On the Geometry of the Space of Monopole-Clusters

by

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Declaration

The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

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Abstract

We review the results pertaining to the space of monopole-clusters, $M_{k,l}$, which was first proposed by Roger Bielawski. In particular, it has a pseudo-hyperkähler metric which approximates the metric of the moduli space of $SU(2)$-monopoles on $\mathbb{R}^3$ with exponential accuracy. We define actions of the groups $\mathbb{R}^3$, T^2 and $SO(3)$ on $M_{k,l}$, and show that they are all isometry groups. In the case $(k,l) = (1,2)$, we express the monopole-clusters in terms of elliptic functions, and verify that they approach the true monopoles with rate inversely proportional to the separation distance between the clusters. For some $\widetilde{SO(2)} \subset SO(3)$, the subgroups of $\widetilde{SO(2)} \times T^2$ that admit a fixed point in the asymptotic region of $M_{1,2}$ will be classified; their fixed point sets will be parametrized in terms of real coordinates and hence are manifolds. Finally, we compute the induced metric on an axially symmetric manifold in such family of manifolds, and show that it is asymptotically flat.

Contents

Chapter 1

Introduction

Generally speaking, *magnetic monopole* is the concept of a hypothetical particle in physics that admits an isolated point source of magnetic charge. In mathematics, the BPS-monopoles are solutions of the Bogomolny equations: the latter is a non-linear system of partial differential equations and they describe static monopoles in $\mathbb{R}^3$. In addition, BPS-monopoles depend on a Lie group, called the gauge group. This thesis will only be concerned with static monopoles in $\mathbb{R}^3$ with the non-abelian gauge group $SU(2)$, hence the word "monopole" should always be understood to refer to this particular type, unless otherwise stated.

Physicists began to study a more general type of monopoles in the early half of the 1970s and made discoveries of some special solutions. Then in the 1980s, mathematicians stepped in and made significant contributions to the theory of monopoles. In particular, it turned out to be useful to study the metric on certain moduli space of static monopoles M_k, as it was suggested by Manton [36] that the geodesic motions of these static monopoles were good approximations of the monopole dynamics when traveling with low velocity. This suggestion certainly requires analytical justification. Fortunately it was later proved rigorously by Stuart in [42]. However, the monopole metric is very complicated, and in general will be difficult to study directly on the whole moduli space. A proposition in [2] says the following:

Proposition 1.1. *For any sequence $\{m_i\}_{i=1}^{\infty}$ of M_k, there exists a subsequence $\{m_{i_j}\}_{j=1}^{\infty}$, a partition $k = \sum_{l=1}^{n} k_l$ with $n, k_l \in \mathbb{Z}_+$, and a sequence of points $\{x_{i_j}^l\}_{j\in\mathbb{N}}$ in $\mathbb{R}^3$ for each $l = 1, \ldots, n$ such that*

(i) *if m_i^l denotes the point $m_i \in M_k$ translated by x_i^l, then the sequence $\{m_{i_j}^l\}_{j=1}^{\infty}$ converges weakly to a k_l-monopole m^l with centre at the origin;*

(ii) *as $j \to \infty$ the distances between any pair of points $x_j^l, x_j^{l'}$ tend to infinity and the direction of the line $x_j^l x_j^{l'}$ converges to a fixed direction.*

The proposition suggests the idea that, when restricted to various asymptotic regions of the moduli space, the metric should be "close" to a product of monopole metrics with lower charges. Supposing this is true, then the description using product-metric is valid only when monopole-clusters are infinitely far apart, since there are interactions in the finite region, for example, the relative electric charges generated by the motion of clusters. Gibbons and Manton [17] studied the moduli space of dyons on $\mathbb{R}^3$: they are particles with both magnetic charge and electric charge. They found a metric on this space, now called Gibbons-Manton metric, and conjectured that this metric is exponentially close to the monopole metric in the asymptotic region where monopoles break down into charge 1 monopoles. In the above notation, this corresponds to the case where $k_l = 1$ for all l. Bielawski proved the conjecture [9], and went on to study the asymptotic region where monopoles break down into two monopole-clusters, i.e. the $n = 2$ case. To do this, he defined the space $M_{k,l}$, where the numbers k and l correspond to the charges of the clusters. By equipping $M_{k,l}$ with the right metric, Bielawski proved that such metric also approximates the monopole metric exponentially when the monopole-clusters become widely separated. Note that the above proposition does not assert anything in regard to the existence of any given partition of the monopole charges, though it seems to have been suggested by Taubes' analytic results in [43].

This thesis focuses primarily on the study of $M_{1,2}$, which should contribute to the understanding of the moduli space of charge 3 monopoles: Proposition (1.1) says that these monopoles break down into either three charge 1 monopoles, or into a charge 1 and a charge 2 monopoles. Our original motivation was to first gain a good understanding of $M_{1,2}$, in the hope to extend some of the results to the situation where there is a single charge 2 monopole with an arbitrary number of charge 1 monopoles.

There are a total of five chapters. This chapter will begin with a review of the relevant results in the literature about $SU(2)$-monopoles on $\mathbb{R}^3$; the notions and notations will be set in a way that can readily be adapted for the theory of monopole-clusters.

The definition of $M_{k,l}$, the space of monopole-clusters, and the relevant existing results, will be given in the next chapter. In particular, we show that there is a bijection between the generic elements between $M_{k,l}$ and $N_{k,l}/\mathcal{G}_0$, where the latter is a moduli space of solutions to Nahm's equations. There are statements given in [7] which are incorrect, we have mended them and completed the proofs where appropriate. Moreover, some of the results will be extended.

In Chapter 3, we show that there are natural actions of $\mathbb{R}^3$, T^2 and $SO(3)$ on $M_{k,l}$, and that they are isometry groups with respect to the monopole-cluster metric. In addition, there are also natural actions of the same groups on $N_{k,l}/\mathcal{G}_0$; we show such group actions on the two spaces are essentially equivalent.

We begin Chapter 4 by showing that the general elements of $M_{1,2}$ can be described in terms of elliptic functions, thereby that the monopole-clusters exhibit asymptotic behaviour when clusters become widely separated. Besides, for certain $\widetilde{SO(2)} \subset SO(3)$, we classify the subgroups of $\widetilde{SO(2)} \times T^2$ that admit a fixed point in the asymptotic region of $M_{1,2}$; their fixed point sets will be parametrized in terms of real coordinates.

In the final chapter, the induced metric on the fixed point set of certain $SO(2)$-subgroup in $\widetilde{SO(2)} \times T^2$ will be computed, and expressed explicitly in terms of real local coordinates. Moreover, we show that such metric shown is asymptotically flat.

1.1 Yang-Mills-Higgs Theory

Let $\mathbb{R}^3$ be the three dimensional Euclidean space equipped with the standard Euclidean metric on its tangent bundle $T\mathbb{R}^3$. It can be made oriented by decreeing that the 3-form $\eta = dx_1 \wedge dx_2 \wedge dx_3$ gives the positive orientation, hence η becomes a volume form. Denote by $\langle \cdot, \cdot \rangle$ the induced metric on $\bigwedge^p T^*\mathbb{R}^3$. The Hodge star operator with respect to the volume form η is the linear map $* : \Gamma(\bigwedge^p T^*\mathbb{R}^3) \to \Gamma(\bigwedge^{(3-p)} T^*\mathbb{R}^3)$ defined by

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \, \eta.$$

Let $SU(2)$ be the special unitary group, and $\mathfrak{su}(2)$, its Lie algebra. Since $\mathfrak{su}(2)$ is semi-simple, it has a nondegenerate symmetric bilinear form, namely, the Killing form. This form, together with the Hodge star operator, defines a norm on $\Gamma\left(\bigwedge^p T^*\mathbb{R}^3 \otimes \mathfrak{su}(2)\right)$: if α is a $\mathfrak{su}(2)$-valued p-form, then its norm is defined to be $|\alpha|^2 := -\frac{1}{2}\mathrm{tr}\ \alpha \wedge (*\alpha)$.

Consider the space $\mathcal{A}$ consisting of elements of the form (A, Φ), where A is a smooth section of $T^*\mathbb{R}^3 \otimes \mathfrak{su}(2)$ and $\Phi : \mathbb{R}^3 \to \mathfrak{su}(2)$ is a smooth map. Let V be a rank two hermitian bundle on $\mathbb{R}^3$, then Φ may be viewed as a section of $End(V)$, called a *Higgs field*; A is a connection form on V, acting on Φ via the exterior covariant derivative D_A:

$$(D_A\Phi)_i := \frac{\partial \Phi}{\partial x_i} + \mathrm{ad}(A_i)(\Phi) \quad \text{for } i = 1, 2, 3, \tag{1.1}$$

where $\mathrm{ad} : \mathfrak{su}(2) \to \mathrm{End}\ (\mathfrak{su}(2))$ is the adjoint representation of $\mathfrak{su}(2)$. Denoting the curvature of A by F_A. Then the *Yang-Mill-Higgs energy functional*, given by

$$\mathcal{U}(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^3} |F_A|^2 + |D_A\Phi|^2, \tag{1.2}$$

is a functional defined on $\mathcal{A}$. It is natural to seek for the critical points: any such points must satisfy the variational equations

$$\begin{aligned} D_A * F_A &= - * [\Phi, D_A\Phi], \\ * (D_A * (D_A\Phi)) &= 0. \end{aligned} \tag{1.3}$$

Taubes' analytic results [43] show that finiteness of energy implies that

$$|\Phi| \to c \in \mathbb{R} \tag{1.4}$$

as $|x| \to \infty$. Note that if $c \neq 0$, then by rescaling we may assume $c = 1$. We are now ready to give the definition of monopoles:

Definition 1.2. *Any pair* $(A, \Phi) \in \mathcal{A}$ *satisfying* (1.4) *with* $c = 1$ *and*

$$D_A\Phi = *F_A$$

is called an $SU(2)$*-monopole on* $\mathbb{R}^3$.

The equation in the above definition is called the *Bogomolny equations*; it is plural since it gives rise to a system of partial differential equations. Any solution to the Bogomolny equations is a critical point of the energy functional, i.e. the solution satisfies the above variational equations.

It turns out that condition (1.4) with $c = 1$ alone, is actually equivalent to the following seemingly stronger condition [23]:

$$\begin{aligned} |\Phi| &= 1 - \frac{k}{2r} + O(r^{-2}), \\ \frac{\partial |\Phi|}{\partial \Omega} &= O(r^{-2}), \\ |D_A\Phi| &= O(r^{-2}), \end{aligned} \tag{1.5}$$

where $k \in \mathbb{Z}$; the angular derivative $\frac{\partial}{\partial \Omega}$ is defined by

$$\frac{\partial |\Phi|}{\partial \Omega} = \left(\left(\frac{\partial |\Phi|}{\partial \theta} \right)^2 + \sin^2\theta \left(\frac{\partial |\Phi|}{\partial \phi} \right)^2 \right)^{1/2},$$

for (r, θ, ϕ) the spherical coordinates. Suppose $(A, \Phi) \in \mathcal{A}$ satisfies the conditions in (1.5), then the number k is called the *magnetic charge* of (A, Φ). It can be shown that degree of the map

$$\Phi/|\Phi| : S^2_R \to S^2 \subset \mathfrak{su}(2), \tag{1.6}$$

where S^2_R is the sphere with a large radius R, coincides with the magnetic charge. In addition, if $\mathcal{A}'$ is the set of elements in $\mathcal{A}$ that satisfy the conditions in (1.5), then $\mathcal{A}'$ can be partitioned into the disjoint union

$$\bigsqcup_{k \in \mathbb{Z}} \mathcal{A}'_k,$$

indexed by the magnetic charge. The charge k monopoles are the absolute minima to (1.2) within the class $\mathcal{A}'_k$, having energy $4\pi|k|$. There are critical points in $\mathcal{A}'_k$ that are non-minimizing, and they were first discovered by Taubes [43]. Except mathematical conventions, there is no significant difference between the study of monopoles with positive charges and with negative charges, hence from now on we shall restrict our attention to the former case, and assume that $k > 0$.

Let $\mathcal{B}_k \subset \mathcal{A}'_k$ denote the set of monopoles of charge k. Suppose $g : \mathbb{R}^3 \to SU(2)$ is a smooth function, then its action on the complex bundle V induces an action on $\mathcal{A}$ as follows:

$$\begin{aligned} &A \mapsto gAg^{-1} - dgg^{-1}, \\ &\Phi \mapsto g\Phi g^{-1}. \end{aligned} \tag{1.7}$$

Let $\mathcal{G}$ be the set of all such g. The $SU(2)$-group gives rise to a group structure to $\mathcal{G}$; it is called a *gauge group* and its elements are called *gauge transformations*. It is readily shown that, for each k, the gauge group $\mathcal{G}$ descends to an action on $\mathcal{B}_k$, i.e. the gauge transformations preserve both the Bogomolny equations and the charges of the monopoles. Recall that A is a connection form, hence its definition depends on a particular choice of trivialization of V. Since monopole is a physical quantity, its description is physically meaningful only if it is independent of the choice of coordinates systems. As an action of gauge transformation is equivalent to a change of trivialization, it is therefore natural to identify monopoles that differ by elements in $\mathcal{G}$:

Definition 1.3. *The moduli space M'_k of monopoles of charge k is defined to be the quotient $\mathcal{B}_k/\mathcal{G}$.*

1.1.1 Existence of Monopoles

So far, the existence of monopoles has not been discussed. By imposing spherical symmetry, the solutions may be written down explicitly and are necessarily of charge 1; they were first discovered by Bogomolny, Parasad and Sommerfield. For higher charges, it was Taubes who gave the first rigorous proof about their existence: roughly speaking, k-monopoles were constructed by "gluing" together the charge 1 monopoles. Hence, one would incline to envisage that a monopole is comprised of k individual particles on $\mathbb{R}^3$, giving a total of $3k$ parameters. However, Taubes showed that M'_k forms a smooth noncompact manifold of dimension $4k - 1$: yielding an extra $k - 1$ parameters. Unfortunately, his proof does not provide a method to producing solutions to the Bogomolny equations. Nonetheless, it was Ward [45] who first found the exact solution for the case $k = 2$, which involves elliptic functions.

1.1.2 The Moduli Space of Framed Monopoles M_k

Instead of tackling directly the $(4k-1)$-dimensional space M_k', it turns out to be more convenient to consider its circle bundle, denoted M_k. One way to define this is as follows. Let $* \in S^2$ be a direction in $\mathbb{R}^3$. First of all, we assume $*$ is the x_1-direction. Then up to gauge transformations, any monopole is gauge equivalent to the pair (A, Φ) such that

$$\Phi(x_1, x_2, x_3) \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{as } x_1 \to \infty. \tag{1.8}$$

Let σ be the matrix in the limit above. If $\tilde{\mathcal{G}}$ is the subgroup of $\mathcal{G}$ whose elements tend to a diagonal matrix as $x_1 \to \infty$, it can be seen that $M_k'/\mathcal{G}$ is isomorphic to the set of k-monopoles satisfying (1.8) modulo $\tilde{\mathcal{G}}$. We define $M_k(*)$ to be the set of k-monopoles satisfying (1.8) modulo $\mathcal{G}_0$, where $\mathcal{G}_0$ is the set of elements in $\tilde{\mathcal{G}}$ such that $g \to Id$ as $x_1 \to \infty$. As the centralizer of σ is isomorphic to $U(1)/\pm$, i.e. a circle, it is easily seen that $M_k(*)$ is indeed an S^1-bundle of M_k'. Although we have assumed $*$ to be the x_1-direction, the same construction can be carried out equally well for other directions, all the spaces obtained are diffeomorphic to each other.

Definition 1.4. *The moduli space of charge k framed monopoles, M_k, is defined to be $M_k(*)$, up to diffeomorphisms.*

Observe that the formulation of $M_k(*)$ above breaks the symmetry; there is actually a more sophisticated way to define M_k, which does not rely on a particular choice of direction, see [2]. The resulting space of this definition will also be diffeomorphic to $M_k(*)$ for any $* \in S^2$. We shall often say that M_k itself is the moduli space of monopoles, since it is mathematically more tractable and has richer properties.

1.1.3 Metric of M_k

There is the L^2-inner product on $\mathcal{A}$: for any $(a, \Psi) \in \mathcal{A}$, it is given by

$$h((a, \Psi), (a, \Psi)) = \int_{\mathbb{R}^3} |a|^2 + |\Psi|^2. \tag{1.9}$$

Observe that h is preserved by $\mathcal{G}$. For $c = (A, \Phi)$, we let $[c]$ denote the $\mathcal{G}$-equivalence class of c. It is known that:

Proposition 1.5. [2] *The moduli space M_k is a $4k$-dimensional smooth manifold. Its tangent space at $[c] = [(A, \Phi)]$ can be identified with the space $T_{[c]}$ of L^2-integrable solutions (a, Ψ) in $\mathcal{A}$, satisfying*

$$\begin{aligned} *D_A a - D_A \Psi + [\Phi, a] &= 0, \\ *D_A * a + [\Phi, \Psi] &= 0. \end{aligned}$$

The first equation is the linearization of the Bogomolny equations, whereas the second equation is equivalent to the L^2-orthogonality condition for (a, Ψ) to the $\mathcal{G}$-orbit of (A, Φ). As $\mathbb{R}^3$ is noncompact, a priori it is not obvious that $T_{[c]}$ is non-empty. Taubes [44] showed that $T_{[c]}$ is $4k$-dimensional, then Atiyah and Hitchin went on to apply this result to monopoles, yielding the above proposition.

The moduli space M_k has a natural Riemannian metric, namely, by taking the L^2-inner product on each tangent space $T_{[c]}$. Such metric was shown to be complete [2], which is physically (and mathematically) a desirable property: as geodesics in the moduli space approximate the monopole motion, completeness ensures that monopoles do not disappear in finite time. $T_{[c]}$ admits a quaternionic structure in the following way: writing $a = \alpha dx_1 + \beta dx_2 + \gamma dx_3$, then any element $(a, \Psi) \in T_{[c]}$ can be identified with

$$\Psi \otimes Id + \alpha \otimes I + \beta \otimes J + \gamma \otimes K$$

in $\mathfrak{su}(2) \otimes \mathbb{H}$, where $\{Id, I, J, K\}$ is the usual basis of $\mathbb{H}$. The basis elements act by right-multiplications, each of which preserves the equations in Proposition (1.5), hence $T_{[c]}$ is indeed an $\mathbb{H}$-module. It is then readily seen that I, J and K are *almost complex structures* on M_k, i.e. they are automorphisms of TM_k which square to $-Id$. Moreover, they all preserve the metric.

Before proceeding further, we shall recall some important notions from complex geometry. Suppose (X, g) is a Riemannian manifold with an almost complex structure J which preserves g, then there is an associated 2-form on X defined by $\omega_J(\,\cdot\,,\,\cdot\,) = g(\,J\cdot\,,\,\cdot\,)$. If ω is closed and J is integrable, then X is called a *Kähler manifold* and ω is called a *Kähler form.* If X possesses three almost complex structures I, J and K, satisfying the quaternionic relations, then X, in fact, has a whole sphere worth of almost complex structures: for every unit vector (x_1, x_2, x_3) in $\mathbb{R}^3$,

$$(x_1 I + x_2 J + x_3 K)^2 = -Id.$$

X is said to be *hypercomplex* if I, J and K are integrable. Furthermore, if (X, g) is Kähler with respect to the complex structures I, J and K, and all of which preserve the metric, then it is called a *hyperkähler.* Its holonomy group is a subgroup of $Sp(k)$, so such manifold must have vanishing Ricci curvature. There is the following lemma in [2]:

Lemma 1.6. *Suppose (X, g) is a Riemannian manifold such that it admits almost complex structures I, J and K that preserve g and satisfy the quaternionic relations. Then the metric is hyperkähler if and only if the associated forms $\omega_I, \omega_J, \omega_K$ are closed.*

One direction is trivial, the content lies in the converse statement. As a consequence of Lemma (1.6), Atiyah and Hitchin showed that M_k is hyperkähler by proving that the three associated 2-forms are closed.

As M_k is a circle bundle of M_k', it has a natural action of S^1 given by the group of constant diagonal gauge transformations. Any isometry group G on $\mathbb{R}^3$ induces an action on $\mathcal{A}$ by means of pullback: if $f : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry, then the action is given by

$$(A, \Phi) \mapsto (f^*A, f^*\Phi). \tag{1.10}$$

In general, G may not preserve the Bogomolny equations, namely, any group that contains an orientation-reversing element. However, if all the elements in G are orientation-preserving, then the group does descend to an action on M_k. It is well-known that the only orientation-preserving isometres on $\mathbb{R}^3$ are either translations or rotations. Hence M_k admits an action of $\mathbb{R}^3$ and $SO(3)$. It can be checked that all these actions on M_k are isometric with respect to the hyperkähler metric. Note that when the $SO(3)$-group acts on M_k, the complex structures of M_k get rotated simultaneously.

In general, it will be difficult to compute the metric directly; knowing the symmetries of M_k may help to determine the form of the metric. It is known [2] that M_k admits a decomposition as the isometric product

$$M_k = \mathbb{R}^3 \times (S^1 \times M_k^0)/\mathbb{Z}_k, \tag{1.11}$$

where the space M_k^0 is a simply-connected, connected, irreducible hyperkähler manifold of dimension $4k-4$, called the space of *strongly centred* monopoles [25]. Decomposition (1.11) tells us that the topologically nontrivial information lies in M_k^0. For $k = 1$, the moduli space M_1 is $\mathbb{R}^3 \times S^1$ and has a flat metric, which is not too complicated. The first non-trivial case is $k = 2$: it has been shown [28] that M_2^0 is diffeomorphic to $T\mathbb{RP}^2/\mathbb{Z}_2$. As any four dimensional manifold is hyperkähler if and only if it is anti-self-dual Einstein, hence M_2^0 is an anti-self-dual Einstein manifold. In addition, the metric is $SO(3)$-invariant. It turns out that these two facts alone is sufficient to determine the form of the metric:

$$ds^2 = f(r)^2 dr^2 + a(r)^2\sigma_1^2 + b(r)^2\sigma_2^2 + c(r)^2\sigma_3^2, \tag{1.12}$$

where $f^2 = (abc)^2$, $\sigma_1, \sigma_2, \sigma_3$ are the invariant one-forms of $SO(3)$ and r is a separation parameter of the monopoles. Atiyah and Hitchin [2] computed the values of the coefficients explicitly for the metric on M_2^0, which is commonly referred to as the *Atiyah-Hitchin manifold.*

1.2 Nahm's Equations

Rather than dealing with the moduli space directly, Nahm took a different approach: in his paper [37], he suggests that solutions to the Bogomolny equations can be constructed from solutions to some non-linear ordinary differential equations, called the *Nahm's Equations*. The method he used is an adaptation of the ADHM construction, which was originally invented by Atiyah, Drinfeld, Hitchin and Manin in [3] to construct instantons using methods of linear algebra. While Nahm solved half of the story, the other half was due to Hitchin: in [23] he showed that, for any given charge, there is a bijective correspondence between solutions to Nahm's equations and monopoles. We shall now review Nahm's approach.

Let $\mathcal{I} \subset \mathbb{R}$ be an interval and $\mathcal{A}_k(\mathcal{I})$ be the space of quadruples $T := (T_0, T_1, T_2, T_3)$, where $T_i : \mathcal{I} \to \mathfrak{u}(k)$, $i = 0, 1, 2, 3$, are analytic functions on $\mathcal{I}$. The *Nahm's equations* is the system of ordinary differential equations given by

$$\begin{aligned} \frac{dT_1}{dt} + [T_0, T_1] + [T_2, T_3] = 0, \\ \frac{dT_2}{dt} + [T_0, T_2] + [T_3, T_1] = 0, \\ \frac{dT_3}{dt} + [T_0, T_3] + [T_1, T_2] = 0. \end{aligned} \tag{1.13}$$

Suppose $g : \mathcal{I} \to U(k)$ is an analytic function. Then g acts on $\mathcal{A}_k(\mathcal{I})$ in the following way:

$$\begin{aligned} T_0 &\mapsto gT_0g^{-1} - \frac{dg}{dt}g^{-1}, \\ T_i &\mapsto gT_ig^{-1}, \qquad\qquad i = 1, 2, 3. \end{aligned} \tag{1.14}$$

It can be checked that this map preserves the solutions to (1.13). Let us denote the set of all such g by $\mathcal{G}$. It is a gauge group: the group structure is induced by $U(k)$.

Definition 1.7. *The space N_k is defined to be the set of $T = (T_0, T_1, T_2, T_3)$ in $\mathcal{A}_k((0, 1])$ satisfying the following conditions:*

(a) *T is a solution to Nahm's equations.*

(b) *T_0 is analytic on $[0, 1]$ and T_i, $i = 1, 2, 3$, are analytic on $(0, 1]$.*

(c) *For each $t \in (0, 1]$, any of the T_i satisfies $T_i(t)^* = -T_i(t)$.*

(d) *T_i are symmetric at $t = 1$ for $i = 0, 1, 2, 3$.*

(e) *T_i, $i = 1, 2, 3$, have simple pole at $t = 0$, and their residues r_i define a standard k-dimensional irreducible representation of $\mathfrak{su}(2)$.*

We shall now explain what we meant by a standard representation of $\mathfrak{su}(2)$.

Let $\sigma_i^{(n)}$, $i = 1, 2, 3$, be given by

$$\begin{aligned}
\sigma_1^{(n)} &= \operatorname{diag}\left(\frac{(n-1)i}{2}, \ldots, -\frac{(n-1)i}{2}\right), \\
\sigma_2^{(n)} &= \frac{\sqrt{n-1}}{2}\begin{pmatrix} 0 & -1 & & 0 \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & 1 & 0 \end{pmatrix}, \\
\sigma_3^{(n)} &= -\frac{\sqrt{n-1}}{2}\begin{pmatrix} 0 & i & & 0 \\ i & 0 & \ddots & \\ & \ddots & \ddots & i \\ 0 & & i & 0 \end{pmatrix}.
\end{aligned} \tag{1.15}$$

In particular, the matrices

$$\sigma_1^{(2)} = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

form a basis of $\mathfrak{su}(2)$. Then the residues r_1, r_2, r_3 are said to define a *standard k-dimensional irreducible representation of* $\mathfrak{su}(2)$ if $r_i = \sigma_i^{(k)}$ for all $i = 1, 2, 3$; the linear representation is given by

$$\sum_{i=1}^{3} x_i \sigma_i^{(2)} \mapsto \sum_{i=1}^{3} x_i \sigma_i^{(2)}, \quad x_1, x_2, x_3 \in \mathbb{R}.$$

Let

$$\mathcal{G}_0 = \{\, g \in \mathcal{G} \,|\, g(0) = Id,\ g(1) \in O(k, \mathbb{R}) \,\}. \tag{1.16}$$

Then $\mathcal{G}_0$ is a gauge group of N_k. It is the quotient $N_k/\mathcal{G}_0$ that is important:

Proposition 1.8. *There is a one-to-one correspondence between M_k and $N_k/\mathcal{G}_0$.*

Suppose T is an element of N_k. We shall linearize Nahm's equations about the point T. Consider the set of $Y = (Y_0, Y_1, Y_2, Y_3) \in \mathcal{A}_k((0, 1])$ that satisfy

$$T + \epsilon Y + O(\epsilon^2) \in N_k$$

for all sufficiently small $\epsilon > 0$. As solutions in N_k have fixed residues at $t = 0$, it can be seen that Y_i must be analytic on $[0, 1]$, and satisfy the linearization of Nahm's equations:

$$
\begin{aligned}
\frac{dY_1}{dt} + [T_0, Y_1] + [Y_0, T_1] + [T_2, Y_3] + [Y_2, T_3] = 0 \\
\frac{dY_2}{dt} + [T_0, Y_2] + [Y_0, T_2] + [T_3, Y_1] + [Y_3, T_1] = 0 \\
\frac{dY_3}{dt} + [T_0, Y_3] + [Y_0, T_3] + [T_1, Y_2] + [Y_1, T_2] = 0.
\end{aligned}
\tag{1.17}
$$

Moreover, they are symmetric at $t = 1$. There is an L^2-inner product on N_k given by

$$
h(Y, Y) = -\frac{1}{2} \sum_{i=0}^{3} \int_0^1 \text{tr } Y_i^2 dt. \tag{1.18}
$$

This is well-defined as the Y_i are analytic on the compact set $[0, 1]$.

Let $[T]$ denote the $\mathcal{G}_0$-orbit of T. To define $T_{[T]}(N_k/\mathcal{G}_0)$, we need to compute the infinitesimal gauge transformations of $\mathcal{G}_0$ about T. Suppose $g \in \mathcal{G}_0$ is any element that is sufficiently close to the identity element of $\mathcal{G}_0$, then for some small $\epsilon > 0$, g can be written as $Id + \epsilon\Psi + O(\epsilon^2)$, where Ψ is in the Lie algebra of $\mathcal{G}_0$. If $g \cdot T$ belongs to N_k for all sufficiently small ϵ, it implies that

$$
\left(-\frac{d\Psi}{dt} + [\Psi, T_0], [\Psi, T_1], [\Psi, T_2], [\Psi, T_3] \right) \tag{1.19}
$$

is a solution to (1.17). These are the infinitesimal gauge transformations generated by $\mathcal{G}_0$. Formally, assuming $N_k/\mathcal{G}_0$ is a smooth manifold, then its tangent vectors are the variations that are "orthogonal" to these infinitesimal gauge transformations. Consider the set of variations of N_k that are L^2-orthogonal to the infinitesimal gauge transformations: the orthogonality condition turns out to be equivalent to

$$
\frac{dY_0}{dt} + [T_0, Y_0] + [T_1, Y_1] + [T_2, Y_2] + [T_3, Y_3] = 0. \tag{1.20}
$$

It is known in [38] that

Proposition 1.9. *The moduli space $N_k/\mathcal{G}_0$ is a smooth manifold. For any $T \in N_k$, the tangent space of $N_k/\mathcal{G}_0$ at $[T]$ can be identified with the set of $Y = (Y_0, Y_1, Y_2, Y_3) \in \mathcal{A}_k([0, 1])$ that are symmetric at $t = 1$ and satisfy*

$$
\begin{aligned}
\frac{dY_1}{dt} + [T_0, Y_1] + [Y_0, T_1] + [T_2, Y_3] + [Y_2, T_3] = 0 \\
\frac{dY_2}{dt} + [T_0, Y_2] + [Y_0, T_2] + [T_3, Y_1] + [Y_3, T_1] = 0 \\
\frac{dY_3}{dt} + [T_0, Y_3] + [Y_0, T_3] + [T_1, Y_2] + [Y_1, T_2] = 0 \\
\frac{dY_0}{dt} + [T_0, Y_0] + [T_1, Y_1] + [T_2, Y_2] + [T_3, Y_3] = 0.
\end{aligned}
$$

Hence, we can define a Riemannian metric on $N_k/\mathcal{G}_0$ by taking the L^2-inner product on each tangent space. There are three almost complex structures of

$N_k/\mathcal{G}_0$ defined by

$$\begin{aligned} I(Y_0, Y_1, Y_2, Y_3) &= (-Y_1, Y_0, -Y_3, Y_2) \\ J(Y_0, Y_1, Y_2, Y_3) &= (-Y_2, Y_3, Y_0, -Y_1) \\ K(Y_0, Y_1, Y_2, Y_3) &= (-Y_3, -Y_2, Y_1, Y_0). \end{aligned} \tag{1.21}$$

Clearly they all preserve the metric. In fact, all these almost complex structures are integrable and their associated forms are closed, hence $N_k/\mathcal{G}_0$ is a hyperkähler manifold by Lemma (1.6). Nakajima [38] proved that the correspondence in Proposition (1.8) between M_k and $N_k/\mathcal{G}_0$ is in fact an isometry.

1.3 Hyperkähler Quotient Construction

The existence of hyperkähler structure on the moduli space M_k and $N_k/\mathcal{G}_0$ can be seen from an infinite-dimensional version of the hyperkähler quotient construction in [24], which we shall review here.

Suppose G is a compact Lie group which acts freely and isometrically on a finite-dimensional hyperähler manifold M^{4n} and preserves the Kähler forms $\omega_1, \omega_2, \omega_3$. If X is a vector field generated by this action, then we have

$$0 = \mathcal{L}_X \omega_i = d(i_X \omega_i) \quad i = 1, 2, 3,$$

where $\mathcal{L}_X$ and i_X are the Lie derivative and interior product along X respectively. Suppose G is a Hamiltonian action on the symplectic manifold (M, ω_i) for each i, then (1.3) implies that there are Hamiltonian functions μ_i^X such that

$$d\mu_i^X = i_X \omega_i.$$

Note that each μ_i is defined up to addition of a constant on each connected component of M. A *moment map* for the Hamiltonian action of G on the symplectic manifold (M, ω_i) is an equivariant map

$$\mu_i : M \to \mathfrak{g}^*$$

defined by

$$\langle \mu_i(m), \xi \rangle = \mu_i^X(m),$$

where $m \in M$, $\xi \in \mathfrak{g}$ is the element which generates X. The existence of μ_i, particularly the property of equivariance, is ensured by the compactness of G [19].

The three moment maps can be combined into a single function:

$$\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3.$$

Consider the set $\mu^{-1}(0) = \bigcap_{i=1}^3 \mu_i^{-1}(0)$: it is a submanifold of M with dimension $4n - 3\dim G$. Since μ_i are equivariant, the group G also acts on $\mu^{-1}(0)$; the quotient $\mu^{-1}(0)/G$, called the *hyperkähler quotient* of M by the group G, is a manifold of dimension $4n - 4\dim G$, possessing a hyperkähler metric [24]. In the monopole case, both M_k and $N_k/\mathcal{G}_0$ are formally a hyperkähler quotient of an infinite-dimensional hyperkähler manifold, where the group is the gauge group and the moment map is given by the defining functions of either the Bogomolny equations, or the Nahm's equations.

1.4 Anti-Self-Duality Equations

The correspondence between solutions to the Bogomolny equations and solutions to Nahm's equations should be viewed as an analogue to the Fourier transform and it occurs as a general phenomenon, which we shall give a brief review here.

Let G be a compact, connected semi-simple Lie group. Suppose A is a connection form on a principal G-bundle over $\mathbb{R}^4$. Let F_A be the curvature of A. Then A is called an *anti-self-dual connection*, or *instanton*, if it satisfies

$$*F_A = -F_A, \tag{1.22}$$

where $*$ denotes the Hodge star-operator on $\mathbb{R}^4$. The equation above is called the *anti-self-duality equations*. Suppose Λ is a subgroup of the translation group of $\mathbb{R}^4$. Then, roughly speaking, the *Nahm's transform* is a one-to-one correspondence between Λ-invariant instantons on $\mathbb{R}^4$ and Λ^*-invariant instantons on $(\mathbb{R}^4)^*$, where Λ^* is the dual subgroup of Λ, i.e. the group of real linear functionals of Λ. Various subgroups have been considered, for a survey on this, see [33]. We shall only mention the Nahm's transform for certain subgroups isomorphic to $\mathbb{R}$ or $\mathbb{Z}$, which are most relevant to us.

Let us write

$$A = A_0 dx_0 + A_1 dx_1 + A_2 dx_2 + A_3 dx_3.$$

First suppose that $\Lambda = \mathbb{R}$ with generators belong to the x_0-direction. This means that A is invariant under the action given by

$$(x_0, x_1, x_2, x_3) \mapsto (x_0 + c, x_1, x_2, x_3)$$

for any $c \in \mathbb{R}$. If A satisfies the anti-self-duality equations, then one may check that

$$(A_1 dx_1 + A_2 dx_2 + A_3 dx_3, A_0)$$

is a solution to the Bogomolny equations.

Using the Euclidean metric on $\mathbb{R}^4$, we have the identifications $(\mathbb{R}^4)^* \simeq \mathbb{R}^4$

and $\Lambda^* \simeq \mathbb{R}^3$, thereby the latter group is generated by

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1 + c_1, x_2 + c_2, x_3 + c_3),$$

where $c_1, c_2, c_3 \in \mathbb{R}$. Suppose A is an instanton over $\mathbb{R}^4$ which is invariant under Λ^*, then the anti-self-duality equation implies that $A = (A_0, A_1, A_2, A_3)$ satisfies the Nahm's equations. This illustrates the correspondence in proposition (1.8).

In the case $\Lambda = \mathbb{Z}$, the Λ-invariant instantons are called *calorons*, and their transform are periodic solutions to Nahm's equations. The Nahm's transform between calorons and Nahm data has been proved: Nye and Singer did most of the work on the proof in [39][40], Charbonneau and Hurtubise [10] completed it using complex geometry.

1.5 Spectral Point of View

By the work of Hitchin [21], any monopole can be associated to a holomorphic bundle on $T\mathbb{P}^1$, which was then used to introduce an algebraic curve in $T\mathbb{P}^1$; it turns out that this curve actually determines the bundle and yields back the original monopole up to gauge equivalence. That means, in principle, questions about monopoles can be tackled by the study of algebraic geometry. We shall give an outline of the construction of these objects.

1.5.1 Preliminary

Let us begin by reviewing some basic notions of complex geometry. Let X be a complex manifold. Consider the exponential sheaf sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \overset{\exp}{\to} \mathcal{O}^* \to 0,$$

where $\mathbb{Z}$ is the sheaf of locally constant functions on X with integer values; $\mathcal{O}$ is the sheaf of holomorphic functions on X; $\mathcal{O}^*$ is the multiplicative sheaf of non-zero holomorphic functions on X. The above sheaf sequence induces a long exact sequence on the cohomology groups:

$$\cdots \to H^1(X, \mathcal{O}) \overset{\exp}{\to} H^1(X, \mathcal{O}^*) \overset{\delta}{\to} H^2(X, \mathbb{Z}) \to \cdots . \qquad (1.23)$$

We say that E is a *holomorphic line bundle* of X if it is a locally free sheaf of $\mathcal{O}^*$-modules of rank 1, i.e. E is an invertible sheaf. The group $H^1(X, \mathcal{O}^*)$ is called the *Picard group* of X. It may be identified with the group of isomorphism classes of holomorphic line bundles on X, where the group operation is given by tensor product, and the inverse operation, by taking the dual. The connecting homomorphism δ sends the Picard group to $H^2(X, \mathbb{Z})$, called the *first Chern class* of X. If X is a compact connected Riemann surface, then

$$H^2(X, \mathbb{Z}) \simeq \mathbb{Z}.$$

In which case, the *degree* of a holomorphic line bundle E on X is the integer $\delta(E)$. The space of isomorphism classes of degree zero line bundles of X is called the *Jacobian* of X, which we denote by $J(X)$. Note that the notion of degree zero line bundle still makes sense for general complex manifolds.

Let us continue with the assumption that X is a complex manifold, but not necessarily compact. Denote by $\mathcal{M}^*$ the multiplicative sheaf of non-zero meromorphic functions on X that are not identically zero. Consider the following exact sheaf sequence:

$$0 \to \mathcal{O}^* \xrightarrow{i} \mathcal{M}^* \xrightarrow{j} \mathcal{M}^*/\mathcal{O}^* \to 0.$$

It induces the long exact sequence

$$\cdots \to H^0(X, \mathcal{M}^*) \xrightarrow{j_*} H^0(X, \mathcal{M}^*/\mathcal{O}^*) \xrightarrow{\delta'} H^1(X, \mathcal{O}^*) \to \cdots . \tag{1.24}$$

Each element D in $H^0(X, \mathcal{M}^*/\mathcal{O}^*)$ is called a *Cartier divisor* on X; it can be described by $\{(U_i, f_i)\}$, where $\{U_i\}$ is an open cover of X, $f_i \in H^0(U_i, \mathcal{M}^*)$ which satisfy

$$f_i/f_j \in \mathcal{O}^*(U_i \cap U_j). \tag{1.25}$$

Let $D = \{(U_i, f_i)\}$, $D' = \{(U_j', f_j)\}$ be any divisors on X. $H^0(X, \mathcal{M}^*/\mathcal{O}^*)$ has a structure of an abelian group:

$$\begin{aligned} D + D' &= \{(U_i \cap U_j', f_i f_j')\}, \\ -D &= \{(U_i, f_i^{-1})\}. \end{aligned}$$

A divisor D is called *principal* if $D = j_*(f)$ for some meromorphic function $f \not\equiv 0$ on X, whereby we write $D = (f)$. D and D' are said to be *linearly equivalent* if $D - D'$ is a principal divisor. D is called *effective* if all the f_i are holomorphic; in which case, one may associate to D the subset of X given by the zero-set of the f_i, and is usually denoted by the same symbol.

The connecting homomorphism δ' in (1.24) sends linear equivalence classes of divisors to line bundles: let $[D]$ denotes $\delta'(D)$, then it is called the *associated line bundle* of the divisor D, and its transition functions are given by $g_{ij} = f_i/f_j$ over $U_i \cap U_j$. One may check that δ' is well-defined.

Let $\{U_i\}$ be any open cover of X, and E be a holomorphic line bundle over X with transition functions g_{ij} over $U_i \cap U_j$. Suppose f is a global meromorphic section of E. Using a fixed local trivializations of E, f can be determined by $\{(U_i, f_i)\}$, where f_i are some meromorphic functions on U_i satisfying (1.25), i.e. it is a Cartier divisor. Thus one can define the principal divisor of f to be the divisor $\{(U_i, f_i)\}$, and we denote it by (f).

For any divisor D, let us write $E[D] = E \otimes [D]$. The bundle E is isomorphic to $[D]$ if and only if there exists a global meromorphic section f of E such that $D = (f)$. Let s_0 be a holomorphic section of $[D]$. It follows that

$$f \mapsto f \otimes s_0$$

gives a one-to-one correspondence between the space of meromorphic sections of E such that $(f) + D$ is effective, and $H^0(X, E[D])$. We shall often make use of this identification implicitly. Finally, the *linear system* of E, denoted $|E|$, is the set of linear equivalence classes of effective divisors such that their associated line bundle is E.

1.5.2 Line Bundles on $\mathbb{P}^1$ and $T\mathbb{P}^1$

Now we consider the complex projective line $\mathbb{P}^1$. One way to define this is by gluing two complex planes in a certain way: if we take two copies $U, \tilde{U}$ of the complex plane and let $\zeta, \tilde{\zeta}$ be the standard complex coordinates of them, then $\mathbb{P}^1$ is obtained by taking the disjoint union of $U, \tilde{U}$, identifying points over $U \cap \tilde{U}$ by the equation

$$\tilde{\zeta} = 1/\zeta. \tag{1.26}$$

Naturally, (U, ζ), $(\tilde{U}, \tilde{\zeta})$ are charts of $\mathbb{P}^1$ and (1.26) is the transition function between these charts, showing that $\mathbb{P}^1$ is a complex manifold. As $\mathbb{P}^1 \simeq S^2$, the antipodal map of S^2 induces the anti-holomorphic involution $\tau : \mathbb{P}^1 \to \mathbb{P}^1$: in local coordinates it is given by $\tau(\zeta) = -1/\bar{\zeta}$. The map τ is a *real structure* of $\mathbb{P}^1$; in general, for any complex manifold X, any anti-holomorphic involution of X is called a real structure.

Let $T\mathbb{P}^1$ denote the holomorphic tangent bundle of $\mathbb{P}^1$, and $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$ be the standard projection map. Analogous to $\mathbb{P}^1$, the tangent bundle can be viewed as the disjoint union of two copies of $\mathbb{C}^2$, with points over $\pi^{-1}(U) \cap \pi^{-1}(\tilde{U})$ being identified by the equations

$$\tilde{\zeta} = 1/\zeta, \quad \tilde{\eta} = -\eta/\zeta^2. \tag{1.27}$$

The real structure τ above lifts to one on $T\mathbb{P}^1$: if we denote it by the same symbol τ, then $\tau : T\mathbb{P}^1 \to T\mathbb{P}^1$ is locally defined by

$$\tau(\zeta, \eta) = (-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2). \tag{1.28}$$

The holomorphic vector bundles over $\mathbb{P}^1$ have been classified: for any rank k holomorphic vector bundle over $\mathbb{P}^1$, the Birkhoff-Grothendieck theorem says that it must be isomorphic to

$$\mathcal{O}(n_1) \oplus \cdots \oplus \mathcal{O}(n_k) \tag{1.29}$$

for some $n_1, \ldots, n_k \in \mathbb{Z}$, where $\mathcal{O}(n_i)$ are holomorphic line bundles of degree n_i, and the n_i are unique up to permutations. $\mathcal{O}(n)$ can be characterized as follows: it admits local trivializations $(U, \chi), (\tilde{U}, \tilde{\chi})$ such that they satisfy

$$\tilde{\chi} = \zeta^n \chi \tag{1.30}$$

over $U \cap \tilde{U}$. We say that the bundle $\mathcal{O}(n)$ is given by the transition function ζ^n from U to $\tilde{U}$. We may deduce that the group $H^1(\mathbb{P}^1, \mathcal{O}^*)$ is generated by $\mathcal{O}(n)$, and

$$\mathcal{O}(m) \otimes \mathcal{O}(n) \simeq O(m+n),$$
$$\mathcal{O}(n)^* \simeq \mathcal{O}(-n),$$

for any $m, n \in \mathbb{Z}$. If we pullback the bundle $\mathcal{O}(2)$ by π, it is not hard to see that $T\mathbb{P}^1 \simeq \pi^*\mathcal{O}(2)$.

Consider the set of degree zero line bundles of $T\mathbb{P}^1$. From the long exact sequence (1.23), such bundles are given by the image of $H^1(T\mathbb{P}^1, \mathcal{O})$ under the exponential map:

Proposition 1.10. *$H^1(T\mathbb{P}^1, \mathcal{O})$ is generated by the cocycles $[\eta^i/\zeta^j]$ for $i > 0$, $0 < j < 2i$. The corresponding line bundles have transition function $\exp(\eta^i/\zeta^j)$ from $\pi^{-1}(U)$ to $\pi^{-1}(\tilde{U})$.*

We shall be particularly interested in the case where $i = j = 1$: for any $t \in \mathbb{R}$, the bundle L^t has local trivializations χ and $\tilde{\chi}$ over $\pi^{-1}(U)$ and $\pi^{-1}(\tilde{U})$ respectively, such that they are related by

$$\tilde{\chi} = \exp(-t\eta/\zeta)\chi \tag{1.31}$$

on the overlap. We shall often write L^1 as L, and $L^t \otimes \pi^*\mathcal{O}(n)$, as $L^t(n)$.

Suppose F is a holomorphic line bundle over $T\mathbb{P}^1$, let J be the corresponding complex structure. As a real bundle, F is just a rank 2 real vector bundle. Let us write $\overline{F}$ for F with the opposite complex structure, i.e. $-J$. Suppose τ lifts to an anti-holomorphic isomorphism σ on F, that is, the following commutative diagram holds:

$$\begin{array}{ccc} F & \xrightarrow{\sigma} & \overline{\tau^* F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ T\mathbb{P}^1 & \xrightarrow{\tau} & T\mathbb{P}^1 \end{array}, \tag{1.32}$$

where p is the projection map from F onto $T\mathbb{P}^1$. σ induces the natural anti-linear isomorphism

$$\sigma : H^0(T\mathbb{P}^1, F) \to H^0(T\mathbb{P}^1, \overline{\tau^* F}) \tag{1.33}$$

defined by $\sigma(s) := \sigma \circ s \circ \tau$. For the holomorphic bundles $\pi^*\mathcal{O}(n)$ or L^t, there *does* exist an anti-holomorphic isomorphism σ; it is unique up to a holomorphic function of $T\mathbb{P}^1$. One may check that

$$\overline{\tau^*(\pi^*\mathcal{O}(n))} = \pi^*\mathcal{O}(n), \quad \overline{\tau^* L^t} = L^{-t}. \tag{1.34}$$

1.5.3 Holomorphic Vector Bundle

Twistor theory, first proposed by Penrose [41], suggests to consider the space of oriented geodesics on X, where X is any Riemannian manifold. In the case of monopole, the manifold is taken to be $\mathbb{R}^3$, so that oriented geodesics are just oriented lines. The space T of oriented geodesics has the following parametrization:

$$\left\{(\mathbf{u}, \mathbf{v}) \in S^2 \times \mathbb{R}^3 \middle| \langle \mathbf{u}, \mathbf{v} \rangle = 0\right\}. \tag{1.35}$$

From this it is not hard to see that T is diffeomorphic to the tangent bundle of S^2. Moreover, from [21], T can be given a complex structure so that it becomes biholomorphic to the holomorphic tangent bundle $T\mathbb{P}^1$. It can be seen that the real structure $\tau : T\mathbb{P}^1 \to T\mathbb{P}^1$ given before induces the natural involution on T that reverses the orientation of the geodesics.

Let E be a rank 2 complex vector bundle on $\mathbb{R}^3$. To each pair $(A, \Phi) \in \mathcal{A}$, we associate to it a vector bundle on $T\mathbb{P}^1$, with fibres given by

$$\tilde{E_z} = \{s \in \Gamma(\gamma_z, E) | ((D_A)_{\mathbf{u}} - i\Phi)s = 0\}, \tag{1.36}$$

where γ_z is the oriented geodesic in $\mathbb{R}^3$ corresponding to $z \in T$ and $\mathbf{u}$ is the unit tangent vector along γ_z. This bundle is holomorphic if (A, Φ) satisfies the Bogomolny equations. In addition, if (A, Φ) satisfies the boundary condition (1.5), then there are two distinguished holomorphic subbundles L^+, L^- of $\tilde{E}$, whose fibre at $z \in T$ are given by

$$L_z^{\pm} = \{s \in \tilde{E}_z \,|\, s(t) \to 0 \text{ as } t \to \pm\infty\}. \tag{1.37}$$

Thus, if (A, Φ) is a monopole, then there is a curve $S \subset T$ associated to it by

$$S = \{z \in T \,|\, L_z^+ = L_z^-\}. \tag{1.38}$$

In other words, S is a family of oriented lines in $\mathbb{R}^3$ such that over each line, there is a solution to the differential operator $(D_A)_{\mathbf{u}} - i\Phi$ which decays in both directions. We refer to S as *spectral curve* of the monopole (A, Φ) for now, though there are other conditions that this curve must satisfy, as we shall see later. Note that S is preserved by the real structure τ.

