Integrable lattice equations: Connection to the Möbius group, Bäcklund transformations and solutions

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The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

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Abstract

We consider scalar integrable lattice equations which arise as the natural discrete counterparts to KdV-type PDEs. Several results are reported.

We identify a new and natural connection between the 'Schwarzian' (Möbius invariant) integrable lattice systems and the Möbius group itself. The lattice equation in some sense describes dynamics of fixed-points as they change under composition between transformations.

A classification result is given for lattice equations which are linear but also consistent on the cube. Such systems lie outside previous classification schemes.

New Bäcklund transformations (BTs) for some known integrable lattice equations are given. As opposed to the natural auto-BT inherent in every such equation, these BTs are of two other kinds. Specifically, it is found that some equations admit additional auto-BTs (with Bäcklund parameter), whilst some pairs of apparently distinct equations admit a BT which connects them.

Adler's equation has come to hold the status of 'master equation' among the integrable lattice equations. Solutions of this equation are derived which are associated with 1-cycles and 2-cycles of the BT. They were the first explicit solutions written for Adler's equation. We also apply the BT to the 1-cycle solution in order to construct a soliton-type solution.

Contents

Notational conventions

s.t.	Abbreviation for 'such that'
w.r.t.	Abbreviation for 'with respect to'
$\exists$	Abbreviation for 'there exists'
$\forall$	Abbreviation for 'for all'
$a \in A$	a is a member of A
$A \subset B$	A is a subset of B
$A \cup B$	union of the sets A and B
$A \cap B$	intersection of the sets A and B
$\mathbb{C}$	The complex numbers
$\mathbb{Z}$	The integers
$\mathbb{N}$	The non-negative integers
∂_x	Partial derivative operator w.r.t. x
u_x	Partial derivative of u w.r.t. x, $u_x = \partial_x u$

Acronyms

KdV	Korteweg de-Vries
KN	Krichever-Novikov
KP	Kadomtsev-Petviashvili
ABS	Adler-Bobenko-Suris
NQC	Nijhoff-Quispel-Capel
BT	Bäcklund transformation
PDE, ODE	Partial, Ordinary differential equation
ODE	Ordinary differential equation
IVP	Initial value problem
LHS, RHS	Left, right hand side

Index of notation

Symbol	Description	Page
$\widehat{\mathbb{C}}$	The extended complex plane $\mathbb{C} \cup \{\infty\}$	19
$\Delta[f, x]$	The discriminant of f w.r.t x, $f_x^2 - 2ff_{xx}$	
$\phi(w, x, y, z)$	The cross-ratio of four points $w, x, y, z \in \widehat{\mathbb{C}}$	37,121
M	The group of Möbius transformations	19,122
$\mathsf{F}(u, v, \ldots)$	The fixed-point subgroup of M on the points $u, v, \ldots \in \widehat{\mathbb{C}}$	23,37
$\mathsf{S}(u, v, \ldots)$	The stabilizer subgroup of M on the points $u, v, \ldots \in \widehat{\mathbb{C}}$	41,123
G	Always denotes a subgroup of M	
$\mathsf{l}, \mathsf{m}, \mathsf{n}, \mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{a}, \mathsf{b}$	Always denote Möbius transformations	19
e	The identity Möbius transformation	19
i, j	Always denote Möbius involutions	24
$\langle \mathsf{m}, \mathsf{n}, \ldots \rangle$	The subgroup generated by $\mathsf{m}, \mathsf{n}, \ldots \in \mathsf{M}$	
α, β	The normal-form isomorphisms for the groups $\mathsf{F}(u_1, u_2)$ and $\mathsf{F}(u_1)$ respectively	36,37
π_v^u	The natural projection from $\mathsf{S}(u)$ to $\mathsf{F}(u, v)$	44
α_u, β_u	The extended normal-form homomorphism resp. isomorphism for $\mathsf{S}(u)$ and $\mathsf{F}(u)$	45
Γ	The set of all points on a particular elliptic curve	
$\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u}$	Always denote points on an elliptic curve	
$\mathfrak{e}$	The identity point on an elliptic curve	

Chapter 1

Introduction

In this chapter we will begin by setting the scene with a very brief history of the subject matter. We will then give a detailed technical example to introduce the concepts of greatest importance for understanding the material in the thesis. Finally we will give an overview of the thesis itself, highlighting the main results and describing the factors which have motivated the research.

1.1 Background

The celebrated transformation of Bäcklund, published in 1883 [12], provided a method in differential geometry of generating surfaces of constant negative curvature. In a particular fixed co-ordinate system (and in two dimensions) these surfaces are exactly the solutions of a particular two dimensional PDE which is now known as the sine-Gorden equation. The transformation discovered by Bäcklund generates a new solution of the PDE from a known solution using (one dimensional) quadrature. The commutativity property between two such transformations, distinguished by the choice of a parameter (the Bäcklund parameter), was later discovered by Bianchi [13]. This property revealed a *superposition principle* for solutions related by the Bäcklund transformation (the permutability condition) which as a practical upshot enables the algebraic construction of new solutions by the superposition of known solutions. (cf. [21, 67] for a discussion of the geometrical aspects of BTs and a more comprehensive history and references.)

Transformations of the type originally discovered by Bäcklund, which have also come to bear his name, play a central role in the theory of *integrable systems* (cf. the collection [46]). A step of particular importance in establishing this was made by Wahlquist and Estabrook [83] in 1973 when they reported their discovery of a BT for the (potential) Korteweg-de Vries (KdV) equation (famously solved in [24]). Among the several important facts they observed was a direct connection between their BT and the remarkable multi-soliton solutions discovered by Hirota [29]. Specifically, it was found that the N-soliton solution is related to the $(N+1)$-soliton solution by the BT.

Integrable semi-discrete and fully discrete equations, which are the analog of integrable PDEs but with one or more of the variables discrete, arose naturally from several areas early on in the development of integrable systems. For example from physical modelling [74] (cf. [22, 33]), discretization of the Zakaharov-Shabat [85] inverse scattering scheme [1, 44], discretization of PDEs in bilinear form [29] (cf. also [48]) and consideration of BTs themselves as semi-discrete equations [43, 42]. In 1983 Nijhoff, Quispel and Capel (NQC) [50] (cf. [66, 52, 54]) added to this tally, they proposed a fully discrete lattice equation (the NQC equation) based on a singular linear integral equation which generalized that of Fokas and Ablowitz [23] (the direct linearisation approach). The resulting lattice equation was shown to coincide with the superposition principle for a quite general PDE of KdV type. They demonstrated that the PDE and its BT could be recovered from the lattice equation in a particular continuum limit and, by construction, were able to give multi-soliton solutions for the lattice equation.

In 1998 Adler [3] found a BT for the Krichever-Novikov (KN) equation [37, 38]. The KN equation is a PDE of KdV type which holds a distinguished position because essentially all other non-linear equations of KdV type are degenerate cases of this one [72, 73] (cf. also [31] for a connection to the Landau-Lifshitz equation). In particular the PDE discretized by NQC is a parameter sub-case of the KN equation. As a result the superposition principle found by Adler [3] for the KN BT (we refer to this superposition principle as Adler's lattice equation) provided a generalisation of the NQC equation. Note that this was a generalisation for which the direct linearising transformation, and in particular the soliton solutions, were not known.

In a paper of Nijhoff and Walker [58] in 2001 (cf. also [15]) that the property of *consistency on the cube*, which was actually implicit in the construction of the NQC equation, was identified as a property of the lattice equation alone, without reference to other features of integrability. This property should be expected for any superposition principle, and was observed for Adler's equation by Nijhoff in [61].

It was the promotion of consistency on the cube to a criteria for integrability by Adler, Bobenko and Suris which led them to important classification results for integrable lattice equations in 2002 and 2007 [4, 6]. They provided a list of canonical forms for integrable lattice equations (denoted $Q1, Q2, Q3, Q4, A1, A2, H1, H2$ and $H3$) which have since become common terminology in the field. Basically though, they confirmed that Adler's lattice equation (denoted $Q4$) holds the same status among integrable quadrilateral lattice equations as does the KN equation among PDEs of KdV type, i.e., everything within the classification is a degenerate case of Adler's equation. Adler and Suris [5] have also reported connections between Adler's lattice equation and several other important (elliptic) integrable systems [70, 40], giving it the status of an 'integrable master equation'. There are some technical challenges in studying this equation which makes it the least understood of the integrable scalar quadrilateral lattice equations, but it is also of course the most interesting.

1.2 Example

The notion of Bäcklund transformation (BT) can be seen as a unifying one as far as this thesis is concerned. First, almost without exception, each integrable lattice equation considered here will be the superposition principle for BTs of some known integrable PDE. Second, the notion of integrability will be that of higher-dimensional consistency, which is basically a discrete analog for the existence of a BT. The superposition principle, higher dimensional consistency, and several more ideas (and terms used in section 1.1) which are well established components of the integrable systems theory will be explained in this section by means of an example. We remark at this point that any attempt at a universal definition of a BT is unlikely to satisfy everyone (unless it were overly vague), so using a relevant but particular example is a practical way to establish in the mind of the reader what we mean by this term. In particular we distinguish the notion of BT considered here from the more general notion of a Lie-Bäcklund transformation (which is the subject of for example [7]).

The Schwarzian KdV equation and its Bäcklund transformation

The example we will consider starts with an integrable PDE known as the *Schwarzian KdV equation*

$$SKdV(u) := \frac{u_y}{u_x} - \frac{u_{xxx}}{u_x} + \frac{3}{2}\frac{u_{xx}^2}{u_x^2} = 0, \tag{1.1}$$

which was first written in this form by Weiss in 1982 [79], but which can actually be transformed to a parameter sub-case of the equation found by Krichever and Novikov [37, 38] in 1979. Among the known two-dimensional integrable PDEs of KdV type, i.e., equations of the form $u_y = u_{xxx} + F(u, u_x, u_{xx})$ for some function F, the principal distinguishing feature of (1.1) is its invariance under Möbius transformations,

$$u \to (au + b)/(cu + d), \quad ad \neq bc. \tag{1.2}$$

The BT for (1.1) may be written as the following coupled system of equations

$$u_x\widetilde{u}_x = \frac{1}{2p}\left(u-\widetilde{u}\right)^2, \qquad SKdV(u) + SKdV(\widetilde{u}) = 0 \tag{1.3}$$

where $u = u(x, y)$, $\widetilde{u} = \widetilde{u}(x, y)$ and $p \in \mathbb{C}$ is a constant parameter. This system is perhaps *not* as good as some others for an introductory discussion of BTs because, like all BTs for equations of KdV type, the system (1.3) contains one ODE in x and one PDE (involving both x and y derivatives). In this respect the original system of Bäcklund, which contains one first order ODE in x and one first order ODE in y, involves easier calculation and is neater conceptually. The difference can be understood because the sine-Gorden equation, unlike (1.1), is first order in, and invariant under the interchange of, the two independent variables x and y. The example (1.1) does have the advantage that it is typical of the equations of KdV type. But we have made it our particular choice because the superposition principle for the BT (1.3) also emerges quite naturally from an altogether different origin, which will be one of the principal new observations contained in this thesis. For this reason it is useful for the reader to encounter it in the present context first.

So consider the system (1.3). If we fix u for example, then the *single* variable $\widetilde{u}$ is determined by a system of *two* equations. Rearranging the first equation of (1.3) for $\widetilde{u}_x$ and the second for $\widetilde{u}_y$, we can write a condition for the compatibility of this system, a direct calculation yields

$$\partial_y\widetilde{u}_x - \partial_x\widetilde{u}_y = \frac{2(\widetilde{u}_x^2 u_{xx} + u_x^2\widetilde{u}_{xx})}{u_x(u_x - \widetilde{u}_x)} SKdV(u), \tag{1.4}$$

where we have used (1.3) to simplify the expression on the RHS. This reveals that the constraint (1.1) is sufficient for the compatibility of (1.3) in $\widetilde{u}$, moreover it is clear from (1.3) that $\widetilde{u}$ which then emerges *also* satisfies (1.1). The new solution $\widetilde{u}$ will depend not only on the original solution u but also on the parameter p and a single constant of integration. We say that two solutions u and $\widetilde{u}$ of (1.1) satisfying (1.3) are related by the BT and for convenience we will write

$$u \overset{p}{\sim} \widetilde{u}. \tag{1.5}$$

Inspecting (1.3) we see that the BT actually defines a *symmetric* relation on the set of solutions to (1.1). But this is not where the story ends because the relation defined by the BT finds itself endowed with an elegant property.

The superposition principle

The parameter p is a free parameter of the transformation (1.3) which we will refer to as the Bäcklund parameter, transformations with different choices of this parameter satisfy a remarkable commutativity property (Bianchi permutability) which we will now describe. If we suppose that $u = u(x,y)$, $\widetilde{u} = \widetilde{u}(x,y)$ and $\widehat{u} = \widehat{u}(x,y)$ are solutions of (1.1) for which

$$u \overset{p}{\sim} \widetilde{u} \quad \text{and} \quad u \overset{q}{\sim} \widehat{u} \tag{1.6}$$

for some other parameter q, and define $\widehat{\widetilde{u}} = \widehat{\widetilde{u}}(x,y)$ by the relation

$$\mathcal{Q}_{pq}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) := p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}}) - q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}}) = 0, \tag{1.7}$$

then the new function $\widehat{\widetilde{u}}$ is a solution of (1.1), moreover the BT relations

$$\widehat{u} \overset{p}{\sim} \widehat{\widetilde{u}} \quad \text{and} \quad \widetilde{u} \overset{q}{\sim} \widehat{\widetilde{u}} \tag{1.8}$$

also hold. The relation (1.7) at the core of this notion of commutativity is the *superposition principle*. The assertions about the function $\widehat{\widetilde{u}}$ determined by (1.7) can be verified by direct calculation, so it remains to show how (1.7) can be found from (1.3). The idea is to start by writing necessary conditions for (1.8) assuming (1.6). Certainly the following should hold

$$\begin{aligned} u_x\widetilde{u}_x &= \frac{1}{p}(u-\widetilde{u})^2, & u_x\widehat{u}_x &= \frac{1}{q}(u-\widehat{u})^2, \\ \widehat{u}_x\widehat{\widetilde{u}}_x &= \frac{1}{p}(\widehat{u}-\widehat{\widetilde{u}})^2, & \widetilde{u}_x\widehat{\widetilde{u}}_x &= \frac{1}{q}(\widetilde{u}-\widehat{\widetilde{u}})^2. \end{aligned} \tag{1.9}$$

Elimination of $u_x, \widetilde{u}_x, \widehat{u}_x$ and $\widehat{\widetilde{u}}_x$ from this system leads to the condition

$$\frac{1}{q^2}(u-\widehat{u})^2(\widetilde{u}-\widehat{\widetilde{u}})^2 - \frac{1}{p^2}(u-\widetilde{u})^2(\widehat{u}-\widehat{\widetilde{u}})^2 = 0. \tag{1.10}$$

The factorization of this expression is immediate,

$$\mathcal{Q}_{pq}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}})\mathcal{Q}_{pq'}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = 0, \tag{1.11}$$

where $q' = -q$, which introduces the expression defined in (1.7) as well as a related competing expression. Whereas the vanishing of the second factor of (1.11) determines a function $\widehat{\widetilde{u}}$ which does *not* satisfy (1.8), a direct calculation reveals that the vanishing of the first factor determines a function $\widehat{\widetilde{u}}$ which (beyond usual expectations) *does*. The superposition principle determines algebraically a new solution $\widehat{\widetilde{u}}$ from known solutions u, $\widetilde{u}$ and $\widehat{u}$ satisfying (1.6). Observe that if $\widetilde{u}$ and $\widehat{u}$ are constructed by application of the BT to an arbitrary *seed* solution u, so each contains an arbitrary constant of integration, then the new solution $\widehat{\widetilde{u}}$ constructed by superposition contains (in principle) *two* arbitrary constants. But the consecutive application of two BTs to the seed solution u also results in a new solution which contains *two* constants of integration. What this reveals is that the construction of new solutions algebraically using the superposition principle does not lose out in any way to the iterative application of the BT itself, which requires quadrature at both steps.

Consistency on the cube

There is one more essential ingredient present in the BT apparatus, to reveal it we need to focus our attention on the superposition principle (1.7) alone. Consider this expression as a polynomial of degree one in the four variables $u, \widetilde{u}, \widehat{u}$ and $\widehat{\widetilde{u}}$ without, for the moment, any reference to the continuous variables x and y. Now it turns out that the consistency of the following system can be verified,

$$\begin{aligned}
&\mathcal{Q}_{pq}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = 0, && \mathcal{Q}_{pq}(\overline{u},\overline{\widetilde{u}},\overline{\widehat{u}},\overline{\widehat{\widetilde{u}}}) = 0,\\
&\mathcal{Q}_{qr}(u,\widehat{u},\overline{u},\overline{\widehat{u}}) = 0, && \mathcal{Q}_{qr}(\widetilde{u},\widehat{\widetilde{u}},\overline{\widetilde{u}},\overline{\widehat{\widetilde{u}}}) = 0,\\
&\mathcal{Q}_{rp}(u,\overline{u},\widetilde{u},\overline{\widetilde{u}}) = 0, && \mathcal{Q}_{rp}(\widehat{u},\overline{\widehat{u}},\widehat{\widetilde{u}},\overline{\widehat{\widetilde{u}}}) = 0.
\end{aligned} \tag{1.12}$$

By consistency here we mean that given initial data $u, \widetilde{u}, \widehat{u}$ and $\overline{u}$, and after evaluation of the intermediate variables $\widehat{\widetilde{u}}, \overline{\widehat{u}}$ and $\overline{\widetilde{u}}$ using the equations on the left, the remaining equations on the right determine the *same* value for $\widehat{\widetilde{\overline{u}}}$. By assigning the variables involved to the vertices of a cube as in figure 1.1, we may associate each equation in (1.12) to a face of the same cube. This provides a convenient geometrical configuration to visualise this notion of consistency. It also gives rise to the name of this property - we say that the expression (1.7) is *consistent on the cube*. This consis-

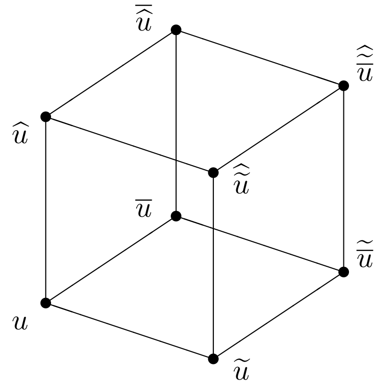

Figure 1.1: Variables associated to the vertices of a cube

tency property naturally augments the commutativity property already described for the BT (1.3). Specifically, if we suppose $u, \widetilde{u}, \widehat{u}$ and $\overline{u}$ are solutions of (1.1) for which

$$u \stackrel{p}{\sim} \widetilde{u}, \qquad u \stackrel{q}{\sim} \widehat{u}, \qquad u \stackrel{r}{\sim} \overline{u}, \tag{1.13}$$

and use the consistent system (1.12) to define $\widehat{\widetilde{u}}, \overline{\widehat{u}}, \widetilde{\overline{u}}$ and $\widehat{\widetilde{\overline{u}}}$, then all of the following relations hold

$$\begin{array}{lll} \widehat{u} \stackrel{p}{\sim} \widehat{\widetilde{u}}, & \overline{u} \stackrel{q}{\sim} \overline{\widehat{u}}, & \widetilde{u} \stackrel{r}{\sim} \widetilde{\overline{u}}, \\ \overline{u} \stackrel{p}{\sim} \widetilde{\overline{u}}, & \widetilde{u} \stackrel{q}{\sim} \widehat{\widetilde{u}}, & \widehat{u} \stackrel{r}{\sim} \overline{\widehat{u}}, \\ \overline{\widehat{u}} \stackrel{p}{\sim} \widehat{\widetilde{\overline{u}}}, & \widetilde{\overline{u}} \stackrel{q}{\sim} \widehat{\widetilde{\overline{u}}}, & \widehat{\widetilde{u}} \stackrel{r}{\sim} \widehat{\widetilde{\overline{u}}}. \end{array} \tag{1.14}$$

With reference to the cube in figure 1.1, we may associate the relations in (1.13) and (1.14) to the *edges*. It is natural to ask if the permutability we have described, of two and three BTs, extends to a higher number of BTs, say with Bäcklund parameters $p, q, r, s, \ldots$, and the answer turns out to be in the affirmative. In fact we require only one further property of the expression (1.7) and that is *covariance*, by which we will mean the symmetry

$$\mathcal{Q}_{pq}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) = 0 \quad \Leftrightarrow \quad \mathcal{Q}_{qp}(u, \widehat{u}, \widetilde{u}, \widehat{\widetilde{u}}) = 0. \tag{1.15}$$

Covariance together with consistency on the cube implies consistency also on any higher dimensional hypercube. That is, if we give initial data $u, \widetilde{u}, \widehat{u}, \overline{u}, \dot{u}, \ldots$ then the values at each other vertex on the hypercube are determined uniquely by imposing the appropriate copy of the equation (1.7) to each face. Once again, if

$$u \overset{p}{\sim} \widetilde{u}, \qquad u \overset{q}{\sim} \widehat{u}, \qquad u \overset{r}{\sim} \overline{u}, \qquad u \overset{s}{\sim} \dot{u}, \qquad \ldots \tag{1.16}$$

are related solutions of (1.1), then the superposition principle can be used to consistency populate the remaining vertices of the hypercube with new solutions of (1.1), related by the BT along each edge.

The associated lattice equation and its Bäcklund transformation

As we have seen, the consistency on the cube is a property of the polynomial expression (1.7) on its own, without any reference to the continuous variables x and y. This property has significant consequences when we use (1.7) as a starting point to define a *quadrilateral lattice equation*. The idea is to consider the equation

$$\mathcal{Q}_{pq}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) = 0, \tag{1.17}$$

but now we suppose $u = u(n, m)$, $\widetilde{u} = u(n + 1, m)$, $\widehat{u} = u(n, m + 1)$ and $\widehat{\widetilde{u}} = u(n + 1, m + 1)$ are values of a dependent variable u as a function of independent variables $n, m \in \mathbb{Z}$, so (1.17) holds on each elementary quadrilateral of the $\mathbb{Z}^2$ lattice. In this context the parameters p and q will be referred to as *lattice parameters* of the equation. This particular discrete equation, known as the lattice Schwarzian KdV, was first written down in 1995 by Nijhoff and Capel [52], but can be identified as a parameter subcase of the earlier equation of NQC [50]. Like its continuous counterpart, this equation is distinguished among the known integrable quadrilateral lattice equations because of its invariance under Möbius transformations (1.2).

The following system constitutes a BT for for the lattice equation (1.17),

$$\mathcal{Q}_{rp}(u, v, \widetilde{u}, \widetilde{v}) = 0, \qquad \mathcal{Q}_{qr}(u, \widehat{u}, v, \widehat{v}) = 0, \tag{1.18}$$

where $u = u(n, m)$, $v = v(n, m)$ will be the related solutions of (1.17) and $r \in \mathbb{C}$ takes the role of the Bäcklund parameter. With u fixed, (1.18) is an overdetermined system for v. Now it turns out that (1.17) is a sufficient condition on u for the compatibility of (1.18) in v $((\widetilde{v})^{\widehat{}} - (\widehat{v})^{\widetilde{}} = 0)$, and moreover the solution v which then emerges *also* satisfies (1.17). To verify this assertion we don't actually need to do any further calculation, it holds because the BT (1.18), like the equation (1.17) itself, is constructed from the expression (1.7), and the assertion amounts to nothing more than its consistency on the cube.

If we inspect the BT (1.18) for the lattice equation we see it has a somewhat simpler appearance than the BT (1.3) for the PDE in that it consists of a pair of first order ordinary difference equations, which is actually more reminiscent of Bäcklund's original system for the sine-Gorden equation. However in most respects the BT (1.18) is very similar to the BT (1.3) described earlier for the PDE. As in that case the new solution v will depend upon u as well as the choice of the parameter r and one constant of integration. Also it is clear that, like before, the BT defines a symmetric relation on the set of solutions of (1.17) - our attention is now firmly on the lattice equation and its BT so we allow ourselves to recycle the previous notation - we will write

$$u \overset{r}{\sim} v \tag{1.19}$$

to indicate that u and v are solutions of (1.17) related by the BT (1.18).

The superposition principle

As in the continuous case, BTs (1.18) with different choices of Bäcklund parameter commute in the sense that a superposition principle exists. Specifically, if we suppose that $u = u(n,m)$ is a solution of (1.17) and that $\overline{u} = \overline{u}(n,m)$ and $\dot{u} = \dot{u}(n,m)$ are related solutions,

$$u \overset{r}{\sim} \overline{u} \quad \text{and} \quad u \overset{s}{\sim} \dot{u}, \tag{1.20}$$

then the new function $\dot{\overline{u}} = \dot{\overline{u}}(n,m)$ defined by the relation

$$\mathcal{Q}_{rs}(u, \overline{u}, \dot{u}, \dot{\overline{u}}) = 0 \tag{1.21}$$

also satisfies (1.17), moreover it is related to $\overline{u}$ and $\dot{u}$ by the BTs

$$\dot{u} \overset{r}{\sim} \dot{\overline{u}} \quad \text{and} \quad \overline{u} \overset{s}{\sim} \dot{\overline{u}}. \tag{1.22}$$

The equation (1.21) is the superposition principle for solutions of the lattice equation (1.17) related by its BT (1.18). Again it can be written in terms of the same expression (1.7). To verify the stated commutativity property turns out to be equivalent to checking for the consistency of (1.7) on the faces of the four dimensional hypercube, which we have already observed is just a consequence of covariance plus consistency on the cube. Similarly, the permutability of any higher number of BTs for the lattice equation is assured because of consistency on the higher dimensional hypercube. So structurally the relation defined by the BT on the solution space is just the same for the lattice equation as for the PDE.

