Magic Squares of Lie Algebras

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Abstract

This thesis is an investigation of the relation between Tits's magic square of Lie algebras and certain Lie algebras of 3×3 and 6×6 matrices with entries in alternative algebras. We show that when the columns of the magic square are labelled by the real division algebras and the rows by their split versions, then the rows can be interpreted as analogues of the matrix Lie algebras $\mathfrak{su}(3)$, $\mathfrak{sl}(3)$ and $\mathfrak{sp}(6)$ defined for each division algebra. We also define another magic square based on 2×2 and 4×4 matrices and prove that it consists of various orthogonal or (in the split case) pseudo-orthogonal Lie algebras. We then show that the exceptional Lie algebras can consequently be expressed in terms of 3×3 and 6×6 matrices with octonionic entries. We then explore the application of these representations to a problem from classical invariant theory, namely the problem of an explicit statement of Casimir identities for the exceptional Lie algebras. Invariant tensors are found for the exceptional Lie algebra G_2 and an indication is made as to how this may be followed through into higher rank exceptional Lie algebras.

Contents

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Introduction

Classical investigations of semi-simple Lie groups and Lie algebras depend mainly on the particularly simple matrix description that can be given for the classical Lie algebras. This creates a distinction between the classical Lie algebras and the exceptional ones which, although unified through common features of their underlying structures (found by examination of their root systems), can create a problem when attempting to extend results from the classical Lie algebras to the exceptional ones. This problem has been highlighted in several recent pieces of work (e.g. [**17, 25, 26**]).

This thesis is motivated by the desire to extend the matrix description of the classical Lie algebras to the exceptional Lie algebras and to show how these descriptions may enable certain results (such as those of Capelli and Casimir) to be calculated in a unifiable way for the exceptional Lie algebras.

To achieve this aim we draw on work first begun by Tits and Freudenthal [**38, 13**] between 1955 and 1958 and expanded upon by Ramond in 1976 in unpublished work [**32**]. Tits and Freudenthal proposed that the vector space construction

$$M(\mathbb{J},\mathbb{K}_1) = \mathrm{Der}\,\mathbb{J} \dot{+} (\mathbb{J}' \otimes \mathbb{K}_1') \dot{+} \mathrm{Der}\,\mathbb{K}_1 \qquad (1)$$

(where $\mathbb{J}$ is a Jordan algebra, $\mathbb{K}_1$ is an alternative algebra and A' is the algebra A with its centre factored out) would yield a table of related algebras. Furthermore if $\mathbb{J}$ is taken to be a Jordan algebra of 3×3 hermitian matrices, $H_3(\mathbb{K}_2)$, with entries from another alternative algebra the resulting algebra (written $L_3(\mathbb{K}_1,\mathbb{K}_2)$) will be a Lie algebra for any pair of alternative algebras $\mathbb{K}_1$, $\mathbb{K}_2$. If $\mathbb{K}_1$ and $\mathbb{K}_2$ are both taken to be division algebras (as classified by Hurwitz) then the resulting 'magic square' will be a 4×4 table of compact

Lie algebras which has the remarkable ('magic') properties of not only being symmetric but also containing the compact real forms of four of the five exceptional algebras, namely F_4, E_6, E_7 and E_8. By adding an initial row where $\mathbb{J}$ is taken to be the Jordan algebra of real numbers, the compact real form of the fifth exceptional Lie algebra, G_2, can also be included.

We show that if the division algebra $\mathbb{K}_2$ is replaced by its split form, the composition algebra $\widetilde{\mathbb{K}}_2$, a non-compact magic square of Lie algebras is obtained. This square is non-symmetric, but contains non-compact real forms of F_4, E_6, E_7 and E_8. We then show that the first three rows of the non-compact magic square can be described as 3×3 and 6×6 matrices over the octonions, i.e. the rows can be labelled as

$$\begin{aligned} L_3(\mathbb{K}, \mathbb{R}) &= \mathfrak{sa}(3, \mathbb{K}) \\ L_3(\mathbb{K}, \widetilde{\mathbb{C}}) &= \mathfrak{sl}(3, \mathbb{K}) \\ L_3(\mathbb{K}, \widetilde{\mathbb{H}}) &= \mathfrak{sp}(6, \mathbb{K}). \end{aligned} \tag{2}$$

To be precise, we define for each alternative algebra $\mathbb{K}$, a Lie algebra $\mathfrak{sa}(3, \mathbb{K})$, such that $\mathfrak{sa}(3, \mathbb{C}) = \mathfrak{su}(3)$ and $\mathfrak{sa}(3, \mathbb{O})$ is the compact real form of F_4; a Lie algebra $\mathfrak{sl}(3, \mathbb{K})$ equal to $\mathfrak{sl}(3, \mathbb{C})$ for $\mathbb{K} = \mathbb{C}$ and a non-compact real form of E_6 for $\mathbb{K} = \mathbb{O}$; and a Lie algebra $\mathfrak{sp}(6, \mathbb{K})$ such that $\mathfrak{sp}(6, \mathbb{C})$ is the set of 6×6 complex matrices X satisfying $X^\dagger J = -JX$, (where J is an antisymmetric real 6×6 matrix and $X^\dagger$ denotes the hermitian conjugate of X), and such that $\mathfrak{sp}(6, \mathbb{O})$ is a non-compact real form of E_7.

Since the algebra of $n \times n$ matrices over $\mathbb{O}$ is a Jordan algebra not only for $n = 3$ but also for $n = 2$ we also consider the magic square obtained by taking the Jordan algebra in a slightly adjusted form of the Tits construction to be $H_2(\mathbb{K})$. We show that if $\mathbb{K}_2$ is again taken to be its split form then the definitions for the non-compact 3×3 magic square can be modified to give the algebras $\mathfrak{sa}(2, \mathbb{K})$, $\mathfrak{sl}(2, \mathbb{K})$ and $\mathfrak{sp}(4, \mathbb{K})$. We also show that the Lie algebra $L_2(\mathbb{K}_1, \mathbb{K}_2)$ is isomorphic to the pseudo-orthogonal Lie algebra $\mathfrak{so}(\nu_1 + \frac{1}{2}\nu_2, \frac{1}{2}\nu_2)$, where $\dim \mathbb{K}_1 = \nu_1$ and $\dim \mathbb{K}_2 = \nu_2$.

Finding the Casimir and Capelli elements are two methods of establishing the generators of the centre of the universal enveloping algebra of a Lie algebra. They are also important in mathematical physics where Casimir elements appear as conserved quantities (or at least as being related to these quantites) in symmetry theories. This is because Casimir elements commute with all generators of an algbra.

For both the Capelli and Casimir elements the results are well established [**22, 25, 26, 27**] for the classical Lie algebras but remain more elusive [**23, 22**] in the case of the exceptional algebras. We will begin to show how considering the exceptional Lie algebras as matrices with octonionic entries may enable such work to be continued to the exceptional Lie algebras.

In particular we show how considering the exceptional Lie algebra G_2 to be the algebra consisting of derivations of the octonions, $\mathrm{Der}\,\mathbb{O}$, enables results on the sixth order Casimir for G_2 to be calculated easily, showing that it is primitive and giving relations between d-tensors which demonstrate the nature of the centre of the universal enveloping algebra. Finally we survey a series of techniques that may enable the Casimir elements of the exceptional Lie algebras to be found in a unified way.

Organisation of thesis

The organisation of the thesis is as follows. In Chapter 1 we establish notation and recall the definitions of the various kinds of algebra with which we will be concerned. We also introduce in some detail the representation of the exceptional Lie algebra G_2 used throughout this thesis, and give a basis for this in terms of elements of $\mathfrak{so}(7)$.

In Chapter 2 we give Tits's definition of the Lie algebras $L_3(\mathbb{K}_1, \mathbb{K}_2)$ and state the main properties of the magic square; we also give the definition and properties of the 2×2 magic square $L_2(\mathbb{K}_1, \mathbb{K}_2)$. We give a review of work completed by other authors [**29, 33**] on the symmetry properties of the magic square and then show a new vector space construction which makes apparent the underlying symmetry by means of introducing a new algebra, known as the *triality algebra*. This result is then extended to the case of the 2×2 magic square.

Chapter 3 contains proofs of the properties of the 2×2 magic square which were stated in Chapter 2, whilst Chapter 4 contains the proofs of the corresponding properties of the 3×3 magic square. This chapter also addresses a notational issue that arises when comparing our notation with that of Helgason [**16**] and other authors.

In Chapter 5 we introduce the concept of invariance and the Casimir elements and Capelli Identities. We explain the notion of a d-tensor and remark that Casimir elements have a one-one correspondance with such tensors. Invariant tensors are found for the representation of G_2 introduced in Chapter 1. Various results about these tensors are proven.

Finally in Chapter 6 we start to give an indication as to how such concepts may be extended to the other exceptional Lie algebras and an idea of the methods and techniques that may be required.

CHAPTER 1

Algebraic Background

In this Chapter we review the algebraic background to the magic square. We begin by introducing composition and division algebras and then go on to explain the notion of hypercomplex numbers, and their derivation from complex numbers via the Cayley-Dickson process. We also show how split forms of hypercomplex numbers arise from the same process.

We then explain the concept of a Lie algebra and in particular the notation used for such algebras throughout this thesis. We define a Jordan algebra and give some particular examples which are of interest.

We derive a representation of the exceptional Lie algebra G_2 in terms of generators of $\mathfrak{so}(7)$ and give a detailed description of this representation.

Finally we define a Clifford algebra and make explicit a particular relation between the Jordan algebra of conformal maps and the related Clifford algebra.

1. Composition and Division algebras

An algebra $\mathbb{K}$ (over $\mathbb{R}$) with identity, $\mathbb{1}$, which has a non-degenerate quadratic form, denoted by $x \mapsto |x|^2$, satisfying

$$|xy|^2 = |x|^2 \, |y|^2 \quad x, y \in \mathbb{K}, \tag{3}$$

is known as a *composition algebra.* Consider $\mathbb{R}$ to be embedded in $\mathbb{K}$ as the set of scalar multiples of the identity element, and denote by $\mathbb{K}'$ the subspace of $\mathbb{K}$ orthogonal to $\mathbb{R}$. It can then be shown [20] that $\mathbb{K} = \mathbb{R} \dot{+} \mathbb{K}'$ (where the symbol $\dot{+}$ is used to denote the direct sum of vector spaces) and the conjugation which fixes each element of $\mathbb{R}$ and multiplies every element of $\mathbb{K}'$ by -1, denoted $x \mapsto \overline{x}$, satisfies

$$\overline{xy} = \overline{y}\,\overline{x} \tag{4}$$

and

$$x\overline{x} = |x|^2 . \tag{5}$$

The notation $[x, y, z]$ will be used for the function that measures lack of associativity, known as the associator, which is defined to be

$$[x, y, z] = (xy)z - x(yz). \tag{6}$$

Any composition algebra $\mathbb{K}$ satisfies the *alternative law*, i.e. the associator is an alternating function of x, y and z for all $x, y, z \in \mathbb{K}$. If $|x|^2$ is positive definite then $\mathbb{K}$ is a *division algebra*, i.e. a *division algebra* is defined as an algebra in which

$$xy = 0 \Rightarrow x = 0 \text{ or } y = 0,$$

namely an algebra in which there are no zero divisors. A composition algebra which is not a division algebra is called a *split* algebra.

2. Hypercomplex Numbers

Hypercomplex numbers have been known since the mid nineteenth century, when Hamilton discovered the quaternions, denoted $\mathbb{H}$, around 1843 and Cayley the octonions, denoted $\mathbb{O}$, in 1845. They can be derived, along with the complex numbers, from the real numbers by means of the *Cayley-Dickson process.*

2.1. The Cayley-Dickson Process. Start by considering a pair of numbers $(a, b) \in \mathbb{K}$. Note that if $\mathbb{K} = \mathbb{R}$ this is equivalent to $(a, b) \mapsto a + ib$. Then define the multiplication

$$(a, b)(c, d) = (ac + \alpha b\bar{d}, \bar{a}d + cb), \tag{7}$$

where α is some real constant. Call this algebra $\mathcal{C}^2$. Taking $a, b, c, d \in \mathbb{R}$ and $\alpha = -1$ gives the complex numbers, $\mathbb{C}$, with the usual multiplication

$$(a, b)(c, d) = (ac - bd, ad + cb)$$

which can be written as

$$(a+ib)(c+id) = (ac-bd) + i(ad+cb)$$

in the more commonly used notation. Thus $\mathbb{C}$ can be written as $\mathbb{C} = \mathbb{R} + i\mathbb{R}$. If α is taken to be -1 instead of $+1$ the split form of the complex numbers, written $\widetilde{\mathbb{C}}$, are obtained. Here the basis of the algebra $\widetilde{\mathbb{C}}$ is $\{1, \tilde{i}\}$ where $\tilde{i}^2 = 1$.

Now take pair of numbers to be $a, b \in \mathcal{C}^2$ giving the multiplication

$$(a,b)(c,d) = (ac + \beta b\bar{d}, \bar{a}d + cb).$$

Putting $a = (a_1, a_2), b = (b_1, b_2), c = (c_1, c_2), d = (d_1, d_2)$ with $a_i, b_i, c_i, d_i \in \mathbb{R}$ such that a, b, c, d multiply as in equation (7), the equation

$$(a,b)(c,d) = (e,f)$$

where

$$e = (a_1c_1 + \alpha a_2c_2 + \beta b_1d_1 - \alpha\beta b_2d_2, a_1c_2 + c_1a_2 - \beta d_1b_2 + \beta b_1d_2)$$

and

$$f = (a_1d_1 - \alpha d_2a_2 + \beta c_1b_1 + \alpha\beta b_2c_2, a_1d_2 - d_1a_2 + \beta c_1b_2 + \beta b_1c_2)$$

defines the multiplication in a further, four dimensional algebra, where β is again taken to be some real constant. Let the algebra generated by this multiplication be labelled $\mathcal{C}^4$. Let the basis of this algebra be $\{e_0, e_1, e_2, e_3\}$, then the multiplication of $\mathcal{C}^4$ can be written as the table

	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	αe_0	e_3	αe_2
e_2	e_2	$-e_3$	βe_0	$-\beta e_1$
e_3	e_3	$-\alpha e_2$	βe_1	$-\alpha\beta e_0$

Taking $\alpha = \beta = -1$ (i.e. $\mathcal{C}^4 = \mathbb{C}^2$) and considering $\mathbb{H} = \mathbb{C} + j\mathbb{C}$, i.e. $\mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + ij\mathbb{R}$, then $\mathbb{H}$ is the set of *quaternions*. Relabelling $ij = k$

gives the more usual definition of the four dimensional algebra of quaternions. with addition defined by

$$(t_1 + x_1 i + y_1 j + z_1 k) + (t_2 + x_2 i + y_2 j + z_2 k) = (t_1 + t_2) + (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$$

and multiplication of the base elements defined by

$$\begin{aligned} i^2 &= j^2 = k^2 = -1 \\ ij &= k = -ji \\ jk &= i = -kj \\ ki &= j = -ik. \end{aligned} \tag{8}$$

This multiplication can be demonstrated pictorially with the following useful diagram, where multiplication in the direction of the arrows gives a positive

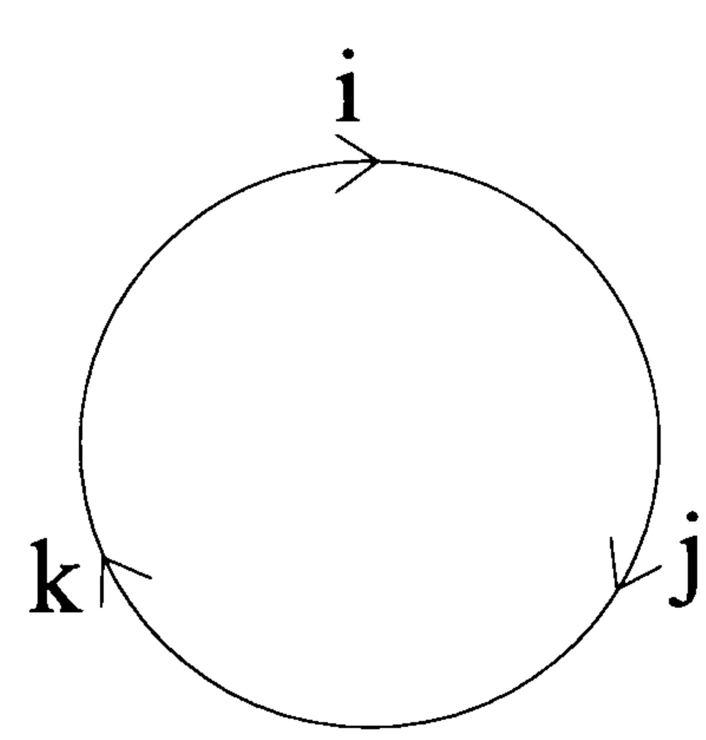


FIGURE 1. Mnemonic for quaternionic multiplication.

coefficient.

LEMMA 1. The three algebras obtained by taking $\alpha = \beta = 1$, $\alpha = 1$, $\beta = -1$ and $\alpha = -1$, $\beta = 1$ in the Cayley-Dickson process are all isomorphic.

PROOF. Label the three algebras as follows:

1. $\widetilde{\mathbb{H}}_1$ is the algebra obtained by taking $\alpha = \beta = 1$,
2. $\widetilde{\mathbb{H}}_2$ is the algebra obtained by taking $\alpha = 1$, $\beta = -1$.
3. $\widetilde{\mathbb{H}}_3$ is the algebra obtained by taking $\alpha = -1$, $\beta = 1$.

Let $\widetilde{\mathbb{H}}_i$ have units $\tilde{i}_i$, $\tilde{j}_i$ and $\tilde{k}_i$. Then the multiplication of each of the three algebras are given by

$\widetilde{\mathbb{H}}_1$	$\tilde{k}_1$	$\tilde{j}_1$	$\tilde{i}_1$
$\tilde{k}_1$	-1	$\tilde{i}_1$	$-\tilde{j}_1$
$\tilde{j}_1$	$-\tilde{i}_1$	1	$-\tilde{k}_1$
$\tilde{i}_1$	$\tilde{j}_1$	$\tilde{k}_1$	1

$\widetilde{\mathbb{H}}_2$	$\tilde{j}_2$	$\tilde{k}_2$	$\tilde{i}_2$
$\tilde{j}_2$	-1	$\tilde{i}_2$	$-\tilde{k}_2$
$\tilde{k}_2$	$-\tilde{i}_2$	1	$-\tilde{j}_2$
$\tilde{i}_2$	$\tilde{k}_2$	$\tilde{j}_2$	1

$\widetilde{\mathbb{H}}_3$	$\tilde{i}_3$	$\tilde{j}_3$	$\tilde{k}_3$
$\tilde{i}_3$	-1	$\tilde{k}_3$	$-\tilde{j}_3$
$\tilde{j}_3$	$-\tilde{k}_3$	1	$-\tilde{i}_3$
$\tilde{k}_3$	$\tilde{j}_3$	$\tilde{i}_3$	1

From these tables it is clear that $\widetilde{\mathbb{H}}_1 \cong \widetilde{\mathbb{H}}_2 \cong \widetilde{\mathbb{H}}_3$. □

These are the three split form of the quaternions, $\widetilde{\mathbb{H}}$. Throughout this thesis we will take the split algebra, $\widetilde{\mathbb{H}}$ to be the algebra defined as $\widetilde{\mathbb{H}}_3$ above. Thus $\widetilde{\mathbb{H}}$ has a complex subalgebra spanned by $\{1, i\}$ and consider $\widetilde{\mathbb{H}}$ to have a basis defined by $\{1, i, \tilde{j}, \tilde{k}\}$. Note that all other values of α and β give algebras isomorphic to either $\mathbb{H}$ or $\widetilde{\mathbb{H}}$ by normalising the squares of the units to give ± 1.

A further application of the Cayley-Dickson process, introducing a third real constant γ, gives an eight dimensional Cayley-Dickson algebra, $\mathcal{C}^8$. Multiplication in $\mathcal{C}^8$ can be seen most clearly by considering the following table, where the basis elements of $\mathcal{C}^4$ are denoted by e_i, where $i = 0, \ldots, 7$,

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	αe_0	e_3	αe_2	e_5	αe_4	$-e_7$	$-\alpha e_6$
e_2	e_2	$-e_3$	βe_0	$-\beta e_1$	e_6	e_7	βe_4	βe_5
e_3	e_3	$-\alpha e_2$	βe_1	$-\alpha\beta e_0$	e_7	αe_6	$-\beta e_5$	$-\alpha\beta e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	γe_0	$-\gamma e_1$	$-\gamma e_2$	$-\gamma e_3$
e_5	e_5	$-\alpha e_4$	$-e_7$	$-\alpha e_6$	γe_1	$-\gamma\alpha e_0$	γe_3	$\gamma\alpha e_2$
e_6	e_6	e_7	$-\beta e_4$	βe_5	γe_2	$-\gamma e_3$	$-\beta\gamma e_0$	$-\beta\gamma e_1$
e_7	e_7	αe_6	$-\beta e_5$	$\alpha\beta e_4$	γe_3	$-\gamma\alpha e_2$	$\gamma\beta e_1$	$\alpha\beta\gamma e_0$

If the three Cayley-Dickson constants are taken to be -1 the algebra obtained is called the octonions and written $\mathbb{O}$. The octonions are added in the same style as the quaternions, i.e. if $a = a_i e_i$ and $b = b_i e_i$ are octonions,

$$a + b = (a_i + b_i)e_i$$

(using the Eistein summation convention). The basis elements of $\mathbb{O}$ are written as either $\{1, i, j, k, l, il, jl, kl\}$ or $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, with multiplication of the basis elements now defined as

$$(9) \qquad e_i^2 = -1, \quad i = 1, \ldots, 7$$
$$e_i e_j = -\delta_{ij} + \psi_{ijk} e_k$$

where ψ_{ijk} is completely antisymmetric, non zero and equal to 1 for

$$(10) \qquad ijk = (123), (246), (435), (367), (651), (572), (714).$$

Again the multiplication for the imaginary octonionic units can be expressed diagramatically using Figure 2. In this diagram positive multiplication follows

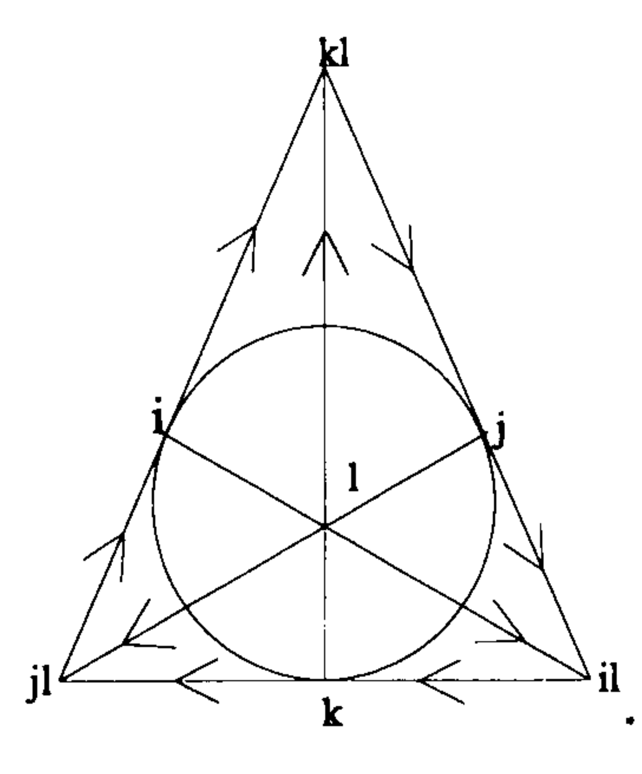


FIGURE 2. Mnemonic for Octonionic Multiplication

the direction of the arrows and every straight line can be thought of as a circle. Elements lying on a straight line form what is often referred to as a *quaternionic triple*, i.e. they are a basis for a quaternionic subalgebra of the octonions. (Note that since every straight line can be thought of as a circle in this diagram, the central circle can also be thought of as a straight line.)

LEMMA 2. Write (α, β, γ) as the three constants used in the Cayley-Dickson process to obtain an eight dimensional algebra. The following seven choices of

(α, β, γ) give isomorphic forms of the split octonionic algebra, $\widetilde{\mathbb{O}}$:
$(1,1,1), (1,1,-1), (1,-1,1), (-1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)$.

PROOF. Use the same method as for Lemma 1. As an example two cases will be shown to be isomorphic, that of $(1,1,1)$ ($\widetilde{\mathbb{O}}_1$) and $(-1,-1,1)$ ($\widetilde{\mathbb{O}}_2$). Label the imaginary basis elements in $\widetilde{\mathbb{O}}_1$ by e_i for $i = 1, \ldots, 7$ and in $\widetilde{\mathbb{O}}_2$ by f_i for $i = 1, \ldots 7$. Then the multiplication table for the e_i is

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	1	e_3	e_2	e_5	e_4	$-e_7$	$-e_6$
e_2	$-e_3$	1	$-e_1$	e_6	e_7	e_4	e_5
e_3	$-e_2$	e_1	-1	e_7	e_6	$-e_5$	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	1	$-e_1$	$-e_2$	$-e_3$
e_5	$-e_4$	$-e_7$	$-e_6$	e_1	-1	e_3	e_2
e_6	e_7	$-e_4$	e_5	e_2	$-e_3$	-1	$-e_1$
e_7	e_6	$-e_5$	e_4	e_3	$-e_2$	e_1	1

The multiplication table for the f_i is

	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_1	-1	f_3	$-f_2$	f_5	$-f_4$	$-f_7$	f_6
f_2	$-f_3$	-1	f_1	f_6	f_7	$-f_4$	$-f_5$
f_3	f_2	$-f_1$	-1	f_7	$-f_6$	f_5	$-f_4$
f_4	$-f_5$	$-f_6$	$-f_7$	1	$-f_1$	$-f_2$	$-f_3$
f_5	f_4	$-f_7$	f_6	f_1	1	f_3	$-f_2$
f_6	f_7	f_4	$-f_5$	f_2	$-f_3$	1	f_1
f_7	$-f_6$	f_5	f_4	f_3	f_2	$-f_1$	1

Let $\psi : \widetilde{\mathbb{O}}_2 \to \widetilde{\mathbb{O}}_1$ by

$$\psi(f_1) = e_1 \qquad \psi(f_2) = e_5 \qquad \psi(f_3) = e_6 \qquad \psi(f_4) = e_4$$

$$\psi(f_5) = e_7 \qquad \psi(f_6) = e_1 \qquad \psi(f_7) = e_2.$$

Then by inspection of the above tables it can be seen that ψ is a Lie algebra isomorphism. □

Throughout this thesis take the split form of the octonions to be defined by the triplet of constants $(-1, -1, 1)$. In this way $\widetilde{\mathbb{O}}$ contains a complex subalgebra spanned by $\{1, i\}$ and a quaternionic subalgebra spanned by $\{1, i, j, k\}$. Thus we will denote the basis of $\widetilde{\mathbb{O}}$ either by $\{1, i, j, k, \tilde{l}, \tilde{il}, \tilde{jl}, \tilde{kl}\}$, or using the more convenient notation $\{1, e_1, e_2, e_3, \tilde{e}_4, \tilde{e}_5, \tilde{e}_6, \tilde{e}_7\}$.

In general $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ will be denoted by $\mathbb{K}$. Denote the dimension of $\mathbb{K}$ by ν, thus $\nu = 1, 2, 4$ or 8. Then $\mathbb{K}$ is a division algebra with the quadratic form defined by conjugation, as explained in equation (4). The split forms $\widetilde{\mathbb{C}}$, $\widetilde{\mathbb{H}}$ and $\widetilde{\mathbb{O}}$ are not division algebras but are composition algebras (as implied by their name) and will be denoted generally by $\widetilde{\mathbb{K}}$.

The following Theorem is well established [**35**].

THEOREM 1 (Hurwitz). The only real positive definite composition algebras (division algebras) are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.

3. Lie algebras

A *Lie algebra*, L, is an algebra over a field, F, with a product, known as a *Lie bracket*, $[,]$, such that

1. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$
2. $[x, y] = -[y, x]$.

A Lie algebra is called *real* if the field F is the field of real numbers. It is called *complex* if the field is the complex numbers.

The *Killing Form* of a Lie algebra is the symmetric bilinear form $K : L \times L \to F$ defined by

$$K(x, y) = \mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) \tag{11}$$

for $x, y \in L$, where $(\mathrm{ad}\, x)(z) = [x, z]$.

A complex Lie algebra, L, is called *simple* if it contains no ideals (subalgebras closed under subtraction and Lie bracket multiplication) except 0 and L and $\dim L > 1$. It is called *semi-simple* if its maximal solvable ideal (its *radical*) is 0, i.e. there are no abelian ideals. Cartan gave two important results for complex semi-simple Lie algebras, known as Cartan's First and Second Criterion. These are stated below.

THEOREM 2 (Cartan's First Criterion). A complex Lie algebra is solvable if and only if the Killing form of $[L, L]$ (the *derived algebra*) is identically zero.

THEOREM 3 (Cartan's Second Criterion). A complex Lie algebra is semi-simple if and only if its Killing form is non-degenerate.

Every finite dimensional Lie algebra has a particularly useful subalgebra for determining its structure. A *Cartan sub-algebra*, H, is defined to be a subalgebra satisfying

$$H = \{x \in L : (\operatorname{ad} x)^r h = 0, \text{ for some } r \in \mathbb{R} \text{ and all } h \in H\}. \tag{12}$$

This can also be written as $H = \{x \in L : \operatorname{ad} x(H) \subset H\}$ (see, for example, [**14**]).

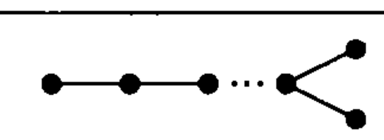
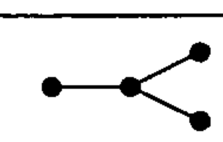
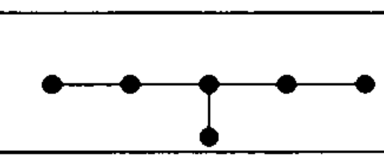
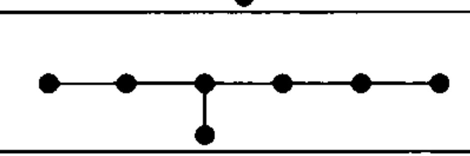
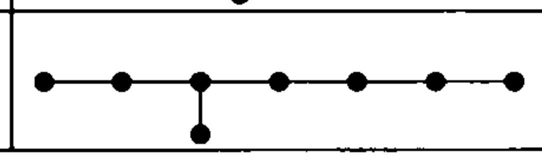
A complex semi-simple Lie algebra, L, can be described in terms of its root system, P (up to isomorphism). The root system is a closed set under addition where the elements of the set are the *roots* of L. The roots of L are the weights (eigenvalues), α, of $\operatorname{ad} H$ which are regarded as linear maps $\alpha : H \to \mathbb{C}$. For each $\alpha \in P$ there corresponds a non zero element $h_\alpha \in H$ such that $\alpha(x) = K(h_\alpha, x)$. Then the h_α span H. The root system has a *root basis* which is a set of linearly independent vectors $\alpha_i \in P$. Then any $\alpha \in P$ can be written in the form $\alpha = \sum m_i \alpha_i$ where the m_i are either all non-positive or non-negative integers. The root basis can be described pictorially in terms of the *Dynkin diagram*, which is a diagram where each element in the root basis is represented by a circle. The circles representing two basis elements α and β are joined by

1. • • no line if α and β are orthogonal
2. •—• one line if the angle between α and β is $\frac{2\pi}{3}$

3. two lines if the angle between α and β is $\frac{3\pi}{4}$

4. three lines if the angle between α and β is $\frac{5\pi}{6}$.

Complex Lie algebras can then be classified into nine different types, as shown by Cartan and described in (for example) [**18**], the first four of which are known as the *classical* Lie algebras. This classification is shown in the following table. The middle row is included in the diagram since D_4 has a particularly interesting type of symmetry in relation to the octonions which will be commented on whilst considering the symmetry property of the 3×3 magic square. The final five Lie algebras are known as the *exceptional* Lie algebras. These are all shown, along with their Dynkin diagrams, in the table below.

A_n	$\mathfrak{sl}(n+1, \mathbb{C})$	
B_n	$\mathfrak{o}(2n+1, \mathbb{C})$	
C_n	$\mathfrak{sp}(n, \mathbb{C})$	
D_n	$\mathfrak{o}(2n, \mathbb{C})$	
D_4	$\mathfrak{o}(8, \mathbb{C})$	
G_2	14	
F_4	52	
E_6	78	
E_7	133	
E_8	248	

The notation for Lie algebras is that used in [**37**]. The notation $A^\dagger$ will be used for the hermitian conjugate of the matrix A with entries in $\mathbb{K}$, defined in analogy to the complex case by

$$X^\dagger = \overline{X}^t.$$

The Lie algebra of the pseudo-unitary group will be denoted $\mathfrak{su}(s, t)$, where

$$\mathfrak{su}(s, t) = \{A \in \mathbb{C}^{n \times n} : A^\dagger G + GA = 0\} \tag{13}$$

and $G = \text{diag}(-1, \ldots, -1, +1, \ldots, +1)$ with s $-$ signs and t $+$ signs: $\mathfrak{sq}(n)$ will denote the Lie algebra of antihermitian quaternionic matrices A.

$$\mathfrak{sq}(n) = \{A \in \mathbb{H}^{n \times n} : A^\dagger = -A\}. \tag{14}$$

The notation $\mathfrak{sp}(2n, \mathbb{K})$ will be used for the Lie algebra of the symplectic group of $2n \times 2n$ matrices with entries in $\mathbb{K}$, i.e.

$$\mathfrak{sp}(2n, \mathbb{K}) = \{A \in \mathbb{K}^{2n \times 2n} : A^\dagger J + JA = 0\} \tag{15}$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The Lie algebra of the pseudo-orthogonal group $\text{SO}(s,t)$, is given by

$$\mathfrak{so}(s,t) = \{A \in \mathbb{R}^{n \times n} : A^T G + GA = 0\} \tag{16}$$

where G is defined as before. The group of linear maps of the vector space V preserving the non-degenerate quadratic form q is written $\text{O}(V,q)$, whilst its unimodular (or special) subgroup is written $\text{SO}(V,q)$. Their Lie algebras are respectively $\mathfrak{o}(V,q)$ and $\mathfrak{so}(V,q)$. The quadratic form q is omitted if it is understood from the context. Thus for any division algebra, $\text{SO}(\mathbb{K})$ and $\mathfrak{so}(\mathbb{K})$ are respectively its special orthogonal group and Lie algebra. These may often be written $\text{SO}(\nu)$ and $\mathfrak{so}(\nu)$ where $\nu = \dim \mathbb{K}$.

