\gamma - \tilde\gamma|_D. \quad (6.2.13)$$

Term II in (6.2.9) can be estimated as follows

$$|II| \leq |\nabla\psi^{\tilde\gamma}|_{L^\infty}|\tilde\gamma|^2_{\mathcal{D}_X}|\gamma - \tilde\gamma|_D \leq C_X^3|\phi|_{C^3}(1 + |\gamma|_{\mathcal{D}_X} + |\tilde\gamma|_{\mathcal{D}_{\tilde X}})^3|\gamma - \tilde\gamma|_D. \quad (6.2.14)$$

Thus it remains to estimate third term of equality (6.2.9). We have by inequality (6.1.20)

$$\begin{aligned}|Q(0,1) - \tilde Q(0,1)| &\leq ||Q - \tilde Q||_{\tilde C^{3\nu}} \leq C_X((|\psi^{\tilde\gamma}(\tilde\gamma)|_{\mathcal{D}_{\tilde X}} + |\psi^{\gamma}(\gamma)|_{\mathcal{D}_X})|\gamma - \tilde\gamma|_D \\ &\quad + (|\tilde\gamma|_{\mathcal{D}_{\tilde X}} + |\tilde\gamma|_{\mathcal{D}_{\tilde X}})|\psi^{\tilde\gamma}(\tilde\gamma) - \psi^{\gamma}(\gamma)|_D \\ &+ (|\psi^{\tilde\gamma}(\tilde\gamma)|_{\mathcal{D}_{\tilde X}} + |\psi^{\gamma}(\gamma)|_{\mathcal{D}_X})(|\tilde\gamma|_{\mathcal{D}_{\tilde X}} + |\tilde\gamma|_{\mathcal{D}_{\tilde X}})(|X - \tilde X|_{C^\nu} + |\mathbb{X}^2 - \tilde{\mathbb{X}}^2|_{\tilde C^{2\nu}})) \quad (6.2.15)\end{aligned}$$

Term $|\psi^{\tilde\gamma}(\tilde\gamma)|_{\mathcal{D}_X}$ is bounded by the constant $C = C(\nu, \mathbb{X}, |\tilde\gamma|_{\mathcal{D}_X})$ by inequality (6.1.22). Therefore, to prove estimate (6.2.5) it is enough to show that there exists constant $C =$

$C(\nu, \mathbb{X}, R)$ such that for $\gamma, \tilde{\gamma} \in \mathcal{D}_X$ with $|\gamma|_{\mathcal{D}_X}, |\tilde{\gamma}|_{\mathcal{D}_X} \leq R$

$$|\psi^{\tilde{\gamma}}(\tilde{\gamma}) - \psi^{\gamma}(\gamma)|_D \leq C|\tilde{\gamma} - \gamma|_D. \tag{6.2.16}$$

By triangle inequality we have

$$|\psi^{\tilde{\gamma}}(\tilde{\gamma}) - \psi^{\gamma}(\gamma)|_D \leq |\psi^{\tilde{\gamma}}(\tilde{\gamma}) - \psi^{\tilde{\gamma}}(\gamma)|_D + |\psi^{\tilde{\gamma}}(\gamma) - \psi^{\gamma}(\gamma)|_D = I + II. \tag{6.2.17}$$

The first term can be estimated using inequality (6.1.9) as follows

$$|I| \leq KC_X^3 |\psi^{\tilde{\gamma}}|_{C^3}(1 + |\tilde{\gamma}|_{\mathcal{D}_{\tilde{X}}} + |\gamma|_{\mathcal{D}_X})^2 |\tilde{\gamma} - \gamma|_D. \tag{6.2.18}$$

By inequality (6.1.22) we have

$$|\psi^{\tilde{\gamma}}|_{C^3} \leq C|\phi|_{C^5} |\tilde{\gamma}|^2_{\mathcal{D}_X}(1 + |\tilde{\gamma}|_{\mathcal{D}_X}) \tag{6.2.19}$$

Combining (6.2.18) and (6.2.19) we get necessary estimate for I. It remains to find an estimate for term II. By inequalities (6.1.8) and (6.1.24) we have

$$\begin{aligned} II = |\psi^{\tilde{\gamma}}(\gamma) - \psi^{\gamma}(\gamma)|_D &\leq (1 + |X|_\nu)|\psi^{\tilde{\gamma}}(\gamma) - \psi^{\gamma}(\gamma)|_{\mathcal{D}_X} \leq \\ &K|\nabla\psi^{\tilde{\gamma}} - \nabla\psi^{\gamma}|_{C^1}|\gamma|_{\mathcal{D}_X}(1 + |\gamma|_{\mathcal{D}_X})(1 + |X|_\nu)^3 \leq \\ &K|\phi|_{C^5} C_X^7 (1 + |\gamma|_{\mathcal{D}_X} + |\tilde{\gamma}|_{\mathcal{D}_{\tilde{X}}})^5 |\tilde{\gamma} - \gamma|_D. \end{aligned} \tag{6.2.20}$$

