On Orthogonal Polynomials and Related Discrete Integrable Systems

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Abstract

Orthogonal polynomials arise in many areas of mathematics and have been the subject of interest by many mathematicians. In recent years this interest has often arisen from outside the orthogonal polynomial community after their connection with integrable systems was found. This thesis is concerned with the different ways these connections can occur. We approach the problem from both perspectives, by looking for integrable structures in orthogonal polynomials and by using an integrable structure to relate different classes of orthogonal polynomials.

In Chapter 2, we focus on certain classes of semi-classical orthogonal polynomials. For the classical orthogonal polynomials, the recurrence relations and differential equations are well known and easy to calculate explicitly using an orthogonality relation or generating function. However with semi-classical orthogonal polynomials, the recurrence coefficients can no longer be expressed in an explicit form, but instead obeys systems of non-linear difference equations. These systems are derived by deriving compatibility relations between the recurrence relation and the differential equation. The compatibility problem can be approached in two ways; the first is the direct approach using the orthogonality relation, while the second introduces the Laguerre method, which derives a differential system for semi-classical orthogonal polynomials. We consider some semi-classical Hermite and Laguerre weights using the Laguerre method, before applying both methods to a semi-classical Jacobi weight. While some of the systems derived will have been seen before, most of them (at least not to our knowledge) have not been acquired from this approach.

Chapter 3 considers a singular integral transform that is related to the Gel'fand-Levitan equation, which provides the inverse part of the inverse scattering method (a solution method of integrable systems). These singular integral transforms constitute a dressing method between elementary (bare) solutions of an integrable system to more complicated solutions of the same system. In the context of this thesis we are interested in adapting

this method to the case of polynomial solutions and study dressing transforms between different families of polynomials, in particular between certain classical orthogonal polynomials and their semi-classical deformations.

In chapter 4, a new class of orthogonal polynomials are considered from a formal approach: a family of two-variable orthogonal polynomials related through an elliptic curve. The formal approach means we are interested in those relations that can be derived, without specifying a weight function. Thus, we are mainly concerned with recursive structures, particularly on their explicit derivation so that a series of elliptic polynomials can be constructed. Using generalized Sylvester identities, recurrence relations are derived and we consider the consistency of their coefficients and the compatibility between the two relations. Although the chapter focuses on the structure of the recurrence relations, some applications are also presented.

For my parents, who provided unconditional support

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" *The first rule of discovery is to have brains and good luck. The second rule of discovery is to sit tight and wait till you get a good idea*" - George Polya

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Contents

Chapter 1

Introduction

As the title suggests, this thesis is concerned with the connection between orthogonal polynomials and discrete integrable systems. The focus will be from the orthogonal polynomials point of view, looking at the kind of integrable systems that occur after studying the underlying structure of orthogonal polynomials. Because I will be using this approach, most of the introductory material in this first chapter will be concerned with some of the general theory of orthogonal polynomials and introduce the best known families and classes of orthogonal polynomials. By comparison, discrete integrable systems will be introduced when they first arise in Chapter 2. This secondary introduction, will give a brief look at some of the key equations and systems studied in discrete integrable systems as well as an introduction into the Painlevé equations.

In the literature, there have been many connections found between certain classes of orthogonal polynomials and discrete integrable systems and our focus lies in two specific areas. They are the study of semi-classical orthogonal polynomials and the introduction of a new formal class of two-variable orthogonal polynomials defined through an elliptic curve. Although two very different approaches to orthogonal polynomials it is the focus on the recursive properties of both these classes that yields the connections to discrete integrability. At this point in the thesis I will not endeavor to give a comprehensive description of the terminology since this will be provided later in the chapter.

The motivation for this research is that recursive structures, particularly the recurrence coefficients have been found to have many connections with integrable systems. By considering recurrence coefficients defined through an elliptic curve, we gain further insight into the connections between elliptic functions and discrete integrable systems as well as with orthogonal polynomials.

The subject of orthogonal polynomials finds its origins in the 18th century, thanks to the works of Legendre, Laplace and Lagrange. While these three brilliant mathematicians are best remembered for their work in elliptic functions, the theory of differential equations and mathematical astronomy, they also developed the first examples of orthogonal polynomials, before any general theory existed. The development of the general theory of orthogonal polynomials began in the 19th century after investigations into Stieltjes continued fractions [155, 156] by Chebyshev [36]. Other important results found independent of the general theory were given by Gauss, Abel, Jacobi, Hermite and Laguerre, of whom the latter three gave their name to what became the *classical* orthogonal polynomials.

The classical orthogonal polynomials (referred to as the very classical polynomials in modern literature) were the first families of orthogonal polynomials to be established and are important because they were discovered to possess many more properties than other orthogonal polynomial systems of the time. Orthogonal polynomials have since been found to have connections with trigonometric, hypergeometric, Bessel and elliptic functions; they have significance in helping to solve certain problems in quantum mechanics and mathematical statistics; and are related to important problems of interpolation and mechanical quadrature. One example of their breadth of interest, is the bibliography [149] up to 1940, which consists of 1952 papers by 643 authors. Their use in the solution and application of other mathematical problems, has led to our continued interest in the theory to this day.

Up until the late 20th century, there were only a few authorative texts on the subject of orthogonal polynomials. These included the book [159] by Szegö (1939) on orthogonal polynomials, that covered most of the general theory along with all the standard formulae for the three very classical orthogonal polynomials. The monograph [66] by G. Freud (1971) also gave a detailed view of the classical orthogonal polynomials in the context of asymptotics. The text [41] by Chihara (1978) was meant to introduce those unfamiliar with orthogonal polynomials to the subject, by focusing on the elementary theory and aiming at a less advanced audience. The focus was often on recurrence relations using the justification that "a great deal can be developed only using elementary tools". Recently though, there has been a renewed interest in orthogonal polynomials, especially since the connection with integrable systems has been found. Amongst these I mention the books by B. Simon [150, 151] (2004), which has developed the general theory from [159] to become authorative texts on orthogonal polynomials on the unit circle, and the monograph [84] (2005), which approaches orthogonal polynomials from the viewpoint of special functions.

Since the days when the classical theory was established we have since seen a split in the field, into multiple strands. Of these, the main strands and their key contributors are

- *special functions*, includes the work of Ismail [84] and Carlitz. Their interests lie in the connections that different special functions have with orthogonal polynomials, such as elliptic functions.

- *Freud and asymptotics*, the work of van Assche [165] and Nevai [122]. Their work often involves the work begun by Géza Freud in asymptotics and Freud weights.

- *formal orthogonal polynomials*, which is based in a French school of numerical analysis. A class of formal orthogonal polynomials is one where the weight function is not defined. As such the focus is on recursive structures. The main contributors include Draux [50], who first coined the term formal orthogonal polynomials; Brezinski [27], who often approaches formal orthogonal polynomials using determinant structures and Maroni [111, 112, 113, 114], who has been involved with formal semi-classical orthogonal polynomials.

The theory of formal orthogonal polynomials plays a central role in modern numerical analysis, in particular in connection with the theory of Padé approximants, with the QD-algorithm and in the development of convergence acceleration algorithms, cf. [25, 27, 50]. Recently it was pointed out in [139] that also these matters are intimately connected to integrable discrete systems. The famous ε-algorithm of Wynn [170], specified by the partial difference equation (1.0.1),

$$(\varepsilon_{n+1}^{(m)} - \varepsilon_{n-1}^{(m+1)})(\varepsilon_n^{(m+1)} - \varepsilon_n^{(m)}) = 1, \tag{1.0.1}$$

turns out to be identical to a well-known exactly integrable lattice system related to the Korteweg-de Vries (KdV) equation (a soliton system defined on the space-time lattice). This allows us to interpret the numerical algorithm as a symplectic dynamical system with extremely rich behaviour from the point of view of analytical solutions. Similarly, the famous “missing identity of Frobenius” [171] (found by Wynn in 1966) in the theory of Padé approximants,

$$\frac{1}{r_{m+1,n} - r_{m,n}} + \frac{1}{r_{m-1,n} - r_{m,n}} = \frac{1}{r_{m,n+1} - r_{m,n}} + \frac{1}{r_{m,n-1} - r_{m,n}}$$

can be regarded as an exactly solvable lattice system closely related to discretisations of the KdV equation and intimately connected to the Toda lattice.

In recent years there has also been more interest in establishing a solid connection between the theory of matrix models and orthogonal polynomials, and the theory of discrete integrable systems. The manifestations of this connection is manifold: nonlinear integrable systems, in particular equations of Painlevé type, arise as the governing equations for the partition functions of matrix models and the hierarchies of soliton type equations sit on the background of the main algebraic structures for these models.

Furthermore, the Riemann-Hilbert approach, which has already found its key role in the construction of analytic solutions of integrable PDEs and ODEs, has recently become a powerful new tool in studying the asymptotic properties of orthogonal polynomials [48].

1.1 Basic Properties of Orthogonal Polynomials

This introductory chapter will give a brief account on the standard theory of orthogonal polynomials, focusing on the construction of a recurrence relation using determinants. We then describe hypergeometric functions and some of its associated relations. Most "classical" orthogonal polynomials can be written as terminating hypergeometric series and during the twentieth century people have been working on a classification of all such hypergeometric orthogonal polynomial and their characterizations. Of the classical orthogonal polynomials, we state some of the standard formulae of the *very classical* orthogonal polynomials (those named after Jacobi, Laguerre and Hermite [166]) and then consider several other classes of orthogonal polynomials, including the discrete, multi-variable and q-orthogonal polynomials. We emphasize the specific relations of the very classical orthogonal polynomial families, since they will be used in greater detail in later chapters of the thesis. Moving into the applications of orthogonal polynomials, we will demonstrate how quantum mechanics and matrix models use the theory of orthogonal polynomials to aid in the solution of some of their problems, which in turn shows the broader world which orthogonal polynomials exists in.

1.1.1 System of Orthogonal Functions

With a specific interval (a, b) (on the real line $\mathbb{R}$) and a fixed weight function, we can define an inner product for a pair of functions [16]. An inner product may be defined by a Stieltjes integral

$$\langle \phi_1, \phi_2 \rangle = \int_a^b \phi_1(x)\phi_2(x)d\mu(x) \tag{1.1.1a}$$

where $\mu(x)$ is a non-decreasing function. If $\mu(x)$ is absolutely continuous then (1.1.1a) reduces to

$$\langle \phi_1, \phi_2 \rangle = \int_a^b \phi_1(x)\phi_2(x)w(x)dx \tag{1.1.1b}$$

where the integral is assumed to exist in Lebesgue's sense [159]. However if $\mu(x)$ is a jump function that is constant except for jumps of the magnitude w_i at $x = x_i$, then (1.1.1a) reduces to

$$\langle \phi_1, \phi_2 \rangle = \sum_i w_i \phi_1(x_i)\phi_2(x_i), \tag{1.1.1c}$$

the definition for functions of a discrete variable.

Two functions are said to be orthogonal to one another if their inner product is zero, hence a family of functions forms an orthogonal system on an interval (a, b) with a weight function $w(x)$ if for any two distinct members of the family $\langle \phi_1, \phi_2 \rangle = 0$. An orthogonal system can be written as a sequence of functions $\{\phi_n\}_{n=0}^{\infty}$ and the corresponding orthogonal property can be expressed as $\langle \phi_n, \phi_m \rangle = 0 \quad \text{for} \quad n \neq m$. Assuming that $\{\phi_n\}$ doesn't contain any null function, then $\langle \phi_n, \phi_n \rangle$ is positive for all n and consequently the functions of any finite subset of an orthogonal system are linearly independent. Then the functions $\{\phi_n\}$ form an orthonormal system if

$$\langle \phi_n, \phi_m \rangle = \begin{cases} 0 & \text{if} \quad n \neq m \\ 1 & \text{if} \quad n = m \end{cases} . \tag{1.1.2}$$

Although individually these functions are unimportant, collectively this property of orthogonality with a given weight, is a decisive property that fixes the functions uniquely.

1.1.2 Single-Variable Orthogonal Polynomials

Moving from general orthogonal functions to polynomials, we consider a single variable polynomial $P_n(x)$

$$P_n(x) = \sum_{j=0}^{n} a_{n,j}x^j = a_{n,n}x^n + a_{n,n-1}x^{n-1} + \ldots + a_{n,1}x + a_{n,0}, \tag{1.1.3}$$

and for convenience we will deal monic polynomials of order n ($a_{n,n} = 1$) with some given coefficients $a_{n,0}, a_{n,1}, \ldots, a_{n,n}$. What makes these rather simple functions interesting is their orthogonality property, by which they form a family of orthogonal functions i.e. a family of polynomials organized according to their degree and within the family, due to the *orthogonality* to each other. Formally we define an *orthogonal polynomial sequence* $\{P_n(x)\}_{n=0}^{\infty}$ [41], with respect to a moment functional $\mathfrak{L}$ provided for all non-negative integers m and n,

- $P_n(x)$ is a polynomial of degree n,
- $\mathfrak{L}[P_n(x)P_m(x)] = \langle P_n, P_m\rangle = 0$ for $m \neq n$,

where

$$\mathfrak{L}[P_n(x)] = \int w(x)P_n(x)dx. \tag{1.1.4}$$

In many cases the inner product $\langle P_n, P_m\rangle$ can be expressed explicitly in terms of an integral with a certain measure, which leads to

$$\langle P_n(x), P_m(x)\rangle \quad = \int_a^b P_n(x)P_m(x)w(x)dx = h_n\delta_{nm}, \tag{1.1.5}$$

$$\text{where the Kronecker delta } \delta_{nm} = \begin{cases} 0 & \text{when} \quad n \neq m \\ 1 & \text{when} \quad n = m \end{cases}.$$

This applies for the case of a non-zero continuous weight function w, which is non-negative, is integrable on an interval $[a, b]$ and where $h_n \neq 0$. In such a case we have

$$\int_a^b w(x)dx > 0$$

and $\mathcal{L}[P_n^2(x)] = \langle P_n, P_n \rangle \neq 0$.

We use this particular case, because this will be most used in subsequent chapters. Whenever we have such a family of polynomials, with a given weight function $w(x)$ and interval $[a, b]$ a large number of properties follow for the polynomials in the family, such as a recurrence relation.

1.1.3 Recursive Structure

The orthogonality condition (1.1.5) implies the existence of a three point recurrence relation, which can be seen by considering the inner product relation:

$$\langle xP_n, P_m \rangle = \langle P_n, xP_m \rangle, \tag{1.1.6}$$

and an expression for $xP_n(x)$:

$$xP_n = P_{n+1} + \sum_{j=0}^{n} a_j^{(n)} P_j \tag{1.1.7}$$

(a consequence of xP_n being a polynomial). Using these two expressions it is possible to acquire a general from of a recurrence relation for orthogonal polynomials. We begin by expanding the left side of (1.1.6) using (1.1.7)

$$\begin{aligned} \langle P_{n+1} + \sum_{j=0}^{n} a_j^{(n)} P_j, P_m \rangle &= \sum_{j=0}^{n} a_j^{(n)} h_m \delta_{jm} \quad \text{for} \quad m \leq n, \\ &= a_m^{(n)} \end{aligned} \tag{1.1.8}$$

then consider expanding the right side of (1.1.6) using (1.1.7)

$$\langle P_n, P_{m+1} + \sum_{j=0}^{m} a_j^{(m)} P_j \rangle = 0 \quad \text{for} \quad m \leq n-2. \tag{1.1.9}$$

By comparing these two expressions we can conclude that $a_m^{(n)} = 0$ for $m < n-1$ and thus we have only three terms in the recurrence relation (1.1.7). We refer to this relation as the monic recurrence relation, since it gives rise to monic polynomials.

$$xP_n = P_{n+1} + S_n P_n + R_n P_{n-1} \tag{1.1.10}$$

where

$$a_n^{(n)} = S_n \quad , \quad a_{n-1}^{(n)} = R_n$$

We can evaluate the values of S_n and R_n by considering certain inner products [41]. To derive S_n, we consider $\langle P_n, P_{n+1}\rangle$ and expand with the recurrence relation to get

$$\begin{aligned}
\langle P_n, P_{n+1}\rangle &= \langle P_n, xP_n\rangle - S_n\langle P_n, P_n\rangle - R_n\langle P_n, P_{n-1}\rangle, \\
\Rightarrow 0 &= \langle xP_n, P_n\rangle - S_n\langle P_n, P_n\rangle, \\
\Rightarrow S_n &= \frac{\langle xP_n, P_n\rangle}{\langle P_n, P_n\rangle}. \qquad (1.1.11)
\end{aligned}$$

To derive R_n, we consider $\langle x^{n-1}, P_{n+1}\rangle$ and expand with the recurrence relation to get

$$\begin{aligned}
\langle x^{n-1}, P_{n+1}\rangle &= \langle x^n, P_n\rangle - S_n\langle x^{n-1}, P_n\rangle - R_n\langle x^{n-1}, P_{n-1}\rangle, \\
\Rightarrow 0 &= \langle x^n, P_n\rangle - R_n\langle x^{n-1}, P_{n-1}\rangle, \\
\Rightarrow R_n &= \frac{\langle x^n, P_n\rangle}{\langle x^{n-1}, P_{n-1}\rangle} = \frac{h_n}{h_{n-1}} \qquad (1.1.12)
\end{aligned}$$

since $\langle P_n, P_n\rangle = h_n$ and we take $R_0 = h_0$ setting $h_{-1} = 1$ [41].

1.1.4 Determinant Representation of Orthogonal Polynomials

While the orthogonality condition can be used to prove the existence of a recurrence relation, the previous method does not give us explicit expressions for the recurrence coefficients derived in terms of moments. So using the orthogonality we construct a determinant representation for the polynomials $P_n(x)$, which can then be used to derive an explicit form for the recurrence relation. This method is well known and can be found in [16].

A weight function $w(x)$ on an interval a, b determines a system of orthogonal polynomials P_n uniquely, apart from a constant factor in each polynomial, c_n. These numbers are the *moments* of the weight function,

$$c_n = \int_a^b w(x)x^n dx \qquad (1.1.13a)$$

with the following scalar product representation

$$c_{m+n} = \langle x^n, x^m \rangle. \tag{1.1.13b}$$

Proposition

With this definition for the moments, we can construct the determinant representation $P_n(x)$.

$$P_n(x) = \frac{\begin{vmatrix} c_0 & \dots & c_j & \dots & c_n \\ c_1 & \dots & c_{j+1} & \dots & c_{n+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n-1} & \dots & c_{j+n-1} & \dots & c_{2n-1} \\ 1 & \dots & x^j & \dots & x^n \end{vmatrix}}{\Delta_{n-1}} \tag{1.1.14}$$

Proof

A sequence of linearly independent functions can be orthogonalized with respect to the inner product $\langle P_n, P_m \rangle$ by the formation of suitable linear combinations. This leads to the following triangular structure:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= \alpha_{10} P_0(x) + x \\ \vdots \quad & \qquad \qquad \vdots \\ P_n(x) &= \alpha_{n0} P_0(x) + \alpha_{n1} P_1(x) + \ldots + \alpha_{n,n-1} P_{n-1}(x) + x^n \end{aligned}$$

Now the inner product of the polynomial $P_n(x)$ (1.1.3) with x^m (after rearrangement) can be found for values of $m = 0, 1, \ldots, n-1$ and for $m = n$:

$$\begin{aligned} \langle x^m, P_n \rangle &= a_0 c_m + a_1 c_{m+1} + \ldots + a_{n-1} c_{m+n-1} + a_n c_{m+n} = 0 \text{ unless } n = m \\ \langle x^n, P_n \rangle &= a_0 c_n + a_1 c_{n+1} + \ldots + a_{n-1} c_{2n-1} + a_n c_{2n} = h_n \end{aligned}$$

where the only contributor is $\langle P_n, P_n \rangle = h_n$ and this is due to the orthogonality between polynomials. Then these equations can be written in a matrix form

$$\begin{pmatrix} c_0 & \dots & c_n \\ c_1 & \dots & c_{n+1} \\ \vdots & \ddots & \vdots \\ c_n & \dots & c_{2n} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ h_n \end{pmatrix} \qquad a_n = 1, \tag{1.1.15}$$

which can be solved by using Cramer's rule.

Lemma 1.1.1 - *Cramer's Rule Given the expression $A\underline{x} = \underline{b}$, where $A = (\underline{a}_1, \dots, \underline{a}_n)$ is an $n \times n$ matrix and $\underline{x}$ and $\underline{b}$ are n-component column vectors. Then the elements of $\underline{x}_i$ can be represented as*

$$x_i = \frac{|\underline{a}_1, \dots, \overset{i\downarrow}{\underline{b}}, \dots, \underline{a}_n|}{\det(A)} \tag{1.1.16}$$

and where $\det(A) \neq 0$.

That is we replace the i^{th} column with the right side of (1.1.15) and then divide it by itself. This equation is multiplied by x^j and summed over n to give $P_n(x)$

$$\sum_{j=0}^{n} x^j a_j = h_n \frac{\begin{vmatrix} c_0 & \dots & c_j & \dots & c_n \\ c_1 & \dots & c_{j+1} & \dots & c_{n+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n-1} & \dots & c_{j+n-1} & \dots & c_{2n-1} \\ 1 & \dots & x^j & \dots & x^n \end{vmatrix}}{\Delta_n} = P_n(x), \tag{1.1.17}$$

with the *Hankel determinant* Δ_n:

$$\Delta_n = \begin{vmatrix} c_0 & \dots & c_n \\ \vdots & \ddots & \vdots \\ c_n & \dots & c_{2n} \end{vmatrix}$$

and where a necessary and sufficient condition is $\Delta_n \neq 0$ [41]. $\square$

If this determinant is expanded in terms of the Hankels, the first term x^n in the polynomial P_n is

$$P_n(x) = x^n h_n \frac{\Delta_{n-1}}{\Delta_n} + \dots \quad , \tag{1.1.18}$$

and since we assume this polynomial is monic:

$$h_n = \frac{\Delta_n}{\Delta_{n-1}} \ , \tag{1.1.19}$$

where we supplement this equation by imposing $\Delta_{-1} = 1$ when $n = 0$.

1.1.5 A Recurrence Relation from Hankel Determinants

Returning to the general form of a recurrence relation along with (1.1.14) (which is altered through the inclusion of an upper index), allows the creation of a recurrence relation (1.1.10), with S_n and R_n defined in terms of Hankel determinants. Thus the family of adjacent polynomials is introduced:

$$P_n^{(m)}(x) \equiv \frac{1}{\Delta_{n-1}^{(m)}} \begin{vmatrix} c_m & \dots & \dots & c_{n+m} \\ \vdots & & & \vdots \\ c_{n+m-1} & & & c_{2n+m-1} \\ 1 & \dots & \dots & x^n \end{vmatrix} \tag{1.1.20a}$$

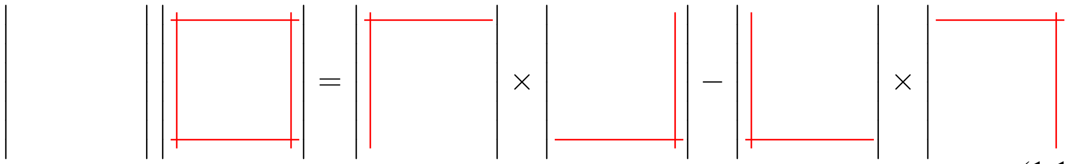
with the corresponding Hankel determinant:

$$\Delta_n^{(m)}(x) \equiv \begin{vmatrix} c_m & \dots & \dots & c_{n+m} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ c_{n+m} & \dots & \dots & c_{2n+m} \end{vmatrix} \tag{1.1.20b}$$

(where we assume $\Delta_n^{(m)}(x) \neq 0$). From this, it is possible to acquire a pair of relations by using two different forms of the corresponding two row/column Sylvester[1] determinant

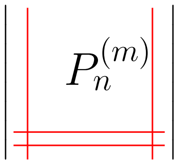
[1]this identity has many different names including the Jacobi identity, Lewis Carroll's identity and the window-pane identity, however we will refer to it as the Sylvester identity throughout the thesis

identity [23, 119, 137], which we apply onto the polynomial $P_n^{(m)}(x)$. Although in this chapter we only make use of the two row/column Sylvester Identity, we will consider a more generalized form (where we extend the removal of two rows and columns to m rows and columns) in Chapter 4 and a derivation is presented in Appendix B.

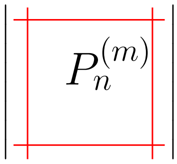


(1.1.21)

We use two different forms of the Sylvester identity and applying them to $P_n(x)$ leads to



$$\Rightarrow \quad P_n^{(m)}(x) = xP_{n-1}^{(m+1)}(x) - \frac{\Delta_{n-1}^{(m+1)}\Delta_{n-2}^{(m)}}{\Delta_{n-2}^{(m+1)}\Delta_{n-1}^{(m)}}P_{n-1}^{(m)}(x), \tag{1.1.22a}$$



$$\Rightarrow \quad P_n^{(m)}(x) = xP_{n-1}^{(m+2)}(x) - \frac{\Delta_{n-1}^{(m+1)}\Delta_{n-2}^{(m+1)}}{\Delta_{n-1}^{(m)}\Delta_{n-2}^{(m+2)}}P_{n-1}^{(m+1)}(x), \tag{1.1.22b}$$

which can also be expressed in the following form:

$$\begin{aligned} P_{n+1}^{(m)} &= xP_n^{(m+1)} - V_n^{(m)}P_n^{(m)} && (1.1.23a) \\ P_{n+1}^{(m)} &= xP_n^{(m+2)} - W_n^{(m)}P_n^{(m+1)} \quad , && (1.1.23b) \end{aligned}$$

with

$$V_n^{(m)} = \frac{\Delta_n^{(m+1)}\Delta_{n-1}^{(m)}}{\Delta_{n-1}^{(m+1)}\Delta_n^{(m)}} \quad , \quad W_n^{(m)} = \frac{\Delta_n^{(m+1)}\Delta_{n-1}^{(m+1)}}{\Delta_n^{(m)}\Delta_{n-1}^{(m+2)}} \quad .$$

These two equations can be combined to leave an equation in terms of $P_n^{(m+1)}$:

$$P_{n+1}^{(m)} = P_{n+1}^{(m+1)} + V_n^{(m+1)}P_n^{(m+1)} - W_n^{(m)}P_n^{(m+1)}, \tag{1.1.24}$$

which in turn can be eliminated to give an equation just in terms of P:

$$\begin{aligned} xP_{n+1}^{(m)} &= (P_{n+2}^{(m)} + V_{n+1}^{(m)}P_{n+1}^{(m)}) + (V_n^{(m+1)} - W_n^{(m)})(P_{n+1}^{(m)} + V_n^{(m)}P_n^{(m)}) \\ xP_n^{(m)} &= P_{n+1}^{(m)} + (V_n^{(m)} + V_{n-1}^{(m+1)} - W_n^{(m)})P_n^{(m)} + (V_{n-1}^{(m+1)} - W_{n-1}^{(m)})V_{n-1}^{(m)}P_{n-1}^{(m)}. \end{aligned}$$

This final equation gives the recurrence relation and since all the P_n have the same order of m, we can omit this in the final relation

$$xP_n = P_{n+1} + S_n P_n + R_n P_{n-1}. \tag{1.1.25}$$

The coefficients S_n and R_n can be further simplified to the following:

$$S_n = \frac{\Delta_n^{(m+1)}\Delta_{n-1}^{(m)}}{\Delta_n^{(m)}\Delta_{n-1}^{(m+1)}} + \frac{\Delta_{n-2}^{(m+1)}\Delta_n^{(m)}}{\Delta_{n-1}^{(m+1)}\Delta_{n-1}^{(m)}} = \frac{h_n^{(m+1)}}{h_n^{(m)}} + \frac{h_n^{(m)}}{h_{n-1}^{(m+1)}}, \tag{1.1.26a}$$

$$R_n = \frac{\Delta_n^{(m)}\Delta_{n-2}^{(m)}}{\Delta_{n-1}^{(m)}\Delta_{n-1}^{(m)}} = \frac{h_n^{(m)}}{h_{n-1}^{(m)}}, \tag{1.1.26b}$$

(where we have suppressed the m-dependence in the symbols R_n and S_n). We achieve this simplification by making use of a bilinear relation that exists between the Hankel determinants Δ_n. We find this relation by applying the Sylvester identity to Δ_n:

$$\left| \Delta_n^{(m)} \right| \Rightarrow \Delta_n^{(m)}\Delta_{n-2}^{(m+2)} = \Delta_{n-1}^{(m+2)}\Delta_{n-1}^{(m)} - \Delta_{n-1}^{(m+1)}\Delta_{n-1}^{(m+1)}. \tag{1.1.27}$$

By incorporating (1.1.27) with the Hankel forms of $(V_n^{(m)} + V_{n-1}^{(m+1)} - W_n^{(m)})$ and $(V_{n-1}^{(m+1)} - W_{n-1}^{(m)})V_{n-1}^{(m)}$, gives rise to the simplified forms of R_n and S_n.

We compare this relation with a special case (one with no parameters) of the discrete-time Toda equation [81]

$$\tau_{n-1}^{(m-1)}\tau_{n+1}^{(m+1)} - \tau_{n+1}^{(m-1)}\tau_{n-1}^{(m+1)} + \tau_n^{(m)}\tau_n^{(m)} = 0 \tag{1.1.28}$$

and it is clear to see that they are very similar with regards to their shifts. This pattern of shifts (which the Toda equation satisfies) demonstrates a bilinear Hirota form and as such is an example of a discrete integrable system. This simple case illustrates how the shadows of integrability already appear in the underlying structure of the standard theory of orthogonal polynomials. We explore this connection in the subsequent chapters of the thesis. While the similarities between the two equations are clear to see, it is also possible to transform one into the other, thus we introduce the simple transformation of $\tau_n^{(m)} = \tilde{\tau}_N^m$, where $N = n + m$:

$$\tilde{\tau}_{N-2}^{(m-1)}\tilde{\tau}_{N+2}^{(m+1)} - \tilde{\tau}_N^{(m-1)}\tilde{\tau}_N^{(m+1)} + \tilde{\tau}_N^{(m)}\tilde{\tau}_N^{(m)} = 0. \tag{1.1.29}$$

We can then let $\tilde{\tau}_N^m = \Delta_{N/2}^{-m}$ and after a simple transformation, this gives the same result as (1.1.27). Any terminology mentioned here will be introduced in greater detail in Chapter 2, together with references to the relevant literature.

Further equations can be found from Hankel determinants using similar Sylvester identities, which lead to bilinear relations of a similar form to the above.

While this derivation is not widely used in the literature, examples of it can be found in [11]. This method demonstrates one way to derive an explicit form of a recurrence relation in terms of Hankel identities. In Chapter 4, this example will be extended by using a generalized version of the Sylvester identity applied to a determinant constructed for polynomials in two variables.

1.1.6 The Christoffel-Darboux Identity

The Christoffel-Darboux identity [44, 46] can be seen as a direct consequence of the recurrence relation, although it is possible to derive it independently of the recurrence relation by using a similar method to the above [29].

The Christoffel-Darboux identity is found using the monic recurrence relation (1.1.25) and a corresponding monic recurrence relation in terms of y

$$yP_n(y) = P_{n+1}(y) + S_nP_n(y) + R_nP_{n-1}(y).$$

We multiply the former by $P_n(y)$ and the latter by $P_n(x)$ and take the difference

$$\begin{aligned} xP_n(x)P_n(y) &= (P_{n+1}(x)+S_nP_n(x)+R_nP_{n-1}(x))P_n(y) \\ yP_n(y)P_n(x) &= (P_{n+1}(y)+S_nP_n(y)+R_nP_{n-1}(y))P_n(x) \\ \Rightarrow (x-y)P_n(x)P_n(y) &= P_{n+1}(x)P_n(y)-P_{n+1}(y)P_n(x) \\ &\quad +R_n(P_{n-1}(x)P_n(y)-P_{n-1}(y)P_n(x)) \end{aligned} \tag{1.1.30}$$

Eliminating R_n by using $R_n = \frac{h_n}{h_{n-1}}$ (1.1.26b), (1.1.30) can be rewritten as

$$\begin{aligned} (x-y)\frac{P_n(x)P_n(y)}{h_n} &= \frac{1}{h_n}(P_{n+1}(x)P_n(y)-P_{n+1}(y)P_n(x)) \\ &\quad -\frac{1}{h_{n-1}}(P_{n-1}(y)P_n(x)-P_{n-1}(x)P_n(y)) \end{aligned}$$

and we apply a discrete integration to give a sum.

$$\begin{aligned} \Rightarrow \quad & \sum_{j=0}^{n}(x-y)\frac{P_j(x)P_j(y)}{h_j} = \frac{1}{h_n}(P_{n+1}(x)P_n(y)-P_{n+1}(y)P_n(x)) \\ \Rightarrow \quad & \sum_{j=0}^{n}\frac{P_j(x)P_j(y)}{h_j} = \frac{(P_{n+1}(x)P_n(y)-P_{n+1}(y)P_n(x))}{h_n(x-y)} \end{aligned} \tag{1.1.31}$$

This identity has many uses in the theory of orthogonal polynomials, particularly when eliminating a sum from an equation, which has particular use in continuous integral equations. There is also a confluent form of (1.1.31), which can be obtained by taking the limit $y \longrightarrow x$ and applying l'Hôpital rule to get

$$\sum_{j=0}^{n}\frac{P_j^2(x)}{h_j} = \frac{(P'_{n+1}(x)P_n(x)-P_{n+1}(x)P'_n(x))}{h_n} \tag{1.1.32}$$

where $P'_n(x) = \frac{d}{dx}P_n(x)$. A consequence of this is that

$$(P'_{n+1}(x)P_n(x)-P_{n+1}(x)P'_n(x)) > 0 \quad \text{for all } \ x \tag{1.1.33}$$

and this has particular use in exploring the zeros of orthogonal polynomials [41], specifically that the zeros of $P_n(x)$ and $P_{n+1}(x)$ separate each other. To prove this we denote the zeros of $P_n(x)$ in increasing order by $x_{1n} < x_{2n} < \cdots < x_{nn}$. Given the fact that $x_{k,n+1}$ is a zero of $P_{n+1}(x)$ and using (1.1.33) we get

$$P_n(x_{k,n+1})P'_{n+1}(x_{k,n+1}) > 0. \tag{1.1.34}$$

The simplicity of zeros implies that $P'_{n+1}(x_{k,n+1})$ and $P'_{n+1}(x_{k+1,n+1})$ have different signs. It follows that $P_n(x_{k,n+1})$ and $P_n(x_{k+1,n+1})$ have different signs. By the continuity of P_n we know it has a zero between $x_{k,n+1}$ and $x_{k+1,n+1}$ for $k = 1, 2, \ldots, n$ and the result follows.

1.2 The Hypergeometric Series and Associated Special Functions

It is possible to express almost all elementary functions of mathematics as hypergeometric functions or ratios of hypergeometric functions and truncations of hypergeometric functions lead to orthogonal polynomials [9]. First though, we introduce two special functions, namely the Gamma and the Beta functions $\Gamma(x)$ and $B(x, y)$ respectively.

1.2.1 The Gamma and Beta Functions

The Gamma function extends the factorial function $n!$ to complex numbers and can be defined as

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{(x)_{n+1}} \tag{1.2.1a}$$

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \tag{1.2.1b}$$

(where if the real part of the complex number x is positive, the integral converges absolutely) and we introduce the notation of the Pochhammer symbol or rising factorial

$(x)_n$

$$(x)_n = x(x+1)(x+2)\dots(x+n-2)(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}. \tag{1.2.2}$$

Also by using integration by parts we find the difference equation

$$\Gamma(x+1) = x\Gamma(x) \tag{1.2.3}$$

and we define the Beta function (also referred to as the Euler integral of the first kind) as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{1.2.4}$$

The two special functions are related through the expression

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\ . \tag{1.2.5}$$

An alternate way to introduce the Gamma function is as the infinite product

$$\Gamma(x) = \frac{e^{-\gamma x}}{x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1} e^{\frac{x}{n}} \tag{1.2.6}$$

where γ is Euler's constant

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log n\right) \tag{1.2.7}$$

and takes an approximate value of 0.57722. This product is valid for all complex numbers x, which are not negative integers or zero. Their connection to hypergeometric functions, occurs in its integral representation, first defined by Euler.

1.2.2 The Hypergeometric Function

We say a series $\sum c_n$ is hypergeometric if the ratio $\frac{c_{n+1}}{c_n}$ is a rational function of n. Thus, factorizing polynomials in n, we obtain

$$\frac{c_{n+1}}{c_n} = \frac{(n+a_1)(n+a_2)\dots(n+a_p)}{(n+b_1)(n+b_2)\dots(n+b_q)(n+1)}. \tag{1.2.8a}$$

This relation leads to

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{x^n}{n!} = {}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; x \right) \tag{1.2.8b}$$

where ${}_pF_q$ is commonly referred to as the hypergeometric series. The series ${}_pF_q$ is absolutely convergent for all x if $p \leq q$ and for $|x| < 1$ if $p = q + 1$. It diverges for all $x \neq 0$ if $p > q + 1$ and the series does not terminate. Thus we define hypergeometric functions ${}_2F_1(a, b; c; x)$ as

$${}_2F_1(a, b; c; x) = {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; x \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \tag{1.2.9}$$

which is convergent for $|x| < 1$. Most elementary functions are special cases of hypergeometric series, for example

$$\begin{aligned} \sin x &= x \, {}_0F_1 \left(\begin{matrix} - \\ \frac{3}{2} \end{matrix} ; \frac{-x^2}{4} \right), \quad \cos x = {}_0F_1 \left(\begin{matrix} - \\ \frac{1}{2} \end{matrix} ; \frac{-x^2}{4} \right) \\ e^x &= {}_0F_0 \left(\begin{matrix} - \\ - \end{matrix} ; x \right) \end{aligned}$$

(where - represents a blank space) and hypergeometric functions can have different representations including the Euler integral representation (which makes use of the Gamma and Beta functions)

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; x \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt, \tag{1.2.10}$$

and summation theorems such as the Chu-Vandermonde Sum.

$${}_2F_1 \left(\begin{matrix} -n, b \\ c \end{matrix} ; 1 \right) = \frac{(c-b)_n}{(c)_n} \tag{1.2.11}$$

While the methods to derive these two equations are straightforward, they are also long so I will not include them. A full derivation can be found in [9], where they are used to simplify equations including the Jacobi orthogonality relation (which is detailed later).

Hypergeometric functions also generalize many special functions, including the Bessel functions, the Gamma function, the error function, the elliptic integrals and the orthogonal polynomials. Equation (1.2.11) in particular is used when proving the orthogonality of the Jacobi polynomials. This is in part because hypergeometric functions are solutions of the hypergeometric differential equation, which is a (Fuchsian) second-order ordinary differential equation, with three regular singular points.

$$x(1-x)\frac{d^2w}{dx^2} + (c - (a+b+1)x)\frac{dw}{dx} - abw = 0 \tag{1.2.12}$$

The theory of Fuchsian differential equations is a very broad theory, but is not one that we deal with in this thesis.

1.2.3 The Heun and Lamé equations

Going beyond the hypergeometric differential equation (in terms of complexity) is the Heun equation [145, 152], which has Lamé [169] as a special case. As we have mentioned, an alternative definition of hypergeometric functions would be to define it as a solution of a Fuchsian differential equation with at most three regular singularities, $0, 1, \infty$. Heun functions, are defined as special solutions of a generic linear second order Fuchsian differential equation with four regular singularities, $0, 1, a, \infty$, where a is the additional singularity. Then we present the Heun equation as

$$\frac{d^2w}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\frac{dw}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-a)}w = 0 \tag{1.2.13a}$$

where

$$\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0. \tag{1.2.13b}$$

Of these seven parameters, $\alpha, \beta, \gamma, \delta, \epsilon$ are referred to as the exponent parameters (since they determine the exponents at the four singularities), a is the singularity parameter and q is the accessory parameter. Many important subclasses are found choosing specific values of these parameters. Heun's equation was originally constructed as a deliberate generalization of the hypergeometric equation, so unsurprisingly there are three ways in which the former reduces to the latter [145], e.g.

$$x(x-1)(x-a)y''(x)+[\gamma(x-1)(x-a)+\delta x(x-a)+\epsilon x(x-1)]y'(x)$$
$$+(\alpha\beta x-q)y(x)=0$$

and set $$a=1, \quad q=\alpha\beta$$

then a factor $(x-1)$ can be taken out, leaving

$$x(1-x)\frac{d^2y}{dx^2}+(c-(a+b+1)x)\frac{dy}{dx}-aby=0$$

the hypergeometric equation (1.2.12). The Heun equation has many uses in mathematics, but for this work we are primarily interested in its connections with orthogonal polynomials. One such class is known as the Stieltjes-Carlitz polynomials, which will be mentioned in Chapter 4.

One particular case of the Heun's general (non-confluent), which has seen a lot of attention in recent years [108], is the case where $\gamma=\delta=\epsilon=\frac{1}{2}$. The equation then becomes the Lamé equation

$$\begin{aligned} y''(x) &+ \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-a}\right)y'(x) \\ &+ \frac{ah-\nu(\nu+1)x}{4x(x-1)(x-a)}y=0 \end{aligned} \tag{1.2.14}$$

of which there are several forms, but here we just consider the Jacobi and *Weierstrass* forms.

$$[-\frac{d^2}{d\alpha^2}+l(l+1)k\,\mathrm{sn}^2(\alpha|k)]\Psi = E\Psi \tag{1.2.15a}$$

$$\left\{\frac{d^2}{du^2}-[l(l+1)\wp(u;g_2,g_3)+B]\right\}\Psi = 0 \tag{1.2.15b}$$

The Weierstrass form may be further rearranged to the *elliptic-curve algebraic form*

$$\left\{\left(y\frac{d}{dx}\right)^2-[l(l+1)x+B]\right\}\Psi=0 \tag{1.2.16a}$$

where

$$B = -E + \frac{1}{3}l(l+1)(m+1). \tag{1.2.16b}$$

Both of the forms (1.2.15a,1.2.15b) that are introduced here make use of elliptic functions: in the Jacobi form there is the Jacobi sine function and in the Weierstrass there is the $\wp$ function. The solutions of the Lamé equation are the Lamé polynomials, which are polynomials in the Jacobi elliptic functions sn$(\alpha|m)$, cn$(\alpha|m)$ and dn$(\alpha|m)$ (See appendix A).

1.3 The 'very' Classical Orthogonal Polynomials

In the modern theory the following are referred to as the very classical orthogonal polynomials [166],

1. *Hermite* polynomials
2. (generalised) *Laguerre* polynomials
3. *Jacobi* or hypergeometric polynomials

(of which, these are all characterized by their different weight functions and integration intervals), where it is important to highlight the distinction between these and other orthogonal polynomials. But where does that distinction lie?

The *classical* orthogonal polynomials can be defined [16, 41] as those orthogonal polynomials satisfying the properties:

1. $\{P_n'(x)\}$ is a system of orthogonal polynomials;
2. $P_n(x)$ satisfies a differential equation of the form $A(x)y'' + B(x)y' + \lambda_n y = 0$, where $A(x)$ and $B(x)$ are independent of n and λ_n is independent of x;

3. there is a generalized Rodrigues' Formula

$$P_n(x) = \frac{1}{K_n w(x)} \frac{d^n}{dx^n}(w(x)X^n)$$

where K_n is a constant and X is a polynomial in x, whose coefficients are independent of n.

Conversely any of these three properties characterizes the classical orthogonal polynomials in the sense that any system of orthogonal polynomials which has one of these properties can be reduced to a classical system. For (1) this has been proved by Hahn [78] and Krall [97]; for (2) by Bochner [22] (in this case there are some trivial exceptions); and for (3) by Tricomi [161].

However in recent times there have been a number of families that satisfy these conditions, but are not called *classical*. Thus we refer to those mentioned above as the very classical orthogonal polynomials.

Looking more closely at the properties that these families have, it can be shown that they all have a *generating function*, a *recursion* and *differential relation* (which can be combined to give a second order ODE) and a *Rodrigues' formula*. These equations are easier to derive for the Hermite and Legendre polynomials and more difficult for the other families with the Jacobi polynomials providing large equations, which although solvable, require some mathematical tricks and techniques. Families of orthogonal polynomials have a lot of beautiful mathematical structure behind them, which make them into interesting objects to study.

The generating function for the family Hermite can be used to calculate the other three properties, ie. the recurrence and differential relations and the Rodrigues' formula, and is also useful in proving the orthogonality of the family. To emphasis this point the Hermite family will be defined through the use of its generating function. However it is not the generating function alone that shares this property, since the Rodrigues' formula can also be used to define the recursion and differential relations, and prove the orthogonality.

1.3.1 The Hermite Polynomials

Hermite polynomials are orthogonal polynomials associated with the interval $(-\infty, \infty)$ and the exponential weight function $w(x) = e^{-x^2}$. The Hermite polynomials, denoted as $H_n(x)$, can be represented as

$$H_n(x) = (2x)^n \, {}_2F_0 \left(\begin{matrix} -n/2, -(n-1)/2 \\ - \end{matrix} ; -\frac{1}{x^2} \right) \tag{1.3.1}$$

and satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = 2^n n! \sqrt{\pi}\, \delta_{n,m} \tag{1.3.2}$$

The Hermite polynomials have the generating function:

$$S(x,t) = e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x), \tag{1.3.3}$$

which are functions of x as well as an additional "dummy" variable t. One method of deriving (1.3.2) is to integrating the left and right sides of (1.3.3). The generating function (1.3.3) also allows the calculation of the recurrence relation

$$H_{n+1} = 2xH_n - 2nH_{n-1} \qquad n = 0, 1, 2, \ldots \tag{1.3.4}$$

and the differential relation

$$H_n' = 2nH_{n-1} \qquad n = 1, 2, \ldots, \tag{1.3.5}$$

where $H_0' = 0$. The recurrence relation (1.3.4) and the differential relation (1.3.5) can be combined to give a second-order differential equation, the *Hermite equation*:

$$H_n'' - 2xH_n' + 2nH_n. \tag{1.3.6}$$

Also n differentiations of the generating function, leads to Rodrigues' formula,

$$H_n(x) = e^{x^2} \left(-\frac{\partial}{\partial x} \right)^n e^{-x^2} \tag{1.3.7}$$

which can also be written in the form of the recurrence relation:

$$H_{n+1}(x) = e^{x^2}\left(-\frac{\partial}{\partial x}\right)e^{-x^2}H_n(x).$$

Hermite polynomials can also be expressed by a truncating hypergeometric series,

$$H_n(x) = (2x)^n\ {}_2F_0\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix}; -\frac{1}{x^2}\right) \tag{1.3.8}$$

where the truncation occurs, because of the negative n in the expression.

1.3.2 The (associated) Laguerre Polynomials

The (associated) Laguerre polynomials are orthogonal with respect to the weight function $w(x) = x^\alpha e^{-x}$, on the interval $(0, \infty)$. They have the explicit representation

$$L_n^\alpha(x) = \frac{(\alpha+1)_n)}{n!}\ {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right) \tag{1.3.9}$$

and satisfy the orthogonality relation

$$\int_0^\infty L_n^\alpha(x)L_m^\alpha(x)x^\alpha e^{-x}dx = \frac{\Gamma(\alpha+n+1)}{n!}\,\delta_{n,m}, \quad \alpha > -1 \tag{1.3.10}$$

(which is expressed in terms of the Gamma function). The (associated) Laguerre Polynomials also have the generating function:

$$\sum_{n=0}^{\infty} L_n^\alpha(x)t^n = (1-x)^{-\alpha-1}e^{-xt/(1-t)} \tag{1.3.11}$$

which is a function of x and the "dummy" variable t. The three term recurrence relation can be expressed as

$$(n+1)L_{n+1}^\alpha(x) - (x-\alpha-2n-1)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0, \tag{1.3.12}$$

where for $n = 0,\ L_{-1} = 0$ and a differential relation

$$x\frac{dL_n^\alpha(x)}{dx} = nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x). \tag{1.3.13}$$

Like the Hermite equation, we derive the (associated) Laguerre equation from (1.3.12) and (1.3.13)

$$x\frac{d^2L_n^\alpha(x)}{dx^2} + (\alpha+1-x)\frac{dL_n^\alpha(x)}{dx} + nL_n^\alpha(x) = 0\,, \quad \text{for } n \geq 0. \tag{1.3.14}$$

(associated) Laguerre polynomials can be expressed by the truncating hypergeometric series

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array}; x\right), \tag{1.3.15}$$

where the truncation occurs, because of the negative n in the expression.

1.3.3 The Jacobi Polynomials

Jacobi polynomials, also known as hypergeometric polynomials, occur in the study of rotation groups and in the solution to the equations of motion of the symmetric top. They are associated with the interval $(-1,1)$ and the weight function

$$w(x) = (1-x)^\alpha(1+x)^\beta.$$

For certain values of α and β, the Jacobi polynomials reduce to other orthogonal polynomials including Legendre (for $\alpha = \beta = 0$) and Gegenbauer (for $\alpha = \beta = \lambda - \frac{1}{2}$).

The hypergeometric representation of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, is

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array}; \frac{1-x}{2}\right) \tag{1.3.16}$$

and satisfy the orthogonality relation

$$\begin{aligned}&\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta dx \\ &= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)n!}\delta_{nm}.\end{aligned} \tag{1.3.17}$$

The Jacobi polynomials also have a generating function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) z^n = 2^{\alpha+\beta} R^{-1} (1 - z + R)^{-\alpha} (1 + z + R)^{-\beta} \tag{1.3.18}$$
$$\text{where} \quad R = (1 - 2xz + z^2)^{\frac{1}{2}}$$

which is a function of x as well as of an additional "dummy" variable z. These relations make use of both the hypergeometric function and the Gamma function.

Using the generating function (1.3.18), allows us to calculate the the recurrence relation

$$\begin{aligned}
&2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x) \\
&= (2n+\alpha+\beta+1)[(2n+\alpha+\beta)(2n+\alpha+\beta+2)x + \alpha^2 - \beta^2]P_n^{(\alpha,\beta)}(x) \\
&-2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(x)
\end{aligned} \tag{1.3.19}$$

and a differential relation

$$\begin{aligned}
&(2n+\alpha+\beta)(1-x^2)\frac{d}{dx}P_n^{(\alpha,\beta)}(x) = n[(\alpha-\beta) - (2n+\alpha+\beta)x]P_n^{(\alpha,\beta)}(x) \\
&2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(x)
\end{aligned} \tag{1.3.20}$$

with $P_{-1} = 0$ and $P_0' = 0$. The recurrence relation (1.3.19) and the differential relation (1.3.5) can be combined to give a second-order differential equation:

$$(1-x^2)\frac{d^2}{dx^2}P_n^{(\alpha,\beta)}(x) + [\beta - \alpha - (\alpha+\beta+2)x]\frac{d}{dx}P_n^{(\alpha,\beta)}(x) + n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) = 0. \tag{1.3.21}$$

and n differentiations of the generating function, leads to Rodrigues' formula.

$$2^n n! P_n^{(\alpha,\beta)}(x) = (-1)^n (1-x)^{-\alpha}(1+x)^{-\beta}\left(-\frac{\partial}{\partial x}\right)^n [(1-x)^{\alpha+n}(1+x)^{\beta+n}]. \tag{1.3.22}$$

All these results can be proved by making use of the hypergeometric function and an explicit derivation can be found in [84].

1.4 Further Classes of Orthogonal Polynomials

Having described the standard theory and given examples of the classical orthogonal polynomials, we now give a brief description of other classes of orthogonal polynomials, which are the subject of current research.

1. The *discrete orthogonal polynomials*, include the Hahn and Meixner [135] polynomials and are orthogonal with respect to a discrete measure. Both these classes of polynomials are expressed in a hypergeometric form and have the usual set of relations, common to the very classical orthogonal polynomials except their orthogonality relation has the discrete form (1.1.1c) and they satisfy a second order *difference* equation in the variable x.

2. *Multi-variable orthogonal polynomials*, usually consist either of extending the univariate case to the multivariate case or of a class orthogonal polynomials defined in terms of multiple variables [57]. For examples we can consider the multiple Hermite polynomials as an extension from the univariate case and the Jack and MacDonald [104] polynomials as orthogonal polynomials in n variables.

   Recently greater interest has been paid to specific numbers of variables such as the book by Suetin [157], who takes a detailed look into orthogonal polynomials in two variables. Importantly we stress the difference between this study and the study of bi-orthogonality. The study of two-variable orthogonal polynomials will be looked at in greater detail in Chapter 4.

3. *Bi-orthogonal Polynomials* consist of two families of orthogonal polynomials $\{P_n\}$ and $\{Q_n\}$ related by a weight function and defined through a biorthogonal relation such as

$$\int P_n(x)Q_m(x)w(x)dx = h_n\delta_{n,m}. \tag{1.4.1}$$

The study of this research finds its origins with Hermite and Appell, who looked into it while also considering two variable orthogonal polynomials.

4. *Multiple orthogonal polynomials* are polynomials of one variable which satisfy orthogonality conditions with respect to p different measures $\mu_1, \mu_2, \ldots, \mu_p$. Multiple orthogonal polynomials [12] are intimately related to Hermite-Padé approximants [136] and often they are also called Hermite-Padé polynomials.

   Typically there are two types of multiple orthogonal polynomial. For **type I** let $w_1, w_2, \ldots, w_p$ be p weights on the real line and let $\vec{n} = n_1, \ldots, n_p$ be a multi-index consisting of non-negative integers. If $A_1, \ldots, A_p$ are polynomials and

$$Q(x) = \sum_{j=1}^{p} A_j(x) w_j(x), \quad \deg A_j \leq n_j - 1, \tag{1.4.2a}$$

   such that

$$\int Q(x) x^j dx = 0 \quad \text{for } j = 1, \ldots, |\vec{n}| - 2 \tag{1.4.2b}$$

   (where $|\vec{n}| = \sum_{j=1}^{p} n_j$), then the A_j are called multiple orthogonal polynomials of type I and Q is the linear form built out of the multiple orthogonal polynomials of type I.

   For **type II**, let $w_1, w_2, \ldots, w_q$ be q weights on the real line and let $\vec{m} = m_1, \ldots, m_q$ be a multi-index of length q. If P is a polynomial of degree $|\vec{m}|$ such that

$$\int P(x) x^j w_j(x) dx = 0 \quad \text{for } j = 0, \ldots, m_k - 1 \quad \text{and } k = 1, \ldots, q, \tag{1.4.2c}$$

   then P is called a multiple orthogonal polynomial of type II. Although it usually type I or type II that are studied, there has recently been a generalization of the two [45]. There are also the classical multiple orthogonal polynomials (including multiple Hermite and multiple Jacobi), which have been studied extensively by van Assche [166].

5. The *q-orthogonal polynomials*

With the advent of quantum groups, q-orthogonal polynomials are objects of special interest in both mathematics and physics. For instance, the q-deformed harmonic oscillator provides a group-theoretic setting for the q-Hermite and the q-Laguerre polynomials. The q-orthogonal polynomials involve the use of an additional parameter q in their original formulas, so the easiest way to describe them is by considering the hypergeometric series and its q-analogue [71]. Since all known orthogonal polynomials (in a single variable) can be expressed in terms of a hypergeometric series, this seems the best approach. The modern definition of the q-hypergeometric function is

$$ {}_r\phi_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \ldots (a_r;q)_n}{(b_1;q)_n \ldots (b_s;q)_n} \frac{x^n}{(q;q)_n} [(-1)^n q^{\frac{1}{2}n(n-1)}]^{1+s-r} \tag{1.4.3} $$

where $(a;q)_n$ is the q-Pochhammer symbol defined by

$$ (a;q)_n = (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}), \tag{1.4.4a} $$

$$ (a;q)_0 = 1. \tag{1.4.4b} $$

We also state the special case $r = s+1$, since this is the form most commonly used for orthogonal polynomials.

$$ {}_{s+1}\phi_s \left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| q; x \right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \ldots (a_r;q)_n}{(b_1;q)_n \ldots (b_s;q)_n} \frac{x^n}{(q;q)_n} \tag{1.4.5} $$

We can then consider the q-orthogonal polynomials in terms of the Askey-Wilson classification. This classification (of which a comprehensive report is found in [92]), provides a list of all the hypergeometric polynomials and their q-analogues (both continuous and discrete). The scheme is referred to as the Askey-Wilson scheme, since these are the polynomials that rank at the top of the q-hypergeometric list and are defined as

$$ \frac{a^n W_n(x;a,b,c,d|q)}{(ab,ac,ad;q)_n} =_4 \phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q; q \right), \; x = \cos\theta. \tag{1.4.6} $$

1.5 Applications of Orthogonal Polynomials

The connection of orthogonal polynomials with other branches of mathematics is truly impressive. Without even trying to be complete, we mention continued fractions, operator theory (Jacobi operators), moment problems, analytic functions (Bieberbachs conjecture), interpolation, Pade approximation, quadrature, approximation theory, numerical analysis, electrostatics, statistical quantum mechanics, special functions, number theory (irrationality and transcendence), graph theory (matching numbers), combinatorics, random matrices, stochastic processes (birth and death processes; prediction theory), data sorting and compression, Radon transform and computer tomography.

The main areas from the point of view of physics, are quantum mechanics and matrix models, to which we will now give a brief introduction to.

1.5.1 Quantum Mechanics

In quantum mechanics [147] one studies the Hamiltonian of a system, which is the operator that defines the model under certain assumptions (canonical commutation relations, choice of Hilbert space, etc.) and the operator represents the observable of energy. The stationary Schrödinger equation is an eigenvalue problem for the Hamiltonian; the corresponding eigenvalues are the allowed energy values. The eigenfunctions which are the solutions of the eigenvalue problem correspond to the energy states of the system (each eigenvalue has at least one state of the system) and these are often solved in terms of special functions. Many classes of orthogonal polynomials arise in this context.

As an example, we look at the Hamiltonian for the one-dimensional quantum harmonic oscillator. This a model of key importance, since it represents the first basic step in studying the quantum mechanics of systems of vibrating particles. It is defined by the Hamiltonian

$$\widehat{H} = \frac{1}{2m}\widehat{p}^2 + \frac{1}{2}m\omega^2\widehat{x}^2 \tag{1.5.1}$$

which in turn prescribes the eigenvalue problem

$$\widehat{H}\Phi = E\Phi \tag{1.5.2}$$

(a stationary Schrödinger equation with eigenvalue E), which has the explicit form

$$-\frac{\hbar^2}{2m}\frac{d^2\Phi}{dx^2} + \frac{1}{2}m\omega^2x^2\Phi = E\Phi. \tag{1.5.3}$$

This differential equation can be simplified by making the change of variables $x = \xi\sqrt{\frac{\hbar}{m\omega}}$ and using the substitution

$$\Phi(\xi) = e^{-\frac{1}{2}\xi^2}H(\xi) \tag{1.5.4}$$

(where $H(\xi)$ should not be confused with the Hamiltonian), leading to the differential equation (1.5.3)

$$H'' - 2\xi H' + \left(\frac{2E}{\hbar\omega} - 1\right)H = 0 \tag{1.5.5}$$

the *Hermite equation* (1.3.6). This second order differential equation, for special values $E = \hbar\omega(n + \frac{1}{2})$, $n \in \mathbb{N}$ has polynomial solutions which are the Hermite polynomials. These are the physically relevant solutions leading to the spectrum and eigenstates of the quantum model. Although this is the simplest model to describe, the structure of the solutions as in (1.5.4), a ground state times a polynomial, which is common to many systems in quantum mechanics.

Other families of orthogonal polynomials occur from solving other models in quantum mechanics. A more complex model is the hydrogen atom, which is solved by separating the problem into 3 parts, as in polar coordinates there is one radial variable and two angular variables; by separation of variables the former (radial part) is solved with Laguerre polynomials and the latter (spherical harmonics) are solved with Legendre polynomials and exponentials.

1.5.2 Random Matrix Models

Random matrix models [115] arise from, and have important applications to, number theory, probability, combinatorics, representation theory, quantum mechanics, solid state physics, quantum field theory, quantum gravity, and many other areas of physics and mathematics, but here we are interested in their connection with orthogonal polynomials. The theory first began in the 1960's with Dyson [51, 52, 53] and then with Metha [54, 55] and is described in detail in their series of papers on "Statistical Theory of the Energy Levels of Complex Systems". Then at the end of the 1980s, interest was renewed in matrix models after the connection with quantum gravity and string theory was discovered [24, 74]. One approach to this theory was presented by Bilal [19], from which the connection with orthogonal polynomials is now explained. This class of orthogonal polynomials is different from what we have already seen and is referred to as semi-classical orthogonal polynomials, an area which will be covered in more detail in the next chapter.

The main tool of use in the theory is the partition function, (a notion taken from statistical mechanics) and is defined as

$$Z = \int [dA] e^{-\beta(\text{tr } V(A))}, \tag{1.5.6}$$

where the integration can be defined over all $N \times N$ antisymmetric matrices A and

$$\text{tr } V(A) = \sum_{j=1}^{\frac{N}{2}} V(x_j) \quad , \quad V(x) = \frac{1}{2}x^2 + \sum_{n=2}^{m} g_{2n} x^{2n} \tag{1.5.7}$$

where the x_j are the eigenvalues. The eigenvalues of an antisymmetric matrix always come in pairs $\pm\lambda_i$ and the trace of such a matrix will always be zero. Thus we consider the even elements and the trace sums to $\frac{N}{2}$.