1.5.4 Spectral Curve of Monopoles

Suppose (A, Φ) is a monopole of charge k with spectral curve S. Then Hitchin [21] showed that $L^+ \simeq L^1(-k)$ and $L^- \simeq L^{-1}(-k)$ over S. For the condition $L^+ = L^-$ to hold, it is necessary to have $L^2 \simeq \mathcal{O}$, which imposes a constraint on S. Moreover, the same condition corresponds to the vanishing of ψ below:

$$L^- \subset \tilde{E} \xrightarrow{\psi} \tilde{E}/L^+ \simeq (L^+)^*.$$

Since $\mathrm{Hom}(L^-, (L^+)^*)$ is isomorphic to $\mathcal{O}(2k)$, the curve S is given by the zero-set of the holomorphic section $\psi \in H^0(T\mathbb{P}^1, \pi^*\mathcal{O}(2k))$. Equivalently we have $S = (\psi)$ and $\pi^*\mathcal{O}(2k) \simeq [(\psi)]$, thus we may describe S as a divisor in the linear system $|\pi^*\mathcal{O}(2k)|$.

The following proposition can be deduced from [21]:

Proposition 1.11. *Every section $s \in H^0(T\mathbb{P}^1, \pi^*\mathcal{O}(2n))$ may be written uniquely in the form*

$$s = a_0\eta^n + a_1\eta^{n-1} + \ldots + a_n,$$

where $a_i \in \pi^*(H^0(\mathbb{P}^1, \mathcal{O}(2i)))$, $\eta \in H^0(T\mathbb{P}^1, \pi^*\mathcal{O}(2))$ is the tautological section.

Here the tautological section means the following: under the identification $T\mathbb{P}^1 \simeq \mathcal{O}(2)$, η is the natural section of $\pi^*T\mathbb{P}^1$ defined by $\eta(z) = z$ for all $z \in T\mathbb{P}^1$. Using the coordinates (ζ, η) over $\pi^{-1}(U)$, S is locally described by the polynomial

$$P(\zeta, \eta) = \eta^k + a_1(\zeta)\eta^{k-1} + \ldots + a_k(\zeta) = 0,$$

where $a_i(\zeta)$ are polynomials of degree $2i$ in ζ. From this description, it can be seen that S is compact and connected. Moreover, Proposition (1.10) and (1.12) together imply that the arithmetic genus of S, i.e. the dimension of $H^1(S, \mathcal{O})$, is $(k-1)^2$.

Proposition 1.12. [1] *If S is a divisor of the linear system $|\pi^*\mathcal{O}(2n)|$, then the map*

$$H^1(T\mathbb{P}^1, \mathcal{O}) \to H^1(S, \mathcal{O})$$

induced by the natural inclusion $S \hookrightarrow T\mathbb{P}^1$ is surjective.

We shall show that the reality condition of S can be translated into a constraint on $P(\zeta, \eta)$. Recall that the real structure τ on $T\mathbb{P}^1$ can be lifted to an anti-holomorphic isomorphism σ on $\pi^*\mathcal{O}(m)$. One may check that this map must satisfy $\sigma^2 = (-1)^m Id$, so σ is a real structure precisely when m is even. Suppose ψ is a section of $\pi^*\mathcal{O}(2k)$ which gives rise to the divisor S. Then S is said to be *real* if the section given by $\sigma(\psi) = \sigma \circ \psi \circ \tau$ also gives rise to the divisor S: it is equivalent to the condition

$$(-\zeta)^{2k}\overline{(P \circ \tau)(\zeta, \eta)} = P(\zeta, \eta).$$

Recall that, for any $t \in \mathbb{R}$, the real structure τ can be lifted to an antiholomorphic isomorphism $\sigma : L^t \to L^{-t}$. Since the line bundle L^2 is trivial over the spectral curve, there exists a global trivialization of L^2, which we denote by ν. Consider the line bundle $L(k-1)$: if σ now denotes an anti-holomorphic isomorphism $\sigma : L^1(k-1) \to L^{-1}(k-1)$ which lifts τ, then we define

$$\sigma' : L^1(k-1) \to L^1(k-1) \tag{1.39}$$

by $\sigma'(s) = \nu\sigma(s)$. Note that $(\sigma')^2(s) = (-1)^{k-1}\nu\sigma(\nu)s$, where the quantity $\nu\sigma(\nu)$ is a holomorphic section of the trivial bundle over S. As S is compact and connected, $\nu\sigma(\nu)$ must be a non-zero complex number; in fact, it is a real number. By normalizing it so that it takes values ± 1, any σ' given by (1.39) defines a real structure on $L^1(k-1)$ if and only if $\nu\sigma(\nu) = (-1)^{k-1}$. We say that $L(k-1)$ over S is *real* if there exists ν such that the associated σ' is a real structure. Note that any such ν is unique up to a multiplicative unit complex number, so σ' must be the same and the notion is well-defined. Deducing from the fact that any such spectral curve corresponds to a unique gauge equivalence class of a (non-framed) monopole, we have:

Proposition 1.13. [23] *There is a one-to-one correspondence between the moduli space of framed monopoles M_k and the space of pairs (S, ν), where $S \subset T\mathbb{P}^1$ is a compact real curve in the linear system $|\pi^*\mathcal{O}(2k)|$ satisfying that following conditions:*

(i) *S has no multiple component.*

(ii) *L^2 is trivial and $L^1(k-1)$ is real over S.*

(iii) *$H^0(S, L^t(k-2)) = 0$ for all $t \in (0, 2)$.*

ν is a global holomorphic section of L^2 over S such that $\nu\sigma(\nu) = (-1)^{k-1}$.

Condition (iii) is the condition on the curves which ensures that the corresponding monopoles are non-singular. Any compact real curve that satisfies the conditions (i) to (iii) in the above proposition is therefore called the *spectral curve* of a monopole.

1.6 Flow of Line Bundles

As a consequence of Propositions (1.8) and (1.13), we see that any pair (S, ν) determines a unique gauge equivalence class of solutions to Nahm's equations. We shall demonstrate that there is a correspondence between solutions to Nahm's equations and curves with flow of line bundles over them, which follows from the general theory of Lax equations given in [1].

Let S be a smooth curve given by a divisor in the linear system $|\pi^*\mathcal{O}(2k)|$, then it has genus $g = (k-1)^2$. Denote by $J(S)^{g-1}$ the set of isomorphism classes of line bundles of degree $g-1$ over S. There is an isomorphism between the Jacobian $J(S)$ and $J(S)^{g-1}$: the map is given by

$$E \mapsto E \otimes \pi^* \mathcal{O}(k-2).$$

Let K_S be the canonical bundle of S. By the adjunction formula [18], it is easy to check that $K_S \simeq \pi^* \mathcal{O}(2k-4)$. Let Θ, called the *theta divisor* of S, denote the set of line bundles in $J(S)^{g-1}$ that has at least a non-zero section. Any E in the theta divisor that satisfies $E^2 \simeq K_S$ is called a *theta-characteristic*. Observe that, from Proposition (1.12), every element in $J(S)$ is given by the restriction of a degree zero line bundle over $T\mathbb{P}^1$.

A matrix $M \in \mathfrak{gl}(n, \mathbb{C})$ is called *regular* if it has a cyclic vector, i.e. there exists a vector $v \in \mathbb{C}^n$ such that $v, Mv, \ldots, M^{n-1}v$ span $\mathbb{C}^n$. There are other useful characterizations:

Lemma 1.14. *The following conditions are equivalent:*

(a) *M is a regular matrix.*

(b) *The characteristic polynomial of M is equal to its minimal polynomial.*

(c) *The geometric multiplicity of each of its eigenvalue is* 1*, i.e.* $\dim \ker(\lambda - M) = 1$ *for every eigenvalue λ of M.*

(d) *M is similar to a companion matrix, i.e.*

$$\begin{pmatrix} 0 & \cdots & 0 & b_1 \\ 1 & & 0 & \vdots \\ & \ddots & & \vdots \\ 0 & & 1 & b_n \end{pmatrix}. \tag{1.40}$$

Let (T_0, T_1, T_2, T_3) be any solution to Nahm's equations over an interval. Putting

$$\begin{aligned} \beta(\zeta, t) &= (T_2(t) + iT_3(t)) + 2iT_1(t)\zeta + (T_2(t) - iT_3(t))\zeta^2, \\ \alpha(\zeta, t) &= (T_0(t) + iT_1(t)) + (T_2(t) - iT_3(t))\zeta, \end{aligned} \tag{1.41}$$

then $\alpha(\zeta, t), \beta(\zeta, t)$ satisfy the *Lax equation*

$$\frac{d\beta(\zeta)}{dt} = [\beta(\zeta), \alpha(\zeta)]. \tag{1.42}$$

As a consequence of the Lax equation, regularity of $\beta(\zeta, \cdot)$ at one point implies regularity at any other points:

Lemma 1.15. *Suppose $\alpha, \beta : [a, b] \to \mathfrak{gl}(k)$ are continuously differentiable functions satisfying the Lax equation. If β is regular at some point $t_0 \in [a, b]$, then it is regular at every $t \in [a, b]$.*

Proof. Let $g : [a,b] \to GL(k,\mathbb{C})$ be the unique solution to

$$\frac{dg}{dt} + \alpha g = 0$$

with $g(t_0) = Id$. If v is a cyclic vector for $\beta(t_0)$, then for any $t \in [a,b]$, gv is a cyclic vector for $\beta(t)$. □

The following result comes from [4]:

Proposition 1.16. *Let S be a divisor in $|\pi^*\mathcal{O}(2k)|$. There is a one-to-one correspondence between the set of $(S,\mathcal{L}_t)$, where S has no multiple components and $\mathcal{L}_t$ is a flow of degree $g-1+k$ line bundles satisfying $\mathcal{L}_t \otimes \pi^*\mathcal{O}(-1) \in J(S)^{g-1} \setminus \Theta$ for $t \in (0,2)$, and solutions (T_0,T_1,T_2,T_3) to Nahm's equations over $(0,2)$ such that $\beta(\zeta)$ is regular for all $\zeta \in \mathbb{C}$.*

Proof. In what follows, we write $\mathcal{L}_t(-1) = \mathcal{L}_t \otimes \pi^*\mathcal{O}(-1)$. Let $t \in (0,2)$. For any point $a \in \mathbb{P}^1$, there is an exact sequence over S given by

$$0 \to \mathcal{L}_t(-1) \to \mathcal{L}_t \to \mathcal{L}_t|_{S \cap T_a\mathbb{P}^1} \to 0.$$

In general, suppose Y is a complex algebraic curve and E is a holomorphic line bundle on X. The Riemann-Roch theorem states that

$$\dim H^0(Y,E) - \dim H^1(Y,E) = \deg E - g + 1, \tag{1.43}$$

where g is the arithmetic genus of Y. Since it is known that the degree of $\mathcal{L}_t(-1)$ over S is $g-1$, and that $H^0(S,\mathcal{L}_t(-1)) = 0$, the Riemann-Roch theorem yields $H^1(S,\mathcal{L}_t(-1)) = 0$. If we let $V_t = H^0(S,\mathcal{L}_t)$, then the long exact sequence of cohomology implies that

$$V_t \simeq H^0(S \cap T_a\mathbb{P}^1, \mathcal{L}_t). \tag{1.44}$$

For a generic $a \in \mathbb{P}^1$, S intersects with $T_a\mathbb{P}^1$ at k distinct points, so V may be viewed as a rank k vector bundle over $(0,2)$. Let (U,ζ), $(\tilde{U},\tilde{\zeta})$ be standard affine charts of $\mathbb{P}^1$ such that U contains a. We define the section $\underline{\beta}(a,\cdot)$ of the bundle $End(V) \to (0,2)$ via the following commutative diagram:

$$\begin{tikzcd} V_t \arrow[r] \arrow[d, "\underline{\beta}(a,t)"'] & H^0(S \cap T_a\mathbb{P}^1, \mathcal{L}_t) \arrow[d, "\times\eta"] \\ V_t \arrow[r] & H^0(S \cap T_a\mathbb{P}^1, \mathcal{L}_t) \end{tikzcd} \tag{1.45}$$

where the horizontal arrows denote the restriction map. Suppose $s \in H^0(S \cap T_a\mathbb{P}^1, \mathcal{L}_t)$. Let $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$ be the usual projection map. If (ζ_0,η_0) represents a point in $S \cap T_a\mathbb{P}^1$ with multiplicity m over $\pi^{-1}(U)$, then $s(\zeta_0,\eta_0)$ is given by the truncated power series:

$$\sum_{j=0}^{m-1} \alpha_j (\eta - \eta_0)^j.$$

From the above commutative diagram, it can be seen that the eigenvalues of $\underline{\beta}(a,t)$ are the points in $S \cap T_a\mathbb{P}^1$. Moreover, since S has no multiple components, V_t must be a line bundle and hence $\dim\ker\,(\eta - \underline{\beta}(a,t)) \leq 1$. According to [23], using the chart (U,ζ), $\underline{\beta}(t,\cdot)$ may be expressed as a quadratic polynomial in ζ:

$$\underline{\beta}(t,\zeta) = \underline{A_0}(t) - \underline{A_1}(t)\zeta - \underline{A_2}(t)\zeta^2. \tag{1.46}$$

To represent $\underline{A_i}(t)$ as matrices, one needs to trivialize the bundle V. Since the base space is 1-dimensional, up to an overall change of basis, this is equivalent to giving a connection to V. Hitchin in [23] constructed a connection ∇ on V: if s is an element in $V_t = H^0(S, \mathcal{L}_t)$ such that, over $S \cap \pi^{-1}(U)$ (resp. $S \cap \pi^{-1}(\tilde{U})$), it is represented by the function f (resp. $\tilde{f}$), then the covariant derivative of s is defined by

$$\nabla_t s = \begin{cases} \frac{\partial f}{dt} + (\frac{1}{2}\underline{A_1} + \zeta \underline{A_2}) s\big|_{S\cap\pi^{-1}(U)} & \text{over } S\cap\pi^{-1}(U) \\ \frac{\partial \tilde{f}}{dt} - (\tilde{\zeta}\underline{A_0} + \frac{1}{2}\underline{A_1}) s\big|_{S\cap\pi^{-1}(\tilde{U})} & \text{over } S\cap\pi^{-1}(\tilde{U}). \end{cases} \tag{1.47}$$

With respect to a covariant basis, $\underline{A_i}(t)$ are each represented by a flow of $(k\times k)$-matrices $A_i(t)$. Let

$$\begin{aligned} \beta(\zeta,t) &= A_0(t) + A_1(t)\zeta + A_2(t)\zeta^2, \\ \alpha(\zeta,t) &= A_1(t)/2 + A_2(t)\zeta. \end{aligned}$$

Over S, we know that $(\eta - \beta(t,\zeta))s = 0$ for all $s \in V_t$. If s is covariant constant, we have

$$\left(\frac{d\beta(\zeta)}{dt} + [\beta(\zeta), \alpha(\zeta)] \right) s = 0.$$

Since this is true for all covariant constant s, $(\alpha(\zeta), \beta(\zeta))$ must satisfy the Lax equation. As a is generic, the equation is true for all ζ. By writing A_i as

$$A_0 = T_2 + iT_3, \quad A_1 = 2iT_1, \quad A_2 = T_2 - iT_3,$$

one obtains the solution $T = (0, T_1, T_2, T_3)$ to Nahm's equations on $(0,2)$.

Conversely, suppose we have a solution $T = (T_0, T_1, T_2, T_3)$ to Nahm's equations over $(0,2)$, where T_i are all $(k\times k)$ matrices. For $\zeta \in \mathbb{C}$, we construct $\alpha(\zeta,t), \beta(\zeta,t)$ as in (1.41), then they satisfies the Lax equation. Note that $\beta(\zeta,t)$ is *isospectral*: one may check that, as a consequence of Lax equation, $\frac{d}{dt}\mathrm{tr}\ (\beta(\zeta)^n)$ is zero for all $n \geq 0$, then since the coefficients of the characteristic polynomial of $\beta(\zeta, t)$ depend on tr (β^n) algebraically, the expression

$$\det(\eta - \beta(\zeta, t)) = 0 \tag{1.48}$$

is independent of t. Let S be the curve in $\mathbb{C}^2$ defined by this equation, then by viewing such $\mathbb{C}^2$ as the chart neighbourhood $\pi^{-1}(U)$ of $T\mathbb{P}^1$, we may extend S uniquely to a compact curve in $T\mathbb{P}^1$, which is again denoted by S. Note that S may be viewed as a divisor in $|\pi^*\mathcal{O}(2k)|$.

For each $t \in (0, 2)$, we define the sheaf $\mathcal{L}_t$ over $T\mathbb{P}^1$ via the exact sequence

$$0 \to \pi^*\mathcal{O}(-2)^{\oplus k} \overset{\eta-\beta(\zeta,t)}{\to} \mathcal{O}^{\oplus k} \to \mathcal{L}_t \to 0. \tag{1.49}$$

Then $\mathcal{L}_t$ is supported over S. If ker $(\eta - \beta(\zeta, t))$ is everywhere one dimensional over S, then $\mathcal{L}_t$ has degree $g - 1 + k$. Moreover, it is a line bundle when S is smooth. Multiplying the exact sequence above by $\pi^*\mathcal{O}(-1)$, we consider the associated long sequence

$$\begin{aligned} 0 \to H^0(T\mathbb{P}^1, \pi^*\mathcal{O}(-1)^{\oplus k}) &\to H^0(S, \mathcal{L}_t(-1)) \\ &\to H^1(T\mathbb{P}^1, \pi^*\mathcal{O}(-3)^{\oplus k}) \overset{\eta-\beta(\zeta,t)}{\to} H^1(T\mathbb{P}^1, \pi^*\mathcal{O}(-1)^{\oplus k}) \to 0. \end{aligned}$$

Since $H^0(T\mathbb{P}^1, \pi^*\mathcal{O}(-1)^{\oplus k}) = 0$ and the map $\eta - \beta(\zeta, t)$ is injective on the cohomology groups, $\mathcal{L}_t(-1) \in J(S)^{g-1} \setminus \Theta$. It remains to show that S has no multiple components: suppose not, $\mathcal{L}_t$ would have rank greater than 1 at any point on the multiple components, then $H^0(S \cap T_a\mathbb{P}^1, \mathcal{L}_t)$ would contain a direct sum of at least two copies of a vector subspace, implying that dim ker $(\eta - \beta(a, t)) > 1$, a contradiction. $\square$

Proposition 1.17. [6] *Under the correspondence in Proposition* (1.16)*,* $\mathcal{L}_1(-1)$ *is a theta-characteristic if and only if* $T_i(1)$*,* $i = 0, 1, 2, 3$*, are symmetric.*

Let $S \in |\pi^*\mathcal{O}(2k)|$ be the spectral curve of a monopole. It is known that

$$\mathcal{L}_t = L^t(k-1), \tag{1.50}$$

where $t \in (0, 2)$, is the correct flow which corresponds to Nahm data for monopoles. Since $L^2 \simeq \mathcal{O}$ over S, we have

$$(\mathcal{L}_1(-1))^2 \simeq (L^1(k-2))^2 \simeq \pi^*\mathcal{O}(2k-4) \simeq K_S,$$

i.e. $\mathcal{L}_1(-1)$ is a theta-characteristic. By the above Proposition (1.17), the corresponding solution to Nahm's equations are symmetric at $t = 1$, which is consistent with the definition of Nahm data for monopoles. Moreover, there is a hermitian form on $V_t = H^0(S, \mathcal{L}_t))$ given as follows. For any $s, s' \in V_t$, denote by

$$\sigma : \mathcal{L}_t \to \mathcal{L}_{-t}$$

the anti-holomorphic isomorphism which lifts the real structure τ of $T\mathbb{P}^1$. The quantity $s\sigma(s)$ is a section of $\pi^*\mathcal{O}(2k-2)$ over S.

Lemma 1.18. [31] *Let S be a divisor in the linear system $|\pi^*\mathcal{O}(n)|$. Then the restriction map*

$$H^0(T\mathbb{P}^1, \pi^*\mathcal{O}(j)) \to H^0(S, \pi^*\mathcal{O}(j))$$

is a surjection.

This Lemma, together with Proposition (1.11), implies that $s\sigma(s)$ admits the unique expression

$$a_0\eta^{k-1} + a_1\eta^{k-2} + \ldots + a_k.$$

In particular, the coefficient a_0 is a constant. The hermitian form is defined by

$$\langle s, s' \rangle_t = a_0. \tag{1.51}$$

The connection ∇ given in (1.47) is compatible with this hermitian form. From the proof of Proposition (1.16), the flow of line bundles $\mathcal{L}_t$ gives rise to the endomorphisms $\underline{A_i}(t)$, $i = 1, 2, 3$, of V_t. From [23], the endomorphisms that arises from (1.50) have a pole at $t = 0$ (and $t = 2$) with residues defining an irreducible representation of $\mathfrak{su}(2)$, which actually implies that the hermitian form constructed in (1.51) is positive-definite. If the endomorphisms $\underline{A_i}(t)$ are trivialized by a covariant basis which is unitary, then the corresponding solutions to Nahm's equations are skew-hermitian (cf. Definition (1.7)).

1.7 Rational Maps

Analogous to his work on anti-self-duality equations on compact 4-manifolds, Donaldson proposed to interpret solutions to Nahm's equations as complex object: analogous to the isomorphism

$$(x_0, x_1, x_2, x_3) \mapsto (x_0 + ix_1, x_2 + ix_3)$$

which identifies $\mathbb{R}^4 \cong \mathbb{C}^2$, for any $T = (T_0, T_1, T_2, T_3) \in \mathcal{A}_k((0, 1])$, we set

$$\alpha = T_0 + iT_1, \quad \beta = T_2 + iT_3. \tag{1.52}$$

Conversely, for any analytic functions $\alpha, \beta : (0, 1] \to \mathfrak{gl}(k, \mathbb{C})$, we can decompose them as in (1.52) to find T_i. Hence we shall identify T with (α, β) implicitly; it is easy to see that T satisfies Nahm's equations if and only if

(α, β) satisfies the following:

$$\frac{d\beta}{dt} + [\beta, \alpha] = 0, \tag{1.53}$$

$$\frac{d}{dt}(\alpha + \alpha^*) + [\alpha, \alpha^*] + [\beta, \beta^*] = 0. \tag{1.54}$$

These equations are called the *complex equation* and the *real equation* respectively. Recall that there is a gauge group $\mathcal{G}$ acting on $\mathcal{A}_k((0,1])$: under the identification above, the action is given by

$$\begin{aligned} \alpha &\mapsto g\alpha g^{-1} - \frac{dg}{dt}g^{-1}, \\ \beta &\mapsto g\beta g^{-1}. \end{aligned} \tag{1.55}$$

Observe that both (1.53) and (1.54) are preserved by $\mathcal{G}$.

Lemma 1.19. *The space N_k can be identified with the set of $(\alpha, \beta) \in \mathcal{A}_k((0,1])$ that satisfies the following conditions:*

(a) *α, β are $\mathfrak{gl}(k, \mathbb{C})$-valued analytic functions on $(0,1]$.*

(b) *They are solutions to both the complex and real equation.*

(c) *α, β are symmetric at $t = 1$.*

(d) *α, β have a simple pole at $t = 0$, with residue a, b given by*

$$a = \operatorname{diag}\left(-\frac{(n-1)}{2}, \ldots, \frac{(n-1)}{2}\right), \quad b = \sqrt{n-1}\begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Definition 1.20. *The space $\mathcal{N}_k$ is defined to be the set of solutions $(\alpha, \beta) \in \mathcal{A}_k((0,1])$ to the complex equation that satisfy the conditions* (a) *and* (d) *in Lemma* (1.19)*, and that each β* (*but not necessarily α*) *is symmetric at $t = 1$.*

Let $\mathcal{G}_0^{\mathbb{C}}$ be the complexification of the gauge group $\mathcal{G}_0$: it is given by

$$\mathcal{G}_0^{\mathbb{C}} = \{\, g \in C^{\omega}\left([0,1], GL(k, \mathbb{C})\right) \mid g(0) = Id,\ g(1) \in O(k, \mathbb{C})\}\,. \tag{1.56}$$

Note that although the action of $\mathcal{G}_0^{\mathbb{C}}$ preserves the complex equation, it does not in general preserve the real equation. Let us consider the quotient $\mathcal{N}_k / \mathcal{G}_0^{\mathbb{C}}$. Observe that the $\mathcal{G}_0^{\mathbb{C}}$-orbits are similar to complex vector bundles in the sense that they are locally flat: for each (α, β), there is always a gauge transformation g such that $g \cdot (\alpha, \beta) = (0, g\beta g^{-1})$, the complex equation then implies that $\beta_0 = g\beta g^{-1}$ is constant. This is only a local classification: the nontrivial part to classifying globally comes from the pole at $t = 0$. A suitable analysis on this has been carried out by Donaldson in [15], there he showed that the $\mathcal{G}_0^{\mathbb{C}}$-orbit of every element in $\mathcal{N}_k$ contains a unique $\mathcal{G}_0$-orbit of an element in N_k. In other words, he proved that:

Proposition 1.21. *There is a one-to-one correspondence between $N_k/\mathcal{G}_0$ and $\mathcal{N}_k/\mathcal{G}_0^{\mathbb{C}}$.*

Definition 1.22. *$\mathcal{V}_k$ is defined to be the set of elements $(B, w) \in GL(k, \mathbb{C}) \times \mathbb{C}^k$ such that B is a symmetric matrix and w is a cyclic vector for B.*

$\mathcal{V}_k$ admits an $O(k, \mathbb{C})$-action given by

$$(B, w) \mapsto (ABA^{-1}, Aw).$$

Suppose $(\alpha, \beta) \in \mathcal{N}_k$. Let u be the unique solution to

$$\frac{du}{dt} + \alpha u = 0$$

such that $t^{-(k-1)/2}u(t) \to 0$ as $t \to 0$. Then Donaldson proved that there is a one-to-one correspondence between $\mathcal{N}_k/\mathcal{G}_0^{\mathbb{C}}$ and $\mathcal{V}_k/O(k, \mathbb{C})$: the map is given by

$$(\alpha, \beta) \mapsto (\beta(1), u(1)). \tag{1.57}$$

Moreover:

Proposition 1.23. *The map defined by*

$$(B, w) \mapsto w^T(z - B)^{-1}w$$

induces a one-to-one correspondence between $\mathcal{V}_k/O(k, \mathbb{C})$ and $Rat_k(\mathbb{P}^1)$.

$Rat_k(\mathbb{P}^1)$, the space of based rational maps of degree k, consists of elements of the form

$$\frac{\sum_{i=0}^{k-1} a_i z^i}{z^k + \sum_{i=0}^{k-1} b_i z^i}, \quad a_i, b_i \in \mathbb{C},$$

where the numerator and denominator of each rational map have no common factor; the latter condition is equivalent to the nonvanishing of the *resultant*:

$$\begin{vmatrix}
a_0 & a_1 & \cdots & a_{k-1} & & & & \\
 & a_0 & a_1 & \cdots & a_{k-1} & & & \\
 & & \ddots & \ddots & \cdots & \ddots & & \\
 & & & a_0 & a_1 & \cdots & a_{k-1} & \\
b_0 & b_1 & \cdots & b_{k-1} & b_k & & & \\
 & b_0 & b_1 & \cdots & b_{k-1} & b_k & & \\
 & & \ddots & \ddots & \cdots & \ddots & \ddots & \\
 & & & b_0 & b_1 & \cdots & b_{k-1} & b_k
\end{vmatrix}.$$

As $Rat_k(\mathbb{P}^1)$ is an open subset in $\mathbb{C}^{2k}$, it is a complex manifold. This shows that, via the correspondences above, the moduli space $N_k/\mathcal{G}_0$ is a complex manifold. Actually, the fact that M_k admits a complex structure was first realized in this way.

By varying the complex structures, we may deduce the following:

Corollary 1.24. *For each $a \in \mathbb{P}^1$, there is a one-to-one correspondence between an M_k and $Rat_k(\mathbb{P}^1)$.*

Recall that any framed monopole is uniquely determined by their spectral data (S, ν): S is a curve in $T\mathbb{P}^1$ and ν is a holomorphic section of the bundle L^2 over S. Then rational maps can be given directly via its spectral data:

Proposition 1.25. [29] *Given $a \in \mathbb{P}^1$, let (U, ζ) be an affine chart about a, so that $\zeta(a) = 0$. For any $p(z)/q(z) \in Rat_k(\mathbb{P}^1)$, under the correspondence given in Corollary* (1.24)*, the corresponding pair (S, ν) characterizes the rational map in the following way:*

$$\begin{aligned} q(z) &= P(0, z), \\ p(z) &= f(0, z) \pmod{q(z)}, \end{aligned}$$

where the vanishing of $P(\zeta, \eta)$ represents S over $\pi^{-1}(U)$, and f is the holomorphic function which represents ν over $S \cap \pi^{-1}(U)$.

Remark that the rational maps can also be constructed directly from framed monopoles [2]. There is actually an alternative approach to the rational map construction, which was first suggested by Atiyah but worked out by Jarvis [32], and the idea is as follows. From Proposition (1.25), it can be seen that Donaldson's rational map depends on a choice of complex structure, which corresponds to a direction in $\mathbb{R}^3$. To define Jarvis's rational map, some choice also needs to be made: one requires to pick a point in $\mathbb{R}^3$, serving as some kind of origin. Clearly, both approaches break the natural symmetry of the space of solutions. It is neither known how Donaldson's rational map changes if the direction is varied, nor the Jarvis's rational map when the origin gets shifted; the relationship between the two approaches is also unknown. Here, we shall only be concerned with Donaldson's approach, since the rational map construction for monopole-clusters may be viewed as its generalization, as we shall see in the next chapter.

1.8 Twistor Space

As mentioned earlier, any hyperkähler manifold M has a 2-sphere of complex structures. The idea of twistor theory in our setting is that the metric information of M can be encoded into the holomorphic structure of some complex manifold Z, hence in principal, by finding the right holomorphic data of Z, one should be able to not only recover the metric information, but given in terms of simpler descriptions. The relevant details will be reviewed in this section.

Suppose M is a hyperkähler manifold of real dimension $4k$. Let I, J and K denote the standard complex structures of M satisfying the quaternionic relations; let I_0 be the complex structure of $\mathbb{P}^1$. We shall identify $\mathbb{P}^1$ with the unit 2-sphere S^2 implicitly.

Definition 1.26. *The twistor space Z of a hyperkähler manifold M is the complex manifold*

$$Z = M \times \mathbb{P}^1,$$

where its complex structure is given as follows: restricting to the tangent space of Z at $(m, (a, b, c))$, the (almost) complex structure is given by $I_Z = (aI + bJ + cK, I_0)$.

A priori, I_Z may not be integrable, but it is shown in [24] that this does give Z a structure of complex manifold with $2k+1$ complex dimensions. Let $p : Z \to \mathbb{P}^1$ be the projection map onto $\mathbb{P}^1$, it is clear that the map is holomorphic. In addition, each fibre $p^{-1}(a, b, c)$ may be viewed as the complex manifold M with respect to the complex structure $aI + bJ + cK$. To each $m \in M$, we can associate a *twistor line* P_m: it is the unique holomorphic section of $p : Z \to \mathbb{P}^1$ with image $\{m\} \times \mathbb{P}^1$. Let us consider the normal bundle of the twistor lines in TZ. In the differential geometric viewpoint, Z is just the product space, so the normal bundle of P_m is just the trivial bundle $T_m M \times S^2$. However, as a holomorphic bundle, it is not trivial: it is holomorphically equivalent to

$$p^* \mathcal{O}(1)^{\oplus 2k}.$$

Let T_F denote the vertical bundle of the projection p. In other words, it is defined by

$$T_F = \ker \left(dp : TZ \to T\mathbb{P}^1 \right).$$

Let ω_1, ω_2 and ω_3 be the Kähler forms of M corresponding to the complex structures I, J and K respectively. There is a holomorphic section Ω of the bundle $\Lambda^2 T_F^* \otimes p^* \mathcal{O}(2)$: if $(U, \zeta), (\tilde{U}, \tilde{\zeta})$ are affine charts of $\mathbb{P}^1$, then

$$\Omega = \begin{cases} \omega \otimes \frac{d}{d\zeta} & \text{over } p^{-1}(U) \\ \tilde{\omega} \otimes \frac{d}{d\tilde{\zeta}} & \text{over } p^{-1}(\tilde{U}) \end{cases}$$

is given by

$$\begin{aligned} \omega(\zeta) &= (\omega_2 + i\omega_3) + 2i\omega_1 \zeta - (\omega_2 - i\omega_3)\zeta^2, \\ \tilde{\omega}(\tilde{\zeta}) &= (\omega_2 - i\omega_3) - 2i\omega_1 \tilde{\zeta} - (\omega_2 + i\omega_3)\tilde{\zeta}^2. \end{aligned}$$

In the above we have made the identification $\mathcal{O}(2) \simeq T\mathbb{P}^1$ implicitly. Note that for each $a \in \mathbb{P}^1$, $\Omega|_{p^{-1}(a)}$ may be viewed as a *complex* symplectic form on the fibre $p^{-1}(a) \simeq (M, I_a)$.

There is a real structure on Z: it is given by

$$\tau : (m, (a, b, c)) \mapsto (m, -(a, b, c)).$$

Observe that τ is compatible with the above holomorphic objects in the following sense:

(a) τ is a lift of the antipodal map of S^2.

(b) The twistor lines P_m and their holomorphic normal bundle NP_m are preserved by τ, i.e.

$$\overline{\tau(P_m)} = P_m, \quad \overline{\tau_*(NP_m)} = NP_m.$$

(c) Let $\sigma : p^*\mathcal{O}(2) \to p^*\mathcal{O}(2)$ be a lift of the antipodal map of $\mathbb{P}^1$. If

$$\Xi : \Lambda^2 T_F^* \otimes p^*\mathcal{O}(2) \to \Lambda^2 T_F^* \otimes p^*\mathcal{O}(2)$$

is defined by $\Xi = \tau^* \otimes \sigma$, then we have

$$\Xi \circ \Omega \circ \tau = \Omega.$$

The converse is also true:

Theorem 1.27. [24] *Suppose Z is a complex manifold of $(2k+1)$ complex dimensions with the following properties:*

(i) *Z is the total space of a holomorphic fibre bundle $p : Z \to \mathbb{P}^1$;*

(ii) *There is a family of holomophic sections of Z, each with normal bundle isomorphic to $p^*\mathcal{O}(1)^{\oplus 2k}$;*

(iii) *There exists a holomorphic section of $\bigwedge^2 T_F^* \otimes \mathcal{O}(2)$ such that it is a complex symplectic form on each fibre;*

(iv) *There exists a real structure τ on Z compatible with* (i), (ii) *and* (iii), *and which induces the antipodal map on $\mathbb{P}^1$.*

Then the parameter space of real sections is a $4k$-dimensional hyperkähler manifold such that Z is its twistor space.

Let Z_k be the twistor space of M_k. The right holomorphic data for Z_k have been constructed in [2]; we shall review particularly the holomorphic symplectic form since its formula is similar to the description of the hyperkähler metric for $M_{k,l}$.

Recall from Donaldson's theorem that, for each $a \in \mathbb{P}^1$, there is a biholomorphism between M_k and $Rat_k(\mathbb{P}^1)$ with respect to the corresponding complex structure, thus fibres of Z_k may be identified with $Rat_k(\mathbb{P}^1)$. Let $p(z)/q(z)$ be any element in $Rat_k(\mathbb{P}^1)$. Suppose the roots of $q(z)$, denoted by $\beta_1, \ldots, \beta_k$, are distinct, then β_i, $p(\beta_i)$, $i = 1, \ldots, k$, are local complex coordinates of $Rat_k(\mathbb{P}^1)$. Over such region, let

$$\omega = \sum_{i=1}^{k} \frac{dp(\beta_i) \wedge d\beta_i}{p(\beta_i)}.$$

Clearly ω is a closed 2-form. It can be shown that ω extends everywhere over $Rat_k(\mathbb{P}^1)$ and is non-degenerate, hence is a symplectic form. The formula (1.8) gives rise to the holomorphic symplectic form Ω for Z_k as follows. Let $(S, \nu) \in M_k$, then for each $a \in U$, the rational map $p(z)/q(z)$ corresponding to (S, ν) is given, according to Proposition (1.25), by $P(\zeta_0, z) = q(z)$ and $f(\zeta_0, z) = p(z) \pmod{q(z)}$. Suppose $P(\zeta_0, z)$ have distinct roots $\beta_i(\zeta_0)$, $i = 1, \ldots, k$. If we view $\{\beta_i(\zeta_0), f(\zeta_0, \beta_i(\zeta_0))\}_{i=1}^k$ as complex coordinates of $p^{-1}(a)$, then by letting $\zeta = \zeta_0$ to vary,

$$(\beta_1(\zeta), \ldots, \beta_k(\zeta), f(\zeta, \beta_1(\zeta)), \ldots, f(\zeta, \beta_k(\zeta)), \zeta)$$

becomes a system of *local* coordinates for Z_k; the holomophic symplectic form for Z_k is given by

$$\omega(\zeta) = \sum_{i=1}^{k} \frac{df(\zeta, \beta_i(\zeta)) \wedge d\beta_i(\zeta)}{f(\zeta, \beta_i(\zeta))} \tag{1.58}$$

over $p^{-1}(U)$. In the next chapter, we shall appeal to Theorem (1.27) to acquire the hyperkähler structure for $M_{k,l}$.

Chapter 2

Monopole-Clusters

As mentioned at the beginning of Chapter 1, there are asymptotic regions in the monopole moduli space where monopoles decompose into ones with lower charges. In this chapter, we shall review the theory pertaining to the space of monopole-clusters $M_{k,l}$, which was proposed by Bielawski in [7]. The motivation for studying this is that there is an asymptotic region in $M_{k,l}$ corresponding to the region in M_{k+l} where monopoles breakdown into a charge k and a charge l monopole, such that the monopole-cluster metric approximates the monopole metric with exponential accuracy as the monopoles separate.

2.1 Basic Settings

The definition of $M_{k,l}$ is rather technical; it will be necessary to begin with some definitions.

2.1.1 Hermitian Form

Recall that $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$ is the natural projection map, $\tau : T\mathbb{P}^1 \to T\mathbb{P}^1$ is the usual lift of the antipodal map on $\mathbb{P}^1$, and $L^s(n)$ denotes the tensor product $L^s \otimes \pi^*\mathcal{O}(n)$.

Let $(S^-, S^+) \in |\pi^*\mathcal{O}(2k)| \times |\pi^*\mathcal{O}(2l)|$ be a pair of compact, real curves in $T\mathbb{P}^1$. Due to the real structure, the intersection of the curves has the following splitting:

$$S^- \cap S^+ = D \cup \tau(D),$$

where D is some collection of points in $T\mathbb{P}^1$. Generically, the number of points in $S^- \cap S^+$ is $2kl$ counting with multiplicity, in which case there are 2^{kl} choices of D. Suppose there exists an effective divisor D on S^+ such that the equality $S^- \cap S^+ = D + \tau(D)$ holds as divisors on S^+. For each $t \in \mathbb{R}$, let

$$\mathcal{L}_t = L^t(k + l - 1)[-D] \tag{2.1}$$

over S^+. We shall construct a hermitian form on the vector space $V_t^+ =$

$H^0(S^+, \mathcal{L}_t)$. Fixing t and suppose $u, v \in V_t^+$. The real structure τ gets lifted to $\sigma : \mathcal{L}_t \to \overline{\tau^* \mathcal{L}_t}$, thereby induces the anti-linear map

$$\sigma : H^0(S^+, \mathcal{L}_t) \to H^0(S^+, \overline{\tau^* \mathcal{L}_t}) \tag{2.2}$$

defined by $\sigma(v) = \sigma \circ v \circ \tau$. Then $u\sigma(v)$ is a section of the bundle

$$\mathcal{L}_t \otimes \overline{\tau^* \mathcal{L}_t} \simeq \pi^* \mathcal{O}(2k + 2l - 2)[-D - \tau(D)]$$

over S^+. By identifying $u\sigma(v)$ as a section of $\pi^*\mathcal{O}(2k + 2l - 2)$ over S^+ which vanishes along $S^- \cap S^+$, Lemma (1.18) implies that $u\sigma(v)$ can be extended over $T\mathbb{P}^1$. As $S^- \cap S^+$ can be viewed as a divisor on S^- or S^+ cut out by the opposite curve, according to [31], there exists $a \in H^0(T\mathbb{P}^1, \pi^*\mathcal{O}(2k-2))$ and $b \in H^0(T\mathbb{P}^1, \pi^*\mathcal{O}(2l-2))$ such that

$$u\sigma(v) = ah_+ + bh_-,$$

where h_-, h_+ are the unique sections whose principal divisor are S^-, S^+; uniqueness follows from the fact that they are locally represented by monic polynomial. Clearly, we have

$$u\sigma(v) = bh_-$$

over S^+. By Proposition (1.11), b can be decomposed uniquely as $b_0\eta^{l-1} + \cdots + b_{l-1}(\zeta)$. The hermitian form on V_t^+ is defined by

$$\langle u, v \rangle_t := b_0. \tag{2.3}$$

Suppose there is an effective divisor on S^- which is also set theoretically equal to D above, and is denoted by the same symbol. If the equality $S^- \cap S^+ = D + \tau(D)$ holds as divisors on S^-, then the bundle $\mathcal{L}_t$ and the hermitian form on $V_t^- = H^0(S^-, \mathcal{L}_t)$ can be defined in a similar way.

2.1.2 Real Line Bundles

Let S^-, S^+ be a pair of curves given as above. Suppose the bundle $L^2[\tau(D) - D]$ is holomorphically trivial over S^+, then there exists a global meromorphic section ν^+ of L^2 over S^+ such that $(\nu^+) = D - \tau(D)$. It is known that τ can be lifted to an anti-holomophic isomorphism $\sigma : \mathcal{L}_2 \to \overline{\tau^* \mathcal{L}_{-2}}$, thereby gives rise to the map

$$\sigma' : H^0(S^+, \mathcal{L}_1) \to H^0(S^+, \mathcal{L}_1) \tag{2.4}$$

defined by $\sigma'(a) = \nu^+ \sigma(a)$. Observe that

$$(\sigma')^2(a) = \nu^+ \sigma(\nu^+)(-1)^{k+l-1} a.$$

Since S^+ is compact, the quantity $\nu^+\sigma(\nu^+)$ is a scalar; when it is equal to $(-1)^{k+l-1}$, we say that the bundle $\mathcal{L}_1$ over S^+ is *real*. Note that the definition is independent of the choice of ν^+, hence is well-defined (cf. Chapter 1).

Similarly, we say that $\overline{\tau^*\mathcal{L}_{-1}} = L(k+l-1)[-\tau(D)]$ is real if there is a meromorphic section ν^- on L^2 whose associated divisor is $\tau(D) - D$ and satisfies $\nu^-\sigma(\nu^-) = (-1)^{k+l-1}$.

2.2 The Space of Monopole-Clusters $M_{k,l}$

Definition 2.1. *$\Sigma_{k,l}$ is defined to be the set of pairs of compact, real curves $(S^-, S^+) \in |\pi^*\mathcal{O}(2k)| \times |\pi^*\mathcal{O}(2l)|$ such that there exists an effective divisor on S^- and S^+, both of which are denoted by the same symbol D, satisfying the following conditions:*

(i) *$D + \tau(D) = S^- \cap S^+$ as divisors on S^- and S^+.*

(ii) *$L^2[D - \tau(D)] \simeq \mathcal{O}$ and $\overline{\tau^*\mathcal{L}_{-1}}$ is real on S^-;*

$L^2[\tau(D) - D] \simeq \mathcal{O}$ and $\mathcal{L}_1$ is real on S^+.

(iii) *$H^0(S^-, \overline{\tau^*(\mathcal{L}_{-t}(-1))}) = 0$ for $t \in (0,2)$, and also for $t = 0$ if $k \leq l$;*

$H^0(S^+, \mathcal{L}_t(-1)) = 0$ for $t \in (0,2)$, and also for $t = 0$ if $k \geq l$.

(iv) *The hermitian form $\langle \cdot, \cdot \rangle_t$ is positive-definite on both $H^0(S^-, \overline{\tau^*\mathcal{L}_{-t}})$ and $H^0(S^+, \mathcal{L}_t)$ for $t \in (0,2)$.*

Suppose $(S^-, S^+) \in \Sigma_{k,l}$ and let $\nu^\pm \in H^0(S^\pm, L^2)$. We can now give the definition of our central object:

Definition 2.2. *The quadruple*

$$(S^-, \nu^-, S^+, \nu^+)$$

is said to be a Monopole-Cluster of charge (k,l) if it satisfies the following properties:

(i) *$(S^-, S^+) \in \Sigma_{k,l}$.*

(ii) *$\nu^-\sigma(\nu^-) = (-1)^{k+l-1}$ and $\nu^+\sigma(\nu^+) = (-1)^{k+l-1}$.*

We denote the space of monopole-clusters of charge (k,l) by $M_{k,l}$.

Condition (ii) implies that ν^-, ν^+ are defined up to a unit complex number, so $M_{k,l}$ should be thought of as a T^2-bundle over $\Sigma_{k,l}$. Note that our definition of $M_{k,l}$ is somewhat different to the one given in [7], there it requires the condition $\nu^\pm\sigma(\nu^\pm) = 1$, which is incorrect: for example, it is easy to check that on $M_{1,1}$, one always gets $\nu^\pm\sigma(\nu^\pm) = -1$.

Intuitively, each quadruple (S^-, ν^-, S^+, ν^+) should be viewed as follows: the pair (S^-, ν^-) (resp. (S^+, ν^+)) corresponds to a framed-monopole with charge k (resp. l) respectively. These monopole-clusters are closer to become actual monopoles when they are widely separated. We shall make this statement more precise in the last section of this chapter.

2.2.1 The Moduli Space of Nahm Data $N_{k,l}/\mathcal{G}_0$

As in Chapter 1, we write $T = (T_0, T_1, T_2, T_3)$ for elements in $\mathcal{A}_k(\mathcal{I})$.

Definition 2.3. *$\mathcal{F}_{k,l}$ is defined to be the set of elements*

$$(T^-, T^+) \in \mathcal{A}_k([-1,0)) \times \mathcal{A}_l((0,1])$$

that satisfy the following conditions:

(a) *T^- (resp. T^+) is a solution to Nahm's equations on* $[-1,0)$ (*resp.* $(0,1]$).

(b) *$T_i^-(t)^* = -T_i^-(t)$ for $t \in [-1,0)$ and $T_i^+(t)^* = -T_i^+(t)$ for $t \in (0,1]$.*

(c) *For $k \le l$, the limits $\lim_{t\to 0-} T_i^-(t)$, $i = 0,1,2,3$, $\lim_{t\to 0+} T_0^+(t)$ exist and are finite. In addition, T_j^+ has the following expansion near $t = 0$:*

$$T_j^+(t) = \left(\begin{array}{c|c} X_j + O(t) & O\left(t^{(l-k-1)/2}\right) \\ \hline O\left(t^{(l-k-1)/2}\right) & r_j/t + O(t) \end{array} \right), \tag{2.5}$$

where $X_j \in \mathfrak{su}(k)$, $r_j \in \mathfrak{su}(l-k)$. The residues r_j define the standard irreducible $(l-k)$-dimensional representation of $\mathfrak{su}(2)$ (cf. Chapter 1).

In the case $k \ge l$, the condition is the same except that the signs $\pm$ are interchanged.

(d) (Patching condition) *If $k < l$* (*resp.* $k > l$), *then $\lim_{t\to 0-} T_j^-(t) = X_j$* (*resp.* $\lim_{t\to 0+} T_j^+(t) = X_j$) *for $j = 1,2,3$.*

For $k = l$, letting $\alpha_\pm = T_0^\pm + iT_1^\pm$, $\beta_\pm = T_2^\pm + iT_3^\pm$, then there exists column vectors $U, W \in \mathbb{C}^k$ such that

$$\lim_{t\to 0^+} \beta_+(t) - \lim_{t\to 0^-} \beta_-(t) = -UW^T,$$
$$\lim_{t\to 0^+} (\alpha_+(t) + \alpha_+(t)^*) - \lim_{t\to 0^-} (\alpha_-(t) + \alpha_-(t)^*) = -U\bar{U}^T + \bar{W}W^T.$$

Define the group $\mathcal{G}$ to be the set of elements $g = (g_-, g_+)$, where $g_- : [-1,0] \to U(k)$, $g_+ : [0,1] \to U(l)$ are analytic functions such that if $k \le l$ (resp. $k \ge l$), then the upper $k \times k$ (resp. $l \times l$) diagonal block of $g_+(0)$ (resp. $g_-(0)$) is equal to $g_-(0)$ (resp. $g_+(0)$), and its lower diagonal block is the identity. Moreover, near $t = 0$, the off-diagonal blocks of g_+ (resp. g_-) have derivatives $O(t^{(k+l-1)/2})$. $\mathcal{G}$ acts on $A_k([-1,0)) \times \mathcal{A}_l((0,1])$ in a natural way:

$$\begin{aligned} T_0^\pm &\mapsto g_\pm T_0^\pm g_\pm^{-1} - \frac{dg_\pm}{dt} g_\pm^{-1}, \\ T_i^\pm &\mapsto g_\pm T_i^\pm g_\pm^{-1}, \qquad\qquad i = 1,2,3. \end{aligned} \tag{2.6}$$

It can be seen that the action of $\mathcal{G}$ preserves the conditions in Definition (2.3), hence is a gauge group of $\mathcal{F}_{k,l}$.

Definition 2.4. *The space $N_{k,l}$ is defined to be the set of elements $(T^-, T^+) \in \mathcal{F}_{k,l}$ such that $T_i^{\pm}(\pm 1)$ are symmetric for all $i = 0, 1, 2, 3$.*

The relevant gauge group for $N_{k,l}$ is given by

$$\mathcal{G}_0 = \{ \, g \in \mathcal{G} \,|\, g_-(-1) \in O(k, \mathbb{R}), \; g_+(1) \in O(l, \mathbb{R}) \, \} \, . \tag{2.7}$$

It is clear that such group acts on $N_{k,l}$.