The symbol $\oplus$ will be used to denote the direct sum of Lie algebras, i.e. $A \oplus B$ implies that $[A, B] = 0$.

3.1. Structure Constants. Two particular types of tensor arise naturally in connection with any Lie algebra, antisymmetric tensors and symmetric tensors. In this subsection we introduce the concept of an f-tensor, the antisymmetric tensor associated with the structure constants.

Define a Lie algebra $\mathfrak{g}$ of dimension n with generators X_i such that

$$[X_i, X_j] = f_{ij}^k X_k \tag{17}$$

(using the Einstein summation convention) and

$$\langle X_i, X_j \rangle = -\delta_{ij} \tag{18}$$

where $\langle , \rangle$ is the Cartan-Killing form on $\mathfrak{g}$. The f-tensor is called a *structure constant*. (Note that in some texts an f-tensor is defined by

$$[X_i, X_j] = 2f_{ij}^k X_k,$$

but we find it more convenient to include the factor 2 within the constant). Furthermore the structure constants obey

$$f_{ij}^k = -f_{ji}^k \tag{19}$$

by the antisymmetric property of the Lie bracket, and

$$f_{ij}^k f_{kl}^m + f_{jl}^k f_{ki}^m + f_{li}^k f_{kj}^m = 0$$

by the Jacobi Identity. When $\mathfrak{g}$ is semi-simple the Cartan-Killing form ($g_{ij} = \langle \operatorname{ad} X_i, \operatorname{ad} X_j \rangle$) has an inverse, i.e. there exists g^{jk} such that

$$g_{ij} g^{jk} = \delta_i^k = g^{kj} g_{ji}.$$

The Cartan-Killing form is used to raise and lower the indices for the structure constants, i.e.

$$f_{ijk} = f_{ij}^l g_{lk}.$$

If $\mathfrak{g}$ is compact then $g_{ij} = \delta_{ij}$ and consequently in this case

$$f_{ijk} = f_{ij}^k.$$

The structure constants can also be used to provide an $n \times n$ matrix representation for $\mathfrak{g}$, although it is generally unfaithful. This is obtained from the constants by associating the elements of an $n \times n$ matrix, M_a, with the structure constants as follows

$$(M_a)_\alpha^\beta = -f_{a\alpha}^\beta.$$

This can also be obtained by considering $M_a = \operatorname{ad} X_a$. This representation is known as the *adjoint* or *regular* representation. The structure constants provide us with a particularly simple formula for the Cartan-Killing form. namely

$$\langle X_i, X_j \rangle_{\text{reg}} = f^s_{jr} f^r_{is}.$$

Finally we note that under a change of basis the structure constants transform as a third order tensor that is covariant of the second rank and contravariant of the first, i.e.

$$\begin{array}{lr} \text{if} & Y_r = V^i_r X_i \\ \text{and} & [Y_r, Y_s] = F^t_{rs} Y_t \\ \text{then} & F^t_{rs} = V^i_r V^j_s f^k_{ij} (V^{-1})^t_k. \end{array}$$

From now on the totally antisymmetrised structure constant f_{ijk} will be referred to as an f-tensor.

One example of f-tensors arising from the Gell-Mann matrix description of $\mathfrak{su}(3)$ will be of particular use throughout this thesis. In this representation a set of hermitian generators are obtained by multiplying an orthogonal set of generators of $\mathfrak{su}(3)$ by the complex unit i. Thus a set of eight matrices are obtained which are then labelled as follows;

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

These have the commutation relations

$$[\lambda_i, \lambda_j] = f_{ijk}\lambda_k \tag{20}$$

where

$$\begin{aligned} f_{ijk} &= -\tfrac{i}{4}\operatorname{tr}(\lambda_i\lambda_j\lambda_k - \lambda_j\lambda_i\lambda_k) \\ &= -\tfrac{i}{4}\operatorname{tr}(\lambda_k\lambda_i\lambda_j - \lambda_i\lambda_k\lambda_j) \\ &= f_{kij}, \end{aligned}$$

i.e. f_{ijk} is a totally antisymmetric tensor. Thus the f-tensors for $\mathfrak{su}(3)$ can be written explicitly as

$f_{123} = 2$	$f_{246} = 1$	$f_{367} = -1$
$f_{147} = 1$	$f_{257} = 1$	$f_{458} = \sqrt{3}$
$f_{156} = -1$	$f_{345} = 1$	$f_{678} = \sqrt{3}$

.

The f-tensors are totally isotropic, which means that their components remain unchanged under an arbitary change of basis transformation. In the case of $\mathfrak{su}(3)$ various identites can be found for f_{ijk}, these are not stated here but can be found in [**21**].

4. Jordan algebras

A Jordan algebra $\mathbb{J}$ is defined to be a commutative algebra (over a field $\mathbb{K}$) in which all products satisfy the Jordan identity

$$(xy)x^2 = x(yx^2). \tag{21}$$

Let $L_n(\mathbb{K})$ (where $\mathbb{K}$ is now a composition algebra) be the set of all $n \times n$ matrices with entries in $\mathbb{K}$, and let $H_n(\mathbb{K})$ and $A_n(\mathbb{K})$ be the sets of all $n \times n$ hermitian and antihermitian matrices with entries in $\mathbb{K}$ respectively. The subspaces of traceless matrices of $H_n(\mathbb{K})$ and $A_n(\mathbb{K})$ are denoted $H'_n(\mathbb{K})$ and $A'_n(\mathbb{K})$ respectively. Thus $L_n(\mathbb{K}) = H_n(\mathbb{K}) \dot{+} A_n(\mathbb{K})$ and $L'_n(\mathbb{K}) = H'_n(\mathbb{K}) \dot{+} A'_n(\mathbb{K})$. The following Lemma will be extremely useful throughout the remainder of this thesis.

LEMMA 3. The algebra $H_n(\mathbb{K})$ is a Jordan algebra for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ for all n and for $\mathbb{K} = \mathbb{O}$ when $n = 2, 3$, with the Jordan product as the anticommutator

$$X \cdot Y = XY + YX. \tag{22}$$

This is a commutative but non-associative product.

Proof of this Lemma can be found in [**35**].

5. The Derivation Algebra

A derivation, D, of an algebra, A, is an automorphism $D : A \to A$ such that

$$D(xy) = D(x)y + xD(y).$$

Thus define the derivation algebra of A, $\operatorname{Der} A$, to be the algebra of all such maps, namely

$$\operatorname{Der} A = \{D \mid D(xy) = D(x)y + xD(y)\} \tag{23}$$

for $x, y \in A$.

LEMMA 4. The derivation algebras of the four positive definite composition algebras are as follows:

$$\operatorname{Der} \mathbb{R} = \operatorname{Der} \mathbb{C} = 0$$

$$\operatorname{Der} \mathbb{H} = \mathfrak{so}(3)$$

$\operatorname{Der} \mathbb{O}$ is an exceptional Lie algebra of type G_2.

The next five sub-sections prove the results for each of these cases and also show how the proof that $\operatorname{Der} \mathbb{O} \cong G_2$ can be adapted for use with the split octonions.

5.1. Derivations of $\mathbb{R}$. In this case $\operatorname{Der}\mathbb{R} = 0$ since

$$\begin{aligned} D(1) &= D(1.1) \\ &= D(1).1 + 1.D(1) \quad \text{by the definition of a derivation.} \\ &= 2.D(1). \end{aligned}$$

For this to be true it must be the case that $D(1) = 0$ and thus the set of derivations of $\mathbb{R}$ consists only of the zero map.

5.2. Derivations of $\mathbb{C}$. Take a basis of $\mathbb{C}$ to be the set $\{1, i\}$. Then consider

$$\begin{aligned} D(i.i) &= D(i).i + i.D(i) && \text{by the definition of a derivation,} \\ &= 2i.D(i) && \text{since } \mathbb{C} \text{ is commutative.} \end{aligned}$$

However,

$$D(i.i) = D(-1) = -D(1) = 0$$

which leads to the conclusion that

$$2i.D(i) = 0.$$

Thus the set of all derivations of $\mathbb{C}$ will also consist only of the zero map.

5.3. Derivations of $\mathbb{H}$. Take the set $\{e_0 = 1, e_1, e_2, e_3\}$ to be a basis of the quaternions, $\mathbb{H}$. Then $D(1) = 0$ by exactly the same reasoning as before. Since a derivation, by definition, maps $\mathbb{H}$ to itself, pick one of the imaginary basis elements and let $D(e_i) = a_{i0} + a_{im}e_m$ (using the Einstein summation convention to sum over m). Now

$$\begin{aligned} D(e_ie_i) &= D(e_i)e_i + e_iD(e_i) \\ &= (a_{i0} + a_{in}e_n)e_i + e_i(a_{io} + a_{in}e_n) \\ &= 2a_{i0}e_i + a_{in}(e_ie_n + e_ne_i) \\ &= 0 \quad \text{by the same reasoning as above.} \end{aligned}$$

Since $e_ie_n + e_ne_i = 0$ if $i \neq n$ and $e_ie_n + e_ne_i = -2$ if $i = n$ this reduces to the statement

$$D(e_ie_i) = 2(a_{i0}e_i - a_{ii}) = 0. \tag{24}$$

Further

$$\begin{aligned} D(e_ie_j) &= (a_{i0} + a_{im}e_m)e_j + e_i(a_{j0} + a_{jn}e_n) \\ &= -(a_{ij} + a_{ji}) + (a_{j0} - a_{ik})e_i + (a_{i0} - a_{jk})e_j + (a_{ii} + a_{jj})e_k \end{aligned}$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. However,

$$\begin{aligned} D(e_ie_j) &= D(e_k) \\ &= a_{k0} + a_{ki}e_i + a_{kj}e_j + a_{kk}e_k. \end{aligned}$$

By equating coefficients of e_n the following set of identifications can be made:

$$a_{k0} = -a_{ij} - a_{ji} \tag{25}$$

$$a_{j0} = a_{ik} + a_{ki} \tag{26}$$

$$a_{i0} = a_{jk} + a_{kj} \tag{27}$$

$$a_{kk} = a_{ii} + a_{jj}. \tag{28}$$

Now,

$$\begin{aligned} D(e_je_i) &= (a_{j0} + a_{jn}e_n)e_i + e_j(a_{i0} + a_{im}e_m) \\ &= -(a_{ij} + a_{ji}) + (a_{j0} + a_{ik})e_i + (a_{i0} + a_{jk})e_j - (a_{ii} + a_{jj})e_k \end{aligned}$$

and

$$\begin{aligned} D(e_je_i) &= D(-e_k) \\ &= -a_{k0} - a_{ki}e_i - a_{kj}e_j - a_{kk}e_k. \end{aligned}$$

Again by equating coefficients of e_n a further set of identifications are made:

$$a_{k0} = a_{ij} + a_{ji} \tag{29}$$

$$a_{j0} = -a_{ik} - a_{ki} \tag{30}$$

$$a_{i0} = -a_{jk} - a_{kj} \tag{31}$$

$$a_{kk} = a_{ii} + a_{jj}. \tag{32}$$

By comparing (25) with (29), (26) with (30) and (27) with (31), deduce that $a_{k0} = a_{j0} = a_{i0} = 0$. Substituting $a_{i0} = 0$ into (24) further gives $a_{ii} = a_{jj} = a_{kk} = 0$. The final deduction that can be made is that since $a_{k0} = 0$, $a_{ij} = -a_{ji}$. Thus

$$D(e_1) = a_{12}e_2 - a_{31}e_3$$

$$D(e_2) = -a_{12}e_1 + a_{23}e_3$$

$$D(e_3) = a_{31}e_1 - a_{23}e_2.$$

Thus any derivation of $\mathbb{H}$ is of the form

$$D = a_{23}D_1 - a_{12}D_2 - a_{31}D_3$$

where

$$D_1(e_1) = 0, \qquad D_1(e_2) = e_3, \qquad D_1(e_3) = -e_2$$

$$D_2(e_1) = -e_2, \qquad D_2(e_2) = 0, \qquad D_2(e_3) = e_1$$

$$D_3(e_1) = e_3, \qquad D_3(e_2) = -e_1, \qquad D_1(e_3) = 0.$$

This is equivalent to the elements of $\mathfrak{so}(3)$ acting on $\mathbb{H}'$ as vectors, where the basis $\{D_1, D_2, D_3\}$ for $\operatorname{Der}\mathbb{H}$ defined above is clearly isomorphic to the basis of $\mathfrak{so}(3)$, and consequently $\operatorname{Der}\mathbb{H} \cong \mathfrak{so}(3)$. This method can be easily adapted to show that $\operatorname{Der}\widetilde{\mathbb{H}} \cong \mathfrak{so}(2,1)$ by taking into account the changes in the multiplication table of $\widetilde{\mathbb{H}}$. Further if the basis of the algebra of commutator maps of $\mathbb{H}'$, $C(\mathbb{H}')$, (maps of the form C_a where $C_a x = ax - xa$ and $a \in \mathbb{H}'$)

is taken to be $\{C_{e_1}, C_{e_2}, C_{e_3}\}$ then the action of these three maps on the basis elements of $\mathbb{H}'$ is given by

$$\begin{array}{lll} C_{e_1}(e_1) = 0 & C_{e_1}(e_2) = 2e_3 & C_{e_1}(e_3) = -2e_2 \\ C_{e_2}(e_1) = -2e_3 & C_{e_2}(e_2) = 0 & C_{e_2}(e_3) = 2e_1 \\ C_{e_3}(e_1) = 2e_2 & C_{e_3}(e_2) = -2e_1 & C_{e_1}(e_3) = 0. \end{array}$$

Clearly this is isomorphic to $\mathfrak{so}(3)$ and consequently $\operatorname{Der}\mathbb{H} \cong C(\mathbb{H}')$.

5.4. Derivations of $\mathbb{O}$. $\operatorname{Der}\mathbb{O}$ is the Lie algebra of the group of automorphisms of $\mathbb{O}$, which is written $\operatorname{Aut}\mathbb{O}$. The easiest way to obtain the elements of the derivation algebra of $\mathbb{O}$ is to first look at the elements of $\operatorname{Aut}\mathbb{O}$.

Decompose $\mathbb{O}$ into $\mathbb{O} = \mathbb{H} + i\mathbb{H}$ where i is an imaginary unit. Then the following Lemma will be required.

LEMMA 5. Automorphisms of $\mathbb{O}$ fixing the complex subspace $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ leave the subspace $\mathbb{C}^3 = j\mathbb{C} + l\mathbb{C} + jl\mathbb{C}$ invariant.

PROOF. Throughout this proof let $\alpha \in \operatorname{Aut}\mathbb{O}$ and x, $y \in \mathbb{O}$. The proof is given in three stages, each stage leading to the next.

1. Automorphisms preserve conjugation, i.e. $\alpha(\bar{x}) = \overline{\alpha(x)}$.

   Let $x = \operatorname{Re} x + \operatorname{Im} x$. Then

$$\alpha(x) = \alpha(\operatorname{Re} x) + \alpha(\operatorname{Im} x).$$

   Thus

$$\begin{aligned} \alpha(\bar{x}) &= \alpha(\operatorname{Re} x) + \alpha(-\operatorname{Im} x) \\ &= \alpha(\operatorname{Re} x) - \alpha(\operatorname{Im} x) \qquad \text{since } \alpha(-1) = -1 \\ &= \overline{\alpha(x)}. \end{aligned}$$

2. Automorphism are orthogonal maps, i.e. $\langle\alpha(x), \alpha(y)\rangle = \langle x, y\rangle$.

   Now,

$$\begin{aligned}\langle\alpha(x),\alpha(y)\rangle &= \tfrac{1}{2}\left(\alpha(x)\overline{\alpha(y)} + \alpha(y)\overline{\alpha(x)}\right)\\ &= \tfrac{1}{2}\left(\alpha(x)\alpha(\bar{y}) + \alpha(y)\alpha(\bar{x})\right) && \text{since } \alpha(\bar{x}) = \overline{\alpha(x)}\\ &= \alpha\left(\tfrac{1}{2}\{x\bar{y} + y\bar{x}\}\right)\\ &= \alpha\left(\langle x, y\rangle\right)\\ &= \langle x, y\rangle && \text{since } \langle x, y\rangle \in \mathbb{R}.\end{aligned}$$

3. Automorphisms fixing $\mathbb{C}$ leave $\mathbb{C}^3$ invariant, i.e. $\alpha(\mathbb{C}^3) = \mathbb{C}^3$. Since $\mathbb{C}^3 = \mathbb{C}^\perp$,

$$\begin{aligned}\alpha(\mathbb{C}^3) &= \alpha(\mathbb{C}^\perp)\\ &= \alpha(\mathbb{C})^\perp && \text{since } \alpha \text{ is orthogonal}\\ &= \mathbb{C}^\perp && \text{since } \mathbb{C} \text{ is fixed}\\ &= \mathbb{C}^3.\end{aligned}$$

□

Since $\mathbb{C}$ is fixed α must be $\mathbb{C}$-linear on $\mathbb{C}^3$. Using the fact that the product of two 3-vectors is

$$\mathbf{u}\mathbf{v} = -\mathbf{u}.\bar{\mathbf{v}} + \bar{\mathbf{u}} \times \bar{\mathbf{v}},$$

then if $\alpha \in \operatorname{Aut}\mathbb{O}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^3$,

$$\begin{aligned}\alpha(\mathbf{u}\mathbf{v}) &= \alpha(\mathbf{u})\alpha(\mathbf{v})\\ &= \alpha(-\mathbf{u}.\bar{\mathbf{v}} + \bar{\mathbf{u}} \times \bar{\mathbf{v}}) = -\mathbf{u}.\bar{\mathbf{v}} + \alpha(\bar{\mathbf{u}} \times \bar{\mathbf{v}})\\ &= -\alpha(\mathbf{u})\overline{\alpha(\mathbf{v})} + \overline{\alpha(\mathbf{u})} \times \overline{\alpha(\mathbf{v})}.\end{aligned}$$

Consequently $\alpha(\mathbf{u})\overline{\alpha(\mathbf{v})} = \mathbf{u}.\bar{\mathbf{v}}$ which implies that α is unitary. Furthermore α has determinant 1 since

$$\alpha_{ij}\epsilon_{jlm} = \epsilon_{ijk}\overline{\alpha_{jl}}\;\overline{\alpha_{km}}$$

and thus

$$\epsilon_{nlm} = \epsilon_{ijk}\overline{\alpha_{jl}}\ \overline{\alpha_{km}}\ \overline{\alpha_{in}}.$$

Therefore $\alpha \in \mathrm{SU}(3)$. Also any $\alpha \in \mathrm{SU}(3)$ satisfies

$$\alpha(\mathbf{u})\overline{\alpha(\mathbf{v})} = \mathbf{u}.\bar{\mathbf{v}}$$

$$\overline{\alpha(\mathbf{u})} \times \overline{\alpha(\mathbf{v})} = \alpha(\bar{\mathbf{u}} \times \bar{\mathbf{v}})$$

and consequently $\mathrm{SU}(3) \subset \mathrm{Aut}\,\mathbb{O}$.

Pictorially, if we redraw Figure 2 with i in the centre we obtain Figure 3. Then the eight basis elements of SU(3) describle the eight possible rearrange-

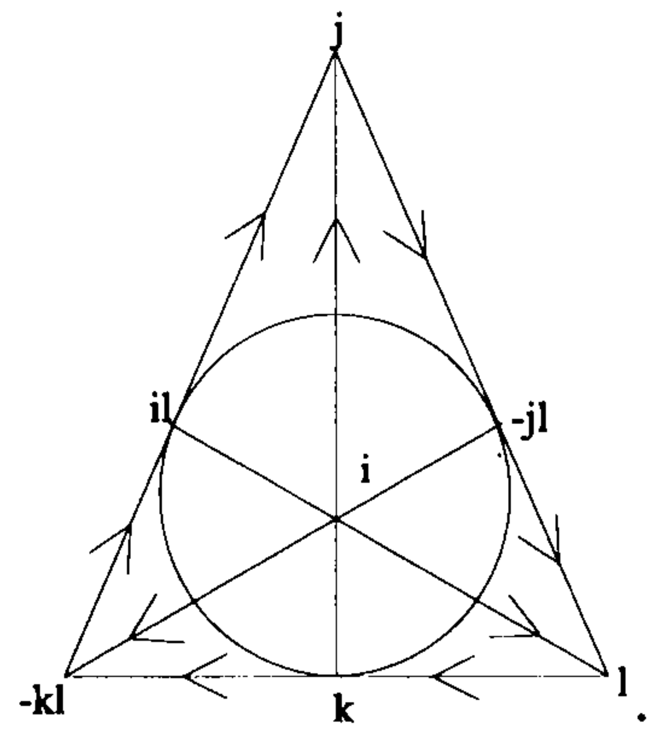


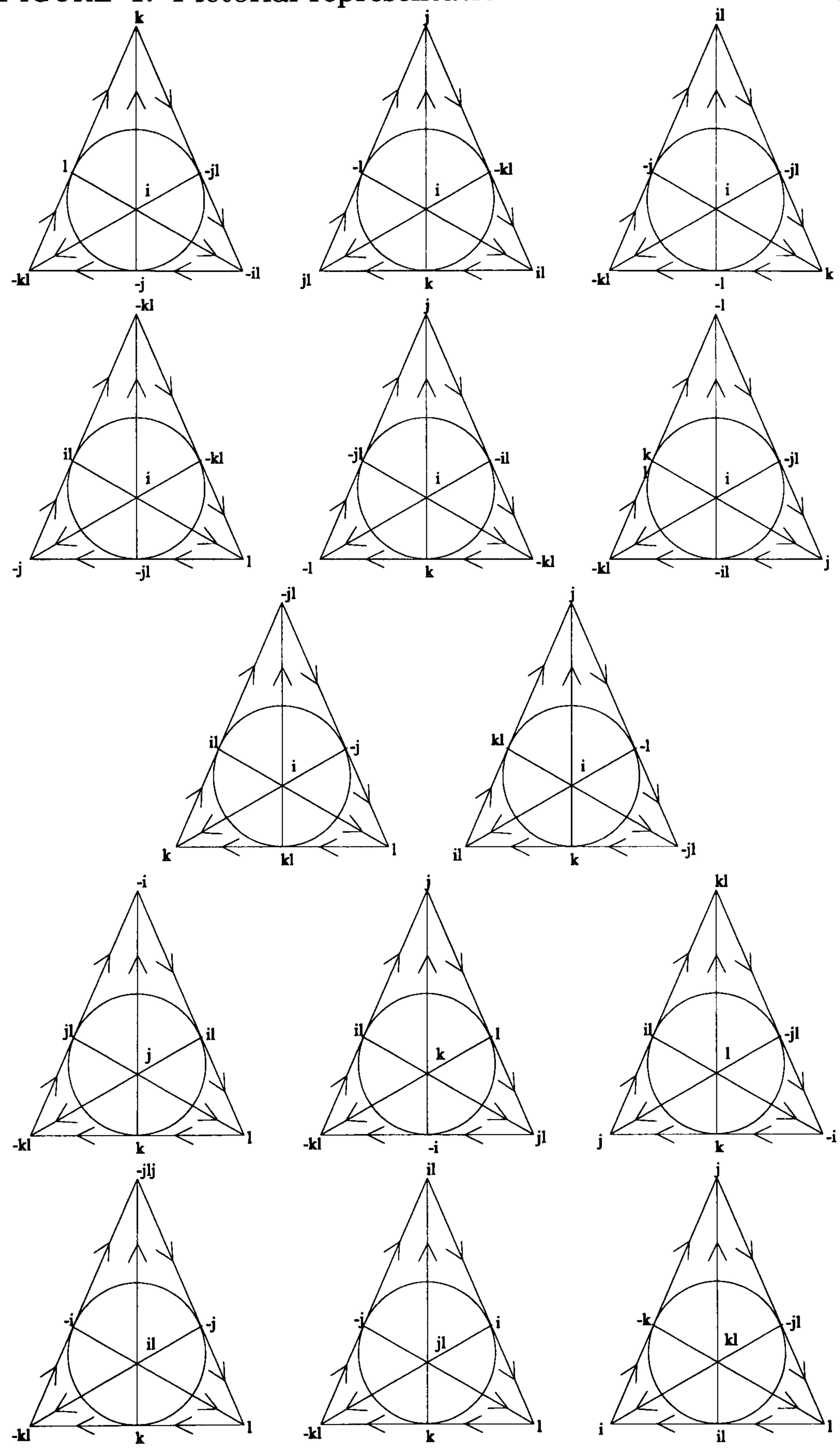
FIGURE 3. Revised Mnemonic

ments of the octonionic units j, k, l, il, jl, kl leaving i fixed and preserving octonionic multiplication, i.e. preserving the direction of the arrows on the diagram. The remaining six elements of $\mathrm{Aut}\,\mathbb{O}$ are the operations of taking the unit i to each of the other six octonionic units whilst again still preserving the directions of the arrows. Then the SU(3) subalgebra can be thought of as the maps taking Figure 3 to each of the first eight pictures in Figure 4.

We can now adapt the description of $\mathrm{Aut}\,\mathbb{O}$ by thinking of its infinitesimal transformations to obtain a complete picture of $\mathrm{Der}\,\mathbb{O}$.

The action of $\mathfrak{su}(3)$ in terms of elements of $\mathfrak{so}(7)$ can also be calculated by considering the action of the eight basis elements of $\mathfrak{su}(3)$ on a vector $\mathbf{v} = (\alpha + i\beta)j + (\gamma + i\delta)l + (\epsilon + i\phi)jl = (\alpha + i\beta, \gamma + i\delta, \epsilon + i\phi)$ in $\mathbb{C}^3$. The eight

FIGURE 4. Pictorial representation of the action of Aut $\mathbb{O}$



basis elements of $\mathfrak{su}(3)$ will be labelled b_i as follows

$$b_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$b_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad b_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \qquad b_6 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$b_7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad b_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then the $\mathbb{C}^3$ elements $b_1\mathbf{v}, \ldots, b_8\mathbf{v}$ are identified with octonions as shown below:

$$b_1\mathbf{v} \mapsto k\alpha - j\beta - il\gamma + l\delta \qquad b_2\mathbf{v} \mapsto il\gamma - l\delta + kl\epsilon + jl\phi$$

$$b_3\mathbf{v} \mapsto k\gamma - j\delta + il\alpha - l\beta \qquad b_4\mathbf{v} \mapsto k\epsilon - j\phi - kl\alpha - jl\beta$$

$$b_5\mathbf{v} \mapsto il\epsilon - l\phi - kl\gamma - jl\delta \qquad b_6\mathbf{v} \mapsto j\gamma + k\delta - l\alpha - il\beta$$

$$b_7\mathbf{v} \mapsto j\epsilon + k\phi - jl\alpha + kl\beta \qquad b_8\mathbf{v} \mapsto l\epsilon + il\phi - jl\gamma + kl\delta$$

so that $b_1, \ldots, b_8$ represent the linear maps of $\mathbb{O}$ as below:

$$b_1 \rightsquigarrow s_{23} - s_{45} \qquad b_2 \rightsquigarrow s_{45} + s_{67}$$

$$b_3 \rightsquigarrow s_{25} - s_{34} \qquad b_4 \rightsquigarrow -s_{27} - s_{36}$$

$$b_5 \rightsquigarrow -s_{47} - s_{56} \qquad b_6 \rightsquigarrow -s_{24} - s_{35}$$

$$b_7 \rightsquigarrow -s_{26} + s_{37} \qquad b_8 \rightsquigarrow -s_{46} + s_{57}.$$

The remaining six elements of Aut $\mathbb{O}$ are the maps taking Figure 3 to each of the last six diagrams shown in Figure 4. These are the six maps taking e_1 to each of the other imaginary units in turn. These give the elements

$$b_9 = s_{12} + s_{56} \qquad b_{10} = s_{13} + s_{46}$$

$$b_{11} = s_{14} + s_{27} \qquad b_{12} = s_{15} - s_{26}$$

$$b_{13} = s_{16} + s_{25} \qquad b_{14} = s_{17} + s_{35}.$$

Thus the 14 elements of $\mathrm{Der}\,\mathbb{O} \cong G_2$ can have a basis defined by elements of $\mathfrak{so}(7)$ obtained from the b_i, relabelled as d_i, as follows

$$d_1 = s_{25} - s_{34} \qquad d_2 = -s_{24} - s_{35}$$
$$d_3 = s_{23} - s_{45} \qquad d_4 = s_{27} - s_{36}$$
$$d_5 = -s_{26} - s_{37} \qquad d_6 = s_{47} - s_{56}$$
$$d_7 = -s_{46} - s_{57} \qquad d_8 = s_{45} + s_{67}$$

in $\mathfrak{su}(3)$ and

$$d_9 = s_{12} - \tfrac{1}{2}s_{47} + \tfrac{1}{2}s_{56} \qquad d_{10} = s_{13} + \tfrac{1}{2}s_{46} + \tfrac{1}{2}s_{57}$$
$$d_{11} = s_{14} + \tfrac{1}{2}s_{27} - \tfrac{1}{2}s_{36} \qquad d_{12} = s_{15} - \tfrac{1}{2}s_{26} - \tfrac{1}{2}s_{37}$$
$$d_{13} = s_{16} + \tfrac{1}{2}s_{25} + \tfrac{1}{2}s_{34} \qquad d_{14} = s_{17} - \tfrac{1}{2}s_{24} + \tfrac{1}{2}s_{35}$$

to complete the algebra, where

$$s_{ij}(e_i) = e_j$$
$$s_{ij}(e_j) = -e_i$$
$$s_{ij}(e_k) = 0$$

for $k \neq i, j$, i.e.

$$s_{ij}(e_k) = \langle e_i, e_k\rangle e_j - \langle e_j, e_k\rangle e_i. \tag{33}$$

Note that this basis is chosen purely for the particularly simple structure constants that it yields.

Computing the Killing form for all of the above matrices we find that

$$\langle d_i, d_j\rangle = 0 \qquad \text{if } i \neq j$$
$$\langle d_i, d_i\rangle = -4 \qquad \text{if } i = 1, \ldots, 8$$
$$\langle d_i, d_i\rangle = -3 \qquad \text{if } i = 9, \ldots, 14.$$

So that the structure constants are totally antisymmetric divide d_i by $\frac{1}{2}$ for $i = 1, \ldots, 8$ and d_j by $\frac{1}{\sqrt{3}}$ for $j = 9, \ldots, 14$. Thus the basis elements of G_2 now become

$$d_1 = \tfrac{1}{2}(s_{25} - s_{34}) \qquad d_2 = \tfrac{1}{2}(-s_{24} - s_{35})$$
$$d_3 = \tfrac{1}{2}(s_{23} - s_{45}) \qquad d_4 = \tfrac{1}{2}(s_{27} - s_{36})$$
$$d_5 = \tfrac{1}{2}(-s_{26} - s_{37}) \qquad d_6 = \tfrac{1}{2}(s_{47} - s_{56})$$
$$d_7 = \tfrac{1}{2}(-s_{46} - s_{57}) \qquad d_8 = \tfrac{1}{2}(s_{45} + s_{67})$$
$$d_9 = \tfrac{1}{\sqrt{3}}(s_{12} - \tfrac{1}{2}s_{47} + \tfrac{1}{2}s_{56}) \qquad d_{10} = \tfrac{1}{\sqrt{3}}(s_{13} + \tfrac{1}{2}s_{46} + \tfrac{1}{2}s_{57})$$
$$d_{11} = \tfrac{1}{\sqrt{3}}(s_{14} + \tfrac{1}{2}s_{27} - \tfrac{1}{2}s_{36}) \qquad d_{12} = \tfrac{1}{\sqrt{3}}(s_{15} - \tfrac{1}{2}s_{26} - \tfrac{1}{2}s_{37})$$
$$d_{13} = \tfrac{1}{\sqrt{3}}(s_{16} + \tfrac{1}{2}s_{25} + \tfrac{1}{2}s_{34}) \qquad d_{14} = \tfrac{1}{\sqrt{3}}(s_{17} - \tfrac{1}{2}s_{24} + \tfrac{1}{2}s_{35})$$

and the Cartan-Killing form can now be written

$$\langle d_i, d_j \rangle = -\delta_{ij}.$$

Calculating a multiplication table for $[d_i, d_j]$ using the computer algebra package MAPLE to generate all of the Lie brackets gives the following non-zero structure constants (f-tensors) for G_2.