Hence the inequality (6.2.6) follows. Inequality (6.2.5) is a consequence of inequality (6.2.6). □

Corollary 6.2.9. *Under assumptions of Lemma 6.2.8 and assumption 6.1.9 energy* $\mathcal{H}_X : \mathcal{D}_X \to \mathbb{R}$ *is a continuous function on* $\mathcal{D}_X$. *Furthermore, for any* $\gamma \in \mathcal{D}_X$

$$0 \leq \mathcal{H}_X(\gamma) < \infty.$$

Proof of Corollary 6.2.9. We only need to show that $\mathcal{H}_X(\gamma) \geq 0$, for any $\gamma \in \mathcal{D}_X$. Other statements of the Corollary easily follow from Lemma 6.2.8. Let $q_n \in C^\infty(S^1, L(\mathbb{R}^3, \mathbb{R}^3))$ and $X_n \in C^\infty(S^1, \mathbb{R}^3)$ be such that

$$\gamma' = \lim_{n\to\infty} q_n \text{ w.r.t. the topology of } C^\nu(S^1, L(\mathbb{R}^3, \mathbb{R}^3)),$$

$$X = \lim_{n\to\infty} X_n \text{ w.r.t. the topology of } C^\nu(S^1, \mathbb{R}^3).$$

and

$$\mathbb{X}^2(\xi, \eta) = \lim_{n\to\infty} \mathbb{X}_n^2 \text{ w.r.t. the topology of } \tilde{C}^{2\nu}((S^1)^2, \mathbb{R}^3)$$

where

$$\mathbb{X}_n^2(\xi, \eta) = \int_\xi^\eta (X_\rho - X_\xi) dX_\rho, \xi, \eta \in S^1.$$

Existence of such X_n follows from Assumption 6.1.9. Existence of q_n follows from density of $C^\infty(S^1, L(\mathbb{R}^3, \mathbb{R}^3))$ in $C^\nu(S^1, L(\mathbb{R}^3, \mathbb{R}^3))$. Define

$$\gamma_n(\xi) = \gamma(0) + \int_0^\xi q_n dX_n, \xi \in S^1.$$

Then $\gamma_n \in D_{X_n} \subset C^\infty(S^1, \mathbb{R}^3)$ and

$$\lim_{n\to\infty} |\gamma_n - \gamma|_D = 0.$$

Therefore, by inequality (6.2.6) it is enough to show that $\mathcal{H}_X$ is nonnegative for $\gamma_n \in C^\infty(S^1, \mathbb{R}^3)$ and the result follows from Lemma 1 of [7]. □

Now we are going to show that energy is a local integral of motion for problem (6.0.1)-(6.0.3).

Lemma 6.2.10. *Let $\gamma \in \mathcal{D}_{\mathbb{X},T_0}$ be a local solution of problem* (6.0.1)-(6.0.3) *(such a solution exists by Theorem 6.2.3). Then*

$$\frac{d\mathcal{H}_X(\gamma(s))}{ds} = 0, s \in [0, T_0).$$

Proof of Lemma 6.2.10. Let $\gamma(0) = \gamma_0 \in \mathcal{D}_X$. We can construct (in the same way as in the Corollary 6.2.9) sequence $\{\gamma_0^n\}_{n=1}^{\infty} \in C^{\infty}(S^1, \mathbb{R}^3)$ such that

$$|\gamma_0^n - \gamma_0|_D \to 0, n \to \infty.$$

Denote $\gamma^n \in C([0,\infty), \mathbf{H}^1(S^1,\mathbb{R}^3))$ the global solution of problem (6.0.1)-(6.0.3) with initial condition γ_0^n. Existence of such solution proved in Theorem 2 of [7]. Then according to [9] (Theorem 4, p.1846) we have

$$\sup_{t\in[0,T_0]} |\gamma^n(t) - \gamma(t)|_D \to 0, n \to \infty.$$

Therefore, by continuity of energy functional $\mathcal{H}_X$ we have

$$\mathcal{H}_X(\gamma(s)) = \lim_{n\to\infty} \mathcal{H}_{X_n}(\gamma^n(s)), s \in [0, T_0]. \tag{6.2.21}$$

Furthermore, by Lemma 2 of [7], we have

$$\mathcal{H}_{X_n}(\gamma^n(s)) = \mathcal{H}_{X_n}(\gamma_0^n), s \in [0, T_0]. \tag{6.2.22}$$