Recall that, to each Nahm data $(\alpha, \beta) \in N_k$, one can associate a curve in the linear system $|\pi^*\mathcal{O}(2k)|$. In the same way, we can find a pair of curves $(S^-, S^+) \in |\pi^*\mathcal{O}(2k)| \times |\pi^*\mathcal{O}(2l)|$ to each element $((\alpha_-, \beta_-), (\alpha_+, \beta_+)) \in N_{k,l}$. Suppose they satisfy the following conditions:

$$\begin{aligned} &\text{(i)} \quad S^- \text{ and } S^+ \text{ have no common components.} \\ &\text{(ii)} \quad S^- \text{ and } S^+ \text{ have no multiple components.} \end{aligned} \tag{2.8}$$

Any element in $M_{k,l}$ or $N_{k,l}$ is said to be *generic* if their associated curves satisfy (2.8).

Proposition 2.5. [7] *There is a one-to-one correspondence between the generic elements in $M_{k,l}$ and the space of $\mathcal{G}_0$-orbit of generic elements in $N_{k,l}$.*

Proof. The model argument is given in [31], so we shall only apply it appropriately. Suppose $(S^-, \nu^-, S^+, \nu^+) \in \Sigma_{k,l}$, we shall consider the flow of line bundles

$$\mathcal{L}_t = L^t(k + l - 1)[-D] \tag{2.9}$$

over the spectral curves. Since $\mathcal{L}_t$ satisfies $\mathcal{L}_t(-1) \in J(S^+)^{g-1} \backslash \Theta$ for $t \in (0, 2)$, by Proposition (1.16), it gives rise to a solution T^+ to Nahm's equations over $(0, 2)$. Similarly, the flow $\mathcal{L}_t$ over S^- yields a solution T^- to Nahm's equations over $(-2, 0)$. Let $V^{\pm} = H^0(S^{\pm}, \mathcal{L}_t)$. There is a canonical connection ∇ on $V^{\pm}$ constructed as in (1.47), them $T_i^{\pm}$ are simply the matrix representation of some endomorphisms on $V^{\pm}$ with respect to a covariant constant basis. By definition, there is a hermitian metric on $V^{\pm}$ which the connection is compatible with. Thus, using a covariant constant basis that is unitary, the corresponding $T_i^{\pm}$ must be skew-hermitian. Recall that the sections $\nu^{\pm}$ determine the real structure on $\mathcal{L}_{\pm 1}$, which are also compatible with the connection.

If $k = l$, then [31] implies that the boundary condition is always satisfied. In particular, using unitary covariant constant bases of $V^{\pm}$ that are real at $t = \pm 1$, the solutions to Nahm's equations are symmetric at $t = \pm 1$. Hence $(T^-, T^+) \in N_{k,l}$.

If $k < l$, then the T_i^+, have a pole at $t = 0$ for $i = 1, 2, 3$; it is shown in [31] that the patching condition at $t = 0$ and the hermitian structure are compatible, that is, the block decomposition (2.5) can be obtained in some unitary basis. We can, as for the case with $k = l$, use some appropriate covariant constant basis for $V^{\pm}$ to obtain $T^{\pm}$ that are skew-hermitian and are symmetric at $t = \pm 1$. However, the resultant solutions may then not satisfy the boundary condition. The remedy is by a change to some appropriate *local* unitary gauge around $t = 0$, then T_i^+ will again have the correct block decomposition around $t = 0$ for each $i = 1, 2, 3$. This yields a unique point in $N_{k,l}/\mathcal{G}_0$ since any two such gauge differ by an element in $\mathcal{G}_0$.

Conversely, suppose $(T^-, T^+) \in N_{k,l}$ is a generic element. Proposition (1.16) says that there is a curve S^+ with a flow of line bundles $\mathcal{L}_t^+$ which satisfies $\mathcal{L}_t^+(-1) \in J(S^+)^{g-1} \setminus \Theta$ for $(0, 2)$. Similarly, there is a flow of line bundles $\mathcal{L}_t^-$ which satisfies $\mathcal{L}_t^-(-1) \in J(S^-)^{g-1} \setminus \Theta$ for $(-2, 0)$. The patching condition of $T_i^{\pm}$ at $t = 0$ implies that $\mathcal{L}_t^{\pm}$ are both holomorphically equivalent to (2.9), for some effect divisors $D = D^{\pm}$ on $S^{\pm}$, satisfying $S^- \cap S^+ = D^{\pm} + \tau(D^{\pm})$ as divisors along $S^{\mp}$. Let us denote both divisors by the same symbol D. The hermitian structure of $T_i^{\pm}$ implies that $S^{\pm}$ are real and that the hermitian form constructed on $V_t^{\pm} = H^0(S^{\pm}, \mathcal{L}_t)$, given by (2.3), is positive-definite.

Let us consider $T_i^+(1)$: since they are symmetric, Proposition (1.17) implies that $\mathcal{L}_1(-1)$ is a theta-characteristic, i.e. $(\mathcal{L}_1(-1))^2 \simeq K_{S^+}$. So we have

$$L^2(2k + 2l - 4)[-2D] \simeq (\mathcal{L}_1(-1))^2 \simeq K_S \simeq \pi^* \mathcal{O}(2l - 4),$$

and $L^2(2k)[-2D] \simeq \mathcal{O}$ over S^+. The isomorphism $\pi^*\mathcal{O}(2k) \simeq [D + \tau(D)]$ then yields $L^2[\tau(D) - D] \simeq \mathcal{O}$ over the same curve. Suppose ν^+ is a section of this bundle, then it is shown in [31] that, up to normalizations, the value of $\nu^+\sigma(\nu^+)$ always has sign $(-1)^{k+l-1}$, implying that $\mathcal{L}_1$ is real. By the same argument, the symmetricity of $T_i^-(-1)$ implies $L^2[D - \tau(D)]$ is trivial and that $\mathcal{L}_{-1}$ is real over S^-.

Since the constructions above come from Proposition (1.16), they are mutual inverses of each other. Hence there is a one-to-one correspondence between $\Sigma_{k,l}$ and spectral curves that arise from $N_{k,l}$. The choice of isomorphisms $L^2[\tau(D) - D] \simeq \mathcal{O}$ over S^+ and $L^2[D - \tau(D)] \simeq \mathcal{O}$ over S^- give rise to the correspondence between $M_{k,l}$ and $N_{k,l}/\mathcal{G}_0$. □

Hence the generic elements in $N_{k,l}$ are said to be the Nahm data for clusters. Although it has been claimed in [7] that there is a full bijection between $M_{k,l}$ and $N_{k,l}/\mathcal{G}_0$, it is not obvious from the proof therein.

Remark that $N_{k,l}/\mathcal{G}_0$ is isomorphic to the moduli space of $SU(2)$-calorons of charge (k, l). Since the latter space can be given a parametrization [11], the existence of calorons implies the existence of clusters via the above proposition.

As moduli space of solutions to Nahm's equations can be seen formally as a hyperkähler quotient of some flat hyperkähler manifold, it is expected that $N_{k,l}/\mathcal{G}_0$ also has a hyperkähler structure, given by the L^2-metric with the usual hypercomplex structure. Indeed, it is known [5] that the moduli space $\mathcal{F}_{k,l}/\mathcal{G}_c$, where

$$\mathcal{G}_c = \{\, g \in \mathcal{G} \,|\, g_-(-1) = g_+(1) = Id \,\}\,,$$

is a finite-dimensional hyperkähler manifold. Then $N_{k,l}/\mathcal{G}_0$ is the hyperkähler quotient of $\mathcal{F}_{k,l}/\mathcal{G}_c$ by the group $\mathcal{G}_0/\mathcal{G}_c \simeq O(k,\mathbb{R}) \times O(l,\mathbb{R})$. Despite the correspondence given in Proposition (2.5), the hyperkähler structure of $M_{k,l}$ that gives rise to the asymptotic monopole metric is a different one, according to [7]; we shall see later that such hyperkähler structure is nonetheless defined in terms of the natural one on $N_{k,l}/\mathcal{G}_0$ via twistor theory.

2.3 Rational Map Descriptions

Recall that for monopoles, the space N_k can be identified with solutions to

$$\begin{aligned}
&\frac{d\beta}{dt} + [\beta, \alpha] = 0 && \text{(complex equation)},\\
&\frac{d}{dt}(\alpha + \alpha^*) + [\alpha, \alpha^*] + [\beta, \beta^*] = 0 && \text{(real equation)}.
\end{aligned}$$

Then $N_k/\mathcal{G}_0$ in one-to-one correspondence with $\mathcal{N}_k/\mathcal{G}_0^{\mathbb{C}}$, where $\mathcal{N}_k$ is the set of solutions to the complex equation. Similarly, one could identify $N_{k,l}$ with solutions to the complex and real equations, then defined the analogous space $\mathcal{N}_{k,l}$ in the hope to show that $\mathcal{N}_{k,l}/\mathcal{G}_0^{\mathbb{C}} \simeq N_{k,l}/\mathcal{G}_0$. However, such proposition has only been proved partially. Now, let us go through the details.

For any $(T^-, T^+) \in N_{k,l}$, let

$$\alpha_\pm = T_0^\pm + iT_1^\pm, \quad \beta_\pm = T_2^\pm + iT_3^\pm, \tag{2.10}$$

and we write (α, β) for $(\alpha_-, \beta_-, \alpha_+, \beta_+)$. Then

Lemma 2.6. *$N_{k,l}$ is in one-to-one correspondence with the space of (α, β) satisfying the following conditions:*

(a) *α_-, β_- are $GL(k,\mathbb{C})$-valued analytic functions on $[-1, 0)$; α_+, β_+ are $GL(l,\mathbb{C})$-valued analytic functions on $(0, 1]$.*

(b) *(α_-, β_-), (α_+, β_+) are solutions to both the complex and real equation.*

(c) *α_-, β_- are symmetric at $t = -1$; α_+, β_+ are symmetric at $t = 1$.*

(d) *For $k \le l$, the limits $\lim_{t\to 0-} \alpha_-(t)$ and $\lim_{t\to 0-} \beta_-(t)$ exist and are finite. In addition, α_+, β_+ have the following expansion near $t = 0$:*

$$\alpha_+(t) = \left(\begin{array}{c|c} Y + O(t) & O\left(t^{(l-k-1)/2}\right) \\ \hline O\left(t^{(l-k-1)/2}\right) & a/t + O(t) \end{array}\right),$$

$$\beta_+(t) = \left(\begin{array}{c|c} X + O(t) & O\left(t^{(l-k-1)/2}\right) \\ \hline O\left(t^{(l-k-1)/2}\right) & b/t + O(t) \end{array}\right),$$

where $X, Y \in \mathfrak{gl}(k)$, $a, b \in \mathfrak{gl}(l-k)$. If $k < l$, then a, b are given by

$$a = \operatorname{diag}\left(-\frac{(l-k-1)}{2}, \dots, \frac{(l-k-1)}{2}\right),$$

$$b = \sqrt{l-k-1}\begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

(e) (Patching condition) *If $k < l$, then*

$$\lim_{t \to 0^-} \beta_-(t) = X,$$
$$\lim_{t \to 0^-} \alpha_-(t) + \alpha_-(t)^* = Y + Y^*.$$

If $k = l$, then there exists column vectors $U, W \in \mathbb{C}^k$ such that

$$\lim_{t \to 0^+} \beta_+(t) - \lim_{t \to 0^-} \beta_-(t) = -UW^T,$$
$$\lim_{t \to 0^+} (\alpha_+(t) + \alpha_+(t)^*) - \lim_{t \to 0^-} (\alpha_-(t) + \alpha_-(t)^*) = -U\bar{U}^T + \bar{W}W^T.$$

The conditions for the case $k > l$ is the same as $k < l$ except that the signs $\pm$ are interchanged.

Remark that the above identification involved choosing a particular complex structure, given by the choice (2.10). As we know, space of solutions to Nahm's equations has complex structures parametrized by the 2-sphere, Lemma (2.6) works just as well for other choice of complex structure.

Any element $g \in \mathcal{G}_0$ acts on $(\alpha, \beta) \in N_{k,l}$ by

$$\begin{aligned} \alpha_\pm &\mapsto g_\pm \alpha_\pm g_\pm^{-1} - \frac{dg_\pm}{dt} g_\pm^{-1}, \\ \beta_\pm &\mapsto g_\pm \beta_\pm g_\pm^{-1}. \end{aligned} \tag{2.11}$$

Hence $N_{k,l}/\mathcal{G}_0$ can be identified with the space of $\mathcal{G}_0$-orbit of elements (α, β) satisfying the conditions in Lemma (2.6).

Definition 2.7. *Let $\mathcal{N}_{k,l}$ be the set of (α, β) satisfying the conditions (a), (c) and (d) in Lemma (2.6). In addition, (α_-, β_-), (α_+, β_+) are solutions to the complex equation, and β_-, β_+ (but not necessarily α_-, α_+) satisfy the patching condition in (e). We denote the set of elements in $\mathcal{N}_{k,l}$ (resp. $N_{k,l}$) that with $\beta_\pm(\pm 1)$ regular by $\mathcal{N}^r_{k,l}$ (resp. $N^r_{k,l}$).*

It is clear that the complexification of $\mathcal{G}_0$, denoted by $\mathcal{G}_0^{\mathbb{C}}$, defines a group action on both $\mathcal{N}_{k,l}$ and $\mathcal{N}^r_{k,l}$. Then the following result is analogous to Proposition (1.21) in Chapter 1:

Proposition 2.8. [7] *There is a one-to-one correspondence between $N^r_{k,l}/\mathcal{G}_0$ and $\mathcal{N}^r_{k,l}/\mathcal{G}_0^{\mathbb{C}}$.*

2.3.1 Normal Form

Henceforth, we shall assume $k < l$ implicitly unless otherwise stated since this is mainly the case that we shall have to deal with. Let $(\alpha, \beta) \in N_{k,l}$, and (U, ζ), $(\tilde{U}, \tilde{\zeta})$ be affine charts of $\mathbb{P}^1$. Viewing ζ as a general element in $\mathbb{C}$, we construct

$$\begin{aligned} \beta_\pm(\zeta) &= \beta_\pm + (\alpha_\pm + \alpha_\pm^*)\zeta - \beta_\pm^*\zeta^2, \\ \alpha_\pm(\zeta) &= \alpha_\pm - \beta_\pm^*\zeta, \end{aligned} \tag{2.12}$$

as before. They satisfy the Lax's equation

$$\frac{d\beta_\pm(\zeta)}{dt} = [\beta_\pm(\zeta), \alpha_\pm(\zeta)].$$

Suppose ζ is fixed, then the elements $g \in \mathcal{G}_0^{\mathbb{C}}$ act on $(\alpha(\zeta), \beta(\zeta))$ by

$$\begin{aligned} \alpha_\pm(\zeta) &\mapsto g_\pm \alpha_\pm(\zeta) g_\pm^{-1} - \frac{dg_\pm}{dt} g_\pm^{-1}, \\ \beta_\pm(\zeta) &\mapsto g_\pm \beta_\pm(\zeta) g_\pm^{-1}. \end{aligned} \tag{2.13}$$

Note that such an action is not in general equivalent to constructing $(\alpha(\zeta), \beta(\zeta))$ from $g \cdot (\alpha, \beta)$. The following can be deduced from [30]:

Proposition 2.9. (Normal form) *Let $(\alpha, \beta) \in N_{k,l}$ and $\alpha(\zeta), \beta(\zeta)$ are constructed as in (2.12). Then for each ζ, $(\alpha_+(\zeta), \beta_+(\zeta))$ is locally $\mathcal{G}_0^{\mathbb{C}}$-equivalent to the following form near $t = 0$:*

$$\alpha_+(\zeta, t) = \frac{1}{t}\left(\begin{array}{c|ccccc} 0 & & & 0 & & \\ \hline & a_1 & -c\zeta & & & \\ 0 & & a_2 & \ddots & & \\ & & & \ddots & -c\zeta & \\ & & & & a_{l-k} & \end{array}\right),$$

$$\beta_+(\zeta, t) = \left(\begin{array}{c|ccccc} \beta_-(\zeta, 0) + O(t) & & 0 & & & t^{(l-k-1)/2} g(\zeta) \\ \hline t^{(l-k-1)/2} f(\zeta) & a_1\zeta/t & -c\zeta^2/t & 0 & \cdots & t^{(l-k-1)} e_0(\zeta) \\ & c/t & a_2\zeta/t & \ddots & & t^{(l-k-2)} e_1(\zeta) \\ 0 & 0 & \ddots & \ddots & & \vdots \\ & \vdots & & & & -c\zeta^2/t + t e_{l-k-2}(\zeta) \\ & 0 & & & c/t & a_{l-k}\zeta/t + e_{l-k-1}(\zeta) \end{array}\right),$$

where $e_i \in \mathbb{C}$, $f(\zeta) = (f_1(\zeta), \ldots, f_k(\zeta))$, $g(\zeta)^T = (g_1(\zeta), \ldots, g_k(\zeta))$ are row vectors in $\mathbb{C}^k$, $c = \sqrt{l-k-1}$ and $a_n = -c^2/2 + n - 1$.

Proof. Let us consider the equation

$$\frac{dw}{dt} + \alpha_+(\zeta)w = 0. \tag{2.14}$$

Following [31], there is a unique solution $w_1(\zeta, t)$ to the above equation with

$$\lim_{t \to 0^+} (t^{-(l-k-1)/2} w_1(\zeta, t) - E_{k+1}) = 0,$$

where $E_1, \ldots, E_l$ are the standard basis of $\mathbb{C}^l$. For each $1 \leq i \leq l-k$, we set

$$w_i(\zeta, t) = \beta_+^{i-1}(\zeta, t) w_1(\zeta, t).$$

Then each $w_i(\zeta, t)$ is the unique solution to (2.14) with

$$\lim_{t \to 0^+} (t^{(i-1)-(l-k-1)/2} w_i(\zeta, t) - b(\zeta)^{i-1} E_{k+1}) = 0.$$

Moreover, there are solutions $u_1(\zeta, t), \ldots, u_k(\zeta, t)$ whose last k components vanish at $t = 0$ to order $(k+1)/2$, and which are linearly independent at $t = 0$. If we let Q be a gauge transformation such that Q_+^{-1} is locally of the form

$$\left(u_1(\zeta, t), \ldots, u_k(\zeta, t), t^{-(l-k-1)/2} w_1(\zeta, t), \ldots, t^{(l-k-1)/2} w_{l-k}(\zeta, t)\right)$$

near $t = 0$, then in this gauge, $\beta_+(\zeta)$ is locally of the form

$$\beta_+(\zeta, t) = \left(\begin{array}{c|ccc|c} \beta_-(\zeta, 0) + O(t) & & 0 & & t^{(l-k-1)/2} g \\ \hline t^{(l-k-1)/2} f & 0 & \ldots & 0 & t^{(l-k-1)} e_0 \\ \hline & 1 & & & t^{(l-k-2)} e_1 \\ 0 & & \ddots & & \vdots \\ & & & 1 & e_{l-k-1} \end{array}\right).$$

If R is a gauge transformation such that R_+ is equal to $Q_+^{-1}(0)$ in a neighbourhood of $t = 0$, then RQ belongs to $\mathcal{G}_0^{\mathbb{C}}$ and gauges α_+, β_+ into the required forms near $t = 0$. $\square$

2.3.2 Intersection Points

The spectral curves $S^{\pm}$ of any element in $N_{k,l}$ are given by the equations $\det(\eta - \beta_{\pm}(\zeta)) = 0$ over the chart neighbourhood $\pi^{-1}(U)$ of $T\mathbb{P}^1$. We shall characterize the intersection points of these curves in terms of the following:

Definition 2.10. *For any $(\alpha, \beta) \in N_{k,l}$, we say that $\beta_-(\zeta)$ and $\beta_+(\zeta)$ have a common eigenvector with common eigenvalue η at $t = 0$ if there exists a* *column vector $v \in \mathbb{C}^k$ such that*

$$(\eta - \beta_-(\zeta, 0))v = 0, \quad \lim_{t \to 0^+} (\eta - \beta_+(\zeta, 0)) \begin{pmatrix} v \\ 0 \end{pmatrix} = 0.$$

Similar, $\beta_-(\zeta, 0)$ and $\beta_+(\zeta, 0)$ are said to have a common eigen-covector with common eigenvalue η if there exists a row vector $v \in \mathbb{C}^k$ such that

$$v(\eta - \beta_-(\zeta)) = 0, \quad \lim_{t \to 0^+} \begin{pmatrix} v & 0 \end{pmatrix} (\eta - \beta_+(\zeta, 0)) = 0.$$

Note that the above definition extends the one given in [7]. Then:

Lemma 2.11. *Let $(\alpha, \beta) \in N_{k,l}$. Suppose for a fixed ζ, $(\alpha(\zeta), \beta(\zeta))$ is in the normal form given by Proposition (2.9) near $t = 0$, then $(\zeta, \eta) \in \pi^{-1}(U)$ corresponds to an intersection point of S^-, S^+ if and only if one of the following conditions holds:*

(a) *There exists a common eigenvector v_- of $\beta_-(\zeta)$ and $\beta_+(\zeta)$ at $t = 0$ with eigenvalue η such that $fv_- = 0$.*

(b) *There exists a common eigen-covector v_- of $\beta_-(\zeta)$ and $\beta_+(\zeta)$ at $t = 0$ with eigenvalue η such that $v_- g = 0$.*

Proof. Let us first suppose that we are in the generic case where $\beta_-(\zeta), \beta_+(\zeta)$ are regular.

If (a) or (b) holds, then for some sufficiently small t, both $\eta - \beta_\pm(\zeta, 0)$ have corank at least 1 so $\det(\eta - \beta_\pm(\zeta, t)) = 0$, i.e. (ζ, η) corresponds to a point in $S^- \cap S^+$ over $\pi^{-1}(U)$. Conversely, let us expand $\det(\eta - \beta_+(\zeta))$:

$$\det(\eta - \beta_+(\zeta)) = \det(\eta - \beta_-(\zeta)) \times p(\eta) - f(\eta - \beta_-(\zeta, 0))_{\mathrm{adj}} g,$$

where $p(\eta)$ is some polynomial. If (ζ, η) is a point such that $\det(\eta - \beta_\pm(\zeta, t)) = 0$, then

$$f(\eta - \beta_-(\zeta, 0))_{\mathrm{adj}} g = 0.$$

We claim that one of $f(\eta - \beta_-(\zeta, 0))_{\mathrm{adj}}$ or $(\eta - \beta_-(\zeta, 0))_{\mathrm{adj}} g$ must be zero. As $\beta_-(\zeta)$ is regular, $\eta - \beta_-(\zeta)$ must have corank one, which implies that $(\eta - \beta_-(\zeta, 0))_{\mathrm{adj}}$ has rank one and may be written as uw^T for some $u, w \in \mathbb{C}^k \setminus \{0\}$. From here it is clear that fu or $w^T g$ must be zero.

Suppose $f(\eta - \beta_-(\zeta, 0))_{\mathrm{adj}} = 0$, then $\det(\eta - \beta_-(\zeta)) = 0$ implies that there exists an eigenvector $v \in \mathbb{C}^k$ of $\beta_-(\zeta, 0)$ with eigenvalue η. Since $(\eta - \beta_-(\zeta, 0))_{\mathrm{adj}}$ has rank one and

$$(\eta - \beta_-(\zeta))(\eta - \beta_-(\zeta))_{\mathrm{adj}} = \det(\eta - \beta_-(\zeta)) = 0,$$

w is nonzero implies $(\eta - \beta_-(\zeta))u = 0$, so $u = \lambda v$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and hence fv must be zero. By the same argument, $(\eta - \beta_-(\zeta, 0))_{\mathrm{adj}} g = 0$ implies condition (b).

As the lemma is true for generic intersections, by varying continuously, the result holds also for non-generic curves . □

Lemma 2.12. *Suppose* $(\zeta, \eta) \in \mathbb{C}^2$*,* $\zeta \neq 0$*, corresponds to an intersection point of* S^-*,* S^+ *in* $\pi^{-1}(U)$*, then it satisfies condition* (a) *if and only if* $(-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2)$ *satisfies condition* (b) *in the above lemma.*

Proof. Suppose v_- is an eigenvector of $\beta_-(\zeta, 0)$ with eigenvalue η and $f(\zeta)v_- = 0$. Since

$$\beta_-(\zeta)^* = -\bar{\zeta}^2\beta_-(-1/\bar{\zeta}),$$

by taking the conjugate transpose of $\beta_-(\zeta, 0)v_- = \eta v_-$ and re-arranging yields

$$v_-^*\beta_-(-1/\bar{\zeta}, 0) = (-\bar{\eta}/\bar{\zeta}^2)v_-^*.$$

In other words, v_-^* is an eigen-covector of $\beta_-(-1/\bar{\zeta}, 0)$. It remains to show that $v_-^*g(-1/\bar{\zeta}) = 0$. Observe that $g(-1/\bar{\zeta}) = (-1/\bar{\zeta}^2)f(\zeta)^*$, so

$$v_-^*g(-1/\bar{\zeta}) = v_-^*(-1/\bar{\zeta}^2)f(\zeta)^* = (-1/\bar{\zeta}^2)\left(f(\zeta)v_-\right)^* = 0.$$

The converse is similar. □

2.3.3 Construction of Rational Maps

Suppose S^-, S^+ are spectral curves arising from Nahm data in $N_{k,l}$. Let D (resp. D') denote the set of points in $T\mathbb{P}^1$ that corresponds to $(\zeta, \eta) \in \mathbb{C}^2$ satisfying condition (b) (resp. (a)). Then Lemma (2.11) implies that, set theoretically,

$$S^- \cap S^+ = D \cup D'.$$

Moreover, Lemma (2.12) yields $\tau(D) = D'$. Recall that (U, ζ) is the standard affine chart of $\mathbb{P}^1$, then

Definition 2.13. *The space* $\mathcal{N}_{k,l}^{rr}$ *consists of elements in* $\mathcal{N}_{k,l}^{r}$ *such that*

$$0 \notin \zeta\left(\pi(\tau(D))\right).$$

Equivalently, $\mathcal{N}_{k,l}^{rr}$ is the subset such that the condition (a) in Lemma (2.11) fails whenever $\zeta = 0$.

Proposition 2.14. [7] *There is a one-to-one correspondence between* $\mathcal{N}_{k,l}^{rr}/\mathcal{G}_0^{\mathbb{C}}$ *and* $Rat_k\left(\mathbb{P}^1\right) \times Rat_l\left(\mathbb{P}^1\right)$*.*

Note that the proof in [7] only defined a map from $\mathcal{N}_{k,l}^{rr}/\mathcal{G}_0^{\mathbb{C}}$ to $Rat_k\left(\mathbb{P}^1\right) \times Rat_l\left(\mathbb{P}^1\right)$, we shall complete it by showing that such map is well-defined and that it is indeed a bijection.

Proof. Any element $(\alpha, \beta) \in \mathcal{N}^{rr}_{k,l}$ can be extended by symmetry: there exists a gauge in $\mathcal{G}^{\mathbb{C}}_0$ so that the resulting $\alpha_\pm$ are symmetric at $t = \pm 1$. Then (α, β) can be extended to $(-2, 0) \cup (0, 2)$ by

$$\alpha_\pm(\pm 2 - t) = \alpha_\pm(t)^T, \quad \beta_\pm(\pm 2 - t) = \beta_\pm(t)^T. \tag{2.15}$$

We denote the image of such construction by $\tilde{\mathcal{N}}^{rr}_{k,l}$. The corresponding gauge group $\tilde{\mathcal{G}}^{\mathbb{C}}_0$ consists of the set of elements in $\mathcal{G}^{\mathbb{C}}_0$ that are extended by

$$g_\pm(\pm 2 - t)^T = g_\pm(t)^{-1}. \tag{2.16}$$

It is not hard to see that $\mathcal{N}^{rr}_{k,l}/\mathcal{G}^{\mathbb{C}}_0 \simeq \tilde{\mathcal{N}}^{rr}_{k,l}/\tilde{\mathcal{G}}^{\mathbb{C}}_0$. It is more convenient to get rid of the pole of β_+: we define the space $\tilde{\mathcal{B}}^{rr}_{k,l}$ just as for $\tilde{\mathcal{N}}^{rr}_{k,l}$ except that β_+ now has no pole at $t = 0$ and the patching condition is replaced by

$$\beta_+(0) = \left(\begin{array}{c|ccc|c} \beta_-(0) & & 0 & & g \\ \hline f & 0 & \dots & 0 & e_0 \\ \hline & 1 & & & e_1 \\ 0 & & \ddots & & \vdots \\ & & & 1 & e_{l-k-1} \end{array}\right). \tag{2.17}$$

Lemma 2.15. *There is a one-to-one correspondence between $\tilde{\mathcal{N}}^{rr}_{k,l}/\tilde{\mathcal{G}}^{\mathbb{C}}_0$ and $\tilde{\mathcal{B}}^{rr}_{k,l}/\tilde{\mathcal{G}}^{\mathbb{C}}$.*

Proof of Lemma (2.15)*.* Assuming $(\alpha, \beta) \in \tilde{\mathcal{N}}^{rr}_{k,l}$ have the normal form near $t = 0$. Let $\rho(t)$ be a smooth function for $t \in (0, 1)$ such that $\rho(t) = t$ in a neighbourhood of $t = 0$ and $\rho(t) = 1$ in a neighbourhood of $t = 1$. Let Q be the singular gauge transformation defined by $Q_- \equiv Id$ and

$$Q_+(t) = \operatorname{diag}\left(1, \dots, 1, c^{-1}\rho(t)^{-(l-k-1)/2}, \dots, c^{-1}\rho(t)^{(l-k-1)/2}\right)$$

for $t \in (0, 1]$, extending to $(0, 2)$ by $Q_+(2 - t) = Q_+(t)^T$. Then the correspondence is given by $(\alpha, \beta) \mapsto Q \cdot (\alpha, \beta)$. It is well-defined since if $g \in \tilde{\mathcal{G}}^{\mathbb{C}}_0$, then $QgQ^{-1} \in \tilde{\mathcal{G}}^{\mathbb{C}}_0$. Such map is easily seen to be a bijection. $\square$

Proposition 2.16. [7] *Let B is a companion matrix and $u \in \mathbb{C}^k$ be a row-vector. There exists an invertible matrix Y that centralizes B with $uY^{-1} = (0, \dots, 0, 1)$ if and only if $uv \neq 0$ for any eigenvector v of B. If Y exists, then it is unique.*

Let $(\alpha, \beta) \in \tilde{\mathcal{B}}^{rr}_{k,l}$. It is known that is $g \in \tilde{\mathcal{G}}^{\mathbb{C}}$ which gauges $\beta_-(0)$ into a companion matrix. From Proposition (2.16), one may deduce that there is $X \in \tilde{\mathcal{G}}^{\mathbb{C}}_0$ that makes $\beta_+(0)$ into the form (2.17) with $\beta_-(0)$ a companion matrix and $f = (0, \dots, 0, 1)$. The same proposition tells us that X is unique up to gauge transformations $g \in \tilde{\mathcal{G}}^{\mathbb{C}}_0$ with $g_\pm(0) = Id$. By finding the unique solution h to the equations

$$((hX) \cdot \alpha)_{\pm} = 0$$

with $h_{\pm}(0) = Id$, we define $F : \tilde{\mathcal{B}}_{k,l}^{rr}/\tilde{\mathcal{G}}_0^{\mathbb{C}} \to \mathfrak{gl}(k,\mathbb{C})^2 \times \mathfrak{gl}(l,\mathbb{C})^2$ by

$$(\alpha, \beta) \mapsto ((X \cdot \beta)_-(0), h_-(-2), (X \cdot \beta)_+(0), h_+(2)). \tag{2.18}$$

Let $\mathcal{K}$ be the set of elements $(B_-, h_-, B_+, h_+) \in \mathfrak{gl}(k,\mathbb{C})^2 \times \mathfrak{gl}(l,\mathbb{C})^2$ such that

(i) B_- is a companion matrix; B_+ is a matrix of the form (2.17) with $f = (0, \ldots, 0, 1)$ and the upper diagonal block equals B_-;

(ii) Both h_- and h_+ are symmetric;

(iii) The equations $h_- B_-^T h_-^{-1} = B_-$ and $h_+ B_+^T h_+^{-1} = B_+$ are satisfied.

Proposition 2.17. *F is a well-defined bijection onto $\mathcal{K}$.*

Proof of Proposition (2.17). F is well-defined: since (i) and (iii) are clear, we shall only check condition (ii). Suppose h is a solution to

$$\frac{dh}{dt} = h\alpha. \tag{2.19}$$

on $[0,1]$. Because of the symmetry condition $\alpha(2-t) = \alpha(t)^T$, $\tilde{h}$ defined by

$$\tilde{h}(t) = A\left(h(2-t)^T\right)^{-1}$$

is a solution to (2.19) on $[1,2]$ for any constant matrix A. If $A = h(1)h(1)^T$, then $\tilde{h}(1) = h(1)$. By uniqueness of solutions to ordinary differential equations, $h \equiv \tilde{h}$ on $[1,2]$. In particular, $h(2) = \tilde{h}(2) = h(1)h(1)^T$, hence it is symmetric.

To see surjectivity, let $(B_-, h_-, B_+, h_+) \in \mathcal{K}$. Since h_+ is a complex symmetric matrix, we can write $h_+ = C_+ C_+^T$ for some $C_+ \in GL(l,\mathbb{C})$ (Corollary (4.4.6) in [26]). Let $P_+ : [0,1] \to GL(l,\mathbb{C})$ be a smooth function with $P_+(0) = Id$ and equals to C_+^{-1} in a neighbourhood of $t = 1$. Extending P_+ by symmetry and let $(\alpha_+, \beta_+) = P_+ \cdot (0, B_+)$. Condition (iii) implies $\beta_+(1) = C_+^{-1} B_+ C_+$ is symmetric. Similarly, we can define P_- and the corresponding solution (α_-, β_-) to the complex equation on $[-2,0]$. We have found (α, β) with $\beta_{\pm}(0) = B_{\pm}$ and the unique $P = (P_-, P_+)$ with $P_{\pm} = Id$, $P_{\pm}(2) = h_{\pm}$ such that $P \cdot (\alpha, \beta) = 0$, hence $F(\alpha, \beta) = (B_-, h_-, B_+, h_+)$.

For injectivity, suppose there are $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \tilde{\mathcal{B}}_{k,l}^{rr}$ such that

$$F(\alpha_1, \beta_1) = F(\alpha_2, \beta_2).$$

This means we have

$$h_1 \cdot (X_1 \cdot \alpha_1) = h_2 \cdot (X_2 \cdot \alpha_2) = 0,$$

with $(h_1)_\pm(\pm 2) = (h_2)_\pm(\pm 2)$ and $(\beta_1)_\pm(0) = (\beta_2)_\pm(0)$. It is sufficient to show that $h_1^{-1}h_2 \in \tilde{\mathcal{G}}_0^{\mathbb{C}}$ since this would then imply (α_1, β_1) and (α_2, β_2) differ only by an element in $\tilde{\mathcal{G}}_0^{\mathbb{C}}$. We only need to check that $\left(h_1^{-1}h_2\right)_\pm(\pm 1)$ are orthogonal. Indeed, write $(h_j)_+(2) = (h_j)_+(1)(h_j)_+(1)^T$ for $j = 1, 2$. Then it follows directly from the equality $(h_1)_+(2) = (h_2)_+(2)$ that $\left(h_1^{-1}h_2\right)_+(1)$ is orthogonal. The same argument implies that $\left(h_1^{-1}h_2\right)_-(-1)$ is also orthogonal.
□

In the above proof, we have used the following fact: for any complex symmetric matrix $h \in GL(n, \mathbb{C})$, there exists $C \in GL(n, \mathbb{C})$ such that $h = CC^T$.

Proposition 2.18. *Let $(B_-, h_-, B_+, h_+) \in \mathcal{K}$. By writing $h_\pm = C_\pm C_\pm^T$ for some $C_\pm$, the map $G : \mathcal{K} \to \mathcal{V}_k/O(k, \mathbb{C}) \times \mathcal{V}_l/O(l, \mathbb{C})$ given by*

$$(B_-, h_-, B_+, h_+) \mapsto \left(C_-^{-1}B_-C_-, C_-^{-1}E_1, C_+^{-1}B_+C_+, C_+^{-1}E_1\right)$$

is well-defined and is a bijection.

Recall that the space $\mathcal{V}_n$ consists of pairs $(B, w) \in \mathfrak{gl}(n, \mathbb{C}) \times \mathbb{C}^n$ such that B is symmetric and w is a cyclic vector for B; it admits an $O(n, \mathbb{C})$-action defined by

$$(B, w) \mapsto (gBg^{-1}, gw).$$

Proof of Proposition (2.18). We first check that G is well-defined. Suppose $h_\pm = C_\pm C_\pm^T = \tilde{C}_\pm \tilde{C}_\pm^T$, we must show that

$$(C_\pm^{-1}B_\pm C_\pm, C_\pm^{-1}E_1),\ (\tilde{C}_\pm^{-1}B_\pm \tilde{C}_\pm, \tilde{C}_\pm^{-1}E_1)$$

are orthogonally equivalent. It suffices to show that $C_\pm^{-1}\tilde{C}_\pm \in O(l, \mathbb{C})$, but this is straightforward.

For surjectivity, suppose $(M_-, w_-, M_+, w_+) \in \mathcal{V}_k \times \mathcal{V}_l$. Consider first the pair (M_-, w_-). Let $g_-^{-1} = (w_-, M_-w_-, \ldots, M_-^{k-1}w_-)$. Then $B_- := g_-M_-g_-^{-1}$ is a companion matrix. Since S^- is symmetric, this implies $h_-B_-^Th_-^{-1} = B_-$, where $h_- = g_-g_-^T$. It is clear that we have $(g_-^{-1}B_-g_-, g_-^{-1}E_1) = (M_-, w_-)$.

Let us now consider (M_+, w_+). Let $g_+^{-1} = (w_+, M_+w_+, \ldots, M_+^{l-1}w_+)$, then $g_+M_+g_+^{-1}$ is a companion matrix. There is the following lemma:

Lemma 2.19. [7] *Let P_+ be a matrix of the form* (2.17) *whose upper diagonal block P_- is a companion matrix and $f = (0, \ldots, 0, 1)$. Then there exists an invertible matrix, which depends only on P_-, conjugates P_+ into a companion matrix.*

By the lemma above, there exists an invertible matrix A, which depends only on B_-, such that it conjugates $g_+M_+g_+^{-1}$ into the matrix of the form (2.17) with upper diagonal block equals B_- and $f = (0, \ldots, 0, 1)$. Let $B_+ :=$

$(Ag_+)M_+(Ag_+)^{-1}$. Symmetricity of M_+ implies that

$$Ag_+(Ag_+)^T B_+^T \left(Ag_+(Ag_+)^T\right)^{-1} = B_+.$$

Let $h_+ = Ag_+(Ag_+)^T$. It is clear that $(Ag_+)^{-1}B_+(Ag_+) = M_+$. Moreover, by the construction of A (see [7]) it satisfies $AE_1 = E_1$. It follows that $(Ag_+)^{-1}E_1 = w_+$. Therefore we have $G(B_-, h_-, B_+, h_+) = (M_-, w_-, M_+, w_+)$, showing that G is surjective.

Suppose

$$G(B_-, h_-, B_+, h_+) = G(\tilde{B}_-, \tilde{h}_-, \tilde{B}_+, \tilde{h}_+).$$

We shall first show that $(B_-, h_-) = (\tilde{B}_-, \tilde{h}_-)$. By definition, there exists $U_- \in O(k, \mathbb{C})$ such that $U_-^{-1}\left(C_-^{-1}B_-C_-\right)U_- = \left(\tilde{C}_-^{-1}\tilde{B}_-\tilde{C}_-\right)$. Since any companion matrix is unique in its conjugacy class, we have $B_- = \tilde{B}_-$. In addition, we have $(C_-U_-)^{-1}E_1 = \tilde{C}_-^{-1}E_1$. This implies

$$\begin{aligned}
\tilde{C}_-^{-1} &= \left(\tilde{C}_-^{-1}E_1, \left(\tilde{C}_-^{-1}\tilde{B}_-\tilde{C}_-\right)\tilde{C}_-^{-1}E_1, \ldots, \left(\tilde{C}_-^{-1}\tilde{B}_-\tilde{C}_-\right)^{k-1}\tilde{C}_-^{-1}E_1\right) \\
&= (C_-U_-)^{-1}\left(E_1, B_-E_1, \ldots, B_-^{k-1}E_1\right) \\
&= (C_-U_-)^{-1}.
\end{aligned}$$

It is now easy to see that $h_- = C_-C_-^T = \tilde{C}_-\tilde{C}_-^T = \tilde{h}_-$.

Similarly, there is $U_+ \in O(l, \mathbb{C})$ such that

$$U_+^{-1}\left(C_+^{-1}B_+C_+\right)U_+ = \left(\tilde{C}_+^{-1}\tilde{B}_+\tilde{C}_+\right).$$

Since $B_+, \tilde{B}_+$ are not companion matrix, we cannot use the same argument to conclude that they are equal. Instead, we consider their characteristic functions:

$$\begin{aligned}
\det(\eta - B_+) = \det(\eta - B_-)&\left(\eta^{l-k} - e_{l-k}\eta^{l-k-1} - \cdots - e_1\right) \\
&- (0, \ldots, 0, 1)(\eta - B_-)_{\mathrm{adj}}g
\end{aligned}$$

and respectively the tilde version for $\tilde{B}_+$. Since $B_- = \tilde{B}_-$ and the second summand is a polynomial of degree $k-1$, by comparing the coefficients of the two characteristic functions with degree $\geq k$, it obtain $e_j = \tilde{e}_j$ for all $1 \leq j \leq l-k$. Thus, in the above expansion, the first summand of $\det(\eta - B_+), \det(\eta - \tilde{B}_+)$ are the same. For the second summand, observe that

$$(0, \ldots, 0, 1)(\eta - B_-)_{\mathrm{adj}} = (0, \ldots, 0, 1)(\eta - \tilde{B}_-)_{\mathrm{adj}} = \left(1, \eta, \ldots, \eta^k\right).$$

Thus $\det(\eta - B_+) = \det(\eta - \tilde{B}_+)$ yields $g = \tilde{g}$, thereby $B_+ = \tilde{B}_+$.

The argument for $h_+ = \tilde{h}_+$ is similar: let A be the invertible matrix from the above lemma conjugating B_+ into a companion matrix $A^{-1}B_+A$, then proceed as before with $(AC_+U_+)^{-1}E_1 = (A\tilde{C}_+)^{-1}E_1$. At the end, one obtains $(AC_+U_+)^{-1} = (A\tilde{C}_+)^{-1}$. This implies $h_+ = C_+C_+^T = \tilde{C}_+\tilde{C}_+^T = \tilde{h}_+$. Therefore we have $(B_-, h_-, B_+, h_+) = (\tilde{B}_-, \tilde{h}_-, \tilde{B}_+, \tilde{h}_+)$ and hence G is injective.

□

Finally, by Proposition (1.23), we know that there is the bijection between $\mathcal{V}_n/O(n,\mathbb{C})$ and the space of rational maps $Rat_n(\mathbb{P}^1)$. This, together with the correspondences given above, shows that there is a bijection between $\mathcal{N}^{rr}_{k,l}/\mathcal{G}^{\mathbb{C}}_0$ and $Rat_k(\mathbb{P}^1) \times Rat_l(\mathbb{P}^1)$. □

Since there is a one-to-one correspondence between the set of generic elements in $M_{k,l}$, and in $N_{k,l}$, by varying the complex structures, we deduce the following:

Corollary 2.20. *For each $a \in \mathbb{P}^1$, there is a one-to-one correspondence between an open dense subset of $M_{k,l}$ and an open subset of $Rat_k(\mathbb{P}^1)\times Rat_l(\mathbb{P}^1)$.*

2.4 Construction of Meromorphic Sections

Let us review the construction of the rational maps from the proof of Proposition (2.14). As usual, we let (U,ζ), $(\tilde{U},\tilde{\zeta})$ be affine charts of $\mathbb{P}^1$. If $(\alpha,\beta) \in \mathcal{N}^{rr}_{k,l}$, then there exists a unique (singular) gauge transformation h which takes $\alpha_\pm$ to zero, β_+ to the form

$$\left(\begin{array}{cccc|cccc}
0 & \cdots & 0 & c_0 & & & & g_0 \\
1 & & & c_1 & & 0 & & g_1 \\
 & \ddots & & \vdots & & & & \vdots \\
0 & & 1 & c_{k-1} & & & & g_{k-1} \\
\hline
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & e_0 \\
 & & & & 1 & & & e_1 \\
 & 0 & & & & \ddots & & \vdots \\
 & & & & 0 & & 1 & e_{l-k-1}
\end{array}\right)$$

and β_- to the upper diagonal block of this matrix. Then the rational maps

$$(R_-, R_+) \in Rat_k(\mathbb{P}^1) \times Rat_l(\mathbb{P}^1)$$

are given by

$$R_\pm(\eta) = \left(h_\pm(\pm 1)^{-1}E_1\right)^T (\eta - \beta_\pm(\pm 1))^{-1}\left(h_\pm(\pm 1)^{-1}E_1\right), \tag{2.20}$$

where $E_1 = (1,0,\ldots,0)^T$. Note that (α,β) is the special case of $(\alpha_\pm(\zeta),\beta_\pm(\zeta))$, with $\zeta = 0$. If $(\alpha,\beta) \in N_{k,l}$ is generic, then the above construction also go through for $(\alpha_\pm(\zeta), \beta_\pm(\zeta))$ in the same manner; we shall proceed by first rephrasing it in a more elementary way.

Recall that α_+, β_+ have simple pole at $t = 0$, let us denote their residue by a, b. The triple $\{a, b, b^*\}$ defines an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$. For any fixed ζ, $\alpha_+(\zeta), \beta_+(\zeta)$ also have simple pole at $t = 0$, whose residue are denoted by $a(\zeta), b(\zeta)$. Let $e = E_{k+1}$. Observe that e is a lowest weight vector of a, i.e. $ae = -\frac{1}{2}(l-k-1)e$. In fact, e also belongs to the $-\frac{1}{2}(l-k-1)$-eigenspace of $a(\zeta)$:

$$\begin{aligned} a(\zeta)e &= (a - b^*\zeta)e \\ &= ae - \zeta b^* e \\ &= -\frac{1}{2}(l-k-1)e, \end{aligned}$$

where the last equality holds since $b^*e = 0$. Let us consider the equation

$$\frac{dw}{dt} + \alpha_+(\zeta)w = 0. \tag{2.21}$$

It has a unique solution $w(\zeta, t)$ satisfying

$$w(\zeta, t)t^{-(l-k-1)/2} \to e \tag{2.22}$$

as $t \to 0$. Existence of the gauge transformation h above implies that the same equation has the unique solution $u_+(\zeta, t)$ with

$$P_-(\beta_+(\zeta))u_+(\zeta) = w(\zeta), \tag{2.23}$$

where $P_-(z) = \det(z - \beta_-(\zeta))$. Then the rational map $R_+(\eta)$ is given by

$$u_+(\zeta, 1)^T(\eta - \beta_+(\zeta, 1))^{-1}u_+(\zeta, 1). \tag{2.24}$$

Let $v_+(\zeta) = \lim_{t\to 0} u_+(\zeta, t)$. The limit is a column vector in $\mathbb{C}^l$ with vanishing last $(l-k)$-entries, so that it is of the form $(v_-(\zeta)^T, 0, \ldots, 0)^T$. Let $u_-(\zeta, t)$ be the unique solution to

$$\frac{dw}{dt} + \alpha_-(\zeta)w = 0 \tag{2.25}$$

with $u_-(\zeta, 0) = v_-(\zeta)$. Then the rational map $R_-(\eta)$ is given by

$$u_-(\zeta, -1)^T(\eta - \beta_-(\zeta, -1))^{-1}u_-(\zeta, -1). \tag{2.26}$$

As $\alpha_\pm(\zeta), \beta_\pm(\zeta)$ are constructed over U, there are analogues on $\tilde{U}$ given by

$$\begin{aligned} \tilde{\beta}_\pm(\tilde{\zeta}) &= \beta_\pm^* - (\alpha_\pm + \alpha_\pm^*)\tilde{\zeta} - \beta_\pm\tilde{\zeta}^2 \\ \tilde{\alpha}_\pm(\tilde{\zeta}) &= -\alpha_\pm^* - \beta_\pm\tilde{\zeta}. \end{aligned} \tag{2.27}$$

One may check that $\tilde{\alpha}(\tilde{\zeta}), \tilde{\beta}(\tilde{\zeta})$ also satisfy the complex equation. Moreover, they are related to $\alpha(\zeta), \beta(\zeta)$ by

$$\begin{aligned}\tilde{\beta}_{\pm}(\tilde{\zeta}) &= -\beta_{\pm}(\zeta)/\zeta^2 \\ \tilde{\alpha}_{\pm}(\tilde{\zeta}) &= \alpha_{\pm}(\zeta) - \beta_{\pm}(\zeta)/\zeta.\end{aligned} \tag{2.28}$$

Let $\tilde{a}(\tilde{\zeta}), \tilde{b}(\tilde{\zeta})$ be the residue of $\tilde{\alpha}_{+}(\tilde{\zeta}), \tilde{\beta}_{+}(\tilde{\zeta})$ at $t = 0$, and

$$\tilde{e} = \frac{1}{(l-k-1)!} b^{l-k-1} e. \tag{2.29}$$

Since $\tilde{e}$ is an eigenvector of a with eigenvalue $(l-k-1)/2$, we have

$$\begin{aligned}\tilde{a}(\tilde{\zeta})\tilde{e} &= (-a^* - b\tilde{\zeta})\tilde{e} \\ &= -ae - \tilde{\zeta} b\tilde{e} \\ &= -\frac{1}{2}(l-k-1)\tilde{e},\end{aligned}$$

i.e. $\tilde{e}$ is in the $-(l-k-1)/2$-eigenspace of $\tilde{a}(\tilde{\zeta})$. There is a unique solution $\tilde{w}(\tilde{\zeta}, t)$ to the equation

$$\frac{dw}{dt} + \tilde{\alpha}_{+}(\tilde{\zeta})w = 0, \tag{2.30}$$

such that $\tilde{w}(\tilde{\zeta}, t)t^{-(l-k-1)/2} \to \tilde{e}$ as $t \to 0$. Let $\tilde{u}_{+}(\tilde{\zeta}, t)$ be the unique solution to the same equation such that

$$\tilde{p}_{-}(\tilde{\beta}_{+}(\tilde{\zeta}))\tilde{u}_{+}(\tilde{\zeta}) = \tilde{w}(\tilde{\zeta}), \tag{2.31}$$

where $\tilde{p}_{-}(z) = \det(z - \tilde{\beta}_{-}(\tilde{\zeta}))$. Let

$$\tilde{v}_{+}(\tilde{\zeta}) = \lim_{t \to 0} \tilde{u}_{+}(\tilde{\zeta}, t) = (\tilde{v}_{-}(\zeta)^T, 0, \ldots, 0)^T, \tag{2.32}$$

and $\tilde{u}_{-}(\tilde{\zeta}, t)$ be the unique solution to

$$\frac{du}{dt} + \tilde{\alpha}_{-}(\tilde{\zeta})u = 0 \tag{2.33}$$

satisfying $\tilde{u}(\tilde{\zeta}, 0) = \tilde{v}_{-}(\tilde{\zeta})$. The corresponding rational maps

$$(\tilde{R}_{-}, \tilde{R}_{+}) \in Rat_k(\mathbb{P}^1) \times Rat_l(\mathbb{P}^1)$$

are defined by

$$\tilde{R}_{\pm}(\tilde{\eta}) = \tilde{u}_{\pm}(\tilde{\zeta}, \pm 1)^T(\tilde{\eta} - \tilde{\beta}_{\pm}(\tilde{\zeta}, \pm 1))^{-1}\tilde{u}_{\pm}(\tilde{\zeta}, \pm 1). \tag{2.34}$$

Now let ζ to vary. We shall prove the following result:

Proposition 2.21. *Suppose $(\alpha, \beta) \in N_{k,l}$ is a generic element. Then the* *functions $f^+, \tilde{f}^+$ given by*

$$
\begin{aligned}
f^+(\zeta,\eta) &= u_+(\zeta,1)^T(\eta - \beta_+(\zeta,1))_{\mathrm{adj}} u_+(\zeta,1),\\
\tilde{f}^+(\tilde{\zeta},\tilde{\eta}) &= \tilde{u}_+(\tilde{\zeta},1)^T(\tilde{\eta} - \tilde{\beta}_+(\tilde{\zeta},1))_{\mathrm{adj}} \tilde{u}_+(\tilde{\zeta},1),
\end{aligned}
$$

define a section on $L^2(-2k)$ over S^+.

