Properties of Delay Systems and Diffusive Systems

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The candidate confirms that the work submitted is her own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

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Abstract

In this thesis, we investigate questions about the properties of delay systems and diffusive systems as well as Hankel and weighted Hankel operators. After detailing the necessary background in Chapter 1, in Chapter 2 the focus is on the development of methods to study the stability of delay and fractional systems. This analysis is carried forward using some BIBO and H^∞ stability tests. Generalisation of the Walton-Marshall method [38] enable us to move from the single and multi-delay cases to fractional delay systems. This method gives procedures for finding stability windows as the delay varies.

Chapter 3 is concerned with diffusive systems. Via convenient adaptations of some tests due to Howland [19], it becomes possible to give necessary and sufficient conditions for the Hankel operator and the weighted Hankel operator to be nuclear. Also, in this Chapter we introduce more general weighted Hankel operators and discuss their boundedness. Here the reproducing kernel test plays an essential role in testing boundedness. Some fundamental examples are given to support our work.

In Chapter 4 here we investigate questions regarding approximating infinite-dimensional linear system by finite-dimensional ones. Moreover, we develop more research on the rate of decay of singular values of the associated Hankel operator.

In Chapter 5 we mainly focus on diffusive systems defined by holomorphic distributions and measures on a half plane. In particular we look at the nuclearity (trace class) and Hilbert-Schmidt properties of such systems. Moreover, we begin further study of explicit examples of weighted Hankel operators for which we did not know whether they were bounded, those examples already introduced in Chapter 3.

In Chapter 6 the boundedness of weighted Hankel corresponding to diffusive

systems is analysed using the theory of Carleson measures.

Chapter 7 gives some suggestions for further work.

Contents

Chapter 1

Background

1.1 Introduction

We begin by recalling some necessary background. There are two main themes with which familiarity will be assumed in later chapters: operator theory and systems theory. This chapter will by no means provide an exhaustive summary of any of these but rather it will serve to equip the reader with basic concepts and results used later. It will rather serve to provide the reader with much of the terminology and conventions that are adopted throughout. There will be no new results in this chapter and so all theorems are stated without proof. Suitable references are [4], [6], [18], [28], [30], [31], [34] and [35].

1.1.1 Notation

$\mathbb{R}_+$ denotes the set of all the real numbers that are greater than zero, $\mathbb{C}_+$ denotes the set of complex numbers with real part strictly greater than zero, and L^∞ denotes the complex-valued measurable functions on the non-negative real axis such that $\operatorname{ess\,sup}_{t\in\mathbb{R}_+} |f(t)| < \infty$. Also $L^p(\mathbb{R}_+)$ denotes the complex-valued measurable functions on the non-negative real axis with $\int_0^\infty |f(t)|^p \, \mathrm{d}t < \infty$.

1.2 Results from Functional Analysis

1.2.1 Banach Spaces

A normed space is a vector space B (assumed to be over the complex number field $\mathbb{C}$) provided with a norm $\|.\|$ satisfying

- $\|f\| \geq 0$,
- $\|f\| = 0$ implies $f = 0$,
- $\|\alpha f\| = |\alpha| \, \|f\|$,
- $\|f + g\| \leq \|f\| + \|g\|$, for all $\alpha \in \mathbb{C}$ and $f, g \in$ B

$\|.\|$ is a seminorm if it satisfies all the axioms except the second.
A Banach space is defined to be a normed space B which is complete in sense that every Cauchy sequence in B converges to a limit in B. Every normed space B has a completion $\overline{B}$, which is a Banach space in which B is embedded isometrically and densely. (An isometric embedding is a linear, norm-preserving (and hence one-one) map of one normed space into another in which every element of the first space is identified with its image in the second).
We now move on to the Hardy spaces, which are in the unit disc $\mathbb{D}$ or the right half-plane $\mathbb{C}_+$ and extended, respectively, to the unit circle $\mathbb{T}$ or the imaginary axis $i\mathbb{R}$.

Definition 1.2.1. *(Inner product).* An inner product space is a vector space V over the field F together with an inner product, i.e., with a map

$$\langle ., . \rangle : V \times V \to F$$

that satisfies the following axioms for all vectors $x, y, z \in V$ and all scalars $a \in F$:

- $\langle x, y \rangle = \overline{\langle x, y \rangle}$.
- $\langle ax, y \rangle = a\langle x, y \rangle$ and $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.
- $\langle x, y \rangle \geq 0$ and $\langle x, y \rangle = 0 \Rightarrow x = 0$.

An inner product $\langle .,. \rangle$ on a vector space induces a norm by means of the formula $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$, and a complete inner-product space is called a Hilbert space.
A linear operator T from a normed space X to a normed space Y is just a linear mapping, that is, it satisfies

$$T(a_1x_1 + a_2x_2) = a_1Tx_1 + a_2Tx_2 \quad \text{for all} \quad x_1, x_2 \in X \quad \text{and} \quad a_1, a_2 \in \mathbb{C}.$$

The operator T is said to be bounded, if there is a constant $k > 0$ such that

$$\|Tx\| \leq k\,\|x\| \quad \text{for all vectors} \quad x \in X.$$

The least k that holds for all x is the norm of T, written

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|\,.$$

1.2.2 Hardy space on the half-plane

For $1 \leq p < \infty$ the Hardy space $H^p(\mathbb{C}_+)$ of the right half-plane $\mathbb{C}_+$ may be defined as the set of all analytic functions $f : \mathbb{C}_+ \to \mathbb{C}$ such that

$$\|f\|_p = (\sup_{x>0} \int_{-\infty}^{\infty} |f(x+iy)|^p \,\mathrm{d}y)^{1/p} < \infty.$$

Likewise, the space $H^\infty(\mathbb{C}_+)$ consists of all analytic and bounded functions in $\mathbb{C}_+$, and the norm is given by

$$\|f\|_\infty = \sup_{z\in\mathbb{C}_+} |f(z)|\,.$$

Those functions have boundary values $\tilde{f}(iy) = \lim_{x\to 0^+} f(x+iy)$ almost everywhere, and the boundary function $\tilde{f}$ lies in $L^p(i\mathbb{R})$ and satisfies

$$\left\|\tilde{f}\right\|_{L^p} = \|f\|_{H^p}\,.$$

We may identify f and $\tilde{f}$, and thus $H^p(\mathbb{C}_+)$ can naturally be regarded as a closed subspace of $L^P(i\mathbb{R})$ and hence a Banach space.
The Laplace transform $\mathcal{L} : L^2(0,\infty) \to H^2(\mathbb{C}_+)$ plays an important role. Let $f(t)$ be a function of t specified for $t > 0$. Then the Laplace transform of $f(t)$, denoted by $(\mathcal{L}f)(s)$, is defined by

$$F(s) = (\mathcal{L}f)(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t.$$

The parameter s is a complex number: $s = \sigma + i\omega$, with real numbers σ and ω, and up to a constant factor gives an isometric isomorphism between the two spaces, since it is bijective and satisfies $\|\mathcal{L}g\|_{H^2} = \sqrt{2\pi}\,\|g\|_{L^2}$, see [13, p, 1-2] and [31, p 1-7]. Also, one can define the Laplace transform of a finite Borel measure μ by the integral

$$(\mathcal{L}\mu)(s) = \int_{[0,\infty)} e^{-st}\mathrm{d}\mu(t),$$

see [35].

Theorem 1.2.2. (Cauchy integral formula). *Let $f(z)$ be analytic on and in the interior of a simple closed contour C . Let a be a point in the interior of C. Then*

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\mathrm{d}z}{(z-a)}.$$

Moreover,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\mathrm{d}z}{(z-a)^{n+1}},$$

(see [12], p. 182,184).

1.2.3 Elementary properties of measures

Definition 1.2.3. (a) A collection $\Re$ of subsets of a set X is said to be a $\sigma -$ algebra in X if $\Re$ has the following properties:

(i) $X \in \Re$.

(ii) If $A \in \Re$, then $A^c \in \Re$, where A^c is the complement of A relative to X.

(ii) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \Re$ for $n = 1, 2, 3, ...$, then $A \in \Re$.

(b) If $\Re$ is a $\sigma -$ algebra in X, then X is called a measurable space, and the members of $\Re$ are called the measurable sets in X.

(c) If X is a measurable space, Y is a toplogical space, and f is a mapping of X into Y, then f is said to be measurable provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

Definition 1.2.4. (a) A *positive measure* is a function μ, defined on a $\sigma -$ algebra, whose range is in $[0, \infty]$ and which is countably additive. This means that if A_i is a disjoint countable collection of members of $\Re$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

(b) A *measure space* is a measurable space which has a positive measure defined on a $\sigma -$ algebra of its measurable sets.

(c) A *complex measure* is a complex-valued countably additive function defined on a $\sigma -$ algebra .

See [34, p. 8-30].

Theorem 1.2.5. Theorem (Fatou's lemma). *Let (f_n) be a sequence of measurable functions $X \to [0, \infty)$, and define*

$$f(x) = \begin{cases} \liminf_n f_n(x) & \textit{if } \liminf_n f_n(x) < \infty \\ 0 & \textit{otherwise} \end{cases} \tag{1.2.1}$$

Then f is measurable, and

$$\int_X f \mathrm{d}\mu \leq \liminf_n \int_X f_n \mathrm{d}\mu.$$

1.2.4 Linear Operators

Definition 1.2.6. *(Spectral radius).* Let X be a complex Banach space. For an operator $T : X \to X$, the spectrum of T is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

and $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$, and thus we can define the spectral radius

$$\rho(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\},$$

and then

$$\rho(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf\{\|T^n\|^{1/n} : n \geq 1\}.$$

In particular $\rho(T) \leq \|T\|$, (see [31, p. 2]).

Lemma 1.2.7. *Let the convolution operator on $L^1(0,\infty)$ be defined by*

$$(f \star g)(x) = \int_0^x f(x-t)g(t)\mathrm{d}t.$$

Then, for $f, g \in L_1(0,\infty)$, one has $f \star g \in L_1(0,\infty)$ and

$$\|f \star g\|_1 \leq \|f\|_1 \|g\|_1 .$$

Moreover, the Laplace transforms are related by

$$(\mathcal{L}(f \star g))(s) = (\mathcal{L}f)(s)(\mathcal{L}g)(s).$$

Definition 1.2.8. Let $\phi \in L^\infty(\mathbb{T})$. Then the Laurent (or multiplication operator) $M_\phi : L^2(\mathbb{T}) \mapsto L^2(\mathbb{T})$ is given by

$$(M_\phi f)(e^{it}) = \phi(e^{it})f(e^{it}).$$

Theorem 1.2.9. *Let $\phi \in L^\infty(\mathbb{T})$. Then M_ϕ is bounded operator and its norm is given by $\|M_\phi\| = \|\phi\|_\infty$. Moreover*

$$\sup\left\{\|M_\phi f\| : f \in L^2, \|f\|_2 = 1\right\} = \|\phi\|_\infty .$$

If ϕ is a measurable function on $\mathbb{T}$ which is not in $L_\infty(\mathbb{T})$, then M_ϕ is not a bounded operator on L_2.

Definition 1.2.10. *(Definition of Hankel operator).* If $h(x) \in L^1(0,\infty) \bigcap L^2(0,\infty)$, then the Hankel operator
$\Gamma_h : L^2(0,\infty) \to L^2(0,\infty)$ given by

$$(\Gamma_h u)(x) = \int_0^\infty h(x+y)u(y)\mathrm{d}y$$

is well-defined and bounded, with $\|\Gamma_h\| \leq \|h\|_1$, (see [28, p. 42]).

Theorem 1.2.11. (Schmidt expansion of a compact operator) *An operator T is compact if and only if there exist orthonormal sequences $(\nu_i), (\omega_i), i \geqslant 1$, and scalars (σ_i) decreasing to 0, such that*

$$Tx = \Sigma_1^\infty \sigma_i(x, \nu_i)\omega_i.$$

The numbers are called singular values, (see [28, p. 6]).

Definition 1.2.12. We say that a compact operator T is in class C_p, $(1 \leqslant p \leqslant \infty)$ if and only if $\Sigma_1^\infty \sigma_i(T)^p < \infty$.
Important values of p are:
C_1: The nuclear or trace-class operators, and C_2: The Hilbert-Schmidt operators, (see [28, p. 9]).

Corollary 1.2.13. *If $h \in L^1$, then Γ_h is a compact operator, (see [28, p. 67]).*

Theorem 1.2.14. *If $h \in L^1$ determines the bounded Hankel operator Γ, then Γ is Hilbert-Schmidt if and only if $t^{1/2}h(t) \in L^2(0,\infty)$, and if so $\|\Gamma\|_{HS} = \left\|t^{1/2}h\right\|_{L^2}$ (see [28, p. 67]).*

Definition 1.2.15. If Γ is compact, then

$$\sigma_n(\Gamma) = \inf\{\|\Gamma - S\| : rank(S) < n\},$$

(see [28]).

Remark 1.2.16. If the Hankel operator is nuclear then, $h \in L^1$ and

$$\|h\|_1 \leqslant 2\,\|\Gamma\|_N$$

where $\|\Gamma\|_N = \Sigma_{i=1}^\infty \sigma_i(\Gamma)$, (see [18, p. 68]).

Remark 1.2.17. *(Relationship between classes).* We give the inclusions between different classes of operators on H where $H = L^2(X, d\mu)$ with X a locally compact Hausdorff space and $d\mu$ is Borel measure, and sometimes H is a general Hilbert space.
Finite rank $\Rightarrow$ trace class $\Rightarrow$ Hilbert-Schmidt $\Rightarrow$ compact $\Rightarrow$ bounded, (see [13], p. 151).

1.3 Systems

Definition 1.3.1. *Transfer function* is a compact description of the input-output relation for a linear system, it is a function of complex variables. In other word the transfer function of a linear dynamic system is the ratio of the Laplace transform of its output to the Laplace transform of its input.

We consider two types of systems:-

- Discrete time linear system. These can be regarded as linear operators T on $\ell^p(\mathbb{Z}_+), 1 \le p \le \infty$ with the variable indexed by $0, 1, 2, ...$

- Continuous time linear system. These can be regarded as linear operators T on $L^p(0, \infty), 1 \le p \le \infty$.
  Conventionally we write $y = Tu$, where $u, y \in L^p(0, \infty)$ and u is called the input and y the output of the system.
  Convolution operators in discrete time on ℓ^p are defined by

$$y(t) = (T_h u)(t) = (h * u)(t) = \sum_{s=0}^{t} h(t-s)u(s)$$

  and in continuous time on L^p by

$$y(t) = (T_h u)(t) = (h * u)(t) = \int_0^t h(t-\tau)u(\tau)\mathrm{d}\tau.$$

  See the book of Partington [30] .

1.3.1 BIBO Stability

BIBO stands for Bounded-Input Bounded-Output, and if a system is BIBO stable, then the output will be bounded for every input to the system that is bounded.
The condition for BIBO stability for continuous time linear systems is

$$\int_0^\infty |h(t)| \, \mathrm{d}t = \|h\|_1 < \infty.$$

For discrete time linear systems the condition is

$$\sum_{n=0}^{\infty} |h(n)| = \|h\|_1 < \infty.$$

More generally, we have convolution operators defined in continuous time by measures,

$$y(t) = \int_0^t u(t-\tau)\mathrm{d}\mu(\tau),$$

and these are BIBO stable if and only if

$$\|\mu\| := \int_0^\infty \mathrm{d}\,|\mu|\,(t) < \infty.$$

1.3.2 H^∞ Stability

H^∞ stability is, the property that $\mathcal{L}h$ or $\mathcal{L}\mu$ (the transfer function) is bounded and analytic in $\mathbb{C}_+$.
The notion of BIBO stability is stronger than H^∞ Stability, and the following diagram shows the relationship between them,
BIBO stability $\Rightarrow$ H^∞ stability $\Rightarrow$ no poles in the right half plane.

Theorem 1.3.2. *For $p = 1$ and ∞, the (continuous-time) operator*

$$T_h : L^p(0,\infty) \to L^p(0,\infty)$$

or

$$u \mapsto h * u$$

is bounded if and only if $h \in L_1(0,\infty)$: if so, then $\|T_h\| = \|h\|_1$. For $p = 2$, the operator T_h is bounded if and only if $H(s) \in H_\infty(\mathbb{C}_+)$: if so, then $\|T_h\| = \|H\|_\infty$.

1.3.3 The poles of the systems

We look at a time-delay systems with transfer functions of form

$$G(s) = \frac{\sum_{k=1}^{M} p_k(s) e^{-T_k s}}{\sum_{l=1}^{N} q_l(s) e^{-u_l s}}$$

where $T_k > 0$ and $u_l > 0$, and $p_k(s), q_l(s)$ are real polynomials. As in Bellman and Cooke [4] and Partington [32] we can divide the poles of the systems into three types of chains:

- Chains of retarded type, where the poles (s_n) satisfy $\operatorname{Re} s_n \to -\infty$, and thus there are only finitely many poles in any right half-plane.
- Chains of neutral type, where the poles lie in a band centred on the imaginary axis.
- Chains of advanced type, where the poles (s_n) satisfy $\operatorname{Re} s_n \to \infty$.

1.3.4 Generalizing the Walton-Marshall method

Bonnet and Partington in [6] extended the Walton-Marshall technique with very few modifications to the case of fractional delay systems and we also use it as well. This method is shown in the following proposition.

Proposition 1.3.3. *Let $A(s)$ and $B(s)$ be real polynomials. If $P_h(s) = A(s) + B(s)e^{-sh}$ has a zero at a point $s \in i\mathbb{R}$, and $A(s)$ and $B(s)$ are not zero there, then such an s satisfies the equation*

$$A(s)A(-s) = B(s)B(-s).$$

Moreover, at such a point s we have

$$\operatorname{sgn} Re\frac{ds}{dh} = \operatorname{sgn} Re\frac{1}{s}[\frac{B'(s)}{B(s)} - \frac{A'(s)}{A(s)}].$$

Chapter 2

Delay and Fractional Systems

2.1 Introduction

In this chapter we deal with various stability notions of linear time invariant systems, specified in the frequency domain by their transfer functions. The class of systems that we shall consider contains delay systems of neutral type, as well as fractional delay systems of neutral and retarded type: that is, systems whose transfer function may contain polynomials in fractional powers of s combined with delay terms. The three versions of stability that we shall consider (decreasing strength) are BIBO (i.e., bounded-input bounded-output) stability, H^∞ stability (i. e., finite $L^2 - L^2$ gain), and asymptotic stability (no poles in the closed right-hand half-plane $\overline{\mathbb{C}_+}$).

In Section 2, we give a new test for BIBO stability of delay systems of neutral type, and use it to give answers to some delicate questions raised in [5] and [32].

In Section 3 we shall consider fractional systems, those whose transfer functions involve fractional powers of s.

Moreover, we develop a generalization of the Walton-Marshall test (see [38]), which finds stability intervals for delay systems with variable delay. The theory is motivated by an example before being stated in detail.

2.2 Delay systems

In this section we shall analyse linear systems with transfer functions of the form

$$G(s) = \frac{f(s)}{p(s) + q(s)e^{-hs}}, \quad s \in \mathbb{C}_+,$$

where $h > 0$ and p, q and f are polynomials. (In fact we need to consider just the case $h = 1$, since the general case reduces to this by a trivial change of variable.)

More generally, p, q and f may be quasi-polynomials, that is, of the form $a_0 s^{\alpha_0} + ... + a_n s^{\alpha_n}$, where $0 \leq \alpha_0 < ... < \alpha_n$. Throughout this chapter, we regard s^α as being a single-valued holomorphic function defined on the cut plane $\{s = re^{i\theta} : r \geq 0 : -\pi < \theta < \pi\}$ as $s^\alpha = r^\alpha e^{i\alpha\theta}$, with the obvious convention that $0^\alpha = 0$.

If $\deg p > \deg q$, the system said to be of retarded type: if $\deg p = \deg q$ it is said to be neutral type, and if $\deg p < \deg q$ it is of advanced type. (See for instance [4], [31].)

Stability questions are well understood for delay systems of retarded and advanced type: in this section we shall concentrate on systems of neutral type, which are more difficult to analyse. Also we necessarily assume that the system is proper, i. e., $\deg f \leq \deg p$; see [31].

We begin with a motivating example, which has been considered in several papers such as [5] and [32]; we consider

$$G_l = \frac{1}{(s+1)^l(s+1+se^{-s})}, \quad l = 0, 1, 2, ...$$

This transfer function is asymptotically stable (i. e., no poles in the closed right-hand half-plane); it is known that it does not lie in H^∞ for $l = 0$, but it is H^∞ stable for $l \geq 1$, (see [32]). The question of BIBO stability is far more difficult: G_l is clearly not BIBO stable for $l = 0$, but following the results of [5] and [32] it is known to be BIBO stable for $l = 4$. The remaining cases were open, but new methods enable us to resolve the cases $l = 2$ and $l = 3$. Now before stating a more general result, we shall analyse G_l for $l \geq 2$, as the method is easiest to explain with this example.

Lemma 2.2.1. *For $k \geq 0$ let $h_k \in L^1(0, \infty)$ satisfy $\mathcal{L}h_k(s) = \frac{s^k}{(1+s)^{k+3}}$. Then $\|h_k\|_{L_1} = O(k^{\frac{-5}{4}})$ as $k \to \infty$.*

Proof. Take $g_k = e^{t/4}h_k(t)$. Note that $\mathcal{L}g_k(s) = \frac{(s-\frac{1}{4})^k}{(s+\frac{3}{4})^{k+3}}$. Then, by the Cauchy-Schwarz inequality we have

$$\|h_k\| \leq \left\|e^{-1/4}\right\|_{L^2} \|g_k\|_{L^2}.$$

Now $\|g_k\|_{L^2} = \frac{1}{\sqrt{2\pi}} \|\mathcal{L}g_k\|_{H^2}$, and

$$\begin{aligned} \|\mathcal{L}g_k\|_{H^2}^2 &= 2\int_0^\infty \frac{\left|iy-\frac{1}{4}\right|^{2k}}{\left|iy+\frac{3}{4}\right|^{2k+6}}\mathrm{d}y \\ &= 2\left(\int_0^{\sqrt{k}} + \int_{\sqrt{k}}^\infty\right) \frac{(y^2+\frac{1}{16})^k}{(y^2+\frac{9}{16})^{k+3}}\mathrm{d}y. \end{aligned}$$

We may estimate the first integral as at most $\sqrt{k}$ times the maximum value of the integrand on $[0, \sqrt{k}]$, or $O(k^{1/2}k^{-3})$, since the integrand is an increasing function of y. The second integral is at most $\int_{\sqrt{k}}^\infty y^{-6}\mathrm{d}y$, which is also $O(k^{-5/2})$. This gives the result. □

Theorem 2.2.2. *Let $G_l(s) = \frac{1}{(s+1)^l(s+1+se^{-s})}$ be the transfer function of a delay system; then it is BIBO stable for $l \geq 2$.*

Proof. It is sufficient to consider the case $l = 2$, as higher-order G_l are simply cascades of G_2 with BIBO-stable finite-dimensional systems. Now, we have

$$G_2 = \sum_{k=0}^\infty (-1)^k e^{-sk}\frac{s^k}{(s+1)^{k+3}},$$

converging point-wise in $\mathbb{C}_+$, and it is easy to notice that the inverse Laplace transforms converge point-wise on $(0, \infty)$, since the kth term vanishes on $[0, k)$. Then if $\mathcal{L}h = G_2$, we have

$$\|h\|_1 \leq \sum_{k=0}^\infty \left\|(-1)^k e^{-sk}\frac{s^k}{(s+1)^{k+3}}\right\|_{BIBO} = \sum_{k=0}^\infty \left\|\frac{s^k}{(s+1)^{k+3}}\right\|_{BIBO},$$

by Fatou's lemma 1.2.5 (in the form that asserts that if $f_n \to f$ point-wise then $\|f\|_1 \leq \liminf \|f_n\|_1$). Using Lemma 2.2.1, we can conclude that $h \in L^1$, and the system G_2 is BIBO stable. □

A more general result can be proved by the same method. Also, note that one necessary condition on p and q for the neutral system $\frac{1}{p(s)+q(s)e^{-s}}$ to be asymptotically stable is that

$$\lim_{|s|\to\infty} \left| \frac{q(s)}{p(s)} \right| \leq 1, \tag{2.2.1}$$

(see [32], Proposition 2.1), as otherwise the poles are asymptotic to a vertical line strictly in $\mathbb{C}_+$.

The following theorem gives conditions for stability of neutral systems (see [32]).

Theorem 2.2.3. *See [32]. Let $G(s) = \frac{f(s)}{p(s)+q(s)e^{-sh}}$ be a neutral delay system satisfying*

- *$h > 0$ and p, q and f are real polynomials.*
- *$\deg p = \deg g$ (neutral type) and $\deg f \leq \deg p$,*

and suppose that

$$\frac{p(s)}{q(s)} = \alpha + \frac{\beta}{s} + \frac{\gamma}{s^2} + O(\frac{1}{s^3}) \quad as \quad |s| \to \infty,$$

where α, β and γ are constants, with $\alpha = \pm 1$. For sufficiently large integers n let $\lambda_n = 2ni\pi$ if $\alpha = -1$ and let $\lambda_n = (2n+1)i\pi$ if $\alpha = 1$. Then the poles s_n of G satisfy

$$s_n = \frac{\lambda_n}{h} - \frac{\beta}{\alpha\lambda_n} + \frac{h}{\lambda_n^2}(\frac{\beta^2}{2} - \frac{\gamma}{\alpha}) + o(\frac{1}{n^2}).$$

The system has infinitely many unstable poles if $\gamma/\alpha > \beta^2/2$, and infinitely many stable poles if $\gamma/\alpha < \beta^2/2$. In the latter case there can be at most finitely many unstable poles, and if there are none, then the transfer function G lies in H^∞ if and only if $\deg p \geq \deg f + 2$. If $\gamma/\alpha = \beta^2/2$, then the condition $\deg p \geq \deg f + 2$ is still necessary for stability.

Theorem 2.2.4. *Let $G(s) = \frac{1}{p(s)+q(s)e^{-s}}$ be the transfer function of a neutral delay system. Suppose that*

- *$\deg p = \deg q = N \geq 3$;*

- *all roots of* p *in* $\mathbb{C}_-$;
- $|\mathrm{Re}(\textit{zeros of}\quad \tilde{q}(s-c))| < |\mathrm{Re}(\textit{zeros of}\quad \tilde{p}(s-c))|$, *where* $c > 0$, *and* $G(s) = \frac{1}{r(s)[\tilde{p}+\tilde{q}e^{-s}]}$, *with* $r(s)$ *is the greatest common divisor of* p *and* q;
- $\frac{\left|(\tilde{q}(iy-c))^k\right|}{|r(iy-c)|\left|(\tilde{p}(iy-c))^{k+1}\right|}$ *is an increasing function on* $[0, \delta_k]$, *where* $\delta_k \asymp k^\alpha$ *and* $\alpha > \frac{2}{5}$.

Then $G(s)$ *is BIBO stable, and hence* H^∞ *stable.*

Proof. We have

$$\begin{aligned} G(s) &= \frac{1}{r(s)(\tilde{p}(s)+\tilde{q}(s)e^{-s})} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{r(s)\tilde{p}(s)}(\frac{\tilde{q}(s)e^{-s}}{\tilde{p}(s)})^k. \end{aligned}$$

Take $\mathcal{L}h_k(s) = \frac{\tilde{q}^k(s)}{r(s)\tilde{p}^{k+1}(s)}$.

Let $h_k(t) = e^{-ct}g_k(t)$, where $c > 0$. Then by the Cauchy-Schwarz inequality

$$\|h_k\|_{L_1} \le \left\|e^{-ct}\right\|_{L_2} \|g_k\|_{L_2}.$$

Since $g_k(t) = e^{ct}h_k(t)$, then $\mathcal{L}g_k(s) = \mathcal{L}h_k(s-c) = \frac{\tilde{q}^k(s-c)}{r(s-c)\tilde{p}^{k+1}(s-c)}$.

We have $\|\mathcal{L}g_k\|_{H^2} = \sqrt{2\pi}\,\|g_k\|_{L_2}$.

Now let $s = iy$, then

$$\begin{aligned} \|g_k\|^2_{L_2} &= (\frac{1}{\sqrt{2\pi}})^2 \int_{-\infty}^{\infty} \left|\frac{\tilde{q}^k(s-c)}{r(s-c)\tilde{p}^{k+1}(s-c)}\right|^2 \mathrm{d}s \\ &= \frac{1}{\pi}\int_0^\infty \frac{\tilde{q}^{2k}(iy-c)}{r^2(iy-c)\tilde{p}^{2k+2}(iy-c)}\mathrm{d}y \\ &= \frac{1}{\pi}\int_0^{\delta_k} \frac{\tilde{q}^{2k}(iy-c)}{r^2(iy-c)\tilde{p}^{2k+2}(iy-c)}\mathrm{d}y + \frac{1}{\pi}\int_{\delta_k}^\infty \frac{\tilde{q}^{2k}(iy-c)}{r^2(iy-c)\tilde{p}^{2k+2}(iy-c)}\mathrm{d}y \\ &\le \frac{\delta_k}{\pi}(\text{max value on}\quad [0,\delta_k]) + \frac{1}{\pi}\int_{\delta_k}^\infty \frac{\tilde{q}^{2k}(iy-c)}{r^2(iy-c)\tilde{p}^{2k+2}(iy-c)}\mathrm{d}y \\ &= O(\delta_k^{-2N+1}) + O(\delta_k^{-2N+1}) \\ &= O(k^{\frac{-2N+1}{2}}). \end{aligned}$$

Then $\|g_k\|_{L_2} = O(k^{\frac{-2N+1}{4}})$.

Since

$$G(s) = \sum_{k=0}^{\infty} (-1)^k \mathcal{L}h_k(s),$$

by Fatou's lemma,

$$\|G\|_{BIBO} \leq \sum_{k=0}^{\infty} \left\| \frac{\tilde{q}^k(s)}{r(s)\tilde{p}^{k+1}(s)} \right\| < \infty.$$

Then $G(s)$ is BIBO stable and so it is H^∞ stable. $\square$

The following is a more general result.

Theorem 2.2.5. *Let $G(s) = \frac{f(s)}{p(s)+q(s)e^{-s}}$ be the transfer function of a neutral delay system. Suppose that*

- $\deg p = \deg q = N \geq 3 + \deg f$*, where* $\deg f = N'$;
- *all roots of p in $\mathbb{C}_-$;*
- $|\mathrm{Re}($ *zeros of* $\tilde{q}(s-c))| < |\mathrm{Re}($ *zeros of* $\tilde{p}(s-c))|$*, where $c > 0$, and* $G(s) = \frac{f(s)}{r(s)[\tilde{p}+\tilde{q}e^{-s}]}$*, with $r(s)$ is the greatest common divisor of p and q;*
- $\frac{|f(iy-c)|\left|(\tilde{q}(iy-c))^k\right|}{|r(iy-c)|\left|(\tilde{p}(iy-c))^{k+1}\right|}$ *is an increasing function on $[0, \delta_k]$, where $\delta_k \asymp k^\alpha$ and $\alpha > \frac{2}{5}$.*

Then $G(s)$ is BIBO stable, and hence H^∞ stable.

Proof. Take $\mathcal{L}h_k(s) = \frac{f(s)\tilde{q}^k(s)}{r(s)\tilde{p}^{k+1}(s)}$.

Let $h_k(t) = e^{-ct}g_k(t)$. Then by the Cauchy-Schwarz inequality

$$\|h_k\|_{L_1} \leq \left\|e^{-ct}\right\|_{L_2} \|g_k\|_{L_2}.$$

Since $g_k(t) = e^{ct}h_k(t)$,then $\mathcal{L}g_k(s) = \mathcal{L}h_k(s-c) = \frac{f(s-c)\tilde{q}^k(s-c)}{r(s-c)\tilde{p}^{k+1}(s-c)}$.

We have $\|\mathcal{L}g_k\|_{H^2} = \sqrt{2\pi}\,\|g_k\|_{L_2}$.

Now let $s = iy$, then

$$\begin{aligned}
\|g_k\|_{L_2}^2 &= \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} \left| \frac{f(s-c)\tilde{q}^k(s-c)}{r(s-c)\tilde{p}^{k+1}(s-c)} \right|^2 \mathrm{d}s \\
&= \frac{1}{\pi} \int_0^{\infty} \frac{(f(iy-c))^2 \tilde{q}^{2k}(iy-c)}{r^2(iy-c)\tilde{p}^{2k+2}(iy-c)} \mathrm{d}y \\
&= \frac{1}{\pi} \int_0^{\delta_k} \frac{(f(iy-c))^2 \tilde{q}^{2k}(iy-c)}{r^2(iy-c)\tilde{p}^{2k+2}(iy-c)} \mathrm{d}y + \frac{1}{\pi} \int_{\delta_k}^{\infty} \frac{(f(iy-c))^2 \tilde{q}^{2k}(iy-c)}{r^2(iy-c)\tilde{p}^{2k+2}(iy-c)} \mathrm{d}y \\
&\leq \frac{\delta_k}{\pi}(\text{max value on} \quad [0, \delta_k]) + \frac{1}{\pi} \int_{\delta_k}^{\infty} \frac{(f(iy-c))^2 \tilde{q}^{2k}(iy-c)}{r^2(iy-c)\tilde{p}^{2k+2}(iy-c)} \mathrm{d}y \\
&= O(\delta_k^{(2N'-2N+1)}) + O(\delta_k^{(2N'-2N+1)}) \\
&= O(k^{(2N'-2N+1)\alpha}).
\end{aligned}$$

Therefore

$$\|g_k\|_{L_2} = O(k^{\frac{(2N'-2N+1)\alpha}{2}}).$$

By Fatou's lemma,

$$\|G\|_{BIBO} \leq \sum_{k=0}^{\infty} \left\| \frac{f(s)\tilde{q}^k(s)}{r(s)\tilde{p}^{k+1}(s)} \right\| < \infty.$$

Then $G(s)$ is BIBO stable and hence H^∞ stable. □

Example 2.2.6. Let $G(s) = \frac{1}{(s+3)(s+2)^2 + (s-\frac{1}{2})s^2 e^{-s}}$ be the transfer function of a neutral delay system. Then $G(s)$ is BIBO stable and hence H^∞ stable.

Proof. From 2.2.5 we can deduce that

$$\|g_k\|_{L_2} = O(k^{\frac{-5}{4}})$$

and

$$\|G\|_{BIBO} \leq \sum_{k=0}^{\infty} \left\| \frac{(s-\frac{1}{2})^k s^{2k}}{(s+3)^{k+3}(s+2)^{2k+2}} \right\| < \infty.$$

Then $G(s)$ is BIBO stable and so it is H^∞ stable. □

Remark 2.2.7. In Example 2.2.6, the transfer function does not have poles in the right half plane (see [32]). Take,

$$\frac{p(s)}{q(s)} = \frac{(s+3)(s+2)^2}{(s-\frac{1}{2})s^2} = \frac{1}{s^2}\left[s^2 + \frac{15}{2}s + \frac{79}{4} + \frac{(\frac{79}{16}+12)}{s-\frac{1}{2}}\right],$$

then

$$\frac{p(s)}{q(s)} = 1 + \frac{15}{2s} + \frac{79}{4s^2} + O(\frac{1}{s^3}) \quad as \quad |s| \to \infty.$$

So

$\alpha = 1, \quad \beta = \frac{15}{2}, \quad \gamma = \frac{79}{4}, \quad h = 1$, and then, $\lambda_n = (2n+1)i\pi$.

Thus, the poles s_n of G satisfy

$$s_n = \frac{\lambda_n}{h} - \frac{\beta}{\alpha\lambda_n} + \frac{h}{\lambda_n^2}(\frac{\beta^2}{2} - \frac{\gamma}{\alpha}) + o(\frac{1}{n^2}).$$

Also,

$$s_n = (2n+1)i\pi - \frac{15}{2(2n+1)i\pi} - \frac{1}{(2n+1)^2\pi^2}(\frac{225}{16} - \frac{79}{4}).$$

Because $\frac{\gamma}{\alpha} < \frac{\beta^2}{2}$ the system has infinitely many stable poles (in $\operatorname{Re} s < 0$). Moreover, there are no small poles in the right-half plane, since for

$$(s+3)(s+2)^2 + (s - \frac{1}{2})s^2e^{-s},$$

if $\operatorname{Re} s \geq 0$, then

$$\left|(s+3)(s+2)^2\right| > \left|(s - \frac{1}{2})s^2e^{-s}\right|,$$

and then

$$(s+3)(s+2)^2 + (s - \frac{1}{2})s^2e^{-s} \neq 0.$$

Another more elementary result is also useful.

Theorem 2.2.8. *Let $G(s) = \frac{1}{g(s)+h(s)}$ be transfer function. Suppose that $\frac{1}{g}$ is BIBO stable and $\rho(\frac{h}{g}) < 1$ (ρ denotes the spectral radius) . Then $G(s)$ is BIBO stable.*

Proof. We have

$$\begin{aligned} G(s) &= \frac{1}{g+h} \\ &= \frac{1}{g(1+\frac{h}{g})} \\ &= \frac{1}{g}\sum_{k=0}^{\infty}\left(\frac{h}{g}\right)^k(-1)^k. \end{aligned}$$

So

$$\|G\|_{BIBO} \leq \left\|\frac{1}{g}\right\|_{BIBO} \sum_{k=0}^{\infty} \left\|\left(\frac{h}{g}\right)\right\|_{BIBO}^{k} < \infty.$$

Then $G(s)$ is BIBO stable, since $\rho(\frac{h}{g}) < 1$. □

Example 2.2.9. Let

$$G(s) = \frac{1}{(s+1)^4 + s(s+1)^3 e^{-s} + h(s)e^{-Ts}},$$

be a transfer function. We know that $\frac{1}{g} = \frac{1}{(s+1)^4+s(s+1)^3e^{-s}}$ is BIBO stable. Also we have

$$\left\|\frac{1}{(s+1)^4 + s(s+1)^3 e^{-s}}\right\|_{\infty} \leq \left\|\frac{1}{(s+1)}\right\|_{\infty}^{2} \left\|\frac{1}{(s+1)^2 + s(s+1)e^{-s}}\right\|_{\infty} \leq 2,$$

then $\left\|\frac{h(s)e^{-Ts}}{(s+1)^4+s(s+1)^3e^{-s}}\right\|_{\infty} < 1$ if $\|h\|_{\infty} < \frac{1}{2}$. Then G is BIBO stable as in Theorem 2.2.8.