As a result, combining (6.2.21) and (6.2.22) we get statement of the Lemma. □

Let us recall definition of along with ν-rough path Y

$$u^Y(x) = \int_Y \nabla\phi(x-y) \times dy, Y \in \mathcal{D}_X. \tag{6.2.23}$$

Now we show that if energy functional of Y is bounded then associated velocity is a smooth function. We have

Lemma 6.2.11 (See Lemma 3 in [7]). *For any $n \in \mathbb{Z}, n \geq 0$, we have following bound*

$$||\nabla^n u^\gamma||_{L^\infty} \leq \frac{1}{(2\pi)^{3/2}} \left[\int_{\mathbb{R}^3} |\vec{k}|^{2(1+n)} \hat{\phi}(\vec{k}) d\vec{k} \right]^{1/2} \mathcal{H}_X^{1/2}(\gamma), \gamma \in \mathcal{D}_X. \tag{6.2.24}$$

provided that the integral $\int_{\mathbb{R}^3} |\vec{k}|^{2(1+n)} \hat{\phi}(\vec{k}) d\vec{k}$ is finite and $\phi \in C^{n+4}(\mathbb{R}^3, \mathbb{R}^3)$.

Proof of Lemma 6.2.11. For smooth γ Lemma has been proved in [7] Lemma 3. In general case, when $\gamma \in \mathcal{D}_X$, it is enough to notice that both sides of inequality (6.2.24) are locally Lipshitz and therefore, continuous w.r.t. distance $d(Y, \tilde{Y}) := |Y - \tilde{Y}|_D, Y \in \mathcal{D}_X, \tilde{Y} \in \mathcal{D}_{\tilde{X}}$. Indeed, continuity of $\mathcal{H}_X$ has been proven in Lemma 6.2.8 and continuity of $||\nabla^n u^\gamma||_{L^\infty}$ follows from Lemma 6.1.14. □

Now we are ready to prove Theorem 6.2.4.

Proof of Theorem 6.2.4. According to Theorem 6.2.3 there exists unique local solution of problem (6.0.1)-(6.0.3). Then, we can find $T^* > 0$ such that there exists unique maximal local solution $\gamma : [0, T^*) \to \mathcal{D}_X$ and

$$\lim_{t \uparrow T^*} ||\gamma(t)||_{\mathcal{D}_X} = \infty, \tag{6.2.25}$$

see e.g. [19]. Notice that we will have

$$\frac{d\mathcal{H}_X(\gamma(s))}{ds} = 0, s \in [0, T^*). \tag{6.2.26}$$

Indeed, by Theorem 6.2.3 for any $t_0 \in [0, T^*)$ there exists unique local solution $\tilde{\gamma}$ of problem (6.0.1), (6.0.3) with initial condition $\gamma(t_0)$ on segment $[t_0, t_0 + \varepsilon_{t_0}]$ for some $\varepsilon_{t_0} > 0$. Therefore, $\gamma = \tilde{\gamma}$ on the segment $[t_0, t_0 + \varepsilon_{t_0}]$. Hence,

$$\frac{d\mathcal{H}_X(\gamma(s))}{ds} = 0, s \in [t_0, t_0 + \varepsilon_{t_0}], t_0 \in [0, T^*),$$

and identity (6.2.26) follows. We need to show that $T^* = \infty$. Therefore, it is enough to prove

$$\sup_{t \in [0, T^*)} ||\gamma(t)||_{\mathcal{D}_X} < \infty.$$

Indeed, by contradiction with (6.2.25), the result will follow. In the rest of the proof we show such estimate. We recall that

$$\gamma(t) = \gamma_0 + \int_0^t u^{\gamma(s)}(\gamma(s))ds. \tag{6.2.27}$$

Firstly we have

$$|\gamma(t)|_{L^\infty} \le |\gamma_0|_{L^\infty} + \int_0^t |u^{\gamma(s)}|_{L^\infty} ds \le$$

$$|\gamma_0|_{L^\infty} + C\int_0^t \mathcal{H}_X(\gamma(s))ds \le |\gamma_0|_{L^\infty} + C\mathcal{H}_X(\gamma_0)t, t \in [0,T^*). \tag{6.2.28}$$

It follows from (6.2.27) that

$$\gamma'(t) = \gamma_0' + \int_0^t \nabla u^{\gamma(s)}(\gamma(s))\gamma'(s)ds, t \in [0,T^*). \tag{6.2.29}$$

Therefore, by Lemmas 6.2.10 and 6.2.11

$$|\gamma'(t)|_{L^\infty} \le |\gamma_0'|_{L^\infty} + \int_0^t |\nabla u^{\gamma(s)}|_{L^\infty}|\gamma'(s)|_{L^\infty} ds \le$$

$$|\gamma_0'|_{L^\infty} + \int_0^t C\mathcal{H}_X^{1/2}(\gamma(s))|\gamma'(s)|_{L^\infty} ds =$$

$$|\gamma_0'|_{L^\infty} + \int_0^t C\mathcal{H}_X^{1/2}(\gamma_0)|\gamma'(s)|_{L^\infty} ds, t \in [0,T^*). \tag{6.2.30}$$