$f_{123} = 1$	$f_{246} = \frac{1}{2}$	$f_{367} = -\frac{1}{2}$
$f_{147} = \frac{1}{2}$	$f_{257} = \frac{1}{2}$	$f_{458} = \frac{1}{2}$
$f_{156} = -\frac{1}{2}$	$f_{345} = \frac{1}{2}$	$f_{678} = \frac{1}{2}$
$f_{19(13)} = \frac{1}{2}$	$f_{49(14)} = \frac{1}{2}$	$f_{7(10)(11)} = -\frac{1}{2}$
$f_{1(10)(12)} = \frac{1}{2}$	$f_{4(11)(12)} = -\frac{1}{2}$	$f_{7(13)(14)} = -\frac{1}{2}$
$f_{29(10)} = \frac{1}{2}$	$f_{59(11)} = -\frac{1}{2}$	$f_{89(12)} = \frac{1}{2}$
$f_{2(12)(13)} = \frac{1}{2}$	$f_{5(12)(14)} = -\frac{1}{2}$	$f_{8(10)(13)} = \frac{1}{2}$
$f_{39(12)} = \frac{1}{2}$	$f_{6(10)(14)} = \frac{1}{2}$	$f_{8(11)(14)} = 1$
$f_{3(10)(13)} = -\frac{1}{2}$	$f_{6(11)(13)} = -\frac{1}{2}$	
$f_{9(10)(11)} = \frac{1}{\sqrt{3}}$	$f_{(10)(12)(14)} = -\frac{1}{\sqrt{3}}$	
$f_{9(13)(14)} = \frac{1}{\sqrt{3}}$	$f_{(11)(12)(13)} = -\frac{1}{\sqrt{3}}$	

5.5. Derivations of $\widetilde{\mathbb{O}}$. Derivations of $\widetilde{\mathbb{O}}$ can be found by adapting the work shown above. Consider the split algebra obtained by taking $\alpha = \beta = -1$ and $\gamma = 1$ in the Cayley-Dickson process (i.e. $\tilde{i} = i, \tilde{j} = j$ and $\tilde{k} = k$ form a non-split quaternionic subalgebra) since this is convenient for following work on maximal compact subalgebras. Then $\mathrm{Der}\,\widetilde{\mathbb{O}}$ will have a subalgebra $\mathfrak{su}(2,1)$ with metric $\mathcal{G}_1 = \mathrm{diag}(-1,1,1)$ and will be expressed in terms of elements of $\mathfrak{so}(4,3)$ with metric $\mathcal{G}_2 = \mathrm{diag}(-1,-1,-1,1,1,1,1)$. Then calculating directly the action of $\mathfrak{su}(2,1)$ in terms of elements of $\mathfrak{so}(4,3)$ we obtain

$$\tilde{d}_1 = \tfrac{1}{2}(\tilde{s}_{25} - \tilde{s}_{34}) \qquad \tilde{d}_2 = \tfrac{1}{2}(-\tilde{s}_{24} - \tilde{s}_{35})$$

$$d_3 = \tfrac{1}{2}(s_{23} - s_{45}) \qquad \tilde{d}_4 = \tfrac{1}{2}(\tilde{s}_{27} - \tilde{s}_{36})$$

$$\tilde{d}_5 = \tfrac{1}{2}(-\tilde{s}_{26} - \tilde{s}_{37}) \qquad d_6 = \tfrac{1}{2}(s_{47} - s_{56})$$

$$d_7 = \tfrac{1}{2}(-s_{46} - s_{57}) \qquad d_8 = \tfrac{1}{2}(s_{45} + s_{67})$$

where if S_{ij} is the matrix representation of s_{ij} then $\mathcal{G}_2 S_{ij}$ is the matrix representation of $\tilde{s_{ij}}$. Thus $\tilde{s}_{ij}$ is given by equation (33) with $\langle , \rangle$ given by $\widetilde{\mathbb{O}}$ instead of $\mathbb{O}$. Applying this knowledge the remaining six elements can be calculated to be

$$d_9 = \frac{1}{\sqrt{3}}(s_{12} - \tfrac{1}{2}s_{47} + \tfrac{1}{2}s_{56}) \qquad d_{10} = \frac{1}{\sqrt{3}}(s_{13} + \tfrac{1}{2}s_{46} + \tfrac{1}{2}s_{57})$$
$$\tilde{d}_{11} = \frac{1}{\sqrt{3}}(\tilde{s}_{14} + \tfrac{1}{2}\tilde{s}_{27} - \tfrac{1}{2}\tilde{s}_{36}) \qquad \tilde{d}_{12} = \frac{1}{\sqrt{3}}(\tilde{s}_{15} - \tfrac{1}{2}\tilde{s}_{26} - \tfrac{1}{2}\tilde{s}_{37})$$
$$\tilde{d}_{13} = \frac{1}{\sqrt{3}}(\tilde{s}_{16} + \tfrac{1}{2}\tilde{s}_{25} + \tfrac{1}{2}\tilde{s}_{34}) \qquad \tilde{d}_{14} = \frac{1}{\sqrt{3}}(\tilde{s}_{17} - \tfrac{1}{2}\tilde{s}_{24} + \tfrac{1}{2}\tilde{s}_{35})$$

Thus we also have a complete expression of the elements of $\operatorname{Der}\widetilde{\mathbb{O}}$ in terms of elements of $\mathfrak{so}(4,3)$.

6. Structure and Conformal Algebras

The structure algebra $\operatorname{Str} A$ of any algebra A is defined to be the Lie algebra generated by left and right multiplication maps L_a and R_a for each $a \in A$, where the Lie bracket is the usual commutator. For Jordan algebras this can be shown to have the vector space [**35**]

$$\operatorname{Str}\mathbb{J} = \operatorname{Der}\mathbb{J} \dot{+} L(\mathbb{J}) \tag{34}$$

where $L(\mathbb{J})$ is the set of all L_a with $a \in \mathbb{J}$. The algebra denoted by $\operatorname{Str}'\mathbb{J}$ is the structure algebra with its centre factored out (i.e. $\operatorname{Str}'\mathbb{J} = \operatorname{Der}\mathbb{J}\dot{+}L(\mathbb{J}')$). For $D \in \operatorname{Der}\mathbb{J}$ and $L_a \in L(\mathbb{J})$ the Lie brackets can be stated explicitly for $\operatorname{Str}\mathbb{J}$ as follows

$$[D, D'] = DD' - D'D$$
$$[L_a, L_b] = L_a L_b - L_b L_a$$
$$[D, L_a] = L_{D(a)}.$$

Another Lie algebra associated with a Jordan algebra is also required, namely the conformal algebra, as constructed by Kantor (1973) and Koecher (1967). The underlying vector space of this is

$$\operatorname{Con}\mathbb{J} = \operatorname{Str}\mathbb{J}\dot{+}\mathbb{J}^2. \tag{35}$$

Consider $\operatorname{Str}\mathbb{J}$ as a Lie subalgebra of $\operatorname{Con}\mathbb{J}$ and define the rest of the Lie brackets as follows. We first define $R \mapsto R^\star$ to be an involutive automorphism acting on $\operatorname{Str}\mathbb{J}$ such that it is the identity of $\operatorname{Der}\mathbb{J}$ and multiplies every element of $L(\mathbb{J})$ by -1. Then if $T \in \operatorname{Str}\mathbb{J}$ and $(X, Y) \in \mathbb{J}^2$ the other brackets are

$$[T, (X, Y)] = (TX, T^\star Y)$$

$$[(X, 0), (Y, 0)] = [(0, X), (0, Y)] = 0$$

$$[(X, 0), (0, Y)] = 2L_{XY} + 2[L_X, L_Y].$$

The conformal algebra can also be defined in a more geometric fashion. A conformal map is defined to be a smooth map $f : X \to Y$ where X and Y are finite dimensional smooth manifolds such that the differential $df(x)$ of f at a point $x \in X$ is a non-zero real multiple of an orthogonal map. Generally speaking this will be an angle preserving map. For example conformal maps of $\mathbb{R}^2$ $(= \mathbb{C})$ are any complex analytic function $f : \mathbb{C} \to \mathbb{C}$. The algebra of all such maps for $\mathbb{R}^n$ with $n > 2$ can be described in terms of a *Clifford algebra.*

Consider a vector space, $V = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ where $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = g_{ij}$ and g_{ij} is some diagonal (metric) matrix. Then if the basis elements of V can be normalised to the set $\{1, e_1, \ldots e_n\}$ whereby

$$e_i e_j + e_j e_i = -2g_{ij}$$

i.e.

$$\mathbf{v}.\mathbf{w} + \mathbf{w}.\mathbf{v} = -2\langle \mathbf{v}, \mathbf{w} \rangle$$

then $Cl(V)$, the Clifford algebra of V, has a basis defined by $\{1, e_1, \ldots, e_n,$ $e_i e_j (i < j), e_i e_j e_k (i < j < k), \ldots e_1 e_2 \ldots e_n\}$ where $i, j, \ldots k = 1, \ldots, n$. The dimension of $Cl(V)$ is 2^n. The group of conformal maps over the quadratic space $\mathbb{R}^{p,q}$ is isomorphic to the Clifford algebra of 2×2 matrices with entries in $\mathbb{R}_{p,q-1}$ (see **[31]**).

CHAPTER 2

The Magic Square and its Symmetry

In this chapter we introduce the 3×3 magic square and its compression to the 2×2 case. We begin with the Tits-Freudenthal construction of the 3×3 magic square and then examine how other constructions make more obvious the underlying symmetry of the magic square. In particular we introduce a new symmetric construction of the underlying vector space utilising a Lie algebra called the triality algebra. We then introduce the 2×2 extension to the magic square and show that the new triality algebra construction of the 3×3 magic square yields a directly equivalent construction for the 2×2 magic square using orthogonal algebras.

1. The 3×3 Magic Square

Let $\mathbb{K}$ be a real composition algebra and $\mathbb{J}$ a real Jordan algebra, with $\mathbb{K}'$ and $\mathbb{J}'$ the quotients of their vector spaces by the subspaces of scalar multiples of the identity. Define a vector space

(36) $$M(\mathbb{J}, \mathbb{K}) = \operatorname{Der} \mathbb{J} \dot{+} (\mathbb{J}' \otimes \mathbb{K}') \dot{+} \operatorname{Der} \mathbb{K}.$$

Then define

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = M(H_3(\mathbb{K}_1), \mathbb{K}_2).$$

Explicitly this is the vector space

(37) $$L_3(\mathbb{K}_1, \mathbb{K}_2) = \operatorname{Der} H_3(\mathbb{K}_1) \dot{+} H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2' \dot{+} \operatorname{Der} \mathbb{K}_2.$$

This is a Lie algebra when taken with the brackets

$$
\begin{aligned}
[D, A\otimes x] &= D(A)\otimes x\\
[E, A\otimes x] &= A\otimes E(x)\\
[D,E] &= 0 \qquad (38)\\
[A\otimes x, B\otimes y] &= \tfrac{1}{6}\langle A,B\rangle D_{x,y} + (A*B)\otimes \tfrac{1}{2}[x,y] - \langle x,y\rangle [L_A, L_B]
\end{aligned}
$$

where $D \in \operatorname{Der} H_3(\mathbb{K}_1)$; $A, B \in H_3'(\mathbb{K}_1)$; $x, y \in \mathbb{K}_2'$; $E \in \operatorname{Der}\mathbb{K}_2$, and $\operatorname{Der} H_3(\mathbb{K}_1)$ and $\operatorname{Der}\mathbb{K}_2$ are taken to be Lie subalgebras.

These brackets are obtained from Schafer's description of the Tits construction [**35**]. They require some explanation. The notations $\langle A,B\rangle$ and $\langle x,y\rangle$ denote the symmetric bilinear forms on $H_3(\mathbb{K}_1)$ and $\mathbb{K}_2$ respectively, given by

$$
\begin{aligned}
\langle A,B\rangle &= \operatorname{Re}(\operatorname{tr}(A\cdot B)) = 2\operatorname{Re}(\operatorname{tr}(AB))\\
\langle x,y\rangle &= \tfrac{1}{2}(|x+y|^2 - |x|^2 - |y|^2) = \operatorname{Re}(x\overline{y}).
\end{aligned}
$$

The derivation $D_{x,y}$ is defined as

$$D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y] \quad \in \operatorname{Der}\mathbb{K}_2. \qquad (39)$$

For future reference we note that

$$D_{x,y}(z) = [[x,y],z] - 3[x,y,z] \qquad (40)$$

which shows that $D_{x,y} = -D_{y,x}$, since all division algebras are alternative. Finally $(A*B)$ is the traceless part of the Jordan product of A and B, which is

$$A*B = A\cdot B - \tfrac{1}{3}\operatorname{tr}(A\cdot B) \qquad (41)$$

in the 3×3 case.

Tits [**38**] (see also [**13, 35**]) showed that this gives a unified construction leading to the so-called magic square of Lie algebras of 3×3 matrices whose complexifications are

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	B_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

i.e. the Lie algebras with compact real forms

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{su}(2)$	$\mathfrak{su}(3)$	$\mathfrak{sq}(3)$	F_4
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(6)$	E_6
$\mathbb{H}$	$\mathfrak{sq}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

.

The striking properties of this square are (a) its symmetry and (b) the fact that four of the five exceptional Lie algebras occur in its last row. The explanation of the symmetry property is the subject of the following section. The fifth exceptional Lie algebra, G_2, can be included by adding an extra row corresponding to the Jordan algebra $\mathbb{R}$ in $M(\mathbb{J}, \mathbb{K})$.

In [**37**] it is asserted without proof that we can write this in a slightly different form and that we can include the isomorphisms listed in (2). This involves a different set of real forms obtained by taking the split composition algebras $\mathbb{R}, \widetilde{\mathbb{C}}, \widetilde{\mathbb{H}}$, and $\widetilde{\mathbb{O}}$ rather than $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ as the second algebra, i.e. we consider $L_3(\mathbb{K}_1, \widetilde{\mathbb{K}}_2)$. Thus the split magic square for three by three matrices looks like

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\operatorname{Der} H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \mathbb{R}) \cong \mathfrak{su}(3, \mathbb{K})$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sq}(3)$	$F_{4,1}$
$\operatorname{Str}' H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \widetilde{\mathbb{C}}) \cong \mathfrak{sl}(3, \mathbb{K})$	$\mathfrak{sl}(3, \mathbb{R})$	$\mathfrak{sl}(3, \mathbb{C})$	$\mathfrak{sl}(3, \mathbb{H})$	$E_{6,1}$
$\operatorname{Con} H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \widetilde{\mathbb{H}}) \cong \mathfrak{sp}(6, \mathbb{K})$	$\mathfrak{sp}(6, \mathbb{R})$	$\mathfrak{su}(3, 3)$	$\mathfrak{sp}(6, \mathbb{H})$	$E_{7,1}$
$L_3(\mathbb{K}, \widetilde{\mathbb{O}})$	$F_{4,2}$	$E_{6,2}$	$E_{7,2}$	$E_{8,1}$

where the notation $_{,1}$ and $_{,2}$ (in the style of [**13**]) is used to distinguish between different real forms of the exceptional Lie algebras in the last row and column. These are identified by their maximal compact subalgebras as follows:

Exceptional Lie Algebra	Maximal Compact Subalgebra
$E_{6,1}$	F_4
$E_{7,1}$	$E_6 \oplus \mathfrak{so}(2)$
$E_{8,1}$	$E_7 \oplus \mathfrak{so}(3)$
$F_{4,2}$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$
$E_{6,2}$	$\mathfrak{su}(6) \oplus \mathfrak{so}(3)$
$E_{7,2}$	$\mathfrak{so}(12) \oplus \mathfrak{so}(3)$

2. Symmetry Property of the 3×3 Magic Square

In this section we will examine several explanations for the symmetry of the 3×3 Magic Square, namely those of Vinberg [**29**], Santander and Herranz [**33**] and an alternative description of the magic square symmetry involving a new algebra, the triality algebra, which is a result of joint work with A. Sudbery. We begin by presenting a definition of the *triality algebra* and then a series of results which will be used throughout this section.

DEFINITION 1. Let $\mathbb{K}$ be a composition algebra over $\mathbb{R}$. The *triality algebra* of $\mathbb{K}$, $\operatorname{Tri}\mathbb{K}$, is defined to be

$$\operatorname{Tri}\mathbb{K} = \{(A, B, C) \in \mathfrak{so}(\mathbb{K})^3 | A(xy) = x(By) + (Cx)y, \forall x, y \in \mathbb{K}\}. \tag{42}$$

Then $\operatorname{Tri}\mathbb{K}$ is a Lie algebra with brackets defined such that $\operatorname{Tri}\mathbb{K}$ is a Lie subalgebra of $\mathfrak{so}(\mathbb{K}) \oplus \mathfrak{so}(\mathbb{K}) \oplus \mathfrak{so}(\mathbb{K})$. Any two elements $x, y \in \mathbb{K}$ have an element of $\operatorname{Tri}\mathbb{K}$ associated with them as follows.

LEMMA 6. For any $x, y \in \mathbb{K}$, let

$$T_{x,y} = (4S_{x,y},\ R_y R_{\bar{x}} - R_x R_{\bar{y}},\ L_y L_{\bar{x}} - L_x L_{\bar{y}}).$$

Then $T_{x,y} \in \operatorname{Tri}\mathbb{K}$.

The proof requires the following Lemma.

LEMMA 7. For any composition algebra $\mathbb{K}$,

$$\operatorname{Tri}\mathbb{K} = \operatorname{Der}\mathbb{K} \dot{+} 2\mathbb{K}'$$

in which $\operatorname{Der}\mathbb{K}$ is a Lie subalgebra and $2\mathbb{K}' = \mathbb{K}' \dot{+} \mathbb{K}'$,

$$[D,(a,b)] = (Da, Db) \in 2\mathbb{K}'$$
$$[(a,0),(b,0)] = \tfrac{2}{3}D_{a,b} + \left(\tfrac{1}{3}[a,b], -\tfrac{2}{3}[a,b]\right),$$
$$[(a,0),(0,b)] = \tfrac{1}{3}D_{a,b} - \left(\tfrac{1}{3}[a,b], \tfrac{1}{3}[a,b]\right),$$
$$[(0,a),(0,b)] = \tfrac{2}{3}D_{a,b} + \left(-\tfrac{2}{3}[a,b], \tfrac{1}{3}[a,b]\right).$$

PROOF. Define $T : \operatorname{Der}\mathbb{K} \dot{+} 2\mathbb{K}' \to \operatorname{Tri}\mathbb{K}$ by

$$T(D,a,b) = \qquad (43)$$
$$(D + L_a - R_b,\ D - L_a - L_b - R_b,\ D + L_a + R_a + R_b).$$

The fact that this is a Lie algebra isomorphism follows from the brackets

$$[L_x, L_y] = \tfrac{2}{3}D_{x,y} + \tfrac{1}{3}L_{[x,y]} + \tfrac{2}{3}R_{[x,y]}$$
$$[L_x, R_y] = -\tfrac{1}{3}D_{x,y} + \tfrac{1}{3}L_{[x,y]} - \tfrac{1}{3}R_{[x,y]}$$
$$[R_x, R_y] = \tfrac{2}{3}D_{x,y} - \tfrac{2}{3}L_{[x,y]} - \tfrac{1}{3}R_{[x,y]}$$

and the definition of the inverse map $T^{-1} : \operatorname{Tri}\mathbb{K} \to \operatorname{Der}\mathbb{K} \dot{+} 2\mathbb{K}'$ which can be defined as

$$T^{-1}(A,B,C) = (A - L_a + R_b, a, b)$$

where $a = \frac{1}{3}B(1) + \frac{2}{3}C(1)$ and $b = -\frac{2}{3}B(1) - \frac{1}{3}C(1)$. □

PROOF OF LEMMA 6. Write the action of $S_{x,y}$ as

$$2S_{x,y}z = (x\bar{z} + z\bar{x})y - x(\bar{z}y + \bar{y}z) \qquad (44)$$
$$= -[x,y,z] + z(\bar{x}y) - (x\bar{y})z \qquad (45)$$

using the alternative law and the relation $[x, y, \bar{z}] = -[x, y, z]$. Since $\mathrm{Re}(\bar{x}y) = \mathrm{Re}(x\bar{y})$, we can write the last two terms as

$$z(\bar{x}y) - (x\bar{y})z = z\,\mathrm{Im}(\bar{x}y) - \mathrm{Im}(x\bar{y})z \tag{46}$$

$$= \tfrac{1}{2}z(\bar{x}y - \bar{y}x) - \tfrac{1}{2}(x\bar{y} - y\bar{x})z. \tag{47}$$

Now, by equation (40), we have

$$S_{x,y} = \tfrac{1}{6}D_{x,y} + L_a - R_b \tag{48}$$

with

$$a = -\tfrac{1}{6}[x, y] - \tfrac{1}{4}(x\bar{y} - y\bar{x}) \in \mathbb{K}'$$

$$b = -\tfrac{1}{6}[x, y] - \tfrac{1}{4}(\bar{x}y - \bar{y}x) \in \mathbb{K}'.$$

Hence there is an element $(A, B, C) \in \mathrm{Tri}\,\mathbb{K}$ with $A = S_{x,y}$ and

$$B = \tfrac{1}{6}D_{x,y} - L_a - L_b - R_b = S_{x,y} - L_{2a+b},$$

$$C = \tfrac{1}{6}D_{x,y} + L_a + R_a + R_b = S_{x,y} + R_{a+2b},$$

Writing $[x, y] = -\frac{1}{2}([\bar{x}, y] + [x, \bar{y}])$ gives

$$a + 2b = \tfrac{1}{4}(\bar{y}x - \bar{x}y)$$

$$2a + b = \tfrac{1}{4}(y\bar{x} - x\bar{y})$$

thus equations (44) and (46) imply that

$$S_{x,y} = -\tfrac{1}{2}Q_{x,y} - R_{a+2b} + L_{2a+b} \tag{49}$$

where $Q_{x,y}z = [x, y, z]$. Hence

$$Cz = -\tfrac{1}{2}[x, y, z] + \tfrac{1}{4}(y\bar{x} - x\bar{y})z$$

$$= \tfrac{1}{4}y(\bar{x}z) - \tfrac{1}{2}x(\bar{y}z)$$

i.e.

$$C = \tfrac{1}{4}(L_y L_{\bar{x}} - L_x L_{\bar{y}})$$

and similarly

$$B = \tfrac{1}{4}(R_y R_{\bar{x}} - R_x R_{\bar{y}}).$$

Thus $T_{x,y} = (4S_{x,y}, 4C, 4B)$ is an element of $\mathrm{Tri}\,\mathbb{K}$. □

Define an automorphism of $\mathrm{Tri}\,\mathbb{K}$ as follows. For any linear map $A : \mathbb{K} \to \mathbb{K}$ let $\overline{A} = KAK$ where $K : \mathbb{K} \to \mathbb{K}$ is the conjugation $x \mapsto \bar{x} \in \mathbb{K}$, i.e.

$$\overline{A}(x) = \overline{A(\bar{x})}.$$

LEMMA 8. Given $T = (A, B, C) \in \mathrm{Tri}\,\mathbb{K}$, let

$$\theta(T) = (\overline{B}, C, \overline{A}).$$

Then $\theta(T) \in \mathrm{Tri}\,\mathbb{K}$ and θ is a Lie algebra automorphism.

PROOF. By Lemma 7, $T = T(D, a, b)$ for some $D \in \mathrm{Der}\,\mathbb{K}$ and $a, b \in \mathbb{K}'$. Then

$$A = D + L_a - R_b$$

$$B = D - L_a - L_b - R_b$$

$$C = D + L_a + R_a + R_b.$$

It follows that

$$\overline{B} = D + R_a + R_b + L_b = D + L_{a'} - R_{b'}$$

which is the first component of $T' = (A', B', C') \in \mathrm{Tri}\,\mathbb{K}$, where

$$\begin{aligned} B' &= D - L_{a'} - L_{b'} - R_{b'} \\ &= D - L_b + L_{a+b} + R_{a+b} = C \\ C' &= D + L_{a'} + R_{a'} + R_{b'} \\ &= D + L_b + R_b - R_{a+b} = \overline{A}, \end{aligned}$$

i.e. $T' = (\overline{B}, C, \overline{A}) = \theta(T)$. It is clear that θ is a Lie algebra automorphism. □

THEOREM 4. For any composition algebra $\mathbb{K}$,

$$\operatorname{Der} H_3(\mathbb{K}) = \operatorname{Tri}\mathbb{K} \dot{+} 3\mathbb{K}$$

in which $\operatorname{Tri}\mathbb{K}$ is a Lie subalgebra, and the brackets in $[\operatorname{Tri}\mathbb{K}, 3\mathbb{K}]$ are

$$[T, F_i(x)] = F_i(T_i x) \quad \in 3\mathbb{K}, \tag{50}$$

if $T = (T_1, \overline{T}_2, \overline{T}_3) \in \operatorname{Tri}\mathbb{K}$ and $F_1(x) + F_2(y) + F_3(z) = (x, y, z) \in 3\mathbb{K}$; and the brackets in $[\operatorname{Tri}\mathbb{K}, \operatorname{Tri}\mathbb{K}]$ are given by

$$[F_i(x), F_j(y)] = F_k(\bar{y}\bar{x}) \in 3\mathbb{K}, \tag{51}$$

if $x, y \in \mathbb{K}$ and (i, j, k) is a cyclic permutation of $(1, 2, 3)$; and

$$[F_i(x), F_i(y)] = \theta^{i-1}(T_{x,y}) \in \operatorname{Tri}\mathbb{K}. \tag{52}$$

PROOF. Define elements $e_i, P_i(x)$ of $H_3(\mathbb{K})$ (where $i = 1, 2, 3$; $x \in \mathbb{K}$) by the equation

$$\begin{pmatrix} \alpha & z & \bar{y} \\ \bar{z} & \beta & x \\ y & \bar{x} & \gamma \end{pmatrix} = \alpha e_1 + \beta e_2 + \gamma e_3 + P_1(x) + P_2(y) + P_3(z) \tag{53}$$

for $\alpha, \beta, \gamma \in \mathbb{R}$; $x, y, z \in \mathbb{K}$. The Jordan product in $H_3(\mathbb{K})$ is given by

$$e_i \cdot e_j = 2\delta_{ij} e_i \tag{54a}$$

$$e_i \cdot P_j(x) = (1 - \delta_{ij}) P_j(x) \tag{54b}$$

$$P_i(x) \cdot P_i(y) = 2(x, y)(e_j + e_k) \tag{54c}$$

$$P_i(x) \cdot P_j(y) = P_k(\bar{y}\,\bar{x}) \tag{54d}$$

where in each of the last two equations (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Now let $D : H_3(\mathbb{K}) \to H_3(\mathbb{K})$ be a derivation of this algebra. First suppose that

$$De_i = 0, \quad i = 1, 2, 3.$$

Then

$$e_i \cdot DP_i(x) = 0$$
$$e_i \cdot DP_j(x) = DP_j(x) \quad \text{if } i \neq j$$

Thus $DP_j(x)$ is an eigenvector of each of the multiplication operators L_{e_i}, with eigenvalue 0 if $i = j$ and 1 if $i \neq j$. It follows that

$$DP_j(x) = P_j(T_j x) \tag{55}$$

for some $T_j : \mathbb{K} \to \mathbb{K}$. Now

$$DP_j(x) \cdot P_j(y) + P_j(x) \cdot DP_j(y) = 0$$

gives $T_j \in \mathfrak{so}(\mathbb{K})$; and the derivation property of D applied to equation (54d) gives

$$T_k(\bar{y}\bar{x}) = \bar{y}(\overline{T_i x}) + (\overline{T_j y})\bar{x}$$

i.e. $(T_k, \overline{T_i}, \overline{T_j}) \in \operatorname{Tri} \mathbb{K}$ and therefore $(T_1, \overline{T_2}, \overline{T_3}) \in \operatorname{Tri} \mathbb{K}$.

If $De_i \neq 0$, then from equation (54a) with $i = j$,

$$2e_i \cdot De_i = 2De_i$$

so De_i is an eigenvector of the multiplication L_{e_i} with eigenvalue 1, i.e. $De_i \in P_j(\mathbb{K}) + P_k(\mathbb{K})$ where (i, j, k) are distinct. Write

$$De_i = P_j(x_{ij}) + P_k(x_{ik});$$

then equation (54a) with $i \neq j$ gives

$$e_i \cdot P_k(x_{jk}) + e_i \cdot P_i(x_{ji}) + P_j(x_{ij}) \cdot e_j + P_k(x_{ik}) \cdot e_j = 0.$$

Thus

$$P_k(x_{jk} + x_{ik}) = 0.$$

It follows that the action of any derivation on the e_i must be of the form $F_1(x) + F_2(y) + F_3(z)$ where

$$F_i(x)e_i = 0$$

(56) $$F_i(x)e_j = -F_i(x)e_k = P_i(x),$$

(i, j, k) being a cyclic permutation of $(1, 2, 3)$. Hence Der $H_3(\mathbb{K}) \subseteq \mathrm{Tri}\,\mathbb{K} \oplus 3\mathbb{K}$.

To show that such derivations $F_i(x)$ exist and therefore the inclusion just mentioned is an equality, consider the operation of commutation with the matrix

$$X = \begin{pmatrix} 0 & -z & \bar{y} \\ \bar{z} & 0 & -x \\ -y & \bar{x} & 0 \end{pmatrix} = X_1(x) + X_2(y) + X_3(z)$$

i.e. define $F_i(x) = C_{X_i(x)}$ where $C_X : H_3(\mathbb{K}) \to H_3(\mathbb{K})$ is the commutator map

(57) $$C_X(H) = XH - HX.$$

This satisfies equation (56) and also

$$F_i(x)P_i(y) = -2(x, y)(e_j - e_k)$$

(58) $$F_i(x)P_j(y) = -P_k(\bar{y}\,\bar{x})$$

$$F_i(x)P_k(y) = P_j(\bar{x}\,\bar{y}).$$

It is a derivation of $H_3(\mathbb{K})$ by virtue of the matrix identity

(59) $$[X, \{H, K\}] = \{[X, H], K\} + \{H, [X, K]\}$$

(in which square brackets denote commutators and chain brackets denote anticommutators), which will be proved separately in Appendix A.

The Lie brackets of these derivations follow from another matrix identity which is also proved in Appendix A,

(60) $$[X, [Y, H]] - [Y, [X, H]] = [[X, Y], H] + E(X, Y)H$$

where $E(X, Y) \in \mathfrak{so}(\mathbb{K}')$ is defined by

$$E(X,Y)z = \sum_{ij} [x_{ij}, y_{ji}, z],$$

x_{ij}, y_{ji} being the matrix elements of X and Y. If $X = X_i(x)$ and $Y = X_j(y)$ we have $D(X,Y) = 0$ and

$$[X_i(x), X_j(y)] = X_k(\bar{y}\bar{x})$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$. This yields the Lie bracket (51). If $X = X_i(x)$ and $Y = X_j(y)$, the matrix commutator $Z = [X,Y]$ is diagonal with $z_{ii} = 0$, $z_{jj} = y\bar{x} - x\bar{y}$ and $z_{kk} = \bar{y}x - \bar{x}y$ $(i,j,k$ cyclic). Hence the action of the commutator $[F_i(x), F_i(y)] = C_z + E(X,Y)$ on $H_3(\mathbb{K})$ is

$$
\begin{aligned}
{[F_i(x), F_i(y)]e_m} &= 0 \quad (m = i,j,k) \\
[F_i(x), F_i(y)]P_i(w) &= P_i(z_{jj}w - wz_{kk} - 2[x,y,w]) = P_i(T_1 w) \\
&= 4P_i(S_{xy}w) \quad \text{by equation (73)}. \\
[F_i(x), F_i(y)]P_j(w) &= P_j(z_{kk}w - 2[x,y,w]) \\
&= P_j(\bar{y}(xw) - \bar{x}(yw)) \\
[F_i(x), F_i(y)]P_k(w) &= P_k(-wz_{jj} - 2[x,y,w]) \\
&= P_k((wx)\bar{y} - (wy)\bar{x}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
{[F_i(x), F_i(y)]P_i(w)} &= P_i(T_1 w) = P_i(T_i' w) \\
[F_i(x), F_i(y)]P_j(w) &= P_j(\overline{T_2}w) = P_j(\overline{T_j'}w) \\
[F_i(x), F_i(y)]P_k(w) &= P_k(\overline{T_3}w) = P_k(\overline{T_k'}w)
\end{aligned}
$$

where $(T_1, T_2, T_3) = T_{xy}$, so that $T' = \theta^{i-1}(T_{xy})$. This establishes the Lie bracket (52). □

THEOREM 5. For any composition algebra $\mathbb{K}$,

(61) $$\operatorname{Der} H_3(\mathbb{K}) = \operatorname{Der} \mathbb{K} \dot{+} A_3'(\mathbb{K})$$

in which $\operatorname{Der}\mathbb{K}$ is a Lie subalgebra, the Lie brackets between $\operatorname{Der}\mathbb{K}$ and $A_3'(\mathbb{K})$ are given by the elementwise action of $\operatorname{Der}\mathbb{K}$ on 3×3 matrices, and

$$[X,Y] = (XY - YX)' + \tfrac{1}{3}D(X,Y)$$

where $X, Y \in A_3'(\mathbb{K})$,

$$(XY - YX)' = XY - YX - \tfrac{1}{3}\operatorname{tr}(XY - YX)\mathbb{1} \quad \in A_3'(\mathbb{K})$$

and

$$D(X,Y) = \sum_{ij} D_{x_{ij},y_{ji}} \quad \in \operatorname{Der}\mathbb{K}$$

where x_{ij} and y_{ji} are the matrix elements of X and Y.

PROOF. By Lemma 7 and Theorem 4

$$\operatorname{Der} H_3(\mathbb{K}) = \operatorname{Der}\mathbb{K} \dot{+} 2\mathbb{K}' \dot{+} 3\mathbb{K}. \tag{62}$$

Identify $(a,b) + (x,y,z) \in 2\mathbb{K}' \dot{+} 3\mathbb{K}$ with the traceless antihermitian matrix

$$X = \begin{pmatrix} -a-b & -z & \bar{y} \\ \bar{z} & a & -x \\ -y & \bar{x} & b \end{pmatrix} \quad \in A_3'(\mathbb{K});$$

then the actions of $2\mathbb{K}'$ and $3\mathbb{K}$ on $H_3(\mathbb{K})$ defined in Theorem 4 are together equivalent to the commutator action C_x defined by equation (57). By the matrix identities defined in Appendix A,

$$[C_X, C_Y] = C_{(XY-YX)'} + C_{t\mathbb{1}} + E(X,Y)$$

where

$$\begin{aligned} t &= \tfrac{1}{3}\operatorname{tr}(XY - YX) \\ &= \tfrac{1}{3}\sum_{ij}(x_{ij}y_{ji} - y_{ji}x_{ij}). \end{aligned}$$

Now $C_{t\mathbb{1}} + E(X,Y)$ acts elementwise on matrices in $H_3(\mathbb{K})$ according to the map $D : \mathbb{K} \to \mathbb{K}$ given by

$$\begin{aligned} Dz &= [t,z] + E(X,Y)z \\ &= \sum_{ij} \left(\tfrac{1}{3}[[x_{ij},y_{ji}],z] - [x_{ij},y_{ji},z]\right) \\ &= \tfrac{1}{3}D(X,Y)Z. \end{aligned} \tag{63}$$

Hence the bracket $[X,Y]$ is as stated in Theorem 5. □

2.1. The Vinberg Construction. For this construction let $\mathbb{K}_1$ and $\mathbb{K}_2$ be composition algebras. For these two algebras define the tensor algebra $\mathbb{K}_1 \otimes \mathbb{K}_2$ to be the algebra in which the inner product $x \mapsto \bar{x}$ is defined by $a \otimes b \mapsto \bar{a} \otimes \bar{b}$, where conjugation in $\mathbb{K}_1$ and $\mathbb{K}_2$ is the usual conjugation. Then the vector space

$$V_3(\mathbb{K}_1,\mathbb{K}_2) = A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2) \dot{+} \operatorname{Der}\mathbb{K}_1 \dot{+} \operatorname{Der}\mathbb{K}_2 \tag{64}$$

is clearly symmetric. This is a Lie algebra when taken with the Lie brackets defined by the statements:

1. $\operatorname{Der}\mathbb{K}_1 \dot{+} \operatorname{Der}\mathbb{K}_2$ is a Lie subalgebra.
2. For $D \in \operatorname{Der}\mathbb{K}_1 \oplus \operatorname{Der}\mathbb{K}_2$ and $X \in A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2)$,

   $$[D,X] = D(X)$$

   where $D(X)$ is the matrix formed by making D act elementwise on X.
3. For $X = (x_{ij})$, $Y = (y_{ij}) \in A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2)$,

   $$[X,Y] = (XY - YX)' + \tfrac{1}{3}\sum_{ij} D_{x_{ij}y_{ij}}.$$

The definition of $D_{a_{ij}b_{ij}}$ is such that if $a_{ij} = x \otimes y$ and $b_{ij} = u \otimes v$ then

$$D_{x\otimes y, u\otimes v} = \langle x,u\rangle D_{y,v} + \langle y,v\rangle D_{x,u}$$

Furthermore, we will now prove that the algebra $V_3(\mathbb{K}_1,\mathbb{K}_2)$ is a Lie algebra isomorphic to the algebra $L_3(\mathbb{K}_1,\mathbb{K}_2)$ defined by the Tits-Freudenthal construction since no proof of this is readily available.