From this it is clear that $e^{-\beta(\text{tr } V(A))}$ depends only on the eigenvalues and it can be shown that the measure dA factorizes into an integration over the eigenvalues and an integration over the parameters of the diagonalizing matrix U. This is shown by example, by taking lower order values of N, say at $N = 2$ and $N = 4$,

$$N = 2 \;\; \Rightarrow \;\; [dA] = da = dx_1 \tag{1.5.8a}$$

$$N = 4 \;\; \Rightarrow \;\; [dA]_{N=4} = (x_2^2 - x_1^2)^2 dx_1 dx_2 d\Omega_1 d\Omega_2 \tag{1.5.8b}$$

which allows us to extrapolate an alternate form for Z. Thus the partition function Z can be expressed as

$$Z = \int \left(\prod_{i=1}^{\frac{N}{2}} dx_i e^{-\beta(\mathrm{tr}\ V(x_i))} \right) \prod_{i>j=1}^{\frac{N}{2}} (x_i^2 - x_j^2)^2. \tag{1.5.9}$$

Initially the use of orthogonal polynomials as a solution is not obvious, until the van der Monde determinant is introduced, which can be expressed as

$$\Delta(x_i^2) = \prod_{i>j=1}^{\frac{N}{2}} (x_i^2 - x_j^2)^2 = \begin{vmatrix} 1 & \cdots & 1 \\ x_1^2 & \cdots & x_{\frac{N}{2}}^2 \\ x_1^4 & \cdots & x_{\frac{N}{2}}^4 \\ \cdots & & \cdots \\ x_1^{N-2} & \cdots & x_{\frac{N}{2}}^{N-2} \end{vmatrix}. \tag{1.5.10}$$

The van der Monde determinant can then be manipulated so that its contents can be expressed as polynomials. First though, it is necessary to introduce the polynomials P_n, which are orthogonal with respect to the weight

$$d\mu = dx e^{-\beta V(x)} \tag{1.5.11}$$

and as before the P_n have the following standard form and orthogonality condition.

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0, \qquad \int P_n(x) P_m(x) d\mu(x) = h_n \delta_{nm} \tag{1.5.12}$$

Since $V(x)$ is defined as an even function, the P_n is an even polynomial in x, if n is even and an odd polynomial in x, if n is odd. So we redefine P_n

$$P_n(x) = x^n + \sum_{k=1}^{[\frac{n}{2}]} c_{n-2k}^{(n)} x^{n-2k}, \tag{1.5.13}$$

where $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$.

The van der Monde determinant is now manipulated, with the aim of re-expressing it in terms of polynomials instead of powers of x. By making use of (1.5.13)

$$\begin{aligned}
n = 0 &\quad \Rightarrow 1 = P_0 \\
n = 2 &\quad \Rightarrow x_1^2 + c_0^{(2)} = P_2(x_1) \\
n = 4 &\quad \Rightarrow x_1^4 + c_0^{(4)} x_1^2 + c_0^{(2)} = P_4(x_1) \\
&\vdots
\end{aligned}$$

it is possible to rewrite (1.5.10) as

$$\Delta(x_i^2) = \det[p_{2(j-1)}(x_i)]_{i,j=1,\dots,\frac{N}{2}} = \begin{vmatrix} P_0 & \cdots & P_0 \\ P_2(x_1) & \cdots & P_2(x_{\frac{N}{2}}) \\ P_4(x_1) & \cdots & P_4(x_{\frac{N}{2}}) \\ \cdots & & \cdots \\ P_{N-2}(x_1) & \cdots & P_{N-2}(x_{\frac{N}{2}}) \end{vmatrix}. \tag{1.5.14}$$

Referring back to the partition function for an antisymmetric matrix (1.5.9), we have

$$\begin{aligned}
Z &= \int \prod_{i=1}^{\frac{N}{2}} d\mu(x_i) (\det[P_{2(j-1)}(x_i)]_{i,j=1,\dots,\frac{N}{2}})^2 \\
&= \left(\frac{N}{2}\right)! \prod_{i=1}^{\frac{N}{2}} h_{2(j-1)}
\end{aligned} \tag{1.5.15}$$

a result of the orthogonality (1.5.12) of $P_{2(j-1)}$. If we had considered the Hermitian case the partition function Z would have had the form $Z = (N)! \prod_{i=1}^{N} h_{j-1}$. In order to compute Z we need to compute h (in Chapter 2 this is a problem that arises in Painlevé equations), and it is this understanding which concerns physicists in the study of matrix models.

1.6 Organization of the Thesis

The opening chapter has introduced and derived some of the standard ingredients found in formal orthogonal polynomials: a recurrence relation, determinant expressions and the Christoffel-Darboux formula. The focus has been on these relations since they will have further involvement later on in the thesis (Chapter 4), particularly the determinant expression and the recurrence relation, which see extensive use. These general relations have been backed up with some of the main relations from the important very classical orthogonal polynomials. These will provide a good comparison with the relations derived in Chapter 2, which is concerned with their semi-classical counterparts.

The **second chapter** is concerned with semi-classical orthogonal polynomials, with particular relevance to any connections with discrete integrable systems. So the chapter begins with some history and examples of what is meant by an integrable system, before providing an illustration of a semi-classical orthogonal polynomial that has some well established connections with discrete integrable systems. Thus we use the simple case of semi-classical Hermite polynomials, which also demonstrates one method of calculating the compatibility between a differential equation and a recurrence relation. The resulting equations lead to the derivation of a discrete Painlevé equation, a discrete $\mathrm{P_I}$. The methods for deriving other discrete integrable systems are also introduced. Then we introduce our own approach to the Laguerre method, which derives a differential system for semi-classical orthogonal polynomials. This system is compatible with a recurrence relation (expressed in a matrix form) and leads to a a pair of compatibility relations. On completion of the derivation of this method, we can consider its application to specific deformed weights of semi-classical orthogonal polynomials. So we begin with the deformed weights of Hermite and Laguerre polynomials, before considering the more complex case of the deformed Jacobi weight. As a comparison for the benefits of this method for the Jacobi weight (both in deriving a differential equation and compatibility relations), the consistency relations are also derived by only using the orthogonality relation.

While the second chapter is concerned with the integrable systems structures coming from orthogonal polynomials, the **third chapter** is concerned with orthogonal polynomials coming from integrable systems. Thus our interest moves into the domain of singular integral transforms, specifically one analogous to the Gel'fand-Levitan integral equation. The chapter is essentially separated into two sections where the first is concerned with the presentation of a general singular integral transform and the second is concerned with applying the said singular integral transform to some examples. Thus, we begin by approaching the problem from a general point of view, considering the dressing method between two $n \times n$ matrices Φ_k^0 and Φ_k^1, present some key notation and introducing an associated Lax-type linear equation. Through this association we derive a discrete Lax equation, that is written in terms of an integral expression, which is explored further by applying a differential and difference operator to it. This leads to the application of the singular integral transform, where our first example uses the matrix representation of the recurrence relation (derived in Chapter 2) to derive singular integral transforms between the recurrence coefficients of a general class of classical orthogonal polynomials and the recurrence coefficients of semi-classical orthogonal polynomials. We also consider the differential system derived in Chapter 2. As the second application we consider a singular integral transform for the lattice Gel'fand-Dikii $N \times N$ matrix hierarchy, which for the case of $N = 2$ reduces to the Lax representation of the KdV equation. We present a vector reduction of the general $N \times N$ case and show how the singular integral transform satisfies the discrete Lax equation (derived in the first section). We then use the vector reduction for the KdV case ($N = 2$) and derive a singular integral transform for the KdV equation. Given the existence of a gauge transformation that relates KdV to a Volterra linear problem (which incidentally satisfies the recurrence relation for orthogonal polynomials), we present a singular integral transform for a class of orthogonal polynomials related to the KdV transform and present results for a specific example.

The **fourth chapter** considers formal orthogonal polynomials, where we define a new class of orthogonal polynomials: two-variable orthogonal polynomials whose variables are related through the equation of an elliptic curve in Weierstrass form. Our interest lies in the formal structure and most of the chapter is concerned with recursive structures. Since we are dealing with two-variable orthogonal polynomials, there exists a recurrence relation for the x variable and the y variable. We can state these recurrence relations by using inner products of $\langle xP_k, P_l\rangle$ and $\langle yP_k, P_l\rangle$ (such as with (1.1.6)), where P_k is an orthogonal polynomial in two variables and x and y are of different order. We can then present compatibility relations between these recurrence relations and give consistency conditions between the recurrence coefficients of the x-recurrence relation and the y-recurrence relation. Further, by using an analogue of the determinant form from Chapter 1 (1.1.14) and applying the generalized Sylvester identity (B.4) to it, we can derive an explicit x-recurrence relation, where the coefficients consist of Hankel determinants c.f. (1.1.26b, 1.1.26a). Applications of recurrence relations, such as the generation of a sequence of polynomials and the derivation of Christoffel-Darboux relations are also presented and we end the chapter with some speculations concerning the non-formal case (where an explicit weight function is given).

Chapter 2

Semi-Classical Orthogonal Polynomials

As we have seen in Section 1.3, classical orthogonal polynomials are governed by a set of conditions, which lead to a number of explicitly defined equations and relations; they have a fixed weight function, generating function, recurrence relation, differential equation, etc. On the other hand, semi-classical orthogonal polynomials occur when the conditions are less restrictive ie. when some of the properties are relaxed. This results in less equations than found in the classical case and these equations cannot be explicitly derived, instead their coefficients contain transcendental functions.

In 1929, Bochner [22] gave a characterization of the classical orthogonal polynomials Hermite, Laguerre or Jacobi type. If $\{P_n\}$ is a sequence of classical orthogonal polynomials, then $P_n(x)$ is a solution of the second-order differential equation

$$\phi(z)\frac{d^2y}{dz^2} + \psi(z)\frac{dy}{dz} = \lambda_n y \tag{2.0.1}$$

where $\phi(z)$ and $\psi(z)$ are fixed polynomials of degree≤ 2 and ≤ 1 respectively, and λ_n is a real number depending on the degree of the polynomial solution. As a consequence of this the weights of classical orthogonal polynomials satisfy a first order differential equation called the Pearson differential equation

$$\frac{d}{dz}(\phi(z)w(z)) = \psi(z)w(z)\ , \tag{2.0.2}$$

when the degrees of ϕ and ψ satisfy deg $\phi \leq 2$ and deg $\psi = 1$. However when the deg $\phi > 2$ and\or deg $\psi > 1$ then the weight function produces a class of semi-classical orthogonal polynomials. Thus by extension, this equation also implies that the weight functions of the semi-classical orthogonal polynomials are different from their classical counterparts. We write the Pearson equation in the following form [79, 80]

$$\frac{1}{w(z)}\frac{dw(z)}{dz} = \frac{\psi - \phi'}{\phi} = \frac{V(z)}{W(z)}, \tag{2.0.3}$$

where equation (2.0.3) expresses the logarithmic derivative of the weight function $w(z)$ as a ratio of the polynomials $V(z)$ and $W(z)$. In this case the weight function satisfies classical orthogonal polynomials if deg $V \leq 1$ and deg $W \leq 2$ and semi-classical arise for deg $W > 2$ and\or deg $V > 1$. For instance the weight functions for Hermite polynomials, Laguerre polynomials, and Jacobi polynomials are the classical weights for W of degree zero, one and two, respectively. As an example we look at the weight function of Hermite $w(z) = e^{-z^2}$, then the Pearson differential equation is

$$e^{z^2}(-2z)e^{-z^2} = -2z \tag{2.0.4a}$$

a polynomial of order 1 (V has order 1 and W has order 0). Therefore this weight satisfies classical orthogonal polynomials. Alternatively, if we consider the altered weight function $w(z) = e^{-z^2-z^4}$, then from the Pearson equation we have

$$e^{z^2+z^4}(-2z-4z^3)e^{-z^2-z^4} = -2z-4z^3 \tag{2.0.4b}$$

a polynomial of order 3, therefore this is a semi-classical orthogonal polynomial. Although this weight function is not the Hermite weight function it is very similar in form therefore we call orthogonal polynomials satisfying it, semi-classical Hermite orthogonal polynomials. In particular when considering general exponential weights, we are reminded of Freud weights [65], although they have the form $w_\rho(x) = |x|^\rho \exp(-|x|^m)$.

The very classical orthogonal polynomials and all the other classes that occur in the Askey-Wilson scheme, have been the subject of great interest and have therefore been explored in detail. By comparison, semi-classical orthogonal polynomials have had less attention and are therefore less developed. One of the key consequences of moving from classical to semi-classical is that you lose the explicit nature of the relations, but you do gain more connections. This allows us to take the study of orthogonal polynomials in new directions.

The beginnings of semi-classical orthogonal polynomials are unclear, but the derivation of a differential relation for a general class of orthogonal polynomials by Shohat in 1939 [42, 148], provides a starting point for classes of semi-classical orthogonal polynomials to be formed. Since that time, semi-classical orthogonal polynomials have found use with matrix models (1.5.11) and recently there has been the emergence of integrable systems. This is a very broad area of mathematics, that has found connections in many areas of pure and applied mathematics and also in mathematical physics, biology and engineering.

In this chapter we will begin with an introduction to some discrete integrable systems as well as the origin of the Painlevé equations. This introduction will be complemented by the semi-classical Hermite orthogonal polynomials, which will demonstrate the types of rich connections found between semi-classical orthogonal polynomials and discrete integrable systems, by using the orthogonality relation. Following this is a section on the Laguerre method [100], which will provide an alternate way (compared to using the orthogonality relation) of approaching semi-classical orthogonal polynomials, by introducing a general theory that results in a matricial differential system. Then introducing a Lax pair between the differential system and a recurrence relation leads to a Lax equation, from which compatibility relations are derived. This will lead to our work into applying the Laguerre method to specific semi-classical weights, focusing on the discrete relations found from the compatibility relations and their connections to discrete integrable systems. While semi-classical weights from Hermite and Laguerre will be considered, our main effort will be with the semi-classical Jacobi weight and compatibility relations will be derived from both the Laguerre method and the orthogonality relation approach.

2.1 Painlevé equations and Integrable Systems

The modern theory of integrable systems finds its origins in 1965, after a study by Gardner, Greene, Kruskal and Miura [69, 70] into the Korteweg-de Vries equation [95]

$$\partial_t u = \partial_x^3 u + 6u\partial_x u, \tag{2.1.1}$$

which is a mathematical model of waves on shallow water surfaces. Their study showed that the KdV can be exactly solved by (what is now called) the inverse scattering method. This method is only applicable to a certain class of equations, that we now refer to as soliton equations or exactly integrable equations.

Since then, there has been increased interest in the study of the KdV equation and other systems deemed to be integrable, but providing a proper definition of what is meant by integrability has not proven to be straightforward. Integrable systems can be mappings, ordinary differential equations (ODEs), partial differential equations (PDEs), ordinary difference equations (OΔEs), partial difference equations (PΔEs) and differential-difference equations (DΔE's); hence they cover most types of equations. This makes it a little difficult to give a general definition of an integrable system. What we can say is that there are a number of characteristics which describe an integrable system, although the contributors to this subject have different perspectives on how they should be defined. The characteristics include the existence of a rich solution structure; admitting exact solutions and solution methods like the inverse scattering method [3, 4]; there exist hierarchies of compatible equations; and there are associated linear systems including Lax pairs [102]. Lax pairs (which can exist for both discrete and continuous systems), involve expressing an equation in terms of matrices, that satisfy a compatibility condition. We can consider three types of Lax equation covering the differential, differential-difference and difference types equations. For a differential system a Lax pair would be of the form

$$L\phi = \lambda\phi \tag{2.1.2}$$

$$\frac{d\phi}{dt} = A\phi \tag{2.1.3}$$

where L and A are differential operators and satisfy the compatibility condition

$$\frac{dL}{dt} = [A, L] = AL - LA. \tag{2.1.4}$$

In a difference system we can consider the system of two 2×2 matrix equations

$$\phi_{n+1,m} = L_{nm}\phi_{nm}\ ,\ \phi_{n,m+1} = M_{nm}\phi_{nm} \tag{2.1.5a}$$

which is satisfied by the consistency relation

$$L_{n,m+1}M_{n,m} = M_{n+1,m}L_{n,m} \tag{2.1.5b}$$

and for a differential-difference system we consider differential and recurrence structures

$$\begin{aligned} \partial_x\psi_n &= M_n\psi_n, \\ \psi_{n+1} &= L_n\psi_n, \end{aligned} \tag{2.1.6}$$

whose compatibility leads to the semi-discrete Lax equation

$$\partial_x L_n = M_{n+1}L_n - L_nM_n. \tag{2.1.7}$$

2.1.1 Discrete Integrable Systems

The study of integrable systems is usually separated into the continuous and the discrete equations, of which we are interested in the latter. Discrete integrable systems usually consist of two types of equations; PΔEs, which are viewed on the two-dimensional space-time lattice and DΔEs, which are discrete in space but continuous in time.

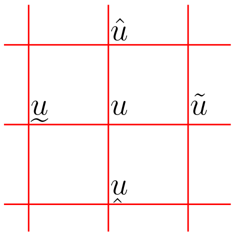
We can consider a class of PΔEs on the lattice (which are called quadrilateral PΔEs), to have the form

$$f(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}, p, q) = 0 \tag{2.1.8a}$$

where we adopt the canonical notation of vertices surrounding an elementary plaquette on a regular lattice:

$$\begin{aligned} u = u_{nm} \quad &, \quad \tilde{u} = u_{n+1,m} \\ \hat{u} = u_{n,m+1} \quad &, \quad \hat{\tilde{u}} = u_{n+1,m+1} \end{aligned} \tag{2.1.8b}$$

Here we see that a $\tilde{}$ represents a shift forward (if raised) or backward (if lowered) on the horizontal line and a $\hat{}$ represents a shift up (if raised) or down (if lowered) on the vertical line.



Apart from the independent discrete variables n, m (on which u_{nm} depends), there are the lattice parameters p, q which we associate with these independent variables. If n, m denote the direction which u moves on the lattice, then p, q denote the width of the grid.

By letting the parameters p, q be variable rather than fixed we can define a whole parameter-family of equations, however we must attach each parameter to a specific discrete variable such as p associated with n and q with m. This allows us to place the PΔE in a multi-dimensional lattice with multiple independent variables, though for practical reasons it is easiest to deal with the three-dimensional lattice. Then we have a shift $\bar{}$ in another direction, which leads to eight points on a cube. The interesting property is that there is consistency around the cube ie. the value of $\hat{\tilde{u}}$ can be achieved in three separate ways

$$u \to \bar{u} \to \tilde{\bar{u}} \to \hat{\tilde{\bar{u}}}, \tag{2.1.9}$$

$$u \to \tilde{u} \to \hat{\tilde{u}} \to \bar{\hat{\tilde{u}}}, \tag{2.1.10}$$

$$u \to \hat{u} \to \bar{\hat{u}} \to \tilde{\bar{\hat{u}}}. \tag{2.1.11}$$

This property is the main hallmark of the integrability of a quadrilateral P∆E [21, 132]. Another consequence of this condition being satisfied is it gives rise to the existence of a Lax pair (2.1.5a), [133]. As an example consider the lattice potential KdV

$$(p - q + \hat{u} - \tilde{u})(p + q + u - \hat{\tilde{u}}) = p^2 - q^2 \tag{2.1.12}$$

which satisfies the consistency around the cube property and has the following matricial Lax pair:

$$(p-k)\phi_{n+1,m} = L_{n,m}\phi_{n,m} \quad , \quad (q-k)\phi_{n,m+1} = M_{n,m}\phi_{n,m}, \tag{2.1.13}$$

where the matrices L and M are given by

$$L_{n,m} = \begin{pmatrix} p - u_{n+1,m} & 1 \\ k^2 - p^2 + (p - u_{n+1,m})(p + u_{n,m}) & p + u_{n,m} \end{pmatrix}, \tag{2.1.14a}$$

$$M_{n,m} = \begin{pmatrix} q - u_{n,m+1} & 1 \\ k^2 - q^2 + (q - u_{n,m+1})(q + u_{n,m}) & q + u_{n,m} \end{pmatrix}. \tag{2.1.14b}$$

From the lattice potential KdV equation, we can derive the lattice KdV [81]

$$\hat{Q} - \tilde{Q} = \frac{a}{Q} - \frac{a}{\hat{\tilde{Q}}}\ , \tag{2.1.15}$$

for $Q = (p + q + u - \hat{\tilde{u}}),\ a = p^2 - q^2$. However, this equation does not satisfy the consistency around the cube in the strict sense because of the choice of the variable Q, since it has lost its covariance with respect to the interchange of lattice directions. Nevertheless it has a Lax representation of the form

$$\hat{\hat{\psi}} \;=\; Q\hat{\psi} + \lambda\psi, \tag{2.1.16a}$$

$$\tilde{\psi} \;=\; \hat{\psi} + \frac{a}{Q}\psi, \tag{2.1.16b}$$

where λ is the spectral parameter (where this form is first written in [134] as far as we are aware). The discrete-time Toda equation is an example of an integrable P∆E on a five point stencil, that is related to the above KdV systems and is given by

$$(p-q)^2 \underset{\underset{\sim}{\wedge}}{\tau}\,\hat{\tilde{\tau}} - (p+q)^2 \underset{\wedge}{\tilde{\tau}}\,\underset{\sim}{\hat{\tau}} + 4pq\tau^2 = 0. \tag{2.1.17}$$

It provides an example of a bilinear equation of Hirota type [81], of which we saw a parameter less example (1.1.28) in Chapter 1.

The D∆Es are also defined on the lattice as well as being differential with respect to the variable time t. Thus they have aspects common to continuous as well as discrete systems. Examples of D∆Es equations include the Volterra system [90]

$$\partial_t u_n = u_n(u_{n+1} - u_{n-1}), \tag{2.1.18}$$

and the Toda lattice system [160],

$$\partial_t^2 y_n = e^{y_{n+1}-y_n} - e^{y_n - y_{n-1}}. \tag{2.1.19}$$

Like integrable systems that are fully continuous, the Volterra system and the Toda lattice system both possess Lax representations and an infinite number of commuting flows, where the family of these flows form a hierarchy of equations.

One interesting fact about the structure of the KdV equation and other soliton equations, is that they all admit a substitution that will bring the equation into a bilinear form [82] (a fact first discovered by Hirota). Then in the 1980s Jimbo, Kashiwara, Date and Miwa [47, 88] discovered that there is a close connection between the Hirota forms and the underlying algebraic structure (in terms of infinite dimensional Lie algebras) of soliton systems. This is an intriguing connection, which has led to a great deal of study in this area.

2.1.2 Painlevé Equations

Another important class of nonlinear differential equation (that arose to have connections with integrable systems) are the Painlevé equations. Following the work of Picard [83] in classifying first-order ordinary differential equations, Painlevé studied second order ordinary differential equations of the form

$$\frac{d^2y}{dx^2} = F(y', y, x) \tag{2.1.20a}$$

where F is analytic in x and rational in y and y'. Painlevé found 50 types whose only movable singularities are poles, where a movable singularity of an equation is one whose location is dependent on the constants of integration involved in its solution. This characteristic is known as the Painlevé property. Out of this list there were six equations emerging, which were called the Painlevé transcendents (they cannot be integrated in terms of any of the known classical functions [169]). The remaining 44 can be integrated in terms of classical transcendents, quadratures, or are directly related to one of the other six Painlevé transcendents.

Five of the Painlevé list were discovered by Painlevé and his students, but the sixth transcendent was found by R. Fuchs (1905) and contains the other five as limiting cases. Hence the sixth Painlevé transcendent is one of the most important nonlinear differential equations for defining new transcendental functions.

$$\begin{aligned}\frac{d^2y}{dx^2} \quad = \quad & \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx}\\ &+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\frac{\beta x}{y^2}+\frac{\gamma(x-1)}{(y-1)^2}+\frac{\delta x(x-1)}{(y-x)^2}\right)\end{aligned} \tag{2.1.20b}$$

These transcendental equations occur in many areas of mathematics and physics, hence the great interest in their study.

As an example we can consider a Painlevé equation as a de-autonomization of elliptic functions specifically $\mathrm{P_I}$, that is

$$\frac{d^2y}{dx^2} = 6y^2 + x. \tag{2.1.20c}$$

If the x on the right side of the equation is replaced with a constant, then we are left with the second order differential equation for the Weierstrass elliptic function $\wp(x)$ (see (A.7)).

The search for discrete analogues of these transcendents, has been an outstanding problem for many years and only recently has there been any progress made in this direction. Discretizations of the Painlevé equations have been the result of a variety of methods including orthogonal polynomials [60, 105]. A discrete analogue of $\mathrm{P_I}$ will be given later in this chapter (2.1.20c).

Many, if not all integrable systems possess symmetry reductions to one or more of the Painlevé equations. It follows that the Painlevé equations themselves are each one dimensional integrable systems.

2.2 Semi-Classical Hermite Polynomials

The Hermite orthogonal polynomials provide the simplest case to demonstrate the types of connections that can be found between semi-classical orthogonal polynomials and discrete integrable systems. We obtain semi-classical Hermite polynomials through a deformation of the classical Hermite weight function (since most classical orthogonal polynomials are defined through the weight function and integration interval via the orthogonality relation).

The Hermite weight function is dependent only on a single variable x, $w(x) = e^{-x^2}$. However, this can be changed by allowing x^2 to be expressed as $V(x)$. Thus we have $w(x) = e^{-V(x)}$, where

$$V(x) = \frac{a}{2}x^2 + \frac{b}{4}x^4 + \frac{c}{6}x^6 + \dots \tag{2.2.1}$$

where $a, b, c, \dots$ are positive parameters. This is an increasing function, which is now also dependent on the parameters as well as x. Like before, when we provided the example of a semi-classical Hermite weight, the Pearson equation (2.0.2) implies that a weight function of the form $e^{-V(x)}$ will lead to semi-classical orthogonal polynomials. Even weights (such as this), have been studied in great detail, beginning with Laguerre [101] and Freud [107]. Recently these results and their connections to discrete Painlevé equations have been further explored by Magnus [106] and van Assche [167].

2.2.1 Discrete Integrable Systems from Recurrence Coefficients

We use the recurrence relation (1.1.25) defined in chapter 1, except here we can take $S_n = 0$, a consequence of the weight function being even. Thus we redefine the monic recurrence relation

$$xP_n = P_{n+1} + R_n P_{n-1}. \tag{2.2.2}$$

The orthogonality relation is defined as:

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)e^{-V(x)}dx = \langle P_n, P_m\rangle = h_n\delta_{nm} \tag{2.2.3}$$

where $V(x) = \frac{a}{2}x^2 + \frac{b}{4}x^4 + \frac{c}{6}x^6 + \ldots$ and $P_n(x)$ is a monic Hermite polynomial. Since every polynomial can express itself in terms of lower order polynomials, consider $P_n' = \sum_{j=0}^{n-1}\alpha_{nj}P_j$ where $P_n' = \frac{dP_n}{dx}$. Now consider

$$\int_{-\infty}^{\infty} P_n' P_m e^{-V(x)}dx = [P_nP_me^{-V(x)}]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} P_n(P_me^{-V(x)})'dx,$$

where $[P_nP_me^{-V(x)}]\big|_{-\infty}^{\infty} = 0$, where the boundary terms vanish since $V(x)$ only consists of even powers of x and hence will dominate the limit as $x \to \pm\infty$. Now expanding the remainder gives:

$$\int_{-\infty}^{\infty} P_n' P_m e^{-V(x)}dx = -\int_{-\infty}^{\infty} P_nP_m' e^{-V(x)}dx + \int_{-\infty}^{\infty} P_nP_mV'(x)e^{-V(x)}dx,$$

which can in turn be expressed as an inner product:

$$\langle P_n', P_m\rangle = -\langle P_n, P_m'\rangle + \langle P_n, (V'P_m)\rangle, \tag{2.2.4}$$

so for $m \leq n-1$

$$\langle P_n', P_m \rangle = 0 + \langle P_n, V' P_m \rangle = \langle (V' P_n), P_m \rangle \text{ by symmetry.}$$

Of course this relation in its current form is of little use as a differential equation, so we introduce the recurrence relation $x\boldsymbol{P}_n = (\boldsymbol{L}\boldsymbol{P})_n$ [106], where L is the semi-infinite matrix

$$\boldsymbol{L} = \begin{pmatrix} 0 & 1 & & & \\ R_1 & 0 & 1 & & \\ & R_2 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.2.5}$$

and $\boldsymbol{P}$ is the (semi-infinite) column vector of $P_0, P_1, P_2, \ldots, P_n, \ldots$. Thus we can re-express the recurrence relation as

$$x\boldsymbol{P}_n = \boldsymbol{P}_{n+1} + R_n \boldsymbol{P}_{n-1} = (\boldsymbol{L}\boldsymbol{P})_n \tag{2.2.6}$$

where we can think of $\boldsymbol{L} = \boldsymbol{\Sigma} + R\boldsymbol{\Sigma}^T$ with the $\boldsymbol{\Sigma}$ shift operators represented by $\boldsymbol{\Sigma}\boldsymbol{P}_n = \boldsymbol{P}_{n+1}$ and $\boldsymbol{\Sigma}^T \boldsymbol{P}_n = \boldsymbol{P}_{n-1}$:

$$(\boldsymbol{\Sigma}\boldsymbol{P})_n = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_n \\ \vdots \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n+1} \\ \vdots \end{pmatrix} \tag{2.2.7a}$$

$$(\boldsymbol{\Sigma}^T\boldsymbol{P})_n = \begin{pmatrix} 0 & 0 & & & \\ 1 & 0 & 0 & & \\ & 1 & 0 & 0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_n \\ \vdots \end{pmatrix} = \begin{pmatrix} P_{-1} \\ P_0 \\ \vdots \\ P_{n-1} \\ \vdots \end{pmatrix} \tag{2.2.7b}$$

(where we set $P_{-1} = 0$).

Now for any general polynomial $F(x)$ one has

$$(F(\boldsymbol{L})\boldsymbol{P})_n = F(x)P_n, \tag{2.2.8}$$

in particular

$$(V'(\boldsymbol{L})\boldsymbol{P})_n = V'P_n. \tag{2.2.9}$$

It follows from (2.2.4) that $P_n' - V'P_n = P_n' - (V'(\boldsymbol{L})\boldsymbol{P})_n$ is orthogonal to $P_0, P_1, \ldots, P_{n-1}$, thus it is a linear combination of P_j with $j \geq n$. This can be expressed as the differential equation

$$P_n' = ((V'(\boldsymbol{L}))_-\boldsymbol{P})_n. \tag{2.2.10}$$

The $(V'(\boldsymbol{L}))_-$ refers to the lower order shifts (in $\boldsymbol{L}$), since a differential of P would only yield lower order terms eg. $(\boldsymbol{L})_- = R\Sigma^t$.

As an example, we choose the weight function

$$w(x) = e^{-(\frac{1}{2}ax^2+\frac{1}{4}bx^4)}, \tag{2.2.11}$$

where we have the value $V(x) = \frac{1}{2}ax^2 + \frac{1}{4}bx^4$ (where a and b are parameters). Inserting this into the differential equation (2.2.10) and setting $x = L$ implies

$$P_n' = ((V'(\boldsymbol{L}))_-\boldsymbol{P})_n \quad , \quad (V'(\boldsymbol{L}))_- = a(\boldsymbol{L})_- + b(\boldsymbol{L}^3)_-, \tag{2.2.12}$$

where $(\boldsymbol{L})_-$ and $(\boldsymbol{L}^3)_-$ *only* involve terms which contain Σ^t. Therefore only polynomials of degree m, where $m \leq n-1$, are considered. In order to determine $\frac{dP}{dx} = ((V'(\boldsymbol{L}))_-\boldsymbol{P})_n(x)$, the terms $(\boldsymbol{L})_-$ and $(\boldsymbol{L}^3)_-$ must first be established and for notational purposes $\underline{R}$ will represent R_{n-1} and $\overline{R}$ will represent R_{n+1}. For example, we would be reorder the term $\Sigma^t R\Sigma^t P$ as

$$\underline{R}\,\underline{\underline{P}}\Sigma^t\Sigma^t = R_{n-1}P_{n-2}. \tag{2.2.13}$$

Since the value of $(L)_-$ has already been stated, we only require the $(\boldsymbol{L}^3)_-$ term:

$$\begin{aligned} L^3 &= (\Sigma^2 + \overline{R} + R + R\underline{R}\Sigma^{t^2})(\Sigma + R\Sigma^t) \\ &= \Sigma^3 + \overline{R}\Sigma + R\Sigma + R\underline{R}\Sigma^{t^2}\Sigma + \Sigma^2 R\Sigma^t + \overline{R}R\Sigma^t + RR\Sigma^t + R\underline{R}\Sigma^{t^2}R\Sigma^t \\ \Rightarrow (L^3)_- &= R\underline{R}\Sigma^t + \overline{R}R\Sigma^t + RR\Sigma^t + R\underline{R}\,\underline{\underline{R}}\Sigma^{t^3} \end{aligned} \tag{2.2.14}$$

and thus the two terms give us a differential equation of the form:

$$P_n' = (aR\Sigma^t P + bR\overline{R}\Sigma^t P + bRR\Sigma^t P + bR\underline{R}\Sigma^t P + bR\underline{R}\,\underline{\underline{R}}\Sigma^{t^3} P)_n.$$

This equation can now be transformed to an equation in terms of P_n using $\frac{dP}{dx} = (V'(L))_- P_n$.

$$P_n' = A_n P_{n-1} + B_n P_{n-3} \tag{2.2.15}$$

and hence the values of A_n and B_n:

$$A_n = (a + b(R_{n+1} + R_n + R_{n-1}))R_n, \quad B_n = bR_n R_{n-1} R_{n-2}. \tag{2.2.16}$$

Now since we consider a monic polynomial $P_n = \sum_{j=0}^{n} a_{nj} x^j$ (with coefficients $a_{n,0}, a_{n,1}, \ldots, a_{n,n}$ and $a_{nn} = 1$), then we also have that $P_n' = nx^{n-1} + \ldots$, therefore $A_n = n$ and hence

$$(a + b(R_{n+1} + R_n + R_{n-1}))R_n = n \tag{2.2.17}$$

which is a discrete Painlevé first equation $\mathrm{P_I}$ [142, 143]. This was first found by Shohat [148], then later rediscovered by Freud [66], as a consequence of the Laguerre-Freud equations (which are similar in content to (2.2.10)), although historically it was Laguerre who has been the first who introduced a method to obtain nonlinear recursion for the coefficients of a L matrix associated with some semi-classical weights of orthogonal polynomials.

It is also possible to derive the Volterra and Toda equations (along with their hierarchies), from semi-classical Hermite-type weights. These equation result from differentiating the recurrence relation coefficients with respect to the parameters introduced in the new weight function. If we had let $V(x)$ to be of a higher order $V(x) = \sum_{j=1}^{\infty} a_j x^{2j}$ then differentiating R_n with respect to the parameter a_1, leads to the Volterra system

$$\partial_{a_1} R_n = -\frac{1}{2} R_n (R_{n+1} - R_{n-1}) \tag{2.2.18a}$$

Further differentiations with respect to the other parameters $a_2, a_3, a_4, \ldots$ leads to the corresponding hierarchy of Volterra equations. If we were to consider the case when

$V(x) = \sum_{j=1}^{\infty} t_j x^j$, we are now dealing with both odd and even powers of x so the recurrence relation would now be cast as (1.1.25) with both $S_n, R_n \neq 0$. It is still possible to acquire differentiations of R_n with respect to the parameters t_j of which t_1 leads to the (modified) Toda equation

$$\partial_{t_1}(\partial_{t_1}(\ln R_n)) = (R_{n-1} - R_n) - (R_n - R_{n+1}) \tag{2.2.18b}$$

and $t_2, t_3, t_4, \ldots$ gives the corresponding hierarchy of equations. These systems along with the Painlevé equations describe an intimate connection between the semi-classical Hermite polynomials and integrable systems.

2.3 The Laguerre Method

We now consider a general approach to semi-classical orthogonal polynomials, by making use of the Laguerre method [100]. This method derives a pair of first order differential equations, after the reduction of continued fractions. Since then this method has been used in conjunction with semi-classical orthogonal polynomials [105], by associating the semi-classical weight function $w(x)$ of the polynomials with a Pearson equation (2.0.3).

While our aim and approach is different, the Laguerre method has been used to find connections with integrable systems, including continuous Painlevé equations, recently. Magnus [105], found a continuous Painlevé equation of the sixth kind from the recurrence coefficients of a semi-classical Jacobi polynomial and Forrester and Witte [62, 63], found a Painlevé equation of the fifth kind, also using the Laguerre method, but extended over a bi-orthogonal framework.

This work will use the same semi-classical Jacobi weight that Magnus used, with the exception being that while he was interested on deriving continuous equations our interest lies in the discrete.

2.3.1 The Basic Principles

Initially we begin by introducing the generating function of moments (a Cauchy-like integral representation of a formal series), analytic outside a set S made of contours and arcs

$$f(z) = \int_S \frac{w(x)}{z-x} dx \tag{2.3.1}$$

and the weight function $w(x)$, which satisfies a differential equation (2.0.3)

$$W(z)\frac{dw(z)}{dz} = V(z)w(z).$$

where $W(z)$ and $V(z)$ are polynomials. This differential equation is an analogue to the Pearson equation, since semi-classical orthogonal polynomials are orthogonal with respect to it. We then introduce the formal semi-classical orthogonal polynomials $P_n(z)$, $n = 0, \ldots, \infty$ which are orthogonal with respect to some weight function $w(x)$ on a support S (2.2.3)

$$\int_S P_n P_m w(x) dx = \langle P_n, P_m \rangle,$$

and a recurrence relation

$$P_{n+1} = (x - S_n)P_n + R_n P_{n-1}. \tag{2.3.2}$$

Multiplying $P_n(z)$ with $f(z)$, gives the integral:

$$\begin{aligned} f(z)P_n(z) &= \int_S dx \frac{P_n(z)}{z-x} w(x) \\ &= \int_S dx \left(\frac{P_n(z) - P_n(x)}{z-x} w(x) + \frac{P_n(x)}{z-x} w(x) \right) \end{aligned}$$

from which we can separate the integrals and define as:

$$\int_S \frac{P_n(z) - P_n(x)}{z-x} w(x) dx = P^{(1)}_{n-1}(z), \tag{2.3.3a}$$

$$\int_S \frac{P_n(x)}{z-x} w(x) dx = \epsilon_n(z), \tag{2.3.3b}$$

or:

$$f(z)P_n(z) = P^{(1)}_{n-1}(z) + \epsilon_n(z), \tag{2.3.4}$$

where $P_n^{(1)}(z)$ is an associated polynomial to $P_n(z)$ of degree $n-1$, but $\epsilon_n(z)$ is not a polynomial. However they both satisfy the recurrence relation (1.1.25).

Now $P_n(z)$ satisfies a first order linear differential equation [18]

$$W(z)\partial_z f(z) = V(z)f(z) + U(z) \tag{2.3.5}$$

which we solve using $f(z)$ (2.3.1), to get expressions for V and U (which are polynomials in x).

$$\begin{aligned} W(z)(\partial_z f(z)) = -\int_S \frac{W(z)w(x)}{(z-x)^2}dx &= -\int_S \frac{d}{dx}\left(\frac{1}{z-x}W(z)w(x)\right)dx + \int_S \frac{W(z)}{z-x}\partial_x w(x) \\ &= \int_S \frac{W(z)}{W(x)}V(x)\frac{1}{z-x}w(x)dx \\ &= V(z)f(z) + W(z)\int_S \left(\frac{V(x)}{W(x)} - \frac{V(z)}{W(z)}\right)\frac{w(x)}{z-x}dx \end{aligned}$$

On the first line we assume that $W(z)w(x) \to 0$ at the end points of S and the second term reduces using (2.0.3), then this leaves an expression for $U(z)$.

$$U(z) = W(z)\int_S \left(\frac{V(x)}{W(x)} - \frac{V(z)}{W(z)}\right)\frac{w(x)}{z-x}dx$$

Another piece of information are the following relations between $P_n, P_n^{(1)}$ and ϵ_n

$$P_n P_{n-2}^{(1)} - P_{n-1}P_{n-1}^{(1)} = -h_{n-1} \tag{2.3.6a}$$

$$P_{n-1}\epsilon_n - P_n\epsilon_{n-1} = -h_{n-1} \tag{2.3.6b}$$

The first of these two equations (2.3.6a), is found using the integral representation of $P_n^{(1)}$, (2.3.3a):

$$(P_n P_{n-2}^{(1)} - P_{n-1}P_{n-1}^{(1)})(z) = \int_S \frac{P_{n-1}(z)P_n(x) - P_{n-1}(x)P_n(z)}{z-x}\, w(x)dx \tag{2.3.7}$$

which by Christoffel-Darboux (1.1.31) gives us a sum.

$$\begin{aligned} (P_n P_{n-2}^{(1)} - P_{n-1}P_{n-1}^{(1)})(z) &= -h_{n-1}\sum_{j=0}^{n-1}\int_S \frac{P_j(z)P_j(x)}{h_j}\, w(x)dx \\ &= -h_{n-1}\sum_{j=0}^{n-1} P_j(z)\frac{h_0}{h_j}\,\delta_{j,0} \\ &= -h_{n-1} \end{aligned} \tag{2.3.8}$$

The second equation (2.3.6b), merely involves rearranging $P_n P^{(1)}_{n-2} - P_{n-1}P^{(1)}_{n-1}$ using (2.3.4).

2.3.2 Explicit Derivation of $P_n(z)$ and $\epsilon_n(z)$

The coefficients of the monic polynomial $P_n(z)$ can be expressed in terms of the coefficients of the recurrence relation (2.3.2) and introduce a monic polynomial $P_n(z) = z^n + p_{n,n-1}z^{n-1} + p_{n,n-2}z^{n-2} + \dots$ into both sides of the expression. Then comparing the lead coefficients, we find:

$$\begin{aligned} p_{n+1,n} - p_{n,n-1} &= -S_n && (2.3.9a) \\ p_{n+1,n-1} - p_{n,n-2} &= -S_n p_{n,n-1} - R_n && (2.3.9b) \end{aligned}$$

from which (2.3.9a) is solved by integrating up and since $P_1(z) = z - S_0$, then $p_{10} = -S_0$:

$$p_{n,n-1} = -\sum_{j=0}^{n-1} S_j. \tag{2.3.10}$$

Moving on to (2.3.9b), we integrate to get:

$$p_{n,n-2} = -\sum_{j=2}^{n-1}(S_j p_{j,j-1} + R_j) + p_{20} \tag{2.3.11}$$

and since

$$P_2(z) = (z - S_1)P_1 - R_1P_0 = (z - S_1)(z - S_0)P_0 - R_1P_0$$

then we have an expression for $p_{n,n-2}$:

$$\begin{aligned} p_{n,n-2} &= \sum_{j=2}^{n-1}\sum_{k=0}^{j-1} S_jS_k - \sum_{j=2}^{n-1} R_j + S_1S_0 - R_1 \\ &= \sum_{j=1}^{n-1}\sum_{k=0}^{j-1} S_jS_k - \sum_{j=1}^{n-1} R_j && (2.3.12) \end{aligned}$$

and thus:

$$P_n(z) = z^n - \left(\sum_{j=0}^{n-1} S_j\right) z^{n-1} + \sum_{j=1}^{n-1}\left(\sum_{k=0}^{j-1} S_j S_k - R_j\right) + \cdots. \tag{2.3.13}$$

Now the function $\epsilon_n(z)$ also satisfies the same recurrence relation,

$$z\epsilon_n = \epsilon_{n+1} + S_n\epsilon_n + R_n\epsilon_{n-1} \qquad n \geq 1 \tag{2.3.14}$$

where this is found by integrating the recurrence relation in P_n over $z - x$ with respect to $w(x)$ and using (2.3.3b).

$$\begin{aligned} & \int_S \frac{xP_n}{z-x} dw(x) = \int_S \frac{P_{n+1} + S_n P_n + R_n P_{n-1}}{z-x}\, dw(x) \\ \Rightarrow \quad & z\epsilon_n(z) - h_n\delta_{n0} = \epsilon_{n+1}(z) + S_n\epsilon_n(z) + R_n\epsilon_{n-1}(z) \end{aligned}$$

However this is not a polynomial, as can be seen from the expansion of (2.3.3b):

$$\epsilon_n(z) = h_n\left(\frac{1}{z^{n+1}} + \frac{e_{n,n+2}}{z^{n+2}} + \frac{e_{n,n+3}}{z^{n+3}} + \cdots\right)$$

so again we substitute this into both sides of the recurrence relation (2.3.14) and upon comparing the lead coefficients we find:

$$h_n e_{n,n+2} = h_n S_n + h_n e_{n-1,n+1} \tag{2.3.15a}$$

$$h_n e_{n,n+3} = h_{n+1} + h_n S_n e_{n,n+2} + h_n e_{n-1,n+2} \tag{2.3.15b}$$

The first relation (2.3.15a) contains a total difference of $e_{n,n+2}$, but has the problem of introducing an integration constant. By solving the difference equation we get:

$$e_{n,n+2} = \sum_{j=1}^{n} S_j + e_{02} \tag{2.3.16}$$

and the integration problem is solved by using $\epsilon_n(z)$ for $n = 0$ and using (1.1.25) to eliminate powers of x:

$$\begin{aligned}
\epsilon_0(z) &= \int_S dx \frac{P_0(x)}{z-x} w(x) = \int_S dx P_0(x) \left(\frac{1}{z} + \frac{x}{z^2} + \frac{x^2}{z^3} + \cdots \right) w(x), \\
&= \frac{h_0}{z} + \frac{1}{z^2} \int_S dx P_0(x)(P_1(x) + S_0 P_0(x)) w(x) \\
&\quad + \frac{1}{z^3} \int_S dx w(x) P_0(x)(P_2(x) + (S_1 + S_0)P_1(x) + (R_1 + S_0^2)P_0(x)) + \cdots, \\
&= h_0 \left(\frac{1}{z} + \frac{S_0}{z^2} + \frac{R_1 + S_0^2}{z^3} + \cdots \right).
\end{aligned} \tag{2.3.17}$$

From this expression we can see that $e_{02} = S_0$, so we have

$$e_{n,n+2} = \sum_{j=0}^{n} S_j. \tag{2.3.18}$$

Moving on to (2.3.15b) we have a difference equation and after using what we have learned above an answer is easily forthcoming.

$$\begin{aligned}
e_{n,n+3} &= \sum_{j=1}^{n} (S_j e_{j,j+2} + R_{j+1}) + e_{03} \\
&= \sum_{j=1}^{n} \left(\sum_{i=0}^{j} S_j S_i + R_{j+1} \right) + R_1 + S_0^2 \\
&= \sum_{j=0}^{n} \left(R_{j+1} + \sum_{i=0}^{j} S_j S_i \right)
\end{aligned} \tag{2.3.19}$$

We now have the necessary components to construct the first few terms of the $\epsilon_n(z)$ expansion.

$$\epsilon_n(z) = h_n \left(\frac{1}{z^{n+1}} + \left(\sum_{j=0}^{n} S_j \right) \frac{1}{z^{n+2}} + \sum_{j=0}^{n} \left(R_{j+1} + \sum_{i=0}^{j} S_j S_i \right) \frac{1}{z^{n+3}} + \cdots \right) \tag{2.3.20}$$

Having established the full derivations of ϵ_n and $P_n^{(1)}$, we can now make use of them by using Laguerre's method.

2.3.3 The Fundamental Linear System for Semi-Classical Orthogonal Polynomials

We start with the expression for fP_n (2.3.4), differentiate it and multiply by W, so that we can then make use of the first order linear differential equation (2.3.5) (with the exception, that for this case we will consider the x variable to be dominant).

$$\begin{aligned} Wf\partial_x P_n + (Vf+U)P_n &= W(\partial_x P^{(1)}_{n-1} + \partial_x \epsilon_n) \\ W\partial_x P_n(P^{(1)}_{n-1}+\epsilon_n) + VP_n(P^{(1)}_{n-1}+\epsilon_n) + UP_n^2 &= W(\partial_x P^{(1)}_{n-1} + \partial_x \epsilon_n)P_n \end{aligned} \tag{2.3.21}$$

We then go about separating the polynomial expression $P^{(1)}_{n-1}$ and ϵ_n so we get the following expression:

$$\Theta_n = W(\partial_x P^{(1)}_{n-1}P_n - \partial_x P_n P^{(1)}_{n-1}) - UP_n^2 - VP_nP^{(1)}_{n-1} \tag{2.3.22a}$$

$$= W(\partial_x P_n \epsilon_n - \partial_x \epsilon_n P_n) + VP_n\epsilon_n \; , \tag{2.3.22b}$$

which is a polynomial bounded by a constant.

We try the same method again except this time we use fP_{n-1}, which is again differentiated and multiplied by W. This will lead to a second object, which will be called Ω_n.

$$\begin{aligned} \partial_x f P_{n-1} + f\partial_x P_{n-1} &= \partial_x P^{(1)}_{n-2} + \partial_x \epsilon_{n-1} \\ VP_{n-1}(P^{(1)}_{n-1}+\epsilon_n) + UP_nP_{n-1} + W\partial_x P_{n-1}(P^{(1)}_{n-1}+\epsilon_n) &= W(\partial_x P^{(1)}_{n-2} + \partial_x \epsilon_{n-1})P_n \end{aligned} \tag{2.3.23}$$

Again we separate the polynomial expression $P^{(1)}_{n-1}$ and ϵ_n to get:

$$\Omega_n = W(P_n\partial_x P^{(1)}_{n-2} - P^{(1)}_{n-1}\partial_x P_{n-1}) - VP_{n-1}P^{(1)}_{n-1} - UP_nP_{n-1} \tag{2.3.24a}$$

$$= W(\epsilon_n \partial_x P_{n-1} - P_n \partial_x \epsilon_{n-1}) + V\epsilon_n P_{n-1} \tag{2.3.24b}$$

Since the recurrence relation (1.1.25) can be expressed in a matrix form

$$\psi_{n+1}(x) = \begin{pmatrix} x - S_n & -R_n \\ 1 & 0 \end{pmatrix} \psi_n(x), \text{ where } \psi_n(x) = \begin{pmatrix} P_n(x) \\ P_{n-1}(x) \end{pmatrix} \tag{2.3.25}$$

we collect the important relations we have derived so far and put them in a matrix form so that our intended differential system can be written as one expression. We begin with the two expressions (2.3.22a) and (2.3.24a), written in matrix form:

$$\begin{pmatrix} P_{n-1} & -P^{(1)}_{n-2} \\ P_n & -P^{(1)}_{n-1} \end{pmatrix} \begin{pmatrix} W\partial_x P^{(1)}_{n-1} \\ W\partial_x P_n(x) \end{pmatrix} = \begin{pmatrix} \Omega_n + VP_{n-1}P^{(1)}_{n-1} + UP_nP_{n-1} \\ \Theta_n + VP_nP^{(1)}_{n-1} + UP_n^2 \end{pmatrix}, \tag{2.3.26a}$$

which can easily be solved making use of (2.3.6a) to give:

$$\begin{pmatrix} W\partial_x P^{(1)}_{n-1} \\ W\partial_x P_n(x) \end{pmatrix} = \frac{1}{h_{n-1}} \begin{pmatrix} P^{(1)}_{n-1} & -P^{(1)}_{n-2} \\ P_n & -P_{n-1} \end{pmatrix} \begin{pmatrix} \Omega_n + VP_{n-1}P^{(1)}_{n-1} + UP_nP_{n-1} \\ \Theta_n + VP_nP^{(1)}_{n-1} + UP_n^2 \end{pmatrix}, \tag{2.3.26b}$$

so that we have an expression for $W\partial_x P_n$:

$$W\partial_x P_n = \frac{1}{h_{n-1}}(\Omega_n P_n - \Theta_n P_{n-1}) \tag{2.3.27a}$$

and $W\partial_x P^{(1)}_{n-1}$:

$$W\partial_x P^{(1)}_{n-1} = (\Omega_n P^{(1)}_{n-1} - \Theta_n P^{(1)}_{n-2} + Vh_{n-1}P^{(1)}_{n-1} + Uh_{n-1}P_n) \tag{2.3.27b}$$

We now look for a second differential relation in P_n, so we take (2.3.27a) with a reduced index in conjunction with the recurrence relation (1.1.25), which leads to

$$W(\partial_x P_{n-1}) = \frac{1}{h_{n-2}} \left(\Omega_{n-1}P_{n-1} - \frac{\Theta_{n-1}}{R_{n-1}}((x - S_{n-1})P_{n-1} - P_n) \right). \tag{2.3.28}$$

However we have no expression to remove the x from the equation, so we consider the problematic part of the expression:

$$(x - S_n)\Theta_n = (x - S_n)\left(W(\epsilon_n \partial_x(P_n) - \partial_x(\epsilon_n)P_n) + V\epsilon_n P_n\right), \tag{2.3.29}$$

which we expand using (1.1.25) and the differential of (2.3.6b)

$$\partial_x P_{n-1}\epsilon_n + \partial_x \epsilon_n P_{n-1} - \partial_x P_n \epsilon_{n-1} - \partial_x \epsilon_{n-1} P_n = 0,$$

to get:

$$\begin{aligned}
(x - S_n)\Theta_n &= W(-\partial_x\epsilon_n(P_{n+1} + R_n P_{n-1}) + \partial_x P_n(\epsilon_{n+1} +' //R_n\epsilon_{n-1})) \\
&\quad +V P_n(\epsilon_{n+1} + R_n\epsilon_{n-1}) \\
&= \Omega_{n+1} + R_n(-W(\partial_x\epsilon_{n-1}P_n - \partial_x P_{n-1}\epsilon_n) + V h_{n-1} + V\epsilon_n P_{n-1}) \\
&= \Omega_{n+1} + R_n\Omega_n + V h_n
\end{aligned} \tag{2.3.30}$$

This allows us to remove x from (2.3.28) to give a second differential equation.

$$W\partial_x P_{n-1} = \frac{1}{h_{n-1}}(\Theta_{n-1}P_n - \Omega_n P_{n-1}) - V P_{n-1} \tag{2.3.31}$$

We now have a *differential system*

$$W\partial_x \begin{pmatrix} P_n(x) \\ P_{n-1}(x) \end{pmatrix} = \frac{1}{h_{n-1}} \begin{pmatrix} \Omega_n(x) & -\Theta_n(x) \\ \Theta_{n-1}(x) & -(\Omega_n(x) + V(x)h_{n-1}) \end{pmatrix} \begin{pmatrix} P_n(x) \\ P_{n-1}(x) \end{pmatrix}, \tag{2.3.32}$$

which can also be written in terms of ψ, where $\psi = \begin{pmatrix} P_n(x) \\ P_{n-1}(x) \end{pmatrix}$. Thus if we give the recurrence and differential equations in a semi-discrete Lax representation (2.1.6) we have

$$\begin{aligned}
\psi_{n+1}(x) &= L_n(x)\psi_n(x) \\
\partial_x\psi_n(x) &= M_n(x)\psi_n(x)
\end{aligned}$$

where

$$L_n = \begin{pmatrix} x - S_n & -R_n \\ 1 & 0 \end{pmatrix} \quad , \quad M_n = \frac{1}{W h_{n-1}} \begin{pmatrix} \Omega_n(x) & -\Theta_n(x) \\ \Theta_{n-1}(x) & -(\Omega_n(x) + V(x)h_{n-1}) \end{pmatrix}$$

2.3.4 Compatibility Relations

We now use the differential system (2.3.32) with the matrix form of the recurrence relation (2.3.25) in order to create a compatibility relation so that relations between Ω_n and Θ_n can be derived. Thus we consider the compatibility between the semi-discrete Lax pair

$$\begin{aligned}\partial_x \psi_{n+1} &= \partial_x(L_n\psi_n) = M_{n+1}\psi_{n+1},\\ &= (\partial_x L_n)(x)\psi_n(x) + L_n(x)M_n(x)\psi_n(x) = M_{n+1}L_n(x)\psi_n(x),\end{aligned} \tag{2.3.34}$$

which leads to the semi-discrete Lax equation (2.1.7)

$$\partial_x L_n = M_{n+1}L_n - L_n M_n.$$

So we begin by differentiating the new form of the recurrence relation (2.3.25)

$$\begin{aligned}\partial_x\psi_{n+1}(x) &= \partial_x \begin{pmatrix} x-S_n & -R_n \\ 1 & 0 \end{pmatrix} \psi_n(x) + \begin{pmatrix} x-S_n & -R_n \\ 1 & 0 \end{pmatrix} \partial_x\psi_n(x)\\ &= \frac{1}{Wh_{n-1}} \begin{pmatrix} x-S_n & -R_n \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Omega_n(x) & -\Theta_n(x) \\ \Theta_{n-1}(x) & -(\Omega_n(x)+V(x)h_{n-1}) \end{pmatrix} \psi_n(x)\\ &\quad + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \psi_n(x)\end{aligned} \tag{2.3.35}$$

and equating this with the differential of (2.3.32)

$$\partial_x\psi_{n+1}(x) = \frac{1}{Wh_n} \begin{pmatrix} \Omega_{n+1}(x) & -\Theta_{n+1}(x) \\ \Theta_n(x) & -(\Omega_{n+1}(x)+V(x)h_n) \end{pmatrix} \begin{pmatrix} x-S_n & -R_n \\ 1 & 0 \end{pmatrix} \psi_n(x), \tag{2.3.36}$$

we can identity two distinct relations.

$$(x-S_n)\left(\frac{\Omega_{n+1}}{h_n} - \frac{\Omega_n}{h_{n-1}}\right) = R_{n+1}\frac{\Theta_{n+1}}{h_{n+1}} - R_n\frac{\Theta_{n-1}}{h_{n-1}} + W \tag{2.3.37}$$

$$(x-S_n)\frac{\Theta_n}{h_n} = \frac{\Omega_{n+1}}{h_n} + \frac{\Omega_n}{h_{n-1}} + V \tag{2.3.38}$$

2.4 Applying the Laguerre Method

This method can be demonstrated by using the semi-classical Hermite and Laguerre families of orthogonal polynomials. The Pearson equation will provide the values of $V(x), W(x)$ and by substituting the expressions for P_n (2.3.13) and ϵ_n (2.3.20) into Ω_n (2.3.24b) and Θ_n (2.3.22b), we have all the necessary components.

$$\begin{aligned}
\Theta_n \;=\;& W(x)h_n\left\{\left[\frac{1}{x^{n+1}}+\left(\sum_{j=0}^{n}S_j\right)\frac{1}{x^{n+2}}+\cdots\right]\right.\\
&\times\left[nx^{n-1}-\left(\sum_{j=0}^{n-1}S_j\right)(n-1)x^{n-2}+\cdots\right]\\
&\left.+\left[\frac{n+1}{x^{n+2}}+\left(\sum_{j=0}^{n}S_j\right)\frac{n+2}{x^{n+3}}+\cdots\right]\times\left[x^n-\left(\sum_{j=0}^{n-1}S_j\right)x^{n-1}+\cdots\right]\right\}\\
&+V(x)\\
&\times h_n\left[\frac{1}{x^{n+1}}+\left(\sum_{j=0}^{n}S_j\right)\frac{1}{x^{n+2}}+\cdots\right]\times\left[x^n-\left(\sum_{j=0}^{n-1}S_j\right)x^{n-1}+\cdots\right]
\end{aligned} \tag{2.4.1}$$

$$\begin{aligned}
\Omega_n \;=\;& W(x)\left\{h_n\left[\frac{1}{x^{n+1}}+\left(\sum_{j=0}^{n}S_j\right)\frac{1}{x^{n+2}}+\sum_{j=0}^{n}\left(R_{j+1}+\sum_{k=0}^{j}S_jS_k\right)\frac{1}{x^{n+3}}+\cdots\right]\right.\\
&\times\left[(n-1)x^{n-2}-(n-2)\left(\sum_{j=0}^{n-2}S_j\right)x^{n-3}+(n-3)\sum_{j=1}^{n-2}\left(\sum_{k=0}^{j-1}S_jS_k-R_j\right)x^{n-4}+\cdots\right]\\
&+h_{n-1}\left[x^n-\left(\sum_{j=0}^{n-1}S_j\right)x^{n-1}+\sum_{j=1}^{n-1}\left(\sum_{k=0}^{j-1}S_jS_k-R_j\right)x^{n-2}+\cdots\right]\\
&\left.\times\left[\frac{n}{x^{n+1}}+\left(\sum_{j=0}^{n-1}S_j\right)\frac{(n+1)}{x^{n+2}}+\sum_{j=0}^{n-1}\left(R_{j+1}+\sum_{k=0}^{j}S_jS_k\right)\frac{(n+2)}{x^{n+3}}+\cdots\right]\right\}\\
&+V(x)\\
&\times h_n\left[\frac{1}{x^{n+1}}+\left(\sum_{j=0}^{n}S_j\right)\frac{1}{x^{n+2}}+\sum_{j=0}^{n}\left(R_{j+1}+\sum_{k=0}^{j}S_jS_k\right)\frac{1}{x^{n+3}}+\cdots\right]\\
&\times\left[x^{n-1}-\left(\sum_{j=0}^{n-2}S_j\right)x^{n-2}+\sum_{j=1}^{n-2}\left(\sum_{k=0}^{j-1}S_jS_k-R_j\right)x^{n-3}+\cdots\right],
\end{aligned} \tag{2.4.2}$$

Once a specific weight is chosen, then the recurrence coefficients will be different for each case considered.

2.4.1 Semi-Classical Hermite Polynomials

From the orthogonality relation (1.3.2) we can see that the classical weight of the Hermite polynomials is $w(x) = e^{-x^2}$. Any deformations of this weight, to the semi-classical case, will involve altering the degree of the polynomial in the exponential. As an example that this method works we use the the even semi-classical Hermite weight (2.2.11) from before and show that it yields discrete $\mathrm{P_I}$. Then we consider an odd semi-classical Hermite weight.