Then by Gronwall inequality we infer our second estimate

$$|\gamma'(t)|_{L^\infty} \le |\gamma_0'|_{L^\infty} e^{C\mathcal{H}_X^{1/2}(\gamma_0)t}, t \in [0,T^*). \tag{6.2.31}$$

We will need one auxiliary estimate. We have

$$|\gamma(t)|_{C^\nu} \le |\gamma_0|_{C^\nu} + \int_0^t |u^{\gamma(s)}(\gamma(s))|_{C^\nu}| ds \le$$

$$|\gamma_0|_{C^\nu} + \int_0^t |\nabla u^{\gamma(s)}|_{L^\infty}|\gamma(s)|_{C^\nu} ds \le |\gamma_0|_{C^\nu} + \int_0^t C\mathcal{H}_X^{1/2}(\gamma(s))|\gamma(s)|_{C^\nu} ds =$$

$$= |\gamma_0|_{C^\nu} + \int_0^t C\mathcal{H}_X^{1/2}(\gamma_0)|\gamma(s)|_{C^\nu} ds, t \in [0,T^*). \tag{6.2.32}$$

Thus, by Gronwall inequality we get

$$|\gamma(t)|_{C^\nu} \leq |\gamma_0|_{C^\nu} e^{C\mathcal{H}_X^{1/2}(\gamma_0)t}, t \in [0, T^*). \tag{6.2.33}$$

Now we can estimate C^ν norm of γ'. We have

$$|\gamma'(t)|_{C^\nu} \leq |\gamma_0'|_{C^\nu} + \int_0^t |\nabla u^{\gamma(s)}(\gamma(s))\gamma'(s)|_{C^\nu} ds \leq$$

$$|\gamma_0'|_{C^\nu} + \int_0^t (|\nabla u^{\gamma(s)}|_{L^\infty}|\gamma'(s)|_{C^\nu} + |\gamma'(s)|_{L^\infty}|\nabla u^{\gamma(s)}(\gamma(s))|_{C^\nu}) ds \leq$$

$$|\gamma_0'|_{C^\nu} + \int_0^t (|\nabla u^{\gamma(s)}|_{L^\infty}|\gamma'(s)|_{C^\nu} + |\gamma'(s)|_{L^\infty}|\nabla^2 u^{\gamma(s)}|_{L^\infty}|\gamma(s)|_{C^\nu}) ds \leq$$

$$|\gamma_0'|_{C^\nu} + \int_0^t (C\mathcal{H}_X^{1/2}(\gamma_0)(|\gamma'(s)|_{C^\nu} + |\gamma_0'|_{L^\infty}|\gamma_0|_{C^\nu} e^{C\mathcal{H}_X(\gamma_0)s})) ds, t \in [0, T^*), \tag{6.2.34}$$

where last inequality follows from Lemmas 6.2.10 and 6.2.11. Then by Gronwall inequality we get third estimate

$$|\gamma'(t)|_{C^\nu} \leq (|\gamma_0'|_{C^\nu} + |\gamma_0'|_{L^\infty}|\gamma_0|_{C^\nu}) e^{C\mathcal{H}_X(\gamma_0)t}, t \in [0, T^*). \tag{6.2.35}$$

It remains to find an estimate for $|R^{\gamma(t)}|_{2\nu}$. We have

$$R^{\gamma(t)} = R^{\gamma_0} + \int_0^t R^{u^{\gamma(s)}(\gamma(s))} ds, t \in [0, T^*). \tag{6.2.36}$$

By (6.1.7) we have

$$R^{u^{\gamma(s)}(\gamma(s))}(\xi, \eta) = \nabla u^{\gamma(s)}(\gamma(s, \xi)) R^{\gamma(s)}(\xi, \eta) + \sum_k (\gamma^k(s, \eta) - \gamma^k(s, \xi)) \times$$

$$\int_0^1 \left[\frac{\partial u^{\gamma(s)}}{\partial x_k}(\gamma(s, \xi) + r(\gamma(s, \eta) - \gamma(s, \xi))) - \frac{\partial u^{\gamma(s)}}{\partial x_k}(\gamma(s, \xi))\right] dr, s \in [0, T^*). \tag{6.2.37}$$

Therefore,

$$|R^{u^{\gamma(s)}(\gamma(s))}|_{\tilde{C}^{2\nu}} \leq |\nabla u^{\gamma(s)}|_{L^\infty}|R^{\gamma(s)}|_{\tilde{C}^{2\nu}} + \frac{1}{2}|\gamma(s)|^2_{C^\nu}|\nabla^2 u^{\gamma(s)}|_{L^\infty}, s \in [0, T^*). \quad (6.2.38)$$