2.3 Fractional Systems

Definition 2.3.1. Fractional systems are those which in the frequency domain have transfer functions involving fractional powers of s, such as $\sqrt{s}$ and $s^{\frac{1}{3}}$. For $\alpha > 0$ we choose a single-valued analytic branch of s^{α} on $\mathbb{C} \setminus (-\infty, 0]$ with $1^{\alpha} = 1$, i. e; $s^{\alpha} = (re^{i\theta})^{\alpha} = r^{\alpha}e^{i\alpha\theta}$ where $-\pi < \theta < \pi$ and $r > 0$.

Example 2.3.2. There are many examples of fractional systems. Several examples are linked to the heat equation.

(i) Heat equation with Neumann boundary control: $G(s) = \cosh\sqrt{s}x_0/\sqrt{s}\sinh\sqrt{s}$;

(ii) Heat equation with Dirichlet boundary control: $G(s) = \sinh\sqrt{s}x_0/\sinh\sqrt{s}$;

(iii) Arising in the theory of transmission lines: $G(s) = e^{-a\sqrt{s}}/s$, with $a > 0$;

in each case with $0 < x_0 < 1$ a fixed number. These examples are given in [5] and [9].

Some more examples can be found in [10]:

(iv) $G(s) = (\tanh(\sqrt{s}/2))/(s\sqrt{s})$;

(v) $G(s) = (\cosh(sx_0)/(s\sinh s), \quad 0 < x_0 < 1$;

(vi) $G(s) = (\cosh(\sqrt{s}x_0)/(\sqrt{s}\sinh\sqrt{s}), \quad 0 \leqslant x_0 \leqslant 1$, linked to the heat equation;

(vii) $G(s) = (2e^{-a\sqrt{s}})/(b(1-e^{-2a\sqrt{s}}))$, linked to the heat equation, see [11].

We begin with an example.

Proposition 2.3.3. *Let* $G(s) = \frac{1}{s^\alpha + e^{-sh}}$ *be a transfer function. We fix* $0 < h < \frac{\pi}{2}$ *and vary* α. *Then the system is asymptotically stable for* $0 < \alpha < 2(1-\frac{h}{\pi})$.

Proof. $G(s) = \frac{1}{s^\alpha + e^{-sh}}$; it is known to be stable at $\alpha = 1$ see [31].
As α varies, the poles move continuously, and cross the axis when $s^\alpha + e^{-sh} = 0$ on $i\mathbb{R}$. It is enough to consider $y > 0$ so

$$e^{i\frac{\pi}{2}\alpha}y^\alpha + e^{-iyh} = 0,$$

and the conjugate equation is

$$e^{-i\frac{\pi}{2}\alpha}y^\alpha + e^{iyh} = 0.$$

Then, $y = 1$ (since $\left|e^{-i\frac{\pi}{2}\alpha}\right| = \left|e^{iyh}\right| = 1$), so we have $e^{-i\frac{\pi}{2}\alpha} + e^{ih} = 0$, so $e^{-i\frac{\pi}{2}\alpha} = -e^{ih}$, thus, $e^{-i\frac{\pi}{2}\alpha} = e^{-i\pi + ih}$ and then

$$-i\frac{\pi}{2}\alpha = -i\pi + ih + 2ik\pi.$$

Hence, the first crossing is at $\alpha = 2(1-\frac{h}{\pi})$. □

Remark 2.3.4. For $G(s) = \frac{1}{s^\alpha + e^{-sh}}$, with $\alpha = 2(1-\frac{h}{\pi})$ and $0 < h < \frac{\pi}{2}$, then $\operatorname{Re}\frac{\partial s}{\partial \alpha} > 0$, so the system become unstable as α increases.

Here is a more general result.

Proposition 2.3.5. *Let A and B be real polynomials. If $P_h(s^\alpha) = A(s^\alpha) + B(s^\alpha)e^{-sh}$ has a zero at a point $s \in i\mathbb{R}$, and $A(s^\alpha)$ and $B(s^\alpha)$ are not zero there, then such an s satisfies the equation*

$$A(s^\alpha)A((-s)^\alpha) = B(s^\alpha)B((-s)^\alpha).$$

and,

$$\frac{\mathrm{d}s}{\mathrm{d}\alpha} = \frac{-s\log s(\frac{A'(s^\alpha)}{A(s^\alpha)} - \frac{B'(s^\alpha)}{B(s^\alpha)})}{\alpha(\frac{A'(s^\alpha)}{A(s^\alpha)} - \frac{B'(s^\alpha)}{B(s^\alpha)}) + \frac{h}{s^{\alpha+1}}}.$$

Proof. From the equation $A(s^\alpha) + B(s^\alpha)e^{-sh} = 0$ with $s \in i\mathbb{R}$, we obtain $A((-s)^\alpha) + B((-s)^\alpha)e^{sh} = 0$ by conjugation, and by eliminating the exponential term from two equations we get $A(s^\alpha)A((-s)^\alpha) = B(s^\alpha)B((-s)^\alpha)$. We have

$$A(s^\alpha) + B(s^\alpha)e^{-sh} = 0. \tag{2.3.1}$$

By differentiating with respect to α,
$A'(s^\alpha)s^\alpha \log s + A'(s^\alpha)\alpha s^{\alpha-1}\frac{\mathrm{d}s}{\mathrm{d}\alpha} + e^{-sh}s^\alpha B'(s^\alpha)\log s + \alpha s^{\alpha-1}e^{-sh}B'(s^\alpha)\frac{\mathrm{d}s}{\mathrm{d}\alpha} + B(s^\alpha)e^{-sh}(-h)\frac{\mathrm{d}s}{\mathrm{d}\alpha} = 0$,
and, after simplification

$$\frac{\mathrm{d}s}{\mathrm{d}\alpha} = \frac{-s\log s(\frac{A'(s^\alpha)}{A(s^\alpha)} - \frac{B'(s^\alpha)}{B(s^\alpha)})}{\alpha(\frac{A'(s^\alpha)}{A(s^\alpha)} - \frac{B'(s^\alpha)}{B(s^\alpha)}) + \frac{h}{s^{\alpha+1}}}.$$

If $\frac{\mathrm{d}s}{\mathrm{d}\alpha} > 0$, then zeroes of P_h cross from left to right; however if $\frac{\mathrm{d}s}{\mathrm{d}\alpha} < 0$, then zeroes cross from right to left. □

Remark 2.3.6. This condition is not sufficient for $P_h(s^\alpha)$ to have roots on $i\mathbb{R}$ (e.g if $P_h(s^\alpha) = s^\alpha - \frac{1}{2} + e^{\frac{-\pi s}{4}}$).
In the following work we will find necessary and sufficient conditions. We use a different method where α is fixed and h varies. This is used for different values of α until we find the α for which the critical value of h is $\frac{\pi}{4}$.

Example 2.3.7. Let $G(s) = \frac{1}{s^\alpha - \frac{1}{2} + e^{\frac{-\pi s}{4}}}$ be the transfer function.
Take $\alpha = 1$, and use the Walton-Marshall-Bonnet-Partington method to find

$h > 0$, making $\frac{1}{s-\frac{1}{2}+e^{-sh}}$ unstable (where $G(s)$ is stable when $h = 0$).
So, now, $s - \frac{1}{2} + e^{-sh} = 0, \quad A(s) = s - \frac{1}{2}, \quad B(s) = 1$, then

$$A(s)A(-s) = B(s)B(-s),$$

thus $s^2 = -\frac{3}{4}$ and then, $s = \pm\frac{\sqrt{3}i}{2}$.
$e^{-sh} = -\frac{A(s)}{B(s)}$ then $e^{-\frac{\sqrt{3}}{2}ih} = \frac{1}{2} - \frac{\sqrt{3}i}{2} = e^{\frac{-\pi i}{3}}$ so, $h = \frac{2\pi}{3\sqrt{3}} + \frac{4n\pi}{\sqrt{3}}$ with $n \geq 0$.
The system is stable for $0 < h < \frac{2\pi}{3\sqrt{3}}$ because

$$\operatorname{sgn}\operatorname{Re}\frac{\mathrm{d}s}{\mathrm{d}h} = \operatorname{sgn}\operatorname{Re}\frac{1}{s}[\frac{B'(s)}{B(s)} - \frac{A'(s)}{A(s)}],$$

and then

$$\operatorname{sgn}\operatorname{Re}\frac{\mathrm{d}s}{\mathrm{d}h} = \operatorname{sgn}\operatorname{Re}\frac{1}{\frac{\sqrt{3}i}{2}}[\frac{-1}{\frac{\sqrt{3}i}{2}-\frac{1}{2}}] = \frac{12-4\sqrt{3}i}{12} > 0.$$

So, the poles cross from left to right.
In general we have the equation $s^\alpha - \frac{1}{2} + e^{-sh} = 0$ on $i\mathbb{R}$, so let $s = iy$; then $(iy)^\alpha - \frac{1}{2} + e^{-sh} = 0$, so we obtain $(-iy)^\alpha - \frac{1}{2} + e^{sh} = 0$ by conjugation, and it follows easily on eliminating the exponential term from the equations,

$$y^{2\alpha} - \frac{1}{2}(e^{\frac{-\pi\alpha i}{2}} + e^{\frac{\pi\alpha i}{2}})y^\alpha + \frac{1}{4} = 1$$

and then

$$y^{2\alpha} - y^\alpha\cos(\frac{\pi\alpha}{2}) - \frac{3}{4} = 0.$$

Then

$$y^\alpha = \frac{\cos(\frac{\pi\alpha}{2}) \pm \sqrt{\cos^2(\frac{\pi\alpha}{2}) + 3}}{2},$$

or

$$y = (\frac{\cos(\frac{\pi\alpha}{2}) \pm \sqrt{\cos^2(\frac{\pi\alpha}{2}) + 3}}{2})^{\frac{1}{\alpha}}.$$

By substituting the value of y in $e^{\frac{\pi\alpha i}{2}}y^\alpha - \frac{1}{2} + e^{-iyh} = 0$, we have

$$h = \frac{\log[\frac{1}{2} - e^{\frac{\pi\alpha i}{2}}(\frac{\cos(\frac{\pi\alpha}{2})\pm\sqrt{\cos^2(\frac{\pi\alpha}{2})+3}}{2})]}{-i[\frac{\cos(\frac{\pi\alpha}{2})\pm\sqrt{\cos^2(\frac{\pi\alpha}{2})+3}}{2}]^{\frac{1}{\alpha}}}.$$

When $\alpha = 1$, then

$$h = \frac{2\pi}{3\sqrt{3}} + \frac{4\pi n}{\sqrt{3}}.$$

We now use a different method where α is fixed and h varies. This is used for different values of α until we find the α for which the critical value of h is $\frac{\pi}{4}$.
When $h = \frac{\pi}{4}$ then we have two values of α such that the poles of $G(s)$ lie on the axis, $\alpha_1 \simeq 1.3650$ and $\alpha_2 \simeq 0.3082$. We vary h and use the Walton-Marshall-Bonnet-Partington method. For each α we plot the minimum h we find for which $\frac{1}{P_h(s^\alpha)}$ is unstable (see Figure 2.1). Then $\operatorname{sgn} \operatorname{Re} \frac{\mathrm{d}s}{\mathrm{d}\alpha}|_{\alpha_1 \simeq 1.3650} \simeq 0.7441931 > 0$. which means that the poles move from left to right, and $\operatorname{sgn} \operatorname{Re} \frac{\mathrm{d}s}{\mathrm{d}\alpha}|_{\alpha_2 \simeq 0.3082} \simeq -2.3611552 < 0$, which means that the poles move from right to left.
So, $\frac{1}{s^\alpha - 0.5 + e^{\frac{-\pi s}{4}}}$ is stable for $\alpha_2 < \alpha < \alpha_1$.
The transfer functions $G(iy)$ in Figures 2.2 and 2.3 have singularities and are unbounded.
We use Matlab to create these figures.

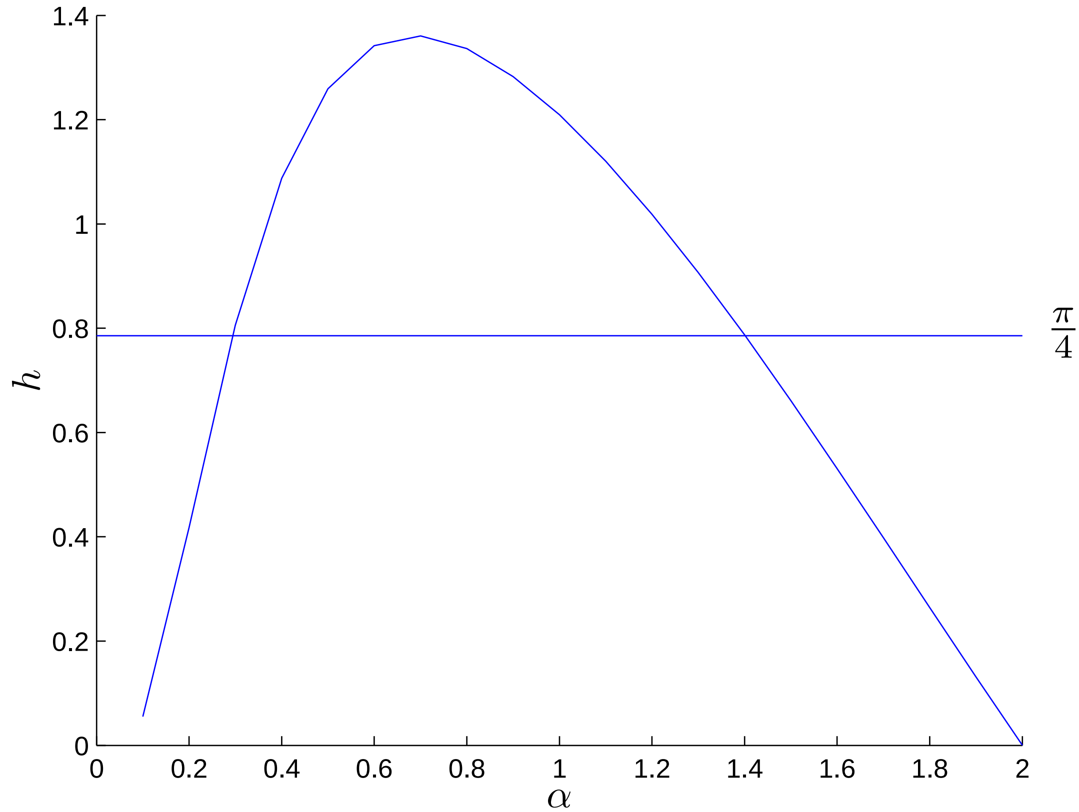

Figure 2.1: Relationship between α and h. Example (2.3.7)

Now we consider $|G(iy)|$ for α near to the critical values to show where the pole crosses the axis (for $y > 0$).

Example 2.3.8. Consider $P_h(s^\alpha) = s^\alpha - s^\alpha e^{-sh} + (s^\alpha - 2)e^{-2sh} = 0$, (see Fioravanti [16]) which for $h = 0$ has zeroes in the right half plane. Suppose now that $h > 0$ and that s is a point on the imaginary axis such that

$$s^\alpha - s^\alpha e^{-sh} + (s^\alpha - 2)e^{-2sh} = 0, \tag{2.3.2}$$

and hence

$$(-s)^\alpha - (-s)^\alpha e^{sh} + ((-s)^\alpha - 2)e^{2sh} = 0, \tag{2.3.3}$$

by complex conjugation. We wish to eliminate the exponential terms from these equations. A simple way to do this is to multiply (2.3.2) by e^{sh} and

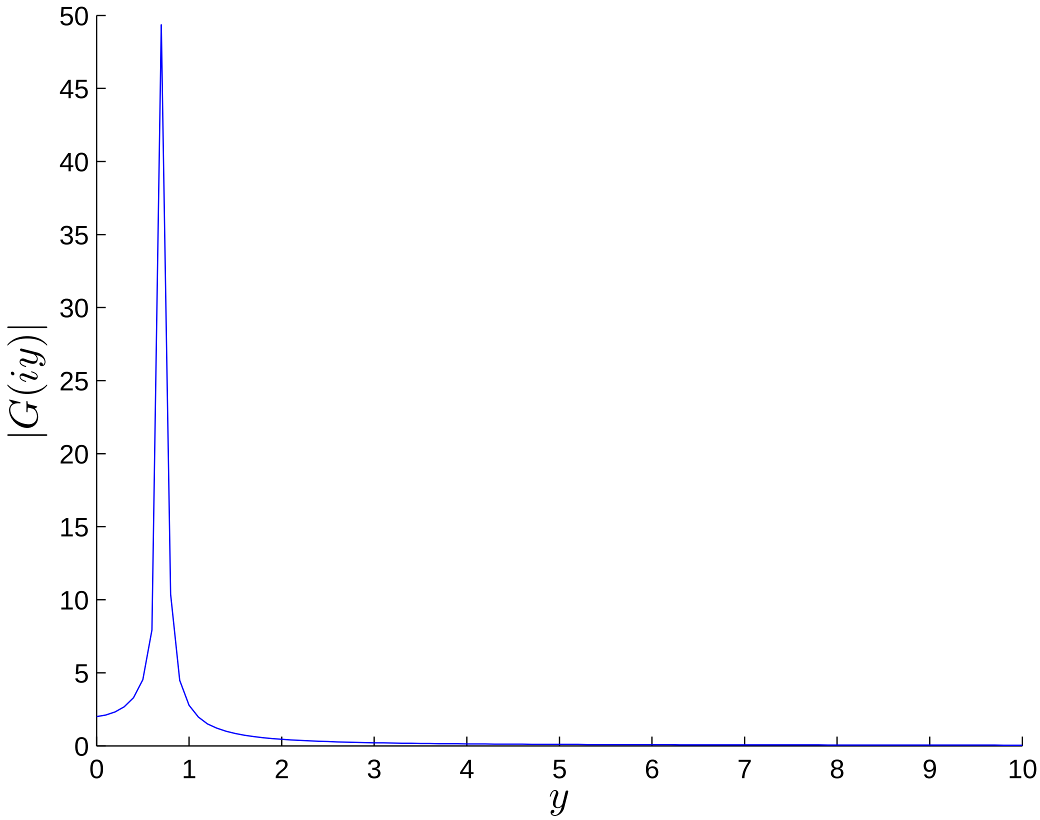

Figure 2.2: $|G(iy)|$ against y for $\alpha = 1.3650$. Example (2.3.7)

multiply (2.3.3) by e^{-sh} to produce

$$s^{\alpha}e^{sh} - s^{\alpha} + (s^{\alpha} - 2)e^{-sh} = 0, \tag{2.3.4}$$

$$(-s)^{\alpha}e^{-sh} - (-s)^{\alpha} + ((-s)^{\alpha} - 2)e^{sh} = 0. \tag{2.3.5}$$

From the equation (2.3.4)

$$e^{sh} = \frac{s^{\alpha} - (s^{\alpha} - 2)e^{-sh}}{s^{\alpha}}, \tag{2.3.6}$$

and substituting in (2.3.5) we produce

$$s^{\alpha} + (2 - (-s)^{\alpha} - s^{\alpha})e^{-sh} = 0, \tag{2.3.7}$$

and hence

$$(-s)^{\alpha} + (2 - s^{\alpha} - (-s)^{\alpha})e^{sh} = 0, \tag{2.3.8}$$

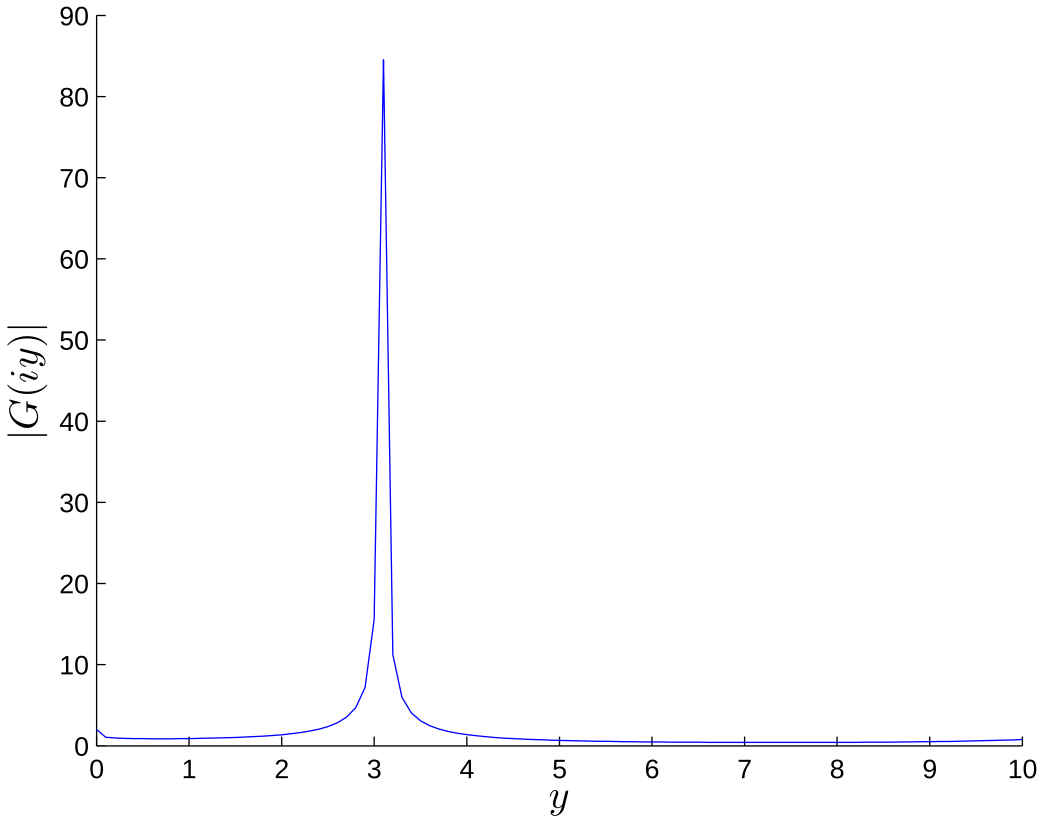

Figure 2.3: $|G(iy)|$ against y for $\alpha = 0.3082$. Example (2.3.7)

and by conjugation, finally the polynomial equation

$$4 - 4s^{\alpha} - 4(-s)^{\alpha} + s^{\alpha}(-s)^{\alpha} + (-s)^{2\alpha} + s^{2\alpha} = 0. \tag{2.3.9}$$

Taking $s = iy$ and $(-s)^{\alpha} = s^{\alpha}e^{-\pi i\alpha}$, we have

$$s^{2\alpha}e^{-i\alpha\pi} - 4s^{\alpha} - 4s^{\alpha}e^{-i\alpha\pi} + s^{2\alpha}e^{-2i\alpha\pi} + s^{2\alpha} + 4 = 0, \tag{2.3.10}$$

so

$$y^{2\alpha}(1 + e^{-i\alpha\pi} + e^{i\pi\alpha}) + y^{\alpha}(-4e^{i\alpha\frac{\pi}{2}} - 4e^{-i\alpha\frac{\pi}{2}}) + 4 = 0, \tag{2.3.11}$$

and

$$y = \left[\frac{4\cos(\frac{\pi\alpha}{2}) \mp 2\sqrt{4\cos^2(\frac{\pi}{2}\alpha) - 2\cos(\pi\alpha) - 1}}{1 + 2\cos(\pi\alpha)}\right]^{1/\alpha}. \tag{2.3.12}$$

By substituting in (2.3.7), we get,

$$h = \frac{1}{-iy}\log(\frac{-(iy)^{\alpha}}{2-(-iy)^{\alpha}-(iy)^{\alpha}}). \tag{2.3.13}$$

When $\alpha = 1$, then $y = \pm 2$ and $h = \frac{\pi}{4} + n\pi$.

Also, $s = 2i$ is a solution to $s - se^{-sh} + (s-2)e^{-2sh} = 0$; then the poles cross from left to right at this point and for $0 \leq h \leq \frac{\pi}{4}$ and $\alpha = 1$ the system is unstable. If $\alpha = 0.5$, then $h \geqslant 1$, so this G is asymptotically stable. We still do not know if it is H_∞ stable.

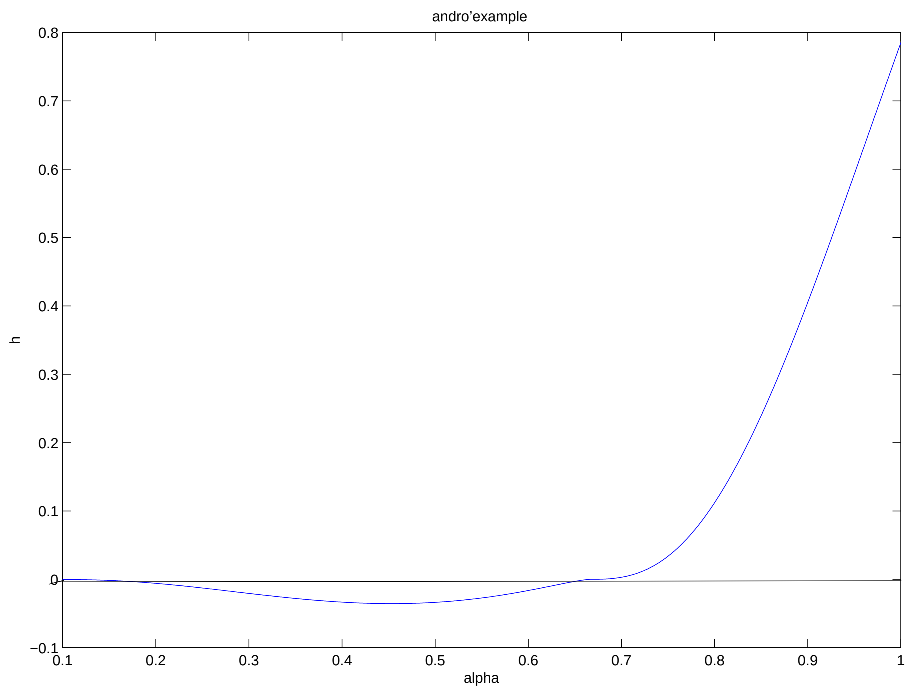

Figure 2.4: Relationship between α and h. Example (2.3.8)

2.3.1 Systematic method (1 step) for fractional systems

We now consider the general case of fractional systems with n delays, $n \geq 2$, the characteristic equation of which will be.

$$F(s^\alpha, h) \equiv \sum_{k=0}^{n} A_k(s^\alpha)e^{-ksh} = \sum_{k=0}^{n} A_k(s^\alpha)z^k = 0.$$

Firstly there is the usual preliminary step of examining the stability at $h = 0$. The next step is the determination of any potential crossing point, i.e. we seek solutions with $s = i\omega$. We therefore seek solutions of

$$F(s^\alpha, h) \equiv \sum_{k=0}^{n} A_k(s^\alpha)z^k = 0,$$

and, replacing s by $-s$,

$$F((-s)^\alpha, h) \equiv \sum_{k=0}^{n} A_k((-s)^\alpha)z^{-k} = 0.$$

This method gives a procedure for the systematic reduction in degree by elimination of the highest power of z. This iterative scheme eventually yields an equation independent of z.
Define

$$\begin{aligned}
F^{(1)}(s^\alpha, h) &= A_0(-s^\alpha)F(s^\alpha, h) - A_n(s^\alpha)z^n F((-s)^\alpha, h) \\
&= A_0((-s)^\alpha)\sum_{k=0}^{n} A_k(s^\alpha)z^k - A_n(s^\alpha)z^n \sum_{k=0}^{n} A_k((-s)^\alpha)z^{-k} \\
&= A_0((-s)^\alpha)A_n(s^\alpha)z^n + A_0((-s)^\alpha)\sum_{k=0}^{n-1} A_k(s^\alpha)z^k - A_n(s^\alpha)z^n A_0((-s)^\alpha)z^0 \\
&\quad -\sum_{k=1}^{n} A_n(s^\alpha)A_k((-s)^\alpha)z^{-k}z^n \\
&= \sum_{k=0}^{n-1} A_0((-s)^\alpha)A_k(s^\alpha)z^k - \sum_{k=0}^{n-1} A_n(s^\alpha)A_{n-k}((-s)^\alpha)z^k \\
&= \sum_{k=0}^{n-1}[A_0((-s)^\alpha)A_k(s^\alpha) - A_n(s^\alpha)A_{n-k}((-s)^\alpha)]z^k,
\end{aligned}$$

and
$F^{(1)}((-s)^\alpha, h) = \sum_{k=0}^{n-1} A_0(s^\alpha)A_k((-s)^\alpha) - A_n((-s)^\alpha)A_{n-k}(s^\alpha)z^{-k}$. We now define $F^{(2)}$ similarly as in the next example.

Example 2.3.9. Consider $F(s^\alpha, h) = s^\alpha - s^\alpha e^{-sh} + (s^\alpha - 2)e^{-2sh} = 0$, by using the systematic method. We have

$$\begin{aligned}
F^{(1)}(s^\alpha, h) &= (-s)^\alpha (s)^\alpha - (s^\alpha - 2)((-s)^\alpha - 2) + [(-s)^\alpha(-(s)^\alpha) - (s^\alpha - 2)(-(-s)^\alpha)]z \\
&= (2s^\alpha(1 + e^{-\pi i\alpha}) - 4) + (-2e^{-\pi i\alpha} s^\alpha)z \\
&= -2 + s^\alpha(1 + e^{-\pi i\alpha}) - e^{-\pi i\alpha} s^\alpha z \\
&= 0,
\end{aligned}$$

then

$$F^{(1)}((-s)^\alpha, h) = ((s)^\alpha(1 + e^{-\pi i\alpha}) - 2) - s^\alpha z^{-1},$$

Let

$$F^{(2)}(s^\alpha, h) = A_0^{(1)}(-s)F^{(1)}(s^\alpha, h) - A_1^{(1)}(s)z^1 F^{(1)}((-s)^\alpha, h).$$

Then

$$\begin{aligned}
F^{(2)}(s^\alpha, h) &= (s^\alpha(1 + e^{-\pi i\alpha}) - 2)(s^\alpha(1 + e^{-\pi i\alpha}) - 2) - (-(s)^\alpha)(-(-s)^\alpha) \\
&= s^{2\alpha}(1 + e^{-\pi i\alpha})^2 - 4s^\alpha(1 + e^{-\pi i\alpha}) + 4 - s^{2\alpha}e^{-\pi i\alpha} \\
&= s^{2\alpha}(1 + 2e^{-\pi i\alpha} + e^{-2\pi i\alpha}) - 4s^\alpha(1 + e^{-\pi i\alpha}) + 4 - s^{2\alpha}e^{-\pi i\alpha} \\
&= s^{2\alpha}(1 + e^{-\pi i\alpha} + e^{-2\pi i\alpha}) - 4s^\alpha(1 + e^{-\pi i\alpha}) + 4 \\
&= 0.
\end{aligned}$$

And we get

$$s^\alpha = \frac{2(1 + e^{-\pi i\alpha}) \pm 2\sqrt{(1 + e^{-\pi i\alpha})^2 - (1 + e^{-\pi i\alpha} + e^{-2\pi i\alpha})}}{(1 + e^{-\pi i\alpha} + e^{-2\pi i\alpha})}$$

$$s = [\frac{2(1 + e^{-\pi i\alpha}) \pm 2\sqrt{(1 + e^{-\pi i\alpha})^2 - (1 + e^{-\pi i\alpha} + e^{-2\pi i\alpha})}}{(1 + e^{-\pi i\alpha} + e^{-2\pi i\alpha})}]^{\frac{1}{\alpha}}.$$

so

$z = \frac{-2 + s^\alpha(1 + e^{-i\pi\alpha})}{s^\alpha e^{-i\pi\alpha}}$, where $z = e^{-sh}$, and

$$h = -\frac{1}{s}\log z.$$

When $\alpha = 1$, $s = \pm 2i$ and $h = \frac{\pi}{4}$,

and when $\alpha = 0.5$, $s = \pm 0.686i$ and $h = 3.433$.

Example 2.3.10. Let $G(s) = \frac{1}{s-(s-1)e^{-s}+(s-0.5)e^{-2s}}$ be a transfer function. By using the systematic method, we have

$$\begin{aligned}
F^{(1)}(s,h) &= \sum_{k=0}^{n-1}[A_0(-s)A_k(s) - A_n(s)A_{n-k}(-s)]z^k \\
&= A_0(-s)A_0(s) - A_2(s)A_2(-s) + [A_0(-s)A_1(s) - A_2(s)A_1(-s)]z \\
&= (-s)(s) - (s-0.5)(-s-0.5) + [(-s)(-(s-1)) - (s-0.5) \\
&\quad (-(-s-1))]z \\
&= -s^2 - (s-0.5)(-s-0.5) + [s(s-1) + (s-0.5)(-s-1)]z \\
&= -s^2 - [-s^2 - 0.5s + 0.5s + 0.25] + [s^2 - s - s^2 - s + 0.5s + 0.5]z \\
&= -0.25 + [-1.5s + 0.5]z.
\end{aligned}$$

Then

$z = \frac{0.25}{0.5-1.5s}$.

$$\begin{aligned}
F^{(1)}(-s,h) &= \sum_{k=0}^{n-1}[A_0(s)A_k(-s) - A_n(-s)A_{n-k}(s)]z^{-k} \\
&= A_0(s)A_0(-s) - A_2(-s)A_2(s) + [A_0(s)A_1(-s) - A_2(-s)A_1(s)]z^{-1} \\
&= s(-s) - (-s-0.5)(s-0.5) + [s(-(-s-1)) - (-s-0.5)(-(s-1))]z^{-1} \\
&= -s^2 - [-s^2 + 0.5s - 0.5s + 0.25] + [s^2 + s - s^2 + s - 0.5s + 0.5]z^{-1} \\
&= -0.25 + [1.5s + 0.5]z^{-1},
\end{aligned}$$

thus

$$\begin{aligned}
F^{(2)}(s,h) &= \sum_{k=0}^{n-r-1} A_k^{(r+1)}(s)z^k \\
&= A_0^{(1)}(-s)A_0^{(1)}(s) - A_1^{(1)}(s)A_1^{(1)}(-s) \\
&= (-0.25)(-0.25) - (-1.5s + 0.5)(1.5s + 0.5) \\
&= 0.0625 + (-(1.5)^2s^2 - 0.5(1.5)s + 0.5(1.5)s + 0.25) \\
&= 0.0625 + (1.5)^2s^2 - 0.25.
\end{aligned}$$

Then

$s^2 = \frac{0.25(1-0.25)}{(1.5)^2} \simeq 0.83333$ then $s = \pm 0.289$.

Since s is not purely imaginary the poles do not cross the axis.

Example 2.3.11. Let $G(s) = \frac{1}{s^\alpha - (s^\alpha - 1)e^{-sh} + (s^\alpha - 0.5)e^{-2sh}}$ be a transfer function. This example was considered by Nguyen for $h = 1$. Using Theorem 2 [17], Nguyen [27] found asymptotic expressions for the poles of G.

For $\alpha = 0.2$ there are neutral chains of poles of $G(s)$ located in the left-hand half-plane. However, for $\alpha = 0.5$ there are neutral chains of poles of $G(s)$ located on both sides.

Now by using the systematic method we have,

$$\begin{aligned}
F^{(1)}(s,h) &= \sum_{k=0}^{n-1} [A_0((-s)^\alpha)A_k(s^\alpha) - A_n(s^\alpha)A_{n-k}((-s)^\alpha)]z^k \\
&= A_0((-s)^a)A_0(s^\alpha) - A_2(s^\alpha)A_2((-s)^\alpha) + [A_0((-s)^\alpha)A_1(s^\alpha) \\
&\quad -A_2(s^\alpha)A_1((-s)^\alpha)]z \\
&= (-s)^\alpha(s^\alpha) - (s^\alpha - 0.5)((-s)^\alpha - 0.5) + [(-s)^\alpha(-(s^\alpha - 1)) \\
&\quad -(s^\alpha - 0.5)(-((-s)^\alpha - 1))]z \\
&= s^{2\alpha}e^{\pi i\alpha} - (s^{2\alpha}e^{\pi i\alpha} - 0.5s^\alpha - 0.5s^\alpha e^{\pi i\alpha} + 0.25) + \\
&\quad [-s^{2\alpha}e^{\pi i\alpha} + s^\alpha e^{\pi i\alpha} + s^{2\alpha}e^{\pi i\alpha} - s^\alpha - 0.5s^\alpha e^{\pi i\alpha} + 0.5]z \\
&= 0.5s^\alpha + 0.5s^\alpha e^{\pi i\alpha} - 0.25 + [0.5s^\alpha e^{\pi i\alpha} - 0.5s^\alpha + 0.5]z \\
&= 0.
\end{aligned}$$

Then

$$z = \frac{0.25 - 0.5s^\alpha(1 + e^{\pi i\alpha})}{0.5s^\alpha e^{\pi i\alpha} - s^\alpha + 0.5}.$$

Hence

$$F^{(1)}(-s,h) = 0.5s^\alpha e^{\pi i\alpha} + 0.5s^\alpha - 0.25 + (0.5s^\alpha - s^\alpha e^{\pi i\alpha} + 0.5)z^{-1} = 0,$$

and,

$$
\begin{aligned}
F^{(2)}(s,h) &= \sum_{k=0}^{n-r-1} A_k^{(r+1)}(s^\alpha)z^k \\
&= A_0^{(1)}((-s)^\alpha)A_0^{(1)}(s^\alpha) - A_1^{(1)}(s^\alpha)A_1^{(1)}((-s)^\alpha) \\
&= (0.5s^\alpha e^{\pi i\alpha} + 0.5s^\alpha - 0.25)(0.5s^\alpha + 0.5s^\alpha e^{\pi i\alpha} - 0.25) - \\
&\quad [(0.5s^\alpha e^{\pi i\alpha} - s^\alpha - 0.5)(0.5s^\alpha - s^\alpha e^{\pi i\alpha} - 0.5)] \\
&= -s^{2\alpha}e^{\pi i\alpha} + s^{2\alpha}e^{2\pi i\alpha} + s^{2\alpha} - 0.25 \\
&= 0.
\end{aligned}
$$

Then

$$s^{2\alpha} = \frac{0.25}{e^{2\pi i\alpha} - e^{\pi i\alpha} + 1},$$

so

$$s = [\frac{0.25}{e^{2\pi i\alpha} - e^{\pi i\alpha} + 1}]^{\frac{1}{2\alpha}}.$$

In the particular case when $\alpha = 1$, then $s = [\frac{0.25}{-(-1)+1+1}]^{0.5} \simeq 0.289$ and $h = \frac{-1}{s}\log(z)$.

In this system we notice that:

1. The asymptotes show a change at $\alpha = \frac{1}{3}$.

2. Analysis of the small poles shows a change at $\alpha = 0.297$.

   So we have three cases:

   - For $0 < \alpha < 0.297$ the system has no unstable poles.
   - For $0.297 < \alpha < 0.333$ the system has finitely many unstable poles.
   - For $0.333 < \alpha$ the system has infinitely many unstable poles.