Proof. We shall first establish the relationship between $w(\zeta,t)$ and $\tilde{w}(\tilde{\zeta},t)$:

Lemma 2.22. *Over $U \cap \tilde{U}$, we have*

$$
\tilde{w}(\tilde{\zeta},t) = \exp(t\beta_+(\zeta,t)/\zeta)\zeta^{(l-k-1)}w(\zeta,t).
$$

Proof of Lemma (2.22). Let us first observe the following:

$$
[a(\zeta), b(\zeta)^n] = nb(\zeta)^n,
$$

and

$$
\begin{aligned}
[a, \exp(b/\zeta)] &= \sum_{n=0}^{\infty} \frac{1}{n!\zeta^n}[a, b^n]\\
&= \frac{b}{\zeta}\sum_{n=1}^{\infty} \frac{(b/\zeta)^{n-1}}{(n-1)!}\\
&= (b/\zeta)\exp(b/\zeta).
\end{aligned}
$$

From these we deduce that

$$
\tilde{a}(\tilde{\zeta})\exp(b(\zeta)/\zeta)e = -\frac{(k+l-1)}{2}\exp(b(\zeta)/\zeta)e, \tag{2.35}
$$

i.e. $\tilde{e}$ must be proportional to $\exp(b(\zeta)/\zeta)e$. Note that $b(\zeta)^{l-k} = 0$, thus

$$
\begin{aligned}
\exp(b(\zeta)/\zeta)e &= \sum_{n=0}^{l-k-1} \frac{\zeta^{-n}}{n!}b(\zeta)^n e\\
&= \frac{\zeta^{-(l-k-1)}}{(l-k-1)!}b^{l-k-1}e\\
&= \zeta^{-(l-k-1)}\tilde{e}.
\end{aligned}
$$

This means the quantity

$$
\exp(t\beta_+(\zeta,t)/\zeta)\zeta^{(l-k-1)}w(\zeta,t)
$$

satisfies the same boundary condition as $\tilde{w}(\tilde{\zeta},t)$. Moreover, as a consequence of the identity

$$
[\alpha_+(\zeta), \beta_+(\zeta)^n] = -\frac{d}{dt}\left(\beta_+(\zeta)^n\right), \tag{2.36}
$$

one obtains

$$\left(\frac{d}{dt}+\alpha_+(\zeta,t)\right)\exp(t\beta_+(\zeta,t)/\zeta)\zeta^{(l-k-1)}w(\zeta,t)$$
$$=(\beta_+(\zeta,t)/\zeta)\exp(t\beta_+(\zeta,t)/\zeta)\zeta^{(l-k-1)}w(\zeta,t).$$

Thus

$$\left(\frac{d}{dt}+\tilde{\alpha}_+(\tilde{\zeta},t)\right)\exp(t\beta_+(\zeta,t)/\zeta)\zeta^{(l-k-1)}w(\zeta,t)=0.$$

As $\tilde{w}(\tilde{\zeta},t)$ satisfies the same equation, the result follows from uniqueness. □

Recall that

$$\begin{aligned}\tilde{w}(\tilde{\zeta})&=\tilde{P}_-(\tilde{\beta}_+(\tilde{\zeta}))\tilde{u}_+(\tilde{\zeta}),\\ w(\zeta)&=P_-(\beta_+(\zeta))u_+(\zeta).\end{aligned}$$

Since $\tilde{P}_-(\tilde{\eta})=\zeta^{-2k}P_-(\eta)$, we obtain

$$P_-(\beta_+(\zeta))\xi(\zeta,\cdot)=0,$$

where $\xi(\zeta,t)=\tilde{u}_+(\tilde{\zeta})-\zeta^{(l+k-1)}\exp(tb(\zeta)/\zeta)u_+(\zeta)$. As it is the unique solution to (2.21) satisfying $t^{-(l-k-1)}P_-(\beta_+(\zeta))\xi(\zeta,t)\to 0$, hence by uniqueness, ξ must be zero. Therefore

$$\tilde{u}_+(\tilde{\zeta},t)=\zeta^{(l+k-1)}\exp(t\beta_+(\zeta,t)/\zeta)u_+(\zeta,t). \tag{2.37}$$

Let us consider $\tilde{f}^+$:

$$\begin{aligned}\tilde{f}^+(\tilde{\zeta},\tilde{\eta})&=\tilde{u}_+(\tilde{\zeta},1)^T(\tilde{\eta}-\tilde{\beta}_+(\tilde{\zeta},1))_{\mathrm{adj}}u_+(\tilde{\zeta},1)\\ &=\zeta^{2(l+k-1)}u_+(\zeta,1)^T\exp(\beta_+(\zeta,1)^T/\zeta)\left((\beta_+(\zeta,1)-\eta)/\zeta^2\right)_{\mathrm{adj}}\\ &\qquad\times\exp(\beta_+(\zeta,1)/\zeta)u_+(\zeta,1).\end{aligned}$$

On S^+, we have

$$(\eta-\beta_+(\zeta))(\eta-\beta_+(\zeta))_{\mathrm{adj}}=\det(\eta-\beta_+(\zeta))=0. \tag{2.38}$$

Furthermore, bearing in mind that $\beta_+(\zeta)$ is symmetric at $t=1$,

$$\begin{aligned}\tilde{f}^+(\tilde{\zeta},\tilde{\eta})&=\zeta^{2(l+k-1)}u_+(\zeta,1)^T\exp(2\eta/\zeta)\left((\beta_+(\zeta,1)-\eta)/\zeta^2\right)_{\mathrm{adj}}u_+(\zeta,1)\\ &=\zeta^{2k}\exp(2\eta/\zeta)u_+(\zeta,1)^T\left(\eta-\beta_+(\zeta,1)\right)_{\mathrm{adj}}u_+(\zeta,1)\\ &=\zeta^{2k}e^{2\eta/\zeta}f^+(\zeta,\eta).\end{aligned}$$

Therefore $\tilde{f}^+,f^+$ define a section of $L^2(-2k)$ over S^+. □

Proposition 2.23. *Suppose $(\alpha,\beta)\in N_{k,l}$ is a generic element. Then the* *functions* $f^-, \tilde{f}^-$ *given by*

$$\begin{aligned} f^-(\zeta,\eta) &= u_-(\zeta,-1)^T(\eta-\beta_-(\zeta,-1))_{\mathrm{adj}}u_-(\zeta,-1),\\ \tilde{f}^-(\tilde{\zeta},\tilde{\eta}) &= \tilde{u}_-(\tilde{\zeta},-1)^T(\tilde{\eta}-\tilde{\beta}_-(\tilde{\zeta},-1))_{\mathrm{adj}}\tilde{u}_-(\tilde{\zeta},-1), \end{aligned}$$

define a section on $L^{-2}(-2l)$ *over* S^-*.*

Proof. From (2.37), one gets $\tilde{v}_+(\tilde{\zeta}) = \zeta^{(l+k-1)}\exp(b(\zeta)/\zeta)v_+(\zeta)$ by letting $t \to 0$. Hence

$$\tilde{v}_-(\tilde{\zeta}) = \zeta^{(l+k-1)}v_-(\zeta).$$

If $u_-(\zeta,t), \tilde{u}_-(\tilde{\zeta},t)$ are the unique solutions such that

$$\begin{aligned} u_-(\zeta,t) &\to v_-(\zeta),\\ \tilde{u}_-(\tilde{\zeta},t) &\to \tilde{v}_-(\tilde{\zeta}), \end{aligned}$$

they must then satisfy

$$\tilde{u}_-(\tilde{\zeta},t) = C\zeta^{(l+k-1)}\exp(t\beta_-(\zeta,t)/\zeta)u_-(\zeta,t).$$

Following the same line of argument as before yields

$$\tilde{f}^-(\tilde{\zeta}) = \zeta^{2l}e^{-2\eta/\zeta}f^-(\zeta),$$

i.e. they define a section of $L^{-2}(-2l)$ over S^-. □

Proposition 2.24. *Let* $(\alpha,\beta) \in N_{k,l}$ *be a generic element. Suppose* $(\alpha(\zeta),\beta(\zeta))$ *satisfies condition* (*a*) *in Lemma* (2.11) *at* $(\zeta_0,\eta_0) \in \mathbb{C}^2$*. If the multiplicity of the eigenvalue* η_0 *of* $\beta_-(\zeta_0)$ *and* $\beta_+(\zeta_0)$ *are respectively* m_- *and* m_+*, then* $u_-(\zeta)$ *and* $u_+(\zeta)$ *have a pole at* ζ_0 *with order at most* m_-m_+*.*

Proof. Suppose $\beta_+(\zeta,t)$ is already of the normal form near $t=0$:

$$\left(\begin{array}{c|c} \beta_-(\zeta,0)+O(t) & t^{(l-k-1)/2}G(\zeta) \\ \hline t^{(l-k-1)/2}F(\zeta) & R(\zeta,t) \end{array}\right),$$

where the off-diagonal blocks $F(\zeta), G(\zeta)$ are of the form

$$F(\zeta) = \left(\begin{array}{c} f(\zeta) \\ \hline \\ \end{array}\right), \quad g(\zeta) = \left(\begin{array}{c|c} & g(\zeta) \end{array}\right).$$

We need to expand $P_-(\beta_+(t))$; observe that, for each $0 \le m \le k$, we have

$$\beta_+(\zeta,t)^m = \left(\begin{array}{c|c} \beta_-(\zeta,0)^m + B(\zeta,t)F(\zeta) & \\ \hline t^{(l-k-1)/2}\sum_{j=0}^{m-1}C_j(\zeta,t)F(\zeta)\beta_-(\zeta,0)^j & \end{array}\right), \tag{2.39}$$

where $B(\zeta,\cdot), C_j(\zeta,\cdot)$ are some matrices which possibly have a pole at $t=0$.

Then $P_-(\beta_+(t))$ is equal to

$$\left(\begin{array}{c|c} \dfrac{\tilde{B}(\zeta,t)F(\zeta)}{t^{(l-k-1)/2}\left(F(\zeta)(\beta_-(\zeta,0))_{\mathrm{adj}}+\sum_{j=0}^{k-2}\tilde{C}_j(\zeta,t)F(\zeta)\beta_-(\zeta,0)^j\right)} & \end{array}\right).$$

Suppose u_+ is the solution to the differential equation

$$\frac{dw}{dt}+\alpha_+(\zeta)w=0 \tag{2.40}$$

with

$$t^{-(l-k-1)/2}P_-(\beta_+(\zeta))u_+(\zeta)\to E_{k+1}.$$

Let us first suppose that $\beta_-(\zeta_0,0)$ has distinct eigenvalues, so that it has k linearly independent eigenvectors $v_i(\zeta_0)$. Since $\beta_-(\zeta,0)$ depends on ζ holomorphically, there exists a neighbourhood $N(\zeta_0)$ of ζ_0 such that it also has k linearly independent eigenvectors $v_i(\zeta)$ over $N(\zeta_0)$. These eigenvectors can be chosen so that they vary smoothly in ζ. In fact, they can be chosen to be holomorphic: we shall show that there are smooth functions f_j so that f_jv_j are holomorphic.

First observe that, if $v_j(\zeta)$ is an eigenvector of $\beta_-(\zeta,0)$ with eigenvalue $\eta_j(\zeta)$, then $\partial_{\bar{\zeta}}v_j$ is also an eigenvector of $\beta_-(\zeta,0)$ with the same eigenvalue. Since eigenvalues are distinct, $\partial_{\bar{\zeta}}v_j$ must be proportional to v_j and there is a smooth function g_j on a neighbourhood of ζ such that $\partial_{\bar{\zeta}}v_j=g_jv_j$. Then

$$\frac{\partial(f_jv_j)}{\partial\bar{\zeta}}=\frac{\partial f_j}{\partial\bar{\zeta}}v_j+f_j\frac{\partial v_j}{\partial\bar{\zeta}}=\left(\frac{\partial f_j}{\partial\bar{\zeta}}+f_jg_j\right)v_j=0.$$

As v_j is non-zero, this implies

$$\frac{\partial h_j}{\partial\bar{\zeta}}=-g_j,$$

where $f_j=\exp h_j$. By the $\bar{\partial}$-Poincaré lemma, on a sufficiently small neighbourhood $N_j(\zeta_0)$, there exists a solution h_j to this equation. Thus for each j, f_jv_j is holomorphic on some $N_j(\zeta_0)$.

Let $N(\zeta_0)$ be contained in each $N_j(\zeta_0)$ and satisfy $N(\zeta_0)\cap\pi(\tau(D))=\{\zeta_0\}$. For any fixed $\zeta\in N(\zeta_0)\setminus\{\zeta_0\}$, since $u_+(\zeta,0)=(v_-(\zeta)^T,0,\ldots,0)^T$, we can write

$$v_-(\zeta)=\sum_{j=1}^{k}a_j(\zeta)v_j(\zeta) \tag{2.41}$$

for some $a_j(\zeta)\in\mathbb{C}$, $j=1,\ldots,k$. Let $\hat{v}_j(\zeta,t)$ be the solution to (2.40) with the boundary condition $\hat{v}_j(\zeta,0)=(v_j(\zeta)^T,0,\ldots,0)^T$. Note that these solutions are constant near $t = 0$; uniqueness implies

$$u_+(\zeta, t) = \sum_{j=1}^{k} a_j(\zeta)\hat{v}_j(\zeta, 0)$$

for all t sufficiently close to zero.

Consider $t^{-(l-k-1)/2}P_-(\beta_+(\zeta))\hat{v}_i(\zeta)$ for t sufficiently small: it is equal to

$$\left(\frac{t^{-(l-k-1)/2}\tilde{B}(\zeta,t)F(\zeta)v_i(\zeta)}{\left(\prod_{j\neq i}\eta_j(\zeta)\right)F(\zeta)v_i(\zeta) + \sum_{j=0}^{k-2}\tilde{C}_j(\zeta,t)\eta_i(\zeta)^j F(\zeta)v_i(\zeta)} \right).$$

Thus $t^{-(l-k-1)/2}P_-(\beta_+(\zeta,t))u_+(\zeta,t)$ is given by

$$\left(\frac{t^{-(l-k-1)/2}\tilde{B}(\zeta,t)\sum_{i=1}^{k} a_i(\zeta)F(\zeta)v_i(\zeta)}{\sum_{i=1}^{k} a_i(\zeta)\left(\prod_{j\neq i}\eta_j\right)F(\zeta)v_i(\zeta) + \sum_{j=0}^{k-2}\tilde{C}_j(\zeta,t)\sum_{i=1}^{k} a_i(\zeta)\eta_i^j F(\zeta)v_i(\zeta)} \right).$$

We need to find $a_i(\zeta)$ so that the above has limit E_{k+1}; observe that the $(k+1)$-th entry is due to the first summand of the lower block alone, hence we must solve for $a_i(\zeta)$ in the following system of equations:

$$\begin{pmatrix} (fv_1)(\zeta) & \cdots & (fv_k)(\zeta) \\ \eta_1(\zeta)(fv_1)(\zeta) & \cdots & \eta_k(\zeta)(fv_k)(\zeta) \\ \vdots & & \\ \eta_1(\zeta)^{k-2}(fv_1)(\zeta) & \cdots & \eta_k(\zeta)^{k-2}(fv_k)(\zeta) \\ \left(\prod_{j\neq 1}\eta_1(\zeta)\right)(fv_1)(\zeta) & \cdots & \left(\prod_{j\neq k}\eta_k(\zeta)\right)(fv_k)(\zeta) \end{pmatrix} \begin{pmatrix} a_1(\zeta) \\ a_2(\zeta) \\ \vdots \\ a_k(\zeta) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Since $\eta_i(\zeta)$ are distinct, the system has a unique solution given by

$$a_i(\zeta) = \frac{(-1)^k}{(fv_i)(\zeta)\prod_{j\neq i}(\eta_i(\zeta) - \eta_j(\zeta))}, \quad i = 1, \ldots, k. \tag{2.42}$$

By assumption, $(fv_i)(\zeta) \to 0$ as $\zeta \to \zeta_0$ for some i, hence we see that $u_+(\zeta)$ has a pole at ζ_0. It remains to show that this pole is at most a simple pole. By the implicit function theorem, the graph $(\zeta, \eta_i(\zeta))$ locally parametrizes S^- in a sufficiently small neighbourhood of ζ_0. Then

$$\begin{aligned} \det(\eta_i(\zeta) - \beta_+(\zeta)) &= (1, 0, \ldots, 0)F(\zeta)(\eta_i(\zeta) - \beta_-(\zeta, 0))_{\mathrm{adj}}G(\zeta)(0, \ldots, 0, 1)^T \\ &= f(\zeta)(\eta_i(\zeta) - \beta_-(\zeta, 0))_{\mathrm{adj}}G(0, \ldots, 0, 1)^T \\ &= (f(\zeta)v_i(\zeta))\left(w(\zeta)^T G(\zeta)(0, \ldots, 0, 1)^T\right). \end{aligned}$$

Since the left-hand-side vanishes to order 1 as $\zeta \to \zeta_0$, $f(\zeta)v_i(\zeta)$ cannot vanish to order greater than 1, therefore we have the result. As $u_-(\zeta)$ is determined by $v_-(\zeta)$, one can deduce that $u_-(\zeta)$ also has a simple pole at ζ_0.

In the other extreme, if ζ_0 is a point such that all eigenvalues of $\beta_-(\zeta_0, 0)$

coincide and equal to $\eta(\zeta_0)$. Since $\beta_-(\zeta_0, 0)$ is regular, there exists k linearly independent vectors $v_i(\zeta_0) \in \mathbb{C}^k \setminus \{0\}$ such that

$$\begin{aligned} (\beta_-(\zeta_0, 0) - \eta(\zeta_0))v_1(\zeta_0) &= 0, \\ (\beta_-(\zeta_0, 0) - \eta(\zeta_0))v_i(\zeta_0) &= v_{i-1}(\zeta_0), \quad i = 2, \dots, k, \end{aligned} \tag{2.43}$$

i.e. v_1 is an eigenvector and v_i, $i = 2, \dots, k$, are generalized eigenvectors of $\beta_-(\zeta_0, 0)$. For ζ sufficiently close to ζ_0, $\beta_-(\zeta, 0)$ has eigenvalues $\eta_i(\zeta)$ with vectors $v_i(\zeta) \in \mathbb{C}^k \setminus \{0\}$ satisfying

$$\begin{aligned} (\beta_-(\zeta, 0) - \eta_1(\zeta))v_1(\zeta) &= 0, \\ (\beta_-(\zeta, 0) - \eta_i(\zeta))v_i(\zeta) &= v_{i-1}(\zeta), \quad i = 2, \dots, k, \end{aligned} \tag{2.44}$$

where $\eta_1(\zeta_0) = \eta(\zeta_0)$. As in the previous case, $v_i(\zeta)$ may be chosen so that they vary holomorphically with respect to ζ. Moreover, let $v_-(\zeta)$, $\hat{v}(\zeta)$ be defined as before. We need to consider

$$t^{-(l-k-1)/2} P_-(\beta_+(\zeta))u_+(\zeta)$$

for t sufficiently small: this is equivalent to solving the equations

$$\left(\begin{pmatrix} (fv_1)(\zeta) & (fv_2)(\zeta) & \cdots & (fv_k)(\zeta) \\ 0 & (fv_1)(\zeta) & \cdots & (fv_{k-1})(\zeta) \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & (fv_1)(\zeta) \end{pmatrix} + B(\zeta) \right) \begin{pmatrix} a_1(\zeta) \\ a_2(\zeta) \\ \vdots \\ a_k(\zeta) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{2.45}$$

where $B(\zeta)$ is some singular matrix with the top row zero. The $a_i(\zeta)$ are found to be of the form

$$a_i(\zeta) = \frac{c_i(\zeta)}{(fv_1)(\zeta)^i}, \quad i = 1, \dots, k, \tag{2.46}$$

where $c_i(\zeta)$ are constants which depend on the eigenvalues $\eta_j(\zeta)$ and $(fv_j)(\zeta)$ algebraically. Since v_1 is the eigenvector of $\beta_-(\zeta, 0)$, by assumption, (fv_1) has a zero at ζ_0. Suppose the order of such zero is m, then we see that $u_+(\zeta)$ has a pole of order at most km. We shall show that m is at most equal to the multiplicity of the eigenvalue $\eta_1(\zeta_0)$ of $\beta_+(\zeta_0)$. Indeed, as

$$\det(\eta_1(\zeta) - \beta_+(\zeta)) = (f(\zeta)v_1(\zeta)) \left(w(\zeta)^T G(\zeta)(0, \dots, 0, 1)^T \right),$$

we see that the left-hand-side vanishes to order of the multiplicity of the eigenvalue $\eta_1(\zeta_0)$ of $\beta_+(\zeta_0)$, m say, as $\zeta \to \zeta_0$. Hence $f(\zeta)v_1(\zeta)$ vanishes to order at most m.

For the general case, suppose $\beta_-(\zeta, 0)$ has eigenvalues $\eta_i(\zeta_0)$ with algebraic multiplicities k_i, $i = 1, \dots, n$, where $\sum_{i=1}^n k_i = k$. In a neighbourhood of ζ_0, for each i we can find linearly independent vectors $v_{ij}(\zeta)$ satisfying the analogous condition to (2.44). Then we need to solve for the coefficients b_{ij} in

$$v_-(\zeta) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} b_{ij}(\zeta) v_{ij}(\zeta),$$

so that $t^{-(l-k-1)/2} P_-(\beta_+(\zeta)) u_+(\zeta) \to E_{k+1}$. As in the previous two cases, this is equivalent to solving a system of equations. Moreover, the solution can be given in terms of the solution obtained before: if $a_{ij}(\zeta)$ is the solution to (2.45) with k, v_j and a_j replaced by k_i, $v_{ij}(\zeta)$ and $a_{ij}(\zeta)$ respectively, then for each $i = 1, \ldots, n$, we have

$$b_{ij}(\zeta) = \frac{(-1)^n a_{ij}(\zeta)}{\prod_{l \neq i}(\eta_i(\zeta) - \eta_l(\zeta))}, \quad j = 1, \ldots, k_i. \tag{2.47}$$

We deduce as in the previous case that $v_-(\zeta)$ has a pole of order $k_i m$, where m is the multiplicity of the eigenvalue $\eta_i(\zeta_0)$ of $\beta_+(\zeta)$. □

From Proposition (2.23) we know that $f^-, \tilde{f}^-$ define a section of $L^{-2}(-2l)$ over S^-, then the above proposition implies that such section is meromorphic with poles occurring only along $\tau(D)$, and have 2 times the multiplicity. Let us denote the principal divisor of such section by Δ. Since $L^{-2}(-2l) \simeq [\Delta]$, by considering the degree of the bundles, Δ must have degree $-2kl$, implying that the section cannot have any zero. Hence we write $\Delta = -2\tau(D)$. By the same argument, $f^+, \tilde{f}^+$ define a meromorphic section of $L^2(-2k)$ over S^+, with its principal divisor also written as $-2\tau(D)$ so that $L^2(-2k) \simeq [-2\tau(D)]$. Note that since S^+ (resp. S^-) is defined by a holomorphic section of $\pi^*\mathcal{O}(2l)$ (resp. $\pi^*\mathcal{O}(2k)$), it has zeros along $D \cup \tau(D)$ over S^- (resp. S^+), hence on either curve, we denote the principal divisor of such section by $D + \tau(D)$. Let

$$\begin{aligned} \kappa^\pm &= P_\pm f^\pm, \\ \tilde{\kappa}^\pm &= \tilde{P}_\pm \tilde{f}^\pm, \end{aligned} \tag{2.48}$$

then $\kappa^+, \tilde{\kappa}^+$ (resp. $\kappa^-, \tilde{\kappa}^-$) give rise to a section ϱ^+ (resp. ϱ^+) of the line bundle $L^2[\tau(D) - D]\big|_{S^+}$ (resp. $L^{-2}[\tau(D) - D]\big|_{S^-}$). Furthermore, if we put

$$\nu^- = \sigma(\varrho^-), \quad \nu^+ = \varrho^+, \tag{2.49}$$

then after rescaling $\nu^\pm$ by some positive real numbers if necessary, we have $(S^-, \nu^-, S^+, \nu^+) \in M_{k,l}$. Note that it was claimed in [7] that $f^\pm, \tilde{f}^\pm$ themselves define meromorphic section of $L^{\pm 2}[\tau(D) - D]\big|_{S^\pm}$, which is not quite correct.

Corollary 2.25. *Given $a \in \mathbb{P}^1$, let (U, ζ) be an affine chart about a, so that*

$\zeta(a) = 0$. *For any*

$$\left(\frac{p_-(z)}{q_-(z)}, \frac{p_+(z)}{q_+(z)}\right) \in Rat_k(\mathbb{P}^1) \times Rat_l(\mathbb{P}^1), \tag{2.50}$$

under the correspondence given in Corollary (2.20)*, the corresponding monopole-cluster* $(S^-, \nu^-, S^+, \nu^+) \in M_{k,l}$ *characterizes the rational maps in the following way:*

$$\begin{aligned} q_\pm(z) &= P_\pm(0, z), \\ p_+(z) &= f^+(0, z)/P_-(0, z) \pmod{q_+(z)}, \\ p_-(z) &= \overline{(\tau^* \tilde{f}^-)}(0, z)/P_+(0, z) \pmod{q_-(z)}, \end{aligned}$$

where the vanishing of $P_\pm(\zeta, \eta)$ *represent* $S^\pm$ *over* $\pi^{-1}(U)$*, and* $f^\pm$ *(resp.* $\tilde{f}^\pm$*) are the holomorphic functions that represent* $\nu^\pm$ *over* $S^\pm \cap \pi^{-1}(U)$ *(resp.* $S^\pm \cap \pi^{-1}(\tilde{U})$*).*

2.5 Hyperkähler Structure of $M_{k,l}$

We shall only outline the principle behind the construction of the hyperkähler structure on $M_{k,l}$, the details of which are given in [7]. As mentioned before, $N_{k,l}$ has a natural hyperkähler structure, hence admitting a twistor space $Z(N_{k,l})$ with $p : Z(N_{k,l}) \to \mathbb{P}^1$ its holomorphic projection. The twistor space has a natural real structure τ lifting the antipodal map on $\mathbb{P}^1$. Each quadruple $(S^-, \varrho^-, S^+, \varrho^+)$, where $(S^-, S^+) \in \Sigma_{k,l}$ and $\varrho^\pm$ are section of $L^{\pm 2}[\tau(D) - D]$, is a real section of this twistor space, and its normal bundle is isomorphic to $p^*\mathcal{O}(1)^{\oplus 2(k+l)}$. There is a natural holomorphic symplectic form on $Z(N_{k,l})$: a holomorphic section of $\Lambda^2 T_F \otimes p^*\mathcal{O}(2)$, where $T_F = \ker(dp : TZ(N_{k,l}) \to T\mathbb{P}^1)$. According to [7], there is an involutive fibre map

$$T : Z(N_{k,l}) \to Z(N_{k,l}) \tag{2.51}$$

which gives rise to the new real structure $\tau' = T \circ \tau \circ T$. Then $M_{k,l}$ becomes the parameter space of real sections of the twistor space with respect to τ'. Changing an appropriate parameter of the holomorphic symplectic form, one obtains a new holomorphic symplectic form Ω which is compatible with τ'. More explicitly, let us identify $T\mathbb{P}^1 \simeq \mathcal{O}(2)$ and write

$$\Omega = \omega \otimes \frac{d}{d\zeta} \tag{2.52}$$

over $p^{-1}(U)$, using an affine chart (U, ζ) on $\mathbb{P}^1$. Then ω is locally given by

$$\omega(\zeta) = \left(\sum_{i=1}^{k} \frac{df^-(\zeta, \eta_i^-)}{\nu^-(\zeta, \eta_i^-)} \wedge d\eta_i^- + \sum_{i=1}^{l} \frac{df^+(\zeta, \eta_i^+)}{\nu^+(\zeta, \eta_i^+)} \wedge d\eta_i^+\right), \tag{2.53}$$

where $\eta_i^\pm$ are the roots of $P_\pm(\zeta,\eta)$ and $f^\pm$ are the local representative functions of $\nu^\pm$ over $\pi^{-1}(U)$. One obtains a (pseudo)-hyperkähler metric on $M_{k,l}$ by an application of Theorem (1.27).

Definition 2.26. *The (pseudo)-hyperkähler metric on $M_{k,l}$ is characterized as follows. For each $a \in \mathbb{P}^1$, let (U,ζ) be an affine chart of $\mathbb{P}^1$ around a, so that $\zeta(a) = 0$. If I is the complex structure that corresponds to a, then the Kähler form ω_I of the metric with respect to I is given by*

$$\omega_I = \frac{d}{d\zeta}\omega\bigg|_{\zeta=0}.$$

2.6 Relations to Monopoles

We shall review some of the most important results about the relationship between monopoles and monopole-clusters, including a description of the asymptotic region of $M_{k,l}$ where the monopole-cluster metric approximates the monopole metric at large separation. What follows is true for any general charge (k,l).

Let d_E be the metric on the fibres of $T\mathbb{P}^1$ induced by the Riemannian round metric of diameter π on $\mathbb{P}^1 \simeq S^2$. Let $S, S' \in |\pi^*\mathcal{O}(2n)|$, then we define a distance on $|\pi^*\mathcal{O}(2n)|$ by

$$d(S,S') = \max_{\zeta\in\mathbb{P}^1}\left\{d_E(S\cap\pi^{-1}(\zeta), S'\cap\pi^{-1}(\zeta))\right\}.$$

Next, let us take a pair of compact, real curves $(S^-,S^+) \in |\pi^*\mathcal{O}(2k)| \times |\pi^*\mathcal{O}(2l)|$, and (U,ζ) is an affine chart of $\mathbb{P}^1$. Recall that S^- may be expressed as the vanishing of

$$a_0\eta^n + a_1\eta^{n-1} + \ldots + a_n,$$

where $a_i \in \pi^*(H^0(\mathbb{P}^1,\mathcal{O}(2i)))$ and $\eta \in H^0(T\mathbb{P}^1,\pi^*\mathcal{O}(2))$ is the tautological section. Then the *centre* c^- of S^- is given by $c^- = a_1$. The centre c^+ of S^+ is defined similarly. Over U, $c^\pm$ can be written as

$$c^\pm(\zeta) = z_\pm + 2x_\pm\zeta - \bar{z}_\pm\zeta^2.$$

We associate the curves $C(S^\pm) \subset T\mathbb{P}^1$ to $S^\pm$: over $\pi^{-1}(U)$, they are given by

$$(\eta + c^\pm(\zeta))^k = 0. \tag{2.54}$$

Let

$$R = R(S^-,S^+) = \sqrt{(x_- - x_+)^2 + |z_- - z_+|^2} \tag{2.55}$$

be the distance between $S^\pm$; a_{12}, a_{21} are the intersection points of $c^\pm$, given over U by

$$\zeta(a_{12}) = \frac{x_- - x_+ + R}{\bar{z}_- - \bar{z}_+}, \quad \zeta(a_{21}) = \frac{x_- - x_+ - R}{\bar{z}_- - \bar{z}_+}. \tag{2.56}$$

Definition 2.27. *For each $K > 0$, the space $\Sigma_{k,l}(K)$ is defined to be the set of elements $(S^-, S^+) \in \Sigma_{k,l}$ satisfying the following conditions:*

(i) $d\left(S^{\pm}, C(S^{\pm})\right) \leq K$.

(ii) *Suppose $d\left(\pi(S^- \cap S^+), \min\{a_{12}, a_{21}\}\right) \leq 1$, then the divisor D can be chosen so that $d(\pi(D), a_{21}) \leq 1$.*

Proposition 2.28. [7] *Let $\{(S_n^-, S_n^+)\}_{n\in\mathbb{N}}$ be a sequence in $\Sigma_{k,l}(K)$. For each $n \in \mathbb{Z}_+$, let $P_{\pm}^n(\zeta, \eta)$ be the defining polynomial of $S_n^{\pm}$ over $\pi^{-1}(U)$, and $c_n^{\pm}$ are the corresponding centre of the curves. Suppose $R_n = R_n(S_n^-, S_n^+)$ tends to infinity as $n \to \infty$, then the sequence of centred curves defined by the equations $P_{\pm}^n(\zeta, \eta - c_n^{\pm}(\zeta)) = 0$ over $\pi^{-1}(U)$, has a convergent subsequence such that its limit is (S_∞^-, S_∞^+), where S_∞^- and S_∞^+ are spectral curves corresponding to monopole of charge k and l respectively.*

The proposition says that, up to framing, there is an asymptotic region where clusters of charge (k, l) converge to a pair of monopoles with corresponding charges. It is known that for each $a \in \mathbb{P}^1$, there is a region $M_{k,l}^a(K) \subset M_{k,l}$ (which shall be defined below) such that the map

$$M_{k,l}^a(K) \hookrightarrow N_{k,l}/\mathcal{G}_0$$

is asymptotically close to being an isometry with rate $O(1/R)$. These facts together suggests that over $M_{k,l}^a(K)$, the monopole-cluster metric is $O(1/R)$-close to the product metric on

$$M_k \times M_l \simeq Rat_k(\mathbb{P}^1) \times Rat_l(\mathbb{P}^1).$$

However, a proof has yet to be given.

Recall that for each $a \in \mathbb{P}^1$, there is a bijection between an open dense subset of $M_{k,l}$ and an open dense subset of $Rat_k(\mathbb{P}^1) \times Rat_l(\mathbb{P}^1)$. For any pair of rational maps, let us write it in the form (2.50). Moreover, let $\beta_1^-, \ldots, \beta_k^-$ (resp. $\beta_1^-, \ldots, \beta_l^-$) be the roots of $q_-(z)$ (resp. $q_+(z)$). Then

Definition 2.29. *Let $a \in \mathbb{P}^1$ and $K > 0$. $M_{k,l}^a(K)$ is defined to be the subset of $M_{k,l}$ such that, under the correspondence given in Corollary (2.20), rational maps satisfy the following conditions:*

(i) $|\beta_i^- - \beta_j^+| \geq 1$ *for all* $1 \leq i \leq k$, $1 \leq j \leq l$.

(ii) $|\beta_i^- - \beta_j^-| \leq 2K$ *for* $1 \leq i, j \leq k$;
$|\beta_i^+ - \beta_j^+| \leq 2K$ *for* $1 \leq i, j \leq l$.

(iii) $\left|\log|p_-(\beta_i^-)| - \log|p_-(\beta_j^-)|\right| \leq 2K$ *for* $1 \leq i, j \leq k$;
$\left|\log|p_+(\beta_i^+)| - \log|p_+(\beta_j^+)|\right| \leq 2K$ *for* $1 \leq i, j \leq l$.

From now on, we shall often identify the spaces in the correspondence given by Corollary (2.20) whenever appropriate. Then, $M^a_{k,l}(K)$ may be viewed as the space of pairs of rational maps such that each of its elements is within a fixed distance from the points

$$\left(\frac{\exp\left(\sum_{i=1}^k \log|p_-(\beta_i^-)|/k\right)}{(z - \sum_{i=1}^k \beta_i^-/k)^k}, \frac{\exp\left(\sum_{i=1}^l \log|p_+(\beta_i^+)|/l\right)}{(z - \sum_{i=1}^l \beta_i^+/l)^l} \right). \tag{2.57}$$

If $\rho : M_{k,l} \to \Sigma_{k,l}$ is the natural projection, then it can be seen that $\rho(M^a_{k,l}(K)) \subset \Sigma_{k,l}(K)$.

For each $a \in \mathbb{P}^1$, there is the map

$$\Phi_a : \left\{ (S^-, \nu^-, S^+, \nu^+) \in M^a_{k,l} \,\middle|\, a \notin \pi(S^- \cap S^+) \right\} \to M_{k+l}$$

given by

$$\left(\frac{p_-(z)}{q_-(z)}, \frac{p_+(z)}{q_+(z)} \right) \mapsto \frac{P(z)}{Q(z)}, \tag{2.58}$$

where $Q(z) = q_-(z)q_+(z)$, and $P(z)$ is the unique polynomial of degree $k+l-1$ such that $P(z) \equiv p_\pm(z) \pmod{q_\pm(z)}$. The condition $a \notin \pi(S^- \cap S^+)$ implies $q_-(z), q_+(z)$ are coprime, ensuring that Φ_a is well-defined. Moreover, the map is holomorphic for the complex structure corresponding to a, and it preserves the corresponding symplectic form. For $m \in M^a_{k,l}$, we let

$$R^a(m) = \min \left\{ |\beta_i^- - \beta_j^+| \,\middle|\, 1 \le i \le k,\, 1 \le j \le l \right\}. \tag{2.59}$$

Also, denote the (pseudo)-hyperkähler metrics on M_{k+l} and $M_{k,l}$ by g_M and g_C respectively. We can now state the main theorem:

Theorem 2.30. *Let $a \in \mathbb{P}^1$. For each $K > 0$, there exists positive constants R_0, α, C such that, over the region where elements $m \in M^a_{k,l}(K)$ satisfy the conditions*

(i) $R^a(m) \ge R_0$

(ii) $d\left(a, \min\{a_{12}, a_{21}\}\right) \ge 1/2$,

we have

$$\|\Phi_a^* g_M - g_C\| \le Ce^{-\alpha R}.$$

As we might expect, g_C may not be everywhere positive-definite over $M_{k,l}$. The theorem tells us that, when the monopole-clusters are sufficiently far apart within the asymptotic region described therein, g_C must become positive-definite and is exponentially close to g_M.

Chapter 3

Isometries of $M_{k,l}$

In this chapter, we shall define actions of $\mathbb{R}^3$, T^2 and $SO(3)$ on both $M_{k,l}$ and $N_{k,l}/\mathcal{G}$. Then, those actions on $M_{k,l}$ are shown to be isometries with respect to the monopole-cluster metric. Lastly, we show that the actions of the two spaces are essentially equivalent.

3.1 Group Actions on $N_{k,l}/\mathcal{G}_0$

Let us begin by considering the following actions on $N_{k,l}/\mathcal{G}$:

(a) $\mathbb{R}^3$-action: for any $(x_1, x_2, x_3) \in \mathbb{R}^3$,

$$\begin{aligned} T_0^{\pm} &\mapsto T_0^{\pm} \\ T_k^{\pm} &\mapsto T_k^{\pm} + ix_j Id, \quad j = 1, 2, 3. \end{aligned} \tag{3.1}$$

(b) T^2-action: for any $(\theta_-, \theta_+) \in (\mathbb{R}/2\pi i\mathbb{Z})^2 \simeq T^2$, let $p_{(\theta_-,\theta_+)}$ be the gauge transformation in $\mathcal{G}$ given by

$$p_{(\theta_-,\theta_+)_{\pm}}(t) = e^{\pm i\theta_{\pm} t} Id. \tag{3.2}$$

Then $p_{(\theta_-,\theta_+)}$ acts on $N_{k,l}/\mathcal{G}_0$ by gauge transformation.

It is easy to see that the above actions preserve all the conditions for Nahm data in $N_{k,l}$. Because the actions commute with $\mathcal{G}_0$, they indeed act on $N_{k,l}/\mathcal{G}$.

There is an $SO(3)$-action on $N_{k,l}/\mathcal{G}_0$, but before we can give the definition, it is necessary to recall some facts. Suppose $l \geq k$. For any $(T^-, T^+) \in N_{k,l}$, the T_j^+ have the following expansion near $t = 0$:

$$\left(\begin{array}{c|c} T_j^-(0) + O(t) & O(t^{(l-k-1)/2}) \\ \hline O(t^{(l-k-1)/2}) & \sigma_j^{(l-k)}/t + O(1) \end{array}\right), \quad j = 1, 2, 3,$$

where $\sigma_j^{(n)}$ are matrices given by (1.15) in Chapter 1. Let $\rho_* : \mathfrak{su}(2) \to \mathfrak{gl}(\mathbb{C}^{l-k})$ be the standard $(l-k)$-dimensional irreducible representation of $\mathfrak{su}(2)$. Since

$SU(2)$ is simply-connected, there is a unique Lie group representation $\rho : SU(2) \to GL(\mathbb{C}^{l-k})$ whose differential is ρ_*. Let $V \in SU(2)$. First note that

$$\rho_*(VXV^{-1}) = \rho(V)\rho_*(X)\rho(V)^{-1}$$

holds for any $X \in \mathfrak{su}(2)$. Moreover, V descends to an element $A = (a_{ij})$ in $SO(3)$ via

$$\mathrm{Ad}(V^{-1})\sigma_i^{(2)} = \sum_{j=1}^{3} a_{ij}\sigma_j^{(2)}, \quad i = 1, 2, 3. \tag{3.3}$$

Then we see that

$$\rho(U)\left(\sum_{j=1}^{3} a_{ij}\sigma_j^{(l-k)}\right)\rho(U)^{-1} = \sigma_i^{(l-k)} \tag{3.4}$$

is true for each i. We are now ready to define the $SO(3)$-action on $N_{k,l}/\mathcal{G}_0$.

(c) $SO(3)$-action: suppose $l \geq k$. For any $A = (a_{ij})$, let $V \in SU(2)$ be an element in the preimage of A under (3.3). Then the action is defined by

$$\begin{aligned} T_0^{\pm} &\mapsto g_{\pm}^V T_0^{\pm}(g_{\pm}^V)^{-1} - \frac{dg_{\pm}^V}{dt}(g_{\pm}^V)^{-1}, \\ T_i^{\pm} &\mapsto g_{\pm}^V\left(\sum_{j=1}^{3} a_{ij}T_j^{\pm}\right)(g_{\pm}^V)^{-1} \quad i = 1, 2, 3, \end{aligned} \tag{3.5}$$

where g^V is a unitary-valued gauge transformation in $\mathcal{G}$ satisfying $g_-^V \equiv Id$, $g_+^V(1) = Id$ and $g_+^V(0) = Id \oplus \rho(V)$.

For $l < k$, the action is defined in an analogous manner, with the role of $\pm$ interchanged.

Note that for any $(a_{ij}) \in SO(3)$, although the map defined by

$$\begin{aligned} T_0^{\pm} &\mapsto T_0^{\pm}, \\ T_i^{\pm} &\mapsto \sum_{j=1}^{3} a_{ij}T_j^{\pm} \quad i = 1, 2, 3, \end{aligned}$$

preserves (T^-, T^+) as solutions to Nahm's equations, it does not in general preserve the boundary conditions of the Nahm data, and this is the reason that the action needs the coupling gauge transformation g^V in (3.5). Note that the $SO(3)$-action does not depend on the choice of V in the preimage; for such action to be well-defined, it remains to check that for each g^V and $g \in \mathcal{G}_0$, there exists a $g' \in \mathcal{G}_0$ such that

$$g'g^V = g^V g.$$

But this is easy since it is not hard to see that $g^V g(g^V)^{-1} \in \mathcal{G}_0$.

3.2 Group Actions on $M_{k,l}$

We shall show that there are actions of $SO(3)$ and $\mathbb{R}^3$ on the monopole-cluster space $M_{k,l}$, which are defined in terms of some lifts of their natural action on $\mathbb{R}^3$. In addition, there is a T^2-action which acts on the fibres of $M_{k,l} \to \Sigma_{k,l}$. As monopole-clusters are comprised of curves and sections, we shall first need to know how these groups act on such data before we can define actions on $M_{k,l}$.

In Chapter 1, we gave a *local* definition for the spaces $\mathbb{P}^1$, $T\mathbb{P}^1$, and the line bundles $\mathcal{O}(n)$, L^t, namely, they are given either by local charts or local trivializations. As the action of $SO(3)$ moves the chart neighbourhoods, the original charts may no longer be valid after the action, hence it is more convenient to have also a *global* definition.

The Riemann sphere $\mathbb{P}^1$ can be viewed as the space of all complex lines in $\mathbb{C}^2$ through the origin. By equipping it with homogeneous coordinates, the elements in $\mathbb{P}^1$ are represented by equivalence classes of the form $[Z_0 : Z_1]$, where Z_0, Z_1 are not both zero. The equivalence relation is given by

$$[Z_0 : Z_1] \sim [\lambda Z_0 : \lambda Z_1], \quad \lambda \in \mathbb{C}^*.$$

$\mathbb{P}^1$ is covered by the charts $(U, \zeta), (\tilde{U}, \tilde{\zeta})$, where

$$U = \mathbb{P}^1 \setminus \{[0:1]\}, \quad \tilde{U} = \mathbb{P}^1 \setminus \{[1:0]\},$$

and

$$\begin{aligned} \zeta : U \to \mathbb{C}, &\quad \zeta([Z_0 : Z_1]) = Z_1/Z_0, \\ \tilde{\zeta} : \tilde{U} \to \mathbb{C}, &\quad \tilde{\zeta}([Z_0 : Z_1]) = Z_0/Z_1. \end{aligned}$$

On the overlap, the coordinate functions are related by $\tilde{\zeta} = 1/\zeta$.

Let $T\mathbb{P}^1$ be the holomorphic tangent bundle of $\mathbb{P}^1$ and $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$ be the natural projection map. As $T\mathbb{P}^1$ is trivialized by the coordinate vector fields $\frac{d}{d\zeta}$, $\frac{d}{d\tilde{\zeta}}$ on the open sets $\pi^{-1}(U)$, $\pi^{-1}(\tilde{U})$, any point $z \in \pi^{-1}(U)$ can be written as

$$\eta(z) \left.\frac{d}{d\zeta}\right|_{\pi(z)},$$

where $\eta : \pi^{-1}(U) \to \mathbb{C}$ is a holomorphic function such that its restriction to each fibre is a complex linear isomorphism. In other words, $T\mathbb{P}^1$ can be given coordinates (ζ, η) over $\pi^{-1}(U)$. Similarly, there are also coordinates $(\tilde{\zeta}, \tilde{\eta})$ on $\pi^{-1}(\tilde{U})$ such that, they are relate to (ζ, η) by $(\tilde{\zeta}, \tilde{\eta}) = \big(1/\zeta, -\eta/\zeta^2\big)$ over $\pi^{-1}(U) \cap \pi^{-1}(\tilde{U})$.

We shall now give a global definition to the bundle $\mathcal{O}(n)$. Let $\mathbb{P}^1 \times \mathbb{C}^2$ be the trivial rank two complex bundle on $\mathbb{P}^1$. The tautological bundle of $\mathbb{P}^1$, $\mathcal{O}(-1)$, is the subbundle of the trivial bundle whose fibre at each point

$[Z_0, Z_1] \in \mathbb{P}^1$ is the complex line

$$\left\{(\alpha Z_0, \alpha Z_1) \in \mathbb{C}^2 \,\middle|\, \alpha \in \mathbb{C}\right\}. \tag{3.6}$$

There are natural local trivializations χ, $\tilde{\chi}$ of $\mathcal{O}(-1)$ given by

$$\begin{aligned} \chi([Z_0, Z_1]) &= ([Z_0, Z_1], (1, Z_1/Z_0)) \quad \text{over } U, \\ \tilde{\chi}([Z_0, Z_1]) &= ([Z_0, Z_1], (Z_0/Z_1, 1)) \quad \text{over } \tilde{U}. \end{aligned} \tag{3.7}$$

Clearly they are holomorphic and are related by

$$\tilde{\chi} = \zeta^{-1} \chi \quad \text{on } U \cap \tilde{U},$$

i.e. the transition function of $\mathcal{O}(-1)$ is given by ζ^{-1}. For any $n \in \mathbb{Z}$, the bundle $\mathcal{O}(n)$ is a tensor power of either $\mathcal{O}(-1)$ or its dual $(\mathcal{O}(-1))^* \simeq \mathcal{O}(1)$. Hence we have also an explicit description of the local holomorphic trivializations of $\mathcal{O}(n)$, namely, by taking a tensor product of χ (resp. $\tilde{\chi}$) or with its dual. Then it is not hard to see that $\mathcal{O}(n)$ has transition function ζ^n.