Semi-Classical Hermite Weight: $h_0(x) = e^{-\frac{a}{2}x^2-\frac{b}{4}x^4}$

We choose the semi-classical Hermite weight $e^{-\frac{a}{2}x^2-\frac{b}{4}x^4}$ with $a, b > 0$ and where the support S is an arc from $(-\infty \to \infty)$, then from the Pearson equation (2.0.2) we have

$$V(x) = -(ax + bx^3) \quad , \quad W(x) = 1. \tag{2.4.3}$$

When we substitute $V(x), W(x)$ into the relations above and then make use of the consistency relations, we must be reminded that a weight function of this form satisfies a simplified recurrence relation, specifically one where $S_n = 0$. As a result these relations are greatly reduced in size. We find that Θ_n and Ω_n have the following forms respectively

$$\frac{\Theta_n}{h_n} = -(bx^2 + (R_{n+1} + R_n)b + a) \quad , \quad \frac{\Omega_n}{h_{n-1}} = -bR_n x. \tag{2.4.4}$$

Upon substitution into (2.3.37) all the relations are trivial and in (2.3.38), there is only one non-trivial equation

$$\left(R_{n+1}(R_{n+2} + R_{n+1}) - R_n(R_n + R_{n-1})\right) b + a(R_{n+1} - R_n) = 1 \tag{2.4.5}$$

which is clearly a pure difference equation after inserting the term R_nR_{n+1}. Then after integrating up we are left with

$$R_n(b(R_{n+1} + R_n + R_{n-1}) + a) = n + c \tag{2.4.6}$$

(where c is a an integration constant), which is a discrete form of Painlevé I, d-$\mathrm{P_I}$, of which a similar example was derived earlier (2.2.17) using the same weight function. The difference between the two relations (2.2.17, 2.4.6) is the inclusion of an integration constant in (2.4.6).

Semi-Classical Hermite Weight: $h_1(x) = e^{-a_1x-\frac{a_2}{2}x^2-\frac{a_3}{3}x^3}$

Now we try a weight that does makes use of the recurrence coefficient S_n. So we consider the semi-classical Hermite weight $e^{-a_1x-\frac{a_2}{2}x^2-\frac{a_3}{3}x^3}$ with $a_1, a_2, a_3 > 0$ and where the support S is an arc from $(-\infty \to \infty)$, then from the Pearson equation we have

$$V(x) = -(a_1 + a_2x + a_3x^2) \quad , \quad W(x) = 1. \tag{2.4.7}$$

From these values of $V(x), W(x)$ we have the following forms for Θ_n and Ω_n respectively

$$\frac{\Theta_n}{h_n} = -(a_3x + a_2 + S_na_3) \quad , \quad \frac{\Omega_n}{h_{n-1}} = -a_3R_n. \tag{2.4.8}$$

Then in the consistency relations we have two non-trivial equations

$$R_{n+1}(a_3(S_{n+1} + S_n + a_2)) - R_n(a_3(S_n + S_{n-1}) + a_2) = 1 \tag{2.4.9a}$$

$$S_n(a_2 + S_na_3) = -a_3(R_{n+1} + R_n) - a_1 \tag{2.4.9b}$$

of which the first is a pure difference equation and implies that

$$R_n = \frac{n}{a_3(S_n + S_{n-1}) + a_2} + c_1 \tag{2.4.10}$$

(where c_1 is a constant) hence we have

$$S_n^2a_3 + S_na_2 + a_1 = -a_3\left(\frac{n+1}{a_3(S_{n+1} + S_n) + a_2} + \frac{n}{a_3(S_n + S_{n-1}) + a_2} + 2c_1\right). \tag{2.4.11}$$

which is alternate expression for discrete $\mathrm{P_I}$ [64]. Exponential weights of this type (odd weights), are not usually considered since most are interested in Freud weights, which are all even.

Both of these examples have yielded different forms of discrete $\mathrm{P_I}$, however now we try semi-classical Laguerre, whose weight function combines the exponential part of Hermite with the linear part of Jacobi.

2.4.2 Semi-Classical Laguerre Polynomials

From the orthogonality relation (1.3.10) we can see that the classical weight of the Laguerre is $w(x) = x^{\alpha}e^{-x}$. Our choice of deformations for this weight, involve altering the order of the polynomial in the exponential and/or multiplying the weight by another x. So we consider all three cases.

Semi-Classical Laguerre Weight: $l_0(x) = (x-t)^{\alpha}e^{-(a_1x+\frac{a_2}{2}x^2)}$

We first consider a deformation in the exponential part of the weight function, the semi-classical weight $w(x) = (x-t)^{\alpha}e^{-(a_1x+\frac{a_2}{2}x^2)}$ with $\alpha, a_1, a_2 > 0$ and where the support S is an arc from $(t \to \infty)$. Then from the Pearson equation, we have

$$V(x) = \alpha - (a_1 + a_2x)(x-t) \quad , \quad W(x) = x - t. \tag{2.4.12}$$

From these values of $V(x), W(x)$ we have the following forms for Θ_n and Ω_n respectively

$$\frac{\Theta_n}{h_n} = -(a_2x + a_1 + a_2(S_n - t)) \quad , \quad \frac{\Omega_n}{h_{n-1}} = (n - a_2R_n). \tag{2.4.13}$$

Then in the consistency relations we have two non-trivial equations

$$a_2(R_{n+1} + R_n) = -S_n(a_2S_n + (a_1 - a_2t)) + (2n + 1 + a_1t + \alpha), \tag{2.4.14a}$$

$$R_{n+1}(a_2(S_{n+1} + S_n) + (a_1 - a_2t)) - R_n(a_2(S_n + S_{n-1}) - (a_1 - a_2t)) = S_n - t. \tag{2.4.14b}$$

Although the second equation could be integrated to give a sum for S_j (since it is a pure difference equation), it serves no practical purpose. However we can consider this to be a nonlinear system in terms of the recurrence coefficients R_n and S_n, which we find to be analogues to the Laguerre-Freud equations acquired in [18]. As was mentioned earlier, the Laguerre-Freud equations arise through the differential equation of semi-classical orthogonal polynomials. They are usually studied for Freud weights $|x|^\rho e^{-|x|^{2m}}$, where the case of $\rho = 0$ are also the semi-classical Hermite polynomials.

Due to the ordering of these two equations, it is possible to generate the sequence of R_n, S_n after allowing for the introduction of certain initial conditions. We begin by setting $S_{-1} = 0$ and also make the consideration that $h_{-1} = 1$ and $P_0 = 1$. Then using the inner product expressions for R_n (1.1.11) and S_n (1.1.12) we have the transcendental functions:

$$R_0 = h_0 = \langle P_0, P_0 \rangle = \int_t^\infty (x-t)^\alpha e^{-(a_1 x + \frac{a_2}{2}x^2)} dx \tag{2.4.15a}$$

$$S_0 = \frac{\langle xP_0, P_0 \rangle}{\langle P_0, P_0 \rangle} = \frac{\int_t^\infty x(x-t)^\alpha e^{-(a_1 x + \frac{a_2}{2}x^2)} dx}{\int_t^\infty (x-t)^\alpha e^{-(a_1 x + \frac{a_2}{2}x^2)} dx} \tag{2.4.15b}$$

Using these initial conditions we can initially generate R_1 from (2.4.14a) and S_1 from (2.4.14b), then generate the rest iteratively.

Semi-Classical Laguerre Weight: $l_1(x) = x^\alpha (t-x)^\beta e^{-x}$

Alternatively we can consider a deformation of the non-exponential part of the weight, thus we pose the weight function $x^\alpha (t-x)^\beta e^{-x}$ with $\alpha, \beta > 0$ and where the support S joins the points $0, t$ and ∞ in some way, such as an arc from $0 \to \infty$. From the Pearson equation we have

$$\begin{aligned} V(x) &= x^2 - x(t+\alpha+\beta) + \alpha t, \\ W(x) &= x(t-x). \end{aligned}$$

From these values of $V(x), W(x)$ we have the following forms for Θ_n and Ω_n respectively

$$\begin{aligned}\frac{\Theta_n}{h_n} &= x + S_n - (2n+1+t+ar+\beta)\\ \frac{\Omega_n}{h_{n-1}} &= -\left(nx + \sum_{j=0}^{n-1} S_j - R_n - nt\right).\end{aligned}$$

Then in the consistency relations we have two non-trivial equations

$$\begin{aligned}&S_n(S_n - t) - R_{n+1}(S_{n+1}+S_n) + R_n(S_n + S_{n-1})\\ &= -R_{n+1}(2n+3+t+\alpha+\beta) + R_n(2n-1+t+\alpha+\beta), \qquad (2.4.17a)\end{aligned}$$

$$2\sum_{j=0}^{n-1} S_j - S_n^2 + S_n(2n+2+t+\alpha+\beta) = R_{n+1} + R_n + (2n+1+\alpha)t. \qquad (2.4.17b)$$

The sum in the second equation can be eliminated by subtracting (2.4.17)$_n$−(2.4.17)$_{n-1}$, which leaves us with

$$S_{n-1}^2 - S_n^2 + S_n(2n+2+t+\alpha+\beta) - S_{n-1}(2n-2+t+\alpha+\beta) = R_{n+1} - R_{n-1} + 2t. \qquad (2.4.18)$$

Again we are left with a non-linear system, which can iteratively generate a sequence of R_n, S_n (after the input of specific initial conditions $R_{-1} = S_{-1} = 0,\ h_{-1} = 1,\ P_0 = 1$),

$$\begin{aligned}R_0 &= \langle P_0, P_0\rangle = \int_0^\infty x^\alpha (t-x)^\beta e^{-x} dx,\\ S_0 &= \frac{\langle xP_0, P_0\rangle}{\langle P_0, P_0\rangle} = \frac{\int_0^\infty x^{\alpha+1}(t-x)^\beta e^{-x} dx}{\int_0^\infty x^\alpha (t-x)^\beta e^{-x} dx}.\end{aligned}$$

Semi-Classical Laguerre Weight: $l_2(x) = x^\alpha (t-x)^\beta e^{-(a_1 x + \frac{a_2}{2}x^2)}$

Finally we can deform both parts of the weight and have a weight function of the form $l_2(x) = x^\alpha (t-x)^\beta e^{-(a_1 x + \frac{a_2}{2}x^2)}$ with $\alpha, \beta, a_1, a_2 > 0$ and where the support S joins the points $0, t$ and ∞ in some way, such as an arc from $0 \to \infty$. Then from the Pearson equation, we have

$$\begin{aligned}V(x) &= \alpha - (a_1 + a_2 x + a_3 x^2)(x-t), \qquad (2.4.19a)\\ W(x) &= x - t. \qquad (2.4.19b)\end{aligned}$$

From these values of $V(x), W(x)$ we have the following forms for Θ_n and Ω_n respectively

$$\begin{aligned}\frac{\Theta_n}{h_n} &= a_2x^2 + (a_1 + a_2(S_n - t))x + a_1(S_n - t) - a_2tS_n - (2n + 1 + \alpha + \beta)\\ &\quad +a_2\left(R_{n+1} + R_n + \sum_{j=0}^{n}\sum_{k=0}^{j} S_jS_k + \sum_{j=1}^{n-1}\sum_{k=0}^{j-1} S_jS_k - \left(\sum_{j=0}^{n} S_j\right)\left(\sum_{j=0}^{n-1} S_j\right)\right)\end{aligned} \tag{2.4.20a}$$

$$\frac{\Omega_n}{h_{n-1}} = (a_2R_n - n)x + R_n(a_2(S_n - t + S_{n-1}) + a_1) + nt - \sum_{j=0}^{n-1} S_j. \tag{2.4.20b}$$

In order to group the two double sums $\sum_{j=0}^{n-1}\sum_{k=0}^{j} S_jS_k$ and $\sum_{j=1}^{n-1}\sum_{k=0}^{j-1} S_jS_k$ together into a single double sum, it is necessary to introduce an extra term:

$$\sum_{j=0}^{n-1}\sum_{k=0}^{j} S_jS_k + \left(\sum_{j=1}^{n-1}\sum_{k=0}^{j-1} S_jS_k + \sum_{j=0}^{n-1} S_j^2\right) = 2\sum_{j=0}^{n-1}\sum_{k=0}^{j} S_jS_k,$$

which can be further reduced by subtracting the squared sum $\left(\sum_{j=0}^{n-1} S_j\right)\left(\sum_{j=0}^{n-1} S_j\right)$

$$2\sum_{j=0}^{n-1}\sum_{i=0}^{j} S_jS_i - \left(\sum_{j=0}^{n-1} S_j\right)\left(\sum_{i=0}^{n-1} S_i\right) = \sum_{j=0}^{n-1} S_j^2. \tag{2.4.21}$$

This allows (2.4.20a) to be written in the following way:

$$\begin{aligned}\frac{\Theta_n}{h_n} &= a_2x^2 + (a_1 + a_2(S_n - t))x + a_1(S_n - t) + a_2(R_{n+1} + R_n + S_n(S_n - t))\\ &\quad -(2n + 1 + \alpha + \beta).\end{aligned} \tag{2.4.22}$$

Then from the consistency relations we have the non-trivial equations

$$\begin{aligned}&R_{n+1}(a_1 + a_2(S_{n+1} + 2S_n - t)) + R_n(a_1 + a_2(2S_n + S_{n-1} - t)) + (2n + 1 + \alpha t)\\ &= 2\sum_{j=0}^{n-1} S_j - S_na_1(S_n - t) - a_2S_n^2(S_n - t) + S_n(2n + 2 + \alpha + \beta),\end{aligned} \tag{2.4.23a}$$

$$\begin{aligned}&R_{n+1}(a_1(S_{n+1} + S_n - t) + a_2(R_{n+2} + R_n + S_{n+1}^2 + S_n^2 + S_{n+1}S_n\\ &-t(S_{n+1} + S_n)) - (2n + 3 + \alpha + \beta))\\ &-R_n(a_1(S_n + S_{n-1} - t) + a_2(R_{n+1} + R_{n-1} + S_n^2 + S_{n-1}^2 + S_nS_{n-1}\\ &-t(S_n + S_{n-1})) - (2n + 1 + \alpha + \beta))\\ &= S_n(S_n - t) + 2R_n.\end{aligned} \tag{2.4.23b}$$

This non-linear system is more complicated than the previous systems, since the first equation is first order in R_n and second order in S_n, while the second is third order in R_n and second order in S_n. Consequently it is not possible to iteratively generate the sequence of R_n, S_n, unless we lower the order of the higher order equation and introduce further initial conditions including $R_{-2} = S_{-2} = 0$. An alternative approach is to integrate the second equation (since it is a pure difference equation) and then lower the order of the resulting equation

$$\begin{aligned} & R_n(a_1(S_n + S_{n-1} - t) + a_2(R_{n+1} + R_{n-1} + S_n^2 + S_{n-1}^2 + S_nS_{n-1} \\ & -t(S_n + S_{n-1})) - (2n + 3 + \alpha + \beta)) \\ = \; & \sum_{j=0}^{n-1} S_j^2 - t\sum_{j=0}^{n-1} S_j + 2\sum_{j=0}^{n-1} R_j. \end{aligned} \tag{2.4.24}$$

Then we can iteratively generate the sequence of R_n, S_n, using the same initial conditions (from the previous weights).

From the weights that have been used, a semi-classical Hermite weight leads directly to discrete Painlevé equations, while a semi-classical Laguerre weight leads to a system of two equations in terms of the two recurrence coefficients. These systems consist of one relation first order in R_n and second order S_n, and another relation first order S_n and second order R_n, that can generate a sequence of R_n, S_n after suitable initial conditions are applied. Unsurprisingly a more complex weight leads to a more complex system of relations. All these systems would be of interest if they were investigated further such as looking into the possibility of express them in terms of a single recurrence coefficient and taking continuum limits to work out their corresponding continuous cases.

2.5 The Orthogonality Relation Approach to Semi-Classical Jacobi Polynomials

Here we introduce, the semi-classical Jacobi polynomials, which will be approached using two different methods. First we will look at the derivation of a differential equation using the orthogonality relation for semi-classical orthogonal polynomials. This approach is very similar to the approach used for the semi-classical Hermite polynomials. We are interested in the compatibility between this relation and the recurrence relation, particularly what relations can be derived. The advantages of this theory are that one does not require any prior knowledge, since you are just working with the orthogonality relation, however it does take a long time to calculate the necessary information. Also with this approach there is always the possibility that information is lost. The second way involves the Laguerre method just derived, this is much more accurate method, but takes a lot longer to set up. However, knowledge of this method allows for the quick and easy derivation of compatibility relations between the differential system and the recurrence relation. The matrix method allows more information to be calculated.

Of all the classical families, the Jacobi polynomials are some of the most interesting, not least because they are a general case of some of the other classical families, most notably Legendre and Gegenbauer families. They consist of an integration interval of (-1,1) and a weight function of $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ so that

$$\langle P_n, P_m \rangle = \int_{-1}^{1} P_n(x)P_m(x)w(x)dx = 0 \qquad \text{for all } n \neq m. \tag{2.5.1}$$

Our choice of deformation to the semi-classical case, consists of rewriting the weight function as $w(x) = (1-x)^{\alpha}x^{\beta}(t-x)^{\gamma}$, where a second variable t has been included with addition of another parameter γ. Using the Pearson equation we can demonstrate whether the weight functions satisfy classical or semi-classical orthogonal polynomials. First we consider the Jacobi weight function $w(x) = (1-x)^\alpha(1+x)^\beta$, from which we have

$$\frac{(-\alpha(1-x)^{\alpha-1}(1+x)^\beta + \beta(1-x)^\alpha(1+x)^{\beta-1})}{(1-x)^\alpha(1+x)^\beta} = \frac{-x(\alpha+\beta)-\alpha+\beta}{(1-x^2)}. \tag{2.5.2a}$$

The numerator has deg $=1$ and the denominator has deg $=2$, therefore the weight is classical. Now we consider $w(x) = (1-x)^\alpha x^\beta (t-x)^\gamma$, from which we have

$$\begin{aligned} & \frac{(-\alpha(1-x)^{\alpha-1}x^\beta(t-x)^\gamma + \beta(1-x)^\alpha x^{\beta-1}(t-x)^\gamma - \gamma(1-x)^\alpha x^\beta(t-x)^\gamma)}{(1-x)^\alpha x^\beta (t-x)^\gamma} \\ = & \frac{x^2(\alpha+\beta+\gamma) - x(\alpha t + \beta(t+1)+\gamma) + \beta t}{(1-x)x(t-x)}. \end{aligned} \tag{2.5.2b}$$

In this case the numerator has deg $= 2$ and the denominator has deg $= 3$,ie.

$$V(x) = x^2(\alpha+\beta+\gamma) - x(\alpha t + \beta(t+1)+\gamma) + \beta t \tag{2.5.3a}$$

$$W(x) = (1-x)x(t-x) \tag{2.5.3b}$$

therefore the weight is semi-classical.

Since there are now two variables it is possible to derive two differential equations in terms of t and x:

$$(t-x)\partial_t P_n = c_{nn}P_n + c_{n,n-1}P_{n-1} \tag{2.5.4a}$$

where ∂_t is simply defined as $\frac{\partial}{\partial_t}$ and

$$(1-x)x(t-x)\partial_x P_n = nP_{n+2} + a_{n,n+1}P_{n+1} + a_{nn}P_n + a_{n,n-1}P_{n-1} + a_{n,n-2}P_{n-2}. \tag{2.5.4b}$$

These two equations and (2.3.2)

$$xP_n = P_{n+1} + S_nP_n + R_nP_{n-1}$$

are the primary equations involved in the deformed Jacobi. The form of both these equations can be proven and their coefficients defined explicitly.

2.5.1 The t-Differential Equation

Although the focus is on the compatibility between the x-differential equation an the recurrence relation, we also have a t-differential equation for this particular weight function, so we also consider its corresponding compatibility with the recurrence relation. Although the relations will not be discrete, they may provide interesting relations none the less. We take the general form for the t-differential equation as

$$(t-x)\partial_t P_n = \sum_{j=0}^{n} c_{nj} P_j. \tag{2.5.5}$$

Then differentiate the orthogonality relation (2.5.1) with respect to t

$$\begin{aligned} \partial_t \langle (t-x)P_n, P_m \rangle &= \langle (t-x)\partial_t P_n, P_m \rangle + \langle (t-x)P_n, \partial_t P_m \rangle \\ &\quad + (\gamma+1)\langle P_n, P_m \rangle \end{aligned} \tag{2.5.6}$$

and substitute in the value (2.5.5) to give a relation,

$$\begin{aligned} &\sum_{j=0}^{n} c_{nj} h_m \delta_{jm} + \sum_{j=0}^{m} c_{mj} h_n \delta_{jn} + (\gamma+1) h_n \delta_{nm} \\ = \ &\frac{d}{dt}(t h_n \delta_{nm} - h_m(\delta_{n+1,m} + S_n \delta_{nm} + R_n \delta_{n-1,m})) \end{aligned} \tag{2.5.7}$$

which must be satisfied for all values of m. We begin with $m = n+1$, since that is the value of the highest order polynomial and reduce the order until no more relations are found

$$m = n+1 \ \Rightarrow \ h_n c_{n+1,n} = -\frac{d}{dt} h_{n+1}, \tag{2.5.8a}$$

$$m = n \ \Rightarrow \ 2c_{nn} h_n + (\gamma+1) h_n = \frac{d}{dt}((t - S_n) h_n), \tag{2.5.8b}$$

$$m = n-1 \ \Rightarrow \ c_{n,n-1} h_{n-1} = -\frac{d}{dt}(R_n h_{n-1}), \tag{2.5.8c}$$

$$m = n-2 \ \Rightarrow \ c_{n,n-2} = 0.$$

For $m \leq n-2$ we have no more relations and since the relations for $m = n+1$ and $m = n-1$ are identical, we only have two relations or two coefficients. Thus, we can explicitly define the t-differential equation (2.5.4a) as

$$(t-x)\partial_t P_n = c_{nn}P_n + c_{n,n-1}P_{n-1} \tag{2.5.9}$$

where we have the coefficients

$$\begin{aligned} c_{nn} &= \frac{1}{2}\left(\frac{1}{h_n}\frac{d}{dt}[(t-S_n)h_n] - (\gamma+1)\right), && (2.5.10a)\\ c_{n,n-1} &= -\frac{1}{h_{n-1}}\frac{d}{dt}h_n. && (2.5.10b)\end{aligned}$$

2.5.2 Compatibility Between the Recurrence Relation and the t-Differential Equation

Since the compatibility relations (2.3.38,2.3.37) are generated by a differential system with respect to x, they cannot be used for calculating the compatibility between the t-differential equation (2.5.4a) and recurrence relation (2.3.2) and so a different approach must be used. Thus consider $(t-x)\partial_t(xP_n)$, which we can expand using the recurrence or the t-differential equations respectively.

$$\begin{aligned} (t-x)\partial_t(xP_n) &= \left[\partial_t P_{n+1} + \left(\frac{d}{dt}S_n\right)P_n + S_n\partial_t P_n + \left(\frac{d}{dt}R_n\right)P_n + R_n\partial_t P_{n-1}\right](t-x)\\ &= c_{nn}P_{n+1} + (c_{nn}S_n + c_{n,n-1})P_n + (c_{nn}R_n + c_{n,n-1}S_{n-1})P_{n-1}\\ &\quad + c_{n,n-1}R_{n-1}P_{n-2}\end{aligned}$$

The left side of this relation can be further expanded by making use of the (2.5.4a) again, so that we have

$$\begin{aligned} (t-x)\partial_t(xP_n) &= \left(c_{n+1,n+1} - \frac{d}{dt}S_n\right)P_{n+1} + \left(c_{n+1,n} + c_{nn}S_n + (t-S_n)\frac{d}{dt}S_n - \frac{d}{dt}R_n\right)P_n\\ &\quad + \left(c_{n,n-1}S_n + c_{n-1,n-1}R_n + (t-S_{n-1})\frac{d}{dt}R_n - R_n\frac{d}{dt}S_n\right)P_{n-1}\\ &\quad + \left(c_{n-1,n-2}R_n - R_{n-1}\frac{d}{dt}R_n\right)P_{n-2}\end{aligned}$$

then we can compare the coefficients of P_n. We begin the comparison with substituting the values of c_{nn} and c_{nn-1} into the relations, after which we notice that the coefficients of P_{n+1} and P_{n-1} give the same result and the coefficient of P_{n-2} is trivial. Thus we have only two non-trivial relations

$$\begin{aligned} \frac{d}{dt}S_{n+1} + \frac{d}{dt}S_n &= (t - S_{n+1})\frac{1}{h_{n+1}}\frac{dh_{n+1}}{dt} - (t - S_n)\frac{1}{h_n}\frac{dh_n}{dt}, && (2.5.11a) \\ (t - S_n)\frac{1}{h_n}\frac{d}{dt}S_n &= \frac{1}{h_n^2}\frac{dh_{n+1}}{dt} - \frac{1}{h_{n-1}^2}\frac{dh_{n-1}}{dt}. && (2.5.11b) \end{aligned}$$

These two relations consist of many differentiations of two separate variables, the recurrence coefficients and it would be interesting to see if they can be expressed as one differential in terms of the other variable. So to eliminate the $\partial_t S_n$ we take (2.5.11b), rearrange it and substitute it into (2.5.11a) (after reducing the order of (2.5.11a) by one).

$$\begin{aligned} &\frac{R_{n+1}}{h_{n+1}(t - S_n)}\frac{dh_{n+1}}{dt} - \frac{R_n}{h_{n-1}(t - S_n)}\frac{dh_{n-1}}{dt} \\ &+\frac{R_n}{h_n(t - S_{n-1})}\frac{dh_n}{dt} - \frac{R_{n-1}}{h_{n-2}(t - S_{n-1})}\frac{dh_{n-2}}{dt} = \frac{dh_n}{dt}\frac{(t - S_n)}{h_n} - \frac{dh_{n-1}}{dt}\frac{(t - S_{n-1})}{h_{n-1}} \end{aligned}$$

Now we are left with differentiations of h_n, but of different order. In order for the relation to have the same order of h_n, we make use of the fact that $R_n = \frac{h_n}{h_{n-1}}$,

$$\frac{1}{h_{n-1}}\frac{dh_{n-1}}{dt} = \frac{1}{h_n}\frac{dh_n}{dt} - \frac{1}{R_n}\frac{dR_n}{dt} \qquad (2.5.12)$$

which in turn brings in differentials of R_n with respect to t.

$$\begin{aligned} &\frac{1}{h_n}\frac{dh_n}{dt}\left(\frac{R_{n+1} - R_n}{t - S_n} - (t - S_n) + \frac{R_n - R_{n-1}}{t - S_{n-1}} - (t - S_{n-1})\right) \\ &+\frac{1}{t - S_n}\frac{dR_{n+1}}{dt} + \frac{1}{R_n}\frac{dR_n}{dt}\left(\frac{R_n}{t - S_n} + \frac{R_{n-1}}{t - S_{n-1}} - (t - S_{n-1})\right) + \frac{1}{t - S_{n-1}}\frac{dR_{n-1}}{dt} = 0 \end{aligned} \qquad (2.5.13)$$

This is a differential closed-form system in terms of both h_n and S_n.

2.5.3 The x-Differential Equation

The previous method considered the compatibility between a 2 point differential system and a recurrence relation. Here we will consider the compatibility between a 5 point differential relation and recurrence relation.

We begin by considering the general form of the x-differential relation, $(1-x)x(t-x)\partial_x P_n$, which when expanded would look something like:

$$(1-x)x(t-x)\partial_x P_n = nP_{n+2} + \sum_{j=0}^{n+1} a_{nj} P_j \tag{2.5.14}$$

where the leading coefficient is obviously n. We look to orthogonality relations to confirm the rest of the relation:

$$\begin{aligned}\langle (1-x)x(t-x)\partial_x P_n, P_m\rangle &= \int (\partial_x P_n) P_m (1-x)^{\alpha+1} x^{\beta+1} (t-x)^{\gamma+1} dx \\ &= -\langle P_n, (1-x)x(t-x)\partial_x P_m\rangle + \int P_n P_m [(\alpha+1)x(t-x) \\ &\quad -(\beta+1)(1-x)(t-x) + (\gamma+1)(1-x)x] w(x) dx\end{aligned} \tag{2.5.15}$$

From here it is possible to substitute (2.5.14) into (2.5.15) to give:

$$\begin{aligned}n\delta_{n+2,m}h_m + \sum_{j=0}^{n+1} a_{nj}\delta_{jm}h_m &= -m\delta_{m+2,n}h_n - \sum_{j=0}^{m+1} a_{mj}\delta_{jn}h_n - (\beta+1)t\delta_{nm}h_m \\ &\quad -h_m(\alpha+\beta+\gamma+3)(\delta_{n+2,m} + (S_{n+1}+S_n)\delta_{n+1,m} \\ &\quad +(R_{n+1}+S_n^2+R_n)\delta_{nm} + R_n(S_n+S_{n-1})\delta_{n-1,m} \\ &\quad +R_nR_{n-1}\delta_{n-2,m}) \\ &\quad +h_m((\alpha+1)t + (\beta+1)(t+1) + (\gamma+1)) \\ &\quad \times(\delta_{n+1,m} + S_n\delta_{nm} + R_n\delta_{n-1,m})\end{aligned} \tag{2.5.16}$$

Using this equation it is possible to derive a series of relations and determine the shape of the x-differential relation by substituting in values of m. Beginning with $m = n+2$, the value of the highest order polynomial and decreasing the value of m, one finds that after $m = n - 2$, there are no more relations to be found. This is of the form stated at the beginning (2.5.4b), with the following unique relations giving the coefficients.

$$a_{n,n-2} = -(\alpha+\beta+\gamma+n+1)R_nR_{n-1} \tag{2.5.17a}$$

$$a_{n,n+1} + \frac{a_{n+1,n}}{R_{n+1}} = -(\alpha+\beta+\gamma+3)(S_{n+1}+S_n) + (\alpha+\beta+2)t + (\beta+\gamma+2) \tag{2.5.17b}$$

$$\begin{aligned} 2a_{nn} &= -(\beta+1)t - (\alpha+\beta+\gamma+3)(R_{n+1}+R_n+S_n^2) \\ &\quad +((\alpha+\beta+2)t+(\beta+\gamma+2))S_n \end{aligned} \tag{2.5.17c}$$

It is easy to see, however, that while a_{nn-2} and a_{nn} can be explicitly defined, there is no such result for a_{nn+1} and a_{nn-1}. Thus, by using this approach we are able to show that there is a finite number of terms in the x-differential equation, but not give an explicit formula of the coefficients. We can now consider the compatibility between the recurrence relation and differential equation, to see if further information can be acquired.

2.5.4 Compatibility Between the Recurrence Relation and the x-Differential Equation

Consider using the x-differential relation with the recurrence relation and the following relation is achieved:

$$x(1-x)(t-x)\partial_x(xP_n) = x(1-x)(t-x)(\partial_xP_{n+1} + S_n\partial_xP_n + R_n\partial_xP_{n-1}), \tag{2.5.18}$$

which can then be expanded by substituting in (2.5.4b):

$$\begin{aligned} & x(1-x)(t-x)P_n + x(nP_{n+2} + a_{nn+1}P_{n+1} + a_{nn}P_n + a_{nn-1}P_{n-1} + a_{nn-2}P_{n-2}) \\ = \; & (n+1)P_{n+3} + a_{n+1n+2}P_{n+2} + a_{n+1n+1}P_{n+1} + a_{n+1n}P_n + a_{n+1n-1}P_{n-1} \\ & +S_n(nP_{n+2} + a_{nn+1}P_{n+1} + a_{nn}P_n + a_{nn-1}P_{n-1} + a_{nn-2}P_{n-2}) \\ & +R_n\left((n-1)P_{n+1} + a_{n-1n}P_n + a_{n-1n-1}P_{n-1} + a_{n-1n-2}P_{n-2} + a_{n-1n-3}P_{n-3}\right) \end{aligned}$$

then order-by-order, in powers of x, we get the following set of relations for the coefficients by substituting in (2.5.4b) (and extended versions), which lead to the following series of relations:

$$R_n a_{n-1,n-3} = R_n R_{n-1} R_{n-2} + a_{n,n-2} R_{n-2} \quad (2.5.19a)$$

$$a_{n+1,n+2} - a_{n,n+1} = S_n + S_{n+1} + S_{n+2} - (1+t) + n(S_{n+2} - S_n), \quad (2.5.19b)$$

$$\begin{aligned} a_{n+1,n+1} - a_{nn} = \; & R_n + R_{n+1} + R_{n+2} + S_{n+1}^2 + S_n S_{n+1} + S_n^2 + t - (n-1)R_n \\ & +nR_{n+2} + a_{n,n+1}(S_{n+1} - S_n) - (1+t)(S_{n+1} + S_n), \end{aligned} \quad (2.5.19c)$$

$$\begin{aligned} a_{n+1,n} - a_{n,n-1} \; & = R_{n+1}S_{n+1} + R_n S_{n-1} + S_n(S_n - 1)(S_n - t) \\ +R_n a_{n-1,n} - R_{n+1} a_{n,n+1} \; & \quad +2S_n(R_{n+1} + R_n) - (1+t)(R_{n+1} + R_n), \end{aligned} \quad (2.5.19d)$$

$$\begin{aligned} a_{n+1,n-1} - a_{n,n-2} \; & = R_n(R_{n+1} + R_n + R_{n-1}) + tR_n - (1+t)R_n(S_n + S_{n-1}) \\ -(a_{nn} - a_{n-1,n-1}) \; & \quad +R_n(S_n^2 + S_{nn-1} + S_{n-1}^2) + a_{n,n-1}(S_{n-1} - S_n) \end{aligned} \quad (2.5.19e)$$

$$\begin{aligned} a_{n-1,n-2}R_n - a_{n,n-1}R_{n-1} = \; & (S_{n-2} - S_n)a_{n,n-2} + R_n R_{n-1}(S_n + S_{n-1} + S_{n-2}) \\ & -(t+1)R_n R_{n-1}, \end{aligned} \quad (2.5.19f)$$

which can be solved by two methods. The first, the method of substitution, defines the coefficients very specifically, defining them in terms of the parameters α, β, γ, but produces no new information. The second method involves arranging the equations to give a total difference and then integrating up. The resulting equations contain integration coefficients, which on comparison with (2.5.17a) and (2.5.17c) must contain the parameters α, β, γ.

2.5.5 Difference equations for R_n and S_n

From these equations, five are used to give explicit derivations of the coefficients, leaving one equation to play around with. Of the former equations the easiest to calculate are described as follows:

The solution of (2.5.19a) comes from moving the $a_{n,n-2}, a_{n-1,n-3}$ to one side, dividing by $R_nR_{n-1}R_{n-2}$ and integrating up

$$a_{n,n-2} = (-n + b_{-2})R_nR_{n-1}, \tag{2.5.20}$$

where the value b_{-2} represents the integration constant, for the coefficient of $a_{n,n-2}$.

The solution of (2.5.19b) involves similar manipulation, except the term nS_{n+1} is included to give a pure difference equation

$$(a_{n+1,n+2} - (n+1)S_{n+2} - nS_{n+1}) - (a_{n,n+1} - nS_{n+1} - (n-1)S_n) = S_{n+1} - (1+t),$$

which can then be integrated to give:

$$a_{n,n+1} = \sum_{j=0}^{n} S_j - (1+t)(n+1) + nS_{n+1} + (n-1)S_n + b_1 \tag{2.5.21}$$

where the value b_1 represents the integration constant. The remaining two equations involve the use of $a_{n,n-2}$ and $a_{n,n+1}$ to give their solutions.

The solution of (2.5.19f) involves the substitution of (2.5.20), then dividing by R_nR_{n-1} and introducing the term $(n - b_{-2})S_{n-1}$ gives another pure difference equation

$$\begin{aligned} &\left(\frac{a_{n,n-1}}{R_n} + (n+1-b_{-2})S_n + (n-b_{-2})S_{n-1}\right) \\ -&\left(\frac{a_{n-1,n-2}}{R_{n-1}} + (n-b_{-2})S_{n-1} + (n-1-b_{-2})S_{n-2}\right) \quad = \quad (t+1) - S_{n-1}, \end{aligned}$$

which integrates to give:

$$\frac{a_{n,n-1}}{R_n} = -\sum_{j=0}^{n-1} S_j + (1+t)n - (n+1-b_{-2})S_n - (n-b_{-2})S_{n-1} + b_{-1} \tag{2.5.22}$$

with b_{-1} the integration constant.

The final coefficient is the most difficult to find and involves using equations (2.5.19c) and (2.5.19e). Noting that increasing the index by one in (2.5.19e) gives the same difference of a_{nn} as in (2.5.19c) allows two combinations, from which one will eliminate a_{nn} and the other will keep a_{nn}. Thus $(2.5.19c)_n - (2.5.19e)_{n+1} \rightarrow$ a pure difference in $a_n$and $(2.5.19c)_n + (2.5.19e)_{n+1} \rightarrow a_n$ disappearing

$$\begin{aligned} a_{n+1,n+1} - a_{nn} \quad &= \quad R_n - R_{n+2} + S_n^2 - S_{n+1}^2 + \frac{b_{-2}}{2}(R_{n+2} - R_n + S_{n+1}^2 - S_n^2), \\ &\quad + \frac{1}{2}(b_1 - b_{-1})(S_{n+1} - S_n) \end{aligned}$$

which after some inclusions can be integrated to give:

$$a_{nn} = (R_{n+1} + R_n + S_n^2)(\frac{b_{-2}}{2} - 1) + \frac{1}{2}(b_1 - b_{-1})S_n + b_0 \tag{2.5.23}$$

with b_0 the integration constant.

A parameterization of the coefficients is given by the following set

$$b_{-2} \quad = \quad -(\alpha + \beta + \gamma + 1) \tag{2.5.24a}$$

$$b_1 - b_{-1} \quad = \quad ((\alpha + \beta + 2)t + (\beta + \gamma + 2)) \tag{2.5.24b}$$

$$b_0 \quad = \quad -\frac{1}{2}(\beta + 1)t \tag{2.5.24c}$$

but we only acquire this result by comparing the two solution sets. While we can state b_0, b_{-2}, we have a certain amount of freedom with regards to the value of b_1, b_{-1}.

This of course leaves two equations remaining (2.5.19d) and $(2.5.19c)_n + (2.5.19e)_{n+1}$. We now have all the necessary ingredients for (2.5.19d) so substituting in and cancelling gives us:

$$\begin{aligned}
& 2R_{n+1}\left(\sum_{j=0}^{n} S_j-(1+t)(2n+3)+S_{n+1}(2n+3-b_{-2})-b_{-1}\right) \\
& -2R_n\left(\sum_{j=0}^{n-1} S_j-(1+t)(2n-1)+S_{n-1}(2n-3-b_{-2})+b_1\right) \\
= \quad & -S_n\left(R_{n+1}(2n+2-b_{-2})-R_n(2n-2-b_{-2})+(S_n-1)(S_n-t)\right).
\end{aligned} \tag{2.5.25a}$$

Moving onto the second, we find:

$$\begin{aligned}
& S_n\left(2\sum_{j=0}^{n-1} S_j-(1+t)2n+(2n-b_{-2})S_n+(b_1-b_{-1})\right) \\
& -S_{n+1}\left(\sum_{j=0}^{n-1} S_j-(1+t)(2n+4)+(2n+4-b_{-2})S_{n+1}+(b_1-b_{-1})\right) \\
= \quad & (2n+4-b_{-2})R_{n+2}+2R_{n+1}-(2n-2-b_{-2})R_n+2t.
\end{aligned} \tag{2.5.25b}$$

Thus we have a pair of relations resulting from the consistency between the differential and recursion relations. This method, while correct, is not as powerful compared with the method earlier established. We now have two paths of exploration, the first requires re-expressing the five point differential relation as a two-point differential relation and then comparing the coefficients with Ω_n and Θ_n or we calculate Ω_n and Θ_n using their respective formulas (2.3.24b) and (2.3.22b).

2.6 The Laguerre Method Approach to Semi-Classical Jacobi Polynomials

The use of the Laguerre method has already been demonstrated with semi-classical Hermite and Laguerre so now consider the more complex case of Jacobi.

2.6.1 Explicit Derivation of V_n, Θ_n and Ω_n

By using the Pearson equation (2.0.2), we were able to derive the polynomials V (2.5.3a) and W (2.5.3b) respectively.

$$\begin{aligned} V(x) &= \big((\alpha+\beta+\gamma)x^2 - (\alpha t + \beta(t+1) + \gamma)x + \beta t\big) \\ W(x) &= (1-x)x(t-x) \end{aligned}$$

Given that we can express both Ω_n and Θ_n in terms of P_n and ϵ_n, (2.3.22b) and (2.3.24b), it is possible to derive explicit forms for Ω_n and Θ_n by using the explicit derivations of P_n (2.3.13) and ϵ_n (2.3.20) using simple substitution and then looking for powers of x.

$$\begin{aligned} P_n(x) &= x^n - \left(\sum_{j=0}^{n-1} S_j\right) x^{n-1} + \sum_{j=1}^{n-1}\left(\sum_{k=0}^{j-1} S_j S_k - R_j\right) x^{n-2} + \cdots, \\ \epsilon_n(x) &= h_n\left(\frac{1}{x^{n+1}} + \left(\sum_{j=0}^{n} S_j\right)\frac{1}{x^{n+2}} + \sum_{j=0}^{n}\left(R_{j+1} + \sum_{k=0}^{j} S_j S_k\right)\frac{1}{x^{n+3}} + \cdots\right). \end{aligned}$$

Beginning with the expression for Θ_n,

$$\Theta_n = W(\epsilon_n(x)\partial_x P_n(x) - P_n(x)\partial_x \epsilon_n(x)) + V\epsilon_n(x)P_n(x)$$

and expanding:

$$\begin{aligned} \Theta_n &= (1-x)x(t-x)h_n\left\{\left[\frac{1}{x^{n+1}} + \left(\sum_{j=0}^{n} S_j\right)\frac{1}{x^{n+2}} + \cdots\right]\right. \\ &\quad \times\left[nx^{n-1} - \left(\sum_{j=0}^{n-1} S_j\right)(n-1)x^{n-2} + \cdots\right] \\ &\quad \left.+\left[\frac{n+1}{x^{n+2}} + \left(\sum_{j=0}^{n} S_j\right)\frac{n+2}{x^{n+3}} + \cdots\right] \times \left[x^n - \left(\sum_{j=0}^{n-1} S_j\right)x^{n-1} + \cdots\right]\right\} \\ &\quad +((\alpha+\beta+\gamma)x^2 - (\alpha t + \beta(t+1) + \gamma)x + \beta t) \\ &\quad \times h_n\left[\frac{1}{x^{n+1}} + \left(\sum_{j=0}^{n} S_j\right)\frac{1}{x^{n+2}} + \cdots\right] \times \left[x^n - \left(\sum_{j=0}^{n-1} S_j\right)x^{n-1} + \cdots\right] \end{aligned}$$

which after expanding and cancelling terms reduces to an equation of the form:

$$\Theta_n = \Theta_n^{(1)} x + \Theta_n^{(0)} \tag{2.6.1}$$

where we have:

$$\begin{aligned}
\Theta_n^{(1)} &= h_n(2n+1+\alpha+\beta+\gamma) && (2.6.2a)\\
\Theta_n^{(0)} &= h_n\left(2\sum_{j=0}^{n-1} S_j - (1+t)(2n+1) + (2n+2+\alpha+\beta+\gamma)S_n\right.\\
&\quad \left. -(\alpha t+\beta(t+1)+\gamma)\right) && (2.6.2b)
\end{aligned}$$

Continuing with Ω_n,

$$\Omega = W(x)(\epsilon_n(x)\partial_x P_{n-1}(x) - P_n(x)\partial_x\epsilon_{n-1}(x)) + V(x)\epsilon_n(x)P_{n-1}(x)$$

and expanding:

$$\begin{aligned}
\Omega_n &= (1-x)x(t-x)\left\{h_n\left[\frac{1}{x^{n+1}} + \left(\sum_{j=0}^{n} S_j\right)\frac{1}{x^{n+2}} + \sum_{j=0}^{n}\left(R_{j+1} + \sum_{k=0}^{j} S_j S_k\right)\frac{1}{x^{n+3}} + \cdots\right]\right.\\
&\quad \times\left[(n-1)x^{n-2} - (n-2)\left(\sum_{j=0}^{n-2} S_j\right)x^{n-3} + (n-3)\sum_{j=1}^{n-2}\left(\sum_{k=0}^{j-1} S_j S_k - R_j\right)x^{n-4} + \cdots\right]\\
&\quad + h_{n-1}\left[x^n - \left(\sum_{j=0}^{n-1} S_j\right)x^{n-1} + \sum_{j=1}^{n-1}\left(\sum_{k=0}^{j-1} S_j S_k - R_j\right)x^{n-2} + \cdots\right]\\
&\quad \left.\times\left[\frac{n}{x^{n+1}} + \left(\sum_{j=0}^{n-1} S_j\right)\frac{(n+1)}{x^{n+2}} + \sum_{j=0}^{n-1}\left(R_{j+1} + \sum_{k=0}^{j} S_j S_k\right)\frac{(n+2)}{x^{n+3}} + \cdots\right]\right\}\\
&\quad + \big((\alpha+\beta+\gamma)x^2 - (\alpha t+\beta(t+1)+\gamma)x + \beta t\big)\\
&\quad \times h_n\left[\frac{1}{x^{n+1}} + \left(\sum_{j=0}^{n} S_j\right)\frac{1}{x^{n+2}} + \sum_{j=0}^{n}\left(R_{j+1} + \sum_{k=0}^{j} S_j S_k\right)\frac{1}{x^{n+3}} + \cdots\right]\\
&\quad \times\left[x^{n-1} - \left(\sum_{j=0}^{n-2} S_j\right)x^{n-2} + \sum_{j=1}^{n-2}\left(\sum_{k=0}^{j-1} S_j S_k - R_j\right)x^{n-3} + \cdots\right],
\end{aligned}$$

which after expanding and cancelling terms reduces to an equation of the form:

$$\Omega_n = \Omega_n^{(2)} x^2 + \Omega_n^{(1)} x + \Omega_n^{(0)}, \tag{2.6.3}$$

where we have

$$\Omega_n^{(2)} = nh_{n-1}, \tag{2.6.4a}$$

$$\Omega_n^{(1)} = \left(\sum_{j=0}^{n-1} S_j - (1+t)n\right) h_{n-1}, \tag{2.6.4b}$$

$$\begin{aligned}\Omega_n^{(0)} &= (n-1+\alpha+\beta+\gamma)h_n + h_{n-1}\left(nt - (1+t)\sum_{j=0}^{n-1} S_j - (n+1)\sum_{j=0}^{n-1} S_j \sum_{j=0}^{n-1} S_j\right.\\ &\left. + n\sum_{j=1}^{n-1}\left(\sum_{k=0}^{j-1} S_j S_k - R_j\right) + (n+2)\sum_{j=0}^{n-1}\left(R_{j+1} + \sum_{k=0}^{j} S_j S_k\right)\right).\end{aligned} \tag{2.6.4c}$$

$\Omega_n^{(0)}$ can be simplified further by making use of (2.4.21) to remove the relations involving double sums.

$$2(n+1)\sum_{j=0}^{n-1}\sum_{k=0}^{j} S_j S_k - (n+1)\left(\sum_{j=0}^{n-1} S_j\right)\left(\sum_{k=0}^{n-1} S_k\right) - n\sum_{j=0}^{n-1} S_j^2 = \sum_{j=0}^{n-1} S_j^2$$

Then (2.6.4c) can be expressed as

$$\Omega_n^{(0)} = \left((2n+1+\alpha+\beta+\gamma)R_n + 2\sum_{j=1}^{n-1} R_j + tn + \sum_{j=0}^{n-1} S_j^2 - (1+t)\sum_{j=0}^{n-1} S_j\right) \tag{2.6.5}$$

2.6.2 Solving the Compatibility Relations Explicitly

With the combined expressions for Θ_n (2.6.1), Ω_n (2.6.3) and $V(x)$ (2.5.3a) it is possible to solve equations (2.3.38) and (2.3.37). Thus we begin (2.3.38), since its the smaller of the two equations and expand in powers of x.

$$(x - S_n)\frac{\Theta_n}{h_n} = \frac{\Omega_{n+1}}{h_n} + \frac{\Omega_n}{h_{n-1}} + V$$

The first two expressions, that is the coefficients of x^2 and x, which leaves the following relation:

$$\begin{aligned}
& -S_n 2\sum_{j=0}^{n-1} S_j + (1+t)(2n+2)S_n - (2n+3+\alpha+\beta+\gamma)S_n^2 + S_n(\alpha t + \beta(t+1)+\gamma) \\
= \quad & (2n+3+\alpha+\beta+\gamma)R_{n+1} + (2n-1+\alpha+\beta+\gamma)R_n + 4\sum_{j=1}^{n} R_j + t(2n+1+\beta) \\
& +2\sum_{j=0}^{n-1} S_j^2 - 2(1+t)\sum_{j=0}^{n-1} S_j,
\end{aligned} \tag{2.6.6}$$

which doesn't simplify any further. However, the expression can be reduced by considering the difference between itself with a raised index ie. $(2.6.6)_{n+1} - (2.6.6)_n$, which removes $\sum_{j=1}^{n-1} R_j$ and $\sum_{j=0}^{n-1} S_j^2$.

$$\begin{aligned}
& -S_{n+1}\left(2\sum_{j=0}^{n} S_j - (1+t)(2n+4) + (2n+5+\alpha+\beta+\gamma)S_{n+1} - (\alpha t + \beta(t+1)+\gamma)\right) \\
& +S_n\left(2\sum_{j=0}^{n} S_j - (1+t)2n + (2n-1+\alpha+\beta+\gamma)S_n - (\alpha t + \beta(t+1)+\gamma)\right) \\
& = (2n+5+\alpha+\beta+\gamma)R_{n+2} + 2R_{n+1} - (2n-1+\alpha+\beta+\gamma)R_n + 2t
\end{aligned} \tag{2.6.7}$$

If we compare this equation with (2.5.25b), we find them to have the same order S and R, which if we were interested in the value of the integration constants would allow us to speculate. So this equation can be derived using both ways, but the expression (2.6.6) cannot. Moving onto (2.3.37), we find that the only non-trivial term is the coefficient of x^0,

$$(x - S_n)\left(\frac{\Omega_{n+1}}{h_n} - \frac{\Omega_n}{h_{n-1}}\right) = R_{n+1}\frac{\Theta_{n+1}}{h_{n+1}} - R_n\frac{\Theta_{n-1}}{h_{n-1}} + W$$

which leads to an equation of a similar form to (2.6.7):

$$
\begin{aligned}
&-S_n\left((2n+3+\alpha+\beta+\gamma)R_{n+1}-(2n-1+\alpha+\beta+\gamma)R_n+t+S_n^2-(1+t)S_n\right)= \\
&R_{n+1}\left(2\sum_{j=0}^{n}S_j-(1+t)(2n+3)+(2n+4+\alpha+\beta+\gamma)S_{n+1}-(\alpha t+\beta(t+1)+\gamma)\right) \\
&-R_n\left(2\sum_{j=0}^{n-1}S_j-(1+t)(2n-1)+(2n-2+\alpha+\beta+\gamma)S_{n-1}-(\alpha t+\beta(t+1)+\gamma)\right)
\end{aligned}
\tag{2.6.8}
$$

This approach has produced a a coupled system of nonlinear difference equations, similar in content to (2.5.25b) and (2.5.25a), but there is also the relation (2.6.6) (showing what could be described as an earlier form), and unlike the previous method (which has the unknown integration constants), this method leads to explicit derivations from the compatibility relations. While (2.6.6) clearly contains more information it also less manageable since it contains sums of R_j, S_j and S_j^2, so we focus our attention on (2.6.7). With the exception of $\sum_{j=0}^{n-1} S_j$, (2.6.7) is a first order relation in S_n and second order in R_n, while (2.6.8) is a first order relation in R_n and second order in S_n. This is exactly the same situation as the semi-classical Laguerre weights, thus an introduction of initial values allows us to generate the sequence of R_n, S_n. From (1.1.26b) and (1.1.26a) we can take $R_{-1} = S_{-1} = 0$, (using the definitions of the Hankel determinants)and we also make the consideration that $h_{-1} = 1$ and $P_0 = 1$. Then we consider the value of R_0, S_0 using the inner product expressions for R_n (1.1.11) and S_n (1.1.12), thus we have the transcendental functions:

$$
R_0 = h_0 = \langle P_0, P_0\rangle = \int_{-1}^{1}(1-x)^\alpha x^\beta (t-x)^\gamma dx \tag{2.6.9a}
$$

$$
S_0 = \frac{\langle xP_0, P_0\rangle}{\langle P_0, P_0\rangle} = \frac{\int_{-1}^{1}(1-x)^\alpha x^{\beta+1}(t-x)^\gamma dx}{\int_{-1}^{1}(1-x)^\alpha x^\beta (t-x)^\gamma dx} \tag{2.6.9b}
$$

Using these initial conditions we can initially generate R_1 from (2.4.14a) and S_1 from (2.4.14b), then generate the rest iteratively.

While it is possible to express the two difference equations as a single equation, using the current information we cannot eliminate one recurrence coefficient in favor of the another. Thus we leave the expression in this form.

2.7 Summary

This chapter highlights the connections between semi-classical orthogonal polynomials and discrete integrable systems. Semi-classical orthogonal polynomials are obtained by deforming the weight function of the classical orthogonal polynomials and given any weight function we can determine whether the weight is classical or not by using the Pearson equation (2.0.2). Thus, we started with a well known example (Section 2.2.1) which considered the compatibility between the differential equation (2.2.10) and the recurrence relation (2.2.2) for a class of semi-classical Hermite polynomials. The compatibility led to the derivation of a discrete P_{I} (2.2.17). We also described the methodology from which further discrete integrable systems can be derived.

For a more precise way of calculating the compatibility between a differential equation and a recurrence relation we introduced the Laguerre method, which generates a differential system (2.3.32) for a general class of semi-classical orthogonal polynomials. From the point of view of integrable systems this differential system can be seen as a semi-discrete Lax equation (2.1.7) of which the compatibility with the recurrence relation (2.3.25) leads to a pair of non-linear relations, which can be seen as Laguerre-Freud equations (2.3.37, 2.3.38). As a simple example to demonstrate this method, we used the same semi-classical Hermite weight from before, which again led to a discrete P_{I}. While even exponential weights are often studied (since they satisfy Freud weights), odd exponential weights are not so we introduced an alternate semi-classical Hermite weight, which we found satisfied an alternate discrete P_{I} (2.4.11). Following this three Laguerre weights were considered,

$$
\begin{aligned}
l_0(x) &= (x-t)^{\alpha} e^{-(a_1 x + \frac{a_2}{2} x^2)} && (2.7.10a)\\
l_1(x) &= x^{\alpha}(t-x)^{\beta} e^{-x} && (2.7.10b)\\
l_2(x) &= x^{\alpha}(t-x)^{\beta} e^{-(a_1 x + \frac{a_2}{2} x^2)} && (2.7.10c)
\end{aligned}
$$

where each weight leads to a well defined non-linear closed system (2.4.14a, 2.4.14b), (2.4.17) and (2.4.23) respectively (Laguerre-Freud equations) for R_n, S_n, after the introduction of some initial conditions. We consider these systems to be new discrete Painlevé-type equations, where they are new in the sense that we have not already come across these systems in the literature. One interesting problem to consider in the future would be to specify which continuous Painlevé equations these systems correspond to; a problem that could be approached by looking at their continuum limits.

Of the very classical orthogonal polynomials, the Jacobi polynomials are the richest, since they have a number of special cases and Hermite and Laguerre are limiting cases of Jacobi. Thus we approached the compatibility problem using two separate methods: the direct method (which is covered in Section 2.5) and the Laguerre method (which is covered in Section 2.6). Then we compared the results of both approaches. The first approach calculated the compatibility between the recurrence relation and differential equation directly by using substitutions. Of the resulting six equations, four led to the explicit derivation of the coefficients in the differential equation and the remaining two led to two coupled nonlinear difference equations (2.5.25a, 2.5.25b) (of Freud Laguerre type), for the recurrence coefficients R_n, S_n. Using a Jacobi weight with the Laguerre method leads to two nonlinear relations, similar in content to the results found using the Laguerre weights, with the exception that these relations were of a higher order. At first glance, of the two relations found through the Laguerre method (2.6.6) and (2.6.8) only one relation was the same as the direct approach. However after taking a difference, equation (2.6.6) was reduced to (2.6.7), thus showing that ultimately both relations (2.5.25a, 2.5.25b) can also be derived using the Laguerre method.

Chapter 3

Singular Integral Transforms and Orthogonal Polynomials

Connections between orthogonal polynomials and integrable systems, have usually been found by studying the structure of orthogonal polynomials, often finding relations between the recurrence coefficients. However this is not always the case. There have been attempts to find more subtle connections, for instance between orthogonal polynomials and the inverse scattering method (a key characteristic of an integrable system).

The inverse scattering method involved three separate steps; a direct transform, a time evolution of scattering data and an inverse transform. The Gel'fand-Levitan equation [72] is the important inversion transform and is defined

$$K(x,y) + F(x,y) + \int_0^x K(x,t)F(x,t)dt = 0, \tag{3.0.1}$$

where $K(x,y)$is an unknown $d \times d$ matrix valued function and $F(x,y)$ is a known function of two variables which is constructed on the basis of the scattering data.

We note that the Gel'fand-Levitan equation is a linear integral equation in configuration space (the space of the spatial x variable of the system). As an alternative approach to the inverse problem, rather than in configuration space, one works in spectral space (i.e., the complex space of the spectral parameter) using singular integral transforms based on Cauchy's integral theorem [1, 172]. The latter approach, which could be considered as a nonlinear Fourier transform method, is relevant to the Riemann-Hilbert problem formulation of integrable systems theory [173].

The main contributor to the research connecting orthogonal polynomials and the inverse scattering method has been Case, whose work in scattering theory began in 1972 after collaborating with Kac on deriving a discrete version of the inverse scattering transform [32]. This work was followed by papers [33, 34] concerning the close parallels between the theory of a class of orthogonal polynomials and scattering theory. This work was often based in a general setting, with the connection between the two coming from an intermediary object. One particular paper from 1978 [35], found that the three topics orthogonal polynomials, inverse scattering and linear estimation could all be described by the same equation, which was coined the generalized Gel'fand-Levitan equation. Case describes the classical approach (from this paper) as considering a polynomial $P_n(\lambda)$ of order n which satisfies the following orthogonality relation:

$$\int P_n(\lambda)P_m(\lambda)d\rho(\lambda) = \delta(n,m).$$

We also consider $P_i^0(\lambda)$ which is any linearly independent polynomial of degree i (where $i = 0, 1, 2, \ldots, n, \ldots$) and where

$$P_n(\lambda) = \sum_{m=0}^{n} K(n,m)P_m^0(\lambda). \tag{3.0.2}$$

Since $P_n(\lambda)$ is orthogonal to all $P_r(\lambda)$ for $r \leq n-1$, it is orthogonal to all polynomials of degree less than $n-1$. Thus we integrate both sides to get the following expression:

$$\begin{aligned}\int P_n(\lambda)P_r^0(\lambda)d\rho(\lambda) &= \sum_{m=0}^{n} K(n,m)\int P_m^0(\lambda)P_r^0(\lambda)d\rho(\lambda)\\ &= 0, \quad r \leq n-1.\end{aligned}$$

Then if we define $\mu(m,r)$ and $\kappa(n,m)$ as

$$\mu(m,r) = \int P_m^0(\lambda)P_r^0(\lambda)d\rho(\lambda) \quad , \quad \kappa(n,m) = \frac{K(n,m)}{K(n,n)}$$

we obtain

$$\sum_{m=0}^{n-1} \kappa(n,m)\mu(m,r) = -\mu(n,r). \tag{3.0.3}$$

Case considers this equation to be a discrete version of the Gel'fand-Levitan equation and for convenience Case also states the generalized version:

$$\sum_{m=0}^{n-1} \overline{\kappa}(n,m)\mu(m,r) = -\overline{\mu}(n,r), \quad 0 \leq r \leq n-1. \tag{3.0.4}$$

Thus the connections Case found between orthogonal polynomials and inverse scattering came through this intermediary equation, the generalized Gel'fand-Levitan equation (GGLE). He then shows how this equation can be derived from both the inverse scattering method and linear estimation.

While we are also interested in the connections between orthogonal polynomials and the inverse scattering method we are not interested in the GGLE and it's connections. Thus using these ideas of Case as inspiration, we set up a framework using formal singular integral transforms, to make the connection between integrable systems and orthogonal polynomials. One aim is to understand the transition from classical orthogonal polynomials to semi-classical orthogonal polynomials. in terms of a dressing method. We will begin by reviewing the general framework of singular integral transforms for linear problems associated with integrable hierarchies [124]. Then we will specify to the case of 2×2 matrix Lax systems of the type that arise from the Laguerre method in chapter 2. The special case of recurrence relations with even weights is closely related to the KdV-Volterra system. In order to derive the relevant integral transforms, we first consider the more general $N \times N$ matrix case leading to the Gel'fand-Dikii hierarchy, which reduces to the KdV case for $N = 2$.