But perhaps the most appealing feature of the fully discrete system is that the lattice equation itself, its BT and the superposition principle, are all expressible in terms of the same quadrilateral expression, and the gears which make the BT apparatus work are turned entirely by its properties of covariance and consistency on the cube. This simplicity may leave the reader with the impression that the lattice system is perhaps a complementary addition to the underlying integrable PDE, but that the separated discrete part can only contain a residue of the richness present in the full system. However we are about to finish this section by describing one last feature which will reveal that this impression is essentially a false one, for although the discrete system remains one step away from any physics described the PDE, both the PDE itself and its BT may be recovered from the discrete system by a *continuum limit*.

The continuum limit

We will reintroduce continuous variables x and y through the operator

$$C_p = e^{\sqrt{2p}(\partial_x + \frac{p}{6}\partial_y)}, \tag{1.23}$$

which depends also on a parameter p. (This could be referred to as a truncated vertex operator [78].) Acting on an analytic function $f = f(x, y)$ this operator simply increments the arguments by a particular amount (which depends on the value of p),

$$[C_p f](x, y) = f(x + \sqrt{2p}, y + p\sqrt{2p}/6), \tag{1.24}$$

and may be referred to as an *analytic difference operator*. Note also, for example, that

$$\lim_{p\to 0} \frac{1}{\sqrt{2p}}(C_p - 1)f = f_x. \tag{1.25}$$

Now, if we suppose $u = u(x, y)$ then the following is straightforward to verify

$$\lim_{p,q\to 0} \frac{-3}{pq(p-q)} \mathcal{Q}_{pq}(u, C_p u, C_q u, C_p C_q u) = u_x^2 SKdV(u). \tag{1.26}$$

So in this limit the PDE (1.1) emerges from the lattice equation (1.17). (According [78] we would recover a hierarchy of PDEs if the vertex operator (1.23) were not truncated.) The limit of the BT (1.18) is slightly more subtle. Given $u = u(x, y)$ and $v = v(x, y)$, we want to find the limit of the system

$$\mathcal{Q}_{rp}(u, v, C_p u, C_p v) = 0, \qquad \mathcal{Q}_{qr}(u, C_q u, v, C_q v) = 0, \tag{1.27}$$

as p and q go to zero. But these two expressions have expansions with precisely the same terms,

$$\begin{aligned}
&\mathcal{Q}_{rp}(u,v,C_pu,C_pv) = p[2ru_xv_x - (u-v)^2] + \\
&\quad p\sqrt{2p}[r(u_xv_{xx} + u_{xx}v_x) - (u-v)(u_x - v_x)] + \\
&p^2[\frac{r}{3}((u_t + 2u_{xxx})v_x + (v_t + 2v_{xxx})u_x) + ru_{xx}v_{xx} - (u-v)(u_{xx} - v_{xx})] \\
&+ \dots,
\end{aligned} \tag{1.28}$$

the only difference between them being the small parameter. What we do in this circumstance is take the limit of the system (1.27) to be the first two independent terms in the expansion (1.28). The second term vanishes as a consequence of the first, so we take the first and the third terms, a rearrangement of the resulting two equations yields the system

$$u_xv_x = \frac{1}{2r}(u-v)^2, \qquad SKdV(u) + SKdV(v) = 0, \tag{1.29}$$

which up to the choice of variables and Bäcklund parameter is exactly (1.3). So in this limit we also find that the continuous BT emerges from the discrete BT. The continuum limit of the discrete system thus brings us full circle, back to the PDE and its BT.

Conclusion

We have given an introductory run-through of an integrable PDE and its lattice counterpart. The facts presented are self-contained in the sense of being verifiable without the need for additional material. The aspect of integrability we have described in this example is not much exceeded throughout the thesis, and everything else which is new will be thoroughly introduced. The hope of the author is that accessibility remains as wide as possible throughout.

1.3 Synopsis

It is useful to consider the thesis as being in two distinct parts, each with a different underlying motivation and each presented in a different style.

Part I

The first part explores a new connection between integrable lattice equations on the one hand and dynamical aspects of Möbius transformations on the other. This exploration begins firmly in the realm of the latter, with an introduction to the very basics of Möbius transformations in chapter 2. Of course to study Möbius transformations is far from being a new idea, so some justification for the presence of this introduction is required. Our purpose is to give a perspective which is non-standard, from a vantage somewhere in-between two others. The first is a consideration of the group structure, appropriately the matrix group $\mathrm{GL}_2(\mathbb{C})$ which is homomorphic to the Möbius group (or projective $\mathrm{GL}_2(\mathbb{C})$ which is isomorphic). The second is a discrete dynamical systems point of view, where a Möbius transformation can be considered as a simple mapping, which is the discrete analog of the autonomous Riccati ODE.

The in-between perspective we take will turn out to be quite natural for the results we wish to state. Importantly though, the principal generalisation we have in mind for these notions is to the *biquadratic correspondence* where a similar approach is natural.

In chapter 3 we describe a very simple lattice equation naturally associated with subgroups of Möbius transformations that share their fixed-points. Although the lattice equation is linear, the connection is shown to expose other strong solveability properties for the equation, including consistency on the cube.

Chapter 4 introduces another quadrilateral lattice system, this time though some effort is needed to establish the equation, which turns out to be the Schwarzian example we used in the previous section. The embedding in three dimensions will establish a connection with the system already discussed in chapter 3. Note that the 'Schwarzian' lattice systems have been connected to classical results in geometry by Konopelchenko and Schief [36].

It turns out that a crucial link between the systems discussed in chapters 3 and 4 is a higher dimensional lattice equation, specifically the lattice Schwarzian KP equation, which is the subject of chapter 5. The connection to Möbius transformations here is used to characterize the reduction from this equation to the lattice Schwarzian KdV equation of chapter 4.

Part II

In chapter 6 we introduce Adler's equation and review the ABS classification results discussed in section 1.1. An important theme which is brought out by this material is a natural connection between the integrable (scalar) quadrilateral lattice equations and the *biquadratic*, a polynomial of degree at most two in two variables.

Equations which are consistent on the cube, and which appear not to be degenerations of Adler's equation, are known. In fact the equation introduced in chapter 3 is one such example, also there is an interesting system discovered by Hietarinta [26]. However (as far as the author is aware) none are known which are not linearisable (the Hietarinta equation was linearised in [69]). In chapter 7 we give a new but modest classification result which aims to clarify the situation for the linear lattice equations. An interesting feature of the main equation given is that the lattice parameters (explained in section 1.2) naturally have two components.

There are two ways to view the results we present in chapter 8. Within this chapter new systems are given which are consistent on a cube, but the systems are non-symmetric in the sense of having functionally distinct equations on the various faces. However on at least two opposite faces the equations present coincide with previously known equations which are themselves consistent on the cube (in the conventional sense as described in section 1.2). A natural way to think of these systems is as BTs, which are distinct from the conventional BT, and that connect these previously known equations. ABS gave one such system in [4], the only other example of which the author is aware was a BT between the lattice Schwarzian and modified KdV equations attributable to [59], which is discussed (among other things) in appendix C.

In chapter 9 we give (to the author's knowledge) the first explicit solutions of Adler's equation. Given that this equation has a natural BT (like the one described in section 1.2) one is tempted to think that the construction of solutions is almost immediate. However the construction of a *seed* solution to start a Bäcklund chain turns out to be a non-trivial problem for Adler's equation.

We start the chapter by describing the natural connection between Adler's equation, the group on an elliptic curve and the elliptic functions, which is all orchestrated by the associated *symmetric biquadratic correspondence*. With almost no further consideration this leads to an elementary solution of Adler's equation, however we go on to discover that this solution is only trivially altered by an application of the BT. In other words this solution constitutes a *non-germinating* seed. We resolve this problem by constructing a generalisation of the non-germinating seed solution as what we refer to as a 1-cycle (or fixed-point) of the BT (this terminology is ambiguous because the BT is not injective, but we do give a precise definition), a result given originally in [8]. Application of the BT (with different Bäcklund parameter) to this solution then yields a non-trivial soliton type solution.

The 1-cycle is actually the simplest case of the more general problem (considered first by Weiss [81, 82]) to find N-cycles of the BT. We finish chapter 9 by solving the quite technical problem of constructing explicitly the 2-cycles of the BT, this result was published originally in [9]. The construction we apply is closely related to the issue of periodic reductions of the quadrilateral lattice. This idea was first explored in the context of periodic 'staircase' reductions of integrable lattice equations of KdV type, cf [65], where they led to mappings integrable in the sense of Liouville

[18] (cf. [77]). A two-step reduction of this type for Adler's equation was studied recently in the work of Joshi et. al. [32]. We will explain the connection between this mapping and the mapping associated with the 2-cycle of the BT considered here. More generally, periodic reductions on the lattice can be considered the analog of finite-gap solutions [57] (cf also [81]) and this connection leads us to expect that parametrisation of N-cycles of the BT with $N > 2$ will need Abelian functions associated with hyper-elliptic curves, which we do not consider here.

Ending

In the final chapter (10) we draw-out two important themes which have emerged during the course of the thesis. These themes suggest practical future research, and we provide some technical details about one direction in particular. The appendices contain some additional material which complements the results of the thesis.

Chapter 2

Möbius transformations

In this chapter we give an introductory account of the Möbius transformations, that is transformations of the form

$$\mathsf{m} : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}, \qquad u \mapsto \mathsf{m}(u) = \frac{au+b}{cu+d}, \quad ad \neq bc. \tag{2.1}$$

Here $\widehat{\mathbb{C}}$ denotes the extended complex plane, $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The parameters $a, b, c, d \in \mathbb{C}$ will be called the *coefficients* of the transformation. The additional constraint $ad \neq bc$ is equivalent to demanding the transformation be invertible, so m permutes $\widehat{\mathbb{C}}$. The set of all Möbius transformations,

$$\mathsf{M} = \left\{ u \mapsto \frac{au+b}{cu+d} \;\middle|\; a, b, c, d \in \mathbb{C}, ad \neq bc \right\},$$

form a group under composition, we will refer to the identity in M as e.

Through this chapter we will collect together some elementary facts about Möbius transformations. These facts will all be stated as theorems with a uniform numbering system. The main reason is that they are then available for convenient reference in later chapters, we remark that nothing in this introductory chapter is actually a theorem in the sense of being a new result.

2.1 The linear system for the coefficients

Consider the action of $\mathsf{m} \in \mathsf{M}$ on four points in $\widehat{\mathbb{C}}$

$$\mathsf{m}(u_k) = \widetilde{u}_k, \quad k \in \{1, 2, 3, 4\}. \tag{2.2}$$

These equations can be written as a linear system for the coefficients of m,

$$\begin{bmatrix} u_1\widetilde{u}_1 & \widetilde{u}_1 & u_1 & 1 \\ u_2\widetilde{u}_2 & \widetilde{u}_2 & u_2 & 1 \\ u_3\widetilde{u}_3 & \widetilde{u}_3 & u_3 & 1 \\ u_4\widetilde{u}_4 & \widetilde{u}_4 & u_4 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \\ -a \\ -b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.3}$$

The vanishing of the determinant of the matrix in this system imposes that

$$\frac{(\widetilde{u}_4 - \widetilde{u}_3)(\widetilde{u}_2 - \widetilde{u}_1)}{(\widetilde{u}_4 - \widetilde{u}_2)(\widetilde{u}_3 - \widetilde{u}_1)} = \frac{(u_4 - u_3)(u_2 - u_1)}{(u_4 - u_2)(u_3 - u_1)}, \tag{2.4}$$

which is therefore necessary for the existence of $\mathsf{m} \in \mathsf{M}$ satisfying (2.2). The condition (2.4) leads to the following basic theorem.

Theorem 1 *Given two sets of distinct points* $\{u_1, u_2, u_3\}$ *and* $\{\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3\}$ *taken from* $\widehat{\mathbb{C}}$*, there exists a unique* $\mathsf{m} \in \mathsf{M}$ *for which* $\mathsf{m}(u_k) = \widetilde{u}_k$, $k \in \{1, 2, 3\}$*. Moreover, given any point* $u_4 \in \widehat{\mathbb{C}}$ *the action of* m *on* u_4 *is the point* $\widetilde{u}_4$ *determined uniquely by* (2.4)*.*

Proof If such $\mathsf{m} \in \mathsf{M}$ existed and its action on an arbitrary point u_4 were $\widetilde{u}_4$, then $\widetilde{u}_4$ should indeed satisfy (2.4) which has been established as a necessary condition on m. Furthermore, the assumption that the sets each contain three *distinct* points ensures the point $\widetilde{u}_4$ is fixed unambiguously by (2.4), so any such m is unique.

But the mapping $u_4 \mapsto \widetilde{u}_4$ defined by (2.4) is invertible (again, we need that the points in each set are distinct) and therefore constitutes a Möbius transformation. Moreover this Möbius transformation sends u_k to $\widetilde{u}_k$ for $k \in \{1, 2, 3\}$ so amounts to a construction of m and the theorem is proved.

□

There is an issue which we need to address, specifically the interpretation of theorem (1) and in particular the equation (2.4) given that the points involved are taken from $\widehat{\mathbb{C}}$. If one or more of the points in (2.4) are equal to ∞ it may not be obvious at a glance that the point $\widetilde{u}_4$ is uniquely determined. However, by substiting $1/\epsilon$ for those points which are equal to ∞ and rearranging for $\widetilde{u}_4$ it will be seen that $\widetilde{u}_v \in \widehat{\mathbb{C}}$ is uniquely determined in the limit $\epsilon \longrightarrow 0$ (provided we stay within the hypotheses of the theorem). On a few later occasions we will write down equations which will also involve points taken from $\widehat{\mathbb{C}}$, and although we will not mention it explicitly, we mean for them to be interpreted in the way we have described here when one or more of the points involved is equal to ∞.

2.2 Fixed-points

For any $\mathsf{m} \in \mathsf{M}$ the equation for v

$$\mathsf{m}(v) = v \tag{2.5}$$

has at least one solution in $\widehat{\mathbb{C}}$. To see this it is sufficient to observe that when (2.5), written as a polynomial equation for v, does not have a solution in $\mathbb{C}$, then it has the solution 0 as a polynomial equation for $1/v$. We refer to any solution of (2.5) as a *fixed-point* of m, hence every $\mathsf{m} \in \mathsf{M}$ has at least one fixed-point. This can be complemented by the observation that any $\mathsf{m} \in \mathsf{M}$ with more than two fixed-points coincides with e by theorem 1, upon which we can formulate the following theorem.

Theorem 2 *Every non-identity Möbius transformation has one or two fixed-points in* $\widehat{\mathbb{C}}$*.*

It turns out to be natural to parametrise the group of Möbius transformations by their fixed-points.

If $\mathsf{m} \in \mathsf{M}$ satisfies (2.2) with $\widetilde{u}_1 = u_1$ and $\widetilde{u}_2 = u_2$, i.e., u_1 and u_2 are fixed-points of m, then provided $u_2 \neq u_1$ the condition (2.4) simplifies to the relation

$$\frac{(\widetilde{u}_4-\widetilde{u}_3)(u_4-u_2)(u_1-u_3)}{(\widetilde{u}_4-u_2)(u_4-u_3)(u_1-\widetilde{u}_3)}=1 \quad\Leftrightarrow\quad \begin{vmatrix} u_1u_2 & u_1+u_2 & 1\\ u_3\widetilde{u}_4 & u_3+\widetilde{u}_4 & 1\\ \widetilde{u}_3u_4 & \widetilde{u}_3+u_4 & 1\end{vmatrix}=0. \quad (2.6)$$

(The alternative determinant form makes the remarkable symmetries of this equation plain to the eye.) In fact the $u_2 \neq u_1$ proviso can be removed leaving us with a stronger specialisation of theorem 1.

Theorem 3 *Given a (possibly coinciding) pair* $u_1, u_2 \in \widehat{\mathbb{C}}$ *and a pair of distinct points* $u_3, \widetilde{u}_3 \in \widehat{\mathbb{C}} \setminus \{u_1, u_2\}$, *there exists a unique* $\mathsf{m} \in \mathsf{M}$ *the only fixed-points of which are* u_1 *and* u_2, *and for which* $\mathsf{m}(u_3) = \widetilde{u}_3$. *Moreover, given any point* $u_4 \in \widehat{\mathbb{C}}$ *the action of* m *on* u_4 *is the point* $\widetilde{u}_4$ *determined uniquely by* (2.6).

Proof If $u_2 \neq u_1$, theorem 1 assures the existence of a unique $\mathsf{m} \in \mathsf{M}$ for which $\mathsf{m}(u_1) = u_1$, $\mathsf{m}(u_2) = u_2$ and $\mathsf{m}(u_3) = \widetilde{u}_3$. If $u_2 \neq u_1$ (2.6) is equivalent to (2.4) with $\widetilde{u}_1 = u_1$ and $\widetilde{u}_2 = u_2$, so according to theorem 1 m sends u_4 to $\widetilde{u}_4$ fixed by (2.6). Finally, the condition that $\widetilde{u}_3 \neq u_3$ implies that $\mathsf{m} \neq \mathsf{e}$ so by theorem 2 $\{u_1, u_2\}$ are the *only* fixed-points of m and the theorem is proved in the case $u_2 \neq u_1$.

Now consider the statement of the theorem in the case $u_2 = u_1$. We may choose $\widetilde{u}_2 = u_2 = \widetilde{u}_1 = u_1$ so the condition (2.4) is satisfied. Let us then proceed, using (2.3), to find the coefficients (up to a common factor) of m. By construction m satisfies $\mathsf{m}(u_1) = u_1$, $\mathsf{m}(u_3) = \widetilde{u}_3$ and $\mathsf{m}(u_4) = \widetilde{u}_4$, however the assertion that u_1 is the *only* fixed-point of m puts one additional additional constraint on the coefficients of m (the vanishing of the discriminant of (2.5) as a quadratic equation in v), specifically this condition reads

$$(a+d)^2 = 4(ad-bc).$$

Imposing this on the calculated coefficients with the assumption $\widetilde{u}_3 \neq u_3$ yields nothing but (2.6) with $u_2 = u_1$. Hence (2.6) is necessary for the existence of m. Now, $\widetilde{u}_4$, the action of m on the point u_4, is fixed unambiguously by (2.6) so any such m is unique. Moreover the mapping $u_4 \mapsto \widetilde{u}_4$ defined by (2.6) is invertible and therefore constitutes a Möbius transformation which, by construction, has only one fixed-point u_1 and sends u_3 to $\widetilde{u}_3$, so we are done.

□

Note that theorem 3 provides a converse of theorem 2, i.e., a Möbius transformation is uniquely determined by a set of (one or two) fixed-points and its action on one other point.

It is obvious to ask if a family of Möbius transformations with shared fixed-points forms a subgroup within M, the following theorem, which is essentially a consequence of theorem 3, answers this question in the affirmative (although we do need to add e to complete this family).

Theorem 4 *Given a (possibly coinciding) pair of points $u_1, u_2 \in \widehat{\mathbb{C}}$, let*

$$\mathsf{F}(u_1, u_2) = \{ \ \mathsf{m} \in \mathsf{M} \mid \mathsf{m}(u) = u \ \Leftrightarrow \ u \in \{u_1, u_2\} \ \} \cup \{\mathsf{e}\}. \tag{2.7}$$

Then $\mathsf{F}(u_1, u_2)$ is a subgroup of M.

Proof Clearly $\mathsf{m}(u) = u \Leftrightarrow u = \mathsf{m}^{-1}(u)$, so from the definition of $\mathsf{F}(u_1, u_2)$, $\mathsf{m} \in \mathsf{F}(u_1, u_2) \Rightarrow \mathsf{m}^{-1} \in \mathsf{F}(u_1, u_2)$. It remains to demonstrate closure under composition, for which it is sufficient to consider the composition of non-identity Möbius transformations within $\mathsf{F}(u_1, u_2)$, say m and n, which therefore share their set of fixed-points $\{u_1, u_2\}$. Now, the set of fixed-points of $\mathsf{m} \cdot \mathsf{n}$ must contain u_1 and u_2, if it contains no other points then we are done. Suppose then that it also contains another point w, so that $\mathsf{m}(\mathsf{n}(w)) = w$ and hence $\mathsf{n}(w) = \mathsf{m}^{-1}(w)$. But this means n and m^{-1} agree on the point w as well as sharing their set of fixed-points $\{u_1, u_2\}$. By theorem 3 n and m^{-1} must therefore coincide, so $\mathsf{m} \cdot \mathsf{n} = \mathsf{e}$ and the theorem is proved.

□

These fixed-point subgroups constitute a 'group partition' of M, specifically

$$\{u_1, u_2\} \neq \{\widetilde{u}_1, \widetilde{u}_2\} \quad \Leftrightarrow \quad \mathsf{F}(u_1, u_2) \cap \mathsf{F}(\widetilde{u}_1, \widetilde{u}_2) = \{\mathsf{e}\}$$

and

$$\bigcup_{u_1,u_2\in\widehat{\mathbb{C}}} \mathsf{F}(u_1,u_2) = \mathsf{M}.$$

For convenience we will refer to the points $\{u_1, u_2\}$ as fixed-points of the *group* $\mathsf{F}(u_1, u_2)$. We may also write $\mathsf{F}(u_1)$ for $\mathsf{F}(u_1, u_1)$.

Finally we remark that given some $\mathsf{m} \in \mathsf{M}$

$$\mathsf{m}\mathsf{F}(u_1, u_2)\mathsf{m}^{-1} = \mathsf{F}(\mathsf{m}(u_1), \mathsf{m}(u_2)), \tag{2.8}$$

which is clear from the definition (2.7). Hence two subgroups $\mathsf{F}(u_1, u_2)$ and $\mathsf{F}(\widetilde{u}_1, \widetilde{u}_2)$ are conjugate provided we can find $\mathsf{m} \in \mathsf{M}$ for which $\{\mathsf{m}(u_1), \mathsf{m}(u_2)\} = \{\widetilde{u}_1, \widetilde{u}_2\}$. If the groups have the same number of fixed-points then theorem 1 assures us that such $\mathsf{m} \in \mathsf{M}$ can always be found. So up to conjugacy there are exactly two types of fixed-point subgroups, those with one fixed-point and those with two fixed-points. (Conjugation obviously preserves the number of fixed-points.)

2.3 Involutions

Suppose a pair of distinct points in $\widehat{\mathbb{C}}$ are transposed by some Möbius transformation m, then these are fixed-points of m^2. But m has at least one fixed-point and this is *also* a fixed-point of m^2, giving m^2 at least three distinct fixed-points. It follows that $\mathsf{m}^2 = \mathsf{e}$ by theorem 1. We will refer to Möbius transformations $\mathsf{m} \in \mathsf{M} \setminus \{\mathsf{e}\}$ for which $\mathsf{m}^2 = \mathsf{e}$ as *involutions*, thus we have established the following:

Theorem 5 *If any pair of distinct points in* $\widehat{\mathbb{C}}$ *are transposed by* $\mathsf{m} \in \mathsf{M}$*, then* m *is an involution.*

In this section we will state some basic results relating to Möbius involutions. Involutions will have an important, if technical role to play on several occasions. For notational emphasis, $\mathsf{i}, \mathsf{j} \in \mathsf{M}$ will always denote involutions. We begin with a reappraisal of theorems 1, 2 and 3.

Theorem 6 *Given two disjoint pairs of points* $\{u_1, \widetilde{u}_1\}$ *and* $\{u_2, \widetilde{u}_2\}$ *taken from* $\widehat{\mathbb{C}}$*, there exists a unique involution* $\mathsf{i} \in \mathsf{M}$ *for which* $\mathsf{i}(u_1) = \widetilde{u}_1$ *and* $\mathsf{i}(u_2) = \widetilde{u}_2$*. Moreover, given any point* $u_3 \in \widehat{\mathbb{C}}$ *the action of* i *on* u_3 *is the point* $\widetilde{u}_3$ *determined uniquely by the equation*

$$\begin{vmatrix} u_1\widetilde{u}_1 & u_1 + \widetilde{u}_1 & 1 \\ u_2\widetilde{u}_2 & u_2 + \widetilde{u}_2 & 1 \\ u_3\widetilde{u}_3 & u_3 + \widetilde{u}_3 & 1 \end{vmatrix} = 0. \tag{2.9}$$

Proof A direct calculation shows that choosing $u_4 = \widetilde{u}_3$ and $\widetilde{u}_4 = u_3$ reduces (2.4) to (2.9). The theorem follows from theorems 1 and 5.
□

Theorem 7 *Every involution in* M *has two distinct fixed-points in* $\widehat{\mathbb{C}}$*.*

Proof Setting $u_4 = \widetilde{u}_3$ and $\widetilde{u}_4 = u_3$ in (2.6) we find that

$$\frac{(\widetilde{u}_3 - u_2)(u_1 - u_3)}{(\widetilde{u}_3 - u_1)(u_2 - u_3)} = -1, \tag{2.10}$$

which according to theorem 3 is necessary for the existence of $\mathsf{i} \in \mathsf{M}$ for which $\mathsf{i}(u_3) = \widetilde{u}_3$, $\mathsf{i}(\widetilde{u}_3) = u_3$ and such that its only fixed-points are $\{u_1, u_2\}$. If $u_1 = u_2$ this condition is violated so no such i exists, i.e., there is no involution with a single fixed-point.
□

Theorem 8 *Given two distinct points* $u_1, u_2 \in \widehat{\mathbb{C}}$*, there exists a unique involution* $\mathsf{i} \in \mathsf{M}$ *for which these are fixed-points. Moreover, the action of* i *on an arbitrary point* u_3 *is the point* $\widetilde{u}_3$ *determined uniquely by* (2.10)*.*

Proof The theorem is a corollary of theorem 6, the equation (2.10) appears by setting $\widetilde{u}_1 = u_1$ and $\widetilde{u}_2 = u_2$ in (2.9).
□

Note that the equations (2.6) and (2.9) take the same form. This observation can be cast more formally as a theorem, to do so it turns out to be natural to first give the following more technical result.