We shall now describe the line bundle L^t more explicitly. Recall from Chapter 1 that, the space T of oriented geodesics in $\mathbb{R}^3$ is biholomorphic to $T\mathbb{P}^1$. Viewing T as an embedded subspace of $S^2 \times \mathbb{R}^3$ given by (1.35), we define $p : S^2 \times \mathbb{R}^3 \to T$ by

$$p(\mathbf{u}, \mathbf{x}) = (\mathbf{u}, \mathbf{x} - \langle \mathbf{u}, \mathbf{x} \rangle \mathbf{u}). \tag{3.8}$$

Let p_1 (resp. p_2) denote the first (resp. second) projection map of $S^2 \times \mathbb{R}^3$. For any point $z \in T$,

$$\gamma_z := p^{-1}(z) \subset \{p_1(z)\} \times \mathbb{R}^3 \tag{3.9}$$

is viewed as the corresponding oriented line in $\mathbb{R}^3$. For any $t \in \mathbb{R}$, L^t is defined to be the vector bundle on T with fibre

$$L^t_z := \left\{ r \in C^\infty(p_2(\gamma_z), \mathbb{C}) \,\middle|\, \frac{dr}{ds} + tr = 0 \right\}, \tag{3.10}$$

where $r = \langle p_1(z), \mathbf{x} \rangle$, $\mathbf{x}$ is the affine parameter on γ_z. Clearly, any point r which belongs to the fibre L^t_z is of the form $r(s) = ce^{-ts}$ for some constant $c \in \mathbb{C}$, this implies r is determined by c and hence L^t is a complex line bundle. We now show that L^t is in fact a *holomorphic* line bundle.

Consider the function $\hat{l} : S^2 \times \mathbb{R}^3 \to \mathbb{C}$ defined by

$$\hat{l}(\mathbf{u}, \mathbf{x}) = e^{-t\langle \mathbf{u}, \mathbf{x} \rangle}. \tag{3.11}$$

As $\hat{l}|_{\gamma_z}$ defines a solution to the differential equation over $p_2(\gamma_z)$, it belongs to L^t_z. Thus by restricting $\hat{l}$ to each γ_z, it induces a global smooth section l of

L^t: if $i_z : p_2(\gamma_z) \hookrightarrow \gamma_z$ is the natural embedding, then $l(z)$ is given by

$$l(z) := \hat{l} \circ i_z. \tag{3.12}$$

Moreover, the section is non-vanishing, thus l trivializes L^t as a smooth trivial bundle $\mathbb{R}^2 \times TS^2$. From [21], L^t admits a holomorphic structure with local holomorphic trivializations χ, $\tilde{\chi}$ defined by

$$\begin{aligned} \chi(\zeta, \eta) &= \exp\left(\frac{t\eta\bar{\zeta}}{(1+|\zeta|^2)}\right) l := fl \quad \text{over } \pi^{-1}(U), \\ \tilde{\chi}(\tilde{\zeta}, \tilde{\eta}) &= \exp\left(\frac{t\tilde{\eta}\bar{\tilde{\zeta}}}{(1+|\tilde{\zeta}|^2)}\right) l := \tilde{f}l \quad \text{over } \pi^{-1}(\tilde{U}). \end{aligned} \tag{3.13}$$

On the intersection $\pi^{-1}(U) \cap \pi^{-1}(\tilde{U})$, the trivializations are related by

$$\tilde{\chi} = \exp\left(-t\eta/\zeta\right) \chi.$$

3.2.1 $SO(3)$-Actions

We shall show that there are actions of $SO(3)$ on both $\mathcal{O}(n)$ and L^t, and that are both holomorphic. First consider the natural $SO(3)$-action on $\mathbb{P}^1 \simeq S^2$: let

$$g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

be an element in $SU(2)/\pm \simeq SO(3)$, then it acts on $\mathbb{P}^1$ by

$$R_g : \quad [Z_0 : Z_1] \mapsto [-\bar{b}Z_1 + \bar{a}Z_0 : aZ_1 + bZ_0]. \tag{3.14}$$

The open sets U, $\tilde{U}$ get mapped to

$$R_g(U) = \mathbb{P}^1 \setminus \{[-\bar{b} : a]\}, \quad R_g(\tilde{U}) = \mathbb{P}^1 \setminus \{[\bar{a} : b]\},$$

under R_g, which have coordinates ζ_g , $\tilde{\zeta}_g$ defined by

$$\begin{aligned} \zeta_g([Z_0 : Z_1]) &:= \zeta \circ R_{g^{-1}} = \frac{\bar{a}Z_1 - bZ_0}{\bar{b}Z_1 + aZ_0}, \\ \tilde{\zeta}_g([Z_0 : Z_1]) &:= \tilde{\zeta} \circ R_{g^{-1}} = \frac{\bar{b}Z_1 + aZ_0}{\bar{a}Z_1 - bZ_0}. \end{aligned} \tag{3.15}$$

It is not hard to see that

$$\begin{aligned} \zeta_g &= \frac{\bar{a}\zeta - b}{\bar{b}\zeta + a} \qquad \text{on } U \cap R_g(U), \\ \tilde{\zeta}_g &= \frac{a\tilde{\zeta} + \bar{b}}{-b\zeta + \bar{a}} \quad \text{on } \tilde{U} \cap R_g(\tilde{U}). \end{aligned}$$

We now define the $SO(3)$-action on $T\mathbb{P}^1$: for any $g \in SO(3)$, the action $R_g : T\mathbb{P}^1 \to T\mathbb{P}^1$ simply given by its differential:

$$\eta(z) \left.\frac{d}{d\zeta}\right|_{\pi(z)} \mapsto \eta(R_g(z)) \left.\frac{d}{d\zeta_g}\right|_{\pi(R_g(z))}. \tag{3.16}$$

If we identify $T\mathbb{P}^1$ with T, then $R_g : T \to T$ is induced by the natural rotation: viewing $SO(3)$ as the group of rotations on $\mathbb{R}^3$, then R_g is given by

$$(\mathbf{u}, \mathbf{v}) \mapsto (g(\mathbf{u}), g(\mathbf{v})). \tag{3.17}$$

We claim that the $SO(3)$-action on $\mathbb{P}^1$ lifts to a holomorphic action on $\mathcal{O}(n)$; it is sufficient to consider only the case $n = -1$. Any point $r \in \mathcal{O}(-1)$ is of the form

$$([Z_0, Z_1], c(Z_0, Z_1)), \quad c \in \mathbb{C},$$

then the action $S_g : \mathcal{O}(-1) \to \mathcal{O}(-1)$ is defined to be the map which sends r to

$$\left(R_g([Z_0, Z_1]), c(-\bar{b}Z_1 + \bar{a}Z_0, aZ_1 + bZ_0)\right). \tag{3.18}$$

Clearly S_g is a lift of R_g and is holomorphic. Let V be any open subset of $\mathbb{P}^1$, then S_g acts on section ν of $\mathcal{O}(-1)$ by

$$\nu_g = S_g \circ \nu \circ R_{g^{-1}}. \tag{3.19}$$

In particular, we have

$$\chi_g = S_g \circ \chi \circ R_{g^{-1}} = \begin{cases} \Gamma_{00}\chi & \text{on } R_g(U) \cap U \\ \Gamma_{01}\tilde{\chi} & \text{on } R_g(U) \cap \tilde{U} \end{cases},$$

$$\tilde{\chi}_g = S_g \circ \tilde{\chi} \circ R_{g^{-1}} = \begin{cases} \Gamma_{10}\chi & \text{on } R_g(\tilde{U}) \cap U \\ \Gamma_{11}\tilde{\chi} & \text{on } R_g(\tilde{U}) \cap \tilde{U} \end{cases},$$

where χ, $\tilde{\chi}$ are the trivializations of $\mathcal{O}(-1)$ given by (3.7), and

$$\begin{aligned} \Gamma_{00} &= (\bar{b}\zeta + a)^{-1}, \quad \Gamma_{01} = (\bar{b} + a\tilde{\zeta})^{-1}, \\ \Gamma_{10} &= (\bar{a}\zeta - b)^{-1}, \quad \Gamma_{11} = (\bar{a} - b\tilde{\zeta})^{-1}. \end{aligned} \tag{3.20}$$

In other words, the action on χ, $\tilde{\chi}$ is determined by the functions Γ_{ij}. But because such functions are all holomorphic, the map $S_g : \mathcal{O}(-1) \to \mathcal{O}(-1)$ must be holomorphic. We may now deduce that, as $\mathcal{O}(n)$ is a tensor product of $\mathcal{O}(-1)$ or of its dual, there is also a holomorphic action on $\mathcal{O}(n)$, denoted by the same symbol S_g.

We now show that the $SO(3)$-action on $T\mathbb{P}^1$ may be lifted to a holomorphic action on L^t. Recall that $q : L^t \to T\mathbb{P}^1$ is the natural projection map and l is the global smooth trivialization of L^t given in (3.12). For any $g \in SO(3)$, the action $S_g : L^t \to L^t$ is defined by

$$r \mapsto r \circ g^{-1}\big|_{p_2\left(\gamma_{R_g(z)}\right)}, \tag{3.21}$$

where $z = q(r)$. It can be seen that S_g is a lift of $R_g : T\mathbb{P}^1 \to T\mathbb{P}^1$.

Lemma 3.1. *The section l of L^t is $SO(3)$-invariant.*

Proof. For any $g \in SO(3)$, its action on l is given by $l_g := S_g \circ l \circ R_{g^{-1}}$; we wish to show that $l_g = l$. If l_g is evaluated at $z \in T$, we have

$$\begin{aligned}
l_g(z) &= \left(S_g \circ l \circ R_{g^{-1}}\right)(z) \\
&= S_g(l(R_{g^{-1}}(z))) \\
&= l(R_{g^{-1}}(z)) \circ g^{-1}\big|_{p_2(\gamma_z)} \\
&= \left(\hat{l} \circ i_{R_{g^{-1}}(z)}\right) \circ g^{-1}\big|_{p_2(\gamma_z)}.
\end{aligned}$$

Then for any $\mathbf{v} \in p_2(\gamma_z)$,

$$\begin{aligned}
\left(l_g(z)\right)(v) &= \left(\hat{l} \circ i_{R_{g^{-1}}(z)}\right) \circ g^{-1}(\mathbf{v}) \\
&= \hat{l}\left(p_1(R_{g^{-1}}(z)), g^{-1}(\mathbf{v})\right) \\
&= \hat{l}\left(g^{-1}(p_1(z)), g^{-1}(\mathbf{v})\right) \\
&= e^{-t\langle g^{-1}(p_1(z)), g^{-1}(\mathbf{v})\rangle} \\
&= e^{-t\langle p_1(z), \mathbf{v}\rangle} \\
&= \left(\hat{l} \circ i_z\right)(p_1(z), \mathbf{v}) \\
&= (l(z))(\mathbf{v}),
\end{aligned}$$

which implies that $l_g(z) = l(z)$. Since this is true for all $z \in T$, the result follows. □

Recall that there are holomorphic trivializations χ, $\tilde{\chi}$ given by (3.13). Applying the $SO(3)$-action to χ, $\tilde{\chi}$, we see that

$$\chi_g := S_g \circ \chi \circ R_{g^{-1}} = \begin{cases} h_{00}\chi & \text{on } R_g(\pi^{-1}(U)) \cap U \\ h_{01}\tilde{\chi} & \text{on } R_g(\pi^{-1}(U)) \cap \tilde{U} \end{cases},$$

$$\tilde{\chi}_g := S_g \circ \tilde{\chi} \circ R_{g^{-1}} = \begin{cases} h_{10}\chi & \text{on } R_g(\pi^{-1}(\tilde{U})) \cap U \\ h_{11}\tilde{\chi} & \text{on } R_g(\pi^{-1}(\tilde{U})) \cap \tilde{U} \end{cases},$$

where

$$\begin{aligned}
h_{00} &= \left(f \circ R_{g^{-1}}\right)/f, \quad & h_{01} &= \left(f \circ R_{g^{-1}}\right)/\tilde{f}, \\
h_{10} &= \left(\tilde{f} \circ R_{g^{-1}}\right)/f, \quad & h_{11} &= \left(\tilde{f} \circ R_{g^{-1}}\right)/\tilde{f}.
\end{aligned} \tag{3.22}$$

In terms of local coordinates, they are computed to be

$$\begin{aligned} h_{00} &= \exp\left(\frac{-t\bar{b}\eta}{(\bar{b}\zeta + a)}\right), \quad h_{10} = \exp\left(\frac{-t\bar{a}\eta}{(\bar{a}\zeta - b)}\right), \\ h_{01} &= \exp\left(\frac{-ta\tilde{\eta}}{(\bar{b} + a\tilde{\zeta})}\right), \quad h_{11} = \exp\left(\frac{tb\tilde{\eta}}{(\bar{a} - b\tilde{\zeta})}\right). \end{aligned} \tag{3.23}$$

Clearly all the h_{ij} are holomorphic, hence the $SO(3)$-action on L^t is holomorphic.

The $SO(3)$-action on $\mathcal{O}(n)$ lifts to an action on the pullback bundle $\pi^*\mathcal{O}(n)$ over $T\mathbb{P}^1$. In particular, this induces an action on $|\pi^*\mathcal{O}(2k)|$. We shall see in the final section of this chapter that the $SO(3)$-action on $N_{k,l}$ induces the same $SO(3)$-action on the linear system $|\pi^*\mathcal{O}(2k)|$. Granting this, we deduce that the $SO(3)$-action on $\Sigma_{k,l}$ given by

$$(S^-, S^+) \mapsto (R_g(S^-), R_g(S^+))$$

is well-defined. We can now give our $SO(3)$-action on $M_{k,l}$:

Definition 3.2. *For any $g \in SO(3)$, let $R_g : T\mathbb{P}^1 \to T\mathbb{P}^1$ and $S_g : L^2 \to L^2$ be the maps given above. Then the $SO(3)$-action ψ_g on $M_{k,l}$ is defined by*

$$(S^-, \nu^-, S^+, \nu^+) \mapsto (R_g(S^-), \nu_g^-, R_g(S^+), \nu_g^+),$$

where $\nu_g^\pm = S_g \circ \nu^\pm \circ R_{g^{-1}}$.

For ψ_g to be well-defined, it remains to check that the conditions $\nu^\pm\sigma(\nu^\pm) = (-1)^{k+l-1}$ are preserved, where $\sigma : L^2 \to L^{-2}$ is the anti-holomorphic isomorphism that lifts the real structure τ on $T\mathbb{P}^1$. Noting that the $SO(3)$-action on L^t commutes with σ, we have

$$\begin{aligned} \nu_g^\pm\sigma(\nu_g^\pm) &= \left(S_g \circ \nu^\pm \circ R_{g^{-1}}\right)\left(\sigma \circ \left(S_g \circ \nu^\pm \circ R_{g^{-1}}\right) \circ \tau\right) \\ &= \left(S_g \circ \nu^\pm \circ R_{g^{-1}}\right)\left(S_g \circ \left(\sigma \circ \nu^\pm \circ \tau\right) \circ R_{g^{-1}}\right) \\ &= S_g \circ \left(\nu^\pm\sigma(\nu^\pm)\right) \circ R_{g^{-1}} \\ &= \left(\nu^\pm\sigma(\nu^\pm)\right)_g \\ &= (-1)^{k+l-1}. \end{aligned}$$

3.2.2 $\mathbb{R}^3$-Actions

We first show that there is a holomorphic $\mathbb{R}^3$-action on $T\mathbb{P}^1$. For any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, let $T_\mathbf{x} : \mathbb{R}^3 \to \mathbb{R}^3$ be the natural translation $\mathbf{v} \mapsto \mathbf{v} + \mathbf{x}$. The map on $S^2 \times \mathbb{R}^3$, defined by $Id \times T_x$, will also be denoted by $T_\mathbf{x}$. There is an $\mathbb{R}^3$-action on T: for any $\mathbf{x} \in \mathbb{R}^3$, there is the map $R_\mathbf{x} : T \to T$ given by

$$(\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{u}, \mathbf{v} + \mathbf{x} - \langle\mathbf{u}, \mathbf{x}\rangle\mathbf{u}). \tag{3.24}$$

If we identify $T \simeq T\mathbb{P}^1$, the action is given locally by

$$\begin{aligned}
(\zeta, \eta) &\mapsto (\zeta, \eta + \xi(\zeta)) \quad \text{on } \pi^{-1}(U), \\
(\tilde{\zeta}, \tilde{\eta}) &\mapsto (\tilde{\zeta}, \tilde{\eta} + \tilde{\xi}(\tilde{\zeta})) \quad \text{on } \pi^{-1}(\tilde{U}),
\end{aligned}$$

where

$$\begin{aligned}
\xi(\zeta) &:= i[(x_2 + ix_3) + 2ix_1\zeta + (x_2 - ix_3)\zeta^2], \\
\tilde{\xi}(\tilde{\zeta}) &:= -i[(x_2 - ix_3) + 2ix_1\tilde{\zeta} + (x_2 + ix_3)\tilde{\zeta}^2].
\end{aligned} \tag{3.25}$$

Since ξ, $\tilde{\xi}$ are both holomorphic, the $\mathbb{R}^3$-action on $T\mathbb{P}^1$ must be holomorphic.

Next, we define an $\mathbb{R}^3$-action on L^t and show that it is holomorphic. Recall that $q : L^t \to T$ is the projection map. For any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, there is the map $S_{\mathbf{x}} : L^t \to L^t$ defined by

$$r \mapsto r \circ T_{-\mathbf{x}}\big|_{p_2\left(\gamma_{R_g(z)}\right)}. \tag{3.26}$$

It can be seen that $S_{\mathbf{x}}$ is a lift of $R_{\mathbf{x}}$. Let us consider the $\mathbb{R}^3$-action on l, the global smooth trivialization of L^t given in (3.12).

Lemma 3.3. *For any $\mathbf{x} \in \mathbb{R}^3$, the $\mathbb{R}^3$-action of L^t acts on l by*

$$l \mapsto e^{t\langle p_1(\cdot), \mathbf{x}\rangle} l.$$

Proof. First let us write $l_{\mathbf{x}} = S_{\mathbf{x}} \circ l \circ R_{-\mathbf{x}}$. Then for $z \in T$,

$$\begin{aligned}
l_{\mathbf{x}}(z) &= (S_{\mathbf{x}} \circ l \circ R_{-\mathbf{x}})(z) \\
&= S_{\mathbf{x}}(l(R_{-\mathbf{x}}(z))) \\
&= l(R_{-\mathbf{x}}(z)) \circ T_{-\mathbf{x}}\big|_{p_2(\gamma_z)} \\
&= \left(\hat{l} \circ i_{R_{-\mathbf{x}}(z)}\right) \circ T_{-\mathbf{x}}\big|_{p_2(\gamma_z)}.
\end{aligned}$$

If $\mathbf{v} \in p_2(\gamma_z)$, we have

$$\begin{aligned}
(l_{\mathbf{x}}(z))(\mathbf{v}) &= \left(\hat{l} \circ i_{R_{-\mathbf{x}}(z)}\right)(T_{-\mathbf{x}}(\mathbf{v})) \\
&= \hat{l}(p_1(z), T_{-\mathbf{x}}(\mathbf{v})) \\
&= e^{-t\langle p_1(z), T_{-\mathbf{x}}(\mathbf{v})\rangle} \\
&= e^{-t\langle p_1(z), \mathbf{v}-\mathbf{x}\rangle} \\
&= e^{t\langle p_1(z), \mathbf{x}\rangle} e^{-t\langle p_1(z), \mathbf{v}\rangle} \\
&= e^{t\langle p_1(z), \mathbf{x}\rangle} (l(z))(\mathbf{v}).
\end{aligned}$$

Since this is true for any $\mathbf{v} \in p_2(\gamma_z)$ and $z \in T\mathbb{P}^1$, the result follows. $\square$

Recall that there are holomorphic trivializations χ, $\tilde{\chi}$ given by (3.13). Applying the $\mathbb{R}^3$-action to χ, $\tilde{\chi}$ yields

$$\begin{aligned}\chi_{\mathbf{x}} &:= S_{\mathbf{x}} \circ \chi \circ R_{-\mathbf{x}} = h\chi \quad \text{on } \pi^{-1}(U),\\ \tilde{\chi}_{\mathbf{x}} &:= S_{\mathbf{x}} \circ \tilde{\chi} \circ R_{-\mathbf{x}} = \tilde{h}\tilde{\chi} \quad \text{on } \pi^{-1}(\tilde{U}),\end{aligned}$$

where

$$\begin{aligned}h &= e^{t\langle p_1(\,\cdot\,),\mathbf{x}\rangle}(f \circ R_{-\mathbf{x}})/f,\\ \tilde{h} &= e^{t\langle p_1(\,\cdot\,),\mathbf{x}\rangle}(\tilde{f} \circ R_{-\mathbf{x}})/\tilde{f}.\end{aligned} \tag{3.27}$$

We now show that $h, \tilde{h}$ are holomorphic. Let $z \in T\mathbb{P}^1$ and suppose $p_1(z) \in U$. Using a stereographic projection on $\mathbb{P}^1 \simeq S^2 \subset \mathbb{R}^3$, we may write

$$p_1(z) = \left(\frac{1-|\zeta|^2}{1+|\zeta|^2}, \frac{2\Im\{\zeta\}}{1+|\zeta|^2}, -\frac{2\Re\{\zeta\}}{1+|\zeta|^2}\right).$$

Then it is readily checked that

$$h(\zeta) = \exp\left\{t[x_1 - (x_3 + ix_2)\zeta]\right\}, \tag{3.28}$$

where $\mathbf{x} \in \mathbb{R}^3$. By the same procedures, one finds

$$\tilde{h}(\tilde{\zeta}) = \exp\left\{t[-x_1 + (-x_3 + ix_2)\tilde{\zeta}]\right\}. \tag{3.29}$$

Clearly both h, $\tilde{h}$ are holomorphic functions. Therefore our $\mathbb{R}^3$-action on L^t is holomorphic.

The $\mathbb{R}^3$-action on $T\mathbb{P}^1 \simeq \mathcal{O}(2)$ lifts to an action on the pullback bundle $\pi^*\mathcal{O}(2)$ over $T\mathbb{P}^1$. In particular, this induces an action on $|\pi^*\mathcal{O}(2k)|$. As for the $SO(3)$-action, the actions of $\mathbb{R}^3$ on $N_{k,l}$ and the linear system $|\pi^*\mathcal{O}(2k)|$ will be shown equivalent in the last section. Thus, we deduce that the $\mathbb{R}^3$-action on $\Sigma_{k,l}$ given by

$$\left(S^-, S^+\right) \mapsto \left(R_{\mathbf{x}}(S^-), R_{\mathbf{x}}(S^+)\right)$$

is well-defined.

Definition 3.4. *For any $\mathbf{x} \in \mathbb{R}^3$, let $R_{\mathbf{x}} : T\mathbb{P}^1 \to T\mathbb{P}^1$ and $S_{\mathbf{x}} : L^2 \to L^2$ be the maps given above. Then the $\mathbb{R}^3$-action $\psi_{\mathbf{x}}$ on $M_{k,l}$ is defined by*

$$(S^-, \nu^-, S^+, \nu^+) \mapsto (R_{\mathbf{x}}(S^-), \nu^-_{\mathbf{x}}, R_{\mathbf{x}}(S^+), \nu^+_{\mathbf{x}}),$$

where $\nu^\pm_{\mathbf{x}} = S_{\mathbf{x}} \circ \nu^\pm \circ R_{-\mathbf{x}}$.

The above action is well-defined since, from the commutativity of $S_{\mathbf{x}}$ and σ, the conditions $\nu^\pm \sigma\left(\nu^\pm\right) = (-1)^{k+l-1}$ are preserved.

3.2.3 T^2-Actions

The action of T^2 on $M_{k,l}$ is much easier to define:

Definition 3.5. *For any* $(\theta_-, \theta_+) \in T^2$*, the* T^2*-action* $\psi_{(\theta_-,\theta_+)}$ *on* $M_{k,l}$ *is given by*

$$(S^-, \nu^-, S^+, \nu^+) \mapsto (S^-, e^{-2i\theta_-}\nu^-, S^+, e^{-2i\theta_+}\nu^+).$$

Note that $\psi_{(\theta_-,\theta_+)}$ is well-defined since the conditions $\nu^\pm\sigma(\nu^\pm) = (-1)^{k+l-1}$ are preserved:

$$\begin{aligned}\left(e^{-i\theta_j}\nu^\pm\right)\sigma\left(e^{-i\theta_j}\nu^\pm\right) &= e^{-i\theta_j}\overline{e^{-i\theta_j}}\nu^\pm\sigma\left(\nu^\pm\right)\\ &= \nu^\pm\sigma\left(\nu^\pm\right)\\ &= (-1)^{k+l-1}.\end{aligned}$$

3.3 Isometries of $M_{k,l}$

In this section, we shall show that the $\mathbb{R}^3$, T^2 and $SO(3)$-actions on $M_{k,l}$ are all isometries with respect to the monopole-cluster metric, given in Chapter 2.

Let us first review how the metric on $M_{k,l}$ is defined. Recall that the twistor space $Z_{k,l}$ of $M_{k,l}$, as a real manifold, is just the product $M_{k,l} \times \mathbb{P}^1$. Let $p : Z_{k,l} \to \mathbb{P}^1$ be the map given by the second projection, then $Z_{k,l}$ has a natural complex structure which arises from the hyperkähler structure of $M_{k,l}$, such that p becomes a holomorphic map. From Chapter 2, there is a holomorphic section Ω of the bundle $\bigwedge^2 T_F^* \otimes p^*\mathcal{O}(2)$ over $Z_{k,l}$, where

$$T_F = \mathrm{Ker}\{dp : TZ_{k,l} \to T\mathbb{P}^1\}.$$

Using affine charts (U, ζ), $(\tilde{U}, \tilde{\zeta})$ of $\mathbb{P}^1$, $\frac{d}{d\zeta}, \frac{d}{d\tilde{\zeta}}$ may be viewed as local trivializations of $\mathcal{O}(2)$ under the identification $\mathcal{O}(2) \simeq T\mathbb{P}^1$. Let $(m, u) \in Z_{k,l}$, where $m = (S^-, \nu^-, S^+, \nu^+)$. $S^\pm$ are given by the vanishing of the polynomials $P_\pm = P_\pm(\zeta, \eta)$ over $\pi^{-1}(U)$, whereas $\nu^\pm$ are represented by the meromorphic functions $f^\pm = f^\pm(\zeta, \eta)$ over $S^\pm \cap \pi^{-1}(U)$. The roots of $P_\pm(\zeta, \eta)$ are denoted by $\eta_i^\pm(\zeta)$ and we let $\nu_i^\pm(\zeta) = f^\pm(\zeta, \eta_i^\pm(\zeta))$ for $i = 1, \ldots, k_\pm$, where $(k_-, k_+) = (k, l)$. Then Ω is given by

$$\Omega = \begin{cases}\Omega^- + \Omega^+ & \text{on } p^{-1}(U)\\ \tilde{\Omega}^- + \tilde{\Omega}^+ & \text{on } p^{-1}(\tilde{U})\end{cases},$$

where $\Omega^\pm, \tilde{\Omega}^\pm$ are of the form

$$\begin{aligned}\Omega^\pm(\zeta) &:= \sum_{i=1}^{k_\pm} d\log\nu_i^\pm(\zeta) \wedge d\eta_i^\pm(\zeta) \otimes \frac{d}{d\zeta},\\ \tilde{\Omega}^\pm(\tilde{\zeta}) &:= \sum_{i=1}^{k_\pm} d\log\tilde{\nu}_i^\pm(\tilde{\zeta}) \wedge d\tilde{\eta}_i^\pm(\tilde{\zeta}) \otimes \frac{d}{d\tilde{\zeta}}.\end{aligned} \tag{3.30}$$

The metric information is encoded in Ω: if $a \in U$ such that $\zeta(a) = 0$, then the Kähler form of the monopole-cluster metric with respect to the complex structure associated to a is given by the linear term of the power series expansion of $\Omega^- + \Omega^+$ around $\zeta = 0$. We shall show that the actions of $SO(3)$, $\mathbb{R}^3$ and T^2 on $M_{k,l}$ can be lifted to actions on $\bigwedge^2 T_F^* \otimes p^*\mathcal{O}(2)$ that preserve the holomorphic symplectic form Ω. This turns out to be sufficient for proving that the actions on $M_{k,l}$ are isometries.

Let us give a more global interpretation for the data $\eta_i^\pm, \nu_i^\pm$. Define

$$\begin{aligned} S^\pm(m) &= S^\pm, \\ \nu^\pm(m) &= \nu^\pm, \\ \Sigma^\pm(m) &= \nu^\pm(S^\pm). \end{aligned} \tag{3.31}$$

As before, we let $q : L^2 \to T\mathbb{P}^1$ be the projection map; χ, $\tilde{\chi}$ are the local holomorphic trivializations of L^2 given by (3.13), whose duals are denoted by $\vartheta, \tilde{\vartheta}$. Then by definition, $\vartheta, \tilde{\vartheta}$ are local holomorphic trivializations of L^{-2}. If $(m, u) \in Z_{k,l}$ with $u \in U$, then observe that there are bijections between $S^\pm(m) \cap T_u\mathbb{P}^1$ and $\{\eta_1^\pm(m,u), \ldots, \eta_{k_\pm}^\pm(m,u)\}$, where η is the tautological section. Thus, for any $s \in \Sigma^\pm(m) \cap (\pi \circ q)^{-1}(u)$, there is a $z \in S^\pm \cap T_u\mathbb{P}^1$ such that $s = \nu^\pm(z)$. In addition,

$$\vartheta(s) = \vartheta(\nu^\pm(z)) = \vartheta((f^\pm\chi)(z)) = f^\pm(z) = \nu_i^\pm(m,u)$$

for some i. As both $\eta_i^\pm$ and $\nu_i^\pm$ depend on (m,u), we have shown the following:

$$\begin{aligned} \eta_i^\pm(m,u) &\in \eta\left(S^\pm(m) \cap T_u\mathbb{P}^1\right), \\ \nu_i^\pm(m,u) &\in \vartheta\left(\Sigma^\pm(m) \cap (\pi \circ q)^{-1}(u)\right). \end{aligned} \tag{3.32}$$

In the case where $\eta_i^\pm$ are distinct, the functions $\eta_i^\pm, \nu_i^\pm$ may be considered as local coordinate functions of the twistor space $Z_{k,l}$; these functions depend on $u \in \mathbb{P}^1$ and are holomorphic with respect to the complex structure corresponding to u, which is most easily seen from the rational map construction.

3.3.1 $SO(3)$-Actions on Holomorphic Symplectic Form

For any $g \in SO(3)$, the map defined by

$$\Xi_g := \psi_g \times R_g \tag{3.33}$$

acts naturally on the twistor space. We claim that Ξ_g lifts to an action on $\bigwedge^2 T_F^* \otimes p^*\mathcal{O}(2)$. Indeed, it is clear that the dual of the differential of Ξ_g induces an action on $\bigwedge^2 T_F^*$. For the action on $p^*\mathcal{O}(2)$, first recall that the bundle is defined to be

$$p^*\mathcal{O}(2) = \{\, (u,v) \in Z_{k,l} \times \mathcal{O}(2) \,|\, p(u) = q(v) \,\},$$

where $q : \mathcal{O}(2) \to \mathbb{P}^1$ is the natural projection. As $\Xi_g : Z_{k,l} \to Z_{k,l}$ and $S_g : \mathcal{O}(2) \to \mathcal{O}(2)$ are lifts of the rotational map R_g on $\mathbb{P}^1$ and $T\mathbb{P}^1$ respectively, it is not hard to see that the action on $p^*\mathcal{O}(2)$ given by

$$(u, v) \mapsto (\Xi_g(u), S_g(v)) \tag{3.34}$$

is well-defined. Hence the $SO(3)$-action on $\bigwedge^2 T_F^* \otimes p^*\mathcal{O}(2)$ is defined by the tensor product of the above actions, which will be denoted by Υ.

Proposition 3.6.

$$\Omega_g := \Upsilon_g \circ \Omega \circ \Xi_{g^{-1}} = \Omega.$$

Proof. By linearity, it is sufficient to show that

$$\begin{aligned} \Omega_g^{\pm} &:= \Upsilon_g \circ \Omega^{\pm} \circ \Xi_{g^{-1}} = \Omega^{\pm}, \\ \tilde{\Omega}_g^{\pm} &:= \Upsilon_g \circ \tilde{\Omega}^{\pm} \circ \Xi_{g^{-1}} = \tilde{\Omega}^{\pm}, \end{aligned} \tag{3.35}$$

for each $n = 1, 2$. We shall only demonstrate this for the first case, as the other case is completely analogous. Let us expand $\Omega_g^{\pm}$:

$$\begin{aligned} \Omega_g^{\pm} &= \Upsilon_g \circ \Omega^{\pm} \circ \Xi_{g^{-1}} \\ &= \Xi_{g^{-1}}^* \left(\sum_{i=1}^{k_{\pm}} d\log \nu_i^{\pm} \wedge d\eta_i^{\pm} \right) \otimes \left(\Upsilon_g \circ \left(\frac{d}{d\zeta} \right) \circ \Xi_{g^{-1}} \right) \\ &= \sum_{i=1}^{k_{\pm}} \Xi_{g^{-1}}^* \left(d\log \nu_i^{\pm} \right) \wedge \Xi_{g^{-1}}^* \left(d\eta_i^{\pm} \right) \otimes \frac{d}{d\zeta_g} \\ &= \sum_{i=1}^{k_{\pm}} d\log \Xi_{g^{-1}}^* \nu_i^{\pm} \wedge d\Xi_{g^{-1}}^* \eta_i^{\pm} \otimes \frac{d}{d\zeta_g}. \end{aligned}$$

Let us consider the quantities $\Xi_{g^{-1}}^* \eta_i^{\pm}$ and $\Xi_{g^{-1}}^* \nu_i^{\pm}$. First, for any $(m, u) \in p^{-1}(U)$, we have

$$\begin{aligned} \left(\Xi_{g^{-1}}^* \eta_i^{\pm} \right)(m, u) &= \left(\eta_i^{\pm} \circ \Xi_{g^{-1}} \right)(m, u) \\ &= \eta_i^{\pm}(\psi_{g^{-1}} m, R_{g^{-1}} u). \end{aligned}$$

But

$$\begin{aligned} \eta_i^{\pm}(\psi_{g^{-1}} m, R_{g^{-1}} u) &\in \eta \left(S^{\pm}(\psi_{g^{-1}}(m)) \cap T_{R_{g^{-1}}(u)} \mathbb{P}^1 \right) \\ &= \eta \left(R_{g^{-1}}(S^{\pm}(m)) \cap R_{g^{-1}} \left(T_u \mathbb{P}^1 \right) \right) \\ &= \eta \left(R_{g^{-1}} \left(S^{\pm}(m) \cap T_u \mathbb{P}^1 \right) \right) \\ &= (\eta \circ R_{g^{-1}}) \left(S^{\pm}(m) \cap T_u \mathbb{P}^1 \right) \\ &= \eta_g \left(S^{\pm}(m) \cap T_u \mathbb{P}^1 \right), \end{aligned}$$

which implies

$$\Xi_{g^{-1}}^* \eta_i^\pm = \left(\eta_g^\pm\right)_{\sigma^\pm(i)}, \tag{3.36}$$

where $\sigma^\pm$ are some permutations of the set $\{1, \ldots, k_\pm\}$. Next, we consider $\Xi_{g^{-1}}^* \nu_i^\pm$: for $(m,u) \in p^{-1}(U)$, we have

$$\left(\Xi_{g^{-1}}^* \nu_i^\pm\right)(m,u) = \nu_i^\pm\left(\psi_{g^{-1}} m, R_{g^{-1}} u\right) \tag{3.37}$$

for each i. As $g \in SO(3)$ acts on L^2 by $S_g : L^2 \to L^2$, it induces an action on its dual by

$$S_g^* : s \mapsto s \circ S_{g^{-1}}. \tag{3.38}$$

Let $\vartheta_g := S_g^* \circ \vartheta \circ R_{g^{-1}}$. It is clear that ϑ_g is the dual of χ_g. Note that the right-hand-side of (3.37) is equal to $\left(\nu_g^\pm\right)_{\sigma^\pm(i)}(m,u)$:

$$\begin{aligned}
\nu_i^\pm\left(\psi_{g^{-1}} m, R_{g^{-1}} u\right) &\in \vartheta|_{\pi^{-1}(R_{g^{-1}}(u))}\left(\Sigma^\pm(\psi_{g^{-1}}(m)) \cap (\pi \circ q)^{-1}(R_{g^{-1}}(u))\right)\\
&= \vartheta|_{R_{g^{-1}}(\pi^{-1}(u))}\left(S_{g^{-1}}\left(\Sigma^\pm(m)\right) \cap S_{g^{-1}}\left((\pi \circ q)^{-1}(u)\right)\right)\\
&= \vartheta|_{R_{g^{-1}}(\pi^{-1}(u))}\left(S_{g^{-1}}\left(\Sigma^\pm(m) \cap (\pi \circ q)^{-1}(u)\right)\right)\\
&= \left(\vartheta|_{R_{g^{-1}}(\pi^{-1}(u))} \circ S_{g^{-1}}\right)\left(\Sigma^\pm(m) \cap (\pi \circ q)^{-1}(u)\right)\\
&= \left(S_{g^{-1}}^* \circ \vartheta\right)\Big|_{R_{g^{-1}}(\pi^{-1}(u))}\left(\Sigma^\pm(m) \cap (\pi \circ q)^{-1}(u)\right)\\
&= \left(S_{g^{-1}}^* \circ \vartheta \circ R_{g^{-1}}\right)\Big|_{\pi^{-1}(u)}\left(\Sigma^\pm(m) \cap (\pi \circ q)^{-1}(u)\right)\\
&= \vartheta_g|_{\pi^{-1}(u)}\left(\Sigma^\pm(m) \cap (\pi \circ q)^{-1}(u)\right),
\end{aligned}$$

where the first equality follows from

$$\begin{aligned}
\Sigma^\pm(\psi_{g^{-1}}(m)) &= \left(\nu^\pm\left(\psi_{g^{-1}}(m)\right)\right)\left(S^\pm\left(\psi_{g^{-1}}(m)\right)\right)\\
&= \left(\nu^\pm(m)\right)_{g^{-1}}\left(R_{g^{-1}}\left(S^\pm(m)\right)\right)\\
&= \left(S_{g^{-1}} \circ \left(\nu^\pm(m)\right) \circ R_g\right)\left(R_{g^{-1}}\left(S^\pm(m)\right)\right)\\
&= S_{g^{-1}}\left(\left(\nu^\pm(m)\right)\left(S^\pm(m)\right)\right)\\
&= S_{g^{-1}}\left(\Sigma^\pm(m)\right).
\end{aligned}$$

Therefore

$$\Xi_{g^{-1}}^* \nu_i^\pm = \left(\nu_g^\pm\right)_{\sigma^\pm(i)}. \tag{3.39}$$

Putting (3.36) and (3.39) into $\Omega_g^\pm$ we obtain

$$\Omega_g^\pm = \sum_{i=1}^{k_\pm} d\log\left(\nu_g^\pm\right)_i \wedge d\left(\eta_g^\pm\right)_i \otimes \frac{d}{d\zeta_g}. \tag{3.40}$$

It remains to express $\Omega_g^\pm$ in terms of the coordinates $\zeta, \nu_j^\pm, \eta_j^\pm$ (resp. $\tilde{\zeta}, \tilde{\nu}_j^\pm, \tilde{\eta}_j^\pm$) over $p^{-1}(R_g(U)) \cap p^{-1}(U)$ (resp. $p^{-1}(R_g(U)) \cap p^{-1}(\tilde{U})$). We shall only do this for the former case, as the other case follows from the same argument.

Recall that χ, χ_g are related by

$$\chi_g = h_{00}\chi$$

over $R_g(\pi^{-1}(U)) \cap \pi^{-1}(U)$, where h_{00} is some holomorphic function. Their duals ϑ, ϑ_g then satisfy

$$\vartheta_g = h_{00}^{-1}\vartheta.$$

It follows that for each i,

$$\left(\nu_g^{\pm}\right)_i = \left(h_{00}^{-1}\right)_i \nu_i^{\pm} \tag{3.41}$$

holds over $p^{-1}(R_g(U)) \cap p^{-1}(U)$, where $\left(h_{00}^{-1}\right)_i = h_{00}(z)^{-1}$ for some $z \in S^{\pm} \cap \pi^{-1}(U)$. Similarly, η_g and η are related by

$$\eta_g = \Gamma_{00}\eta$$

over $R_g(\pi^{-1}(U)) \cap \pi^{-1}(U)$, where Γ_{00} is some holomorphic function. Then over the same region we have

$$\left(\eta_g^{\pm}\right)_i = \left(\Gamma_{00}\right)_i \eta_i^{\pm}, \tag{3.42}$$

where $(\Gamma_{00})_i = \Gamma_{00}(z)$ for some $z \in S^{\pm} \cap \pi^{-1}(U)$. In local coordinates, h_{00} and Γ_{00} are given by

$$h_{00}(\zeta, \eta) = \exp\left(\frac{-2\bar{b}\eta}{\bar{b}\zeta + a}\right), \quad \Gamma_{00}(\zeta) = \frac{1}{(\bar{b}\zeta + a)^2},$$

which yield

$$\left(\nu_g^{\pm}\right)_i = \exp\left(\frac{2\bar{b}\eta_{\sigma(i)}^{\pm}}{\bar{b}\zeta + a}\right)\nu_{\sigma(i)}^{\pm}, \quad \left(\eta_g^{\pm}\right)_i = \frac{\eta_{\sigma(i)}^{\pm}}{(\bar{b}\zeta + a)^2}. \tag{3.43}$$

Substituting into $\Omega_g^\pm$, we have

$$\begin{aligned}
\Omega_g^\pm &= \sum_{i=1}^{k_\pm} d\log\left(\nu_g^\pm\right)_i \wedge d\left(\eta_g^\pm\right)_i \otimes \frac{d}{d\zeta_g} \\
&= \sum_{i=1}^{k_\pm} d\log\left(\exp\left(\frac{2\bar{b}\eta_i^\pm}{\bar{b}\zeta_0+a}\right)\nu_i^\pm\right) \wedge d\left(\frac{\eta_i^\pm}{(\bar{b}\zeta+a)^2}\right) \otimes \left((\bar{b}\zeta+a)^2\frac{d}{d\zeta}\right) \\
&= \sum_{i=1}^{k_\pm} \left(d\log\nu_i^\pm + \frac{2\bar{b}}{(\bar{b}\zeta+a)}d\eta_i^\pm\right) \wedge d\eta_i^\pm \otimes \frac{d}{d\zeta} \\
&= \sum_{i=1}^{k_\pm} d\log\nu_i^\pm \wedge d\eta_i^\pm \otimes \frac{d}{d\zeta} \\
&= \Omega^\pm.
\end{aligned}$$

A similar argument shows that $\tilde{\Omega}_g^\pm = \tilde{\Omega}^\pm$, and the proof is complete. □

3.3.2 $\mathbb{R}^3$-Actions on Holomorphic Symplectic Form

There is a natural $\mathbb{R}^3$-action on $Z_{k,l}$, namely, for any $\mathbf{x} \in \mathbb{R}^3$, the action is given by

$$\Xi_{\mathbf{x}} := \psi_{\mathbf{x}} \times Id. \tag{3.44}$$

$\Xi_{\mathbf{x}}$ lifts to an action on the bundle $\bigwedge^2 T_F^* \otimes p^*\mathcal{O}(2)$, which is given by the tensor product with the dual of the differential of $\Xi_{\mathbf{x}}$ on the first factor and the identity map on the second factor. We denote such action by $\Upsilon_{\mathbf{x}}$.

Proposition 3.7.

$$\Omega_{\mathbf{x}} := \Upsilon_{\mathbf{x}} \circ \Omega \circ \Xi_{-\mathbf{x}} = \Omega.$$

Proof. Again, it is sufficient to show that

$$\begin{aligned}
\Omega_{\mathbf{x}}^\pm &:= \Upsilon_{\mathbf{x}} \circ \Omega^\pm \circ \Xi_{-\mathbf{x}} = \Omega^\pm, \\
\tilde{\Omega}_{\mathbf{x}}^\pm &:= \Upsilon_{\mathbf{x}} \circ \tilde{\Omega}^\pm \circ \Xi_{-\mathbf{x}} = \tilde{\Omega}^\pm.
\end{aligned} \tag{3.45}$$

Since the proof is similar to the one for the $SO(3)$-case, we shall omit the details. By expanding $\Omega_{\mathbf{x}}^\pm$, it is easy to see that

$$\Omega_{\mathbf{x}}^\pm = \sum_{i=1}^{k_\pm} d\log\Xi_{-\mathbf{x}}^*\nu_i^\pm \wedge d\Xi_{-\mathbf{x}}^*\eta_i^\pm \otimes \frac{d}{d\zeta}.$$

One finds

$$\begin{aligned}
\Xi_{-\mathbf{x}}^*\nu_i^\pm &= (\nu_{\mathbf{x}})_{\sigma^\pm(i)}^\pm, \\
\Xi_{-\mathbf{x}}^*\eta_i^\pm &= (\eta_{\mathbf{x}})_{\sigma^\pm(i)}^\pm,
\end{aligned} \tag{3.46}$$

for each i, where $\sigma^\pm$ is a permutation of the set $\{1, \ldots, k_\pm\}$. Putting these into $\Omega^{\pm}_{\mathbf{x}}$ yields

$$\Omega^{\pm}_{\mathbf{x}} = \sum_{i=1}^{k_{\pm}} d\log(\nu_{\mathbf{x}})^{\pm}_i \wedge d(\eta_{\mathbf{x}})^{\pm}_i \otimes \frac{d}{d\zeta}.$$

It remains to express $\Omega^{\pm}_{\mathbf{x}}$ in terms of the coordinates $\zeta, \nu^{\pm}_j, \eta^{\pm}_j$ (and $\tilde{\zeta}, \tilde{\nu}^{\pm}_j, \tilde{\eta}^{\pm}_j$); we shall do this only for the first case. Recall that χ, $\chi_{\mathbf{x}}$ are related by

$$\chi_{\mathbf{x}} = h\chi,$$

where h is some holomorphic function. Their duals ϑ, $\vartheta_{\mathbf{x}}$ then satisfy

$$\vartheta_{\mathbf{x}} = h^{-1}\vartheta.$$

Similarly, η, $\eta_{\mathbf{x}}$ satisfy

$$\eta_{\mathbf{x}} = \eta - \xi.$$

Thus, over $p^{-1}(U)$, we have

$$\begin{aligned}(\nu_{\mathbf{x}})^{\pm}_i &= h^{-1}\nu^{\pm}_i,\\ (\eta_{\mathbf{x}})^{\pm}_i &= \eta^{\pm}_i - \xi.\end{aligned} \tag{3.47}$$

As h and ξ are given by

$$\begin{aligned}h(\zeta) &= \exp\left\{2[x_1 - (x_3 + ix_2)\zeta]\right\},\\ \xi(\zeta) &= i[(x_2 + ix_3) + 2ix_1\zeta + (x_2 - ix_3)\zeta^2],\end{aligned} \tag{3.48}$$

substituting them into $\Omega^{\pm}_{\mathbf{x}}$ one gets

$$\begin{aligned}\Omega^{\pm}_{\mathbf{x}} &= \sum_{i=1}^{k_{\pm}} d\log \Xi^*_{-\mathbf{x}}\nu^{\pm}_i \wedge d\Xi^*_{-\mathbf{x}}\eta^{\pm}_i \otimes \frac{d}{d\zeta}\\ &= \sum_{i=1}^{k_{\pm}} d\log\left(h(\zeta)^{-1}\nu^{\pm}_i\right) \wedge d\left(\eta^{\pm}_i - \xi(\zeta)\right) \otimes \frac{d}{d\zeta}\\ &= \sum_{i=1}^{k_{\pm}} d\log \nu^{\pm}_i \wedge d\eta^{\pm}_i \otimes \frac{d}{d\zeta}\\ &= \Omega^{\pm}.\end{aligned}$$

A similar argument shows that $\tilde{\Omega}^{\pm}_{\mathbf{x}} = \tilde{\Omega}^{\pm}$. Therefore we have the result. □

3.3.3 T^2-Actions on Holomorphic Symplectic Form

The T^2-action on $Z_{k,l}$ is defined as follows: for any $(\theta_-, \theta_+) \in T^2$, the action is given by

$$\Xi_{(\theta_-,\theta_+)} := \psi_{(\theta_-,\theta_+)} \times Id. \tag{3.49}$$

$\Xi_{(\theta_-,\theta_+)}$ lifts to an action on $\bigwedge^2 T^*_F \otimes p^*\mathcal{O}(2)$, it is a tensor product whose first factor is given by the dual of the differential of $\Xi_{(\theta_-,\theta_+)}$, and the second factor is the identity map on $p^*\mathcal{O}(2)$. We denote such action by $\Upsilon_{(\theta_-,\theta_+)}$.