3.1 Matrix Singular Integral Transforms

In this section we will study a very general but formal setup of integral transforms, which preserve linear systems associated with integrable equations. We begin by considering a transformation $\Phi^0 \to \Phi^1$ from a given $N \times N$ matrix Φ^0 to a new Φ^1, which are functions of a spectral parameter $k \,\epsilon\, \mathbb{C}$. Rather than writing $\Phi(k)$ we prefer to write Φ_k for the matrix function, highlighting the dependence on the argument k as a suffix in order to make clearly visible which functions Φ depend on which argument. The transformation $\Phi^0 \to \Phi^1$ is defined through a singular integral equation of the form

$$\Phi^0_k + \int_{C^{10}} \Phi^1_l d\Lambda^{10}(l) \frac{(\Phi^0_l)^{-1}\Phi^0_k}{k-l} = \Phi^1_k \tag{3.1.5}$$

which is a generalization of an integral transform proposed in [146], c.f [124]. In (3.1.5) $d\Lambda^{10}(l)$ denotes an $N \times N$ matrix measure with components $d\Lambda_{ij}(l),\ \ i,j = 1,\ldots,N$ each component of which is associated with a contour C_{ij} in the complex l-plane over which the integration is performed (this is symbolically indicated by the "matrix contour" $C^{10} = (C_{ij})$). The inverse of the integral transform (3.1.5) is

$$(\Phi^1_k)^{-1} + \int_{C^{10}} \frac{(\Phi^1_k)^{-1}\Phi^1_l}{k-l} d\Lambda^{10}(l)(\Phi^0_l)^{-1} = (\Phi^0_k)^{-1}. \tag{3.1.6}$$

We now introduce the kernel G_{lk}, which will aid us in more advanced calculations, by simplifying the equation.

$$G_{lk} = \frac{(\Phi_l)^{-1}\Phi_k}{l-k} \tag{3.1.7}$$

In order to have a equation in terms of G_{lk} only, we expand (3.1.5) and (3.1.6)

$$\begin{aligned}(\Phi^1_{k'})^{-1}\Phi^1_k \;\;=\;\; & \left((\Phi^0_{k'})^{-1} - \int_{C^{10}} G^1_{k'l} d\Lambda^{10}(l)(\Phi^0_l)^{-1}\right) \\ & \times \left(\Phi^0_k - \int_{C^{10}} \Phi^1_l d\Lambda^{10}(l) G^0_{lk}\right),\end{aligned} \tag{3.1.8}$$

which in turn leads to:

$$G^1_{k'k}(k'-k) = G^0_{k'k}(k'-k) - \int_{C^{10}} G^1_{k'l} d\Lambda^{10}(l) G^0_{lk}(l-k) - \int_{C^{10}} G^1_{k'l}(k'-l) d\Lambda^{10}(l) G^0_{lk}.$$

dividing by $(k'-k)$ the two integrals reduce to a single integral, that can be rewritten as

$$G^1_{k'k} = G^0_{k'k} - \int_{C^{10}} G^1_{k'l} d\Lambda^{10}(l) G^0_{lk} \tag{3.1.9}$$

which is an integral transform for the kernel.

3.1.1 Composition Formulae

We now consider compositions of subsequent transformations of the form (3.1.5) with different integration measures $d\Lambda^{10}$ and $d\Lambda^{21}$, each associated with their respective matrix contours C_{10} and C_{21}. Thus we obtain the following *dressing chain*, of subsequent matrix functions

$$\Phi^0_k \xrightarrow{\Lambda^{10}} \Phi^1_k \xrightarrow{\Lambda^{21}} \Phi^2_k. \tag{3.1.10}$$

We consider an integral transform between Φ^1_k and Φ^2_k :

$$\Phi^1_k - \int_{C^{21}} \Phi^2_l d\Lambda^{21}(l) G^1_{lk} = \Phi^2_k. \tag{3.1.11}$$

defined in terms of G_{lk} (3.1.7). This can be expanded using (3.1.9) to give an equation in terms of the new function and the original function only, removing the intermediary function.

$$\begin{aligned} \Phi^2_k &= \Phi^0_k - \int_{C^{10}} \Phi^1_{l'} d\Lambda^{10}(l') G^0_{l'k} - \int_{C^{21}} \Phi^2_l d\Lambda^{21}(l) \left(G^0_{lk} - \int_{C^{10}} G^1_{ll'} d\Lambda^{10}(l') G^0_{l'k} \right) \\ &= \Phi^0_k - \int_{C^{10}} \left(\Phi^2_{l'} + \int_{C^{21}} \Phi^2_l d\Lambda^{21}(l) G^1_{ll'} \right) d\Lambda^{10}(l') G^0_{l'k} \\ &- \int_{C^{21}} \Phi^2_l d\Lambda^{21}(l) \left(G^0_{lk} - \int_{C^{10}} G^1_{ll'} d\Lambda^{10}(l') G^0_{l'k} \right) \end{aligned} \tag{3.1.12}$$

Since we treat this mechanism as a formal structure we assume that the integrals can be interchanged and then the double integrals will cancel, leaving:

$$\Phi^0_k - \int_{C^{10}} \Phi^2_{l'} d\Lambda^{10}(l') G^0_{l'k} - \int_{C^{21}} \Phi^2_l d\Lambda^{21}(l) G^0_{lk} = \Phi^2_k. \tag{3.1.13}$$

We conclude from above that in the transformation from $\Phi^0 \to \Phi^2$ there is no longer a dependency on the intermediary matrix Φ^1 and hence the transformation from $\Phi^0 \to \Phi^2$ remains of the same form as its constitutive steps but with a combined integration of the form $\int_{C^{21}} d\Lambda^{21}(l) + \int_{C^{10}} d\Lambda^{10}(l)$. This exhibits the group theoretical structure of the integral transforms. In particular as a corollary, a formula for the inverse integral transform is obtained by setting

$$\int_{C^{21}} d\Lambda^{21}(l) + \int_{C^{10}} d\Lambda^{10}(l) = 0$$

which then implies that $\Phi_k^2 = \Phi_k^0$. Thus, the inverse integral transform $\Phi^1 \to \Phi^0$ is obtained by setting

$$\int_{C^{01}} d\Lambda^{01}(l) = -\int_{C^{10}} d\Lambda^{10}(l). \tag{3.1.14}$$

3.1.2 Transformation Properties for Lax Forms

We use the standard definition for a linear integral transform

$$\Phi_k^0 + \int_{C^{10}} \Phi_l^1 d\Lambda^{10}(l) \frac{(\Phi_l^0)^{-1}\Phi_k^0}{k-l} = \Phi_k^1 \tag{3.1.15}$$

and impose a linear dependence of Φ_k on k

$$\tilde{\Phi}_k = (kJ + Q)\Phi_k = L(k)\Phi_k, \tag{3.1.16}$$

where J and Q are $N \times N$ matrices, J a constant diagonal matrix, Q a matrix potential (under suitable boundary conditions on the real line) and the tilde represents a discrete shift in some arbitrary variable. Then (3.1.16) is preserved under the integral equation (3.1.15) provided that the matrix J is invariant $(J^1 = J^0)$ under transformations. This means that if we impose (3.1.16) on the reference state Φ_k^0 with a potential Q^0, then Φ_k^1 also obeys (3.1.16) with a new potential Q^1 related to the old one Q^0. To show this we begin with $\tilde{\Phi}_k = L(k)\Phi_k$ and its inverse $(\Phi_k)^{-1} = (\tilde{\Phi}_k)^{-1}L(k)$ and consider $(\Phi_l)^{-1}\Phi_k$:

$$\begin{aligned}(\Phi_l)^{-1}\Phi_k &= (\tilde{\Phi}_l)^{-1}L(l)\Phi_k \\ &= (\tilde{\Phi}_l)^{-1}(L(l)-L(k))\Phi_k + (\tilde{\Phi}_l)^{-1}\tilde{\Phi}_k \\ &= (\tilde{\Phi}_l)^{-1}\tilde{\Phi}_k - (k-l)(\tilde{\Phi}_l)^{-1}J\Phi_k\end{aligned}$$

consider (3.1.15) with an index increased by one and substitute the above into the integral:

$$\tilde{\Phi}^0_k + \int_{C^{10}} \tilde{\Phi}^1_l d\Lambda^{10}(l)\frac{(k-l)(\tilde{\Phi}^0_l)^{-1}J\Phi^0_k + (\Phi^0_l)^{-1}\Phi^0_k}{k-l} = \tilde{\Phi}^1_k. \tag{3.1.17}$$

where we assume that $d\Lambda^{10}(l) = d\tilde{\Lambda}^{10}(l)$. This can be expanded using (3.1.16):

$$\begin{aligned}(kJ+Q^0)\Phi^0_k + \int_{C^{10}} \tilde{\Phi}^1_l d\Lambda^{10}(l)(\tilde{\Phi}^0_l)^{-1}J\Phi^0_k &+ \int_{C^{10}}(lJ+Q^1)\Phi^1_l d\Lambda^{10}(l)\frac{(\Phi^0_l)^{-1}\Phi^0_k}{k-l} \\ &= (kJ+Q^1)\Phi^1_k\end{aligned}$$

and then include a correction term (to reduce the equation).

$$\begin{aligned}(kJ+Q^0)\Phi^0_k + \int_{C^{10}} \tilde{\Phi}^1_l d\Lambda^{10}(l)(\tilde{\Phi}^0_l)^{-1}J\Phi^0_k &+ \int_{C^{10}}(kJ+Q^1)\Phi^1_l d\Lambda^{10}(l)\frac{(\Phi^0_l)^{-1}\Phi^0_k}{k-l} \\ &- J\int_{C^{10}}\frac{k-l}{k-l}\Phi^1_l d\Lambda^{10}(l)(\Phi^0_l)^{-1}\Phi^0_k \\ &= (kJ+Q^1)\Phi^1_k\end{aligned}$$

Using (3.1.15), terms can be collected to reduce to a single term $(kJ+Q^1)\Phi^0_k$, then the expression becomes:

$$Q^0\Phi^0_k + \int_{C^{10}} \tilde{\Phi}^1_l d\Lambda^{10}(l)(\tilde{\Phi}^0_l)^{-1}J\Phi^0_k - J\int_{C^{10}} \Phi^1_l d\Lambda^{10}(l)(\Phi^0_l)^{-1}\Phi^0_k = Q^1\Phi^0_k. \tag{3.1.18}$$

For symmetrical purposes we define the following

$$H^1 - H^0 = \int_{C^{10}} \Phi^1_l d\Lambda^{10}(l)(\Phi^0_l)^{-1} \tag{3.1.19}$$

so that the relation reduces to:

$$Q^1 - Q^0 = (\tilde{H}^1 - \tilde{H}^0)J - J(H^1 - H^0). \tag{3.1.20}$$

and thus

$$Q = \tilde{H}J - JH + \text{invariant} \tag{3.1.21}$$

where the invariant is any kind of object that doesn't change under the dressing transform. Clearly the potential H obeys a compatibility equation and we consider $\tilde{\Phi}_k = (kJ+Q)\Phi_k$ to be a Lax representation where we would have the Lax pair between $\tilde{\phi} = (kJ+Q)\phi$ and $\hat{\phi} = (kJ+R)\phi$.

We have found an integral expression for the Q, the leading term in the linear expression. However this term consists of the newly defined $H^1 - H^0$, which we investigate further, specifically for how it reacts under a differential or difference operator.

3.1.3 Squared Eigenfunction Expansions

Although we have an expression for $H^1 - H^0$, (3.1.19), it has a mixed integrand consisting of the new function Φ^1 and the original function Φ^0, thus, we require both expressions to determine its value. When we apply an arbitrary differential operator (δ) or an arbitrary difference operator (Δ) (in terms of some yet unspecified additional variable on which Φ and H may depend) to $H^1 - H^0$, the result can be expressed in terms of the action of these operators on the Φ^0 only.

If we consider (3.1.5) expressed in terms of G_{kl}

$$\Phi_k^0 = \Phi_k^1 + \int_{C^{10}} \Phi_l^1 d\Lambda^{10}(l) G_{lk}^0 \tag{3.1.22}$$

and compare it with

$$\Phi_k^1 = \Phi_k^0 - \int_{C^{10}} \Phi_l^0 d\Lambda^{10}(l) G_{lk}^1 \tag{3.1.23}$$

(where the 0 and 1 are interchanged and $\int_{C^{01}} d\Lambda^{01} = -\int_{C^{10}} d\Lambda^{10}$ (3.1.14)), this leads to a new inverse equation:

$$(\Phi_k^1)^{-1} + \int_{C^{10}} G_{kl}^0 d\Lambda^{10}(l)(\Phi_l^1)^{-1} = (\Phi_k^0)^{-1}. \tag{3.1.24}$$

This information allows us to expand (3.1.19).

(a) The Continuous Case

The only assumption we need on δ is the Leibniz rule for differentiation i.e.,

$$\delta(AB) = (\delta A)B + A(\delta B) \tag{3.1.25}$$

where A and B are matrix functions on which δ acts by differentiation. In agreement with the previous section, taking (3.1.14) it is possible to express $H^1 - H^0$ in two separate forms, its original and inverted form (which are expressed in terms of k, the spectral parameter of the transforms).

$$H^1 - H^0 = \int_{C^{10}} \Phi_k^1 d\Lambda^{10}(k)(\Phi_k^0)^{-1} = \int_{C^{10}} \Phi_k^0 d\Lambda^{10}(k)(\Phi_k^1)^{-1} \tag{3.1.26}$$

We begin with substituting (3.1.24) into (3.1.19) and apply the differential operator δ:

$$\begin{aligned}\delta(H^1 - H^0) \;=\; & \int_{C^{10}} (\delta\Phi_k^1) d\Lambda^{10}(k)(\Phi_k^1)^{-1} + \int_{C^{10}} \Phi_k^1 d\Lambda^{10}(k)\delta(\Phi_k^1)^{-1} \\ & + \int_{C^{10}}\int_{C^{10}} \big((\delta\Phi_k^1) d\Lambda^{10}(k) G_{kl}^0 d\Lambda^{10}(l)(\Phi_l^1)^{-1} \\ & + \Phi_k^1 d\Lambda^{10}(k)(\delta G_{kl}^0) d\Lambda^{10}(l)(\Phi_l^1)^{-1} + \Phi_k^1 d\Lambda^{10}(k) G_{kl}^0 d\Lambda^{10}(l)\delta(\Phi_l^1)^{-1}\big)\end{aligned} \tag{3.1.27}$$

then (3.1.24) helps reduce the first and third term, while (3.1.22) helps reduce the second and fifth term so that we have :

$$\begin{aligned}\delta(H^1 - H^0) \;=\; & \int_{C^{10}} (\delta\Phi_k^1) d\Lambda^{10}(k)(\Phi_k^0)^{-1} + \int_{C^{10}} \Phi_k^0 d\Lambda^{10}(k)\delta(\Phi_k^1)^{-1} \\ & + \int_{C^{10}}\int_{C^{10}} \Phi_k^1 d\Lambda^{10}(k)(\delta G_{kl}^0) d\Lambda^{10}(l)(\Phi_l^1)^{-1}\end{aligned} \tag{3.1.28}$$

and using the differential of (3.1.26) allows us to bring in $\delta(H^1 - H^0)$ for the first two terms.

$$\begin{aligned}\delta(H^1 - H^0) \;=\; & \delta(H^1 - H^0) - \int_{C^{10}} \Phi_k^1 d\Lambda^{10}(k)\delta(\Phi_k^0)^{-1} \\ & + \delta(H^1 - H^0) - \int_{C^{10}} (\delta\Phi_k^0) d\Lambda^{10}(k)(\Phi_k^1)^{-1} \\ & + \int_{C^{10}}\int_{C^{10}} \Phi_k^1 d\Lambda^{10}(k)(\delta G_{kl}^0) d\Lambda^{10}(l)(\Phi_l^1)^{-1}\end{aligned} \tag{3.1.29}$$

All that remains now is for the last term to be dealt with where the main problem is G_{kl}, so we reexpress G_{kl} in terms of its components and then apply the delta to them.

$$\begin{aligned}\delta G^0_{kl} = \delta\left(\frac{(\Phi^0_k)^{-1}\Phi^0_l}{k-l}\right) &= (\delta\Phi^0_k)^{-1}\Phi^0_k\frac{(\Phi^0_k)^{-1}\Phi^0_l}{k-1} + \frac{(\Phi^0_k)^{-1}\Phi^0_l}{k-1}(\Phi^0_l)^{-1}\delta\Phi^0_l \\ &= (\delta\Phi^0_k)^{-1}\Phi^0_k G^0_{kl} + G^0_{kl}(\Phi^0_l)^{-1}\delta\Phi^0_l \qquad (3.1.30)\end{aligned}$$

If we replace this in the previous expression, then the last term now can now be expressed:

$$\int_{C^{10}}\int_{C^{10}} \Phi^1_k\, d\Lambda^{10}(k)\left((\delta\Phi^0_k)^{-1}\Phi^0_k G^0_{kl} + G^0_{kl}(\Phi^0_l)^{-1}\delta\Phi^0_l\right) d\Lambda^{10}(l)(\Phi^1_l)^{-1} \qquad (3.1.31)$$

then by using (3.1.24) and (3.1.22) we can eliminate the double integrals:

$$\begin{aligned}\delta(H^1 - H^0) &= \delta(H^1 - H^0) - \int_{C^{10}} \Phi^1_k d\Lambda^{10}(k)\delta(\Phi^0_k)^{-1} \\ &\quad + \delta(H^1 - H^0) - \int_{C^{10}} (\delta\Phi^0_k) d\Lambda^{10}(k)(\Phi^1_k)^{-1} \\ &\quad + \int_{C^{10}} \Phi^1_k d\Lambda^{10}(k)\delta(\Phi^0_k)^{-1}\Phi^0_k\left((\Phi^0_k)^{-1} - (\Phi^1_k)^{-1}\right) \\ &\quad + \int_{C^{10}} (\Phi^0_l - \Phi^1_l)(\Phi^0_l)^{-1}\delta\Phi^0_l d\Lambda^{10}(l)(\Phi^1_l)^{-1} \qquad (3.1.32)\end{aligned}$$

and this can be simplified to:

$$\begin{aligned}\delta(H^1 - H^0) &= \int_{C^{10}} \Phi^1_k d\Lambda^{10}(k)\delta(\Phi^0_k)^{-1}\Phi^0_k(\Phi^1_k)^{-1} \\ &\quad + \int_{C^{10}} \Phi^1_l(\Phi^0_l)^{-1}\delta\Phi^0_l d\Lambda^{10}(l)(\Phi^1_l)^{-1}.\end{aligned}$$

or

$$\delta(H^1 - H^0) = \int_{C^{10}} \Phi^1_k[(\Phi^0_k)^{-1}(\delta\Phi^0_k), d\Lambda^{10}(k)](\Phi^1_k)^{-1} \qquad (3.1.33)$$

In the integral the only differentiations are of Φ^0_k, which are located in the middle of both the integrals. If we consider these middle terms (the $d\Lambda^{10}(k)$ and Φ^0_k) to be an extended interpolating measure, then the differential of $H^1 - H^0$ is dependent on Φ^1_k. This is the matrix analogue of the squared matrix expansions (sometimes referred to as trace formulae), which have arisen in the inverse scattering method cf. eg. [49].

(b) The Discrete Case

Now we have considered the simpler continuous case we move onto the discrete case where we use a difference operator Δ instead of differential operator δ, and where $\Delta A = \tilde{A} - A$. The discrete Leibniz rule for the difference of two arbitrary functions of a variable x can therefore be given as

$$\begin{aligned} \Delta(AB) &= A\Delta(B) + \Delta(A)\tilde{B} && \text{(3.1.34a)} \\ &= \tilde{A}\Delta(B) + \Delta(A)B. && \text{(3.1.34b)} \end{aligned}$$

Working again from the equation involving $H^1 - H^0$ (3.1.19),

$$H^1 - H^0 = \int_{C^{10}} \Phi_k^1 d\Lambda^{10}(k)(\Phi_k^0)^{-1}$$

we apply the difference operator Δ to get:

$$\begin{aligned} \Delta(H^1 - H^0) &= \int_{C^{10}} (\Delta\Phi_k^1) d\Lambda^{10}(k)(\Phi_k^1)^{-1} + \int_{C^{10}} \widetilde{\Phi}_k^1 d\Lambda^{10}(k)\Delta[(\Phi_k^1)^{-1}] \\ &+ \int_{C^{10}}\int_{C^{10}} \big((\Delta\Phi_k^1) d\Lambda^{10}(k) G_{kl}^0 d\Lambda^{10}(l)(\Phi_l^1)^{-1} \\ &+ \widetilde{\Phi}_k^1 d\Lambda^{10}(k)(\Delta G_{kl}^0) d\Lambda^{10}(l)(\Phi_l^1)^{-1} + \widetilde{\Phi}_k^1 d\Lambda^{10}(k)\widetilde{G}_{kl}^0 d\Lambda^{10}(l)\Delta(\Phi_l^1)^{-1}\big) \end{aligned} \tag{3.1.35}$$

and using the same equations as before (3.1.22) and (3.1.24) we reduce the equation to

$$\begin{aligned} \Delta(H^1 - H^0) &= \int_{C^{10}} (\Delta\Phi_k^1) d\Lambda^{10}(k)(\Phi_k^0)^{-1} + \int_{C^{10}} \widetilde{\Phi}_k^0 d\Lambda^{10}(k)\Delta[(\Phi_k^1)^{-1}] \\ &+ \int_{C^{10}}\int_{C^{10}} \widetilde{\Phi}_k^1 d\Lambda^{10}(k)(\Delta G_{kl}^0) d\Lambda^{10}(l)(\Phi_l^1)^{-1} \end{aligned} \tag{3.1.36}$$

then using (3.1.26) just leaves us with the problematic ΔG_{lk}.

$$\begin{aligned} \Delta(H^1 - H^0) &= \Delta(H^1 - H^0) - \int_{C^{10}} \widetilde{\Phi}_k^1 d\Lambda^{10}(k)\Delta[(\Phi_k^0)^{-1}] \\ &= +\Delta(H^1 - H^0) - \int_{C^{10}} (\Delta\Phi_k^0) d\Lambda^{10}(k)(\Phi_k^1)^{-1} \\ &\quad + \int_{C^{10}}\int_{C^{10}} \widetilde{\Phi}_k^1 d\Lambda^{10}(k)(\Delta G_{kl}^0) d\Lambda^{10}(l)(\Phi_l^1)^{-1} \end{aligned} \tag{3.1.37}$$

Being mindful of the shift caused by the application of the Δ

$$\begin{aligned}\Delta G^0_{kl} = \Delta\frac{(\Phi^0_k)^{-1}\Phi^0_l}{k-l} &= \Delta[(\Phi^0_k)^{-1}]\Phi^0_k\frac{(\Phi^0_k)^{-1}\Phi^0_l}{k-l} + \frac{(\widetilde{\Phi}^0_k)^{-1}\widetilde{\Phi}^0_l}{k-l}(\widetilde{\Phi}^0_l)^{-1}\Delta\Phi^0_l \\ &= \Delta[(\Phi^0_k)^{-1}]\Phi^0_k G^0_{kl} + \widetilde{G}^0_{kl}(\widetilde{\Phi}^0_l)^{-1}\Delta\Phi^0_l \end{aligned} \tag{3.1.38}$$

then putting all the pieces together:

$$\begin{aligned}\Delta(H^1 - H^0) &= \Delta(H^1 - H^0) - \int_{C^{10}} \widetilde{\Phi}^1_k d\Lambda^{10}(k)\Delta[(\Phi^0_k)^{-1}] \\ &+\Delta(H^1 - H^0) - \int_{C^{10}} (\Delta\Phi^0_k)d\Lambda^{10}(k)(\Phi^1_k)^{-1} \\ &+ \int_{C^{10}} \widetilde{\Phi}^1_k d\Lambda^{10}(k)\Delta[(\Phi^0_k)^{-1}]\Phi^0_k\left((\Phi^0_k)^{-1} - (\Phi^1_k)^{-1}\right) \\ &+ \int_{C^{10}} (\widetilde{\Phi}^0_l - \widetilde{\Phi}^1_l)(\widetilde{\Phi}^0_l)^{-1}\Delta[\Phi^0_l]d\Lambda^{10}(l)(\Phi^1_l)^{-1} \end{aligned} \tag{3.1.39}$$

and after cancelling leaves us with:

$$\begin{aligned}\Delta(H^1 - H^0) &= \int_{C^{10}} \widetilde{\Phi}^1_k d\Lambda^{10}(k)\Delta[(\Phi^0_k)^{-1}]\Phi^0_k(\Phi^1_k)^{-1} \\ &+ \int_{C^{10}} \widetilde{\Phi}^1_l(\widetilde{\Phi}^0_l)^{-1}\Delta[\Phi^0_l]d\Lambda^{10}(l)(\Phi^1_l)^{-1}. \end{aligned} \tag{3.1.40}$$

or

$$\Delta(H^1 - H^0) = \int_{C^{10}} \tilde{\Phi}^1_k[d\Lambda^{10}(k), (\tilde{\Phi}^0_k)^{-1}\Phi^0_k](\Phi^1_k)^{-1} \tag{3.1.41}$$

Again we see that the difference operator only acts on Φ^0_k, so if we consider the middle terms as an extended interpolating measure, then the difference of $H^1 - H^0$ is dependent on Φ^1_k.

We mention these expressions because of the role they play in determining orthogonality conditions for eigenfunctions of linear problems.

3.2 Applications of the Singular Integral Transform

In order to see the applications of the singular integral transform, we present two possible approaches. The first is a direct use of this transform for a specific Φ, while the second considers a singular integral transform related to another system of equations: the Gel'fand-Dikii hierarchy .

3.2.1 Integral Transforms and 2×2 Matrix Recurrence Relation

In chapter 2, we looked at a differential system (2.3.32) consisting of $P_n(x)$. This system was equally valid for $\epsilon_n(x)$ as well, although it was not necessary to consider it, since we were only interested in $P_n(x)$. Thus, we consider a general polynomial system, where Φ is a 2×2 matrix consisting of consisting of a polynomial P_n and a Laurent expansion ϵ_n.

$$\Phi_n(x) = \begin{pmatrix} P_n(x) & \epsilon_n(x) \\ P_{n-1}(x) & \epsilon_{n-1}(x) \end{pmatrix}. \tag{3.2.1}$$

In comparison with Φ_k, the x variable in $\Phi_n(x)$ replaces the spectral parameter k, hence the tilde, which was an arbitrary shift for Φ_k now becomes a shift in the discrete variable n. Initially our interest lies with the 2×2 matrix interpolating measure Λ and whether any relations can be derived between the individual components. This will allow us some freedom when considering initial conditions for this example. While we treat the polynomial route as a possible route to follow, we don't pretend to have made great progress in this direction.

We use the specific Lax acquired from the Laguerre method (2.3.25), the recurrence relation,

$$\begin{aligned} \Phi_{n+1} &= \begin{pmatrix} x - S_n & -R_n \\ 1 & 0 \end{pmatrix} \Phi_n \\ &= (x\sigma_+ + Q_n)\Phi_n \end{aligned} \tag{3.2.2}$$

where

$$\sigma_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad Q_n = \begin{pmatrix} -S_n & -R_n \\ 1 & 0 \end{pmatrix}. \tag{3.2.3}$$

Upon comparison with the linear relation $\Phi_{k+1} = (kJ + Q)\Phi_k$, we now have a specific value for J, Q and thus an expression for Q (3.1.20) in terms of $H^1 - H^0$, an integral transform. Through the use of this example we expect transforms to exist for the recurrence coefficients R_n and S_n.

Since we are already looking at the consistency between the transforms and the recurrence relation we introduce (3.2.1) into (3.1.19),

$$H^1 - H^0 = \int \Phi_n^1(x) d\Lambda^{10}(x)(\Phi_n^0(x))^{-1}$$

which leads to the following expression,

$$\begin{aligned} H^1 - H^0 &= -\int \begin{pmatrix} P_n^1 & \epsilon_n^1 \\ P_{n-1}^1 & \epsilon_{n-1}^1 \end{pmatrix} \begin{pmatrix} d\Lambda_{11} & d\Lambda_{12} \\ d\Lambda_{21} & d\Lambda_{22} \end{pmatrix} \begin{pmatrix} \epsilon_{n-1}^0 & -\epsilon_n^0 \\ -P_{n-1}^0 & P_n^0 \end{pmatrix} \frac{1}{h_{n-1}^0} \\ &= \begin{pmatrix} a_n^1 - a_n^0 & b_n^1 - b_n^0 \\ c_n^1 - c_n^0 & d_n^1 - d_n^0 \end{pmatrix} \end{aligned}$$

(where we omit the (x) for convenience) and with

$$H_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}. \tag{3.2.4}$$

Making further use of this form of H_n, we consider (3.1.20) simplified:

$$\begin{aligned} Q_n &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} - \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \text{invariant} \\ &= \begin{pmatrix} a_n - a_{n+1} & b_n \\ -c_{n+1} & 0 \end{pmatrix} + \text{invariant} \\ &= \begin{pmatrix} -S_n & -R_n \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{3.2.5}$$

Of these relations, the easiest to deal with are the corresponding integrals for c_n and d_n. Unfortunately we learn nothing about the composition of d_n from Q, so can say nothing about $d_n^1 - d_n^0$. Since $c_{n+1} = -1$, we can take $c^1 - c^0 = 0$ and then evaluate the corresponding integral. Through the use of (3.1.26), we can derive two equations for c_n:

$$\begin{aligned}\int \left(P_{n-1}^1(d\Lambda_{11}\epsilon_{n-1}^0 - d\Lambda_{12}P_{n-1}^0) + \epsilon_{n-1}^1(d\Lambda_{21}\epsilon_{n-1}^0 - d\Lambda_{22}P_{n-1}^0)\right) &= 0 \\ \int \left(P_{n-1}^0(d\Lambda_{11}\epsilon_{n-1}^1 - d\Lambda_{12}P_{n-1}^1) + \epsilon_{n-1}^0(d\Lambda_{21}\epsilon_{n-1}^1 - d\Lambda_{22}P_{n-1}^1)\right) &= 0\end{aligned}$$

and taking the difference between the two leaves:

$$\begin{aligned}\int \left(P_{n-1}^1\epsilon_{n-1}^0(d\Lambda_{11} + d\Lambda_{22}) - \epsilon_{n-1}^1P_{n-1}^0(d\Lambda_{22} + d\Lambda_{11})\right) &= 0 \\ \Rightarrow \int (P_{n-1}^1\epsilon_{n-1}^0 - \epsilon_{n-1}^1P_{n-1}^0)(d\Lambda_{11} + d\Lambda_{22}) &= 0. \end{aligned} \tag{3.2.6}$$

From this integral it would be fair to assume that

$$d\Lambda_{11} + d\Lambda_{22} = 0. \tag{3.2.7}$$

This relation gives us a dependency between $d\Lambda_{11}$ and $d\Lambda_{22}$, so we now take a look at the remaining relations to determine if any further dependency can be derived. So we consider the relation formed from b_n,

$$b_n^1 - b_n^0 = -\frac{1}{h_{n-1}^0}\int \left(P_n^1(d\Lambda_{12}P_n^0 - d\Lambda_{11}\epsilon_n^0) + \epsilon_n^1(d\Lambda_{22}P_n^0 - d\Lambda_{21}\epsilon_n^0)\right)$$

where this can be reduced using (3.2.7) and the fact that $b_n = -R_n$

$$R_n^1 - R_n^0 = -\frac{1}{h_{n-1}^0}\int \left(d\Lambda_{11}(P_n^1\epsilon_n^0 + P_n^0\epsilon_n^1) + d\Lambda_{21}\epsilon_n^1\epsilon_n^0 - d\Lambda_{12}P_n^1P_n^0\right). \tag{3.2.8}$$

This is an integral transform for the recurrence coefficient R_n, we can also consider the inverse of this expression by using (3.1.26), but we learn nothing new, so we consider two relations for $a_n^1 - a_n^0$ instead.

$$a_n^1 - a_n^0 = -\frac{1}{h_{n-1}^0}\int \left(P_n^1(d\Lambda_{11}\epsilon_{n-1}^0 - d\Lambda_{12}P_{n-1}^0) + \epsilon_n^1(d\Lambda_{21}\epsilon_{n-1}^0 - d\Lambda_{22}P_{n-1}^0)\right) \tag{3.2.9a}$$

$$a_n^1 - a_n^0 = -\frac{1}{h_{n-1}^1}\int \left(P_n^0(d\Lambda_{11}\epsilon_{n-1}^1 - d\Lambda_{12}P_{n-1}^1) + \epsilon_n^0(d\Lambda_{21}\epsilon_{n-1}^1 - d\Lambda_{22}P_{n-1}^1)\right) \tag{3.2.9b}$$

Taking the difference between these two equations leaves:

$$\begin{aligned} 0 &= \int \frac{1}{h^0_{n-1}h^1_{n-1}} \left(d\Lambda_{11}(h^1_{n-1}P^1_n\epsilon^0_{n-1} - h^0_{n-1}P^0_n\epsilon^1_{n-1}) - d\Lambda_{22}(h^1_{n-1}P^0_{n-1}\epsilon^1_n - h^0_{n-1}P^1_{n-1}\epsilon^0_n)\right) \\ &- \int \frac{1}{h^0_{n-1}h^1_{n-1}} \left(d\Lambda_{12}(h^1_{n-1}P^1_nP^0_{n-1} - h^0_{n-1}P^1_{n-1}P^0_n) + d\Lambda_{21}(h^1_{n-1}\epsilon^1_n\epsilon^0_{n-1} - h^0_{n-1}\epsilon^0_n\epsilon^1_{n-1})\right) \end{aligned} \tag{3.2.10}$$

from which further reductions cannot be made. However we know that $S_n = a_{n+1} - a_n$ so by taking $S^1_n - S^0_n = (a_{n+1} - a_n)^1 - (a_{n+1} - a_n)^0$ and consider (3.2.9a), we have the following expression for S_n:

$$\begin{aligned} S^1_n - S^0_n &= -\frac{1}{h^0_n}\int \left(P^1_{n+1}(d\Lambda_{11}\epsilon^0_n - d\Lambda_{12}P^0_n) + \epsilon^1_{n+1}(d\Lambda_{21}\epsilon^0_n - d\Lambda_{22}P^0_n)\right) \\ &\quad + \frac{1}{h^0_{n-1}}\int \left(\epsilon^0_{n-1}(d\Lambda_{11}P^1_n + d\Lambda_{21}\epsilon^1_n) - P^0_{n-1}(d\Lambda_{12}P^1_n - d\Lambda_{22}\epsilon^1_n)\right) \end{aligned}$$

an integral transform for S_n. We try to reduce the relation further by rearranging the second term so that we can introduce the recurrence relation (1.1.25) (which is satisfied by both P_{n-1} and ϵ_{n-1}), and including (3.2.7), leads to

$$\begin{aligned} S^1_n - S^0_n &= -\frac{1}{h^0_n}\int \left(d\Lambda_{11}(P^1_{n+1}\epsilon^0_n + P^0_n\epsilon^1_{n+1}) - d\Lambda_{12}P^1_{n+1}P^0_n + d\Lambda_{21}\epsilon^1_{n+1}\epsilon^0_n\right) \\ &\quad + \frac{1}{h^0_n}\int (x - S^0_n)\left(d\Lambda_{11}(P^1_n\epsilon^0_n + P^0_n\epsilon^1_n) - d\Lambda_{12}P^1_nP^0_n + d\Lambda_{21}\epsilon^1_n\epsilon^0_n\right) \\ &\quad - \frac{1}{h^0_n}\int \left(d\Lambda_{11}(P^1_n\epsilon^0_{n+1} + P^0_{n+1}\epsilon^1_n) - d\Lambda_{12}P^1_nP^0_{n+1} + d\Lambda_{21}\epsilon^0_n\epsilon^1_{n+1}\right). \end{aligned} \tag{3.2.11}$$

This can simplify by bringing in (3.2.8) and the inverse property of (3.2.9)

$$\begin{aligned} S^1_nR^0_n - S^0_nR^1_n &= \frac{1}{h^0_{n-1}}\int x\left(d\Lambda_{11}(P^1_n\epsilon^0_n + P^0_n\epsilon^1_n) + d\Lambda_{21}\epsilon^1_n\epsilon^0_n - d\Lambda_{12}P^1_nP^0_n\right) \\ &- \left(\frac{h^1_n + h^0_n}{h^1_nh^0_{n-1}}\right)\int \left(d\Lambda_{11}(P^1_n\epsilon^0_{n+1} + P^0_{n+1}\epsilon^1_n) - d\Lambda_{12}P^1_nP^0_{n+1} + d\Lambda_{21}\epsilon^0_n\epsilon^1_{n+1}\right) \end{aligned} \tag{3.2.12}$$

Although we have have acquired an integral transform for both R_n and S_n and derived a relation between two of the interpolating measures $d\Lambda_{11}$ and $d\Lambda_{22}$, we still don't have enough information. Especially since we know nothing about how the $d\Lambda_{12}$ and $d\Lambda_{21}$ are related. So to continue this problem we would also have to consider the differential Lax from Chapter 2 (2.3.32).

The differential Lax equation consisted of the following form

$$\begin{aligned}\partial_x \Phi_n(x) &= \frac{1}{Wh_{n-1}} \begin{pmatrix} \Omega_n(x) & -\Theta_n(x) \\ \Theta_{n-1}(x) & -(\Omega_n(x) + V(x)h_{n-1}) \end{pmatrix} \Phi_n(x) \quad , \\ \text{where} \quad \Phi_n(x) &= \begin{pmatrix} P_n(x) & \epsilon_n(x) \\ P_{n-1}(x) & \epsilon_n(x) \end{pmatrix}\end{aligned} \tag{3.2.13}$$

and by comparison with $\tilde{\Phi}_k = (kJ + Q)\Phi_k$, we can take

$$\partial_x \Phi_n(x) = \left[-\frac{V}{W} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{Wh_{n-1}} \begin{pmatrix} \Omega_n(x) & -\Theta_n(x) \\ \Theta_{n-1}(x) & -\Omega_n(x) \end{pmatrix} \right] \Phi_n(x), \tag{3.2.14}$$

where $\frac{V}{W}$ is a polynomial in x. While it is not necessary to state specifics for the recurrence relation (since all single-variable polynomials satisfy a three term recurrence relation), this is not the case for the differential equation. It is necessary to state a weight so that the values of V, W, Ω and Θ can be determined.

3.2.2 The Singular Integral Transform of the Lattice Gel'fand-Dikii Hierarchy

We will consider a particular class of $N \times N$ matrix problems associated with singular integral transforms of a specific type. In the case $N = 2$, this reduces to a singular integral transform associated with the lattice KdV case, which in turn is connected to the Volterra lattice. It is this latter example that we will associate with some special case of orthogonal polynomials, but first we will treat the general $N \times N$ case of this specific reduction, which is associated with so-called lattice Gel'fand-Dikii hierarchy of equations [126]. Thus, we present the Gel'fand-Dikii matrix L_k, where

$$L_k = \begin{pmatrix} p - \tilde{u}_{00} & 1 & & & \emptyset \\ -\tilde{u}_{10} & p & 1 & & \\ \vdots & & \ddots & \ddots & \\ -\tilde{u}_{N-2,0} & \emptyset & & p & 1 \\ k^N + T & \omega^{N-2} u_{0,N-2} & & \omega u_{0,1} & p + u_{00} \end{pmatrix} \tag{3.2.15}$$

and $T = \omega^{N-1} u_{0,N-1} - \tilde{u}_{N-1,0}$. This matrix together with the linear equation

$$\begin{aligned} \tilde{\Phi}_k D_k &= L_k \Phi_k \qquad (3.2.16) \\ \text{where } D_k &= \operatorname{diag}(p + \omega k, p + \omega^2 k, \ldots, p + \omega^N k) \end{aligned}$$

(where $\omega = e^{\frac{2\pi i}{N}}$ an Nth root of unity) forms part of a Lax representation for a coupled set of lattice equations exhibited in [126]. If we consider the determinant of this Lax, we have

$$\begin{aligned} \det(\tilde{\Phi}_k) \det(D_k) &= \det(L_k) \det(\Phi_k) \\ \det(\tilde{\Phi}_k)(p^N + e^{\pi i (N-1)} k^N) &= (p^N - (-k)^N) \det(\Phi_k) \qquad (3.2.17) \end{aligned}$$

which implies that $\det(\Phi_k)$ is a constant.

The singular integral transform for this equation is

$$\Phi_k^0 + \int_{C^{10}} \Phi_l^1 d\Lambda^{10}(l) \frac{(\Phi_l^0)^{-1} \Phi_k^0}{k^N - l^N} = \Phi_k^1 \tag{3.2.18}$$

and we want to write this equation in a more explicit form, in terms of the column vectors of the matrix Φ_k, where $\Phi_k = (\boldsymbol{\phi}_{k_1}, \ldots, \boldsymbol{\phi}_{k_N})$, in which the vectors $\boldsymbol{\phi}_{k_j}$ ($k_j = \omega^{j-1} k$), form a set of N independent vector solutions of (3.2.16).

From (3.2.17) we can without loss of generality set $\det(\Phi_k) = 1$, in which case we can express the entries of the matrix $(\Phi_l^0)^{-1}$ as $N \times N$ determinants consisting of the column vectors $\boldsymbol{\phi}_{l_i}$ $(i = 1, \dots, N)$, as follows

$$((\Phi_l^0)^{-1})_{ij} = \begin{vmatrix} \boldsymbol{\phi}_{l_1}^0 & \cdots & \overset{i\downarrow}{\boldsymbol{e}_j} & \cdots & \boldsymbol{\phi}_{l_N}^0 \end{vmatrix} \tag{3.2.19}$$

where the ith column of the determinant is replaced with e_j and the transpose of the unit vector e_j is $e_j^T = (0, \dots, 0, 1, 0, \dots, 0)$. The matrix product of $(\Phi_l^0)^{-1}\Phi_k^0$ using Cramer's rule can then be expressed as

$$((\Phi_l^0)^{-1}\Phi_k^0)_{ij} = \begin{vmatrix} \boldsymbol{\phi}_{l_1}^0 & \cdots & \overset{i\downarrow}{\boldsymbol{\phi}_{k_j}^0} & \cdots & \boldsymbol{\phi}_{l_N}^0 \end{vmatrix} \tag{3.2.20}$$

replacing the ith column with Φ_k^0. Then our scalar integral transform becomes

$$\begin{aligned} (\boldsymbol{\phi}_{k_1}^1, \dots, \boldsymbol{\phi}_{k_N}^1)_{.j} &= (\boldsymbol{\phi}_{k_1}^0, \dots, \boldsymbol{\phi}_{k_N}^0)_{.j} \\ &+ \sum_{p,q=1}^{N} \int_{C_{p,q}} (\boldsymbol{\phi}_{l_1}^1, \dots, \boldsymbol{\phi}_{l_N}^1)_{.p} (d\boldsymbol{\Lambda}_{p,q}) \frac{\begin{vmatrix} \boldsymbol{\phi}_{l_1}^0 & \cdots & \overset{q\downarrow}{\boldsymbol{\phi}_{k_j}^0} & \cdots & \boldsymbol{\phi}_{l_N}^0 \end{vmatrix}}{k^N - l^N} \end{aligned} \tag{3.2.21}$$

where the $\boldsymbol{\phi}_{k_j}^0$ is the qth column in the determinant and C_{pq} denotes the contour of integration associated with the measure $d\Lambda_{pq}$. We first consider only the column $\boldsymbol{\phi}_k$ (i.e. set $j = 1$) which gives a formula from which all the other j values can be derived as well. Identifying $k_j = \omega^{j-1}k$

$$\boldsymbol{\phi}_{\omega^{j-1}k}^1 = \boldsymbol{\phi}_{\omega^{j-1}k}^0 + \sum_{p,q=1}^{N} \int_{C_{p,q}} \boldsymbol{\phi}_{\omega^{p-1}l}^1 d\boldsymbol{\Lambda}_{pq}(l) \frac{\begin{vmatrix} \boldsymbol{\phi}_l^0 & \cdots & \overset{q\downarrow}{\boldsymbol{\phi}_{\omega^{j-1}k}^0} & \cdots & \boldsymbol{\phi}_{\omega^{N-1}l}^0 \end{vmatrix}}{k^N - l^N} \tag{3.2.22}$$

we can without loss of generality write this as an integral equation for $\boldsymbol{\phi}_k$. We make the change $l \to \omega^{1-p}l$ so that we can reevaluate the sum over p, by a single object

$$\boldsymbol{\phi}_k^1 = \boldsymbol{\phi}_k^0 + \sum_{q=1}^{N} \int_{C_q} \boldsymbol{\phi}_l^1 d\boldsymbol{\lambda}_q(l) \frac{\begin{vmatrix} \boldsymbol{\phi}_l^0 & \cdots & \overset{q\downarrow}{\boldsymbol{\phi}_{\omega^{j-1}k}^0} & \cdots & \boldsymbol{\phi}_{\omega^{N-1}l}^0 \end{vmatrix}}{k^N - l^N} \tag{3.2.23}$$

where $d\boldsymbol{\lambda}_q$ represents a sum over p of the $d\boldsymbol{\Lambda}_{p,q}$.

By applying the Lax form to the vector reduction of the integral transform and writing the Lax matrix as $L_k = pI + \Sigma_{k^N} + Q$ (where I is the unit matrix, Σ is the matrix with 1 on the upper semi-diagonal and k^N in the $(N,1)$ entry, and where Q is the matrix containing only a first column and a last row), we can identify that

$$Q_{\cdot 1} = \widetilde{\boldsymbol{h}}_N - E_{N,1}\boldsymbol{h}_1, \tag{3.2.24}$$

(i.e. the first column of the matrix Q) where $E_{N,1}$ is the matrix with the only nonzero entry being the $(N,1)$ entry, which is equal to 1 and with

$$Q_{\cdot i} = -E_{N,1}\boldsymbol{h}_i \text{ for } i = 2, ..., N \tag{3.2.25}$$

where h_i is the ith vector of h. Here $\boldsymbol{h}_i$ $(i = 1, ..., N)$ denote a collection of N, N-component vectors (with components $(\boldsymbol{h}_i)_j$) which transform as:

$$(\boldsymbol{h}_i^1)_j - (\boldsymbol{h}_i^0)_j = \sum_{q=1}^{N} \int_{C_q} d\boldsymbol{\lambda}_q(l)\rho_{l,q}(\boldsymbol{\phi}_l^1)_j |\boldsymbol{\phi}_l^0, \ldots, \overset{q\downarrow}{\boldsymbol{e}_i}, \ldots, \boldsymbol{\phi}_{\omega^{N-1}l}^0| \tag{3.2.26}$$

in which the vector $\boldsymbol{e}_i$ enters at the qth place in the determinant. The factors $\rho_{l,q}$ are represented by

$$\rho_{l,q} = \frac{(p+\omega l)^n}{(p+\omega^q l)^n}, \tag{3.2.27}$$

and arise from the diagonal matrix D_k after separating the n-dependent factor from the measure $d\lambda_q(l)$. While (3.2.24) and (3.2.25) are not difficult to prove, it is notationally technical, so we leave the general case and now we consider a specific example.

3.2.3 A KdV Integral Transform

When we consider the Gel'fand-Dikii matrix for $N = 2$ we have the KdV equation. Thus from (3.2.21) for $N = 2$ we can consider the following singular integral transform for the lattice KdV

$$\boldsymbol{\phi}_k^1 = \boldsymbol{\phi}_k^0 + \int_{\boldsymbol{C}} (\boldsymbol{\phi}_l^1, \boldsymbol{\phi}_{-l}^1) \begin{pmatrix} d\boldsymbol{\Lambda}_{11}(l) & d\boldsymbol{\Lambda}_{12}(l) \\ d\boldsymbol{\Lambda}_{21}(l) & d\boldsymbol{\Lambda}_{22}(l) \end{pmatrix} \frac{\begin{pmatrix} |\boldsymbol{\phi}_k^0\ \boldsymbol{\phi}_{-l}^0| \\ |\boldsymbol{\phi}_l^0\ \boldsymbol{\phi}_k^0| \end{pmatrix}}{k^2 - l^2}, \tag{3.2.28}$$

where $\boldsymbol{C}$ denotes a matrix contour, each entry of which goes with the corresponding entry of the matrix measure. We have also chosen one column with $k \to -k$ and the numerator of the fraction is a column vector consisting of two determinants. We can expand this expression to the following

$$\begin{aligned}\boldsymbol{\phi}_k^1 = \boldsymbol{\phi}_k^0 \quad &+ \int_{C_{11}} \boldsymbol{\phi}_l^1 d\boldsymbol{\Lambda}_{11}(l) \frac{|\boldsymbol{\phi}_k^0\, \boldsymbol{\phi}_{-l}^0|}{k^2-l^2} + \int_{C_{12}} \boldsymbol{\phi}_l^1 d\boldsymbol{\Lambda}_{12}(l) \frac{|\boldsymbol{\phi}_l^0\, \boldsymbol{\phi}_k^0|}{k^2-l^2} \\ &+ \int_{-C_{21}} \boldsymbol{\phi}_l^1 d\boldsymbol{\Lambda}_{21}(-l) \frac{|\boldsymbol{\phi}_k^0\, \boldsymbol{\phi}_l^0|}{k^2-l^2} + \int_{-C_{22}} \boldsymbol{\phi}_l^1 d\boldsymbol{\Lambda}_{22}(-l) \frac{|\boldsymbol{\phi}_{-l}^0\, \boldsymbol{\phi}_k^0|}{k^2-l^2}\end{aligned} \tag{3.2.29}$$

where we have let $l \to -l$ in the last two terms so that the Φ_l all have the same order. This has the consequence that the sign of the contour changes for these two cases.

Now, from the KdV's Lax representation (2.1.13) we may consider the following 2×2 matrix

$$\tilde{\Phi}_k D_k = \left[(k^2-p^2) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ p+u \end{pmatrix} \begin{pmatrix} p-\tilde{u} & 1 \end{pmatrix} \right] \Phi_k \tag{3.2.30}$$

where we have introduced a specific Φ and the normalization D_k,

$$\Phi_k = \begin{pmatrix} u_{k1} & u_{-k1} \\ u_{k2} & u_{-k2} \end{pmatrix} \quad , \quad D_k = \begin{pmatrix} p-k & 0 \\ 0 & p+k \end{pmatrix} \tag{3.2.31}$$

If we take the determinant of both sides, then like the general case we find that the determinant of Φ_k is a constant. The introduction of the normalization means that when we consider the shifted singular integral transform (3.2.18) with D_k and let (3.2.30) be written as $\tilde{\Phi}_k D_k = ((k^2-p^2)J + Q)\Phi_k$ then following the same path as Section 3.1.3, we have

$$\begin{aligned} \tilde{\Phi}_k^1 D_k &= \tilde{\Phi}_k^0 D_k + \int_{C_{10}} \tilde{\Phi}_l^1 D_l D_l^{-1} d\tilde{\boldsymbol{\Lambda}}^{10}(l) \frac{(\tilde{\Phi}_l^0)^{-1} \tilde{\Phi}_k^0 D_k}{k^2-l^2} \\ \Rightarrow ((k^2-p^2)J + Q^1)\Phi_k^1 &= ((k^2-p^2)J + Q^0)\Phi_k^0 + \int_{C_{10}} \left(\tilde{\Phi}_l^1 d\tilde{\boldsymbol{\Lambda}}^{10}(l) (\tilde{\Phi}_l^0)^{-1} \right) J \Phi_k^0 \\ &\quad + \int_{C_{10}} ((l^2-p^2)J + Q^1)\Phi_l^1 D_l^{-1} d\tilde{\boldsymbol{\Lambda}}^{10}(l) D_l \frac{(\Phi_l^0)^{-1}\Phi_k^0}{k^2-l^2} \end{aligned} \tag{3.2.32}$$

this has the consequence that

$$
\begin{aligned}
& d\tilde{\mathbf{\Lambda}}(l) = D_l d\mathbf{\Lambda}(l) D_l^{-1} \\
\Rightarrow \quad & d\mathbf{\Lambda}(l) = D_l^n d\mathbf{\Lambda}^0(l) D_l^{-n}
\end{aligned} \tag{3.2.33}
$$

and the rest of the derivation would lead to (3.1.20).

Returning to (3.2.29) we can re-express this equation, by grouping the determinants together and introducing a normalization factor, thus:

$$
\begin{aligned}
\boldsymbol{\phi}_k^1 = \boldsymbol{\phi}_k^0 \quad & + \quad \int_{\mathbf{\Gamma}_1} \boldsymbol{\phi}_l^1 (d\mathbf{\Lambda}_{11}^0(l) - d\mathbf{\Lambda}_{22}^0(-l)) \frac{|\boldsymbol{\phi}_k^0 \; \boldsymbol{\phi}_{-l}^0|}{k^2 - l^2} \\
& + \quad \int_{\mathbf{\Gamma}_2} \boldsymbol{\phi}_l^1 \left(\frac{p-l}{p+l} \right)^n (d\mathbf{\Lambda}_{12}^0(l) - d\mathbf{\Lambda}_{21}^0(-l)) \frac{|\boldsymbol{\phi}_l^0 \; \boldsymbol{\phi}_k^0|}{k^2 - l^2}.
\end{aligned} \tag{3.2.34}
$$

Looking at the explicit derivation of the determinant allows us to express (3.2.34) in terms of the KdV function u, so given that $(p-k)\tilde{u}_{k1} = (p-\tilde{u})u_{k1} + u_{k2}$ and

$$
\begin{vmatrix} u_{l1} & u_{k1} \\ (p-l)\tilde{u}_{l1} - (p-\tilde{u})u_{l1} & (p-k)\tilde{u}_{k1} - (p-\tilde{u})u_{k1} \end{vmatrix} = (p-k)\tilde{u}_{k1} u_{l1} - (p-l)\tilde{u}_{l1} u_{k1}
$$

then we have the following singular integral transform for u

$$
\begin{aligned}
u_{k1}^1 = u_{k1}^0 \quad & + \quad \int_{\mathbf{\Gamma}_1} u_{l1}^1 d\boldsymbol{\lambda}_1(l) \frac{(p-k)\tilde{u}_{k1}^0 u_{-l1}^0 - (p+l)\tilde{u}_{-l1}^0 u_{k1}^0}{k^2 - l^2} \\
& + \quad \int_{\mathbf{\Gamma}_2} u_{l1}^1 d\boldsymbol{\lambda}_2(l) \left(\frac{p-l}{p+l} \right)^n \frac{(p-k)\tilde{u}_{k1}^0 u_{l1}^0 - (p-l)\tilde{u}_{l1}^0 u_{k1}^0}{k^2 - l^2}.
\end{aligned} \tag{3.2.35}
$$

While the integral transform for the KdV is an important result and useful in providing further connections between the KdV and inverse scattering, there is no obvious connection to orthogonal polynomials here, except that the KdV can be reduced to a linear problem for Volterra, which satisfies the recurrence relation for orthogonal polynomials. Thus, we consider a singular integral transform for orthogonal polynomials by reducing KdV to Volterra.

3.3 An Integral Transform for Hermite Polynomials

When we consider semi-classical and classical orthogonal polynomials, they are looked at as two separate classes of orthogonal polynomials, where their only connections lie in the family name and consequently have similar weight functions. However using this approach it should be possible to relate a classical orthogonal polynomial with a semi-classical orthogonal polynomial (of the same family), by using a transform integral. While the possibility of a transform existing for relating a classical orthogonal polynomial to a semi-classical polynomial of the same family seems possible, that does not seem likely for orthogonal polynomials of different families. Thus we consider the problem of what kind of interpolating measure would allow a semi-classical Hermite polynomial to be transformed into a classical Hermite polynomial and vice-versa?

In this instance we consider the classical orthogonal polynomials P_n^0 and the semi-classical orthogonal polynomials P_n^1, where no bi-orthogonality exists between them. The classical Hermite polynomials have an even weight function, hence they have a recurrence relation of the form:

$$xP_n(x) = P_{n+1}(x) + R_nP_{n-1}(x), \qquad R_n = \frac{h_n}{h_{n-1}}. \tag{3.3.1}$$

and a differential relation defined as:

$$\partial_x P_n(x) = 2aR_nP_{n-1}(x). \tag{3.3.2}$$

We are interested in the dressing transform between this classical case and the next semi-classical case where the weight function is still even (so that the structure of the recurrence relation is preserved).

3.3.1 Reduction to the Volterra Equation

Having considered the integral transform for the KdV, we will now consider the integral transform for Volterra, but first we must consider the gauge transformation that leads to the eigenvalue problem for the Volterra equation.

Although we have an eigenfunction u_{k1} from the KdV, it isn't the eigenfunction we require for this transformation. Thus we introduce another eigenfunction $u_k(a)$ for an arbitrary parameter a (not necessarily p) which is related with u_{k1} via a set of relations (where the framework arises from [134, 123, 141])

$$(p-k)\widetilde{u}_k(a) = (p-a)u_k(a) + \widetilde{V}(a)u_{k1} \tag{3.3.3a}$$

$$(p+k)\underset{\sim}{u}_k(a) = (p+a)u_k(a) - \underset{\sim}{V}(a)u_{k1} \tag{3.3.3b}$$

and this simplifies if either $a = p$ or $a = -p$. So we rewrite the equations for $a = -p$

$$(p-k)\widetilde{u}_k(-p) = 2pu_k(-p) + \widetilde{V}(-p)u_{k1}, \tag{3.3.4a}$$

$$(p+k)\underset{\sim}{u}_k(-p) = -\underset{\sim}{V}(-p)u_{k1}, \tag{3.3.4b}$$

and then eliminate u_{k1} by substitution

$$(p-k)\widetilde{u}_k(-p) = 2pu_k(-p) - \frac{\widetilde{V}(-p)}{\underset{\sim}{V}(-p)}(p+k)\underset{\sim}{u}_k(-p). \tag{3.3.5}$$

Then, we make the substitution $u_k(-p) = \left(\frac{p+k}{p-k}\right)^{\frac{n}{2}}\psi$ and we get:

$$\widetilde{\psi} = \frac{2p}{\sqrt{(p^2-k^2)}}\psi - \frac{\widetilde{V}(-p)}{\underset{\sim}{V}(-p)}\underset{\sim}{\psi}, \tag{3.3.6}$$

the linear problem (recurrence relation) for the polynomials. Upon comparison with the recurrence relation for Hermite orthogonal polynomials we would have

$$R = \frac{\widetilde{V}(-p)}{\underset{\sim}{V}(-p)} \quad , \quad x = \frac{2p}{\sqrt{(p^2-k^2)}}. \tag{3.3.7}$$

By using this transformation it is possible to transform (3.2.35) into a singular integral transform for Volterra and from Volterra to orthogonal polynomials. Thus, we present the following singular integral transform for polynomials P_n

$$P_n^1(x) + \int_C d\mu(y)P_n^1(y)\frac{W_n^0(x,y)}{x^2-y^2} = P_n^0(x) \tag{3.3.8}$$

where

$$W_n^0(x,y) = \frac{1}{h_{n-1}^0}(yP_n^0(x)P_{n-1}^0(y) - xP_n^0(y)P_{n-1}^0(x)), \tag{3.3.9a}$$
$$= \frac{1}{h_{n-2}^0}(P_n^0(x)P_{n-2}^0(y) - P_n^0(y)P_{n-2}^0(x)). \tag{3.3.9b}$$

and the term $d\mu(y)$ is the transform measure and $W_n^0(x,y)$ has the form of a discrete Wronskian (a Casorati determinant). The second of these two expressions has just used (3.3.1) to eliminate the x and y.

The relation (3.3.8) is an analogue of (3.2.35), where we assume that one of the integrals does not contribute namely the one consisting of u_{-l}. We take a chance with the omission of one of the integrals, since the form of (3.3.8) is now comparable with the original linear integral transform (3.1.5) presented in Section 3.1. Hence we can consider this to be one possible choice, which we now investigate.

3.3.2 Recurrence Relations

The form of the recurrence relation for the semi-classical Hermite polynomials is the same as the classical Hermite polynomials, the only difference between the two is the polynomials generated and the recurrence coefficient. As such for this transform, we are only interested in how the recurrence coefficient R_n^1 from a new class of polynomials is related to the original recurrence coefficient R_n^0, with particular interest in the interpolating measure (which governs the transform).