Thus, by inequalities (6.2.38) and (6.2.33)

$$|R^{\gamma(t)}|_{\tilde{C}^{2\nu}} \leq |R^{\gamma_0}|_{\tilde{C}^{2\nu}} + \int_0^t (|\nabla u^{\gamma(s)}|_{L^\infty}|R^{\gamma(s)}|_{\tilde{C}^{2\nu}} + \frac{1}{2}|\gamma(s)|^2_{C^\nu}|\nabla^2 u^{\gamma(s)}|_{L^\infty})ds \leq$$

$$|R^{\gamma_0}|_{\tilde{C}^{2\nu}} + \int_0^t (|\nabla u^{\gamma(s)}|_{L^\infty}|R^{\gamma(s)}|_{\tilde{C}^{2\nu}} + |\gamma_0|_{C^\nu}e^{C\mathcal{H}_X^{1/2}(\gamma_0)t}|\nabla^2 u^{\gamma(s)}|_{L^\infty})ds \leq$$

$$|R^{\gamma_0}|_{\tilde{C}^{2\nu}} + C(|\gamma_0|_{C^\nu}, \mathcal{H}_X(\gamma_0))e^{C\mathcal{H}_X^{1/2}(\gamma_0)t} + \int_0^t C\mathcal{H}_X^{1/2}(\gamma_0)|R^{\gamma(s)}|_{\tilde{C}^{2\nu}}ds, t \in [0, T^*) \quad (6.2.39)$$

where in last inequality we used Lemmas 6.2.10 and 6.2.11. As the result, by Gronwall Lemma we get

$$|R^{\gamma(t)}|_{\tilde{C}^{2\nu}} \leq (|R^{\gamma_0}|_{\tilde{C}^{2\nu}} + C(|\gamma_0|_{C^\nu}, \mathcal{H}_X(\gamma_0))e^{C\mathcal{H}_X^{1/2}(\gamma_0)t}))e^{C\mathcal{H}_X^{1/2}(\gamma_0)t}, t \in [0, T^*), \quad (6.2.40)$$

and combining estimates (6.2.28), (6.2.31), (6.2.35), and (6.2.40) we prove following a'priori estimate

$$|\gamma(t)|_{\mathcal{D}_X} \leq K(1 + \mathcal{H}_X(\gamma_0))(1 + |\gamma_0|_{\mathcal{D}_X})|\gamma_0|_{\mathcal{D}_X}e^{C\mathcal{H}_X(\gamma_0)t}, t \in [0, T^*), \quad (6.2.41)$$

and the result follows. □

6.3 Future directions of research

It would be interesting to consider problem (6.0.1)-(6.0.3) with added white noise i.e. to consider problem

$$d\gamma(t) = u^{\gamma(t)}(\gamma(t))dt + \sqrt{2\nu}dw_t, \nu > 0, t \in [0, T] \quad (6.3.1)$$

$$\gamma(0) = \gamma_0 \in \mathcal{D}_X. \quad (6.3.2)$$

where vector field of velocity u^Y is given by (6.0.2) and w_t is $\mathcal{D}_X$–valued Wiener process. This model would correspond to Navier-Stokes equations rather than Euler equations. There are two possible mathematical frameworks for the model.

First one is to make change of variables

$$\alpha(t) = \gamma(t) - \sqrt{2\nu}w_t, t \in [0,T].$$

Then we can fix $\{w_t\}_{t\geq 0}$ and system (6.3.1)-(6.3.2) is reformulated as follows

$$\frac{d\alpha}{dt} = u^{\alpha(t)+\sqrt{2\nu}w_t}(\alpha(t)+\sqrt{2\nu}w_t), \nu > 0, t \in [0,T] \quad (6.3.3)$$

$$\alpha(0) = \gamma_0 \in \mathcal{D}_X. \quad (6.3.4)$$

Now the problem (6.3.3)-(6.3.4) is ordinary differential equation (ODE) with random coefficients in $\mathcal{D}_X$ and it can be studied by methods of the theory of random dynamical systems, see [3] and [28]. This approach works only in the case of additive noise.

Second approach is to consider problem (6.3.1)-(6.3.2) as SDE in Banach space $\mathcal{D}_X$. Then, we can consider more general system with multiplicative noise:

$$d\gamma(t) = u^{\gamma(t)}(\gamma(t))dt + \sqrt{2\nu}G(\gamma)dw_t, \nu > 0, t \in [0,T] \quad (6.3.5)$$

$$\gamma(0) = \gamma_0 \in \mathcal{D}_X. \quad (6.3.6)$$

The problem which appear here is to define Stochastic integral in the Banach space $\mathcal{D}_X$. Stochastic calculus in M-type 2 Banach spaces developed in works [25]-[27], [12], [13] does not work in this situation. It seems that it is necessary to try to alter definition of $\mathcal{D}_X$ to be able to apply the theory.