Theorem 9 *Given* $\mathsf{m} \in \mathsf{M} \setminus \{\mathsf{e}\}$ *with (possibly coinciding) fixed-points* $u_1, u_2 \in \widehat{\mathbb{C}}$ *and an involution* $\mathsf{i} \in \mathsf{M}$*, then* $\mathsf{i} \cdot \mathsf{m}$ *is an involution if and only if* $\mathsf{i}(u_1) = u_2$*.*

Proof Suppose first that i and $\mathsf{i} \cdot \mathsf{m}$ are involutions, then $\mathsf{i} \cdot \mathsf{m} \cdot \mathsf{i} = \mathsf{m}^{-1}$. It follows that $\mathsf{m}(v) = v \Rightarrow [\mathsf{i} \cdot \mathsf{m} \cdot \mathsf{i}](v) = v \Rightarrow \mathsf{m}(\mathsf{i}(v)) = \mathsf{i}(v)$, so i permutes the fixed-points of m, i.e., i permutes the points $\{u_1, u_2\}$. There are two cases, either $\mathsf{i}(u_1) = u_2$ or $\mathsf{i} \in \mathsf{F}(u_1, u_2)$. We will show that the latter case cannot hold. Clearly if $\mathsf{i} \in \mathsf{F}(u_1, u_2)$ then $\mathsf{i} \cdot \mathsf{m} \in \mathsf{F}(u_1, u_2)$ as well. But according to theorem 8 there is only one involution in $\mathsf{F}(u_1, u_2)$, so we conclude that $\mathsf{i} \cdot \mathsf{m} = \mathsf{i}$. This implies $\mathsf{m} = \mathsf{e}$ contradicting our choice of m.

Now suppose that i is an involution, $\mathsf{m} \in \mathsf{M} \setminus \{\mathsf{e}\}$ has fixed-points $\{u_1, u_2\}$ and $\mathsf{i}(u_1) = u_2$. If $u_1 \neq u_2$ then $\{u_1, u_2\}$ are clearly transposed by $\mathsf{i} \cdot \mathsf{m}$, so $\mathsf{i} \cdot \mathsf{m}$ is an involution by theorem 5. It remains to consider the case $u_1 = u_2$, when $\mathsf{m} \in \mathsf{F}(u_1)$ and u_1 is a fixed-point of the involution i. Observe first that $\mathsf{i} \cdot \mathsf{m} \notin \mathsf{F}(u_1)$, for if it were then $\mathsf{i} \in \mathsf{F}(u_1)$ which, according to theorem 7, contradicts i being an involution. This means that $\mathsf{i} \cdot \mathsf{m} \in \mathsf{F}(u_1, v)$ for some $v \neq u_1$. On the other hand, $(\mathsf{i} \cdot \mathsf{m})^2 = [\mathsf{i} \cdot \mathsf{m} \cdot \mathsf{i}^{-1}] \cdot \mathsf{m}$, i.e., $(\mathsf{i} \cdot \mathsf{m})^2$ can be written as a product of Möbius transformations from $\mathsf{F}(u_1)$, and hence $(\mathsf{i} \cdot \mathsf{m})^2 \in \mathsf{F}(u_1)$. It follows that $(\mathsf{i} \cdot \mathsf{m})^2 \in \mathsf{F}(u_1) \cap \mathsf{F}(u_1, v) = \{\mathsf{e}\}$ so $\mathsf{i} \cdot \mathsf{m}$ must be an involution.
□

So we are now in a position to state an equivalence which formalises the observation that the equations (2.6) and (2.9) take the same form.

Theorem 10 *Given a (possibly coinciding) pair* $u_1, u_2 \in \widehat{\mathbb{C}}$ *let* $u, \widetilde{u}, v, \widetilde{v} \in \widehat{\mathbb{C}} \setminus \{u_1, u_2\}$ *be arbitrary. Then there exists* $\mathsf{m} \in \mathsf{F}(u_1, u_2)$ *such that* $\mathsf{m}(u) = \widetilde{u}$ *and* $\mathsf{m}(v) = \widetilde{v}$ *if and only if there exists an involution* $\mathsf{i} \in \mathsf{M}$ *for which* $\mathsf{i}(u_1) = u_2$*,* $\mathsf{i}(u) = \widetilde{v}$ *and* $\mathsf{i}(v) = \widetilde{u}$*.*

Proof Suppose first that m described exists. Then by theorem 6 we may construct an involution $\mathsf{i} \in \mathsf{M}$ for which $\mathsf{i}(u_1) = u_2$ and $\mathsf{i}(u) = \widetilde{v}$. According to theorem 9 $\mathsf{i} \cdot \mathsf{m} = \mathsf{m}^{-1} \cdot \mathsf{i}$, so $\mathsf{i}(\widetilde{u}) = \mathsf{i}(\mathsf{m}(u)) = \mathsf{m}^{-1}(\mathsf{i}(u)) = \mathsf{m}^{-1}(\widetilde{v}) = v$ and hence i is the described involution.

Suppose now that i described exists. Then by theorem 3 we may construct $\mathsf{m} \in \mathsf{F}(u_1, u_2)$ for which $\mathsf{m}(u) = \widetilde{u}$. According to theorem 9 $\mathsf{i} \cdot \mathsf{m} = \mathsf{m}^{-1} \cdot \mathsf{i}$, so $\mathsf{m}(v) = \mathsf{m}(\mathsf{i}(\widetilde{u})) = \mathsf{i}(\mathsf{m}^{-1}(\widetilde{u})) = \mathsf{i}(u) = \widetilde{v}$ and hence m is the described Möbius transformation.

□

One practical upshot of this observation is that if we know the action of some $\mathsf{m} \in \mathsf{M}$ on two points (which are not fixed-points of m), we may, by theorem 6, construct an involution which will transpose the fixed-points of m.

Actually, although the need for the more technical theorem 9 has been motivated here by theorem 10, it is the former result which will play a very important role in establishing the connection between M and the lattice equations in the subsequent chapters.

2.4 Commutativity

This is the final section of our introduction to the Möbius transformation. We give necessary and sufficient conditions for commutativity.

Theorem 11 *A pair of Möbius transformations commute if and only if they permute each others set of fixed-points. Moreover, this can happen only if one of the following statements hold:*

(i) One of them is the identity.

(ii) They have the same set of fixed-points.

(iii) Each has two distinct fixed-points that are transposed by the other.

Proof If $\mathsf{m}, \mathsf{n} \in \mathsf{M}$ commute and $\mathsf{m}(v) = v$ then $\mathsf{m}(\mathsf{n}(v)) = \mathsf{n}(v)$. That is, n permutes the fixed-points of m and we have demonstrated necessity.

For sufficiency we will first show that if a pair in M permute each others set of fixed-points, then one of (i), (ii) or (iii) hold. To complete the proof we then establish that any of (i), (ii) or (iii) are sufficient for a pair in M to commute.

Suppose that $\mathsf{m}, \mathsf{n} \in \mathsf{M}$ permute each others fixed-points, we will classify all cases. If, say, $\mathsf{m} = \mathsf{e}$, then m trivially permutes the fixed-points of any $\mathsf{n} \in \mathsf{M}$, moreover the set of fixed-points of e is $\widehat{\mathbb{C}}$ which is permuted by any $\mathsf{n} \in \mathsf{M}$. This is case (i). It remains to consider $\mathsf{m}, \mathsf{n} \in \mathsf{M} \setminus \{\mathsf{e}\}$, we argue on the number of fixed-points of m which, according to theorem 2, is either one or two.

If m has only one fixed-point then its permutation by n is the identity, this implies it is a fixed-point of n as well. If n has a second fixed-point not shared by m, then m must transpose it with the first, but the first was a fixed-point of m so n does not have a second fixed-point. Thus m and n have one shared fixed-point as in case (ii).

If m has two fixed-points then n must either share them as fixed-points as in case (ii), or transpose them. If the fixed-points of m are transposed by n, then they are not fixed-points of n, so the sets of fixed-points of n and m are disjoint. Moreover if n had just one fixed-point it would be shared by m, so n has two fixed-points that are transposed by m. This is exactly case (iii).

To complete the proof we show that (i), (ii) or (iii) are sufficient for commutativity. Case (i) is obvious. Suppose case (ii) holds so that $\mathsf{m}, \mathsf{n} \in \mathsf{F}(u_1, u_2)$ for some $u_1, u_2 \in \widehat{\mathbb{C}}$. Then by theorem 8 we may construct an involution $\mathsf{i} \in \mathsf{M}$ for which $\mathsf{i}(u_1) = u_2$, and by theorem 9 $\mathsf{i} \cdot \mathsf{m}$, $\mathsf{i} \cdot \mathsf{n}$ and $\mathsf{i} \cdot \mathsf{m} \cdot \mathsf{n}$ are also involutions. We then have that $\mathsf{i} \cdot \mathsf{m} \cdot \mathsf{n} = \mathsf{n}^{-1} \cdot \mathsf{m}^{-1} \cdot \mathsf{i} = \mathsf{n}^{-1} \cdot \mathsf{i} \cdot \mathsf{m} = \mathsf{i} \cdot \mathsf{n} \cdot \mathsf{m}$ and hence $\mathsf{m} \cdot \mathsf{n} = \mathsf{n} \cdot \mathsf{m}$. Finally suppose case (iii) holds so $\mathsf{m}, \mathsf{n} \in \mathsf{M}$ transpose each others fixed-points, then $\mathsf{m} \cdot \mathsf{n}$ and $\mathsf{n} \cdot \mathsf{m}$ clearly agree on the union of their sets of fixed-points. This means they agree on a set of four distinct points and must therefore coincide by theorem 1.
□

This theorem reveals, by case (ii), the important result that the fixed-point subgroup $\mathsf{F}(u_1, u_2)$ is abelian.

In order to answer the natural question about the existence of pairs satisfying case (iii) of theorem 11 we prove the following specialisation.

Theorem 12 *For a pair of distinct involutions in* M *to commute it is necessary and sufficient that one transpose the fixed-points of the other.*

Proof Suppose $\mathsf{i}, \mathsf{j} \in \mathsf{M}$ are involutions and $\mathsf{i} \cdot \mathsf{j} \neq \mathsf{e}$ (i.e., they are distinct). Then $\mathsf{i} \cdot \mathsf{j} = \mathsf{j} \cdot \mathsf{i} \Leftrightarrow (\mathsf{i} \cdot \mathsf{j})^2 = \mathsf{e}$, so i and j commute if and only if $\mathsf{i} \cdot \mathsf{j}$ is an involution. The theorem therefore follows from the particular instance of theorem 9 when $\mathsf{m} = \mathsf{j}$, an involution.
□

The immediate corollary to theorem 12 (given theorem 11) is that if one involution were to transpose the fixed-points of another, then the other would reciprocate and transpose the fixed-points of the first.

Perhaps the main point we want to make at the end of this section is that if two Möbius transformations commute we cannot straightaway conclude that they lie in the same fixed-point subgroup $\mathsf{F}(u_1, u_2)$. However it will be seen later that sharing fixed-points, which is slightly stronger than commutativity, is perhaps a more useful notion to us than the comutativity itself.

Chapter 3

Fixed-point subgroups and lattice systems

It is a straightforward consequence of theorem 3 that the mapping from the fixed-point subgroup $\mathsf{F}(u_1, u_2)$ to its action on a single point, say $u \in \widehat{\mathbb{C}} \setminus \{u_1, u_2\}$,

$$\begin{array}{ccc} \mathsf{F}(u_1, u_2) & \longrightarrow & \widehat{\mathbb{C}} \setminus \{u_1, u_2\}, \\ \mathsf{m} & \mapsto & \mathsf{m}(u), \end{array} \tag{3.1}$$

is a bijection. (3.1 is the *evaluation* map.) The composition of Möbius transformations within $\mathsf{F}(u_1, u_2)$ therefore induces a product on $\widehat{\mathbb{C}} \setminus \{u_1, u_2\}$. In fact given $\mathsf{m}, \mathsf{n} \in \mathsf{F}(u_1, u_2)$ let us suppose that $\mathsf{m}(u) = \widetilde{u}$, $\mathsf{n}(u) = \widehat{u}$ and $[\mathsf{m} \cdot \mathsf{n}](u) = \widehat{\widetilde{u}}$, then we may write an explicit equation for this product,

$$\begin{vmatrix} u_1 u_2 & u_1 + u_2 & 1 \\ u\widehat{\widetilde{u}} & u + \widehat{\widetilde{u}} & 1 \\ \widetilde{u}\widehat{u} & \widetilde{u} + \widehat{u} & 1 \end{vmatrix} = 0. \tag{3.2}$$

Here $\widehat{\widetilde{u}}$ is the product of $\widetilde{u}$ and $\widehat{u}$ in $\widehat{\mathbb{C}} \setminus \{u_1, u_2\}$, the point $u \in \widehat{\mathbb{C}} \setminus \{u_1, u_2\}$ ends up being the identity in this group. The relation (3.2) appears by using theorem 3, i.e., equation (2.4), to find $\widehat{\widetilde{u}} = \mathsf{m}(\widehat{u}) = \mathsf{n}(\widetilde{u})$.

In this chapter we will exploit the affiliation between the fixed-point subgroup $\mathsf{F}(u_1, u_2)$ and the relation (3.2) to reveal important properties of

(3.2) when it is considered on its own merit as a lattice equation. These properties yield a solution method for (3.2) and reveal its consistency on the cube.

3.1 A solvable lattice equation

We begin by supposing that $u_1, u_2 \in \widehat{\mathbb{C}}$ are fixed and introduce the quadrilateral expression $\mathcal{Q}$

$$\mathcal{Q}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = \begin{vmatrix} u_1u_2 & u_1+u_2 & 1 \\ u\widehat{\widetilde{u}} & u+\widehat{\widetilde{u}} & 1 \\ \widetilde{u}\widehat{u} & \widetilde{u}+\widehat{u} & 1 \end{vmatrix}. \tag{3.3}$$

We wish to consider the lattice equation defined by $\mathcal{Q}$ which we write as

$$\mathcal{Q}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = 0 \tag{3.4}$$

where $u = u(n,m)$, $\widetilde{u} = u(n+1,m)$, $\widehat{u} = u(n,m+1)$ and $\widehat{\widetilde{u}} = u(n+1,m+1)$ are values of the dependent variable u as a function of the independent variables $n, m \in \mathbb{Z}$.

At the end of section 2.2 we made the observation that the generic group $\mathsf{F}(u_1,u_2)$ is conjugate to, for example, one of $\mathsf{F}(\infty,0)$ or $\mathsf{F}(\infty)$. This means that, up to a Möbius change of variables, the equation (3.4) is equivalent to one of the two cases $u_1 = \infty, u_2 = 0$ or $u_1 = u_2 = \infty$, when it reduces to

$$u\widehat{\widetilde{u}} = \widetilde{u}\widehat{u} \qquad \text{and} \qquad u+\widehat{\widetilde{u}} = \widetilde{u}+\widehat{u} \tag{3.5}$$

respectively. In other words the equation is basically linear.

However the more remarkable property is that given u satisfying (3.4) and not equal to u_1 or u_2 on any lattice site, it can be verified that

$$\mathcal{Q}(u(0,0),u(n,0),u(0,m),u(n,m)) = 0 \tag{3.6}$$

for all $n, m \in \mathbb{Z}$. Let us give the key observation which makes this property apparent. If $u, \widetilde{u}, v, \widetilde{v} \in \widehat{\mathbb{C}} \setminus \{u_1, u_2\}$ then

$$\mathcal{Q}(u,\widetilde{u},v,\widetilde{v}) = 0 \quad \Leftrightarrow \quad \exists\ \mathsf{m} \in \mathsf{F}(u_1,u_2) \text{ s.t. } \mathsf{m}(u) = \widetilde{u}, \mathsf{m}(v) = \widetilde{v}, \tag{3.7}$$

which is immediate from theorem 3. It is then clear that given $u, \widetilde{u}, \widetilde{\widetilde{u}}, v, \widetilde{v}, \widetilde{\widetilde{v}} \in \widehat{\mathbb{C}} \setminus \{u_1, u_2\}$,

$$\begin{array}{c} \mathcal{Q}(u, \widetilde{u}, v, \widetilde{v}) = 0 \\ \mathcal{Q}(\widetilde{u}, \widetilde{\widetilde{u}}, \widetilde{v}, \widetilde{\widetilde{v}}) = 0 \end{array} \quad \Rightarrow \quad \mathcal{Q}(u, \widetilde{\widetilde{u}}, v, \widetilde{\widetilde{v}}) = 0. \tag{3.8}$$

Now the property (3.6) follows by induction from (3.8). The property (3.6) yields the solution of (3.4) directly from initial data $u(n, 0), u(0, n), n \in \mathbb{Z}$.

When u takes the value u_1 or u_2 on a lattice site we say u is singular on that site, we now consider such an occurrence in a solution of (3.4), which we have so far assumed does not happen. A direct calculation shows that

$$\begin{aligned} \mathcal{Q}(u_1, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) &= (\widetilde{u} - u_1)(\widehat{u} - u_1)(\widehat{\widetilde{u}} - u_2), \\ \mathcal{Q}(u, u_1, \widehat{u}, \widehat{\widetilde{u}}) &= -(u - u_1)(\widehat{u} - u_2)(\widehat{\widetilde{u}} - u_1), \\ \mathcal{Q}(u, \widetilde{u}, u_1, \widehat{\widetilde{u}}) &= -(u - u_1)(\widetilde{u} - u_2)(\widehat{\widetilde{u}} - u_1), \\ \mathcal{Q}(u, \widetilde{u}, \widehat{u}, u_1) &= (u - u_2)(\widetilde{u} - u_1)(\widehat{u} - u_1). \end{aligned} \tag{3.9}$$

This reveals that if (3.4) holds on a quadrilateral and u is singular on one vertex, then u is singular also on at least one other vertex on the same quadrilateral. By induction we see that passing through any singular point on the lattice there must be a line of singularities. The possible path of such a *shock line* is governed by the relations (3.9) (and their counterparts with $u_1 \leftrightarrow u_2$), these paths for an example solution are illustrated in figure 3.1. The shock lines divide $\mathbb{Z}^2$ into regions, and it is clear that within each region a similar relation to (3.6) will hold (the point of origin now being somewhere also within the same region). We also remark that it would be quite natural to *impose* (3.6) to resolve non-uniqueness in a solution which may occur due to the presence of a shock line.

3.2 Embedding in three dimensions part I

We now discuss how the lattice equation defined by $\mathcal{Q}$ in (3.3) can be embedded in three dimensions. Again, this property is straightforward to see from the affiliation between this equation and the group $\mathsf{F}(u_1, u_2)$.

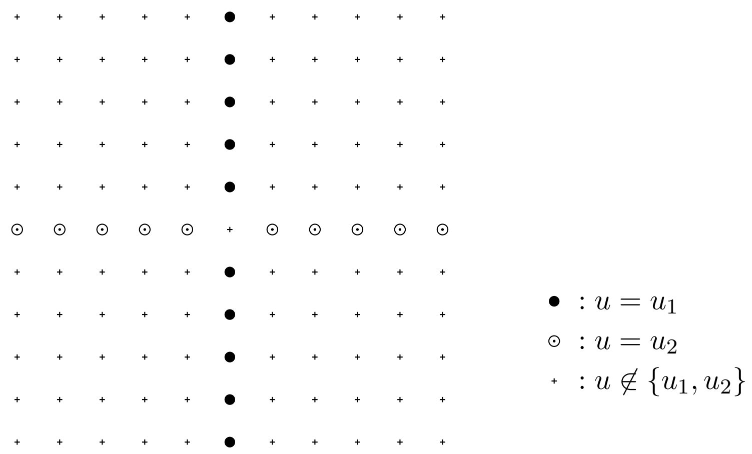

Figure 3.1: Part of a lattice on which an example solution of (3.4) contains shock lines. There can be no elementary quadrilaterals with $u \in \{u_1, u_2\}$ on a single vertex. When a shock line passes through an elementary quadrilateral the other two vertices are unconstrained by the equation on it.

Consider initial data $u, \widetilde{u}, \widehat{u}, \overline{u} \in \widehat{\mathbb{C}} \setminus \{u_1, u_2\}$. This fixes three Möbius transformations $\mathsf{l}, \mathsf{m}, \mathsf{n} \in \mathsf{F}(u_1, u_2)$ uniquely through the relations

$$\mathsf{l}(u) = \widetilde{u}, \quad \mathsf{m}(u) = \widehat{u}, \quad \mathsf{n}(u) = \overline{u}. \tag{3.10}$$

We may then define the points $\widehat{\widetilde{u}}, \widetilde{\overline{u}}, \overline{\widehat{u}}$ and $\widehat{\widetilde{\overline{u}}}$ by the relations

$$\widehat{\widetilde{u}} = \mathsf{l}(\mathsf{m}(u)), \quad \widetilde{\overline{u}} = \mathsf{m}(\mathsf{n}(u)), \quad \overline{\widehat{u}} = \mathsf{n}(\mathsf{l}(u)), \quad \widehat{\widetilde{\overline{u}}} = \mathsf{l}(\mathsf{m}(\mathsf{n}(u))). \tag{3.11}$$

Now, given (3.10) and (3.11), we observe using (3.7) that all of the following equations hold

$$\begin{aligned} &\mathcal{Q}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) = 0, \qquad && \mathcal{Q}(\overline{u}, \widetilde{\overline{u}}, \overline{\widehat{u}}, \widehat{\widetilde{\overline{u}}}) = 0, \\ &\mathcal{Q}(u, \widehat{u}, \overline{u}, \overline{\widehat{u}}) = 0, && \mathcal{Q}(\widetilde{u}, \widehat{\widetilde{u}}, \widetilde{\overline{u}}, \widehat{\widetilde{\overline{u}}}) = 0, \\ &\mathcal{Q}(u, \overline{u}, \widetilde{u}, \widetilde{\overline{u}}) = 0, && \mathcal{Q}(\widehat{u}, \overline{\widehat{u}}, \widehat{\widetilde{u}}, \widehat{\widetilde{\overline{u}}}) = 0. \end{aligned} \tag{3.12}$$

On the face of it, this system of equations overdetermines $\widehat{\widetilde{\overline{u}}}$ from the initial data. The existence of the solution $\widehat{\widetilde{\overline{u}}} = \mathsf{l}(\mathsf{m}(\mathsf{n}(u)))$ means the quadrilateral expression $\mathcal{Q}$ is consistent on a cube.

As a result of this property, a natural IVP can be posed for the three dimensional lattice system defined by the equations

$$\mathcal{Q}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}})=0,\quad \mathcal{Q}(u,\widehat{u},\overline{u},\overline{\widehat{u}})=0,\quad \mathcal{Q}(u,\overline{u},\widetilde{u},\widetilde{\overline{u}})=0 \tag{3.13}$$

where $u=u(n,m,l)$, $\widetilde{u}=u(n{+}1,m,l)$, $\widehat{u}=u(n,m{+}1,l)$, $\overline{u}=u(n,m,l{+}1)$ etc. are values of u as a function of three independent variables $n,m,l\in\mathbb{Z}$. That is, given $u(n,0,0),u(0,n,0),u(0,0,n),\ n\in\mathbb{Z}$, a solution of (3.13) can then be determined throughout the lattice. From (3.9) we see that shocks form planes (rather than lines) in this three dimensional lattice system, again dividing the lattice into regions. To extend the property (3.6) to three dimensions we first introduce the *normal-form* for Möbius transformations.

3.3 The normal-form

If we consider the action of the particular groups $\mathsf{F}(\infty,0)$ and $\mathsf{F}(\infty)$ on the points 1 and 0 respectively, (3.2) reduces as follows (cf. the discussion at the end of section 2.1):

$$\left.\begin{array}{l}u_1=\infty\\ u_2=0\\ u=1\end{array}\right\}\quad \widehat{\widetilde{u}}=\widetilde{u}\widehat{u},\qquad \left.\begin{array}{l}u_1=\infty\\ u_2=\infty\\ u=0\end{array}\right\}\quad \widehat{\widetilde{u}}=\widetilde{u}+\widehat{u}. \tag{3.14}$$

In other words, evaluation at the point 1 is a natural isomorphism between $\mathsf{F}(\infty,0)$ and $\mathbb{C}\setminus\{0\}$ under multiplication, and evaluation at 0 is a natural isomorphism between $\mathsf{F}(\infty)$ and $\mathbb{C}$ under addition. (by the end of this section we will establish the fairly obvious fact that $\mathsf{F}(\infty,0)=\{u\mapsto pu, p\in\mathbb{C}\setminus\{0\}\}$ and $\mathsf{F}(\infty)=\{u\mapsto u+p, p\in\mathbb{C}\}$.)