Since we are interested in bringing the recurrence relation (3.3.1) into (3.3.8) we multiply it by x:

$$xP_n^1(x) + \int_C d\mu(y) y P_n^1(y) \frac{x}{y} \frac{W_n^0(x,y)}{x^2 - y^2} = xP_n^0(x), \tag{3.3.10}$$

then we need to find an expression for $\frac{x}{y}W_n^0(x,y)$. This requires the following manipulation of (3.3.9a):

$$\begin{aligned}\frac{x}{y}W_n^0(x,y) &= \frac{1}{h_{n-1}^0}\left(xP_n^0(x)P_{n-1}^0(y) - yP_n^0(y)P_{n-1}^0(x) - \left(\frac{x^2}{y}-y\right)P_n^0(y)P_{n-1}^0(x)\right)\\ &= \frac{1}{h_n^0}[xP_n^0(x)(yP_n^0(y)-P_{n+1}^0(y)) - yP_n^0(y)(xP_n^0(x)-P_{n+1}^0(x))]\\ &\quad -\frac{x^2-y^2}{h_{n-1}^0 y}P_n^0(y)P_{n-1}^0(x)\\ &= W_{n+1}^0(x,y) - \frac{x^2-y^2}{y}\frac{P_n^0(y)P_{n-1}^0(x)}{h_{n-1}^0} \end{aligned} \tag{3.3.11}$$

Alternatively if we consider the expansion of $\frac{y}{x}W_n^0(x,y)$ instead ($x \leftrightarrow y$ and $W_n(x,y) = -W_n(y,x)$), we would get the following equation:

$$\frac{y}{x}W_n^0(x,y) = W_{n+1}^0(x,y) + \frac{y^2-x^2}{x}\frac{P_n^0(x)P_{n-1}^0(y)}{h_{n-1}^0},$$

which can be rearranged to give another expression for $\frac{x}{y}W_n^0(x,y)$.

$$\begin{aligned}\frac{x}{y}W_n^0(x,y) &= W_{n-1}^0(x,y) + \frac{x^2-y^2}{y}\frac{P_{n-1}^0(x)P_{n-2}^0(y)}{h_{n-2}^0}\\ &= W_{n-1}^0(x,y) + \frac{x^2-y^2}{y}\frac{P_{n-1}^0(x)(yP_{n-1}^0(y)-P_n^0(y))}{h_{n-1}^0}. \end{aligned} \tag{3.3.12}$$

We can then expand (3.3.10) using the recurrence relation (3.3.1):

$$\begin{aligned}P_{n+1}^1(x) + R_n^1 P_{n-1}^1(x) &+ \int_C d\mu(y)(P_{n+1}^1(y) + R_n^1 P_{n-1}^1(y))\frac{x}{y}\frac{W_n^0(x,y)}{x^2-y^2}\\ &= P_{n+1}^0(x) + R_n^0 P_{n-1}^0(x)\end{aligned}$$

and then introduce the two forms of $\frac{x}{y}W_n^0(x,y)$, (3.3.11) and (3.3.12),

$$\begin{aligned}P_{n+1}^1(x) + R_n^1 P_{n-1}^1(x) &+ \int_C d\mu(y)P_{n+1}^1(y)\frac{W_{n+1}^0(x,y) - \left(\frac{x^2-y^2}{y}\frac{P_n^0(y)P_{n-1}^0(x)}{h_{n-1}^0}\right)}{x^2-y^2}\\ &+ \int_C d\mu(y)R_n^1 P_{n-1}^1(y)\frac{W_{n-1}^0(x,y) + \left(\frac{x^2-y^2}{y}\frac{P_{n-1}^0(x)(yP_{n-1}^0(y)-P_n^0(y))}{h_{n-1}^0}\right)}{x^2-y^2}\\ &= P_{n+1}^0(x) + R_n^0 P_{n-1}^0(x)\end{aligned}$$

from which we can expand and cancel terms. For instance the terms W^0_{n+1} and W^0_{n-1} cancel since they fit (3.3.8) with other terms and thus we have:

$$\begin{aligned} R^1_n P^0_{n-1}(x) - \int_C d\mu(y) P^1_{n+1}(y) \frac{\left(\frac{x^2-y^2}{y} \frac{P^0_n(y)P^0_{n-1}(x)}{h^0_{n-1}}\right)}{x^2-y^2} & \\ + \int_C d\mu(y) R^1_n P^1_{n-1}(y) \frac{\left(\frac{x^2-y^2}{y} \frac{P^0_{n-1}(x)(yP^0_{n-1}(y)-P^0_n(y))}{h^0_{n-1}}\right)}{x^2-y^2} &= R^0_n P^0_{n-1}(x) \end{aligned} \tag{3.3.13}$$

then cancelling $x^2 - y^2$ in the quotient terms and taking everything as a common factor of $P^0_{n-1}(x)$ leads to:

$$\begin{aligned} R^1_n - R^0_n &= \frac{1}{h^0_{n-1}} \left(\int_C d\mu(y) \frac{1}{y} P^1_{n+1}(y) P^0_n(y) \right. \\ &\quad \left. - \int_C d\mu(y) R^1_n P^1_{n-1}(y) \frac{1}{y} (yP^0_{n-1}(y) - P^0_n(y)) \right) \end{aligned} \tag{3.3.14}$$

then recombining terms of R^1_n and R^0_n (on the left and right sides), and simplifying the right side leads to:

$$\begin{aligned} & R^1_n \left(1 + \frac{1}{h^0_{n-1}} \int_C d\mu(y) P^1_{n-1}(y) P^0_{n-1}(y) \right) \\ = \; & R^0_n \left(1 + \frac{1}{h^0_n} \int_C d\mu(y) P^1_n(y) P^0_n(y) \right), \end{aligned} \tag{3.3.15}$$

so that we are simply left with:

$$\frac{R^1_n}{R^0_n} = \frac{\left(1 + \frac{1}{h^0_n} \int_C d\mu(y) P^1_n(y) P^0_n(y)\right)}{\left(1 + \frac{1}{h^0_{n-1}} \int_C d\mu(y) P^1_{n-1}(y) P^0_{n-1}(y)\right)} . \tag{3.3.16}$$

This can then be integrated to give

$$\frac{h^1_n}{h^0_n} = \left(1 + \frac{1}{h^0_n} \int_C d\mu(y) P^1_n(y) P^0_n(y) \right) c \tag{3.3.17}$$

where $c = 1$ is an integration constant (since we take $h_{-1} = 1$). This gives us a simple transform for h_n.

$$\int_C d\mu(y) P^1_n(y) P^0_n(y) = h^1_n - h^0_n \tag{3.3.18}$$

From this equation it can be shown that there is a realization of the integration over the interpolating measure $d\mu(y)$, which we take to have the form

$$\int_C d\mu(y) = \int_{C^1} w_n^1(y)dy - \int_{C^0} w_n^0(y)dy, \tag{3.3.19}$$

where $w_n^1(y)$ is the weight of semi-classical orthogonal polynomials. If we were to consider the case where the contour is the same for both weights, then we can set $C^1 = C^0$. There might be other choices for the interpolating measure, but we do not consider those other choices at this time.

3.3.3 A Differential Relation for a General Weight Function

When considering the integral transform for the differential equation, we note that the classical and semi-classical weights yield different equations, a consequence of the latter weight bringing in additional terms. As we have already mentioned (in Section 2.2.1), the size of the weight will then determine the number of terms to include in the differential equation. However for this particular case we will consider differential equations, whose solutions are not necessarily polynomials. Since the integral transform preserves the similarity constraint of the differential equation, we can view the compatibility of the recurrence relation and differential equation as a Lax pair, and hence an integrable systems problem. After considering a general setting, differential equations with polynomial solutions could also be looked at. Thus we present the following general expressions for a differential equation and it's transposed counterpart (since we can use the recurrence relation to reduce the size of any differential equation),

$$x\frac{d}{dx}P_n^0(x) = A_n^0(x)P_n^0(x) + B_n^0(x)P_{n-1}^0(x), \tag{3.3.20a}$$
$$x\frac{d}{dx}P_n^1(x) = A_n^1(x)P_n^1(x) + B_n^1(x)P_{n-1}^1(x), \tag{3.3.20b}$$

where the coefficients A^0, A^1 are even polynomials in x and the coefficients B^0, B^1 are odd polynomials in x

Since we are interested in a differential transform we consider $x\partial_x$ of (3.3.8):

$$\begin{aligned} x\partial_x P_n^1(x) &+ \int d\mu(y)P_n^1(y)(x\partial_x + \partial_y y)\left(\frac{W_n^0(x,y)}{x^2-y^2}\right) = x\partial_x P_n^0(x) \\ &- \int d\mu(y)P_n^1(y)\partial_y y\left(\frac{W_n^0(x,y)}{x^2-y^2}\right) \end{aligned}$$

where we assume that the derivative with respect to x can be brought into the integral. The second integral is a correction term and can be solved through separation by parts:

$$\begin{aligned} x\partial_x P_n^1(x) &+ \int d\mu(y)P_n^1(y)(x\partial_x + \partial_y y)\left(\frac{W_n^0(x,y)}{x^2-y^2}\right) = x\partial_x P_n^0(x) \\ &+ \int d\mu(y)y(\partial_y P_n^1(y) + \partial_y(\ln w^{10}(y)))\left(\frac{W_n^0(x,y)}{x^2-y^2}\right), \end{aligned} \tag{3.3.21}$$

where we have introduced $d\mu(y) = w^{10}(y)dy$ to allow for the differential of $d\mu(y)$ and we assume that $W_n^0(x,y)$ disappears at the boundary. We would expect any extra terms to appear through the interpolation measure $d\mu(y)$.

If we consider the derivative of the integral (3.3.21), we find that $(x\partial_x + \partial_y y)\frac{W_n^0(x,y)}{x^2-y^2}$ must be calculated, thus we let

$$(x\partial_x + \partial_y y)\left(\frac{W_n^0(x,y)}{x^2-y^2}\right) = \frac{(x\partial_x + y\partial_y)W_n^0(x,y)}{x^2-y^2} - \frac{W_n^0(x,y)}{x^2-y^2}, \tag{3.3.22}$$

which can be computed using the explicit form of $W_n^0(x,y)$ (3.3.9a) and the recurrence relation (3.3.1):

$$\begin{aligned} (x\partial_x + \partial_y y)\left(\frac{W_n^0(x,y)}{x^2-y^2}\right) &= \frac{1}{h_{n-1}^0}\Bigg[\ (yB_n^0(x) - xB_n^0(y))P_{n-1}^0(x)P_{n-1}^0(y) \\ &+ \frac{1}{R_{n-1}^0}(xB_{n-1}^0(x) - yB_{n-1}^0(y))P_n^0(x)P_n^0(y) \\ &+ \left(A_{n-1}^0(y) + A_n^0(y) + \frac{y}{R_{n-1}^0}B_{n-1}^0(y)\right)\frac{h_{n-1}W_n^0(x,y)}{x^2-y^2} \\ &+ \frac{A_n^0(x) - A_n^0(y)}{x^2-y^2}xP_n^0(y)P_{n-1}^0(x) \\ &+ \frac{A_n^0(y) - A_n^0(y)}{x^2-y^2}yP_n^0(x)P_{n-1}^0(y) \\ &+ \frac{1}{R_{n-1}^0}\frac{yB_{n-1}^0(y) - xB_{n-1}^0(x)}{x^2-y^2}xP_n^0(y)P_{n-1}^0(x)\Bigg] \end{aligned} \tag{3.3.23}$$

We now insert (3.3.23) into the differential equation (3.3.21) and we choose to order it as:

$$
\begin{aligned}
& \left(x\frac{d}{dx}P_n^1(x) - A_n^1 P_n^1(x) + B_n^1 P_{n-1}^1(x) \right) \\
+ \quad & \int d\mu(y) \left(x\frac{d}{dx}P_n^1(y) - A^1 P_n^1(y) + B_n^1 P_{n-1}^1(y) \right) \frac{W_n^0(x,y)}{x^2-y^2} \\
+ \quad & \int d\mu(y) \left[A_{n-1}^0(y) + A_n^0(y) + \frac{y}{R_{n-1}^0} B_{n-1}^0(y) + y\partial_y(\ln w^{10}(y)) \right] P_n^1(y) \frac{W_n^0(x,y)}{x^2-y^2} \\
= \quad & \left(x\frac{d}{dx}P_n^0(x) - A_n^1 P_n^0(x) + B_n^1 P_{n-1}^0(x) \right) \\
- \quad & \frac{1}{h_{n-1}^0} \left(\int d\mu(y) P_n^1(y) P_{n-1}^0(y) \frac{yB_n^0(x) - xB_n^0(y)}{x^2-y^2} \right) P_{n-1}^0(x) \\
- \quad & \frac{1}{h_{n-1}^0} \left(\int d\mu(y) P_n^1(y) P_n^0(y) \frac{1}{R_{n-1}^0} \frac{xB_{n-1}^0(x) - yB_{n-1}^0(y)}{x^2-y^2} \right) P_n^0(x) \\
+ \quad & \frac{1}{h_{n-1}^0} \left(\int d\mu(y) P_n^1(y) P_n^0(y) \frac{A_{n-1}^0(x) - A_{n-1}^0(y)}{x^2-y^2} \right) xP_{n-1}^0(x) \\
- \quad & \frac{1}{h_{n-1}^0} \left(\int d\mu(y) P_n^1(y) P_{n-1}^0(y) y \frac{A_n^0(x) - A_n^0(y)}{x^2-y^2} \right) P_n^0(x) \\
+ \quad & \frac{1}{h_{n-1}^0} \left(\int d\mu(y) P_n^1(y) P_n^0(y) \frac{1}{R_{n-1}^0} \frac{xB_{n-1}^0(x) - yB_{n-1}^0(y)}{x^2-y^2} \right) xP_{n-1}^0(x) \\
+ \quad & \frac{1}{h_{n-1}^0} \left(\int d\mu(y) P_n^1(y) P_{n-1}^0(y) y \frac{A_n^1(x) - A_n^1(y)}{x^2-y^2} \right) P_n^0(x) \\
- \quad & \frac{1}{h_{n-1}^0} \left(\int d\mu(y) P_n^1(y) P_n^0(y) \frac{A_n^1(x) - A_n^1(y)}{x^2-y^2} \right) xP_{n-1}^0(x) \\
+ \quad & \frac{1}{h_{n-1}^0} \left(\int d\mu(y) P_{n-1}^1(y) P_{n-1}^0(y) \frac{xB_n^1(x) - yB_n^1(y)}{x^2-y^2} \right) P_n^0(x) \\
- \quad & \frac{1}{h_{n-1}^0} \left(\int d\mu(y) P_{n-1}^1(y) P_n^0(y) \frac{1}{y} \frac{xB_n^1(x) - yB_n^1(y)}{x^2-y^2} \right) xP_{n-1}^0(x) \\
- \quad & \frac{1}{h_{n-2}^0} B_n^1(x) \left(\int d\mu(y) \frac{1}{y} P_{n-1}^1(y) P_{n-2}^0(y) \right) P_{n-1}^0(x)
\end{aligned} \tag{3.3.24}
$$

so that the right side is in terms of P_n, P_{n-1}. This equation gives a general form for an integral transform between two different differential equations, from which we can determine the integration transform weight w^{10}. First of all though, we consider the third integral on the left side of (3.3.24), since it is this integral that contains the unknown integral transform weight w^{10}. If we set the term contained by the brackets equal to zero, then we can calculate w^{10} in terms of the coefficients of a differential equation

$$A^0_{n-1}(y) + A^0_n(y) + \frac{y}{R^0_{n-1}} B^0_{n-1}(y) + y\partial_y(\ln w^{10}(y)) = 0. \tag{3.3.25}$$

Immediately we are confronted by the fact that w^{10} only depends on coefficients associated with the initial differential equation A^0_n, B^0_n, where we would expect it to also have some dependency on A^1_n, B^1_n. However we will try some examples by choosing specific values of A^0_n, B^0_n.

We begin with the standard differential equation for Hermite orthogonal polynomials (3.3.2),

$$x\partial_x P_n(x) = 2aR_n x P_{n-1}(x)$$

which has the corresponding coefficients $A_n = 0, B_n = 2axR_n$. Substituting these values into (3.3.25) leads to

$$w^{10}(y) = e^{-ay^2}, \tag{3.3.26}$$

the classical weight for Hermite orthogonal polynomials. Alternatively, if we consider our deformed differential equation from Chapter 2 as our starting differential equation (2.2.15)

$$x\partial_x P_n(x) = -x^2 bR_n P_n(x) + (x^3 b + x(a + b(R_{n+1} + R_n)))R_n P_{n-1}(x), \tag{3.3.27}$$

then from (3.3.25) we have

$$w^{10}(y) = e^{-(ay^2+by^4)}, \tag{3.3.28}$$

a deformed Hermite weight for semi-classical polynomials. It is interesting to see that both times this relation has yielded the weight function for its corresponding differential equation. However, since (3.3.25) has a log function and the Volterra reduction is synonymous with the Hermite polynomials, I expect (3.3.25) is only applicable for deformed Hermite polynomials rather than a general class.

In order to learn more about the relationship between the coefficients of our source differential equation P^0 and our transformed differential equation P^1, we substitute some general values of A_n, B_n into (3.3.24). Since $A_n(x)$ are even polynomials and $B_n(x)$ are odd polynomials we consider the following general polynomials:

$$A_n(x) = a_n + b_n x^2 \quad , \quad B_n(x) = c_n x + d_n x^3, \tag{3.3.29}$$

then we can rewrite (3.3.24) as

$$\begin{aligned}
& (A_n^1(x) - A_n^0(x))P_n^0(x) + (B_n^1(x) - B_n^0(x))P_{n-1}^0(x) \\
= & -\frac{1}{h_{n-1}^0}\left(\int d\mu(y) y P_n^1(y) P_{n-1}^0(y)\right) d_n^0 x P_{n-1}^0(x) \\
+ & \frac{1}{h_{n-1}^0}\left(\int d\mu(y) P_n^1(y) P_n^0(y)\right) b_{n-1}^0 x P_{n-1}^0(x) \\
- & \frac{1}{h_{n-1}^0}\left(\int d\mu(y) P_n^1(y) P_{n-1}^0(y) y\right) b_n^0 P_n^0(x) \\
- & \frac{1}{h_{n-1}^0}\left(\int d\mu(y) P_n^1(y) P_n^0(y)\right) c_{n-1}^0 R_{n-1}^0 P_{n-2}^0(x) \\
+ & \frac{1}{h_{n-1}^0}\left(\int d\mu(y) P_n^1(y) P_n^0(y)(x^2+y^2)\right) d_{n-1}^0 P_{n-2}^0(x) \\
+ & \frac{1}{h_{n-1}^0}\left(\int d\mu(y) P_n^1(y) P_{n-1}^0(y) y\right) b_n^1 P_n^0(x) \\
- & \frac{1}{h_{n-1}^0}\left(\int d\mu(y) P_n^1(y) P_n^0(y)\right) b_n^1 x P_{n-1}^0(x) \\
+ & \frac{1}{h_{n-1}^0}\left(\int d\mu(y) P_{n-1}^1(y) P_{n-1}^0(y)\right) c_n^1 P_n^0(x) \\
+ & \frac{1}{h_{n-1}^0}\left(\int d\mu(y) P_{n-1}^1(y) P_{n-1}^0(y)(x^2+y^2)\right) d_n^1 P_n^0(x) \\
- & \frac{1}{h_{n-1}^0}\left(\int d\mu(y) \frac{1}{y} P_{n-1}^1(y) P_n^0(y)\right) c_n^1 x P_{n-1}^0(x) \\
- & \frac{1}{h_{n-1}^0}\left(\int d\mu(y) \frac{1}{y} P_{n-1}^1(y) P_n^0(y)(x^2+y^2)\right) d_n^1 x P_{n-1}^0(x) \\
- & \frac{1}{h_{n-2}^0}(c_n^1 x + d_n^1 x^3)\left(\int d\mu(y) \frac{1}{y} P_{n-1}^1(y) P_{n-2}^0(y)\right) P_{n-1}^0(x).
\end{aligned} \tag{3.3.30}$$

This equation can be further expanded by using expansions of the recurrence relation

$$\begin{aligned} xP_n &= P_{n+1} + R_n P_{n-1} \\ x^2 P_n &= P_{n+2} + (R_{n+1} + R_n)P_n + R_n R_{n-1} P_{n-2} \\ x^3 P_n &= P_{n+3} + (R_{n+2} + R_{n+1} + R_n)P_{n+1} + (R_{n+1} + R_n + R_{n-1})R_n P_{n-1} \\ &\quad + R_n R_{n-1} R_{n-2} P_{n-3} \end{aligned}$$

which leads to the following identities between the coefficients:

$$b_n^1 - b_n^0 + d_n^1 - d_n^0 = 0 \tag{3.3.31a}$$

$$\begin{aligned} a_n^1 - a_n^0 &+ (b_n^1 - b_n^0)(R_{n+1}^0 + R_n^0) + c_n^1 - c_n^0 + (d_n^1 - d_n^0)(R_{n+1}^0 + R_n^0 + R_{n-1}^0) \\ &= \frac{1}{h_{n-1}^0}(h_n^1 - h_n^0)(b_{n-1}^0 - b_n^0 + d_{n-1}^0 - d_n^0) \\ &\quad + \frac{1}{h_{n-2}^0}\int d\mu(y) P_n^1(y) P_{n-2}^0(y)(b_n^1 - b_n^0 + d_n^1 - d_n^0) \\ &\quad + \frac{1}{h_{n-2}^0}\left[(h_{n-2}^1 - h_{n-2}^0)R_{n-1}^1 - (h_{n-1}^1 - h_{n-1}^0)\right] d_n^1 \\ &= \frac{1}{h_{n-1}^0}(h_n^1 - h_n^0)(b_{n-1}^0 - b_n^0 + d_{n-1}^0 - d_n^0) - (R_{n-1}^1 - R_{n-1}^0)d_n^1 \end{aligned} \tag{3.3.31b}$$

$$\begin{aligned} (b_n^1 - b_n^0)R_n^0 R_{n-1}^0 &+ (c_n^1 - c_n^0)R_{n-1} + (d_n^1 - d_n^0)R_n(R_{n+1}^0 + R_n^0 + R_{n-1}^0) \\ &= \frac{1}{h_{n-1}^0}(R_{n-1}^0 + R_{n-2}^0)((h_n^1 - h_n^0)d_{n-1}^0 - (h_{n-1}^1 - h_{n-1}^0)d_n^1 R_{n-1}^0) \\ &\quad - \frac{1}{h_{n-1}^0}(h_{n-1}^1 - h_{n-1}^0)(R_n^1 R_{n-1}^0 d_n^0 - R_n^1 R_n^0 d_{n-1}^0 + R_n^0 R_{n-1}^0 d_n^1) \\ &\quad + \frac{1}{h_{n-1}^0}(h_{n+1}^1 - h_{n+1}^0)d_{n-1}^0 + \frac{1}{h_{n-2}^0}(h_n^1 - h_n^0)(b_n^1 - b_{n-1}^0) \\ &\quad - \frac{1}{h_{n-2}^0}(h_n^1 - h_n^0)c_{n-1}^0 - \frac{1}{h_{n-2}^0}(h_{n-1}^1 - h_{n-1}^0)c_n^1 \\ &\quad + (h_{n-1}^1 - h_{n-1}^0)R_n^0 d_n^1 - (h_{n-1}^1 - h_{n-1}^0)d_n^1(R_n^0 + R_{n-1}^0 + R_{n-2}^0) \\ &\quad + \frac{1}{h_{n-1}^0}\int d\mu(y) P_{n+1}^1(y) P_{n-1}^0(y)(R_n^0 d_{n-1}^0 - R_{n-1}^0 d_n^0) \\ &\quad + \frac{1}{h_{n-1}^0}\int d\mu(y) P_{n-1}^1(y) P_{n+1}^0(y)(R_n^1 d_{n-1}^0 - R_{n-1}^0 d_n^1) \end{aligned} \tag{3.3.31c}$$

$$
\begin{aligned}
(d_n^1 - d_n^0) &= \frac{1}{R_{n-1}^0 h_{n-1}^0}(h_n^1 - h_n^0)d_{n-1}^0 - \frac{1}{h_{n-1}^0}(h_{n-1}^1 - h_{n-1}^0)d_n^1 \\
\Rightarrow \quad R_n^0 d_{n-1}^0 - R_{n-1}^0 d_n^0 &= (R_n^1 d_{n-1}^0 - R_{n-1}^0 d_n^1)\frac{h_{n-1}^1}{h_{n-1}^0} \qquad (3.3.31d)
\end{aligned}
$$

from the functions $P_{n+2}, P_n, P_{n-2}, P_{n-4}$ respectively. The second identity (3.3.31b) is reduced using the first identity (3.3.31a) and it would appear that the third identity (3.3.31c) can be reduced using the fourth identity (3.3.31d), although we have been unable to find out how. From these identities we can then derive the coefficients of the transformed differential equation in terms of the source coefficients.

It is not clear if the interpolating measure $w_1 - w_0$ (as mentioned below (3.3.17)) is an appropriate choice in this section and whether it leads to a closed-form system from the integral transform. One possible route to follow would be to derive the squared eigenfunction expressions in the reduction from the formulae derived in Section (3.1.4) to the scalar case and see if this leads to expressions where the required orthogonality can be seen to emerge.

3.4 Summary

We looked at a singular integral transform that is related to the Gel'fand-Levitan equation the all important inversion formula in the inverse scattering transform.

The first section was concerned with the presentation of the singular integral transform. The singular integral transform consists of a dressing method from a function Φ^0 to a corresponding function Φ^1 by using an interpolating measure $d\Lambda$. We introduced the idea of composition formulae, which removed the dependence of intermediary functions and by associating the transform with a lax type relation, we derived a discrete Lax equation

$$Q^1 = \tilde{H}^1 J - JH^1$$

where the potential H satisfied the compatibility condition. Although $H^1 - H^0$ had a mixed integration (Φ^0, Φ^1), when differential and difference operators were applied to it we could write the results in terms of the new function Φ^1, while the operators only acted on the original function Φ^0. This provided a framework for future investigation into determining the orthogonality conditions for the eigenfunctions of linear problems.

The second sections deals with two applications of singular integral transform, where both seemingly different approaches had the same objective in mind: to relate classical to semi-classical orthogonal polynomials. For the first application, we used the singular integral transform (3.1.5) and chose a 2×2 matrix value for Φ. Using this with the Lax matrices from the recurrence relation (2.3.25) and differential equation (2.3.32) derived in Chapter 2, we derived singular integral transforms for the recurrence coefficients and looked for relations between the interpolating measures $d\Lambda_{ij}$ for $i, j = 1, 2$. The possibility of exploring the differential Lax was introduced, however we were unable to deal with a general differential relation, but this case can certainly be investigated for specific cases. In the second application, we first consider the singular integral transform for the lattice Gel'fand-Dikii $N \times N$ matrix hierarchy, which we write in more explicit form by performing a scalar reduction, that reexpresses the integral transform in terms of column vectors of the function Φ_k. We find that this vector reduction also satisfies the $Q^1 = \tilde{H}^1 J - JH^1$ Lax equation from earlier and we present the explicit form for the column vectors $(h_i^1)_j - (h_i^0)_j$. This reevaluation allows us to present the integral transform in a simplified form that is easier to calculate, which we demonstrate by using the KdV ($N = 2$) as an example. The resulting equation is thus, the scalar integral transform for the KdV equation.

We also mention the existence of a gauge transformation that relates the KdV function u to the Volterra linear problem, which satisfies the recurrence relation for a class of orthogonal polynomials. Thus, we choose a form for the scalar reduction of a singular integral transform for orthogonal polynomials, which we use with the recurrence relation and differential equation of an even weighted orthogonal polynomial (such as Hermite). Like the case when dealing with the standard singular integral transform we found a relation between a new recurrence coefficient R^1 and the old recurrence coefficient R^0 in terms of a very simple integral. When considering the singular integral transform for a differential equation we present two general differential equations (whose solutions are not necessarily polynomials) of differing order and derive a method for calculating the transformed coefficients A_n^1, B_n^1 in terms of the original coefficients A_n^0, B_n^0. Included in this methodology is formula which determines the corresponding weight function for a deformed Hermite differential equation and examples of this are given. Whether this formula is a natural result of the equation or fortuitous luck, further investigation would still be required.

Chapter 4

Formal Elliptic Polynomials

In the formal approach to orthogonal polynomials the notions of bi-orthogonality, adjacency, vector orthogonalities and vector Padé approximants (see the exposition in Chapter 1), are studied in certain areas of numerical analysis. In this context the issue of formal orthogonal polynomials associated with an algebraic curve, has arisen in the literature cf [28]. However, it seems that topologically nontrivial curves have so far not been explicitly studied in detail. This chapter focuses on the latter problem, in particular the construction of formal orthogonal polynomials associated with an elliptic curve. We are motivated by possible connections with integrable systems, which already (as in Chapter 1, (1.1.27)) appear in standard orthogonal polynomials. Although the main thrust of this chapter is to focus on formal aspects, at the end of the chapter we will develop some ideas on extending the formal results to the case where we have explicit weights.

It is important to point out that the notion of "elliptic orthogonal polynomials" has already surfaced in various forms in the literature. Akhiezer's generalization of the Chebyshev polynomials, cf. [6] and also [40, 154, 168], is one way to introduce polynomials associated with an elliptic curve. In the work of Carlitz [31], continued by Ismail and Valent [85], [86], another construction of orthogonal polynomials related to elliptic curves is created.

Carlitz constructed orthogonal polynomials after studying four continued fractions derived by Stieltjes [156] and his research focused on recurrences implied by these continued fractions. He found that certain elliptic function formulas could be utilized to derive relations among the polynomials and that they occur in the multiplication formulas of the Jacobi elliptic function. Thus his approach was to look at orthogonal polynomials and then found that elliptic functions could be used to connect them together.

The Stieltjes-Carlitz polynomials also have connections with the Heun differential equation, specifically the generating functions, which give a finite set of exact solutions of Heun's differential equation. It was Valent [164] who found that the associated Stieltjes-Carlitz polynomials, lead to a new differential equation which he called associated Heun.

Our approach to elliptic polynomials is different, because we are using two-variable orthogonal polynomials, where the two variables are related through an elliptic curve, and as a consequence the polynomials are equivalent to algebraic functions in one of the variables.

4.1 Polynomials in two Variables, Orthogonal over an Algebraic Curve

4.1.1 Two Variable Orthogonal Polynomials

Historically, orthogonal polynomials in two or more variables, is an area less studied than the univariate case. In recent times we have seen plenty of studies of the multivariate case, although their main definitions and simplest properties were dealt with in the latter part of the 19th century. Among the first to study them were Hermite and Appell [10], who considered biorthogonal systems in two variables and Orlov (1881) [138], who looked at an analogue of the Rodrigues formula for two variable orthogonal polynomials. Bateman and Erdélyi [17] did a detailed survey, which covered many of these results including a paper by Jackson [87] in 1938 on the simplest properties of two variables orthogonal over a domain with arbitrary weight. Other papers worthy of note include the one by Krall and Sheffer [98], which considered some linear partial differential operators of the second order for which orthogonal polynomials in two variables are solutions and Engelis [58] who derived similar results but from a different approach.

The book by Suetin [157], provides a comprehensive overview of 2 variable orthogonal polynomials, compared with the more general text by Dunkl and Xu, on orthogonal polynomials in several variables [57]. It covers a great deal of the history surrounding the subject, including the aforementioned material as well as some results by Tom Koornwinder, who has obtained a considerable number of results for new systems of orthogonal polynomials in two variables [93, 94]. The majority of the research done on two-variable orthogonal polynomials is limited to the open domain, where the book of Suetin [157] is a good example of this. While our own research is not on the open domain, these works are presented to provide context of how our own work fits in.

Most of [157] deals with orthogonal polynomials in two variables over a open domain in the x, y-plane, where a set of monomials is constructed consisting of products of a pair of independent variables x and y. and the ordering usually consists of the following form:

$$1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots, x^n, x^{n-1}y, \ldots, y^n, \ldots. \tag{4.1.1}$$

However there is also a chapter dealing with orthogonal polynomials over an algebraic curve ([157], Chapter 7), although this only focuses on such trivial curves as a linear curve $y = ax + b$ or unit circle $x^2 + y^2 = 1$. For the line $y = ax + b$, the list of monomials is reduced to the set $\{x_n\}$ and for the unit circle we have $1, x, y, x^2, xy, x^3, \ldots, x^n, x^{n-1}y, \ldots$, although x and y have equal ordering.

In contrast to this treatment ([157], Chapter 7), we will consider an x and y that do not have the same degree, because we will consider a non-trivial elliptic curve. By using the

Weierstrass gap sequence [14] we are provided with a natural ordering for the monomials associated with this elliptic curve. In principle we could consider curves of a higher genus and use the Weierstrass gap sequence in the construction that follows, but we are not going to venture into that direction and restrict ourselves to genus 1 in this chapter.

We model our construction on the case of the Weierstrass elliptic curve (an elliptic curve in Weierstrass form), and details of the corresponding Weierstrass elliptic functions can be found in Appendix A.

4.2 Polynomials Orthogonal over an Elliptic Curve

As a starting point for our construction we start from the Weierstrass elliptic curve:

$$y^2 = 4x^3 - g_2 x - g_3 \; , \tag{4.2.1}$$

and develop a sequence of elementary monomials associated with this curve:

$$\boldsymbol{e}_0 = 1 \quad , \quad \boldsymbol{e}_2 = x \quad , \quad \boldsymbol{e}_3 = y \quad , \quad \boldsymbol{e}_4 = x^2 \quad , \quad \boldsymbol{e}_5 = xy \quad , \quad \boldsymbol{e}_6 = x^3 \quad , \quad \cdots$$

or, in general:

$$\boldsymbol{e}_0(x,y) = 1 \quad , \quad \boldsymbol{e}_{2k}(x,y) = x^k \qquad , \qquad \boldsymbol{e}_{2k+1}(x,y) = x^{k-1}y \qquad , \quad k = 1, 2, \ldots$$

From the *Weierstrass gap sequence* theorem we can read off the genus of the underlying curve as $g = 1$ as a consequence of the omission of one order (namely order 1) in the corresponding sequence of weights of the monomials. Obviously, the monomials $\boldsymbol{e}_k$ are subject to the algebraic relations:

$$\begin{aligned} \boldsymbol{e}_k \cdot \boldsymbol{e}_l &= \boldsymbol{e}_{k+l} \qquad , \qquad k, l \ \text{not both odd} \\ \boldsymbol{e}_{2k+1} \cdot \boldsymbol{e}_{2l+1} &= 4\boldsymbol{e}_{2(k+l+1)} - g_2 \boldsymbol{e}_{2(k+l-1)} - g_3 \boldsymbol{e}_{2(k+l-2)} \; . \end{aligned}$$

the latter relations being a consequence of the algebraic curve (4.2.1).

We use the sequence $\{\boldsymbol{e}_k\ ,\ k = 0, 2, 3 \ldots\}$ as our basis of monomials for the expansion of polynomials in the two variables x,y related through the algebraic equation (4.2.1), thus taking the form:

$$P_k(x,y) = \sum_{j=0}^{k} p_j^{(k)} \boldsymbol{e}_j(x,y)\ , \tag{4.2.2}$$

and we will call them *monic* if the leading coefficient $p_k^{(k)} = 1$.

4.2.1 Recursive Structures

From the point of view of formal orthogonal polynomials the key relations to be considered are the recurrence relations, which can be derived irrespective of a choice of the weight function. By considering the monomial sequence of two variables, we expect there to be two recurrence relations for the x and y respectively. From a basic perspective we can learn the form of the relations.

We consider the case for k even and k odd respectively.

$$\begin{aligned} P_{2n} &= \boldsymbol{e}_{2n} + p_{2n-1}^{(2n)} \boldsymbol{e}_{2n-1} + p_{2n-2}^{(2n)} \boldsymbol{e}_{2n-2} + p_{2n-3}^{(2n)} \boldsymbol{e}_{2n-3} + p_{2n-4}^{(2n)} \boldsymbol{e}_{2n-4} + \ldots \\ &= x^n + p_{2n-1}^{(2n)} x^{n-2} y + p_{2n-2}^{(2n)} x^{n-1} + p_{2n-3}^{(2n)} x^{n-3} y + p_{2n-4}^{(2n)} x^{n-2} + \ldots \end{aligned} \tag{4.2.3a}$$

$$\begin{aligned} P_{2n+1} &= \boldsymbol{e}_{2n+1} + p_{2n}^{(2n+1)} \boldsymbol{e}_{2n} + p_{2n-1}^{(2n+1)} \boldsymbol{e}_{2n-1} + p_{2n-2}^{(2n+1)} \boldsymbol{e}_{2n-2} + p_{2n-3}^{(2n+1)} \boldsymbol{e}_{2n-3} + \ldots \\ &= x^{n-1} y + p_{2n}^{(2n+1)} x^n + p_{2n-1}^{(2n+1)} x^{n-2} y + p_{2n-2}^{(2n+1)} x^{n-1} + p_{2n-3}^{(2n+1)} x^{n-3} y + \ldots \end{aligned} \tag{4.2.3b}$$

We multiply these polynomials by x and y, to show how the order of the polynomials changes.

$$\begin{aligned}
xP_{2n} &= x^{n+1} + \dots \quad , \quad xP_{2n+1} = x^n y + \dots \\
yP_{2n} &= x^n y + p^{(2n)}_{2n-1} x^{n-2} y^2 + p^{(2n-2)}_{2n-2} x^{n-1} y + p^{(2n)}_{2n-3} x^{n-3} y^2 + p^{(2n)}_{2n-4} x^{n-2} y + \dots \\
yP_{2n+1} &= x^{n-1} y^2 + p^{(2n+1)}_{2n} x^n y + p^{(2n+1)}_{2n-1} x^{n-2} y^2 + p^{(2n+1)}_{2n-2} x^{n-1} y + p^{(2n+1)}_{2n-3} x^{n-3} y^2 + \dots
\end{aligned} \tag{4.2.4}$$

The x does not present a problem, since this is merely absorbed into the equation, but the y^2 requires the substitution of the curve $y^2 = 4x^3 - g_2 x - g_3$.

$$\begin{aligned}
yP_{2n} &= x^n y + 4p^{(2n)}_{2n-1} x^{n+1} + p^{(2n)}_{2n-2} x^{n-1} y + 4p^{(2n)}_{2n-3} x^n + p^{(2n)}_{2n-4} x^{n-2} y + \dots \\
yP_{2n+1} &= 4x^{n+2} + p^{(2n+1)}_{2n} x^n y + 4p^{(2n+1)}_{2n-1} x^{n+1} + p^{(2n+1)}_{2n-2} x^{n-1} y + (4 - g_2) p^{(2n+1)}_{2n-3} x^n + \dots
\end{aligned} \tag{4.2.5}$$

At this point we learn nothing about the $p^{(j)}_k$, however we do learn that for odd yP_k the leading coefficient is 4.

Next, we assume the existence of an inner product $\langle , \rangle$ on the space $\mathcal{V}$ spanned by the monomials $\boldsymbol{e}_k$, such that

$$\langle xP, Q \rangle = \langle P, xQ \rangle \qquad , \qquad \langle yP, Q \rangle = \langle P, yQ \rangle \qquad ,$$

for any two elements $P, Q \in \mathcal{V}$. Since x has order 2, $e_2 = x$ and y has order 3, $e_3 = y$, we have

$$xP_k = P_{k+2} + X^{(1)}_k P_{k+1} + X^{(0)}_k P_k + X^{(-1)}_k P_{k-1} + X^{(-2)}_k P_{k-2}, \tag{4.2.6a}$$

$$\begin{aligned}
yP_k = {} & 4^{\epsilon_k} P_{k+3} + Y^{(2)}_k P_{k+2} + Y^{(1)}_k P_{k+1} + Y^{(0)}_k P_k + Y^{(-1)}_k P_{k-1} \\
& + Y^{(-2)}_k P_{k-2} + Y^{(-3)}_k P_{k-3}
\end{aligned} \tag{4.2.6b}$$

where

$$\epsilon_k = \begin{cases} 1 \quad , & k \text{ odd} , \\ 0 \quad , & k \text{ even} . \end{cases} \tag{4.2.7}$$

The inclusion of the 4 for k odd is straightforward, since the leading term in the expansion was a 4. For k even, the 4 will be included in the Y_k coefficients.

4.2.2 Moments and the Determinant Form

In the spirit of the formal approach to orthogonal polynomials, cf. e.g. [25, 26], we assume that the bilinear form $\langle , \rangle$ derives from a linear functional $\mathcal{L}$ and consequently we can define the associated moments (as in Chapter 1, (1.1.13a)) by

$$c_k = \mathcal{L}(\boldsymbol{e}_k)\ . \tag{4.2.8}$$

Under the assumption of orthogonality the standard Gram-Schmidt orthogonalisation, through the use of Cramer's rule (1.1.14), leads to the following expression for the polynomials:

$$P_k(x,y) = \frac{1}{\Delta_{k-1}} \begin{vmatrix} \langle \boldsymbol{e}_0, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_0, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_0, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_2, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_2, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_2, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{k-1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{k-1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{k-1}, \boldsymbol{e}_k \rangle \\ \boldsymbol{e}_0 & \boldsymbol{e}_2 & \cdots & \cdots & \boldsymbol{e}_k \end{vmatrix}\ , \tag{4.2.9a}$$

in which we have the elliptic Hankel determinants:

$$\Delta_k = \begin{vmatrix} \langle \boldsymbol{e}_0, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_0, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_0, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_2, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_2, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_2, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_k, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_k, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_k, \boldsymbol{e}_k \rangle \end{vmatrix}\ . \tag{4.2.9b}$$

4.3 Recurrence relations

We shall now derive a closed set of recurrence relations for the elliptic orthogonal polynomials introduced in the previous section starting from the determinantal form

(4.2.9a). To do that we will need a number of determinantal identities that have been derived using a generalized form of the Sylvester identity, and are introduced in Appendix B. There are also a series of Hankel identities that have been derived using different variations of the standard Sylvester identity.

To perform the derivations in an elegant way, we find it convenient to introduce a slightly generalized form for the polynomials.

4.3.1 Extended Polynomials

For convenience we introduced adjacent orthogonal polynomials (1.1.20a) in Chapter 1, which aided in the derivation of a recurrence relation for single variable orthogonal polynomials. Using a similar approach (for this case), we introduce the following adjacent 2-variable polynomials associated with the curve (4.2.1):

$$P_k^{(l)}(x,y) \equiv \begin{vmatrix} \langle \boldsymbol{e}_l, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k-2}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+k-2}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-2}, \boldsymbol{e}_k \rangle \\ \boldsymbol{e}_0 & \boldsymbol{e}_2 & \cdots & \cdots & \boldsymbol{e}_k \end{vmatrix} / \Delta_{k-1}^{(l)} \quad , \quad l \neq 0,1 \quad , \tag{4.3.1a}$$

together with the corresponding Hankel determinant:

$$\Delta_k^{(l)} = \begin{vmatrix} \langle \boldsymbol{e}_l, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_k \rangle \end{vmatrix} \quad , \quad l \neq 0,1 \quad , \tag{4.3.1b}$$

and for $l = 0$:

$$P_k^{(0)}(x,y) \equiv \begin{vmatrix} \langle \boldsymbol{e}_0, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_0, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_0, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_2, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_2, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_2, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{k-1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{k-1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{k-1}, \boldsymbol{e}_k \rangle \\ \boldsymbol{e}_0 & \boldsymbol{e}_2 & \cdots & \cdots & \boldsymbol{e}_k \end{vmatrix} \Big/ \Delta_{k-1}^{(0)} \quad , \tag{4.3.1c}$$

with

$$\Delta_k^{(0)} = \begin{vmatrix} \langle \boldsymbol{e}_0, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_0, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_0, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_2, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_2, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_2, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_k, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_k, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_k, \boldsymbol{e}_k \rangle \end{vmatrix} \quad . \tag{4.3.1d}$$

Remark: We note that the shift by one step in the ordered sequence of monomials: $\boldsymbol{e}_0, \boldsymbol{e}_2, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_l, \ldots$ can be realised through a shift operator $\widehat{\cdot}$, which shifts the series by one step:

$$\boldsymbol{e}_0 \ , \ \boldsymbol{e}_2 \ , \ \boldsymbol{e}_3 \ , \ \ldots \qquad \rightsquigarrow \qquad \widehat{\boldsymbol{e}}_0 = \boldsymbol{e}_2 \ , \ \widehat{\boldsymbol{e}}_2 = \boldsymbol{e}_3 \ , \ \widehat{\boldsymbol{e}}_3 = \boldsymbol{e}_4 \ , \ \ldots$$

or in other words:

$$\widehat{\boldsymbol{e}}_l = \begin{cases} \boldsymbol{e}_{l+2} & , \quad l = 0 \\ \boldsymbol{e}_{l+1} & , \quad l \neq 0 \end{cases} \quad .$$

In addition to the polynomials (4.3.1a) we also need to introduce the polynomials:

$$Q_k^{(l)}(x,y) \equiv \begin{vmatrix} \langle \boldsymbol{e}_l, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_k \rangle \\ \boldsymbol{e}_0 & \boldsymbol{e}_2 & \cdots & \cdots & \boldsymbol{e}_k \end{vmatrix} / \Theta_{k-1}^{(l)} \quad , \quad l \neq 1 \tag{4.3.2a}$$

together with the corresponding Hankel determinant:

$$\Theta_k^{(l)} = \begin{vmatrix} \langle \boldsymbol{e}_l, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+k}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k}, \boldsymbol{e}_k \rangle \end{vmatrix} \quad , \quad l \neq 1 \tag{4.3.2b}$$

noting that

$$Q_k^{(0)} = P_k^{(0)} \qquad , \qquad \Theta_k^{(0)} = \Delta_k^{(0)} \ .$$

Remark: We note that for $l \neq 0, 1$ the polynomials $P_k^{(l)}$ do *not* form an orthogonal family, but rather a biorthogonal one. In fact, from the determinantal definition (4.3.1a) we immediately observe that

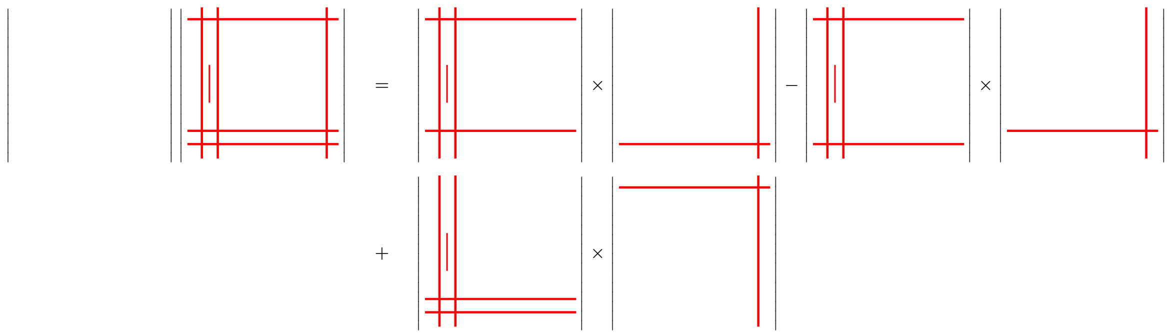
$$\langle \boldsymbol{e}_l, P_k^{(l)} \rangle = \langle \boldsymbol{e}_{l+1}, P_k^{(l)} \rangle = \cdots = \langle \boldsymbol{e}_{l+k-2}, P_k^{(l)} \rangle = 0 \quad , \quad \langle \boldsymbol{e}_{l+k-1}, P_k^{(l)} \rangle = \frac{\Delta_k^{(l)}}{\Delta_{k-1}^{(l)}} \ ,$$

whereas

$$\langle \boldsymbol{e}_{l-1}, P_k^{(l)} \rangle = (-1)^{k-1} \frac{\Delta_k^{(l-1)}}{\Delta_{k-1}^{(l)}} \ .$$

4.3.2 The $xP_k^{(l)}$ Recurrence Relation

The aim now is to use the determinantal identities of a Sylvester type (Appendix B) to derive a recurrence relation in a similar way as in Chapter 1 (1.1.25). The first step is to acquire a recurrence relation in which the variable x is extracted from the determinant as a common factor, in order to get a relation of the form (4.2.6). Thus of the monomials in the last row, we need to remove $\boldsymbol{e}_0 = 1$ and $\boldsymbol{e}_3 = y$. So we implement a 3-row/column Sylvester identity (B.7) from Appendix B on the matrix for $P_k^{(l)}$. For this case it is necessary to fix the columns so that $\boldsymbol{e}_0$ and $\boldsymbol{e}_3$ are removed from the determinant and the position of the row removal is dependent on restricting the introduction of new objects. Hence we apply the following the cutting of three rows and columns according to the diagram:



leads to the following recurrence relation:

$$P_k^{(l)} = xP_{k-2}^{(l+3)} - \frac{\Delta_{k-2}^{(l)}\Delta_{k-2}^{(l+3)}}{\Delta_{k-1}^{(l)}\Delta_{k-3}^{(l+3)}}\, P_{k-1}^{(l)} + \frac{\Delta_{k-2}^{(l+1)}\Delta_{k-2}^{(l+2)}}{\Delta_{k-1}^{(l)}\Delta_{k-3}^{(l+3)}}\, P_{k-1}^{(l+1)} \quad , \quad l \neq 0, 1 \, , \tag{4.3.3a}$$

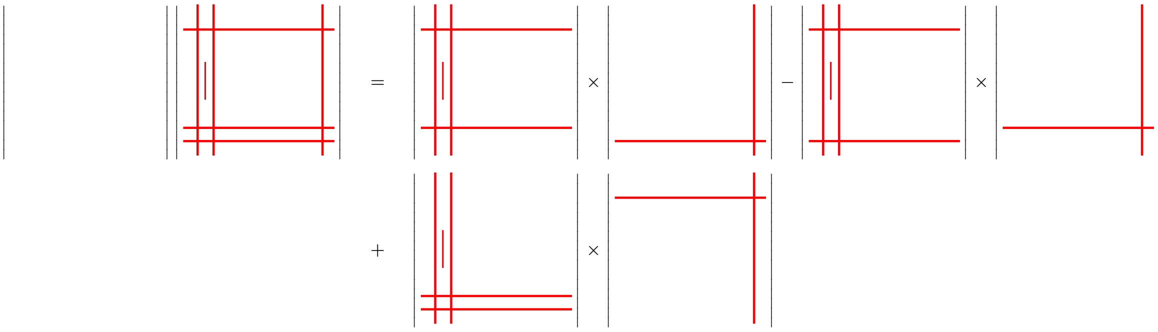
whereas for $l = 0$ we have:

$$P_k^{(0)} = xP_{k-2}^{(4)} - \frac{\Delta_{k-2}^{(0)}\Delta_{k-2}^{(4)}}{\Delta_{k-1}^{(0)}\Delta_{k-3}^{(4)}}\, P_{k-1}^{(0)} + \frac{\Delta_{k-2}^{(2)}\Theta_{k-2}^{(2)}}{\Delta_{k-1}^{(0)}\Delta_{k-3}^{(4)}}\, P_{k-1}^{(2)} \quad , \tag{4.3.3b}$$

whilst obviously, since $P_k^{(1)}$ is not defined, there is no relation for $l = 1$. Applying the same cutting of rows and columns and implementing the same Sylvester identity but now on the matrix for $Q_k^{(l)}$ we obtain a different relation, namely:

$$Q_k^{(l)} = xP_{k-2}^{(l+4)} - \frac{\Theta_{k-2}^{(l)}\Delta_{k-2}^{(l+4)}}{\Theta_{k-1}^{(l)}\Delta_{k-3}^{(l+4)}}\, Q_{k-1}^{(l)} + \frac{\Theta_{k-2}^{(l+2)}\Delta_{k-2}^{(l+2)}}{\Theta_{k-1}^{(l)}\Delta_{k-3}^{(l+4)}}\, P_{k-1}^{(l+2)} \quad , \quad l \neq 0, 1\ . \tag{4.3.3c}$$

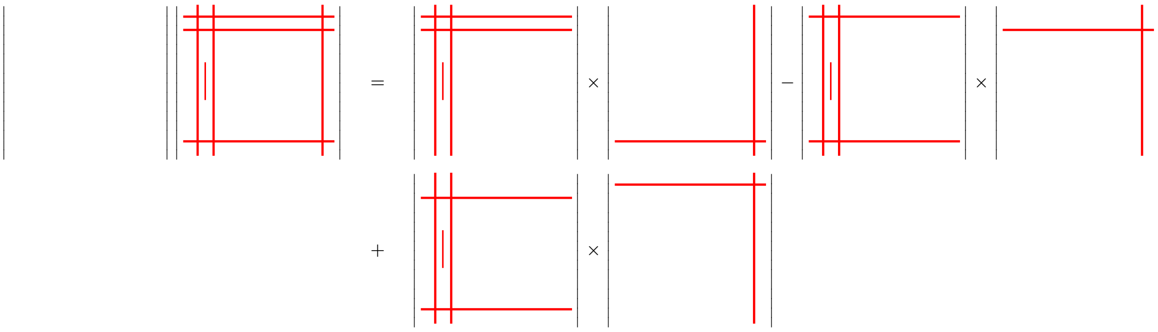
Furthermore, implementing a different cutting of rows and columns on the matrix for $P_k^{(l)}$ according to:



and again applying the 3-row/column Sylvester identity in that situation, we obtain:

$$P_k^{(l)} = xQ_{k-2}^{(l+2)} - \frac{\Delta_{k-2}^{(l)}\Theta_{k-2}^{(l+2)}}{\Delta_{k-1}^{(l)}\Theta_{k-3}^{(l+2)}} P_{k-1}^{(l)} + \frac{\Theta_{k-2}^{(l)}\Delta_{k-2}^{(l+2)}}{\Delta_{k-1}^{(l)}\Theta_{k-3}^{(l+2)}}\, Q_{k-1}^{(l)} \quad , \quad l \neq 0, 1\ . \tag{4.3.3d}$$

Finally, implementing a third way of cutting of rows and columns on the matrix for $P_k^{(l)}$ according to:



and again applying the 3-row/columm Sylvester identity in that situation, we obtain:

$$P_k^{(l)} = xP_{k-2}^{(l+4)} - \frac{\Theta_{k-2}^{(l)}\Delta_{k-2}^{(l+3)}}{\Delta_{k-1}^{(l)}\Delta_{k-3}^{(l+4)}}\, Q_{k-1}^{(l)} + \frac{\Delta_{k-2}^{(l+1)}\Theta_{k-2}^{(l+2)}}{\Delta_{k-1}^{(l)}\Delta_{k-3}^{(l+4)}}\, P_{k-1}^{(l+1)} \quad , \quad l \neq 0, 1\ . \tag{4.3.3e}$$

The relations (4.3.3) form the lowest-order set of relations between two types of objects, the $P_k^{(l)}$ and the $Q_k^{(l)}$, leading to a recursive structure on the polynomials. At this point we do not have enough information to remove the $Q_k^{(l)}$, but this will be dealt with later.

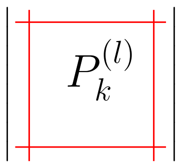
Although we have a series of relations of an $xP_k^{(l)}$ form we require an additional relation to eliminate the need for an upper index in the $P_k^{(l)}$, thereby obtaining a closed-form recurrence relation (namely one in which the super-index l remains fixed). So we introduce the intermediate quantity:

$$T_{k-1}^{(l)}(x,y) \equiv \begin{vmatrix} \langle \boldsymbol{e}_l, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_l, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k-3}, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_{l+k-3}, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-3}, \boldsymbol{e}_k \rangle \\ \boldsymbol{e}_2 & \boldsymbol{e}_3 & \cdots & \cdots & \boldsymbol{e}_k \end{vmatrix} / \Pi_{k-2}^{(l)} \quad , \tag{4.3.4a}$$

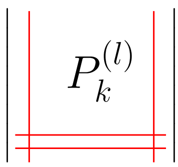
together with its corresponding Hankel determinant:

$$\Pi_{k-1}^{(l)} \equiv \begin{vmatrix} \langle \boldsymbol{e}_l, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_l, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k-2}, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_{l+k-2}, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-2}, \boldsymbol{e}_k \rangle \end{vmatrix} . \tag{4.3.4b}$$

Using the usual Sylvester identity we can now derive the following two equations



$$\Rightarrow \quad P_k^{(l)} = T_{k-1}^{(l+1)} - \frac{\Delta_{k-2}^{(l+1)} \Pi_{k-1}^{(l)}}{\Delta_{k-1}^{(l)} \Pi_{k-2}^{(l+1)}} P_{k-1}^{(l+1)} \,, \tag{4.3.5a}$$



$$\Rightarrow \quad P_k^{(l)} = T_{k-1}^{(l)} - \frac{\Delta_{k-2}^{(l)} \Pi_{k-1}^{(l)}}{\Delta_{k-1}^{(l)} \Pi_{k-2}^{(l)}} P_{k-1}^{(l)} \,, \tag{4.3.5b}$$

which by eliminating the $T_k^{(l)}$ polynomials, collecting like terms

$$\begin{aligned} P_k^{(1)} &= P_k^{(l+1)} + \frac{\Delta_{k-2}^{(l+1)}\Pi_{k-1}^{(l+1)}}{\Delta_{k-1}^{(l+1)}\Pi_{k-2}^{(l+1)}}P_{k-1}^{(l+1)} - \frac{\Delta_{k-2}^{(l+1)}\Pi_{k-1}^{(l)}}{\Delta_{k-1}^{(l)}\Pi_{k-2}^{(l+1)}}P_{k-1}^{(l+1)} \\ &= P_k^{(l+1)} + \frac{\Delta_{k-2}^{(l+1)}}{\Pi_{k-2}^{(l+1)}\Delta_{k-1}^{(l+1)}\Delta_{k-1}^{(l)}}(\Delta_{k-1}^{(l)}\Pi_{k-1}^{(l+1)} - \Delta_{k-1}^{(l+1)}\Pi_{k-1}^{(l)})P_{k-1}^{(l+1)} \quad (4.3.6) \end{aligned}$$

and using the Hankel determinant identity (B.9a)

$$\Delta_k^{(l)}\Pi_{k-2}^{(l+1)} = \Delta_{k-1}^{(l)}\Pi_{k-1}^{(l+1)} - \Delta_{k-1}^{(l+1)}\Pi_{k-1}^{(l)}$$

leads to:

$$P_k^{(l)} = P_k^{(l+1)} + \frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}}P_{k-1}^{(l+1)} \quad , \tag{4.3.7a}$$

or simply

$$P_k^{(l)} = P_k^{(l+1)} + A_k^{(l)}P_{k-1}^{(l+1)}. \tag{4.3.7b}$$

The existence of this linear relation is very helpful, since it allows us to reduce the order of (4.3.3a) and hence is an important relation in the derivation of the $xP_k^{(l)}$. Although it required the introduction of a new object $T_k^{(l)}$ and corresponding Hankel determinant $\Pi_k^{(l)}$, neither of them have appeared in the end result. So now we combine the two equations (4.3.3a) and (4.3.7a), so that we obtain closed-form recurrence relation for the polynomials $P_k^{(l)}$.