Proposition 3.8.

$$\Omega_{(\theta_-,\theta_+)} := \Upsilon_{(\theta_-,\theta_+)} \circ \Omega \circ \Xi_{(-\theta_-,-\theta_+)} = \Omega.$$

Proof. Again, it is sufficient to show that

$$\begin{aligned} \Omega^{\pm}_{(\theta_-,\theta_+)} &:= \Upsilon_{(\theta_-,\theta_+)} \circ \Omega^{\pm} \circ \Xi_{(-\theta_-,-\theta_+)} = \Omega^{\pm}, \\ \tilde{\Omega}^{\pm}_{(\theta_-,\theta_+)} &:= \Upsilon_{(\theta_-,\theta_+)} \circ \tilde{\Omega}^{\pm} \circ \Xi_{(-\theta_-,-\theta_+)} = \tilde{\Omega}^{\pm}. \end{aligned} \tag{3.50}$$

As before, we shall demonstrate it for the first case. Expand $\Omega^{\pm}_{(\theta_-,\theta_+)}$:

$$\begin{aligned} \Omega^{\pm}_{(\theta_-,\theta_+)} &= \sum_{i=1}^{k_\pm} d\log \Xi^*_{(-\theta_-,-\theta_+)}\nu_i^{\pm} \wedge d\Xi^*_{(-\theta_-,-\theta_+)}\eta_i^{\pm} \otimes \frac{d}{d\zeta} \\ &= \sum_{i=1}^{k_\pm} d\log e^{2i\theta_\pm}\nu_i^{\pm} \wedge d\eta_i^{\pm} \otimes \frac{d}{d\zeta} \\ &= \sum_{i=1}^{k_\pm} d\log \nu_i^{\pm} \wedge d\eta_i^{\pm} \otimes \frac{d}{d\zeta} \\ &= \Omega^{\pm}. \end{aligned}$$

A similar argument shows that $\tilde{\Omega}^{\pm}_{(\theta_-,\theta_+)} = \tilde{\Omega}^{\pm}$, therefore the proposition is proved. $\square$

To summarize, we have shown that the lifts of the $\mathbb{R}^3$, T^2 and $SO(3)$-actions on $M_{k,l}$ preserve the holomorphic symplectic form Ω. Recall that, to each $a \in \mathbb{P}^1$, Ω gives rise to the Kähler form of the metric on $M_{k,l}$, with respect to the complex structure that corresponds to a. Since the action of $\mathbb{R}^3$ and T^2 do not change the complex structures of $M_{k,l}$, our result implies that they must preserve all the Kähler forms, hence the metric. The $SO(3)$-action, in contrast, acts on the 2-sphere of complex structures by rotation; the Kähler form that corresponds to u is only preserved by an $SO(2)$-subgroup, which shows that the elements in such subgroup are isometries. But since any element in $SO(3)$ preserves a complex structure and hence a Kähler form, the whole $SO(3)$ group must act by isometries. We have deduced that:

Corollary 3.9. *$M_{k,l}$ admits the following isometry groups: $\mathbb{R}^3$, T^2 and $SO(3)$.*

3.4 Equivalence of Group Actions on $M_{k,l}$ and $N_{k,l}/\mathcal{G}_0$

From here onwards, we shall always assume $k < l$, since this is the only case that will concern us in the next two chapters. This section is devoted to showing that the actions of $\mathbb{R}^3$, T^2, and $SO(2) \subset SO(3)$ on the set of generic elements in $M_{k,l}$ and $N_{k,l}$ are equivalent.

Let (U,ζ), $(\tilde{U},\tilde{\zeta})$ be affine charts of $\mathbb{P}^1$, and $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$ is the usual projection map. Recall that if $(\alpha,\beta) \in \mathcal{A}_k((a,b))$ is a solution to Nahm's equations, then there is an associated curve S in $T\mathbb{P}^1$, called spectral curve, such that it is given by the equation

$$P(\zeta,\eta) = \det(\eta - \beta(\zeta)) = 0 \tag{3.51}$$

over $\pi^{-1}(U)$, where

$$\beta(\zeta) = (T_2 + iT_3) + 2iT_1\zeta + (T_2 - iT_3)\zeta^2. \tag{3.52}$$

If (α,β) is a generic Nahm data in $N_{k,l}$, then we can find spectral curves S^-, S^+. In addition, from the last chapter, we see that there are associated meromorphic sections $\nu^\pm$ of L^2 over $S^\pm$ given by the functions

$$f^- = \overline{\tau \circ \kappa^-}, \quad f^+ = \kappa^+, \tag{3.53}$$

over $\pi^{-1}(U) \cap S^\pm$, where

$$\begin{aligned}\kappa^+(\zeta,\eta) &= u_+(\zeta,1)^T(\eta - \beta_+(\zeta,1))_{\mathrm{adj}} u_+(\zeta,1) P_-(\eta),\\ \kappa^-(\zeta,\eta) &= u_-(\zeta,-1)^T(\eta - \beta_-(\zeta,-1))_{\mathrm{adj}} u_-(\zeta,-1) P_+(\eta).\end{aligned}$$

In the above, $P_\pm(z)$ are the characteristic polynomial of $\beta_\pm(\zeta)$; $u_\pm(\zeta,\cdot)$ are the unique solutions to

$$\frac{dw}{dt} + \alpha_\pm(\zeta)w = 0, \tag{3.54}$$
$$\frac{dw}{dt} + \alpha_-(\zeta)w = 0, \tag{3.55}$$

satisfying the boundary conditions

$$\begin{aligned} t^{-(l-k-1)/2} P_-\left(\beta_+(\zeta,t)\right) u_+(\zeta,t) &\to E_{k+1},\\ u_-(\zeta,t) &\to v_-(\zeta),\end{aligned}$$

as $t \to 0$, where $u_+(\zeta,0) = (v_-(\zeta)^T, 0, \ldots, 0)^T$.

Lemma 3.10. *The $\mathbb{R}^3$-action on Nahm data and spectral data coincide on spectral curves.*

Proof. Let $(\alpha,\beta) \in \mathcal{A}_k((a,b))$. For $\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$, the $\mathbb{R}^3$-action on Nahm data takes $\beta(\zeta)$ to

$$\hat{\beta}(\zeta) := \beta(\zeta) + i[(x_2 + ix_3) + 2ix_1\zeta + (x_2 - ix_3)\zeta^2]Id,$$

so that the resulting curve has the equation $\det\left(\eta - \hat{\beta}(\zeta)\right) = 0$. On the other hand, the $\mathbb{R}^3$-action on spectral data takes S to $R_{\mathbf{x}}(S)$, which is given by the equation $(P \circ R_{-\mathbf{x}})(\zeta, \eta) = 0$ over $\pi^{-1}(U)$. But

$$\begin{aligned}
(P \circ R_{-\mathbf{x}})(\zeta, \eta) &= P\left(\zeta, \eta - i[(x_2 + ix_3) + 2ix_1\zeta + (x_2 - ix_3)\zeta^2]\right) \\
&= \det\left(\eta - i[(x_2 + ix_3) + 2ix_1\zeta + (x_2 - ix_3) - \beta(\zeta)\right) \\
&= \det\left(\eta - \hat{\beta}(\zeta)\right).
\end{aligned}$$

□

We deduce from the lemma that the $\mathbb{R}^3$-action on $N_{k,l}$ and $M_{k,l}$ are equivalent as actions on $\Sigma_{k,l}$.

Proposition 3.11. *The $\mathbb{R}^3$-action on the generic elements of the spaces $M_{k,l}$ and $N_{k,l}/\mathcal{G}_0$ are equivalent.*

Proof. In the light of Lemma (3.10), it remains to prove that both the $\mathbb{R}^3$-actions are equivalent on the meromorphic sections of L^2. For $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, let

$$\begin{aligned}
h(\zeta) &= e^{2[x_1 - (x_3 + ix_2)\zeta]}, \\
\xi(\zeta) &= i[(x_2 + ix_3) + 2ix_1\zeta + (x_2 - ix_3)\zeta^2].
\end{aligned} \tag{3.56}$$

While the $\mathbb{R}^3$-action on $M_{k,l}$ takes $f^{\pm}(\zeta, \eta)$ to

$$h(\zeta) f^{\pm}(\zeta, \eta - \xi(\zeta)), \tag{3.57}$$

the $\mathbb{R}^3$-action on $N_{k,l}/\mathcal{G}_0$ is given by

$$(\alpha(\zeta), \beta(\zeta)) \mapsto \left(\alpha_{\mathbf{x}\pm}(\zeta), \beta_{\mathbf{x}\pm}(\zeta)\right), \tag{3.58}$$

where

$$\begin{aligned}
\alpha_{\mathbf{x}\pm}(\zeta) &= \alpha_{\pm}(\zeta) + [-x_1 + (x_3 + ix_2)\zeta] Id, \\
\beta_{\mathbf{x}\pm}(\zeta) &= \beta_{\pm}(\zeta) + \xi(\zeta) Id.
\end{aligned} \tag{3.59}$$

We wish to show that the meromorphic functions given by $\left(\alpha_{\mathbf{x}\pm}(\zeta), \beta_{\mathbf{x}\pm}(\zeta)\right)$ are the same as those in (3.57). First note that $u_+(\zeta, t)$ is a solution to (3.54) if and only if

$$u_{\mathbf{x}+}(\zeta, t) := e^{[x_1 - (x_3 + ix_2)\zeta]t} u_+(\zeta, t) \tag{3.60}$$

is a solution to

$$\frac{dw}{dt} + \alpha_{\mathbf{x}+}(\zeta) w = 0.$$

Let $P_{\mathbf{x}\pm}$ be the characteristic polynomial of $\beta_{\mathbf{x}\pm}(\zeta)$. We claim that

$$t^{-(l-k-1)/2} P_{\mathbf{x}-}\left(\beta_{\mathbf{x}+}(\zeta)\right) u_{\mathbf{x}+}(\zeta) \to E_{k+1} \tag{3.61}$$

as $t \to 0$, so that $u_{\mathbf{x}+}(\zeta, t)$ is indeed the unique solution to (3.54) which satisfies the correct boundary condition. It is sufficient to notice that $P_{\mathbf{x}-}(\beta_{\mathbf{x}+}(\zeta)) =$

$P_-(\beta_+(\zeta))$: writing

$$P_-(z) = \det(z - \beta_-(\zeta)) = z^k + c_{k-1}z^{k-1} + \cdots + c_0,$$

then $P_{\mathbf{x}-}(z) = \det(z - \beta_{\mathbf{x}-}(\zeta))$ is given by

$$(z - \xi(\zeta))^k + c_{k-1}(z - \xi(\zeta))^{k-1} + \cdots + c_0,$$

which yields

$$\begin{aligned} P_{\mathbf{x}-}(\beta_{\mathbf{x}+}(\zeta)) &= (\beta_{\mathbf{x}+}(\zeta) - \xi(\zeta)Id)^k + c_{k-1}(\beta_{\mathbf{x}+}(\zeta) - \xi(\zeta)Id)^{k-1} + \cdots + c_0 \\ &= \beta_+(\zeta)^k + c_{k-1}\beta_+(\zeta)^{k-1} + \cdots + c_0 \\ &= P_-(\beta_+(\zeta)). \end{aligned}$$

Hence, the section $\nu_{\mathbf{x}}^+$ given by $\left(\alpha_{\mathbf{x}\pm}(\zeta), \beta_{\mathbf{x}\pm}(\zeta)\right)$ is the function

$$f_{\mathbf{x}}^+(\zeta, \eta) := u_{\mathbf{x}+}(\zeta, 1)^T(\eta - \beta_{\mathbf{x}+}(\zeta, 1))_{\mathrm{adj}} u_{\mathbf{x}+}(\zeta, 1) P_{\mathbf{x}-}(\eta),$$

which is equal to

$$e^{2[x_1 - (x_3 + ix_2)\zeta]} u_+(\zeta, 1)^T(\eta - \xi(\zeta) - \beta_+(\zeta, 1))_{\mathrm{adj}} u_+(\zeta, 1) P_-(\eta - \xi(\zeta)).$$

One may check that this is the same as (3.57). Similarly, one can show that

$$f_{\mathbf{x}}^- = h(\zeta) f^-(\zeta, \eta - \xi(\zeta)).$$

Hence, the $\mathbb{R}^3$-actions coincide on f^-, f^+. Repeating the argument for $\tilde{f}^-, \tilde{f}^+$, one concludes that the two actions are indeed equivalent. $\square$

Next, we prove the analogous statement for T^2-actions:

Proposition 3.12. *The T^2-action on the generic elements of the spaces $N_{k,l}/\mathcal{G}_0$ and $M_{k,l}$ are equivalent.*

Proof. As the actions of T^2 act trivially on the spectral curves, it remains to check that they are equivalent on the meromorphic sections. Let $(\theta_-, \theta_+) \in T^2$. Then the T^2-action of $M_{k,l}$ on $f^\pm$ is given by

$$f^\pm \mapsto e^{-i\theta_\pm} f^\pm,$$

whereas $p_{(\theta_-, \theta_+)}$, the T^2-action of $N_{k,l}/\mathcal{G}_0$, acts by

$$(\alpha(\zeta), \beta(\zeta)) \mapsto \left(\alpha_{(\theta_-, \theta_+)}(\zeta), \beta_{(\theta_-, \theta_+)}(\zeta)\right),$$

where $\beta_{(\theta_-,\theta_+)_\pm}(\zeta) := \beta_\pm(\zeta)$ and

$$\begin{aligned} \alpha_{(\theta_-,\theta_+)_-}(\zeta) &:= \alpha_-(\zeta) - i\theta_- Id, \\ \alpha_{(\theta_-,\theta_+)_+}(\zeta) &:= \alpha_+(\zeta) + i\theta_+ Id. \end{aligned} \tag{3.62}$$

Let $f^\pm_{(\theta_-,\theta_+)}$ be the representative functions of L^2 over $\pi^{-1}(U) \cap S^\pm$, associated to $\big(\alpha_{(\theta_-,\theta_+)}(\cdot), \beta_{(\theta_-,\theta_+)}(\cdot)\big)$. We need to show that

$$f^\pm_{(\theta_-,\theta_+)} = e^{-i\theta_\pm} f^\pm. \tag{3.63}$$

Observe that $u_+(\zeta, t)$ is a solution to (3.54) if and only if

$$u_{(\theta_-,\theta_+)_+}(\zeta, t) := e^{-i\theta_+ t} u_+(\zeta, t) \tag{3.64}$$

is a solution to

$$\frac{dw}{dt} + \alpha_{(\theta_-,\theta_+)_+}(\zeta) w = 0.$$

As $u_+(\zeta, t)$ satisfies the boundary condition, one may check that $u_{(\theta_-,\theta_+)_+}(\zeta, t)$ also does. Hence,

$$\begin{aligned} e^{-i\theta_+} f^+(\zeta, \eta) &= e^{-i\theta_+} u_+(\zeta, 1)^T (\eta - \beta_+(\zeta, 1))_{\text{adj}} u_+(\zeta, 1) p_-(\eta) \\ &= \left(e^{-i\theta_+/2} u_+(\zeta, 1)\right)^T (\eta - \beta_+(\zeta, 1))_{\text{adj}} \left(e^{-i\theta_+/2} u_+(\zeta, 1)\right) P_-(\eta) \\ &= u_{(\theta_-,\theta_+)_+}(\zeta, 1)^T (\eta - \beta_{(\theta_-,\theta_+)_+}(\zeta, 1))_{\text{adj}} u_{(\theta_-,\theta_+)_+}(\zeta, 1) P_-(\eta) \\ &= f^+_{(\theta_-,\theta_+)}(\zeta, \eta). \end{aligned}$$

Similarly, one can show that $f^-_{(\theta_-,\theta_+)} = e^{-i\theta_-} f^-$. Repeating the argument for $\tilde{f}^-, \tilde{f}^+$, we conclude that the two actions do indeed coincide. $\square$

Let $S \subset T\mathbb{P}^1$ be the curve defined by the Nahm data $(\alpha, \beta) \in \mathcal{A}_k((a, b))$ via (3.51), and

$$g = \begin{bmatrix} \bar{\lambda} & -\mu \\ \bar{\mu} & \lambda \end{bmatrix} \in SU(2)/\pm \simeq SO(3).$$

Over $\pi^{-1}(R_g(U)) \cap \pi^{-1}(U)$, $R_g(S)$, the image of S under $R_g : T\mathbb{P}^1 \to T\mathbb{P}^1$, is given by the equation

$$\det\left(\eta - (-\bar{\mu}\zeta + \bar{\lambda})^2 \beta\left(\frac{\lambda\zeta + \mu}{-\bar{\mu}\zeta + \bar{\lambda}}\right)\right) = 0. \tag{3.65}$$

On the other hand, for any $A = (a_{ij}) \in SO(3)$, its action on Nahm data takes S to the curve defined by

$$\det(\eta - A \cdot \beta(\zeta)) = 0, \tag{3.66}$$

where

$$A \cdot \beta(\zeta) := \sum_{j=1}^{3} \left((a_{2j}T_j + ia_{3j}T_j) + 2ia_{1j}T_j\zeta + (a_{2j}T_j - ia_{3j}T_j)\zeta^2 \right).$$

Let $\Phi : SU(2) \to SO(3)$ be the double covering map defined by

$$\begin{pmatrix} \lambda & \mu \\ -\bar{\mu} & \bar{\lambda} \end{pmatrix} \mapsto \begin{pmatrix} |\lambda|^2 - |\mu|^2 & -2\Im\{\bar{\lambda}\bar{\mu}\} & -2\Re\{\bar{\lambda}\bar{\mu}\} \\ -2\Im\{\bar{\lambda}\mu\} & \Re\{\bar{\lambda}^2 + \mu^2\} & -\Im\{\bar{\lambda}^2 - \mu^2\} \\ 2\Re\{\bar{\lambda}\mu\} & \Im\{\bar{\lambda}^2 + \mu^2\} & \Re\{\bar{\lambda}^2 - \mu^2\} \end{pmatrix}.$$

Lemma 3.13. *The above $SO(3)$-actions are equivalent on the space of spectral curves.*

Proof. It suffices to show that the equality

$$\Phi(g) \cdot \beta(\zeta) = (-\bar{\mu}\zeta + \bar{\lambda})^2 \beta\left(\frac{\lambda\zeta + \mu}{-\bar{\mu}\zeta + \bar{\lambda}}\right) \tag{3.67}$$

holds for any $g \in SU(2)$. We begin by expanding the right-hand-side:

$$(T_2 + iT_3)(-\bar{\mu}\zeta + \bar{\lambda})^2 + 2iT_1(\lambda\zeta + \mu)(-\bar{\mu}\zeta + \bar{\lambda}) + (T_2 - iT_3)(\lambda\zeta + \mu)^2.$$

It is not difficult to see that this is equivalent to

$$\sum_{j=1}^{3} \left((\Phi(g)_{2j}T_j + i\Phi(g)_{3j}T_j) + 2i\Phi(g)_{1j}T_j\zeta + (\Phi(g)_{2j}T_j - i\Phi(g)_{3j}T_j)\zeta^2 \right),$$

which is equal to $\Phi(g) \cdot \beta(\zeta)$ by definition. $\square$

We deduce from the lemma that the two $SO(3)$-actions are equivalent on $\Sigma_{k,l}$. Although such actions should also be equivalent on $M_{k,l}$, it seems to be very difficult to show. We have, however, the following weaker result:

Proposition 3.14. *The $\widetilde{SO(2)}$-actions on the generic elements of the spaces $N_{k,l}/\mathcal{G}_0$ and $M_{k,l}$ are equivalent.*

The group $\widetilde{SO(2)}$ is the subgroup of $SO(3)$ generated by elements of the form

$$V = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix}, \quad \theta \in [0, 2\pi).$$

Proof. Let $(\alpha, \beta) \in N_{k,l}$ be a generic element. Due to Lemma (3.13), it remains to prove that the two actions are equivalent on the meromorphic functions $f^{\pm}$. Observe that the element V gives rise to the map

$$f^{\pm}(\zeta, \eta) \mapsto f^{\pm}(e^{-i\theta}\zeta, e^{-i\theta}\eta)$$

under the action on $M_{k,l}$, and the map

$$(\alpha_\pm(\zeta), \beta_\pm(\zeta)) \mapsto (\alpha_\pm^V(\zeta), \beta_\pm^V(\zeta)),$$

under the action on $N_{k,l}/\mathcal{G}_0$, where

$$\begin{aligned} \beta_\pm^V(\zeta) &= e^{i\theta} g^V \beta_\pm(e^{-i\theta}\zeta)(g^V)^{-1}, \\ \alpha_\pm^V(\zeta) &= g^V \alpha_\pm(e^{-i\theta}\zeta) g^V - \frac{dg^V}{dt}(g^V)^{-1}, \end{aligned} \tag{3.68}$$

as V descends to the $SO(3)$ element

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Let f_V^+ denote the meromorphic function that represents the section of L^2 associated to these Nahm data: it is given by

$$f_V^+(\zeta, \eta) := u_+^V(\zeta, 1)^T (\eta - \beta_+^V(\zeta, 1))_{\mathrm{adj}} u_+^V(\zeta, 1) P_-^V(\eta),$$

where $u_+^V(\zeta, \cdot)$ is the unique solution to

$$\frac{dw}{dt} + \alpha_+^V(\zeta) w = 0, \tag{3.69}$$

such that

$$t^{-(l-k-1)/2} P_-^V\left(\beta_+^V(\zeta, t)\right) u_+^V(\zeta, t) \to E_{k+1} \tag{3.70}$$

as $t \to 0$, where $P_-^V(z)$ is the characteristic polynomial of $\beta_-^V(\zeta)$. We shall show that

$$f_V^+(\zeta, \eta) = f^+(e^{-i\theta}\zeta, e^{-i\theta}\eta).$$

First observe that we have

$$u_+^V(\zeta, \cdot) = g^V u_+(e^{-i\theta}\zeta, \cdot)$$

if $u_+(\zeta, \cdot)$ a solution to (3.54). Let us consider the quantity

$$P_-^V\left(\beta_+^V(\zeta)\right) g^V u_+(e^{-i\theta}\zeta, \cdot). \tag{3.71}$$

Lemma 3.15.

$$P_-^V\left(\beta_+^V(\zeta)\right) = e^{ik\theta} g^V P_-(\beta_+(e^{-i\theta}\zeta))(g^V)^{-1}.$$

Proof of Lemma. Let $P_-(z) = \det(z - \beta_-(e^{-i\theta}\zeta))$. Then

$$\begin{aligned}
P_-^V(z) &= \det(z - \beta_-^V(\zeta)) \\
&= \det(z - e^{i\theta} g^V \beta_-(e^{-i\theta}\zeta)(g^V)^{-1}) \\
&= e^{ik\theta} \det(e^{-i\theta} z - \beta_-(e^{-i\theta}\zeta)) \\
&= e^{ik\theta} P_-(e^{-i\theta} z).
\end{aligned}$$

Substituting $\beta_+^V(\zeta)$ into $P_-^V(z)$, we have

$$\begin{aligned}
P_-^V(\beta_+^V(\zeta)) &= e^{ik\theta} P_-(e^{-i\theta}\beta_+^V(\zeta)) \\
&= e^{ik\theta} P_-(e^{-i\theta} e^{i\theta} g^V \beta_+(e^{-i\theta}\zeta)(g^V)^{-1}) \\
&= e^{ik\theta} g^V P_-(\beta_+(e^{-i\theta}\zeta))(g^V)^{-1}.
\end{aligned}$$

□

By the lemma, we see that (3.71) is equal to

$$e^{ik\theta} g^V P_-(\beta_+(e^{-i\theta}\zeta)) u_+(e^{-i\theta}\zeta, \cdot).$$

If $u_+(e^{-i\theta}\zeta, \cdot)$ satisfies the boundary condition

$$t^{-(l-k-1)/2} P_- \left(\beta_+(e^{-i\theta}\zeta, t)\right) u_+(e^{-i\theta}\zeta, t) \to E_{k+1}$$

as $t \to 0$, then in the same limit we have

$$P_-^V \left(\beta_+^V(\zeta)\right) g^V u_+(e^{-i\theta}\zeta) \to e^{ik\theta} \left(\begin{array}{c|c} 0 & \\ \hline & \rho(V) \end{array}\right) E_{k+1}. \tag{3.72}$$

But since

$$\rho(V) = \begin{pmatrix} e^{i(l-k-1)\theta/2} & & & 0 \\ & e^{i(l-k-3)\theta/2} & & \\ & & \ddots & \\ 0 & & & e^{-i(l-k-1)\theta/2} \end{pmatrix},$$

the solution

$$u_+^V(\zeta) = e^{-i(l+k-1)\theta/2} g^V u_+(e^{-i\theta}\zeta) \tag{3.73}$$

satisfies both (3.69) and (3.70). Thus, $f_V^+(\zeta, \eta)$ is given by

$$e^{-i(l-1)\theta} u_+(e^{-i\theta}\zeta, 1)^T (\eta - e^{i\theta}\beta_+(e^{-i\theta}\zeta, 1))_{\mathrm{adj}} u_+(e^{-i\theta}\zeta, 1) P_-(e^{-i\theta}\eta),$$

which is easily seen to be equal to $f^+(e^{-i\theta}\zeta, e^{-i\theta}\eta)$. In addition, since

$$u_-^V(\zeta) = e^{-i(l+k-1)\theta/2} u_-(e^{-i\theta}\zeta). \tag{3.74}$$

one may show that

$$f_V^-(\zeta, \eta) = f^-(e^{-i\theta}\zeta, e^{-i\theta}\eta).$$

along the same line. Repeating the argument for $\tilde{f}^-, \tilde{f}^+$, one concludes that the two actions are indeed equivalent. $\square$

Chapter 4

Monopole-Clusters of Charge $(1, 2)$

In the first section, we show that any general solution in $N_{1,2}$ can be reduced to some standard solution, where it is given in terms of Jacobi elliptic functions and Pauli matrices. As the procedure is invertible, it implies that any element in $N_{1,2}$ must take a particular form. Next, we define a 1-parameter family of regions $N(\delta)$ in the moduli space $N_{1,2}/\mathcal{G}_0$, each of which contains the asymptotic region where monopole-clusters are widely separated. Then we prove that, within any given $N(\delta)$, the intersection points of the spectral curves come into pairs with rate $1/R$, where R is the separation distance between the monopole-clusters; such result is consistent with the postulate that the metric of $M_{k,l}$ becomes $(1/R)$-close to the product metric when monopole-clusters separate. Afterwards, we write down explicitly the spectral data for $M_{1,2}$ in terms of familiar functions, and also the constraints that the spectral curves must satisfy.

The last section is devoted to classifying the action of the subgroups of $\widetilde{SO(2)} \times T^2$ that have at least one fixed point in $N(\delta)$ with $\delta = 1$. In particular, we show that the fixed point sets of certain subgroups are given, in some fixed gauge, by families of Nahm data parametrized by real coordinates; the nontrivial part is to show that any fixed point actually lies in such a family. As both the $\widetilde{SO(2)}$-action and the T^2-action are isometries with respect to the monopole-cluster metric, all such families of Nahm data represent totally geodesic submanifolds of $M_{1,2}$.

4.1 The Moduli Space $N_{1,2}/\mathcal{G}_0$

Recall from Definition (2.4) that the space $N_{1,2}$ consists of a collection of solutions (T^-, T^+) to Nahm's equation satisfying the following:

(a) T_j^-, $j = 0, 1, 2, 3$, are $\mathfrak{u}(1)$-valued on $[-1, 0]$ and T_k^+, $k = 0, 1, 2, 3$, are $\mathfrak{u}(2)$-valued on $[0, 1]$.

(b) $T_j^{\pm}$ is analytic everywhere for $j = 0, 1, 2, 3$.

(c) For each $j = 1, 2, 3$, T_j^- patches with the $(1,1)$-entry of T_j^+ at $t = 0$, i.e. $T_j^-(0) = T_j^+(0)_{1,1}$.

(d) The solutions are symmetric at $t = 1$.

The gauge group $\mathcal{G}_0$ consists of gauge transformations g, where g_- is $U(1)$-valued on $[-1, 0]$ with $g_-(-1) \in O(1, \mathbb{R})$, and g_+ is $U(2)$-valued on $[0, 1]$ with $g_+(1) \in O(2, \mathbb{R})$. In addition, g_- and g_+ must satisfy the patching condition

$$g_+(0) = \begin{pmatrix} g_-(0) & 0 \\ 0 & 1 \end{pmatrix}.$$

We shall be interested in the moduli space $N_{1,2}/\mathcal{G}_0$.

It is evident that if (T_0, T_1, T_2, T_3) is a one-dimensional solution to Nahm's equations, then T_1, T_2, T_3 must all be constants. Note also that, to completely determine a point $(T^-, T^+) \in N_{1,2}$, it is sufficient to know T^+ on $[0, 1]$ together with T_0^- on $[-1, 0]$: for $i = 1, 2, 3$, T_i^- are one-dimensional and they coincide with the $(1,1)$-entry of T_i^+ at $t = 0$.

4.1.1 Parametrizations

From the last chapter, we know that in general, there are $\mathbb{R}^3$, T^2 and $SO(3)$-actions on $N_{k,l}/\mathcal{G}_0$. In particular, the $SO(3)$-action on $N_{1,2}/\mathcal{G}_0$ becomes less complicated: for any $A = (a_{ij}) \in SO(3)$, it acts on $N_{1,2}/\mathcal{G}_0$ merely by

$$\begin{aligned} T_0^{\pm} &\mapsto T_0^{\pm}, \\ T_i^{\pm} &\mapsto \sum_{j=1}^{3} a_{ij} T_j^{\pm}, \quad i = 1, 2, 3, \end{aligned}$$

i.e. it has no coupling gauge transformation. We shall show that, using the actions mentioned above, any general element in $N_{k,l}/\mathcal{G}_0$ can be reduced to some standard solution. But first we have the following:

Lemma 4.1. *Up to the action of $\mathbb{R}^3$ and T^2, any point in $N_{1,2}$ can be given by Nahm data (T^-, T^+), where T_j^+ is $\mathfrak{su}(2)$-valued for each $j = 0, 1, 2, 3$, and $T_0^- \equiv 0$.*

Proof. Let $(T^-, T^+) \in N_{1,2}$ and write $T^{\pm} = (T_0^{\pm}, T_1^{\pm}, T_2^{\pm}, T_3^{\pm})$. Using the $\mathbb{R}^3$-action, one can make T_1^+, T_2^+, T_3^+ trace-free by means of the map

$$T_j^{\pm} \mapsto T_j^{\pm} - \frac{1}{2}\mathrm{tr}\, \left(T_j^+(1)\right) Id.$$

Since Nahm's equations imply that the trace of T_1^+, T_2^+, T_3^+ are constant in t, they are actually $\mathfrak{su}(2)$-valued functions. Moreover, the action given by

$$g_{\pm}(t) = \exp\left(-\frac{1}{2}\int_0^t \mathrm{tr}\; T_0^{\pm}\right)$$

makes T_0^- identically zero and the trace of T_0^+ zero. This action is actually equivalent to a T^2-action on $N_{1,2}/\mathcal{G}_0$: if $h_{(\theta_-,\theta_+)}$ is the T^2-action with

$$(\theta_-,\theta_+) = \left(\frac{i}{2}\int_{-1}^0 \mathrm{tr}\; T_0^-, \frac{i}{2}\int_0^1 \mathrm{tr}\; T_0^+\right),$$

then $h_{(\theta_-,\theta_+)}g^{-1} \in \mathcal{G}_0$. $\square$

The set of elements in $N_{1,2}$ that have the property in the above lemma is denoted by $N_{1,2}^0$.

We shall now use the $SO(3)$-action to reduce the elements in $N_{1,2}^0$ further. The following uses the argument given by Dancer [14]. We define

$$\langle T_i^+, T_j^+\rangle := -\frac{1}{2}\mathrm{tr} T_i^+ T_j^+,$$

where $\langle\, ,\, \rangle$ is the Killing form of $\mathfrak{su}(2)$. Let us consider the following quantities:

$$\begin{aligned}
\alpha_1 &= \langle T_1^+, T_1^+\rangle - \langle T_2^+, T_2^+\rangle,\\
\alpha_2 &= \langle T_1^+, T_1^+\rangle - \langle T_3^+, T_3^+\rangle,\\
\alpha_3 &= \langle T_1^+, T_2^+\rangle,\\
\alpha_4 &= \langle T_1^+, T_3^+\rangle,\\
\alpha_5 &= \langle T_2^+, T_3^+\rangle.
\end{aligned} \tag{4.1}$$

Note that the Nahm's equations imply that all the α_i are constant in t. The map defined by

$$(T^-,T^+) \mapsto \begin{pmatrix} \frac{1}{3}(\alpha_1+\alpha_2) & \alpha_3 & \alpha_4 \\ \alpha_3 & \frac{1}{3}(\alpha_2 - 2\alpha_1) & \alpha_5 \\ \alpha_4 & \alpha_5 & \frac{1}{3}(\alpha_1 - 2\alpha_2) \end{pmatrix} \tag{4.2}$$

sends elements in $N_{1,2}^0$ to the space of real traceless 3×3 symmetric matrices. This is an $SO(3)$-equivariant map: the action of $SO(3)$ on the target space acts by conjugation. Since symmetric matrices are diagonalizable, the image of any element in $N_{1,2}^0$ can be conjugated to a diagonal matrix by some matrix $A \in SO(3)$. In addition, A may be chosen so that the entries of the diagonal matrix are arranged in ascending order. The resulting Nahm data satisfy

$$\begin{aligned}
&\langle T_1^+, T_2^+\rangle = \langle T_2^+, T_3^+\rangle = \langle T_3^+, T_1^+\rangle = 0,\\
&\langle T_1^+, T_1^+\rangle \le \langle T_2^+, T_2^+\rangle \le \langle T_3^+, T_3^+\rangle.
\end{aligned} \tag{4.3}$$

Let

$$\chi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \chi_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

be a basis of $\mathfrak{su}(2)$. Since T_3^+ is symmetric at $t = 1$, there is a matrix $P \in SO(2)$ which diagonalizes $T_3^+(1)$, i.e. $RT_3^+(1)R^{-1}$ is diagonal. Let $g : [0, 1] \to SU(2)$ be the unique solution to

$$\frac{dg}{dt} = gT_0^+ \tag{4.4}$$

with $g(1) = P$. We claim that

Lemma 4.2.

$$(gT_j^+ g^{-1})(1) = \lambda_j \chi_j$$

with $\lambda_j = \langle T_j^+(1), T_j^+(1)\rangle^{1/2}$ *for* $j = 2, 3$ *and* $\lambda_1 = 0$*.*

Proof. We see immediately that this is true for $j = 3$. Then as orthogonality is preserved under conjugations, (4.3) implies that

$$\begin{aligned}(gT_1^+ g^{-1})(1) &= a\chi_1 + b\chi_2, \\ (gT_2^+ g^{-1})(1) &= c\chi_1 + d\chi_2,\end{aligned}$$

for some constants $a, b, c, d \in \mathbb{R}$. The fact that the solutions are symmetric at $t = 1$ means a and c must both be zero, so the claim is true for $j = 2$. Finally, orthogonality implies b or d is zero, but since $b^2 \leq d^2$, we must have $b = \lambda_1 = 0$ and the lemma is proved. □

Thus, the quadruple

$$(0, gT_1^+ g^{-1}, gT_2^+ g^{-1}, gT_3^+ g^{-1}) \tag{4.5}$$

is the unique solution to Nahm's equations over $[0, 1]$, satisfying $T_i(1) = \lambda_i \chi_i$. It turns out that such solutions must be given by the following ansatz:

$$T_0 \equiv 0, \quad T_i = -\frac{f_i}{2}\chi_i, \quad \text{for } i = 1, 2, 3, \tag{4.6}$$

where $\{f_j\}_{j=1}^3$ is a solution to the Euler's equations:

$$\frac{df_1}{dt} = f_2 f_3 \tag{4.7}$$

and two other equations are obtained by cyclic permutations. For the solutions given by (4.5), f_i must additionally satisfy

$$f_1^2 \leq f_2^2 \leq f_3^2, \quad f_i(1) = -2\lambda_i, \quad i = 1, 2, 3. \tag{4.8}$$

There are two cases: either all the f_j are constant, or they are all non-constant:

Lemma 4.3. *Suppose $\{f_j\}_{j=1}^3$ is a solution to the Euler's equations satisfying* (4.8)*. If f_i is constant for some i, then f_3 must also be constant. Moreover, $(f_1, f_2, f_3) = (0, 0, -2\lambda_3)$ is the only constant solution to the Euler's equations.*

Proof. Suppose f_1 is constant. Since $f_1(1) = -2\lambda_1 = 0$, f_1 must be identically zero. Then equations (4.7) imply f_3 is constant.

Suppose f_2 is constant, then from (4.7) we have $f_3(t)f_1(t) = 0$ for all t. Suppose $f_3(\tilde{t}) = 0$ for some $\tilde{t}$. Then by the inequalities, $f_2(\tilde{t})$ is zero and so $f_2 \equiv f_1 \equiv 0$. Now f_1 is constant, so by the first case, f_3 is constant. If f_3 is never zero, then $f_1 \equiv 0$ and once again f_3 must be constant.

Now suppose f_3 is constant, so $f_3 \equiv -2\lambda_3$. The Euler's equations give $f_1 f_2 \equiv 0$ and

$$\begin{aligned}
\frac{df_1}{dt} &= -2\lambda_3 f_2, \\
\frac{df_2}{dt} &= -2\lambda_3 f_1.
\end{aligned}$$

The general solution to these equations is of the form

$$\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = \begin{pmatrix} a\cosh(-2\lambda_3 t) + b\sinh(-2\lambda_3 t) \\ a\sinh(-2\lambda_3 t) + b\cosh(-2\lambda_3 t) \end{pmatrix}.$$

Suppose f_2 vanishes at some point $\tilde{t}$, then $f_1(\tilde{t}) = 0$. Multiplying these equations and rearranging, we obtain $(a^2 - b^2)\sinh(\lambda_3 \tilde{t}) = 0$, i.e. $a^2 = b^2$ or $\tilde{t} = 0$. If $a^2 \neq b^2$ holds, then $\tilde{t} = 0$ and this implies $a = b = 0$, a contradiction. Therefore $a^2 = b^2$. For $a = \pm b$, The equations $f_1(1) = 0$ and $f_1(\tilde{t}) = 0$ imply

$$\begin{aligned}
a(1 \pm \tanh(-2\lambda_3)) &= 0, \\
a(1 \pm \tanh(-2\lambda_3 \tilde{t})) &= 0.
\end{aligned}$$

Solving the equations yields $a = b = 0$ or $\tilde{t} = 1$. The former case implies $f_1 \equiv f_2 \equiv 0$. In the latter case, it implies f_2 is only zero at $t = 1$. Bear in mind that $f_1 f_2 = 0$, we must have $f_1 \equiv 0$. From (4.7), f_1 is constant means that $f_3 f_2 = -2\lambda_3 f_2 = 0$, which implies $f_2 \equiv 0$.

Suppose f_2 is nowhere vanishing. Then $f_1 \equiv 0$, but it is constant means that $f_3 f_2 = -2\lambda_3 f_2 = 0$, which implies $f_2 \equiv 0$, a contradiction. □

Now suppose $\{f_j\}_{j=1}^3$ is a non-constant solution to the Euler's equations satisfying (4.8). We shall show that they are given in terms of the Jacobi elliptic functions. Since $f_1^2 \leq f_2^2 \leq f_3^2$, there are constants of integration given by

$$\begin{aligned}
c_{21}^2 &= f_2^2 - f_1^2, \\
c_{31}^2 &= f_3^2 - f_1^2, \\
c_{32}^2 &= f_3^2 - f_2^2.
\end{aligned} \tag{4.9}$$

Suppose f_j has the form

$$f_j(t) = DF_j(u),$$

where $D > 0$ and F_j satisfy the Euler's equations. For $p \in \mathbb{R}$, we let

$$u = D(t + p).$$

Then

$$\begin{aligned} F_1^2 &= -(c_{31}/D)^2 + F_3^2, \\ F_2^2 &= -(c_{32}/D)^2 + F_3^2. \end{aligned}$$

For some appropriate $0 \leq k \leq 1$, we may set

$$c_{21}^2 = D^2(1 - k^2), \quad c_{31}^2 = D^2, \quad c_{32}^2 = D^2 k^2,$$

which yield

$$\left(\frac{dF_3}{du}\right)^2 = (F_3^2 - 1)(F_3^2 - k^2).$$

Put $y = -1/F_3$, then the equation can be re-written in terms of y:

$$\left(\frac{dy}{du}\right)^2 = (1 - y^2)(1 - k^2 y^2). \tag{4.10}$$

Let

$$v = \int_0^\phi = \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \quad k \in [0, 1].$$

It is well-known that the *Jacobi elliptic function*

$$\mathrm{sn}_k(v) = \sin \phi \tag{4.11}$$

solves (4.10). The conclusion is that f_i must have the form

$$\begin{aligned} f_1(t) &= \pm_a \frac{D\mathrm{cn}_k(D(t+p))}{\mathrm{sn}_k(D(t+p))}, \\ f_2(t) &= \pm_b \frac{D\mathrm{dn}_k(D(t+p))}{\mathrm{sn}_k(D(t+p))}, \\ f_3(t) &= \pm_c \frac{D}{\mathrm{sn}_k(D(t+p))}, \end{aligned} \tag{4.12}$$

where

$$\mathrm{cn}_k(v) = \cos \phi, \quad \mathrm{dn}_k(v) = \sqrt{1 - k^2 \sin^2 \phi}. \tag{4.13}$$

The indices a, b, c in (4.12) are inserted to emphasise that the signs do not necessarily share the same index; the possibilities are either all negative or two positive. Note that the f_j have a pole at $-p$.

Let us now consider the ansatz given in (4.6), where f_i are of the form in (4.12). By virtue of the $SO(3)$-action, we may assume that all the signs in f_j are negative. As the solutions need to be non-singular throughout $[0, 1]$, we must either have $p > 0$ or $p < -1$. But the symmetricity condition at $t = 1$

implies that

$$K(k) = D(1+p), \tag{4.14}$$

where

$$K = K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n-1)!!}{(2n)!!}\right]^2 k^{2n} \tag{4.15}$$

is *the complete elliptic integral of the first kind*. Since K, D are both positive, p can never be negative. Hence $p > 0$. Note that the constant solutions correspond to the ansatz by taking the limit $k \to 1$ and $p \to \infty$ simultaneously. More precisely, we have just shown the following:

Proposition 4.4. *Up to the actions of $\mathbb{R}^3$, T^2 and $SO(3)$, any point in $N_{1,2}$ is given by*

$$\begin{aligned} \tilde{T}_0^- &\equiv 0, & \tilde{T}_0^+ &= \frac{dg}{dt}g^{-1}, \\ \tilde{T}_j^- &\equiv (T_j^+)_{11}, & \tilde{T}_j^+ &= -\frac{f_j}{2}g\chi_j g^{-1}, \quad j = 1,2,3, \end{aligned}$$

where g is some $SU(2)$-valued function on $[0,1]$ such that $g(1) \in SO(2)$;

$$\begin{aligned} f_1(t) &= -\frac{D\mathrm{cn}_k(D(t+p))}{\mathrm{sn}_k(D(t+p))}, \\ f_2(t) &= -\frac{D\mathrm{dn}_k(D(t+p))}{\mathrm{sn}_k(D(t+p))}, \\ f_3(t) &= -\frac{D}{\mathrm{sn}_k(D(t+p))}, \end{aligned}$$

with $0 \le k \le 1$, $0 < p \le \infty$ and $D(k,p) = K(k)/(1+p)$. Note that, as $k \to 1$, we have $p \to \infty$, so that the limit of D is finite.

For the interpretation of g, see below. Observe that if we let $p \to 0$, the Nahm data T^+ approaches a monopole of charge 2: the solution would then have pole at $t = 0$ with residues defining a two-dimensional irreducible representation of $\mathfrak{su}(2)$. Therefore, we may view p as a separation parameter which controls the distance between the two monopoles.

4.1.2 Special Action on $N_{1,2}$

In the proof of the above proposition, one uses a function

$$g : [0,1] \to SU(2)$$

with $g(1) \in SO(2)$ to gauge T_0^+ to zero. We shall give a geometrical interpretation of such map. Let $\mathcal{E}$ be the group of all such g. Then $\mathcal{E}$ acts on $N_{1,2}$ as follows: for any $g \in \mathcal{E}$, its action on T^+ is by gauge transformations, whereas

T^- gets mapped by

$$\begin{aligned} T_0^- &\mapsto T_0^-, \\ T_j^- &\mapsto (gT_j^+g^{-1})_{11}, \quad j = 1,2,3. \end{aligned} \tag{4.16}$$

In particular, $\mathcal{E}$ acts on the spectral curves that arise from $N_{1,2}$. Notice that such action *does not* descend to an action on $N_{1,2}/\mathcal{G}_0$, since in general it does not preserve the $\mathcal{G}_0$-orbits. Nevertheless, it does act on $\Sigma_{1,2}$, and can be described in terms of a nice picture, which we shall illustrate now.

Given $(T^-, T^+) \in N_{1,2}$, the location of S^- is given by

$$-i(T_1^+(0)_{11}, T_2^+(0)_{11}, T_3^+(0)_{11}) \in \mathbb{R}^3. \tag{4.17}$$

Suppose the solutions are in the form given by Proposition (4.4). If we identify $\mathbb{R}^3 \cong \mathfrak{su}(2)$ by the isomorphism $\phi(e_i) := \chi_i/2$, then the adjoint action on $\mathfrak{su}(2)$ is equivalent to the $SO(3)$-action on $\mathbb{R}^3$. That means, there exists an $A \in SO(3)$ such that $\phi(Ae_i) = \mathrm{Ad}(g(0))\chi_i$. Writing $A = (a_{ij})$, the location of S^- is then given by

$$-\frac{1}{2}(f_1(0)a_{13}, f_2(0)a_{23}, f_3(0)a_{33}).$$

As T_j^+ are trace-free for $j = 1,2,3$, the centre of S^+ is at the origin, hence the distance between S^-, S^+ is simply the Euclidean norm of the centre of S^-:

$$R^2 = \frac{D^2}{4\mathrm{sn}_k^2(Dp)}\left[a_{13}^2\mathrm{cn}_k^2(Dp) + a_{23}^2\mathrm{dn}_k^2(Dp) + a_{33}^2\right]. \tag{4.18}$$

If $(x_1, x_2, x_3) \in \mathbb{R}^3$ represents the position of S^-, then it satisfies

$$\frac{x_1^2}{(f_1(0))^2} + \frac{x_2^2}{(f_2(0))^2} + \frac{x_3^2}{(f_3(0))^2} = \frac{1}{4}. \tag{4.19}$$

That means, if we view the curves as objects in $\mathbb{R}^3$, then for fixed k, p, the $\mathcal{E}$-orbit of (S^-, S^+) fixes S^+ since it acts on T^+ by gauge transformation, but the position of S^- traces out an ellipsoid. In particular, for $k = 0$ we have $f_1(0) < f_2(0) = f_3(0)$, the ellipsoid then becomes axially symmetric about the x_1-axis, and it is called *oblate ellipsoid of revolution.* If we are restricted to the region where $0 \le k < \delta < 1$ for some fixed δ, then from (4.18), we see that $p \to 0$ and $R^2 \sim (1/2p)^2$ as $R \to \infty$. Hence

$$x_1^2 + x_2^2 + x_3^2 \sim R^2, \tag{4.20}$$

i.e. the ellipsoid is asymptotically a sphere. Therefore, for any fixed $(S_0^-, S_0^+) \in \Sigma_{1,2}$, the $g \in \mathcal{E}$ may be viewed as a parametrization of

$$\left\{ (S^-, S^+) \in \Sigma_{1,2} \,\middle|\, S^+ = S_0^+ \right\}$$

as an ellipsoid.

4.1.3 Asymptotic Behaviour

In (4.20), we have considered the following region:

Definition 4.5. *Let $0 < \delta < 1$ be any fixed number. Then $N_{1,2}(\delta)$ is defined to be*

$$\{ T \in N_{1,2} \,|\, 0 < p < \infty,\, 0 \leq k < \delta < 1 \} .$$

Note that each $N_{1,2}(\delta)$ contains the asymptotic region of $N_{1,2}$ where p is sufficiently small and that the monopole-cluster metric is positive-definite. We shall prove the following:

Proposition 4.6. *Let $0 < \delta < 1$ be fixed, and let (U, ζ) be an affine chart of $\mathbb{P}^1$. For any $(S^-, S^+) \in \Sigma_{1,2}$ that arises from $N_{1,2}(\delta)$. Then, up to the $SO(3)$-action, we have*

$$|\zeta_+ - \zeta_-| = O(1/R) \quad \textit{as } R \to \infty,$$

where $\zeta_-, \zeta_+ \in \mathbb{C}$ correspond to a pair of non-antipodal points of $\pi(S^- \cap S^+)$ in U; R is the Euclidean distance between the centres of S^- and S^+.