In this section we introduce the *normal-form* for the Möbius transformations. This is a parametrisation of the fixed-point subgroup $\mathsf{F}(u_1,u_2)$ that exploits the fact that this group is conjugate to either $\mathsf{F}(\infty,0)$ or $\mathsf{F}(\infty)$ (cf. section 2.2) together with (3.14) in order to bring composition within the group to multiplication (in the case $u_2\neq u_1$) or addition (when $u_2=u_1$) of the parameter.

In the construction we need to conjugate $\mathsf{F}(u_1, u_2)$ by some element to arrive at either $\mathsf{F}(\infty, 0)$ or $\mathsf{F}(\infty)$, a weakness is that the element by which we conjugate is not unique. This is partially resolved in the following theorem and will be completly resolved in chapter 4.

Theorem 13 *Let u_1 and u_2 be distinct points in $\widehat{\mathbb{C}}$, let $\mathsf{a}, \mathsf{b} \in \mathsf{M}$ be such that*

$$\mathsf{a}(u_1) = \infty, \quad \mathsf{a}(u_2) = 0, \qquad \mathsf{b}(u_1) = \infty, \tag{3.15}$$

and define the mappings α and β by

$$\alpha : \begin{array}{l} \mathsf{F}(u_1, u_2) \longrightarrow \mathbb{C} \setminus \{0\} \\ \mathsf{m} \mapsto \mathsf{a}(\mathsf{m}(\mathsf{a}^{-1}(1))) \end{array}, \qquad \beta : \begin{array}{l} \mathsf{F}(u_1) \longrightarrow \mathbb{C} \\ \mathsf{m} \mapsto \mathsf{b}(\mathsf{m}(\mathsf{b}^{-1}(0))) \end{array}. \tag{3.16}$$

Then α is an isomorphism between $\mathsf{F}(u_1, u_2)$ and $\mathbb{C} \setminus \{0\}$ under multiplication and any other choice of a satisfying (3.15) yields the same isomorphism α. The mapping β is an isomorphism between $\mathsf{F}(u_1)$ and $\mathbb{C}$ under addition. Any other choice of b satisfying (3.15), say b_*, yields a mapping β_* related to β by

$$\beta_*(\mathsf{m}) = \beta(\mathsf{n} \cdot \mathsf{m} \cdot \mathsf{n}^{-1}) \tag{3.17}$$

for some n with fixed-point u_1.

Proof Conjugation by a takes $\mathsf{F}(u_1, u_2)$ to $\mathsf{F}(\infty, 0)$ and α is nothing but the composition of this with evaluation at 1 which, given (3.14), reveals that α is the described isomorphism. Any different choice of a satisfying (3.15), say a_*, yields another isomorphism α_*. Clearly $\alpha_*(\mathsf{m}) = \alpha(\mathsf{n} \cdot \mathsf{m} \cdot \mathsf{n}^{-1})$ where $\mathsf{n} = \mathsf{a}^{-1} \cdot \mathsf{a}_*$. We observe directly that u_1 and u_2 are fixed-points of n, so by theorem 8 $\mathsf{n} \in \mathsf{F}(u_1, u_2)$. Now for any $\mathsf{m} \in \mathsf{F}(u_1, u_2)$, $\alpha_*(\mathsf{m}) = \alpha(\mathsf{m})$ as claimed because $\mathsf{F}(u_1, u_2)$ is abelian.

Conjugation by b takes $\mathsf{F}(u_1)$ to $\mathsf{F}(\infty)$ and β is just the composition of this with evaluation at 0, so again (3.14) reveals that β is the described isomorphism. A different choice of b, say b_*, yields β_* related to β by $\beta_*(\mathsf{m}) = \beta(\mathsf{n} \cdot \mathsf{m} \cdot \mathsf{n}^{-1})$ where $\mathsf{n} = \mathsf{b}^{-1} \cdot \mathsf{b}_*$, which clearly has fixed-point u_1 as claimed.

□

We will refer to α and β as the normal-form isomorphism for the groups $\mathsf{F}(u_1, u_2)$ and $\mathsf{F}(u_1)$ respectively. They can be found in concrete form by a direct calculation. If we introduce the expressions ϕ and χ,

$$\phi(w,x,y,z) = \frac{(w-x)(y-z)}{(w-y)(x-z)}, \quad \chi(w,x,y) = \frac{1}{w-x} - \frac{1}{y-x} \tag{3.18}$$

then given $\mathsf{m} \in \mathsf{F}(u_1, u_2)$ and $\mathsf{n} \in \mathsf{F}(u_1)$ we find

$$\begin{aligned} &\alpha(\mathsf{m}) = \phi(\mathsf{m}(u), u_1, u_2, u), \quad (u_1 \neq u_2), \\ &\beta(\mathsf{n}) = b_0 \chi(\mathsf{n}(u), u_1, u), \quad (u_1 \neq \infty), \\ &\beta(\mathsf{n}) = b_0(\mathsf{n}(u) - u), \quad (u_1 = \infty), \end{aligned} \tag{3.19}$$

where $b_0 \neq 0$ is arbitrary. The presence of the constant b_0 reflects the non-uniqueness of the normal-form isomorphism for $\mathsf{F}(u_1)$, a precise statement connecting this with (3.17) will be made when we consider a natural extension of the normal-form in the next chapter.

Now, fixing $u \in \widehat{\mathbb{C}} \setminus \{u_1, u_2\}$ we can compute $\alpha(\mathsf{m})$ by evaluating m at the point u. Similarly, fixing $u \in \widehat{\mathbb{C}} \setminus \{u_1\}$ we can compute $\beta(\mathsf{n})$ by evaluating n at the point u. Conversely, fixing $\alpha(\mathsf{m})$ and $\beta(\mathsf{n})$ the above relations can be rearranged to give $\mathsf{m}(u)$ and $\mathsf{n}(u)$ in terms of u and these fixed parameters. This is what we will mean by the normal-form, specifically

$$\begin{aligned} \mathsf{F}(u_1, u_2) &= \left\{ u \mapsto \frac{(pu_2 - u_1)u + u_1 u_2(1-p)}{u_2 - pu_1 - (1-p)u} \;\Big|\; p \in \mathbb{C} \setminus \{0\} \right\}, \\ \mathsf{F}(u_1) &= \left\{ u \mapsto \frac{(1 + pu_1)u - pu_1^2}{1 - pu_1 + pu} \;\Big|\; p \in \mathbb{C} \right\}, \qquad u_1 \neq \infty, \\ \mathsf{F}(\infty) &= \{ \, u \mapsto u + p \mid p \in \mathbb{C} \, \}. \end{aligned} \tag{3.20}$$

We remark that this parameterisation is equivalent to eigenvalue decomposition in the associated linear system, also one can descend through the list (3.20) by *degeneration*, for example substitution of $u_2 \to u_1 + \epsilon$ and $p \to 1 + \epsilon p$ in the first expression yields the second.

3.4 Embedding in three dimensions part II

Let us now return to consider the three dimensional lattice system introduced in section 3.2. We will give an expression for $\widehat{\widetilde{\overline{u}}}$ in terms of the initial data $\{u, \widetilde{u}, \widehat{u}, \overline{u}\}$ (i.e., eliminate $\{\widehat{\widetilde{u}}, \overline{\widehat{u}}, \widetilde{\overline{u}}\}$ from (3.12)) by exploiting the normal-form. We consider two cases, in the first instance suppose $u_1 \neq u_2$ so using (3.19) with (3.10) and (3.11) we see that $\alpha(\mathsf{l}) = \phi(\widetilde{u}, u_1, u_2, u)$, $\alpha(\mathsf{l}\cdot\mathsf{m}) = \phi(\widehat{\widetilde{u}}, u_1, u_2, u)$ etc. Now the equation $\alpha(\mathsf{l}\cdot\mathsf{m}) = \alpha(\mathsf{l})\alpha(\mathsf{m})$ leads directly to

$$\phi(\widehat{\widetilde{u}}, u_1, u_2, u) = \phi(\widetilde{u}, u_1, u_2, u)\phi(\widehat{u}, u_1, u_2, u) \tag{3.21}$$

which it can be verified is equivalent to (3.4) in the case $u_1 \neq u_2$, so this is an alternative form for (3.4) in that case. The equation $\alpha(\mathsf{l}\cdot\mathsf{m}\cdot\mathsf{n}) = \alpha(\mathsf{l})\alpha(\mathsf{m})\alpha(\mathsf{n})$ yields the desired expression for $\widehat{\widetilde{\overline{u}}}$ in terms of the initial data,

$$\phi(\widehat{\widetilde{\overline{u}}}, u_1, u_2, u) = \phi(\widetilde{u}, u_1, u_2, u)\phi(\widehat{u}, u_1, u_2, u)\phi(\overline{u}, u_1, u_2, u). \tag{3.22}$$

It is this relation which extends to yield the solution of the natural IVP on the three dimensional lattice,

$$\begin{aligned}\phi(u(n,m,l), u_1, u_2, u(0,0,0)) = {} & \phi(u(n,0,0), u_1, u_2, u(0,0,0))\times \\ & \phi(u(0,m,0), u_1, u_2, u(0,0,0))\times \\ & \phi(u(0,0,l), u_1, u_2, u(0,0,0)).\end{aligned} \tag{3.23}$$

In the other case, when $u_2 = u_1$ in (3.3), we use the isomorphism β in a similar way to α above and conclude that

$$\chi(\widehat{\widetilde{\overline{u}}}, u_1, u) = \chi(\widetilde{u}, u_1, u) + \chi(\widehat{u}, u_1, u) + \chi(\overline{u}, u_1, u). \tag{3.24}$$

And the extension of this relation is just

$$\begin{aligned}\chi(u(n,m,l), u_1, u(0,0,0)) = {} & \chi(u(n,0,0), u_1, u(0,0,0))+ \\ & \chi(u(0,m,0), u_1, u(0,0,0))+ \\ & \chi(u(0,0,l), u_1, u(0,0,0)).\end{aligned} \tag{3.25}$$

3.5 Möbius transformations on the cube

The following equivalence provides a converse to the three dimensional lattice construction discussed in sections 3.2 and 3.4.

Theorem 14 *Given distinct points* $u, \widetilde{u}, \widehat{u}, \overline{u}, \overline{\widehat{u}}, \widehat{\widetilde{u}}, \widetilde{\overline{u}} \in \widehat{\mathbb{C}}$ *let* $\mathsf{l}, \mathsf{m}, \mathsf{n} \in \mathsf{M}$ *be the unique (by theorem 1) Möbius transformations for which*

$$\begin{aligned} &\mathsf{l}(u) = \widetilde{u}, \qquad &&\mathsf{m}(u) = \widehat{u}, \qquad &&\mathsf{n}(u) = \overline{u}, \\ &\mathsf{l}(\widehat{u}) = \widehat{\widetilde{u}}, &&\mathsf{m}(\overline{u}) = \overline{\widehat{u}}, &&\mathsf{n}(\widetilde{u}) = \widetilde{\overline{u}}, \\ &\mathsf{l}(\overline{u}) = \widetilde{\overline{u}}, &&\mathsf{m}(\widetilde{u}) = \widehat{\widetilde{u}}, &&\mathsf{n}(\widehat{u}) = \overline{\widehat{u}}. \end{aligned} \tag{3.26}$$

Then for $\mathsf{l}, \mathsf{m}, \mathsf{n}$ *to share their fixed-points it is necessary and sufficient that there exists an involution* $\mathsf{i} \in \mathsf{M}$ *for which*

$$\mathsf{i}(\widetilde{u}) = \overline{\widehat{u}}, \quad \mathsf{i}(\widehat{u}) = \widetilde{\overline{u}}, \quad \mathsf{i}(\overline{u}) = \widehat{\widetilde{u}}. \tag{3.27}$$

Moreover, if such i *exists and we define* $\widehat{\widetilde{\overline{u}}} = \mathsf{l}(\mathsf{m}(\mathsf{n}(u)))$*, then* $\mathsf{i}(u) = \widehat{\widetilde{\overline{u}}}$*.*

Proof Suppose first that $\mathsf{l}, \mathsf{m}, \mathsf{n}$ satisfying (3.26) have shared fixed-points $\{u_1, u_2\}$. By theorem 6 we may construct an involution i for which $\mathsf{i}(u_1) = u_2$ and $\mathsf{i}(\widetilde{u}) = \overline{\widehat{u}}$. Now $\mathsf{m} \cdot \mathsf{l}^{-1}$ also has fixed-points $\{u_1, u_2\}$, so by theorem 9 $\mathsf{i} \cdot \mathsf{m} \cdot \mathsf{l}^{-1}$ is an involution, which means $\mathsf{i} \cdot \mathsf{m} \cdot \mathsf{l}^{-1} = \mathsf{l} \cdot \mathsf{m}^{-1} \cdot \mathsf{i}$ and it follows that

$$\begin{aligned} \mathsf{i}(\widehat{u}) &= \mathsf{i}(\mathsf{m}(\mathsf{l}^{-1}(\widetilde{u}))), \\ &= \mathsf{l}(\mathsf{m}^{-1}(\mathsf{i}(\widetilde{u}))), \\ &= \mathsf{l}(\mathsf{m}^{-1}(\overline{\widehat{u}})), \\ &= \widetilde{\overline{u}}. \end{aligned} \tag{3.28}$$

Clearly $\mathsf{i}(\overline{u}) = \widehat{\widetilde{u}}$ can be verified in a similar way, so the involution i we have constructed satisfies (3.27) and we have demonstrated necessity.

Suppose now that the involution i satisfying (3.27) exists and that $\mathsf{l}, \mathsf{m}, \mathsf{n}$ are defined by (3.26). Then clearly $[\mathsf{i} \cdot \mathsf{l}](\overline{u}) = \mathsf{i}(\widetilde{\overline{u}}) = \widehat{u}$ and $[\mathsf{i} \cdot \mathsf{l}](\widehat{u}) = \mathsf{i}(\widehat{\widetilde{u}}) = \overline{u}$, so the points $\widehat{u}$ and $\overline{u}$ are transposed by $\mathsf{i} \cdot \mathsf{l}$ and it follows that $\mathsf{i} \cdot \mathsf{l}$ is an involution by theorem 5. Similarly $\mathsf{i} \cdot \mathsf{m}$ and $\mathsf{i} \cdot \mathsf{n}$ are also involutions. Using this we see that $[\mathsf{i}\cdot\mathsf{l}\cdot\mathsf{m}](\overline{u}) = [\mathsf{l}^{-1}\cdot\mathsf{i}](\overline{\widehat{u}}) = \mathsf{l}^{-1}(\widetilde{u}) = u$ and given $[\mathsf{i}\cdot\mathsf{l}\cdot\mathsf{m}](u) = \mathsf{i}(\widehat{\widetilde{u}}) = \overline{u}$ we have demonstrated u and $\overline{u}$ are transposed by ilm, so by theorem 5 $\mathsf{i}\cdot\mathsf{l}\cdot\mathsf{m}$ is also an involution. So $\mathsf{i}\cdot\mathsf{l}$, $\mathsf{i}\cdot\mathsf{m}$ and $\mathsf{i}\cdot\mathsf{l}\cdot\mathsf{m}$ are all involutions, from which we deduce l and m commute: $\mathsf{i}\cdot\mathsf{l}\cdot\mathsf{m} = \mathsf{m}^{-1}\cdot\mathsf{l}^{-1}\cdot\mathsf{i} = \mathsf{m}^{-1}\cdot\mathsf{i}\cdot\mathsf{l} = \mathsf{i}\cdot\mathsf{m}\cdot\mathsf{l}$. Similarly l and m both commute with n. Using theorem 11 and the fact that $\mathsf{l}, \mathsf{m}, \mathsf{n}$ are distinct which is clear from (3.26), we conclude that either $\mathsf{l}, \mathsf{m}, \mathsf{n}$ share their fixed-points, or they are three distinct commuting involutions. To conclude the proof we will show that the latter case cannot hold. By theorem 9 we know that i will transpose the fixed-points of m and l. But if $\mathsf{l}, \mathsf{m}, \mathsf{n}$ are distinct commuting involutions, then l must also transpose the fixed-points of m and n. But then l and i agree on four distinct points, so by theorem 1 $\mathsf{l} = \mathsf{i}$, which given (3.26) contradicts (3.27).

Finally, supposing again that i exists, we have already established that $\mathsf{i}\cdot\mathsf{l}$ is an involution. It follows that $\mathsf{i}(u) = \mathsf{i}(\mathsf{l}^{-1}(\widetilde{u})) = \mathsf{l}(\mathsf{i}(\widetilde{u})) = \mathsf{l}(\overline{\widehat{u}}) = \mathsf{l}(\mathsf{m}(\mathsf{n}(u)))$ which proves the final statement of the theorem.
□

3.6 Conclusion

We have investigated a quadrilateral lattice equation which bears an intimate conection to the abelian group of Möbius transformations which share their fixed-points. Although this equation is basically linear, its essential solvability property can be seen to arise from this intimate connection, moreover it shows that the equation is consistent on the cube (cf. section 1.2).

Finally, it is straightforward to deduce from the final theorem (theorem 14) that a necessary and sufficient condition for this equation (the equation (3.2) for some $u_1, u_2 \in \widehat{\mathbb{C}}$) to be satisfied on each face of a cube (as in (3.12), see figure 1.1) is the existance of a Möbius involution i for which

$$\mathsf{i}(\widetilde{u}) = \overline{\widehat{u}}, \quad \mathsf{i}(\widehat{u}) = \overline{\widetilde{u}}, \quad \mathsf{i}(\overline{u}) = \widehat{\widetilde{u}}, \quad \mathsf{i}(u) = \overline{\widehat{\widetilde{u}}}. \tag{3.29}$$

Chapter 4

Stabilizer subgroups and lattice systems

Here we will introduce the stabilizer subgroups of M, which are larger than the fixed-point subgroups studied in chapter 3. We will begin by collecting some basic results to enable an extension of the normal-form to these larger groups. This extended normal-form will be the basic tool we use to construct a lattice system related to composition in these groups. In fact this turns out to be the lattice Schwarzian KdV equation described in section 1.2. The embedding in three dimensions will provide an unexpected connection between this lattice system and the 'trivial' lattice system described in chapter 3.

4.1 Stabilizer subgroups

Let us denote by $\mathsf{S}(u_1 \dots u_k)$ the group of Möbius transformations that permute the points $u_1 \dots u_k \in \widehat{\mathbb{C}}$,

$$\mathsf{S}(u_1 \dots u_k) = \{ \ \mathsf{m} \in \mathsf{M} \mid v \in \{u_1 \dots u_k\} \ \Rightarrow \ \mathsf{m}(v) \in \{u_1 \dots u_k\} \ \}, \quad (4.1)$$

we refer to this group as the *stabilizer* of $\{u_1 \dots u_k\}$ in M. It is clear from the definitions (2.7) and (4.1) that given any $\mathsf{m} \in \mathsf{M}$

$$\mathsf{m}\mathsf{F}(u_1, u_2)\mathsf{m}^{-1} = \mathsf{F}(u_1, u_2) \quad \Leftrightarrow \quad \mathsf{m} \in \mathsf{S}(u_1, u_2), \tag{4.2}$$

so in particular $\mathsf{F}(u_1, u_2)$ is a normal subgroup of $\mathsf{S}(u_1, u_2)$. In fact more is true as we see in the following basic theorem which clarifies the structure of $\mathsf{S}(u_1, u_2)$.

Theorem 15 *Let u_1 and u_2 be arbitrary (possibly coinciding) points in $\widehat{\mathbb{C}}$ and choose some $v \in \widehat{\mathbb{C}} \setminus \{u_1, u_2\}$. Then each $\mathsf{m} \in \mathsf{S}(u_1, u_2)$ has a unique decomposition $\mathsf{m} = \mathsf{f} \cdot \mathsf{g}$ with $\mathsf{f} \in \mathsf{F}(u_1, u_2)$ and $\mathsf{g} \in \mathsf{G} = \mathsf{S}(v) \cap \mathsf{S}(u_1, u_2)$. Moreover,*

$$\begin{aligned} u_1 = u_2 \quad &\Rightarrow \quad \mathsf{G} = \mathsf{F}(u_1, v), \\ u_1 \neq u_2 \quad &\Rightarrow \quad \mathsf{G} = \{\mathsf{e}, \mathsf{i}\} \end{aligned} \tag{4.3}$$

where i is the unique Möbius involution for which

$$\mathsf{i}(v) = v, \quad \mathsf{i}(u_1) = u_2, \quad \mathsf{i}(u_2) = u_1. \tag{4.4}$$

Proof Given $\mathsf{m} \in \mathsf{S}(u_1, u_2)$, choose $\mathsf{f} \in \mathsf{F}(u_1, u_2)$ so that $\mathsf{f}(v) = \mathsf{m}(v)$. If $\mathsf{m}(v) = v$ then it is sufficient to choose $\mathsf{f} = \mathsf{e}$ and otherwise the existence of f is assured by theorem 3. Then g defined as $\mathsf{g} = \mathsf{f}^{-1} \cdot \mathsf{m}$ leaves v fixed, so that $\mathsf{g} \in \mathsf{S}(v)$ and hence $\mathsf{g} \in \mathsf{G}$. This demonstrates the existence of the decomposition.

For uniqueness it is sufficient to observe that $\mathsf{G} \cap \mathsf{F}(u_1, u_2) = \mathsf{S}(v) \cap \mathsf{F}(u_1, u_2) = \{\mathsf{e}\}$ which follows from the definitions (2.7) and (4.1), because e is the only Möbius transformation with three distinct fixed-points.

The equations (4.3) for G follow by considering the definition (4.1). In the case $u_1 = u_2$, $\mathsf{G} = \mathsf{S}(v) \cap \mathsf{S}(u_1)$ contains all Möbius transformations that have u_1 *and* v as fixed-points, given theorem 2 this is exactly $\mathsf{F}(u_1, v)$. In the case $u_1 \neq u_2$ Möbius transformations within $\mathsf{G} = \mathsf{S}(v) \cap \mathsf{S}(u_1, u_2)$ fix v and permute the pair $\{u_1, u_2\}$. There are two such permutations and (by theorem 1) to each permutation there corresponds a unique Möbius

transformation which also fixes v, these are the identity e and the involution i satisfying (4.4).
□

Previously, in the construction of the normal-form (section 3.3), it was apparent that the element by which we conjugate to get from one fixed-point subgroup to another is not unique. The definition 4.1 and theorem 15 put us in a position to make a precise comment about this non-uniqueness. If we say that the action by conjugation of $\mathsf{m} \in \mathsf{M}$ is a *natural automorphism* of $\mathsf{F}(u_1, u_2)$ when $\mathsf{m}\mathsf{F}(u_1, u_2)\mathsf{m}^{-1} = \mathsf{F}(u_1, u_2)$, then the non-uniqueness to which we refer is exactly (the group of) all natural automorphisms of $\mathsf{F}(u_1, u_2)$. The observation (4.2) reveals that $\mathsf{S}(u_1, u_2)$ contains all natural automorphisms of $\mathsf{F}(u_1, u_2)$.

On the other hand, if $\mathsf{m} \in \mathsf{M}$ commutes with all of $\mathsf{F}(u_1, u_2)$ then the natural automorphism associated with m is trivial in that it just sends each element to itself. Given that $\mathsf{F}(u_1, u_2)$ is abelian, it is only the action of the quotient-group $\mathsf{S}(u_1, u_2)/\mathsf{F}(u_1, u_2)$ which gives rise to non-trivial natural automorphisms of $\mathsf{F}(u_1, u_2)$. Theorem 15 reveals that the action of this quotient-group is exactly equivalent to the action of the group G given in the theorem. The non-uniqueness of the normal-form arises from non-trivial natural automorphisms. Hence in the case $u_2 \neq u_1$, the only non-trivial natural automorphisms of $\mathsf{F}(u_1, u_2)$ arise from the action of i defined in (4.4). Essentially, when the fixed-points are distinct, $u_2 \neq u_1$, the normal-form is unique up to their ordering. In the case $u_2 = u_1$ the situation is very different, in fact any non-identity pair in $\mathsf{F}(u_1)$ are conjugate. Theorem 15 shows that the non-trivial natural automorphisms arise in this case by the action of $\mathsf{F}(u, v)$ for some $v \neq u$, later, when we extend the normal-form, we will see that the action of this group effectively changes the value of b_0 in (3.19).

In the rest of this chapter we will principally be interested in the stabilizer of a single point, $\mathsf{S}(u)$ for some $u \in \widehat{\mathbb{C}}$. This chapter is similar in spirit to chapter 3 in that we will establish a natural connection between composition of Möbius transformations within this subgroup and some particular lattice equations.

We will say little else about $\mathsf{S}(u_1, u_2)$ with $u_1 \neq u_2$, some observations about the stabilizer of a set of three and a set of four points are made in appendix A.