2.4 Fractional systems with particular values of α

2.4.1 Fractional systems with $\alpha = 0.5$

Example 2.4.1. Let

$$G_h(s) = \frac{1}{\sqrt{s} + e^{-h\sqrt{s}}}$$

where $h \geq 0$. Then G_h is stable for $0 \leq h < \frac{3\pi}{2\sqrt{2}} e^{3\pi/4}$. As h increases, the poles cross the axis from left to right.

Proof. We consider the variation of the zeros of $\sqrt{s} + e^{-h\sqrt{s}}$ as h increases: in particular the values of h at which they cross the y-axis. Equivalently, we consider the values of $h > 0$ for which $G_h(u) = \frac{1}{u+e^{-hu}}$ has a zero on the line $\{u \in \mathbb{C} : \arg u = \pi/4\}$. Accordingly, suppose that $e^{-hu} = -u$, and let $u = xe^{i\pi/4}$, where $x > 0$.

We have

$$xe^{i\pi/4} + e^{-hxe^{i\pi/4}} = 0,$$

and so

$$xe^{-i\pi/4} + e^{-hxe^{-i\pi/4}} = 0.$$

Then

$$e^{2hx\cos(\pi/4)} = x^2,$$

and

$$e^{-2ihx\sin(\pi/4)} = e^{-i\pi/2}.$$

We now eliminate h and solve for x, so that

$$i \log x^2 = \frac{i\pi}{2} + 2in\pi \quad (n \in \mathbb{Z}),$$

whence $x = e^{\pi/4+n\pi}$, and

$$h = \frac{\frac{\pi}{2} + 2n\pi}{\sqrt{2}e^{\pi/4+n\pi}}.$$

The smallest positive value of h occurs at $n = -1$, giving $h = \frac{3\pi}{2\sqrt{2}} e^{3\pi/4}$.

Now, it is straightforward to check that for very small positive values of h

the transfer function G_h is asymptotically stable, and so it remains stable until the first pole-crossing, which is at $h = \frac{3\pi}{2\sqrt{2}}e^{3\pi/4}$.
It is possible to show that the poles cross from left to right as h increases by calculating $\frac{\partial s}{\partial h}$ at a point where $\sqrt{s} + e^{-h\sqrt{s}} = 0$. Similar calculations are done for delay systems in [31] and [39].
We have

$$\frac{1}{2\sqrt{s}}\frac{\partial s}{\partial h} - [\sqrt{s}e^{-h\sqrt{s}} + \frac{h}{2\sqrt{s}}\frac{\partial s}{\partial h}]e^{-h\sqrt{s}} = 0,$$

now it is easy to deduce a formula for $\frac{\partial s}{\partial h}$. □

Also, we have another argument to solve this example.
Take $\sqrt{s} = u$,
then

$$G(u) = \frac{1}{u + e^{-uh}}.$$

By Lemma 6.1.2 ([31]), we have,
$u + e^{-uh} = 0$ so, $ue^{uh} = -1$, let $z = hu$ then $u = \frac{z}{h}$ and thus, $ze^z = -h$.
Suppose that $z_n = x_n + iy_n$, then

$$x_n = -\log(2n\pi) + \log|-h| + o(1) = -\log(2n\pi) + \log(h) + o(1),$$

and

$$y_n = \pm 2n\pi \mp \frac{\pi}{2} + \arg(-h) + o(1).$$

Here $u_n = \frac{z_n}{h}$, then $u_n = \frac{x_n}{h} + i\frac{y_n}{h}$, and $s_n = u_n^2$, so

$$u_n^2 = (\frac{x_n}{h} + i\frac{y_n}{h})^2 = ((\frac{x_n}{h})^2 - (\frac{y_n}{h})^2) + \frac{2x_n}{h}\frac{y_n}{h}i,$$

so

$$u^2 \sim -n^2.$$

Then $|\operatorname{Re} s_n| \sim n^2$, and $|\operatorname{Im} s_n| \sim n \log n$, with $\operatorname{Re} s_n < 0$.

Theorem 2.4.2. *Let $G(s) = \frac{1}{p(\sqrt{s}) + q(\sqrt{s})e^{-h\sqrt{s}}}$ be the transfer function of a neutral fractional exponential system. Then the poles (s_n) of G satisfy*

$$s_n = \frac{\lambda_n^2}{h^2} - \frac{2\beta}{\alpha h} + \frac{\beta^2}{\alpha^2\lambda_n^2} + \frac{\beta^2}{\lambda_n} - \frac{2\gamma}{\alpha\lambda_n} - \frac{h\beta^3}{\alpha\lambda_n^3} + \frac{2h\beta\gamma}{\alpha^2\lambda_n^3} + \frac{h^2}{\lambda_n^4}[\frac{\beta^4}{4} - \frac{\beta^2\gamma}{\alpha} + \frac{\gamma^2}{\alpha^2}] + o(\frac{1}{n^2}).$$

and hence $|\operatorname{Re} s_n| \sim n^2$, and $|\operatorname{Im} s_n| \sim n^{-1}$, with $\operatorname{Re} s_n < 0$, for large n and $\alpha = \pm 1$.

Proof. Let $\sqrt{s} = u$ so, $s = u^2$, then

$$G(u) = \frac{1}{p(u) + q(u)e^{-hu}}.$$

By Theorem 2.1 [32], we have

$$\frac{p(u)}{q(u)} = \alpha + \frac{\beta}{u} + \frac{\gamma}{u^2} + O(\frac{1}{u^3}) \quad \text{as} \quad |u| \to \infty,$$

for constants α, β and γ with $\alpha = \pm 1$.

For sufficiently large integers n let $\lambda_n = 2ni\pi$ if $\alpha = -1$ and let $\lambda_n = (2n+1)i\pi$ if $\alpha = -1$.

Then the poles u_n of G satisfy

$$u_n = \frac{\lambda_n}{h} - \frac{\beta}{\alpha\lambda_n} + \frac{h}{\lambda_n^2}(\frac{\beta^2}{2} - \frac{\gamma}{\alpha}) + o(\frac{1}{n^2}),$$

so,

$$\begin{aligned} s_n = u_n^2 &= [\frac{\lambda_n}{h} - \frac{\beta}{\alpha\lambda_n} + \frac{h}{\lambda_n^2}(\frac{\beta^2}{2} - \frac{\gamma}{\alpha}) + o(\frac{1}{n^2})]^2 \\ &= \frac{\lambda_n^2}{h^2} - \frac{2\beta}{\alpha h} + \frac{\beta^2}{\alpha^2\lambda_n^2} + \frac{\beta^2}{\lambda_n} - \frac{2\gamma}{\alpha\lambda_n} - \frac{h\beta^3}{\alpha\lambda_n^3} + \frac{2h\beta\gamma}{\alpha^2\lambda_n^3} + \frac{h^2}{\lambda_n^4}[\frac{\beta^4}{4} - \frac{\beta^2\gamma}{\alpha} + \\ &\quad \frac{\gamma^2}{\alpha^2}] + o(\frac{1}{n^2}). \end{aligned}$$

Then $|\operatorname{Re} s_n| \sim n^2$, and $|\operatorname{Im} s_n| \sim n^{-1}$, with $\operatorname{Re} s_n < 0$. ☐

Proposition 2.4.3. *Let $G(s) = \frac{1}{p(\sqrt{s}) + q(\sqrt{s})e^{-h\sqrt{s}}}$ be the transfer function of a neutral fractional exponential system. Then*

$$s_n = \frac{1}{h^2}\log^2(-\alpha) - \frac{i4n\pi}{h^2}\log(-\alpha) - \frac{4n^2\pi^2}{h^2}.$$

and

$|\operatorname{Re} s_n| \sim n^2$, *and* $|\operatorname{Im} s_n| \sim n$, *with* $\operatorname{Re} s_n < 0$ *and* $|\alpha| > 1$.

Proof. Let $\sqrt{s} = u$ so, $s = u^2$, then

$$G(u) = \frac{1}{p(u) + q(u)e^{-hu}}.$$

By Proposition 2.1 [32], then $\alpha \neq \pm 1$.

At poles of $G(u)$, we have $\frac{p(u)}{q(u)} = -e^{-uh}$, and thus $e^{-uh} = -\alpha + O(\frac{1}{u})$.

A standard argument involving Rouche's Theorem ([4], chapter 12) shows that the poles of G are asymptotic to the roots of $e^{-uh} = -\alpha$ or $-uh = \log(-\alpha) + i2n\pi$, $n \in \mathbb{N}$, sufficiently large. Taking real parts, we have, for a pole of G, $u_n = \frac{-1}{h}\log(-\alpha) - \frac{i2n\pi}{h}$, $\quad n \in \mathbb{Z}$, sufficiently large.
Then

$$s_n = u_n^2 = (\frac{-1}{h}\log(-\alpha))^2 + \frac{2}{h}\log(-\alpha)(\frac{i2n\pi}{h}) - \frac{4n^2\pi^2}{h^2},$$

therefore,

$$s_n = \frac{1}{h^2}\log^2(-\alpha) + \frac{i4n\pi}{h^2}\log(-\alpha) - \frac{4n^2\pi^2}{h^2}.$$

As a result, $|\operatorname{Re} s_n| \sim n^2$, and $|\operatorname{Im} s_n| \sim n$, with $\operatorname{Re} s_n < 0$.

□

Example 2.4.4. Let $G(s) = \frac{1}{\sqrt{s}(\sqrt{s}+1)+se^{-\sqrt{s}}}$ be the transfer function of a neutral fractional exponential system.
From Theorem 2.4.2, then $p(s) = \sqrt{s}(\sqrt{s}+1)$, $\quad q(s) = s$ and $h = 1$.
Let $u = \sqrt{s}$, Thus,

$$G(u) = \frac{1}{u(u+1) + u^2 e^{-u}},$$

and then, $p(u) = u(u+1)$, and $\quad q(u) = u^2$

thus

$\frac{p(u)}{q(u)} = \frac{u(u+1)}{u^2} = 1 + \frac{1}{u}$, so $\alpha = 1$, $\quad \beta = 1$ and $\gamma = 0$, then $\lambda_n = (2n+1)i\pi$.

therefore

$$u_n = \frac{\lambda_n}{h} - \frac{\beta}{\alpha\lambda} + \frac{h}{\lambda_n^2}(\frac{\beta^2}{2} - \frac{\gamma}{\alpha}) + o(\frac{1}{n^2}),$$

and then

$$u_n = \frac{(2n+1)i\pi}{1} - \frac{1}{(2n+1)i\pi} + \frac{1}{(2(2n+1)^2\pi^2} + o(\frac{1}{n^2}).$$

But we have

$$s_n = u_n^2 = [\frac{(2n+1)i\pi}{1} - \frac{1}{(2n+1)i\pi} + \frac{1}{(2(2n+1)^2\pi^2} + o(\frac{1}{n^2})]^2,$$

thus

$$s_n = -(2n+1)^2\pi^2 + 2 - \frac{1}{(2n+1)^2\pi^2} + \frac{2i}{2(2n+1)\pi} - \frac{1}{(2n+1)^3\pi^3 i} + \frac{1}{4(2n+1)^4\pi^4},$$

therefore

$$\operatorname{Re} s_n = -(2n+1)^2\pi^2 + 2 - \frac{1}{(2n+1)^2\pi^2} + \frac{1}{4(2n+1)^4\pi^4} \quad \text{as} \quad n \to \infty.$$

As result $\operatorname{Re} s_n = O(n^2)$, with $\operatorname{Re} s_n < 0$.

Theorem 2.4.5. *Let* $G(s) = \frac{1}{p(\sqrt{s})+q(\sqrt{s})e^{-h\sqrt{s}}}$ *be the transfer function of an advanced fractional exponential system. Then*

$$\operatorname{Re} s_n \asymp n^2 \quad \text{and} \quad \operatorname{Im} s_n \asymp \log n \quad \text{with} \quad \operatorname{Re} s_n < 0.$$

Proof. As G is of a advanced type $\deg p = d_0 < \deg q = d_1$.

Let $\sqrt{s} = u$, so $s = u^2$, and then

$$G(u) = \frac{1}{p(u) + q(u)e^{-hu}}.$$

By the Theorem 6.1.4 [31] the roots of $p(u) + q(u)e^{-hu} = 0$ are asymptotic to the roots of $u^{d_0} + u^{d_1}e^{-uh} = 0$.

Then, $u^{d_0-d_1} = -e^{-uh}$, or $u^{d_0-d_1}e^{uh} = -1$.

Let $z = \frac{hu}{d_0-d_1}$, then $ze^z = \frac{h(-1)^{1/(d_0-d_1)}}{d_0-d_1}$.

So, by Lemma 6.1.2 [31] with $z = x + iy$ the solutions are

$$x_n = -\log(2n\pi) + \log\left(\left|\frac{h(-1)^{1/(d_0-d_1)}}{d_0-d_1}\right|\right) + o(1),$$

and

$$y_n = \pm 2n\pi \mp \frac{\pi}{2} + \arg\left(\frac{h(-1)^{1/(d_0-d_1)}}{d_0-d_1}\right) + o(1).$$

Hence $z = \frac{hu}{d_0-d_1}$, so $u = \frac{d_0-d_1}{h}(x+iy)$, and then $\operatorname{Re} u_n \asymp \log n$, with $\operatorname{Re} u_n < 0$.

However, we have $s_n = u_n^2$, then

$$s_n = \left(\frac{d_0-d_1}{h}\right)^2(x_n+iy_n)^2 = \left(\frac{d_0-d_1}{h}\right)^2(x_n^2 + 2x_ny_ni - y_n^2),$$

and therefore $\operatorname{Re} s = (\frac{d_0-d_1}{h})^2(x^2-y^2)$.

Then

$$\operatorname{Re} s_n = \left(\frac{d_0-d_1}{h}\right)^2(x^2-y^2),$$

and then

$$\operatorname{Re} s_n \asymp n^2 \quad \text{and} \quad \operatorname{Im} s_n \asymp \log n \quad \text{with} \quad \operatorname{Re} s_n < 0.$$

$\square$

Theorem 2.4.6. *Let $G(s) = \frac{1}{p(\sqrt{s})+q(\sqrt{s})e^{-h\sqrt{s}}}$ be the transfer function of a retarded fractional exponential system.*
Then

$$\operatorname{Re} s_n \asymp n^2 \quad \textit{and} \quad \operatorname{Im} s_n \asymp n \log n \quad \textit{with} \quad \operatorname{Re} s_n < 0.$$

Proof. As G is of a retarded type $\deg p = d_0 > \deg q = d_1$.
Let $\sqrt{s} = u$ so, $s = u^2$, then

$$G(u) = \frac{1}{p(u) + q(u)e^{-hu}}.$$

By the Theorem 6.1.4 [31] the roots of $p(u) + q(u)e^{-hu} = 0$ are asymptotic to the roots of $u^{d_0} + u^{d_1}e^{-uh} = 0$.
Then $u^{d_0-d_1} = -e^{-uh}$, or $u^{d_0-d_1}e^{uh} = -1$.
Let $z = \frac{hu}{d_0-d_1}$ then, $ze^z = \frac{h(-1)^{1/(d_0-d_1)}}{d_0-d_1}$. So, by Lemma 6.1.2 [31] with $z = x + iy$ the solutions are

$$x_n = -\log(2n\pi) + \log(\left|\frac{h(-1)^{1/(d_0-d_1)}}{d_0 - d_1}\right|) + o(1),$$

and

$$y_n = \pm 2n\pi \mp \frac{\pi}{2} + \arg(\frac{h(-1)^{1/(d_0-d_1)}}{d_0 - d_1}) + o(1).$$

Hence $z = \frac{hu}{d_0-d_1}$, so $u = \frac{d_0-d_1}{h}(x+iy)$, and then $\operatorname{Re} u_n \asymp \log n$, with $\operatorname{Re} u_n < 0$.
But $s = u^2$, so

$$s_n = (\frac{d_0 - d_1}{h})^2(x_n + iy_n)^2 = (\frac{d_0 - d_1}{h})^2(x_n^2 + 2x_ny_ni - y_n^2),$$

then

$$\operatorname{Re} s = (\frac{d_0 - d_1}{h})^2(x^2 - y^2).$$

Therefore,

$$\operatorname{Re} s_n \asymp n^2 \quad \text{and} \operatorname{Im} s_n \asymp n \log n \quad \text{with} \operatorname{Re} s_n < 0.$$

Here we illustrate one method for finding the h where the poles cross the axis. $\square$

Example 2.4.7. Let $G(s) = \frac{1}{\sqrt{s}+e^{-h\sqrt{s}}}$.
Let $\sqrt{s} = x + ix$,
then,

$$x + ix + e^{-h(x+ix)} = 0,$$

and the conjugate form is

$$(x - ix) + e^{-h(x-ix)} = 0.$$

The real part is

$$x + e^{-hx}\cos(hx) = 0,$$

and the imaginary part is

$$x - e^{-hx}\sin(hx) = 0.$$

Thus we have $\tan(hx) = -1$, so $hx = \frac{3\pi}{4} + n\pi$, thus $x = \frac{(\frac{3\pi}{4}+n\pi)}{h}$.
By substituting the value of x in $x + e^{-hx}\cos(hx) = 0$, we get an infinite number of solutions for h but the smallest h is $h = \frac{-3\pi e^{\frac{3\pi}{4}}}{4\cos(\frac{3\pi}{4})} \simeq 34.817$.

2.4.2 Procedure for finding zero-crossings

Let $G(s) = \frac{1}{p(\sqrt{s})+q(\sqrt{s})e^{-h\sqrt{s}}}$ be the transfer function of a fractional delay system.
Let $\sqrt{s} = x + ix$. Thus we have

$$p(x + ix) + q(x + ix)e^{-h(x+ix)} = 0,$$

and the conjugate form is

$$p(x - ix) + q(x - ix)e^{-h(x-ix)} = 0.$$

Now we have to eliminate h, then find $x > 0$,
so

$$e^{-hx-ihx} = -\frac{p(x+ix)}{q(x+ix)},$$

and

$$e^{-hx+ihx} = -\frac{p(x-ix)}{q(x-ix)},$$

then

$$e^{-2hx} = A(x), \quad \text{where} \quad A(x) = \frac{p(x+ix)p(x-ix)}{q(x+ix)q(x-ix)},$$

and

$$e^{-i2hx} = B(x), \quad \text{where} \quad B(x) = \frac{p(x+ix)q(x-ix)}{p(x-ix)q(x+ix)},$$

thus $\log(B) = i\log(A)$. From this equality we can find the value of x then substituting in $e^{-2hx} = A(x)$ to find the value of h.

Example 2.4.8. Let $G(s) = \frac{1}{\sqrt{s}+e^{-h\sqrt{s}}}$.
Let $\sqrt{s} = x + ix$,
Thus we have

$$e^{-hx-ihx} = -(\frac{x+ix}{1}),$$

and

$$e^{-hx+ihx} = -(\frac{x-ix}{1}),$$

then

$$e^{-2hx} = (\frac{x+ix}{1})(\frac{x-ix}{1}) = 2x^2,$$

and

$$e^{-i2hx} = \frac{x+ix}{x-ix} = \frac{1+i}{1-i} = i.$$

Then

$$-2hx = \log(2x^2) + 2in\pi,$$

and

$$-2hix = i\frac{\pi}{2} + 2im\pi.$$

Then, $i = \frac{i(\frac{\pi}{2}+2m\pi)}{\log(2x^2)+2in\pi}$, so $\log(2x^2) + 2in\pi = \frac{\pi}{2} + 2m\pi$ and then, $n = 0$,
Therefore $2x^2 = e^{\frac{\pi}{2}+2m\pi}$ so

$$x = \frac{1}{\sqrt{2}}e^{\frac{\pi}{4}+m\pi}.$$

Thus

$-2hx = \frac{\pi}{2} + 2m\pi \Rightarrow \quad h = \frac{\frac{\pi}{2}+2m\pi}{-2(\frac{1}{\sqrt{2}}e^{\frac{\pi}{4}+m\pi})}$, for the value $m=-1$, then

$$h = \frac{3\pi}{2\sqrt{2}}e^{\frac{3\pi}{4}}.$$

2.4.3 More general procedure for finding zero-crossings

In this section we are going to give a more general procedure to find zero-crossing. Let

$$G(s) = \frac{1}{p(s^\alpha) + q(s^\alpha)e^{-hs^\alpha}}.$$

Let $s = ye^{\frac{\pi i}{2}}$, then

$$s^\alpha = y^\alpha e^{\frac{\pi i\alpha}{2}}.$$

Thus we have,

$$p(y^\alpha e^{\frac{\pi i\alpha}{2}}) + q(y^\alpha e^{\frac{\pi i\alpha}{2}})e^{-hy^\alpha e^{\frac{\pi i\alpha}{2}}} = 0,$$

and the conjugate form is

$$p(y^\alpha e^{\frac{-\pi i\alpha}{2}}) + q(y^\alpha e^{-\frac{\pi i\alpha}{2}})e^{-hy^\alpha e^{\frac{-\pi i\alpha}{2}}} = 0.$$

We have to eliminate h, then find $y > 0$, then

$$e^{-2hy^\alpha \cos(\frac{\pi\alpha}{2})} = \frac{p(y^\alpha e^{\frac{\pi i\alpha}{2}})p(y^\alpha e^{\frac{-\pi i\alpha}{2}})}{q(y^\alpha e^{\frac{\pi i\alpha}{2}})q(y^\alpha e^{-\frac{\pi i\alpha}{2}})}.$$

Let

$$I = -2hy^\alpha = \frac{1}{\cos(\frac{\pi\alpha}{2})}\log[\frac{p(y^\alpha e^{\frac{\pi i\alpha}{2}})p(y^\alpha e^{\frac{-\pi i\alpha}{2}})}{q(y^\alpha e^{\frac{\pi i\alpha}{2}})q(y^\alpha e^{\frac{-\pi i\alpha}{2}})}],$$

and

$$e^{-2ihy^\alpha \sin(\frac{\pi\alpha}{2})} = \frac{p(y^\alpha e^{\frac{\pi i\alpha}{2}})q(y^\alpha e^{\frac{-\pi i\alpha}{2}})}{p(y^\alpha e^{\frac{-\pi i\alpha}{2}})q(y^\alpha e^{\frac{\pi i\alpha}{2}})}.$$

Let

$$II = -2hy^\alpha = \frac{1}{i\sin(1\frac{\pi\alpha}{2})}\log[\frac{p(y^\alpha e^{\frac{\pi i\alpha}{2}})q(y^\alpha e^{\frac{-\pi i\alpha}{2}})}{p(y^\alpha e^{\frac{-\pi i\alpha}{2}})q(y^\alpha e^{\frac{\pi i\alpha}{2}})}].$$

Thus, $I = II$.

This equality gives us the value of y, then we can find the value of h from the previous equations after substituting the value of y.

Comment

For $\alpha = 0.5$ the previous procedure gives the same answer.

Example 2.4.9. Let $G(s) = \frac{1}{s^{1/3}+e^{-hs^{1/3}}}$.
Let $s = ye^{\frac{\pi i}{2}}$, then $s^{1/3} = y^{1/3}e^{\frac{\pi i}{6}}$, thus we have

$$y^{1/3}e^{\frac{\pi i}{6}} + e^{-hy^{1/3}(\cos(\frac{\pi}{6})+i\sin(\frac{\pi}{6})} = 0,$$

and the conjugate form is

$$y^{1/3}e^{\frac{-\pi i}{6}} + e^{-hy^{1/3}(\cos(\frac{\pi}{6})-i\sin(\frac{\pi}{6})} = 0.$$

We have to eliminate h, then find $y > 0$,

$$y^{2/3} = e^{-hy^{1/3}\cos(\frac{\pi}{6})+2n\pi i},$$

then $\log(y^{2/3}) = -hy^{1/3}\cos(\frac{\pi}{6}) + 2n\pi i \Rightarrow \quad n = 0$, thus $I = -2hy^{1/3} = \frac{2/3\log(y)}{\cos(\frac{\pi}{6})}$. Also,
$e^{\frac{\pi i}{3}} = e^{-hy^{1/3}i\sin(\frac{\pi}{6})+2m\pi i}$.

Let

$$II = -2hy^{1/3} = \frac{\pi}{3\sin(\frac{\pi}{6})} - \frac{2m\pi}{\sin(\frac{\pi}{6})}.$$

Thus, $I = II$.

Then

$$y = e^{\frac{3\sqrt{3}\pi}{2}(1/3-2m)}.$$

Now, we can calculate the value of h,

$$h = [\frac{\frac{2\pi}{3} - \frac{4m\pi}{\sqrt{3}}}{-2e^{\frac{\sqrt{3}\pi}{2}(1/3-2m)}}],$$

we choose $m = 1$ for the smallest $h > 0$, so

$$h = \frac{\pi(2\sqrt{3}-1)}{3e^{\frac{-5\sqrt{3}\pi}{6}}}.$$

Chapter 3

Diffusive Systems

3.1 Introduction

In this chapter we mainly focus on diffusive systems, the Hankel operator and the Θ operator. We are looking at diffusive systems which are continuous-time linear systems with impulse response $h(t)$ which can be represented as

$$h(t) = \int_0^\infty e^{-t\xi} \mathrm{d}\mu(\xi),$$

and the transfer function $G(s)$, defined as the Laplace transform of the impulse response $h(t)$, is

$$G(s) = \int_0^\infty \frac{\mathrm{d}\mu(\xi)}{s+\xi},$$

where μ is a signed measure defined on $\mathbb{R}$. If μ is absolutely continuous we write $\mathrm{d}\mu(\xi) = f(\xi)\mathrm{d}\xi$. We give a theorem that gives us the necessary and sufficient conditions for diffusive systems to be BIBO and H^∞ stable. Moreover, we consider a system with discrete measure μ where h is given by a series and μ is a sum of point masses, and we give necessary conditions for system to be BIBO and H^∞ stable.

In the theory of approximation of unstable systems, the coprime factor technique is based on coprime factorization of the system as $G(s) = \frac{N}{M}$ where N and M are functions defined on the right half of the complex plane. This technique plays an essential role in some interesting examples.

A number of techniques and tools are available for finding conditions that test properties of the Hankel operator and Θ operator of a diffusive system and in general for other weighted Hankel operators. Two tests in Howland's paper [19] have been adapted to test nuclearity of the Θ operator. The reproducing kernel test has been used see [8] to say that Γ (Hankel operator) is bounded if and only if $\sup_z \frac{\|\Gamma u_z\|_2}{\|u_z\|} < \infty$, where $u_z(t) = e^{-zt}$ for $t > 0$.

3.2 Diffusive Systems

Following Montseny [25] we make the following definition.

Definition 3.2.1. A diffusive system is a continuous-time linear system with impulse response $h(t)$ which can be represented as

$$h(t) = \int_0^\infty e^{-t\xi} \mathrm{d}\mu(\xi).$$

Note that h is real if μ is real. Also, the transfer function $G(s)$, defined as the Laplace transform of the impulse response $h(t)$, is

$$G(s) = \int_0^\infty \frac{\mathrm{d}\mu(\xi)}{s+\xi},$$

where μ is a signed measure defined on $\mathbb{R}$.
If μ is absolutely continuous we write $\mathrm{d}\mu(\xi) = f(\xi)\mathrm{d}\xi$, where f is absolutely continuous function.

Theorem 3.2.2. (See Montseny [25]). *A convolution system $y = h * u$ with diffusive representation μ can be realized as a diffusive equation*

$$\psi_t(\xi, t) = -\xi\psi(\xi, t) + u(t) \tag{3.2.1}$$

$$y(t) = \int_0^\infty f(\xi)\psi(\xi, t)\mathrm{d}\xi, \tag{3.2.2}$$

with $\psi(\xi, t)$ a state variable such that $\psi(\xi, 0) = 0$. Equivalently, as a heat equation

$$\Phi_t(x, t) = \Phi_{xx}(x, t) + \delta(x)u(t) \tag{3.2.3}$$

$$y(t) = \int_{-\infty}^{\infty} m.\Phi \mathrm{d}x, \tag{3.2.4}$$

with $m(x) = 4\pi^2 x f(4\pi^2 x^2)$ *and* $\Phi(x, 0) = 0$, *and equivalently*

$$\Psi_t = -4\pi^2\zeta^2\Psi + u, \quad \zeta \in \mathbb{R}, \quad \Psi(\zeta, 0) = 0 \tag{3.2.5}$$

$$y(t) = \int_{-\infty}^{\infty} \hat{m}.\Psi \mathrm{d}\zeta. \tag{3.2.6}$$

Proof. For a diffusive system we have

$$h(t) = \int_0^{\infty} e^{-t\xi} \mathrm{d}\mu(\xi)$$

and

$$y(t) = (h * u)(t) \quad = \quad \int_0^t \int_0^{\infty} e^{-x\xi} \mathrm{d}\mu(\xi) u(t-x) \mathrm{d}x.$$

By Fubini

$$\begin{aligned} y(t) \quad &= \quad \int_0^t \int_0^{\infty} e^{-x\xi} u(t-x) \mathrm{d}x f(\xi) \mathrm{d}\xi \\ &= \quad \int_0^{\infty} (e^{-t\xi} * u) f(\xi) \mathrm{d}\xi, \end{aligned}$$

and

$$\begin{aligned} G(s) = (\mathcal{L}h)(s) \quad &= \quad \int_0^{\infty} e^{-st} h(t) \mathrm{d}t \\ &= \quad \int_0^{\infty} \int_0^{\infty} e^{-st} e^{-t\xi} \mathrm{d}\mu(\xi) \mathrm{d}t. \end{aligned}$$

By Fubini

$$\begin{aligned} G(s) \quad &= \quad \int_0^{\infty} \int_0^{\infty} e^{-t(s+\xi)} \mathrm{d}t \mathrm{d}\mu(\xi) \\ &= \quad \int_0^{\infty} \frac{1}{s+\xi} \mathrm{d}\mu(\xi). \end{aligned}$$

To prove those three formulas are equivalent we make the change of variables $\xi = 4\pi^2\zeta^2$ and $\mathrm{d}\xi = 8\pi^2\zeta \mathrm{d}\zeta$ and from (3.2.1) and (3.2.2) we would get (3.2.5) and (3.2.6) respectively and from Fourier transform with respect to the $\xi-$ variable: $\Psi = \mathcal{F}\Phi$, it is easly shown that we would change (3.2.3) and (3.2.4) to (3.2.5) and (3.2.6) respectively. □

Theorem 3.2.3. *Let G be a transfer function of a diffusive system, where $h(t) = \int_0^\infty e^{-t\xi}\mathrm{d}\mu(\xi)$, and $\int_0^\infty \frac{\mathrm{d}|\mu|(\xi)}{\xi} < \infty$; then the system is BIBO stable and hence H^∞ stable. Moreover if $\mu \geqslant 0$, the system is BIBO stable and H^∞ stable if and only if $\int_0^\infty \frac{\mathrm{d}\mu(\xi)}{\xi} < \infty$.*

Proof. Part I holds, since,

$$\begin{aligned}\int_0^\infty |h(t)|\,\mathrm{d}t &= \int_{t=0}^\infty \int_{\xi=0}^\infty e^{-t\xi}\mathrm{d}\,|\mu|\,(\xi)\mathrm{d}t \\ &= \int_0^\infty \frac{\mathrm{d}\,|\mu|\,(\xi)}{\xi} \\ &< \infty.\end{aligned}$$

Then G is BIBO stable, hence G is H^∞ stable.

Now if $\mu \geqslant 0$ then, it is BIBO stable from Part I. Moreover, G is H^∞ stable. Conversely, if $\mu \geq 0$, then for $s > 0$

$$\begin{aligned}G(s) &= \int_0^\infty \frac{\mathrm{d}\mu(\xi)}{s+\xi} \\ &\leqslant \|G\|_\infty .\end{aligned}$$

Let $s \to 0$, then

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}\mu(\xi)}{\xi} &\leqslant \|G\|_\infty \\ &< \infty.\end{aligned}$$

Hence, if G is BIBO and H^∞ stable and $\mu \geqslant 0$, then $\int_0^\infty \frac{\mathrm{d}\mu(\xi)}{\xi} < \infty$.

□

Remark 3.2.4. In fact the above condition also implies that the system is also nuclear (see Howland [19]).

3.3 Hankel Operator

We shall consider the Hankel operator Γ_h on $L^2(0,\infty)$ defined by

$$\Gamma_h f(x) = \int_0^\infty h(x+y)f(y)\mathrm{d}y. \tag{3.3.1}$$

Theorem 3.3.1. (Howland Test 1[19, Theorem 2.3]). *If* $h(t) = \int_0^\infty e^{-\xi t} \mathrm{d}\mu(\xi)$ *where* μ *is a positive Borel measure, then* Γ_h *is a nuclear operator if and only if*

$$\int_0^\infty \frac{1}{\xi} \mathrm{d}\mu(\xi) < \infty.$$

Theorem 3.3.2. (Howland test 2[19, Theorem 2.1]). *If* $h(x) = \int_x^\infty k(t)\mathrm{d}t$, *where*

$$\int_0^\infty t^{1/2} (\int_t^\infty |k(s)|^2 \, \mathrm{d}s)^{1/2} \mathrm{d}t < \infty,$$

then $h(x)$ *is finite for* $x > 0$, *and the operator* Γ_h *of (3.3.1) is of nuclear type.*

We require the following notation.

Definition 3.3.3. $E_1(z) = \int_z^\infty \frac{e^{-t}}{t} \mathrm{d}t \qquad (|\arg z| < \pi)$.
$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n} \mathrm{d}t \qquad (n = 0, 1, 2, 3, ...; \operatorname{Re} z > 0)$.
Also we can define the step function $u(x)$,

$$u(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0. \end{cases} \tag{3.3.2}$$

see [1, p. 227, 1020].

Example 3.3.4. We will study some examples of diffusive systems which are BIBO stable or just H^∞ or neither.

1. Let $\mu = \delta_a$ and $h(t) = e^{-at} \in L^1$, then $G(s) = \frac{1}{s+a}$, $a > 0$, so it is BIBO stable and nuclear.

2. Let $f(\xi) = e^{-a\xi}$ and $h(t) = \frac{1}{t+a} \notin L^1$, then $G(s) = \int_0^\infty \frac{e^{-st}}{t+a} \mathrm{d}t \to \infty$ as $s \to \infty \notin H^\infty$, so it is not H^∞ stable hence it is not BIBO stable.

3. Let $f(\xi) = \frac{\xi^{n-1}e^{-a\xi}}{(n-1)!}$ and $h(t) = \frac{1}{(t+a)^n} \in L^1$, then $G(s) = a^{1-n}e^{as}E_n(as), (a > 0, n = 2, 3, ...)$,

   By using the Howland test Theorem 3.3.1,

$$\frac{1}{(n-1)!}\int_0^\infty \xi^{n-2}e^{-a\xi}\mathrm{d}\xi = \frac{1}{(n-1)a^{n-1}}\Gamma(n-1) < \infty,$$

   where Γ is the gamma function. So it is BIBO stable and nuclear.

4. Let $f(\xi) = \frac{\sin(\pi\xi)}{\pi\xi^\alpha}$, where, $0 < \mathrm{Re}\,\alpha < 1$ and $h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \notin L^1$. By using the Howland test Theorem 3.3.2, we can not tell whether the Γ_h operator is nuclear.

5. Let $f(\xi) = \frac{1}{\sqrt{\pi\xi}}$ and $h(t) = \frac{1}{\sqrt{t}} \notin L^1$, then $G(s) = \frac{\sqrt{\pi}}{\sqrt{s}}$ (fractional system), so it is not BIBO nor H^∞ stable.

6. Let $f(\xi) = \frac{u(\xi-1)}{\sqrt{\pi(\xi-1)}}$, where u is the step function (3.3.2), and $h(t) = \frac{e^{-t}}{\sqrt{t}} \in L^1$, then $G(s) = \frac{\sqrt{\pi}}{\sqrt{s+1}}$ (fractional system).
   By using the Howland test Theorem 3.3.1,

$$I = \int_1^\infty \frac{u(\xi-1)}{\xi\sqrt{\pi(\xi-1)}}\mathrm{d}\xi.$$

   Put $\xi = x + 1$, then

$$I \leq \int_0^1 \frac{1}{\sqrt{\pi x}}\mathrm{d}x + \int_1^\infty \frac{1}{\sqrt{\pi}x^{3/2}}\mathrm{d}x < \infty.$$

   So it is BIBO stable and nuclear.

7. Let $f(\xi) = \frac{1}{\sqrt{\pi\xi}}e^{-k^2/4\xi}$ and $h(t) = \frac{1}{\sqrt{t}}e^{-k\sqrt{t}} \in L^1$, then

$$G(s) = \int_0^\infty \frac{1}{\sqrt{t}}e^{-k\sqrt{t}}e^{-st}\mathrm{d}t.$$

   By using the Howland test Theorem 3.3.1, to calculate $\int_0^\infty \frac{1}{\xi\sqrt{\xi}}e^{-k^2/4\xi}$ put $z = \frac{1}{\sqrt{\xi}}$ then

$$\int_0^\infty ze^{-k^2z^2/4}\mathrm{d}z < \infty.$$

   Thus it is BIBO stable and nuclear.

8. Let $f(\xi) = \cos(a\sqrt{\xi})/(\pi\sqrt{\xi})$, a heat kernel with $h(t) = e^{-a^2/4t}/\sqrt{\pi t} \notin L^1$ and $G(s) = e^{-a\sqrt{s}}/\sqrt{s}$. This is not Hilbert-Schmidt, so it not nuclear, since
$$\int_0^\infty t\frac{e^{-a^2/2t}}{\pi t}\mathrm{d}t = \infty.$$
Indeed it is not even H^∞ stable.

Comment 3.3.5. If μ is not a positive measure we can have $\int_0^\infty \frac{\mathrm{d}|\mu(\xi)|}{\xi} = \infty$, but $h \in L^1$ (i. e. it is BIBO stable and H^∞ stable).

Example 3.3.6. Let $f(\xi) = \sin(\xi)$ and if $\sin(\xi)\mathrm{d}\xi = \mathrm{d}\mu(\xi)$, we have
$$\int_0^\infty \frac{\sin(\xi)}{\xi}\mathrm{d}\xi = \frac{\pi}{2} < \infty$$
and
$$\int_0^\infty \frac{|\sin(\xi)|}{\xi}\mathrm{d}\xi = \infty,$$
so $h(t) = \frac{1}{t^2+1} \in L^1$ and $G(s) = [\pi/2 + \mathrm{Si}(s)]\cos(s) + \mathrm{Ci}(s)\sin(s)$, so it is BIBO stable and H^∞ stable, where
$$\mathrm{Si}(z) = \int_0^z \frac{\sin(t)\mathrm{d}t}{t},$$
and
$$\mathrm{Ci}(z) = \gamma + \ln(z) + \int_0^z \frac{\cos(t)\mathrm{d}t}{t}.$$
Also it is nuclear, since, using the Howland test Theorem 3.3.2 we have, $k(t) = -h'(t)$, then, $h(t) = \int_t^\infty k(x)\mathrm{d}x$ and in this example we have, $k(t) = \frac{2t}{(t^2+1)^2}$.