Other possible direction of research is the theory of connections on infinite dimensional manifolds, see [40], [16], [50]. In [14] the authours claimed, see p. 251 therein, that it is possible to define the topological space of Gawędzki's [40] line bundle over the set of rough loops in the sense of Lyons [57]. Since the trajectories of the Brownian loop are almost surely rough paths, this allows us to define the topological space of Gawędzki's line bundle over the Brownian bridge, because it is possible to define the integral of a one-form over a rough path. It would be interesting to write down a complete proof of this claim. The theory presented in this chapter could be seen as a first step in realizing such a programme.

Bibliography

[1] S. Agmon, *Lectures on elliptic boundary value problems*, Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten. Jr. Van Nostrand Mathematical Studies, **No. 2**, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965

[2] S. Albeverio and Ya. I. Belopolskaya, *Probabilistic approach to hydrodynamic equations*, in Probabilistic methods in fluids (ed. I. M. Davies et al .), pp. 1-21 (World Scientific, River Edge, NJ, 2003).

[3] L. Arnold, *Random dynamical systems,* Springer-Verlag, Berlin Heidelberg, New York, 1998

[4] J. T. Beale, T. Kato, and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys. **94**, no. 1, 61–66 (1984)

[5] J. Bell and D. Markus, *Vorticity intensification and the transition to turbulence in the three-dimensional Euler equation*,Comm. Math. Phys. **147**, no. 2, 371–394 (1992)

[6] J. Bergh and J. Löfström, *Interpolation spaces, an introduction*, Springer-Verlag, 1976

[7] L. C. Berselli, M. Gubinelli, *On the global evolution of vortex filaments, blobs, and small loops in 3D ideal flows*,e-print,arXiv:math-ph/0510091.

[8] L. C. Berselli, H. Bessaih, *Some results for the line vortex equation*, Nonlinearity, **15**(2002), no. 6, 1729–1746 (2002)

[9] H. Bessaih, M. Gubinelli, F. Russo, *The evolution of a random vortex filament*, Ann. Probab. **33**, no. 5, 1825–1855 (2005)

[10] A. Bensoussan, *Stochastic control by functional analysis methods*, Studies in Mathematics and its Applications, **11**, North-Holland Publishing Co., Amsterdam-New York, 1982

[11] Z. Brzeźniak, On analytic dependence of solutions of Navier-Stokes equations with respect to exterior force and initial velocity, *Universitatis Iagellonicae Acta Mathematica*, Fasciculus XXVIII, 111-124 (1991)

[12] Z. Brzeźniak, Stochastic partial differential equations in M-type 2 Banach spaces, *Potential Anal.* 4, no. 1, 1–45 (1995)

[13] Z. Brzeźniak, Some remarks on Itô and Stratonovich integration in 2-smooth Banach spaces, *Probabilistic methods in fluids*, proceedings of the Swansea 2002 workshop, 14-19, April 2002, Wales, UK, World Scientific, 48-69 (2003)

[14] Z. Brzeźniak, R. Léandré, *Horizontal Lift of an Infinite Dimensional Diffusion*, Potential Anal. **12**, 249–280 (2000)

[15] Z. Brzeźniak and Y. Li, *Asymptotic compactness and absorbing sets for 2D Stochastic Navier-Stokes equations on some unbounded domains*, Trans. Amer. Math. Soc., Vol.**358**, No. 12, 5587–5629 (2006)

[16] J. L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Progress. in Math. **107**, Birkhauser, Basel, 1992.

[17] B. Busnello, *A probabilistic approach to the two-dimensional Navier-Stokes equations*, Ann. Probab. **27** (1999), no. 4, 1750–1780.

[18] B. Busnello, F. Flandoli, M. Romito, *A probabilistic representation for the vorticity of a three-dimensional viscous fluid and for general systems of parabolic equations*,Proc. Edinb. Math. Soc. (2) **48**, no. 2, 295–336 (2005)

[19] H. Cartan, *Differential Calculus*, Hermann, Paris, 1971

[20] A. Chorin, *Vorticity and Turbulence*, Springer-Verlag, New York, 1994

[21] P. Constantin and C.Foias, *Navier-Stokes equations,* The University of Chikago Press Ltd., London, 1988

[22] P. Constantin, *An Eulerian-Lagrangian approach for incompressible fluids: local theory*, J. Amer. Math. Soc. **14**, no. 2, 263–278 (2001)

[23] P. Constantin, G. Iyer, *A stochastic lagrangian representation of the 3-dimensional incompressible navier-stokes equations*, 2006,Eprint

[24] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions,* Cambridge University Press, Cambridge, 1992

[25] E. Dettweiler, *Stochastic integration of Banach space valued functions*, in *Stochastic Space-Time Models and Limit Theorem*, pp. 33-79, D.Reidel Publ. Co., 1985.