4.2 The single-point stabilizer subgroups

Theorem 15, taken with the observation (4.2), reveals that the stabilizer of a single point, $\mathsf{S}(u)$, can be decomposed as the semi-direct product of the normal subgroup $\mathsf{F}(u)$ and a subgroup $\mathsf{G} = \mathsf{F}(u,v)$ for any $v \neq u$. In this brief section we will define the natural projection from $\mathsf{S}(u)$ to $\mathsf{F}(u,v) \cong \mathsf{S}(u)/\mathsf{F}(u)$.

Theorem 16 *Given $u \in \widehat{\mathbb{C}}$, choose $v \in \widehat{\mathbb{C}} \setminus \{u\}$ and define*

$$\begin{array}{rccl} \pi_v^u: & \mathsf{S}(u) & \longrightarrow & \mathsf{F}(u,v), \\ & \mathsf{m} & \mapsto & \mathsf{f}^{-1} \cdot \mathsf{m}, \end{array} \tag{4.5}$$

where $\mathsf{f} \in \mathsf{F}(u)$ is the unique (by theorem 3) Möbius transformation for which $\mathsf{f}(v) = \mathsf{m}(v)$. Then π_v^u is an idempotent group homomorphism with kernel $\mathsf{F}(u)$.

Proof Given $\mathsf{m}, \mathsf{n} \in \mathsf{S}(u)$ define $\mathsf{f}, \mathsf{g} \in \mathsf{F}(u)$ so that $\mathsf{f}(v) = \mathsf{m}(v)$ and $\mathsf{g}(v) = \mathsf{n}(v)$. Then if we define $\mathsf{h} = [\mathsf{m} \cdot \mathsf{g} \cdot \mathsf{m}^{-1}] \cdot \mathsf{f}$ we see that h is a product of Möbius transformations in $\mathsf{F}(u)$, so $\mathsf{h} \in \mathsf{F}(u)$, moreover $\mathsf{h}(v) = \mathsf{m}(\mathsf{n}(v))$, so in fact

$$\begin{aligned} \pi_v^u(\mathsf{m} \cdot \mathsf{n}) &= \mathsf{h}^{-1} \cdot \mathsf{m} \cdot \mathsf{n}, \\ &= \mathsf{f}^{-1} \cdot \mathsf{m} \cdot \mathsf{g}^{-1} \cdot \mathsf{m}^{-1} \cdot \mathsf{m} \cdot \mathsf{n}, \\ &= [\mathsf{f}^{-1} \cdot \mathsf{m}] \cdot [\mathsf{g}^{-1} \cdot \mathsf{n}], \\ &= \pi_v^u(\mathsf{m}) \cdot \pi_v^u(\mathsf{n}). \end{aligned}$$

That $\mathsf{m} \in \mathsf{F}(u) \Leftrightarrow \pi_v^u(\mathsf{m}) = \mathsf{e}$ and $\pi_v^u(\pi_v^u(\mathsf{m})) = \pi_v^u(\mathsf{m})\ \forall \mathsf{m} \in \mathsf{S}(u)$ are clear from the definition.
□

It turns out that the level-sets of π_v^u (i.e., the cosets of its kernel) are exactly the conjugacy classes of $\mathsf{S}(u)$.

Theorem 17 *Let* $u \in \widehat{\mathbb{C}}$ *be arbitrary. Then given* $\mathsf{m}, \mathsf{n} \in \mathsf{S}(u) \setminus \{\mathsf{e}\}$,

$$\mathsf{m} \cdot \mathsf{n}^{-1} \in \mathsf{F}(u) \quad \Leftrightarrow \quad \exists \mathsf{l} \in \mathsf{S}(u) \ s.t. \ \mathsf{n} = \mathsf{l} \cdot \mathsf{m} \cdot \mathsf{l}^{-1}. \tag{4.6}$$

Proof Begin by choosing $v \in \widehat{\mathbb{C}} \setminus \{u\}$ and suppose that $\mathsf{n} = \mathsf{l} \cdot \mathsf{m} \cdot \mathsf{l}^{-1}$ for some $\mathsf{l} \in \mathsf{S}(u)$. Then $\pi_v^u(\mathsf{m} \cdot \mathsf{n}^{-1}) = \pi_v^u(\mathsf{m}) \cdot \pi_v^u(\mathsf{l}) \cdot \pi_v^u(\mathsf{m})^{-1} \cdot \pi_v^u(\mathsf{l})^{-1} = \mathsf{e}$ because the image of π_v^u is abelian. Hence $\mathsf{m} \cdot \mathsf{n}^{-1} \in \ker(\pi_v^u) = \mathsf{F}(u)$ and we have proved ($\Leftarrow$).

To prove the converse implication we will proceed by construction of l. Consider $\mathsf{m}, \mathsf{n} \in \mathsf{S}(u) \setminus \{\mathsf{e}\}$. We can (by theorem 2) always find $v \in \widehat{\mathbb{C}}$ which is not a fixed-point of m or n and subsequently define the point $\widetilde{v} = \mathsf{n}(v) \neq v$. Then choose $\mathsf{l} \in \mathsf{F}(u, v)$ so that $\mathsf{l}(\mathsf{m}(v)) = \widetilde{v}$ (by theorem 3 this determines l uniquely) and define $\mathsf{n}_* = \mathsf{l} \cdot \mathsf{m} \cdot \mathsf{l}^{-1}$. Suppose now that $\mathsf{m} \cdot \mathsf{n}^{-1} \in \mathsf{F}(u)$, from ($\Leftarrow$) proved above we also have that $\mathsf{m} \cdot \mathsf{n}_*^{-1} \in \mathsf{F}(u)$. Moreover it is clear that $[\mathsf{m} \cdot \mathsf{n}_*^{-1}](\widetilde{v}) = [\mathsf{m} \cdot \mathsf{n}^{-1}](\widetilde{v})$ so by theorem 3 $\mathsf{m} \cdot \mathsf{n}_*^{-1} = \mathsf{m} \cdot \mathsf{n}^{-1}$ and consequently we have shown that $\mathsf{n} = \mathsf{n}_* = \mathsf{l} \cdot \mathsf{m} \cdot \mathsf{l}^{-1}$ which confirms ($\Rightarrow$).

□

4.3 Extension of the normal-form

Now, the normal-form introduced previously (in section 3.3) for the subgroups $\mathsf{F}(u)$ and $\mathsf{F}(u, v)$ can be used to parametrise $\mathsf{S}(u)$ (which is the semidirect product of these groups). Effectively this would bring composition in $\mathsf{S}(u)$ to composition in $\mathsf{S}(\infty) = \{v \mapsto av + b \mid a, b \in \mathbb{C}, a \neq 0\}$. This however is not our intention. Instead we give a mild extension of the normal-form which will turn it into a useful tool for studying $\mathsf{S}(u)$ with a parametrisation based on fixed-points.

Theorem 18 *Given $u \in \widehat{\mathbb{C}}$ choose some $v \in \widehat{\mathbb{C}} \setminus \{u\}$, let $\mathsf{a}, \mathsf{b} \in \mathsf{M}$ be such that*

$$\mathsf{a}(u) = \infty, \quad \mathsf{a}(v) = 0, \qquad \mathsf{b}(u) = \infty \tag{4.7}$$

and define the mappings α_u and β_u by

$$\alpha_u : \begin{array}{l} \mathsf{S}(u) \longrightarrow \mathbb{C} \setminus \{0\} \\ \mathsf{m} \mapsto [\mathsf{a} \cdot \pi_v^u(\mathsf{m}) \cdot \mathsf{a}^{-1}](1) \end{array}, \qquad \beta_u : \begin{array}{l} \mathsf{F}(u) \longrightarrow \mathbb{C} \\ \mathsf{m} \mapsto [\mathsf{b} \cdot \mathsf{m} \cdot \mathsf{b}^{-1}](0) \end{array}. \tag{4.8}$$

Then α_u is a homomorphism from $\mathsf{S}(u)$ to $\mathbb{C} \setminus \{0\}$ under multiplication with kernel $\mathsf{F}(u)$ and any other choice of $v \in \widehat{\mathbb{C}} \setminus \{u\}$ or a satisfying (4.7) yields the same homomorphism α_u. Furthermore, if $\mathsf{m} \in \mathsf{S}(u) \setminus \{\mathsf{e}\}$ and $\mathsf{n} \in \mathsf{S}(\widetilde{u}) \setminus \{\mathsf{e}\}$ then

$$\alpha_u(\mathsf{m}) = \alpha_{\widetilde{u}}(\mathsf{n}) \quad \Leftrightarrow \quad \exists \mathsf{l} \in \mathsf{M} \ s.t. \ \mathsf{l}(u) = \widetilde{u}, \ \mathsf{n} = \mathsf{l} \cdot \mathsf{m} \cdot \mathsf{l}^{-1}. \tag{4.9}$$

The mapping β_u is an isomorphism between $\mathsf{F}(u)$ and $\mathbb{C}$ under addition. Any other choice of b satisfying (4.7), say b_*, yields a mapping β_u^* related to β_u by

$$\beta_u^*(\mathsf{m}) = \beta_u(\mathsf{n} \cdot \mathsf{m} \cdot \mathsf{n}^{-1}) \tag{4.10}$$

for some $\mathsf{n} \in \mathsf{S}(u)$. Furthermore, given any $\mathsf{m} \in \mathsf{F}(u)$ and $\mathsf{n} \in \mathsf{S}(u)$,

$$\beta_u(\mathsf{n} \cdot \mathsf{m} \cdot \mathsf{n}^{-1}) = \alpha_u(\mathsf{n})\beta_u(\mathsf{m}). \tag{4.11}$$

Proof The mapping α_u is nothing but the composition of the projection π_v^u with the normal-form isomorphism for $\mathsf{F}(u, v)$ described in theorem 13, so is clearly the described homomorphism. Any different choice of v and a satisfying (4.7), say v_* and a_*, yields another homomorphism α_u^*. If we define $\mathsf{n} = \mathsf{a}^{-1} \cdot \mathsf{a}_*$ and observe that $\mathsf{n} \in \mathsf{S}(u)$ and $\mathsf{n}(v_*) = v$, then a simple calculation reveals that $\alpha_u^*(\mathsf{m}) = \alpha_u(\mathsf{n} \cdot \mathsf{m} \cdot \mathsf{n}^{-1})$ so using the fact that α_u is the described homomorphism, i.e., $\alpha_u(\mathsf{n} \cdot \mathsf{m} \cdot \mathsf{n}^{-1}) = \alpha_u(\mathsf{n})\alpha_u(\mathsf{m})/\alpha_u(\mathsf{n}) = \alpha_u(\mathsf{m})$, we see that $\alpha_u^* = \alpha_u$. The equivalence (4.9) follows from theorem 17 (because $\ker(\alpha_u) = \mathsf{F}(u)$) together with the observation that for any $\mathsf{l} \in \mathsf{M}$ and $\mathsf{m} \in \mathsf{S}(u)$, $\alpha_u(\mathsf{m}) = \alpha_{\mathsf{l}(u)}(\mathsf{l} \cdot \mathsf{m} \cdot \mathsf{l}^{-1})$.

The mapping β_u is exactly the normal-form isomorphism for $\mathsf{F}(u)$ described in theorem 13, so it only remains to demonstrate the relation

(4.11). We begin by observing that for any $c \in \mathbb{C} \setminus \{0\}$ and $\mathsf{n} \in \mathsf{S}(u)$, $\alpha_u(\mathsf{n})c = [\mathsf{a} \cdot \pi_v^u(\mathsf{n}) \cdot \mathsf{a}^{-1}](c)$. This follows by supposing $c = \alpha_u(\mathsf{h})$ for some h. Now, given $\mathsf{m} \in \mathsf{F}(u)$ and $\mathsf{n} \in \mathsf{S}(u)$, (4.11) clearly holds if $\mathsf{m} = \mathsf{e}$ so it remains to consider the case when $\beta_u(\mathsf{m}) \neq 0$ which enables us to write $\alpha_u(\mathsf{n})\beta_u(\mathsf{m}) = [\mathsf{a} \cdot \pi_v^u(\mathsf{n}) \cdot \mathsf{a}^{-1}](\beta_u(\mathsf{m}))$. Observing that $\mathsf{a} \cdot \pi_v^u(\mathsf{n}) \cdot \mathsf{a}^{-1}$ has 0 as a fixed-point we can then contrive to write

$$
\begin{aligned}
\alpha_u(\mathsf{n})\beta_u(\mathsf{m}) &= [\mathsf{a} \cdot \pi_v^u(\mathsf{n}) \cdot \mathsf{a}^{-1}](\beta_u(\mathsf{m})), \\
&= [[\mathsf{a} \cdot \pi_v^u(\mathsf{n}) \cdot \mathsf{a}^{-1}] \cdot [\mathsf{b} \cdot \mathsf{m} \cdot \mathsf{b}^{-1}]](0), \\
&= [[\mathsf{a} \cdot \pi_v^u(\mathsf{n}) \cdot \mathsf{a}^{-1}] \cdot [\mathsf{b} \cdot \mathsf{m} \cdot \mathsf{b}^{-1}] \cdot [\mathsf{a} \cdot \pi_v^u(\mathsf{n})^{-1} \cdot \mathsf{a}^{-1}]](0), \\
&= \beta_u(\mathsf{g} \cdot \mathsf{m} \cdot \mathsf{g}^{-1}), \qquad \mathsf{g} = \mathsf{b}^{-1} \cdot \mathsf{a} \cdot \pi_v^u(\mathsf{n}) \cdot \mathsf{a}^{-1} \cdot \mathsf{b}.
\end{aligned}
$$

It remains to show that $\mathsf{g} \cdot \mathsf{m} \cdot \mathsf{g}^{-1} = \mathsf{n} \cdot \mathsf{m} \cdot \mathsf{n}^{-1}$. Given $\mathsf{b}^{-1} \cdot \mathsf{a} \in \mathsf{S}(u)$, theorem 17 reveals that $\mathsf{g} \cdot \pi_v^u(\mathsf{n})^{-1} \in \mathsf{F}(u)$. And from the definition of π_v^u (4.5) we see that in fact $\mathsf{g} \cdot \pi_v^u(\mathsf{n})^{-1} = \mathsf{g} \cdot \mathsf{n}^{-1} \cdot \mathsf{f}$ for some $\mathsf{f} \in \mathsf{F}(u)$, therefore $\mathsf{g} \cdot \mathsf{n}^{-1} \in \mathsf{F}(u)$. Given that $\mathsf{n} \cdot \mathsf{m} \cdot \mathsf{n}^{-1} \in \mathsf{F}(u)$ and that $\mathsf{F}(u)$ is abelian it follows that $\mathsf{n} \cdot \mathsf{m} \cdot \mathsf{n}^{-1} = [\mathsf{g} \cdot \mathsf{n}^{-1}] \cdot [\mathsf{n} \cdot \mathsf{m} \cdot \mathsf{n}^{-1}] \cdot [\mathsf{g} \cdot \mathsf{n}^{-1}]^{-1} = \mathsf{g} \cdot \mathsf{m} \cdot \mathsf{g}^{-1}$ which concludes the proof.
□

Observe that the final statement of this theorem clarifies the non-uniqueness of the normal-form for $\mathsf{F}(u)$ which was discussed in section 4.1.

4.4 The lattice Schwarzian KdV equation

Suppose we fix two Möbius transformations $\mathsf{f}, \mathsf{g} \in \mathsf{S}(u)$ by specifying their fixed-points and normal-form parameters,

$$
\begin{aligned}
\mathsf{f} \in \mathsf{F}(u, \widetilde{u}), &\qquad \mathsf{g} \in \mathsf{F}(u, \widehat{u}), \\
\alpha_u(\mathsf{f}) = p, &\qquad \alpha_u(\mathsf{g}) = q.
\end{aligned} \tag{4.12}
$$

This fixes f and g uniquely provided neither $\widetilde{u}$ or $\widehat{u}$ are equal to u. Now consider the composition of f and g. Clearly the shared fixed-point, u, is also a fixed-point of the composed transformation $\mathsf{f} \cdot \mathsf{g}$. It is also clear that

$\alpha_u(\mathsf{f}\cdot\mathsf{g}) = pq$, so to determine $\mathsf{f}\cdot\mathsf{g}$ uniquely it remains to find its other fixed-point. Now, this will depend upon the order in which we take f and g, i.e., $\mathsf{f}\cdot\mathsf{g}$ and $\mathsf{g}\cdot\mathsf{f}$ will in general have different fixed-points because $\mathsf{S}(u)$ is not abelian.

It turns out to be natural to rephrase this question by considering the composition $\mathsf{g}^{-1}\cdot\mathsf{f}$. Although $\alpha_u(\mathsf{g}^{-1}\cdot\mathsf{f}) = p/q$ now depends upon the ordering of f and g, the fixed-points of $\mathsf{g}^{-1}\cdot\mathsf{f}$ are determined by the equation for w,

$$\mathsf{f}(w) = \mathsf{g}(w). \tag{4.13}$$

Clearly the solutions of this equation will not depend on the order in which we take f and g. According to theorem 18, u is the *only* solution of (4.13), i.e., $\mathsf{g}^{-1}\cdot\mathsf{f}\in\mathsf{F}(u)$, if and only if $\alpha_u(\mathsf{g}^{-1}\cdot\mathsf{f}) = 1$, i.e., if and only if $p = q$. So provided $p \neq q$, there is a point $w \neq u$ that satisfies (4.13). Let us denote this point $\widehat{\widetilde{u}}$, in other words we suppose that $\mathsf{g}^{-1}\cdot\mathsf{f}\in\mathsf{F}(u,\widehat{\widetilde{u}})$.

The following theorem provides a formal approach to finding $\widehat{\widetilde{u}}$, the actual quantity determined in the theorem is $\beta_u(\mathsf{z})$ where $\mathsf{z}\in\mathsf{F}(u)$ sends some point v to the point $\widehat{\widetilde{u}}$, but as will be seen this provides a round-about way to find $\widehat{\widetilde{u}}$ itself.

Theorem 19 *Given $u\in\widehat{\mathbb{C}}$ and $\widetilde{u},\widehat{u},\widehat{\widetilde{u}}\in\widehat{\mathbb{C}}\setminus\{u\}$, suppose $\mathsf{f},\mathsf{g}\in\mathsf{S}(u)$ are such that*

$$\mathsf{f}\in\mathsf{F}(u,\widetilde{u}),\quad \mathsf{g}\in\mathsf{F}(u,\widehat{u}),\quad \mathsf{g}^{-1}\cdot\mathsf{f}\in\mathsf{F}(u,\widehat{\widetilde{u}}). \tag{4.14}$$

If for some $v\in\widehat{\mathbb{C}}\setminus\{u\}$ we define $\mathsf{x},\mathsf{y},\mathsf{z}\in\mathsf{F}(u)$ to be the unique (by theorem 3) Möbius transformations for which

$$\mathsf{x}(v) = \widetilde{u},\quad \mathsf{y}(v) = \widehat{u},\quad \mathsf{z}(v) = \widehat{\widetilde{u}}, \tag{4.15}$$

then

$$\beta_u(\mathsf{z})\left(\alpha_u(\mathsf{f}) - \alpha_u(\mathsf{g})\right) = \beta_u(\mathsf{x})\left(\alpha_u(\mathsf{f}) - 1\right) - \beta_u(\mathsf{y})\left(\alpha_u(\mathsf{g}) - 1\right). \tag{4.16}$$

Proof We claim that, given $\mathsf{f}, \mathsf{g}, \mathsf{x}, \mathsf{y}$ and z as described,

$$\begin{aligned} \pi_v^u(\mathsf{f}) &= \mathsf{x}^{-1} \cdot \mathsf{f} \cdot \mathsf{x}, \\ \pi_v^u(\mathsf{g}) &= \mathsf{y}^{-1} \cdot \mathsf{g} \cdot \mathsf{y}, \\ \pi_v^u(\mathsf{g}^{-1} \cdot \mathsf{f}) &= \mathsf{z}^{-1} \cdot \mathsf{g}^{-1} \cdot \mathsf{f} \cdot \mathsf{z}. \end{aligned} \tag{4.17}$$

To see this observe first that $\mathsf{x}^{-1} \cdot \mathsf{f} \cdot \mathsf{x} \in \mathsf{F}(u, v)$ by considering the action of $\mathsf{x}^{-1} \cdot \mathsf{f} \cdot \mathsf{x}$ on the points u and v. But π_v^u is just the projection onto $\mathsf{F}(u, v)$, so it follows that $\pi_v^u(\mathsf{x}^{-1} \cdot \mathsf{f} \cdot \mathsf{x}) = \mathsf{x}^{-1} \cdot \mathsf{f} \cdot \mathsf{x}$. But clearly $\pi_v^u(\mathsf{x}^{-1} \cdot \mathsf{f} \cdot \mathsf{x}) = \pi_v^u(\mathsf{f})$ directly from theorem 16 (either using the fact that $\pi_v^u(\mathsf{x}) = \mathsf{e}$ or that $\mathsf{F}(u, v)$ is abelian). Hence we have established the first equality in 4.17, the other two follow in exactly the same way.

To complete the proof it remains only to consider the following identity

$$\mathsf{g} \cdot \pi_v^u(\mathsf{g}^{-1} \cdot \mathsf{f}) \cdot \mathsf{f}^{-1} = [\mathsf{g} \cdot \pi_v^u(\mathsf{g})^{-1}] \cdot [\mathsf{f} \cdot \pi_v^u(\mathsf{f})^{-1}]^{-1}. \tag{4.18}$$

Substitution from (4.17) reveals that

$$[\mathsf{g} \cdot \mathsf{z}^{-1} \cdot \mathsf{g}^{-1}] \cdot [\mathsf{f} \cdot \mathsf{z} \cdot \mathsf{f}^{-1}] = [\mathsf{g} \cdot \mathsf{y}^{-1} \cdot \mathsf{g}^{-1}] \cdot \mathsf{y} \cdot \mathsf{x}^{-1} \cdot [\mathsf{f} \cdot \mathsf{x} \cdot \mathsf{f}^{-1}], \tag{4.19}$$

where clearly all bracketed terms lie within $\mathsf{F}(u)$. The relation (4.16) follows by application of β_u to both sides of (4.19) and using (4.11) from theorem 18.

□

To find a concrete formula for $\widehat{\widetilde{u}}$, the other fixed-point of $\mathsf{g}^{-1} \cdot \mathsf{f}$, we need only make the following substitutions

$$\begin{aligned} \beta_u(\mathsf{x}) &= b_0 \chi(\widetilde{u}, u, v), & & \\ \beta_u(\mathsf{y}) &= b_0 \chi(\widehat{u}, u, v), & \alpha_u(\mathsf{f}) &= p, \\ \beta_u(\mathsf{z}) &= b_0 \chi(\widehat{\widetilde{u}}, u, v), & \alpha_u(\mathsf{g}) &= q, \end{aligned} \tag{4.20}$$

in (4.16). The three on the left follow by using (3.19) and the two on the right are by definition. The result is that

$$(p - q)\chi(\widehat{\widetilde{u}}, u, v) = (p - 1)\chi(\widetilde{u}, u, v) - (q - 1)\chi(\widehat{u}, u, v). \tag{4.21}$$

The expression χ was defined in (3.18) and using this we observe that, as it should, the dependence on v drops out and we are left with the relation

$$\frac{p-q}{\widehat{\widetilde{u}}-u} = \frac{p-1}{\widetilde{u}-u} - \frac{q-1}{\widehat{u}-u}. \tag{4.22}$$

This relation is perhaps better known in the rearranged form

$$(p-1)(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}}) = (q-1)(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}}). \tag{4.23}$$

Up to a point transformation of the parameters p and q this is the lattice Schwarzian KdV equation (1.7) introduced in section 1.2 as the superposition principle for BTs of the Schwarzian KdV equation. ABS coined (4.22) the three-leg-form for (4.23) in their article [4].

4.5 Fixed-points on the cube

As we know from section 1.2, the lattice Schwarzian KdV equation (4.23) can be embedded naturally into three dimensions. However, unlike for the lattice system described in chapter 3, this embedding is not obvious from the 'fixed-point composition' meaning we have attributed to (4.23). This leads us to expect a theorem about 'fixed-points on the cube'. It turns out that the natural result, which we will prove in this section, connects the 'fixed-points on the cube' to the configuration of Möbius transformations with shared fixed-points described in chapter 3.

Let us begin with a preliminary technical result.