Begin by noting that a matrix in $A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2)$ has a general form given by

$$\begin{pmatrix} x_1 \otimes 1 + 1 \otimes x_2 & a_1 \otimes a_2 & b_1 \otimes b_2 \\ -\bar{a}_1 \otimes \bar{a}_2 & y_1 \otimes 1 + 1 \otimes y_2 & c_1 \otimes c_2 \\ -\bar{b}_1 \otimes \bar{b}_2 & -\bar{c}_1 \otimes \bar{c}_2 & z_1 \otimes 1 + 1 \otimes z_2 \end{pmatrix}$$

where $x_i + y_i + z_i = 0$, $x_i, y_i, z_i \in \mathbb{K}_i'$ and $a_i, b_i, c_i \in \mathbb{K}_i$. Using Theorem 5, write the Tits-Freudenthal vector space as

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = A_3'(\mathbb{K}_1) + \operatorname{Der}\mathbb{K}_1 + H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2' + \operatorname{Der}\mathbb{K}_2.$$

Define a function $\psi : V(\mathbb{K}_1, \mathbb{K}_2) \to L_3(\mathbb{K}_1, \mathbb{K}_2)$ by

$$\psi \begin{pmatrix} x_1 \otimes 1 + 1 \otimes x_2 & a_1 \otimes a_2 & b_1 \otimes b_2 \\ -\bar{a}_1 \otimes \bar{a}_2 & y_1 \otimes 1 + 1 \otimes y_2 & c_1 \otimes c_2 \\ -\bar{b}_1 \otimes \bar{b}_2 & -\bar{c}_1 \otimes \bar{c}_2 & z_1 \otimes 1 + 1 \otimes z_2 \end{pmatrix} =$$

$$\begin{pmatrix} x_1 & a_1 \operatorname{Re}(a_2) & b_1 \operatorname{Re}(b_2) \\ -\bar{a}_1 \operatorname{Re}(a_2) & y_1 & c_1 \operatorname{Re}(c_2) \\ -\bar{b}_1 \operatorname{Re}(b_2) & -\bar{c}_1 \operatorname{Re}(c_2) & z_1 \end{pmatrix} \in A_3'(\mathbb{K}_1)$$

$$+ \begin{pmatrix} 1 \otimes x_2 & a_1 \otimes \operatorname{Im}(a_2) & b_1 \otimes \operatorname{Im}(b_2) \\ \bar{a}_1 \otimes \operatorname{Im}(a_2) & 1 \otimes y_2 & c_1 \otimes \operatorname{Im}(c_2) \\ \bar{b}_1 \otimes \operatorname{Im}(b_2) & \bar{c}_1 \otimes \operatorname{Im}(c_2) & 1 \otimes z_2 \end{pmatrix} \in H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2$$

and for $D \in \operatorname{Der}\mathbb{K}_1$, $E \in \operatorname{Der}\mathbb{K}_2$,

$$\psi(D) = D$$

$$\psi(E) = E.$$

To prove that ψ is an isomorphism we show that

$$[A_3'(\mathbb{K}_1 \otimes \mathbb{R}), A_3'(\mathbb{K}_1 \otimes \mathbb{R})]_{\text{Vin}} = [A_3'(\mathbb{K}_1), A_3'(\mathbb{K}_1)]_{\text{Tits}}$$

$$[A_3'(\mathbb{K}_1 \otimes \mathbb{R}), A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2')]_{\text{Vin}} = [A_3'(\mathbb{K}_1), H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2']_{\text{Tits}}$$

$$[A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2'), A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2')]_{\text{Vin}} = [H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2', H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2']_{\text{Tits}}.$$

where $[,]_{\text{Vin}}$ is used to describe the Lie brackets in the Vinberg construction and $[,]_{\text{Tits}}$ is used to describe the Lie brackets in the Tits-Freudenthal construction.

1. $[A_3'(\mathbb{K}_1 \otimes \mathbb{R}), A_3'(\mathbb{K}_1 \otimes \mathbb{R})]_{\text{Vin}}$. In $V(\mathbb{K}_1, \mathbb{K}_2)$ the bracket is

$$[A, B]_{\text{Vin}} = (AB - BA)' + \tfrac{1}{3}\sum_{ij} D_{a_{ij}b_{ij}}$$

where $A, B \in A_3'(\mathbb{K}_1 \otimes \mathbb{R})$. When y and v are 1, the sum $\frac{1}{3}\sum_{ij} D_{a_{ij}b_{ij}}$ reduces to $D_{x,u}$ which is the case here. In $L_3(\mathbb{K}_1, \mathbb{K}_2)$ the bracket is

$$[A, B]_{\text{Tits}} = (AB - BA)' + \tfrac{1}{3}\sum_{ij} D_{a_{ij}b_{ji}}$$

which is clearly the same as the Vinberg bracket.

2. $[A_3'(\mathbb{K}_1 \otimes \mathbb{R}), A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2')]_{\text{Vin}}$. Let $A \in A_3'(\mathbb{K}_1 \otimes \mathbb{R})$ and $B \in A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2')$. Then the bracket in the Vinberg construction is

$$[A, B]_{\text{Vin}} = (AB - BA)' + \tfrac{1}{3}\sum_{ij} D_{a_{ij}b_{ij}}.$$

In the Tits construction the equivalent bracket is, for $A \in A_3'(\mathbb{K}_1)$ and $B \otimes x \in H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2'$,

$$[A, B \otimes x]_{\text{Tits}} = (AB - BA) \otimes x,$$

since any matrix in $H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2'$ can be written as the sum of (at most eight) matrices of the form $H \otimes a$. Now $\psi^{-1}(A) = A \otimes 1 \in A_3'(\mathbb{K}_1 \otimes \mathbb{R})$ and $\psi^{-1}(B \otimes x) = B \otimes x \in A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2')$. Thus

$$\begin{aligned}
[\psi^{-1}(A), \psi^{-1}(B \otimes x)]_{\text{Vin}} &= [A, B \otimes x]_{\text{Vin}} \\
&= ((A(B \otimes x) - (B \otimes x)A)' + \tfrac{1}{3} D_{A\otimes 1, B\otimes x} \\
&= (AB - BA)' \otimes x + \tfrac{1}{3}\left(\langle 1, x\rangle D_{A,B} + \langle A, B\rangle D_{1,x}\right).
\end{aligned}$$

Since A is anti-hermitian and B is hermitian, cancellation occurs and $D_{A,B} = 0$. Also since $(AB - BA) \in H_3'(\mathbb{K}_1)$, $(AB - BA)' = (AB - BA)$. Thus

$$\begin{aligned}
[\psi^{-1}(A), \psi^{-1}(B \otimes x)]_{\text{Vin}} &= (AB - BA) \otimes x \\
&= \psi^{-1}([A, B \otimes x]_{\text{Tits}}).
\end{aligned}$$

Thus $[A_3'(\mathbb{K}_1 \otimes \mathbb{R}), A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2')]_{\text{Vin}} = [A_3'(\mathbb{K}_1), H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2']_{\text{Tits}}$.

3. $[A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2'), A_3'(\mathbb{K}_1 \otimes \mathbb{K}_2')]_{\text{Vin}}$. In this case the Tits-Freudenthal bracket is

$$[A \otimes x, B \otimes y]_{\text{Tits}} = \tfrac{1}{6}\langle A, B\rangle + A * B \otimes \operatorname{Im} xy - \langle x, y\rangle [L_A, L_B].$$

Consider the Lie bracket where $\psi(A \otimes a) = C = A \otimes a$ and $\psi(B \otimes b) = D = B \otimes b$. Then

$$\begin{aligned}
[C, D]_{Vin} &= (CD - DC)' + \tfrac{1}{3}\sum_{ij} D_{c_{ij}d_{ij}} \\
&= ((A \otimes a)(B \otimes b) - (B \otimes b)(A \otimes a))' + \tfrac{1}{3}\sum_{ij} D_{c_{ij}d_{ij}} \\
&= (AB \otimes \operatorname{Re}(ab) + AB \otimes \operatorname{Im}(ab) - BA \otimes \operatorname{Re}(ba) + BA \otimes \operatorname{Im}(ba))' \\
&\qquad - \tfrac{1}{3}\operatorname{tr}(AB \otimes ab - BA \otimes ba) + \tfrac{1}{3}\sum_{ij} D_{c_{ij}d_{ij}} \\
&= (A * B) \otimes \operatorname{Im}(ab) + (AB - BA)' \otimes \operatorname{Re}(ab) + \tfrac{1}{3}\sum_{ij} D_{c_{ij}d_{ij}} \\
&= (A * B) \otimes \operatorname{Im} ab - [A, B]' \otimes \langle a, b\rangle \\
&\qquad + \tfrac{1}{3}\sum_{ij} \big(\langle a, b\rangle D_{A_{ij},B_{ij}} + \langle A_{ij}, B_{ij}\rangle D_{a,b}\big) \\
&= (A * B) \otimes \operatorname{Im} ab - [A, B]' \otimes \langle a, b\rangle + \tfrac{1}{6}\langle A, B\rangle D_{a,b} \\
&\qquad + \tfrac{1}{3}\sum_{ij} \langle a, b\rangle D_{A_{ij},B_{ij}} \\
&= (A * B) \otimes \operatorname{Im} ab + \tfrac{1}{6}\langle A, B\rangle D_{a,b} - [A, B]' \otimes \langle a, b\rangle \\
&\qquad - \tfrac{1}{3}\sum_{ij} \langle a, b\rangle D_{A_{ij},B_{ji}}
\end{aligned}$$

since $\sum_{ij}\langle A_{ij}, B_{ij}\rangle = \frac{1}{2}\langle A, B\rangle$ and $\operatorname{Re}(ab) = -\langle a, b\rangle$. In Appendix A it is shown that any general element $[L_A, L_B]$ of $\operatorname{Der} H_3(\mathbb{K})$ can be written in the form $[A, B] + \frac{1}{3}\sum D_{A_{ij}B_{ij}}$ where $[A, B] \in A_3'(\mathbb{K})$ and $\frac{1}{3}\sum D_{A_{ij}B_{ji}} \in \operatorname{Der}\mathbb{K}$. Thus $[\psi(A \otimes a), \psi(B \otimes b)]_{Vin} = \psi([A \otimes a, B \otimes b]_{Tits})$.

2.2. The Santander-Herranz Construction. An extension to the Vinberg approach to magic square symmetry is given by Santender and Herranz in their construction of so-called 'Cayley-Klein' (CK) algebras. Start by defining

$I_\omega = \mathrm{diag}(1, \omega_{01}, \omega_{02}, \ldots, \omega_{0N})$ (where $\omega_{ab} = \omega_a \omega_{a+1} \ldots \omega_b$) depending on $N+1$ fixed non-zero co-efficients ω_i and $\mathbb{I}_\omega = \left(\begin{smallmatrix} 0 & I_\omega \\ -I_\omega & 0 \end{smallmatrix}\right)$. A matrix X is defined to be G-antihermitian if $X^\dagger G + GX = 0$. Santander and Herranz use these to define three series of classical CK-algebras. These are:

1. The special antihermitian CK-algebra, $\mathfrak{sa}_{\omega_1 \ldots \omega_N}(N+1, \mathbb{K})$. This is the Lie algebra of I_ω-antihermitian matrices, X, over $\mathbb{K}$ if $\mathbb{K} = \mathbb{R}, \mathbb{H}$ and the subalgebra of such matrices with the condition $\mathrm{tr}\, X = 0$ if $\mathbb{K} = \mathbb{C}$.
2. The special linear CK-algebra, $\mathfrak{sl}_{\omega_1 \ldots \omega_N}(N+1, \mathbb{K})$. This is the Lie algebra of all matrices over $\mathbb{K}$, X, with $\mathrm{tr}\, X = 0$ if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathrm{Re}(\mathrm{tr}\, X) = 0$ if $\mathbb{K} = \mathbb{H}$.
3. The special symplectic CK-algebra, $\mathfrak{sn}_{\omega_1 \ldots \omega_N}(2(N+1), \mathbb{K})$. This is the Lie algebra of all $\mathbb{I}_\omega$-antihermitian matrices over $\mathbb{K}$ if $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$ and the subalgebra of matrices with zero trace if $\mathbb{K} = \mathbb{C}$.

For $N = 1, 2$ these definitions can be extended to include $\mathbb{K} = \mathbb{O}$ by adding the derivations of $\mathbb{O}$ in each case. A fourth CK-algebra can also be added, the metasymplectic CK-algebra, $\mathfrak{mn}(N+1, \mathbb{K})$ although the explicit form of this is not made clear, but is based on the definition of the metasymplectic geometry given in [**13**].

Now define the set of matrices

$$J_{ab} = \begin{pmatrix} & \vdots & & \vdots & \\ \cdot & \cdot & \cdots & -\omega_{ab} & \cdots \\ & \vdots & & \vdots & \\ \cdot & 1 & \cdots & \cdot & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \qquad M_{ab} = \begin{pmatrix} & \vdots & & \vdots & \\ \cdot & \cdot & \cdots & \omega_{ab} & \cdots \\ & \vdots & & \vdots & \\ \cdot & 1 & \cdots & \cdot & \cdots \\ & \vdots & & \vdots & \end{pmatrix}$$

and

$$H_m = \begin{pmatrix} 1 & & & \cdot & \\ & & & \vdots & \\ \cdot & & \cdots & 1 & \cdots \\ & & & \vdots & \end{pmatrix} \qquad E_0 = \begin{pmatrix} 1 & \vdots \\ & \end{pmatrix}.$$

where $a, b = 0, 1, \ldots, N$ with the condition that $a < b$; $m = 1, \ldots, N$ and matrix indices run over the range $0, \ldots, N$. Further if X is one of these matrices then define $X^i = e_i X$ and

$$\mathbb{X} = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \quad \mathbb{X}_1 = \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix} \quad \mathbb{X}_2 = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \quad \mathbb{X}_3 = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}.$$

Also note that an isomorphism can be found such that $J \mapsto \mathbb{J}_{ab}$, $M_{ab} \mapsto \mathbb{M}_{ab;2}$, $M^1_{ab} \mapsto \mathbb{M}^1_{ab}$, $M^2_{ab} \mapsto \mathbb{M}^2_{ab}$.

The first three rows and columns of the Tits-Freudenthal magic square can now be generalised to the $(N+1)$-dimensional case using the three CK-algebra series as follows

Lie Algebra	Lie span of the generators		
	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathfrak{sa}_{\omega_1\ldots\omega_N}(N+1, \mathbb{K})$	J_{ab}	$J_{ab}, M^1_{a,b}$	$J_{ab}, M^1_{a,b}, M^2_{ab}$
$\mathfrak{sl}_{\omega_1\ldots\omega_N}(N+1, \mathbb{K})$	J_{ab}, M_{ab}	$J_{ab}, M_{ab}, M^1_{a,b}$	$J_{ab}, M_{ab}, M^1_{a,b}, M^2_{ab}$
$\mathfrak{sn}_{\omega_1\ldots\omega_N}(N+1, \mathbb{K})$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;1}, \mathbb{M}_{ab;2}$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;1}, \mathbb{M}_{ab;2}, \mathbb{M}^1_{ab}$	$\mathbb{J}_{ab}, \mathbb{M}_{ab;1}, \mathbb{M}_{ab;2}, \mathbb{M}^1_{ab}, \mathbb{M}^2_{ab}$

Then the symmetry of the $(N+1)$ dimensional magic square (and consequently of the 3×3 magic square) can be explained as follows.

Each algebra is a subalgebra of all the algebras to its right and below it. This can also be expressed by saying that as we move from left to right and from top to bottom across the square in each step the same new generators appear. Explicitly, moving from the top algebra ($\mathfrak{sa}$) to the bottom ($\mathfrak{sn}$), in each column M_{ab} appears in the first step ($\mathfrak{sa} \to \mathfrak{sl}$) and $\mathbb{M}_{ab;1}$ appears in the second ($\mathfrak{sl} \to \mathfrak{sn}$). Similarly moving from left to right, M^1_{ab} is the additional generator after the first step and M^2_{ab} is the additional generator after the second.

In more recent work Santander [**34**] has gone on to define the tensor algebra $\mathfrak{sa}_{\omega_1\ldots\omega_N}(N+1, \mathbb{K}_1 \otimes \mathbb{K}_2)$ which he indicates to be an extension of the Vinberg construction which may includes all simple Lie algebras, i.e. *any* simple Lie algebra can be written in the form $\mathfrak{sa}_{\omega_1\ldots\omega_N}(N+1, \mathbb{K}_1 \otimes \mathbb{K}_2)$ for an appropriate choice of ω_i, N, $\mathbb{K}_1$ and $\mathbb{K}_2$. Explicitly this is the algebra of $(N+1)\times(N+1)$ matrices with entries in $\mathbb{K}_1 \otimes \mathbb{K}_2$ and the derivations of $\mathbb{K}_1$ and $\mathbb{K}_2$.

Thus we have a second way of approaching an explanation of the symmetry of the magic square and indeed to classify all simple Lie algebras in terms of matrices with entries in the division algebras.

2.3. The Triality Construction. A third way of giving the magic square a clearly symmetric formulation will be given in terms of the previously defined *triality algebra.* One further definition is required before this formulation can be given.

DEFINITION 2. Let $F_i(x) = C_{X_i(x)}$ where $C_X : H_3(\mathbb{K}) \to H_3(\mathbb{K})$ is the commutator map

$$C_X(H) = XH - HX$$

and X is the antihermitian 3×3 matrix given by

$$\begin{aligned} X &= \begin{pmatrix} 0 & -z & \bar{y} \\ \bar{z} & 0 & -x \\ -y & \bar{x} & 0 \end{pmatrix} \\ &= X_1(x) + X_2(y) + X_3(z). \end{aligned}$$

Then the following Theorem makes explicit the symmetry in the 3×3 magic square.

THEOREM 6. For any two composition algebras $\mathbb{K}_1$ and $\mathbb{K}_2$,

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = \mathrm{Tri}\,\mathbb{K}_1 \oplus \mathrm{Tri}\,\mathbb{K}_2 \dotplus 3\mathbb{K}_1 \otimes \mathbb{K}_2 \tag{65}$$

in which $\operatorname{Tri}\mathbb{K}_1 \oplus \operatorname{Tri}\mathbb{K}_2$ is a Lie subalgebra;

$$[T_1, F_i(x \otimes y)] = F_i(T_{1i}x_1 \otimes x_2) \quad \in 3\mathbb{K}_1 \otimes \mathbb{K}_2 \tag{66}$$

$$[T_2, F_i(x \otimes y)] = F_i(x_1 \otimes T_{2i}x_2) \quad \in 3\mathbb{K}_1 \otimes \mathbb{K}_2 \tag{67}$$

if $T_\alpha = (T_{\alpha 1}, \overline{T}_{\alpha 2}, \overline{T}_{\alpha 3}) \in \operatorname{Tri}\mathbb{K}\,(\alpha = 1, 2)$, and

$$F_1(x_1 \otimes x_2) + F_2(y_1 \otimes y_2) + F_3(z_1 \otimes z_2) = (x_1 \otimes x_2, y_1 \otimes y_2, z_1 \otimes z_2) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2;$$

$$[F_i(x_1 \otimes x_2), F_j(y_1 \otimes y_2)] = F_k(\bar{y}_1\bar{x}_1 \otimes \bar{y}_2\bar{x}_2) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2 \tag{68}$$

if $x_\alpha, y_\alpha \in \mathbb{K}_2$ and (i, j, k) is a cyclic permutation of $(1, 2, 3)$; and

$$[F_i(x_1 \otimes x_2), F_i(y_1 \otimes y_2)] = \langle x_2, y_2\rangle \theta^{i-1} T_{x_1 y_1} + \langle x_1, y_1\rangle \theta^{i-1} T_{x_2 y_2} \in \operatorname{Tri}\mathbb{K}_1 \oplus \operatorname{Tri}\mathbb{K}_2 \tag{69}$$

PROOF. The vector space structure (37) of $L_3(\mathbb{K}_1, \mathbb{K}_2)$ can be written using Theorem 4, as

$$\begin{aligned} L_3(\mathbb{K}_1, \mathbb{K}_2) &= \operatorname{Der} H_3(\mathbb{K}_1) \dot{+} H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2' \dot{+} \operatorname{Der}\mathbb{K}_2 \\ &= (\operatorname{Tri}\mathbb{K} \dot{+} 3\mathbb{K}_1) \dot{+} (2\mathbb{K}_2' \dot{+} 3\mathbb{K}_1 \otimes 2\mathbb{K}_2') \dot{+} \operatorname{Der}\mathbb{K}_2 \\ &= \operatorname{Tri}\mathbb{K}_1 \dot{+} (\operatorname{Der}\mathbb{K}_2 \dot{+} 2\mathbb{K}_2') \dot{+} (3\mathbb{K}_1 \otimes \mathbb{K}_2' \dot{+} 3\mathbb{K}_1) \\ &\cong \operatorname{Tri}\mathbb{K}_1 \dot{+} \operatorname{Tri}\mathbb{K}_2 \dot{+} 3\mathbb{K}_1 \otimes \mathbb{K}_2. \end{aligned}$$

We use the following notation for the elements of the five subspaces of $L_3(\mathbb{K}_1, \mathbb{K}_2)$:

1. $\operatorname{Tri}\mathbb{K} \subset \operatorname{Der} H_3(\mathbb{K}_1)$ contains elements $T = (T_1, \overline{T}_2, \overline{T}_3)$ acting on $H_3'(\mathbb{K}_1)$ as in Theorem 4:

$$Te_i = 0, \quad TP_i(x) = P_i(T_i x) \quad (x \in \mathbb{K}; i = 1, 2, 3)$$

2. $3\mathbb{K}_1$ is the subspace of $\operatorname{Der} H_3(\mathbb{K}_1)$ containing the elements $F_i(x)$ defined in Theorem 4; these will be identified with the elements $F_i(x \otimes 1) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2'$.
3. $2\mathbb{K}_2'$ is the subspace $\Delta \otimes \mathbb{K}_2'$ of $H_3(\mathbb{K}_1) \otimes \mathbb{K}_2'$, where $\Delta \subset H_3'(\mathbb{K}_1)$ is the subspace of real, diagonal, traceless matrices and is identified with the subspace of $\operatorname{Tri}\mathbb{K}$ as described in Lemma 7. We will regard $2\mathbb{K}_2'$ as a subspace of $3\mathbb{K}_2'$, namely

$$2\mathbb{K}_2' = \{(a_1, a_2, a_3) \in 3\mathbb{K}_2' : a_1 + a_2 + a_3 = 0\}$$

and identify $\mathbf{a} = (a_1, a_2, a_3)$ with the 3×3 matrix

$$\Delta(\mathbf{a}) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \in H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2'$$

and with the triality $T(\mathbf{a}) = (T_1, \overline{T}_2, \overline{T}_3)$ where $T_i = L_{a_j} - R_{a_k}$ (see the remark after the proof of Lemma 8).
4. $3\mathbb{K}_1 \otimes \mathbb{K}_2'$ is the subspace of $H_3(\mathbb{K}_1 \otimes \mathbb{K}_2')$ spanned by elements $P_i(x) \otimes a$ $(i = 1, 2, 3 : x \in \mathbb{K}_1, a \in \mathbb{K}_2')$; it is also a subspace of $3\mathbb{K}_1 \otimes \mathbb{K}_2$ in the obvious way.
5. $\operatorname{Der}\mathbb{K}_2$ is a subspace of $\operatorname{Tri}\mathbb{K}_2$, a derivation D being identified with $(D, D, D) \in \operatorname{Tri}\mathbb{K}_2$.

To complete the proof we must verify that the Lie brackets defined by Tits (see Section 1 of this Chapter) coincide with those in the statement of the theorem. The above decomposition of $L_3(\mathbb{K}_1, \mathbb{K}_2)$ into five parts gives us fifteen types of bracket to examine. We will write $[\,,\,]_{\text{Tits}}$ for the bracket defined in section 1 and $[\,,\,]_{\text{here}}$ for that defined above.

1. $[\operatorname{Tri}\mathbb{K}_1, \operatorname{Tri}\mathbb{K}_2]$: For $T_1, T_2 \in \operatorname{Tri}\mathbb{K}_1$, $[T_1, T_2]_{\text{Tits}}$ is the bracket in $\operatorname{Der} H_3(\mathbb{K}_1)$, which by Theorem 4 is the same as $[T_1, T_2]_{\text{here}}$.

2. $[\mathrm{Tri}\,\mathbb{K}_1, 3\mathbb{K}_1]$: For $T \in \mathrm{Tri}\,\mathbb{K}_1$, $F_i(x \otimes 1) \in 3\mathbb{K}_1$,

$$\begin{aligned}[T, F_i(x)]_{\mathrm{Tits}} &= F_1(T_i x) \text{ see Theorem 4} \\ &= F_i(T_i x \otimes 1) = [T, F_i(x \otimes 1)]_{\mathrm{here}}.\end{aligned}$$

3. $[\mathrm{Tri}\,\mathbb{K}_1, 2\mathbb{K}_2]$: For $T_1 \in \mathrm{Tri}\,\mathbb{K}_1$, $(a, b, c) \in 2\mathbb{K}_2'$,

$$[T_1, (a, b, c)]_{\mathrm{Tits}} = [T, e_1 \otimes a + e_2 \otimes b + e_3 \otimes c] = 0$$

since in Theorem 4 $\mathrm{Tri}\,\mathbb{K}_1$ was obtained as the subspace of derivations which annihilate the diagonal matrices e_i. On the other hand,

$$[T_1, (a, b, c)]_{\mathrm{here}} = [T_1, T_2(a, b, c)] = 0.$$

4. $[\mathrm{Tri}\,\mathbb{K}_1, 3\mathbb{K}_1 \otimes \mathbb{K}_2']$: For $T_1 \in \mathrm{Tri}\,\mathbb{K}_1$, $P_i(x \otimes a) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2'$,

$$[T_1, P_i(x \otimes a) = P_i(T_{1i} x \otimes a) = [T_1, P_i(x \otimes a)]_{\mathrm{here}}. \tag{70}$$

5. $[\mathrm{Tri}\,\mathbb{K}_1, \mathrm{Der}\,\mathbb{K}_2]_{\mathrm{Tits}} \subset [\mathrm{Der}\,H_3(\mathbb{K}_1), \mathrm{Der}\,\mathbb{K}_2] = 0$, while $[\mathrm{Tri}\,\mathbb{K}_1, \mathrm{Der}\,\mathbb{K}_2]_{\mathrm{here}} \subset [\mathrm{Tri}\,\mathbb{K}_1, \mathrm{Tri}\,\mathbb{K}_2] = 0$.
6. $[3\mathbb{K}_1, 3\mathbb{K}_1]$: $3\mathbb{K}_1 = 3\mathbb{K}_1 \otimes \mathbb{R}$ is spanned by $F_i(x) = F_i(x \otimes 1)$ $(i = 1, 2, 3; x \in \mathbb{K}_1)$, and $[F_i(x), F_j(y)]_{\mathrm{Tits}}$ is given by Theorem 4, while $[F_i(x \otimes 1), F_j(y \otimes 1)]_{\mathrm{here}}$ is the same since $T_{x_2, y_2} = 0$ if $x_2, y_2 \in \mathbb{R}$.
7. $[3\mathbb{K}_1, 2\mathbb{K}_2']$: For $F_i(x) \in 3\mathbb{K}_1$, $\mathbf{a} = (a_1, a_2, a_3) \in 2\mathbb{K}_2'$ with $(a_1 + a_2 + a_3 = 0)$,

$$\begin{aligned}[F_i(x), \mathbf{a}]_{\mathrm{Tits}} &= [F_i(x), \textstyle\sum e_i \otimes a_i] \in [\mathrm{Der}\,H_3(\mathbb{K}_1), H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2'] \\ &= P_i(x) \otimes (a_j - a_k) \qquad \text{by (56)}\end{aligned}$$

while

$$\begin{aligned}[F_i(x), \mathbf{a}]_{\mathrm{here}} &= [F_i(x \otimes 1), T(\mathbf{a})] \;\in [3\mathbb{K}_1 \otimes \mathbb{K}_2, \mathrm{Tri}\,\mathbb{K}_2] \\ &= F_i(x \otimes (a_j - a_k)).\end{aligned}$$

8. $[3\mathbb{K}_1, 3\mathbb{K}_1 \otimes \mathbb{K}_2']$: For $F_i(x) \in 3\mathbb{K}_1, F_j(y \otimes a) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2'$,

$$\begin{aligned}[F_i(x), F_i(y \otimes a)]_{\text{Tits}} &= [F_i(x), P_i(y) \otimes a] \ \in [\operatorname{Der} H_3(\mathbb{K}_1), H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2'] \\ &= -2\langle x, y\rangle(e_j - e_k) \otimes a \ \in H_3'(\mathbb{K}_1 \otimes \mathbb{K}_2' \\ &= -2\langle x, y\rangle(a_1, a_2, a_3) \ \in 2\mathbb{K}_2'\end{aligned}$$

where $a_i = 0, a_j = a, a_k = -a$ $(i, j, k$ cyclic). On the other hand

$$\begin{aligned}[F_i(x), F_i(y \otimes a)]_{\text{here}} &= [F_i(x \otimes 1), F_i(y \otimes a)] \\ &= -\langle x, y\rangle\theta^{i-1}T_{1,a} \\ &= [F_i(x), F_i(y \otimes a)]_{\text{Tits}}\end{aligned}$$

since $T_{1,a} = (2L_a + R_a, 2R_a, 2L_a)$ which is identified with $(0, 2a, -2a)$ $\in \mathbb{K}_2'$ in paragraph 3 above. If $i \neq j$ and (i, j, k) is a cyclic permutation of $1, 2, 3)$,

$$\begin{aligned}[F_i(x), F_j(y \otimes a)]_{\text{Tits}} &= [F_i(x), P_j(y \otimes a)] \ \in [\operatorname{Der} H_3(\mathbb{K}_1), H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2] \\ &= -P_k(\bar{y}\bar{x}) \otimes a \in H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2' \\ &= -F_k(\bar{y}\bar{x}) \otimes a \in 3\mathbb{K}_1 \otimes \mathbb{K}_2'\end{aligned}$$

while $[F_i(x), F_j(y \otimes a)]_{\text{here}} = -F_k(\bar{y}\bar{x}) \otimes a$ since $\bar{a} = -a$. Similarly,

$$[F_i(x), F_k(y \otimes a)]_{\text{Tits}} = F_k(\bar{x}\bar{y}) \otimes a = [F_i(x), F_k(y \otimes a)]_{\text{here}}.$$

9. $[3\mathbb{K}_1, \operatorname{Der} \mathbb{K}_2]_{\text{Tits}} \in [\operatorname{Der} H_3(\mathbb{K}_1), \operatorname{Der} \mathbb{K}_2] = 0$
and

$$[3\mathbb{K}_1, \operatorname{Der} \mathbb{K}_2]_{\text{here}} \in [3\mathbb{K}_1 \otimes \mathbb{R}, \operatorname{Der} \mathbb{K}_2] = 0.$$

10. $[2\mathbb{K}_2', 2\mathbb{K}_2']$: For $\mathbf{a}, \mathbf{b} \in 2\mathbb{K}_2'$, with $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ where $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 0$,

$$\begin{aligned}
[\mathbf{a}, \mathbf{b}]_{\text{Tits}} &= [\sum e_i \otimes a_i, \sum e_j \otimes b_j] \\
&= \sum_{i,j} \left(\langle e_i, e_j \rangle D_{a_i, b_j} + (e_i * e_j) \otimes \operatorname{Im}(a_i b_j) + \langle a_i, b_j \rangle [L_{e_i}, L_{e_j}] \right) \\
&= \sum_{ij} \left(2\delta_{ij} D_{a_i, b_j} + \delta_{ij}(2e_i - \tfrac{2}{3}\mathbb{1}) \otimes \tfrac{1}{2}[a_i, b_j] \right) \\
&= \sum_{i} \left(2D_{a_i, b_i} + \tfrac{1}{3} e_i \otimes (2[a_i, b_i] - [a_j, b_j] - [a_k, b_k]) \right) \\
&= [\mathbf{a}, \mathbf{b}]_{\text{here}}
\end{aligned}$$

11. $[2\mathbb{K}_2', 3\mathbb{K}_1 \otimes \mathbb{K}_2']$: For $\mathbf{a} = (a_1, a_2, a_3) \in 2\mathbb{K}_2'$ and $F_i(x \otimes b) \in 3\mathbb{K}_1 \otimes \mathbb{K}_2'$,

$$\begin{aligned}
[\mathbf{a}, F_i(x \otimes b)]_{\text{Tits}} &= [e_i \otimes a_i + e_j \otimes a_j + e_k \otimes a_k, P_i(x) \otimes b] \\
&= P_i(x) \otimes \tfrac{1}{2}[a_j, b] + P_i(x) \otimes \tfrac{1}{2}[a_k, b] - \langle a_j - a_k, b \rangle F_i(x).
\end{aligned}$$

The first two terms belong to the subspace $3\mathbb{K}_1 \otimes \mathbb{K}_2'$ of $H_3' \otimes \mathbb{K}_2'$ and the third to the subspace $3\mathbb{K}_1$ of $\operatorname{Der} H_3(\mathbb{K}_1)$, so together they constitute an element of $3\mathbb{K}_1 \otimes \mathbb{K}_2$:

$$\begin{aligned}
[\mathbf{a}, F_i(x \otimes b)]_{\text{Tits}} &= F_i(x \otimes (\tfrac{1}{2}[a_j, b] = \tfrac{1}{2}[a_k, b] - \langle a_j - a_k, b \rangle)) \\
&= F_i(x \otimes (a_j b - b a_k)) \\
&= F_i(x \otimes T(\mathbf{a})_i b) \\
&= [\mathbf{a}, F_i(x \otimes b)]_{\text{here}}
\end{aligned}$$

12. $[2\mathbb{K}_2', \operatorname{Der} \mathbb{K}_2]$: Tits's bracket (38) coincides with the bracket in $\operatorname{Tri} \mathbb{K}_2$ as given by Lemma 7.