Then,
$$\int_0^\infty t^{1/2}(\int_t^\infty \frac{4s^2}{(s^2+1)^4}\mathrm{d}s)^{1/2}\mathrm{d}t \approx \int_0^\infty t^{1/2}\frac{1}{t^{5/2}}\mathrm{d}t < \infty,$$
so it is nuclear.

Proposition 3.3.7. *If $\mu \geqslant 0$ and $h = \mathcal{L}\mu$, then $h \in L^2$ (i. e. $G \in H^2$) if and only if*
$$\begin{aligned}\int_0^\infty |h(t)|^2\,\mathrm{d}t &= \int_0^\infty [\int_0^\infty e^{-t\xi}\mathrm{d}\mu(\xi)\int_0^\infty e^{-tx}\mathrm{d}\mu(x)]\mathrm{d}t \\ &= \int_0^\infty\int_0^\infty \frac{1}{\xi + x}\mathrm{d}\mu(\xi)\mathrm{d}\mu(x) < \infty.\end{aligned}$$

Proposition 3.3.8. *If $\mu \geqslant 0$ and $h = \mathcal{L}\mu$, then the Hankel operator is Hilbert-Schmidt if and only if* $\int_0^\infty \int_0^\infty \frac{d\mu(\xi)d\mu(x)}{(\xi+x)^2} < \infty$.

Proof. According to Theorem (1.2.14) and using Fubini's theorem, we have,

$$\begin{aligned}
&\int_0^\infty t \left| \int_0^\infty e^{-t\xi} \mathrm{d}\mu(\xi) \int_0^\infty e^{-tx} \mathrm{d}\mu(x) \right| \mathrm{d}t \\
&= \int_0^\infty t(\left| \int_0^\infty e^{-t\xi} \mathrm{d}\mu(\xi) \right| \left| \int_0^\infty e^{-tx} \mathrm{d}\mu(x) \right|) \mathrm{d}t \\
&= \int_0^\infty t(\int_0^\infty \left| e^{-t\xi} \right| \mathrm{d}\mu(\xi) \int_0^\infty \left| e^{-tx} \right| \mathrm{d}\mu(x)) \mathrm{d}t \\
&= \int_0^\infty t(\int_0^\infty \int_0^\infty e^{-t\xi - tx} \mathrm{d}\mu(\xi) \mathrm{d}\mu(x)) \mathrm{d}t \\
&= \int_0^\infty t e^{-t(\xi+x)} \mathrm{d}t \int_0^\infty \int_0^\infty \mathrm{d}\mu(\xi) \mathrm{d}\mu(x) \\
&= \int_0^\infty \int_0^\infty \frac{\mathrm{d}\mu(\xi)\mathrm{d}\mu(x)}{(\xi+x)^2}.
\end{aligned}$$

This yields the result. □

3.4 Θ Operator

In this section we shall consider the scaled Hankel operator Θ on $L^2(0,\infty)$ given by

$$(\Theta u)(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/4} h(t+\tau) \tau^{-1/4} u(\tau) \mathrm{d}\tau. \tag{3.4.1}$$

Proposition 3.4.1. (See for instance [28]). *Θ is a Hilbert-Schmidt operator if and only if $h \in L^2(0,\infty)$ moreover, $\|\Theta\|_{HS} = \|h\|_2$.*

Theorem 3.4.2. *If Θ has the form (3.4.1) and $\mu \geqslant 0$, then Θ is of trace class (nuclear) if and only if*

$$\int_0^\infty \frac{1}{\sqrt{p}} \mathrm{d}\mu(p) < \infty.$$

Proof. We modify the proof of Theorem 2.3 in [19].

Let $\psi_p(t) = t^{-1/4} e^{-pt}$ and define,

$$T_0 = \int_0^\infty \langle ., \psi_p \rangle \psi_p \mathrm{d}\mu(p).$$

This integral clearly converges in trace norm to a non-negative operator, with

$$\begin{aligned} I &= trT_0 \\ &= \int_0^\infty \|\psi_p\|^2 \,\mathrm{d}\mu(p) \\ &= \int_0^\infty \int_0^\infty \left|t^{-1/4}e^{-pt}\right|^2 \mathrm{d}t\mathrm{d}\mu(p) \\ &= \int_0^\infty (\int_0^\infty t^{-1/2}e^{-2pt}\mathrm{d}t)\mathrm{d}\mu(p), \end{aligned}$$

letting $(2pt)^{1/2} = z$, so $\frac{1}{2}\sqrt{2p}t^{-1/2}\mathrm{d}t = \mathrm{d}z$,

and then

$$\begin{aligned} I &= \int_0^\infty (\int_0^\infty \sqrt{\frac{2}{p}}e^{-z^2}\mathrm{d}z)\mathrm{d}\mu(p) \\ &= \int_0^\infty \int_0^\infty \sqrt{\frac{2}{p}}e^{-z^2}\mathrm{d}z\mathrm{d}\mu(p) \\ &= \int_0^\infty \sqrt{\frac{2}{p}}\frac{\sqrt{\pi}}{2}\mathrm{d}\mu(p) \\ &= \sqrt{\frac{\pi}{2}}\int_0^\infty \frac{1}{\sqrt{p}}\mathrm{d}\mu(p) < \infty. \end{aligned}$$

After a simple computation with Fubini's theorem, we conclude that $T = T_0$, where T is that given in 3.4.1.

Moreover,

$$T_n = \int_{1/n}^n \langle ., \psi_p\rangle\psi_p\mathrm{d}\mu(p) \quad (n > 0).$$

In fact this is increasing sequence of nuclear operator with $T_n \leqslant T$, and thus,

$$0 \leqslant \int_{1/n}^n \frac{\mathrm{d}\mu(p)}{\sqrt{p}} = \sqrt{\frac{2}{\pi}}trT_n \leqslant \sqrt{\frac{2}{\pi}}trT < \infty.$$

This yields the result by letting $n \to \infty$. □

Theorem 3.4.3. *Define the operator* $(Tu)(t) = \int_0^\infty \omega(t)h(t+\tau)\omega(\tau)u(\tau)\mathrm{d}\tau$, *where* $\omega \geq 0$ *and* h *corresponds to a measure* $\mu \geq 0$, *and* $\psi_p \in L^2 \quad \forall p > 0$, *where* $\psi_p = e^{-pt}\omega(t)$, *then* T *is of trace class (nuclear) if and only if* $\int_0^\infty \|\psi_p\|^2 \,\mathrm{d}\mu(p) < \infty$.

Proof. Let $\psi_p = e^{-pt}\omega(t) \in L^2$ and define,

$$T_0 = \int_0^\infty \langle ., \psi_p\rangle \psi_p \mathrm{d}\mu(p).$$

This integral clearly converges in trace norm to a non-negative operator with

$$trT_0 = \int_0^\infty \|\psi_p\|_2^2 < \infty.$$

The proof continues by the same argument as in the proof of Theorem 3.4.3.

□

Theorem 3.4.4. *If* $h(x) = \int_x^\infty k(t)\mathrm{d}t$ *and* $k \in L^1(0,\infty)$ *where*

$$\int_0^\infty t^{1/4}(\int_t^\infty (k(x+t))^2 x^{-1/2}\mathrm{d}x)^{1/2}\mathrm{d}t < \infty. \tag{3.4.2}$$

Then $h(x)$ *is finite for* $x > 0$*, and the operator* Θ *of (3.4.1) is of trace class (nuclear).*

Proof. Since $k \in L^1(0,\infty)$, $h(x)$ is finite for $x > 0$. If $f, g \in L^2(0,\infty)$, then we have

$$\begin{aligned}
\langle \Theta f, g\rangle &= \int_0^\infty \overline{g(x)}(\Theta f)(x)\mathrm{d}x \\
&= \int_0^\infty \overline{g(x)} \int_0^\infty x^{-1/4} \int_{x+y}^\infty k(s)f(y)y^{-1/4}\mathrm{d}s\mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^\infty \overline{g(x)} \int_{y=0}^\infty x^{-1/4} \int_{t=y}^\infty k(x+t)\mathrm{d}t f(y)y^{-1/4}\mathrm{d}y\mathrm{d}x \\
&= \int_{x=0}^\infty \overline{g(x)} \int_{t=0}^\infty \int_{y=0}^t x^{-1/4}k(x+t)f(y)y^{-1/4}\mathrm{d}y\mathrm{d}t\mathrm{d}x
\end{aligned}$$

so

$$\begin{aligned}
|\langle \Theta f, g\rangle| &= \left|\int_0^\infty \left(\langle k_t(x)x^{-1/4}, g\rangle\langle f, \chi_{[0,t]}y^{-1/4}\rangle\right)\mathrm{d}t\right| \\
&\leq \|g\|_2 \|f\|_2 \int_0^\infty \left\|k_t(x)x^{-1/4}\right\|_2 \left\|\chi_{[0,t]}(y)y^{-1/4}\right\|_2 \mathrm{d}t \\
&\leq \|g\|_2 \|f\|_2 \int_0^\infty \sqrt{2}t^{1/4} \left\|k_t(x)x^{-1/4}\right\|_2 \mathrm{d}t < \infty.
\end{aligned}$$

Moreover

$$\Theta = \int_0^\infty \langle ., \chi_{[0,t]}y^{-1/4}\rangle x^{-1/4}k_t(x)\mathrm{d}t$$

where this integral converges weakly. However, if we estimate this integral in trace norm, we obtain

$$\begin{aligned}\|\Theta\|_1 &\leq \int_0^\infty \left\|\chi_{[0,t]} y^{-1/4}\right\|_2 \left\|k_t x^{-1/4}\right\|_2 \mathrm{d}t \\ &= \sqrt{2}\int_0^\infty t^{1/4} \left\|k_t x^{-1/4}\right\|_2 \mathrm{d}t\end{aligned}$$

where the integral converges. Thus, the operator Θ of (3.4.1) is of trace class (nuclear).

This proof is similar to Howland's Theorem 2.1 [19].

□

We have a more general result, as follows:

Theorem 3.4.5. *If $h(x) = \int_x^\infty k(t)\mathrm{d}t$ and $k \in L^1(0,\infty)$ define the operator*

$$(Tu)(t) = \int_0^\infty \omega(t)h(t+\tau)\omega(\tau)u(\tau)\mathrm{d}\tau,$$

where $\omega, \mu \geq 0$ and

$$\int_0^\infty \|\omega\| \, \|k_t(x)\omega(x)\| \, \mathrm{d}t < \infty.$$

Then $h(x)$ is finite for $x > 0$, and the operator T is of trace class (nuclear).

Proof. Since $k \in L^1(0,\infty)$, $h(x)$ is finite for $x > 0$. If $f, g \in L^2(0,\infty)$, then we have

$$\begin{aligned}\langle Tf, g\rangle &= \int_0^\infty \overline{g(x)}(Tf)(x)\mathrm{d}x \\ &= \int_0^\infty \overline{g(x)} \int_0^\infty \omega(x)h(x+y)\omega(y)f(y)\mathrm{d}y\mathrm{d}x \\ &= \int_0^\infty \overline{g(x)} \int_0^\infty \omega(x) \int_{x+y}^\infty k(s)f(y)\omega(y)\mathrm{d}s\mathrm{d}y\mathrm{d}x \\ &= \int_{x=0}^\infty \overline{g(x)} \int_{y=0}^\infty \omega(x) \int_{t=y}^\infty k(x+t)\mathrm{d}t f(y)\omega(y)\mathrm{d}y\mathrm{d}x \\ &= \int_{x=0}^\infty \overline{g(x)} \int_{t=0}^\infty \int_{y=0}^t \omega(x)k(x+t)f(y)\omega(y)\mathrm{d}y\mathrm{d}t\mathrm{d}x\end{aligned}$$

so

$$\begin{aligned} |\langle Tf, g\rangle| &= \left| \int_0^\infty \left(\langle k_t(x)\omega(x), g\rangle \langle f, \chi_{[0,t]}\omega(y)\rangle \right) \mathrm{d}t \right| \\ &\leq \|g\|_2 \|f\|_2 \int_0^\infty \|k_t(x)\omega(x)\|_2 \left\|\chi_{[0,t]}(y)\omega(y)\right\|_2 \mathrm{d}t \\ &\leq \|g\|_2 \|f\|_2 \int_0^\infty \|\omega\|_2 \|k_t(x)\omega(x)\|_2 \mathrm{d}t < \infty. \end{aligned}$$

Moreover

$$T = \int_0^\infty \langle ., \chi_{[0,t]}\omega(y)\rangle \omega(x) k_t(x) \mathrm{d}t$$

where this integral converges weakly. However, if we estimate this integral in trace norm, we obtain

$$\|T\|_1 \leq \int_0^\infty \left\|\chi_{[0,t]}\omega(y)\right\|_2 \|k_t\omega(x)\|_2 \mathrm{d}t$$

where the integral converges. Thus, the operator T is of trace class (nuclear). This proof is similar to Howland's Theorem 2.1 [19]. □

Theorem 3.4.6. (Mercer's Theorem), (see [13], Proposition 5.6.9). *If the non-negative, bounded, self-adjoint operator T has the continuous integral kernel $a(.,.)$ then,*

$$tr[T] = \int_X a(x,x)\mathrm{d}x \tag{3.4.3}$$

where the finiteness of either side implies the finiteness of the other.

We now show that Θ is a positive operator.

Theorem 3.4.7. *Let $h(t) = \int_0^\infty e^{-xt}\mathrm{d}\mu(x)$ with $\mu \geq 0$ then, Θ is a positive operator i.e. $\Theta \geq 0$.*

Proof. We have to prove that $\langle \Theta u, u\rangle \geq 0$, thus

$$\begin{aligned} \langle \Theta u, u\rangle &= \int_0^\infty \int_0^\infty t^{-1/4} h(t+\tau)\tau^{-1/4} u(\tau)\overline{u(t)}\mathrm{d}t\mathrm{d}\tau \\ &= \int_0^\infty \int_0^\infty \int_0^\infty t^{-1/4} e^{-x(t+\tau)} \mathrm{d}\mu(x) \tau^{-1/4} u(\tau)\overline{u(t)}\mathrm{d}t\mathrm{d}\tau \\ &= \int_0^\infty [\int_0^\infty t^{-1/4} e^{-xt}\overline{u(t)}\mathrm{d}t \int_0^\infty \tau^{-1/4} e^{-x\tau} u(\tau)\mathrm{d}\tau]\mathrm{d}\mu(x) \\ &= \int_0^\infty \left| \int_0^\infty t^{-1/4} e^{-xt}\overline{u(t)}\mathrm{d}t \right|^2 \mathrm{d}\mu(x) \geqslant 0. \end{aligned}$$

□

According to Mercer's Theorem we can find the trace of the Θ operator, where the kernel of Θ is $t^{-1/4}h(t,\tau)\tau^{-1/4}$.

Theorem 3.4.8. $tr\Theta = \int_0^\infty \sqrt{\frac{\pi}{2x}}\mathrm{d}\mu(x)$.

Proof. We have

$$\begin{aligned}
\Sigma_{n=1}^{\infty}\sigma_n(\Theta) &= \int_0^\infty t^{-1/4}h(t,t)t^{-1/4}\mathrm{d}t \\
&= \int_0^\infty t^{-1/2}h(2t)\mathrm{d}t \\
&= \int_0^\infty t^{-1/2}\int_0^\infty e^{-2tx}\mathrm{d}\mu(x)\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty t^{-1/2}e^{-2xt}\mathrm{d}t\mathrm{d}\mu(x) \\
&= \int_0^\infty \sqrt{\frac{\pi}{2x}}\mathrm{d}\mu(x).
\end{aligned}$$

□

Example 3.4.9. In Examples 3.3.4 and Example 3.3.6 we shall look at the cases where $h \in L^2$ and use Theorems 3.4.2 and 3.4.4 to examine the Θ operators to decide whether they are nuclear or not.

1. Let $\mu = \delta_a$ and $h(t) = e^{-at}$, then $G(s) = \frac{1}{s+a}, a > 0$ and $h \in L^2$ (i.e. $G \in H^2$), where

$$\int_0^\infty |h(t)|^2\,\mathrm{d}t = \int_0^\infty e^{-at}\mathrm{d}t = \frac{1}{2a} < \infty.$$

By using Theorem 3.4.2

$$\int_0^\infty \frac{\mathrm{d}\mu(\xi)}{\sqrt{\xi}} = \int_0^\infty \frac{\mathrm{d}\delta_a(\xi)}{\sqrt{\xi}} = \frac{1}{\sqrt{a}} < \infty.$$

So, the operator Θ is nuclear.

2. Let $f(\xi) = e^{-a\xi}$ and $h(t) = \frac{1}{t+a} \in L^2$, where

$$\int_0^\infty |h(t)|^2\,\mathrm{d}t = \int_0^\infty \frac{1}{t+a)^2}\mathrm{d}t = \frac{1}{a} < \infty.$$

By using Theorem 3.4.2

$$\int_0^\infty \frac{e^{-a\xi}}{\sqrt{\xi}}\mathrm{d}t = 2\int_0^\infty e^{-az^2}\mathrm{d}z < \infty.$$

So the operator Θ is nuclear.

3. Let $f(\xi) = \frac{\xi^{n-1}e^{-a\xi}}{(n-1)!}$ and $h(t) = \frac{1}{(t+a)^n} \in L^2$, where

$$\int_0^\infty |h(t)|^2 \,\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}t}{(t+a)^{2n}} < \infty.$$

In addition by using Theorem 3.4.2

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}\mu(\xi)}{\sqrt{\xi}} &= \int_0^\infty \frac{\xi^{n-1}e^{-a\xi}\mathrm{d}\xi}{\xi^{1/2}(n-1)!} \\ &= \int_0^\infty \frac{\xi^{(n-1/2-1)}e^{-a\xi}}{(n-1)!}\mathrm{d}\xi \\ &= \frac{\Gamma(n-1/2)}{a^{n-1/2}} < \infty.\end{aligned}$$

So, Θ is nuclear.

In Example 3.3.4, Examples 5-7 the operator Θ is not Hilbert-Schmidt ($h \notin L^2(0,\infty)$), so not nuclear.
In addition, in Example 3.3.6, with $h(t) = \frac{1}{t^2+1}$ and $k \in L^1$, the Θ operator is Hilbert-Schmidt but, using Theorem 3.4.4 fails since (using Maple)

$$\int_0^\infty t^{1/4}(\int_t^\infty (k(x+t))^2 x^{-1/2}\mathrm{d}x)^{1/2}\mathrm{d}t = \int_0^\infty t^{1/4}(\int_t^\infty \frac{(x+t)^2}{((x+t)^2+1)^2 x^{1/2}})^{1/2}\mathrm{d}t$$

$$\begin{aligned}&= \int_0^\infty t^{1/4}(\frac{\pi[8t^8+23t^6+23t^4+8t^3(t^2+1)^{5/2}+9t^2+3(t^2+1)^{5/2}t+1]}{(4(\sqrt{t^2+1}+2t)^{3/2}[t^8+4t^6+6t^4+4t^2+1+(t^2+1)^{7/2}t]})^{1/2}\mathrm{d}t \\ &= \infty.\end{aligned}$$

So we can not tell whether Θ is nuclear.

Boundedness of Θ

Theorem 3.4.10. *Write* $\Theta_\omega u(t) = \int_0^\infty \omega(t)h(t+\tau)\omega(\tau)u(\tau)\mathrm{d}\tau$ *and suppose that* $h(t) = \int_0^\infty e^{-tx}\mathrm{d}\mu(x)$, *with* $\mu, \omega \geqslant 0$. *Then* Θ *is bounded if*

$$\int_0^\infty \int_0^\infty (V(\frac{x+y}{2}))^2 V(x)V(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) < \infty,$$

where $V(x) = \|e^{-x\tau}\omega(\tau)\|_2$.

Proof. We first show that,

$$|\Theta u(t)| \leqslant \int_0^\infty \int_0^\infty \omega(t)e^{-xt}e^{-x\tau}\omega(\tau)\,|u(\tau)|\,\mathrm{d}\mu(x)\mathrm{d}\tau,$$

by Cauchy-Schwarz,

$$\leqslant \int_0^\infty \omega(t)e^{-xt}\left\|e^{-x\tau}\omega(\tau)\right\|_2 \|u\|_2\,\mathrm{d}\mu(x).$$

Let $V(x) = \|e^{-x\tau}\omega(\tau)\|_2$, so $V(x)$ can be worked out (depending on ω) in standard examples like $\omega = 1, \omega = t^{-1/4}$.

Moreover, we have

$$\begin{aligned}
\|\Theta u\|_2^2 &= \langle \Theta u, \Theta u\rangle \\
&\leqslant \int_0^\infty \int_0^\infty \omega(t)e^{-xt}V(x)\,\|u\|_2\,\mathrm{d}\mu(x) \times \int_0^\infty \omega(t)e^{-yt}V(y)\,\|u\|_2\,\mathrm{d}\mu(y)\mathrm{d}t \\
&\leqslant \|u\|_2^2 \int_0^\infty \int_0^\infty \int_0^\infty e^{-(x+y)t}(\omega(t))^2\mathrm{d}tV(x)V(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&\leqslant \|u\|_2^2 \int_0^\infty \int_0^\infty \left\|\omega(t)e^{-\frac{x+y}{2}t}\right\|_2^2 V(x)V(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) \\
&= \|u\|_2^2 \int_0^\infty \int_0^\infty (V(\frac{x+y}{2}))^2V(x)V(y)\mathrm{d}\mu(x)\mathrm{d}\mu(y).
\end{aligned}$$

This finishes the proof. □

Corollary 3.4.11. *(i) For the Γ operator, we have $\omega(t) = 1$ and*

$$V(x) = (\int_0^\infty e^{-2x\tau}\mathrm{d}\tau)^{1/2} = \sqrt{\frac{1}{2x}},$$

thus Γ is bounded if

$$\int_0^\infty \int_0^\infty \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{\sqrt{xy}(x+y)} < \infty.$$

(ii) For the Θ operator, we have $\omega = t^{-1/4}$ and

$$V(x) = (\int_0^\infty (e^{-x\tau}\tau^{-1/4})^2\mathrm{d}\tau)^{1/2} = (\frac{\pi}{2x})^{1/4},$$

thus Θ is bounded if

$$\int_0^\infty \int_0^\infty \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{\sqrt[4]{x}\sqrt[4]{y}\sqrt{x+y}} < \infty.$$

3.5 Reproducing Kernel Test

Theorem 3.5.1. *(i) See([8]). If Γ is a Hankel operator and $(\Gamma u)(t) = \int_0^\infty h(t+\tau)u(\tau)\mathrm{d}\tau$, then Γ is bounded if and only if $\sup_u \frac{\|\Gamma u\|_2}{\|u\|_2} < \infty$, where $u \neq 0$ and $u \in L^2$. Moreover, by using the reproducing kernel test for the case $h \geq 0$, Γ is bounded if and only if $\sup_{\mathrm{Re}\, x>0} \frac{\|\Gamma u_x\|_2}{\|u_x\|_2} < \infty$, where $u_x(t) = e^{-xt}$, for $t > 0$.*

(ii) If the operator Θ is bounded, then $\sup_u \frac{\|\Theta u\|}{\|u\|} < \infty$, and so $\sup_{\mathrm{Re}\, x>0} \frac{\|\Theta u_x(t)\|}{\|u_x\|} < \infty$.

We apply this idea on our Examples 3.3.4.

(i) In example 5, $h(t) = \frac{1}{\sqrt{t}}$.
Now let $u_x(t) = e^{-xt} \in L^2$, with $x = 1$ thus

$$
\begin{aligned}
\langle \Theta u, \Theta u \rangle &= \int_0^\infty (\Theta u)(t)\overline{(\Theta u)(t)}\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty t^{-1/4}h(t+\tau)\tau^{-1/4}u(\tau)t^{-1/4}h(t+\tau)\tau^{-1/4}\overline{u(\tau)}\mathrm{d}\tau\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \frac{t^{-1/4}\tau^{-1/4}e^{-\tau}t^{-1/4}\tau^{-1/4}e^{-\tau}}{\sqrt{t+\tau}\sqrt{t+\tau}}\mathrm{d}t\mathrm{d}\tau \\
&= \int_0^\infty \int_0^\infty \frac{t^{-1/2}\tau^{-1/2}e^{-2\tau}}{(t+\tau)}\mathrm{d}t\mathrm{d}\tau \\
&= \int_0^\infty \frac{t^{-1/2}\mathrm{d}t}{t+\tau} \int_0^\infty \tau^{-1/2}e^{-2x}\mathrm{d}\tau,
\end{aligned}
$$

let $\sqrt{t} = z$, then

$$
\begin{aligned}
&= 2\int_0^\infty [\int_0^\infty \frac{\mathrm{d}z}{\tau((z/\sqrt{\tau})^2+1)}]\tau^{-1/2}e^{-2\tau}\mathrm{d}\tau \\
&= 2\int_0^\infty \arctan(z/\sqrt{\tau})|_0^\infty \tau^{-3/2}e^{-2\tau}\mathrm{d}\tau \\
&= 2\int_0^\infty \frac{\pi}{2}\tau^{-3/2}e^{-2\tau}\mathrm{d}\tau,
\end{aligned}
$$

let $\sqrt{2\tau} = \omega$ thus

$$
\begin{aligned}
&= \frac{\pi}{\sqrt{2}}\int_0^\infty \frac{e^{-\omega^2}}{\omega^2}\mathrm{d}\omega \\
&= \infty,
\end{aligned}
$$

and $\|u_x\|_2 = (\int_0^\infty e^{-2xt}\mathrm{d}t)^{1/2} = \frac{1}{\sqrt{2x}}$ and for $x = 1$, then $\|u_x\|_2 = \frac{1}{\sqrt{2}}$.
Hence, $\Theta u \notin L^2$ and so the Θ operator is unbounded.

(ii) Example 6, $h(t) = \frac{e^{-t}}{\sqrt{t}}$.
Now, let $u(t) = e^{-xt}$, with $x = 1$, thus

$$\begin{aligned}
\langle \Theta u, \Theta u \rangle &= \int_0^\infty (\Theta u)(t)\overline{(\Theta u)(t)}\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \frac{t^{-1/2}e^{-2(t+\tau)}\tau^{-1/2}e^{-2\tau}}{t+\tau}\mathrm{d}t\mathrm{d}\tau \\
&= \int_0^\infty \int_0^\infty \frac{t^{-1/2}e^{-2t}\mathrm{d}t}{t+\tau}\tau^{-1/2}e^{-4\tau}\mathrm{d}\tau,
\end{aligned}$$

let $z = \sqrt{t}$, then

$$\begin{aligned}
&= 2\int_0^\infty [\int_0^\infty \frac{e^{-2z^2}\mathrm{d}z}{z^2+\tau}]\tau^{-1/2}e^{-4\tau}\mathrm{d}\tau \\
&= \infty,
\end{aligned}$$

and $\|u_x\|_2 = (\int_0^\infty e^{-2xt}\mathrm{d}t)^{1/2} = \frac{1}{\sqrt{2x}}$, thus for $x = 1$, $\quad \|u_x\|_2 = \frac{1}{\sqrt{2}}$.
Then $\Theta u \notin L^2$ so, Θ is unbounded (so not H-S). For the examples 7 and 8 we can not tell whether Θ is bounded using these methods. We develop further techniques in Chapter 5.

3.6 Special case of a discrete measure

We shall consider a system where h is given by a series and μ is a sum of point masses.

Example 3.6.1. Consider the following heat equation:

$$Z_t = Z_{xx} + b(x)u(t),$$

$$Z_x(0,t) = 0 = Z_x(1,t), Z(x,0) = z_0(x),$$
$$y(t) = \int_0^1 c(x)Z(x,t)\mathrm{d}x,$$

where $b(x)$ and $c(x)$ are L^1 functions for $x \in (0,1)$, see [10, p. 142].
Using analytic solution of partial differential equations, it is readily verified that the transfer function of this heat equation is given by the following infinite series for $s \neq -n^2\pi^2$,

$$G(s) = \frac{\alpha_0\beta_0}{s} + 2\sum_{n=1}^\infty \frac{\alpha_n\beta_n}{s+n^2\pi^2},$$

where,

$$\alpha_n = \int_0^1 b(x)\cos(n\pi x)\mathrm{d}x \quad \text{for} \quad n = 0, 1, 2...$$
$$\beta_n = \int_0^1 c(x)\cos(n\pi x)\mathrm{d}x \quad \text{for} \quad n = 0, 1, 2...$$

and its impulse response is given by

$$h(t) = (\mathcal{L}^{-1}G)(t) = \alpha_0\beta_0 + 2\sum_{n=1}^{\infty}\alpha_n\beta_n e^{-n^2\pi^2 t},$$

so,

$$f(\xi) = (\mathcal{L}^{-1}h)(\xi) = \delta(\xi)\alpha_0\beta_0 + 2\sum_{n=1}^{\infty}\alpha_n\beta_n\delta(\xi - n^2\pi^2).$$

This example has a pole $s = 0$ in the closed right half plane, so it is unstable unless $\alpha_0 = 0$ or $\beta_0 = 0$.

Example 3.6.2. *(General Example).* Consider this example with $x_n > 0$,

$$\mu = \sum_{n=0}^{\infty} c_n\delta(x - x_n),$$

$$h(t) = \mathcal{L}\mu = \sum_{n=0}^{\infty} c_n e^{-x_n t},$$

$$G(s) = \mathcal{L}h = \sum_{n=0}^{\infty}\frac{c_n}{s + x_n}.$$

This system has no poles in the closed right half plane. In addition,

$$\begin{aligned} \|G(s)\|_{H^\infty} &= \sup_{Res>0} |G(s)| \\ &\leq \sup_{Res>0}\sum_{n=0}^{\infty}\left|\frac{c_n}{s + x_n}\right|, \end{aligned}$$

so, the system is H^∞ if $\sum_{n=0}^{\infty}\left|\frac{c_n}{x_n}\right|$ converges.

Now, we test whether the system is BIBO stable,

$$\begin{aligned} \int_0^\infty |h(t)|\,\mathrm{d}t &= \int_0^\infty \left|\sum_{n=0}^{\infty} c_n e^{-x_n t}\right| \mathrm{d}t \\ &\leq \sum_{n=0}^{\infty}\frac{|c_n|}{x_n}, \end{aligned}$$

thus, the system will be BIBO stable if $\sum_{n=0}^{\infty} \frac{|c_n|}{x_n}$ converges.

The operator Γ is nuclear if,

$$\begin{aligned}
\int_0^{\infty} \frac{\mathrm{d}\,|\mu|\,(x)}{x} &= \int_0^{\infty} \frac{\sum_{n=0}^{\infty} |c_n|\,\mathrm{d}\delta(x-x_n)}{x} \\
&\leq \sum_{n=0}^{\infty} \frac{|c_n|}{x_n} \\
&< \infty.
\end{aligned}$$

The operator Γ is a Hilbert-Schmidt operator if,

$$\begin{aligned}
\int_0^{\infty} \left|t^{1/2}h(t)\right|^2 \mathrm{d}t &= \int_0^{\infty} (t \left| (\sum_{n=0}^{\infty} c_n e^{-x_n t} \sum_{m=0}^{\infty} c_m e^{-x_m t}) \right|)\mathrm{d}t \\
&\leq \sum_n \sum_m |c_n|\,|c_m| \frac{1}{(x_n+x_m)^2} \\
&< \infty.
\end{aligned}$$

Similarly, the Θ operator is nuclear if $\sum_{n=0}^{\infty} \frac{|c_n|}{\sqrt{x_n}} < \infty$, moreover, it is a Hilbert-Schmidt operator if $\sum_n \sum_m \frac{|c_n||c_m|}{x_n+x_m} < \infty$. However, if $c_n \geq 0$ for all n, the conditions are also necessary (we get equality).

Example 3.6.3. Consider the following equation:

$$Z_t = Z_{xx},$$

$$Z(0,t) = u(t), Z(1,t) = 0, Z(x,0) = z_0(x),$$
$$y(t) = \int_0^{\infty} Z(x,t)\mathrm{d}x.$$

Using analytic solution of partial differential equations, it is readily verified that the transfer function of this heat equation is given by the following infinite series for $s \neq -(2n+1)^2\pi^2$,

$$G(s) = \frac{1}{2} - 4\sum_{n=0}^{\infty} \frac{s}{(2n+1)\pi(s+(2n+1)^2\pi^2)},$$

and its impulse response is given by

$$h(t) = \frac{\delta(t)}{2} - 4\sum_{n=0}^{\infty} [\frac{\delta(t)}{(2n+1)\pi} - (2n+1)\pi e^{-(2n+1)^2\pi^2 t}].$$

The system is not diffusive, however it could be considered as a diffusive system + feed-through.

3.7 Coprime Factorization

In this section we extend the approximation techniques to unstable system using a coprime factorization $G(s) = \frac{N}{M}$ where N, M are H^∞ functions defined on the right half of the complex plane, as in [37].

3.7.1 The gap metric

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $A : \mathcal{D}(A) \to \mathcal{K}$ and $B : \mathcal{D}(B) \to \mathcal{K}$ be linear operators with domains $\mathcal{D}(A), \mathcal{D}(B) \subset \mathcal{H}$ respectively, (see [31], p. 30).

Definition 3.7.1. Let $A : \mathcal{X} \to \mathcal{Y}$ be mapping between sets. Then its graph is the set of all pairs $(x, A(x))$ with $x \in \mathcal{D}(A)$, namely,

$$\mathcal{G}(A) = \{(x, A(x)) : x \in \mathcal{X}\}$$

(see [31], p. 31).

Definition 3.7.2. As A is linear $\mathcal{G}(A)$ is a subspace of the product Hilbert spaces $\mathcal{H} \times \mathcal{K}$. A is said to be closed if its graph $\mathcal{G}(A)$ is a closed subspace of $\mathcal{H} \times \mathcal{K}$ (see [14]).

Definition 3.7.3. The gap between closed subspaces $\mathcal{V}$ and $\mathcal{W}$ of a Hilbert space $\mathcal{H}$ is given by,

$$\delta(\mathcal{V}, \mathcal{W}) = \|P_v - P_w\|,$$

where P_v and P_w denote the orthogonal projections from $\mathcal{H}$ onto $\mathcal{V}$ and $\mathcal{W}$ respectively.

For a closed operator G, with $\mathcal{D}(G)$ (Domain of G) dense and $\mathcal{G}(G) =$

$\{(Mu, Nu) : u \in \mathcal{H}\}$, where M and N are bounded operators that are strongly right coprime in the sense that $\tilde{X}M + \tilde{Y}N = I$ for some operators $\tilde{X}$ and $\tilde{Y}$, we can write $G = NM^{-1}$ (see [31], p. 30).

Definition 3.7.4. The gap metric between two Hilbert operators A and B which are as above is defined as the gap between their graphs, namely,

$$\delta(A, B) = \delta(\mathcal{G}(A), \mathcal{G}(B))$$

(see [31], p. 31).

Proposition 3.7.5. *(See [31], p. 72). Let $G = NM^{-1}$ be a right coprime factorization of an operator; then there exists $\epsilon > 0$ such that, if $\|N_k - N\| < \epsilon$ and $\|M_k - M\| < \epsilon$, then $G_k = N_k M_k$ is still a right coprime factorization.*

Proposition 3.7.6. *(See [31], p. 72). Assume that $G = NM^{-1}$ and $G_k = N_k M_k^{-1}$ are as in Proposition 3.7.5. Then $\delta(G_k, G) \to 0$ as $\epsilon \to 0$. Conversely, for any $\epsilon > 0$ there exist $\eta > 0$ such that any G_k with $\delta(G_k, G) < \eta$ has a coprime factorization $G_k = N_k M_k^{-1}$ with $\|N_k - N\| < \epsilon$ and $\|M_k - M\| < \epsilon$.*

3.7.2 The chordal metric

Definition 3.7.7. The chordal distance between two points w_1 and w_2 in the complex plane is defined by

$$\kappa(w_1, w_2) = \frac{|w_1 - w_2|}{\sqrt{(1 + |w_1|^2)(1 + |w_2|^2)}}$$

with $\kappa(w, \infty) = 1/\sqrt{1 + |w|^2}$. In other words, the chordal distance between two points in $\mathbb{C} \bigcup \{\infty\}$ is given by measuring the length of the chord between the corresponding points on the Riemann sphere (see [31], p. 82).

Definition 3.7.8. For any meromorphic functions G and H in the open right half plane; the chordal distance between them is given by,

$$\begin{aligned}\kappa(G,H) &= \sup_s\{\kappa(G(s),H(s)) : Res \geq 0\} \\ &= \sup_{Re(s)\geq 0} \frac{|G(s)-H(s)|}{(1+|G(s)|^2)^{1/2}(1+|H(s)|^2)^{1/2}}\end{aligned}$$

(see [15]).

Example 3.7.9. Consider this example with $x_n > 0$

$$\mu(\xi) = c_0\delta(\xi) + \sum_{n=1}^{\infty} c_n\delta(\xi - x_n),$$

$$h(t) = \mathcal{L}\mu = c_0 + \sum_{n=1}^{\infty} c_n e^{-x_n t},$$

$$G(s) = \mathcal{L}h = \frac{c_0}{s} + \sum_{n=1}^{\infty} \frac{c_n}{s+x_n}.$$

This system has a pole $s = 0$ in the closed right half plane, so it is not stable. We here use an approximation technique based on coprime factorization of the system as $G(s) = \frac{N}{M}$ where N, M are H^∞ functions defined on the right half of the complex plane.