Proof. Since the actions of $\mathbb{R}^3$, T^2 and $SO(3)$ on $N_{1,2}$ all preserve the distance R, it is sufficient to consider solutions of the form given in Proposition (4.4). We shall first compare

$$u = K - D$$

with respect to R, and then use this result to compute the rate of convergence of ζ_-, ζ_+ in terms of R. Consider the Taylor expansion of the Jacobi elliptic functions for small u:

$$\begin{aligned}
\mathrm{cn}_k(u) &= 1 - u^2/2 + O(u^4), \\
\mathrm{dn}_k(u) &= 1 - k^2u^2/2 + O(u^4), \\
\mathrm{sn}_k(u) &= u - (1 + k^2)u^3/6 + O(u^4).
\end{aligned}$$

From (4.18) and the relation $u = Dp$, we see that $R \to \infty$ if and only if $u \to 0$. Moreover, R^2 can be expressed in terms of u:

$$R^2 = \frac{D^2}{4u^2}(1 + O(u^2)).$$

After some calculation, we can change the role between u and R:

$$u = \frac{D}{2R}(1 + O(R^{-2})).$$

Using this, we may write

$$T_j^+(0) = R(1 + b_j u^2 + O(u^4))\chi_j, \quad \text{for } j = 1, 2, 3,$$

where the b_j are some constants uniformly bounded in k. Over U, we have

$$\beta_+(\zeta,0) = Ri\begin{pmatrix} i(1-\zeta^2)+O(u^2) & (1-\zeta)^2+O(u^2) \\ (1+\zeta)^2+O(u^2) & -i(1-\zeta^2)+O(u^2) \end{pmatrix}.$$

Using the $SO(3)$-action if necessary, we may assume $\pi(S^-\cap S^+)\subset U$, where $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$ is the projection map. Then each point in $\pi(S^-\cap S^+)$ corresponds to the conditions that $\beta_-(\zeta), \beta_+(\zeta)$ have either a common eigenvector or a common eigen-covector, with the same eigenvalue at $t=0$. In our case, the existence of a common eigenvector (resp. eigen-covector) implies that $\beta_+(\zeta,0)$ is upper-triangular (resp. lower-triangular). Let $h=g(0)$, where g is given in Proposition (4.4). If $h\beta_+(\zeta,0)h^{-1}$ is upper-triangular, then the vanishing of its $(2,1)$-entry gives us the following constraint:

$$\begin{aligned}\left((\bar{h}_{11}-i\bar{h}_{12})^2+O(u^2)\right)-2\left(\bar{h}_{11}^2+\bar{h}_{12}^2+O(u^2)\right)\zeta& \\ +\left((\bar{h}_{11}+i\bar{h}_{12})^2+O(u^2)\right)\zeta^2&=0.\end{aligned}$$

Solving for ζ we obtain

$$\zeta_\pm = \frac{\left(\bar{h}_{11}^2+\bar{h}_{12}^2+O(u^2)\right)\pm\sqrt{O(u^2)}}{(\bar{h}_{11}+i\bar{h}_{12})^2+O(u^2)}.$$

Now compare the distance $|\zeta_+-\zeta_-|$ in terms of R:

$$|\zeta_+-\zeta_-| = \left|\frac{\sqrt{O(u^2)}}{(\bar{h}_{11}+i\bar{h}_{12})^2+O(u^2)}\right| = O(u).$$

But since we proved that $u\sim D/2R$, the result follows. □

4.2 Spectral Data for $M_{1,2}$

Let us recall the spectral data for $M_{1,2}$: if $(S^-,\nu^-,S^+,\nu^+)\in M_{1,2}$, then S^-, S^+ are compact real curves in $T\mathbb{P}^1$, and ν^-, ν^+ are meromorphic sections of L^2 over S^-, S^+. If (U,ζ) is an affine chart, then any curves S^-, S^+ may be represented over $\pi^{-1}(U)$ by

$$\begin{aligned} S^- :&\quad \eta+q(\zeta)=0,\\ S^+ :&\quad \eta^2+a_1(\zeta)\eta+a_2(\zeta)=0,\end{aligned}$$

where $a_j(\zeta)$ is a polynomial of degree j and $q(\zeta)$ is a quadratic polynomial. Generically, the intersection of S^- and S^+ consists of four points in $T\mathbb{P}^1$. Suppose for each point $a\in\pi(S^-\cap S^+)$, we have $\zeta(a)\neq 0$ (or we choose a different affine chart). Let us view $\pi(S^-\cap S^+)$ as a set of points in $\mathbb{C}$, and we denote those points by α_i,β_i, $i=1,2$, where $\beta_i=-1/\bar{\alpha}_i$. By definition, there is a divisor D of S^- (resp. S^+) such that the equality $S^-\cap S^+ = D+\tau(D)$

holds as divisors of S^+ (resp. S^-). Without loss of generality, we may assume $\alpha_1, \alpha_2 \in \pi(D)$ and $\beta_1, \beta_2 \in \pi(\tau(D))$, where D here is viewed as a set of points in $T\mathbb{P}^1$.

4.2.1 Explicit Expression of Meromorphic Sections

As for spectral curves, $\nu^\pm$ can also be represented by functions $f^\pm$ locally, and it is our task to find explicit expressions for them. Let us first compute f^-: if we write

$$q(\zeta) = -i\left((x_2 + ix_3) + 2ix_1\zeta + (x_2 - ix_3)\zeta^2\right),$$

then since ν^- is a section of $L^2 \simeq [\tau(D) - D]$ over S^-, we must have

$$f^-(\zeta,\eta) = \varsigma e^{-2i(ix_1 + (x_2 - ix_3)\zeta)} \frac{(\zeta-\beta_1)(\zeta-\beta_2)}{(\zeta-\alpha_1)(\zeta-\alpha_2)}$$

over $S^- \cap \pi^{-1}(U)$. There is a corresponding function $\tilde{f}^-$ over $S^- \cap \pi^{-1}(\tilde{U})$, which relates to f^- by $\tilde{f}^- = e^{2\eta/\zeta} f^-$ on the overlap. Moreover, as ν^- satisfies $\nu^-\sigma(\nu^-) = 1$, it turns out that such condition is equivalent to

$$|\varsigma| = 1. \tag{4.21}$$

To obtain an expression for ν^+, let us first use the $\mathbb{R}^3$ and $SO(3)$-actions to put S^+ into the form

$$\eta^2 = r_1\zeta^3 - r_2\zeta^2 - r_1\zeta, \tag{4.22}$$

where $r_1 \geq 0$, $r_2 \in \mathbb{R}$. There are two cases to deal with: either S^+ is reducible or is smooth, corresponding to $r_1 = 0$ or $r_1 > 0$ respectively.

Suppose we are in the former case. Then S^+ can be written as

$$P_+(\zeta,\eta) = (\eta - a_1\zeta)(\eta - a_2\zeta) = 0, \tag{4.23}$$

where $a_1 = i\sqrt{r_2}, a_2 = -i\sqrt{r_2}$. It can be seen that S^+ has singularities at $(\zeta,\eta) = (0,0)$ and $(\infty, 0)$; to represent ν^+ on this curve, we first need to understand what is meant by a holomorphic function around these singularities. Let $\mathscr{J} = \langle P_+ \rangle$ be the ideal generated by P_+. A holomorphic function on $S^+ \cap U$ is a local section of the quotient sheaf $\mathcal{O}/\mathscr{J}$ near $(0,0)$: it is represented uniquely by

$$h = g_1(\zeta) + \eta g_2(\zeta),$$

where g_1, g_2 are holomorphic functions in ζ. For each $j = 1, 2$, we have a holomorphic function

$$h_j = g_1(\zeta) + a_j\zeta g_2(\zeta)$$

over the component $\eta = a_j\zeta$. Suppose we are given the holomorphic functions h_1, h_2 on the respective components, we wish to work out the correct condition to impose so that they define a local section of $\mathcal{O}/\mathscr{J}$. Write

$$g_j = g_{j,0} + g_{j,1}\zeta + \dots,$$
$$h_j = h_{j,0} + h_{j,1}\zeta + \dots.$$

We see that

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 1 & a_1\zeta \\ 1 & a_2\zeta \end{pmatrix} \begin{pmatrix} g_{1,0} \\ 0 \end{pmatrix} + \zeta \begin{pmatrix} 1 & a_1 \\ 1 & a_2 \end{pmatrix} \left[\begin{pmatrix} g_{1,1} \\ g_{2,0} \end{pmatrix} + \zeta \begin{pmatrix} g_{1,2} \\ g_{2,1} \end{pmatrix} + \dots \quad \right].$$

For $a_1 \neq a_2$, the matrix

$$\begin{pmatrix} 1 & a_1 \\ 1 & a_2 \end{pmatrix}$$

is non-singular, and there exists constants $g_{1,(i+1)}, g_{2,i}$, $i \geq 0$, solving $h_{1,k}, h_{2,k}$ uniquely for each $k \geq 1$. This is always true for generic monopole-clusters since their spectral curves do not have multiple components, implying that $a_1 \neq a_2$. For $h_{1,0}, h_{2,0}$, there is the following compatibility condition:

$$h_1(0) = h_2(0). \tag{4.24}$$

Any meromorphic function h in a neighbourhood of $(0,0)$ on S^+ is by definition a quotient of two sections of $\mathcal{O}/\mathscr{J}$ near $(0,0)$, so we deduce that h is equivalent to h_1, h_2 such that $h_j = p_j/q_j$, $j = 1,2$, where p_j and $q_j \not\equiv 0$, are holomorphic functions on $\eta = a_j\zeta$ satisfying $p_1(0) = p_2(0)$ and $q_1(0) = q_2(0)$.

For $j = 1,2$, we let

$$p_j(\zeta) = \xi_j(\zeta - \alpha_j),$$
$$q_j(\zeta) = (1 + \bar{\alpha}_j\zeta),$$

where $\xi_1, \xi_2 \in \mathbb{C} \setminus \{0\}$. It is clear that q_j satisfy the compatibility condition; p_j satisfy the condition if and only if

$$\xi_1\alpha_1 = \xi_2\alpha_2. \tag{4.25}$$

Assuming (4.25) holds, then

$$f^+(\zeta,\eta) := \begin{cases} p_1(\zeta)/q_1(\zeta) & \text{if } \eta = a_1\zeta \\ p_2(\zeta)/q_2(\zeta) & \text{if } \eta = a_2\zeta \end{cases} \tag{4.26}$$

defines a meromorphic function on $S^+ \cap \pi^{-1}(U)$. Similarly, providing that the corresponding compatibility condition

$$\xi_1\bar{\alpha}_2 e^{a_1} = \xi_2\bar{\alpha}_1 e^{a_2} \tag{4.27}$$

also holds, there is a meromorphic function $\tilde{f}^+$ over $S^+ \cap \pi^{-1}(\tilde{U})$ which relates to f^+ by $\tilde{f}^+ = e^{2\eta/\zeta} f^+$ on the overlap, and hence f^+, $\tilde{f}^+$ are local representative function of ν^+ on S^+.

Now suppose $r_1 > 0$. We shall parametrize the curve S^+ using the *Weierstrass $\wp$-function*

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega \in \mathbb{L}\backslash\{0\}} \frac{1}{(u+\omega)^2} - \frac{1}{\omega^2} \tag{4.28}$$

and its derivative, where $\mathbb{L}$ is a lattice in the complex plane. Following [28], let

$$\begin{aligned} \eta &= k_1 \wp'(u), \\ \zeta &= \wp(u) + k_2, \end{aligned} \tag{4.29}$$

where $k_1 = \sqrt{r_1/4}$, $k_2 = r_2/3r_1$. It is readily checked that (4.22) becomes

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

with $g_2 = 12k_2^2 + 4$ and $g_3 = 8k_2^3 + 4k_2$. The lattice $\mathbb{L}$ here is determined by S^+: since the curve is real and the real structure τ has no fixed point, the lattice must be rectangular, with positive real and imaginary generators given by ω_r, ω_i. Let us denote the points on the u-plane corresponding to $\tilde{\zeta} = 0$ by $\pm u_\infty$. In addition, let $u_{\alpha_1}, u_{\alpha_2}, u_{\beta_1}, u_{\beta_2}$ be points on the u-plane that represent the intersections points. Then over $\pi^{-1}(U) \cap S^+$, ν_+ is given by

$$\begin{aligned} f^+(\zeta, \eta) = C \exp\{2k_1(\zeta_W(u - u_\infty) + \zeta_W(u + u_\infty) + cu)\} \\ \times \frac{\sigma(u - u_{\alpha_1})\sigma(u - u_{\alpha_2})}{\sigma(u - u_{\beta_1})\sigma(u - u_{\beta_2})}, \end{aligned} \tag{4.30}$$

where $c, C \in \mathbb{C}$; ζ_W and σ are the *Weierstrass ζ-function* and the *Weierstrass σ-function* respectively:

$$\begin{aligned} \zeta_W(u) &= \frac{1}{u} + \sum_{\omega \in \mathbb{L}\backslash\{0\}} \left(\frac{1}{u - \omega} + \frac{1}{\omega} + \frac{u}{\omega^2} \right), \\ \sigma(u) &= u \prod_{\omega \in \mathbb{L}\backslash\{0\}} \left(1 - \frac{u}{\omega}\right) e^{u/\omega + \frac{1}{2}(u/\omega)^2}. \end{aligned} \tag{4.31}$$

Note that c cannot be arbitrary: for f^+ to represent ν^+, it is necessarily doubly-periodic in the u-coordinate, and this imposes a restriction on the set of values that c may take. We shall discuss this constraint shortly.

Recall that ν^+ requires to satisfy $\nu^+\sigma(\nu^+) = 1$. In the reducible case, such condition implies that the constants ξ_1, ξ_2 in (4.26) have a fixed modulus, whereas in the smooth case, the same is true for C in (4.30). In general, the modulus of these constants are difficult to determine explicitly, and their value are particularly important if one were to compute the metric for $M_{1,2}$. Nevertheless, as we shall see in the next chapter, the modulus of ξ_1, ξ_2 can be obtained for certain axially symmetric submanifold of $M_{1,2}$.

4.2.2 Constraints

Let $(S^-, S^+) \in \Sigma_{1,2}$. Note that in general, there is no constraint on S^- for the existence of sections on L^2, which can be seen directly from the Riemann-Roch theorem. Suppose S^+ is a reducible curve with equation of the form (4.23). Then the conditions (4.25), (4.27) are the sufficient and necessary conditions for $L^2[\tau(D) - D] \simeq \mathcal{O}$. On the other hand, such conditions are equivalent to the following constraint on S^+:

$$|\alpha_1|^2 e^{a_2} = |\alpha_2|^2 e^{a_1}. \tag{4.32}$$

In other words, we have

Lemma 4.7. *Let $(S^-, S^+) \in \Sigma_{1,2}$ and suppose S^+ is reducible. Then $L^2[\tau(D) - D] \simeq \mathcal{O}$ over S^+ if and only if* (4.32) *holds.*

Suppose now S^+ is smooth, then the analogous constraint to (4.32) is given by the next lemma:

Lemma 4.8. *Suppose f^+ is a function of the form* (4.30)*, and let*

$$\gamma = 4k_1 + u_{\beta_1} + u_{\beta_2} - u_{\alpha_1} - u_{\alpha_2}.$$

Then the following statements are equivalent:

(a) *f^+ is doubly-periodic for some $c \in \mathbb{C}$.*

(b) *There exists $c \in \mathbb{C}$ such that for all $\omega \in \mathbb{L}$,*

$$\exp\{\eta_W(\omega)\gamma + c\omega\} = 1. \tag{4.33}$$

(c) *$\gamma \in \mathbb{L}$.*

Proof. (a) to (b) is a direct consequence of the following monodromy property of ζ_W and σ: for any $\omega \in \mathbb{L}$, note that

$$\begin{aligned} \zeta_W(u+\omega) &= \zeta_W(u) + \eta_W(\omega), \\ \sigma(u+\omega) &= \sigma(u)\psi(\omega)\exp\left\{\eta_W(\omega)\left(u + \omega/2\right)\right\}, \end{aligned}$$

where ψ is some function on $\mathbb{L}$ which takes values ± 1, and η_W is defined precisely by the first equation.

(b) to (a) is trivial.

For (b) to (c), first suppose (b) holds. Then in particular, (4.33) is true for ω_r and ω_i too. This means there are integers $n_r, n_i \in \mathbb{Z}$ such that

$$\begin{aligned} \eta_W(\omega_r)\gamma + c\omega_r &= 2\pi i n_r, \\ \eta_W(\omega_i)\gamma + c\omega_i &= 2\pi i n_i. \end{aligned}$$

By Legendre's relation

$$\eta_W(\omega_r)\omega_i - \eta_W(\omega_i)\omega_r = 2\pi i, \tag{4.34}$$

we obtain

$$\gamma = n_r\omega_i - n_i\omega_r \in \mathbb{L},$$

hence (c) holds.

Conversely, if $\gamma \in \mathbb{L}$, choose

$$c = -\eta_W(\gamma). \tag{4.35}$$

Since for any lattice points $\omega, \omega' \in \mathbb{L}$, if they are written as

$$\begin{aligned} \omega &= n_r\omega_r + n_i\omega_i, \\ \omega' &= n_r'\omega_r' + n_i'\omega_i', \end{aligned}$$

then the Legendre's relation implies

$$\eta_W(\omega)\omega' - \eta_W(\omega')\omega = 2k\pi i,$$

where $k = n_r n_i' - n_i n_r' \in \mathbb{Z}$. It follows from this that (b) is satisfied. $\square$

By the lemma, if γ is a lattice point, then it implies the existence of a global trivialization of $L^2[\tau(D) - D]$ over S^+. In fact, the converse is also true:

Lemma 4.9. *Let $(S^-, S^+) \in \Sigma_{1,2}$ and suppose S^+ is smooth. Then $L^2[\tau(D) - D] \simeq \mathcal{O}$ on S^+ if and only if $\gamma \in \mathbb{L}$.*

Proof. Suppose $L^2[\tau(D) - D]$ is trivial on S^+. A global meromorphic section of L^2 is equivalent to a pair of meromorphic functions $f^+, \tilde{f}^+$ on $S^+ \cap U$, $S^+ \cap \tilde{U}$ respectively such that $\tilde{f}^+ = e^{2\eta/\zeta} f^+$ on the overlap. Taking $d\log$ yields

$$d\log \tilde{f}^+ = d(2\eta/\zeta) + d\log f^+.$$

Consider the differential $d(2\eta/\zeta)$. Pulling back to $\mathbb{C}/\mathbb{L}$ gives

$$d(2\eta/\zeta) = 2k_1 d(2\wp'/(\wp + k_2)) = (4k_1/u^2 + O(1))du$$

near 0. Hence, in terms of u-coordinate, $d\log f^+$ has precisely a double pole at 0 and single pole at $u_{\alpha_i}, u_{\beta_i}$, $i = 1, 2$, up to periods. As $\frac{d}{du}(\log f^+)$ is a well-defined doubly-periodic function, the general theory of elliptic functions implies that it can be written as

$$-4k_1\wp(u) + \sum_{i=1}^{2} \zeta_W(u - u_{\alpha_i}) - \zeta_W(u - u_{\beta_i}) + c \tag{4.36}$$

for some $c \in \mathbb{C}$. For each $\omega \in \mathbb{L}$, if the period of $d \log f^{+}$ over the corresponding loop Γ is $2\pi n(\omega)i$, then

$$\begin{aligned}
2\pi n(\omega)i &= \int_{\Gamma} d\log f^{+} \\
&= \int_{u_0}^{u_0+\omega} \frac{d}{du}\left(\log f^{+}\right) du \\
&= \left(4k_1\zeta_W(u) + \log\frac{\sigma(u-u_{\alpha_1})\sigma(u-u_{\alpha_2})}{\sigma(u-u_{\beta_1})\sigma(u-u_{\beta_2})} + cu\right)\Bigg|_{u_0}^{u_0+\omega} \\
&= \gamma\eta_W(\omega) + c\omega \pmod{2\pi ni}.
\end{aligned}$$

This implies (4.33) holds. The previous lemma then gives the result. □

The constraint on S^{+}, given by (c) in Lemma (4.8), is equivalent to the following more general looking constraint:

$$\sum_{i=1}^{2}\int_{\alpha_i}^{\beta_i}\frac{d\zeta}{\sqrt{\eta}} = -4 \pmod{\Lambda}, \tag{4.37}$$

where Λ is the lattice generated by the periods of $\frac{d\zeta}{\sqrt{\eta}}$. The nature of the constraint is transcendental, i.e. there is in general no systematic algebraic method to solving it. Nevertheless, since generic monopole-clusters correspond to Nahm data, the spectral curves obtained there will automatically satisfy it. On the other hand, for any given pair of compact real curves $(S^{-}, S^{+}) \in |\pi^{*}\mathcal{O}(2)| \times |\pi^{*}\mathcal{O}(4)|$, one can find a new pair of curves which satisfies the constraint using a rescaling argument, though it is in general difficult to determine the rescaling constant explicitly. We remark that, with the knowledge of the spectral curves for $(1,2)$-clusters, it is actually possible to construct curves that solve the corresponding constraint for the $(1,\ldots,1,2)$-clusters. We remark also that, for smooth spectral curves with charge greater than 2, meromorphic sections of L^2 over them may be given in terms of *theta functions*, which are in general much more complicated than elliptic functions, and in turn giving harder constraints.

4.3 Totally Geodesic Submanifolds of $M_{1,2}$

In this section, we shall classify the subgroups of certain group action that have at least one fixed point in the region $N_{1,2}(1)/\mathcal{G}_0$, and then their fixed point sets will be computed.

According to Proposition (4.4), any element $(T^{-}, T^{+}) \in N_{1,2}/\mathcal{G}_0$ can be represented by the Nahm data

$$\begin{aligned}
T_0^+ &= \theta i Id - \dot{g}g^{-1}, & T_0^- &= \phi i, \\
T_j^+ &= g\left(\sum_{l=1}^{3} a_{jl}\tilde{T}_l^+\right)g^{-1} + x_j i Id, & T_j^- &\equiv \left(T_j^+(0)\right)_{11} \quad j = 1,2,3,
\end{aligned} \tag{4.38}$$

where $(\tilde{T}_0, \tilde{T}_1, \tilde{T}_2, \tilde{T}_3)$ is the standard solution. Recall from Lemma (3.9) that the $\widetilde{SO(2)}$ group is generated by

$$R_\alpha := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \quad \alpha \in [0, 2\pi), \tag{4.39}$$

which acts on $N_{k,l}/\mathcal{G}_0$ by the $SO(3)$-action. Intuitively, this corresponds to a rotational action around the x_1-axis. It has been proved in the last chapter that such action is equivalent to the $SO(2)$-action in $M_{1,2}$, hence it is a group of isometries. In general, for any isometry group on a manifold, it is natural to consider the fixed point set, since it is a totally geodesic submanifold of the manifold. However, there can be no fixed points for the $\widetilde{SO(2)}$-action: as mentioned in Chapter 2 that there is a one-to-one correspondence between $N_{k,l}/\mathcal{G}_0$ and the space of charge (k,l) calarons, it then follows from [20] that the $\widetilde{SO(2)}$-action has no fixed point on the space of $(1,2)$-calarons. From the spectral curve viewpoint, it can be seen that the $\widetilde{SO(2)}$-action fixes a two dimensional subspace in $\Sigma_{1,2}$; we claim that, by coupling with the T^2-action, there are subgroups of

$$\widetilde{SO(2)} \times T^2$$

that have a fixed point in $N_{k,l}(1)/\mathcal{G}_0$. Let $\Delta^0_{SO(2)}$ be the diagonal subgroup of $\widetilde{SO(2)} \times T^2$ generated by elements $R_\alpha \times p_{(\alpha,\alpha/2)}$.

Proposition 4.10. *The fixed point set $\Sigma^0_{SO(2)} \subset N_{1,2}(1)/\mathcal{G}_0$ under the action of $\Delta^0_{SO(2)}$ is given by the set of $\mathcal{G}_0$-orbits that contain the Nahm data in $N_{1,2}(1)$ given by*

$$\begin{aligned}
T_0^+ &= \theta i Id - \dot{g}g^{-1}, & T_0^- &= \phi i, \\
T_1^+ &= g\tilde{T}_1^+ g^{-1} + x_1 i Id, & T_1^- &\equiv \left(T_1^+(0)\right)_{11}, \\
T_2^+ &= g\tilde{T}_2^+ g^{-1}, & T_2^- &\equiv \left(T_2^+(0)\right)_{11}, \\
T_3^+ &= g\tilde{T}_3^+ g^{-1}, & T_3^- &\equiv \left(T_3^+(0)\right)_{11},
\end{aligned}$$

where $k = 0$ and

$$g(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

In particular, $\Sigma^0_{SO(2)}$ is a four-dimensional family with real coordinates given by p, r, θ, ϕ.

Before going into the proof, let us first make some observations. As constant scalar matrices and gauge transformations in $\mathcal{G}_0$ act trivially on the moduli space $N_{1,2}/\mathcal{G}_0$, for any $q \in \mathcal{G}_0$ and T^2-action $p_{(\theta_-,\theta_+)}$, the gauge transformation given by

$$h_{(\theta_-,\theta_+)} := e^{-i\theta_+} q p_{(\theta_-,\theta_+)}$$

is equivalent to $p_{(\theta_-,\theta_+)}$. Conversely, any gauge transformation h that satisfies $h(0) \in U(1) \oplus U(1)$ and $h(1) \in O(2)$ is equivalent to a T^2-action. In the rest of this section, we shall often identify the above two actions implicitly and use them interchangeably. Let

$$h_+(1) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}, \quad \beta \in [0, 2\pi)$$

Also, let $p_1 : \widetilde{SO(2)} \times T^2 \to \widetilde{SO(2)}$ be the first projection. We first show the following:

Proposition 4.11. *Let (T^-, T^+) be an element in $N_{1,2}(1)$ and let G be a subgroup of $\widetilde{SO(2)} \times T^2$ with $p_1(G) = \widetilde{SO(2)}$. If the value of T_j^+, $j = 1, 2, 3$, at $t = 1$ are fixed by G, then G must be generated by $R_\alpha \times h \in G$ with either $\beta = 0, -\alpha/2$ or $\alpha/2$.*

Proof. Let

$$\begin{aligned} \rho : \widetilde{SO(2)} &\to O(2)/\{\pm Id\} \\ R_\alpha &\mapsto [q_\alpha] \end{aligned} \tag{4.40}$$

be a map which satisfies

$$(R_\alpha \times q_\alpha) \cdot T_j^+(1) = T_j^+(1) \quad j = 1, 2, 3 \tag{4.41}$$

and $\rho(Id) = [Id]$. We shall show that ρ is a continuous homomorphism. Let $S_\alpha = R_\alpha \times q_\alpha$. For each $j = 1, 2, 3$,

$$\begin{aligned} S_{\alpha_1} \cdot \left(S_{\alpha_2} \cdot T_j^+(1) \right) &= S_{\alpha_1} \cdot \left(q_{\alpha_2} \left(\sum_{l=1}^{3} (R_{\alpha_2})_{jl} T_l^+(1) \right) q_{\alpha_2}^{-1} \right) \\ &= q_{\alpha_1} q_{\alpha_2} \left(\sum_{m=1}^{3} \sum_{l=1}^{3} (R_{\alpha_1})_{jm} (R_{\alpha_2})_{ml} T_l^+(1) \right) q_{\alpha_2}^{-1} q_{\alpha_1}^{-1} \\ &= q_{\alpha_1} q_{\alpha_2} \left(\sum_{l=1}^{3} (R_{\alpha_1+\alpha_2})_{jl} T_l^+(1) \right) (q_{\alpha_1} q_{\alpha_2})^{-1} \end{aligned}$$

$$\begin{aligned} (S_{\alpha_1} \cdot S_{\alpha_2}) \cdot T_j^+(1) &= (S_{\alpha_1+\alpha_2}) \cdot T_j^+(1) \\ &= q_{\alpha_1+\alpha_2} \left(\sum_{l=1}^{3} (R_{\alpha_1+\alpha_2})_{jl} T_l^+(1) \right) q_{\alpha_1+\alpha_2}^{-1}. \end{aligned}$$

By assumption, they are both equal to $T_j^+(1)$, thus we have

$$QT_j^+(1)Q^{-1} = T_j^+(1) \quad \text{for } j = 1, 2, 3, \tag{4.42}$$

where $Q = q_{\alpha_1} q_{\alpha_2} q_{\alpha_1+\alpha_2}^{-1} \in O(2)$. As the $T_j^+(1)$ belong to $\mathfrak{u}(2)$, they can be written as a linear combination of χ_1, χ_2, χ_3 and iId. Since they are symmetric, their χ_1-component must vanish. There are two cases to consider: $Q \in O(2) \setminus SO(2)$ and $Q \in SO(2)$.

First suppose that $Q \in O(2) \setminus SO(2)$. Then the sum of the χ_2 and χ_3 component of $T_j^+(1)$ must all be proportional, since Q acts as a reflection on the span of χ_2, χ_3. If we make them into the standard form, then (4.3) implies that at least two of the $T_j^+(1)$ are zero, forces $D = 0$ or $k = 1$, a contradiction.

Suppose that $Q \in SO(2)$, then the adjoint action of $SO(2)$ acts freely on the span of χ_2 and χ_3 and leaves iId invariant. If (4.42) were to hold, then we would have either $T_j^+(1) = 0$ or $Q = \pm Id$. It is easy to see that the former case could not happen in the region $N_{1,2}(1)$. Hence $[Q] = [Id]$ and ρ is a homomorphism.

For the proof of continuity, it suffices to show that ρ is continuous at the identity element. Suppose not, then there exists a sequence $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R}$ which converges to zero such that $q_{\alpha_n} \nrightarrow Id$. Since $\{q_{\alpha_n}\}_{n=1}^{\infty}$ is a sequence of the compact set $O(2)$, it has a convergent subsequence $\{q_{\alpha_{nm}}\}_{m=1}^{\infty}$ with limit $q \neq Id$. Now for each integer $m > 0$,

$$q_{\alpha_{nm}} \left(\sum_{l=1}^{3} \left(R_{\alpha_{n_k}} \right)_{jl} T_l^+(1) \right) q_{\alpha_{nm}}^{-1} = T_j^+(1) \quad \text{for } j = 1, 2, 3.$$

Letting $m \to \infty$ to obtain

$$qT_j^+(1)q^{-1} = T_j^+(1)$$

for each j. But again this cannot be true. Therefore ρ is continuous.

We deduce that ρ is a continuous homomorphism of circle groups. It is well-known that such homomorphisms have been completely classified: ρ must be of the form

$$\rho : R_\alpha \mapsto \begin{bmatrix} \cos(l\alpha/2) & -\sin(l\alpha/2) \\ \sin(l\alpha/2) & \cos(l\alpha/2) \end{bmatrix}, \quad l \in \mathbb{Z}. \tag{4.43}$$

The homomorphism may be viewed as a representation of $SO(2)$ on $\mathfrak{u}(2)$, acting by conjugation. Let $\mathbb{R}^3$ be the standard three-dimensional representation of $\widetilde{SO(2)}$ acting by matrix multiplication; E be the one-dimensional trivial representation of $\widetilde{SO(2)}$; Q_m be the two-dimensional irreducible real representation of $\widetilde{SO(2)}$ given by

$$R_\alpha \mapsto \begin{pmatrix} \cos(m\alpha) & -\sin(m\alpha) \\ \sin(m\alpha) & \cos(m\alpha) \end{pmatrix}, \quad m \in \mathbb{Z}. \tag{4.44}$$

As representations, $\mathfrak{u}(2)$ and $\mathbb{R}^3$ can each be written in terms of Q_m and E: since ρ rotates a two-dimensional subspace spanned by χ_2 and χ_3 and fixes the complement, we have

$$\mathfrak{u}(2) \cong Q_l \oplus 2E,$$

whereas for $\mathbb{R}^3$:

$$\mathbb{R}^3 \cong Q_1 \oplus E.$$

It can be shown that, for any $m_1, m_2 \in \mathbb{Z}$, the tensor product representation $Q_{m_1} \otimes Q_{m_2}$ is isomorphic to $Q_{m_1+m_2} \oplus Q_{m_1-m_2}$. Using this result we can compute the representation $\mathbb{R}^3 \otimes \mathfrak{u}(2)$:

$$\mathbb{R}^3 \otimes \mathfrak{u}(2) \cong Q_{1+l} \oplus Q_{1-l} \oplus 2Q_1 \oplus Q_l \oplus 2E. \tag{4.45}$$

The point is to find all the possible maximal invariant subspaces of this representation. Since each summand of the decomposition is an irreducible subrepresentation, there are strictly larger invariant subspaces than $2E$ only when $l = 0$ or ± 1. In other words, it is sufficient to consider only $\beta = 0$ or $\beta = \pm\alpha/2$. $\square$

Proof of Proposition (4.10). Let $h_\alpha := h_{(\alpha,\alpha/2)}$. Suppose we have a point in $N_{1,2}/\mathcal{G}_0$ that is fixed by $\Delta^0_{SO(2)}$, that means in particular, $R_\alpha \times h_\alpha$ fixes the solution at $t = 1$. By proposition (4.11), we only need to consider $\beta = 0$ and $\beta = \pm\alpha/2$.

The idea of the proof is as follows: first of all we show that the case $\beta = 0$ is precluded. Then in the case $\beta = \pm\alpha/2$, as h_α preserves $T_i^+(1)$, $i = 1, 2, 3$, this implies $x_2 = x_3 = 0$, $k = 0$ and determines a set of possible $A \in SO(3)$. After that, the fact that h_α preserves $T_0^\pm$ and $T_i^+(1)$ determines the value of $h_\alpha(0)$. As h_α preserves the Nahm data at $t = 0$, this determines the possible values for $g(0)$. In the end, there will be two families of Nahm data, but they are shown to be gauge equivalent.

Let us write $b_i = -f_i/2$ so that $\tilde{T}_i = b_i \chi_i$. Then

$$h_{\alpha+}(1) \left(\sum_{l=1}^{3} (R_\alpha)_{jl}\, T_l^+(1) \right) h_{\alpha+}(1)^{-1} = T_j^+(1) \tag{4.46}$$

is true for all $\alpha \in [0, 2\pi)$ and $j = 1, 2, 3$. Observe that conjugation by h_α leaves iId invariant, so by taking the trace of (4.46) for $j = 2, 3$ we obtain

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

Since rotations have no fixed point except the origin, we see that both x_2 and x_3 must be zero.

We shall show that β cannot be zero. Suppose not, then ρ is a trivial representation and so we have $q_\alpha = \pm Id$ for all α. Consider (4.46) for $i = 2$; we expand the left-hand-side

$$\cos\alpha T_2^+(1) - \sin\alpha T_3^+(1) = b_2(1)(a_{22}\cos\alpha - a_{32}\sin\alpha)\chi_2 \\ + b_3(1)(a_{23}\cos\alpha - a_{33}\sin\alpha)\chi_3.$$

The right-hand-side is

$$b_2(1)a_{22}\chi_2 + b_3(1)a_{23}\chi_3.$$

As $b_2(1), b_3(1)$ are non-zero and χ_2, χ_3 are linearly independent, the equality of the two sides gives us the equations

$$\begin{aligned} a_{22}(1-\cos\alpha) + a_{32}\sin\alpha &= 0 \\ a_{23}(1-\cos\alpha) + a_{33}\sin\alpha &= 0 \end{aligned}.$$

Similarly, by considering (4.46) for $i = 3$ we obtain

$$\begin{aligned} a_{22}\sin\alpha - a_{32}(1-\cos\alpha) &= 0 \\ a_{23}\sin\alpha - a_{33}(1-\cos\alpha) &= 0 \end{aligned}.$$

Solving for the coefficients a_{ij} we get

$$a_{ij}(1-\cos\alpha) = 0$$

for all $i, j = 2, 3$. This implies each of these a_{ij} is zero, but this contradicts that $A \in SO(3)$. Therefore we must have $\beta = \pm\alpha/2$.

Suppose (4.46) is true for $\beta = \pm\alpha/2$. Note that since $h_\alpha(1)$ and $g(1)$ are both in $SO(2)$, they commute and $g(1)$ can be eliminated from the equations. Let us first consider (4.46) for $i = 1$:

$$h_{\alpha+}(1)T_1^+(1)h_{\alpha+}(1)^{-1} = T_1^+(1).$$

Because the adjoint action fixes χ_1, so $T_1(1)$ must be proportional to χ_1. As $T_1^+(1)$ can be written as

$$T_1^+(1) = a_{11}b_1(1)\chi_1 + a_{12}b_2(1)\chi_2 + a_{13}b_3(1)\chi_3,$$

we have for $a_{12} = a_{13} = 0$ and $a_{11}^2 = 1$, (4.46) is solved for $i = 1$.

The left-hand-side of (4.46) for $i = 2$ can be computed to be

$$
\begin{aligned}
\big(a_{22}b_2(1)\cos^2\alpha \pm a_{23}b_3(1)\cos\alpha\sin\alpha - {}& \\
&a_{32}b_2(1)\sin\alpha\cos\alpha \mp a_{33}b_3(1)\sin^2\alpha\big)\,\chi_2 \\
&+\big(\mp a_{22}b_2(1)\cos\alpha\sin\alpha + a_{23}b_3(1)\cos^2\alpha \\
&\qquad \pm a_{32}b_2(1)\sin^2\alpha - a_{33}b_3(1)\sin\alpha\cos\alpha\big)\,\chi_3.
\end{aligned}
$$

Whereas the right-hand-side is

$$b_2(1)a_{22}\chi_2 + b_3(1)a_{23}\chi_3.$$

Compare the coefficients of χ_2 and χ_3 on both sides yields

$$
\begin{aligned}
b_2(1)\sin\alpha(a_{22}\sin\alpha + a_{32}\cos\alpha) &= \mp b_3(1)\sin\alpha(a_{33}\sin\alpha - a_{23}\cos\alpha) \\
b_2(1)\sin\alpha(a_{32}\sin\alpha - a_{22}\cos\alpha) &= \pm b_3(1)\sin\alpha(a_{23}\sin\alpha + a_{33}\cos\alpha)
\end{aligned}. \tag{4.47}
$$

For any $\alpha \neq 0, \pi$, the trigonometric functions can be eliminated and these equations imply

$$
\begin{aligned}
b_2(1)a_{22} &= \mp b_3(1)a_{33} \\
b_2(1)a_{32} &= \pm b_3(1)a_{23}
\end{aligned}.
$$

Consider the quantity $b_2(1)a_{11}$:

$$
\begin{aligned}
b_2(1)a_{11} &= b_2(1)(a_{22}a_{33} - a_{23}a_{32}) \\
&= \mp b_3(1)(a_{33}^2 + a_{23}^2) \\
&= \mp b_3(1)a_{11}^2.
\end{aligned}
$$

Since a_{11} is non-zero, we have

$$b_2(1) = \mp a_{11}b_3(1).$$

As both $b_2(1)$ and $b_3(1)$ have the same sign, this forces a_{11} to have sign $\mp$. But we know $a_{11}^2 = 1$, hence $a_{11} = \mp 1$.

The equations in (4.47) still need to be solved for some $A \in SO(3)$, this amounts to verify that the (1,1)-minor of A has determinant equal to ∓ 1: re-writing (4.47) in the form

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \begin{pmatrix} a_{32} \mp a_{23} \\ a_{22} \pm a_{33} \end{pmatrix} = 0,$$

it is clear that the equation can indeed be solved. One may check that (4.46) for $i = 3$ is equivalent to (4.47), hence is also solved. Therefore we have proved that all equations in (4.46) are satisfied.

Note that in the above we obtained $b_2(1) = b_3(1)$. Since $b_i = -f_i/2$ and $f_3^2 - f_2^2 = D^2k^2$, this yields $k = 0$.

If $h_{\alpha+}$ preserves T_0^+, then it must take the form

$$g\begin{pmatrix}\cos(\pm\alpha/2) & -\sin(\pm\alpha/2)\\ \sin(\pm\alpha/2) & \cos(\pm\alpha/2)\end{pmatrix}g^{-1}.$$

It is easy to see that $h_{\alpha+}$ also preserves T_i^+ for $i=1,2,3$. At $t=0$, it is equal to

$$\begin{pmatrix}\cos(\alpha/2)\mp 2i\Im\{\lambda\bar{\mu}\}\sin(\alpha/2) & (\lambda^2+\mu^2)\sin(\alpha/2)\\ -(\bar{\lambda}^2+\bar{\mu}^2)\sin(\alpha/2) & \cos(\alpha/2)\pm 2i\Im\{\lambda\bar{\mu}\}\sin(\alpha/2)\end{pmatrix}.$$

On the other hand, by definition of h_α,

$$h_{\alpha+}(0)=e^{-i\alpha/2}\begin{pmatrix}e^{i\gamma} & 0\\ 0 & 1\end{pmatrix}$$

for some $\gamma\in\mathbb{R}$, so this implies $2i\Im\{\lambda\bar{\mu}\}=\pm1$, or equivalently,

$$g(0)=\begin{pmatrix}e^{i\psi} & 0\\ 0 & e^{-i\psi}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mp i\\ \mp i & 1\end{pmatrix}.$$

$h_{\alpha-}$ preserves T_0^- so it must be a constant function. This constant is determined by the (1,1)-entry of $h_{\alpha+}(0)$. Since the gauge transformation h, with

$$h_+(0)=\begin{pmatrix}e^{i\psi} & 0\\ 0 & e^{-i\psi}\end{pmatrix},$$

$h_+(1)\in O(2)$ and $h_-\equiv(h_+(0))_{11}$, is equivalent to a T^2-action, for some appropriate θ_-,θ_+, the function $p_{(\theta_-,\theta_+)}h^{-1}$ is just a trivial action and hence we may assume

$$g(0)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mp i\\ \mp i & 1\end{pmatrix}.$$

Since (T^-,T^+) is a fixed point of the group $\Delta^0_{SO(2)}$, acting by $R_\alpha\times p_{(-\alpha,\alpha/2)}$ for some appropriate α, the solution is given by Nahm data in (4.38) with conditions

$$A=Id,\quad x_2=x_3=0,\quad k=0,\quad g(0)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix},\tag{4.48}$$

or

$$A=\mathrm{diag}(-1,1,-1),\quad x_2=x_3=0,\quad k=0,\quad g(0)=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}.\tag{4.49}$$

These two families of solutions are in fact the same, we shall only show that the latter family is contained in the former one, as the proof of the other inclusion is similar.

Suppose (T^-, T^+) lies in the latter family. Let $\tilde{g} : [0,1] \to SU(2)$ be any function with $\tilde{g}(0) = g(0)^{-1}$ and $\tilde{g}(1) \in SO(2)$. Define q to be the gauge transformation with

$$q_+ = \tilde{g} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} g^{-1}, \quad q_- \equiv Id.$$

It can be seen that q is equivalent to the action $p_{(\pi,\pi/2)}$, so $p^{-1}_{(\pi,\pi/2)} q$ fixes (T^-, T^+) as a point in $N_{1,2}/\mathcal{G}_0$, and it can be represented by

$$\begin{aligned}
T_0^+ &= (\theta - \pi/2) i Id - \dot{\tilde{g}} \tilde{g}^{-1}, & T_0^- &= (\phi + \pi) i, \\
T_1^+ &= \tilde{g} \tilde{T}_1^+ \tilde{g}^{-1} + x_1 i Id, & T_1^- &\equiv \left(T_1^+(0)\right)_{11}, \\
T_2^+ &= \tilde{g} \tilde{T}_2^+ \tilde{g}^{-1}, & T_2^- &\equiv \left(T_2^+(0)\right)_{11}, \\
T_3^+ &= \tilde{g} \tilde{T}_3^+ \tilde{g}^{-1}, & T_3^- &\equiv \left(T_3^+(0)\right)_{11},
\end{aligned}$$

i.e. the solution indeed lies in the former family. It remains to show that the given Nahm data is preserved by $\Delta^0_{SO(2)}$, but this is straightforward. $\square$

Suppose $\Delta^\infty_{SO(2)}$ is the subgroup of $\widetilde{SO(2)} \times T^2$ generated by $R_\alpha \times p_{(-\alpha,-\alpha/2)}$. Then one may deduce that the fixed point set of $\Delta^\infty_{SO(2)}$ in $N_{1,2}(1)/\mathcal{G}_0$, denoted by $\Sigma^\infty_{SO(2)}$, is given in the same way as the fixed point set of $\Sigma^0_{SO(2)}$, except that $g(0)$ is now given by

$$g(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Consider the subgroup $\Delta_{\mathbb{Z}_2} \subset \Delta^0_{SO(2)}$ generated by the element $R_\pi \times p_{(\pi,\pi/2)}$. Then:

Proposition 4.12. *The fixed point set of $\Delta_{\mathbb{Z}_2}$ on $N_{1,2}(1)/\mathcal{G}_0$ is*

$$\Sigma_{\mathbb{Z}_2} = \Sigma^0_{\mathbb{Z}_2} \cup \Sigma^\infty_{\mathbb{Z}_2},$$

where $\Sigma^0_{\mathbb{Z}_2}$ (resp. $\Sigma^\infty_{\mathbb{Z}_2}$) is the set of $\mathcal{G}_0$-orbits that contain Nahm data in $N_{1,2}(1)$ given by

$$\begin{aligned}
T_0^+ &= \theta i Id - \dot{g} g^{-1}, & T_0^- &= \phi i, \\
T_1^+ &= g \tilde{T}_1^+ g^{-1} + x_1 i Id, & T_1^- &\equiv \left(T_1^+(0)\right)_{11}, \\
T_2^+ &= g \left(\cos\gamma \tilde{T}_2^+ - \sin\gamma \tilde{T}_3^+\right) g^{-1}, & T_2^- &\equiv \left(T_2^+(0)\right)_{11}, \\
T_3^+ &= g \left(\sin\gamma \tilde{T}_2^+ + \cos\gamma \tilde{T}_3^+\right) g^{-1}, & T_3^- &\equiv \left(T_3^+(0)\right)_{11},
\end{aligned}$$

where $\gamma \in [0, 2\pi)$ and

$$g(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \quad \left(\textit{resp.}\ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right).$$

In particular, $\Sigma_{\mathbb{Z}_2}$ is a six-dimensional family parametrized by the real coordinates $p, r, k, \gamma, \theta, \phi$.

Proof. The proof follows the same line of argument as in the proof of Proposition (4.10): first define ρ as in Proposition (4.11) except that we now have $p_1(G)$ equals to the $\mathbb{Z}_2$ subgroup of $\tilde{SO}(2)$, then show that there can only be two such homomorphisms, namely, $R_\pi \mapsto [Id]$ and

$$R_\pi \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The rest is just a deduction of the proof of Proposition (4.10). $\square$

It is clear that $\Sigma^0_{SO(2)} \subset \Sigma^0_{\mathbb{Z}_2}$ and $\Sigma^\infty_{SO(2)} \subset \Sigma^\infty_{\mathbb{Z}_2}$, namely by taking $k = 0$ and $\gamma = 0$. We have the following classification result:

Proposition 4.13. *Suppose G is a subgroup of $\widetilde{SO(2)} \times T^2$ such that $\Sigma_G \neq \emptyset$, where $\Sigma_G \subset N_{1,2}(1)/\mathcal{G}_0$ is the fixed point set of G.*

(a) *If $p_1(G) \neq \{Id\}$ or $\mathbb{Z}_2$, then G must be generated by elements either of the form $R_\alpha \times p_{(\alpha,\alpha/2)}$ or $R_\alpha \times p_{(-\alpha,-\alpha/2)}$. Moreover, Σ_G must be equal to either $\Sigma^0_{SO(2)}$ or $\Sigma^\infty_{SO(2)}$.*

(b) *If $p_1(G) = \mathbb{Z}_2$, then G must be generated by $R_\alpha \times p_{(\pi,\pi/2)}$ or $R_\alpha \times p_{(-\pi,-\pi/2)}$. In either case, the fixed point set of G is $\Sigma_{\mathbb{Z}_2}$.*

Proof. Let $(T^-, T^+) \in N_{1,2}(1)/\mathcal{G}_0$ be a fixed point of $G \leq \widetilde{SO(2)} \times T^2$. Then for every element $R_\alpha \times h_{(\theta_-,\theta_+)} \in G$, we shall show that

$$(\theta_-, \theta_+) = (\mp\alpha, \mp\alpha/2).$$

Writing

$$h_{(\theta_-,\theta_+)_+}(1) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

Define $\rho : p_1(G) \to O(2)/\{\pm Id\}$ as in Proposition (4.11). As $p_1(G)$ is a subgroup of the circle group, it is either a finite cyclic group or is dense. We now show that ρ is in fact a homomorphism into $SO(2)/\{\pm Id\}$.

First suppose $p_1(G) \simeq \mathbb{Z}_m$ with $m \geq 2$. If m is odd, then the generator of $p_1(G)$ must be mapped into $SO(2)/\{\pm Id\}$ since otherwise one would have the identity element equal to a reflection, a contradiction. If m be even, suppose the generator of $p_1(G)$ gets mapped to a reflection element through ρ. Then by deducing from the property (4.41), one has $T_2^+(1) = T_3^+(1) = 0$, which contradicts that $(T^-, T^+) \in N_{1,2}(1)/\mathcal{G}_0$. Therefore ρ must be a homomorphism from cyclic group to cyclic group. Since such homomorphisms are completely classified, ρ must be of the form in (4.43).

Suppose $p_1(G)$ is dense. Since the action of the group $\widetilde{SO(2)} \times T^2$ is continuous, the fact that G fixes $(T^-, T^+) \in N_{1,2}(1)/\mathcal{G}_0$ implies that its closure

$\overline{G}$ also fixes the element. Let $\overline{\rho}$ be the corresponding map for $\overline{G}$. The proof of Proposition (4.10) implies that $\overline{\rho}$ is of the form in (4.43). By restricting to $p_1(G)$, the same is true for ρ.

By Proposition (4.11), in either case we only need to consider $\beta = 0$ or $\beta = \pm\alpha/2$. Without loss of generality, we assume (T^-, T^+) is of the form (4.38). The case where $\beta = 0$ can be ruled out since otherwise it would imply that the solution lies outside of $N_{1,2}(1)/\mathcal{G}_0$, contradicting the assumption. Hence we let $\beta = \pm\alpha/2$. Then $R_\alpha \times h_{(\theta_-,\theta_+)}$ preserves T_0^- implies that

$$h_{(\theta_-,\theta_+)_+} = g \begin{pmatrix} \cos(\alpha/2) & \mp\sin(\alpha/2) \\ \pm\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} g^{-1}.$$

Consider its value at $t = 0$:

$$\begin{pmatrix} \cos(\alpha/2) \mp 2i\Im\{\lambda\bar{\mu}\}\sin(\alpha/2) & (\lambda^2+\mu^2)\sin(\alpha/2) \\ -(\bar{\lambda}^2+\bar{\mu}^2)\sin(\alpha/2) & \cos(\alpha/2) \pm 2i\Im\{\lambda\bar{\mu}\}\sin(\alpha/2) \end{pmatrix}.$$

On the other hand, by definition of $h_{(\theta_-,\theta_+)}$, it is equal to

$$e^{-i\theta_+}\begin{pmatrix} e^{i\gamma} & 0 \\ 0 & 1 \end{pmatrix}$$

for some $\gamma \in \mathbb{R}$. Comparing the two, we have either $\mu = -i\lambda$ or $\mu = i\lambda$.