We now express (4.3.3a) in the following form:

$$P_{k+2}^{(l)} = xP_k^{(l+3)} - B_k^{(l)}P_{k+1}^{(l)} + C_k^{(l)}P_{k+1}^{(l+1)} \tag{4.3.8}$$

where

$$A_k^{(l)} = \frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}} \quad , \quad B_k^{(l)} = \frac{\Delta_k^{(l)}\Delta_k^{(l+3)}}{\Delta_{k+1}^{(l)}\Delta_{k-1}^{(l+3)}} \quad , \quad C_k^{(l)} = \frac{\Delta_k^{(l+1)}\Delta_k^{(l+2)}}{\Delta_{k+1}^{(l)}\Delta_{k-1}^{(l+3)}} \tag{4.3.9}$$

The combination occurs by increasing the super-index of (4.3.7a) by 2, multiplying it by x, then substituting in (4.3.3a).

$$xP_k^{(l+2)} = P_{k+2}^{(l)} + B_k^{(l)}P_{k+1}^{(l)} - C_k^{(l)}P_{k+1}^{(l+1)} + A_k^{(l+2)}(P_{k+1}^{(l)} + B_{k-1}^{(l)}P_k^{(l)} - C_{k-1}^{(l)}P_k^{(l+1)}) \tag{4.3.10}$$

This expression $\left(xP_k^{(l+2)}\right)$, can be substituted into (4.3.7a) with a super-index increase of 1, leading to $\left(xP_k^{(l+1)}\right)$, which is substituted into (4.3.7a).

$$
\begin{aligned}
xP_k^{(l)} \quad = \quad & P_{k+2}^{(l)} + B_k^{(l)} P_{k+1}^{(l)} - C_k^{(l)} P_{k+1}^{(l+1)} + A_k^{(l+2)} \left(P_{k+1}^{(l)} + B_{k-1}^{(l)} P_k^{(l)} - C_{k-1}^{(l)} P_k^{(l+1)} \right) \\
& + A_k^{(l+1)} \left(P_{k+1}^{(l)} + B_{k-1}^{(l)} P_k^{(l)} - C_{k-1}^{(l)} P_k^{(l+1)} + A_{k-1}^{(l+2)} \left(P_k^{(l)} + B_{k-2}^{(l)} P_{k-1}^{(l)} - C_{k-2}^{(l)} P_{k-1}^{(l+1)} \right) \right) \\
& + A_k^{(l)} \left\{ P_{k+1}^{(l)} + B_{k-1}^{(l)} P_k^{(l)} - C_{k-1}^{(l)} P_k^{(l+1)} + A_{k-1}^{(l+2)} \left(P_k^{(l)} + B_{k-2}^{(l)} P_{k-1}^{(l)} - C_{k-2}^{(l)} P_{k-1}^{(l+1)} \right) \right. \\
& + A_{k-1}^{(l+1)} \left[P_k^{(l)} + B_{k-2}^{(l)} P_{k-1}^{(l)} - C_{k-2}^{(l)} P_{k-1}^{(l+1)} \right. \\
& \left. \left. + A_{k-2}^{(l+2)} \left(P_{k-1}^{(l)} + B_{k-3}^{(l)} P_{k-2}^{(l)} - C_{k-3}^{(l)} P_{k-2}^{(l+1)} \right) \right] \right\}
\end{aligned}
$$

We can then collect those polynomials of the same order of k (and upper index l) and rearrange any remaining term with an upper index of $(l+1)$, so that they all have the same order, $k-1$. To achieve this, (4.3.7a) is used, which also introduces some more terms of order l, so that the remaining terms are of the form $P_{k-1}^{(l+1)}$.

$$
\begin{aligned}
xP_k^{(l)} \quad = \quad & P_{k+2}^{(l)} + \left(B_k^{(l)} + A_k^{(l+2)} + A_k^{(l+1)} + A_k^{(l)} - C_k^{(l)} \right) P_{k+1}^{(l)} \\
& + \left\{ C_k^{(l)} A_{k+1}^{(l)} + \left(A_k^{(l+2)} + A_k^{(l+1)} + A_k^{(l+1)} \right) \left(B_{k-1}^{(l)} - C_{k-1}^{(l)} \right) \right. \\
& \left. + A_{k-1}^{(l+2)} A_k^{(l+1)} + A_{k-1}^{(l+2)} A_k^{(l)} + A_{k-1}^{(l+1)} A_k^{(l)} \right\} P_k^{(l)} \\
& + \left\{ \left(A_{k-1}^{(l+2)} A_k^{(l+1)} + A_{k-1}^{(l+2)} A_k^{(l)} + A_{k-1}^{(l+1)} A_k^{(l)} \right) B_{k-2}^{(l)} \right. \\
& \left. + A_k^{(l)} A_{k-1}^{(l+1)} A_{k-2}^{(l+2)} \left(1 - \frac{C_{k-3}^{(l)}}{A_{k-1}^{(l)}} \right) \right\} P_{k-1}^{(l)} \\
& + A_k^{(l)} A_{k-1}^{(l+1)} A_{k-2}^{(l+2)} B_{k-3}^{(l)} P_{k-2}^{(l)} \\
& - \left\{ \left(A_{k-1}^{(l+2)} A_k^{(l+1)} + A_{k-1}^{(l+2)} A_k^{(l)} + A_{k-1}^{(l+1)} A_k^{(l)} \right) C_{k-2}^{(l)} - \frac{A_k^{(l)} A_{k-1}^{(l+1)} A_{k-2}^{(l+2)} C_{k-3}^{(l)}}{A_{k-1}^{(l)}} \right. \\
& \left. + \left[C_k^{(l)} A_{k+1}^{(l)} - C_{k-1}^{(l)} \left(A_k^{(l+2)} + A_k^{(l+1)} + A_k^{(l)} \right) \right] A_k^{(l)} \right\} P_{k-1}^{(l+1)}
\end{aligned}
\tag{4.3.11}
$$

We note that (4.3.11) contains polynomials all with upper index l, with the exception of the latter term which contains $P_{k-1}^{(l+1)}$. Amazingly the coefficient of this term by a miraculous cancellation vanishes altogether. In fact, expressing (4.3.9) in terms of the

Hankel determinants, the coefficient of $P_{k-1}^{(l+1)}$ in (4.3.11) takes the form

$$\begin{aligned}&\left(\frac{\Delta_k^{(l+1)}\Delta_{k-2}^{(l+2)}}{\Delta_{k-1}^{(l+1)}\Delta_{k-1}^{(l+2)}}\frac{\Delta_{k-1}^{(l+2)}\Delta_{k-3}^{(l+3)}}{\Delta_{k-2}^{(l+2)}\Delta_{k-2}^{(l+3)}}+\frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}}\frac{\Delta_{k-1}^{(l+2)}\Delta_{k-3}^{(l+3)}}{\Delta_{k-2}^{(l+2)}\Delta_{k-2}^{(l+3)}}\right.\\
&\left.+\frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}}\frac{\Delta_{k-1}^{(l+1)}\Delta_{k-3}^{(l+2)}}{\Delta_{k-2}^{(l+1)}\Delta_{k-2}^{(l+2)}}\right)\frac{\Delta_{k-2}^{(l+1)}\Delta_{k-2}^{(l+2)}}{\Delta_{k-1}^{(l)}\Delta_{k-3}^{(l+3)}}\\
&-\frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}}\frac{\Delta_{k-1}^{(l+1)}\Delta_{k-3}^{(l+2)}}{\Delta_{k-2}^{(l+1)}\Delta_{k-2}^{(l+2)}}\frac{\Delta_{k-2}^{(l+2)}\Delta_{k-4}^{(l+3)}}{\Delta_{k-3}^{(l+2)}\Delta_{k-3}^{(l+3)}}\frac{\Delta_{k-3}^{(l+1)}\Delta_{k-3}^{(l+2)}}{\Delta_{k-2}^{(l)}\Delta_{k-4}^{(l+3)}}\frac{\Delta_{k-2}^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-3}^{(l+1)}}\\
&+\frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}}\left[\frac{\Delta_k^{(l+1)}\Delta_k^{(l+2)}}{\Delta_{k+1}^{(l)}\Delta_{k-1}^{(l+3)}}\frac{\Delta_{k+1}^{(l)}\Delta_{k-1}^{(l+1)}}{\Delta_k^{(l)}\Delta_k^{(l+1)}}-\frac{\Delta_{k-1}^{(l+1)}\Delta_{k-1}^{(l+2)}}{\Delta_k^{(l)}\Delta_{k-2}^{(l+3)}}\left(\frac{\Delta_k^{(l+2)}\Delta_{k-2}^{(l+3)}}{\Delta_{k-1}^{(l+2)}\Delta_{k-1}^{(l+3)}}\right.\right.\\
&\left.\left.+\frac{\Delta_k^{(l+1)}\Delta_{k-2}^{(l+2)}}{\Delta_{k-1}^{(l+1)}\Delta_{k-1}^{(l+2)}}+\frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}}\right)\right]\end{aligned}$$

which can be simplified to the following:

$$\begin{aligned}&\frac{\Delta_k^{(l+1)}\Delta_{k-2}^{(l+2)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}\Delta_{k-2}^{(l+3)}}+\frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}\Delta_{k-1}^{(l+2)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}\Delta_{k-2}^{(l+3)}\Delta_{k-1}^{(l)}}+\frac{\Delta_k^{(l)}\Delta_{k-3}^{(l+2)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l)}\Delta_{k-3}^{(l+3)}}-\frac{\Delta_k^{(l)}\Delta_{k-3}^{(l+2)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l)}\Delta_{k-3}^{(l+3)}}\\
&+\frac{\Delta_{k-2}^{(l+1)}\Delta_k^{(l+2)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+3)}}-\frac{\Delta_{k-2}^{(l+1)}\Delta_k^{(l+2)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+3)}}-\frac{\Delta_{k-2}^{(l+2)}\Delta_k^{(l+1)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-2}^{(l+3)}\Delta_{k-1}^{(l+1)}}-\frac{\Delta_{k-2}^{(l+1)}\Delta_{k-1}^{(l+2)}\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-2}^{(l+3)}\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}}\\
&=0,\end{aligned}$$

where the first and seventh, second and eighth, third and fourth, and fifth and sixth terms cancel. Thus the relation takes the form of a five-point recurrence relation, namely

$$xP_k^{(l)}=P_{k+2}^{(l)}+X_k^{(1)}P_{k+1}^{(l)}+X_k^{(0)}P_k^{(l)}+X_k^{(-1)}P_{k-1}^{(l)}+X_k^{(-2)}P_{k-2}^{(l)}\,. \tag{4.3.12}$$

where $l\neq 0,1$ and in which the coefficients $X_k^{(j)}$ are given by:

$$X_k^{(1)} = A_k^{(l+2)}+A_k^{(l+1)}+A_k^{(l)}+B_k^{(l)}-C_k^{(l)}\,, \tag{4.3.13a}$$

$$\begin{aligned}X_k^{(0)} &= \left(A_k^{(l+2)}+A_k^{(l+1)}+A_k^{(l)}\right)\left(B_{k-1}^{(l)}-C_{k-1}^{(l)}\right)+C_k^{(l)}A_{k+1}^{(l)}\\
&\quad+A_{k-1}^{(l+2)}A_k^{(l+1)}+A_{k-1}^{(l+2)}A_k^{(l)}+A_{k-1}^{(l+1)}A_k^{(l)}\,,\end{aligned} \tag{4.3.13b}$$

$$X_k^{(-1)} = \left(A_{k-1}^{(l+2)}A_k^{(l+1)}+A_{k-1}^{(l+2)}A_k^{(l)}+A_{k-1}^{(l+1)}A_k^{(l)}\right)B_{k-2}^{(l)}+A_{k-2}^{(l+2)}A_{k-1}^{(l+1)}A_k^{(l)}\left(1-\frac{C_{k-3}^{(l)}}{A_{k-1}^{(l)}}\right)\,, \tag{4.3.13c}$$

$$X_k^{(-2)} = A_{k-2}^{(l+2)}A_{k-1}^{(l+1)}A_k^{(l)}B_{k-3}^{(l)}=\frac{\Delta_k^{(l)}\Delta_{k-3}^{(l)}}{\Delta_{k-1}^{(l)}\Delta_{k-2}^{(l)}}\,, \tag{4.3.13d}$$

where

$$A_k^{(l)} = \frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}} \quad , \quad B_k^{(l)} = \frac{\Delta_k^{(l)}\Delta_k^{(l+3)}}{\Delta_{k+1}^{(l)}\Delta_{k-1}^{(l+3)}} \quad , \quad C_k^{(l)} = \frac{\Delta_k^{(l+1)}\Delta_k^{(l+2)}}{\Delta_{k+1}^{(l)}\Delta_{k-1}^{(l+3)}}$$

or expressed fully are given by:

$$\begin{aligned}
X_k^{(1)} &= \frac{\Delta_k^{(l+2)}\Delta_{k-2}^{(l+3)}}{\Delta_{k-1}^{(l+2)}\Delta_{k-1}^{(l+3)}} + \frac{\Delta_k^{(l+1)}\Delta_{k-2}^{(l+2)}}{\Delta_{k-1}^{(l+1)}\Delta_{k-1}^{(l+2)}} + \frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}} + \frac{\Delta_k^{(l)}\Delta_k^{(l+3)}}{\Delta_{k+1}^{(l)}\Delta_{k-1}^{(l+3)}} - \frac{\Delta_k^{(l+1)}\Delta_k^{(l+2)}}{\Delta_{k+1}^{(l)}\Delta_{k-1}^{(l+3)}} , \\
X_k^{(0)} &= \left(\frac{\Delta_k^{(l+2)}\Delta_{k-2}^{(l+3)}}{\Delta_{k-1}^{(l+2)}\Delta_{k-1}^{(l+3)}} + \frac{\Delta_k^{(l+1)}\Delta_{k-2}^{(l+2)}}{\Delta_{k-1}^{(l+1)}\Delta_{k-1}^{(l+2)}} + \frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}} \right) \left(\frac{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+3)}}{\Delta_k^{(l)}\Delta_{k-2}^{(l+3)}} - \frac{\Delta_{k-1}^{(l+1)}\Delta_{k-1}^{(l+2)}}{\Delta_k^{(l)}\Delta_{k-2}^{(l+3)}} \right) \\
&\quad + \frac{\Delta_k^{(l+1)}\Delta_k^{(l+2)}}{\Delta_{k+1}^{(l)}\Delta_{k-1}^{(l+3)}} \frac{\Delta_k^{(l+1)}\Delta_{k-2}^{(l+2)}}{\Delta_{k-1}^{(l+1)}\Delta_{k-1}^{(l+2)}} + \frac{\Delta_{k-3}^{(l+3)}\Delta_k^{(l+1)}}{\Delta_{k-2}^{(l+3)}\Delta_{k-1}^{(l+1)}} \\
&\quad + \frac{\Delta_{k-1}^{(l+2)}\Delta_{k-3}^{(l+3)}}{\Delta_{k-2}^{(l+2)}\Delta_{k-2}^{(l+3)}} \frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}} + \frac{\Delta_{k-3}^{(l+2)}\Delta_k^{(l)}}{\Delta_{k-2}^{(l+2)}\Delta_{k-1}^{(l)}} , \\
X_k^{(-1)} &= \left(\frac{\Delta_{k-3}^{(l+3)}\Delta_k^{(l+1)}}{\Delta_{k-2}^{(l+3)}\Delta_{k-1}^{(l+1)}} + \frac{\Delta_{k-1}^{(l+2)}\Delta_{k-3}^{(l+3)}}{\Delta_{k-2}^{(l+2)}\Delta_{k-2}^{(l+3)}} \frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+1)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}} + \frac{\Delta_{k-3}^{(l+2)}\Delta_k^{(l)}}{\Delta_{k-2}^{(l+2)}\Delta_{k-1}^{(l)}} \right) \frac{\Delta_{k-2}^{(l)}\Delta_{k-2}^{(l+3)}}{\Delta_{k-1}^{(l)}\Delta_{k-3}^{(l+3)}} \\
&\quad + \frac{\Delta_{k-4}^{(l+3)}\Delta_{k-1}^{(l+1)}\Delta_k^{(l)}}{\Delta_{k-3}^{(l+3)}\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}} \left(1 - \frac{\Delta_{k-3}^{(l+2)}\Delta_{k-3}^{(l+1)}}{\Delta_{k-4}^{(l+3)}\Delta_{k-1}^{(l)}} \right) , \\
X_k^{(-2)} &= \frac{\Delta_k^{(l)}\Delta_{k-3}^{(l)}}{\Delta_{k-1}^{(l)}\Delta_{k-2}^{(l)}} \quad .
\end{aligned}$$

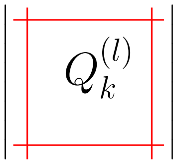
Thus, we have obtained a five point recurrence relation for P_k in terms of the lower index k, with explicit coefficient in terms of the Hankel determinants. It is interesting to see that the last recurrence coefficient $X_k^{(-2)} = \frac{h_k}{h_{k-2}}$, especially when $R_k = \frac{h_k}{h_{k-1}}$ in a standard recurrence relation (1.1.25).

4.3.3 The $xQ_k^{(l)}$ Recurrence Relation

A similar relation to (4.3.12) can be derived for $Q_k^{(l)}$ (4.3.2a) as well. This can be most easily achieved by first deriving a relation between Q and P (namely by employing a 2-row/column Sylvester identity on $Q_k^{(l)}$ directly), then make use of the xQ recurrence relation (4.3.3d). First though, we introduce the following Hankel determinant:

$$\Gamma_{k-1}^{(l)} \equiv \begin{vmatrix} \langle \boldsymbol{e}_l, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_l, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_k \rangle \end{vmatrix}, \tag{4.3.14a}$$

then we obtain the relation



$$\Rightarrow \qquad Q_k^{(l)} = T_{k-1}^{(l+2)} - \frac{\Delta_{k-2}^{(l+2)} \Gamma_{k-1}^{(l)}}{\Theta_{k-1}^{(l)} \Pi_{k-2}^{(l+2)}} P_{k-1}^{(l+2)}, \tag{4.3.14b}$$

which after eliminating the $T_k^{(l)}$ (using (4.3.5a) together with the Hankel determinant identity (B.9c)), leads to

$$\begin{aligned} Q_k^{(l)} &= P_k^{(l+1)} + \frac{\Delta_k^{(l)} \Delta_{k-2}^{(l+2)}}{\Theta_{k-1}^{(l)} \Delta_{k-1}^{(l+1)}} P_{k-1}^{(l+2)} \\ &= P_k^{(l+1)} + D_k^{(l)} P_{k-1}^{(l+2)}, \qquad D_k^{(l)} = \frac{\Delta_k^{(l)} \Delta_{k-2}^{(l+2)}}{\Theta_{k-1}^{(l)} \Delta_{k-1}^{(l+1)}}. \end{aligned} \tag{4.3.14c}$$

While $Q_k^{(l)}$ can be expressed in terms of $P_k^{(l)}$, it is also possible to express $P_k^{(l)}$ in terms of $Q_k^{(l)}$. By introducing the polynomials $S_{k-1}^{(l)}$ and the following sylvester identities on $P_k^{(l)}$ and $Q_k^{(l)}$:

$$S_{k-1}^{(l)}(x,y) \equiv \begin{vmatrix} \langle \boldsymbol{e}_l, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_l, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+2}, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k-2}, \boldsymbol{e}_2 \rangle & \langle \boldsymbol{e}_{l+k-2}, \boldsymbol{e}_3 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-2}, \boldsymbol{e}_k \rangle \\ \boldsymbol{e}_2 & \boldsymbol{e}_3 & \cdots & \cdots & \boldsymbol{e}_k \end{vmatrix} \Big/ \Gamma_{k-2}^{(l)} \quad , \tag{4.3.15a}$$

we subsequently obtain the relations:

$$Q_k^{(l)} \quad \Rightarrow \quad Q_k^{(l)} = S_{k-1}^{(l)} - \frac{\Gamma_{k-1}^{(l)}\Theta_{k-2}^{(l)}}{\Gamma_{k-2}^{(l)}\Theta_{k-1}^{(l)}} Q_{k-1}^{(l)}\,, \tag{4.3.15b}$$

$$P_k^{(l)} \quad \Rightarrow \quad P_k^{(l)} = S_{k-1}^{(l)} - \frac{\Pi_{k-1}^{(l)}\Theta_{k-2}^{(l)}}{\Gamma_{k-2}^{(l)}\Delta_{k-1}^{(l)}} Q_{k-1}^{(l)}\,, \tag{4.3.15c}$$

which after elimination of $S_{k-1}^{(l)}$ and making use of the Hankel determinant identity (B.9d) leaves:

$$\begin{aligned} Q_k^{(l)} &= P_k^{(l)} - \frac{\Delta_k^{(l)}\Theta_{k-2}^{(l)}}{\Delta_{k-1}^{(l)}\Theta_{k-1}^{(l)}} Q_{k-1}^{(l)} \\ &= P_k^{(l)} - W_k^{(l)} Q_{k-1}^{(l)}, \qquad W_k^{(l)} = \frac{\Delta_k^{(l)}\Theta_{k-2}^{(l)}}{\Delta_{k-1}^{(l)}\Theta_{k-1}^{(l)}}. \end{aligned} \tag{4.3.15d}$$

These linear equations involving Q and P provide all the necessary components to derive an xQ recurrence relation. We start with (4.3.3d), the xQ relation established earlier and write it as

$$xQ_k^{(l+2)} = P_{k+2}^{(l)} + U_k^{(l)} P_{k+1}^{(l)} - V_k^{(l)} Q_{k+1}^{(l)} \tag{4.3.16}$$

where

$$U_k^{(l)} = \frac{\Delta_k^{(l)}\Theta_k^{(l+2)}}{\Delta_{k+1}^{(l)}\Theta_{k-1}^{(l+2)}} \qquad , V_k^{(l)} = \frac{\Theta_k^{(l)}\Delta_k^{(l+2)}}{\Delta_{k+1}^{(l)}\Theta_{k-1}^{(l+2)}}$$

then use (4.3.14c) to eliminate the Q and (4.3.7a) $P_k^{(l)} = P_k^{(l+1)} + A_k^{(l)} P_{k-1}^{(l+1)}$ to get the P of the same order as xQ.

$$\begin{aligned} xQ_k^{(l+2)} &= P_{k+2}^{(l+1)} + A_{k+2}^{(l)} P_{k+1}^{(l+1)} + U_k^{(l)}(P_{k+1}^{(l+1)} + A_{k+1}^{(l)} P_k^{(l+1)}) - V_k^{(l)}(P_{k+1}^{(l+1)} + D_{k+1}^{(l)} P_k^{(l+2)}) \\ &= P_{k+2}^{(l+2)} + A_{k+2}^{(l+1)} P_{k+1}^{(l+2)} + (A_{k+2}^{(l)} + U_k^{(l)} - V_k^{(l)})(P_{k+1}^{(l+2)} + A_{k+1}^{(l+1)} P_k^{(l+2)}) \\ &\quad + U_k^{(l)} A_{k+1}^{(l)} (P_k^{(l+2)} + A_k^{(l+1)} P_{k-1}^{(l+2)}) - V_k^{(l)} D_{k+1}^{(l)} P_k^{(l+2)} \end{aligned}$$

Now the order of the relation can just be reduced by 2 and by using (4.3.15d) all the P can be re-expressed as Q and we obtain the five-point recurrence relation for $Q_k^{(l)}$, namely

$$xQ_k^{(l)} = Q_{k+2}^{(l)} + \bar{X}_k^{(1)} Q_{k+1}^{(l)} + \bar{X}_k^{(0)} Q_k^{(l)} + \bar{X}_k^{(-1)} Q_{k-1}^{(l)} + \bar{X}_k^{(-2)} Q_{k-2}^{(l)}\,, \tag{4.3.17}$$

in which the coefficients $\bar{X}_k^{(j)}$ are given by:

$$\begin{aligned}
\bar{X}_k^{(1)} &= A_{k+2}^{(l-1)} + A_{k+2}^{(l-2)} + U_k^{(l-2)} - V_k^{(l-2)} + W_{k+2}^{(l)}\,, && (4.3.18a)\\
\bar{X}_k^{(0)} &= W_{k+1}^{(l)}\left(A_{k+2}^{(l-1)} + A_{k+2}^{(l-2)} + U_k^{(l-2)} - V_k^{(l-2)}\right) + \\
&\quad \left(A_{k+2}^{(l-2)} + U_k^{(l-2)} - V_k^{(l-2)}\right)A_{k+1}^{(l-1)} + U_k^{(l-2)}A_{k+1}^{(l-2)} - V_k^{(l-2)}D_{k+1}^{(l-2)} && (4.3.18b)\\
\bar{X}_k^{(-1)} &= W_k^{(l)}\left((A_{k+2}^{(l-2)} + U_k^{(l-2)} - V_k^{(l-2)})A_{k+1}^{(l-1)} + U_k^{(l-2)}A_{k+1}^{(l-2)} - V_k^{(l-2)}D_{k+1}^{(l-2)}\right)\\
&\quad +U_k^{(l-2)}A_{k+1}^{(l-2)}A_k^{(l-1)} && (4.3.18c)\\
\bar{X}_k^{(-2)} &= U_k^{(l-2)}A_{k+1}^{(l-2)}A_k^{(l-1)}W_{k-1}^{(l)} = \frac{\Theta_k^{(l)}\Theta_{k-3}^{(l)}}{\Delta_{k-1}^{(l)}\Theta_{k-2}^{(l)}}\,. && (4.3.18d)
\end{aligned}$$

which can be expressed explicitly in terms of their respective Hankel determinants

$$\begin{aligned}
\bar{X}_k^{(1)} &= \frac{1}{\Delta_{k+1}^{(l-2)}\Delta_{k+1}^{(l-1)}\Theta_{k-2}^{(l)}}\left(\Delta_k^{(l-1)}\Theta_{k-2}^{(l)}\Delta_{k-1}^{(l)} + \Delta_k^{(l-2)}\Delta_{k+1}^{(l-1)}\Theta_k^{(l)} - \Theta_k^{(l-2)}\Delta_{k+1}^{(l-1)}\Delta_k^{(l)}\right)\\
&\quad +\frac{1}{\Delta_{k+1}^{(l-1)}\Delta_{k+1}^{(l)}\Theta_{k+1}^{(l)}}\left(\Delta_{k+2}^{(l-1)}\Delta_k^{(l)}\Theta_{k+1}^{(l)} + \Delta_{k+2}^{(l)}\Delta_{k+1}^{(l-1)}\Theta_k^{(l)}\right) && (4.3.19a)\\
\bar{X}_k^{(0)} &= \frac{\Delta_{k+2}^{(l-2)}}{\Delta_{k+1}^{(l-2)}\Delta_k^{(l)}}\left(\frac{\Delta_k^{(l-1)}\Theta_{k-1}^{(l)}\Delta_{k+1}^{(l)}}{\Delta_{k+1}^{(l-1)}\Theta_k^{(l)}} + \frac{\Delta_{k-1}^{(l)}\Delta_{k+1}^{(l-1)}}{\Delta_k^{(l-1)}}\right) + \frac{\Delta_{k+2}^{(l-1)}\Theta_{k-1}^{(l)}}{\Delta_{k+1}^{(l-1)}\Theta_k^{(l)}}\\
&\quad +\frac{\Delta_{k+1}^{(l)}}{\Theta_k^{(l)}\Delta_k^{(l)}\Delta_{k+1}^{(l-2)}}\left(\Delta_k^{(l-2)}\Theta_k^{(l)} - \Delta_k^{(l)}\Theta_k^{(l-2)}\right) - \frac{\Delta_k^{(l)}\Delta_{k-1}^{(l)}}{\Delta_k^{(l-1)}\Theta_{k-1}^{(l)}}\\
&\quad +\frac{\Delta_{k+1}^{(l-1)}\Delta_{k-1}^{(l)}}{\Delta_k^{(l-1)}\Delta_k^{(l)}\Delta_{k+1}^{(l-2)}\Theta_{k-1}^{(l)}}\left(\Delta_k^{(l-2)}\Theta_k^{(l)} - \Delta_k^{(l)}\Theta_k^{(l-2)}\right) + \frac{\Theta_k^{(l)}\Delta_{k-1}^{(l-1)}}{\Delta_k^{(l-1)}\Theta_{k-1}^{(l)}} && (4.3.19b)\\
\bar{X}_k^{(-1)} &= \frac{\Theta_k^{(l)}\Theta_{k-2}^{(l)}}{\Theta_{k-1}^{(l)}\Theta_{k-1}^{(l)}}\left(\frac{\Delta_k^{(l-1)}\Delta_{k+1}^{(l-2)}}{\Delta_{k+1}^{(l-1)}\Delta_k^{(l-2)}} + \frac{\Delta_k^{(l)}\Delta_{k-2}^{(l)}}{\Delta_{k-1}^{(l)}\Delta_{k-1}^{(l)}}\right) - \frac{\Theta_{k-2}^{(l)}}{\Theta_{k-1}^{(l)}}\left(\frac{\Delta_{k+2}^{(l-2)}}{\Delta_{k+1}^{(l-2)}} - \frac{\Delta_k^{(l)}\Delta_k^{(l)}}{\Delta_{k-1}^{(l)}\Delta_k^{(l-1)}}\right)\\
&\quad +\frac{\Delta_k^{(l)}}{\Theta_{k-1}^{(l)}\Delta_k^{(l-1)}}\left(\frac{\Theta_k^{(l)}\Delta_{k-1}^{(l-1)}}{\Delta_{k-1}^{(l)}} - \frac{\Delta_{k+1}^{(l-1)}\Theta_k^{(l-2)}}{\Delta_{k+1}^{(l-2)}}\right) && (4.3.19c)\\
\bar{X}_k^{(-2)} &= \frac{\Theta_k^{(l)}\Theta_{k-3}^{(l)}}{\Delta_{k-1}^{(l)}\Theta_{k-2}^{(l)}} && (4.3.19d)
\end{aligned}$$

This time we have obtained a five point recurrence relation for Q_k in terms of the lower order k and where the upper index l remains fixed. The coefficients consist of both the Δ and Θ Hankel determinants, and have been described above.

4.3.4 Bilinear Identities for Hankel Determinants

We have seen that all of the key relations have involved $P_k^{(l)}$ and $Q_k^{(l)}$. Thus by extension they also involve $\Delta_k^{(l)}$ and $\Theta_k^{(l)}$, so it is important to find any relations between them. First we derive a three term bilinear relation, which can be constructed in two separate ways. The first way involves combining three other three-term Hankel determinant relations, thus from (B.9c) and (B.10a) we obtain the following relation:

$$\Delta_{k-1}^{(l+1)}(\Theta_{k-1}^{(l)}\Pi_{k-1}^{(l+2)} - \Theta_k^{(l)}\Pi_{k-2}^{(l+2)}) = \Delta_{k-1}^{(l+2)}(\Theta_{k-1}^{(l)}\Pi_{k-1}^{(l+1)} - \Delta_k^{(l)}\Pi_{k-2}^{(l+2)}) \tag{4.3.20}$$

which is expanded (in the third term) using (B.9a).

$$\Delta_{k-1}^{(l+1)}(\Theta_{k-1}^{(l)}\Pi_{k-1}^{(l+2)} - \Theta_k^{(l)}\Pi_{k-2}^{(l+2)}) = \Theta_{k-1}^{(l)}(\Delta_{k-1}^{(l+1)}\Pi_{k-1}^{(l+2)} - \Delta_k^{(l+1)}\Pi_{k-2}^{(l+2)}) - \Delta_{k-1}^{(l+2)}\Delta_k^{(l)}\Pi_{k-2}^{(l+2)}$$

After we have cancelled the necessary terms, we are left with the following relation.

$$\Delta_k^{(l)}\Delta_{k-1}^{(l+2)} = \Theta_k^{(l)}\Delta_{k-1}^{(l+1)} - \Theta_{k-1}^{(l)}\Delta_k^{(l+1)} \tag{4.3.21}$$

The second way involves using an inner product of (4.3.14c) with $\boldsymbol{e}_{l+k}$ and the inner products:

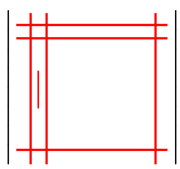
$$\langle Q_k^{(l)}, \boldsymbol{e}_{l+k}\rangle = \frac{\Theta_k^{(l)}}{\Theta_{k-1}^{(l)}}, \tag{4.3.22a}$$

$$\langle P_k^{(l)}, \boldsymbol{e}_{l+k-1}\rangle = \frac{\Delta_k^{(l)}}{\Delta_{k-1}^{(l)}}, \tag{4.3.22b}$$

then we get the same relation (after reducing).

$$\frac{\Theta_k^{(l)}}{\Theta_{k-1}^{(l)}} = \frac{\Delta_k^{(l+1)}}{\Delta_{k-1}^{(l+1)}} + \frac{\Delta_k^{(l)}\Delta_{k-2}^{(l+2)}}{\Theta_{k-1}^{(l)}\Delta_{k-1}^{(l+1)}}\frac{\Delta_{k-1}^{(l+2)}}{\Delta_{k-2}^{(l+2)}} \tag{4.3.23}$$

We find another relation by using the 3 row/column sylvester identity,



from which we derive a four-term bilinear relation involving only the Hankel determinants $\Delta_k^{(l)}$ and $\Theta_k^{(l)}$.

$$\Delta_k^{(l)}\Delta_{k-3}^{(l+4)} = \Delta_{k-1}^{(l)}\Delta_{k-2}^{(l+4)} - \Theta_{k-1}^{(l)}\Delta_{k-2}^{(l+3)} + \Delta_{k-1}^{(l+1)}\Theta_{k-2}^{(l+2)} \tag{4.3.24}$$

It is also possible to derive this relation from the inner product $\langle (4.3.3e), \boldsymbol{e}_{l+k-1}\rangle$. Then (4.3.24) and (4.3.21), provide a **coupled system** for the Hankel determinants Δ and Θ.

We can rewrite these Hankel determinants in order to eliminate the $\Theta_k^{(l)}$. Thus we have

$$\frac{\Delta_k^{(l)}\Delta_{k-1}^{(l+2)}}{\Delta_k^{(l+1)}\Delta_{k-1}^{(l+1)}} = \frac{\Theta_k^{(l)}}{\Delta_k^{(l+1)}} - \frac{\Theta_{k-1}^{(l)}}{\Delta_{k-1}^{(l+1)}} \tag{4.3.25a}$$
$$\frac{\Delta_k^{(l)}\Delta_{k-3}^{(l+4)}}{\Delta_{k-1}^{(l+1)}\Delta_{k-2}^{(l+3)}} - \frac{\Delta_{k-1}^{(l)}\Delta_{k-2}^{(l+4)}}{\Delta_{k-1}^{(l+1)}\Delta_{k-2}^{(l+3)}} = \frac{\Theta_{k-2}^{(l+2)}}{\Delta_{k-2}^{(l+3)}} - \frac{\Theta_{k-1}^{(l)}}{\Delta_{k-1}^{(l+1)}} \tag{4.3.25b}$$

which can be expressed in a simpler way by

$$\begin{aligned} A_k^{(l)} &= \Gamma_k^{(l)} - \Gamma_{k-1}^{(l)}, \\ B_k^{(l)} &= \Gamma_{k-2}^{(l+2)} - \Gamma_{k-1}^{(l)}, \end{aligned}$$

where $\Gamma_k^{(l)} = \frac{\Theta_k^{(l)}}{\Delta_k^{(l+1)}}$. From these two expressions we have

$$\begin{aligned} B_{k+3}^{(l)} + A_{k+2}^{(l)} + A_{k+1}^{(l)} &= \Gamma_{k+1}^{(l+2)} - \Gamma_k^{(l)} \\ A_{k+1}^{(l+2)} + B_{k+2}^{(l)} + A_{k+1}^{(l)} &= \Gamma_{k+1}^{(l+2)} - \Gamma_k^{(l)} \end{aligned}$$

hence

$$\begin{aligned} &\frac{\Delta_{k+3}^{(l)}\Delta_k^{(l+4)}\Delta_{k+1}^{(l+1)} - \Delta_{k+2}^{(l)}\Delta_{k+1}^{(l+4)}\Delta_{k+1}^{(l+1)} + \Delta_{k+2}^{(l)}\Delta_{k+1}^{(l+2)}\Delta_{k+1}^{(l+3)}}{\Delta_{k+1}^{(l+1)}\Delta_{k+1}^{(l+3)}\Delta_{k+2}^{(l+1)}} \\ = \;&\frac{\Delta_{k+1}^{(l+2)}\Delta_k^{(l+4)}\Delta_{k+1}^{(l+1)} + \Delta_{k+2}^{(l)}\Delta_{k-1}^{(l+4)}\Delta_{k+1}^{(l+3)} - \Delta_{k+1}^{(l)}\Delta_k^{(l+4)}\Delta_{k+1}^{(l+3)}}{\Delta_k^{(l+3)}\Delta_{k+1}^{(l+1)}\Delta_{k+1}^{(l+3)}}\,. \end{aligned} \tag{4.3.26}$$

Having seen that we can acquire Hankel determinant relations using inner products we introduce another Hankel determinant Σ_k^l, which like the Θ_k has a row removed, except this row is near the bottom.

$$\Sigma_{k-1}^{(l)} \equiv \begin{vmatrix} \langle \boldsymbol{e}_l, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l, \boldsymbol{e}_{k-1} \rangle \\ \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_{k-1} \rangle \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k-3}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+k-3}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-3}, \boldsymbol{e}_{k-1} \rangle \\ \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_{k-1} \rangle \end{vmatrix} , \quad l \neq 0, 1 \tag{4.3.27}$$

Thus we have the following bilinear relations in terms of $\Sigma_k^{(l)}$ and $\Delta_k^{(l)}$. From the inner product $\langle (4.3.7a), \boldsymbol{e}_{l+k} \rangle$, we have

$$\Delta_k^{(l+1)} \Delta_{k-1}^{(l)} = \Sigma_k^{(l)} \Delta_{k-1}^{(l+1)} - \Delta_k^{(l)} \Sigma_{k-1}^{(l+1)} \tag{4.3.28a}$$

with

$$\langle P_k^{(l)}, e_{l+k} \rangle = \frac{\Sigma_k^{(l)}}{\Delta_{k-1}^{(l)}},$$

and using $\langle (4.3.3a), \boldsymbol{e}_{l+k-1} \rangle$, we have

$$\Delta_k^{(l)} \Delta_{k-3}^{(l+3)} = \Sigma_{k-2}^{(l+3)} \Delta_{k-1}^{(l)} - \Delta_{k-2}^{(l+3)} \Sigma_{k-1}^{(l)} + \Delta_{k-2}^{(l+2)} \Delta_{k-1}^{(l+1)} . \tag{4.3.28b}$$

So in a similar way as before arrange the Σ so we have

$$\frac{\Delta_k^{(l+1)} \Delta_{k-1}^{(l)}}{\Delta_k^{(l)} \Delta_{k-1}^{(l+1)}} = \frac{\Sigma_k^{(l)}}{\Delta_k^{(l)}} - \frac{\Sigma_{k-1}^{(l+1)}}{\Delta_{k-1}^{(l+1)}} \tag{4.3.29a}$$

$$\frac{\Delta_k^{(l)} \Delta_{k-3}^{(l+3)} - \Delta_{k-2}^{(l+2)} \Delta_{k-1}^{(l+1)}}{\Delta_{k-1}^{(l)} \Delta_{k-2}^{(l+3)}} = \frac{\Sigma_{k-2}^{(l+3)}}{\Delta_{k-2}^{(l+3)}} - \frac{\Sigma_{k-1}^{(l)}}{\Delta_{k-1}^{(l)}} \tag{4.3.29b}$$

then eliminate the Σ from the two relations to get

$$\begin{aligned} & \frac{\Delta_k^{(l+3)} \Delta_{k+1}^{(l+1)} \Delta_{k-2}^{(l+4)} - \Delta_{k-1}^{(l+3)} \Delta_k^{(l+3)} \Delta_k^{(l+2)} + \Delta_k^{(l+4)} \Delta_{k-1}^{(l+3)} \Delta_k^{(l+1)}}{\Delta_{k-1}^{(l+4)} \Delta_k^{(l+1)} \Delta_k^{(l+3)}} \\ = \; & \frac{\Delta_{k+2}^{(l)} \Delta_{k-1}^{(l+3)} \Delta_k^{(l+1)} - \Delta_k^{(l+2)} \Delta_{k+1}^{(l+1)} \Delta_k^{(l+1)} + \Delta_{k+1}^{(l+1)} \Delta_k^{(l)} \Delta_k^{(l+3)}}{\Delta_{k+1}^{(l)} \Delta_k^{(l+3)} \Delta_k^{(l+1)}} . \end{aligned} \tag{4.3.30}$$

If we rearrange the grouping of (4.3.26) and (4.3.30) respectively, we can see that they give the same result:

$$\begin{aligned} &\Delta_{k+1}^{(l)}\left(\Delta_k^{(l+4)}\Delta_k^{(l+1)}\Delta_{k-1}^{(l+3)} - \Delta_k^{(l+2)}\Delta_k^{(l+3)}\Delta_{k-1}^{(l+3)} + \Delta_{k-2}^{(l+4)}\Delta_k^{(l+3)}\Delta_{k+1}^{(l+1)}\right) \\ = \ &\Delta_{k-1}^{(l+4)}\left(\Delta_{k+2}^{(l)}\Delta_k^{(l+1)}\Delta_{k-1}^{(l+3)} - \Delta_k^{(l+2)}\Delta_k^{(l+1)}\Delta_{k+1}^{(l+1)} + \Delta_k^{(l)}\Delta_k^{(l+3)}\Delta_{k+1}^{(l+1)}\right) \end{aligned} \tag{4.3.31}$$

((4.3.30) has an index of one less than (4.3.26)). This consistency between the two relations has yielded a single relation for $\Delta_k^{(l)}$.

4.3.5 The $y\bar{P}_k^{(l+3)}$-Recurrence Relation

Having obtained the main five-point recurrence relations (4.3.12) and (4.3.17) for the orthogonal polynomials describing their dependence on the variable x, we still need an additional recurrence relation describing the dependence on the variable y. From the orthogonality we expect the y-recurrence relation to be a seven-point relation in view of the increase in the order of the monomials through the multiplication with y. Furthermore, it will be this relation that will crucially incorporate the dependence on the Weierstrass curve.

In order to derive these y-relations we need to introduce the following intermediate polynomials:

$$\bar{P}_k^{(l+3)}(x,y) \equiv \begin{vmatrix} \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_0\rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_2\rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_k\rangle \\ \langle \boldsymbol{e}_{l+1} \cdot \boldsymbol{e}_3, \boldsymbol{e}_0\rangle & \langle \boldsymbol{e}_{l+1} \cdot \boldsymbol{e}_3, \boldsymbol{e}_2\rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+1} \cdot \boldsymbol{e}_3, \boldsymbol{e}_k\rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k-2} \cdot \boldsymbol{e}_3, \boldsymbol{e}_0\rangle & \langle \boldsymbol{e}_{l+k-2} \cdot \boldsymbol{e}_3, \boldsymbol{e}_2\rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-2} \cdot \boldsymbol{e}_3, \boldsymbol{e}_k\rangle \\ \boldsymbol{e}_0 & \boldsymbol{e}_2 & \cdots & \cdots & \boldsymbol{e}_k \end{vmatrix} \Big/ \bar{\Delta}_{k-1}^{(l+3)} \quad , \tag{4.3.32a}$$

together with the corresponding Hankel determinant:

$$\bar{\Delta}_k^{(l+3)} = \begin{vmatrix} \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_{l+1} \cdot \boldsymbol{e}_3, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+1} \cdot \boldsymbol{e}_3, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+1} \cdot \boldsymbol{e}_3, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k-1} \cdot \boldsymbol{e}_3, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+k-1} \cdot \boldsymbol{e}_3, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-1} \cdot \boldsymbol{e}_3, \boldsymbol{e}_k \rangle \end{vmatrix} . \tag{4.3.32b}$$

The reason for this new determinant is because of the curve and corresponding monomials. Unlike the x, which appears in every monomial (with the exception of the first couple of terms), the y only appears in the odd monomials.

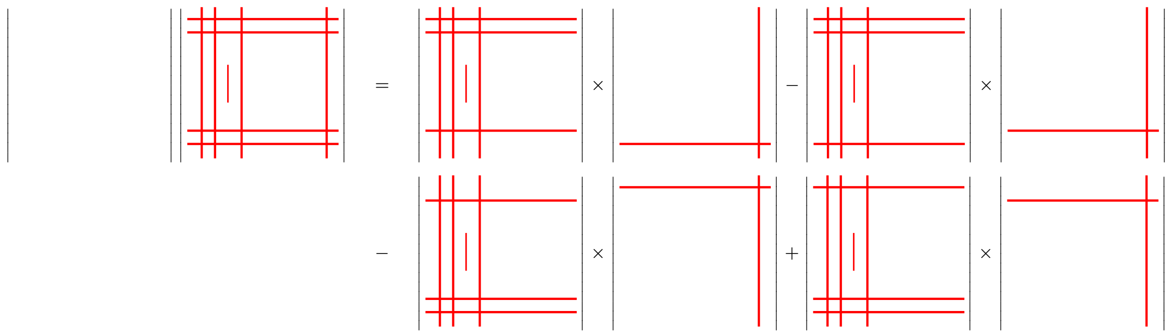
$$\boldsymbol{e}_0 = 1 \quad , \quad \boldsymbol{e}_2 = x \quad , \quad \boldsymbol{e}_3 = y \quad , \quad \boldsymbol{e}_4 = x^2 \quad , \quad \boldsymbol{e}_5 = xy \quad , \quad \boldsymbol{e}_6 = x^3 \quad ,$$
$$\boldsymbol{e}_7 = x^2 y \quad , \quad \boldsymbol{e}_8 = x^4 \quad , \quad \boldsymbol{e}_9 = x^3 y \quad , \quad \boldsymbol{e}_{10} = x^5 \quad , \cdots$$

Thus it is necessary to use the curve to bring y into the even monomials as well. However, a consequence of this is that we are no longer dealing with the original determinant since we have two possible values depending on whether l is odd or even.

$$\boldsymbol{e}_l = \begin{cases} \boldsymbol{e}_{l-3} y \quad l \text{ odd} \quad l \geq 3 \\ \frac{1}{4}(\boldsymbol{e}_{l-3} y + g_2 \boldsymbol{e}_{l-4} + g_3 \boldsymbol{e}_{l-6}) \quad l \text{ even} \quad l \geq 6 \end{cases}$$

Remark: We note that the polynomials $\bar{P}_k^{(l)}$ are orthogonal w.r.t. the functional $\bar{\mathfrak{L}}(\cdot) = \mathfrak{L}(\boldsymbol{e}_3 \cdot)$, with the corresponding Hankel determinants $\bar{\Delta}_k^{(l)}$ being defined accordingly . Similarly we could define associated polynomials $\bar{Q}_k^{(l)}$ and its associated Hankel determinants $\bar{\Theta}_k^{(l)}$ by replacing in the definitions (4.3.2a), (4.3.2b) respectively the brackets associated with the functional $\mathfrak{L}$ by those associated with $\bar{\mathfrak{L}}$. All the relations (4.3.3), together with (4.3.24) that we have derived between the P's and the Q's, as well as the subsequent relations (4.3.3c)-(4.3.3e), hold equally well between the $\bar{P}$'s and $\bar{Q}$'s by replacing everywhere the objects without the bar by those with bars.

In terms of these objects we can now formulate the following recurrence relation which we obtain from the 4-row/column Sylvester identity:



and apply it to (4.3.1a) $P_k^{(l)}$ (except with the curve brought in).

$$\left(\frac{1}{4}\right)^{[\frac{k}{2}]-2}\begin{vmatrix} \langle e_l, e_0\rangle & \langle e_l, e_2\rangle & \langle e_l y, e_0\rangle & \langle e_l, e_4\rangle & \langle e_l y, e_2\rangle & \cdots & \langle e_l y, e_{k-3}\rangle \\ \langle e_{l+1}, e_0\rangle & \langle e_{l+1}, e_2\rangle & \langle e_{l+1} y, e_0\rangle & \langle e_{l+1}, e_4\rangle & \langle e_{l+1} y, e_2\rangle & \cdots & \langle e_{l+1} y, e_{k-3}\rangle \\ \langle e_{l+2}, e_0\rangle & \langle e_{l+2}, e_2\rangle & \langle e_{l+2} y, e_0\rangle & \langle e_{l+2}, e_4\rangle & \langle e_{l+2} y, e_2\rangle & \cdots & \langle e_{l+2} y, e_{k-3}\rangle \\ \vdots & & \vdots & & \vdots & & \vdots \\ \langle e_{l+k-2}, e_0\rangle & \langle e_{l+k-2}, e_2\rangle & \langle e_{l+k-2} y, e_0\rangle & \langle e_{l+k-2}, e_4\rangle & \langle e_{l+k-2} y, e_2\rangle & \cdots & \langle e_l, e_k\rangle \\ e_0 & e_2 & y e_0 & e_4 & y e_2 & \cdots & y e_{k-3} \end{vmatrix} .$$

For this case, $\left(\frac{1}{4}\right)^{[\frac{k}{2}]-2}$ only appears when $(\frac{k}{2}-2)\,\varepsilon\,\mathbb{Z}$. Then we get the following relation (which introduces the $\bar{P}$):

$$\begin{aligned} P_k^{(l)}\Delta_{k-1}^{(l)}\bar{\Delta}_{k-4}^{(l+5)} &= \left(\frac{1}{4}\right)^{[\frac{k}{2}]-2} 4^{[\frac{k-5}{2}]}\Delta_{k-1}^{(l)} y\bar{P}_{k-3}^{(l+5)}\bar{\Delta}_{k-4}^{(l+5)} \\ &- \left(\frac{1}{4}\right)^{[\frac{k}{2}]-2} \bar{\Delta}_{k-3}^{(l+5)} 4^{[\frac{k-5}{2}]} P_{k-1}^{(l)}\Delta_{k-1}^{(l)} \\ &+ \left(\frac{1}{4}\right)^{[\frac{k}{2}]-2} 4^{[\frac{k-5}{2}]} Q_{k-1}^{(l)}\Theta_{k-2}^{(l)}\bar{\Delta}_{k-3}^{(l+4)} \\ &- \left(\frac{1}{4}\right)^{[\frac{k}{2}]-2} \bar{\Theta}_{k-3}^{(l+3)} 4^{[\frac{k-5}{2}]} P_{k-1}^{(l+1)}\Delta_{k-2}^{(l+1)} \end{aligned} \tag{4.3.33}$$

and we reintroduce the 4 when the determinant retains its normal form $P_k^{(l)}$, $Q_k^{(l)}$. This relation can be reduced by making use of the fact that

$$\left(\frac{1}{4}\right)^{[\frac{k}{2}]-2} 4^{[\frac{k-5}{2}]} = 4^{[\frac{k-1}{2}]-[\frac{k}{2}]} = \begin{cases} 1 \quad , \quad k \text{ odd} \,, \\ \frac{1}{4} \quad , \quad k \text{ even} \,, \end{cases} \tag{4.3.34}$$

and we are left with

$$\begin{aligned} 4^{\epsilon_k} P_k^{(l)} \quad &= \quad y\bar{P}_{k-3}^{(l+5)} - \frac{\Delta_{k-2}^{(l)}\bar{\Delta}_{k-3}^{(l+5)}}{\Delta_{k-1}^{(l)}\bar{\Delta}_{k-4}^{(l+5)}} P_{k-1}^{(l)} \\ &\quad + \frac{\Theta_{k-2}^{(l)}\bar{\Delta}_{k-3}^{(l+4)}}{\Delta_{k-1}^{(l)}\bar{\Delta}_{k-4}^{(l+5)}} Q_{k-1}^{(l)} - \frac{\Delta_{k-2}^{(l+1)}\bar{\Theta}_{k-3}^{(l+3)}}{\Delta_{k-1}^{(l)}\bar{\Delta}_{k-4}^{(l+5)}} P_{k-1}^{(l+1)} \,, \end{aligned} \tag{4.3.35}$$

where

$$\epsilon_k = \begin{cases} 0 \quad , \quad k \text{ odd} \,, \\ 1 \quad , \quad k \text{ even} \,. \end{cases}$$

Like before it is possible to get a bilinear relation by taking the inner product of this relation with $\boldsymbol{e}_{l+k-1}$, $\langle(4.3.35), \boldsymbol{e}_{l+k-1}\rangle$,

$$4^{\epsilon_k}\Delta_k^{(l)}\bar{\Delta}_{k-4}^{(l+5)} = \bar{\Sigma}_{k-3}^{(l+5)}\Delta_{k-1}^{(l)} - \Delta_{k-3}^{(l+5)}\Sigma_{k-1}^{(l)} + \bar{\Delta}_{k-3}^{(l+4)}\Theta_{k-1}^{(l)} - \bar{\Theta}_{k-3}^{(l+3)}\Delta_{k-1}^{(l+1)}. \tag{4.3.36}$$

Using (4.3.14c), it is possible to eliminate Q_{k-1} from the relation in order to acquire a $y\bar{P}$ relation, dependent on P and $\bar{P}$ only.

$$\begin{aligned} 4^{\epsilon_k} P_k^{(l)} \quad &= \quad y\bar{P}_{k-3}^{(l+5)} - \frac{\Delta_{k-2}^{(l)}\bar{\Delta}_{k-3}^{(l+5)}}{\Delta_{k-1}^{(l)}\bar{\Delta}_{k-4}^{(l+5)}} P_{k-1}^{(l)} \\ &\quad + \frac{\Theta_{k-2}^{(l)}\bar{\Delta}_{k-3}^{(l+4)}}{\Delta_{k-1}^{(l)}\bar{\Delta}_{k-4}^{(l+5)}} \left(P_{k-1}^{(l+1)} + \frac{\Delta_{k-1}^{(l)}\Delta_{k-3}^{(l+2)}}{\Theta_{k-2}^{(l)}\Delta_{k-2}^{(l+1)}} P_{k-2}^{(l+2)} \right) - \frac{\Delta_{k-2}^{(l+1)}\bar{\Theta}_{k-3}^{(l+3)}}{\Delta_{k-1}^{(l)}\bar{\Delta}_{k-4}^{(l+5)}} P_{k-1}^{(l+1)} \end{aligned} \tag{4.3.37}$$

Then through this relation a closed-form of the $\bar{P}_k^{(l+3)}$ (one in which the superindex l remains fixed) can be derived, using the same method which helped derive the xP relation.

Consider (4.3.7a), except applied to a $\bar{P}_k^{(l+3)}$ instead of a $P_k^{(l)}$, leads to the relation:

$$\bar{P}_k^{(l+3)} = \bar{P}_k^{(l+4)} + \frac{\bar{\Delta}_k^{(l+3)}\bar{\Delta}_{k-2}^{(l+4)}}{\bar{\Delta}_{k-1}^{(l+3)}\bar{\Delta}_{k-1}^{(l+4)}} \bar{P}_{k-1}^{(l+4)} \quad . \tag{4.3.38}$$

Combining the two equations (4.3.35) and (4.3.38) gives way to a four-point recurrence relation, where the coefficient of $P_{n-1}^{(l+1)}$ reduces again to zero.

$$y\bar{P}_k^{(l+3)} = 4^{\epsilon_k} P_{k+3}^{(l)} + y_k^{(2)} P_{k+2}^{(l)} + y_k^{(1)} P_{k+1}^{(l)} + y_k^{(0)} P_k^{(l)} \tag{4.3.39}$$

The explicit forms of the coefficients $y_k^{(j)}$ are given by:

$$y_k^{(2)} = 4^{\epsilon_k}\left(H_k^{(l)} + H_k^{(l+1)}\right) + D_k^{(l)} - E_k^{(l)} \tag{4.3.40a}$$

$$y_k^{(1)} = 4^{\epsilon_k} H_{k-1}^{(l+1)} H_k^{(l)} + D_{k-1}^{(l)}\left(H_k^{(l)} + H_k^{(l+1)}\right) - E_{k-1}^{(l)}\left(H_k^{(l+1)} + H_k^{(l)}\right) - G_k^{(l)} + A_{k+2}^{(l)} E_k^{(l)} \tag{4.3.40b}$$

$$y_k^{(0)} = H_k^{(l)} H_{k-1}^{(l+1)} D_{k-2}^{(l)} \tag{4.3.40c}$$

and in which:

$$H_k^{(l)} = \frac{\bar{\Delta}_k^{(l+3)}\bar{\Delta}_{k-2}^{(l+4)}}{\bar{\Delta}_{k-1}^{(l+3)}\bar{\Delta}_{k-1}^{(l+4)}} \quad , \quad D_k^{(l)} = \frac{\Delta_{k-2}^{(l)}\bar{\Delta}_{k-3}^{(l+5)}}{\Delta_{k-1}^{(l)}\bar{\Delta}_{k-4}^{(l+5)}} \quad , \quad G_k^{(l)} = \frac{\Delta_{k-3}^{(l+2)}\bar{\Delta}_{k-3}^{(l+4)}}{\Delta_{k-2}^{(l+1)}\bar{\Delta}_{k-4}^{(l+5)}}$$

$$E_k^{(l)} = \frac{(\Theta_{k-2}^{(l)}\bar{\Delta}_{k-3}^{(l+4)} - \bar{\Theta}_{k-3}^{(l+3)}\Delta_{k-2}^{(l+1)})}{\Delta_{k-1}^{(l)}\bar{\Delta}_{k-4}^{(l+5)}}.$$

While this relation no-longer has the monic form, the inclusion of the 4 isn't a great surprise, given that the curve itself is not monic in nature. In principal the transformation from $P \to \bar{P}$, just brings an extra y into the determinants, but unlike the x, it is not absorbed into inner product and instead provides us with a choice for the value of $\boldsymbol{e}_l y$. One approach to solving this problem would be to derive a linear relation that relates $\bar{P}_k$ and P_k in a similar way to (4.3.7a), however if this relation does exist we have not been able to find it using current techniques. Such a relation would have provided one way of eliminating $\bar{P}_k^{(l+3)}$ from (4.3.39). As a plan for the future, we could bar this entire equation, which would bar all the normal $P_k^{(l)}$, but have the knock-on effect of adding an additional bar to $\bar{P}_k^{(l)}$. Now the problem is to consider the transformation $\bar{\bar{P}} \to P$, which may require additional determinant identities to break this new structure down.

This equation can also be acquired using similar techniques, except applied to the quantity

$R_k^{(l)}$:

$$R_k^{(l)}(x,y) \equiv \begin{vmatrix} \langle \boldsymbol{e}_l, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+1}, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_{l+k-3}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+k-3}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-3}, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_{l+k-1}, \boldsymbol{e}_k \rangle \\ \boldsymbol{e}_0 & \boldsymbol{e}_2 & \cdots & \cdots & \boldsymbol{e}_k \end{vmatrix} \Big/ \Sigma_{k-1}^{(l)} \quad , \quad l \neq 0,1 \tag{4.3.41a}$$

with corresponding Hankel determinant $\Sigma_k^{(l)}$ (4.3.27), which is the $P_k^{(l)}$ with the row before the penultimate row removed. Then using two row/column identities similar to before with intermediate quantities (identities with the first column removed), it is possible to acquire relations between $R_k^{(l)}$ and $P_k^{(l)}$, of which the most useful is:

$$R_k^{(l)} = P_k^{(l)} - \frac{\Delta_{k-2}^{(l)} \Delta_k^{(l)}}{\Sigma_{k-1}^{(l)} \Delta_{k-1}^{(l)}} P_{k-1}^{(l)}, \tag{4.3.41b}$$

This equation is useful because it removes $R_k^{(l)}$ from a 4 row/column identity on $P_k^{(l)}$. While the $R_k^{(l)}$ provides an alternate approach (from the point of view of analysis), to the $\bar{P}_k^{(l)}$ problem it is not as useful as $Q_k^{(l)}$, since $Q_k^{(l)}$ is valid for $l = 0$.

4.4 Compatibility, Consistency and Elliptic Polynomials

We begin this section with looking at the compatibility between the two recurrence relations (4.2.6). Since (4.2.6a) and (4.2.6b) are connected through the elliptic curve (4.2.1), the coefficients in the corresponding difference operators are not independent, but are related through the curve.

We begin this chapter by stating the consistency relations for the recurrence coefficients X_k, Y_k, which we can describe using inner product relations and can be expressed in terms of h_k.

4.4.1 Consistency in x

The consistency relations involving the X_k are:

$$\begin{aligned}\langle xP_{k+2}, P_k\rangle = \langle P_{k+2}, xP_k\rangle \quad &\Rightarrow \quad X^{(-2)}_{k+2} h_k = h_{k+2} \\ &\Rightarrow \quad X^{(-2)}_k = \frac{h_k}{h_{k-2}} \qquad \text{(4.4.1a)} \\ \langle xP_{k+1}, P_k\rangle = \langle P_{k+1}, xP_k\rangle \quad &\Rightarrow \quad X^{(-1)}_{k+1} h_k = X^{(1)}_k h_{k+1} \\ &\Rightarrow \quad \frac{X^{(-1)}_k}{X^{(1)}_{k-1}} = \frac{h_k}{h_{k-1}} \qquad \text{(4.4.1b)}\end{aligned}$$

where the first (4.4.1a), can be seen in (4.3.13d) and requires no reduction.

$$X^{(-2)}_k = \frac{\Delta^{(l)}_k \Delta^{(l)}_{k-3}}{\Delta^{(l)}_{k-1} \Delta^{(l)}_{k-2}}$$

4.4.2 Consistency in y

Now the relations involving the Y_k:

$$\begin{aligned}\langle yP_{k+3}, P_k\rangle = \langle P_{k+3}, yP_k\rangle \quad &\Rightarrow \quad Y^{(-3)}_{k+3} h_k = h_{k+3} \\ &\Rightarrow \quad Y^{(-3)}_k = 4^{\epsilon_k} \frac{h_k}{h_{k-3}} \qquad \text{(4.4.2a)} \\ \langle yP_{k+2}, P_k\rangle = \langle P_{k+2}, yP_k\rangle \quad &\Rightarrow \quad Y^{(-2)}_{k+2} h_k = Y^{(2)}_k h_{k+2} \\ &\Rightarrow \quad \frac{Y^{(-2)}_k}{Y^{(2)}_{k-2}} = \frac{h_k}{h_{k-2}} \qquad \text{(4.4.2b)} \\ \langle yP_{k+1}, P_k\rangle = \langle P_{k+1}, yP_k\rangle \quad &\Rightarrow \quad Y^{(-1)}_{k+1} h_k = Y^{(1)}_k h_{k+1} \\ &\Rightarrow \quad \frac{Y^{(-1)}_k}{Y^{(1)}_{k-1}} = \frac{h_k}{h_{k-1}} \qquad \text{(4.4.2c)}\end{aligned}$$

where

$$\epsilon_k = \begin{cases} 1 \quad , & k \text{ odd} \ , \\ 0 \quad , & k \text{ even} \ . \end{cases}$$

While we cannot explore the consistency of these relations using explicit derivations of the y recurrence coefficients, we can use them in compatibility relations.

4.4.3 Compatibility Relations

As we have seen in (4.2.6b), the leading monomials in the full expansion of yP_k have different values depending on whether k is odd or even. This distinction is very important, especially when considering the question of compatibility. Here we consider the compatibility between the curve and P_k, $(y^2)P_k = (4x^3 - g_2x - g_3)P_k$ for the separate cases of odd and even.

$$
\begin{aligned}
P_{2n} &= x^n + p_{2n}^{(2n-1)}x^{n-2}y + p_{2n}^{(2n-2)}x^{n-1} + p_{2n}^{(2n-3)}x^{n-3}y + p_{2n}^{(2n-4)}x^{n-2} + \ldots \\
P_{2n+1} &= x^{n-1}y + p_{2n+1}^{(2n)}x^n + p_{2n+1}^{(2n-1)}x^{n-2}y + p_{2n+1}^{(2n-2)}x^{n-1} + p_{2n+1}^{(2n-3)}x^{n-3}y + \ldots
\end{aligned}
$$

Multiplying (4.2.3a) and (4.2.3b) by x^3 and y^2,

$$
\begin{aligned}
x^3P_{2n} &= x^{n+3} + \ldots \quad , \quad x^3P_{2n+1} = x^{n+2}y + \ldots \\
y^2P_{2n} &= x^ny^2 + p_{2n-1}^{(2n)}x^{n-2}y^3 + p_{2n-2}^{(2n)}x^{n-1}y^2 + p_{2n-3}^{(2n)}x^{n-3}y^3 + p_{2n-4}^{(2n)}x^{n-2}y^2 + \ldots \\
y^2P_{2n+1} &= x^{n-1}y^3 + p_{2n}^{(2n+1)}x^ny^2 + p_{2n-1}^{(2n+1)}x^{n-2}y^3 + p_{2n-2}^{(2n+1)}x^{n-1}y^2 + p_{2n-3}^{(2n+1)}x^{n-3}y^3 + \ldots
\end{aligned}
$$

which for the yP_k, reduces to

$$
\begin{aligned}
y^2P_{2n} &= 4x^{n+3} + 4p_{2n-1}^{(2n)}x^{n+1}y + 4p_{2n-2}^{(2n)}x^{n+2} + 4p_{2n-3}^{(2n)}x^ny + (4 - g_2)p_{2n-4}^{(2n)}x^{n+1} + \ldots \\
y^2P_{2n+1} &= 4x^{n+2}y + 4p_{2n}^{(2n+1)}x^{n+3} + 4p_{2n-1}^{(2n+1)}x^{n+1}y + 4p_{2n-2}^{(2n+1)}x^{n+2} + 4p_{2n-3}^{(2n+1)}x^ny + \ldots
\end{aligned}
\tag{4.4.3}
$$

gives an indication of the shape of the two cases, where now a factor of 4 appears in the leading term for both. Thus we consider the inner product relation

$$
\begin{aligned}
\langle y^2P_k, P_k\rangle &= \langle (4x^3 - g_2x - g_3)P_k, P_k\rangle \\
&= \langle 4x^3P_k, P_k\rangle - \langle g_2xP_k, P_k\rangle - \langle g_3P_k, P_k\rangle
\end{aligned}
\tag{4.4.4}
$$

which provide relations between the two sets of coefficients of the two recurrence relations (Appendix D). Of these 13 relations, we see that the first and last terms are either given or derived using the consistency relations for x and y.