We have here,

$$G(s) = \frac{c_0}{s} + \sum_{n=1}^{\infty} \frac{c_n}{s+x_n}.$$

Let

$$G_k(s) = \frac{c_0}{s} + \sum_{n=1}^{k} \frac{c_n}{s+x_n}.$$

We now look at the chordal metric between G and G_k,

$$\kappa(G,G_k) = \sup_s \frac{|G(s)-G_k(s)|}{\sqrt{1+|G(s)|^2}\sqrt{1+|G_k(s)|^2}}$$

If $G(s) = \infty$ and $G_k(s) = \infty$ then $\kappa(G,G_k) = 0$,

otherwise,

$$\kappa(G,G_k) = \sup_{s\in\mathbb{C}_+} \frac{\left|\sum_{n=k+1}^{\infty} \frac{c_n}{s+x_n}\right|}{\sqrt{1+|G(s)|^2}\sqrt{1+|G_k(s)|^2}}.$$

Now write

$$G(s) = \frac{c_0}{s} + H(s), \quad H(s) \in H^\infty,$$
$$G_k(s) = \frac{c_0}{s} + H_k(s), \quad H_k(s) \in H^\infty,$$

and

$$\|H - H_k\|_\infty \to 0.$$

According to Proposition 4.2.2 in [31] we write $G = \frac{N}{M}$ and $G_k = \frac{N_k}{M_k}$, where

$$M_k = \frac{s}{s+1}, \quad M = \frac{s}{s+1},$$
$$N_k = \frac{sG_k(s)}{s+1}, \quad N = \frac{sG(s)}{s+1},$$

thus

$$G(s) = \frac{(s/(s+1))G(s)}{s/(s+1)}$$

and

$$G_k(s) = \frac{(s/(s+1))G_k(s)}{s/(s+1)}.$$

From 3.7.6, since $\|M_k - M\| \to 0$ and $\|N_k - N\| \to 0$ if $\sum_{k+1}^{\infty} \frac{|c_n|}{x_n} < \infty$ it follows that if $c_0 \neq 0$ then, $[M, N]$ with $N, M \in H^\infty$ is a coprime factorization and satisfying the Bézout identity, $XM + YN = I$ for $X, Y \in H^\infty$, so

$$X(\frac{s}{s+1}) + Y(\frac{c_0}{s+1} + \sum \frac{c_n s}{(s+x_n)(s+1)}) = 1$$

let $Y = \frac{1}{c_0}$ then, $X = 1 - \sum \frac{c_n}{(s+x_n)c_0}$,
That $\kappa(G, G_k) \to 0$ as $k \to \infty$ follows from Proposition 4.2.2 in [31].

Chapter 4

Rational Approximation

4.1 Introduction

The problem of approximating infinite dimensional linear systems is considered in this chapter. We work on rational approximation of diffusive systems with transfer function $G(s) = \int_0^\infty \frac{f(x)}{x+s} \mathrm{d}x$ by the Gaussian Quadrature method, which consider the problem of numerical evaluation of the integral $\int_a^b g(t)\mathrm{d}t$. This integral requires changing variables,

$$x = \frac{2}{b-a}(t - \frac{(b+a)}{2})$$

converting the integral $\int_a^b g(t)\mathrm{d}t$ to the one of the form $\int_{-1}^1 \varphi(x)\mathrm{d}x$.
In this chapter we state general theorem for smooth f (including at 0) decaying fast, for which we can find good rational approximants. However, the approximation provides more information; if we have a convergence rate of approximation $\|G - G_n\|_\infty$ then these provide a convergence rate of the Hankel singular values σ_n of the transfer function, since $\sigma_n \leq \|G - G_n\|_\infty$.

4.2 Approximation by polynomials

Theorem 4.2.1. ([23, Theorem 41.2, Gauss]). *If $x_1, x_2, ..., x_n$ are the roots of nth Legendre polynomial P_n, there exist unique $A_1, A_2, ..., A_n$ such that*

$$\int_{-1}^1 P(x)\mathrm{d}x = \sum_{j=1}^n A_j P(x_j),$$

whenever P is a polynomial of degree $2n-1$ or less.

Definition 4.2.2. Let f be a continuous function, then let

$$F_n(f) = \sum_{j=1}^{n} A_j f(x_j)$$

where $x_1, x_2, ..., x_n$ and $A_1, A_2, ..., A_n$ are as in Theorem 4.2.1.

Definition 4.2.3. H_n is the set of all polynomials of degree n or less, i.e, polynomials of the form

$$P(x) = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n$$

where the coefficients $c_0, c_1, ..., c_n$ are arbitrary real numbers (see [26], p. 20).

Definition 4.2.4. Suppose that $g \in C([a,b])$ and $P(x)$ is an arbitrary polynomial, then

$$\Delta(P) = \max_{a \leq x \leq b} |P(x) - g(x)|$$

and

$$E_n = E_n(g) = \inf_{P \in H_n} \{\Delta(P)\}.$$

E_n is considered as the best approximation to $g(x)$ by polynomials from H_n (see [26], p. 20).

Theorem 4.2.5. ([23, Theorem 4.3, Weierstrass]). *If $g : [a,b] \to \mathbb{C}$ is continuous and $\varepsilon > 0$ we can find a polynomial P with*

$$\sup_{t \in [a,b]} |P(t) - g(t)| < \varepsilon.$$

Theorem 4.2.6. ([23, Theorem 41.6, Stieltjes]).

(i) Let $\mathcal{P}_{2n-1}$ be the set of polynomials of degree $2n-1$ or less and $g(x) : [-1,1] \to \mathbb{C}$ is continuous, then

$$\left| F_n(g) - \int_{-1}^{1} g(x) \mathrm{d}x \right| \leq 4 \inf \{ \sup_{-1 \leq t \leq 1} |g(t) - P(t)| : P \in \mathcal{P}_{2n-1} \}.$$

(ii) $F_n(g) \to \int_{-1}^{1} g(x)\mathrm{d}x$ as $n \to \infty$.

Corollary 4.2.7. ([26, Corollary VI. 2. 2]). *Suppose that $g(x)$ possesses a bounded derivative $g^{(p+1)}(x)$ such that $\left\|g^{(p+1)}(x)\right\|_\infty \leq K_{p+1}$ on $[a,b]$ then E_n (the best L^∞ error of approximation by polynomials of degree n or less) is given by*

$$E_n \leq \frac{C_p(b-a)^{p+1}K_{p+1}}{n^{p+1}},$$

where C_p is a constant depending on $p \geq 0$.

Definition 4.2.8. When T is a bounded operator, we define

$$\sigma_n = \inf\{\|T-S\| : rank(S) < n\},$$

(singular values) and $\sigma_n \to 0$ if and only if T is compact, see ([28, 2.34]).

Definition 4.2.9. Let $\tau_n = \inf\{\|G-G_n\|_\infty : degree(G_n) < n\}$, then

$$\sigma_{n+1} \leq \tau_n \leq \sigma_{n+1} + \sigma_{n+2} + \sigma_{n+3} + ...,$$

where σ_n are the approximation numbers of the associated Hankel operator, see([28] and [18]).

4.3 Approximation and diffusive systems

We shall consider the transfer function of a diffusive system given by $G(s) = \int_0^\infty \frac{f(x)}{x+s}\mathrm{d}x$ and then we will use numerical evaluation of $\int_0^M g(x)\mathrm{d}x$, where $g(x) \in C([0,M])$ and the Gaussian quadrature method.

Lemma 4.3.1. *Suppose that f is a measurable function and $f(x) = O(x^{-r})$, for some $r > 0$, as $x \to \infty$, then*

$$\left|\int_M^\infty \frac{f(x)}{x+s}\mathrm{d}x\right| \leq C\int_M^\infty x^{-r-1}\mathrm{d}x = O(M^{-r}) \ \textit{as} \quad M \to \infty.$$

Proof. The proof is clear. □

Theorem 4.3.2. *If $g^{(p+1)}$ bounded on $[0,M]$ by K_{p+1} then*

$$\left|\int_0^M g(x)\mathrm{d}x - \sum_{j=1}^n A_j g((x_j+1)M/2)\right| \leq \frac{M}{2}\frac{C_p 2^{p+1}K_{p+1}}{(2n-1)^{p+1}}.$$

Proof. From 4.2.7. □

Lemma 4.3.3. *Suppose that* $\frac{f(x)}{x+s} \in C([0,M])$ *and* $g_s(t) = \frac{f(Mt/2+M/2)}{Mt/2+M/2+s} \in C([-1,1])$ *and* $g_s(t)$ *possesses a bounded derivative* $g_s^{(p+1)}(t)$ *such that* $\left\|g_s^{(p+1)}(t)\right\|_\infty \leq K_{p+1}$ *then*

$$E_n(g_s) \leq \frac{C_p K_{p+1} 2^{p+1}}{(2n-1)^{p+1}} \approx O(n^{-(p+1)}).$$

Proof. We have

$$g_s(t) = \frac{f(Mt/2+M/2)}{Mt/2+M/2+s},$$

so

$$\int_0^M \frac{f(x)}{x+s}\mathrm{d}x = \frac{M}{2}\int_{-1}^1 g_s(t)\mathrm{d}t,$$

according to Corollary 4.2.7

$$\left\|\int_0^M \frac{f(x)}{x+s}\mathrm{d}x - \frac{M}{2}\sum_{j=1}^n A_j g_s(t_j)\right\|_\infty \leq \frac{M}{2}C_p\frac{2^{p+1}K_{p+1}}{(2n-1)^{p+1}} \approx O(n^{-(p+1)}).$$

This finishes the proof. □

Lemma 4.3.4. *(i) Suppose that* $\frac{f(x)}{x+s} \in C([0,M])$ *and*

$$g_s(t) = \frac{f(Mt/2+M/2)}{Mt/2+M/2+s} \in C([-1,1]),$$

then

$$\left|\frac{M}{2}F_n(g_s) - \int_0^M \frac{f(x)}{x+s}\mathrm{d}x\right| \leq \frac{4M}{2}\inf\{\left\|g_s(t) - \tilde{P}(t)\right\|_\infty\}.$$

(ii) $F_n(g_s) \to \int_{-1}^1 g_s(x)\mathrm{d}x$ *as* $n \to \infty$,

where $\tilde{P}$ *is a polynomial of degree* $2n-1$ *or less.*

Proof. (i) To estimate $\int_0^M \frac{f(x)}{x+s}\mathrm{d}x$, we have to change the variable

$$x = Mt/2 + M/2$$

converts the integral $\int_0^M \frac{f(x)}{x+s}\mathrm{d}x$ to one of the form

$$\int_0^M \frac{f(x)}{x+s}\mathrm{d}x = \frac{M}{2}\int_{-1}^1 \frac{f(Mt/2+M/2)}{Mt/2+M/2+s}\mathrm{d}t \approx \frac{M}{2}\sum_{j=1}^n A_j g_s(t_j)$$

thus, according to Theorem 4.2.6,
$\left|\int_0^M \frac{f(x)}{x+s}\mathrm{d}x - \frac{M}{2}\sum_{j=1}^n A_j g_s(t_j)\right| \leq \frac{4M}{2}\inf\{\sup_{-1\leq t\leq 1}\left|g_s(t)-\tilde{P}(t)\right| : \tilde{P}\in P_{2n-1}\}$.

(ii) The theorem of Weierstrass (Theorem 4.3, Weierstrass) shows

$$\inf\{\sup_{-1\leq t\leq 1}\left|g_s(t)-\tilde{P}(t)\right| : \tilde{P}\in P_{2n-1}\} \to 0 \quad as \quad n\to\infty.$$

□

Lemma 4.3.5. *If $\frac{f(x)}{x+s}$ is a continuous function on $[0, M]$ and*

$$g_s(t) = \frac{f(Mt/2+M/2)}{Mt/2+M/2+s} \in C([-1,1]),$$

possesses a bounded derivative $g_s^{(p+1)}(t)$ such that $\left\|g_s^{(p+1)}(t)\right\|_\infty \leq K_{p+1}$ then

$$\left\|\int_0^M \frac{f(x)}{x+s}\mathrm{d}x - \frac{M}{2}\sum_{j=1}^n A_j g_s(t_j)\right\|_\infty \leq \frac{4MC_p 2^{p+1}K_{p+1}}{2(2n-1)^{p+1}} = \frac{2^{p+2}MC_pK_{p+1}}{(2n-1)^{p+1}}.$$

Proof. By using Lemma 4.3.3 and Lemma 4.3.4, then

$$\begin{aligned}\left|\int_0^M \frac{f(x)}{x+s}\mathrm{d}x - \frac{M}{2}\sum_{j=1}^n A_j g_s(t_j)\right| &\leq \frac{4M}{2}\inf\{\sup_{-1\leq t\leq 1}|g_s(t)-P(t)| : P\in P_{2n-1}\}\\ &\leq \frac{4MC_p2^{p+1}K_{p+1}}{2(2n-1)^{p+1}}\\ &= \frac{2^{p+2}MC_pK_{p+1}}{(2n-1)^{p+1}}.\end{aligned}$$

□

Theorem 4.3.6. *Let $G(s) = \int_0^\infty \frac{f(x)}{x+s}\mathrm{d}x$ the transfer function of a diffusive system, and $\left\|(\frac{f(x)}{x+s})^{(p+1)}\right\|_\infty \leq L_{p+1}$ and in addition $f(x) = O(x^{-r})$ for $r>0$ then*

$$\inf_{\deg(G_n)<n}\|G(s)-G_n(s)\|_\infty = O(n^{\frac{-r(p+1)}{p+r+2}}).$$

Furthermore,

$$\sigma_n = O(n^{\frac{-r(p+1)}{p+r+2}}).$$

Proof. If we take

$$G(s) = \int_0^\infty \frac{f(x)}{x+s}\mathrm{d}x = \int_0^M \frac{f(x)}{x+s}\mathrm{d}x + \int_M^\infty \frac{f(x)}{x+s}\mathrm{d}x,$$

according to Lemma 4.3.1

$$\left|\int_M^\infty \frac{f(x)}{x+s}\mathrm{d}x\right| \leq C\int_M^\infty x^{-r-1}\mathrm{d}x = O(M^{-r}).$$

Set

$$g_s(t) = \frac{f(Mt/2+M/2)}{Mt/2+M/2+s} \in C([-1,1]),$$

then

$$\left\|g_s^{(p+1)}\right\|_\infty \leq M^{p+1}L_{p+1},$$

and, on account of Lemma 4.3.5

$$\left\|\int_0^M \frac{f(x)}{x+s}\mathrm{d}x - \frac{M}{2}\sum_{j=1}^n A_j g(t_j)\right\|_\infty \leq \frac{4MC_pM^{p+1}L_{p+1}2^{p+1}}{2(2n-1)^{p+1}} = \frac{C_pM^{p+2}L_{p+1}2^{p+2}}{(2n-1)^{p+1}}.$$

Combining the previous results then,

$$\inf_{\deg(G_n)<n} \|G(s)-G_n(s)\|_\infty \approx k_1\frac{M^{p+2}}{n^{p+1}} + k_2M^{-r}$$

for some constants k_1 and k_2.

Now we have to choose M to make the error as small as possible thus

$$\min_M(k_1\frac{M^{p+2}}{n^{p+1}} + k_2M^{-r})$$

so,

$$\frac{k_1(p+2)M^{p+1}}{n^{p+1}} = rk_2M^{-(r+1)}$$

then

$$M^{p+r+2} \approx n^{p+1}.$$

It follows that

$$\inf_{\deg(G_n)<n} \|G(s)-G_n(s)\|_\infty = O(n^{\frac{-r(p+1)}{p+r+2}}).$$

Furthermore, since

$$\sigma_n \leq \|G-G_n\|_\infty$$

so

$$\sigma_n = O(n^{\frac{-r(p+1)}{p+r+2}}).$$

□

Theorem 4.3.7. *Suppose that for $p \geq 0$ and $r \geq 0$ there is a constant N_{p+1} such that*

(i) $\left\| \frac{f^{(k)}(x)}{x^{p+2-k}} \right\|_\infty \leq N_{p+1}$ *for all* $0 \leq k \leq p+1$.

(ii) $f(x) = O(x^{-r})$ *at* ∞.

Then

$$\inf_{\deg(G_n)<n} \|G - G_n\|_\infty = O(n^{-\frac{r(p+1)}{p+r+2}}).$$

In addition

$$\sigma_n = O(n^{\frac{-r(p+1)}{p+r+2}}).$$

ey

Proof. We have, on account of the Leibniz rule,

$$\begin{aligned}
\left(\frac{f(x)}{x+s}\right)^{(p+1)} &= \sum_{k=0}^{p+1} \binom{p+1}{k} f^{(k)}(x) (\frac{1}{x+s})^{(p+1-k)} \\
&= \sum_{k=0}^{p+1} \binom{p+1}{k} f^{(k)}(x) \frac{(-1)^{p+1-k}(p+1-k)!}{(x+s)^{p+2-k}}
\end{aligned}$$

then

$$\begin{aligned}
\left| \left(\frac{f(x)}{x+s}\right)^{(p+1)} \right| &= \left| \sum_{k=0}^{p+1} \binom{p+1}{k} f^{(k)}(x) \frac{(-1)^{p+1-k}(p+1-k)!}{(x+s)^{p+2-k}} \right| \\
&\leq \sum_{k=0}^{p+1} \left| \binom{p+1}{k} f^{(k)}(x) \frac{(-1)^{p+1-k}(p+1-k)!}{(x+s)^{p+2-k}} \right| \\
&\leq \sum_{k=0}^{p+1} \left| f^{(k)}(x) \right| \frac{(p+1)!(p+1-k)!}{(p+1-k)! \quad k! \quad x^{p+2-k}} \\
&\leq R_p \sup_{0 \leq k \leq p+1} \frac{\left| f^{(k)}(x) \right|}{x^{p+2-k}}.
\end{aligned}$$

where R_p is a constant depending on p. Since $\frac{f^{(k)}(x)}{x^{p+2-k}}$ is bounded for all $0 \le k \le p+1$, thus $(\frac{f(x)}{x+s})^{(p+1)}$ is bounded. Moreover the assumptions of the Theorem 4.3.6 are available. It follows immediately that

$$\inf_{\deg(G_n)<n} \|G - G_n\|_\infty = O(n^{-\frac{r(p+1)}{p+r+2}}).$$

In addition

$$\sigma_n = O(n^{\frac{-r(p+1)}{p+r+2}}).$$

□

Corollary 4.3.8. *If $rp > p+2$ with $p \ge 1$ in Theorem 4.3.7, then the system is nuclear.*

Example 4.3.9. We shall apply the previous theorems to the Examples 3.3.4.

1. When $\mu = \delta_a$ we have a one dimensional system i. e. finite dimensional.

2. When $f(x) = e^{-ax}$. This is not a good example, because $\frac{f^{(k)}(x)}{x^{p+2-k}} = \frac{(-1)^k a^k e^{-ax}}{x^{p+2-k}}$ is not bounded on the real line for each p with $0 \le k \le p+1$.

3. When $f(x) = \frac{x^m e^{-ax}}{m!}$ where $m = 1, 2, ..,$ in this example r can be any number greater than zero, because $f(x) \to 0$ quickly at ∞.

   Then, on account of Theorem 4.3.7, we may conclude the following:

   For $f(x) = xe^{-ax}$ when $p = 0$ for $k = 0$ we have $f(x)/x^2 = xe^{-ax}/x^2$, which is not bounded on $\mathbb{R}$; and for $k = 1$ then $f'(x)/x = (-axe^{-ax} + e^{-ax})/x$, which is not bounded on $\mathbb{R}$.

   For $f(x) = x^2 e^{-ax}$ when $p = 0$ we have $f(x)/x^2$ and $f'(x)/x$ are bounded on $\mathbb{R}$, then $\tau_n = O(n^{\frac{-r}{r+2}})$ so if r is large but is less than ∞ then $\tau_n = O(n^{-1+\epsilon})$ for any $\epsilon > 0$. In addition if $p = 1$, $f(x)/x^3$ is not bounded on $\mathbb{R}$.

   In general if $f(x) = \frac{x^m e^{-ax}}{m!}$ where $m = 1, 2, ...$ then r can be any number greater than zero, and $p = m - 2$ and $\tau_n = O(n^{\frac{-r(m-1)}{m+r}})$ then as $r \to \infty$, $\tau_n = O(n^{-(m-1)+\epsilon})$. For example for $m = 2$, $\tau_n = O(n^{-1+\epsilon})$

so, we can not conclude that it is nuclear. For $m > 2$ the system would be nuclear.

4. When $f(x) = \frac{\sin(\pi\alpha)}{\pi x^\alpha}$,and $h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $G(s) = \frac{1}{s^\alpha} \notin H^\infty$, where $0 < \operatorname{Re}\alpha < 1$. Then the system is not stable. This example is in [25]

5. When $f(x) = \frac{1}{\sqrt{\pi x}}$ and $h(t) = \frac{1}{\sqrt{t}} \notin L^1$ then $G(s) = \frac{\sqrt{\pi}}{\sqrt{s}}$, which is not BIBO or H^∞ stable.

6. When $f(x) = \frac{u(x-1)}{\sqrt{\pi(x-1)}}$, where u is step function,

   we have

$$G(s) = \int_0^\infty \frac{f(x)}{x+s}\mathrm{d}x = \int_1^\infty \frac{1}{\sqrt{x-1}(x+s)}\mathrm{d}x = \int_0^\infty \frac{1}{\sqrt{x}(x+s+1)}\mathrm{d}x.$$

   Let $x = t^4$, then $\mathrm{d}x = 4t^3\mathrm{d}t$,

   thus

$$\int_0^\infty \frac{f(x)}{x+s}\mathrm{d}x = 4\int_0^\infty \frac{t}{t^4+s+1}\mathrm{d}t = 4\int_0^M \frac{t}{t^4+s+1}\mathrm{d}t + 4\int_M^\infty \frac{t}{t^4+s+1}\mathrm{d}t.$$

   Then

$$\begin{aligned} \left|\int_M^\infty \frac{t}{t^4+s+1}\mathrm{d}t\right| &\le \int_M^\infty \frac{t}{|t^4+s+1|}\mathrm{d}t \\ &\le \int_M^\infty \frac{t}{t^4+1}\mathrm{d}t \\ &\le \int_M^\infty t^{-3}\mathrm{d}t = O(M^{-2}). \end{aligned}$$

   On the other hand, $\left\|\left(\frac{t}{t^4+s+1}\right)^{(p+1)}\right\| \le K_{p+1}$, where K_{p+1} does not depend on M, and

$$\int_0^M \frac{t}{t^4+s+1}\mathrm{d}t = M/2\int_{-1}^1 \frac{Mz/2+M/2}{(Mz/2+M/2)^4+s+1}\mathrm{d}z,$$

   so

$$\left|\int_0^M \frac{t}{t^4+s+1}\mathrm{d}t - M/2\sum_{j=1}^n A_j g(t_j)\right| \le \frac{4MC_{p+1}2^{p+1}K_{p+1}(M/2)^{p+1}}{2(2n-1)^{p+1}}.$$

   Combining the previous results then,

$$\inf_{\deg(G_n)<n} \|G(s) - G_n(s)\|_\infty \approx k_1\frac{M^{p+2}}{n^{p+1}} + k_2M^{-2}.$$

Now we have to choose M to make the error as small as possible thus

$$\min_M (k_1 \frac{M^{p+2}}{n^{p+1}} + k_2 M^{-2}),$$

so

$$\frac{k_1(p+2)M^{p+1}}{n^{p+1}} = -2k_2 M^{-3},$$

then

$$M^{p+4} \approx n^{p+1}.$$

It follows that

$$\inf_{\deg(G_n)<n} \|G(s) - G_n(s)\|_\infty = O(n^{\frac{-2(p+1)}{p+4}}).$$

Furthermore,

$$\tau_n = O(n^{\frac{-2(p+1)}{p+4}}).$$

For $p = 0$, we have $\tau_n = O(n^{-1/2})$.

For $p = 1$, we have $\tau_n = O(n^{-4/5})$.

For $p = 2$, we have $\tau_n = O(n^{-1})$.

For $p = 3$, we have $\tau_n = O(n^{-8/7})$.

Letting $p \to \infty$, we have $\tau_n = O(n^{-2+\epsilon})$ for any $\epsilon > 0$.

7. Let $f(x) = \frac{1}{\sqrt{\pi x}} e^{-k^2/4x}$ then $f(x) = O(x^{-1/2})$ as $x \to \infty$, and

$$\int_0^\infty \frac{f(x)}{x+s} \mathrm{d}x = \int_0^\infty \frac{e^{-k^2/4x}}{\sqrt{\pi x}(x+s)} \mathrm{d}x.$$

Now, $1/\sqrt{x}$ goes to zero very slowly, so we suppose that, $x = z^l$ for $l > 1$, then $\mathrm{d}x = lz^{l-1}\mathrm{d}z$

$$\int_0^\infty \frac{f(x)}{x+s} \mathrm{d}x = \int_0^\infty \frac{lz^{l-1}e^{-k^2/4z^l}}{z^{l/2}(z^l+s)} \mathrm{d}z.$$

We can rewrite our integral in the following expression

$$\int_0^\infty \frac{z^{l-1}e^{-k^2/4z^l}}{z^{l/2}(z^l+s)} \mathrm{d}z = \int_0^M \frac{z^{l-1}e^{-k^2/4z^l}}{z^{l/2}(z^l+s)} \mathrm{d}z + \int_M^\infty \frac{z^{l-1}e^{-k^2/4z^l}}{z^{l/2}(z^l+s)} \mathrm{d}z.$$

Now

$$\left|\int_M^\infty \frac{z^{l-1}e^{-k^2/4z^l}}{z^{l/2}(z^l+s)}\mathrm{d}z\right| \le \int_M^\infty z^{-l/2-1}\mathrm{d}z$$
$$= O(M^{-l/2}).$$

For $\int_0^M \frac{lz^{l-1}e^{-k^2/4z^l}}{z^{l/2}(z^l+s)}\mathrm{d}z$ and $\left\|\left(\frac{lz^{\frac{l}{2}-1}e^{-k^2/4z^l}}{(z^l+s)}\right)^{(p+1)}\right\| \le K_{p+1}$ we have to change the variable,

$z = Mt/2+M/2 \quad t\in[-1,1]$, let, $g(t_j) = \frac{(Mt_j/2+M/2)^{l-1}e^{-k^2/4(Mt_j/2+M/2)^l}}{(Mt_j/2+M/2)^{l/2}((Mt_j/2+M/2)^l+s)}$,

then according to Theorem 4.3.2

$$\left|\int_M^\infty \frac{z^{l/2-1}e^{-k^2/4z^l}}{(z^l+s)}\mathrm{d}z - M/2\sum_{j=1}^n A_j g(t_j)\right| \le \frac{4MC_{p+1}2^{p+1}K_{p+1}(M/2)^{p+1}}{2(2n-1)^{p+1}}.$$

Combining the previous results then

$$\inf_{\deg(G_n)<n}\|G(s)-G_n(s)\| \le 4\frac{MC_{p+1}(M/2)^{p+1}K_{p+1}2^{p+1}}{2(2n-1)^{p+1}} + \frac{2M^{-l/2}}{l},$$

then

$$\inf_{\deg(G_n)<n}\|G(s)-G_n(s)\|_\infty \approx k_1\frac{M^{p+2}}{n^{p+1}} + k_2M^{-l/2}.$$

Now we have to choose M to make the error as small as possible thus

$$\min_M(k_1\frac{M^{p+2}}{n^{p+1}} + k_2M^{-l/2}),$$

so

$$\frac{k_1(p+2)M^{p+1}}{n^{p+1}} = -\frac{l}{2}(k_2M^{-l/2-1}),$$

then

$$M^{p+l/2+2} \approx n^{p+1}.$$

It follows that

$$\|G(s)-G_n(s)\|_\infty = O(n^{\frac{-l/2(p+1)}{p+l/2+2}}).$$

Furthermore,

$$\tau_n = O(n^{\frac{-l/2(p+1)}{p+l/2+2}}).$$

Letting $l\to\infty$ then we have, $\tau_n = O(n^{-p-1+\epsilon})$ for each $p\ge 1$.

8. When $f(x) = \cos(a\sqrt{x})/(\pi\sqrt{x})$, a heat kernel, it is not stable.

Chapter 5

Diffusive systems defined by holomorphic distributions

5.1 Introduction

In this chapter, we introduce diffusive systems defined by holomorphic distributions (and measures on a half plane). We basically start by the easier examples where the distribution can be written as a measure on a compact rectangle. Then we investigate more complicated distributions, where the system is not necessarily stable, although its impulse response and transfer function can be defined.

Diffusive systems have links with the heat equation. For instance, Montseny [25], considered diffusive system as a convolution system $y = h * u$, where $h(t) = \int_0^\infty e^{-t\xi} \mathrm{d}\mu$, and he gave three equivalent formulas. Here we generalize this idea where $h(t) = \langle e_t, \Phi \rangle$ with a diffusive representation Φ (with a measure μ) and these systems can be realized as a diffusive equation (heat equation).

Moreover, we develop more research on rate of decay of singular values of the associated Hankel operator and Θ operator, including nuclearity and the Hilbert-Schmidt property.

5.2 Diffusive systems defined by holomorphic distributions

For $n > 0$ let K_n denotes the compact rectangle

$$K_n := \{z \in \mathbb{C}_+ : z \in [\frac{1}{n}, n] \times [-n, n]\}.$$

Let $\mathbb{C}_+ = \bigcup K_n$ and $H(\mathbb{C}_+)$ denotes the Fréchet space of holomorphic function on K_n equipped with the topology that can be derived from the semi-norms

$$\|f\|_n := \sup\{|f(z)| : z \in K_n\}.$$

Now let $\phi : H(\mathbb{C}_+) \to \mathbb{C}$ be a bounded (continuous) linear functional, then there exists a constant $M > 0$ such that

$$|\langle f, \phi\rangle| = |\phi(f)| \leq M \|f\|_n ,$$

for all $f \in H(\mathbb{C}_+)$.
The Fourier-Borel transform of ϕ, which is the impulse response of a system, can be given by

$$h(t) = FB(\phi)(t) = \langle e_{-t}, \phi\rangle,$$

for $t > 0$, where $e_{-t}(z) = e^{-tz}$ for $z \in K_n$.
The transfer function of a diffusive system is given by Stieltjes's transform,

$$G(s) = S(\phi)(s) = \langle k_s, \phi\rangle, \quad s \in \mathbb{C}_+$$

where $k_s(z) = \frac{1}{s+z}$, see ([24]).

Theorem 5.2.1. *Let $G(s) = S(\phi)(s) = \langle k_s, \phi\rangle, \quad s \in \mathbb{C}_+$ and $h(t) = FB(\phi)(t) = \langle e_{-t}, \phi\rangle$, then $h \in L^1$ and $G(s) \in H^\infty$.*

Proof. We have for some n

$$\begin{aligned} |\langle e_{-t}, \phi\rangle| &\leq M \sup_{z \in K_n} \left|e^{-tz}\right| \\ &\leq Me^{-t/n}, \end{aligned}$$

so, $h \in L^1$.

Similarly

$$\begin{aligned} |G(s)| &\leq M \sup_{z \in K_n, s \in \mathbb{C}_+} \left|\frac{1}{z+s}\right| \\ &\leq M.n, \end{aligned}$$

since, $|z+s| > \mathrm{Re}(z+s) > \frac{1}{n}$ thus $\left|\frac{1}{z+s}\right| < n$, hence $G(s) \in H^\infty$. $\square$

5.2.1 Distribution as a measure

In general for a given distribution ϕ, by the Hahn-Banach theorem one can extend ϕ to $C(K_n)^*$ i.e. there is a measure μ of compact support K_n in $\mathbb{C}_+$ such that

$$|\langle f, \phi\rangle| = \left|\int_{K_n} f \mathrm{d}\mu\right| \leq \|f\|_n |\mu| (K_n) \quad \forall f \in \mathrm{Hol}(\mathbb{C}_+)$$

and it is possible to define the impulse response function and the transfer function as follows:

$$h(t) = \langle e_{-t}, \phi\rangle = \int_{K_n} e^{-tz} \mathrm{d}\mu(z),$$

and

$$G(s) = \langle k_s, \phi\rangle = \int_{K_n} \frac{1}{z+s} \mathrm{d}\mu(z),$$

i. e. the distribution can be written as a measure on the set K_n.

Example 5.2.2. (i) If $\langle f, \phi\rangle = f'(1)$ then $h(t) = -te^{-tz}|_{z=1} = -te^{-t}$ and $G(s) = \frac{-1}{(s+z)^2}|_{z=1} = \frac{-1}{(s+1)^2}$ but by the Cauchy integral formula $f'(1) = \frac{1}{2\pi i}\int_C \frac{f(z)\mathrm{d}z}{(z-1)^2}$ (where C is a circle centred at $(1,0)$), corresponding to μ that is not unique.

$\langle F, \Phi\rangle = \int_{K_n} F(s)\mathrm{d}\mu(s)$, and Φ acts on functions defined on K_n, given by μ where μ is a measure on $K_n \subset \mathbb{C}_+$.
The next result generalizes [25].

Theorem 5.2.3. *A convolution system $y = h*u$, where $h(t) = \langle e_{-t}, \Phi\rangle$, with diffusive representation Φ (with a measure μ) can be realized as a diffusive equation (heat equation)*

$$\begin{aligned} \psi_t(z,t) &= -z\psi(z,t) + u(t). \qquad (5.2.1)\\ y(t) &= \langle (e_{-z} * u)(t), \Phi\rangle_z = \int_{K_n} (e_{-z} * u)(z,t)\mathrm{d}\mu(z) \end{aligned}$$

with $\psi(z,t)$ a state variable such as that $\psi(z,0) = 0$ and then,

$$\Psi(z,s) = \frac{U(s)}{z+s}$$

and

$$Y(s) = \langle \Psi, \Phi \rangle_z = \int_{K_n} \frac{U(s)}{z+s} \mathrm{d}\mu(z)$$

Proof. We have ψ is a solution for the heat equation, from

$$\begin{aligned} \psi_t(z,t) &= -z\psi(z,t) + u(t) \\ y(t) &= \langle \psi, \Phi \rangle = \langle e^{-tz} * u, \Phi \rangle. \end{aligned}$$

Take the Laplace transform for the heat equation then we obtain,

$$s\Psi(z,s) = -z\Psi(z,s) + U(s)$$

and hence

$$\Psi(z,s) = \frac{U(s)}{z+s}$$

thus

$$Y(s) = \int_{K_n} \frac{U(s)}{x+s} \mathrm{d}\mu(x) = \langle \Psi, \Phi \rangle_z = \langle \mathcal{L}(e^{-\tau z} * u)(z,s), \Phi \rangle_z$$

then ψ is a solution for the heat equation.

Moreover $h(t) = \langle e_{-t}, \Phi \rangle_z = \int_{K_n} e^{-tz} \mathrm{d}\mu(z)$, and

$$\begin{aligned} y(t) = (h * u)(t) &= \int_0^t h(\tau) u(t-\tau) \mathrm{d}\tau, \\ &= \int_0^t \langle e^{-\tau z}, \Phi \rangle u(t-\tau) \mathrm{d}\tau \\ &= \int_0^t \int_{K_n} e^{-\tau z} u(t-\tau) \mathrm{d}\mu(z) \mathrm{d}\tau, \end{aligned}$$

by Fubini's theorem

$$\begin{aligned} &= \int_{K_n} \int_0^t e^{-\tau z} u(t-\tau) \mathrm{d}\tau \mathrm{d}\mu(z) \\ &= \langle (e_{-z} * u)(t), \Phi \rangle. \end{aligned}$$

Also, we could express $y(t)$ as the following,

$$\begin{aligned} y(t) &= \int_{K_n} \mathcal{L}^{-1}_{s \to t} \left(\frac{U(s)}{z+s} \right) \mathrm{d}\mu(z) \\ &= \int_{K_n} [\int_{\tau=0}^t e^{-\tau z} u(t-\tau) \mathrm{d}\tau] \mathrm{d}\mu(z) \\ &= \int_{K_n} (e_{-z} * u)(z,t) \mathrm{d}\mu(z) \\ &= \langle (e_{-z} * u)(t), \Phi \rangle. \end{aligned}$$

□

Corollary 5.2.4. *The following diffusive systems are equivalent to (5.2.1).*

$$\begin{aligned} \Psi_t(\zeta,t) &= -4\pi^2\zeta^2\Psi(\zeta,t)+u(t) \\ y(t) &= \langle\Psi,\Phi\rangle. \end{aligned} \tag{5.2.2}$$

with $\Psi(\zeta,0)=0$.

This is also equivalent to

$$\begin{aligned} \Theta_t(x,t) &= \Theta_{xx}(x,t)+\delta(x)u(t) \\ y(t) &= \langle\Theta,\Phi\rangle \end{aligned} \tag{5.2.3}$$

with $\Theta(x,0)=0$.

Proof. We are now in a position to show that (5.2.1) $\Leftrightarrow$ (5.2.2); it is sufficient to make the following substitution $z=4\pi^2\zeta^2$, then $\Psi(\zeta)=\psi(4\pi^2\zeta^2)$, $\Psi_t(\zeta)=\psi_t(4\pi^2\zeta^2)$. It remains to prove that (5.2.2) $\Leftrightarrow$ (5.2.3), thus we only need to observe that $\Psi=\mathcal{F}\Theta$ transforming with respect to the ζ variable.

□

Theorem 5.2.5. *If μ is a measure supported on K_n then the Hankel operator is nuclear.*

Proof. Let

$$Tu(\xi)=\int_0^\infty u(t)h(t+\xi)\mathrm{d}t \quad (0\le\xi<\infty)$$

where $h(t)=\langle e_{-t},\Phi\rangle=\int_{K_n}e^{-zt}\mathrm{d}\mu(z)$.

Assume that $\varphi_z(x)=e^{-zx}$ and define

$$T_0=\int_{K_n}\langle .,\varphi_z\rangle\varphi_z\mathrm{d}\mu(z),$$

this integral converges.

$$\begin{aligned} T_0u(\xi) &= \langle\langle u,\varphi_z\rangle\varphi_z(\xi),\Phi\rangle \\ &= \int_0^\infty u(t)\langle e^{-zt}e^{-z\xi},\Phi\rangle\mathrm{d}t \\ &= \int_0^\infty u(t)h(t+\xi)\mathrm{d}t. \end{aligned}$$

Then, $T = T_0$ by using Fubini's theorem.

$$T_0 u(x) = \langle \langle u, \varphi_z \rangle \varphi_z(x), \Phi \rangle$$

then,

$$tr(\langle u, \varphi_z \rangle \varphi_z) \leq \|\varphi_z\|_{L^2}^2 .$$

T_0 is not a positive operator, but according to Lemma 1.11 [28, p. 11], the nuclear norm of T is

$$tr\,|T_0| \leq \int_{K_n} \|\varphi_z\|_{L^2}^2 \,\mathrm{d}\,|\mu|\,(z) < \infty$$

since $\|\varphi_z\|_{L^2}^2$ is bounded uniformly on K_n.

□

5.3 More general distributions

Definition 5.3.1. Let X be the set of $f : \mathbb{C}_+ \to \mathbb{C}$ analytic such that

$$\sup_{z \in \mathbb{C}_+} \left|(\operatorname{Re} z)^k f^{(j)}(z)\right| < \infty$$

whenever $0 \leq k \leq j+1$.