Theorem 20 *Suppose* $\mathsf{l}, \mathsf{m} \in \mathsf{M} \setminus \{\mathsf{e}\}$ *are distinct but share their fixed-points, and choose some* $u \in \widehat{\mathbb{C}}$ *which is not one of those fixed-points. Then for any* $\mathsf{f} \in \mathsf{F}(u, \mathsf{l}(u)) \setminus \{\mathsf{e}\}$ *and* $\mathsf{g} \in \mathsf{F}(u, \mathsf{m}(u)) \setminus \{\mathsf{e}\}$,

$$\mathsf{g}^{-1}\cdot\mathsf{f} \in \mathsf{F}(u, [\mathsf{l}\cdot\mathsf{m}](u)) \quad \Leftrightarrow \quad \mathsf{f}\cdot\mathsf{l} \textit{ and } \mathsf{g}\cdot\mathsf{m} \textit{ share their fixed-points.} \tag{4.24}$$

Proof Given l, m and u as described, the main idea here is to introduce the unique (by theorem 6) involution i for which

$$\mathsf{i}(u) = [\mathsf{l}\cdot\mathsf{m}](u), \qquad \mathsf{i}(\mathsf{l}(u)) = \mathsf{m}(u). \tag{4.25}$$

Firstly, if f and g are as described, we observe that u is a fixed-point of $\mathsf{g}^{-1} \cdot \mathsf{f}$. Therefore $\mathsf{g}^{-1} \cdot \mathsf{f} \in \mathsf{F}(u, [\mathsf{l} \cdot \mathsf{m}](u))$ if and only if the fixed-points of $\mathsf{g}^{-1} \cdot \mathsf{f}$ are transposed by i. By theorem 9 this is true if and only if $\mathsf{i} \cdot \mathsf{g}^{-1} \cdot \mathsf{f}$ is an involution. From the hypotheses it is easily checked that $\mathsf{i} \cdot \mathsf{g}^{-1} \cdot \mathsf{f} \neq \mathsf{e}$, so this in turn is equivalent to the assertion that

$$(\mathsf{i} \cdot \mathsf{g}^{-1} \cdot \mathsf{f})^2 = \mathsf{e}. \tag{4.26}$$

We will establish that (4.26) is itself in turn equivalent to the commutativity of $\mathsf{f} \cdot \mathsf{l}$ with $\mathsf{g} \cdot \mathsf{m}$. To do this we first observe that the Möbius transformations $\mathsf{i} \cdot \mathsf{l}$ and $\mathsf{i} \cdot \mathsf{m}$ transpose the pairs $\{u, \mathsf{m}(u)\}$ and $\{u, \mathsf{l}(u)\}$ respectively. Hence by theorem 5 they are both involutions

$$\mathsf{i} \cdot \mathsf{l} = \mathsf{l}^{-1} \cdot \mathsf{i}, \qquad \mathsf{i} \cdot \mathsf{m} = \mathsf{m}^{-1} \cdot \mathsf{i}. \tag{4.27}$$

Moreover, by theorem 9, $[\mathsf{i} \cdot \mathsf{m}] \cdot \mathsf{f}$ and $[\mathsf{i} \cdot \mathsf{l}] \cdot \mathsf{g}$ are involutions,

$$[\mathsf{i} \cdot \mathsf{m}] \cdot \mathsf{f} = \mathsf{f}^{-1} \cdot [\mathsf{i} \cdot \mathsf{m}], \qquad [\mathsf{i} \cdot \mathsf{l}] \cdot \mathsf{g} = \mathsf{g}^{-1} \cdot [\mathsf{i} \cdot \mathsf{l}]. \tag{4.28}$$

Now starting from (4.26) and using (4.27) and (4.28) we may write

$$\begin{aligned} \mathsf{g}^{-1} \cdot \mathsf{f} &= \mathsf{i} \cdot \mathsf{f}^{-1} \cdot \mathsf{g} \cdot \mathsf{i}, \\ &= \mathsf{m} \cdot \mathsf{i} \cdot \mathsf{m} \cdot \mathsf{f}^{-1} \cdot \mathsf{g} \cdot \mathsf{i} \cdot \mathsf{l} \cdot \mathsf{l}^{-1}, \\ &= \mathsf{m} \cdot \mathsf{f} \cdot \mathsf{i} \cdot \mathsf{m} \cdot \mathsf{i} \cdot \mathsf{l} \cdot \mathsf{g}^{-1} \cdot \mathsf{l}^{-1}, \\ &= \mathsf{m} \cdot \mathsf{f} \cdot \mathsf{m}^{-1} \cdot \mathsf{l} \cdot \mathsf{g}^{-1} \cdot \mathsf{l}^{-1}. \end{aligned}$$

Then using the fact that l and m commute (by theorem 11) this can be rearranged to reveal

$$[\mathsf{f} \cdot \mathsf{l}] \cdot [\mathsf{g} \cdot \mathsf{m}] = [\mathsf{g} \cdot \mathsf{m}] \cdot [\mathsf{f} \cdot \mathsf{l}]. \tag{4.29}$$

Finally we will show that, due to the hypotheses of the theorem, (4.29) is equivalent to the assertion that $\mathsf{f} \cdot \mathsf{l}$ and $\mathsf{g} \cdot \mathsf{m}$ share their fixed-points. Theorem 11 gives only two other possibilities, that one of $\mathsf{f} \cdot \mathsf{l}$ and $\mathsf{g} \cdot \mathsf{m}$ is equal to e or that $\mathsf{f} \cdot \mathsf{l}$ and $\mathsf{g} \cdot \mathsf{m}$ are distinct commuting involutions, cf. theorem 12. If, say, $\mathsf{f} \cdot \mathsf{l} = \mathsf{e}$, then f and l share fixed-points, which means that u is a fixed-point of l contradicting our choice of u, so neither of $\mathsf{f}\cdot\mathsf{l}$ or $\mathsf{g}\cdot\mathsf{m}$ is equal to e. Suppose now that $\mathsf{f}\cdot\mathsf{l}$ and $\mathsf{g}\cdot\mathsf{m}$ are distinct commuting involutions, clearly they transpose the points $\{u,\mathsf{l}(u)\}$ and $\{u,\mathsf{m}(u)\}$ respectively, but f and g have exactly these as fixed-points. Therefore we conclude that l and m transpose these points and are therefore involutions themselves. But l and m share their fixed-points, so by theorem 8 if they are both involutions they must coincide. This contradicts our hypothesis that l and m are distinct, so $\mathsf{f}\cdot\mathsf{l}$ and $\mathsf{g}\cdot\mathsf{m}$ are not distinct commuting involutions.

□

We now state the main theorem and the proof proceeds quite straightforwardly.

Theorem 21 *Suppose the points* $u,\widetilde{u},\widehat{u},\overline{u},\overline{\widehat{u}},\widehat{\widetilde{u}},\widetilde{\overline{u}}\in\widehat{\mathbb{C}}$ *are all distinct. Then there exist* $\mathsf{f},\mathsf{g},\mathsf{h}\in\mathsf{S}(u)\setminus\{\mathsf{e}\}$ *such that*

$$\begin{array}{ccc} \mathsf{f}\in\mathsf{F}(u,\widetilde{u}), & \mathsf{g}\in\mathsf{F}(u,\widehat{u}), & \mathsf{h}\in\mathsf{F}(u,\overline{u}), \\ \mathsf{g}^{-1}\cdot\mathsf{f}\in\mathsf{F}(u,\widehat{\widetilde{u}}), & \mathsf{h}^{-1}\cdot\mathsf{g}\in\mathsf{F}(u,\overline{\widehat{u}}), & \mathsf{f}^{-1}\cdot\mathsf{h}\in\mathsf{F}(u,\widetilde{\overline{u}}), \end{array} \tag{4.30}$$

if and only if there exists an involution $\mathsf{i}\in\mathsf{M}$ *for which*

$$\mathsf{i}(\widetilde{u})=\overline{\widehat{u}},\quad \mathsf{i}(\widehat{u})=\widetilde{\overline{u}},\quad \mathsf{i}(\overline{u})=\widehat{\widetilde{u}}. \tag{4.31}$$

Proof Suppose first that $\mathsf{f},\mathsf{g},\mathsf{h}\in\mathsf{S}(u)\setminus\{\mathsf{e}\}$ satisfy (4.30). By theorem 6 we may construct a unique involution $\mathsf{i}\in\mathsf{M}$ for which $\mathsf{i}(\widetilde{u})=\overline{\widehat{u}}$ and $\mathsf{i}(\widehat{u})=\widetilde{\overline{u}}$, we may then define the point $\widehat{\widetilde{u}}_*=\mathsf{i}(\overline{u})$. We will show that $\widehat{\widetilde{u}}_*=\widehat{\widetilde{u}}$. This hinges on the construction of $\mathsf{l},\mathsf{m},\mathsf{n}\in\mathsf{M}$ such that

$$\begin{array}{lll} \mathsf{l}(u)=\widetilde{u}, & \mathsf{m}(u)=\widehat{u}, & \mathsf{n}(u)=\overline{u}, \\ \mathsf{l}(\widehat{u})=\widehat{\widetilde{u}}_*, & \mathsf{m}(\overline{u})=\overline{\widehat{u}}, & \mathsf{n}(\widetilde{u})=\widetilde{\overline{u}}, \\ \mathsf{l}(\overline{u})=\widetilde{\overline{u}}, & \mathsf{m}(\widetilde{u})=\widehat{\widetilde{u}}_*, & \mathsf{n}(\widehat{u})=\overline{\widehat{u}}, \end{array} \tag{4.32}$$

which according to theorem 14 share their fixed-points. Now, from the hypotheses of the theorem (and theorem 2) we see that

$$\begin{array}{l} \mathsf{f}^{-1}\cdot\mathsf{h}\in\mathsf{F}(u,\widetilde{\overline{u}})=\mathsf{F}(u,[\mathsf{n}\cdot\mathsf{l}](u)), \\ \mathsf{h}^{-1}\cdot\mathsf{g}\in\mathsf{F}(u,\overline{\widehat{u}})=\mathsf{F}(u,[\mathsf{m}\cdot\mathsf{n}](u)). \end{array} \tag{4.33}$$

So according to theorem 20, $\mathsf{f}\cdot\mathsf{l}$ and $\mathsf{g}\cdot\mathsf{m}$ both share their fixed-points with $\mathsf{h}\cdot\mathsf{n}$. Of course then $\mathsf{f}\cdot\mathsf{l}$ and $\mathsf{g}\cdot\mathsf{m}$ also share their fixed-points with each other, so again by theorem 20 we find

$$\mathsf{g}^{-1}\cdot\mathsf{f}\in\mathsf{F}(u,[\mathsf{l}\cdot\mathsf{m}](u))=\mathsf{F}(u,\widehat{\widetilde{u}}_*). \tag{4.34}$$

However, by supposition $\mathsf{g}^{-1}\cdot\mathsf{f}\in\mathsf{F}(u,\widehat{\widetilde{u}})$, and the intersection of $\mathsf{F}(u,\widehat{\widetilde{u}}_*)$ with $\mathsf{F}(u,\widehat{\widetilde{u}})$ is $\{\mathsf{e}\}$ unless $\{u,\widehat{\widetilde{u}}_*\}=\{u,\widehat{\widetilde{u}}\}$. But $\mathsf{g}^{-1}\cdot\mathsf{f}=\mathsf{e}$ implies f and g share their fixed-points which contradicts our supposition, so we must conclude that $\widehat{\widetilde{u}}_*=\widehat{\widetilde{u}}$.

Suppose now that the involution $\mathsf{i}\in\mathsf{M}$ satisfying (4.31) exists. Choose *any* $\mathsf{f}\in\mathsf{F}(u,\widetilde{u})\setminus\{\mathsf{e}\}$ and fix g,h in terms of f by demanding that

$$\begin{aligned} &\mathsf{g}\in\mathsf{F}(u,\widehat{u}), && \mathsf{h}\in\mathsf{F}(u,\overline{u}),\\ &\mathsf{g}(\widehat{\widetilde{u}})=\mathsf{f}(\widehat{\widetilde{u}}), && \mathsf{h}(\widetilde{\overline{u}})=\mathsf{f}(\widetilde{\overline{u}}), \end{aligned} \tag{4.35}$$

which according to theorem 3 fixes g and h uniquely, and neither is equal to e. This means that, by construction, $\mathsf{g}^{-1}\cdot\mathsf{f}\in\mathsf{F}(u,\widehat{\widetilde{u}})$ and $\mathsf{f}^{-1}\cdot\mathsf{h}\in\mathsf{F}(u,\widetilde{\overline{u}})$. Now if we suppose $\mathsf{h}^{-1}\cdot\mathsf{g}\in\mathsf{F}(u,\overline{\widehat{u}}_*)$ for some point $\overline{\widehat{u}}_*\in\widehat{\mathbb{C}}$, then according to what was proven above there exists an involution $\mathsf{j}\in\mathsf{M}$ for which

$$\mathsf{j}(\widetilde{u})=\overline{\widehat{u}}_*,\quad \mathsf{j}(\widehat{u})=\widetilde{\overline{u}},\quad \mathsf{j}(\overline{u})=\widehat{\widetilde{u}}. \tag{4.36}$$

It then follows from theorem 6 that $\mathsf{j}=\mathsf{i}$, which in turn implies that $\overline{\widehat{u}}_*=\overline{\widehat{u}}$, and so f,g and h as constructed satisfy (4.30).
□

4.6 Embedding in three dimensions

Consider the lattice Schwarzian KdV equation written previously (4.23). Let us for convenience define the quadrilateral expression $\mathcal{Q}_{pq}$,

$$\mathcal{Q}_{pq}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}})=(p-1)(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-(q-1)(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}}), \tag{4.37}$$

and recall from section 4.4 that if $\mathsf{f},\mathsf{g}\in\mathsf{S}(u)$ satisfy (4.12), then $\mathsf{g}^{-1}\cdot\mathsf{f}\in\mathsf{F}(u,\widehat{\widetilde{u}})$ where $\widehat{\widetilde{u}}$ is fixed by the equation

$$\mathcal{Q}_{pq}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}})=0. \tag{4.38}$$

Of course if $\mathsf{f}, \mathsf{g}, \mathsf{h} \in \mathsf{S}(u)$ satisfy (4.30) and we suppose that

$$\alpha_u(\mathsf{f}) = p, \qquad \alpha_u(\mathsf{g}) = q, \qquad \alpha_u(\mathsf{h}) = r, \tag{4.39}$$

then it is immediate that

$$\mathcal{Q}_{pq}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) = 0, \quad \mathcal{Q}_{qr}(u, \widehat{u}, \overline{u}, \overline{\widehat{u}}) = 0, \quad \mathcal{Q}_{rp}(u, \overline{u}, \widetilde{u}, \widetilde{\overline{u}}) = 0. \tag{4.40}$$

Now it turns out that the equations

$$\mathcal{Q}_{pq}(\overline{u}, \widetilde{\overline{u}}, \overline{\widehat{u}}, \widehat{\widetilde{\overline{u}}}) = 0, \quad \mathcal{Q}_{qr}(\widetilde{u}, \widehat{\widetilde{u}}, \widetilde{\overline{u}}, \widehat{\widetilde{\overline{u}}}) = 0, \quad \mathcal{Q}_{rp}(\widehat{u}, \overline{\widehat{u}}, \widehat{\widetilde{u}}, \widehat{\widetilde{\overline{u}}}) = 0, \tag{4.41}$$

all fix the same point $\widehat{\widetilde{\overline{u}}}$, in other words the quadrilateral expression (4.37) is consistent on the cube. In this section we will demonstrate this consistency, principally using theorems 21 and 14. The process will reveal the associated configuration of Möbius transformations, which is actually the main purpose because the consistency itself is well known and straightforward to verify by calculation.

Let us define Möbius transformations $\mathsf{l}, \mathsf{m}, \mathsf{n} \in \mathsf{M}$ by the relations (3.26) which according to theorem 1 fixes them uniquely. The main idea here is to introduce the new Möbius transformations

$$\begin{aligned}
\overline{\mathsf{f}} &= \mathsf{n} \cdot \mathsf{f} \cdot \mathsf{n}^{-1}, & \overline{\mathsf{g}} &= \mathsf{n} \cdot \mathsf{g} \cdot \mathsf{n}^{-1}, \\
\widetilde{\mathsf{g}} &= \mathsf{l} \cdot \mathsf{g} \cdot \mathsf{l}^{-1}, & \widetilde{\mathsf{h}} &= \mathsf{l} \cdot \mathsf{h} \cdot \mathsf{l}^{-1}, \\
\widehat{\mathsf{h}} &= \mathsf{m} \cdot \mathsf{h} \cdot \mathsf{m}^{-1}, & \widehat{\mathsf{f}} &= \mathsf{m} \cdot \mathsf{f} \cdot \mathsf{m}^{-1}.
\end{aligned} \tag{4.42}$$

It is then immediate from (4.30) and (4.39), using (2.8) and (4.9), that $\overline{f}, \overline{g} \in \mathsf{S}(\overline{u})$ satisfy

$$\begin{array}{ll} \overline{\mathsf{f}} \in \mathsf{F}(\overline{u}, \widetilde{\overline{u}}), & \overline{\mathsf{g}} \in \mathsf{F}(\overline{u}, \overline{\widehat{u}}), \\ \alpha_{\overline{u}}(\overline{\mathsf{f}}) = p, & \alpha_{\overline{u}}(\overline{\mathsf{g}}) = q, \end{array} \qquad \overline{\mathsf{g}}^{-1} \cdot \overline{\mathsf{f}} \in \mathsf{F}(\overline{u}, \mathsf{n}(\widehat{\widetilde{u}})). \tag{4.43}$$

Similarly, the pair $\widetilde{\mathsf{g}}, \widetilde{\mathsf{h}} \in \mathsf{S}(\widetilde{u})$ satisfy

$$\begin{array}{ll} \widetilde{\mathsf{g}} \in \mathsf{F}(\widetilde{u}, \widehat{\widetilde{u}}), & \widetilde{\mathsf{h}} \in \mathsf{F}(\widetilde{u}, \widetilde{\overline{u}}), \\ \alpha_{\widetilde{u}}(\widetilde{\mathsf{g}}) = q, & \alpha_{\widetilde{u}}(\widetilde{\mathsf{h}}) = r, \end{array} \qquad \widetilde{\mathsf{h}}^{-1} \cdot \widetilde{\mathsf{g}} \in \mathsf{F}(\widetilde{u}, \mathsf{l}(\overline{\widehat{u}})), \tag{4.44}$$

and the pair $\widehat{\mathsf{h}}, \widehat{\mathsf{f}} \in \mathsf{S}(\widehat{u})$ satisfy

$$\begin{matrix} \widehat{\mathsf{h}} \in \mathsf{F}(\widehat{u}, \overline{\widehat{u}}), & \widehat{\mathsf{f}} \in \mathsf{F}(\widehat{u}, \widehat{\widetilde{u}}), \\ \alpha_{\widehat{u}}(\widehat{\mathsf{h}}) = r, & \alpha_{\widehat{u}}(\widehat{\mathsf{f}}) = p, \end{matrix} \qquad \widehat{\mathsf{f}}^{-1} \cdot \widehat{\mathsf{h}} \in \mathsf{F}(\widehat{u}, \mathsf{m}(\widetilde{\overline{u}})). \tag{4.45}$$

Now using theorems 21 and 14 we see that l, m and n have the same fixed-points, so they commute and hence $\mathsf{n}(\widehat{\widetilde{u}}) = \mathsf{l}(\overline{\widehat{u}}) = \mathsf{m}(\widetilde{\overline{u}}) = [\mathsf{l} \cdot \mathsf{m} \cdot \mathsf{n}](u)$. The three equations (4.41) then follow directly from the relations (4.43), (4.44) and (4.45) if we identify $\widehat{\widetilde{\overline{u}}} = [\mathsf{l} \cdot \mathsf{m} \cdot \mathsf{n}](u)$. Thus the consistency is verified.

Of course by theorem 14 there also exists a Möbius involution $\mathsf{i} \in \mathsf{M}$ which satisfies

$$\mathsf{i}(u) = \widehat{\widetilde{\overline{u}}}, \quad \mathsf{i}(\widetilde{u}) = \overline{\widehat{u}}, \quad \mathsf{i}(\widehat{u}) = \widetilde{\overline{u}}, \quad \mathsf{i}(\overline{u}) = \widehat{\widetilde{u}}. \tag{4.46}$$

We will finish this section by finding an equation for the point $\widehat{\widetilde{\overline{u}}}$ in terms of initial data $\{u, \widetilde{u}, \widehat{u}, \overline{u}\}$ and the parameters p, q and r. The procedure will be similar in spirit to the derivation of the lattice Schwarzian KdV equation itself given in section 4.4.

Let us begin by introducing the following new Möbius transformations

$$\mathsf{f}_* = \mathsf{i} \cdot \mathsf{f} \cdot \mathsf{i}, \qquad \mathsf{g}_* = \mathsf{i} \cdot \mathsf{g} \cdot \mathsf{i}, \qquad \mathsf{h}_* = \mathsf{i} \cdot \mathsf{h} \cdot \mathsf{i}, \tag{4.47}$$

clearly $\mathsf{f}_*, \mathsf{g}_*, \mathsf{h}_* \in \mathsf{S}(\widehat{\widetilde{\overline{u}}})$. Now choose some $v \in \widehat{\mathbb{C}} \setminus \{\widehat{\widetilde{\overline{u}}}\}$ and fix $\mathsf{x}, \mathsf{y}, \mathsf{z} \in \mathsf{F}(\widehat{\widetilde{\overline{u}}})$ by demanding

$$\mathsf{x}(v) = \widetilde{u}, \qquad \mathsf{y}(v) = \widehat{u}, \qquad \mathsf{z}(v) = \overline{u}. \tag{4.48}$$

The following relations can then be verified (in the same way as in the proof of theorem 19)

$$\begin{aligned} \pi_v^{\widehat{\widetilde{\overline{u}}}}(\mathsf{g}_*^{-1} \cdot \mathsf{f}_*) &= \mathsf{z}^{-1} \cdot \mathsf{g}_*^{-1} \cdot \mathsf{f}_* \cdot \mathsf{z}, \\ \pi_v^{\widehat{\widetilde{\overline{u}}}}(\mathsf{h}_*^{-1} \cdot \mathsf{g}_*) &= \mathsf{x}^{-1} \cdot \mathsf{h}_*^{-1} \cdot \mathsf{g}_* \cdot \mathsf{x}, \\ \pi_v^{\widehat{\widetilde{\overline{u}}}}(\mathsf{f}_*^{-1} \cdot \mathsf{h}_*) &= \mathsf{y}^{-1} \cdot \mathsf{f}_*^{-1} \cdot \mathsf{h}_* \cdot \mathsf{y}. \end{aligned} \tag{4.49}$$

Now if we substitute (4.49) into the identity

$$[\mathsf{f}_* \cdot \pi_v^{\widehat{\widetilde{\overline{u}}}}(\mathsf{f}_*^{-1} \cdot \mathsf{h}_*) \cdot \mathsf{h}_*^{-1}] \cdot [\mathsf{h}_* \cdot \pi_v^{\widehat{\widetilde{\overline{u}}}}(\mathsf{h}_*^{-1} \cdot \mathsf{g}_*) \cdot \mathsf{g}_*^{-1}] \cdot [\mathsf{g}_* \cdot \pi_v^{\widehat{\widetilde{\overline{u}}}}(\mathsf{g}_*^{-1} \cdot \mathsf{f}_*) \cdot \mathsf{f}_*^{-1}] = \mathsf{e}, \tag{4.50}$$

we find

$$[\mathsf{f}_*\cdot\mathsf{y}^{-1}\cdot\mathsf{f}_*^{-1}]\cdot[\mathsf{h}_*\cdot\mathsf{y}\cdot\mathsf{h}_*^{-1}]\cdot[\mathsf{h}_*\cdot\mathsf{x}^{-1}\cdot\mathsf{h}_*^{-1}]\cdot[\mathsf{g}_*\cdot\mathsf{x}\cdot\mathsf{g}_*^{-1}]\cdot[\mathsf{g}_*\cdot\mathsf{z}^{-1}\cdot\mathsf{g}_*^{-1}]\cdot[\mathsf{f}_*\cdot\mathsf{z}\cdot\mathsf{f}_*^{-1}] = \mathsf{e}. \tag{4.51}$$

The bracketed terms all lie in $\mathsf{F}(\overline{\widehat{\widetilde{u}}})$ and application of the normal-form isomorphism $\beta_{\overline{\widehat{\widetilde{u}}}}$ to this expression, followed by the substitutions

$$\begin{aligned} \beta_{\overline{\widehat{\widetilde{u}}}}(\mathsf{x}) &= b_0\chi(\widetilde{u},\overline{\widehat{\widetilde{u}}},v), & \alpha_{\overline{\widehat{\widetilde{u}}}}(\mathsf{f}_*) &= p, \\ \beta_{\overline{\widehat{\widetilde{u}}}}(\mathsf{y}) &= b_0\chi(\widehat{u},\overline{\widehat{\widetilde{u}}},v), & \alpha_{\overline{\widehat{\widetilde{u}}}}(\mathsf{g}_*) &= q, \\ \beta_{\overline{\widehat{\widetilde{u}}}}(\mathsf{z}) &= b_0\chi(\overline{u},\overline{\widehat{\widetilde{u}}},v), & \alpha_{\overline{\widehat{\widetilde{u}}}}(\mathsf{h}_*) &= r. \end{aligned} \tag{4.52}$$

which result from (4.48) and (4.47) using (3.19) and (4.9), leads to the desired relation

$$\frac{r-p}{\widehat{u}-\overline{\widehat{\widetilde{u}}}} + \frac{q-r}{\widetilde{u}-\overline{\widehat{\widetilde{u}}}} + \frac{p-q}{\overline{u}-\overline{\widehat{\widetilde{u}}}} = 0. \tag{4.53}$$

Rearranging for $\overline{\widehat{\widetilde{u}}}$ it can be seen that this relation determines $\overline{\widehat{\widetilde{u}}}$ uniquely in terms of the initial data $\{\widetilde{u},\widehat{u},\overline{u}\}$. Note there is no dependence on u, which is the *tetrahedron property* referred to in [4].

4.7 Conclusion

We have developed a bit further the basic results on Möbius transformations started in chapter 2. This has led to an extension of the normal-form introduced in section 3.3 making it suitable for the study of the stabilizer subgroups of M parameterised in a particular way. Composition then gives rise to a quadrilateral expression which (up to a point-transformation of the lattice parameters) we identify as the lattice Schwarzian KdV equation introduced in section 1.2.