Having established the formal structure of the elliptic polynomials (through the recurrence relations), we can now consider how the recurrence relations can be applied. The main uses are deriving the two-variable elliptic polynomials, but we also include a derivation of the elliptic Christoffel-Darboux formula.

4.4.4 Elliptic Polynomials

In order to construct the elliptic polynomials it is necessary to use both of the recurrence relations, along with a few initial conditions. The x and y recurrence relations (4.2.6) are defined as

$$\begin{aligned} xP_k &= P_{k+2} + X_k^{(1)}P_{k+1} + X_k^{(0)}P_k + X_k^{(-1)}P_{k-1} + X_k^{(-2)}P_{k-2}, \\ yP_k &= 4^{\epsilon_k}P_{k+3} + Y_k^{(2)}P_{k+2} + Y_k^{(1)}P_{k+1} + Y_k^{(0)}P_k + Y_k^{(-1)}P_{k-1} + Y_k^{(-2)}P_{k-2} + Y_k^{(-3)}P_{k-3}, \end{aligned}$$

and we consider the initial conditions

$$P_0 = 1 \quad , \quad P_{-1} = P_{-2} = P_{-3} = 0. \tag{4.4.5}$$

While these are the main initial conditions it is also necessary to make a certain allowance. Now since P_1 does not exist, we must let any coefficient that would normally be coupled with P_1 take the value of 0. Moving on to the construction of the polynomials, consider the first four polynomials (which are fully derived in (E)):

$$P_{(0)} = 1 \tag{4.4.6a}$$

$$P_{(2)} = x - X_0^{(0)} \tag{4.4.6b}$$

$$P_{(3)} = y - Y_0^{(2)}x + (X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) \tag{4.4.6c}$$

$$\begin{aligned} P_{(4)} = {} & x^2 - X_2^{(1)}y + (X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)})x - \Big(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) \\ & -(X_2^{(0)}X_0^{(0)} + X_2^{(-2)})\Big) \end{aligned} \tag{4.4.6d}$$

For values of higher order k we have multiple equations since its is possible to create P_5 in two different ways, either through yP_2 or xP_3. Here we provide what we consider to the main form for P_5, P_6 and P_7 where their alternate forms and governing relations are located in Appendix (E).

$$\begin{aligned}
P_5 &= xy - Y_2^{(2)}x^2 + \left(Y_2^{(2)}X_2^{(1)} - Y_2^{(1)} - X_0^{(0)}\right)y \\
&\quad - \left(Y_2^{(2)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) - Y_2^{(1)}Y_0^{(2)} + Y_2^{(0)}\right)x \\
&\quad + \left(Y_2^{(2)}\left(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_0^{(0)}X_2^{(0)} - X_2^{(-2)})\right)\right. \\
&\quad \left. - Y_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) + Y_2^{(0)}X_0^{(0)} - Y_2^{(-2)}\right) \qquad (4.4.7a)
\end{aligned}$$

$$\begin{aligned}
4P_6 &= y^2 - (Y_3^{(2)} + Y_0^{(2)})xy + (Y_3^{(2)}Y_2^{(2)} - Y_3^{(1)})x^2 \\
&\quad - \left(Y_3^{(2)}(Y_2^{(2)}X_2^{(1)} - Y_2^{(1)} - X_0^{(0)}) - Y_3^{(1)}X_2^{(1)} + Y_3^{(0)} - (X_0^{(0)}Y_0^{(2)} - Y_0^{(0)})\right)y \\
&\quad + \left(Y_3^{(2)}(Y_2^{(2)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) - Y_2^{(1)}Y_0^{(2)} + Y_2^{(0)})\right. \\
&\quad \left. - Y_3^{(1)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) + Y_3^{(0)}Y_0^{(2)} - Y_3^{(-1)}\right)x \\
&\quad - \left(Y_3^{(2)}\left(Y_2^{(2)}(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})) - Y_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)})\right.\right. \\
&\quad \left. + Y_2^{(0)}X_0^{(0)} - Y_2^{(-2)}\right) - Y_3^{(1)}\left(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})\right) \\
&\quad \left. + Y_3^{(0)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - Y_3^{(-1)}X_0^{(-0)} + Y_3^{(-3)}\right) \qquad (4.4.7b)
\end{aligned}$$

4.5 Elliptic Christoffel-Darboux Identities

In a similar way as in (1), identities can be constructed of a Christoffel-Darboux form for both the x and y. Since these polynomials are already in two variables , we use the $'$ notation, where the $'$ in this case refers to the alternate variables (x', y').

4.5.1 The x Elliptic Christoffel-Darboux Identity

Using the form of the x-recurrence relation and take differences:

$$\begin{aligned} & (xP_n = P_{n+2} + X^{(1)}P_{n+1} + X^{(0)}P_n + X^{(-1)}P_{n-1} + X^{(-2)}P_{n-2})P'_n \\ - \quad & (x'P'_n = P'_{n+2} + X^{(1)}P'_{n+1} + X^{(0)}P'_n + X^{(-1)}P'_{n-1} + X^{(-2)}P'_{n-2})P_n \end{aligned}$$

(where P' is a polynomial $P(x', y')$ dependent on the variables x' and y') to leave us with the following relation:

$$\begin{aligned} (x - x')P_nP'_n \;&=\; P_{n+2}P'_n - P_nP'_{n+2} - X_n^{(1)}(P_{n+1}P'_n - P_nP'_{n+1}) \\ &\quad + X_n^{(-1)}(P_{n-1}P'_n - P_nP'_{n-1}) + X_n^{(-2)}(P_{n-2}P'_n - P_nP'_{n-2}) \,. \end{aligned}$$

We now make use of the consistency relations (4.4.1b), (4.4.1a) to reduce this relation

$$\begin{aligned} (x - x')\frac{P_nP'_n}{h_n} \;&=\; \frac{1}{h_n}(P_{n+2}P'_n - P_nP'_{n+2}) - \frac{1}{h_{n-2}}(P_nP'_{n-2} - P_{n-2}P'_n) \\ &\quad + \frac{1}{h_{n+1}}X_{n+1}^{(-1)}(P_{n+1}P'_n - P_nP'_{n+1}) - \frac{1}{h_n}X_n^{(-1)}(P_nP'_{n-1} - P_{n-1}P'_n) \end{aligned}$$

and integrate to give the sum:

$$\begin{aligned} \sum_{j=0}^{n} \frac{P_j(x,y)P_j(x',y')}{h_j} \;&=\; \frac{1}{(x - x')}\Bigg(\frac{1}{h_n}(P_{n+2}(x,y)P_n(x',y') - P_n(x,y)P_{n+2}(x',y')) \\ &\quad + \frac{1}{h_{n-1}}(P_{n+1}(x,y)P_{n-1}(x',y') - P_{n-1}(x,y)P_{n+1}(x',y')) \\ &\quad + \frac{1}{h_{n+1}}X_{n+1}^{(-1)}(P_{n+1}(x,y)P_n(x',y') - P_n(x,y)P_{n+1}(x',y'))\Bigg) \end{aligned} \tag{4.5.1}$$

4.5.2 The y Elliptic Christoffel-Darboux Identity

Considering the difference of the y recurrence relation, it is easy to see that:

$$\begin{aligned}(y-y')P_nP_n' &= (P_{n+3}P_n' - P_nP_{n+3}') + Y_n^{(2)}(P_{n+2}P_n' - P_nP_{n+2}') \\ &\quad + Y^{(1)}(P_{n+1}P_n' - P_nP_{n+1}') + Y_n^{(-1)}(P_{n-1}P_n' - P_nP_{n-1}') \\ &\quad + Y_n^{(-2)}(P_{n-2}P_n' - P_nP_{n-2}') + Y_n^{(-3)}(P_{n-3}P_n' - P_nP_{n-3}')\end{aligned}$$

and pairing terms together, making use of the consistency relations and rearranging.

$$\begin{aligned}(y-y')\frac{P_nP_n'}{h_n} &= \frac{1}{h_n}(P_{n+3}P_n' - P_nP_{n+3}') - \frac{1}{h_{n-3}}(P_nP_{n-3}' - P_{n-3}P_n') \\ &\quad + \frac{Y_{n+2}^{(-2)}}{h_{n+2}}(P_{n+2}P_n' - P_nP_{n+2}') - \frac{Y_n^{(-2)}}{h_n}(P_nP_{n-2}' - P_{n-2}P_n') \\ &\quad + \frac{Y_{n+1}^{(-1)}}{h_{n+1}}(P_{n+1}P_n' - P_nP_{n+1}') - \frac{Y_n^{(-1)}}{h_n}(P_nP_{n-1}' - P_{n-1}P_n')\end{aligned}$$

All that remains is to integrate up and we are left with the following relation:

$$\begin{aligned}\sum_{k=0}^{n}\frac{P_k(x,y)P_k(x',y')}{h_k} &= \frac{1}{(y-y')}\Bigg(\frac{1}{h_n}(P_{n+3}(x,y)P_n(x',y') - P_n(x,y)P_{n+3}(x',y')) \\ &\quad + \frac{1}{h_{n-1}}(P_{n+2}(x,y)P_{n-1}(x',y') - P_{n-1}(x,y)P_{n+2}(x',y')) \\ &\quad + \frac{1}{h_{n-2}}(P_{n+1}(x,y)P_{n-2}(x',y') - P_{n-2}(x,y)P_{n+1}(x',y')) \\ &\quad + \frac{Y_{n+2}^{(-2)}}{h_{n+2}}(P_{n+2}(x,y)P_n(x',y') - P_n(x,y)P_{n+2}(x',y')) \\ &\quad + \frac{Y_{n+1}^{(-2)}}{h_{n+1}}(P_{n+1}(x,y)P_{n-1}(x',y') - P_{n-1}(x,y)P_{n+1}(x',y')) \\ &\quad + \frac{Y_{n+1}^{(-1)}}{h_{n+1}}(P_{n+1}(x,y)P_n(x',y') - P_n(x,y)P_{n+1}(x',y'))\Bigg)\end{aligned} \tag{4.5.2}$$

As we explained in Chapter 1, the Christoffel-Darboux relations are very useful in the study of formal orthogonal polynomials, particularly when exploring the zeros. We could develop these relations further, with the study of zeros of the Weierstrass elliptic polynomials in mind. As a further extension, they would also be useful in the construction of an analogue to the Laguerre method for the creation of a differential system for the semi-classical case of two variable orthogonal polynomials related through a curve. However in this instance, we would be moving from formal to non-formal polynomials and bring in the introduction of a weight function and corresponding integration interval.

4.6 Further Outlook

In this section we present a possible extension from the formal case into the non-formal case, where we consider the existence of a weight functional. The introduction of a weight function opens the door to the development of differential equations and the class of semi-classical orthogonal polynomials. While we do introduce some possible avenues of study, this is only a tentative look, so we do not go into any great detail.

We highlight an expression, which evokes a realization of the basic functional $\mathfrak{L}$ in terms of an integral. Thus we assume that this functional takes the form:

$$\mathfrak{L}(P) = \int d\mu(\kappa) P(\wp(\kappa), \wp'(\kappa)) \qquad , \qquad P \in \mathcal{V} \, ,$$

which is some defining integral and measure $d\mu(y)$ in terms of the uniformising variable κ on the curve (4.2.1), i.e. $(x, y) = (\wp(\kappa), \wp'(\kappa))$. Then we can derive the following integral representation for the elliptic polynomials:

$$P_k(x,y) = \frac{\gamma_k\gamma_{k-1}}{(k-1)!\Delta_{k-1}}\int d\mu_1\int d\mu_2\cdots\int d\mu_{k-1}$$
$$\times\begin{vmatrix} 1 & \wp_1 & \cdots & \wp_1^{(k-3)} \\ 1 & \wp_2 & \cdots & \wp_2^{(k-3)} \\ \vdots & \vdots & & \vdots \\ 1 & \wp_{k-1} & \cdots & \wp_{k-1}^{(k-3)} \end{vmatrix}\begin{vmatrix} 1 & \wp_1 & \wp_1' & \cdots & \wp_1^{(k-2)} \\ 1 & \wp_2 & \wp_2' & \cdots & \wp_2^{(k-2)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp_{k-1} & \wp_{k-1}' & \cdots & \wp_{k-1}^{(k-2)} \\ 1 & \wp & \wp' & \cdots & \wp^{(k-2)} \end{vmatrix} \tag{4.6.1a}$$

in which we have abbreviated $\int d\mu_j = \int d\mu(\kappa_j)$, $\wp_j^{(i)} = d^i\wp(\kappa_j)/d\kappa_j^i$, $\wp^{(i)} = d^i\wp(\kappa)/d\kappa^i$ and where γ_k denotes a inconsequential numerical factor (depending on k). Using the so-called Frobenius-Stickelberger formula (A.14), [67],

$$\begin{vmatrix} 1 & \wp(\kappa_0) & \wp'(\kappa_0) & \cdots & \wp^{(n-1)}(\kappa_0) \\ 1 & \wp(\kappa_1) & \wp'(\kappa_1) & \cdots & \wp^{(n-1)}(\kappa_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(\kappa_n) & \wp'(\kappa_n) & \cdots & \wp^{(n-1)}(\kappa_n) \end{vmatrix}$$
$$= (-1)^{\frac{1}{2}n(n-1)}1!2!\cdots n!\frac{\sigma(\kappa_0+\kappa_1+\cdots+\kappa_n)\prod_{i<j}\sigma(\kappa_i-\kappa_j)}{\sigma^{n+1}(\kappa_0)\cdots\sigma^{n+1}(\kappa_n)},$$

we can, thus, derive the following formula for the elliptic polynomials:

$$P_k(x,y) = (-1)^{k-1}[1!2!\cdots(k-2)!]^2\frac{\gamma_k\gamma_{k-1}}{(k-1)!\Delta_{k-1}}\times$$
$$\int d\mu_1\cdots\int d\mu_{k-1}\,\frac{\sigma^2(\kappa_1+\cdots+\kappa_{k-1})}{\sigma^{2k-2}(\kappa_1)\cdots\sigma^{2k-2}(\kappa_{k-1})}\left[\prod_{i<j=1}^{k-1}\sigma^2(\kappa_i-\kappa_j)\right]$$
$$\times\,\Phi_\kappa(\kappa_1+\cdots+\kappa_{k-1})\prod_{j=1}^{k-1}\Phi_{-\kappa}(\kappa_j) \tag{4.6.1b}$$

where $\Phi_n(x) = \frac{\sigma(\kappa+x)}{\sigma(\kappa)\sigma(x)}$.

4.6.1 An Elliptic Weight Function

Having established a formal structure for the elliptic polynomials (that deals mainly with recurrence relations), we now consider a possible informal approach in how to continue the study of these polynomials. The first thing required would be a weight function, which cannot be derived, only guessed at. However this does not mean a random choice, merely careful consideration of what would be an appropriate choice. Upon establishing this we can derive some of the other formulae associated with orthogonal polynomials, including a differential relation.

Given the connection, between Stieltjes-Carlitz polynomials and the Heun equation, we look for inspiration for a weight function by looking at Heun. In the introduction, we defined the Heun differential equation

$$\frac{d^2w}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\frac{dw}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-a)}w = 0 \tag{4.6.2a}$$

where

$$\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0,$$

a Fuchsian equation with four regular singularities at $0, 1, a$ and ∞. In [153], exact solutions of a special case of Heun's equation

$$\frac{d^2y}{dz^2} + \frac{1}{2}\left(\frac{1-2m_1}{z} + \frac{1-2m_2}{z-1} + \frac{1-2m_3}{z-a}\right)\frac{dy}{dz} + \frac{N(N-2m_0-1)z+\lambda}{4z(z-1)(z-a)}y = 0 \tag{4.6.2b}$$

where

$$\begin{aligned} &N = m_0 + m_1 + m_2 + m_3, \\ &m_0, m_1, m_2, m_3 \in \mathbb{Z}, \qquad \lambda, z \in \mathbb{C}. \end{aligned} \tag{4.6.2c}$$

(m_i a non negative integer) are studied and it is shown that they are functions of the following form:

$$Y_{1,2}(m_0, m_1, m_2, m_3; \lambda; z) = \sqrt{\Psi_{g,N}(\lambda, z)}\exp\left(\pm\frac{i\upsilon(\lambda)}{2}\int\frac{z^{m_1}(z-1)^{m_2}(z-a)^{m_3}dz}{\Psi_{g,N}(\lambda, N)\sqrt{z(z-1)(z-a)}}\right), \tag{4.6.2d}$$

where $\Psi_{g,N}$ is is some polynomial of degree N in z and of degree g in λ, $i^2 = -1$ and

$$\upsilon^2 = \prod_{j=1}^{2g+1} (\lambda - \lambda_j)\,, \quad \lambda_j = \lambda(E_j) \tag{4.6.3}$$

where E_j are the gap edges of the finite-gap elliptic potential $u(x)$. Inspired by this result, we propose the following elliptic weight:

$$\begin{aligned} w(x,y) &= \frac{1}{y}(x-e_1)^{\nu_1}(x-e_2)^{\nu_2}(x-e_3)^{\nu_3} && \text{(4.6.4a)} \\ \text{with} \quad y^2 &= 4x^3 - g_2 x - g_3 \\ &= 4(x-e_1)(x-e_2)(x-e_3) && \text{(4.6.4b)} \end{aligned}$$

with integrations between the branch points of the elliptic equation e_i (for instance e_1 and e_2).

In the paper by Fernandez (et al.) [59] they mention orthogonal polynomials in two variables associated with a moment functional u satisfying the two-variable analogue of the Pearson differential equation. However, while we have two variable polynomials, they are related through a curve, so we would expect the Weierstrass elliptic polynomials to satisfy a different analogue of the two-variable Pearson differential equation. We could, however, use this example as a starting point to derive our own analogue of the Pearson equation for two variable orthogonal polynomials related through an algebraic curve.

4.6.2 Differential Equations

As should be expected by the dual nature of these elliptic orthogonal polynomials, a pair of differential equations associated with the elliptic curve can be derived, covering both the odd P_{2n+1} and even P_{2n} cases. Examples of the derivation of a differential equation for another example of elliptic orthogonal polynomials can be found in [144].

At this stage we do not focus on a completed form for the differential relations except to derive their leading terms. We begin by considering P_{2n} and P_{2n+1}, which are even and odd respectively for $n \geq 1$.

$$\begin{aligned} P_{2n} &= x^n + a_{2n-1}x^{n-2}y + a_{2n-2}x^{n-1} + a_{2n-3}x^{n-3}y + \dots && \text{(4.6.5a)} \\ P_{2n+1} &= x^{n-1}y + b_{2n}x^n + b_{2n-1}x^{n-2}y + b_{2n-2}x^{n-1} + \dots && \text{(4.6.5b)} \end{aligned}$$

Since both P_{2n} and P_{2n+1} depend on both x and y, it is necessary to construct a differential relation which will allow for partial differentiation. Thus we begin with

$$\frac{d}{dx}P_{2n} = \left(\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y}\right)P_{2n} \Rightarrow y\frac{d}{dx} = y\frac{\partial}{\partial x} + \left(6x^2 - \frac{g_2}{2}\right)\frac{\partial}{\partial y} \qquad \text{(4.6.6)}$$

where

$$2y\frac{dy}{dx} = \frac{d}{dx}y^2 = 12x^2 - g_2.$$

Thus we have the ingredients to begin our derivation

$$\begin{aligned} y\frac{d}{dx}P_{2n} &= \left(y\frac{\partial}{\partial x} + \left(6x^2 - \frac{g_2}{2}\right)\frac{\partial}{\partial y}\right)\left(x^n + a_{2n-1}x^{n-2}y + a_{2n-2}x^{n-1} + \dots\right) \\ &= nx^{n-1}y + a_{2n-1}(n-2)x^{n-3}y^2 + a_{2n-2}(n-1)x^{n-2}y + \left(6x^2 - \frac{g_2}{2}\right)a_{2n-1}x^{n-2} + \dots \\ &= nx^{n-1}y + \dots && \text{(4.6.7)} \end{aligned}$$

where we get the leading term, followed by lower order terms, hence

$$y\frac{d}{dx}P_{2n} = nP_{2n+1} + \dots \qquad \text{(4.6.8)}$$

As with the even case we have

$$\begin{aligned} y\frac{d}{dx}P_{2n+1} &= \left(y\frac{\partial}{\partial x} + \left(6x^2 - \frac{g_2}{2}\right)\frac{\partial}{\partial y}\right)\left(x^{n-1}y + b_{2n}x^n + b_{2n-1}x^{n-2}y + \dots\right) \\ &= (n-1)x^{n-2}y^2 + \left(6x^2 - \frac{g_2}{2}\right)x^{n-1} + b_{2n}nx^{n-1}y + b_{2n-1}(n-2)x^{n-3}y^2 + \dots \\ &= (n-1)4x^{n+1} + 6x^{n+1} + \dots && \text{(4.6.9a)} \end{aligned}$$

where we have introduced the curve $y^2 = 4x^3 - g_2x - g_3$ and this implies that

$$y\frac{d}{dx}P_{2n+1} = 2(2n+1)P_{2n+2} + \dots \qquad \text{(4.6.10)}$$

4.7 Summary

This chapter was concerned with a new class of orthogonal polynomials associated with the Weierstrass elliptic curve, where the focus was on the formal structure. These polynomials were expressed in a determinantal form, similar to that found in Chapter 1 (1.1.14), except that the last row consisted of a sequence of monomials dependent on x and y (4.3.1a). Then a generalized version of the well known Sylvester Identity (B.4) was applied to (4.3.1a) with the purpose of deriving explicit forms (in terms of Hankel determinants) for the two recursive structures, that would be required to establish a two-variable polynomial structure. For the case of the x recurrence relation, this approach was straightforward since only two terms in the sequence of monomials were not dependent on x. So a 3 row/column Sylvester identity (B.7) was used to take x out of the determinant as a common factor to gain a recurrence type relation in terms of x, $P_k^{(l)}$ and the corresponding Hankel determinant $\Delta_k^{(l)}$. When this relation was coupled with a linear relation in $P_k^{(l)}$ (4.3.7a), a closed form x recurrence relation was derived, where the coefficients were defined in terms of the Hankel determinants. An x recurrence relation was also derived for the $Q_k^{(l)}$ polynomials, which were similar in structure to the $P_k^{(l)}$ except for the omission of a row, which was intended to deal with the problem of the monomial $\boldsymbol{e}_1$ which does not exist (a consequence of the elliptic curve).

This was followed by a section on the Hankel determinants, with particular attention paid to the interaction between the $\Delta_k^{(l)}$ and $\Theta_k^{(l)}$ Hankel determinants. Numerous bilinear relations were derived, of which some had particular use with reducing relations and others were interesting for their distinctly Hirota type bilinear form.

We then considered the y recurrence relation, which was a problem, because in the monomials y only occurred in every other term. Using the relationship between the y and the x from the curve counteracted this problem, except with the consequence of a new determinantal structure being introduced $\bar{P}_k^{(l)}$. After applying a 4 row/colum

Sylvester determinant identity to (4.3.1a), a y could be removed as a common factor and a y recurrence-type relation was derived in terms of $\bar{P}_k^{(l)}$ and $P_k^{(l)}$ (4.3.39). By altering the linear relation (4.3.7a) so that it satisfied $\bar{P}_k^{(l)}$, allowed it to be combined with (4.3.39) to create a four term closed-form semi-y recurrence relation. Essentially this gives a complete recurrence relation for $\bar{P}_k^{(l)}$, but an incomplete recurrence relation for $P_k^{(l)}$.

Then the issue of consistency and compatibility was considered. We gave a series of inner product relations that connected the recurrence coefficients for both X_k and Y_k and for the X_k case, we were able to derive further relations between the Δ_k's that satisfied the consistency between the recurrence coefficients X_k^{-1} and X_k^1. A full derivation of the compatibility between the x-recurrence relation (4.2.6a) and the y-recurrence relation (4.2.6b) was presented in Appendix D, which provides relations between the recurrence coefficients X_k and Y_k respectively. These relations can be reduced by using the corresponding consistency relations.

Having dealt with the derivation, compatibility and consistency of the recurrence relations and corresponding recurrence coefficients we ended the formal part of the chapter with a look at the applications of the recurrence relations. These included the generation of a sequence of elliptic polynomials (where the full derivation was given in Appendix (E)) and the construction of a pair of Christoffel-Darboux relations. Both these sections are important consequences of the recurrence relations and will provide useful tools in the further research of these formal orthogonal polynomials.

The last part of the chapter considers the extension beyond the case of formal orthogonal polynomials to the case where we are dealing with a weight function and corresponding integration interval. A few possible avenues for exploration were established, which we hope to pursue in the future.

Chapter 5

Conclusions and Speculations

Orthogonal polynomials and discrete integrable systems have been the topics of interest in this thesis, where we have been especially interested in connections between the two.

Our opening chapter begins with an introduction to the basic theory of orthogonal polynomials, including the a derivation of the general recurrence relation for orthogonal polynomials using determinants. This derivation is important for two reasons; it is the method used in Chapter 4 to derive explicit recurrence relations (for two variable orthogonal polynomials) and it produces a bilinear relation (1.1.27) that represents an early connection (but not the only one) with integrable systems.

$$\Delta_n^{(m)}\Delta_{n-2}^{(m+2)} = \Delta_{n-1}^{(m+2)}\Delta_{n-1}^{(m)} - \Delta_{n-1}^{(m+1)}\Delta_{n-1}^{(m+1)}$$

Orthogonal polynomials are just one type of special function, so we highlight some others including the Hypergeometric function and the Heun (and Lamé) function, which have connections to orthogonal polynomials. Heun and Lamé have polynomial solutions (for special values of the eigenvalue) and the hypergeometric series reduces to orthogonal polynomials for special values of the parameters. We also introduce two examples of applications of orthogonal polynomials (quantum mechanics, random matrix models), because these also lead to connections to integrable systems.

Chapter 2 is about exploring the connection between semi-classical orthogonal polynomials (which in comparison with the classical orthogonal polynomials, we can no longer express the coefficients of the recurrence in explicit form, but where we still have differential relations and in addition to the recurrence relation which are "order-independent") on the one hand and certain integrable systems (in particular discrete Painlevé equations) on the other hand. Many results already exist in the literature (motivated by connections on Random Matrix Models and going back to Freud [65]) but some novel cases have been treated in this chapter.

Our approach to finding these connections is through the Laguerre method (Section 2.3) and the corresponding compatibility relations (of Laguerre-Freud type) (2.3.37),(2.3.38), which are governed by the Pearson equation (2.0.2). Depending on the choice of weight function w, the Pearson equation will produce a polynomial for V and W, which in turn controls the outcome of Ω (2.4.2) and Θ (2.4.1) (the key entries in the differential structure governing the semi-classical orthogonal polynomials). For example, if the weight is an exponential, then $W = 1$ and the expansions of Ω and Θ are greatly simplified. From the two exponential weights that we use, in both instances it is possible to reduce the compatibility relation(s) to a discrete Painlevé equation, d-$\mathrm{P_I}$.

$$S_n^2 a_3 + S_n a_2 + a_1 = -a_3 \left(\frac{n+1}{a_3(S_{n+1}+S_n)+a_2} + \frac{n}{a_3(S_n+S_{n-1})+a_2} \right)$$

By comparison weights such as the semi-classical Laguerre and the Jacobi (Sections 2.4.2 and 2.6.1) have much more involved expansions, as such their corresponding compatibility relations are more complex. In all four cases we are able to derive two coupled non-linear difference equations for the recurrence coefficients R_n and S_n, where the remaining relations were trivial. For instance the Laguerre weight $l_0 = (x-t)^\alpha e^{-(a_1 x + \frac{a_2}{2}x^2)}$ produces the closed-form system

$$\begin{aligned} &a_2(R_{n+1}+R_n) = -S_n(a_2 S_n + (a_1 - a_2 t)) + (2n+1+a_1 t+\alpha), \\ &R_{n+1}(a_2(S_{n+1}+S_n) + (a_1 - a_2 t)) - R_n(a_2(S_n+S_{n-1}) - (a_1 - a_2 t)) = S_n - t. \end{aligned}$$

With the exception of the systems (2.4.6,2.4.11) derived using deformed Hermite weights, all the other coupled non-linear difference equations are examples of new discrete Painlevé type systems that have not yet appeared in the literature (as far as we are aware). Use of this method for other semi-classical weight functions, may yield further systems. It is interesting to note, that even when the number of relations produced by the compatibility relations for four different weights differed, they always only yielded two closed-form non-trivial relations. Of course the orthogonal polynomials connection gives special solutions of the difference systems related to particular initial value problems.

In many cases in the literature [167], the Freud-Laguerre equations can be reduced to a single second order nonlinear difference equation, but that is not always the case. For the cases we have investigated we have obtained Freud-Laguerre systems in the form of coupled difference equation. It is not always obvious or even easy to establish whether these systems can be further reduced.

In Chapter 3 the emphasis has been reversed: rather than starting with orthogonal polynomials, we start with structures underlying integrable systems, namely (singular) linear integral transforms that preserve the structure of certain linear difference equations, arising in Lax pairs for integrable systems. Such integral transforms amount to dressing transformations (from known solutions (indicated by upper label 0) to new solutions (upper index 1) of the "dressed" system, and the measures are interpolating measures between these solutions) and are related to integral equations arising in the inverse scattering transform. There has been some research done in this direction by Case [33, 34, 35] (who has established a formulation of orthogonal polynomial theory in terms of inverse scattering), but his approach has been in configuration space, whereas our perspective is from the spectral space.

Then there are two ways this is applied to the situation of orthogonal polynomials: first, adapting it to the 2×2 systems arising in the Laguerre method, and second the scalar reduction (applicable to the even weight case), with as a diversion, the construction via the lattice Gel'fand-Dikii hierarchy. The first approach is done in a speculative manner, considering the 2×2 recurrence relation, and while we derive integral transforms between the recurrence coefficients, an explicit form for a polynomial integral transform eludes us. The second approach produces a vector reduction of an integral transform associated with the Gel'fand-Dikii hierarchy.

$$\boldsymbol{\phi}_k^1 = \boldsymbol{\phi}_k^0 + \sum_{q=1}^{N} \int_{C_q} \boldsymbol{\phi}_l^1 d\boldsymbol{\lambda}_q(l) \frac{\begin{vmatrix} \boldsymbol{\phi}_l^0 & \cdots & \overset{q\downarrow}{\boldsymbol{\phi}^0_{\omega^{j-1}k}} & \cdots & \boldsymbol{\phi}^0_{\omega^{N-1}l} \end{vmatrix}}{k^N - l^N}$$

This result for the $N \times N$ case, along with the specific $N = 2$ case (which represents KdV)

$$\begin{aligned} u_{k1}^1 = u_{k1}^0 \quad &+ \quad \int_{\boldsymbol{\Gamma}_1} u_{l1}^1 d\boldsymbol{\lambda}_1(l) \frac{(p-k)\tilde{u}_{k1}^0 u_{-l1}^0 - (p+l)\tilde{u}_{-l1}^0 u_{k1}^0}{k^2 - l^2} \\ &+ \quad \int_{\boldsymbol{\Gamma}_2} u_{l1}^1 d\boldsymbol{\lambda}_2(l) \left(\frac{p-l}{p+l}\right)^n \frac{(p-k)\tilde{u}_{k1}^0 u_{l1}^0 - (p-l)\tilde{u}_{l1}^0 u_{k1}^0}{k^2 - l^2}. \end{aligned}$$

provides an alternate singular integral transform that can be rewritten (after a gauge transform) to give an integral transform for a class of orthogonal polynomials with an even weight (such as the Hermite polynomials). We present a possible transform, which produces a suitable transform for the recurrence coefficients, from which an interpolating measure could be deduced. In addition, the differential part of the linear problem is considered where the focus is on differential equations which may or may not have polynomial solutions. We derive a method that gives the coefficients of a transformed differential equation in terms of the original (source) differential equation. Whether this method also works for polynomial solutions is one area of further study in this topic.

One of the questions that arises is whether the dressing approach allows one to make "dressing transforms" which will effectively lead from classical orthogonal polynomials to their semiclassical (deformed) counterparts through integral transforms involving so-called "interpolating measures".

The remainder of the thesis deals with generalizations of orthogonal polynomials to the elliptic case. The term "elliptic polynomials" has already appeared in the literature in connection with the Carlitz and Lamé polynomials, whereas in Chapter 4 a novel class of formal orthogonal polynomials is introduced, which are 2-variable orthogonal polynomials over an elliptic curve. So, these are really algebraic functions in terms of one variable and different from the other type of elliptic polynomials.

The chapter's main focus is on establishing the recursive structures inherent in any class of orthogonal polynomials and thus, we are mainly interested in a formal construction (no weight function). In Chapter 1 we demonstrated that the recurrence relations for x (4.2.6a) and y (4.2.6b) are easy to derive implicitly (Section 1.1.3), but much more difficult to derive explicitly (Section 1.1.5).

$$\begin{aligned} xP_k &= P_{k+2} + X_k^{(1)} P_{k+1} + X_k^{(0)} P_k + X_k^{(-1)} P_{k-1} + X_k^{(-2)} P_{k-2}, \\ yP_k &= 4^{\epsilon_k} P_{k+3} + Y_k^{(2)} P_{k+2} + Y_k^{(1)} P_{k+1} + Y_k^{(0)} P_k + Y_k^{(-1)} P_{k-1} \\ &\quad + Y_k^{(-2)} P_{k-2} + Y_k^{(-3)} P_{k-3} \end{aligned}$$

In principle we could have a system of commuting difference operators over an algebraic curve with $xP = \Xi P$, $yP = \Upsilon P$ where Ξ and Υ are the difference operators associated with these recurrence relations. The compatibilities follow from $[\Xi, \Upsilon] = 0$ (commutativity) and the relation on the curve $y^2 = 4x^3 - g_2 x - g_3$. This, conjecturally, is a system connected to a discrete version of the Krichever-Novikov system ([99]).

We introduce the generalized Sylvester Identity (B.4) in Appendix B, which can remove m rows and columns rather than the conventional 2 rows and columns. The use of this with the two variable polynomial determinant representation (4.3.1a) leads to the explicit derivation of an xP_k and xQ_k recurrence relation, where the coefficients are defined in terms of the Hankel determinants Δ and Θ. We are able to derive many bilinear relations between the Hankel determinants and these are of particular use in simplifying expressions, although only one expression was found for the Δ (which we are able to derive in two separate ways)

$$\begin{aligned} & \Delta_{k+1}^{(l)}\left(\Delta_{k}^{(l+4)}\Delta_{k}^{(l+1)}\Delta_{k-1}^{(l+3)}-\Delta_{k}^{(l+2)}\Delta_{k}^{(l+3)}\Delta_{k-1}^{(l+3)}+\Delta_{k-2}^{(l+4)}\Delta_{k}^{(l+3)}\Delta_{k+1}^{(l+1)}\right) \\ = \; & \Delta_{k-1}^{(l+4)}\left(\Delta_{k+2}^{(l)}\Delta_{k}^{(l+1)}\Delta_{k-1}^{(l+3)}-\Delta_{k}^{(l+2)}\Delta_{k}^{(l+1)}\Delta_{k+1}^{(l+1)}+\Delta_{k}^{(l)}\Delta_{k}^{(l+3)}\Delta_{k+1}^{(l+1)}\right) \end{aligned}$$

and we can compare, with the Toda-type equation from Chapter 1 (1.1.27).

While the generalized Sylvester identity leads to the derivation of an explicit form for the xP_k-recurrence relation, it produced a new object when trying a similar approach for the yP_k-recurrence relation. Thus, we derive a $y\bar{P}_k$ relation instead. We expect from this relation that we can derive the recurrence y-relation in explicit form. An alternative approach is to work out the equations for the coefficients Y from the condition of the curve (Appendix D) and this can only be really done when we move forward to the case of non-formal elliptic orthogonal polynomials defined through specific weight functions. This is the subject of future research.

When considering extensions beyond the recurrence relations (4.2.6), we can consider the further study of the formal structure, which includes the generation of a sequence of the "Weierstrass elliptic polynomials"(Section 4.4) and the derivation of a pair of Christoffel-Darboux relations (for x and y) (Section 4.5). We can also consider a non-formal structure where we specify weight functions to obtain differential relations and we conjecture discrete Painlevé type equations associated with these structures, but that that is still somewhat speculative.

Appendices

A Elliptic Functions

An elliptic function is a function defined on the complex plane which is periodic in two directions and thus can be compared with the trigonometric functions (which have a single period only). Elliptic functions arise as inverses of elliptic integrals in the study of geometric problems such as the arc length of the ellipse and mechanical problems such as the dynamics of the mathematical pendulum.

In most treatments elliptic functions are defined as doubly periodic meromorphic functions on the complex plane $\mathbb{C}$. According to a theorem of Jacobi a function of one complex variable can have at most two independent primitive periods Ω_1 and Ω_2 such that $\frac{\Omega_1}{\Omega_2}$ is not real.

Throughout the development of the theory of elliptic functions, modern authors mostly follow Karl Weierstrass, since the notations of the Weierstrass's elliptic $\wp$-function are convenient, and any elliptic function can be expressed in terms of these. The elliptic functions introduced by Carl Jacobi, and the auxiliary theta functions (not doubly-periodic), are more complicated but important for the general theory. The main difference between these two theories is that the Weierstrass functions have high-order poles located at the corners of the periodic lattice, whereas the Jacobi functions have simple poles. The development of the Weierstrass theory is easier to present and understand, having fewer complications.

Weierstrass Elliptic Functions

While there are three Weierstrass functions $\wp(z), \zeta(z)$ and $\sigma(z)$, this appendix focuses on the $\wp(z)$ function, which along with its derivative satisfies the Weierstrass elliptic curve. We can state $\wp(z)$, by expressing it in terms of its half-periods (ω_1 and ω_2) or its elliptic invariants (g_2 and g_3). Thus, $\wp(z)$ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}^{\infty} \left(\frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right), \tag{A.1}$$

where the terms in the double sum giving a zero denominator are omitted.

The differential equation satisfied by $\wp(z)$ arises by expanding the function $f(z) = \wp(z) - z^{-2}$ about the origin, but since $f(0) = 0$ and the function is even, $f'(0) = f^{(3)}(0) = 0$ ($\frac{df}{dz} = f'(z)$) and we have

$$f(z) = \wp(z) - z^{-2} = \frac{1}{2!} f''(0) z^2 + \frac{1}{4!} f^{(4)}(0) z^4 + \dots \tag{A.2}$$

Since $f(z)$ is the sum in (A.1), we simply differentiate it the required amount of times, set $z = 0$ and substitute it back into (A.2),

$$\wp(z) - z^{-2} = 3 \sum \Omega_{mn}^{-4} z^2 + 5 \sum \Omega_{mn}^{-6} z^4 + O(z^6) \tag{A.3}$$

where $\Omega_{mn} = 2m\omega_1 + 2n\omega_2$. We now define the elliptic invariants g_2 and g_3 by

$$g_2 = 60 \sum \Omega_{mn}^{-4} \tag{A.4a}$$

$$g_3 = 140 \sum \Omega_{mn}^{-6} \tag{A.4b}$$

then $\wp(z)$ and $\wp'(z)$ can be written as

$$\begin{aligned} \wp(z) &= z^{-2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6), \\ \wp'(z) &= -2z^{-3} + \frac{1}{10} g_2 z + \frac{1}{7} g_3 z^3 + O(z^5), \end{aligned}$$

the two equations can be equated by cubing the first and squaring the last

$$\wp^3(z) = z^{-6} + \frac{3}{20}g_2 z^{-2} + \frac{3}{28}g_3 + O(z^2), \tag{A.5a}$$

$$(\wp')^2(z) = 4z^{-6} - \frac{2}{5}g_2 z^{-2} - \frac{4}{7}g_3 + O(z^2), \tag{A.5b}$$

to leave

$$(\wp')^2(z) - 4\wp^3(z) + g_2 z^{-2} + g_3 = O(z^2).$$

But the Weierstrass elliptic function is analytic at the origin and therefore at all points congruent to the origin. There are no other places where a singularity can occur, so this function is an elliptic function with no singularities. By Liouville's elliptic function theorem, it is therefore a constant. Thus when $z \to 0$, $O(z^2) \to 0$ leaving

$$(\wp')^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3 = 0. \tag{A.6}$$

This first order differential equation can also be differentiated again to give a second order differential equation.

$$\wp''(z) = 6\wp^2(z) - \frac{g_2}{2} \tag{A.7}$$

With $(\wp', \wp) = (y, x)$, the differential equation becomes the Weierstrass cubic equation for an elliptic curve, with the branch points e_1, e_2, e_3, where $e_i = \wp(\omega_i)$

$$y^2 = 4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3) \tag{A.8}$$

where

$$g_2 = -4(e_1e_2 + e_2e_3 + e_3e_1), \tag{A.9a}$$

$$g_3 = 4e_1e_2e_3. \tag{A.9b}$$

The Weierstrass $\wp$ function also satisfies a number of identities, some of which include the other elliptic functions $\zeta(z), \sigma(z)$ and one which involves a determinant. We can relate these other Weierstrass functions using some simple relations

$$\wp(z) = -\frac{d\zeta(z)}{dz} \quad , \quad \zeta(z) = \frac{1}{\sigma(z)}\frac{d\sigma(z)}{dz} \tag{A.10}$$

and some more involved formulae

$$\zeta(x+y) - \zeta(x) - \zeta(y) = \frac{1}{2}\frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)} \tag{A.11}$$

$$\wp(y) - \wp(x) = \frac{\sigma(x-y)\sigma(x+y)}{\sigma^2(x)\sigma^2(y)}. \tag{A.12}$$

$$\wp(x) + \wp(y) + \wp(x+y) = \frac{1}{4}\left(\frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)}\right)^2 \tag{A.13}$$

An elliptic determinantal identity is the Frobenius-Stickelburger formula [67] which can be defined as

$$\begin{vmatrix} 1 & \wp(\kappa_0) & \wp'(\kappa_0) & \cdots & \wp^{(n-1)}(\kappa_0) \\ 1 & \wp(\kappa_1) & \wp'(\kappa_1) & \cdots & \wp^{(n-1)}(\kappa_1) \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \wp(\kappa_n) & \wp'(\kappa_n) & \cdots & \wp^{(n-1)}(\kappa_n) \end{vmatrix}$$
$$= (-1)^{\frac{1}{2}n(n-1)} 1!2!\cdots n! \frac{\sigma(\kappa_0 + \kappa_1 + \cdots + \kappa_n)\prod_{i<j}\sigma(\kappa_i - \kappa_j)}{\sigma^{n+1}(\kappa_0)\cdots\sigma^{n+1}(\kappa_n)} . \tag{A.14}$$

Jacobi Elliptic Functions

The Jacobian elliptic functions correspond to an arrow drawn from one corner of a rectangle to another, where the corners of the rectangle are labelled s, c, d and n. The twelve Jacobian elliptic functions are then pq, where each of p and q is one of the four letters. The most commonly used of these twelve are denoted by cn(u,k), dn(u,k), and sn(u,k), where k is known as the elliptic modulus and u is an incomplete elliptic integral of the first kind. The easiest way to understand Jacobi elliptic functions is as inverses of u

$$u = F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}, \tag{A.15}$$

where $0 < k^2 < 1$, so we consider

$$\phi = F^{-1}(u,k) = \text{am}\,(u,k) \tag{A.16}$$

where am (u, k) = am (u) is the Jacobi amplitude. Then it follows that we have the following list of Jacobi elliptic functions

$$\sin(\phi) = \sin(\text{am}\ (u,k)) = \text{sn}\ (u), \tag{A.17a}$$

$$\cos(\phi) = \cos(\text{am}\ (u,k)) = \text{cn}\ (u), \tag{A.17b}$$

$$\sqrt{1-k^2\sin^2\phi} = \sqrt{1-k^2\sin^2(\text{am}\ (u,k))} = \text{dn}\ (u). \tag{A.17c}$$

where

$$\mathrm{sn}^2(x) + \mathrm{cn}^2(x) = 1 \quad , \quad k^2\mathrm{sn}^2(x) + \mathrm{dn}^2(x) = 1 \tag{A.18}$$

There are also some important addition and differentiation identities involving Jacobian elliptic functions. Thus, we have the addition formula for the Jacobi sn(x) function

$$\mathrm{sn}(x+y) = \frac{\mathrm{sn}(x)\mathrm{cn}(y)\mathrm{dn}(y) + \mathrm{sn}(y)\mathrm{cn}(x)\mathrm{dn}(x)}{1-k^2\mathrm{sn}^2(x)\mathrm{sn}^2(y)} \tag{A.19}$$

the Jacobi cn(x) function

$$\mathrm{cn}(x+y) = \frac{\mathrm{cn}(x)\mathrm{cn}(y) - \mathrm{dn}(x)\mathrm{dn}(y)\mathrm{sn}(x)\mathrm{sn}(y)}{1-k^2\mathrm{sn}^2(x)\mathrm{sn}^2(y)} \tag{A.20}$$

and the Jacobi dn (x) function

$$\mathrm{dn}(x+y) = \frac{\mathrm{dn}(x)\mathrm{dn}(y) - k^2\mathrm{cn}(x)\mathrm{cn}(y)\mathrm{sn}(x)\mathrm{sn}(y)}{1-k^2\mathrm{sn}^2(x)\mathrm{sn}^2(y)}. \tag{A.21}$$

The differentiation formula are

$$\mathrm{sn}'(x) = \mathrm{cn}(x)\mathrm{dn}(x)\ , \quad \mathrm{cn}'(x) = -\mathrm{sn}(x)\mathrm{dn}(x)\ , \quad \mathrm{dn}'(x) = -k^2\mathrm{cn}(x)\mathrm{dn}(x). \tag{A.22}$$

B Some Determinant Identities

In the establishment of the recursive structure of orthogonal polynomials we need a number of identities, which we derive using the Sylvester Identity. So we present a proof of the Sylvester identity, which was first presented by Kowalewski [96], Bareiss [15] and Malaschonok [109, 110] and these seven proofs are presented together in [7].

We consider an $(n+m)\times(n+m)$ matrix R with elements r_{ij} and determinant $|R|$, also written $\det(R)$. Then we partition R and factor by block triangularization such that

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & \mathbf{1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1} & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix} \tag{B.1}$$

where A is a nonsingular square matrix of order n, then

$$|R| = |A|.|D - CA^{-1}B|. \tag{B.2}$$

If we multiply both sides by $|A|^{m-1}$, this becomes

$$|A|^{m-1}|R| = ||A|(D - CA^{-1}B)|$$

because the determinant on the right side of (B.2) is of order m. We can reduce this equation further to

$$|A|^{m-1}|R| = ||A|D - C\widetilde{A}B|, \tag{B.3}$$

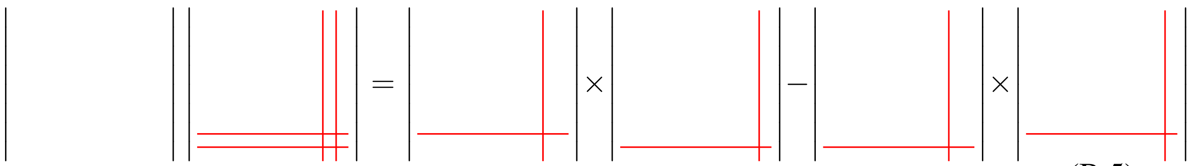
since $A^{-1} = \frac{\widetilde{A}}{|A|}$ (where $\widetilde{A}$ represents the adjugate matrix of the inverse matrix A^{-1}), and the determinant of A is assumed to be $\neq 0$. Specifying some entries in (B.1), taking A to be an $n\times n$ block and D to be an $m\times m$ block, we have the formula:

$$(\det(A))^{m-1} \begin{vmatrix} A & | & b_1 & \dots & b_m \\ - & + & - & - & - \\ c_1^t & | & & & \\ \vdots & | & & D & \\ c_m^t & | & & & \end{vmatrix} = \det_{m\times m}\left\{\det_{n\times n}(A)D_{ij} - (c_i^t\,\widetilde{A}\,b_j)\right\}_{i,j=1,\cdots,m} \tag{B.4}$$

in which the full matrix is supplemented with m n-component column vectors b_i and m n-component row-vectors c_i^t. If we consider the case $m=2$ ie. the removal of two rows and columns, then we get then determinant identity

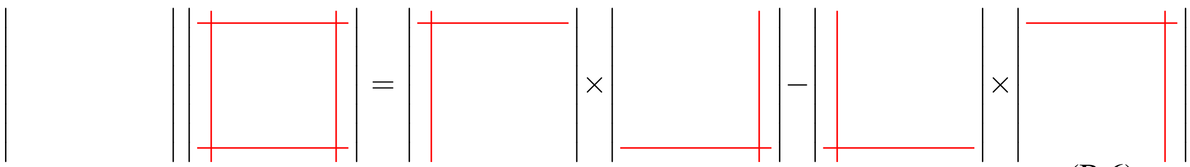
$$
\begin{aligned}
(\det(A))\begin{vmatrix} A & | & b_1 & b_2 \\ - & + & - & - \\ c_1^t & | & d_{11} & d_{12} \\ c_2^t & | & d_{21} & d_{22} \end{vmatrix} &= \det{}_{2\times 2}\left\{(\det(A))\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22}\end{pmatrix} - \begin{pmatrix} c_1^t\widetilde{A}b_1 & c_1^t\widetilde{A}b_2 \\ c_2^t\widetilde{A}b_1 & c_2^t\widetilde{A}b_2 \end{pmatrix}\right\} \\
&= [\det(A)d_{11} - c_1^t\widetilde{A}b_1]\,[\det(A)d_{22} - c_2^t\widetilde{A}b_2] \\
&\quad -[\det(A)d_{21} - c_2^t\widetilde{A}b_1]\,[\det(A)d_{12} - c_1^t\widetilde{A}b_2],
\end{aligned}
$$

which can be symbolically written as:



(B.5)

(where the red lines denote rows and columns omitted from the original determinant). It is then necessary to reorder the position of the row and column to tailor the identity to our requirements. In the case of the derivation of a general recurrence relation for orthogonal polynomials we choose the following alignment:



,

(B.6)

where both the penultimate row and column, have both been shifted $n-1$ places. Since they are both shifted the same distance, it is not necessary to change the sign of the determinant.

While (B.6) is the key identity by which the recurrence structure for ordinary one-variable orthogonal polynomials is obtained, for the elliptic two-variable orthogonal polynomials we need (in addition to (B.6)), determinantal identities involving the simultaneous removal of more than two rows and columns. Thus, the main identities used from the general formula (B.4) will be the cases $m = 3$ and $m = 4$, leading to the different recurrence relations for (4.3.1a) and (4.3.2a).

In the case $m = 3$ we obtain from (B.4) the following 3-row/column Sylvester type identity:

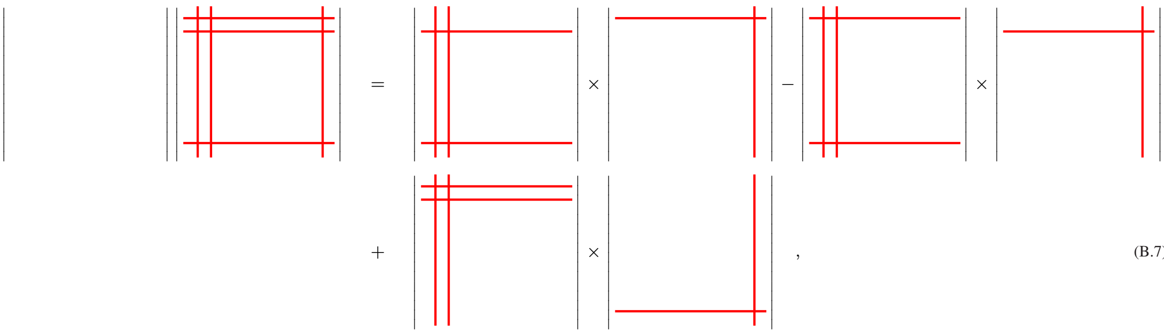
$$
(\det(A))^2 \begin{vmatrix} A & | & b_1 & b_2 & b_3 \\ - & + & - & - & \\ c_1^t & | & d_{11} & d_{12} & d_{13} \\ c_2^t & | & d_{21} & d_{22} & d_{23} \\ c_3^t & | & d_{31} & d_{32} & d_{33} \end{vmatrix} = \det_{3\times 3}\left\{ (\det(A)) \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} - \begin{pmatrix} c_1^t \tilde{A} b_1 & c_1^t \tilde{A} b_2 & c_1^t \tilde{A} b_3 \\ c_2^t \tilde{A} b_1 & c_2^t \tilde{A} b_2 & c_2^t \tilde{A} b_3 \\ c_3^t \tilde{A} b_1 & c_3^t \tilde{A} b_2 & c_3^t \tilde{A} b_3 \end{pmatrix} \right\}
$$

$$
= \begin{vmatrix} A & b_1 \\ c_1^t & d_{11} \end{vmatrix} \times \begin{vmatrix} A & | & b_2 & b_3 \\ - & + & - & - \\ c_2^t & | & d_{22} & d_{23} \\ c_3^t & | & d_{32} & d_{33} \end{vmatrix} (\det(A))
$$

$$
- \begin{vmatrix} A & b_1 \\ c_2^t & d_{21} \end{vmatrix} \times \begin{vmatrix} A & | & b_2 & b_3 \\ - & + & - & - \\ c_1^t & | & d_{12} & d_{13} \\ c_3^t & | & d_{32} & d_{33} \end{vmatrix} (\det(A))
$$

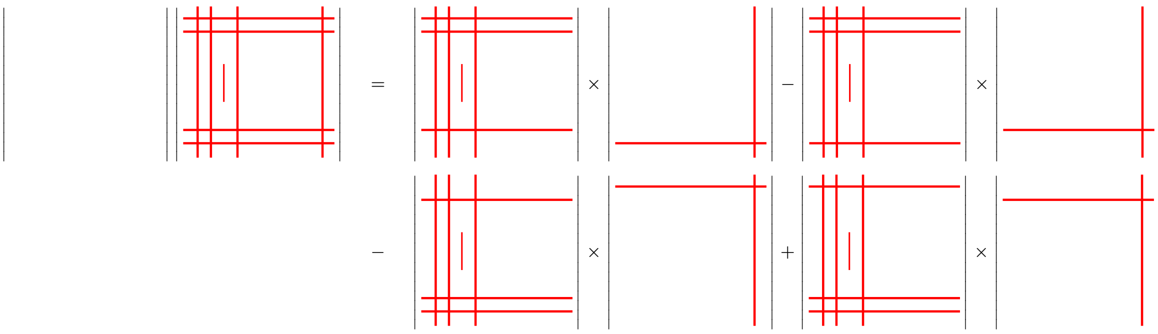
$$
+ \begin{vmatrix} A & b_1 \\ c_3^t & d_{31} \end{vmatrix} \times \begin{vmatrix} A & | & b_2 & b_3 \\ - & + & - & - \\ c_1^t & | & d_{12} & d_{13} \\ c_2^t & | & d_{22} & d_{23} \end{vmatrix} (\det(A))
$$

which can be expressed graphically as:



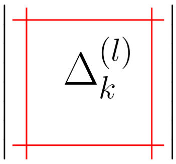
(B.7)

In a similar way the 4-row/column Sylvester identity is obtained:

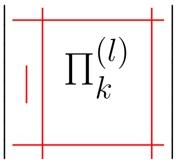


(B.8)

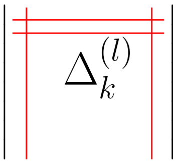
Hankel Identities



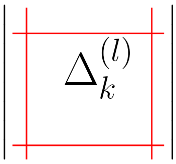
$$\Rightarrow \quad \Delta_k^{(l)}\Pi_{k-2}^{(l+1)} = \Delta_{k-1}^{(l)}\Pi_{k-1}^{(l+1)} - \Delta_{k-1}^{(l+1)}\Pi_{k-1}^{(l)} \tag{B.9a}$$



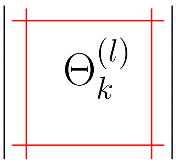
$$\Rightarrow \quad \Pi_k^{(l)}\Delta_{k-2}^{(l+3)} = \Pi_{k-1}^{(l)}\Delta_{k-1}^{(l+3)} - \Pi_{k-1}^{(l+1)}\Delta_{k-1}^{(l+2)} \tag{B.9b}$$



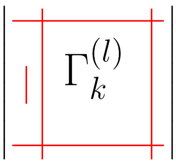
$$\Rightarrow \quad \Delta_k^{(l)}\Pi_{k-2}^{(l+2)} = \Theta_{k-1}^{(l)}\Pi_{k-1}^{(l+1)} - \Delta_{k-1}^{(l+1)}\Gamma_{k-1}^{(l)} \tag{B.9c}$$



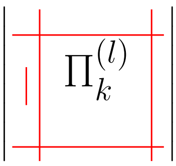
$$\Rightarrow \quad \Delta_k^{(l)}\Gamma_{k-2}^{(l)} = \Delta_{k-1}^{(l)}\Gamma_{k-1}^{(l)} - \Theta_{k-1}^{(l)}\Pi_{k-1}^{(l)} \tag{B.9d}$$



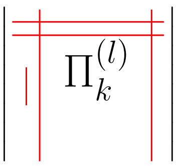
$$\Rightarrow \quad \Theta_k^{(l)}\Pi_{k-2}^{(l+2)} = \Theta_{k-1}^{(l)}\Pi_{k-1}^{(l+2)} - \Delta_{k-1}^{(l+2)}\Gamma_{k-1}^{(l)} \tag{B.10a}$$



$$\Rightarrow \quad \Gamma_k^{(l)}\Delta_{k-2}^{(l+4)} = \Delta_{k-1}^{(l+4)}\Gamma_{k-1}^{(l)} - \Theta_{k-1}^{(l+2)}\Pi_{k-1}^{(l+1)} \tag{B.10b}$$



$$\Rightarrow \quad \Pi_k^{(l)}\Theta_{k-2}^{(l+2)} = \Theta_{k-1}^{(l+2)}\Pi_{k-1}^{(l)} - \Gamma_{k-1}^{(l)}\Delta_{k-1}^{(l+2)} \tag{B.10c}$$



$$\Rightarrow \quad \Pi_k^{(l)}\Delta_{k-2}^{(l+4)} = \Delta_{k-1}^{(l+3)}\Gamma_{k-1}^{(l)} - \Theta_{k-1}^{(l+2)}\Pi_{k-1}^{(l+1)} \tag{B.10d}$$

C Adjacent Orthogonal 2-Polynomials

Here we present an alternative description of the extended polynomials, which, albeit less convenient for the derivation of the recurrence structure developed in section 3, are in a sense more natural since they remain orthogonal at each level indicated by the index l. Noting that the adjacent family of functionals given by the inner product:

$$\mathcal{L}_l(\cdot) = \langle \boldsymbol{e}_l, \cdot \rangle$$

generates in a natural way a set of moments, it is immediate that the family of two-variable polynomials associated with the curve (4.2.1) given by

$$R_k^{(l)}(x,y) \equiv \begin{vmatrix} \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_0, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_0, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_0, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_{k-1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_{k-1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_{k-1}, \boldsymbol{e}_k \rangle \\ \boldsymbol{e}_0 & \boldsymbol{e}_2 & \cdots & \cdots & \boldsymbol{e}_k \end{vmatrix} \Big/ E_{k-1}^{(l)} \quad , \tag{C.1}$$

together with the corresponding Hankel determinant:

$$E_k^{(l)} = \begin{vmatrix} \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_0, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_0, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_0, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_k, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_k, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_k, \boldsymbol{e}_k \rangle \end{vmatrix} \quad , \tag{C.2}$$

for each fixed l, ($l \neq 0, 1$), forms an orthogonal family of polynomials relative to the inner product $\langle \cdot, \cdot \rangle_l = \langle \boldsymbol{e}_l \cdot, \cdot \rangle$.