So X is a Fréchet space with these seminorms

$$\|f\|_{(n)} = \max_{\substack{0 \leq j \leq n \\ 0 \leq k \leq j+1}} \sup_{z \in \mathbb{C}_+} \{\left|(\operatorname{Re} z)^k f^{(j)}(z)\right|\}.$$

The sequence of these seminorms is increasing.

Note: $e_{-t} \in X$ for all $t > 0$ and $k_s \in X$ for all $s \in \mathbb{C}_+$.

Definition 5.3.2. We have $\phi \in X^*$ (the dual space of X) if and only if there are $n \in \mathbb{N}$ and a constant M such that

$$|\langle f, \phi \rangle| \leq M \|f\|_{(n)}\,, \quad \forall f \in X.$$

Proposition 5.3.3. *Let $h(t) = \langle e_{-t}, \phi \rangle$ with $t > 0$ and $G(s) = \langle k_s, \phi \rangle$ with $s \in \mathbb{C}_+$ then $\|e^{-tz}\|_{(n)} < \infty \quad \forall n$ and $\sup_{s \in \mathbb{C}_+} \|k_s\|_{(n)}$ is infinite*

Proof. Let us first notice that

$(e^{-tz})^{(n)} = (-t)^n e^{-tz}$ is bounded on $\overline{\mathbb{C}_+}$,

so

$$\left\|e^{-tz}\right\|_{(n)} = \max_{\substack{0\le j\le n\\ 0\le k\le j+1}} \sup_{z\in\mathbb{C}_+} \{\left|(\operatorname{Re} z)^k (-t)^j e^{-tz}\right|\} < \infty$$

then $e_{-t} \in X$, and $|h(t)| \le M \left\|e^{-tz}\right\|_{(n)} < \infty \quad \forall n$.

Now

$$\|k_s\|_{(n)} = \max_{\substack{0\le j\le n\\ 0\le k\le j+1}} \sup_{z\in\mathbb{C}_+} \{\left|(\operatorname{Re} z)^k \frac{(-1)^j j!}{(z+s)^{j+1}}\right|\}.$$

Here we have two cases.

When $k = j+1$ then $\|k_s\|_{(n)} \simeq j!$, however when $k < j+1$ then $\|k_s\|_{(n)} \simeq (\operatorname{Re} s)^{k-(j+1)}$

then $\|k_s\|_{(n)} < \infty \quad \forall s$ but $\sup_s \|k_s\|_{(n)} = \infty$.

Moreover since

$$G(s) = \langle k_s, \phi\rangle$$

so

$$|G(s)| = |\langle k_s, \phi\rangle| \le M \|k_s\|_{(n)}$$

where M is a constant, since $\|k_s\|_{(n)}$ depends on $\operatorname{Re} s$ so as we shall see later G is not always in H^∞ see for instance, 5.3.11. □

5.3.1 General case

A convolution system $y = h*u$, where $h(t) = \langle e_{-t}, \phi\rangle$, with diffusive representation ϕ (distribution) that acts on analytic functions $f \in X$ as in Definition 5.3.1 can be realized as a diffusive equation (heat equation)

$$\begin{aligned} \psi_t(z,t) &= -z\psi(z,t) + u(t). \\ y(t) &= \langle \psi, \phi\rangle_z = \langle e_{-t} * u, \phi\rangle_z \end{aligned} \tag{5.3.1}$$

with $\psi(z,t)$ a state variable such as that $\psi(z,0) = 0$ and then,

$$\Psi(z,s) = \frac{U(s)}{z+s}$$

and

$$Y(s) = \langle \Psi, \phi\rangle_z = \langle \frac{U(s)}{z+s}, \phi\rangle_z$$

Proof. We have ϕ (distribution) acts on analytic function $f \in X$ is defined in Definition 5.3.1.

ψ is a solution for the heat equation, from

$$\begin{aligned} \psi_t(z,t) &= -z\psi(z,t) + u(t) \\ y(t) &= \langle \psi, \phi \rangle = \langle e^{-tz} * u, \phi \rangle. \end{aligned}$$

Take the Laplace transform for the heat equation then we obtain,

$$s\Psi(z,s) = -z\Psi(z,s) + U(s),$$

and hence

$$\Psi(z,s) = \frac{U(s)}{z+s}.$$

Thus

$$Y(s) = \langle \Psi, \phi \rangle_z = \langle \mathcal{L}(e^{-\tau z} * u)(z,s), \phi \rangle_z = \langle \frac{U(s)}{z+s}, \phi \rangle$$

and ψ is a solution for the heat equation.

Therefore

$$\begin{aligned} y(t) = (h * u)(t) &= \int_0^t h(\tau) u(t-\tau) \mathrm{d}\tau, \\ &= \int_0^t \langle e^{-\tau z}, \phi \rangle u(t-\tau) \mathrm{d}\tau \\ &= \langle \int_0^t e^{-\tau z} u(t-\tau) \mathrm{d}\tau, \phi \rangle \\ &= \langle e_{-t} * u, \phi \rangle_z. \end{aligned}$$

See ([36], p. 52-53). □

It is elementary to show the following.

Lemma 5.3.4. *For $r \geq 0$ then $\|t^r e^{-tx}\|_{L^2} = \frac{c_r}{x^{r+1/2}}$ for some $c_r > 0$.*

Proposition 5.3.5. *(i) If the function $x \mapsto \frac{g(x)}{x+1}$ lies in L^1 and we define $\langle f, \phi \rangle = \int_0^\infty g(x) f(x) \mathrm{d}x$ for $f \in X$, then this gives a bounded ϕ and if $\frac{g(x)}{x} \in L^1$ then $h \in L^1$ and $G \in H^\infty$.*

(ii) *If the function* $x \mapsto \frac{g(x)}{x^2+x+1}$ *lies in* L^1 *and we define* $\langle f, \phi \rangle = \int_0^\infty g(x) f'(x) \mathrm{d}x$ *for* $f \in X$, *then this gives a bounded* ϕ *and if* $\frac{g(x)}{x^2} \in L^1$ *then* $h \in L^1$ *and* $G \in H^\infty$.

(iii) *If the function* $x \mapsto \frac{g(x)}{\sum_{n=0}^{k+1} x^n}$ *lies in* L^1 *and we define* $\phi(f) = \int_0^\infty g(x) f^{(k)}(x) \mathrm{d}x$ *for* $f \in X$, *then this gives a bounded* ϕ *and if* $\frac{g(x)}{x^{n+1}} \in L^1$ *then* $h \in L^1$ *and* $G \in H^\infty$.

Proof. For the case (i) ϕ is a bounded functional since

$$\begin{aligned} |\langle f, \phi \rangle| &= \left| \int_0^\infty g(x) f(x) \mathrm{d}x \right| \\ &\leq \int_0^\infty \left| \frac{g(x)}{x+1} \right| \max |(x+1) f(x)| \, \mathrm{d}x \\ &\leq \left\| \frac{g}{x+1} \right\|_{L^1} [\max |x f(x)| + \max |f(x)|] \\ &\leq C \|f\|_{(0)} . \end{aligned}$$

where C is a constant.

We next claim that $h \in L^1$ and $G \in H^\infty$,

$$h(t) = \int_0^\infty g(x) e^{-tx} \mathrm{d}x$$

then

$$\begin{aligned} \int_0^\infty |h(t)| \, \mathrm{d}t &\leq \int_0^\infty \int_0^\infty |g(x)| \, e^{-tx} \mathrm{d}x \mathrm{d}t \\ &= \int_0^\infty |g(x)| \, \frac{1}{x} \mathrm{d}x < \infty. \end{aligned}$$

Now

$$G(s) = \int_0^\infty g(x) \frac{1}{s+x} \mathrm{d}x$$

so

$$\begin{aligned} |G(s)| &\leq \int_0^\infty \left| g(x) \frac{1}{s+x} \right| \mathrm{d}x \\ &\leq \int_0^\infty |g(x)| \, \frac{1}{x} \mathrm{d}x < \infty. \end{aligned}$$

So, $G \in H^\infty$. Similar arguments apply to prove the case (ii); as in the proof of the first case we have to show that ϕ is a bounded functional

$$\begin{aligned}
|\langle f, \phi\rangle| &= \left|\int_0^\infty g(x) f'(x)\mathrm{d}x\right| \\
&\leq \int_0^\infty \left|\frac{g(x)}{x^2+x+1}\right| \left|(x^2+x+1)f'(x)\right| \mathrm{d}x \\
&\leq \left\|\frac{g(x)}{x^2+x+1}\right\|_{L^1} \left[\max\left|x^2 f'(x)\right| + \max|xf'(x)| + \max|f'(x)|\right] \\
&\leq \left\|\frac{g(x)}{x^2+x+1}\right\|_{L^1} \|f\|_{(1)}.
\end{aligned}$$

We now prove that $h \in L^1$ and $G \in H^\infty$,

$$h(t) = \int_0^\infty g(x)(-t)e^{-tx}\mathrm{d}x,$$

thus

$$\begin{aligned}
\int_0^\infty |h(t)|\,\mathrm{d}t &\leq \int_0^\infty \int_0^\infty |g(x)|\, te^{-tx}\mathrm{d}x\mathrm{d}t \\
&= \int_0^\infty |g(x)|\,\frac{1}{x^2}\mathrm{d}x < \infty.
\end{aligned}$$

Now

$$G(s) = \int_0^\infty g(x)\frac{-1}{(s+x)^2}\mathrm{d}x,$$

thus

$$\begin{aligned}
|G(s)| &\leq \int_0^\infty \left|g(x)\frac{-1}{(s+x)^2}\right| \mathrm{d}x \\
&\leq \int_0^\infty |g(x)|\,\frac{1}{x^2}\mathrm{d}x < \infty.
\end{aligned}$$

Similar arguments apply to prove the case (iii). □

5.3.2 The Hankel operator

We shall consider the Hankel operator Γ_h

$$\Gamma_h u(x) = \int_0^\infty h(x+y)u(y)\mathrm{d}y,$$

where $h(t) = \langle e_{-t}, \phi\rangle$.

Proposition 5.3.6. *(i) If* $\langle f, \phi\rangle = \phi(f) = \int_0^\infty g(x)f(x)\mathrm{d}x$ *and* $h(t) = \langle e_{-t}, \phi\rangle$ *then, the Hankel operator is nuclear if*

$$\int_0^\infty |g(x)| \frac{\mathrm{d}x}{x} < \infty.$$

Moreover, if $g \geq 0$ *the Hankel operator is nuclear if and only if*

$$\int_0^\infty g(x)\frac{\mathrm{d}x}{x} < \infty.$$

(ii) If $\langle f, \phi\rangle = \phi(f) = \int_0^\infty g(x)f'(x)\mathrm{d}x$ *and* $h(t) = \langle e_{-t}, \phi\rangle$ *then, the Hankel operator is nuclear if*

$$\int_0^\infty |g(x)| \frac{1}{x^2}\mathrm{d}x < \infty.$$

(iii) If $\langle f, \phi\rangle = \sum_{k=0}^N \int_0^\infty g_k(x)f^{(k)}(x)\mathrm{d}x$ *then, the Hankel operator is nuclear if*

$$\int_0^\infty |g_k(x)| \frac{\mathrm{d}x}{x^{k+1}} < \infty.$$

for each k.

Proof. We shall use the fact $\|t^n e^{-tx}\|_{L^2} = \frac{C_n}{x^{n+1/2}}, \quad \forall n \geq 0$, with n not necessarily an integer and C a constant see Lemma 5.3.4 .

(i) If $\phi(f) = \int_0^\infty g(x)f(x)\mathrm{d}x$ and $h(t) = \langle e_{-t}, \phi\rangle$ then the Hankel operator is given by

$$\begin{aligned} Tu(\xi) &= \int_0^\infty u(t)h(\xi+t)\mathrm{d}t \\ &= \int_0^\infty u(t)\langle e_{-(t+\xi)}, \phi\rangle\mathrm{d}t \\ &= \int_0^\infty \int_0^\infty u(t)e^{-xt}\mathrm{d}t e^{-x\xi} g(x)\mathrm{d}x. \end{aligned}$$

For a fixed x let

$$\begin{aligned} T_x u(\xi) &= \langle u, e^{-xt}\rangle e^{-x\xi} \\ &= \langle u, e_{-x}\rangle e_{-x}. \end{aligned}$$

This operator has rank 1 and

$$tr\,|T_x| \le \|e_{-x}\|\,\|e_{-x}\| = \frac{1}{2x}.$$

So, if $\int_0^\infty |g(x)|\,\frac{dx}{x} < \infty$ we can write

$$T = \int_0^\infty g(x)T_x \mathrm{d}x$$

and T is a trace class operator. Moreover, if $g \ge 0$ then T is nuclear if and only if $\int_0^\infty \frac{g(x)}{x}\mathrm{d}x < \infty$, since for $x > 0$, $T_x \ge 0$.

(ii) If $\phi(f) = \int_0^\infty g(x)f'(x)\mathrm{d}x$ and $h(t) = \langle e_{-t}, \phi\rangle$ then 5.3.2

$$Tu(\xi) = \int_0^\infty u(t)\langle e_{-(t+\xi)}, \phi\rangle \mathrm{d}t = -\int_0^\infty u(t)\int_0^\infty (t+\xi)g(x)e^{-(t+\xi)x}\mathrm{d}x\mathrm{d}t.$$

Let

$$T_0u(\xi) = \int_0^\infty g(x)[\int_0^\infty u(t)te^{-xt} + u(t)\xi e^{-\xi x}]\mathrm{d}t\mathrm{d}x$$

and

$$\begin{aligned} Tu(\xi) &= \int_0^\infty \frac{\partial}{\partial x}(\int_0^\infty u(t)e^{-xt}\mathrm{d}te^{-x\xi})g(x)\mathrm{d}x \\ &= -\int_0^\infty\int_0^\infty u(t)(t+\xi)e^{-xt}e^{-x\xi}g(x)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Then, $T_0 = T$.

For a fixed x

$$\begin{aligned} T_x &= -\int_0^\infty u(t)(t+\xi)e^{-xt}e^{-x\xi}\mathrm{d}t \\ &= \langle u, te^{-xt}\rangle_t e^{-x\xi} + \langle u, e^{-xt}\rangle_t \xi e^{-x\xi}. \end{aligned}$$

This operator has rank 2 and

$$tr\,|T_x| \le \left\|te^{-xt}\right\|_{L^2}\left\|e^{-\xi x}\right\|_{L^2} + \left\|e^{-\xi x}\right\|_{L^2}\left\|\xi e^{-\xi x}\right\|_{L^2}.$$

So, if $\int_0^\infty |g(x)|\,tr\,|T_x|\,\mathrm{d}x < \infty$ we can write

$$T = \int_0^\infty g(x)T_x \mathrm{d}x$$

and T is a trace class operator. where

$$\begin{aligned} tr\,|T_x| &\leq \left\|te^{-xt}\right\|_{L^2}\left\|e^{-\xi x}\right\|_{L^2} + \left\|e^{-\xi x}\right\|_{L^2}\left\|\xi e^{-\xi z}\right\|_{L^2} \\ &\leq \frac{1}{2(x)^{3/2}}\frac{1}{(2x)^{1/2}} + \frac{1}{2(x)^{3/2}}\frac{1}{(2x)^{1/2}} \\ &= \frac{1}{\sqrt{2}(x)^2}. \end{aligned}$$

Thus, if $\int_0^\infty |g(x)|\, \frac{1}{\sqrt{2}(x)^2}\mathrm{d}x < \infty$ then T is a trace class operator.

(iii) If $\langle f, \phi\rangle = \sum_{k=0}^N \int_0^\infty g_k(x) f^{(k)}(x)\mathrm{d}x$. Then 5.3.2
$Tu(t) = \int_0^\infty u(\tau)\langle e_{-(t+\tau)}, \phi\rangle \mathrm{d}\tau = \int_0^\infty \sum_{k=0}^N \int_0^\infty u(\tau) g_k(x)(-1)^k \frac{(t+\tau)^k}{k!} e^{-(t+\tau)x}\mathrm{d}x\mathrm{d}\tau.$
Then

$$Tu(t) = \sum_{k=0}^N \int_0^\infty \int_0^\infty g_k(x)\frac{(-1)^k}{k!}\sum_{j=0}^k \binom{k}{j} t^j \tau^{k-j} e^{-(t+\tau)x} u(\tau)\mathrm{d}\tau\mathrm{d}x.$$

Write

$$\begin{aligned} (T_x u)(t) &= \int_0^\infty \sum_{j=0}^k \binom{k}{j} t^j \tau^{k-j} e^{-(t+\tau)x} u(\tau)\mathrm{d}\tau \\ &= \sum_{j=0}^k \binom{k}{j} \langle u, t^j e^{-tx}\rangle \tau^{k-j} e^{-\tau x}, \end{aligned}$$

then

$$\begin{aligned} tr\,|T_x| &\leq \sum_{j=0}^k \binom{k}{j} \left\|t^j e^{-tx}\right\|_{L^2(t)} \left\|\tau^{k-j} e^{-\tau x}\right\|_{L^2(t)} \\ &\leq \frac{C}{x^{j+1/2} x^{k-j+1/2}}. \end{aligned}$$

This operator has rank $k+1$.

Then, if

$$\int_0^\infty |g_k(x)|\, \frac{\mathrm{d}x}{x^{k+1}} < \infty,$$

we can write

$$Tu(t) = \sum_{k=0}^N \int_0^\infty g_k(x) T_x \mathrm{d}x,$$

then

$$tr\,|Tu(t)| = \sum_{k=0}^N \int_0^\infty g_k(x) tr\,|T_x|\, \mathrm{d}x.$$

Thus the Hankel operator is nuclear.

Here we used the fact $\|t^n e^{-tx}\|_{L^2} = \frac{C_n}{x^{n+1/2}}, \quad \forall n \geq 0$, with n not necessarily an integer and C is a constant.

□

5.3.3 The Θ operator

In this section we shall consider the scaled Hankel operator Θ on L^2 given by

$$(\Theta u)(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/4} h(t+\tau) \tau^{-1/4} u(\tau) \mathrm{d}\tau,$$

where, $h(t) = \langle e_{-t}, \phi \rangle$.

Proposition 5.3.7. *In the examples 5.3.11 we consider three cases*

(i) If $\langle f, \phi \rangle = \phi(f) = \int_0^\infty g(x) f(x) \mathrm{d}x$ and $h(t) = \langle e_{-t}, \phi \rangle$ then the Θ operator is nuclear if

$$\int_0^\infty \frac{|g(x)|\, \mathrm{d}x}{x^{1/2}} < \infty.$$

(ii) If $\langle f, \phi \rangle = \phi(f) = \int_0^\infty g(x) f'(x) \mathrm{d}x$ and $h(t) = \langle e_{-t}, \phi \rangle$ then the Θ operator is nuclear if

$$\int_0^\infty \frac{|g(x)|\, \mathrm{d}x}{x^{3/2}} < \infty.$$

(iii) If $\langle f, \phi \rangle = \phi(f) = \sum_{k=0}^n \int_0^\infty g_k(x) f^{(k)}(x)$ and $h(t) = \langle e_{-t}, \phi \rangle$ then the Θ operator is nuclear if

$$\int_0^\infty \frac{|g_k(x)|\, \mathrm{d}x}{x^{k+\frac{1}{2}}} < \infty.$$

for each $k = 0, 1, ..., n$.

Proof. (i) In the first case let

$$(Tu)(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/4} h(t+\xi) \xi^{-1/4} u(\xi) \mathrm{d}\xi$$

so

$$(Tu)(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty t^{-1/4} g(x) e^{-(t+\xi)x} \xi^{-1/4} u(\xi) \mathrm{d}x \mathrm{d}\xi.$$

Let

$$\begin{aligned}(T_x u)(t) &= \int_0^\infty t^{-1/4} e^{-(t+\xi)x} \xi^{-1/4} u(\xi) \mathrm{d}\xi \\ &= \langle u, t^{-1/4} e^{-tx} \rangle \xi^{-1/4} e^{-\xi x}.\end{aligned}$$

This operator has rank 1, and

$$tr\,|T_x| \leq \left\| t^{-1/4} e^{-tx} \right\|_{L^2} \left\| \xi^{-1/4} e^{-\xi x} \right\|_{L^2}.$$

where $\left\| t^{-1/4} e^{-tx} \right\|_{L^2} = \frac{c_1}{x^{1/4}}$.
Since $(Tu)(t) = \int_0^\infty g(x) T_x \mathrm{d}x$ then

$$tr\,|T| \leq C \int_0^\infty \frac{|g(x)|\,\mathrm{d}x}{x^{1/2}}.$$

The proof follows as in proof Proposition 5.3.6.

(ii) If $\langle f, \phi \rangle = \phi(f) = \int_0^\infty g(x) f'(x) \mathrm{d}x$, let

$$(Tu)(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/4} h(t+\xi) \xi^{-1/4} u(\xi) \mathrm{d}\xi,$$

so

$$(Tu)(t) = -\frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty t^{-1/4} g(x)(t+\xi) e^{-(t+\xi)x} \xi^{-1/4} u(\xi) \mathrm{d}x \mathrm{d}\xi.$$

Let

$$\begin{aligned}(T_x u)(t) &= -\int_0^\infty t^{-1/4} (t+\xi) e^{-(t+\xi)x} \xi^{-1/4} u(\xi) \mathrm{d}\xi \\ &= -\langle u, t^{3/4} e^{-tx} \rangle \xi^{-1/4} e^{-\xi x} + \langle u, t^{-1/4} e^{-tx} \rangle \xi^{3/4} e^{-\xi x}.\end{aligned}$$

This operator has rank 2, and

$$tr\,|T_x| \leq \left\| t^{3/4} e^{-tx} \right\|_{L^2} \left\| \xi^{-1/4} e^{-\xi x} \right\|_{L^2} + \left\| t^{-1/4} e^{-tx} \right\|_{L^2} \left\| \xi^{3/4} e^{-\xi x} \right\|_{L^2}$$

where $\left\| t^{3/4} e^{-tx} \right\|_{L^2} = \frac{c_2}{x^{5/4}}$.
Since $(Tu)(t) = \int_0^\infty g(x) T_x \mathrm{d}x$ it follows that

$$tr\,|T| \leq C \int_0^\infty \frac{|g(x)|\,\mathrm{d}x}{x^{3/2}}.$$

(iii) If $\langle f,\phi\rangle = \phi(f) = \sum_{k=0}^{n}\int_0^\infty g_k(x)f^{(k)}(x)$ let

$$(Tu)(t) = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/4}h(t+\xi)\xi^{-1/4}u(\xi)\mathrm{d}\xi.$$

So

$$(Tu)(t) = \frac{1}{\sqrt{\pi}}\int_0^\infty\int_0^\infty t^{-1/4}\sum_{k=0}^{n} g_k(x)(-1)^k(t+\xi)^k e^{-(t+\xi)x}\xi^{-1/4}u(\xi)\mathrm{d}x\mathrm{d}\xi.$$

Then

$$(Tu)(t) = \frac{1}{\sqrt{\pi}}\sum_{k=0}^{n}\int_0^\infty g_k(x)(-1)^k\sum_{j=0}^{k}\int_0^\infty \binom{k}{j} t^{j-1/4}\xi^{k-j-1/4}e^{-(t+\xi)x}u(\xi)\mathrm{d}\xi\mathrm{d}x.$$

Let

$$\begin{aligned}(T_x u)(t) &= \sum_{j=0}^{k}\int_0^\infty \binom{k}{j} t^{j-1/4}\xi^{k-j-1/4}e^{-(t+\xi)x}u(\xi)\mathrm{d}\xi\\ &= \sum_{j=0}^{k}\binom{k}{j}\langle u, t^{j-1/4}e^{-tx}\rangle\xi^{k-j-1/4}e^{-\xi x}.\end{aligned}$$

This operator has rank $k+1$, and

$$\begin{aligned} tr\,|T_x| &\le \sum_{j=0}^{k}\binom{k}{j}\left\|t^{j-1/4}e^{-tx}\right\|_{L^2(t)}\left\|\xi^{k-j-1/4}e^{\xi}\right\|_{L^2(\xi)}\\ &\le C\frac{1}{x^{k+1/2}}.\end{aligned}$$

Then, if

$$\int_0^\infty |g_k(x)|\,\frac{\mathrm{d}x}{x^{k+1/2}} < \infty,$$

we can write

$$Tu(t) = \sum_{k=0}^{n}\int_0^\infty g_k(x)T_x\mathrm{d}x.$$

Thus the Θ operator is nuclear.

□

We can consider more general Θ, sometimes written Θ_ω.

Boundedness of Θ

Theorem 5.3.8. *Write* $\Theta_\omega u(t) = \int_0^\infty \omega(t)h(t+\tau)\omega(\tau)u(\tau)\mathrm{d}\tau$ *and suppose that* $h(t) = \langle e_{-t}, \phi\rangle$, *with* $\omega \geqslant 0$ *and* $u \in L^2$.

(i) If $\langle f, \phi\rangle = \int_0^\infty g(x)f(x)\mathrm{d}x$. *Then* Θ *is bounded if*

$$\int_0^\infty V_1^2(x)\,|g(x)|\,\mathrm{d}x < \infty,$$

where $V_1(x) = \|\omega(\tau)e^{-\tau x}\|_{L^2(\tau)}$.

(ii) If $\langle f, \phi\rangle = \int_0^\infty g(x)f'(x)\mathrm{d}x$. *Then* Θ *is bounded if*

$$\int_0^\infty V_1(x)V_2(x)\,|g(x)|\,\mathrm{d}x < \infty,$$

where $V_1(x) = \|\omega(\tau)e^{-\tau x}\|_{L^2(\tau)}$ *and* $V_2(x) = \|\tau\omega(\tau)e^{-\tau x}\|_{L^2(\tau)}$.

(iii) If $\langle f, \phi\rangle = \int_0^\infty g(x)f''(x)\mathrm{d}x$. *Then* Θ *is bounded if*

$$\int_0^\infty V_1(x)V_3(x)\,|g(x)|\,\mathrm{d}x < \infty,$$

where $V_1(x) = \|\omega(\tau)e^{-\tau x}\|_{L^2(\tau)}$, *and* $V_3(x) = \|\tau^2\omega(\tau)e^{-\tau x}\|_{L^2(\tau)}$.

Proof. (i) If $\langle f, \phi\rangle = \int_0^\infty g(x)f(x)\mathrm{d}x$, we have

$$|(\Theta u)(t)| \leq \int_0^\infty \omega(t)e^{-tx}\,|g(x)|\,V_1(x)\,\|u\|_2\,\mathrm{d}x,$$

where $V_1(x) = \|\omega(\tau)e^{-\tau x}\|_{L^2(\tau)}$.

Thus

$$\begin{aligned}
\langle\Theta(u), \Theta(u)\rangle &= \int_0^\infty (\Theta u)(t)\overline{(\Theta u)(t)}\mathrm{d}t \\
&\leq \int_0^\infty [\int_0^\infty \omega(t)e^{-tx}\,|g(x)|\,V_1(x)\,\|u\|_2\,\mathrm{d}x \\
&\quad \int_0^\infty \omega(t)e^{-ty}\,|g(y)|\,V_1(y)\,\|u\|_2\,\mathrm{d}y]\mathrm{d}t \\
&= \|u\|_2^2 \int_0^\infty\int_0^\infty\int_0^\infty \omega(t)^2e^{-tx}e^{-ty}\,|g(x)|\,V_1(x)\,|g(y)|\,V_1(y) \\
&\quad \mathrm{d}t\mathrm{d}x\mathrm{d}y \\
&\leq c\,\|u\|_2^2 \int_0^\infty\int_0^\infty V_1(x)V_1(y)\,|g(x)|\,|g(y)|\,V_1(x)V_1(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

where c is a constant.

The result now follows.

(ii) If $\langle f, \phi \rangle = \int_0^\infty g(x) f'(x) \mathrm{d}x$, we have

$$(\Theta u)(t) = -\int_0^\infty \int_0^\infty \omega(t)(t+\tau)e^{-x(t+\tau)} g(x)\omega(\tau)u(\tau)\mathrm{d}\tau\mathrm{d}x.$$

Then

$$\begin{aligned}
|(\Theta u)(t)| &\leq \int_0^\infty \omega(t)(t+\tau)\,|g(x)|\,e^{-tx}e^{-\tau x}\omega(\tau)\,|u(\tau)|\,\mathrm{d}\tau\mathrm{d}x \\
&= \int_0^\infty [\int_0^\infty t\omega(t)\,|g(x)|\,e^{-tx}e^{-\tau x}\omega(\tau)\,|u(\tau)|\,\mathrm{d}\tau \\
&\quad + \int_0^\infty \tau\omega(t)\,|g(x)|\,e^{-tx}e^{-\tau x}\omega(\tau)\,|u(\tau)|\,\mathrm{d}\tau]\mathrm{d}x \\
&\leq \|u\|_2 \int_0^\infty \left\|\omega(\tau)e^{-\tau x}\right\|_{L^2(\tau)} t\omega(t)\,|g(x)|\,e^{-tx}\mathrm{d}x \\
&\quad + \|u\|_2 \int_0^\infty \left\|\tau\omega(\tau)e^{-\tau x}\right\|_{L^2(\tau)} \omega(t)\,|g(x)|\,e^{-tx}\mathrm{d}x \\
&= \|u\|_2 \int_0^\infty \omega(t)\,|g(x)|\,e^{-tx}[tV_1(x) + V_2(x)]\mathrm{d}x.
\end{aligned}$$

Thus

$$\begin{aligned}
&\langle \Theta(u), \Theta(u) \rangle \\
&= \int_0^\infty (\Theta u)(t)\overline{(\Theta u)(t)}\mathrm{d}t \\
&\leq \int_0^\infty \|u\|_2^2 \int_0^\infty \omega(t)\,|g(x)|\,e^{-tx}[tV_1(x) \\
&\quad +V_2(x)]\mathrm{d}x \int_0^\infty \omega(t)\,|g(y)|\,e^{-ty}[tV_1(y) + V_2(y)]\mathrm{d}y\mathrm{d}t \\
&= \|u\|_2^2 \int_0^\infty \int_0^\infty \int_0^\infty [|g(x)|\,t\omega(t)e^{-tx}V_1(x) + |g(x)|\,\omega(t)e^{-tx}V_2(x)] \\
&\quad [|g(y)|\,t\omega(t)e^{-ty}V_1(y) + |g(y)|\,\omega(t)e^{-ty}V_2(y)]\mathrm{d}t\mathrm{d}x\mathrm{d}y \\
&\leq c\,\|u\|_2^2 \int_0^\infty \int_0^\infty |g(x)|\,|g(y)|\,[V_1(x)V_2(x)V_1(y)V_2(y)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

where c is a constant.

The result now follows.

(iii) If $\langle f, \phi \rangle = \int_0^\infty g(x) f''(x) \mathrm{d}x$, we have

$$(\Theta u)(t) = \int_0^\infty \int_0^\infty \omega(t)(t+\tau)^2 e^{-x(t+\tau)} g(x)\omega(\tau)u(\tau)\mathrm{d}\tau\mathrm{d}x.$$

Then

$$\begin{aligned}|(\Theta u)(t)| &\leq \int_0^\infty \int_0^\infty \omega(t)(t+\tau)^2\,|g(x)|\,e^{-tx}e^{-\tau x}\omega(\tau)\,|u(\tau)|\,\mathrm{d}\tau\mathrm{d}x \\ &= \int_0^\infty \int_0^\infty \omega(t)(t^2+2t\tau+\tau^2)\,|g(x)|\,e^{-tx}e^{-\tau x}\omega(\tau)\,|u(\tau)|\,\mathrm{d}\tau\mathrm{d}x.\end{aligned}$$

Hence

$$\begin{aligned}|(\Theta u)(t)| &\leq \|u\|_2 \int_0^\infty [t^2\omega(t)\,|g(x)|\,e^{-tx}V_1(x) + 2t\omega(t)\,|g(x)|\,e^{-tx}V_2(x) \\ &\quad +\omega(t)\,|g(x)|\,e^{-tx}V_3(x)]\mathrm{d}x.\end{aligned}$$

Now we shall calculate $\|\Theta\|$

$$\begin{aligned}&\langle\Theta(u),\Theta(u)\rangle \\ &\quad= \int_0^\infty (\Theta u)(t)\overline{(\Theta u)(t)}\mathrm{d}t \\ &\quad\leq \|u\|_2^2 \int_0^\infty \int_0^\infty \omega(t)e^{-tx}\,|g(x)|\,[t^2V_1(x)+2tV_2(x)+V_3(x)] \int_0^\infty \omega(t)e^{-ty} \\ &\qquad |g(y)|\,[t^2V_1(y)+2tV_2(y)+V_3(y)]\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &\quad\leq \|u\|_2^2 \int_0^\infty \int_0^\infty \int_0^\infty |g(x)|\,|g(y)|\,[t^4V_1(x)V_1(y)+t^3V_1(x)V_2(y)+t^2V_1(x)V_3(y) \\ &\qquad +2t^3V_2(x)V_1(y)+4t^2V_2(x)V_2(y)+2tV_2(x)V_3(y)+t^2V_3(x)V_1(y) \\ &\qquad +2tV_3(x)V_2(y)+V_3(x)V_3(y)]\mathrm{d}x\mathrm{d}y\mathrm{d}t.\end{aligned}$$

By using Cauchy-Schwarz

$$\begin{aligned}&\leq \|u\|_2^2 \int_0^\infty \int_0^\infty |g(x)|\,|g(y)|\,[V_1(x)V_3(x)V_1(y)V_3(y)+V_1(x)V_3(x)V_2^2(y) \\ &\quad +V_1(x)V_3(x)V_1(y)V_3(y)+2V_2^2(x)V_1(y)V_3(y)+4V_2^2(x)V_2^2(y) \\ &\quad +2V_2^2(x)V_1(y)V_3(y)+V_1(x)V_3(x)V_1(y)V_3(y) \\ &\quad +2V_1(x)V_3(x)V_2(y)^2+V_1(x)V_3(x)V_1(y)V_3(y)]\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Since

$$V_2^2(x) \leq V_1(x)V_3(x),$$

then

$$\langle\Theta(u),\Theta(u)\rangle \leq c\,\|u\|_2^2 \int_0^\infty \int_0^\infty |g(x)|\,|g(y)|\,V_1(x)V_3(x)V_1(y)V_3(y)\mathrm{d}x\mathrm{d}y,$$

where c is a constant.

$\square$

The following results are given in Proposition 5.3.6 and Proposition 5.3.7 but they are special cases of Theorem 5.3.8.

Corollary 5.3.9. *(a) For $\omega(\tau)=1$, then $V_1(x)=\frac{c_1}{x^{1/2}}$, $V_2(x)=\frac{c_2}{x^{3/2}}$ and $V_3(x)=\frac{c_3}{x^{5/2}}$. Thus,*

(•) If $\langle f,\phi\rangle=\int_0^\infty g(x)f(x)\mathrm{d}x$, then Γ is bounded if

$$\int_0^\infty \frac{|g(x)|\,\mathrm{d}x}{x}<\infty.$$

(•) If $\langle f,\phi\rangle=\int_0^\infty g(x)f'(x)\mathrm{d}x$, then Γ is bounded if

$$\int_0^\infty \frac{|g(x)|\,\mathrm{d}x}{x^2}<\infty.$$

(•) If $\langle f,\phi\rangle=\int_0^\infty g(x)f''(x)\mathrm{d}x$, then Γ is bounded if

$$\int_0^\infty \int_0^\infty \frac{|g(x)|}{x^3}\mathrm{d}x<\infty.$$

(b) For $\omega(\tau)=\tau^{-1/4}$, then $V_1(x)=\frac{c_1}{x^{1/4}}$, $V_2(x)=\frac{c_2}{x^{5/4}}$ and $V_3(x)=\frac{c_3}{x^{9/4}}$. Thus,

(•) If $\langle f,\phi\rangle=\int_0^\infty g(x)f(x)\mathrm{d}x$, then Θ is bounded if

$$\int_0^\infty \frac{|g(x)|\,\mathrm{d}x}{x^{1/2}}<\infty.$$

(•) If $\langle f,\phi\rangle=\int_0^\infty g(x)f'(x)\mathrm{d}x$, then Θ is bounded if

$$\int_0^\infty \int_0^\infty \frac{|g(x)|\,|g(y)|\,\mathrm{d}x\mathrm{d}y}{x^{3/2}y^{3/2}}<\infty$$

or

$$\int_0^\infty \frac{|g(x)|\,\mathrm{d}x}{x^{3/2}}<\infty.$$

(•) If $\langle f,\phi\rangle=\int_0^\infty g(x)f''(x)\mathrm{d}x$, then Θ is bounded if

$$\int_0^\infty \frac{|g(x)|}{x^{5/2}}\mathrm{d}x<\infty.$$

5.3.4 Reproducing Kernel test

If the Θ operator is bounded then [8]

$$\sup_{\omega>0} \frac{\left\| \int_0^\infty t^{-1/4} h(t+\tau)\tau^{-1/4} e^{-\omega\tau} \mathrm{d}\tau \right\|_{L^2(0,\infty)}}{\|e^{-\omega t}\|_{L^2(0,\infty)}} < \infty.$$

If we do not know whether Θ is bounded we can use this test and if it does not hold then Θ is not bounded. However, for the Γ operator this test for $\omega \in \mathbb{C}_+$ is necessary and sufficient, i. e. the Γ operator is bounded if and only if

$$\sup_{\omega>0} \frac{\left\| \int_0^\infty h(t+\tau) e^{-\omega\tau} \mathrm{d}\tau \right\|_{L^2(t)}}{\|e^{-\omega t}\|_{L^2(t)}} < \infty,$$

for more details see [8].

Theorem 5.3.10. *If $h(t) = \frac{1}{t}$ then the integral operator (Hankel operator) Γ is a bounded operator which is unitarily equivalent to multiplication by $\pi(\cosh(x\pi/2))^{-1}$ on $L^2(\mathbb{R})$. In particular there are no eigenvalues and the spectrum and essential spectrum are equal to the interval $[0,\pi]$, see [33, p. 18].*

Example 5.3.11. In Proposition 5.3.5

(i) If we take $g(x) = 1$ and $\phi(f) = \int_0^\infty g(x) f(x) \mathrm{d}x$ but $\phi \notin X^*$, $\|f\|_{(0)} = \sup|f(z)|, |\mathrm{Re}\, z f(z)|$, for $\frac{1}{z+1} \in X$ but $\phi(\frac{1}{z+1}) = \infty$ where, (X^* is the dual space of X i. e. the space of all bounded functionals)

$$|\phi(f)| = \left| \int_0^\infty g(x) f(x) \mathrm{d}x \right| \le \|f\|_{(0)} \int_0^\infty \mathrm{d}x = \infty.$$

Next,

$$h(t) = \int_0^\infty 1.e^{-tx} \mathrm{d}x = \frac{1}{t} \notin L^1.$$

then the system is not BIBO stable, hence it is not nuclear. Moreover,

$$G(s) = \int_0^\infty 1.\frac{1}{s+x} \mathrm{d}x = \log(s+x)|_0^\infty,$$

is not defined.