The consistency on the cube of this equation leads us to a result which in turn connects to the lattice equation described in chapter 3. Specifically, we deduce from the embedding in three dimensions described here and from theorem 21, that for the lattice Schwarzian KdV equation (for some p, q and r) to be satisfied on each face of a cube (as in (1.12), see figure 1.1) it is necessary and sufficent that there exists a Möbius involution $\mathsf{i} \in \mathsf{M}$ for which

$$\mathsf{i}(\widetilde{u}) = \overline{\widehat{u}}, \quad \mathsf{i}(\widehat{u}) = \widetilde{\overline{u}}, \quad \mathsf{i}(\overline{u}) = \widehat{\widetilde{u}}, \quad \mathsf{i}(u) = \widehat{\widetilde{\overline{u}}}. \tag{4.54}$$

Chapter 5

The lattice Schwarzian KP equation

The three dimensional embeddings of the quadrilateral lattice equations introduced in chapters 3 and 4 have been connected, perhaps unexpectedly, through theorems 14 and 21. The connecting device is the existence of a Möbius involution $\mathsf{i} \in \mathsf{M}$ satisfying

$$\mathsf{i}(\widetilde{u}) = \overline{\widehat{u}}, \quad \mathsf{i}(\widehat{u}) = \overline{\widetilde{u}}, \quad \mathsf{i}(\overline{u}) = \widehat{\widetilde{u}}. \tag{5.1}$$

According to theorem 6 the necessary and sufficient condition for the existence of this involution can be written

$$\begin{vmatrix} \widetilde{u}\overline{\widehat{u}} & \widetilde{u} + \overline{\widehat{u}} & 1 \\ \widehat{u}\overline{\widetilde{u}} & \widehat{u} + \overline{\widetilde{u}} & 1 \\ \overline{u}\widehat{\widetilde{u}} & \overline{u} + \widehat{\widetilde{u}} & 1 \end{vmatrix} = 0. \tag{5.2}$$

Considered in its own right as an equation on a six-point stencil in the three dimensional lattice this is known as the lattice Schwarzian KP equation. This equation was given originally in [51] it is more commonly written in the form

$$(\widetilde{u} - \widehat{\widetilde{u}})(\widehat{u} - \overline{\widehat{u}})(\overline{u} - \overline{\widetilde{u}}) = (\widetilde{u} - \overline{\widetilde{u}})(\overline{u} - \overline{\widehat{u}})(\widehat{u} - \overline{\widehat{u}}), \tag{5.3}$$

as opposed to the form (5.2) used here. The equation appears as the superposition principle for solutions of the Schwarzian KP equation related by its BT (cf. also [36]). One of the well documented properties of (5.2) is its Möbius invariance, that is, given any $\mathsf{m} \in \mathsf{M}$,

$$\begin{vmatrix} \widetilde{u}\overline{\widehat{u}} & \widetilde{u}+\overline{\widehat{u}} & 1 \\ \widehat{u}\widetilde{\overline{u}} & \widehat{u}+\widetilde{\overline{u}} & 1 \\ \overline{u}\widehat{\widetilde{u}} & \overline{u}+\widehat{\widetilde{u}} & 1 \end{vmatrix} = 0 \quad \Leftrightarrow \quad \begin{vmatrix} \mathsf{m}(\widetilde{u})\mathsf{m}(\overline{\widehat{u}}) & \mathsf{m}(\widetilde{u})+\mathsf{m}(\overline{\widehat{u}}) & 1 \\ \mathsf{m}(\widehat{u})\mathsf{m}(\widetilde{\overline{u}}) & \mathsf{m}(\widehat{u})+\mathsf{m}(\widetilde{\overline{u}}) & 1 \\ \mathsf{m}(\overline{u})\mathsf{m}(\widehat{\widetilde{u}}) & \mathsf{m}(\overline{u})+\mathsf{m}(\widehat{\widetilde{u}}) & 1 \end{vmatrix} = 0. \quad (5.4)$$

Here we see that this is just equivalent to the observation that, given any $\mathsf{m} \in \mathsf{M}$, $\mathsf{m} \cdot \mathsf{i} \cdot \mathsf{m}^{-1}$ is an involution whenever i is.

In this chapter we will describe the integrability property of (5.2) and establish a natural characterisation of the reduction to the lattice systems described in chapters 3 and 4.

5.1 Integrability

The equation (5.2) can be embedded consistently in a four-dimensional lattice. The consistency can be checked by posing an IVP on the four-dimensional hypercube, however the IVP is not unique and a description of them all is quite tedious and does not provide much insight.

However, this consistency is equivalent to the notion of the natural auto-BT for the equation, which is somewhat easier to describe and can be understood more intuitively. Basically, rather than describing the four-dimensional system, we consider only one increment in the fourth direction (the direction of the BT). Moreover, instead of using another accent for that shift, we will introduce a new variable, so we consider two variables $u = u(n, m, l)$ and $v = v(n, m, l)$ each on a three-dimensional lattice. This gives rise to three equations which involve both u and v,

$$\begin{vmatrix} \widetilde{u}\widehat{v} & \widetilde{u}+\widehat{v} & 1 \\ \widehat{u}\widetilde{v} & \widehat{u}+\widetilde{v} & 1 \\ \widehat{\widetilde{u}}v & \widehat{\widetilde{u}}+v & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} \widehat{u}\overline{v} & \widehat{u}+\overline{v} & 1 \\ \overline{u}\widehat{v} & \overline{u}+\widehat{v} & 1 \\ \overline{\widehat{u}}v & \overline{\widehat{u}}+v & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} \overline{u}\widetilde{v} & \overline{u}+\widetilde{v} & 1 \\ \widetilde{u}\overline{v} & \widetilde{u}+\overline{v} & 1 \\ \widetilde{\overline{u}}v & \widetilde{\overline{u}}+v & 1 \end{vmatrix} = 0. \quad (5.5)$$

When u and v satisfy (5.5) on the lattice we will write

$$u \sim v \quad (5.6)$$

and say that u and v are related by the BT (5.5). We claim that, given a solution u of (5.2) the system (5.5) can be solved for v which in turn satisfies (5.2). Even though the system (5.5) is not symmetric in u and v this also works in the other direction. That is, given a solution v of (5.2), the system (5.5) can be solved for u which then satisfies (5.2). Note however that the relation (5.6) is not symmetric.

In what follows we will connect the BT (5.5) with a configuration of Möbius transformations. This will enable us to demonstrate that, if (5.5) holds on a particular lattice cube, then both u and v necessarily satisfy (5.2) on that cube. If $u = v$ on every lattice site then (5.5) is satisfied trivially without any constraint on u, we will exclude this trivial case by making the assumption that $u \neq v$.

First observe from theorem 6 that the BT relations (5.5) imply the existence of Möbius involutions $\mathsf{j}_1, \mathsf{j}_2, \mathsf{j}_3 \in \mathsf{M}$ for which

$$\begin{aligned} &\mathsf{j}_3(\widetilde{u}) = \widehat{v}, &\quad &\mathsf{j}_1(\widehat{u}) = \overline{v}, &\quad &\mathsf{j}_2(\overline{u}) = \widetilde{v},\\ &\mathsf{j}_3(\widehat{u}) = \widetilde{v}, &&\mathsf{j}_1(\overline{u}) = \widehat{v}, &&\mathsf{j}_2(\widetilde{u}) = \overline{v},\\ &\mathsf{j}_3(v) = \widehat{\widetilde{u}}, &&\mathsf{j}_1(v) = \overline{\widehat{u}}, &&\mathsf{j}_2(v) = \widetilde{\overline{u}}. \end{aligned} \tag{5.7}$$

By theorem 1 the relations

$$\mathsf{x}(\widetilde{u}) = \widetilde{v}, \qquad \mathsf{x}(\widehat{u}) = \widehat{v}, \qquad \mathsf{x}(\overline{u}) = \overline{v}, \tag{5.8}$$

fix a unique $\mathsf{x} \in \mathsf{M}$. We then define the point $u_* = \mathsf{x}^{-1}(v)$, and choose $\mathsf{l}, \mathsf{m}, \mathsf{n} \in \mathsf{M}$ to be the unique (by theorem 3) Möbius transformations which share their fixed-points with x and satisfy

$$\mathsf{l}(u_*) = \widetilde{u}, \qquad \mathsf{m}(u_*) = \widehat{u}, \qquad \mathsf{n}(u_*) = \overline{u}. \tag{5.9}$$

Importantly, theorem 10 shows that the fixed-points of x, and thus of l, m and n, are transposed by any of $\mathsf{j}_1, \mathsf{j}_2$ or j_3. By theorem 9 this implies, for example, that $\mathsf{l} \cdot \mathsf{j}_3 = \mathsf{j}_3 \cdot \mathsf{l}^{-1}$. Using also the commutativity of l and x, so that $\mathsf{l}(v) = \mathsf{l}(\mathsf{x}(u_*)) = \mathsf{x}(\mathsf{l}(u_*)) = \mathsf{x}(\widetilde{u}) = \widetilde{v}$, we observe that $\mathsf{l}(\widehat{u}) = \mathsf{l}(\mathsf{j}_3(\widetilde{v})) = \mathsf{j}_3(\mathsf{l}^{-1}(\widetilde{v})) = \mathsf{j}_3(v) = \widehat{\widetilde{u}}$. In fact similar reasoning reveals that all of the following hold

$$\begin{aligned} &\mathsf{l}(\widehat{u}) = \widehat{\widetilde{u}}, \qquad &&\mathsf{m}(\overline{u}) = \overline{\widehat{u}}, \qquad &&\mathsf{n}(\widetilde{u}) = \widetilde{\overline{u}}, \\ &\mathsf{l}(\overline{u}) = \widetilde{\overline{u}}, \qquad &&\mathsf{m}(\widetilde{u}) = \widehat{\widetilde{u}}, \qquad &&\mathsf{n}(\widehat{u}) = \overline{\widehat{u}}. \end{aligned} \tag{5.10}$$

So, from the condition (5.6) we have deduced the existance of $\mathsf{l}, \mathsf{m}, \mathsf{n} \in \mathsf{M}$ which share their fixed-points and satisfy (5.9) and (5.10). We therefore conclude from theorem 14 that there exists an involution $\mathsf{i} \in \mathsf{M}$ satisfying (5.1), in other words that (5.2) holds.

Now observe that the BT relations (5.5) also imply the existance of Möbius involutions $\widetilde{\mathsf{j}}_1, \widehat{\mathsf{j}}_2$ and $\bar{\mathsf{j}}_3$ for which

$$\begin{aligned} &\bar{\mathsf{j}}_3(\widetilde{\overline{u}}) = \overline{\widehat{v}}, \qquad &&\widetilde{\mathsf{j}}_1(\widehat{\widetilde{u}}) = \widetilde{\overline{v}}, \qquad &&\widehat{\mathsf{j}}_2(\overline{\widehat{u}}) = \widehat{\widetilde{v}}, \\ &\bar{\mathsf{j}}_3(\overline{\widehat{u}}) = \widetilde{\overline{v}}, \qquad &&\widetilde{\mathsf{j}}_1(\widetilde{\overline{u}}) = \widehat{\widetilde{v}}, \qquad &&\widehat{\mathsf{j}}_2(\widehat{\widetilde{u}}) = \overline{\widehat{v}}, \\ &\bar{\mathsf{j}}_3(\overline{v}) = \widehat{\widetilde{\overline{u}}}, \qquad &&\widetilde{\mathsf{j}}_1(\widetilde{v}) = \widehat{\widetilde{\overline{u}}}, \qquad &&\widehat{\mathsf{j}}_2(\widehat{v}) = \widehat{\widetilde{\overline{u}}}. \end{aligned} \tag{5.11}$$

Defining $\mathsf{x}_* \in \mathsf{M}$ by the relations

$$\mathsf{x}_*(\widehat{\widetilde{u}}) = \widehat{\widetilde{v}}, \qquad \mathsf{x}_*(\overline{\widehat{u}}) = \overline{\widehat{v}}, \qquad \mathsf{x}_*(\widetilde{\overline{u}}) = \widetilde{\overline{v}}, \tag{5.12}$$

and the new point $\widehat{\widetilde{\overline{v}}}_* = x_*(\widehat{\widetilde{\overline{u}}})$, we can then choose $\mathsf{l}_*, \mathsf{m}_*$ and n_* to share fixed-points with x_* and satisfy

$$\mathsf{l}_*(\overline{\widehat{v}}) = \widehat{\widetilde{\overline{v}}}_*, \qquad \mathsf{m}_*(\widetilde{\overline{v}}) = \widehat{\widetilde{\overline{v}}}_*, \qquad \mathsf{n}_*(\widehat{\widetilde{v}}) = \widehat{\widetilde{\overline{v}}}_*. \tag{5.13}$$

Using (5.11) it can then be verified that all of the following also hold

$$\begin{aligned} &\mathsf{l}_*(\widehat{v}) = \widehat{\widetilde{v}}, \qquad &&\mathsf{m}_*(\overline{v}) = \overline{\widehat{v}}, \qquad &&\mathsf{n}_*(\widetilde{v}) = \widetilde{\overline{v}}, \\ &\mathsf{l}_*(\overline{v}) = \widetilde{\overline{v}}, \qquad &&\mathsf{m}_*(\widetilde{v}) = \widehat{\widetilde{v}}, \qquad &&\mathsf{n}_*(\widehat{v}) = \overline{\widehat{v}}. \end{aligned} \tag{5.14}$$

The relations (5.13) and (5.14) imply, by theorem 14, that there exists an involution $\mathsf{i}_* \in \mathsf{M}$ for which

$$\mathsf{i}_*(\widetilde{v}) = \overline{\widehat{v}}, \quad \mathsf{i}_*(\widehat{v}) = \widetilde{\overline{v}}, \quad \mathsf{i}_*(\overline{v}) = \widehat{\widetilde{v}}. \tag{5.15}$$

By theorem 6 it then follows that (5.2) is also satisfied by v.

So we have shown that if (5.5) holds on a lattice cube, and $v \neq u$, then both u and v satisfy (5.2). The converse statment, that (5.2) is sufficient for the compatibility of (5.5) in v requires that we specify a particular IVP. For a concrete example, consider (5.2) on the domain $\mathbb{N}^3$. In this case fixing a solution u of (5.2) and giving the initial data $v(0,0,n), n \in \mathbb{N}$, the system (5.5) can be solved for v on $\mathbb{N}^3$. On the other hand, fixing a solution v of (5.2) on $\mathbb{N}^3$, then appropriate initial data on the u lattice is $u(n,0,0), u(0,n,0), u(0,0,n), n \in \mathbb{N}$, whence (5.5) can be solved for u on $\mathbb{N}^3$.

Note that iterating the BT establishes a solution of an embedded system in four dimensions, so in this sense the BT as described implies the four dimensional consistency.

5.2 Solutions associated with 2-cycles of the BT

We have already observed that the BT relations (5.5) collapse in the case $u = v$, so in a trivial sense every solution of (5.2) is related to itself by the BT. Here we ask if there exist solutions of (5.2) which return to themselves after two applications of the BT. Specifically, we will look for solutions u for which there exists $v \neq u$ such that

$$u \sim v \qquad \text{and} \qquad v \sim u. \tag{5.16}$$

We will refer to such solutions as 2-cycles of the BT. Clearly if u is a 2-cycle of the BT, then the associated solution v is also a 2-cycle. It turns out that u is a 2-cycle of the BT for (5.2) if and only if, on every lattice cube, there exists an involution $\mathsf{i} \in \mathsf{M}$ for which

$$\mathsf{i}(\widetilde{u}) = \overline{\widehat{u}}, \quad \mathsf{i}(\widehat{u}) = \overline{\widetilde{u}}, \quad \mathsf{i}(\overline{u}) = \widehat{\widetilde{u}}, \quad \mathsf{i}(u) = \overline{\widehat{\widetilde{u}}}. \tag{5.17}$$

This is exactly the condition, discussed in chapters 3 and 4 for the existance of the system of embedded quadrilateral equations in each respectively.

To prove that (5.16) implies the existance of the involution satisfying (5.17) we apply a further the construction of the previous section.

Suppose $\mathsf{x}', \mathsf{l}', \mathsf{m}'$ and n' satisfy the same relations as $\mathsf{x}, \mathsf{l}, \mathsf{m}$ and n as before (5.8), (5.9) and (5.10) but with $u \leftrightarrow v$. Now $\mathsf{l}' \cot \mathsf{l}^{-1} = \mathsf{m}' \cdot \mathsf{m}^{-1} = \mathsf{n}' \cdot \mathsf{n}^{-1}$ because they all send u_* to u as well as share their fixed points (cf. theorem 3).

Then

$$\begin{aligned} \mathsf{x}(\widehat{u}) = \widehat{v}) \Rightarrow & [\mathsf{x} \cdot \mathsf{l}' \cdot \mathsf{l}^{-1}](\widehat{\widetilde{u}}) = \widehat{\widetilde{v}}, \\ & [\mathsf{x} \cdot \mathsf{m}' \cdot \mathsf{m}^{-1}](\overline{\widehat{u}}) = \overline{\widehat{v}}, \\ & [\mathsf{x} \cdot \mathsf{n}' \cdot \mathsf{n}^{-1}](\widetilde{\overline{u}}) = \widetilde{\overline{v}}, \end{aligned} \tag{5.18}$$

because $\mathsf{l}'(\widehat{v}) = \widehat{\widetilde{v}}$ and $l(\widehat{u}) = \widehat{\widetilde{u}}$ etc. Therefore we may also conclude that $\mathsf{l}' \cot \mathsf{l}^{-1} = \mathsf{m}' \cdot \mathsf{m}^{-1} = \mathsf{n}' \cdot \mathsf{n}^{-1} = \mathsf{x}^{-1} \cdot \mathsf{x}_*$ because x_* satisfies (5.12). This means that x_* shares fixed-points with x, so therefore all of $\mathsf{x}_*, \mathsf{l}_*, \mathsf{n}_*$ share fixed-points with all of $\mathsf{x}, \mathsf{l}, \mathsf{m}, \mathsf{n}$. In thurn this means that $\mathsf{l}_* = \mathsf{l}'$, $\mathsf{m}_* = \mathsf{m}'$, $\mathsf{n}_* = \mathsf{n}'$ which follows by considering for example that $\mathsf{l}'(\widehat{v}) = \widehat{\widetilde{v}}$.

Now, according to the previous section, (5.16) implies there exists i satisfying (5.1), so i the fixed-points of all these Möbius transformations are transposed by i, and so by theorem 9 $\mathsf{i} \cdot \mathsf{l} = \mathsf{l}^{-1} \cdot \mathsf{i}$. As a result we observe that

$$\begin{aligned} & \mathsf{l}_*(u) = \widetilde{u} \\ \Rightarrow & \mathsf{i}(\mathsf{l}_*(u)) = \mathsf{i}(\widetilde{u}) \\ \Rightarrow & \mathsf{l}_*^{-1}(\mathsf{i}(u)) = \overline{\widehat{u}} \\ \Rightarrow & \mathsf{i}(u) = \mathsf{l}_*(\overline{\widehat{u}}) = \overline{\widehat{\widetilde{u}}}. \end{aligned} \tag{5.19}$$

Of course by the $u \leftrightarrow v$ symmetry of (5.16) we conclude that i_* which satisfies (5.15) also satisfies $\mathsf{i}_*(v) = \overline{\widehat{\widetilde{v}}}$.

5.3 Conclusion

The natural reduction from the lattice Schwarzian KP equation to the lattice Schwarzian KdV equation is demonstrated to be exactly the reduction to solutions of the Schwarzian KP equation which are 2-cycles of its BT. It is natural to conjecture that higher-rank systems which are consistent on the cube (e.g. Schwarzian Boussinesq lattice systems [53]) arise as N-cycles of the BT, i.e., solutions for which

$$u_1 \sim u_2 \quad , \quad u_2 \sim u_3, \quad u_3 \sim u_4, \quad \ldots, \quad u_{N-1} \sim u_N. \tag{5.20}$$

Chapter 6

Scalar lattice equations which are consistent on the cube

There exist other integrable lattice equations which are as yet un-connected to the story of the previous chapters. In the present chapter, which can be read independantly of what has gone before, we will encounter these other integrable lattice equations. In particular we introduce Adler's lattice equation and give a brief summary of the classification results due to Adler Bobenko and Suris (cf. section 1.1).

6.1 Adler's lattice equation

Here we will introduce Adler's lattice equation as the superposition principle for BTs of the Krichever-Novikov (KN) PDE. From the start Adler's equation is related naturally to a biquadratic correspondence [16, 86], a polynomial of degree two in two variables which can be treated as a two-valued mapping. Later this object will become of great relevance for both the classification and construction of solutions for the lattice equations.

The KN equation [37, 38] (cf. also [49, 31]) may be written in the form

$$\begin{aligned} KN(u) :=& \frac{u_y}{u_x} - \frac{u_{xxx}}{u_x} + \frac{3}{2u_x^2}\left(u_{xx}^2 - \mathcal{R}(u)\right) = 0, \\ \mathcal{R}(u) &= a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4, \end{aligned} \tag{6.1}$$

where $a_0 \dots a_4 \in \mathbb{C}$ are constant parameters of the equation. In the case $\mathcal{R}(u) = 0$ (6.1) reduces to the Schwarzian KdV equation discussed extensively in section 1.2, the effect of the Möbius change of variables $u \to \mathsf{m}(u)$ here is to alter $\mathcal{R}$, $a_0 \dots a_4 \to \widetilde{a}_0 \dots \widetilde{a}_4$, without changing the number or type of its roots. The BT for (6.1) was discovered by Adler in [3], it can be written as the system

$$u_x \widetilde{u}_x = h(u, \widetilde{u}), \qquad KN(u) + KN(\widetilde{u}) = 0 \tag{6.2}$$

where $h = h(u, \widetilde{u})$ is a symmetric polynomial of degree two or less in two variables (henceforth symmetric *biquadratic*, note the symmetry was not required in [3] but will be a convenient and inconsequential assumption here) which satisfies

$$h_u^2 - 2hh_{uu} = \mathcal{R}(\widetilde{u}). \tag{6.3}$$

To proceed we need to specify $\mathcal{R}$ and construct h satisfying (6.3). The symmetric biquadratic h has six coefficents and $\mathcal{R}$ contains five, so we might expect to find a one-parameter family of such biquadratics. Indeed this turns out to be true, but when $\mathcal{R}$ has less than four simple roots this family is in general not unique, in these cases there arise functionally distinct BTs for 6.1. (ABS [6] have in fact devised a notion of degeneracy for the biquadratic h, descirbed later in section 6.3 which when imposed with (6.3) recovers the uniqueness of this family for any $\mathcal{R}$.) Here we will focus on the case when $\mathcal{R}$ has four simple roots (the elliptic case). We may take $\mathcal{R}$ to be in the form

$$\mathcal{R}(u) = 1 - (k + 1/k)u^2 + u^4 \tag{6.4}$$

where $k \in \mathbb{C} \setminus \{0, 1, -1\}$ is arbitrary, which up to a Möbius transformation on u in (6.1) is without loss of generality. We also make the mild assumption that

$$h(0,p) = 0 \tag{6.5}$$

for some p. It turns out (6.5) taken with (6.3) is enough to reconstruct h with no restriction on p

$$h(u,\widetilde{u}) = \mathcal{H}_{\mathfrak{p}}(u,\widetilde{u}) := \frac{1}{2p}\left(u^2 + \widetilde{u}^2 - (1+u^2\widetilde{u}^2)p^2 - 2u\widetilde{u}P\right), \tag{6.6}$$

where we have introduced $\mathfrak{p} = (p,P) \in \Gamma$,

$$\Gamma = \left\{(x,X) \mid X^2 = \mathcal{R}(x)\right\}. \tag{6.7}$$

The reconstruction is unique up to the sign of P and the overall sign of the biquadratic. The point $\mathfrak{p} \in \Gamma$, which can be chosen freely, is the Bäcklund parameter. So to sumarise, the KN equation (6.1) with $\mathcal{R}$ in the form (6.4) admits the BT (6.2) where the biqudratic h is given in (6.6). The superposition principle is as follows

$$\begin{gathered}\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) := \\ p(u\widetilde{u} + \widehat{u}\widehat{\widetilde{u}}) - q(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}) - \frac{pQ - qP}{1-p^2q^2}(u\widehat{\widetilde{u}} + \widetilde{u}\widehat{u} - pq(1 + u\widetilde{u}\widehat{u}\widehat{\widetilde{u}})) = 0,\end{gathered} \tag{6.8}$$

which can be verified explicitly. The factorisation of the necessary condition for commutativity of these BTs is not obvious (unlike in the Schwarzian case (1.10)), but it does still occur,

$$\begin{gathered}\mathcal{H}_{\mathfrak{p}}(u,\widetilde{u})\mathcal{H}_{\mathfrak{p}}(\widehat{u},\widehat{\widetilde{u}}) - \mathcal{H}_{\mathfrak{q}}(u,\widehat{u})\mathcal{H}_{\mathfrak{q}}(\widetilde{u},\widehat{\widetilde{u}}) = \\ \frac{1-p^2q^2}{4p^2q^2}\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}})\mathcal{Q}_{\mathfrak{p},\mathfrak{q}'}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}).\end{gathered} \tag{6.9}$$

where $\mathfrak{q}' = (-q,Q) \in \Gamma$. Equation (6.8), which we will refer to as the *Jacobi form* of Adler's equation, was first given by Hietarinta [27], it is equivalent (by a change of variables) to the *Weierstrass form* given originally by Adler [3] and improved by Nijhoff [61].