13. $[3\mathbb{K}_1 \otimes \mathbb{K}_2', 3\mathbb{K}_1 \otimes \mathbb{K}_2']$: For $P_i(x), P_i(y) \in 3\mathbb{K}_1$ and $a, b \in \mathbb{K}_2'$, if $i \neq j$ and (i, j, k) is a cyclic permutation of $(1, 2, 3)$ then

$$[P_i(x) \otimes a, P_j(y) \otimes b]_{\text{Tits}} = P_k(\bar{y}\bar{x}) \otimes \tfrac{1}{2}[a, b] - \langle a, b \rangle F_k(\bar{y}\bar{x})$$

by equation (54d). This is an element of $3\mathbb{K}_1 \otimes \mathbb{K}_2' \dot{+} 3\mathbb{K}_1 \otimes \mathbb{R}$ which is identified with the following element of $3\mathbb{K}_1 \otimes \mathbb{K}_2$:

$$F_k(\bar{y}\bar{x} \otimes \tfrac{1}{2}\left([a,b] - (a\bar{b} + b\bar{a})\right) = F_k(\bar{y}\bar{x} \otimes \bar{b}\bar{a})$$
$$= [F_i(x \otimes a), F_j(y \otimes b)]_{\text{here}}.$$

If $i = j$, then

$$[P_i(x) \otimes a, P_i(y) \otimes b]_{\text{Tits}} = 4\langle x, y\rangle D_{a,b} +$$
$$\tfrac{1}{2}\langle x, y\rangle(-2e_i + e_j + e_k) \otimes [a,b] - \langle a, b\rangle \theta^{i-1} T_{x,y}.$$

The second term belongs to the subspace $2\mathbb{K}_2'$ and it is to be identified with the triality $\frac{1}{3}\langle x, y\rangle\theta^{i-1}T$ where $T_1 = \frac{1}{3}(L_{[a,b]} - R_{[a,b]})$, $\overline{T}_2 = -\frac{1}{3}(R_{[a,b]} + 2L_{[a,b]})$ and $\overline{T}_3 = \frac{1}{3}(2R_{[a,b]} + L_{[a,b]})$. By (43), $T = T_{a,b}$. Hence

$$[P_i(x) \otimes a, P_i(y) \otimes b]_{\text{Tits}} = \langle x, y\rangle\theta^{i-1}T_{a,b} - \langle a, b\rangle\theta^{i-1}T_{x,y}.$$

14. $[3\mathbb{K}_1 \otimes \mathbb{K}_2', \operatorname{Der}\mathbb{K}_2]$ is given by the action of $\operatorname{Der}\mathbb{K}_2$ on the second factor of the tensor product in both cases.
15. $[\operatorname{Der}\mathbb{K}_2, \operatorname{Der}\mathbb{K}_2]$ is given by the Lie bracket of $\operatorname{Der}\mathbb{K}_2$ in both cases.

□

Notice that if Lemma 7 is applied to the triality symmetric vector space then

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = \operatorname{Der}\mathbb{K}_1 \dot{+} 2\mathbb{K}_1' \dot{+} \operatorname{Der}\mathbb{K}_2 \dot{+} 2\mathbb{K}_2' \dot{+} 3\mathbb{K}_1 \otimes \mathbb{K}_2$$

which can be written as

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = \operatorname{Der}\mathbb{K}_1 \dot{+} 2\mathbb{K}_1' \otimes \mathbb{R} \dot{+} \operatorname{Der}\mathbb{K}_2 \dot{+} 2\mathbb{R} \otimes \mathbb{K}_2' \dot{+} 3\mathbb{K}_1 \otimes \mathbb{K}_2.$$

Applying Theorem 5 gives

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = A_3(\mathbb{K}_1 \otimes \mathbb{K}_2) \dot{+} \operatorname{Der}\mathbb{K}_1 \dot{+} \operatorname{Der}\mathbb{K}_2.$$

The isomorphism of the Lie brackets follows since both spaces are isomorphic to the Tits construction.

Thus all three ways of considering the symmetry of the 3×3 magic square reduce to being different ways of looking at the same underlying vector space.

3. The 2×2 Magic Square

The Tits construction can also be adapted for 2×2 matrix algebras. In this case we take the vector space to be

$$(71) \qquad L_2(\mathbb{K}_1, \mathbb{K}_2) = \operatorname{Der} H_2(\mathbb{K}_1) \dot{+} H_2'(\mathbb{K}_1) \otimes \mathbb{K}_2' \dot{+} \mathfrak{so}(\mathbb{K}_2')$$

which is again a Lie algebra when taken with the brackets

$$(72) \qquad \begin{aligned} [D, A \otimes x] &= D(A) \otimes x \\ [E, A \otimes x] &= A \otimes E(x) \\ [D, E] &= 0 \\ [A \otimes x, B \otimes y] &= \tfrac{1}{4}\langle A, B\rangle D_{x,y} - \langle x, y\rangle [L_A, L_B] \end{aligned}$$

where the symbols used in this set of brackets are defined in the same way as the ones used in the 3×3 case. Let the elements of $\mathfrak{so}(\mathbb{K})$ be $s_{x,y}$ where

$$(73) \qquad s_{x,y}(z) = \langle x, z\rangle y - \langle y, z\rangle x$$

If $\mathbb{K}_1, \mathbb{K}_2$ are division algebras then this gives the compact magic square for 2×2 matrix algebras whose complexifications are

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$L_2(\mathbb{K}, \mathbb{R})$	D_1	$A_1 \cong B_1 \cong C_1$	$C_2 \cong B_2$	B_4
$L_2(\mathbb{K}, \tilde{\mathbb{C}})$	$A_1 \cong B_1 \cong C_1$	$A_1 \oplus A_1$	$A_3 \cong D_3$	D_5
$L_2(\mathbb{K}, \tilde{\mathbb{H}})$	$C_2 \cong B_2$	$A_3 \cong D_3$	D_4	D_6
$L_2(\mathbb{K}, \tilde{\mathbb{O}})$	B_4	D_5	D_6	D_8

.

In Chapter 3 it will be shown that the isomorphism

$$L_2(\mathbb{K}_1, \mathbb{K}_2) \cong \mathfrak{so}(\nu_1 + \nu_2)$$

holds, thus giving the square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$L_2(\mathbb{K},\mathbb{R})$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$L_2(\mathbb{K},\widetilde{\mathbb{C}})$	$\mathfrak{so}(3)$	$\mathfrak{so}(4)$	$\mathfrak{so}(6)$	$\mathfrak{so}(10)$
$L_2(\mathbb{K},\widetilde{\mathbb{H}})$	$\mathfrak{so}(5)$	$\mathfrak{so}(6)$	$\mathfrak{so}(8)$	$\mathfrak{so}(12)$
$L_2(\mathbb{K},\widetilde{\mathbb{O}})$	$\mathfrak{so}(9)$	$\mathfrak{so}(10)$	$\mathfrak{so}(12)$	$\mathfrak{so}(16)$

.

If $\mathbb{K}_2$ is one of the split composition algebras $\widetilde{\mathbb{C}}, \widetilde{\mathbb{H}}$ or $\widetilde{\mathbb{O}}$ this becomes

$$L_2(\mathbb{K}_1,\mathbb{K}_2) = \mathfrak{so}(\nu_1 + \tfrac{1}{2}\nu_2, \tfrac{1}{2}\nu_2).$$

giving the magic square

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$L_2(\mathbb{K},\mathbb{R})$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$L_2(\mathbb{K},\widetilde{\mathbb{C}})$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(3,1)$	$\mathfrak{so}(5,1)$	$\mathfrak{so}(9,1)$
$L_2(\mathbb{K},\widetilde{\mathbb{H}})$	$\mathfrak{so}(3,2)$	$\mathfrak{so}(4,2)$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$L_2(\mathbb{K},\widetilde{\mathbb{O}})$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

.

As in the 3 × 3 case, these Lie algebras can be identified with certain types of 2 × 2 matrix algebras, namely

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathrm{Der}\, H_2(\mathbb{K}) \cong L_2(\mathbb{K},\mathbb{R})$	$\mathfrak{so}(2)$	$\mathfrak{su}(2)$	$\mathfrak{sq}(2)$	$\mathfrak{so}(9)$
$\mathrm{Str}\, H_2(\mathbb{K}) \cong L_2(\mathbb{K},\widetilde{\mathbb{C}})$	$\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{sl}(2,\mathbb{H})$	$\mathfrak{sl}(2,\mathbb{O})$
$\mathrm{Con}\, H_2(\mathbb{K}) \cong L_2(\mathbb{K},\widetilde{\mathbb{H}})$	$\mathfrak{sp}(4,\mathbb{R})$	$\mathfrak{su}(2,2)$	$\mathfrak{sp}(4,\mathbb{H})$	$\mathfrak{sp}(4,\mathbb{O})$
$L_2(\mathbb{K},\widetilde{\mathbb{O}})$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

.

Again this extends the concepts of the Lie algebras $\mathfrak{sa}(2,\mathbb{K})$, $\mathfrak{sl}(2,\mathbb{K})$ and $\mathfrak{sp}(2,\mathbb{K})$ to $\mathbb{K} = \mathbb{H}$ and $\mathbb{O}$. Note that $\mathfrak{su}(2,2) \cong \mathfrak{sp}(4,\mathbb{C})$.

4. Symmetry Property of the 2 × 2 Magic Square

To give a symmetric representation of the 2×2 magic square we first require the following theorem.

THEOREM 7. The derivation algebra of $H_2(\mathbb{K})$ can be expressed in the form

$$\mathrm{Der}\, H_2(\mathbb{K}) = A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}'). \tag{74}$$

PROOF. From Appendix A we see that for each $A \in A_2(\mathbb{K})$ there is a derivation $D(A)$ of $H_2(\mathbb{K})$ given by

$$D(A)(X) = AX - XA,$$

since the identity

$$[A, \{X, Y\}] = \{[A, X], Y\} + \{X, [A, Y]\}$$

holds. We consider $H_2(\mathbb{K})$ as a Jordan algebra with product

$$X \cdot Y = XY + YX$$

We can write a matrix $A \in H_2(\mathbb{K})$ as follows

$$\begin{pmatrix} \alpha & x \\ \bar{x} & \beta \end{pmatrix} = \lambda I + \mu E + P(x)$$

where $\lambda = \frac{1}{2}(\alpha + \beta)$, $\mu = \frac{1}{2}(\alpha - \beta)$, $E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $P(x) = \left(\begin{smallmatrix} 0 & x \\ \bar{x} & 0 \end{smallmatrix}\right)$. Then the Jordan multiplication can be rewritten as

$$E \cdot E = I$$

$$P(x) \cdot P(y) = 2\langle x, y\rangle I$$

$$E \cdot P(x) = 0.$$

Thus $H_2(\mathbb{K})$ can be identified with $\mathbb{J}(V)$, the Jordan algebra associated with the inner product space $V = \mathbb{K} \oplus \mathbb{R}$. $\mathbb{J}(V)$ is a subalgebra of the anticommutator algebra of the Clifford algebra $\mathrm{Cl}(V)$, where $\mathbf{v} \cdot \mathbf{w} = \langle \mathbf{v}, \mathbf{w}\rangle 1$. Derivations of this algebra must satisfy

$$D(1) = 0$$

$$\langle 1, D(\mathbf{v})\rangle = 0.$$

Thus

$$\langle D(\mathbf{v}), \mathbf{w}\rangle + \langle D(\mathbf{w}), \mathbf{v}\rangle = 0 \tag{75}$$

i.e. D is an antisymmetric map of $\mathbf{v}$. Hence $\operatorname{Der} H_2(\mathbb{K}) \subseteq \mathfrak{o}(\mathbb{K}\dot{+}\mathbb{R})$. Since any element of $\mathfrak{o}(\mathbb{K}\oplus\mathbb{R})$ will satisfy equation (75), it is also true that $\operatorname{Der} H_2(\mathbb{K}) \supseteq \mathfrak{o}(\mathbb{K}\dot{+}\mathbb{R})$.

Considering the matrix structure of $\mathfrak{o}(\mathbb{K}\dot{+}\mathbb{R})$ this can be written as $\mathfrak{o}(\mathbb{K})\dot{+}\mathbb{K}$. Consider the action of $\mathbb{K}$ on the $(\nu+1)\times 1$ column vectors $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}x\\0\end{smallmatrix}\right)$. The real coefficients of $k \in \mathbb{K}$ are expressed as the final row and column in a $(\nu+1)\times(\nu+1)$ block matrix, written as the vector $\mathbf{k}$. Then

$$\begin{pmatrix} 0 & \mathbf{k} \\ -\mathbf{k}^t & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{k} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \mathbf{k} \\ -\mathbf{k}^t & 0 \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{k}^t x \end{pmatrix}$$

Thus k maps E to $P(k)$ and $P(x)$ to $-\langle k, x\rangle E$. Now considering the equivalent multiplication where $\left(\begin{smallmatrix}0 & k\\ -\bar{k} & 0\end{smallmatrix}\right)$ is a matrix in $A_2'(\mathbb{K})$ and $\left(\begin{smallmatrix}0 & x\\ \bar{x} & 0\end{smallmatrix}\right)$ is a matrix in $H_2'(\mathbb{K})$,

$$\left[\begin{pmatrix} 0 & k \\ -\bar{k} & 0 \end{pmatrix}, \begin{pmatrix} 1 & x \\ \bar{x} & -1 \end{pmatrix}\right] = 2\begin{pmatrix} \langle k, x\rangle & -k \\ \bar{k} & -\langle k, x\rangle \end{pmatrix}$$

i.e. multiplication by $\left(\begin{smallmatrix}0 & \mathbf{k}\\ -\mathbf{k}^t & 0\end{smallmatrix}\right)$ in $\mathbb{J}(V)$ is equivalent to commutation with $\begin{pmatrix} 0 & -\frac{k}{2} \\ \frac{\bar{k}}{2} & 0 \end{pmatrix}$ in $A_2'(\mathbb{K})$.

Further $\mathfrak{o}(\mathbb{K}) = \mathfrak{o}(\mathbb{K}' \oplus \mathbb{R})$ can be split into $\mathfrak{o}(\mathbb{K}')\dot{+}\mathbb{K}$. Consider the action of the $\nu \times \nu$ matrix $\left(\begin{smallmatrix}0 & \mathbf{l}\\ -\mathbf{l}^t & 0\end{smallmatrix}\right)$, where $\mathbf{l}$ is the vector of the real co-efficients of $l \in \mathbb{K}'$, on the vectors $\left(\begin{smallmatrix}y\\0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ with $y \in \mathbb{K}'$:

$$\begin{pmatrix} 0 & \mathbf{l} \\ -\mathbf{l}^t & 0 \end{pmatrix}\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathbf{l}^t y \end{pmatrix}$$

$$\begin{pmatrix} 0 & \mathbf{l} \\ -\mathbf{l}^t & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{l} \\ 0 \end{pmatrix}$$

and we obtain (by a similar method) that multiplication by $\begin{pmatrix} 0 & 1 \\ -1^t & 0 \end{pmatrix}$ in $\mathbb{J}(V)$ is equivalent to commutation with $\begin{pmatrix} \frac{l}{2} & 0 \\ 0 & -\frac{l}{2} \end{pmatrix}$ in $A_2'(\mathbb{K})$. Further $\mathfrak{so}(\mathbb{K}')$ acts on $\mathbb{J}(V)$ precisely as it does on $H_2(\mathbb{K})$. Thus we have

$$\operatorname{Der} H_2(\mathbb{K}) = A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}')$$

as required. □

The brackets in $A_2'(\mathbb{K}_1) \dot{+} \mathfrak{so}(\mathbb{K}_1')$ are given by

$$\begin{aligned} [A, B] &= (AB - BA)' + S(A, B) \\ [S, A] &= S(A) \\ [S, S'] &= SS' - S'S \end{aligned}$$

with $A, B \in A_2'(\mathbb{K}_1)$ and $S, S' \in \mathfrak{so}(\mathbb{K}_1)$ and $S(A)$ describes S acting element-wise on A as defined in equation (73). Note that as in the 3×3 case $(AB - BA)'$ is the traceless part of $(AB - BA)$. Further $S(A, B)$ is defined in a similar manner to $D(A, B)$, namely

$$S(A, B)C = \sum_{ij} S_{a_{ij}, b_{ji}}(C).$$

Thus

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = A_2'(\mathbb{K}_1) \dot{+} \mathfrak{so}(\mathbb{K}_1') \dot{+} H_2'(\mathbb{K}_1) \otimes \mathbb{K}_2' \dot{+} \mathfrak{so}(\mathbb{K}_2').$$

Consequently we can write the vector space $L_2(\mathbb{K}_1, \mathbb{K}_2)$ in a clearly symmetric form as shown in the next theorem.

THEOREM 8. For any two composition algebras $\mathbb{K}_1$ and $\mathbb{K}_2$,

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = \mathfrak{so}(\mathbb{K}_1) \dot{+} \mathbb{K}_1 \otimes \mathbb{K}_2 \dot{+} \mathfrak{so}(\mathbb{K}_2)$$

where $\mathfrak{so}(\mathbb{K}_1) \dot{+} \mathfrak{so}(\mathbb{K}_2)$ is a Lie subalgebra and the brackets are defined as

$$\begin{aligned} [S_{ij}, a \otimes b] &= S_{ij}(a) \otimes b \\ [T_{ij}, a \otimes b] &= a \otimes T_{ij}(b) \qquad (76) \\ [a \otimes b, c \otimes d] &= \langle b, d \rangle S_{ac} + \langle a, c \rangle T_{bd} \end{aligned}$$

with $S_{ij} \in \mathfrak{so}(\mathbb{K}_1)$, $T_{ij} \in \mathfrak{so}(\mathbb{K}_2)$ and $a \otimes b \in \mathbb{K}_1 \otimes \mathbb{K}_2$.

PROOF. Begin by considering the vector space $L_2(\mathbb{K}_1, \mathbb{K}_2)$. Then

$$\begin{aligned} L_2(\mathbb{K}_1, \mathbb{K}_2) &= A_2'(\mathbb{K}_1) \dot{+} \mathfrak{so}(\mathbb{K}_1') \dot{+} H_2'(\mathbb{K}_1) \otimes \mathbb{K}_2' \dot{+} \mathfrak{so}(\mathbb{K}_2') \\ &= (\mathbb{K}_1' \dot{+} \mathbb{K}_1) \dot{+} \mathfrak{so}(\mathbb{K}_1') \dot{+} (\mathbb{R} \dot{+} \mathbb{K}_1) \otimes \mathbb{K}_2' \dot{+} \mathfrak{so}(\mathbb{K}_2') \\ &= (\mathbb{K}_1' \dot{+} \mathfrak{so}(\mathbb{K}_1')) \dot{+} (\mathbb{K}_1 \dot{+} \mathbb{K}_1 \otimes \mathbb{K}_2') \dot{+} (\mathbb{K}_2' \dot{+} \mathfrak{so}(\mathbb{K}_2')) \\ &= \mathfrak{so}(\mathbb{K}_1) \dot{+} \mathbb{K}_1 \otimes \mathbb{K}_2 \dot{+} \mathfrak{so}(\mathbb{K}_2) \end{aligned}$$

by Theorem 7. Notice that the isomorphism $\psi : L_2(\mathbb{K}_1, \mathbb{K}_2) \to \mathfrak{so}(\mathbb{K}_1) \dot{+} \mathbb{K}_1 \otimes \mathbb{K}_2 \dot{+} \mathfrak{so}(\mathbb{K}_2)$ can be defined as

$$\begin{aligned} \psi \begin{pmatrix} a & b \\ -\bar{b} & -a \end{pmatrix} &= s_{1a} + b \otimes 1 \\ \psi \left(\begin{pmatrix} r & x \\ \bar{x} & -r \end{pmatrix} \otimes y \right) &= t_{ry} + x \otimes y \\ \psi(s_{xy}) &= s_{xy} \\ \psi(t_{xy}) &= t_{xy}, \end{aligned}$$

where t_{xy} is an element of $\mathfrak{so}(\mathbb{K}_2)$.

Consider the four brackets between each part of the $\mathfrak{so}(\mathbb{K}_1) \dot{+} \mathbb{K}_1 \otimes \mathbb{K}_2 \dot{+} \mathfrak{so}(\mathbb{K}_2)$ vector space structure. Using a similar notation to that found in Section 2 the brackets being used are clarified, i.e. the brackets defined in equation (76) will be denoted $[,]_{\text{here}}$ and the brackets previously defined will be denoted $[,]_{\text{Tits}}$.

1. $[\mathfrak{so}(\mathbb{K}_1), \mathfrak{so}(\mathbb{K}_2)]$. The vector space structure used to obtain $\mathfrak{so}(\mathbb{K}_1) \dot{+} \mathfrak{so}(\mathbb{K}_2)$ from $L_2(\mathbb{K}_1, \mathbb{K}_2)$ gives

$$\begin{aligned} [\mathfrak{so}(\mathbb{K}_1), \mathfrak{so}(\mathbb{K}_2)]_{\text{here}} &= [\mathfrak{so}(\mathbb{K}_1') \dot{+} \mathbb{K}_1', \mathfrak{so}(\mathbb{K}_2') \dot{+} \mathbb{K}_2']_{\text{here}} \\ &= [\mathfrak{so}(\mathbb{K}_1'), \mathfrak{so}(\mathbb{K}_2')]_{\text{Tits}} \dot{+} [\mathbb{K}_1', \mathfrak{so}(\mathbb{K}_2')]_{\text{here}} \\ &\quad \dot{+} [\mathfrak{so}(\mathbb{K}_1'), \mathbb{K}_2']_{\text{here}} \dot{+} [\mathbb{K}_1', \mathbb{K}_2']_{\text{here}} \end{aligned}$$

Clearly $[\mathbb{K}_1', \mathfrak{so}(\mathbb{K}_2')]_{\text{here}} = [\mathfrak{so}(\mathbb{K}_1'), \mathbb{K}_2']_{\text{here}} = 0$ (since $\mathfrak{so}(\mathbb{K}_1)$ cannot act on an element of $\mathbb{K}_2$ and vice versa), and by Tits $[\mathfrak{so}(\mathbb{K}_1'), \mathfrak{so}(\mathbb{K}_2')]_{\text{Tits}} = 0$

also. We check that $[\mathbb{K}_1', \mathbb{K}_2'] = 0$ by considering the bracket

$$\left[\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes b\right] = \left[\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \otimes b$$
$$= 0.$$

Thus $[\mathfrak{so}(\mathbb{K}_1), \mathfrak{so}(\mathbb{K}_2)]_{\text{here}} = [\mathfrak{so}(\mathbb{K}_1), \mathfrak{so}(\mathbb{K}_2)]_{\text{Tits}}$.

2. $[\mathfrak{so}(\mathbb{K}_1), \mathbb{K}_1 \otimes \mathbb{K}_2]$. Again by examination of the vector space structure this splits into two parts, $[\mathfrak{so}(\mathbb{K}_1'), \mathbb{K}_1 \otimes \mathbb{K}_2]_{\text{here}}$ which is clearly equal to Tits bracket since in the Tits construction $\mathfrak{so}(\mathbb{K}_1)$ acts elementwise on matrices containing elements of $\mathbb{K}_1$ and $[\mathfrak{so}(\mathbb{R}), \mathbb{K}_1 \otimes \mathbb{K}_2]$. To prove this consider the equivalent brackets in the Tits construction and then apply ψ.

$$\left[\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix} \otimes b\right] = \begin{pmatrix} 0 & xa + ax \\ -x\bar{a} - \bar{a}x & 0 \end{pmatrix} \otimes b$$
$$= s_{1x}(a) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes b.$$

Applying ψ to this gives the required result.

3. $[\mathfrak{so}(\mathbb{K}_2), \mathbb{K}_1 \otimes \mathbb{K}_2]$. Again we need not consider the bracket $[\mathfrak{so}(\mathbb{K}_2'), \mathbb{K}_1 \otimes \mathbb{K}_2]$ since this is clearly the same as the Tits bracket. The same method as with the previous bracket will be used. In this case

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix} \otimes b + \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix}\right] = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix} \right\rangle t_{yb}$$
$$- \langle y, b \rangle \left[L_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}, L_{\left(\begin{smallmatrix} 0 & a \\ \bar{a} & 0 \end{smallmatrix}\right)}\right] + \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix}\right] \otimes y$$
$$= -2\langle y, b \rangle \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix} \otimes y$$

Applying ψ to this gives $a \otimes t_{1y}(b + 1)$ as expected. Thus $[\mathfrak{so}(\mathbb{K}_2), \mathbb{K}_1 \otimes \mathbb{K}_2]_{\text{here}} = [\mathfrak{so}(\mathbb{K}_2), \mathbb{K}_1 \otimes \mathbb{K}_2]_{\text{Tits}}$.

4. $[\mathbb{K}_1 \otimes \mathbb{K}_2, \mathbb{K}_1 \otimes \mathbb{K}_2]$. Again we work from the Tits representation and apply ψ to give the required result. Consider

$$\left[\begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix} \otimes \operatorname{Im} b + \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix} \operatorname{Re} b, \begin{pmatrix} 0 & c \\ \bar{c} & 0 \end{pmatrix} \otimes \operatorname{Im} d + \begin{pmatrix} 0 & c \\ -\bar{c} & 0 \end{pmatrix} \operatorname{Re} d\right]$$

$$= \frac{1}{2} \langle \left(\begin{smallmatrix} 0 & a \\ \bar{a} & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & c \\ \bar{c} & 0 \end{smallmatrix}\right) \rangle \, t_{\operatorname{Im}(b) \operatorname{Im}(d)} - \langle \operatorname{Im} b, \operatorname{Im} d \rangle \left[L_{\left(\begin{smallmatrix} 0 & a \\ \bar{a} & 0 \end{smallmatrix}\right)}, L_{\left(\begin{smallmatrix} 0 & c \\ \bar{c} & 0 \end{smallmatrix}\right)}\right]$$

$$[\left(\begin{smallmatrix} 0 & a \\ -\bar{a} & 0 \end{smallmatrix}\right) \operatorname{Re} b, \left(\begin{smallmatrix} 0 & c \\ \bar{c} & 0 \end{smallmatrix}\right) \otimes \operatorname{Im} d] - [\left(\begin{smallmatrix} 0 & c \\ -\bar{c} & 0 \end{smallmatrix}\right) \operatorname{Re} d, \left(\begin{smallmatrix} 0 & a \\ \bar{a} & 0 \end{smallmatrix}\right) \otimes \operatorname{Im} b]$$

$$+ [\left(\begin{smallmatrix} 0 & a \\ -\bar{a} & 0 \end{smallmatrix}\right) \operatorname{Re} b, \left(\begin{smallmatrix} 0 & c \\ -\bar{c} & 0 \end{smallmatrix}\right) \operatorname{Re} d]$$

$$= \langle a, c \rangle t_{\operatorname{Im}(b) \operatorname{Im}(d)} - \langle \operatorname{Im} b, \operatorname{Im} d \rangle \operatorname{Im}(ac) \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

$$+ \langle a, c \rangle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \otimes (\operatorname{Re}(b) \operatorname{Im}(d) + \operatorname{Im}(b) \operatorname{Re}(d))$$

$$+ \langle \operatorname{Re}(b), \operatorname{Re}(d) \rangle \operatorname{Im}(ac) \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

By application of ψ we obtain $\langle a, c \rangle t_{bd} + \langle b, d \rangle s_{ac}$ as required.

Thus we have shown that all brackets are equivalent and our magic square has the symmetrical vector space construction $\mathfrak{so}(\mathbb{K}_1) \dot{+} \mathbb{K}_1 \otimes \mathbb{K}_2 \dot{+} \mathfrak{so}(\mathbb{K}_2)$. □

CHAPTER 3

Magic Squares of 2×2 Matrices

In this Chapter we present the proofs of the following theorems.

THEOREM 9. For $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$,

$$L_2(\mathbb{K}_1, \mathbb{K}_2) \cong \mathfrak{so}(\mathbb{K}_1 \dot{+} \mathbb{K}_2)$$

$$L_2(\mathbb{K}_1, \widetilde{\mathbb{K}}_2) \cong \mathfrak{so}(\tfrac{1}{2}(\nu_1 + \nu_2), \tfrac{1}{2}\nu_2).$$

THEOREM 10. The following isomorphisms are true for the four division algebras, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.

$$L_2(\mathbb{K}, \mathbb{R}) \cong \operatorname{Der} H_2(\mathbb{K}) \tag{77a}$$

$$L_2(\mathbb{K}, \widetilde{\mathbb{C}}) \cong \operatorname{Str} H_2(\mathbb{K}) \tag{77b}$$

$$L_2(\mathbb{K}, \widetilde{\mathbb{H}}) \cong \operatorname{Con} H_2(\mathbb{K}) \tag{77c}$$

We prove Theorem 9 by extending the symmetric form of the Tits construction given in the previous chapter. We then prove equation (77c) and notice that the proof of this contains the isomorphisms for equations (77a) and (77b). Finally we prove the following theorem

THEOREM 11. If $\mathbb{K}$ is a division algebra then

$$\mathfrak{sa}(2, \mathbb{K}) \cong L_2(\mathbb{K}, \mathbb{R}) \tag{78}$$

$$\mathfrak{sl}(2, \mathbb{K}) \cong L_2(\mathbb{K}, \widetilde{\mathbb{C}}) \tag{79}$$

$$\mathfrak{sp}(4, \mathbb{K}) \cong L_2(\mathbb{K}, \widetilde{\mathbb{H}}) \tag{80}$$

where $\mathfrak{sa}(2, \mathbb{C}) = \mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{C})$ has its usual meaning.

1. The Magic Square of (pseudo)-orthogonal algebras

Using Theorem 7 write $L_2(\mathbb{K}_1, \mathbb{K}_2)$ as

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = A'_2(\mathbb{K}_1)\dot{+}\mathfrak{so}(\mathbb{K}'_1)\dot{+}H'_2(\mathbb{K}_1) \otimes \mathbb{K}'_2\dot{+}\mathfrak{so}(\mathbb{K}'_2).$$

In Theorem 8 it was shown that this is equivalent to

$$\mathfrak{so}(\mathbb{K}_1)\dot{+}\mathbb{K}_1 \otimes \mathbb{K}_2\dot{+}\mathfrak{so}(\mathbb{K}_2).$$

To see how this becomes $\mathfrak{so}(\mathbb{K}_1\dot{+}\mathbb{K}_2)$ it is easiest to consider a matrix in $\mathfrak{so}(\mathbb{K}_1\dot{+}\mathbb{K}_2)$ as consisting of four blocks, of which the two diagonal blocks are $\mathfrak{so}(\mathbb{K}_1)$ and $\mathfrak{so}(\mathbb{K}_2)$ and the two off diagonal blocks are $\mathbb{K}_1 \otimes \mathbb{K}_2$ (considered as vectors) and an antisymmetric reflection of this. A diagramatic representation of this can be found in Figure 1 below.

$\mathfrak{so}(\mathbb{K}_1)$	$\mathbb{K}_1 \otimes \mathbb{K}_2$
	$\mathfrak{so}(\mathbb{K}_2)$

FIGURE 1. Diagramatic representation of $\mathfrak{so}(\mathbb{K}_1\dot{+}\mathbb{K}_2)$

Throughout this Chapter the notation $\mathbf{k}$ will be used to denote the column vector in $\mathbb{R}^{\nu}$ of the real co-efficients of the hypercomplex number $k \in \mathbb{K}$ (where k can be written $k = k_i e_i$ and $i = 0, \ldots, \nu$). Also define G to be the metric matrix for $\mathbb{K}_2$. When $\mathbb{K}_2$ is a division algebra G is merely the identity matrix, whereas when $\mathbb{K}_2$ is a split algebra G is a diagonal matrix consisting of $(\frac{\nu_2}{2} - 1)$ positive 1's and $(\frac{\nu_2}{2} + 1)$ negative 1's. The order of the positive and negative elements in G is determined by the choice of constants in the Cayley-Dickson process. Thus for the choice of constants made in Chapter 1. the matrix G is given by

$$G = (1, -1) \qquad \text{when } \mathbb{K} = \widetilde{\mathbb{C}}$$

$$G = \mathrm{diag}(1, 1, -1, -1) \qquad \text{when } \mathbb{K} = \widetilde{\mathbb{H}}$$

$$G = \mathrm{diag}(1, 1, 1, 1, -1, -1, -1, -1) \qquad \text{when } \mathbb{K} = \widetilde{\mathbb{O}}.$$

These calculations show that the isomorphisms $\varphi : L_2(\mathbb{K}_1, \mathbb{K}_2) \to \mathfrak{so}(\nu_1 + \nu_2)$ and $\psi : L_2(\mathbb{K}_1, \widetilde{\mathbb{K}}_2) \to \mathfrak{so}(\nu_1 + \frac{1}{2}\nu_2, \frac{1}{2}\nu_2)$ are essentially the same and only depend on a change in the metric G, i.e. it is not necessary to consider the division algebra and split algebra cases separately but merely to change the value of G as appropriate.

The following Lemma will be extremely useful in the proof of Theorem 10.

LEMMA 9. If $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^\nu$ are the vector representations of the hypercomplex numbers $a, b \in \mathbb{K}$ then

1. $\mathbf{a}^t\mathbf{b}$ is equivalent to $\langle a, b\rangle = \mathrm{Re}(a\bar{b})$.
2. $\mathbf{c} \mapsto (\mathbf{b}\mathbf{a}^t - \mathbf{a}\mathbf{b}^t)\mathbf{c}$ is the same map as $c \mapsto s_{a,b}(c)$.