Now we will study the Hankel operator and the Θ operator.

Firstly, the Hankel operator is Hilbert-Schmidt if and only if $t^{1/2}h \in L^2$ [28, p. 67] so, it is not a Hilbert-Schmidt operator since,

$$(\int_0^\infty \left| t^{1/2}\frac{1}{t} \right|^2 \mathrm{d}t)^{1/2} = \infty.$$

Hence Γ is not nuclear either.

Secondly, the Θ operator is Hilbert-Schmidt if and only if $h \in L^2(0,\infty)$ because $\|\Theta\|_{HS} = \|h\|_2$ [28, p. 94], so here in this example $h \notin L^2$ so it is not a Hilbert-Schmidt operator. Hence it is not nuclear either.

According to Corollary 5.3.9 we do not know whether the Γ operator is bounded since,

$$\int_0^\infty \frac{|g(x)|}{x}\mathrm{d}x = \int_0^\infty \frac{1}{x}\mathrm{d}x = \infty.$$

However, according to Power [33, Theorem 2.6 p. 18] the Γ operator is bounded. We will see later that the Γ operator is bounded but Θ is not.

(ii) If we take $g(x) = 1$ and $\phi(f) = \int_0^\infty g(x)f'(x)\mathrm{d}x$ (ϕ is a bounded functional) since

$$\left| \int_0^\infty f'(x)\mathrm{d}x \right| = \left| \lim_{t\to\infty} f(t) - f(0) \right| \leq 2\,\|f\|_0\,,$$

thus, $\phi \in X^*$.

$$h(t) = \int_0^\infty 1.(-t)e^{-tx}\mathrm{d}x = 1 \notin L^1$$

then the system is not BIBO stable, hence it is not nuclear. Moreover,

$$G(s) = \int_0^\infty 1.\frac{-1}{(s+x)^2}\mathrm{d}x = \frac{1}{s} \notin H^\infty.$$

The integrator example is $y(t) = (h*u)(t) = \int_0^t u(\tau)\mathrm{d}\tau$.

Now we will study the Hankel operator and the Θ operator.

Similarly, firstly $h(t) = 1$ and the Hankel operator is not Hilbert-Schmidt operator since, $t^{1/2}h \notin L^2$. Hence Γ is not nuclear either.

Secondly, the Θ operator is not Hilbert-Schmidt operator since, $h \notin L^2$.

Hence Θ is not nuclear either.

According to Corollary 5.3.9 we do not know whether the Γ operator is bounded since,

$$\int_0^\infty \frac{|g(x)|}{x^2}\mathrm{d}x = \int_0^\infty \frac{1}{x^2}\mathrm{d}x = \infty.$$

However

$$\left\|e^{-t\omega}\right\|_{L^2(t)} = \frac{1}{\sqrt{2\omega}}$$

and

$$\left\|\int_0^\infty h(t+\tau)e^{-\tau\omega}\mathrm{d}\tau\right\|_{L^2(t)} = \infty$$

then the Γ operator is unbounded.

According to Corollary 5.3.9 we do not know whether the Θ operator is bounded since,

$$\int_0^\infty \frac{|g(x)|}{x^{3/2}}\mathrm{d}x = \int_0^\infty \frac{1}{x^{3/2}}\mathrm{d}x = \infty.$$

However

$$\left\|e^{-t\omega}\right\|_{L^2(t)} = \frac{1}{\sqrt{2\omega}}$$

and

$$\left\|\int_0^\infty t^{-1/4}h(t+\tau)\tau^{-1/4}e^{-\tau\omega}\mathrm{d}\tau\right\|_{L^2(t)} = \infty$$

and so the Θ operator is unbounded. We see this later by another argument.

(iii) If we take $g(x) = e^{-x}$ and $\phi(f) = \int_0^\infty g(x)f(x)\mathrm{d}x$ then,

$$|\phi(f)| \le \int_0^\infty e^{-x}\|f\|_{(0)}\,\mathrm{d}x = \|f\|_{(0)},$$

thus, $\phi \in X^*$. Next,

$$h(t) = \int_0^\infty e^{-x}e^{-tx}\mathrm{d}x = \frac{1}{t+1} \notin L^1.$$

However the transfer function,

$$G(s) = \int_0^\infty \frac{e^{-x}}{x+s}\mathrm{d}x$$

converges, but $G \notin H^\infty$.

Similarly, firstly $h(t) = \frac{1}{t+1}$ and the Hankel operator is not a Hilbert-Schmidt operator since,

$$\int_0^\infty t\left|\frac{1}{t+1}\right|^2 \mathrm{d}t = \int_1^\infty (u-1)u^{-2}\mathrm{d}u = \int_1^\infty \frac{1}{u} - \frac{1}{u^2}\mathrm{d}u = \infty.$$

From 5.3.2 the Hankel operator is nuclear if $\int_0^\infty |g(x)|\,\frac{\mathrm{d}x}{x} < \infty$, but in this case it is not nuclear since

$$\int_0^\infty \frac{e^{-x}}{x}\mathrm{d}x = \infty.$$

According to Proposition 5.3.7, the Θ operator is nuclear, since

$$\int_0^\infty \frac{e^{-x}\mathrm{d}x}{\sqrt{2x}} < \infty.$$

Hence it is a Hilbert-Schmidt operator and we can notice that $h \in L^2(0,\infty)$ and hence Θ is bounded.

(iv) If we take $g(x) = xe^{-x}$ and $\phi(f) = \int_0^\infty g(x)f(x)\mathrm{d}x$ then

$$|\phi(f)| = \left|\int_0^\infty g(x)f(x)\mathrm{d}x\right| \le \|f\|_{(0)} \int_0^\infty xe^{-x}\mathrm{d}x = \|f\|_{(0)} < \infty,$$

thus, $\phi \in X^*$.

Next,

$$h(t) = \int_0^\infty xe^{-x}e^{-tx}\mathrm{d}x = \frac{1}{(t+1)^2},$$

thus $h \in L^1$ and

$$\mathcal{L}(\frac{1}{(t+1)^2}) = 1 - se^sE_1(s) = G(s) = \int_0^\infty xe^{-x}\frac{1}{x+s}\mathrm{d}x.$$

In addition,

$$|G(s)| = \left|\int_0^\infty xe^{-x}\frac{1}{x+s}\mathrm{d}x\right| \le \int_0^\infty xe^{-x}\frac{1}{x}\mathrm{d}x = 1 < \infty,$$

then, $G \in H^\infty$.

The Hankel operator is a Hilbert-Schmidt operator since,

$$\int_0^\infty \frac{t\mathrm{d}t}{(t+1)^4} = \int_1^\infty (u-1)u^{-4}\mathrm{d}u < \infty.$$

From Proposition 5.3.6 the Hankel operator is nuclear since,

$$\int_0^\infty \frac{|g(x)|}{x}\mathrm{d}x = \int_0^\infty \frac{xe^{-x}}{x}\mathrm{d}x < \infty.$$

Also, because $h \in L^2$ then the Θ operator is a Hilbert-Schmidt operator.

According to Proposition 5.3.7, the Θ operator is nuclear, since

$$\int_0^\infty \frac{xe^{-x}\mathrm{d}x}{\sqrt{2x}} < \infty.$$

(v) If we take $g(x) = xe^{-x}$ and $\phi(f) = \int_0^\infty g(x)f'(x)\mathrm{d}x$ then,

$$|\phi(f)| = \left|\int_0^\infty g(x)f(x)\mathrm{d}x\right| \leq \|f\|_{(0)} \int_0^\infty xe^{-x}\mathrm{d}x = \|f\|_{(0)},$$

thus, $\phi \in X^*$.

Next,

$$h(t) = \int_0^\infty xe^{-x}(-t)e^{-tx}\mathrm{d}x = \frac{-t}{(t+1)^2},$$

thus $h \notin L^1$.

Moreover,

$$G(s) = \int_0^\infty xe^{-x}\frac{-1}{(s+x)^2}\mathrm{d}x,$$

since, when $s \to 0+$ then $G(s) \to -\infty$ and so $G \notin H^\infty$.

The Hankel operator is not a Hilbert-Schmidt operator since

$$\int_0^\infty \frac{t^3\mathrm{d}t}{(t+1)^4} = \infty.$$

Hence it is not nuclear.

According to Proposition 5.3.7, the Θ operator is nuclear, since,

$$\int_0^\infty \frac{xe^{-x}\mathrm{d}x}{x^{3/2}} = \sqrt{\pi} < \infty.$$

Hence the Θ operator is a Hilbert-Schmidt operator and bounded operator.

According to Corollary 5.3.9 we do not know whether the Γ operator is bounded since,

$$\int_0^\infty \frac{|g(x)|}{x^2}\mathrm{d}x = \int_0^\infty \frac{xe^{-x}}{x^2}\mathrm{d}x = \infty.$$

However,

$\|e^{-t\omega}\|_{L^2(t)} = \frac{1}{\sqrt{2\omega}}$ and $\left\|\int_0^\infty h(t+\tau)e^{-\tau\omega}\mathrm{d}\tau\right\|_{L^2(t)} = \infty$

and so the Γ operator is unbounded.

(vi) If we take $g(x) = x^2e^{-x}$ and $\phi(f) = \int_0^\infty g(x)f'(x)\mathrm{d}x$ then,

$$|\phi(f)| = \left|\int_0^\infty x^2e^{-x}f'(x)\mathrm{d}x\right| \leq \|f\|_{(0)} \int_0^\infty x^2e^{-x}\mathrm{d}x = 2\,\|f\|_{(0)}$$

thus, $\phi \in X^*$.

Next

$$h(t) = \int_0^\infty x^2e^{-x}(-t)e^{-tx}\mathrm{d}x = \frac{-2t}{(t+1)^3},$$

thus $h \in L^1$.

The Hankel operator is nuclear since,

$$\int_0^\infty \frac{|g(x)|}{x^2}\mathrm{d}x = \int_0^\infty e^{-x}\mathrm{d}x = 1 < \infty.$$

Hence, the Hankel operator is a Hilbert-Schmidt operator.

According to Proposition 5.3.7, the Θ operator is nuclear, since

$$\int_0^\infty \frac{|g(x)|}{x^{3/2}}\mathrm{d}x = \int_0^\infty \frac{x^2e^{-x}}{x^{3/2}}\mathrm{d}x < \infty.$$

Hence the Θ operator is a Hilbert-Schmidt operator.

(vii) If we take $g(x) = x^3e^{-x}$ and $\phi(f) = \int_0^\infty g(x)f''(x)\mathrm{d}x$ then,

$$|\phi(f)| = \left|\int_0^\infty g(x)f(x)\mathrm{d}x\right| \leq \|f\|_{(0)} \int_0^\infty x^3e^{-x}\mathrm{d}x = 6\,\|f\|_{(1)},$$

thus, $\phi \in X^*$.

$$h(t) = \int_0^\infty x^3e^{-x}(t^2)e^{-tx}\mathrm{d}x = \frac{6t^2}{(t+1)^4},$$

thus $h \in L^1$.

The Hankel operator is nuclear since,

$$\int_0^\infty \frac{|g(x)|}{x^3}\mathrm{d}x = \int_0^\infty e^{-x}\mathrm{d}x = 1 < \infty.$$

Hence, the Hankel operator is a Hilbert-Schmidt operator.

$$\int_0^\infty \frac{|g(x)|}{x^{5/2}}\mathrm{d}x = \int_0^\infty x^{1/2}e^{-x}\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Hence the Θ operator is nuclear and Hilbert-Schmidt as well.

5.3.5 Discrete distributions

(i) If $\langle f, \phi \rangle = \sum_{j=1}^{\infty} \lambda_j f(z_j)$ with $z_j \in \mathbb{C}_+$ and $\lambda_j \in \mathbb{C}$.
Since $|\langle f, \phi \rangle| \leq \sum_{j=1}^{\infty} |\lambda_j| \, |f(z_j)| \leq \sum_{j=1}^{\infty} |\lambda_j| \, \|f\|_{(0)}$, and $|\langle f, \phi \rangle| \leq \sum_{j=1}^{\infty} \frac{|\lambda_j|}{\operatorname{Re} z_j} \|f\|_{(0)}$, then, ϕ is a bounded functional if

$$\sum_{j=1}^{\infty} |\lambda_j| < \infty,$$

or

$$\sum_{j=1}^{\infty} \frac{|\lambda_j|}{\operatorname{Re} z_j} < \infty.$$

We have

$$h(t) = \langle e_{-t}, \phi \rangle = \sum_{j=1}^{\infty} \lambda_j e^{-z_j t}$$

and

$$G(s) = \langle k_s, \phi \rangle = \sum_{j=1}^{\infty} \frac{\lambda_j}{z_j + s}.$$

Then $h \in L^1$ if

$$\sum_{j=1}^{\infty} \left\| \lambda_j e^{-z_j t} \right\|_{L^1} = \sum_{j=1}^{\infty} \frac{|\lambda_j|}{\operatorname{Re} z_j} < \infty,$$

and this implies also $G \in H^\infty$.
In other words, $G(s)$ converges in H^∞ if

$$\|G(s)\|_{H^\infty} = \sup_s \left| \sum_{j=1}^{\infty} \frac{\lambda_j}{z_j + s} \right| \leq \sum_{j=1}^{\infty} \frac{|\lambda_j|}{\operatorname{Re} z_j} < \infty.$$

The Hankel operator with symbol $\frac{\lambda_j}{z_j + s}$ has rank 1, and the Hankel operator $\Gamma = \sum_{j=1}^{\infty} \Gamma_j$ is nuclear if,

$$\begin{aligned} \|\Gamma\|_N &\leq \sum_{j=1}^{\infty} \|\Gamma_j\| \\ &\leq \sum_{j=1}^{\infty} \|G_j\| \\ &\leq \sum_{j=1}^{\infty} \frac{|\lambda_j|}{\operatorname{Re} z_j} \\ &< \infty. \end{aligned}$$

We know that Γ is a Hilbert-Schmidt operator if $t^{1/2}h \in L^2$ then,

$$\begin{aligned}
\int_0^\infty \left|t^{1/2}h(t)\right|^2 \mathrm{d}t &= \int_0^\infty t \left(\sum_{n=1}^\infty \lambda_n e^{-z_n t}\right)\left(\sum_{m=1}^\infty \overline{\lambda}_m e^{-\overline{z}_m t}\right) \mathrm{d}t \\
&= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\lambda_n \overline{\lambda}_m}{z_n + \overline{z}_m}.
\end{aligned}$$

In order to use this, we observe that

$$\begin{aligned}
|z_n + \overline{z}_m| &\geq \operatorname{Re} z_n + \operatorname{Re} z_m \\
&\geq (\operatorname{Re} z_n)^{1/2} (\operatorname{Re} z_m)^{1/2},
\end{aligned}$$

thus

$$\sum_n \sum_m \frac{\lambda_n \overline{\lambda}_m}{(z_n + \overline{z}_m)^2} \leq \left(\sum_n \frac{|\lambda_n|}{2^2 \operatorname{Re} z_n}\right)^2.$$

We deduce that the Hankel operator is Hilbert-Schmidt if

$$\sum_n \frac{|\lambda_n|}{\operatorname{Re} z_n} < \infty.$$

Similarly for the Θ operator,

$$\begin{aligned}
\Theta u(t) &= \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/4} h(t+\tau) \tau^{-1/4} u(\tau) \mathrm{d}\tau \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/4} \sum_{j=1}^\infty \lambda_j e^{-(t+\tau)z_j} \tau^{-1/4} u(\tau) \mathrm{d}\tau \\
&= \sum_{j=1}^\infty \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/4} \lambda_j e^{-(t+\tau)z_j} \tau^{-1/4} u(\tau) \mathrm{d}\tau \\
&= \sum_{j=1}^\infty \frac{1}{\sqrt{\pi}} \lambda_j \langle u, e^{-z_j t} t^{-1/4} \rangle e^{-\overline{z}_j \tau} \tau^{-1/4}.
\end{aligned}$$

Then, $\Theta u = \sum_{j=1}^\infty \Theta_j u$, where every Θ_j operator has rank 1.
Since if $\Theta_j : u \to \langle u, v \rangle w$ has rank 1 so, $tr\,|\Theta_j| = \|\Theta_j\|_N = \|v\|_{L^2} \|w\|_{L^2}$, it follows that

$$\begin{aligned}
tr\,|\Theta_j| &= \left\|e^{-z_j t} t^{-1/4}\right\|_{L^2} \left\|e^{-\overline{z}_j \tau} \tau^{-1/4}\right\|_{L^2} \\
&\leq \frac{c}{(\operatorname{Re} z_j)^{-1/4+1/2} (\operatorname{Re} z_j)^{-1/4+1/2}} \\
&= \frac{c}{(\operatorname{Re} z_j)^{1/2}}.
\end{aligned}$$

where c is a constant.
The Θ operator is nuclear if $\sum_{j=1}^{\infty} \frac{c}{(\operatorname{Re} z_j)^{1/2}} < \infty$.
Now the Θ operator is Hilbert-Schmidt if $h \in L^2$, so

$$\begin{aligned}
\int_0^\infty |h(t)|^2 \, \mathrm{d}t &= \int_0^\infty \left(\sum_{n=1}^{\infty} \lambda_n e^{-z_n t} \right) \left(\sum_{m=1}^{\infty} \overline{\lambda}_m e^{-\overline{z}_m t} \right) \mathrm{d}t \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_0^\infty \lambda_n \overline{\lambda}_m e^{-(z_n + \overline{z}_m)t} \mathrm{d}t \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\lambda_n \overline{\lambda}_m}{(z_n + \overline{z}_m)}.
\end{aligned}$$

Thus the Θ operator is Hilbert-Schmidt if $\sum_n \frac{|\lambda_n|}{\sqrt{\operatorname{Re} z_n}} < \infty$, since

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\lambda_n \overline{\lambda}_m}{(z_n + \overline{z}_m)} \leq (\sum_n \frac{|\lambda_n|}{2\sqrt{\operatorname{Re} z_n}})^2.$$

(ii) Suppose $\langle f, \phi \rangle = \sum_{j=1}^{\infty} \lambda_j f'(z_j)$ with $z_j \in \mathbb{C}_+$ and $\lambda_j \in \mathbb{C}$.
Since

$$\begin{aligned}
\|f\|_{(1)} &\geq \sup |f'(z_j)|, \\
\|f\|_{(1)} &\geq \sup |(\operatorname{Re} z_j) f'(z_j)|, \\
\|f\|_{(1)} &\geq \sup \left|(\operatorname{Re} z_j)^2 f'(z_j)\right|,
\end{aligned}$$

it follows that if

$$|\langle f, \phi \rangle| \leq c \sum_{j=1}^{\infty} |\lambda_j| < \infty,$$

or

$$|\langle f, \phi \rangle| \leq c \sum_{j=1}^{\infty} \frac{|\lambda_j|}{\operatorname{Re} z_j} < \infty,$$

or

$$|\langle f, \phi \rangle| \leq c \sum_{j=1}^{\infty} \frac{|\lambda_j|}{(\operatorname{Re} z_j)^2} < \infty,$$

then ϕ is a bounded functional.
We have

$$h(t) = \langle e_{-t}, \phi \rangle = \sum_{j=1}^{\infty} \lambda_j (-t) e^{-z_j t}$$

and

$$G(s) = \langle k_s, \phi \rangle = \sum_{j=1}^{\infty} \frac{-\lambda_j}{(z_j + s)^2}.$$

Then $h \in L^1$ if

$$\sum_{j=1}^{\infty} \left\|\lambda_j(-t)e^{-z_j t}\right\|_{L^1} = \sum_{j=1}^{\infty} \frac{|\lambda_j|}{(\operatorname{Re} z_j)^2} < \infty,$$

and this implies also $G \in H^\infty$.
In other words, $G(s)$ converges in H^∞ if

$$\|G(s)\|_{H^\infty} \leq \sum_{j=1}^{\infty} \frac{|\lambda_j|}{(\operatorname{Re} z_j)^2} < \infty.$$

The Hankel operator with symbol $\frac{\lambda_j}{z_j+s}$ has rank 2.
Since, $\Gamma = \sum_{j=1}^{\infty} \Gamma_j$, so

$$\begin{aligned}
\|\Gamma\|_N &\leq \sum_{j=1}^{\infty} \|\Gamma_j\|_N \\
&\leq 2\sum_{j=1}^{\infty} \|G_j\|_\infty \\
&= 2\sum_{j=1}^{\infty} \frac{|\lambda_j|}{(\operatorname{Re} z_j)^2}.
\end{aligned}$$

Thus the Hankel operator is nuclear if $\sum_{j=1}^{\infty} \frac{|\lambda_j|}{(\operatorname{Re} z_j)^2} < \infty$.
Now the Hankel operator is Hilbert-Schmidt if $t^{1/2}h \in L^2$,
hence

$$\begin{aligned}
\int_0^\infty \left|t^{1/2}h(t)\right|^2 \mathrm{d}t &= \int_0^\infty t(\sum_{n=1}^{\infty} \lambda_n(-t)e^{-z_n t})(\sum_{m=1}^{\infty} \overline{\lambda}_m(-t)e^{-\overline{z}_m t})\mathrm{d}t \\
&= \int_0^\infty t^3(\sum_{n=1}^{\infty} \lambda_n e^{-z_n t})(\sum_{m=1}^{\infty} \overline{\lambda}_m e^{-\overline{z}_m t})\mathrm{d}t \\
&\leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\lambda_n \overline{\lambda}_m}{(z_n + \overline{z}_m)^4} \\
&\leq (\sum_{n=1}^{\infty} \frac{|\lambda_n|}{(2^2 \operatorname{Re} z_n)^2})^2.
\end{aligned}$$

The Hankel operator is Hilbert-Schmidt if $\sum_{n=1}^{\infty} \frac{|\lambda_n|}{(\operatorname{Re} z_n)^2} < \infty$.

Similarly for the Θ operator,

$$\begin{aligned}
\Theta u(t) &= \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/4}h(t+\tau)\tau^{-1/4}u(\tau)\mathrm{d}\tau \\
&= \frac{-1}{\sqrt{\pi}}\int_0^\infty t^{-1/4}\sum_{j=1}^\infty \lambda_j(t+\tau)e^{-(t+\tau)z_j}\tau^{-1/4}u(\tau)\mathrm{d}\tau \\
&= \sum_{j=1}^\infty \frac{-1}{\sqrt{\pi}}\int_0^\infty t^{-1/4}\lambda_j(t+\tau)e^{-(t+\tau)z_j}\tau^{-1/4}u(\tau)\mathrm{d}\tau \\
&= \sum_{j=1}^\infty \frac{-1}{\sqrt{\pi}}[\int_0^\infty t^{3/4}\lambda_j e^{-(t+\tau)z_j}\tau^{-1/4}u(\tau)\mathrm{d}\tau + \\
&\quad \int_0^\infty t^{-1/4}\lambda_j e^{-(t+\tau)z_j}\tau^{3/4}u(\tau)\mathrm{d}\tau] \\
&= \sum_{j=1}^\infty \frac{-1}{\sqrt{\pi}}[\lambda_j\langle u, t^{3/4}e^{-tz_j}\rangle\tau^{-1/4}e^{-\overline{z_j}} + \\
&\quad \lambda_j\langle u, t^{-1/4}e^{-tz_j}\rangle\tau^{3/4}e^{-\overline{z_j}}.
\end{aligned}$$

Thus $\Theta u = \sum_{j=1}^\infty \Theta_j u$, where every Θ_j has rank 2, then

$$\begin{aligned}
tr\,|\Theta_j| &\leq \left\|e^{-z_j t}t^{3/4}\right\|_{L^2}\left\|e^{-\overline{z_j}\tau}\tau^{-1/4}\right\|_{L^2} + \left\|e^{-z_j t}t^{-1/4}\right\|_{L^2}\left\|e^{-\overline{z_j}\tau}\tau^{3/4}\right\|_{L^2} \\
&\leq \frac{c_j}{(\operatorname{Re} z_j)^{-1/4+1/2}(\operatorname{Re} z_j)^{-1/4+1/2}} \\
&= \frac{c_j}{(\operatorname{Re} z_j)^{3/2}}.
\end{aligned}$$

We deduce that $tr\,|\Theta| \leq \sum_{j=1}^\infty \frac{c_j}{(\operatorname{Re} z_j)^{3/2}} < \infty$.

Now the Θ operator is Hilbert-Schmidt if $h \in L^2$.

Now

$$\begin{aligned}
\int_0^\infty |h(t)|^2\,\mathrm{d}t &= \int_0^\infty t^2(\sum_{n=1}^\infty \lambda_n e^{-z_n t})(\sum_{m=1}^\infty \overline{\lambda}_m e^{-\overline{z}_m t})\mathrm{d}t \\
&= \sum_{n=1}^\infty\sum_{m=1}^\infty \int_0^\infty t^2\lambda_n\overline{\lambda}_m e^{-(z_n+\overline{z}_m)t}\mathrm{d}t \\
&= 2\sum_{n=1}^\infty\sum_{m=1}^\infty \frac{\lambda_n\overline{\lambda}_m}{(z_n+\overline{z}_m)^3} \\
&\leq \frac{1}{2^3}\left(\sum_{n=1}^\infty \frac{|\lambda_n|}{(\operatorname{Re} z_n)^{3/2}}\right)^2 \\
&< \infty.
\end{aligned}$$

So Θ is Hilbert-Schmidt if

$$\left(\sum_{n=1}^\infty \frac{|\lambda_n|}{(\operatorname{Re} z_n)^{3/2}}\right)^2 < \infty.$$

(iii) If $\langle f,\phi\rangle = \sum_{k=0}^{N}(\sum_{j=1}^{\infty}\lambda_{k,j}f^{(k)}(z_{k,j}))$ with $z_{k,j}\in\mathbb{C}_+$ and $\lambda_{k,j}\in\mathbb{C}$. In order to get a bounded functional ϕ it is sufficient to have a bounded ϕ_k for each k, where

$$\langle f,\phi_k\rangle = \sum_{j=1}^{\infty}\lambda_{k,j}f^{(k)}(z_{k,j}).$$

Then

$$|\langle f,\phi_k\rangle| \le C_k\,\|f\|_{(k)}\,,\quad k=0,1,...,N.$$

We have

$$h(t)=\langle e_{-t},\phi\rangle = \sum_{k=0}^{N}(\sum_{j=1}^{\infty}\lambda_{k,j}(-t)^k e^{-tz_{k,j}})$$

and

$$G(s)=\langle k_s,\phi\rangle = \sum_{k=0}^{N}(\sum_{j=1}^{\infty}\frac{(-1)^k\lambda_{k,j}}{(z_{k,j}+s)^{k+1}}).$$

Then the system is BIBO stable if $h\in L^1$

$$\begin{aligned}\int_0^{\infty}|h(t)|\,\mathrm{d}t &= \int_0^{\infty}\left|\sum_{k=0}^{N}(\sum_{j=1}^{\infty}\lambda_{k,j}(-t)^k e^{-tz_{k,j}}\right|\mathrm{d}t\\ &\le \sum_{k=0}^{N}\sum_{j=1}^{\infty}\int_0^{\infty}|\lambda_{k,j}|\,t^k\left|e^{-tz_{k,j}}\right|\mathrm{d}t\\ &= \sum_{k=0}^{N}(\sum_{j=1}^{\infty}\frac{|\lambda_{k,j}|}{(\mathrm{Re}\,z_{k,j})^{k+1}})\\ &< \infty.\end{aligned}$$

This implies also $G\in H^\infty$ also, $\|G(s)\|_{H^\infty}\le\sum_{k=0}^{N}(\sum_{j=1}^{\infty}\frac{|\lambda_{k,j}|}{(\mathrm{Re}\,z_{k,j})^{k+1}})$, since,

$$\begin{aligned}\|G(s)\|_{H^\infty} &\le \sum_{k=0}^{N}\sum_{j=1}^{\infty}\left\|\frac{\lambda_{k,j}}{(s+z_{k,j})^{k+1}}\right\|_{H^\infty}\\ &= \sum_{k=0}^{N}\sum_{j=1}^{\infty}\frac{|\lambda_{k,j}|}{(\mathrm{Re}\,z_{k,j})^{k+1}}.\end{aligned}$$

Now the Hankel operator Γ satisfies

$$\begin{aligned}
\Gamma u(t) &= \int_0^\infty h(t+\tau)u(\tau)\mathrm{d}\tau \\
&= \int_0^\infty \sum_{k=0}^{N}\sum_{j=1}^{\infty} \lambda_{k,j}(t+\tau)^k(-1)^k e^{-(t+\tau)z_{k,j}} u(\tau)\mathrm{d}\tau \\
&= \sum_{k=0}^{N}\sum_{j=1}^{\infty} \int_0^\infty \lambda_{k,j}(t+\tau)^k(-1)^k e^{-(t+\tau)z_{k,j}} u(\tau)\mathrm{d}\tau \\
&= \sum_{k=0}^{N}\sum_{j=1}^{\infty} \int_0^\infty \lambda_{k,j}(-1)^k \sum_{i=0}^{k}\binom{k}{i} t^i\tau^{k-i} e^{-(t+\tau)z_{k,j}} u(\tau)\mathrm{d}\tau.
\end{aligned}$$

Then let

$$\begin{aligned}
\Gamma_j^{(k)}u(t) &= \int_0^\infty \lambda_{k,j}(-1)^k \sum_{i=0}^{k}\binom{k}{i} t^i\tau^{k-i} e^{-(t+\tau)z_{k,j}} u(\tau)\mathrm{d}\tau \\
&= \sum_{i=0}^{k} \lambda_{k,j}(-1)^k \binom{k}{i} \langle u, t^i e^{-tz_{k,j}}\rangle \tau^{k-i} e^{-\tau\overline{z}_{k,j}}.
\end{aligned}$$

Then $\Gamma u = \sum_{k=1}^{N}\Gamma^{(k)}u = \sum_{k=1}^{N}\sum_{j=1}^{\infty}\Gamma_j^{(k)}u$ and

$$\begin{aligned}
\sum_{j=1}^{\infty}\left\|\Gamma^{(k)}\right\|_N &\leq c\left\|\Gamma_j^{(k)}\right\| \\
&\leq \frac{C_k}{(\mathrm{Re}\, z_{k,j})^{i+1/2}(\mathrm{Re}\, z_{k,j})^{k-i+1/2}} \\
&= \frac{C_k}{(\mathrm{Re}\, z_{k,j})^{k+1}}.
\end{aligned}$$

Therefore, the Hankel operator Γ is nuclear if

$$\sum_{j=1}^{\infty}\frac{|\lambda_{k,j}|}{(\mathrm{Re}\, z_{k,j})^{k+1}} < \infty,$$

and then Γ will be a Hilbert-Schmidt operator as well.

Now we will consider the Θ operator:

$$\begin{aligned}
\Theta u(t) &= \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/4}h(t+\tau)\tau^{-1/4}u(\tau)\mathrm{d}\tau \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/4}\sum_{k=0}^{N}\sum_{j=1}^{\infty}\lambda_{k,j}(t+\tau)^k(-1)^k e^{-(t+\tau)z_{k,j}}\tau^{-1/4}u(\tau)\mathrm{d}\tau \\
&= \sum_{k=0}^{N}\sum_{j=1}^{\infty}\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/4}\lambda_{k,j}(t+\tau)^k(-1)^k e^{-(t+\tau)z_{k,j}}\tau^{-1/4}u(\tau)\mathrm{d}\tau \\
&= \sum_{k=0}^{N}\sum_{j=1}^{\infty}\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/4}\lambda_{k,j}(-1)^k\sum_{i=0}^{k}\binom{k}{i}t^i\tau^{k-i}e^{-(t+\tau)z_{k,j}}\tau^{-1/4}u(\tau)\mathrm{d}\tau \\
&= \sum_{k=0}^{N}\sum_{j=1}^{\infty}\sum_{i=0}^{k}\frac{1}{\sqrt{\pi}}\lambda_{k,j}(-1)^k\binom{k}{i}\langle u, t^{i-1/4}e^{-tz_{k,j}}\rangle\tau^{k-i-1/4}e^{-\tau\overline{z}_{k,j}}.
\end{aligned}$$

Thus

$$\Theta = \sum_{k=0}^{N}\Theta^{(k)}$$

where $\Theta^{(k)}$ uses k^{th} derivatives.
Then Θ is nuclear if each $\Theta^{(k)}$ is.

$$\begin{aligned}
\Theta^{(k)}u(t) &= \frac{1}{\sqrt{\pi}}\sum_{j=1}^{\infty}\frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/4}\lambda_{k,j}(-1)^k\sum_{i=0}^{k}\binom{k}{i}t^i\tau^{k-i}e^{-(t+\tau)z_{k,j}}u(\tau)\mathrm{d}\tau \\
&= \sum_{j=1}^{\infty}\Theta_j^{(k)}.
\end{aligned}$$

where

$$\Theta_j^{(k)}u = \sum_{i=0}^{k}\frac{1}{\sqrt{\pi}}\lambda_{k,j}(-1)^k\binom{k}{i}\langle u, t^{i-1/4}e^{-tz_{k,j}}\rangle\tau^{k-i-1/4}e^{-\tau\overline{z}_{k,j}}$$

and $\Theta_j^{(k)}$ has rank at most $k+1$, since its range is spanned by $t^{-1/4}e^{-t}t^i$ for $i = 0, 1, ..., k$.
Therefore

$$\left\|\Theta^{(k)}\right\|_N \leq (k+1)\left\|\Theta_j^{(k)}\right\|$$

So, if

$$\sum_{j=1}^{\infty}\left\|\Theta_j^{(k)}\right\| < \infty,$$

thus, $\Theta^{(k)}$ is nuclear.

$$\begin{aligned}\left\|\Theta_j^{(k)}\right\|_N &\leq \frac{1}{\sqrt{\pi}}\sum_{i=0}^{k}|\lambda_{k,j}|\binom{k}{i}\left\|t^{i-1/4}e^{-tz_{k,j}}\right\|_{L^2}\left\|\tau^{k-i-1/4}e^{-\tau\overline{z}_{k,j}}\right\|_{L^2}\\ &\leq \frac{C_k|\lambda_{k,j}|}{(\operatorname{Re} z_{k,j})^{k+1/2}}.\end{aligned}$$

Moreover, Θ is Hilbert-Schmidt as well.

Example 5.3.12. (i) If $\langle f,\phi\rangle = f(\lambda)$ then $h(t) = \langle e_{-t},\phi\rangle = e^{-\lambda t}$ and ϕ is a bounded functional since, $\|f\| = \max_z(|f|, z|f|)$ then

$$|\phi(f)| \leq \|f\|_{(0)}.$$

(ii) If $\langle f,\phi\rangle = f'(\lambda)$ with $\lambda \in \mathbb{C}_+$ then $h(t) = \langle e_{-t},\phi\rangle = (e^{-tx})'(\lambda) = -te^{-t\lambda}$ and ϕ is a bounded functional since

$$\begin{aligned}|\langle f,\phi\rangle| &\leq \sup|f'(z)|\\ &\leq \|f\|_{(1)}.\end{aligned}$$

Lemma 5.3.13. *If an operator A satisfies that $\|A - A_n\|_N = \varepsilon_n$, where $\varepsilon_n \to 0$ for $rank(A_n) = n$, then $\delta_N = \sum_{N+1}^{\infty}\sigma_n \leq \varepsilon_N$ and $\sigma_N = O(\frac{\varepsilon_N}{N})$.*

Proof. We have

$$\sum_{k=n+1}^{\infty}\sigma_k \leq \|A - A_n\|_N$$

then

$$\sigma_{n+1} + \sigma_{n+2} + ... + \sigma_{2n+1} \leq \delta_n,$$

so

$$(n+1)\sigma_{2n+1} \leq \delta_n.$$

Similarly,

$$\sigma_{n+1} + \sigma_{n+2} + ... + \sigma_{2n} \leq \delta_n,$$

so

$$n\sigma_{2n} \leq \delta_n.$$

As a result

$$\sigma_N = O(\frac{\varepsilon_N}{N}).$$

□

Example 5.3.14. (i) Suppose $\langle f, \phi \rangle = \sum_{j=1}^{\infty} f(j)$ for f analytic in $\mathbb{C}_+$ with $\|f\|_{(n)} < \infty \quad \forall n$. Then if $f(z) = \frac{1}{z+1}$,

$$\sum_{j=1}^{\infty} f(j) = \frac{1}{0+1} + \frac{1}{1+1} + \frac{1}{2+1} + ... = \infty$$

hence ϕ is not bounded.

We have

$$h(t) = \sum_{j=1}^{\infty} e^{-jt} = \frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t - 1}.$$

Then $h \notin L^1$.

The system is not BIBO stable, hence is not H^∞ because also $\langle \frac{1}{z+s}, \phi \rangle$ does not converge for $\operatorname{Re} s > 0$.

(ii) Suppose $\langle f, \phi \rangle = \sum_{j=1}^{\infty} f'(j)$. Since

$$\|f\|_{(1)} = \sup_z \max\{|f(z)|, |\operatorname{Re} z| \, |f(z)|, |f'(z)|, |\operatorname{Re} z| \, |f'(z)|, \left|(\operatorname{Re} z)^2\right| |f'(z)|\},$$

so

$$|f'(j)| \leq \frac{1}{j^2} \sup \left|(\operatorname{Re} z)^2\right| |f'(z)|$$

(and equal if $z = j$).

Thus

$$\sum |f'(j)| \leq \sum \frac{1}{j^2} \|f\|_{(1)}.$$

As a result ϕ is a bounded functional.

We have

$$h(t) = \sum_{j=1}^{\infty} -te^{-jt} = \frac{-te^{-t}}{1-e^{-t}}$$

and also, $\lim_{t \to 0} \frac{te^{-t} - e^{-t}}{1-e^{-t}} = -1$ (by L'Hôpital's rule). Then $h \in L^1$, and

$$G(s) = \sum_{j=1}^{\infty} \frac{-1}{(s+j)^2} \in H^\infty.$$

In fact $G(s) = \Psi^{(1)}(s+1)$ where $\Psi^{(1)}$ is a polygamma function [1, p 260].