First suppose $\mu = -i\lambda$, then $2i\Im\{\lambda\bar{\mu}\} = 1$ and $\theta_+ = \mp\alpha/2 \pmod{2\pi}$. On the other hand, as $h_{(\theta_-,\theta_+)}$ preserves T_0^-, $h_{(\theta_-,\theta_+)_-}$ must be a constant. But

$$e^{i(\theta_- - \theta_+)} = h_{(\theta_-,\theta_+)_-}(-1) = \left(h_{(\theta_-,\theta_+)_+}(0)\right)_{11} = e^{\mp i\alpha/2},$$

therefore $\theta_- = \mp\alpha \pmod{2\pi}$. Note that G cannot simultaneously contain elements of the form $R_\alpha \times p_{(\alpha,\alpha/2)}$ and $R_\alpha \times p_{(-\alpha,-\alpha/2)}$: suppose g satisfies $2i\Im\{\lambda\bar{\mu}\} = 1$, the proof of Proposition (4.10) implies that if (T^-, T^+) is fixed by $R_\alpha \times p_{(\alpha,\alpha/2)}$ (resp. $R_\alpha \times p_{(-\alpha,-\alpha/2)}$), then it belongs to $\Sigma^0_{SO(2)}$ (resp. $\Sigma^\infty_{SO(2)}$), but being fixed by both types would imply $\Sigma^0_{SO(2)} \cap \Sigma^\infty_{SO(2)}$ is non-empty, a contradiction. The same argument applies to the case where $\mu = i\lambda$. Therefore we have either $\Sigma_G \subset \Sigma^0_{SO(2)}$ or $\Sigma_G \subset \Sigma^\infty_{SO(2)}$, hence (a) is proved.

The proof for (b) is similar except noting that the elements $R_\alpha \times p_{(\pi,\pi/2)}$ and $R_\alpha \times p_{(-\pi,-\pi/2)}$ define the same action on the moduli space $N_{1,2}/\mathcal{G}_0$. $\square$

Chapter 5

Metric on Axially Symmetric Submanifold

This chapter will be devoted to the calculation of the induced metric on the axially symmetric submanifold $\Sigma^0_{SO(2)}$ of $M_{1,2}$. Recall that the metric is given implicitly via the twistorial approach: roughly speaking, with respect to each complex structure, there is a Kähler form which can be extracted from the given holomorphic symplectic form on twistor space by a power series expansion of the latter form. Since $\Sigma^0_{SO(2)}$ is a complex submanifold of a Kähler manifold, the restriction of the Kähler form of $M_{1,2}$ is the Kähler form associated to the induced metric. First, we express the Kähler form of $\Sigma^0_{SO(2)}$ in terms of coordinates; it is not hard to find local *real* coordinates using spectral data. To get the metric, one needs to act on this Kähler form by the complex structure, however, it is unclear how the almost complex structure might be obtained for spectral data. To bypass the problem, we consider the moduli space of Nahm data: using the L^2-metric, we identify the tangent space of $N_{1,2}/\mathcal{G}_0$ with solutions to the linearization of the Nahm's equations, modulo $\mathrm{Lie}(\mathcal{G}_0)$. As $\Sigma^0_{SO(2)}$ is given by a family of Nahm data parametrized by real coordinates, we can compute the coordinate tangent vectors for these coordinates. Since the almost complex structures are known for $N_{1,2}$, they can be computed in terms of these vectors. As the Kähler form is naturally written in terms of spectral data coordinates, it is necessary to identify them with the Nahm data coordinates, which is done through the rational map construction. Although such construction fails over $\Sigma^0_{SO(2)}$, the identification can still be made for neighbouring points, which then extends continuously to the points in $\Sigma^0_{SO(2)}$. Thus we know how the complex structure acts on spectral data coordinates, and an explicit expression for the Riemannian metric of $\Sigma^0_{SO(2)}$ may be obtained. Finally, the metric is shown to be asymptotically flat with rate $1/R$, where R is a separation parameter.

From the above, although it seems that one could use Nahm data coordinates to do everything, it is not feasible since, from the rational map construction, one would then have to solve an ordinary differential equation *explicitly* for each $\zeta \in \mathbb{C}$, which is very difficult in general, if not impossible.

5.1 Tangent Space of $N_{k,l}/\mathcal{G}_0$

For our purposes, we shall assume $k < l$. Suppose $T = (T^-, T^+)$ is an element in $N_{k,l}$. For $Y^- \in \mathcal{A}_k([-1,0))$ and $Y^+ \in \mathcal{A}_l((0,1])$ (cf. Chapter 1), consider the set of $Y = (Y^-, Y^+)$ satisfying

$$T + \epsilon Y + O(\epsilon^2) \in N_{k,l}$$

for all sufficiently small $\epsilon > 0$. It is easy to see that $Y^{\pm}$ are solutions to the linearization of Nahm's equations:

$$\begin{aligned}
\frac{dY_1^{\pm}}{dt} + [T_0^{\pm}, Y_1^{\pm}] + [Y_0^{\pm}, T_1^{\pm}] + [T_2^{\pm}, Y_3^{\pm}] + [Y_2^{\pm}, T_3^{\pm}] = 0, \\
\frac{dY_2^{\pm}}{dt} + [T_0^{\pm}, Y_2^{\pm}] + [Y_0^{\pm}, T_2^{\pm}] + [T_3^{\pm}, Y_1^{\pm}] + [Y_3^{\pm}, T_1^{\pm}] = 0, \\
\frac{dY_3^{\pm}}{dt} + [T_0^{\pm}, Y_3^{\pm}] + [Y_0^{\pm}, T_3^{\pm}] + [T_1^{\pm}, Y_2^{\pm}] + [Y_1^{\pm}, T_2^{\pm}] = 0.
\end{aligned} \tag{5.1}$$

For each i, Y_i^+ (resp. Y_i^-) can be extended to an analytic function over $[0,1]$ (resp. $[-1,0]$) and are symmetric at $t = 1$ (resp. $t = -1$). Moreover, for $i = 1,2,3$, Y_i^-, Y_i^+ satisfy the patching condition for $N_{k,l}$ at $t = 0$, though here they have no pole. Then the set of such Y form the *virtual* tangent space of $N_{k,l}$, written $T_T N_{k,l}$. The word "virtual" here is supposed to emphasise that, even if T is a singular point, such model of tangent space is still defined.

Let $[T]$ denote the orbit of T. To define the virtual tangent space of $N_{k,l}/\mathcal{G}_0$, we need to compute the infinitesimal gauge transformations of $\mathcal{G}_0$ about T. Suppose $g \in \mathcal{G}_0$ is of the form

$$g_{\pm} = Id + \epsilon \Psi_{\pm} + O(\epsilon^2),$$

where $\Psi = (\Psi_-, \Psi_+)$ is in the Lie algebra of $\mathcal{G}_0$. Then if $g \cdot T \in N_{k,l}$ for all sufficiently small $\epsilon > 0$, it implies that Y of the form

$$\left(-\frac{d\Psi_{\pm}}{dt} + [\Psi_{\pm}, T_0^{\pm}], [\Psi_{\pm}, T_1^{\pm}], [\Psi_{\pm}, T_2^{\pm}], [\Psi_{\pm}, T_3^{\pm}] \right) \tag{5.2}$$

is a solution to (5.1). Let Λ be the vector subspace of $T_T N_{k,l}$ generated by the set of Y of this form, then the virtual tangent space $T_{[T]}(N_{k,l}/\mathcal{G}_0)$ is defined to be $T_T N_{k,l}/\Lambda$.

There is an L^2-metric on $N_{k,l}$ given by

$$h(Y,Y) = -\frac{1}{2} \sum_{i=0}^{3} \left(\int_{-1}^{0} \mathrm{tr}\ (Y_i^-)^2 dt + \int_0^1 \mathrm{tr}\ (Y_i^+)^2 dt \right). \tag{5.3}$$

Let us consider the orthogonal complement of Λ in $T_T N_{k,l}$ with respect to this metric: it is given by the set of $Y \in T_T N_{k,l}$ satisfying

$$\int_{-1}^{0} \operatorname{tr}\left(-Y_0^- \frac{d\Psi_\pm}{dt} + \sum_{i=0}^{3} Y_i^-[\Psi_-, T_0^-]\right) dt \\ + \int_0^1 \operatorname{tr}\left(-Y_0^+ \frac{d\Psi_+}{dt} + \sum_{i=0}^{3} Y_i^+[\Psi_+, T_0^+]\right) dt = 0$$

for all $\Psi \in \mathrm{Lie}(\mathcal{G}_0)$. Integration by parts and application of the fundamental lemma of calculus of variations yield the equations

$$\frac{dY_0^\pm}{dt} + [T_0^\pm, Y_0^\pm] + [T_1^\pm, Y_1^\pm] + [T_1^\pm, Y_1^\pm] + [T_1^\pm, Y_1^\pm] = 0. \tag{5.4}$$

In addition, $Y_0^\pm$ must also satisfy the patching condition at $t = 0$, namely, the upper diagonal block of $Y_0^+(0)$ is equal to $Y_0^-(0)$. We now state the following analytic result without proof:

Proposition 5.1. *Suppose $k < l$. Then*

(a) *$N_{k,l}$ is smooth Banach manifold.*

(b) *Suppose $T \in N_{k,l}$ is a point that admits a neighbourhood $N(T)$, such that the $\mathcal{G}_0$-orbit of every point in $N(T)$ intersects exactly once with the set given by (5.4), then $T_{[T]}N_{k,l}/\mathcal{G}_0$ is the tangent space of $N_{k,l}$, and may be identified with the space of solutions $Y = (Y^-, Y^+)$ to*

$$\begin{aligned}
&\frac{dY_1^\pm}{dt} + [T_0^\pm, Y_1^\pm] + [Y_0^\pm, T_1^\pm] + [T_2^\pm, Y_3^\pm] + [Y_2^\pm, T_3^\pm] = 0,\\
&\frac{dY_2^\pm}{dt} + [T_0^\pm, Y_2^\pm] + [Y_0^\pm, T_2^\pm] + [T_3^\pm, Y_1^\pm] + [Y_3^\pm, T_1^\pm] = 0,\\
&\frac{dY_3^\pm}{dt} + [T_0^\pm, Y_3^\pm] + [Y_0^\pm, T_3^\pm] + [T_1^\pm, Y_2^\pm] + [Y_1^\pm, T_2^\pm] = 0,\\
&\frac{dY_0^\pm}{dt} + [T_0^\pm, Y_0^\pm] + [T_1^\pm, Y_1^\pm] + [T_2^\pm, Y_2^\pm] + [T_3^\pm, Y_3^\pm] = 0,
\end{aligned}$$

such that for each $i = 0, 1, 2, 3$, Y_i^+ and Y_i^- are analytic functions over $[0, 1]$ and $[-1, 0]$ respectively, and satisfy the patching condition at $t = 0$. In addition, $Y_i^\pm$ are symmetric at $t = \pm 1$.

The proof of this proposition will be a modification of the argument given in [35].

Recall that h is the natural hyperkähler metric on the moduli space of solutions to Nahm's equations, with the almost complex structures I, J, K given by

$$\begin{aligned}
I(Y_0, Y_1, Y_2, Y_3) &= (-Y_1, Y_0, -Y_3, Y_2),\\
J(Y_0, Y_1, Y_2, Y_3) &= (-Y_2, Y_3, Y_0, -Y_1),\\
K(Y_0, Y_1, Y_2, Y_3) &= (-Y_3, -Y_2, Y_1, Y_0).
\end{aligned} \tag{5.5}$$

Note that I, J, K commute with the system of equations given in Proposition (5.1), and that they correspond to three complex structures of $T_{[T]}N_{k,l}/\mathcal{G}_0$. Although h is not the monopole-cluster metric, its role above was to help to yield Lemma (5.1), which in turn gives us a formula for the (almost) complex structures. This is particularly important for us since we need to know the action of the complex structure of $\Sigma^0_{SO(2)}$ on the tangent vector fields, and it is known that I above corresponds to such complex structure.

5.2 Complex Structure

We shall compute the almost complex structure of $\Sigma^0_{SO(2)}$ in some given basis. Recall from Proposition (4.10) that, in some gauge, the submanifold $\Sigma^0_{SO(2)}$ can be described by the following family of Nahm data:

$$\begin{aligned}
T_0^+ &= i\theta Id - \frac{dg}{dt}g^{-1}, & T_0^- &= i\phi,\\
T_1^+ &= -\frac{f_1}{2}g\chi_1 g^{-1} + riId, & T_1^- &\equiv \left(T_1^+(0)\right)_{11},\\
T_2^+ &= -\frac{f_2}{2}g\chi_2 g^{-1}, & T_2^- &\equiv \left(T_2^+(0)\right)_{11},\\
T_3^+ &= -\frac{f_3}{2}g\chi_3 g^{-1}, & T_3^- &\equiv \left(T_3^+(0)\right)_{11},
\end{aligned}$$

where $r, \theta, \phi \in \mathbb{R}$ are real parameters; χ_j are a basis of $\mathfrak{su}(2)$ given by

$$\chi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \chi_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix};$$

$g : [0,1] \to SU(2)$ is a fixed smooth function such that $g(1) \in SO(2)$ and

$$g(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix};$$

$f_j : [0,1] \to \mathbb{R}$ are the solutions to the Euler's equations given by

$$\begin{aligned}
f_1(t) &= -D\cot(D(t+p)),\\
f_2(t) = f_3(t) &= -D\csc(D(t+p)),
\end{aligned}$$

with $p > 0$ and $D = \frac{\pi}{2(1+p)}$. We know that $\Sigma^0_{SO(2)}$ is the fixed point set of some $SO(2)$ group of isometries, hence it is a totally geodesic submanifold of $M_{1,2}$. It can be shown that such action is holomorphic with respect to the complex structure I, and following from the lemma below that $\Sigma^0_{SO(2)}$ is actually a complex submanifold of $M_{1,2}$:

Lemma 5.2. *Let (M, J) be a complex manifold and let G be a group of isometries acting holomorphically on M. Then the fixed point set X of G is a complex submanifold of (M, J).*

Proof. We know that X is a (totally geodesic) submanifold of M. The only thing which remains to be checked is the invariance of TX under the complex structure J. Indeed, let $\pi : TM \to M$ denotes the projection map and let $v \in TX$. Since $g_* \circ J = J \circ g_*$ for any $g \in G$, we have $g_*(J(v)) = J(v)$. But then each g preserves pointwise the unique geodesic γ passing through $\pi(v)$ with velocity $J(v)$, γ lies on X and hence $J(v) \in T_{\pi(v)}X$. As this is true for each $v \in TX$ so the result follows. $\square$

Let

$$\begin{aligned} F : \mathbb{R}_{>0} \times \mathbb{R}^3 &\to N_{1,2} \\ (p, r, \theta, \phi) &\mapsto (T^-, T^+) \end{aligned}$$

be the natural map which parametrizes $\Sigma^0_{SO(2)}$. In what follows, we shall *assume* without proof that the hypothesis of Lemma (5.1)(b) holds for each $[T]$ in $\Sigma^0_{SO(2)}$, so that $T_{[T]}N_{1,2}/\mathcal{G}_0$ can be identified with the set of solutions given there. The coordinate vector fields $F_*(\partial_p)$, $F_*(\partial_r)$, $F_*(\partial_\theta)$, $F_*(\partial_\phi)$ on $\Sigma^0_{SO(2)}$ are computed to be

$$\begin{aligned}
F_*(\partial_p) &= \begin{cases} Y_0^+ = 0, & Y_0^- = 0, \\ Y_1^+ = -\dfrac{\partial_p f_1}{2} g\chi_1 g^{-1}, & Y_1^- = -\dfrac{\partial_p f_1(0)}{2} i, \\ Y_2^+ = -\dfrac{\partial_p f_2}{2} g\chi_2 g^{-1}, & Y_2^- = 0, \\ Y_3^+ = -\dfrac{\partial_p f_3}{2} g\chi_3 g^{-1}, & Y_3^- = 0, \end{cases} \\
F_*(\partial_r) &= \begin{cases} Y_0^+ = 0, & Y_0^- = 0, \\ Y_1^+ = iId, & Y_1^- = i, \\ Y_2^+ = 0, & Y_2^- = 0, \\ Y_3^+ = 0, & Y_3^- = 0, \end{cases} \\
F_*(\partial_\theta) &= \begin{cases} Y_0^+ = iId, & Y_0^- = 0, \\ Y_1^+ = 0, & Y_1^- = 0, \\ Y_2^+ = 0, & Y_2^- = 0, \\ Y_3^+ = 0, & Y_3^- = 0, \end{cases} \\
F_*(\partial_\phi) &= \begin{cases} Y_0^+ = 0, & Y_0^- = i, \\ Y_1^+ = 0, & Y_1^- = 0, \\ Y_2^+ = 0, & Y_2^- = 0, \\ Y_3^+ = 0, & Y_3^- = 0. \end{cases}
\end{aligned} \tag{5.6}$$

Under the action of I, the above vector fields are mapped to

$$I\left(F_*\left(\partial_p\right)\right)=\begin{cases} Y_0^+=\dfrac{\partial_p f_1}{2}g\chi_1 g^{-1}, & Y_0^-=\dfrac{\partial_p f_1(0)}{2}i,\\ Y_1^+=0, & Y_1^-=0,\\ Y_2^+=\dfrac{\partial_p f_3}{2}g\chi_3 g^{-1}, & Y_2^-=0,\\ Y_3^+=-\dfrac{\partial_p f_2}{2}g\chi_2 g^{-1}, & Y_3^-=0,\end{cases}$$

$$I\left(F_*\left(\partial_r\right)\right)=\begin{cases} Y_0^+=-iId, & Y_0^-=-i,\\ Y_1^+=0, & Y_1^-=0,\\ Y_2^+=0, & Y_2^-=0,\\ Y_3^+=0, & Y_3^-=0,\end{cases} \tag{5.7}$$

$$I\left(F_*\left(\partial_\theta\right)\right)=\begin{cases} Y_0^+=0, & Y_0^-=0,\\ Y_1^+=iId, & Y_1^-=0,\\ Y_2^+=0, & Y_2^-=0,\\ Y_3^+=0, & Y_3^-=0,\end{cases}$$

$$I\left(F_*\left(\partial_\phi\right)\right)=\begin{cases} Y_0^+=0, & Y_0^-=0,\\ Y_1^+=0, & Y_1^-=i,\\ Y_2^+=0, & Y_2^-=0,\\ Y_3^+=0, & Y_3^-=0.\end{cases}$$

Notice that, although $F_*(\partial_p)$, $F_*(\partial_r)$, $F_*(\partial_\theta)$, $F_*(\partial_\phi)$ satisfy the equations in Lemma (5.1), only the first two actually satisfy the patching condition at $t=0$. Consequently, the $I\left(F_*(\partial_\theta)\right), I\left(F_*(\partial_\phi)\right)$ obtained above do not correspond to tangent vector fields of $\Sigma^0_{SO(2)}$. However, bearing in mind that in the computation above, a particular gauge has been fixed. We claim that, by adding suitable infinitesimal gauge transformations V_θ, V_ϕ of the form (5.2), the vector fields defined by

$$\begin{aligned}\tilde{F}_*(\partial_\theta)&=F_*(\partial_\theta)+V_\theta,\\ \tilde{F}_*(\partial_\phi)&=F_*(\partial_\phi)+V_\phi,\end{aligned} \tag{5.8}$$

will satisfy *all* the conditions in the lemma. Since $T_{[T]}(N_{k,l}/\mathcal{G}_0)\simeq T_T N_{k,l}/\Lambda$, the $\tilde{F}_*(\partial_\theta), \tilde{F}_*(\partial_\phi)$ are in the same coset as $F_*(\partial_\theta), F_*(\partial_\phi)$ and hence representing the same vector fields. Let us begin by exploiting the fact that I preserves the patching condition: writing

$$I(F_*(\partial_p))=a_{pp}F_*(\partial_p)+a_{pr}F_*(\partial_r)+a_{p\theta}F_*(\partial_\theta)+a_{p\phi}F_*(\partial_\phi)+V_p, \tag{5.9}$$

$$I(F_*(\partial_r))=a_{rp}F_*(\partial_p)+a_{rr}F_*(\partial_r)+a_{r\theta}F_*(\partial_\theta)+a_{r\phi}F_*(\partial_\phi)+V_r, \tag{5.10}$$

where a_{pr} etc., are real numbers and V_p, V_r are infinitesimal gauge transformations. We shall solve these equations by finding the correct coefficients and vectors; it turns out that this will give us the information to determine V_θ, V_ϕ.

Consider first the equation (5.9): it is equivalent to

$$
\begin{aligned}
Y_0^+ : &\quad a_{p\theta} i Id - \frac{d\Psi_+}{dt} + [\Psi_+, T_0^+] - \frac{\partial_p f_1}{2} g\chi_1 g^{-1} = 0, && (5.11)\\
Y_1^+ : &\quad [\Psi_+, T_1^+] - a_{pp}\frac{\partial_p f_1}{2} g\chi_1 g^{-1} + a_{pr} i Id = 0, && (5.12)\\
Y_2^+ : &\quad [\Psi_+, T_2^+] - a_{pp}\frac{\partial_p f_2}{2} g\chi_2 g^{-1} - \frac{\partial_p f_3}{2} g\chi_3 g^{-1} = 0, && (5.13)\\
Y_3^+ : &\quad [\Psi_+, T_3^+] - a_{pp}\frac{\partial_p f_3}{2} g\chi_3 g^{-1} + \frac{\partial_p f_2}{2} g\chi_2 g^{-1} = 0, && (5.14)\\
Y_0^- : &\quad -\dot{\Psi}_- + a_{p\phi} i - \frac{\partial_p f_1(0)}{2} i = 0, && (5.15)\\
Y_1^- : &\quad -a_{pp}\frac{\partial_p f_1(0)}{2} i + a_{pr} i = 0. && (5.16)
\end{aligned}
$$

Taking trace of (5.12) implies $a_{pr} = 0$. Then deducing from (5.16) gives $a_{pp} = 0$. Let

$$
\Psi_+ = -\frac{1}{2}\frac{\partial_p f_3}{f_2} g\chi_1 g^{-1} + \frac{1}{2}(t-1)\frac{\partial_p f_3(0)}{f_2(0)} i Id. \tag{5.17}
$$

It can be verified that this solves (5.12), (5.13) and (5.14). Noting that

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial_p f_3}{f_2}\right) &= \frac{1}{f_2}\frac{d}{dt}\left(\partial_p f_3\right) - \frac{\partial_p f_3}{f_2^2}\frac{df_2}{dt}\\
&= \frac{1}{f_2}\partial_p\left(\frac{df_3}{dt}\right) - \frac{\partial_p f_3}{f_2^2}\frac{df_2}{dt}\\
&= \frac{1}{f_2}\partial_p\left(f_1 f_2\right) - \frac{\partial_p f_3}{f_2^2} f_3 f_1 \quad \left(\text{since } \frac{df_i}{dt} = f_j f_k\right)\\
&= \frac{1}{f_2}\left(f_2\partial_p f_1 + f_1\partial_p f_2 - f_1\partial_p f_2\right) \quad (\text{since } f_2 = f_3)\\
&= \partial_p f_1.
\end{aligned}
$$

Thus, if we substitute Ψ_+ into (5.11), then with $a_{p\theta} = \partial_p f_3(0)/(2f_2(0))$, the equation is solved. Finally, it remains to solve the equation (5.15), which requires us to find Ψ_- that patches with Ψ_+ at $t = 0$, and the coefficient $a_{p\phi}$. The appropriate choice is

$$
\Psi_- = -\frac{\partial_p f_3(0)}{f_2(0)}(t+1) i. \tag{5.18}
$$

Substituting into (5.15) forces

$$
a_{p\phi} = -\frac{\partial_p f_3(0)}{f_2(0)} + \frac{1}{2}\partial_p f_1(0).
$$

Therefore we have

$$I(F_*(\partial_p)) = \frac{1}{2}\frac{\partial_p f_3(0)}{f_2(0)}F_*(\partial_\theta) + \left(-\frac{\partial_p f_3(0)}{f_2(0)} + \frac{1}{2}\partial_p f_1(0)\right)F_*(\partial_\phi) + V_p, \quad (5.19)$$

where V_p is given by Ψ_+, Ψ_-. Similarly, solving for (5.10) yields

$$I(F_*(\partial_r)) = -F_*(\partial_\theta) - F_*(\partial_\phi). \quad (5.20)$$

We shall use these equations to determine the infinitesimal gauge transformations V_θ and V_ϕ. Let

$$C = -\frac{3}{2}\frac{\partial_p f_3(0)}{f_2(0)} + \frac{1}{2}\partial_p f_1(0), \quad A = \frac{1}{2}\frac{\partial_p f_3(0)}{f_2(0)}. \quad (5.21)$$

Then (5.19) and (5.20) give

$$\begin{aligned} I(F_*(\partial_p)) + CI(F_*(\partial_r)) &= -CF_*(\partial_\theta) + V_p, \\ I(F_*(\partial_p)) + AI(F_*(\partial_r)) &= CF_*(\partial_\phi) + V_p. \end{aligned}$$

As the left-hand-side of these equations satisfy the patching condition, so does the right-hand-side, hence the vector fields V_θ, V_ϕ in (5.8) are given by

$$\begin{aligned} V_\theta &= -C^{-1}V_p, \\ V_\phi &= C^{-1}V_p. \end{aligned} \quad (5.22)$$

Finally, one may check that V_p, and hence V_θ, V_ϕ, satisfy all the equations in Lemma (5.1), which implies $\tilde{F}_*(\partial_\theta), \tilde{F}_*(\partial_\phi)$ indeed satisfy all the conditions therein. Therefore we have

Proposition 5.3. *With respect to the basis $\{F_*(\partial_p), F_*(\partial_r), \tilde{F}_*(\partial_\theta), \tilde{F}_*(\partial_\phi)\}$, the matrix representation of the almost complex structure I for the complex manifold $\Sigma^0_{SO(2)}$ is given by*

$$I = \begin{pmatrix} 0 & 0 & C^{-1} & -C^{-1} \\ 0 & 0 & C^{-1}(C+A) & -C^{-1}A \\ A & -1 & 0 & 0 \\ (C+A) & -1 & 0 & 0 \end{pmatrix}.$$

5.3 Identification of Coordinates

In this section, we define a six-dimensional space E in $N_{1,2}(1)/\mathcal{G}_0$ which contains $\Sigma^0_{SO(2)}$. Since the Nahm data in E are generic, to each point there are associated spectral curves $S^\pm$ and sections $\varrho^\pm$ of $L^{\pm 2}$. Let (U, ζ), $(\tilde{U}, \tilde{\zeta})$ be affine charts of $\mathbb{P}^1$. Viewing ϱ^+ (resp. ϱ^-) as a pair of meromorphic functions given by $\kappa^+, \tilde{\kappa}^+$ (resp. $\kappa^-, \tilde{\kappa}^-$), we compute the values of κ^+, κ^- for $\zeta = 0$ in two different ways: first, using the rational map construction given in Chapter 2; second, from direct computation of the functions κ^+, κ^-. There are coordinates in E which naturally extend those on $\Sigma^0_{SO(2)}$, we shall compare such coordinates with the spectral data coordinates on the corresponding six-dimensional space in $M_{k,l}$.

5.3.1 Nahm Data

We shall use the rational map construction to compute the values $\kappa^+(0,\eta)$, $\kappa^-(0,\eta)$. Let E be the subset of$N_{1,2}/\mathcal{G}_0$ such that it is given exactly as $\Sigma^0_{SO(2)}$, except that the value of $g : [0,1] \to SU(2)$ at $t = 0$ is now allowed to be arbitrary. Write

$$g(0) = \begin{pmatrix} \lambda & \mu \\ -\bar{\mu} & \bar{\lambda} \end{pmatrix}.$$

Let $T = (T^-, T^+) \in E$. Corresponding to the complex structure I, we put

$$\alpha_\pm = T_0^\pm + iT_1^\pm, \quad \beta_\pm = T_2^\pm + iT_3^\pm.$$

Then (α, β) are given by

$$\begin{aligned} \alpha_+ &= gAg^{-1} - \frac{dg}{dt}g^{-1}, & \alpha_- &= i\phi - f_1(0)\Im\{\lambda\bar{\mu}\} - r, \\ \beta_+ &= -\frac{f_2}{2}g_+(\chi_2 + i\chi_3)g_+^{-1}, & \beta_- &= \frac{f_2(0)}{2}\left(|\lambda|^2 - |\mu|^2 - 2i\Re\{\lambda\bar{\mu}\}\right), \end{aligned} \tag{5.23}$$

where

$$A = \begin{pmatrix} i\theta - r & -\frac{if_1}{2} \\ \frac{if_1}{2} & i\theta - r \end{pmatrix}. \tag{5.24}$$

We need to find an $X \in \mathcal{G}_0^{\mathbb{C}}$ such that the conjugation of $\beta_+(0)$ by $X_+(0)$ has value 1 as its $(2,1)$-entry; it turns out that such $X_+(0)$ must take the form

$$\begin{pmatrix} (\beta_+(0))_{21} & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.25}$$

Let $h_\pm$ be the unique solutions to

$$\frac{dh_\pm}{dt} = h_\pm \alpha_\pm$$

with $h_\pm(0) = X_\pm(0)$. Then the rational map construction says that the values of $\kappa^\pm(0,\eta)$ are given by

$$\kappa^\pm(0,\eta) = \left(h_\pm(\pm 1)^{-1}e_1\right)^T (\eta - \beta_\pm(\pm 1))_{\text{adj}} \left(h_\pm(\pm 1)^{-1}e_1\right) P_\mp(0,\eta),$$

where $P_\pm(\zeta,\eta) = 0$ are the local defining functions of $S^\pm$ over $\pi^{-1}(U)$. We shall compute $\kappa^+(0,\eta)$. Observe that if h_+ is the solution to

$$\frac{dh_+}{dt} = h_+ A$$

with boundary condition $h_+(0) = X_+(0)g_+(0)$, then $\kappa^+(0,\eta)$ is equivalent to

$$\left(h_+(1)^{-1}e_1\right)^T \left(\eta + \frac{f_2}{2}(\chi_2 + i\chi_3)\right)_{\mathrm{adj}} \left(h_+(1)^{-1}e_1\right) P_-(0,\eta), \tag{5.26}$$

Let us first find h_+. Compute the value of $\beta_+(0)$:

$$\begin{aligned}
\beta_+(0) &= \frac{f_2(0)}{2} g(0) \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} g^{-1}(0) \\
&= \frac{f_2(0)}{2} \begin{pmatrix} \lambda & \mu \\ -\bar{\mu} & \bar{\lambda} \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \begin{pmatrix} \bar{\lambda} & -\mu \\ \bar{\mu} & \lambda \end{pmatrix} \\
&= \frac{f_2(0)}{2} \begin{pmatrix} (\lambda - i\mu)(\bar{\lambda} - i\bar{\mu}) & -i(\lambda - i\mu)^2 \\ -i(\bar{\lambda} - i\bar{\mu})^2 & -(\lambda - i\mu)(\bar{\lambda} - i\bar{\mu}) \end{pmatrix}.
\end{aligned}$$

Hence, from (5.25) we have

$$X_+(0) = \begin{pmatrix} -\frac{f_2(0)}{2} i(\bar{\lambda} - i\bar{\mu})^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

To solve for h_+, we first diagonalize A: the eigenvalues are found to be

$$\lambda_\pm(t) = i\theta - r \pm \frac{f_1(t)}{2},$$

with corresponding eigenvectors

$$v_+ = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad v_- = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

If we let

$$C := \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix},$$

then they satisfy

$$A = C\Lambda C^{-1}.$$

Thankfully, since C is independent of t, the solution h_+ is given by

$$h(t) = X(0)g_+(0)C \exp\left(\int_0^t \Lambda\right) C^{-1}. \tag{5.27}$$

It is a routine calculation to check that

$$\exp\left(\int_0^t \Lambda\right) = e^{(i\theta - r)t} \begin{pmatrix} \sqrt{\frac{\sin Dp}{\sin D(t+p)}} & 0 \\ 0 & \sqrt{\frac{\sin D(t+p)}{\sin Dp}} \end{pmatrix}.$$

The value of $h(1)^{-1}e_1$ is computed to be

$$h^{-1}(1)e_1 = \frac{ie^{-(i\theta-r)}}{f_2(0)(\bar{\lambda}-i\bar{\mu})^2}\begin{pmatrix}(\bar{\lambda}-i\bar{\mu}) & (\bar{\lambda}+i\bar{\mu}) \\ i(\bar{\lambda}-i\bar{\mu}) & -i(\bar{\lambda}+i\bar{\mu})\end{pmatrix}\begin{pmatrix}\sqrt{\csc Dp} \\ \sqrt{\sin Dp}\end{pmatrix}.$$

Note that the only root of $P_+(0,\eta)$ is $\eta = 0$, so

$$\left(\eta + \frac{f_2}{2}(\chi_2 + i\chi_3)\right)_{\text{adj}} = -\frac{1}{2}\begin{pmatrix} f_2(1) & if_2(1) \\ if_2(1) & f_2(1)\end{pmatrix}$$

and

$$P_-(0,0) = -\frac{f_2(0)}{2}(\lambda - i\mu)(\bar{\lambda}-i\bar{\mu}).$$

After some calculations, we have

$$\kappa^+(0,0) = \left(\frac{\lambda - i\mu}{\bar{\lambda}-i\bar{\mu}}\right)e^{2r-2\theta i}. \tag{5.28}$$

For $\kappa^-(0,\eta)$, we proceed along a similar line: h_- is computed to be

$$h_-(t) = -\frac{f_2(0)}{2}i(\bar{\lambda}-i\bar{\mu})^2 e^{t(i\phi - r + f_1(0)\Im\{\lambda\bar{\mu}\})}. \tag{5.29}$$

Let η be the root of $P_-(0,\eta)$. It can be seen that classical adjoint of $(\eta - \beta_-)$ is 1, and

$$P_+(0,\eta) = \frac{f_2(0)^2}{4}(\lambda - i\mu)^2(\bar{\lambda}-i\bar{\mu})^2.$$

From these we obtain

$$\kappa^-(0,\eta) = -\left(\frac{\lambda - i\mu}{\bar{\lambda}-i\bar{\mu}}\right)^2 e^{2\phi i - 2r + 2\Im\{\lambda\bar{\mu}\}f_1(0)}. \tag{5.30}$$

5.3.2 Spectral Data

We shall give an explicit expression for the spectral data corresponding to points in E. If S^-, S^+ are curves arise from E, then over $\pi^{-1}(U)$, then their local defining functions are given by

$$\begin{aligned} P_-(\zeta,\eta) = \eta + 2r\zeta - \frac{D}{2}&\left[\sec D(x_3 + ix_2)\right. \\ &\left. + 2\tan Dx_1\zeta - \sec D(x_3 - ix_2)\zeta^2\right] = 0, \end{aligned}$$

$$P_+(\zeta,\eta) = [\eta + (2r - iD)\zeta]\,[\eta + (2r + iD)\zeta] = 0,$$

where x_1, x_2, x_3 are

$$\begin{aligned} x_1 &= -2\Im\{\lambda\bar{\mu}\}, \\ x_2 &= 2\Re\{\lambda\bar{\mu}\}, \\ x_3 &= -\left(|\lambda|^2 - |\mu|^2\right). \end{aligned} \tag{5.31}$$

As before, $\kappa^{\pm}$ are local representative functions of $\varrho^{\pm}$ over $S^{\pm} \cap \pi^{-1}(U)$. Let us first find κ^{+}. Denote the curves with equations

$$\eta = (\pm iD - 2r)\zeta \tag{5.32}$$

by $C_{\pm}$. Then S^{+} is clearly a union of C_{+} and C_{-}. Let α_i, $i = 1, 2$, be the points in U that correspond to $\pi(D)$, then from Chapter 4 we know that

$$\kappa^{+}(\zeta, \eta) = \begin{cases} \xi_{+}\frac{(\zeta-\alpha_1)}{(1+\bar{\alpha}_2\zeta)} & \text{over } C_{+} \cap \pi^{-1}(U) \\ \xi_{-}\frac{(\zeta-\alpha_2)}{(1+\bar{\alpha}_1\zeta)} & \text{over } C_{-} \cap \pi^{-1}(U) \end{cases} \tag{5.33}$$

over $S^{+} \cap \pi^{-1}(U)$, where ξ_{-}, ξ_{+} are non-zero constants satisfying the compatible conditions

$$\begin{aligned} \xi_{+}\alpha_1 &= \xi_{-}\alpha_2, \\ \xi_{+}e^{2iD}\bar{\alpha}_1 &= \xi_{-}e^{-2iD}\bar{\alpha}_2. \end{aligned} \tag{5.34}$$

Note that $\alpha_1 = 0$ if and only if $\alpha_2 = 0$. Supposing they are both non-zero, then by dividing the equations, we can eliminate ξ_{+}, ξ_{-} and arrive at the constraint

$$\alpha_1 e^{-2iD}/\alpha_2 \in \mathbb{R}. \tag{5.35}$$

This is precisely the constraint given in (4.32). Additionally, κ^{+} needs to satisfy the condition $\kappa^{+}\sigma(\kappa^{+}) = 1$, where $\sigma : L^2 \to L^{-2}$ is the standard anti-holomorphic isomorphism which lifts the real structure of $T\mathbb{P}^1$. Since

$$\begin{aligned} \kappa^{+}\sigma(\kappa^{+}) &= \kappa^{+}\left(\overline{\tilde{\kappa}^{+} \circ \tau}\right) \\ &= -|\xi_{+}|^2 e^{-4r}\overline{(\alpha_1 e^{-2iD}/\alpha_2)}, \end{aligned}$$

if $\kappa^{+}\sigma(\kappa^{+}) = 1$ were to hold, then the real number in (5.35) would have to be negative. In fact, we shall show that it is equal to -1.

Let us compute α_1, α_2: in ζ-coordinate, the points in $\pi(S^{-} \cap C_{+})$ are given by

$$\frac{(1 \pm x_1)}{(x_2 + ix_3)} e^{\pm iD}.$$

Similarly, the points in $\pi(S^{-} \cap C_{-})$ are given by

$$-\frac{(1 \pm x_1)}{(x_2 + ix_3)} e^{\mp iD}.$$

There may only be two possibilities for the choice of α_1, α_2: either

$$\alpha_1 = \frac{(1 + x_1)}{(x_2 + ix_3)} e^{iD}, \quad \alpha_2 = -\frac{(1 + x_1)}{(x_2 + ix_3)} e^{-iD},$$

or

$$\alpha_1 = \frac{(1 - x_1)}{(x_2 + ix_3)} e^{-iD}, \quad \alpha_2 = -\frac{(1 - x_1)}{(x_2 + ix_3)} e^{iD}.$$

We preclude the latter possibility since if it were true, then (5.35) would not be negative unless $p = 1$, and hence it is not the generic situation. Therefore we proceed with the former case. Now it is easily checked that we indeed have $\alpha_1 e^{-2iD}/\alpha_2 = -1$, hence the constants ξ_-, ξ_+ must satisfy

$$\begin{aligned} \xi_- &= -\xi_+ e^{2iD}, \\ |\xi_+| &= e^{2r}. \end{aligned} \tag{5.36}$$

On the other hand, the function κ^- on $S^- \cap \pi^{-1}(U)$ is given by

$$\kappa^-(\zeta, \eta) = \varsigma e^{x_1 D \tan D - 2r - D \sec D (x_3 - i x_2)\zeta} \frac{(\zeta - \alpha_1)(\zeta - \alpha_2)}{(1 + \bar{\alpha}_1 \zeta)(1 + \bar{\alpha}_2 \zeta)}. \tag{5.37}$$

The condition $\kappa^- \sigma(\kappa^-) = 1$ is equivalent to $|\varsigma| = 1$. Let

$$\begin{aligned} \xi_+ &= e^{2r + 2\theta' i}, \\ \varsigma &= e^{2\phi' i}, \end{aligned} \tag{5.38}$$

where $\theta', \phi' \in \mathbb{R}$. Gathering the information, we obtain

$$\begin{aligned} \kappa^+(0, \eta) &= -\left(\frac{1 + x_1}{x_2 + i x_3}\right) e^{2\theta' i + 2r + iD} && \text{over } S^+ \cap \pi^{-1}(0), \\ \kappa^-(0, \eta) &= -\left(\frac{1 + x_1}{x_2 + i x_3}\right)^2 e^{2\phi' i - 2r + x_1 D \tan D} && \text{over } S^- \cap \pi^{-1}(0), \end{aligned}$$

or in terms of λ and μ:

$$\begin{aligned} \kappa^+(0, \eta) &= -\left(\frac{\lambda - i\mu}{\lambda + i\mu}\right) i e^{2\theta' i + 2r + iD} && \text{over } S^+ \cap \pi^{-1}(0), \\ \kappa^-(0, \eta) &= \left(\frac{\lambda - i\mu}{\lambda + i\mu}\right)^2 e^{2\phi' i - 2r + 2\Im\{\lambda\bar{\mu}\} f_1(0)} && \text{over } S^- \cap \pi^{-1}(0). \end{aligned}$$

By comparing these values with those computed previously from the rational map construction, we see that θ, ϕ and θ', ϕ' are related by

$$\begin{aligned} \theta &= -\theta' - \frac{D}{2} + \frac{\pi}{4} \quad (\text{mod } 2\pi), \\ \phi &= \phi' + \frac{\pi}{2} + \arg\left(\frac{\bar{\lambda} - i\bar{\mu}}{\lambda + i\mu}\right) \quad (\text{mod } 2\pi). \end{aligned} \tag{5.39}$$

Note that the above is true for all λ, μ that satisfy $|\lambda|^2 + |\mu|^2 = 1$. In particular, it is also true for the limit

$$(\lambda, \mu) \to \frac{1}{2}(1, -i), \tag{5.40}$$

which corresponds to $\Sigma^0_{SO(2)}$.

Recall that our goal is to compute the metric for the axially symmetric submanifold $\Sigma^0_{SO(2)}$, so it is the relationship between the differential of θ, ϕ

and θ', ϕ' that is important; taking the exterior derivative of both sides of (5.39) give us

$$\begin{aligned} d\theta &= -d\theta' - \frac{\partial_p D}{2} dp, \\ d\phi &= d\phi'. \end{aligned} \tag{5.41}$$

5.4 Computation of the Metric

Let $(S^-, \nu^-, S^+, \nu^+) \in \Sigma^0_{SO(2)}$. If $\varrho^\pm$ are sections of the bundles $L^{\pm 2}[\tau(D) - D]$ over $S^\pm$ satisfying $\varrho^\pm \sigma(\varrho^\pm) = 1$, then up to possibly a multiplicative unit complex number, they are related to $\nu^\pm$ by

$$\nu^- = \sigma(\varrho^-), \quad \nu^+ = \varrho^+. \tag{5.42}$$

Let $f^\pm$ be local representative functions of $\nu^\pm$ over $S^\pm \cap \pi^{-1}(U)$, then they are given by

$$f^+(\zeta, \eta) = \begin{cases} -e^{2r+2\theta' i}\zeta & \text{over } C_+ \cap \pi^{-1}(U) \\ e^{2r+2\theta' i + 2Di}\zeta & \text{over } C_- \cap \pi^{-1}(U) \end{cases} \tag{5.43}$$

and

$$f^-(\zeta, \eta) = e^{-f_1(0)+2r-i\phi'}\zeta^{-2}. \tag{5.44}$$

Note that had we taken the limit (5.40) at the beginning, it would be unclear how to determine the constants ξ_-, ξ_+ in (5.33), since the patching conditions (5.34) would not give any information when $\alpha_1 = \alpha_2 = 0$; this is exactly why we needed to consider the family $E \subset M_{1,2}$.

We are now ready to compute the Kähler form with respect to the complex structure I. Recall that (U, ζ), $(\tilde{U}, \tilde{\zeta})$ are affine charts of $\mathbb{P}^1$ so that $\zeta = 0$ corresponds to the complex structure I. On the twistor space $p : Z_{k,l} \to \mathbb{P}^1$ of $M_{k,l}$, there is a holomorphic symplectic form Ω such that, over $p^{-1}(U)$, it is given by $\Omega = \omega \otimes \frac{d}{d\zeta}$, where

$$\omega(\zeta) = \sum_{i=1}^{k} d\log f^-(\zeta, \eta_i^-(\zeta)) \wedge \eta_i^-(\zeta) + \sum_{i=1}^{l} d\log f^+(\zeta, \eta_i^+(\zeta)) \wedge \eta_i^+(\zeta),$$

$\eta_i^\pm(\zeta)$ are the roots of the local defining functions $P_\pm(\zeta, \eta)$ of $S^\pm$. Expanding ω as

$$\sum_{i=0}^{\infty} \Omega_i \zeta^i,$$

then the Kähler form ω_I of $M_{k,l}$ with respect to I is given by

$$\omega_I = \frac{1}{2i}\Omega_1. \tag{5.45}$$

Suppose V is a submanifold of $M_{k,l}$ and $i : V \hookrightarrow M_{k,l}$ is the natural inclusion, then i induces the (non-holomorphic) embedding $j : V \times \mathbb{P}^1 \hookrightarrow Z_{k,l}$, where we view the twistor space as $M_{k,l} \times \mathbb{P}^1$. For each ζ, it is easy to see that

$$i^*(\Omega(\zeta)) = (j^*\Omega)(\zeta). \tag{5.46}$$

As the second projection of j onto $\mathbb{P}^1$ is analytic, $j^*\Omega$ admits a power series expansion with respect to ζ. In particular, $i^*\Omega_1$ may be computed by extracting the linear coefficient of the power series of $j^*\Omega$ around $\zeta = 0$. We shall apply this result to $V = \Sigma^0_{SO(2)}$. As $\Sigma^0_{SO(2)}$ is a complex submanifold, its Kähler form is given by $i^*\omega_I$. Let us consider $(j^*\Omega)(\zeta)$:

$$\begin{aligned} &d\log(-e^{2r+2\theta' i}\zeta) \wedge d((-2r+iD)\zeta) \\ &\qquad + d\log(e^{2r+2\theta' i+2Di}\zeta) \wedge d((-2r-iD)\zeta) \\ &\qquad\qquad + d\log(e^{-f_1(0)+2r-i\phi'}\zeta^{-2}) \wedge d((-2r+f_1(0))\zeta). \end{aligned}$$

Taking the linear term, we see that the Kähler form of $\Sigma^0_{SO(2)}$ is given by

$$i^*\omega_I = 2dr \wedge (2d\theta' - id\phi') + (-\partial_p f_1(0)d\phi' + 2\partial_p D dr) \wedge dp. \tag{5.47}$$

For the metric, we need to act on the second argument of $i^*\omega_I$ by the complex structure I:

$$(i^*g)(\,\cdot\,,\,\cdot\,) = (i^*\omega_I)(\,\cdot\,,\, I\cdot\,),$$

where g is the monopole-cluster metric. Earlier in this chapter, we computed, in term of the basis vector fields $\{F_*(\partial_p), F_*(\partial_r), \tilde{F}_*(\partial_\theta), \tilde{F}_*(\partial_\phi)\}$, the matrix representation of I. The transpose of such matrix tells us the action of I on its dual basis $\{dp, dr, d\theta, d\phi\}$; it is found that, in the basis $\{dp, dr, d\theta', d\phi'\}$, I is represented by

$$\begin{pmatrix} -C^{-1}\partial_p D/2 & 0 & -C^{-1} & -C^{-1} \\ -C^{-1}(C+A)\partial_p D/2 & 0 & -C^{-1}(C+A) & -C^{-1}A \\ -A + C^{-1}(\partial_p D)^2/4 & 1 & C^{-1}\partial_p D/2 & C^{-1}\partial_p D/2 \\ C+A & -1 & 0 & 0 \end{pmatrix}, \tag{5.48}$$

where

$$C = -\frac{3}{2}\frac{\partial_p f_3(0)}{f_2(0)} + \frac{1}{2}\partial_p f_1(0), \quad A = \frac{1}{2}\frac{\partial_p f_3(0)}{f_2(0)}.$$

After a long calculation, we obtain:

Proposition 5.4. *The monopole-cluster metric on $\Sigma^0_{SO(2)}$ is given by*

$$\begin{aligned} i^*g = 6dr^2 + 2(C+A)[(C+3A) + C^{-1}(\partial_p D)^2/2]dp^2 \\ + 4C^{-1}(C+A)d\theta'^2 + 2C^{-1}(C+2A)d\phi'^2 \\ + 4C^{-1}(C+A)dp \odot d\theta' + 4C^{-1}A\partial_p D dp \odot d\phi' \\ + 8C^{-1}Ad\theta' \odot d\phi' - 4(C+3A)dp \odot dr, \end{aligned}$$

where $\odot$ is the symmetric product symbol for tensors.

Let $R = \tan D$, then

$$dp = -\frac{\pi dR}{2(1+R^2)\arctan^2 R}.$$

We may write C and A as

$$\begin{aligned} C &= \frac{\arctan^4 R}{\pi}(1+R^2) + \frac{\arctan R}{\pi}(3 + 4R\arctan R), \\ A &= -\frac{\arctan R}{\pi}(1 + R\arctan R). \end{aligned}$$

Using the coordinates $\{R, r, \theta', \phi'\}$ instead, the metric can be expressed as

$$i^*g = d\left(\frac{\pi^2}{4\sqrt{2}}R + \sqrt{2}r\right)^2 + 4dr^2 + 4d\theta'^2 + 2d\phi'^2 + O\left(1/R\right). \tag{5.49}$$

Observe that, by interpreting R as the separation parameter, the metric is asymptotic to a flat metric with rate $1/R$ for sufficiently large R. Although there is a cross term in the formula, one may easily verify that the asymptotic metric is indeed positive-definite. In particular, this very example shows that in the region $\Sigma^0_{SO(2)}$, the metric is positive-definite providing the separation is sufficiently large, which is consistent with the general theory.

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