To compare the notation provided by (C.1) and (C.2) with the one of section 3, we note that

$$R_k^{(l)} = \begin{cases} Q_k^{(l)} & \text{for } \ l \ \text{ even} \\ \bar{Q}_k^{(l)} & \text{for } \ l \ \text{ odd} \end{cases} \quad , \qquad E_k^{(l)} = \begin{cases} \Theta_k^{(l)} & \text{for } \ l \ \text{ even} \\ \bar{\Theta}_k^{(l)} & \text{for } \ l \ \text{ odd} \end{cases}$$

In order to manipulate this ordered sequence of monomials: $\boldsymbol{e}_0, \boldsymbol{e}_2, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_l, \ldots$ in a convenient way we introduce a shift operator $\widehat{\cdot}$, which shifts the series by one step:

$$\boldsymbol{e}_0 \ , \ \boldsymbol{e}_2 \ , \ \boldsymbol{e}_3 \ , \ \ldots \qquad \widehat{\boldsymbol{e}}_0 = \boldsymbol{e}_2 \ , \ \widehat{\boldsymbol{e}}_2 = \boldsymbol{e}_3 \ , \ \widehat{\boldsymbol{e}}_3 = \boldsymbol{e}_4 \ , \ \ldots$$

thus:

$$\widehat{\boldsymbol{e}}_l = \begin{cases} \boldsymbol{e}_{l+2} & , \quad l = 0 \\ \boldsymbol{e}_{l+1} & , \quad l \neq 0 \end{cases} \quad . \tag{C.3}$$

Furthermore, we need to introduce:

$$Q_k^{(l)}(x,y) \equiv \begin{vmatrix} \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_k, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_k, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_k, \boldsymbol{e}_k \rangle \\ \boldsymbol{e}_0 & \boldsymbol{e}_2 & \cdots & \cdots & \boldsymbol{e}_k \end{vmatrix} \Big/ \Theta_{k-1}^{(l)} \quad , \tag{C.4}$$

together with its corresponding Hankel determinant:

$$\Theta_k^{(l)} = \begin{vmatrix} \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_2, \boldsymbol{e}_k \rangle \\ \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_3, \boldsymbol{e}_k \rangle \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_{k+1}, \boldsymbol{e}_0 \rangle & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_{k+1}, \boldsymbol{e}_2 \rangle & \cdots & \cdots & \langle \boldsymbol{e}_l \cdot \boldsymbol{e}_{k+1}, \boldsymbol{e}_k \rangle \end{vmatrix} \quad . \tag{C.5}$$

The following generalized Sylvester identity:leads to the recurrence relation:

$$P_k^{(l)} = xQ_{k-2}^{(l+2)} - \frac{\Delta_{k-2}^{(l)}\Theta_{k-2}^{(l+2)}}{\Delta_{k-1}^{(l)}\Theta_{k-3}^{(l+2)}}\, P_{k-1}^{(l)} + \frac{\Delta_{k-2}^{(l+2)}\Theta_{k-2}^{(l)}}{\Delta_{k-1}^{(l)}\Theta_{k-3}^{(l+2)}}\, Q_{k-1}^{(l)} \tag{C.6}$$

D Elliptic Polynomial Compatibility

We consider the consistency between the x and y of the Weierstrass elliptic curve $y^2 = 4x^3 - g_2x - g_3$. The expansion of x and y involves the substitution of the recursion relations (4.2.6), however the expansion of y also makes use of (4.4.3), where the curve comes into play. We have two expressions for the y recursion relation

$$\begin{aligned} yP_{2n} &= P_{2n+3} + Y_{2n}^{(2)}P_{2n+2} + Y_{2n}^{(1)}P_{2n+1} + Y_{2n}^{(0)}P_{2n} + Y_{2n}^{(-1)}P_{2n-1} \\ &\quad + Y_{2n}^{(-2)}P_{2n-2} + Y_{2n}^{(-3)}P_{2n-3} \qquad \text{(D.1a)} \\ yP_{2n+1} &= 4P_{2n+4} + Y_{2n+1}^{(2)}P_{2n+3} + Y_{2n+1}^{(1)}P_{2n+2} + Y_{2n+1}^{(0)}P_{2n+1} + Y_{2n+1}^{(-1)}P_{2n} \\ &\quad + Y_{2n+1}^{(-2)}P_{2n-1} + Y_{2n+1}^{(-3)}P_{2n-2} \qquad \text{(D.1b)} \end{aligned}$$

and so must consider both cases separately. Although there is a single expression for the x recursion relation we can express it for k odd or even. First consider y^2P_k for k even,

$$\begin{aligned} y^2P_{2n} &= 4P_{2n+6} + \left(Y_{2n+3}^{(2)} + Y_{2n}^{(2)}\right)P_{2n+5} + \left(Y_{2n+3}^{(1)} + 4Y_{2n}^{(1)} + Y_{2n}^{(2)}Y_{2n+2}^{(2)}\right)P_{2n+4} \\ &+ \left(Y_{2n}^{(2)}Y_{2n+2}^{(1)} + Y_{2n}^{(1)}Y_{2n+1}^{(2)} + Y_{2n+3}^{(0)} + Y_{2n}^{(0)}\right)P_{2n+3} + \Big(Y_{2n+3}^{(-1)} \\ &\quad + 4Y_{2n}^{(-1)} + Y_{2n}^{(2)}Y_{2n+2}^{(0)} + Y_{2n}^{(0)}Y_{2n}^{(2)} + Y_{2n}^{(1)}Y_{2n+1}^{(1)}\Big)P_{2n+2} + \Big(Y_{2n}^{(-2)} \\ &\quad + Y_{2n}^{(1)}Y_{2n+1}^{(0)} + Y_{2n+3}^{(-2)} + Y_{2n}^{(-1)}Y_{2n-1}^{(2)} + Y_{2n}^{(0)}Y_{2n}^{(1)} + Y_{2n}^{(2)}Y_{2n+2}^{(-1)}\Big)P_{2n+1} + \Big((Y_{2n}^{(0)})^2 \\ &\quad + 4Y_{2n}^{(-3)} + Y_{2n+3}^{(-3)} + Y_{2n}^{(-1)}Y_{2n-1}^{(1)} + Y_{2n}^{(-2)}Y_{2n-2}^{(2)} + Y_{2n}^{(2)}Y_{2n+2}^{(-2)} + Y_{2n}^{(1)}Y_{2n+1}^{(-1)}\Big)P_{2n} \\ &+ \Big(Y_{2n}^{(2)}Y_{2n+2}^{(-3)} + Y_{2n}^{(-3)}Y_{2n-3}^{(2)} + Y_{2n}^{(0)}Y_{2n}^{(-1)} + Y_{2n}^{(-2)}Y_{2n-2}^{(1)} + Y_{2n}^{(-1)}Y_{2n-1}^{(0)} \\ &\quad + Y_{2n}^{(1)}Y_{2n+1}^{(-2)}\Big)P_{2n-1} + \Big(Y_{2n}^{(-1)}Y_{2n-1}^{(-1)} + Y_{2n}^{(-3)}Y_{2n-3}^{(1)} + Y_{2n}^{(-2)}Y_{2n-2}^{(0)} + Y_{2n}^{(0)}Y_{2n}^{(-2)} \\ &\quad + Y_{2n}^{(1)}Y_{2n+1}^{(-3)}\Big)P_{2n-2} + \Big(Y_{2n}^{(-1)}Y_{2n-1}^{(-2)} + Y_{2n}^{(0)}Y_{2n}^{(-3)} + Y_{2n}^{(-2)}Y_{2n-2}^{(-1)} \\ &\quad + Y_{2n}^{(-3)}Y_{2n-3}^{(0)}\Big)P_{2n-3} + \left(Y_{2n}^{(-1)}Y_{2n-1}^{(-3)} + Y_{2n}^{(-2)}Y_{2n-2}^{(-2)} + Y_{2n}^{(-3)}Y_{2n-3}^{(-1)}\right)P_{2n-4} \\ &+ \left(Y_{2n}^{(-2)}Y_{2n-2}^{(-3)} + Y_{2n}^{(-3)}Y_{2n-3}^{(-2)}\right)P_{2n-5} + Y_{2n}^{(-3)}Y_{2n-3}^{(-3)}P_{2n-6} \qquad \text{(D.2)} \end{aligned}$$

and the longer expression containing X_k

$$
\begin{aligned}
&(4x^3 - g_2 x - g_3)P_{2n} \\
= \;& 4P_{2n+6} + 4\left(X^{(1)}_{2n+4} + X^{(1)}_{2n+2} + X^{(1)}_{2n}\right) P_{2n+5} \\
+ \;& 4\left(X^{(1)}_{2n+3}\left(X^{(1)}_{2n} + X^{(1)}_{2n+2}\right) + X^{(1)}_{2n}X^{(1)}_{2n+1} + X^{(0)}_{2n+4} + X^{(0)}_{2n+2} + X^{(0)}_{2n}\right) P_{2n+4} \\
+ \;& 4\left(X^{(0)}_{2n+3}\left(X^{(1)}_{2n} + X^{(1)}_{2n+2}\right) + X^{(0)}_{2n+1}X^{(1)}_{2n} + X^{(1)}_{2n+2}\left(X^{(0)}_{2n} + X^{(0)}_{2n+2}\right) + X^{(1)}_{2n}X^{(0)}_{2n} + X^{(1)}_{2n}X^{(1)}_{2n+1}X^{(1)}_{2n+}\right. \\
& \left. + X^{(-1)}_{2n+4} + X^{(-1)}_{2n+2} + X^{(-1)}_{2n}\right) P_{2n+3} + 4\left(X^{(-1)}_{2n+3}\left(X^{(1)}_{2n+2} + X^{(1)}_{2n}\right) + X^{(-1)}_{2n+1}X^{(1)}_{2n}\right. \\
& + X^{(1)}_{2n+1}\left(X^{(-1)}_{2n} + X^{(-1)}_{2n+2}\right) + X^{(1)}_{2n-1}X^{(-1)}_{2n} + (X^{(0)}_{2n})^2 + X^{(0)}_{2n}X^{(0)}_{2n+2} + (X^{(0)}_{2n+2})^2 - \frac{1}{4}g_2 \\
& \left. + X^{(1)}_{2n}X^{(1)}_{2n+1}\left(X^{(0)}_{2n+2} + X^{(0)}_{2n+1} + X^{(0)}_{2n}\right) + X^{(-2)}_{2n+4} + X^{(-2)}_{2n+2} + X^{(-2)}_{2n}\right) P_{2n+2} \\
+ \;& 4\left(X^{(-2)}_{2n+3}\left(X^{(1)}_{2n} + X^{(1)}_{2n+2}\right) + X^{(-2)}_{2n+1}X^{(1)}_{2n} + X^{(1)}_{2n}\left(X^{(-2)}_{2n} + X^{(-2)}_{2n+2}\right) + X^{(1)}_{2n-2}X^{(-2)}_{2n}\right. \\
& + X^{(1)}_{2n}\left((X^{(0)}_{2n})^2 + X^{(0)}_{2n}X^{(0)}_{2n+1} + (X^{(0)}_{2n+1})^2 - \frac{1}{4}g_2\right) + X^{(-1)}_{2n+2}\left(X^{(0)}_{2n+2} + X^{(0)}_{2n+1} + X^{(0)}_{2n}\right) \\
& \left. + X^{(-1)}_{2n}\left(X^{(0)}_{2n+1} + X^{(0)}_{2n} + X^{(0)}_{2n-1}\right) + (X^{(1)}_{2n})^2X^{(-1)}_{2n+1} + X^{(1)}_{2n}X^{(1)}_{2n+1}X^{(-1)}_{2n+2} + X^{(-1)}_{2n}X^{(1)}_{2n-1}X^{(1)}_{2n}\right) P_{2n+} \\
+ \;& 4\left(2X^{(0)}_{2n}X^{(1)}_{2n}X^{(-1)}_{2n+1} + X^{(1)}_{2n}X^{(0)}_{2n+1}X^{(-1)}_{2n+1} + X^{(-1)}_{2n}X^{(-1)}_{2n+1} + X^{(0)}_{2n+2}X^{(-2)}_{2n+2}\right. \\
& + X^{(1)}_{2n}X^{(1)}_{2n+1}X^{(-2)}_{2n+2} + X^{(-1)}_{2n}X^{(-1)}_{2n-1} + 2X^{(0)}_{2n}X^{(-2)}_{2n} + X^{(-1)}_{2n}X^{(0)}_{2n-1}X^{(1)}_{2n-1} \\
& + X^{(-2)}_{2n}X^{(0)}_{2n-2} + X^{(-2)}_{2n}X^{(1)}_{2n-2}X^{(1)}_{2n-1} - \frac{1}{4}g_3 + X^{(0)}_{2n}\left((X^{(0)}_{2n})^2 - \frac{1}{4}g_2\right) \\
& \left. + 2X^{(0)}_{2n}X^{(-2)}_{2n+2} + X^{(-1)}_{2n+2}X^{(-1)}_{2n+1} + 2X^{(-1)}_{2n}X^{(1)}_{2n-1}X^{(0)}_{2n} + X^{(1)}_{2n}X^{(-2)}_{2n+1}X^{(1)}_{2n-1}\right) P_{2n} \\
+ \;& 4\left(X^{(-2)}_{2n+2}X^{(-1)}_{2n} + (X^{(0)}_{2n})^2X^{(-1)}_{2n} + X^{(0)}_{2n}X^{(1)}_{2n}X^{(-2)}_{2n+1} + X^{(-1)}_{2n}X^{(-2)}_{2n+1}\right. \\
& + (X^{(-1)}_{2n})^2X^{(1)}_{2n-1} + X^{(0)}_{2n}X^{(-1)}_{2n}X^{(0)}_{2n-1} + X^{(1)}_{2n}X^{(-1)}_{2n+1}X^{(-1)}_{2n} + X^{(-2)}_{2n}X^{(1)}_{2n-2}X^{(0)}_{2n-1} \\
& + X^{(-2)}_{2n}X^{(-1)}_{2n} + X^{(1)}_{2n}X^{(0)}_{2n+1}X^{(-2)}_{2n+1} + X^{(-1)}_{2n}X^{(-2)}_{2n-1} + X^{(0)}_{2n}X^{(-2)}_{2n}X^{(1)}_{2n-2} \\
& + X^{(-2)}_{2n}X^{(-1)}_{2n-2} - \frac{1}{4}g_2X^{(-1)}_{2n} + X^{(1)}_{2n}X^{(-2)}_{2n+1}X^{(0)}_{2n-1} + X^{(-1)}_{2n}(X^{(0)}_{2n-1})^2 \\
& \left. + X^{(-2)}_{2n}X^{(0)}_{2n-2}X^{(1)}_{2n-2} + X^{(-1)}_{2n+2}X^{(-2)}_{2n+1} + X^{(-1)}_{2n}X^{(-1)}_{2n-1}X^{(1)}_{2n-2}\right) P_{2n-1} \\
+ \;& 4\left(X^{(-2)}_{2n}(X^{(0)}_{2n-2})^2 - \frac{1}{4}g_2X^{(-2)}_{2n} + X^{(-1)}_{2n}X^{(1)}_{2n-1}X^{(-2)}_{2n} + X^{(1)}_{2n}X^{(-1)}_{2n+1}X^{(-2)}_{2n}\right. \\
& + X^{(-2)}_{2n+2}X^{(-2)}_{2n} + (X^{(0)}_{2n})^2X^{(-2)}_{2n} + X^{(-2)}_{2n}X^{(1)}_{2n-2}X^{(-1)}_{2n-1} + X^{(-2)}_{2n}X^{(-2)}_{2n-2} \\
& + X^{(-2)}_{2n}X^{(-1)}_{2n-2}X^{(1)}_{2n-3} + X^{(-1)}_{2n}X^{(-2)}_{2n-1}X^{(1)}_{2n-3} + X^{(0)}_{2n}X^{(-2)}_{2n}X^{(0)}_{2n-2} \\
& + X^{(-1)}_{2n}X^{(0)}_{2n-1}X^{(-1)}_{2n-1} + X^{(-1)}_{2n}X^{(-1)}_{2n-1}X^{(0)}_{2n-2} + X^{(1)}_{2n}X^{(-2)}_{2n+1}X^{(-1)}_{2n-1} \\
& \left. + X^{(0)}_{2n}X^{(-1)}_{2n}X^{(-1)}_{2n-1} + (X^{(-2)}_{2n})^2\right) P_{2n-2} \\
+ \;& 4\left(X^{(0)}_{2n-3}\left(X^{(-1)}_{2n}X^{(-2)}_{2n-1} + X^{(-2)}_{2n}X^{(-1)}_{2n-2}\right) + X^{(0)}_{2n}X^{(-1)}_{2n}X^{(-2)}_{2n-1} + X^{(-2)}_{2n}X^{(1)}_{2n-2}X^{(-2)}_{2n-1}\right. \\
& + X^{(-2)}_{2n}X^{(-2)}_{2n-2}X^{(1)}_{2n-4} + X^{(-1)}_{2n}X^{(-1)}_{2n-1}X^{(-1)}_{2n-2} + X^{(0)}_{2n}X^{(-2)}_{2n}X^{(-1)}_{2n-2} + X^{(-1)}_{2n}X^{(0)}_{2n-1}X^{(-2)}_{2n-1} \\
& \left. + X^{(-2)}_{2n}X^{(0)}_{2n-2}X^{(-1)}_{2n-2} + X^{(1)}_{2n}X^{(-2)}_{2n+1}X^{(-2)}_{2n-1}\right) P_{2n-3} \\
+ \;& 4\left(X^{(-1)}_{2n-3}\left(X^{(-1)}_{2n}X^{(-2)}_{2n-1} + X^{(-2)}_{2n}X^{(-1)}_{2n-2}\right) + X^{(-1)}_{2n}X^{(-1)}_{2n-1}X^{(-2)}_{2n-2}\right. \\
& \left. + X^{(-2)}_{2n}X^{(-2)}_{2n-2}\left(X^{(0)}_{2n} + X^{(0)}_{2n-2} + X^{(0)}_{2n-4}\right)\right) P_{2n-4} + 4\left(X^{(-2)}_{2n-3}\left(X^{(-1)}_{2n}X^{(-2)}_{2n-1} + X^{(-2)}_{2n}X^{(-1)}_{2n-2}\right)\right. \\
& \left. + X^{(-2)}_{2n}X^{(-2)}_{2n-2}X^{(-1)}_{2n-4}\right) P_{2n-5} + 4X^{(-2)}_{2n}X^{(-2)}_{2n-2}X^{(-2)}_{2n-4}P_{2n-6} \qquad \text{(D.}
\end{aligned}
$$

Thus as a simple comparison we see that the first term for both the xP_k and yP_k is 4 and the last term is

$$\begin{aligned} 4X^{(-2)}_{2n}X^{(-2)}_{2n-2}X^{(-2)}_{2n-4} &= 4\frac{h_{2n}}{h_{2n-2}}\frac{h_{2n-2}}{h_{2n-4}}\frac{h_{2n-4}}{h_{2n-6}} = 4\frac{h_{2n}}{h_{2n-6}}, \\ Y^{(-3)}_{2n}Y^{(-3)}_{2n-3} &= \frac{h_{2n}}{h_{2n-3}}4\frac{h_{2n-3}}{h_{2n-6}} = 4\frac{h_{2n}}{h_{2n-6}}, \end{aligned} \tag{D.4}$$

where we make use of (4.4.1a) and (4.4.2a).

$$X^{(-2)}_k = \frac{h_k}{h_{k-2}}, \quad Y^{(-3)}_k = \frac{h_k}{h_{k-3}}$$

Then we can consider the remaining 11 relations.

This equation also contains a great deal of symmetry in it, since we can rewrite the lower order coefficients as the higher order coefficients. So we consider the coefficient of P_{k-5} for k even

$$\begin{aligned} &\left(Y^{(-2)}_{2n}Y^{(-3)}_{2n-2} + Y^{(-3)}_{2n}Y^{(-2)}_{2n-3}\right) \\ = \; &4\left(X^{(-2)}_{2n-3}\left(X^{(-1)}_{2n}X^{(-2)}_{2n-1} + X^{(-2)}_{2n}X^{(-1)}_{2n-2}\right) + X^{(-2)}_{2n}X^{(-2)}_{2n-2}X^{(-1)}_{2n-4}\right) \end{aligned} \tag{D.5}$$

and then substitute in (4.4.1a) and (4.4.2a)

$$\begin{aligned} &\left(Y^{(-2)}_{2n}\frac{h_{2n-2}}{h_{2n-5}} + \frac{h_{2n}}{h_{2n-3}}Y^{(-2)}_{2n-3}\right) \\ = \; &4\left(\frac{h_{2n-3}}{h_{2n-5}}\left(X^{(-1)}_{2n}\frac{h_{2n-1}}{h_{2n-3}} + \frac{h_{2n}}{h_{2n-2}}X^{(-1)}_{2n-2}\right) + \frac{h_{2n}}{h_{2n-4}}X^{(-1)}_{2n-4}\right) \end{aligned}$$

and after a bit of rearranging we find that it is equivalent to the coefficient of P_{2n+5}.

$$\begin{aligned} &\left(Y^{(-2)}_{2n}\frac{h_{2n-2}}{h_{2n}} + \frac{h_{2n-5}}{h_{2n-3}}Y^{(-2)}_{2n-3}\right) \\ = \; &4\left(\frac{h_{2n-3}}{h_{2n}}\left(X^{(-1)}_{2n}\frac{h_{2n-1}}{h_{2n-3}} + \frac{h_{2n}}{h_{2n-2}}X^{(-1)}_{2n-2}\right) + \frac{h_{2n}}{h_{2n-4}}X^{(-1)}_{2n-4}\right) \end{aligned}$$

then substitute back in for h_{2n}

$$\left(Y^{(2)}_{2n-2} + Y^{(2)}_{2n-5}\right) = 4\left(X^{(1)}_{2n-1} + X^{(1)}_{2n-3} + X^{(1)}_{2n-5}\right)$$

We try substituting relations into one another in order to discover more about the interactivity between X_k and Y_k. The best approach is to start at the bottom and work the way up, primarily because the terms $X_k^{(-2)}$ and $Y_k^{(-3)}$ appear in the lower relations and they can be expressed in terms of known functions.

Now we consider $y^2 P_k$ for k odd,

$$\begin{aligned}
y^2 P_{2n+1} \;=\;& 4P_{2n+7} + \left(4Y_{2n+4}^{(2)} + 4Y_{2n+1}^{(2)}\right) P_{2n+6} + \left(4Y_{2n+4}^{(1)} + Y_{2n+1}^{(1)} + Y_{2n+1}^{(2)}Y_{2n+3}^{(2)}\right) P_{2n+5} \\
+\;& \left(Y_{2n+1}^{(2)}Y_{2n+3}^{(1)} + Y_{2n+1}^{(1)}Y_{2n+2}^{(2)} + 4Y_{2n+4}^{(0)} + 4Y_{2n+1}^{(0)}\right) P_{2n+4} + \left(4Y_{2n+4}^{(-1)}\right. \\
& \left. +Y_{2n+1}^{(-1)} + Y_{2n+1}^{(2)}Y_{2n+3}^{(0)} + Y_{2n+1}^{(0)}Y_{2n+1}^{(2)} + Y_{2n+1}^{(1)}Y_{2n+2}^{(1)}\right) P_{2n+3} + \left(4Y_{2n+1}^{(-2)}\right. \\
& \left. +Y_{2n+1}^{(1)}Y_{2n+2}^{(0)} + 4Y_{2n+4}^{(-2)} + Y_{2n+1}^{(-1)}Y_{2n}^{(2)} + Y_{2n+1}^{(0)}Y_{2n+1}^{(1)} + Y_{2n+1}^{(2)}Y_{2n+3}^{(-1)}\right) P_{2n+2} \\
+\;& \left((Y_{2n+1}^{(0)})^2 + Y_{2n+1}^{(-3)} + 4Y_{2n+4}^{(-3)} + Y_{2n+1}^{(-1)}Y_{2n}^{(1)} + Y_{2n+1}^{(-2)}Y_{2n-1}^{(2)} + Y_{2n+1}^{(2)}Y_{2n+1}^{(-2)}\right. \\
& \left. +Y_{2n+1}^{(1)}Y_{2n+2}^{(-1)}\right) P_{2n+1} + \left(Y_{2n+1}^{(2)}Y_{2n+3}^{(-3)} + Y_{2n+1}^{(-3)}Y_{2n-2}^{(2)} + Y_{2n+1}^{(0)}Y_{2n+1}^{(-1)}\right. \\
& \left. +Y_{2n+1}^{(-2)}Y_{2n-1}^{(1)} + Y_{2n+1}^{(-1)}Y_{2n}^{(0)} + Y_{2n+1}^{(1)}Y_{2n+2}^{(-2)}\right) P_{2n} \\
+\;& \left(Y_{2n+1}^{(-1)}Y_{2n}^{(-1)} + Y_{2n+1}^{(-3)}Y_{2n-2}^{(1)} + Y_{2n+1}^{(-2)}Y_{2n-1}^{(0)} + Y_{2n+1}^{(0)}Y_{2n+1}^{(-2)} + Y_{2n+1}^{(1)}Y_{2n+2}^{(-3)}\right) P_{2n-1} \\
+\;& \left(Y_{2n+1}^{(-1)}Y_{2n}^{(-2)} + Y_{2n+1}^{(0)}Y_{2n+1}^{(-3)} + Y_{2n+1}^{(-2)}Y_{2n-1}^{(-1)} + Y_{2n+1}^{(-3)}Y_{2n-2}^{(0)}\right) P_{2n-2} \\
+\;& \left(Y_{2n+1}^{(-1)}Y_{2n}^{(-3)} + Y_{2n+1}^{(-2)}Y_{2n-1}^{(-2)} + Y_{2n+1}^{(-3)}Y_{2n-2}^{(-1)}\right) P_{2n-3} \\
+\;& \left(Y_{2n+1}^{(-2)}Y_{2n-1}^{(-3)} + Y_{2n+1}^{(-3)}Y_{2n-2}^{(-2)}\right) P_{2n-4} + Y_{2n+1}^{(-3)}Y_{2n-2}^{(-3)}P_{2n-5}
\end{aligned} \tag{D.6}$$

and the longer expression containing X_k

$$\begin{aligned}
& (4x^3 - g_2 x - g_3)P_{2n} \\
=\;& 4P_{2n+7} + 4\left(X_{2n+5}^{(1)} + X_{2n+3}^{(1)} + X_{2n+1}^{(1)}\right) P_{2n+6} + 4\left(X_{2n+4}^{(1)}\left(X_{2n+1}^{(1)} + X_{2n+3}^{(1)}\right)\right. \\
& \left. +X_{2n+1}^{(1)}X_{2n+2}^{(1)} + X_{2n+5}^{(0)} + X_{2n+3}^{(0)} + X_{2n+1}^{(0)}\right) P_{2n+5} + 4\left(X_{2n+4}^{(0)}\left(X_{2n+1}^{(1)} + X_{2n+3}^{(1)}\right)\right. \\
& +X_{2n+2}^{(0)}X_{2n+1}^{(1)} + X_{2n+3}^{(1)}\left(X_{2n+1}^{(0)} + X_{2n+3}^{(0)}\right) + X_{2n+1}^{(1)}X_{2n+1}^{(0)} + X_{2n+1}^{(1)}X_{2n+2}^{(1)}X_{2n+3}^{(1)} \\
& \left. +X_{2n+5}^{(-1)} + X_{2n+3}^{(-1)} + X_{2n+1}^{(-1)}\right) P_{2n+4} + 4\left(X_{2n+4}^{(-1)}\left(X_{2n+3}^{(1)} + X_{2n+1}^{(1)}\right) + X_{2n+2}^{(-1)}X_{2n+1}^{(1)}\right. \\
& +X_{2n+2}^{(1)}\left(X_{2n+1}^{(-1)} + X_{2n+3}^{(-1)}\right) + X_{2n}^{(1)}X_{2n+1}^{(-1)} + (X_{2n+1}^{(0)})^2 + X_{2n+1}^{(0)}X_{2n+3}^{(0)} + (X_{2n+3}^{(0)})^2 - \frac{1}{4}g_2 \\
& \left. +X_{2n+1}^{(1)}X_{2n+2}^{(1)}\left(X_{2n+3}^{(0)} + X_{2n+2}^{(0)} + X_{2n+1}^{(0)}\right) + X_{2n+5}^{(-2)} + X_{2n+3}^{(-2)} + X_{2n+1}^{(-2)}\right) P_{2n+3}
\end{aligned}$$

$$
\begin{aligned}
+ \quad & 4\Big(X^{(-2)}_{2n+4}\left(X^{(1)}_{2n+1}+X^{(1)}_{2n+3}\right)+X^{(-2)}_{2n+2}X^{(1)}_{2n+1}+X^{(1)}_{2n+1}\left(X^{(-2)}_{2n+1}+X^{(-2)}_{2n+3}\right)+X^{(1)}_{2n-1}X^{(-2)}_{2n+1} \\
& +X^{(1)}_{2n+1}\left((X^{(0)}_{2n+1})^2+X^{(0)}_{2n+1}X^{(0)}_{2n+2}+(X^{(0)}_{2n+2})^2-\frac{1}{4}g_2\right)+X^{(-1)}_{2n+3}\left(X^{(0)}_{2n+3}+X^{(0)}_{2n+2}+X^{(0)}_{2n+1}\right) \\
& +X^{(-1)}_{2n+1}\left(X^{(0)}_{2n+2}+X^{(0)}_{2n+1}+X^{(0)}_{2n}\right)+(X^{(1)}_{2n+1})^2X^{(-1)}_{2n+2}+X^{(1)}_{2n+1}X^{(1)}_{2n+2}X^{(-1)}_{2n+3}+X^{(-1)}_{2n+1}X^{(1)}_{2n}X^{(1)}_{2n+1}\Big)\,P_{2n+2} \\
+ \quad & 4\Big(2X^{(0)}_{2n+1}X^{(1)}_{2n+1}X^{(-1)}_{2n+2}+X^{(1)}_{2n+1}X^{(0)}_{2n+2}X^{(-1)}_{2n+2}+X^{(-1)}_{2n+1}X^{(-1)}_{2n+2}+X^{(0)}_{2n+3}X^{(-2)}_{2n+3} \\
& +X^{(1)}_{2n+1}X^{(1)}_{2n+2}X^{(-2)}_{2n+3}+X^{(-1)}_{2n+1}X^{(-1)}_{2n}+2X^{(0)}_{2n+1}X^{(-2)}_{2n+1}+X^{(-1)}_{2n+1}X^{(0)}_{2n}X^{(1)}_{2n} \\
& +X^{(-2)}_{2n+1}X^{(0)}_{2n-1}+X^{(-2)}_{2n+1}X^{(1)}_{2n-1}X^{(1)}_{2n}-\frac{1}{4}g_3+X^{(0)}_{2n+1}\left((X^{(0)}_{2n+1})^2-\frac{1}{4}g_2\right) \\
& +2X^{(0)}_{2n+1}X^{(-2)}_{2n+3}+X^{(-1)}_{2n+3}X^{(-1)}_{2n+2}+2X^{(-1)}_{2n+1}X^{(1)}_{2n}X^{(0)}_{2n+1}+X^{(1)}_{2n+1}X^{(-2)}_{2n+2}X^{(1)}_{2n}\Big)\,P_{2n+1} \\
+ \quad & 4\Big(X^{(-2)}_{2n+3}X^{(-1)}_{2n+1}+(X^{(0)}_{2n+1})^2X^{(-1)}_{2n+1}+X^{(0)}_{2n+1}X^{(1)}_{2n+1}X^{(-2)}_{2n+2}+X^{(-1)}_{2n+1}X^{(-2)}_{2n+2} \\
& +(X^{(-1)}_{2n+1})^2X^{(1)}_{2n}+X^{(0)}_{2n+1}X^{(-1)}_{2n+1}X^{(0)}_{2n}+X^{(1)}_{2n+1}X^{(-1)}_{2n+2}X^{(-1)}_{2n+1}+X^{(-2)}_{2n+1}X^{(1)}_{2n-1}X^{(0)}_{2n} \\
& +X^{(-2)}_{2n+1}X^{(-1)}_{2n+1}+X^{(1)}_{2n+1}X^{(0)}_{2n+2}X^{(-2)}_{2n+2}+X^{(-1)}_{2n+1}X^{(-2)}_{2n}+X^{(0)}_{2n+1}X^{(-2)}_{2n+1}X^{(1)}_{2n-1} \\
& +X^{(-2)}_{2n+1}X^{(-1)}_{2n-1}-\frac{1}{4}g_2X^{(-1)}_{2n+1}+X^{(1)}_{2n+1}X^{(-2)}_{2n+2}X^{(0)}_{2n}+X^{(-1)}_{2n+1}(X^{(0)}_{2n})^2 \\
& +X^{(-2)}_{2n+1}X^{(0)}_{2n-1}X^{(1)}_{2n-1}+X^{(-1)}_{2n+3}X^{(-2)}_{2n+2}+X^{(-1)}_{2n+1}X^{(-1)}_{2n}X^{(1)}_{2n-1}\Big)\,P_{2n} \\
+ \quad & 4\Bigg(X^{(-2)}_{2n+1}(X^{(0)}_{2n-1})^2-\frac{1}{4}g_2X^{(-2)}_{2n+1}+X^{(-1)}_{2n+1}X^{(1)}_{2n}X^{(-2)}_{2n+1}+X^{(1)}_{2n+1}X^{(-1)}_{2n+2}X^{(-2)}_{2n+1} \\
& +X^{(-2)}_{2n+3}X^{(-2)}_{2n+1}+(X^{(0)}_{2n+1})^2X^{(-2)}_{2n+1}+X^{(-2)}_{2n+1}X^{(1)}_{2n-1}X^{(-1)}_{2n}+X^{(-2)}_{2n+1}X^{(-2)}_{2n-1} \\
& +X^{(-2)}_{2n+1}X^{(-1)}_{2n-1}X^{(1)}_{2n-2}+X^{(-1)}_{2n+1}X^{(-2)}_{2n}X^{(1)}_{2n-2}+X^{(0)}_{2n+1}X^{(-2)}_{2n+1}X^{(0)}_{2n-1} \\
& +X^{(-1)}_{2n+1}X^{(0)}_{2n}X^{(-1)}_{2n}+X^{(-1)}_{2n+1}X^{(-1)}_{2n}X^{(0)}_{2n-1}+X^{(1)}_{2n+1}X^{(-2)}_{2n+2}X^{(-1)}_{2n} \\
& +X^{(0)}_{2n+1}X^{(-1)}_{2n+1}X^{(-1)}_{2n}+(X^{(-2)}_{2n+1})^2\Bigg)\,P_{2n-1} \\
+ \quad & 4\Big(X^{(0)}_{2n-2}\left(X^{(-1)}_{2n+1}X^{(-2)}_{2n}+X^{(-2)}_{2n+1}X^{(-1)}_{2n-1}\right)+X^{(0)}_{2n+1}X^{(-1)}_{2n+1}X^{(-2)}_{2n}+X^{(-2)}_{2n+1}X^{(1)}_{2n-1}X^{(-2)}_{2n} \\
& +X^{(-2)}_{2n+1}X^{(-2)}_{2n-1}X^{(1)}_{2n-3}+X^{(-1)}_{2n+1}X^{(-1)}_{2n}X^{(-1)}_{2n-1}+X^{(0)}_{2n+1}X^{(-2)}_{2n+1}X^{(-1)}_{2n-1}+X^{(-1)}_{2n+1}X^{(0)}_{2n}X^{(-2)}_{2n} \\
& +X^{(-2)}_{2n+1}X^{(0)}_{2n-1}X^{(-1)}_{2n-1}+X^{(1)}_{2n+1}X^{(-2)}_{2n+2}X^{(-2)}_{2n}\Big)\,P_{2n-2} \\
+ \quad & 4\Big(X^{(-1)}_{2n-2}\left(X^{(-1)}_{2n+1}X^{(-2)}_{2n}+X^{(-2)}_{2n+1}X^{(-1)}_{2n-1}\right)+X^{(-1)}_{2n+1}X^{(-1)}_{2n}X^{(-2)}_{2n-1} \\
& +X^{(-2)}_{2n+1}X^{(-2)}_{2n-1}\left(X^{(0)}_{2n+1}+X^{(0)}_{2n-1}+X^{(0)}_{2n-3}\right)\Big)\,P_{2n-3}+4\left(X^{(-2)}_{2n-2}\left(X^{(-1)}_{2n+1}X^{(-2)}_{2n}+X^{(-2)}_{2n+1}X^{(-1)}_{2n-1}\right)\right. \\
& \left.+X^{(-2)}_{2n+1}X^{(-2)}_{2n-1}X^{(-1)}_{2n-3}\right)\,P_{2n-4}+4X^{(-2)}_{2n+1}X^{(-2)}_{2n-1}X^{(-2)}_{2n-3}P_{2n-5}
\end{aligned}
\tag{D.7}
$$

Again we consider the first and last terms of both sides of the relation, where the former is 4 and the latter is

$$\begin{aligned} 4X^{(-2)}_{2n+1}X^{(-2)}_{2n-1}X^{(-2)}_{2n-3} &= 4\frac{h_{2n+1}}{h_{2n-5}}, \\ Y^{(-3)}_{2n+1}Y^{(-3)}_{2n-2} &= 4\frac{h_{2n+1}}{h_{2n-2}}\frac{h_{2n-2}}{h_{2n-5}} = 4\frac{h_{2n+1}}{h_{2n-5}}. \end{aligned}$$

So at the very least the initial and final terms are identical for k odd or even.

E Elliptic Polynomials

The following list of equations are the elliptic polynomials, where we assume that $P_{(1)}$ does not exist (since $\boldsymbol{e}_1$ does not exist). To create a list of polynomials we require both recursion relations (4.2.6) since for lower order polynomials we require one or the other

$$\begin{aligned} xP_k &= P_{k+2} + X^{(1)}_k P_{k+1} + X^{(0)}_k P_k + X^{(-1)}_k P_{k-1} + X^{(-2)}_k P_{k-2}, \\ yP_k &= 4^{\epsilon_k} P_{k+3} + Y^{(2)}_k P_{k+2} + Y^{(1)}_k P_{k+1} + Y^{(0)}_k P_k + Y^{(-1)}_k P_{k-1} \\ &\quad + Y^{(-2)}_k P_{k-2} + Y^{(-3)}_k P_{k-3}, \end{aligned}$$

and where

$$\epsilon_k = \begin{cases} 1 \quad , & k \text{ odd} , \\ 0 \quad , & k \text{ even} . \end{cases}$$

Beginning with establishing some initial conditions,

$$P_0 = 1 \quad , \quad P_{-1} = P_{-2} = P_{-3} = 0 \tag{E.1}$$

we then move on to constructing the polynomials. Now since P_1 does not exist, we must make the allowance that any coefficient that would normally be coupled with P_1 takes the value of 0. Thus the first two polynomials $P_{(2)}$ and $P_{(3)}$, which are formed by taking $k = 0$ in xP_k and yP_k

$$P_2 = x - X^{(0)}_0 \tag{E.2}$$

$$P_3 = y - Y^{(2)}_0 x + (X^{(0)}_0 Y^{(2)}_0 - Y^{(0)}_0) \tag{E.3}$$

exist when $X_0^{(1)} = 0$ and $Y_0^{(1)} = 0$ For $k = 2$, we gain $P_{(4)}$ from the xP_k relation

$$\begin{aligned} P_4 &= x^2 - X_2^{(1)}y + (X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)})x - \Big(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) \\ &\quad -(X_2^{(0)}X_0^{(0)} + X_2^{(-2)})\Big) \end{aligned} \tag{E.4}$$

and $X_2^{(-1)} = 0$. After this point there will be more than one approach to derive a polynomial, since for values of higher order k we have multiple equations. To illustrate this, first consider P_5 which can be constructed in two separate ways, either through yP_2 or xP_3. Thus we have yP_2 from $k = 2$ and xP_3 from $k = 3$ respectively

$$\begin{aligned} P_5 &= xy - Y_2^{(2)}x^2 + \Big(Y_2^{(2)}X_2^{(1)} - Y_2^{(1)} - X_0^{(0)}\Big)\, y \\ &\quad - \Big(Y_2^{(2)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) - Y_2^{(1)}Y_0^{(2)} + Y_2^{(0)}\Big)\, x \\ &\quad + \Big(Y_2^{(2)}\Big(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_0^{(0)}X_2^{(0)} - X_2^{(-2)})\Big) \\ &\quad -Y_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) + Y_2^{(0)}X_0^{(0)} - Y_2^{(-2)}\Big) \end{aligned} \tag{E.5a}$$

$$\begin{aligned} P_5 &= xy - \Big(X_3^{(1)} + Y_0^{(2)}\Big)\, x^2 + \Big(X_3^{(1)}X_2^{(1)} - X_3^{(0)}\Big)\, y \\ &\quad - \Big(X_3^{(1)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) - Y_0^{(2)}(X_3^{(0)} + X_0^{(0)}) + X_3^{(-1)} + Y_0^{(0)}\Big)\, x \\ &\quad + \Big(X_3^{(1)}\Big(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_0^{(0)}X_2^{(0)} - X_2^{(-2)})\Big) \\ &\quad -X_3^{(0)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) + X_3^{(-1)}X_0^{(0)}\Big) \end{aligned} \tag{E.5b}$$

and where $Y_2^{(-1)} = 0$ and $X_3^{(-2)} = 0$ respectively. Despite the differences we have here, both these relations are equal, which is proved by the consistency between the x and y

relations. This leads to a whole new set of relations between the coefficients X_k and Y_k

$$Y_2^{(2)} = X_3^{(1)} + Y_0^{(2)}, \tag{E.6a}$$

$$Y_2^{(2)} X_2^{(1)} - Y_2^{(1)} - X_0^{(0)} = X_3^{(1)} X_2^{(1)} - X_3^{(0)}, \tag{E.6b}$$

$$\begin{aligned}&Y_2^{(2)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) - Y_2^{(1)}Y_0^{(2)} + Y_2^{(0)} = \\ &X_3^{(1)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) - Y_0^{(2)}(X_3^{(0)} + X_0^{(0)}) + X_3^{(-1)} + Y_0^{(0)},\end{aligned} \tag{E.6c}$$

$$\begin{aligned}&\left(Y_2^{(2)}\left(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_0^{(0)}X_2^{(0)} - X_2^{(-2)})\right)\right. \\ &\left.-Y_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) + Y_2^{(0)}X_0^{(0)} - Y_2^{(-2)}\right) = \\ &\left(X_3^{(1)}\left(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_0^{(0)}X_2^{(0)} - X_2^{(-2)})\right)\right. \\ &\left.-X_3^{(0)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) + X_3^{(-1)}X_0^{(0)}\right).\end{aligned} \tag{E.6d}$$

The smaller relations can reduce the larger relations

$$\tag{E.7}$$

and with the aid of these we can consider a single relation for P_5. For reasons of simplicity I will use the derivation from yP_2.

Next we consider P_6, which raises a further issue, since we have reached a level where the curve has a direct involvement.

For P_6 we use yP_3 with the yP_2 value of P_5,

$$\begin{aligned}4P_6 \;=\;& y^2 - (Y_3^{(2)} + Y_0^{(2)})xy + (Y_3^{(2)}Y_2^{(2)} - Y_3^{(1)})x^2 \\ &- \left(Y_3^{(2)}(Y_2^{(2)}X_2^{(1)} - Y_2^{(1)} - X_0^{(0)}) - Y_3^{(1)}X_2^{(1)} + Y_3^{(0)} - (X_0^{(0)}Y_0^{(2)} - Y_0^{(0)})\right) y \\ &+ \left(Y_3^{(2)}(Y_2^{(2)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) - Y_2^{(1)}Y_0^{(2)} + Y_2^{(0)})\right. \\ &\left.-Y_3^{(1)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) + Y_3^{(0)}Y_0^{(2)} - Y_3^{(-1)}\right) x \\ &- \left(Y_3^{(2)}\left(Y_2^{(2)}(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})) - Y_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)})\right.\right. \\ &\left.+Y_2^{(0)}X_0^{(0)} - Y_2^{(-2)}\right) - Y_3^{(1)}\left(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})\right) \\ &\left.+Y_3^{(0)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - Y_3^{(-1)}X_0^{(-0)} + Y_3^{(-3)}\right)\end{aligned} \tag{E.8a}$$

and similarly for xP_4 with yP_2

$$\begin{aligned}
P_6 \;=\;& x^3 - (X_4^{(1)} + X_2^{(1)})xy + \left(X_4^{(1)}Y_2^{(2)} - X_4^{(0)} + (X_2^{(1)}Y_2^{(2)} - X_2^{(0)} - X_0^{(0)})\right)x^2 \\
& - \left(X_4^{(1)}(Y_2^{(2)}X_2^{(1)} - Y_2^{(1)} - X_0^{(0)}) - X_4^{(0)}X_2^{(1)} + X_4^{(-1)}\right)y \\
& + \left(X_4^{(1)}\left(Y_2^{(2)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) - Y_2^{(1)}Y_0^{(2)} + Y_2^{(0)}\right)\right. \\
& -X_4^{(0)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) + X_4^{(-1)}Y_0^{(2)} + X_4^{(-2)} \\
& \left. -X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) + (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})\right)x \\
& - \left(X_4^{(1)}\left(Y_2^{(2)}(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})) - Y_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)})\right.\right. \\
& \left. +Y_2^{(0)}X_0^{(0)} - Y_2^{(-2)}\right) - X_4^{(0)}\left(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})\right) \\
& \left. +X_4^{(-1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - X_4^{(-2)}X_0^{(-0)}\right). \qquad \text{(E.8b)}
\end{aligned}$$

In the former P_6, we introduce the curve $y^2 = 4x^3 - g_2x - g_3$ and since these two polynomials are equal, a further set of relations arises

$$\frac{1}{4}(Y_3^{(2)} + Y_0^{(2)}) = X_4^{(1)} + X_2^{(1)}, \tag{E.9a}$$

$$\frac{1}{4}(Y_3^{(2)}Y_2^{(2)} - Y_3^{(1)}) = X_4^{(1)}Y_2^{(2)} - X_4^{(0)} + (X_2^{(1)}Y_2^{(2)} - X_2^{(0)} - X_0^{(0)}), \tag{E.9b}$$

$$\frac{1}{4}\left(Y_3^{(2)}(Y_2^{(2)}X_2^{(1)} - Y_2^{(1)} - X_0^{(0)}) - Y_3^{(1)}X_2^{(1)} + Y_3^{(0)} - (X_0^{(0)}Y_0^{(2)} - Y_0^{(0)})\right) =$$
$$X_4^{(1)}(Y_2^{(2)}X_2^{(1)} - Y_2^{(1)} - X_0^{(0)}) - X_4^{(0)}X_2^{(1)} + X_4^{(-1)}, \tag{E.9c}$$

$$\frac{1}{4}\left(Y_3^{(2)}(Y_2^{(2)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) - Y_2^{(1)}Y_0^{(2)} + Y_2^{(0)})\right.$$
$$\left.-Y_3^{(1)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) + Y_3^{(0)}Y_0^{(2)} - Y_3^{(-1)} - g_2\right) =$$
$$\left(X_4^{(1)}\left(Y_2^{(2)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) - Y_2^{(1)}Y_0^{(2)} + Y_2^{(0)}\right)\right.$$
$$-X_4^{(0)}(Y_0^{(2)}X_2^{(1)} - X_2^{(0)} - X_0^{(0)}) + X_4^{(-1)}Y_0^{(2)} + X_4^{(-2)}$$
$$\left.-X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) + (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})\right), \tag{E.9d}$$

$$\frac{1}{4}\left(Y_3^{(2)}\left(Y_2^{(2)}(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})) - Y_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)})\right.\right.$$
$$\left.+Y_2^{(0)}X_0^{(0)} - Y_2^{(-2)}\right) - Y_3^{(1)}\left(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})\right)$$
$$\left.+Y_3^{(0)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - Y_3^{(-1)}X_0^{(-0)} + Y_3^{(-3)} - g_3\right) =$$
$$\left(X_4^{(1)}\left(Y_2^{(2)}(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})) - Y_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)})\right.\right.$$
$$\left.+Y_2^{(0)}X_0^{(0)} - Y_2^{(-2)}\right) - X_4^{(0)}\left(X_2^{(1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - (X_2^{(0)}X_0^{(0)} - X_2^{(-2)})\right)$$
$$\left.+X_4^{(-1)}(X_0^{(0)}Y_0^{(2)} - Y_0^{(0)}) - X_4^{(-2)}X_0^{(-0)}\right), \tag{E.9e}$$

which introduce the non-zero curve constants g_2 and g_3.

As one last example we show the derivation of P_7 where we use xP_5 and yP_4 to generate the two polynomials, where the values of P_6 from yP_2 and P_5 from yP_3 are used. These two polynomials also satisfy the relations above, since if all the P_7 are individually worked out there are 16 representations. Initially we present the form of P_7 acquired using xP_5

$$
\begin{aligned}
P_7 \;=\;& yx^2 - (X_5^{(1)} + X_3^{(1)} + Y_0^{(2)})x^3 + (X_5^{(1)}(X_4^{(1)} + X_2^{(1)}) - X_5^{(0)} + X_3^{(1)}X_2^{(1)} - X_3^{(0)})yx \\
& - \Big(X_5^{(1)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)} + X_4^{(1)}(Y^2_{(0)} + X_3^{(1)}) - X_4^{(0)}) - X_5^{(0)}(Y_0^{(2)} + X_3^{(1)}) \\
& + X_5^{(-1)} + X_3^{(1)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) - X_3^{(0)}Y_0^{(2)} + X_3^{(-1)} - Y_0^{(2)}X_0^{(0)} + Y_0^{(0)}\Big)\, x^2 \\
& + \Big(X_5^{(1)}(X_4^{(1)}(X_3^{(0)} - X_3^{(1)}X_2^{(1)}) - X_4^{(0)}X_2^{(1)} + X_4^{(-1)}) - X_5^{(0)}(X_3^{(1)}X_2^{(1)} - X_3^{(0)}) \\
& + X_5^{(-1)}X_2^{(1)} - X_5^{(-2)}\Big)\, y - \Big[X_5^{(1)}\, \Big(X_4^{(1)}\, \Big(X_3^{(1)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) - X_3^{(0)}Y_0^{(2)} \\
& +X_3^{(-1)} - Y_0^{(2)}X_0^{(0)} + Y_0^{(0)}\Big) - X_4^{(0)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) + X_4^{(-1)}Y_0^{(2)} - X_4^{(-2)} \\
& -X_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)})X_2^{(0)}X_0^{(0)} - X_2^{(-2)}\Big) - X_5^{(0)}\, \Big(X_3^{(1)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) \\
& -X_3^{(0)}Y_0^{(2)} + X_3^{(-1)} - Y_0^{(2)}X_0^{(0)} + Y_0^{(0)}\Big) + X_5^{(-1)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) \\
& -X_5^{(-2)}Y_0^{(2)} - X_3^{(1)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) - X_2^{(0)}X_0^{(0)} + X_2^{(-2)}) \\
& +X_3^{(0)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) - X_3^{(-1)}X_0^{(0)}\Big]\, x \\
& +X_5^{(1)}\, \Big(X_4^{(1)}(X_3^{(1)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) - X_2^{(0)}X_0^{(0)} + X_2^{(-2)}) - X_3^{(0)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) \\
& +X_3^{(-1)}X_0^{(0)}) - X_4^{(0)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) - X_2^{(0)}X_0^{(0)} + X_2^{(-2)}) \\
& +X_4^{(-1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) - X_4^{(-2)}X_0^{(0)}\Big) - X_5^{(0)}\, \Big(X_3^{(1)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) \\
& -X_2^{(0)}X_0^{(0)} + X_2^{(-2)}) - X_3^{(0)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) + X_3^{(-1)}X_0^{(0)}\Big) \\
& +X_5^{(-1)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) - X_2^{(0)}X_0^{(0)} + X_2^{(-2)}) - X_5^{(-2)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)})
\end{aligned}
\tag{E.10}
$$

and then we use yP_4 with the same values

$$
\begin{aligned}
P_7 \;=\;& yx^2 - (X_2^{(1)} + Y_4^{(2)})y^2 + \Big(Y_4^{(2)}(Y_0^{(2)} + Y_3^{(2)}) - Y_4^{(1)} + X_2^{(1)}Y_0^{(2)} \\
& - X_2^{(0)} - X_0^{(0)}\Big)\, yx - \Big(Y_4^{(2)}(Y_3^{(2)}Y_2^{(2)} - Y_3^{(1)}) - Y_4^{(1)}Y_2^{(2)} + Y_4^{(0)}\Big)\, x^2 \\
& + \Big(Y_4^{(2)}(Y_3^{(2)}(Y_2^{(2)}X_2^{(1)} - Y_2^{(1)} - X_0^{(0)}) - Y_3^{(1)}X_2^{(1)} + Y_3^{(0)} - Y_0^{(2)}X_0^{(0)} \\
& + Y_0^{(0)}) - Y_4^{(1)}(Y_2^{(2)}X_2^{(1)} - Y_2^{(1)} - X_0^{(0)}) + Y_4^{(0)}X_2^{(1)} - Y_4^{(-1)} \\
& - X_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) + X_2^{(0)}X_0^{(0)} - X_2^{(-2)}\Big)\, y \\
& - \Big(Y_4^{(2)}(Y_3^{(2)}(Y_2^{(2)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) - Y_2^{(1)}Y_0^{(2)} + Y_2^{(0)}) \\
& - Y_3^{(1)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) + Y_3^{(0)}Y_0^{(2)} - Y_3^{(-1)}) \\
& - Y_4^{(1)}(Y_2^{(2)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) - Y_2^{(1)}Y_0^{(2)} + Y_2^{(0)}) \\
& + Y_4^{(0)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) - Y_4^{(-1)}Y_0^{(2)} + Y_4^{(-2)}\Big)\, x \\
& + Y_4^2\, \Big(Y_3^{(2)}(Y_2^{(2)}(X_2^1(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) - X_2^{(0)}X_0^{(0)} + X_2^{(-2)}) \\
& - Y_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) + Y_2^{(0)}X_0^{(0)} - Y_2^{(-2)}) - Y_3^{(1)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) \\
& - X_2^{(0)}X_0^{(0)} + X_2^{(-2)}) + Y_3^{(0)}(Y_0^{(2)}X_0^{(0)} - Y_0^{0}) - Y_3^{(-1)}X_0^{(0)} + Y_3^{(-3)}\Big) \\
& - Y_4^{(1)}(Y_2^{(2)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) - X_2^{(0)}X_0^{(0)} + X_2^{(-2)}) \\
& - Y_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) + Y_2^{(0)}X_0^{(0)} - Y_2^{(-2)}) + Y_4^{(0)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) \\
& - X_2^{(0)}X_0^{(0)} + X_2^{(-2)}) - Y_4^{(-1)}(Y_0^{(2)}X_0^{(0)} - Y_0^{(0)}) + Y_4^{(-2)}X_0^{(0)}
\end{aligned}
\tag{E.11}
$$

These polynomials satisfy the following series of relations (found by equating the P_7),

$$
(X_5^{(1)} + X_3^{(1)} + Y_0^{(2)}) = 4(X_2^{(1)} + Y_4^{(2)}), \tag{E.12a}
$$

$$
\begin{aligned}
&\Big(X_5^{(1)}(X_4^{(1)} + X_2^{(1)}) - X_5^{(0)} + X_3^{(1)}X_2^{(1)} - X_3^{(0)}\Big) \\
&= \Big(Y_4^{(2)}(Y_0^{(2)} + Y_3^{(2)}) - Y_4^{(1)} + X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}\Big),
\end{aligned}
\tag{E.12b}
$$

$$
\begin{aligned}
&\Big(X_5^{(1)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)} + X_4^{(1)}(Y_{(0)}^2 + X_3^{(1)}) - X_4^{(0)}) - X_5^{(0)}(Y_0^{(2)} + X_3^{(1)}) \\
&+ X_5^{(-1)} + X_3^{(1)}(X_2^{(1)}Y_0^{(2)} - X_2^{(0)} - X_0^{(0)}) - X_3^{(0)}Y_0^{(2)} + X_3^{(-1)} - Y_0^{(2)}X_0^{(0)} + Y_0^{(0)}\Big) \\
&= \Big(Y_4^{(2)}(Y_3^{(2)}Y_2^{(2)} - Y_3^{(1)}) - Y_4^{(1)}Y_2^{(2)} + Y_4^{(0)}\Big),
\end{aligned}
\tag{E.12c}
$$

$$
\begin{aligned}
&\Big(X_5^{(1)}(X_4^{(1)}(X_3^{(0)}-X_3^{(1)}X_2^{(1)})-X_4^{(0)}X_2^{(1)}+X_4^{(-1)})-X_5^{(0)}(X_3^{(1)}X_2^{(1)}-X_3^{(0)})\\
&+X_5^{(-1)}X_2^{(1)}-X_5^{(-2)}\Big)\\
&=\Big(Y_4^{(2)}(Y_3^{(2)}(Y_2^{(2)}X_2^{(1)}-Y_2^{(1)}-X_0^{(0)})-Y_3^{(1)}X_2^{(1)}+Y_3^{(0)}-Y_0^{(2)}X_0^{(0)}\\
&+Y_0^{(0)})-Y_4^{(1)}(Y_2^{(2)}X_2^{(1)}-Y_2^{(1)}-X_0^{(0)})+Y_4^{(0)}X_2^{(1)}-Y_4^{(-1)}\\
&-X_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})+X_2^{(0)}X_0^{(0)}-X_2^{(-2)}\Big),
\end{aligned}
\tag{E.12d}
$$

$$
\begin{aligned}
&\Big[X_5^{(1)}\Big(X_4^{(1)}\Big(X_3^{(1)}(X_2^{(1)}Y_0^{(2)}-X_2^{(0)}-X_0^{(0)})-X_3^{(0)}Y_0^{(2)}\\
&+X_3^{(-1)}-Y_0^{(2)}X_0^{(0)}+Y_0^{(0)}\Big)-X_4^{(0)}(X_2^{(1)}Y_0^{(2)}-X_2^{(0)}-X_0^{(0)})+X_4^{(-1)}Y_0^{(2)}-X_4^{(-2)}\\
&-X_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})X_2^{(0)}X_0^{(0)}-X_2^{(-2)}\Big)-X_5^{(0)}\Big(X_3^{(1)}(X_2^{(1)}Y_0^{(2)}-X_2^{(0)}-X_0^{(0)})\\
&-X_3^{(0)}Y_0^{(2)}+X_3^{(-1)}-Y_0^{(2)}X_0^{(0)}+Y_0^{(0)}\Big)+X_5^{(-1)}(X_2^{(1)}Y_0^{(2)}-X_2^{(0)}-X_0^{(0)})\\
&-X_5^{(-2)}Y_0^{(2)}-X_3^{(1)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})-X_2^{(0)}X_0^{(0)}+X_2^{(-2)})\\
&+X_3^{(0)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})-X_3^{(-1)}X_0^{(0)}\Big]\\
&=\Big(Y_4^{(2)}(Y_3^{(2)}(Y_2^{(2)}(X_2^{(1)}Y_0^{(2)}-X_2^{(0)}-X_0^{(0)})-Y_2^{(1)}Y_0^{(2)}+Y_2^{(0)})\\
&-Y_3^{(1)}(X_2^{(1)}Y_0^{(2)}-X_2^{(0)}-X_0^{(0)})+Y_3^{(0)}Y_0^{(2)}-Y_3^{(-1)})\\
&-Y_4^{(1)}(Y_2^{(2)}(X_2^{(1)}Y_0^{(2)}-X_2^{(0)}-X_0^{(0)})-Y_2^{(1)}Y_0^{(2)}+Y_2^{(0)})\\
&+Y_4^{(0)}(X_2^{(1)}Y_0^{(2)}-X_2^{(0)}-X_0^{(0)})-Y_4^{(-1)}Y_0^{(2)}+Y_4^{(-2)}\Big),
\end{aligned}
\tag{E.12e}
$$

$$
\begin{aligned}
&X_5^{(1)}\Big(X_4^{(1)}(X_3^{(1)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})-X_2^{(0)}X_0^{(0)}+X_2^{(-2)})-X_3^{(0)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})\\
&+X_3^{(-1)}X_0^{(0)})-X_4^{(0)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})-X_2^{(0)}X_0^{(0)}+X_2^{(-2)})\\
&+X_4^{(-1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})-X_4^{(-2)}X_0^{(0)}\Big)-X_5^{(0)}\Big(X_3^{(1)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})\\
&-X_2^{(0)}X_0^{(0)}+X_2^{(-2)})-X_3^{(0)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})+X_3^{(-1)}X_0^{(0)}\Big)\\
&+X_5^{(-1)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})-X_2^{(0)}X_0^{(0)}+X_2^{(-2)})-X_5^{(-2)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})\\
&=Y_4^{2}\Big(Y_3^{(2)}(Y_2^{(2)}(X_2^{1}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})-X_2^{(0)}X_0^{(0)}+X_2^{(-2)})\\
&-Y_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})+Y_2^{(0)}X_0^{(0)}-Y_2^{(-2)})-Y_3^{(1)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})\\
&-X_2^{(0)}X_0^{(0)}+X_2^{(-2)})+Y_3^{(0)}(Y_0^{(2)}X_0^{(0)}-Y_0^{0})-Y_3^{(-1)}X_0^{(0)}+Y_3^{(-3)}\Big)\\
&-Y_4^{(1)}(Y_2^{(2)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})-X_2^{(0)}X_0^{(0)}+X_2^{(-2)})\\
&-Y_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})+Y_2^{(0)}X_0^{(0)}-Y_2^{(-2)})+Y_4^{(0)}(X_2^{(1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})\\
&-X_2^{(0)}X_0^{(0)}+X_2^{(-2)})-Y_4^{(-1)}(Y_0^{(2)}X_0^{(0)}-Y_0^{(0)})+Y_4^{(-2)}X_0^{(0)}.
\end{aligned}
\tag{E.12f}
$$

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