PROOF. 1. Notice that $\mathbf{b}^t\mathbf{a} = \sum_{i=1}^{\nu} b_i a_i$. But

$$\begin{aligned}(a, b) &= \mathrm{Re}(a\bar{b}) \\ &= -\sum_{i=1}^{\nu} a_i b_i (e_i)^2 \\ &= \sum_{i=1}^{\nu} a_i b_i\end{aligned}$$

as required. Clearly $\mathrm{Re}(ab) = -\langle a, b\rangle$.

2. First observe that the i-jth component of $\mathbf{a}\mathbf{b}^t - \mathbf{b}\mathbf{a}^t$ is $a_i b_j - b_i a_j$. If this acts on a further vector representation of a hypercomplex number, say $\mathbf{c}$ then the ith element of the resulting vector will be

$$\sum_j (a_i b_j - b_i a_j) c_j.$$

Now consider the action of the element $s_{a,b} \in \mathfrak{so}(\mathbb{K})$ on the hypercomplex number c. Then

$$s_{a,b}(c) = \langle a, c\rangle b - \langle b, c\rangle a$$
$$= \sum_i a_i c_i b - \sum_i b_i c_i a$$

Thus the co-efficient of e_j will be $\sum_i (a_i b_j - b_i a_j) c_i$.

Thus the Lemma is proved. □

PROOF OF THEOREM 10. Let s be a typical element of $\mathfrak{so}(\mathbb{K}_1)$, t a typical element of $\mathfrak{so}(\mathbb{K}_2)$ and $k_1 \otimes k_2$ a typical element of $\mathbb{K}_1 \otimes \mathbb{K}_2$. Label the representation of $L_2(\mathbb{K}_1, \mathbb{K}_2)$ given by $\mathfrak{so}(\mathbb{K}_1) \dot{+} \mathbb{K}_1 \otimes \mathbb{K}_2 \dot{+} \mathfrak{so}(\mathbb{K}_2)$ as $L_2(\mathbb{K}_1, \mathbb{K}_2)_{\text{sym}}$ with Lie brackets denoted by $[,]_{\text{sym}}$, and the representation of $L_2(\mathbb{K}_1, \mathbb{K}_2)$ given by $\mathfrak{so}(\mathbb{K}_1 \dot{+} \mathbb{K}_2)$ as $L_2(\mathbb{K}_1, \mathbb{K}_2)_{\text{po}}$ with Lie brackets denoted by $[,]_{\text{po}}$. Then define a map $\psi : L_2(\mathbb{K}_1, \mathbb{K}_2)_{\text{sym}} \to L_2(\mathbb{K}_1, \mathbb{K}_2)_{\text{po}}$ by

$$\psi(s + k_1 \otimes k_2 + t) = \begin{pmatrix} s & \mathbf{k}_1 \mathbf{k}_2^t G \\ -G\mathbf{k}_2 \mathbf{k}_1 & t \end{pmatrix}.$$

To prove that this is indeed a Lie algebra isomorphism (since it is clearly a vector space isomorphism) consider the six Lie brackets that arise between the vector spaces of $L_2(\mathbb{K}_1, \mathbb{K}_2)_{\text{sym}}$.

1. $[\mathfrak{so}(\mathbb{K}_1), \mathfrak{so}(\mathbb{K}_1)]$. For $s, s' \in \mathfrak{so}(\mathbb{K}_1)$,

$$[s, s']_{\text{sym}} = ss' - s's$$

and

$$[\psi(s), \psi(s')]_{\text{po}} = \begin{pmatrix} ss' - s's & 0 \\ 0 & 0 \end{pmatrix}.$$

Clearly these are equivalent.

2. $[\mathfrak{so}(\mathbb{K}_1), \mathfrak{so}(\mathbb{K}_2)]$. In both cases the brackets are zero.
3. $[\mathfrak{so}(\mathbb{K}_1), \mathbb{K}_1 \otimes \mathbb{K}_2]$. In this case

$$[s, k_1 \otimes k_2]_{\text{sym}} = s(k_1) \otimes k_2,$$

whereas,

$$
\begin{aligned}
[\psi(s), \psi(k_1 \otimes k_2)]_{\text{po}} &= \left[\begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{k}_1\mathbf{k}_2^t G \\ -G\mathbf{k}_2\mathbf{k}_1^t & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 & s\mathbf{k}_1\mathbf{k}_2^t G \\ G\mathbf{k}_2\mathbf{k}_1^t s & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (s\mathbf{k}_1)\mathbf{k}_2^t G \\ -G\mathbf{k}_2(\mathbf{k}_1^t s^t) & 0 \end{pmatrix} \\
&= \psi\left([s, k_1 \otimes k_2]_{\text{sym}}\right).
\end{aligned}
$$

4. $[\mathbb{K}_1 \otimes \mathbb{K}_2, \mathbb{K}_1 \otimes \mathbb{K}_2]$. In this case

$$[k_1 \otimes k_2, l_1 \otimes l_2]_{\text{sym}} = \langle k_1, l_1 \rangle s_{k_2,l_2} + \langle k_2, l_2 \rangle s_{k_1,l_1}.$$

Now,

$$
\begin{aligned}
&[\psi(k_1 \otimes k_2), \psi(l_1 \otimes l_2)]_{\text{po}} = \\
&\left[\begin{pmatrix} 0 & \mathbf{k}_1\mathbf{k}_2^t G \\ -G\mathbf{k}_2\mathbf{k}_1^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{l}_1\mathbf{l}_2^t G \\ -G\mathbf{l}_2\mathbf{l}_1^t & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} -\mathbf{k}_1\mathbf{k}_2^t G G \mathbf{l}_2\mathbf{l}_1^t + \mathbf{l}_1\mathbf{l}_2^t G G \mathbf{k}_2\mathbf{k}_1^t & 0 \\ 0 & -G\mathbf{k}_2\mathbf{k}_1^t\mathbf{l}_1\mathbf{l}_2^t G + G\mathbf{l}_2\mathbf{l}_1^t\mathbf{k}_1\mathbf{k}_2^t G \end{pmatrix} \\
&= \begin{pmatrix} \langle l_2, k_2 \rangle s_{k_1,l_1} & 0 \\ 0 & \langle l_1, k_1 \rangle s_{k_2,l_2} \end{pmatrix} \qquad \text{since } GG = I. \\
&= \psi\left([k_1 \otimes k_2, l_1 \otimes l_2]_{\text{sym}}\right)
\end{aligned}
$$

5. $[\mathfrak{so}(\mathbb{K}_2), \mathbb{K}_1 \otimes \mathbb{K}_2]$, $[\mathfrak{so}(\mathbb{K}_2), \mathfrak{so}(\mathbb{K}_2)]$. The method for the final two brackets is exactly the same as that of the first two.

□

2. The Conformal Algebra

Start by defining the structure and conformal algebras as they apply to the 2×2 hermitian matrix case. The structure algebra is a subalgebra of the conformal algebra, and the derivation algebra a subalgebra of the structure algebra. This means that the isomorphism for the conformal algebra includes the isomorphisms for the structure algebra, and trivially for the derivation algebra. From equations (34) and (74) deduce that

$$\mathrm{Str}'\, H_2(\mathbb{K}) = H_2'(\mathbb{K}) \dot{+} A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}'). \tag{81}$$

This is a Lie algebra with brackets defined by the statements

1. $\mathrm{Der}\, H_2(\mathbb{K}) = A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}')$ is a subalgebra of $\mathrm{Str}'\, H_2(\mathbb{K})$.
2. Denoting the elements of the algebra by

$$D \in \mathrm{Der}\, H_2(\mathbb{K})$$
$$H \in H_2'(\mathbb{K})$$

   the brackets are

$$[D, H] = D(H)$$
$$[H, H'] = [L_H, L_{H'}]$$

   where the derivation $[L_H, L_{H'}]$ is an element of $\mathrm{Der}\, H_2(\mathbb{K})$.

For the conformal algebra given by (35) define the Lie brackets by taking $\mathrm{Str}\, \mathbb{J}$ as a subalgebra and the brackets as below. Take the elements of $\mathrm{Con}\, H_2(\mathbb{K})$ to be

$$T \in \mathrm{Str}\, \mathbb{J}$$
$$(X, Y) \in [H_2(\mathbb{K})]^2.$$

Then if $R \to R^*$ is an involutive automorphism which is the identity on $\mathrm{Der}\, \mathbb{J}$ and multiplies elements of $\mathbb{J}^2$ by -1 the Lie brackets are

$$[T, (X, Y)] = (TX, T^*Y)$$
$$[(X, 0), (Y, 0)] = [(0, X), (0, Y)] = 0$$
$$[(X, 0), (0, Y)] = 2L_{XY} + 2[L_X, L_Y].$$

In the case of the conformal algebra for 2×2 hermitian matrices substitute $H_2(\mathbb{K})$ for $\mathbb{J}$ in equation (35) to obtain

$$\mathrm{Con}\, H_2(\mathbb{K}) = \mathrm{Str}\, H_2(\mathbb{K}) \dot{+} [H_2(\mathbb{K})]^2.$$

This can be expanded to

$$\mathrm{Con}\, H_2(\mathbb{K}) = H_2'(\mathbb{K}) \dot{+} A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}') \dot{+} \mathbb{R} \dot{+} [H_2(\mathbb{K})]^2. \tag{82}$$

Note that as a vector space this is equivalent to $L_2(\mathbb{K}, \widetilde{\mathbb{H}})$ since

$$\begin{aligned} L_2(\mathbb{K}, \widetilde{\mathbb{H}}) &= A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}) \dot{+} H_2'(\mathbb{K}) \otimes \widetilde{\mathbb{H}}' \dot{+} \mathfrak{so}(2,1) \\ &= A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}) \dot{+} 3H_2'(\mathbb{K}) \dot{+} 3\mathbb{R} \\ &= A_2'(\mathbb{K}) \dot{+} \mathfrak{so}(\mathbb{K}) \dot{+} H_2'(\mathbb{K}) \dot{+} \mathbb{R} \dot{+} [H_2(\mathbb{K})]^2. \end{aligned}$$

Define the Lie brackets explicitly for $\mathrm{Con}\, H_2(\mathbb{K})$ by considering the following elements

$$\begin{aligned} D &\in \mathrm{Der}\, H_2(\mathbb{K}) \\ H &\in H_2'(\mathbb{K}) \\ r, r' &\in \mathbb{R} \\ (X, Y) &\in [H_2(\mathbb{K})]^2. \end{aligned}$$

Then the brackets are

$$\begin{aligned} [D, r] = [r, H] = [r, r'] &= 0 \\ [D, H] &= D(H) \\ [D, (X, Y)] &= (D(X), D(Y)) \\ [r, (X, Y)] &= (rX, -rY) \\ [H, (X, Y)] &= (H \cdot X, -H \cdot Y) \end{aligned}$$

with the brackets for (X, Y) defined as above. The algebras $\mathrm{Str}\, H_2(\mathbb{K})$ and $\mathrm{Con}\, H_2(\mathbb{K})$ can also be thought of in terms of 2×2 matrices over $H_2(\mathbb{K})$.

Writing $\operatorname{Str} H_2(\mathbb{K})$ and $\operatorname{Con} H_2(\mathbb{K})$ in this way gives

$$\operatorname{Str} H_2(\mathbb{K}) = \operatorname{Der} H_2(\mathbb{K}) \dot{+} \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \quad A \in H_2(\mathbb{K}) \right\}$$

$$\operatorname{Con} H_2(\mathbb{K}) = \operatorname{Der} H_2(\mathbb{K}) \dot{+} \left\{ \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \quad A, B, C \in H_2(\mathbb{K}) \right\}.$$

In this section denote the elements of $L_2(\mathbb{K}, \widetilde{\mathbb{H}})$ by

$$A \in A_2'(\mathbb{K})$$

$$S \in \mathfrak{so}(\mathbb{K}')$$

$$B \otimes \tilde{i}, C \otimes \tilde{j}, D \otimes \tilde{k} \in H_2'(\mathbb{K}) \otimes \widetilde{\mathbb{H}}'$$

along with the basis elements of $\mathfrak{so}(\widetilde{\mathbb{H}}') \cong \mathfrak{so}(2,1)$ which will be labelled s_{12}, s_{13} and s_{23}, where

$$s_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$s_{13} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$s_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Now define an isomorphism $\psi : L_2(\mathbb{K}, \widetilde{\mathbb{H}}) \to \mathrm{Con}\, H_2(\mathbb{K})$ by

$$\psi(A) = A \qquad \psi(S) = S$$
$$\psi(B \otimes \tilde{\imath}) = B$$
$$\psi(C \otimes \tilde{\jmath}) = \tfrac{1}{2}(C, C), \quad \psi(D \otimes \tilde{k}) = \tfrac{1}{2}(D, -D)$$
$$\psi(s_{12}) = \tfrac{1}{2}(I, -I), \quad \psi(s_{13}) = \tfrac{1}{2}(I, I)$$
$$\psi(s_{23}) = 1.$$

This can also be defined in terms of 4×4 matrices as

$$\psi(A) = A \qquad \psi(S) = S$$
$$\psi(B \otimes \tilde{\imath}) = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}$$
$$\psi(C \otimes \tilde{\jmath}) = \begin{pmatrix} 0 & \frac{1}{2}C \\ 0 & 0 \end{pmatrix}, \quad \psi(D \otimes \tilde{k}) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}D & 0 \end{pmatrix}$$
$$(s_{12}) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}I & 0 \end{pmatrix}, \quad \psi(s_{13}) = \begin{pmatrix} 0 & \frac{1}{2}I \\ 0 & 0 \end{pmatrix}$$
$$\psi(s_{23}) = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The proof is a series of routine calculations of each product in both algebras showing that the isomorphism holds in all cases, the results of which can be found in tabular form on the next page.

Now $L_2(\mathbb{K}, \widetilde{\mathbb{C}})$ is embedded in $L_2(\mathbb{K}, \widetilde{\mathbb{H}})$ because $\widetilde{\mathbb{C}}$ is embedded in $\widetilde{\mathbb{H}}$ and ψ maps $L_2(\mathbb{K}, \widetilde{\mathbb{C}})$ to $\mathrm{Str}'\, H_2(\mathbb{K})$. Thus if we define $\psi_1 = \psi \mid L_2(\mathbb{K}, \widetilde{\mathbb{C}})$ then ψ_1 is an isomorphism between $L_2(\mathbb{K}, \widetilde{\mathbb{C}})$ and $\mathrm{Str}'\, H_2(\mathbb{K})$. Similarly we can define $\psi_2 = \psi \mid L_2(\mathbb{K}, \mathbb{R})$ as an isomorphism between $L_2(\mathbb{K}, \mathbb{R})$ and $\mathrm{Der}\, H_2(\mathbb{K})$ by restricting ψ to $L_2(\mathbb{K}, \mathbb{R})$. Thus we have obtained proofs for all the 2×2 isomorphisms.

	A'	D'	$B'\otimes\tilde{i}$	$C'\otimes\tilde{j}$	$E'\otimes\tilde{k}$	s_{12}	s_{13}	s_{23}
A	$(AA' - A'A)' + s(A,A')$	$-D'(A)$	$(AB'-B'A) \otimes\tilde{i}$	$(AC'-C'A) \otimes\tilde{j}$	$(AE'-E'A) \otimes\tilde{k}$	0	0	0
D	$D(A')$	$DD'-D'D$	$D(B')\otimes\tilde{i}$	$D(C')\otimes\tilde{j}$	$D(E')\otimes\tilde{k}$	0	0	0
$B\otimes\tilde{i}$	$(BA'-A'B) \otimes\tilde{i}$	$-D'(B)\otimes\tilde{i}$	$[L_B, L_{B'}]$	$\frac{1}{2}\langle B,C'\rangle s_{12}+ (B*C')\otimes\tilde{k}$	$\frac{1}{2}\langle B,E'\rangle s_{13}+ (B*E')\otimes\tilde{j}$	$B\otimes\tilde{j}$	$B\otimes\tilde{k}$	0
$C\otimes\tilde{j}$	$(CA'-A'C) \otimes\tilde{j}$	$-D'(C)\otimes\tilde{j}$	$-\frac{1}{2}\langle C,B'\rangle s_{12}- (C*B')\otimes\tilde{k}$	$[L_C, L_{C'}]$	$-\frac{1}{2}\langle C,E'\rangle s_{23}- (C*E')\otimes\tilde{i}$	$-C\otimes\tilde{i}$	0	$C\otimes\tilde{k}$
$E\otimes\tilde{k}$	$(EA'-A'E) \otimes\tilde{k}$	$-D'(E)\otimes\tilde{k}$	$-\frac{1}{2}\langle E,B'\rangle s_{13}- (E*B')\otimes\tilde{j}$	$\frac{1}{2}\langle E,C'\rangle s_{23}+ (E*C')\otimes\tilde{i}$	$[L_E, L_{E'}]$	0	$E\otimes\tilde{i}$	$E\otimes\tilde{j}$
s_{12}	0	0	$-B'\otimes\tilde{j}$	$C'\otimes\tilde{i}$	0	0	s_{23}	$-s_{13}$
s_{13}	0	0	$-B'\otimes\tilde{k}$	0	$-E'\otimes\tilde{i}$	$-s_{23}$	0	$-s_{12}$
s_{23}	0	0	0	$-C'\otimes\tilde{k}$	$-E'\otimes\tilde{j}$	s_{13}	s_{12}	0

FIGURE 2. Multiplication Tables for $L_2(\mathbb{K},\mathbb{H})$ and $\mathrm{Con}(H_2(\mathbb{K}))$.

	A'	D'	B'	$\frac{1}{2}(C',C')$	$\frac{1}{2}(E',-E')$	$\frac{1}{2}(I,-I)$	$\frac{1}{2}(I,I)$	1
A	$(AA'-A'A)'+$ $s(A,A')$	$-D'(A)$	$AB'-B'A$	$\frac{1}{2}(AC'-C'A,$ $AC'-C'A)$	$\frac{1}{2}(AE'-E'A,$ $E'A-AE')$	0	0	0
D	$D(A')$	$DD'-D'D$	$D(B')$	$\frac{1}{2}(D(C'),D(C'))$	$\frac{1}{2}(D(E'),-D(E'))$	0	0	0
B	$BA'-A'B$	$-D'(B)$	$[L_B,L_{B'}]$	$\frac{1}{4}\langle B,C'\rangle(I,-I)+$ $(B*C',B*C')$	$\frac{1}{4}\langle B,E'\rangle(I,I)+$ $(B*E',B*E')$	$\frac{1}{2}(B,B)$	$\frac{1}{2}(B,-B)$	0
$\frac{1}{2}(C,C)$	$\frac{1}{2}(CA'-A'C,$ $CA'-A'C)$	$-\frac{1}{2}(D'(C),D'(C))$	$-\frac{1}{4}\langle C,B'\rangle(I,-I)-$ $(C*B',C*B')$	$[L_C,L_{C'}]$	$-\frac{1}{2}\langle C,E'\rangle 1-C*E'$	$-C$	0	$\frac{1}{2}(C,-C)$
$\frac{1}{2}(E,-E)$	$\frac{1}{2}(EA'-A'E,$ $A'E-EA')$	$-\frac{1}{2}(D'(E),D'(E))$	$-\frac{1}{4}\langle E,B'\rangle(I,I)-$ $(E*B',E*B')$	$\frac{1}{2}\langle E,C'\rangle 1+E*C'$	$[L_E,L_{E'}]$	0	E	$\frac{1}{2}(E,E)$
$\frac{1}{2}(I,-I)$	0	0	$-\frac{1}{2}(B',B')$	C'	0	0	1	$-\frac{1}{2}(I,I)$
$\frac{1}{2}(I,I)$	0	0	$-\frac{1}{2}(B',-B')$	0	$-E'$	-1	0	$-\frac{1}{2}(I,-I)$
1	0	0	0	$-\frac{1}{2}(C',-C')$	$-\frac{1}{2}(E',E')$	$\frac{1}{2}(I,I)$	$\frac{1}{2}(I,-I)$	0

3. 2 × 2 Matrix Algebras

Begin by defining the 2 × 2 matrix algebras as follows

$$\mathfrak{sa}(2,\mathbb{K}) = A_2'(\mathbb{K})\dot{+}\mathfrak{so}(\mathbb{K})$$
$$\mathfrak{sl}(2,\mathbb{K}) = L_2'(\mathbb{K})\dot{+}\mathfrak{so}(\mathbb{K})$$
$$\mathfrak{sp}(4,\mathbb{K}) = \{A : A^\dagger J + JA = 0\}\dot{+}\mathfrak{so}(\mathbb{K}).$$

This is a reasonable definition since then $\mathfrak{sa}(2,\mathbb{C}) = \mathfrak{su}(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})$ has its usual definition. Define the Lie brackets for each of $\mathfrak{sa}(2,\mathbb{K})$, $\mathfrak{sl}(2,\mathbb{K})$ and $\mathfrak{sp}(4,\mathbb{K})$ as follows. In each case let s, s' be elements of $\mathfrak{so}(\mathbb{K})$ and let A and B be matrices in $A_2'(\mathbb{K})$, $L_2'(\mathbb{K})$ or $\{A : A^\dagger J + JA = 0\}$ respectively. Then the Lie brackets can be defined as

$$[s, s'] = ss' - s's$$
$$[s, A] = s(A)$$
$$[A, B] = (AB - BA)' + s(A, B).$$

Now clearly $\mathfrak{sa}(2,\mathbb{K}) \cong L_2(\mathbb{K},\mathbb{R})$ since

$$L_2(\mathbb{K},\mathbb{R}) = \operatorname{Der} H_2(\mathbb{K})$$
$$= A_2'(\mathbb{K})\dot{+}\mathfrak{so}(\mathbb{K})$$

by Theorem 7. Further $L_2(\mathbb{K},\widetilde{\mathbb{C}}) = A_2'(\mathbb{K})\dot{+}\mathfrak{so}(\mathbb{K})\dot{+}H_2'(\mathbb{K}) \otimes \tilde{i}$. Consequently defining $\psi : L_2(\mathbb{K},\widetilde{\mathbb{C}}) \to \mathfrak{sl}(2,\mathbb{K})$ by

$$\psi(A) = A$$
$$\psi(s) = s$$
$$\psi(H \otimes \tilde{i}) = H$$

gives a Lie algebra isomorphism (by simple calculations to check the Lie brackets).

Finally to see that $L_2(\mathbb{K}, \widetilde{\mathbb{H}}) \cong \mathfrak{sp}(4, \mathbb{K})$ define the isomorphism $\phi : L_2(\mathbb{K}, \widetilde{\mathbb{H}}) \to \mathfrak{sp}(4, \mathbb{K})$ by

$$\phi(A + D + H_1 \otimes \tilde{i} + H_2 \otimes \tilde{j} + H_3 \otimes \tilde{k} + r_1 C_{\tilde{i}} + r_2 C_{\tilde{j}} + r_3 C_{\tilde{k}})$$

$$\begin{pmatrix} A + H_1 + \frac{1}{2} r_1 I & H_2 + r_2 I - H_3 - \frac{1}{2} r_3 I \\ H_2 + r_2 I + H_3 + \frac{1}{2} r_3 I & A - H_1 - \frac{1}{2} r_1 I \end{pmatrix} + D.$$

Again, checking the brackets gives that this is also a Lie algebra isomorphism.

CHAPTER 4

Magic Squares of 3×3 Matrices

In this Chapter we extend the results of the last chapter to the 3×3 matrix case. We then develop these ideas by showing what the maximal compact subalgebras are for each of the exceptional Lie algebras that appear in the magic square.

We begin by showing that

$$L_3(\mathbb{K}, \widetilde{\mathbb{H}}) \cong \operatorname{Con} H_3(\mathbb{K}) \tag{83}$$

and then show

$$\begin{aligned} \mathfrak{sa}(3,\mathbb{K}) &\cong L_3(\mathbb{K},\mathbb{R}), \\ \mathfrak{so}(3,\mathbb{K}) &\cong L_3(\mathbb{K},\widetilde{\mathbb{C}}), \\ \mathfrak{sp}(6,\mathbb{K}) &\cong L_3(\mathbb{K},\widetilde{\mathbb{H}}). \end{aligned} \tag{84}$$

We then show that our notation is consistant with the notation used by Helgason in [**16**] Finally, we prove that the maximal compact subalgebras are those stated in Chapter 2.

1. The 3×3 Conformal Algebra

In the case of 3×3 matrices it is known from (34) and Theorem 5 that

$$\operatorname{Str}' H_3(\mathbb{K}) = L'_3(\mathbb{K}) \dotplus \operatorname{Der} \mathbb{K}$$

and also from (35),

$$\operatorname{Con} H_3(\mathbb{K}) = \operatorname{Str} H_3(\mathbb{K}) \dotplus [H_3(\mathbb{K})]^2.$$

Thus

$$\operatorname{Con} H_3(\mathbb{K}) = \operatorname{Der} \mathbb{K} \dotplus A'_3(\mathbb{K}) \dotplus H'_3(\mathbb{K}) \dotplus \mathbb{R} \dotplus [H_3(\mathbb{K})]^2.$$

This is equivalent to $L_3(\mathbb{K}, \widetilde{\mathbb{H}})$ as a vector space since

$$\begin{aligned} L_3(\mathbb{K}, \widetilde{\mathbb{H}}) &= \operatorname{Der} H_3(\mathbb{K}) \dot{+} H_3'(\mathbb{K}) \otimes \widetilde{\mathbb{H}}' \dot{+} \operatorname{Der} \widetilde{\mathbb{H}} \\ &\cong A_3'(\mathbb{K}) \dot{+} \operatorname{Der} \mathbb{K} \dot{+} [H_3'(\mathbb{K})]^3 \dot{+} C(\widetilde{\mathbb{H}}') \\ &\cong \operatorname{Der} \mathbb{K} \dot{+} A_3'(\mathbb{K}) \dot{+} H_3'(\mathbb{K}) \dot{+} [H_3'(\mathbb{K})]^2 \dot{+} \mathbb{R}^3 \\ &\cong \operatorname{Der} \mathbb{K} \dot{+} A_3'(\mathbb{K}) \dot{+} H_3'(\mathbb{K}) \dot{+} [H_3(\mathbb{K})]^2 \dot{+} \mathbb{R} \\ &\cong \operatorname{Con} H_3(\mathbb{K}). \end{aligned}$$

Consider the elements of $L_3(\mathbb{K}, \widetilde{\mathbb{H}})$ to be

$$\begin{gathered} A \in A_3'(\mathbb{K}) \\ D \in \operatorname{Der}(\mathbb{K}) \\ B \otimes \tilde{i}, C \otimes \tilde{j}, E \otimes \tilde{k} \in H_3'(\mathbb{K}) \otimes \widetilde{\mathbb{H}}' \\ C_{\tilde{i}}, C_{\tilde{j}}, C_{\tilde{k}} \in C(\widetilde{\mathbb{H}}') \end{gathered}$$

and the elements of $\operatorname{Con} H_3(\mathbb{K})$ to be

$$\begin{gathered} A \in A_3'(\mathbb{K}) \\ B \in H_3'(\mathbb{K}) \\ D \in \operatorname{Der}(\mathbb{K}) \\ r \in \mathbb{R} \\ (X, Y) \in [H_3(\mathbb{K})]^2 \end{gathered}$$

then define an isomorphism $\phi : L_3(\mathbb{K}, \widetilde{\mathbb{H}}) \to \operatorname{Con} H_3(\mathbb{K})$ by :

$$\phi(A) = A \in A_3'(\mathbb{K}) \qquad \phi(D) = D \in \operatorname{Der} \mathbb{K}$$

$$\phi(B \otimes \tilde{i}) = B \in H_3'(\mathbb{K})$$

and

$$\phi(C \otimes \tilde{j}) = \tfrac{1}{2}(C, C) \qquad \phi(E \otimes \tilde{k}) = \tfrac{1}{2}(E, -E)$$

$$\phi(C_{\tilde{k}}) = (I, -I) \qquad \phi(C_{\tilde{j}}) = (I, I)$$

in $[H_3(\mathbb{K})]^2$. The final part of the map is

$$\phi(C_{\tilde{i}}) = 2 \in \mathbb{R}.$$

Again $\mathrm{Con}\, H_3(\mathbb{K})$ and $\mathrm{Str}'\, H_3(\mathbb{K})$ can be expressed in terms of 2×2 block matrices over $H_3(\mathbb{K})$ given by

$$\mathrm{Str}'\, H_3(\mathbb{K}) = \mathrm{Der}\, H_3(\mathbb{K}) \dot{+} \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \quad A \in H_3'(\mathbb{K}) \right\}$$

$$\mathrm{Con}\, H_3(\mathbb{K}) = \mathrm{Der}\, H_3(\mathbb{K}) \dot{+} \left\{ \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \quad A, B, C \in H_3(\mathbb{K}) \right\}.$$

Further the isomorphism can also be defined in terms of these matrices,

$$\phi(A) = A \qquad \phi(D) = D$$

$$\phi(B \otimes \tilde{i}) = \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}$$

$$\phi(C \otimes \tilde{j}) = \begin{pmatrix} 0 & \frac{1}{2}C \\ 0 & 0 \end{pmatrix} \qquad \phi(E \otimes \tilde{k}) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}E & 0 \end{pmatrix}$$

$$\phi(C_{\tilde{k}}) = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \qquad \phi(C_{\tilde{j}}) = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$$

$$\phi(C_{\tilde{i}}) = \begin{pmatrix} 2I & 0 \\ 0 & -2I \end{pmatrix}.$$

Again, this can be shown to be a Lie algebra isomorphism by comparisons of the multiplication tables on the following pages for $L_3(\mathbb{K}, \widetilde{\mathbb{H}})$ and $\mathrm{Con}\, H_3(\mathbb{K})$.

	A'	D'	$B' \otimes \tilde{i}$	$C' \otimes \tilde{j}$	$E' \otimes \tilde{k}$	$C_{\tilde{k}}$	$C_{\tilde{j}}$	$C_{\tilde{i}}$
A	$(AA' - A'A)' + \frac{1}{3}D(A, A')$	$-D'(A)$	$(AB' - B'A) \otimes \tilde{i}$	$(AC' - C'A) \otimes \tilde{j}$	$(AE' - E'A) \otimes \tilde{k}$	0	0	0
D	$D(A')$	$DD' - D'D$	$D(B') \otimes \tilde{i}$	$D(C') \otimes \tilde{j}$	$D(E') \otimes \tilde{k}$	0	0	0
$B \otimes \tilde{i}$	$(BA' - A'B) \otimes \tilde{i}$	$-D'(B) \otimes \tilde{i}$	$[L_B, L_{B'}]$	$\frac{1}{6}\langle B, C'\rangle C_{\tilde{k}} + (B * C') \otimes \tilde{k}$	$\frac{1}{6}\langle B, E'\rangle C_{\tilde{j}} + (B * E') \otimes \tilde{j}$	$2B \otimes \tilde{j}$	$2B \otimes \tilde{k}$	0
$C \otimes \tilde{j}$	$(CA' - A'C) \otimes \tilde{j}$	$-D'(C) \otimes \tilde{j}$	$-\frac{1}{6}\langle C, B'\rangle C_{\tilde{k}} - (C * B') \otimes \tilde{k}$	$[L_C, L_{C'}]$	$-\frac{1}{6}\langle C, E'\rangle C_{\tilde{i}} - (C * E') \otimes \tilde{i}$	$-2C \otimes \tilde{i}$	0	$2C \otimes \tilde{k}$
$E \otimes \tilde{k}$	$(EA' - A'E) \otimes \tilde{k}$	$-D'(E) \otimes \tilde{k}$	$-\frac{1}{6}\langle E, B'\rangle C_{\tilde{j}} - (E * B') \otimes \tilde{j}$	$\frac{1}{6}\langle E, C'\rangle C_{\tilde{i}} + (E * C') \otimes \tilde{i}$	$[L_E, L_{E'}]$	0	$2E \otimes \tilde{i}$	$2E \otimes \tilde{j}$
$C_{\tilde{k}}$	0	0	$-2B' \otimes \tilde{j}$	$2C' \otimes \tilde{i}$	0	0	$2C_{\tilde{i}}$	$-2C_{\tilde{j}}$
$C_{\tilde{j}}$	0	0	$-2B' \otimes \tilde{k}$	0	$-2E' \otimes \tilde{i}$	$-2C_{\tilde{i}}$	0	$-2C_{\tilde{k}}$
$C_{\tilde{i}}$	0	0	0	$-2C' \otimes \tilde{k}$	$-2E' \otimes \tilde{j}$	$2C_{\tilde{j}}$	$2C_{\tilde{k}}$	0

FIGURE 1. Multiplication Tables for $L_3(\mathbb{K}, \mathbb{H})$ and $\mathrm{Con}(H_3(\mathbb{K}))$.

	A'	D'	B'	$\frac{1}{2}(C', C')$	$\frac{1}{2}(E', -E')$	$(I, -I)$	(I, I)	2
A	$(AA' - A'A) + \frac{1}{3}D(A, A')$	$-D'(A)$	$AB' - B'A$	$\frac{1}{2}(AC' - C'A, AC' - C'A)$	$(AE' - E'A, -AE' + E'A)$	0	0	0
D	$D(A')$	$DD' - D'D$	$D(B')$	$\frac{1}{2}(D(C'), D(C'))$	$\frac{1}{2}(D(E'), -D(E'))$	0	0	0
B	$BA' - A'B$	$-D'(B)$	$[L_B, L_{B'}]B$	$\frac{1}{6}\langle B, C'\rangle(I, -I) + (B * C', B * C')$	$\frac{1}{6}\langle B, E'\rangle(I, I) + (B * E', B * E')$	(B, B)	$(B, -B)$	0
$\frac{1}{2}(C, C)$	$\frac{1}{2}(CA' - A'C, CA' - A'C)$	$-\frac{1}{2}(D'(C), D'(C))$	$-\frac{1}{6}\langle C, B'\rangle(I, -I) - (C * B', C * B')$	$[L_C, L_{C'}]$	$-\frac{1}{3}\langle C, E'\rangle - C * E'$	$-2C$	0	$(C, -C)$
$\frac{1}{2}(E, -E)$	$\frac{1}{2}(EA' - A'E, -EA' + A'E)$	$-\frac{1}{2}(D'(E), D'(E))$	$-\frac{1}{6}\langle E, B'\rangle(I, I) - (E * B', E * B')$	$\frac{1}{3}\langle E, C'\rangle + E * C'$	$[L_E, L_{E'}]$	0	$2E$	(E, E)
$(I, -I)$	0	0	$-(B', B')$	$2C'$	0	0	4	$-2(I, I)$
(I, I)	0	0	$-(B', B')$	0	$-2E'$	-4	0	$-2(I, -I)$
2	0	0	0	$-(C', -C')$	$-(E', E')$	$2(I, I)$	$2(I, -I)$	0

Invoking the same method as used in the similar proof for 2×2 matrices note that $\phi_1 : L_3(\mathbb{K}, \widetilde{\mathbb{C}}) \to \mathrm{Str}' H_3(\mathbb{K})$ can be defined by:

$$\phi_1(A) = A, \quad \phi_1(D) = D$$

$$\phi_1(B \otimes \tilde{\imath}) = B$$

and also, trivially, $\phi_2 : L_3(\mathbb{K}, \mathbb{R}) \to \mathrm{Der}\, H_3(\mathbb{K})$ by :

$$\phi_2(A) = A, \quad \phi_2(D) = D$$

Thus the following is proved.