[26] E. Dettweiler, *On the martingale problem for Banach space valued SDE*, J. Theoret. Probab. **2**, no. 2, 159–197 (1989)

[27] E. Dettweiler, *Stochastic Integration relative to Brownian Motion on a general Banach space*, Boga– Tr. J. of Mathematics **15**, 6–44 (1991)

[28] J. Duan, K. Lu, B. Schmalfuss, *Invariant manifolds for stochastic partial differential equations*, The Annals of Probability **31**, no. 4, 2109–2135 (2003)

[29] J. Dugundji and A. Granas, *Fixed point theory*, Vol. **I**, Monografie Matematyczne (Mathematical Monographs), **61**, Państwowe Wydawnictwo Naukowe (PWN), Warsaw, 1982

[30] K. D. Elworthy and Xue-Mei Li, *Formulae for the derivatives of heat semigroups*, J. Funct. Anal. **125**, No. 1, 252–286 (1994)

[31] G. Falkovich, K. Gawędzki, M. Vergassola, *Particles and fields in fluid turbulence*. Rev. Mod. Phys. **73**, 913–975 (2001)

[32] F. Flandoli, *On a probabilistic description of small scale structures in 3D fluids*, Annales Inst. Henri Poincare, Probab. & Stat. **38**, 207–228 (2002)

[33] F. Flandoli, M. Gubinelli, *The Gibbs ensamble of a vortex filament*, Probab. Theory Rel. Fields **122**, 317–340 (2002)

[34] C. Foias, J.-C. Saut, *Asymptotic behavior, as $t \to +\infty$, of solutions of Navier-Stokes equations and nonlinear spectral manifolds*, Indiana Univ. Math. J. **33**, no. 3, 459–477 (1984)

[35] C. Foias, J.-C. Saut, *On the smoothness of the nonlinear spectral manifolds associated to the Navier-Stokes equations*, Indiana Univ. Math. J. **33**, no. 6, 911–926 (1984)

[36] A. Friedman, *Partial differential equations,* Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1969

[37] A. Friedman, *Foundations of modern analysis*, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London 1970

[38] P. Friz, *Continuity of the Ito-map for Holder rough paths with applications to the support theorem in Holder norm*, Probability and partial differential equations in modern applied mathematics, 117–135, IMA Vol. Math. Appl., **140**, Springer, New York, 2005.

[39] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*, Vol. **I**, Linearized steady problems, Springer Tracts in Natural Philosophy, **38**, Corrected second printing, Springer-Verlag, New York, 1998

[40] K. Gawędzki, *Topological actions in two-dimensional quantum field theory*. In: Gooft Gt., Jaffe A., Mack G., Mitter P. K., Stora R. (eds.),Nonperturbative Quantum Field Theory, NATO series **185**, Plenum Press, 101-142, 1988

[41] K. Gawędzki, M. Vergassola, *Phase transition in the passive scalar advection*, Physica D **138**, 63–90 (2000)

[42] J.-M. Ghidaglia, *Some Backward Uniqueness Results,* Nonlinear Analysis. Theory, Methods and Applications,Vol.**10**,No.8, 777–790 (1986)

[43] M. Gubinelli, *Controlling rough paths,* Journal of Functional Analysis, Vol.**216**,86–140 (2004)

[44] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes,* John Wiley and Sons, New York, 1981

[45] K. Engø , *On the BCH-formula in* $so(3)$,BIT. **41**, no. 3, 629–632 (2001)

[46] A. N. Kolmogorov, S. V. Fomin, ELEMENTS OF THE THEORY OF FUNCTIONS AND FUNCTIONAL ANALYSIS(in russian), Nauka, Moscow, 4th edition, 1976.

[47] R. H. Kraichnan, *Small-scale structure of a scalar field convected by turbulence,* Phys. Fluids **11**, 945–963 (1968)

[48] N. V. Krylov, *On Kolmogorovs equations for finite dimensional diffusions*, in Stochastic PDEs and Kolmogorov equations in infinite dimensions (ed. G. Da Prato), Lecture Notes in Mathematics, vol. **1715**. Springer, 1999.