Now we consider the Γ operator:

$$\begin{aligned}
\Gamma u(t) &= \int_0^\infty h(t+\tau)u(\tau)\mathrm{d}\tau \\
&= \int_0^\infty \sum_{j=1}^{\infty}(-(t+\tau))e^{-(t+\tau)j}u(\tau)\mathrm{d}\tau \\
&= \sum_{j=1}^{\infty}\int_0^\infty (-(t+\tau))e^{-(t+\tau)j}u(\tau)\mathrm{d}\tau \\
&= \sum_{j=1}^{\infty}[\int_0^\infty -te^{-(t+\tau)j}u(\tau)\mathrm{d}\tau - \int_0^\infty \tau e^{-(t+\tau)j}u(\tau)\mathrm{d}\tau] \\
&= -\sum_{j=1}^{\infty}[\langle u, te^{-tj}\rangle e^{-\tau j} + \langle u, e^{-tj}\rangle \tau e^{-\tau j}],
\end{aligned}$$

thus, $\Gamma u = \sum_{j=1}^{\infty}\Gamma_j u$ where every Γ_j has rank 2, and then

$$\begin{aligned}
\|\Gamma_j\|_N &\leq \left\|te^{-tj}\right\| \left\|e^{\tau j}\right\| + \left\|e^{-tj}\right\| \left\|\tau e^{-\tau j}\right\| \\
&= \frac{c_1}{j^{3/2}j^{1/2}} + \frac{c_2}{j^{1/2}j^{3/2}} \\
&= \frac{c}{j^2}.
\end{aligned}$$

In addition,

$$\|\Gamma\|_N \leq \sum_{j=1}^{\infty}\|\Gamma_j\|_N \leq \sum_{j=1}^{\infty}\frac{c}{j^2}$$

and

$$\sigma_{n+1}(\Gamma) \leq \sum_{j=n+1}^{\infty}\frac{1}{j^2} \leq \frac{1}{n},$$

where

$$\left\|\Gamma - \sum_{j=1}^{n}\Gamma_j\right\|_N \leq \sum_{j=n+1}^{\infty}\frac{c}{j^2} \leq \frac{c'}{n} \quad \text{for some} \quad c' > 0.$$

According to Lemma 5.3.13,

$$\sigma_N = O(\frac{1}{N^2}).$$

Now the Θ operator satisfies

$$\begin{aligned}
\Theta u(t) &= \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/4}h(t+\tau)\tau^{-1/4}u(\tau)\mathrm{d}\tau \\
&= \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/4}\sum_{j=1}^{\infty}(-(t+\tau))e^{-(t+\tau)j}\tau^{-1/4}u(\tau)\mathrm{d}\tau \\
&= \frac{-1}{\sqrt{\pi}}\sum_{j=1}^{\infty}[\int_0^\infty t^{3/4}e^{t+\tau)j}\tau^{-1/4}u(\tau)\mathrm{d}\tau + \int_0^\infty t^{-1/4}e^{t+\tau)j}\tau^{3/4}u(\tau)\mathrm{d}\tau \\
&= \frac{-1}{\sqrt{\pi}}\sum_{j=1}^{\infty}[\langle u, t^{3/4}e^{-tj}\rangle\tau^{-1/4}e^{-\tau j} + \langle u, t^{-1/4}e^{-tj}\rangle\tau^{3/4}e^{-\tau j}.
\end{aligned}$$

Thus, $\Theta u(t) = \sum_{j=1}^{\infty}\Theta_j u(t)$ where every Θ_j has rank 2.

Then

$$\|\Theta\|_N \leq \sum_{j=1}^{\infty}\|\Theta_j\|_N\,.$$

Thus,

$$\begin{aligned}
\|\Theta_j\|_N &\leq \left\|t^{3/4}e^{-tj}\right\|_{L^2}\left\|\tau^{-1/4}e^{\tau j}\right\|_{L^2} + \left\|t^{-1/4}e^{-tj}\right\|_{L^2}\left\|\tau^{3/4}e^{-\tau j}\right\|_{L^2} \\
&\leq \frac{c}{j^{3/2}}.
\end{aligned}$$

Hence

$$\|\Theta\| \leq \sum_{j=1}^{\infty}\frac{c}{j^{3/2}}$$

and

$$\left\|\Theta - \sum_{j=1}^{n}\Theta_j\right\|_N \leq \sum_{j=n+1}^{\infty}\left\|\frac{c}{j^{3/2}}\right\| \leq \frac{1}{\sqrt{n}}.$$

According to Lemma 5.3.13,

$$\sigma_N = O(\frac{1}{N^{3/2}}),$$

so, Θ is a nuclear operator.

Note that $h \in L^2$ since Θ is a Hilbert-Schmidt operator.

(iii) (General case). Suppose $\langle f, \phi\rangle = \sum_{j=1}^{\infty} f^{(k)}(j)$ for some $k \geq 1$.

In order to get a bounded functional ϕ it is convenient to have a bounded functional for each k,

$$\left|f^{(k)}(j)\right| \leq \frac{1}{j^{k+1}}\sup\left|(\operatorname{Re} z)^{k+1}\right|\left|f^{(k)}(z)\right|$$

(with equality if $z = j$).

Thus

$$\sum \left|f^{(k)}(j)\right| \leq \sum \frac{1}{j^{k+1}} \|f\|_{(k)}$$

As a result ϕ is a bounded functional, and

$$h(t) = \langle e_{-t}, \phi\rangle = \sum_{j=1}^{\infty} f^{(k)}(j) = \sum_{j=1}^{\infty}(-t)^k e^{-tj}.$$

Also,

$$\begin{aligned}
\int_0^\infty |h(t)|\,\mathrm{d}t &= \int_0^\infty \left|\sum_{j=1}^{\infty}(-t)^k e^{-tj}\right| \mathrm{d}t \\
&\leq \sum_{j=1}^{\infty} \int_0^\infty t^k e^{-tj}\mathrm{d}t \\
&= \sum_{j=1}^{\infty} \frac{1}{j^{k+1}}.
\end{aligned}$$

So, since $\sum_{j=1}^{\infty} \frac{1}{j^{k+1}} < \infty$, the system is BIBO stable and hence is H^∞.

We can define the Γ operator for each k by

$$\begin{aligned}
\Gamma u(t) &= \int_0^\infty h(t+\tau)u(\tau)\mathrm{d}\tau \\
&= \int_0^\infty \sum_{j=1}^{\infty}(-(t+\tau))^k e^{-(t+\tau)j} u(\tau)\mathrm{d}\tau \\
&= \sum_{j=1}^{\infty} \int_0^\infty (-(t+\tau))^k e^{-(t+\tau)j} u(\tau)\mathrm{d}\tau \\
&= \sum_{j=1}^{\infty} \int_0^\infty \sum_{i=0}^{k} (-1)^k \binom{k}{i} t^i \tau^{k-i} e^{-(t+\tau)j} u(\tau)\mathrm{d}\tau \\
&= \sum_{j=1}^{\infty} \sum_{i=0}^{k} (-1)^k \binom{k}{i} \langle u, t^i e^{-tj}\rangle \tau^{k-i} e^{-\tau j}.
\end{aligned}$$

Then $\Gamma = \sum_{j=1}^{\infty} \Gamma_j$, where each Γ_j has rank at most $k+1$ and so

$$\|\Gamma_j\|_N \leq (k+1)\,\|\Gamma_j\|,$$

where

$$\begin{aligned}\|\Gamma_j\| &\leq \sum_{i=0}^{k} \left\|t^i e^{-tj}\right\| \left\|\tau^{k-i} e^{-\tau j}\right\| \\ &\leq \frac{c}{j^{i+1/2} j^{k-i+1/2}} \\ &= \frac{c}{j^{k+1}}.\end{aligned}$$

Thus if $\sum_{j=1}^{\infty} \frac{1}{j^{k+1}} < \infty$, the Γ operator is nuclear.

We have

$$\begin{aligned}\left\|\Gamma - \sum_{j=1}^{n} \Gamma_j\right\|_N &\leq \sum_{j=n+1}^{\infty} \|\Gamma_j\| \\ &\leq \sum_{j=n+1}^{\infty} \frac{c}{j^{k+1}} \\ &= O(n^{-k}).\end{aligned}$$

Then by Lemma 5.3.13 we have $\sigma_N = O(\frac{1}{N^{k+1}})$.

Now the Θ operator satisfies

$$\begin{aligned}\Theta u(t) &= \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/4} h(t+\tau) \tau^{-1/4} u(\tau) \mathrm{d}\tau \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/4} \sum_{j=1}^{\infty} (-(t+\tau))^k e^{-(t+\tau)j} \tau^{-1/4} u(\tau) \mathrm{d}\tau \\ &= \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\infty} \int_0^\infty t^{-1/4} \sum_{i=0}^{k} (-1)^k \binom{k}{i} t^i \tau^{k-i} e^{-(t+\tau)j} \tau^{-1/4} u(\tau) \mathrm{d}\tau \\ &= \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\infty} \sum_{i=0}^{k} (-1)^k \binom{k}{i} \langle u, t^{i-1/4} e^{-tj} \rangle \tau^{k-i-1/4} e^{-\tau j},\end{aligned}$$

so $\Theta = \sum_{j=1}^{\infty} \Theta_j$ where each Θ_j has rank at most $k+1$ and

$$\|\Theta_j\|_N \leq (k+1) \|\Theta_j\|$$

$$\begin{aligned}\|\Theta_j\| &\leq \sum_{i=0}^{k} \left\|t^{i-1/4} e^{-tj}\right\| \left\|\tau^{k-i-1/4} e^{-\tau j}\right\| \\ &\leq \frac{c_k}{j^{i-1/4+1/2} j^{k-i-1/4+1/2}} \\ &= \frac{c_k}{j^{k+1/2}}.\end{aligned}$$

Thus if $\sum_{j=1}^{\infty} \frac{c_k}{j^{k+1/2}} < \infty$ then Θ is nuclear.

However

$$\begin{aligned} \left\| \Theta - \sum_{j=1}^{n} \Theta_j \right\|_N &\leq \sum_{j=n+1}^{\infty} \|\Theta_j\| \\ &\leq \sum_{j=n+1}^{\infty} \frac{c_k}{j^{k+1/2}} \\ &= O(n^{-k+\frac{1}{2}}). \end{aligned}$$

Hence, by Lemma 5.3.13

$$\sigma_N = O(\frac{c_k}{N^{k+1/2}}),$$

and so, Θ is a nuclear operator.

Note that $h \in L^2$ since Θ is a Hilbert-Schmidt operator.

Chapter 6

Carleson Measures

6.1 Introduction

The boundedness of weighted Hankel operators and Θ operators is the main outstanding problem in this chapter. Therefore, we use the proof in Power's book [33] to get results about boundedness of Hankel operators, via Carleson measures. Also, we prove a new theorem for Θ operators using Carleson embeddings; and this requires Theorem 3.11 in [21]. We look at those explicit examples of Θ operators for which we have not yet determined whether they are bounded.

6.2 Boundedness theorems

Lemma 6.2.1. *Let f, g be continuous functions supported on a closed subinterval of $(0, \infty)$. Then*

$$\langle \Gamma_h f, g \rangle = \langle Z_\mu f, Z_\mu g \rangle$$

where, μ is a measure on $\mathbb{R}_+$, $Z_\mu : L^2(0, \infty) \to L^2(\mu)$ and $Z_\mu f(x) = \int_0^\infty e^{-xy} f(y) \mathrm{d}y$ (the Laplace transform) and Γ_h is the Hankel operator, (see [33, p 13]).

The classical Carleson theorem is to do with finding simple condition for the boundedness of the canonical injection $H^2(\mathbb{C}_+) \to L^2(\mu)$ where μ is a Borel measure on $\mathbb{C}_+$ and $h(x) = \int_{\mathbb{C}_+} e^{-xy} \mathrm{d}\mu(y)$, see ([20]).

Theorem 6.2.2. (Carleson embedding theorem). *Let μ be a positive regular Borel measure on the right half-plane $\mathbb{C}_+$. Then the following are equivalent:*

(i) The natural embedding

$$J_\mu : H^p(\mathbb{C}_+) \to L^p(\mathbb{C}_+, \mu)$$

is bounded for some (or equivalently, for all) $1 \le p < \infty$.

(ii) There exists a constant $C > 0$ such that

$$\int_{\mathbb{C}_+} |k_\lambda(z)|^2 \,\mathrm{d}\mu(z) \le C \,\|k_\lambda\|^2_{H^2} \text{ for all } \lambda \in \mathbb{C}_+,$$

where $k_\lambda(z) = \frac{1}{2\pi}\frac{1}{z+\overline{\lambda}}$ for $\lambda, z \in \mathbb{C}_+$.

(iii)

$$\mu(Q_I) \le c\,|I| \text{ for all intervals } I \in i\mathbb{R},$$

where Q_I denotes the Carleson square

$$Q_I = \{z = x + iy \in \mathbb{C}_+ : iy \in I, 0 < x < |I|\}.$$

In this case, μ is called a Carleson measure, see ([20, theorem 1.1]).

We just use part (i) and (iii) in this theorem.

We now extend Lemma 6.2.1 to measures on $\mathbb{C}_+$.

Lemma 6.2.3. (Extension of Lemma 6.2.1). *Let f, g be continuous functions supported on a closed subinterval of $(0, \infty)$, and μ a positive Borel measure on $\mathbb{C}_+$. Define*

$$h(x) = \int_{\mathbb{C}_+} e^{-xy} \mathrm{d}\mu(y)$$

and

$$Z_\mu : L^2(0, \infty) \to L^2(\mu),$$

by

$$Z_\mu f(x) = \int_0^\infty e^{-xy} f(y) \mathrm{d}y,$$

for

$$\Gamma_h : L^2(0,\infty) \to L^2(0,\infty) \text{ is the Hankel operator.}$$

Then

$$\langle \Gamma_h f, g\rangle = \langle Z_\mu f, Z_\mu g\rangle.$$

Proof. We have

$$\begin{aligned}
\langle \Gamma_h f, g\rangle &= \int_0^\infty \int_0^\infty h(x+y)f(y)\overline{g(x)}\mathrm{d}y\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty \int_{\mathbb{C}_+} e^{-(x+y)z} f(y)\overline{g(x)}\mathrm{d}y\mathrm{d}x\mathrm{d}\mu(z) \\
&= \int_{\mathbb{C}_+} [\int_0^\infty e^{-yz} f(y)\mathrm{d}y][\int_0^\infty e^{-xz}\overline{g(x)}\mathrm{d}x]\mathrm{d}\mu(z) \\
&= \langle Z_\mu f, Z_\mu g\rangle.
\end{aligned}$$

□

Since $Z_\mu = J_\mu \mathcal{L} : L^2(0,\infty) \to L^2(\mathbb{C}_+, \mu)$, it is bounded if and only if J_μ is a bounded operator.

Theorem 6.2.4. (Carleson Theorem). *μ (a positive Borel measure) is a Carleson measure for $H^2(\mathbb{C}_+)$ if and only if $\mu(Q_I) = O(|I|)$ as $|I| \to 0$ or $|I| \to \infty$, where Q_I is a Carleson square, see([22]).*

We now consider the Θ_ω operator.

Lemma 6.2.5. *Let f, g be continuous functions supported on a closed subinterval of $(0,\infty)$, $\omega \in L^2$ and $\mu \geq 0$ on $\mathbb{C}_+$. Define,*

$$h(x) = \int_{\mathbb{C}_+} e^{-xy}\mathrm{d}\mu(y)$$

and

$$Z_\mu : L^2(0,\infty) \to L^2(\mathbb{C}_+, \mu),$$

by

$$Z_\mu f(x) = \int_0^\infty \omega(y)e^{-xy}f(y)\mathrm{d}y,$$

and

$$\Theta_\omega f(x) = \int_0^\infty \omega(x)h(x+y)\omega(y)f(y)\mathrm{d}y.$$

Then

$$\langle \Theta_\omega f, g \rangle = \langle Z_\mu f, Z_\mu g \rangle.$$

Proof. We have

$$\begin{aligned}
\langle \Theta_\omega f, g \rangle &= \int_0^\infty \Theta_\omega f(x)\overline{g(x)}\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty \omega(x)h(x+y)f(y)\omega(y)\overline{g(x)}\mathrm{d}y\mathrm{d}x \\
&= \int_0^\infty \int_0^\infty \int_{\mathbb{C}_+} \omega(x)e^{-(x+y)z}f(y)\omega(y)\mathrm{d}y\mathrm{d}x\mathrm{d}\mu(z) \\
&= \int_{\mathbb{C}_+} [\int_0^\infty \omega(y)e^{-yz}f(y)\mathrm{d}y][\int_0^\infty \omega(x)e^{-xz}\overline{g(x)}\mathrm{d}x]\mathrm{d}\mu(z) \\
&= \langle Z_\mu f, Z_\mu g \rangle.
\end{aligned}$$

□

Theorem 6.2.6. *Let* $\Theta_\omega f(x) = \int_0^\infty \omega(x)h(x+y)\omega(y)f(y)\mathrm{d}y$ *and define,*

$$Z_\mu : L^2(0,\infty) \to L^2(\mathbb{C}_+, \mu),$$

by

$$Z_\mu f(x) = \int_0^\infty \omega(y)e^{-xy}f(y)\mathrm{d}y,$$

where $x \in \mathbb{C}_+$.

Then Θ_ω *is bounded if and only if* Z_μ *bounded and* Z_μ *is bounded if and only if*

$$\mathcal{L} : L^2\left(0, \infty, \frac{\mathrm{d}y}{\omega(y)^2}\right) \to L^2(\mathbb{C}_+, \mu)$$

is bounded.

Proof. We have

$$\langle \Theta_\omega f, g \rangle = \langle Z_\mu f, Z_\mu g \rangle.$$

Take $\|g\| = 1$ so

$$\|\Theta_\omega f\| \leq \|Z_\mu\|^2 \|f\|$$

and also putting $f = g$

$$\|Z_\mu f\|^2 \leq \|\Theta_\omega\| \, \|f\|^2$$

So Θ_ω is bounded if and only if Z_μ is bounded. Take $g = f\omega$, then $\mathcal{L}g = Z_\mu f$, and

$$\|g\|_{L^2(\frac{\mathrm{d}y}{\omega(y)^2})} = \|f\|_{L^2(0,\infty)} \, .$$

So,

$$\|\mathcal{L}g\| \leq C \, \|g\|_{L^2(\frac{\mathrm{d}y}{\omega(y)^2})}$$

if and only if

$$\|Z_\mu f\| \leq C \, \|f\|_{L^2} \, .$$

Special cases

- [i] If $\omega(y) = 1$ then the operator is the Hankel operator Γ and, the operator is bounded if and only if $\mathcal{L} : L^2(0,\infty) \to L^2(\mathbb{C}_+, \mu)$ is a Carleson operator.

- [ii] If $\omega(y) = y^{-1/4}$ then the operator is the Θ operator and it is bounded if and only if

$$\mathcal{L} : L^2(0,\infty; y^{1/2}\mathrm{d}y) \to L^2(\mathbb{C}_+, \mu),$$

  is bounded.

$\square$

6.2.1 Zen space

Let $\tilde{\nu}$ be a positive regular Borel measure on $[0,\infty)$ and satisfying the following (Δ_2)-condition:

$$R := \sup_{t>0} \frac{\tilde{\nu}[0,2t)}{\tilde{\nu}[0,t)} < \infty. \qquad (\Delta_2)$$

Let ν be the positive regular Borel measure on $\mathbb{C}_+ = [0,\infty) \times i\mathbb{R}$ given by $\mathrm{d}\nu = \mathrm{d}\tilde{\nu} \otimes \mathrm{d}\lambda$, where λ denotes Lebesgue measure. In the case of $1 \leq p < \infty$, we call

$$A^p_\nu = \{f : \mathbb{C}_+ \to \mathbb{C} \quad \text{analytic} : \sup_{\varepsilon>0} \int_{\overline{\mathbb{C}_+}} |f(z+\varepsilon)|^p \, \mathrm{d}\nu(z) < \infty\}$$

a Zen space on $\mathbb{C}_+$. If $\tilde{\nu}(0) > 0$, then by standard Hardy space theory, f has a well-defined boundary function $\tilde{f}$, and we can give meaning to the expression. Therefore, we can write

$$\|f\|_{A^p_\nu} = (\int_{\overline{\mathbb{C}_+}} |f(z)|^p \, \mathrm{d}\nu(z))^{1/p}.$$

Note that this expression makes sense in the case that $\tilde{\nu}(0) = 0$ (e.g. the Bergman space, since f is still defined ν-a.e. on $\overline{\mathbb{C}_+}$ Clearly the space A^2_ν is a Hilbert space. In addition, it is known that examples of Zen spaces are the Hardy spaces $H^p(\mathbb{C}_+)$, where ν is the Dirac measure in 0, or the standard Bergman spaces A^p_ν, where $\mathrm{d}\tilde{\nu}(t) = t^\alpha, \alpha > -1$, see ([20]).

6.2.2 Carleson measure on Zen spaces

Proposition 6.2.7. *Let A^2_ν be a Zen space, and let $\omega : (0, \infty) \to \mathbb{R}_+$ be given by*

$$\omega(t) = 2\pi \int_0^\infty e^{-2rt} \mathrm{d}\tilde{\nu}(r) \quad (r > 0).$$

Then the Laplace transform defines an isometric map $\mathcal{L} : L^2_\omega(0, \infty) \to A^2_\nu$, see ([20, Proposition 2.3]).

Theorem 6.2.8. *Let A^2_ν be a Zen space, $\nu = \tilde{\nu} \otimes \lambda$, and let $\omega : (0, \infty) \to \mathbb{R}_+$ be defined as following*

$$\omega(t) = 2\pi \int_0^\infty e^{-2rt} \mathrm{d}\tilde{\nu}(r) \quad (r > 0).$$

Then the following are equivalent:

1. *The Laplace transform $\mathcal{L}$ given by $\mathcal{L}f(z) = \int_0^\infty e^{-tz} f(t) \mathrm{d}t$ defines a bounded linear map*

$$\mathcal{L} : L^2_\omega(0, \infty) \to L^2(\mathbb{C}_+, \mu),$$

where

$$L^2_\omega(0, \infty) = L^2(0, \infty; \omega(t)\mathrm{d}t).$$

2. *There exists a constant $C > 0$ such that*

$$\mu(Q_I) \leq C\nu(Q_I) \quad \textit{for each Carleson square} \quad Q_I,$$

see ([20, Theorem 2.4]).

We now have a new result about boundedness of weighted Hankel operators.

Theorem 6.2.9. *Let μ be a positive Borel measure on $\mathbb{C}_+$, $h(x) = \int_{\mathbb{C}_+} e^{-xy}\mathrm{d}\mu(y)$ and $\nu = \tilde{\nu} \otimes \lambda$.*
Also let $\alpha : (0, \infty) \to \mathbb{R}_+$ be given by

$$\alpha(t) = 2\pi \int_0^\infty e^{-2rt}\mathrm{d}\tilde{\nu}(r) \quad (r > 0).$$

Then the weighted Hankel operator

$$\Theta_\omega f(x) = \int_0^\infty \alpha(x)^{-1/2} h(x+y)\alpha(y)^{-1/2} f(y)\mathrm{d}y,$$

is bounded if and only if

$$\mathcal{L} : L^2(0, \infty; \alpha(y)\mathrm{d}y) \to L^2(\mathbb{C}_+, \mu)$$

is bounded. This happens if and only if

$$\mu(Q_I) \leq C.\nu(Q_I).$$

Proof. From Proposition 6.2.7 and Theorems 6.2.8 and 6.2.6 the result comes immediately. □

Example 6.2.10. (i) Let $\alpha(y) = \frac{1}{y}$, and $\tilde{\nu}$ be Lebesgue measure then the space A^2_ν would be the Bergman space, and

$$\alpha(t) = 2\pi \int_0^\infty e^{-2rt}\mathrm{d}\,|r| = \frac{\pi}{t} \quad (t > 0).$$

and

$$R := \sup_{I>0} \frac{\tilde{\nu}[0, I)}{\tilde{\nu}[0, I/2)} = \frac{I}{I/2} = 2 < \infty.$$

Then the (Δ_2)-condition is satisfied.

The operator Θ_ω defined by

$$\Theta_\omega u(x) = \frac{1}{\pi} \int_0^\infty x^{1/2} h(x+y) y^{1/2} u(y)\mathrm{d}y$$

is equivalent to

$$\mathcal{L} : L^2(0,\infty; \frac{\mathrm{d}y}{y}) \to L^2(\mathbb{C}_+, \mu),$$

and is bounded if and only if

$$\mu(Q_I) \leq C.\nu(Q_I) \leq C\,|I|^2 < \infty,$$

where

$$h(x) = \int_{\mathbb{C}_+} e^{-xy} \mathrm{d}\mu(y).$$

(ii) Let $\alpha(t) = 2\pi \int_0^\infty e^{-2rt} \mathrm{d}\delta_0(r) = 2\pi$, and $\tilde{\nu} = \delta_0$ be the Dirac measure in 0, then the space A^2_ν would be the Hardy space.

Then

$$\nu(Q_I) = \tilde{\nu}[0, I] \times I = 1 \times I = I,$$

and

$$R := \sup_{I>0} \frac{\tilde{\nu}[0, I)}{\tilde{\nu}[0, I/2)} = 1 < \infty,$$

then the (Δ_2)-condition is satisfied.

Then

$$\Theta_\omega u(x) = \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}} h(x+y) \frac{1}{\sqrt{2\pi}} u(y) \mathrm{d}y,$$

is bounded if and only if

$$\mathcal{L} : L^2(0,\infty; 2\pi\mathrm{d}y) \to L^2(\mathbb{C}_+, \mu)$$

is bounded and this happens if and only if

$$\mu(Q_I) \leq C.\nu(Q_I) = C.I < \infty.$$

(iii) The case $\alpha(t) = t^{1/2}$ is not covered by the above methods however, from [21] we can deduce the solution in the case of sectorial measures.

Theorem 6.2.11. *Let μ be a positive Borel measure supported in a sector $S(\theta) \subset \mathbb{C}_+$, where $S(\theta) = \{z \in \mathbb{C} : |\arg z| < \theta\}$, and let $0 < \alpha < 1$. The following are equivalent.*

1. *The Laplace-Carleson embedding*

$$\mathcal{L} : L^2(0,\infty; t^\alpha \mathrm{d}t) \to L^2(\mathbb{C}_+, \mu), \quad f \mapsto \mathcal{L}f$$

is well-defined and bounded.

2. *There exists a constant γ such that*

$$\mu(T_I) \leq \gamma \, |I|^{1-\alpha}$$

for all intervals in $I \subset i\mathbb{R}$ which are symmetric about 0, where T_I is the right half of the Carleson square Q_I.

3. *There exists a constant $k > 0$ such that*

$$\left\| \mathcal{L} t^\alpha e^{-rt} \right\|_{L^2(\mathbb{C}_+,\mu)} \leq k \left\| t^\alpha e^{-rt} \right\|_{L^2(0,\infty;t^\alpha \mathrm{d}t)}$$

for all $r \in \mathbb{R}_+$,

see ([21, Theorem 3.11])

The case $\alpha = \frac{1}{2}$ is important here.

Corollary 6.2.12. *If $h(t) = \frac{1}{t}$ and μ is Lebesgue measure, then the Γ operator is bounded, however the Θ operator is not.*

Proof. Firstly, according to Theorem 2.6 [33, page 18] the Γ operator is bounded.

Secondly, we show that the Θ operator is not bounded.

Let

$$\mathcal{L} : L^2(0,\infty; y^{1/2} \mathrm{d}y) \to L^2(\mathbb{C}_+, \mu),$$

and

$$k_r(t) = e^{-rt} \quad r > 0.$$

then

$$\mathcal{L}(k_r) = \frac{1}{x+r}$$

$$\|k_r\|_{L^2(0,\infty;t^{1/2}\mathrm{d}t)} = \left(\int_0^\infty e^{-2rt}t^{1/2}\mathrm{d}t\right)^{1/2} = \frac{1}{4r\sqrt{2r\pi}}$$

and

$$\|\mathcal{L}(k_r)\|^2_{L^2(\mathbb{C}_+,\mu)} = \int_0^\infty \frac{1}{(x+r)^2}\mathrm{d}x = \frac{1}{r}.$$

Then

$$\sup_r \frac{\|\mathcal{L}(k_r)\|_{L^2(\mathbb{C}_+,\mu)}}{\|k_r\|_{L^2(0,\infty;y^{1/2}\mathrm{d}y)}} = \sup_r \frac{r^{-1/2}}{r^{-3/2}} = \infty.$$

Therefore

$$L^2(0,\infty;y^{1/2}\mathrm{d}y) \to L^2(\mathbb{C}_+,\mu),$$

is unbounded. Also, by using Theorem 6.2.11, then the Θ operator is not bounded since

$$\mu(T_I) = \frac{1}{2}|I| \not< \gamma\, |I|^{1/2}$$

where γ is a constant. □

Example 6.2.13. (i) If $h(t)=1$ and $\mu=\delta_0$ to use the previous test take $k_r(t)=e^{-rt}$.

Let

$$\mathcal{L} : L^2_\omega(0,\infty) \to L^2(\mathbb{C}_+,\mu).$$

Firstly, for the Γ operator

$$\|k_r\|_{L^2(0,\infty)} = \left(\int_0^\infty e^{-2rt}\mathrm{d}t\right)^{1/2} = \frac{1}{\sqrt{2r}},$$

and

$$\|\mathcal{L}(k_r)\|_{L^2(\mathbb{C}_+,\mu)} = \left\|\frac{1}{x+r}\right\|_{\delta_0} = \frac{1}{r}.$$

Then

$$\sup_{r>0} \frac{\|\mathcal{L}(k_r)\|_{L^2(\mathbb{C}_+,\mu)}}{\|k_r\|_{L^2(0,\infty)}} = \sup_{r>0} \frac{\frac{1}{r}}{\frac{1}{\sqrt{2r}}} = \infty.$$

Then, the Γ operator is unbounded.

Secondly, for the Θ operator

Let

$$\mathcal{L} : L^2(0,\infty;y^{1/2}\mathrm{d}y) \to L^2(\mathbb{C}_+,\mu),$$

then

$$\|k_r\|_{L^2(0,\infty;t^{1/2}\mathrm{d}t)} = \left(\int_0^\infty e^{-2rt} t^{1/2} \mathrm{d}t\right)^{1/2} = \frac{1}{4r\sqrt{2r\pi}},$$

and

$$\left\|\frac{1}{x+r}\right\|_{\delta_0} = \frac{1}{r}.$$

Then

$$\sup_{r>0} \frac{\left\|\frac{1}{x+r}\right\|_{\delta_0}}{\|k_r\|_{L^2(0,\infty;y^{1/2}\mathrm{d}y)}} = \sup_{r>0} \frac{\frac{1}{r}}{\frac{1}{4r\sqrt{2r\pi}}} = \infty.$$

Then, the Θ operator is unbounded. Here we can not use Theorem 6.2.11, because $\mu = \delta_0$ is not sectorial measure.

(ii) If $h(t) = \frac{1}{1+t}$ and $\mathrm{d}\mu = e^{-x}\mathrm{d}x$ then by using Theorem 6.2.11, the Θ operator is bounded since

$$\begin{aligned}\mu(T_I) &= \int_{\frac{1}{2}|I|}^{|I|} e^{-x}\mathrm{d}x \\ &= e^{\frac{1}{2}|I|} - e^{|I|} \\ &\lesssim |I|^{1/2}.\end{aligned}$$

(iii) If $h(t) = e^{-\lambda t}$ and $\mu = \delta_\lambda$ then the Γ operator is bounded as well as the Θ operator (we knew that from Example 3.3.4 the Hankel operator is nuclear so it is bounded, and because $h \in L^2$ then the Θ operator is Hilbert-Schmidt hence it is bounded).

Firstly, we have for the Γ operator

$$\mathcal{L} : L^2_\omega(0,\infty) \to L^2(\mathbb{C}_+, \mu).$$

Then

$$|\mathcal{L}f(\lambda)| \leq C\left(\int_0^\infty |f(y)|^2 \mathrm{d}y\right)^{1/2}$$

since, by Cauchy-Schwarz

$$\begin{aligned}\left|\int_0^\infty e^{-\lambda y} f(y)\mathrm{d}y\right| &\leq \left(\int_0^\infty |f(y)|^2 \mathrm{d}y\right)^{1/2} \left(\int_0^\infty e^{-2\lambda y}\mathrm{d}y\right)^{1/2} \\ &= C\left(\int_0^\infty |f(y)|^2 \mathrm{d}y\right)^{1/2}\end{aligned}$$

Secondly, we have for the Θ operator

$$\mathcal{L} : L^2(0,\infty; y^{1/2}\mathrm{d}y) \to L^2(\mathbb{C}_+, \mu),$$

then

$$|\mathcal{L}f(\lambda)| \le C \left(\int_0^\infty |f(y)|^2 \, y^{1/2} \mathrm{d}y \right)^{1/2},$$

since, by Cauchy-Schwarz

$$\begin{aligned} \left| \int_0^\infty e^{-\lambda y} f(y) \mathrm{d}y \right| &\le \left(\int_0^\infty |f(y)|^2 \, y^{1/2} \mathrm{d}y \right)^{1/2} \left(\int_0^\infty e^{-2\lambda y} y^{-1/2} \mathrm{d}y \right)^{1/2} \\ &= C \left(\int_0^\infty |f(y)|^2 \, y^{1/2} \mathrm{d}y \right)^{1/2}. \end{aligned}$$

Chapter 7

Possibilities for further research

First of all, in Chapter 2 we investigated the question of stability. As we know there are three type of stability: BIBO, H^∞ and asymptotic stability. In general the question of BIBO stability of a linear system given in terms of a transfer function is difficult in general; however, our methods now enable us to resolve the question for many systems. For instance, let

$$G_k(s) = \frac{1}{(s+1)^k(s+1+se^{-s})}, \quad k = 0, 1, 2, ...,$$

this transfer function is asymptotically stable, also it is known that it does not lie in H^∞ for $k = 0$, but it is H^∞ stable for $k \geq 1$ see [32].
We have developed new methods that enable us to resolve the cases $k = 2$ and $k = 3$. The case of G_1 as defined in [10] remains open.
Moreover, BIBO stability is a necessary condition for the Hankel operator of a linear system to be nuclear (trace class), a property that has certain implications for model reduction [18], and some related questions remain open. For example, it would be useful to have more precise estimates of Hankel singular values.
In Chapter 2 also, we deal with some specific examples such as 2.3.7. However, we would like to prove results applicable to a wider class of examples and use the Walton-Marshall-Bonnet-Partington method to study these examples and determine the intervals of stability.
A systematic method for fractional systems is considered as an essential method to identify the crossing points and the intervals of stability (asymptotic stability) but on other hand BIBO and H^∞ stability are still open questions when we use this method.

In Chapter 3, we introduce diffusive systems, the Hankel operator and the Θ operator. We look at a wide class of problems involving BIBO and H^∞ stability. In addition, we study the properties of operators such as nuclearity and Hilbert-Schmidt properties. In general, we consider diffusive systems defined with impulse response

$$h(t) = \int_0^\infty e^{-t\xi} \mathrm{d}\mu(\xi)$$

and transfer function

$$G(s) = \int_0^\infty \frac{\mathrm{d}\mu(\xi)}{(s+\xi)}.$$

The majority of results and theorems are with the measure $\mu \geq 0$ thus in further study we can think about all cases, examples and theorems when μ is not necessarily positive.

Moreover, in some examples we could not tell if Γ_h (Hankel operator) is nuclear for instance Example 3.3.4, where $h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ so, it is still an open question. Also, in Example 3.3.6 with $h(t) = \frac{1}{t^2+1}$ the Θ operator is Hilbert-Schmidt but using Theorem 3.4.4 fails to say Θ is nuclear.

The reproducing kernel test gives necessary and sufficient conditions for the Hankel operator to be bounded, however, for Θ this test just gives a necessary condition to be bounded. We do not know whether it is a sufficient condition.

In the Curtain-Zwart book [9] and many other references there are several partial differential equations and systems where h is given by a series and μ is a sum of point masses (discrete systems), therefore more research can be done for these examples.

Chapter 4 focuses mainly on using the Gaussian Quadrature method to approximate irrational transfer function of diffusive system by rational ones. Therefore we can improve the technique of approximation by using other numerical methods. Also, we can develop more research in approximation of unstable systems by coprime factor techniques.

Diffusive systems defined by holomorphic distributions and measures on a half plane is the main subject in Chapter 5. Again the case of non-real and non-positive measures could be investigated further.

In Chapter 6 we mainly concentrate on the boundedness of weighted Hankel operators and Θ operators. Some cases that we solve give boundedness results for the solution to the case of sectorial measures but some difficult questions about non-sectorial measures are still open.

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Appendices

Appendix A

Maple worksheet

This is related to Example 3.3.6

```
> assume(t,'real');
```

$$> \; x = 8\cdot t^8 + 23\cdot t^6 + 23\cdot t^4 + 8\cdot t^3\cdot(t^2+1)^{\frac{5}{2}} + 9\cdot t^2 + 3(t^2+1)^{\frac{5}{2}}\cdot t + 1$$

$$x = 8\,t\sim^8 + 23\,t\sim^6 + 23\,t\sim^4 + 8\,t\sim^3\,(t\sim^2+1)^{5/2} + 9\,t\sim^2 + 3\,(t\sim^2+1)^{5/2}\,t\sim + 1 \qquad \textbf{(1)}$$

$$> \; k = 4\cdot\left((t^2+1)^{\frac{1}{2}} + 2\cdot t\right)^{\frac{3}{2}}\cdot\left(t^8 + 4\cdot t^6 + 6\cdot t^4 + 4\cdot t^2 + 1 + (t^2+1)^{\frac{7}{2}}\cdot t\right)$$

$$k = 4\left(\sqrt{t\sim^2+1} + 2\,t\sim\right)^{3/2}\left(t\sim^8 + 4\,t\sim^6 + 6\,t\sim^4 + 4\,t\sim^2 + 1 + (t\sim^2+1)^{7/2}\,t\sim\right) \qquad \textbf{(2)}$$

$$> \; \int_0^\infty t^{\frac{1}{4}}\cdot\left(\frac{\pi\cdot x}{k}\right)^{\frac{1}{2}}\,\mathrm{d}t$$

$$\sqrt{\frac{\mathrm{signum}(x)}{\mathrm{signum}(k)}}\;\infty \qquad \textbf{(3)}$$

$$> \; evalf\left(\int_0^\infty t^{\frac{1}{4}}\cdot\left(\frac{\pi\cdot x}{k}\right)^{\frac{1}{2}}\,\mathrm{d}t\right)$$

$$Float(\infty)\,\sqrt{\frac{\mathrm{signum}(x)}{\mathrm{signum}(k)}} \qquad \textbf{(4)}$$

```
>
```