THEOREM 12.

$$L_3(\mathbb{K}, \mathbb{R}) \cong \mathrm{Der}\, H_3(\mathbb{K})$$

$$L_3(\mathbb{K}, \widetilde{\mathbb{C}}) \cong \mathrm{Str}'\, H_3(\mathbb{K})$$

$$L_3(\mathbb{K}, \widetilde{\mathbb{H}}) \cong \mathrm{Con}\, H_3(\mathbb{K})$$

hold for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.

2. 3 × 3 Matrix Algebras

Define the following generalisations of 3×3 and 6×6 matrix algebras to all division algebras. Define

$$\mathfrak{sa}(3, \mathbb{K}) = A_3'(\mathbb{K}) \dot{+} \mathrm{Der}\, \mathbb{K}$$

$$\mathfrak{sl}(3, \mathbb{K}) = L_3'(\mathbb{K}) \dot{+} \mathrm{Der}\, \mathbb{K}$$

$$\mathfrak{sp}(6, \mathbb{K}) = \{A \in \mathbb{K}^{3\times3} : A^\dagger J + JA = 0\} \dot{+} \mathrm{Der}\, \mathbb{K}.$$

As in the 2×2 case the Lie brackets for these algebras can be defined in a unified manner by taking D, $E \in \mathrm{Der}\, \mathbb{K}$ and A, $B \in A_3'(\mathbb{K})$, $L_3'(\mathbb{K})$ or $\{A \in \mathbb{K}^{3\times3} : A^\dagger J + JA = 0\}$ respectively. Then the Lie brackets are defined as

$$[D, E] = DE - ED$$

$$[D, A] = D(A)$$

$$[A, B] = (AB - BA)' + \sum_{ij} D_{a_{ij} b_{ij}}.$$

Clearly in the case that $\mathbb{K} = \mathbb{C}$ this leads to the usual definition of $\mathfrak{sl}(3, \mathbb{C})$ and $\mathfrak{sa}(3, \mathbb{C}) = \mathfrak{su}(3)$. In the case that $\mathbb{K} = \mathbb{H}$ this leads also to the definition of $\mathfrak{sq}(3) = \mathfrak{sp}(3, \mathbb{H})$. The following Theorem can now be proved.

THEOREM 13.

$$L_3(\mathbb{K}, \mathbb{R}) \cong \mathfrak{sa}(3, \mathbb{K})$$

$$L_3(\mathbb{K}, \widetilde{\mathbb{C}}) \cong \mathfrak{sl}(3, \mathbb{K})$$

$$L_3(\mathbb{K}, \widetilde{\mathbb{H}}) \cong \mathfrak{sp}(6, \mathbb{K})$$

giving the table

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\operatorname{Der} H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \mathbb{R})$	$\mathfrak{sa}(3, \mathbb{R})$	$\mathfrak{sa}(3, \mathbb{C})$	$\mathfrak{sa}(3, \mathbb{H})$	$\mathfrak{sa}(3, \mathbb{O})$
$\operatorname{Str}' H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \widetilde{\mathbb{C}})$	$\mathfrak{sl}(3, \mathbb{R})$	$\mathfrak{sl}(3, \mathbb{C})$	$\mathfrak{sl}(3, \mathbb{H})$	$\mathfrak{sl}(3, \mathbb{O})$
$\operatorname{Con} H_3(\mathbb{K}) \cong L_3(\mathbb{K}, \widetilde{\mathbb{H}})$	$\mathfrak{sp}(6, \mathbb{R})$	$\mathfrak{sp}(6, \mathbb{C})$	$\mathfrak{sp}(6, \mathbb{H})$	$\mathfrak{sp}(6, \mathbb{O})$

PROOF. 1. In this case we have $L_3(\mathbb{K}, \mathbb{R}) = A_3'(\mathbb{K}) \dot{+} \operatorname{Der} \mathbb{K}$. Take $A \in A_3'(\mathbb{K})$ and $D \in \operatorname{Der} \mathbb{K}$. Clearly this is precisely $\mathfrak{sa}(3, \mathbb{K})$.

2. Now

$$L_3(\mathbb{K}, \widetilde{\mathbb{C}}) = A_3'(\mathbb{K}) \dot{+} \operatorname{Der} \mathbb{K} \dot{+} H_3'(\mathbb{K}) \otimes \widetilde{\mathbb{C}}'.$$

Take $A \in A_3'(\mathbb{K})$, $D \in \operatorname{Der} \mathbb{K}$ and $H \in H_3'(\mathbb{K})$ (since the tensor product $H_3'(\mathbb{K}) \otimes \widetilde{\mathbb{C}}'$ can be regarded as being one copy of $H_3'(\mathbb{K})$), then the isomorphism $\phi : L_3(\mathbb{K}, \widetilde{\mathbb{C}}) \rightarrow \mathfrak{sl}(3, \mathbb{H})$ can be written

$$\phi(A + D + H \otimes \tilde{\imath}) = A + D + H.$$

3. Here

$$L_3(\mathbb{K}, \widetilde{\mathbb{H}}) = A_3'(\mathbb{K}) \dot{+} \operatorname{Der} \mathbb{K} \dot{+} H_3'(\mathbb{K}) \otimes \widetilde{\mathbb{H}}' \dot{+} \operatorname{Der} \widetilde{\mathbb{H}}.$$

Taking $A \in A_3'(\mathbb{K})$, $D \in \operatorname{Der} \mathbb{K}$, $H_1 \otimes \tilde{\imath}$, $H_2 \otimes \tilde{\jmath}$ and $H_3 \otimes \tilde{k} \in H_3'(\mathbb{K}) \otimes \widetilde{\mathbb{H}}$ and $r_1 C_{\tilde{\imath}} + r_2 C_{\tilde{\jmath}} + r_3 C_{\tilde{k}} \in \operatorname{Der} \widetilde{\mathbb{H}}$ then the isomorphism $\chi : L_3(\mathbb{K}, \widetilde{\mathbb{H}}) \rightarrow$

$\mathfrak{sp}(6, \mathbb{K})$ can be written explicitly as

$$\chi(A + D + H_1 \otimes \tilde{i} + H_2 \otimes \tilde{j} + H_3 \otimes \tilde{k} + r_1 C_{\tilde{i}} + r_2 C_{\tilde{j}} + r_3 C_{\tilde{k}}) =$$

$$\begin{pmatrix} A + H_1 + \frac{1}{3} r_1 I & (H_2 + r_2 I - H_3 - \frac{1}{3} r_3 I) \\ (H_2 + r_2 I + H_3 + \frac{1}{3} r_3 I) & A - H_1 - \frac{1}{3} r_1 I \end{pmatrix} + D.$$

□

3. A small matter of notation

In this section we consider the difference between our notation and the more classical style of notation used, for example, in Helgason's book [16]. There, rather than use quaternions, Helgason defines two algebras $\mathfrak{so}^*(2n)$ and $\mathfrak{su}^*(2n)$ as follows

$$\mathfrak{so}^*(2n) = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} : Z_1, Z_2 \in \mathbb{C}^{n \times n}, Z_1 \text{ skew, } Z_2 \text{ hermitian.} \right\} \tag{85}$$

$$\mathfrak{su}^*(2n) = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} : Z_1, Z_2 \in \mathbb{C}^{n \times n}, \operatorname{tr}(Z_1 + \bar{Z}_1) = 0 \right\}. \tag{86}$$

Note that we will re-define $\mathfrak{sl}(n, \mathbb{H})$ and $\mathfrak{sp}(2n, \mathbb{H})$ as follows;

$$\mathfrak{sl}(n, \mathbb{H}) = \{A \in \mathbb{H}^{n \times n} : \operatorname{Re}(\operatorname{tr} A) = 0\}$$

$$\mathfrak{sp}(2n, \mathbb{H}) = \{B \in \mathbb{H}^{2n \times 2n} : B^\dagger J + JB = 0 \text{ and } \operatorname{Re}(\operatorname{tr} B) = 0\}.$$

This is using the fact that $\operatorname{Der} \mathbb{H} \cong \mathfrak{so}(3) \cong \mathbb{H}'$. It is important to realise that this definition does not apply to any of the other division algebras.

The proof of the following lemmas will follow.

LEMMA 10.

$$\mathfrak{su}^*(2n) \cong \mathfrak{sl}(n, \mathbb{H}).$$

LEMMA 11.

$$\mathfrak{so}^*(4n) \cong \mathfrak{sp}(2n, \mathbb{H})$$

(i.e. $\mathfrak{so}^*(2n) \cong \mathfrak{sp}(n, \mathbb{H})$ if and only if n is even).

PROOF OF LEMMA 10. Note that the condition on the block matrix $Z_1 \in \mathfrak{su}^*(2n)$ is the same as the condition on the trace of a matrix in $\mathfrak{sl}(n, \mathbb{H})$. Thus clearly Z_1 is a $n \times n$ matrix with entries in $\mathbb{C}$ and $\mathrm{Re}(\operatorname{tr} Z_1) = 0$. Further the dimension of both algebras is $4n^2 - 1$, indicating that an isomorphism is actually possible. Now for two matrices in $\mathfrak{su}^*(2n)$, say X and Y calculate the Lie brackets. These give

$$\begin{aligned}[X,Y] &= \left[\begin{pmatrix} X_1 & X_2 \\ -\bar{X}_2 & \bar{X}_1 \end{pmatrix}, \begin{pmatrix} Y_1 & Y_2 \\ -\bar{Y}_2 & \bar{Y}_1 \end{pmatrix}\right] \\ &= \begin{pmatrix} X_1Y_1 - Y_1X_1 - X_2\bar{Y}_2 + Y_2\bar{X}_2 & X_1Y_2 - Y_1X_2 + X_2\bar{Y}_1 - Y_2\bar{X}_1 \\ \bar{Y}_2X_1 - \bar{X}_2Y_1 + \bar{Y}_1\bar{X}_2 - \bar{X}_1\bar{Y}_2 & \bar{X}_1\bar{Y}_1 - \bar{Y}_1\bar{X}_1 + \bar{Y}_2X_2 - \bar{X}_2Y_2 \end{pmatrix}\end{aligned}$$

Now take a matrix $A \in \mathfrak{sl}(n, \mathbb{H})$ and write this as $A = A_1 + A_2 j$ where $A_1 \in \mathfrak{sl}(n, \mathbb{C})$ and $A_2 \in \mathfrak{gl}(n, \mathbb{C})$. Recall that $Aj = j\bar{A}$. Then calculating the Lie brackets for $A, B \in \mathfrak{sl}(n, \mathbb{H})$ gives

(87)
$$[A, B] = A_1B_1 - B_1A_1 - A_2\bar{B}_2 + B_2\bar{A}_2 + j(A_1B_2 - B_1A_2 + A_2\bar{B}_1 - B_2\bar{A}_1).$$

Thus we define an isomorphism $\varphi : \mathfrak{su}^*(2n) \to \mathfrak{sl}(n, \mathbb{H})$ by

$$\varphi \begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} = Z_1 + Z_2 j \tag{88}$$

proving that $\mathfrak{su}^*(2n) \cong \mathfrak{sl}(n, \mathbb{H})$. □

PROOF OF LEMMA 11. First notice that the definition of $\mathfrak{so}^*(4n)$ can be re-written as

$$\mathfrak{so}^*(4n) = \{B \in \mathbb{C}^{4n \times 4n} : B^\dagger K + KB = 0\} \tag{89}$$

where K is the $4n \times 4n$ matrix $\begin{pmatrix} 0 & I_{2n} \\ -I_{2n} & 0 \end{pmatrix}$. Recall the definition of $\mathfrak{sp}(2n, \mathbb{H})$:

$$\mathfrak{sp}(2n, \mathbb{H}) = \{A \in \mathbb{H}^{2n \times 2n} : A^\dagger J + JA = 0 \text{ and } \mathrm{Re}(\operatorname{tr} A) = 0\}$$

where $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Define matrices L, M and N such that $K = LN = -NL$ where

$$N = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} \quad L = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \quad M = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Now let $A \in \mathfrak{sp}(2n, \mathbb{H})$ be written as $A = A_1 + jA_2$. Then A satisfies the relation

$$A^\dagger J + JA = 0$$

Thus

$$\begin{aligned} & (A_1 + A_2)^\dagger J + J(A_1 + A_2) = 0 \\ \Rightarrow \quad & (A_1^\dagger - jA_2^t)J + J(A_1 + A_2) = 0 \end{aligned}$$

and equating the complex co-efficients of 1 and j (i.e. considering $\mathbb{H}$ to be $\mathbb{H} = \mathbb{C} + j\mathbb{C}$) gives

$$\begin{aligned} A_1^\dagger J + JA_1 &= 0 \\ A_2^t J - JA_2 &= 0. \end{aligned}$$

Now consider $B \in \mathfrak{so}^*(4n)$ where B is written in the notation used in equation (85). Then

$$\begin{aligned} & B^\dagger K + KB = 0 \\ & \Rightarrow B^\dagger LN - NLB = 0 \\ & \Rightarrow \begin{pmatrix} (B_2^t M)J + J(M\bar{B}_2) & (B_1^\dagger M)J - J(M\bar{B}_1) \\ -(B_1^t M)J + J(MB_1) & (B_2^\dagger M)J + J(MB_2) \end{pmatrix} = 0. \end{aligned}$$

Thus

$$\begin{aligned} (B_2^t M)J + J(M\bar{B}_2) &= 0 \\ (B_1^\dagger M)J - J(M\bar{B}_1) &= 0. \end{aligned}$$

Taking $A_1 = M\bar{B}_2$ gives

$$A_1^\dagger J + JA_1 = 0$$

and taking $A_2 = M\bar{B}_1$ we obtain

$$A_2^t J - J A_2 = 0.$$

Consequently define an isomorphism $\psi : \mathfrak{so}^*(4n) \to \mathfrak{sp}(2n, \mathbb{H})$ by

$$\psi \begin{pmatrix} B_1 & B_2 \\ -\bar{B}_2 & \bar{B}_1 \end{pmatrix} = (M\bar{B}_2) + j(M\bar{B}_1). \tag{90}$$

Then ψ is a Lie algebra isomorphism from $so^*(4n)$ to $\mathfrak{sp}(2n, \mathbb{H})$. □

4. Maximal Compact Subalgebras

A semi-simple Lie algebra is called *compact* if it has a negative-definite Killing form. It is called *non-compact* if its Killing form is not negative-definite.

A non-compact real form, $\mathfrak{g}$, of a semi-simple complex Lie algebra, L, has a *maximal compact subalgebra* $\mathfrak{k}$ with an *orthogonal complementary subspace* $\mathfrak{p}$ such that $\mathfrak{g} = \mathfrak{k} \dot{+} \mathfrak{p}$ and the brackets

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$$

$$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$$

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$$

$$(\mathfrak{k}, \mathfrak{p}) = 0$$

are satisfied (see, for example, [**14**]). Denote by $(,)$ the Killing form of L. There exists an involutive automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{k}$ and $\mathfrak{p}$ are eigenspaces of σ with eigenvalues $+1$ and -1 respectively. A compact real form, $\mathfrak{g}'$, of L will also contain $\mathfrak{k}$ as a compact subalgebra of $\mathfrak{g}'$ but clearly in this case the maximal compact subalgebra will be $\mathfrak{g}'$ itself. We can obtain $\mathfrak{g}'$ from $\mathfrak{g}$ by keeping the same brackets in $[\mathfrak{k}, \mathfrak{k}]$ and $[\mathfrak{k}, \mathfrak{p}]$ but multiplying the brackets in $[\mathfrak{p}, \mathfrak{p}]$ by -1, i.e. by performing the *Weyl unitary trick* (putting $\mathfrak{g}' = \mathfrak{k} \dot{+} i\mathfrak{p}$).

There now follows an overview of the method used to show that the algebras given in the table on page 41 are maximal compact, which is essentially the same in each case. It is known that $L_3(\mathbb{K}_1, \mathbb{K}_2)$ gives a compact real form of each Lie algebra (from, for example [**19**]). Thus if $L_3(\mathbb{K}_1, \widetilde{\mathbb{K}}_2)$ shares a common subalgebra with $L_3(\mathbb{K}_1, \mathbb{K}_2)$, say $\mathfrak{f}$, where

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = \mathfrak{f} \dot{+} \mathfrak{p}_1$$

$$L_3(\mathbb{K}_1, \widetilde{\mathbb{K}}_2) = \mathfrak{f} \dot{+} \mathfrak{p}_2,$$

and the brackets in $[\mathfrak{f}, \mathfrak{p}_1]$ are the same as those in $[\mathfrak{f}, \mathfrak{p}_2]$ but the brackets in $[\mathfrak{p}_1, \mathfrak{p}_1]$ are -1 times the equivalent brackets in $[\mathfrak{p}_2, \mathfrak{p}_2]$, then $\mathfrak{f}$ will be the maximal compact subalgebra of $L_3(\mathbb{K}_1, \widetilde{\mathbb{K}}_2)$ and $\mathfrak{p}_2$ will be its orthogonal complementary subspace. Moreover, because of the nature of the split composition algebras, it will be shown that this sign change in the brackets will reflect precisely the change in sign in the Cayley-Dickson process when moving from the division algebra to the corresponding composition algebra.

Now, $\mathrm{Der}\,\mathbb{O}$ and $\mathrm{Der}\,\widetilde{\mathbb{O}}$ have a common subalgebra $\mathfrak{so}(3) \dot{+} \mathfrak{so}(3)$ (see, for example, [**14**]) which is the subalgebra with basis elements $\{d_3, d_8, d_6, d_9, d_{10}, d_7\}$, these being invariant under multiplication by the metric $\mathcal{G}_2$. We will denote by $\widetilde{G}_2$ the form of the exceptional Lie algebra G_2 isomorphic to $\mathrm{Der}\,\widetilde{\mathbb{O}}$.

We now state explicity the result we are about to prove.

THEOREM 14. The maximal compact subalgebras are of the forms given in the table below.

$\mathfrak{g}$	$\mathfrak{f}$	
$E_{6,1}$	F_4	$\mathrm{Der}\,H_3(\mathbb{O})$
$E_{7,1}$	$E_6 \oplus \mathfrak{so}(2)$	$\mathrm{Der}\,H_3(\mathbb{O}) \dot{+} H_3'(\mathbb{O}) \otimes \langle i \rangle \dot{+} \{C_i\}$
$E_{8,1}$	$E_7 \oplus \mathfrak{so}(3)$	$\mathrm{Der}\,H_3(\mathbb{O}) \dot{+} H_3'(\mathbb{O}) \otimes \mathbb{H}' \dot{+} M_{G_2}$
$F_{4,2}$	$\mathfrak{sq}(3) \oplus \mathfrak{so}(3)$	$\mathrm{Der}\,H_3(\mathbb{R}) \dot{+} H_3'(\mathbb{R}) \otimes \mathbb{H}' \dot{+} M_{G_2}$
$E_{6,2}$	$\mathfrak{su}(3) \oplus \mathfrak{so}(6)$	$\mathrm{Der}\,H_3(\mathbb{C}) \dot{+} H_3'(\mathbb{C}) \otimes \mathbb{H}' \dot{+} M_{G_2}$
$E_{7,2}$	$\mathfrak{so}(12) \oplus \mathfrak{so}(3)$	$\mathrm{Der}\,H_3(\mathbb{H}) \dot{+} H_3'(\mathbb{H}) \otimes \mathbb{H}' \dot{+} M_{G_2}$

where $M_{G_2} (= \mathrm{Span}\{d_3, d_8, d_6, d_9, d_{10}, d_7\})$, is the maximal compact subalgebra of $\tilde{G}_2$.

PROOF. Consider first each of the $_{,1}$ type algebras and then move on to the $_{,2}$ type. Denote by A^N the non- compact form of the algebra A and by A^C the compact form of A.

1. For $E^N_{6,1}$

$$\mathfrak{k} = \text{Der}\, H_3(\mathbb{O})$$
$$\mathfrak{p} = H_3'(\mathbb{O}) \otimes \widetilde{\mathbb{C}}'.$$

E_6^C also has $\mathfrak{k}$ as a subalgebra but in this case $E_6^C/\mathfrak{k}$ is $H_3'(\mathbb{O}) \otimes \mathbb{C}'$. Thus there is only one set of brackets to check. If we consider $H_1 \otimes \tilde{i}, H_2 \otimes \tilde{i} \in H_3(\mathbb{O}) \otimes \widetilde{\mathbb{C}}'$ and $H_1 \otimes i, H_2 \otimes i \in H_3(\mathbb{O}) \otimes \mathbb{C}'$ then clearly, using the definitions for the brackets found on page 39

$$[H_1 \otimes \tilde{i}, H_2 \otimes \tilde{i}] = [L_{H_1}, L_{H_2}]$$
$$[H_1 \otimes i, H_2 \otimes i] = -[L_{H_1}, L_{H_2}],$$

as required.

2. In the case of $E^N_{7,1}$ the orthogonal complementary subspace is

$$\tilde{\mathfrak{p}} = H_3'(\mathbb{O}) \otimes \{\tilde{j}, \tilde{k}\} \dot{+} \text{Span}\{C_{\tilde{j}}, C_{\tilde{k}}\},$$

where $\mathfrak{k}$ is the maximal compact subalgebra given in the table above. Then $\mathfrak{k}$ is also a subalgebra in E_7^C and the remaining subspace is $\mathfrak{p} = H_3'(\mathbb{O}) \otimes \text{Span}\{j, k\} \dot{+} \text{Span}\{C_j, C_k\}$. Now

$$[C_{\tilde{j}}, C_{\tilde{k}}] = -2C_i \qquad [C_j, C_k] = 2C_i$$
$$[C_{\tilde{j}}, H_1 \otimes \tilde{k}] = -H_1 \otimes 2i \qquad [C_j, H_1 \otimes k] = H_1 \otimes 2i$$
$$[C_{\tilde{k}}, H_1 \otimes \tilde{j}] = H_1 \otimes 2i \qquad [C_k, H_1 \otimes j] = -H_1 \otimes 2i.$$

Recall that

$$[H_1 \otimes x, H_2 \otimes y] = 2\,\text{tr}(H_1 H_2) D_{x,y} +$$
$$(H * G) \otimes \text{Im}(xy) - Re(x\bar{y})[L_{H_1}, L_{H_2}].$$

We have to consider the two cases (1) when $x = y$ and (2) when $x \perp y$. Case (1) is considered in E_6. Case (2) gives

$$[H_1 \otimes \tilde{j}, H_2 \otimes \tilde{k}] = -2\,\mathrm{tr}(H_1H_2)C_i - (H_1 * H_2) \otimes i$$

$$[H_1 \otimes j, H_2 \otimes k] = 2\,\mathrm{tr}(H_1H_2)C_i + (H_1 * H_2) \otimes i.$$

Clearly, $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$ are orthogonal complementary subspaces with their brackets with themselves giving opposite signs and thus we can deduce that the choice of maximal compact subalgebra is correct.

3. In E_8^N we have the orthogonal complementary subspace

$$\tilde{\mathfrak{p}} = H_3'(\mathbb{O}) \otimes \mathrm{Span}\{\tilde{l}, \tilde{il}, \tilde{jl}, \tilde{kl}\} \dot{+} \{\tilde{d_a} \mid a = 1, 2, 4, 5, 11, 12, 13, 14\}$$

to the maximal compact subalgebra $\mathfrak{f}$ as shown in the previous table. Then $\mathfrak{f}$ is also a subalgebra of E_8^C and the remaining subspace in E_8^C will be $\mathfrak{p} = H_3'(\mathbb{O}) \otimes \mathrm{Span}\{l, il, jl, kl\} \dot{+} \{d_a \mid a = 1, 2, 4, 5, 11, 12, 13, 14\}$. For convenience we will label the orthogonal complementary subspaces of G_2 $\mathfrak{p}_{G_2}$ and $\tilde{\mathfrak{p}}_{G_2}$ for the compact and non-compact cases respectively. The calculations for E_8 are much the same as those for E_7. For brackets between $H_3'(\mathbb{O}) \otimes \mathrm{Span}\{l, il, jl, kl\}$ and itself we again have two cases with $x = y$ and $x \perp y$. These are resolved in the same way as before. To calculate the brackets between $H_3'(\mathbb{O}) \otimes \mathrm{Span}\{l, il, jl, kl\}$ and $\mathfrak{p}_{G_2}$ and between $\mathfrak{p}_{G_2}$ and itself involves a set of long but relatively simple calculations involving s_{ij} and $\tilde{s_{ij}}$. These produce the signs as expected.

Now notice that these proofs do not in fact involve the matrices in $H_3(\mathbb{K})$ since the change between compactness and non-compactness does not involve $\mathbb{K}_1$ but only $\mathbb{K}_2$. Thus since the orthogonal compact subspaces of $F_{4,2}$, $E_{6,2}$ and $E_{7,2}$ are

$$\tilde{\mathfrak{p}_1} = H_3'(\mathbb{R}) \otimes \{\tilde{l}, \tilde{il}, \tilde{jl}, \tilde{kl}\} \dot{+} \tilde{\mathfrak{p}}_{G_2}$$

$$\tilde{\mathfrak{p}_1} = H_3'(\mathbb{C}) \otimes \{\tilde{l}, \tilde{il}, \tilde{jl}, \tilde{kl}\} \dot{+} \tilde{\mathfrak{p}}_{G_2}$$

$$\tilde{\mathfrak{p}_1} = H_3'(\mathbb{H}) \otimes \{\tilde{l}, \tilde{il}, \tilde{jl}, \tilde{kl}\} \dot{+} \tilde{\mathfrak{p}}_{G_2},$$

the proofs for these maximal compact subalgebras are contained within that of E_8 and can be directly derived from the E_8 proof by changing $\mathbb{O}$ for $\mathbb{C}$ or $\mathbb{H}$ as appropriate.

Thus we have covered all of the subalgebras and our proof is complete. $\square$

CHAPTER 5

Invariant tensors

The branch of mathematics known as classical invariant theory has many applications, especially in the area of mathematical physics. Classical invariant theory (as may be expected) investigates the invariants of an algebra or group. An invariant of an algebra is so-called since it remains invariant (or stays the same) when acted on by the adjoint action of any element of the algebra. This concept can also be extended in the following way. If $\mathfrak{g}$ is a Lie algebra and G is its associated Lie group, then a function $f : \mathfrak{g} \to \mathfrak{g}$ is Lie algebra invariant if

$$f(gXg^{-1}) = gf(X)g^{-1} \tag{91}$$

for all $X \in \mathfrak{g}$ and $g \in G$. There are two key sets of identities concerned with the invariants of an algebra, namely the *Casimir elements* and the *Capelli Identities.*

The Capelli identities, as applied to Lie algebras, are a way of finding the generators of the centre of the *universal enveloping algebra* of the Lie algebra. The universal enveloping algebra of a Lie algebra, $\mathfrak{g}$, is the associative algebra formed by all possible products of the generators of the Lie algebra, denoted $\mathrm{U}(\mathfrak{g})$. The Capelli identites are some of the best known results from classical invariant theory and were introduced by a series of papers published between 1887 and 1890 [**6, 7, 8**]. Casimir operators are also used for a similar purpose, namely to find generators for the centre of the universal enveloping algebra of $\mathfrak{g}$. They were introduced somewhat later, around 1930, [**9**]. and are of great interest in theoretical particle physics.

Both types of element are well known for the classical Lie algebras and are known in part for the exceptional Lie algebras. However no-one has previously attempted to obtain such identites in terms of the tensors given by the matrix representation of the exceptional Lie algebras found within the magic square.

In this chapter we will explain the background to both the Casimir elements and the Capelli identities and how they arise in Lie algebras. As an example we shall find the Casimir elements for G_2 for the representation that has been used throughout this thesis. We start by introducing the concept of a symmetric tensor (known as a d-tensor) of an algebra, by means of the $\mathfrak{su}(3)$ example.

1. d-tensors

A tensor which has the properties that it is

1. symmetric
2. invariant under the adjoint action of the Lie group

is known as a d-tensor. These tensors tend to arise in calculations involving the anti-commutator (mainly since the anticommutator is naturally symmetric). Since it is not possible to define a general formula for a d-tensor that holds in every algebra we consider a specific example.

In the case of $\mathfrak{su}(n,\mathbb{C})$ it is true that for all X_i, X_j that are generators of $\mathfrak{su}(n,\mathbb{C})$ the anticommutator $\{X_i, X_j\}$ is hermitian. Thus $i\{X_i, X_j\}$ must be antihermitian and therefore since the multiplication will be closed, another generator of $\mathfrak{su}(n,\mathbb{C})$. Consequently

$$i\{X_i, X_j\} = d_{ijk}X_k.$$

Then d_{ijk} is a d-tensor because

1. d_{ijk} is symmetric. This is because

$$\begin{aligned} d_{ijk} &= \langle X_k, \{X_i, X_j\}\rangle \\ &= \tfrac{1}{3}\,\mathrm{tr}[X_k\{X_i, X_j\}] \end{aligned}$$

is symmetric since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

2. d_{ijk} is invariant under the adjoint action of $\mathrm{SU}(n)$. This is due to the fact that the equation

$$[X_l, \{X_i, X_j\}] = \{[X_l, X_i], X_j\} + \{X_i, [X_l, X_j]\} \tag{92}$$

holds since $\mathbb{C}$ is associative.

As another example consider $\mathfrak{sa}(3, \mathbb{K})$ for associative $\mathbb{K}$. For $\mathbb{K} = \mathbb{C}$ this is merely the $\mathfrak{su}(3)$ case considered above. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$ the symmetrised product of three matrices will give a symmetric invariant tensor. The totally symmetrised product of three matrices will be written $X_{(i}X_jX_{k)}$. Then $X_{(i}X_jX_{k)}$ will give another generator of $\mathfrak{su}(3, \mathbb{K})$ and

$$X_{(i}X_jX_{k)} = d_{ijkl}X_l.$$

Again d_{ijkl} is a totally symmetric tensor that is invariant under the group action of $\mathrm{SU}(3, \mathbb{K})$.

2. Casimir Operators

Let $\mathfrak{g}$ be a Lie algebra and let $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then a *Casimir element* is an element in the centre of $\mathrm{U}(\mathfrak{g})$, denoted $\mathrm{Z}(\mathfrak{g})$, defined by

$$C(X) = \sum_{i_1,\ldots,i_n} t_{i_1,\ldots,i_n} X_{i_1} \ldots X_{i_n} \tag{93}$$

where $t_{i_1,\ldots,i_n}$ is a symmetric invariant tensor such that

$$[X_i, C(X)] = 0$$

and $[,]$ is the commutator

$$X_iC(X) - C(X)X_i = 0.$$

The statement that $C(X)$ is a Casimir element of $\mathfrak{g}$ is directly equivalent to the equation

$$f_{ii_1}^{k_1} t_{k_1,i_2,\ldots,i_n} + \cdots + f_{ii_n}^{k_n} t_{i_1,\ldots,k_n} = 0. \tag{94}$$

This is due to the fact that

$$[X_j, X_{i_1,\ldots,i_n}] = \sum_{j=i_1}^{i_m} X_{i_1} \ldots X_{i_{m-1}}[X_j, X_{i_m}]X_{i_{m+1}} \ldots X_{i_m}.$$

In some literature reference is made to *the* Casimir element. In this case the Casimir element is defined to be

$$C(X) = \sum_{ij} g^{ij} X_i X_j \tag{95}$$

where g^{ij} is the Cartan-Killing tensor for the Lie algebra $\mathfrak{g}$. This element always exists and is always a Casimir element. We shall not be using this wording in this thesis as this element is of little consequence since it always exists. Instead we shall concentrate on higher order Casimirs since these are unknown.

The *Rank Theorem* (see [**14**]) gives us information about the number of Casimir elements we expect to find in the case that $\mathfrak{g}$ is a semi-simple Lie algebra.

THEOREM 15 (Rank Theorem). Every semi-simple Lie algebra of rank l has exactly l independent invariants C_i called *Casimir Invariants.*

Further it has been shown by Mountain [**23**] that the Casimir operators are in a one-one correspondence with the symmetric invariant tensors, or d-tensors of the algebra. It is this fact that will be used as we start to attempt to calculate Casimir operators for our representations of the exceptional Lie algebras.

The degree of the Casimir elements are well known for both the classical and the exceptional Lie algebras and can be found in the following table which is reproduced from [**1**].

	Algebra dimension	order of invariants
A_n	$n(n+2)$	$2, 3, \ldots, n+1$
B_n	$n(2n+1)$	$2, 4, \ldots, 2n$
C_n	$n(2n+1)$	$2, 4, \ldots, 2n$
D_n	$n(2n-1)$	$2, 4, \ldots, 2n-2, n$
G_2	14	$2, 6$
F_4	52	$2, 6, 8, 12$
E_6	78	$2, 5, 6, 8, 9, 12$
E_7	133	$2, 6, 8, 10, 12, 14, 18$
E_8	248	$2, 8, 12, 14, 18, 20, 24, 30$

Whilst the Casimir elements are well documented for the classical Lie algebras (e.g. [**1, 2, 23**]), little progress has been made on explicit formulae for the exceptional Lie algebras and has hinged mainly around computer based calculations (see for example [**5, 4**]).

3. Casimir elements of G_2

As an example of d-tensors and Casimir elements we consider the representation of G_2 used throughout this thesis and find the Casimir elements for it. The method will require the use of the following well known theorem.