[49] O. A. Ladyzhenskaya, *On uniqueness and smoothness of generalized solutions to the Navier-Stokes equations*, Zapiski Nauchn. Seminar. POMI, **5**, 169–185 (1967)

[50] R. Léandré, *String Structure over the Brownian Bridge*, J. Math. Phys. **40**, no. 1, 454–479 (1999)

[51] Y. Le Jan, O. Raimond, *Integration of brownian vector fields*,The Annals of Probability, **30**, No. 2, 826-873 (2002)

[52] Y. Le Jan, A. S. Sznitman, *Stochastic cascades and 3-dimensional Navier-Stokes equations*, Probab. Theory Related Fields **109**, no. 3, 343–366 (1997)

[53] J. L. Lions and E. Magenes, NON-HOMOGENEOUS BOUNDARY VALUE PROBLEMS AND APPLICATIONS. VOL. I, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 181. Springer-Verlag, New York-Heidelberg, 1972. xvi+357 pp.

[54] J.-L. Lions and E. Magenes, NON-HOMOGENEOUS BOUNDARY VALUE PROBLEMS AND APPLICATIONS. VOL. II, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 182. Springer-Verlag, New York-Heidelberg, 1972. xvi+242 pp.

[55] S. V. Lototsky, B. L. Rozovskii, *Passive scalar equation in a turbulent incompressible Gaussian velocity field*, Russian Math. Surveys **59**, no. 2, 297–312 (2004)

[56] S. V. Lototsky, B. L. Rozovskii, *Wiener chaos solutions of linear stochastic evolution equations*,The Annals of Probability, Vol. **34**, No. 2, 638–662 (2006)

[57] T. J. Lyons, *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana, **14**, no. 2, 215–310 (1998)

[58] T. J. Lyons, Z. Qian, SYSTEM CONTROL AND ROUGH PATHS, Oxford University Press, Oxford, 2002.

[59] C. Marchioro, M. Pulvirenti, MATHEMATICAL THEORY OF INCOMPRESSIBLE NON-VISCOUS FLUIDS, Applied Mathematical Sciences, 96. Springer-Verlag, New York, 1994. xii+283 pp.

[60] H. K. Moffat, *The degree of knottedness of tangled vortex lines*, J. Fluid Mech. **35**, 117–129 (1969)

[61] W. McLean, STRONGLY ELLIPTIC SYSTEMS AND BOUNDARY INTEGRAL EQUATIONS, Cambridge University Press, Cambridge-New York, 2000.

[62] M. Neklyudov, *Equivalence of Navier-Stokes equation and infinite dimensional Burgers equation* Submitted to: Proceedings of Petrovskii Conference on Differential Equations, May 2004, Moscow, Russia. Accepted for publication in 2006.

[63] E. Pardoux, *Stochastic Partial Differential Equations and Filtering of Diffusion Processes*, Stochastics, **3**, 127–167 (1979)

[64] G. Prodi, *Un teorema di unicita per le equazioni di Navier-Stokes*, Ann. Mat. Pura Appl. (4) **48**, 173–182 (1959)

[65] P. Protter, STOCHASTIC INTEGRATION AND DIFFERENTIAL EQUATIONS. Springer, 2005.

[66] D. L. Rapoport, *Closed form integration of the NavierStokes equations with stochastic differential geometry*, Hadronic J. **22**, 577-605 (1999)

[67] D. Revuz and M. Yor, CONTINUOUS MARTINGALES AND BROWNIAN MOTION. 3RD ED. Grundlehren der Mathematischen Wissenschaften. 293. Springer, Berlin, 1999.

[68] L. Rosenhead, *The spread of vorticity in the wake behind a cylinder*, Proc. Royal Soc. **127**, 590–612 (1930)

[69] J. Serrin, THE INITIAL VALUE PROBLEM FOR THE NAVIER-STOKES EQUATIONS. NONLINEAR PROBLEMS. Madison, Wis. pp. 69–98 Univ. of Wisconsin Press, Madison, Wis., 1963.

[70] M. Spivak, CALCULUS ON MANIFOLDS. A MODERN APPROACH TO CLASSICAL THEOREMS OF ADVANCED CALCULUS. W.A. Benjamin Inc., New York-Amsterdam, 1965.

[71] H. Tanabe, EQUATIONS OF EVOLUTION, Monographs and studies in mathematics, 6. Pitman Publishing Ltd., London, 1979.

[72] R. Temam, *Navier-Stokes Equations, Revised edition,* North-Holland Publishing Company, 1979

[73] R. Temam, *Behaviour at time $t = 0$ of the solutions of semilinear evolution equations*,J. Differential Equations **43**, no. 1,73–92 (1982)

[74] H. Triebel, INTERPOLATION THEORY, FUNCTION SPACES, DIFFERENTIAL OPERATORS., North-Holland Publishing Co., Amsterdam-New York, 1978. 528 pp.

[75] A. Vincent, M. Meneguzzi, *The spatial structure and the statistical properties of homogeneous turbulence*, J. Fluid. Mech. **225**, no.1, 1–25 (1991)

[76] L. C. Young, *An inequality of Hölder type, connected with Stieljes integration*, Acta Math. **67**, 251–282 (1936)