THEOREM 16 (Cayley-Hamilton). Any matrix A satisfies its own characteristic equation, i.e.

$$A^n - \Delta_1(A)A^{n-1} + \Delta_2(A)A^{n-2} - \cdots + (-1)^n\Delta_n(A) = 0 \tag{96}$$

where $\Delta_r(x)$ is defined to be the co-efficient of x^r in the expansion of $\det(A - x\mathbb{1})$, which can be calculated from

$$\Delta_r(A) = \tfrac{1}{r}\sum_{s=0}^{r-1}(-1)^{r-s+1}\Delta_s(A)\,\mathrm{tr}(A^{r-s}). \tag{97}$$

Consequently

$$\mathrm{tr}(A^n) - \Delta_1(A)\,\mathrm{tr}(A^{n-1}) + \Delta_2(A)\,\mathrm{tr}(A^{n-2}) - \cdots + (-1)^n\Delta_n(A) = 0.$$

The co-efficients $\Delta_r(A)$ are called matrix invariants of A, since under the action of any matrix $M \in \mathrm{GL}(n)$ such that $A \mapsto MAM^{-1}$, $\Delta_r(A)$ remains invariant. This equation gives the ability to express the trace of higher order powers of matrices in the 7-dimensional representation of G_2 in terms of $\mathrm{tr}(A^2)$ and $\mathrm{tr}(A^6)$, the expected invariants of G_2.

Now using (97) to calculate the values of $\Delta_r(A)$ for any matrix in our representation of G_2 gives

$$\Delta_0 = 1 \qquad \Delta_4 = \tfrac{1}{16}(\mathrm{tr}\, A^2)^2$$

$$\Delta_1 = 0 \qquad \Delta_5 = 0$$

$$\Delta_2 = -\tfrac{1}{2}\,\mathrm{tr}\, A^2 \qquad \Delta_6 = \tfrac{1}{6}(-\,\mathrm{tr}\, A^6 + \tfrac{1}{16}(\mathrm{tr}\, A^2)^3)$$

$$\Delta_3 = 0.$$

(Since this representation of G_2 is antisymmetric all traces of odd powers of any element of G_2 will vanish.) This uses the identity which holds on G_2

$$\mathrm{tr}\, A^4 = \tfrac{1}{4}(\mathrm{tr}\, A^2)^2 \tag{98}$$

which can be found in [**27**].

Thus the Cayley Hamilton equation that any element of G_2 will satisfy is

$$A^7 - \tfrac{1}{2}(\mathrm{tr}\, A^2)A^5 + \tfrac{1}{16}(\mathrm{tr}\, A^2)^2 A^3 - \tfrac{1}{6}(\mathrm{tr}\, A^6 - \tfrac{1}{16}(\mathrm{tr}\, A^2)^3)A = 0. \tag{99}$$

A further theorem is also required.

THEOREM 17. If $X \in G_2$, then $X^5 - \frac{5}{12}(\mathrm{tr}\, X^2)X^3 \in G_2$ also.

PROOF. Use the fact that any element of $\mathfrak{so}(7)$ can be written as $X = \mathrm{diag}(a\epsilon, b\epsilon, c\epsilon, 0)$ where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (so that $\epsilon^2 = -\mathbb{1}$). If $X \in G_2$ then from the $\mathfrak{su}(3)$ isomorphism given on page 33, $a + b + c = 0$. Thus

$$X^5 - \tfrac{5}{12}(\mathrm{tr}\, X^2)X^3 = \mathrm{diag}((a^5 - \tfrac{5}{6}(a^2 + b^2 + c^2)a^3)\epsilon, (b^5 - \tfrac{5}{6}(a^2 + b^2 + c^2)b^3)\epsilon,$$
$$(c^5 - \tfrac{5}{6}(a^2 + b^2 + c^2)c^3)\epsilon, 0).$$

If this is an element of G_2 then the co-efficients of ϵ sum to zero, i.e. it should be true that

$$(a^5 + b^5 + c^5) - \tfrac{5}{6}(a^2 + b^2 + c^2)(a^3 + b^3 + c^3) = 0. \tag{100}$$

Using the fact that $a + b + c = 0$, write $c = (-1)(a + b)$. Then if LHS is the notation used to denote the left hand side of equation (100),

$$\begin{aligned} LHS &= a^5 + b^5 - (a+b)^5 - \tfrac{5}{6}(a^2 + b^2 + (a+b)^2)(a^3 + b^3 - (a+b)^3) \\ &= -(5ab^4 + 5a^4b + 10a^2b^3 + 10a^3b^2) + \tfrac{5}{6}(2a^2 + 2b^2 + 2ab)(3ab^2 + 3ba^2) \\ &= -5ab(b^3 + a^3 + 2ab^2 + 2a^2b) + 5ab(b^3 + a^3 + 2ab^2 + 2a^2b) \\ &= 0. \end{aligned}$$

Thus $X^5 - \frac{5}{12}(\operatorname{tr} X^2)X^3 \in G_2$. □

Now, define a d-tensor for G_2 by the following. If $A = a_i X_i$ where the X_i are a basis for G_2 (previously written d_i) then

$$A^5 - \tfrac{5}{12}(\operatorname{tr} A^2)A^3 = d_{i_1 \dots i_6} a_{i_1} \dots a_{i_5} X_{i_6}. \tag{101}$$

Then $d_{i_1 \dots i_6}$ is a totally symmetric tensor. It is also shown to be invariant in the following way. Define f to be the function $f : G_2 \to G_2$ by $f(X) = X^5 - \frac{5}{12}(\operatorname{tr} X^2)X^3$, then using the definition of invariance given in equation (91),

$$\begin{aligned} f(gXg^{-1}) &= (gXg^{-1})^5 - \tfrac{5}{12}(\operatorname{tr}(gXg^{-1})^2)(gXg^{-1})^3 \\ &= gX^5g^{-1} - \tfrac{5}{12}(\operatorname{tr} gX^2g^{-1})gX^3g^{-1} \end{aligned}$$

where g is an element of the *group* G_2. Since $\operatorname{tr} AB = \operatorname{tr} BA$, it follows that $\operatorname{tr} gX^2g^{-1} = \operatorname{tr} X^2$ and thus

$$\begin{aligned} f(gXg^{-1}) &= gX^5g^{-1} - \tfrac{5}{12}(\operatorname{tr} X^2)gX^3g^{-1} \\ &= g(X^5 - \tfrac{5}{12}(\operatorname{tr} X^2)X^3)g^{-1} \\ &= gf(X)g^{-1}. \end{aligned}$$

Hence f is an invariant function on G_2 and thus the tensor d defined by

$$f(X) = d_{i_1\ldots i_6} a_{i_1\ldots i_5} X_{i_6}$$

is invariant.

Define the rth-rank tensor $d^{(r)}_{i_1\ldots i_r}$ by

$$d^{(r+1)}_{i_1\ldots i_{r+1}} = d^{(r-3)}_{i_1\ldots i_{r-4}j} d_{ji_{r-3}\ldots i_{r+1}}, \tag{102}$$

and the rth rank symmetrical tensor $d^{(r)}_{(i_1\ldots i_r)}$ by the equation

$$d^{(r+1)}_{(i_1\ldots i_{r+1})} = d^{(r-3)}_{(i_1\ldots i_{r-4}j)} d_{(ji_{r-3}\ldots i_{r+1})}, \tag{103}$$

where $_{()}$ indicates the symmetrisation of d in all indices contained within the brackets. The Cayley-Hamilton equation defined in (99) is then used to show that $d_{i_1\ldots i_6}$ combined with $\delta_{i_1 i_2}$ gives all higher order tensors expressed in terms of these rank 6 and rank 2 tensors. To summarise

$$\begin{aligned}
\delta_{(i_1 i_2)} = d^{(2)}_{(i_1 i_2)} &= \operatorname{tr}(X_{(i_1} X_{i_2)}) \\
d^{(4)}_{(i_1 i_2 i_3 i_4)} &= \tfrac{1}{4}\delta_{(i_1 i_2}\delta_{i_3 i_4)} = \tfrac{1}{4}\operatorname{tr}(X_{(i_1} X_{i_2} X_{i_3} X_{i_4)}) \\
d_{(i_1 i_2 i_3 i_4 i_5 i_6)} &= \operatorname{tr}(X_{(i_1} \ldots X_{i_6)})
\end{aligned}$$

and all odd tensors of any order are zero. Consequently the Casimir elements of order 2 and order 6 are primitive and all other Casimirs will be non-primitive (and consequently will comprise of sums of products of order 2 and 6 Casimirs).

For example, from (99)

$$A^8 = \tfrac{1}{2}(\operatorname{tr} A^2)A^6 - \tfrac{1}{16}(\operatorname{tr} A^2)^2 A^4 + \tfrac{1}{6}(\operatorname{tr} A^6 - \tfrac{1}{16}(\operatorname{tr} A^2)^3)A^2.$$

Thus applying all simplification possible,

$$\operatorname{tr} A^8 = \tfrac{2}{3}(\operatorname{tr} A^2)(\operatorname{tr} A^6) - \tfrac{5}{192}(\operatorname{tr} A^2)^4.$$

Writing this in terms of the tensors defined above

$$d^{(8)}_{(i_1 i_2 i_3 i_4 i_5 i_6 i_7 i_8)} = \tfrac{2}{3}\delta_{(i_1 i_2} d_{i_3 i_4 i_5 i_6 i_7 i_8)} - \tfrac{5}{192}\delta_{(i_1 i_2}\delta_{i_3 i_4}\delta_{i_5 i_6}\delta_{i_7 i_8)}. \tag{104}$$

Furthermore any higher rank tensor can be expressed in terms of $d_{i_1 \ldots i_6}$ and δ_{ij} using this formula. Since completing this work it has been independently confirmed by the work of MacFarlane and Pfeiffer in their recent paper [**22**].

Using the recursive definition of a higher rank matrix it can be shown that

$$d^{(10)}_{(i_1 \ldots i_{10})} = d_{(i_1 \ldots i_5 j} d_{j i_6 \ldots i_{10})}.$$

Calculating $\operatorname{tr} A^{10}$ gives

$$\operatorname{tr} A^{10} = \tfrac{15}{48} (\operatorname{tr} A^2)^2 \operatorname{tr} A^6 - \tfrac{1}{64} (\operatorname{tr} A^2)^5$$

which means that we can write the formula for the contraction of one index of the sixth rank d-tensor as

$$d_{(i_1 \ldots i_5 j} d_{j i_6 \ldots i_{10})} = \tfrac{15}{48} \delta_{(i_1 i_2} \delta_{i_3 i_4} d_{i_5 \ldots i_{10})} - \tfrac{111}{2304} \delta_{(i_1 i_2} \cdots \delta_{i_9 i_{10})}. \tag{105}$$

4. The Capelli Identities

Consider the Lie algebra $\mathfrak{gl}(m)$ to have generators E_{ij}. Then the associated group $\mathrm{GL}(m)$ will have a natural action on the tensor product $\mathbb{C}^m \times \mathbb{C}^n$. We consider the tensor product $\mathbb{C}^n \times \mathbb{C}^m$ to be n copies of $\mathbb{C}^m$, which we can consider as an $m \times n$ matrix, $X = x_{ia}$.Then an element A, of $\mathrm{GL}(m)$ acts on X by normal matrix multiplication, i.e. $X \to AX$. Then there will be $l = \min(n, m)$ Capelli identities, one for each element $k = 1, \ldots, l$. We do not intend to derive this identity again, merely to state it and its interest to us. The kth Capelli element, Ω_k (which forms the left hand side of the Capelli Identity) can be written as

$$\frac{1}{k!} \sum_{\sigma \in S_k} \sum_{i_1, \ldots, i_k} \operatorname{sgn}(\sigma) \cdot E_{i_1 \sigma(1)} (E_{i_2 \sigma(2)} + \delta_{i_2 \sigma(2)}) \cdots$$
$$\cdots (E_{i_3 \sigma(3)} + 2\delta_{i_3 \sigma(3)}) \cdots (E_{i_k \sigma(k)} + (k-1)\delta_{i_k \sigma(k)}),$$

where $\operatorname{sgn}(\sigma)$ is the sign of the element σ from the symmetric group S_k, i.e. we are considering all permutations of $1, \ldots, k$, and δ_{ij} is the Kronecker delta.

We can also write Ω_k as

$$\frac{1}{k!}\sum_{\sigma\in S_k}\sum_{i_1,\dots,i_k}\operatorname{sgn}(\sigma)\cdot\prod_{s}\left(E_{i_s\sigma(s)}+(s-1)\delta_{i_s\sigma(s)}\right). \tag{106}$$

Then Ω_k is an element of the universal enveloping algebra of $\mathfrak{gl}(N)$, $U(\mathfrak{gl}(m))$. moreover, it is one of the generators of the centre of $U(\mathfrak{gl}(m))$. which is $Z(\mathfrak{gl}(m))$.

The remarkable thing is yet to come. If we take a representation of $U(\mathfrak{gl}(m))$, $\rho : U(\mathfrak{gl}(m)) \to \mathcal{PD}(x_{ia})$, where $\mathcal{PD}(x_{ia})$ is the ring of polynomial differential operators with coefficients x_{ia}, with

$$\rho(E_{ij}) = \sum_{a=1}^{n} x_{ia}\partial_{ja} \tag{107}$$

then we can apply ρ to Ω_k. This then gives a formula (a Capelli identity) for an invariant in $\mathcal{PD}(x_{ia})$. To be precise, it gives a series of canonical formulas giving some of the invariants in $\mathcal{PD}(x_{ia})$, although there may be more that the Capelli identities do not give. The right hand side of the Capelli identity is then

$$\frac{1}{k!}\sum_{\sigma\in S_k}\sum_{a_1,\dots,a_k}\sum_{i_1,\dots,i_k}\operatorname{sgn}(\sigma)x_{a_1i_1}\dots x_{a_ki_k}\partial_{a_1i_{\sigma(1)}}\dots\partial_{a_ki_{\sigma(k)}}$$

where $a_1,\dots,a_k$ and $i_1,\dots,i_k$ run respectively through $1,\dots,n$ and $\dots,m$. This can also be written as

$$\Omega_k = \sum_{a_1<\dots<a_k}\sum_{i_1<\dots<i_k}\det[x_{a_pi_q}]\det[\partial_{a_pi_q}] \tag{108}$$

where $p,q=1,\dots,k$. Notice that in all these equations $\Omega_k = 0$ for all $k > l$.

CHAPTER 6

Methods for Further Research

In this chapter we present a series of ideas and techniques which may be useful in evaluating the Casimir elements for the remainder of the exceptional Lie algebras, using their 3×3 and 6×6 matrix formulations. We begin by looking at a method for finding d-tensors that is a direct extension of the method used for $\mathfrak{su}(3)$. We then consider the method of finding invariant polynomials as given by Macfarlane and Pfeiffer [**22**] and show how this may possibly be extended to the exceptional Lie algebras, at least in the case of F_4.

We go on to look at the possibility of formulating a Cayley-Hamilton theorem for matrices with octonionic entries, which leads to a consideration of the eigenvalue problem for such matrices. The chapter is finished by giving a suggestion of the best way forward to attack this problem.

1. Completeness relations

In Chapter 5 the concept of a d-tensor was introduced as an invariant, symmetric tensor which arose as a result of the anti-commutator in identites which can be referred to as *completeness relations*. The examples of d-tensors for $\mathfrak{sa}(3,\mathbb{R})$, $\mathfrak{sa}(3,\mathbb{C})$ and $\mathfrak{sa}(3,\mathbb{H})$ were all given.

The problem now is the extension of this to octonions. The octonions are not associative and consequently the identity given in (92) will no longer hold. If this could be adapted in some way to include the octonions (possibly by the use of an "error term" consisting of derivations) then this method could be used to identify d-tensors in F_4. Furthermore the method used for $\mathfrak{sa}(3,\mathbb{K})$ must then be extended to incorporate $\mathfrak{sp}(3,\mathbb{K})$ and $\mathfrak{sl}(3,\mathbb{K})$ which will then yield symmetric invariant tensors for E_6 and E_7, as well as F_4.

We begin by considering $F_4 \cong \mathfrak{sa}(3, \mathbb{O})$. In this case the smallest primitive Casimir element (other than the trivial Casimir element of order 2) is of order 6. From this it can be deduced that the d-tensor should arise from the anti-commutation (or totally symmetrised multiplication) of five elements of F_4. Moreover, since multiplication in $\mathbb{O}$ is not associative this will also have to be accounted for, thus the notation $X_{[i_1} \ldots X_{i_n]}$ will be used to indicate the totally symmetrised product (with respect to multiplication and associativity) of the matrices $X_{i_1}, \ldots, X_{i_n}$. In the case of three matrices this is

$$X_{[i_1}X_{i_2}X_{i_3]} = \{X_{i_1}, \{X_{i_2}, X_{i_3}\}\} + \{\{X_{i_1}, X_{i_2}\}, X_{i_3}\}.$$

An attempt was made to use this method to start trying to find d-type tensors for F_4. By considering the order 4 tensor to try to find an identity involving $d_{i_1 i_2 i_3 i_4}$ we hoped to see the way to extend this to the order 6 tensor. However, when the fourth order tensor was considered the "error term" was not recognisable in terms of derivations. Consequently the equation was quickly going to become difficult to manipulate for any larger number of matrices.

2. An extension to the method of Macfarlane and Pfeiffer

In a recent paper, Macfarlane and Pfeiffer presented a method for finding the invariants of Lie algebras using matrix representations [22]. It appears that this method could be used in the case of octonionic representations. Since any $A \in \mathfrak{su}(3)$ can be diagonalised, a change of basis will give

$$A = \alpha\lambda_3 + \tfrac{\beta}{2}(\lambda_3 + \sqrt{3}\lambda_8) = \mathrm{diag}(\alpha + \beta, -\alpha, -\beta), \tag{109}$$

(with λ_3 and λ_8 defined as in Chapter 1). Macfarlane and Pfeiffer show that this leads to the equations

$$\begin{aligned}
\operatorname{tr} A^2 &= 2(\alpha^2 + \beta^2 + \alpha\beta) \\
\operatorname{tr} A^3 &= 3(\alpha^2\beta + \alpha\beta^2) \\
\operatorname{tr} A^4 &= 3(\alpha^4 + 2\alpha^3\beta + 3\alpha^2\beta^2 + 2\alpha\beta^3 + \beta^4).
\end{aligned}$$

These equations show that $\operatorname{tr} A^2$ and $\operatorname{tr} A^3$ can be taken as the generators of the algebra of invariant polynomials and that

$$\operatorname{tr} A^4 = \tfrac{1}{2}(\operatorname{tr} A^2)^2.$$

Similar calculations show that $\operatorname{tr} A^k$ can be expressed in terms of $\operatorname{tr} A^2$ and $\operatorname{tr} A^3$ for all k. These can be confirmed for $\mathfrak{su}(3)$ by use of the Cayley-Hamilton Theorem.

The extension of this to the octonionic case at first seems obvious. Similar equations to those given above are expected, however because of the non-associativity of $\mathbb{O}$ these will have an additional part which consists of derivations.

For example, in F_4 let $\lambda_3 = \operatorname{diag}(1,-1,0)$ and $\lambda_8 = \operatorname{diag}\frac{1}{\sqrt{3}}(1,1,-2)$. Then any matrix A in the $A_3'(\mathbb{O})$ part of $\mathfrak{su}(3,\mathbb{O})$ can be written as $A = \alpha\lambda_3 + \frac{\beta}{2}(\lambda_3 + \sqrt{3}\lambda_8) = \operatorname{diag}(\alpha+\beta, -\alpha, -\beta)$, where now α and β are octonionic and therefore non-associative and non-commutative. Then

$$\begin{aligned}\operatorname{tr} A^2 &= 2\alpha^2 + 2\beta^2 + \alpha\beta + \beta\alpha\\ \operatorname{tr} A^3 &= \alpha\beta^2 + \alpha^2\beta + \alpha(\beta\alpha) + \beta\alpha^2 + \beta(\alpha\beta) + \beta^2\alpha.\end{aligned}$$

Using this method $\operatorname{tr} A^4$ was also calculated and it was found that it can be expressed as

$$\operatorname{tr} A^4 - \tfrac{1}{2}(\operatorname{tr} A^2)^2 = -\{\alpha^2,\beta^2\} + \tfrac{1}{2}\{\alpha,\beta\}^2. \tag{110}$$

If the right hand side of this equation could be identified in terms of derivations then this could give an identity for $\operatorname{tr} A^4$. Unfortunately no such identification has yet been made. Another calculation yielded an identity for F_4 similar to the identity given in Thoerem 17 for G_2, namely that since

$$\begin{aligned}\operatorname{tr}(A^5 - \tfrac{1}{2}(\operatorname{tr} A^2)A^3) = \alpha^5 + \beta^5 - (\alpha+\beta)^5\\ - \tfrac{1}{2}(2\alpha^2 + 2\beta^2 + \alpha\beta + \beta\alpha)(\alpha^3 + \beta^3 - (\alpha+\beta)^3)\end{aligned}$$

(where the fraction $\frac{1}{2}$ is chosen to minimise the right hand side), then

$$\begin{aligned}\operatorname{tr}(A^5 - \tfrac{1}{2}(\operatorname{tr} A^2)A^3) = &\\ &- \alpha^2\beta^3 - \alpha\beta^4 - \beta\alpha\beta^3 - \beta^2\alpha^3 - \beta^4\alpha\\ &- \tfrac{1}{2}\{\alpha\beta\alpha\beta^2 + \alpha\beta\alpha^2\beta + \alpha\beta\alpha\beta\alpha + \alpha\beta^2\alpha^2 + \alpha\beta^2\alpha\beta + \alpha\beta^3\alpha\}\\ &- \tfrac{1}{2}\{\beta\alpha\beta^2 + \beta\alpha^3\beta + \beta\alpha^2\beta\alpha + \beta\alpha\beta\alpha^2 + \beta\alpha\beta\alpha\beta + \beta\alpha\beta^2\alpha\}.\end{aligned}$$

Again an identification of the right hand side of this equation in terms of derivations cannot be made.

This lack of such an identification may however be due to the fact that the elements A under consideration are only elements of the matrix part of the algebra and do not include the derivations. More explicitly, it is shown in Appendix A that any element of $\mathfrak{sa}(3,\mathbb{K})$ can be written in the form $(AB - BA)' + \frac{1}{3}D(A,B)$ where A and B are matrices in $H_3(\mathbb{O})$. Consequently the way forward with this technique may be to consider these things as a whole rather than in two totally separate parts.

3. The Cayley-Hamilton Method

The difficulty in using both of the above methods to compute the Casimir elements of the exceptional Lie algebras could be simplified if there were a Cayley-Hamilton theorem for matrices that have octonionic entries, i.e. a Cayley Hamilton equation for non-associative and non-commutative matrices. If such an equation could be worked out then the method used to find the invariant elements of G_2 in the previous chapter could be applied again to find the invariant elements of the remaining exceptional Lie algebras. However, such a theorem would be complicated and at present does not exist.

The reason for this complication is due to the non-commutivity of the octonions, which means that the equation defining an eigenvalue of an octonionic matrix could be

$$A\mathbf{v} = \lambda\mathbf{v} \tag{111}$$

but could also be

$$A\mathbf{v} = \mathbf{v}\lambda \tag{112}$$

and since the octonions are non-commutative these are two totally separate problems.

The eigenvalue problem normally considered to be most important is the right eigenvalue problem. This is because, whereas left multiples of eigenvectors by octonions are not eigenvectors, right multiples of eigenvectors by octonions are still eigenvectors [**10**]. Dray and Manogue [**10**] have published a series of results (computed using the computer algebra package Mathematica) for hermitian matrices and it is these results we present here. These have been confirmed, in part, by hand calculations performed by Okubo, which can be found in [**28**]. As far as we are aware there are no published results for antihermitian matrices. We give a brief survey of the known results for 3×3 hermitian matrices, taken from [**10**].

3.1. The Right Hermitian Octonionic Eigenvalue Problem. Firstly, note that 3×3 octonionic hermitian matrices will satisfy the Cayley Hamilton equation for 3×3 matrices, in the sense that if $A \in H_3(\mathbb{O})$, then

$$A^3 - (\operatorname{tr} A)A^2 + \sigma(A)A - (\det A)I = 0 \tag{113}$$

where $A^3 = A \cdot (A \cdot A) + (A \cdot A) \cdot A$, $A^2 = A \cdot A$ and $\sigma(A) = \frac{1}{2}((\operatorname{tr} A)^2 - \operatorname{tr}(A^2))$. It may be suprising that A has a uniquely defined determinant. This is defined in terms of the Freudenthal product, which is

$$A \star B = A \cdot B - \tfrac{1}{2}(A \operatorname{tr} B + B \operatorname{tr} A) + \tfrac{1}{2}(\operatorname{tr} A \operatorname{tr} B - \operatorname{tr} B \operatorname{tr} A). \tag{114}$$

Then a determinant can be defined for any matrix $A \in H_3(\mathbb{O})$ by

$$\det A = \tfrac{1}{3}\operatorname{tr}((A \star A) \cdot A). \tag{115}$$

Explicitly if $A = \begin{pmatrix} \alpha & a & \bar{b} \\ \bar{a} & \beta & c \\ b & \bar{c} & \gamma \end{pmatrix}$, then

$$\det A = \alpha\beta\gamma + b(ac) + \overline{b(ac)} - \gamma|a|^2 - \beta|b|^2 - \alpha|c|^2. \tag{116}$$

Whilst the matrix A satisfies its characteristic equation, its eigenvalues do not. They do, however, satisfy a modified version, namely

$$\lambda^3 - (\operatorname{tr} A)\lambda^2 + \sigma(A)\lambda - \det A = r, \tag{117}$$

where if A has the matrix elements as given above, r is one of the two solutions of the equation

$$r^2 - 4\Phi(a, b, c)r - |[a, b, c]|^2 = 0$$

where $\Phi(a, b, c) = \frac{1}{2}\operatorname{Re}([a, \bar{b}]c)$. Thus there will be two sets of three real eigenvalues of A. Each of the eigenvalues will admit four independent (over $\mathbb{R}$) eigenvectors. For a proof of this see [**10**].

These eigenvectors will be orthogonal in a loose sense, as defined in the following Lemma.

LEMMA 12. If $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of the 3×3 octonionic hermitian matrix A corresponding to different real eigenvalues in the same r family (i.e. both are solutions of the modified characteristic equation with the same value of r) then $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal in the sense that

$$(\mathbf{v}\mathbf{v}^\dagger)\mathbf{w} = 0.$$

Furthermore a hermitian 3×3 octonionic matrix can be expressed in the form

$$A = \sum_{m=1}^{3} \lambda_m \mathbf{v}_m \mathbf{v}_m^\dagger$$

where $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are mutually orthogonal (in the sense given in the previous lemma) eigenvectors of A corresponding to the real eigenvalues λ_m which have the same r value.

Thus we have a Cayley-Hamilton equation for 3×3 hermitian matrices, and also a method of diagonalisation.

4. Discussion

As demonstrated throughout this chapter none of these methods have so far bought any useful results. The non-associativity of $\mathbb{O}$ leads to large error terms, which whilst they would be manageable if they could be expressed in terms of derivations, so far have not yielded to such an expression. Whilst consideration of the matrices in F_4 in terms of functions $[L_A, L_B]$ may lead to a result for F_4 it is not at all clear that this would then extend further to any of the other exceptional Lie algebras. As one of the motivations behind this thesis was to provide a unified method for finding Casimirs of exceptional Lie algebras this would not seem promising.

Another problem with extending the Macfarlane-Pfeiffer method to the exceptional algebras may be the lack of uniqness in diagonalisation. If this could be solved with a method similar to that of Dray and Manogue by using eigenvalues of anti-hermitian matrices, these methods combined may give a way forward. However, this method would rely heavily on a method being developed for finding eigenvalues of octonionic 3×3 matrices, which has not proved simple. Should this method be developed much progress could be made with the solution of this problem.

APPENDIX A

Matrix Identities

This appendix proves various lemmas which are utilised throughout the thesis. They will use the function $E(X,Y) \in \mathfrak{so}(\mathbb{K})$ defined for any 3×3 matrices X, Y by

$$E(X,Y)z = \sum_{ij}[x_{ij}, y_{ji}, z]. \tag{118}$$

LEMMA 13. Let $\mathbb{K}$ be a composition algebra, let H, K and L be hermitian 3×3 matrices with entries from $\mathbb{K}$, and let X, Y be traceless antihermitian matrices over $\mathbb{K}$. In these matrix identities the square brackets denote commutators and the chain brackets denote anticommutators.

$$[X, \{H, K\}] = \{[X, H], K\} + \{H, [X, K]\} \tag{119}$$

$$[X, [Y, H]] - [Y, [X, H]] = [[X, Y], H] + E(X, Y)H \tag{120}$$

$$\{H, \{K, L\}\} - \{K, \{H, L\}\} = [[H, K], L] + E(H, K)L. \tag{121}$$

PROOF. The proof is given only for equation (119) since the method is identical in each case.

The difference between the two sides of equation (119) can be written in terms of matrix associators, where the (i, j)th element is

$$\sum_{mn}([x_{im}, h_{mn}, k_{nj}] + [x_{im}, k_{mn}, h_{nj}] + [k_{im}, h_{mn}, x_{nj}] - [h_{im}, x_{mn}, k_{nj}] - [k_{im}, x_{mn}, h_{nj}]). \tag{122}$$

Suppose $i \neq j$ and let k be the third index. Since the diagonal elements of H and K are real, any associator containing them vanishes. Hence the terms containing x_{ij} or x_{ji} are

$$\sum_n([x_{ij}, h_{jn}, k_{nj}] + [x_{ij}, k_{jn}, h_{nj}]) + \sum_m([h_{im}, k_{mi}, x_{ij}] + [k_{im}, h_{mi}, x_{ij}])$$
$$- [h_{ij}, x_{ji}, k_{ij}] + [k_{ij}, x_{ji}, h_{ij}] = 0$$

by the alternative law, the hermiticity of H and K, and the fact that an associator changes sign when one of its elements is conjugated. The terms containing x_{ik} or x_{ki} are

$$[x_{ik}, h_{ki}, k_{ij}] + [x_{ik}, k_{ki}, h_{ij}] - [h_{ik}, x_{ki}, k_{ij}] - [k_{ik}, x_{ki}, h_{ij}] = 0$$

using also $x_{ki} = -\bar{x}_{ik}$. Similarly, the terms containing x_{jk} or x_{kj} vanish. Finally, the terms containing x_{ii}, x_{jj} and x_{kk} are

$$[x_{ii}, h_{ik}, k_{kj}] + [x_{ii}, k_{ik}, h_{kj}] + [h_{ik}, k_{kj}, x_{jj}] + [k_{ik}, h_{kj}, x_{jj}]$$
$$- [h_{ik}, x_{kk}, k_{kj}] - [k_{ik}, x_{kk}, h_{kj}] = 0$$

since $x_{ii} + x_{jj} + x_{kk} = 0$.

Now consider the (i, i)th element. The last two terms of equation (122) become

$$-\sum_{mn}([h_{im}, x_{mn}, k_{ni}] + [k_{in}, x_{nm}, h_{mi}]) = 0.$$

Let j be one of the other two indices. The terms containing x_{ij} or x_{ji} are

$$[x_{ij}, h_{jk}, k_{ki}] + [x_{ij}, k_{jk}, h_{ki}] + [h_{ik}, k_{kj}, x_{ji}] + [k_{ik}, h_{kj}, x_{ji}] = 0,$$

where k is the third index. There are no terms containing x_{jk} or x_{kj}. The terms containing x_{ii}, x_{jj} or x_{kk} are

$$\sum_n([x_{ii}, h_{in}, k_{ni}] + [x_{ii}, k_{in}, h_{ni}])$$
$$+ \sum_m([h_{im}, k_{mi}, x_{ii}] + [k_{im}, h_{mi}, x_{ii}]) = 0.$$

Thus in all cases the expression (122) vanishes, proving (a). □

LEMMA 14. Let A, B be matrices in $A_2'(\mathbb{K})$ and X, Y, Z be matrices in $H_2'(\mathbb{K})$.

$$[A,\{X,Y\}] = \{[A,X],Y\} + \{X,[A,Y]\} \tag{123}$$

$$[A,[B,X]] - [B,[A,X]] = [[A,B],X] + E(A,B)X \tag{124}$$

$$\{X,\{Y,Z\}\} - \{Y,\{X,Z\}\} = [[X,Y],Z] + E(X,Y)Z \tag{125}$$

PROOF. The 2×2 case can be deduced from the 3×3 proof of equation (119) using the matrices

$$\widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & -\operatorname{tr} A \end{pmatrix}, \quad \widetilde{X} = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{Y} = \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix}.$$

□

LEMMA 15. Let A, $B \in H_3(\mathbb{K})$. Then

$$[[L_A,L_B],C] = [(AB-BA)',C] + \tfrac{1}{3}D(A,B)C \tag{126}$$

(Essentially this is proving that any element of $\operatorname{Der} H_3(\mathbb{K})$, $[L_A,L_B]$ [29] can be written as a sum of elements in $A_3'(\mathbb{K}) \dot{+} \operatorname{Der}\mathbb{K}$.)

PROOF. By equation (60),

$$\begin{aligned}[[L_A,L_B],C] &= [A,[B,C]] - [B,[A,C]] \\ &= [[A,B],C] + E(A,B)C.\end{aligned}$$

Take out the trace from the Lie bracket of A and B and let $t = \frac{1}{3}\operatorname{tr}(AB-BA)$. Then

$$\begin{aligned}[[L_A,L_B],C] &= [(AB-BA)',C] + [t,C] + E(A,B)C \\ &= [(AB-BA)',C] + \tfrac{1}{3}D(A,B)C\end{aligned}$$

by equation (63). □

LEMMA 16. Let A, $B \in H_2(\mathbb{K})$. Then

$$[[L_A,L_B],C] = [(AB-BA)',C] + S(A,B)C \tag{127}$$

PROOF. Use exactly the same method as for the 3×3 case. □

LEMMA 17. Matrices with associative entries are associative.

PROOF. Let $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ be $n \times n$ matrices with entries in an associative algebra. Then the (i, j)th entry in $(AB)C$ is $(a_{ik}b_{kl})c_{lj}$. Since all the entries of A, B and C are associative this is precisely $a_{ik}(b_{kl}c_{lj})$ which is the (i, j)th entry of $A(BC)$. □

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