The continuum limit of Alder's equation and its BT (the BT for (6.8) arises naturally from its consistency on the cube in the way described in section 1.2) recovers the KN equation (6.1) and its BT (6.2), the situation is just as described in section (1.2) and in fact the required analytic differenc operator is the same,

$$C_p = e^{\sqrt{2p}(\partial_x + \frac{p}{6}\partial_y)}. \tag{6.10}$$

6.2 The degenerate cases of Adler's equation

Adler's equation was included in the list of multidimensionally consistent equations given later by Adler, Bobenko and Suris (ABS) in [4] (where it was denoted $Q4$). Here we reproduce the remaining equations in that list:

$$\begin{array}{lll}
Q3^\delta & : & \begin{array}{l}(p-\frac{1}{p})(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-(q-\frac{1}{q})(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})= \\ \qquad (\frac{p}{q}-\frac{q}{p})(\widetilde{u}\widehat{u}+u\widehat{\widetilde{u}}+\frac{\delta^2}{4}(p-\frac{1}{p})(q-\frac{1}{q})),\end{array} \\
Q2 & : & \begin{array}{l}p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})= \\ \qquad pq(q-p)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}}-p^2+pq-q^2),\end{array} \\
Q1^\delta & : & p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})=\delta^2 pq(q-p), \\
A2 & : & \begin{array}{l}(p-\frac{1}{p})(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-(q-\frac{1}{q})(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})= \\ \qquad (\frac{p}{q}-\frac{q}{p})(1+u\widetilde{u}\widehat{u}\widehat{\widetilde{u}}),\end{array} \\
A1^\delta & : & p(u+\widehat{u})(\widetilde{u}+\widehat{\widetilde{u}})-q(u+\widetilde{u})(\widehat{u}+\widehat{\widetilde{u}})=\delta^2 pq(p-q), \\
H3^\delta & : & p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})=\delta(q^2-p^2), \\
H2 & : & (u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})=(p-q)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}}+p+q), \\
H1 & : & (u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})=p-q,
\end{array} \tag{6.11}$$

in each case where it appears δ is a constant parameter of the equation which, by a scaling of u, can be taken as 0 or 1. Equations are listed up to a Möbius transformation of the variable and point-transformation of the lattice parameters. The equations $A1^\delta$ and $A2$ are gauge-related to $Q1^\delta$ and $Q3^0$. Let us give an historical remark (cf. section 1.1), the equations $H1, H3^0, Q1$ and $Q3^0$ and are parameter sub-cases of the NQC equation [50] given in 1983 (in fact NQC is *equivalent* to $Q3^0$ up to a gauge

Eq	u	k	p	P
$Q3^\delta$	$\frac{2i\epsilon}{\delta}u$	$-4\epsilon^2$	$\epsilon(p-1/p)$	$\frac{1}{2}(p+1/p)+O(\epsilon^4)$
$Q2$	$\frac{1}{\epsilon}+\frac{\epsilon}{2}u$	ϵ^2	$\epsilon^2 p$	$1-\frac{\epsilon^2}{2}p^2-\frac{\epsilon^4}{8}p^4+O(\epsilon^6)$
$Q1^\delta$	$\frac{\epsilon}{\delta}u$	k	ϵp	$1+O(\epsilon^2)$
$A2$	u	$-4\epsilon^2$	$\frac{1}{\epsilon}(p-1/p)^{-1}$	$\frac{-1}{2\epsilon^2}(p+1/p)(p-1/p)^{-2}+O(\epsilon^2)$
$A1^\delta$	$\frac{\epsilon}{\delta}u$	k	ϵp	$-1+O(\epsilon^2)$
$H3^\delta$	$1+\frac{\epsilon}{\sqrt{-\delta}}u$	1	$1-\frac{\epsilon^2}{2}p$	$-\epsilon^2 p+O(\epsilon^4)$
$H2$	$\frac{1}{\epsilon}+\epsilon-\frac{\epsilon}{2}u$	$-4\epsilon^4$	$1-\frac{\epsilon^2}{2}p$	$\frac{-1}{2\epsilon^2}+\frac{1}{4}p-2\epsilon^2+\epsilon^4 p+O(\epsilon^{10})$
$H1$	$1+\epsilon u$	k	$1-\frac{\epsilon^2}{2}p$	$\frac{k-1}{\sqrt{-k}}-\epsilon^2\frac{k-1}{2\sqrt{-k}}p+O(\epsilon^4)$

Table 6.1: Substitutions which lead to the indicated degenerate sub case (Eq) of Adler's equation (6.8) in the limit $\epsilon \to 0$. Choose $\delta = \epsilon$ rather than 0 to arrive at Eq with $\delta = 0$.

transformation), $H3^0$ was considered by Hirota [29] in 1977, $H1$ was the superposition principle given by Whalquist and Estabrook [83] in 1973.

It was shown in [4] that all of the equations in the list (6.11) except $H1$ and $H3^0$ appear as the superposition principle for BTs of the KN equation (6.1) (in cases where $\mathcal{R}$ does not have four simple roots) constructed from a biquadratic, h, according to the method of Adler which we described in section 6.1. However all the equations in the list (6.11) (without exception) are degenerate sub-cases of the equation (6.8). Table 6.1 contains the details of these degenerations (some of which are new). To clarify the meaning of the entries in this table we include an example here. Let us make the substitutions

$$u \to \epsilon u, \quad p \to \epsilon p, \quad q \to \epsilon q \tag{6.12}$$

in (6.8) and consider the leading term in the small-ϵ expansion of the resulting expression. For this calculation it is necessary to write the parameters P and Q as a series in ϵ,

$$P = \pm(1-\epsilon^2\frac{1}{2}(k+1/k)p^2+\ldots), \quad Q = \pm(1-\epsilon^2\frac{1}{2}(k+1/k)q^2+\ldots), \tag{6.13}$$

so there is some choice of sign. The rest of the calculation is straightforward and the leading order expression that results is exactly the equation $Q1^1$ or $A1^1$ depending on this choice of sign.

It was pointed out in [4] that one can descend through the lists 'Q', 'A' and 'H' in (6.11) by degeneration from $Q4$, $A2$ and $H3^\delta$ respectively. The degenerations from Adler's equation in Weierstrass form to the equations in the 'Q' list are given explicitly in [5].

6.3 ABS biquadratic non-degeneracy

We have not discussed the extent to which the list (6.11) is exhaustive, which is of course a very important aspect of the ABS result. Suffice it to say that there are several scalar equations which are consistent on the cube, for example the linear equation

$$(\widehat{\widetilde{u}} - u)(p - q) + (\widetilde{u} - \widehat{u})(p + q) = 0, \tag{6.14}$$

which are not included in this list. What we will describe instead, is the result in [6] which superseeded [4], although interestingly the list got shorter in some sense.

A definition of central importance is a notion of non-degeneracy for a biquadratic h (a polynomial of degree at most two in two variables, which here may or may not be symmetric). The biquadratic h is said to be *non-degenerate* if it is of degree two in each variable and either irriducible, or of the form

$$\begin{aligned} h(u, \widetilde{u}) = (c\widetilde{u}u + d\widetilde{u} - au - b)(c_*\widetilde{u}u + d_*\widetilde{u} - a_*u - b_*),& \\ ad \neq bc, \quad a_*d_* \neq b_*c_*.& \end{aligned} \tag{6.15}$$

This is the notion of non-degenaracy which yields uniqueness of the biquadratic family with shared discriminant which we mentioned in section 6.1.

Now, the most general notion of (scalar) consistency on the cube involves arbitrary polynomials of degree one in four variables associated to each face. Such a polynomial takes the general form

$$\begin{aligned}A_0 + A_1 u + A_2\widetilde{u} + A_3\widehat{u} + A_4\widehat{\widetilde{u}} + A_5 u\widetilde{u}\\ +A_6 u\widehat{u} + A_7 u\widehat{\widetilde{u}} + A_8\widetilde{u}\widehat{u} + A_9\widetilde{u}\widehat{\widetilde{u}} + A_{10}\widehat{u}\widehat{\widetilde{u}}\\ +A_{11}u\widetilde{u}\widehat{u} + A_{12}u\widetilde{u}\widehat{\widetilde{u}} + A_{13}u\widehat{u}\widehat{\widetilde{u}} + A_{14}\widetilde{u}\widehat{u}\widehat{\widetilde{u}} + A_{15}u\widetilde{u}\widehat{u}\widehat{\widetilde{u}} = \mathcal{P},\end{aligned} \tag{6.16}$$

where $A_0 \dots A_{15}$ are the *coefficients*. To such $\mathcal{P}$ we may associate four biquadratics,

$$\begin{aligned}&\mathcal{P}_{\widehat{\widetilde{u}}}\mathcal{P}_{\widehat{u}} - \mathcal{P}\mathcal{P}_{\widehat{\widetilde{u}}\widehat{u}} = h_1(u,\widetilde{u}), \qquad &&\mathcal{P}_{\widehat{\widetilde{u}}}\mathcal{P}_{\widetilde{u}} - \mathcal{P}\mathcal{P}_{\widehat{\widetilde{u}}\widetilde{u}} = h_2(u,\widehat{u}),\\ &\mathcal{P}_{\widetilde{u}}\mathcal{P}_{u} - \mathcal{P}\mathcal{P}_{\widetilde{u}u} = h_3(\widehat{u},\widehat{\widetilde{u}}), \qquad &&\mathcal{P}_{\widehat{u}}\mathcal{P}_{u} - \mathcal{P}\mathcal{P}_{\widehat{u}u} = h_4(\widetilde{u},\widehat{\widetilde{u}}),\end{aligned} \tag{6.17}$$

(on the edges of a quadrilateral). Note that $u, \widetilde{u}$ satisfy $h_1(u,\widetilde{u}) = 0$ if and only if the correspondence $\widehat{u} \mapsto \widehat{\widetilde{u}}$ defined by $\mathcal{P} = 0$ is non-invertible (and if $h_1(u,\widetilde{u}) \neq 0$ then this correspondence is a Möbius transformation).

ABS [6] have shown that if $h_1 \dots h_4$ in (6.17) are non-degenerate then, up to Möbius transformations (acting independently on each variable $u, \widetilde{u}, \widehat{u}$ and $\widehat{\widetilde{u}}$), the equation $\mathcal{P} = 0$ coincides with one of the equations $Q1^\delta, Q2, Q3^\delta$ or $Q4^k$ (for some choice of the two lattice parameters and constants δ or k). Such polynomials are said to be of 'type Q'.

They also proved a converse, if a system of equations is consistent on the cube and the polynomial associated to each face is of 'type Q', then up to Möbius transformations (acting independently on each variabe on the cube) the system on the cube is a natural embedding (cf. (1.12) in section 1.2) of one of the equations $Q1^\delta, Q2, Q3^\delta$ or $Q4^k$ (for some choice of the three lattice parameters and constants δ or k).

Let us remark that the biquadratic present in the BT (6.2) from which Adler's equation (6.8) was constructed, is connected to the biquadratics discussed here. Specifically, in the instance that $\mathcal{P} = \mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}})$ defined in (6.8) the associated biquadratics $h_1 \dots h_4$ defined in (6.17) are found to be

$$\begin{aligned}h_1(u,\widetilde{u}) =& h_3(u,\widetilde{u}) = c\mathcal{H}_{\mathfrak{p}}(u,\widetilde{u}),\\ h_2(u,\widehat{u}) =& h_4(u,\widehat{u}) = -c\mathcal{H}_{\mathfrak{q}}(u,\widehat{u}),\end{aligned} \tag{6.18}$$

where the constant $c = 2pq(pQ - qP)/(1 - p^2q^2)$ and $\mathcal{H}$ is the biquadratic defined in (6.6).

6.4 Discussion

The ABS result described (in section 6.3) is decisive for equations of 'type Q', a property which can be algorithmically checked. Systems on the cube where one or more of the faces are *not* of 'type Q' are of course known (for example the 'H' list in (6.11)), and in chapters 7 and 8 we will give some more examples.

Apart from this though, there is one other important issue to consider when we use a polynomial (like $\mathcal{P}$ in 6.16) to define a quadrilateral lattice equation. Effectively, for consistency on a three dimensional lattice, there are additional symmetry constraints over the plain consistency on a cube. For example to construct a BT for an arbitrary 'type Q' polynomial, it should be true that by a (non-autonomous) Möbius change of variables the lattice system can be brought to an embedding of a symmetric (in the sense that opposite faces coincide) system on the cube. (One exception to this is the case of a non-auto-BT where on one pair of opposite faces the equations differ, which we describe in chapter 8.)

Chapter 7

The linear case

The consistency on the cube of a system of quadrilateral expressions can obviously be checked algorithmically. Here we use an ansatz for the quadrilateral expressions and *impose* consistency in order to look for new systems which are consistent on the cube. Given the work of ABS [4] on the classification of such systems (described in chapter 6) this strategy is useful only for the investigation of cases not of 'type Q' (defined in section 6.3) and which are therefore currently outside classification. More restrictive conditions than consistency on the cube have led to classification results for subsets of the 'non-Q' systems [4, 27, 28] (which overlap with 'type Q' systems), here we deal with the general linear equation under a 'lattice parameter' ansatz using a direct (brute force) method.

7.1 The ansatz

Central to the ansatz we will study here is the notion of lattice-parameters, that is, we study a *single* quadrilateral equation with coefficients that depend upon parameters p and q. We will begin by considering the linear homogeneous equation which contains just three essential coefficients,

$$\mathcal{Q}_{pq}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = a(p,q)u + b(p,q)\widetilde{u} + c(p,q)\widehat{u} + \widehat{\widetilde{u}}. \tag{7.1}$$

We then suppose that this quadrilateral expression is consistent on the cube in the sense of (1.12) described in section 1.2.

It turns out that imposing this consistency puts the following single condition on the coefficients,

$$a(p,q) = b(p,q)b(r,p) - b(r,p)c(q,r) + c(q,r)c(p,q). \tag{7.2}$$

Actually, a list of several other conditions also emerge. Remarkably though, they are all a consequence of (7.2) and relations derived from (7.2) by a cyclic permutation of the parameters p, q, r. So, an equation of the form (7.1) is consistent on the cube if and only if the coefficients satisfy the constraint (7.2).

Although from the *integrability* point of view the class of equations (7.1) is perhaps un-interesting (being linear), the constraint itself (7.2) is a good example of the type of functional equation which appears in this approach more generally. Perhaps more importantly, equations of the type (7.1) are degenerate in the ABS sense (not of 'type Q'), so an analysis of this ansatz contributes non-trivially to the full classification of systems which are consistent on the cube.

7.2 A particular solution

To make progress toward solving the constraint equation (7.2) it is useful to understand better what a solution of this equation might look like. This we can illustrate by giving a particular solution. It may be observed (by inspection) that if

$$\begin{aligned} b(p,q) &= f(q), \\ c(p,q) &= g(p), \\ a(p,q) &= f(p)f(q) - f(p)g(q) + g(p)g(q), \end{aligned} \tag{7.3}$$

for arbitrary f and g, the condition (7.2) is satisfied identically. Note that we have not specified the set from which the lattice parameters p, q and

r are taken. Observing that a, b and c depend on p and q only through f and g we may make the identifications

$$\begin{aligned} p &\rightarrow (p_1, p_2) = (f(p), g(p)), \\ q &\rightarrow (q_1, q_2) = (f(q), g(q)), \end{aligned} \tag{7.4}$$

and without loss of generality take $(p_1, p_2), (q_1, q_2) \in \mathbb{C}^2$ to be the (two-component) lattice parameters. From (7.1) we see the resulting lattice equation is

$$(p_2 - p_1)(q_2 - q_1)u + q_1 p_2 u + q_1 \widetilde{u} + p_2 \widehat{u} + \widehat{\widetilde{u}} = 0, \tag{7.5}$$

moreover a calculation shows that $\widehat{\overline{\widetilde{u}}}$ is given by

$$\widehat{\overline{\widetilde{u}}} = q_1 r_2 \widetilde{u} + r_1 p_2 \widehat{u} + p_1 q_2 \overline{u} + (p_1 q_1 r_1 + p_2 q_2 r_2)u. \tag{7.6}$$

the relation (7.6) is invariant under a cyclic permutation of p, q, r and $\widetilde{u}, \widehat{u}, \overline{u}$. Inspecting (1.12) we therefore see that (7.6) is in fact sufficient to verify the consistency because all three equations for $\widehat{\overline{\widetilde{u}}}$ can be generated from any one of them by this permutation.

So we would like to say that (7.3) is a solution of (7.2). The problem then, is to systematically derive *all* solutions of (7.2). The point we would like to draw the readers attention to, about the example solution (7.3), is that the set from which p, q and r are taken remains unspecified. Effectively, the natural number of components for the lattice parameters is a quantity which will be fixed during the solution procedure.

7.3 A solution method

We will begin by supposing that $p, q, r \in \mathrm{S}$ for some unspecified set S, and that a, b and c are functions from $\mathrm{S} \times \mathrm{S}$ to $\mathbb{C}$. The important assumption we make here is that a, b and c are defined throughout the domain. Thus the functional equation (7.2) can be written more precisely as

$$a(p, q) = b(p, q)b(r, p) - b(r, p)c(q, r) + c(q, r)c(p, q), \quad \forall p, q, r \in \mathrm{S}. \tag{7.7}$$

The main property which enables the systematic solution of (7.7) is most easily described if we first rewrite it with the substitutions $(p,q,r) \to (q,s,p)$ and $(p,q,r) \to (s,p,q)$,

$$\begin{aligned} b(p,q) &= \frac{a(q,s) - c(q,s)c(s,p)}{b(q,s) - c(s,p)}, \\ c(p,q) &= \frac{b(q,s)b(s,p) - a(s,p)}{b(q,s) - c(s,p)}, \end{aligned} \tag{7.8}$$

where we have introduced the parameter $s \in \mathrm{S}$ which we suppose to be a particular fixed constant. What we see from the rearrangements (7.8) is that $b(p,q)$ and $c(p,q)$ are rational expressions in functions of a *single* variable (s being a constant parameter).

Now, by writing (7.2) again, but with $(p,q,r) \to (q,s,s)$ and $(p,q,r) \to (s,p,s)$ we see that

$$\begin{aligned} a(q,s) &= b(q,s)b(s,q) - b(s,q)c(s,s) + c(s,s)c(q,s), \\ a(s,p) &= b(s,p)b(s,s) - b(s,s)c(p,s) + c(p,s)c(s,p). \end{aligned} \tag{7.9}$$

Using these relations to substitute for $a(q,s)$ and $a(s,p)$ in (7.8) it is clear that $b(p,q)$ and $c(p,q)$ are rational expressions in just *four* single-variable functions, namely $b(p,s)$, $b(s,p)$, $c(p,s)$ and $c(s,p)$ (here written as functions of p and by convention s is the fixed constant), together with the constants $b(s,s)$ and $c(s,s)$. Of course, using (7.7) we see that $a(p,q)$ is also a rational expression of the same functions and constants.

It turns out that these functions and constants are not all independent, in fact there is one more constraint which connects them. To write down this constraint let us begin by substituting $(p,q,r) \to (s,s,p)$ into (7.7) and rearranging into the form

$$a(s,s) - b(s,s)c(s,s) = (b(s,s) - c(s,p))(b(p,s) - c(s,s)). \tag{7.10}$$

Elimination of the constant $a(s,s) - b(s,s)c(s,s)$ between this relation and itself with the substitution $p \to s$ then yields

$$(b(s,s) - c(s,s))^2 = (b(s,s) - c(s,p))(b(p,s) - c(s,s)). \tag{7.11}$$

This relates $b(p,s)$ and $c(s,p)$, revealing that these functions cannot be chosen independently. However, if we write

$$\begin{aligned} &\frac{b(s,s)-c(s,p)}{b(s,s)-c(s,s)} = \frac{b(s,s)-c(s,s)}{b(p,s)-c(s,s)} = \gamma(p), \\ &b(s,p) = f(p), \\ &c(p,s) = g(p), \end{aligned} \tag{7.12}$$

then the functions f, g and γ *can* be chosen independently. That is, substituting (7.12) back into (7.9), (7.8) and (7.7) we find

$$\begin{aligned} b(p,q) &= \frac{g(q)\gamma(p) - g(q) + f(q)/\gamma(q)}{\gamma(p) - 1 + 1/\gamma(q)}, \\ c(p,q) &= \frac{g(p)\gamma(p) - f(p) + f(p)/\gamma(q)}{\gamma(p) - 1 + 1/\gamma(q)}, \\ a(p,q) &= \frac{g(q)g(p)\gamma(p) - g(q)f(p) + f(p)f(q)/\gamma(q)}{\gamma(p) - 1 + 1/\gamma(q)}, \end{aligned} \tag{7.13}$$

and substitution of (7.13) into (7.7) yields no further constraints on f, g and γ. Note that in the process of substituting back, any dependence on the constants $b(s,s)$ and $c(s,s)$ has dropped out, we were able to arrange this by the particular form we chose for γ.

So, we have established that any three functions a, b and c which map $\mathrm{S} \times \mathrm{S}$ to $\mathbb{C}$ and satisfy (7.7) are necessarily of the form (7.13), and also that (7.13) satisfies (7.7) identically. Note that if we choose $\gamma(p) = 1$ the solution (7.13) reduces to the solution (7.3) we identified before.

7.4 Solutions on a restricted domain

In this section we will uncover solutions of (7.2) which were excluded from consideration in section 7.3 by our assumption that a, b and c were defined throughout $\mathrm{S} \times \mathrm{S}$. In particular, this assumption was needed to ensure the existence of $s \in \mathrm{S}$ for which $b(s,s)$ and $c(s,s)$ were points in $\mathbb{C}$. Here we consider the alternative, that no such s exists. We begin by assuming a, b and c are defined throughout a restricted domain D,

$$\mathrm{D} = \mathrm{S} \times \mathrm{S} \setminus \{\ (p,p) \mid p \in \mathrm{S}\ \}. \tag{7.14}$$

Now, substituting $q = p$ and $s = q$ into (7.8) we find

$$\begin{aligned} b(p,p) &= \frac{a(p,q) - c(p,q)c(q,p)}{b(p,q) - c(q,p)}, \\ c(p,p) &= \frac{b(p,q)b(q,p) - a(q,p)}{b(p,q) - c(q,p)}. \end{aligned} \tag{7.15}$$

Our assumption that a, b and c are defined throughout D means that provided $p \neq q$ the numerators and denominator on the RHS of (7.15) all lie in $\mathbb{C}$. Clearly then, if we can find a pair $(p,q) \in \mathrm{D}$ so that the denominator does not vanish, both $b(p,p)$ and $c(p,p)$ lie in $\mathbb{C}$ and the solution method of section 7.3 continues to apply. Alternatively, when no such pair can be found, i.e., when

$$b(p,q) = c(q,p) \qquad \forall\ (p,q) \in \mathrm{D}, \tag{7.16}$$

that solution method fails. Fortuitously, the additional constraint (7.16) naturally simplifies the problem which leaves it tractable by a similar solution method.

So let us suppose (7.16) holds, this enables us to eliminate c from (7.2) and we are left with the problem of solving the functional equation

$$\begin{aligned} a(p,q) = b(p,q)b(r,p) - b(r,p)b(r,q) + b(r,q)b(q,p), \\ \forall\ \textit{distinct}\ p,q,r \in \mathrm{S}. \end{aligned} \tag{7.17}$$

We proceed by fixing $s \in \mathrm{S}$ and substituting $(p,q,r) \to (q,s,p)$ in (7.17) to find

$$b(p,q) = \frac{a(q,s) - b(s,q)b(p,s)}{b(q,s) - b(p,s)}. \tag{7.18}$$

Then, fixing $t \in \mathrm{S}$ so that $(t,s) \in \mathrm{D}$ and setting $(p,q,r) \to (q,s,t)$ in (7.17),

$$a(q,s) = b(q,s)b(t,q) - b(t,q)b(t,s) + b(t,s)b(s,q). \tag{7.19}$$

Using this to substitute for $a(q,s)$ in (7.18) we conclude that $b(p,q)$ is a rational expression in the single-variable functions $b(t,p)$, $b(s,p)$, $b(p,s)$ and the constant $b(t,s)$. Of course from (7.17) we see that $a(p,q)$ is also a rational expression in the same functions and constant.

It turns out there is no further constraint on these functions, specifically, if we define

$$\begin{aligned} b(p,s) - b(t,s) &= \gamma(p), \\ b(t,p) &= f(p), \\ b(s,p) &= g(p). \end{aligned} \tag{7.20}$$

then γ, f and g may all be chosen independently. As before we substitute (7.20) back through (7.19), (7.18) and (7.17), this time also defining $c(p,q) = b(q,p)$, and we find

$$\begin{aligned} b(p,q) &= \frac{f(q)\gamma(p) - g(q)\gamma(q)}{\gamma(p) - \gamma(q)}, \\ c(p,q) &= \frac{g(p)\gamma(p) - f(p)\gamma(q)}{\gamma(p) - \gamma(q)}, \\ a(p,q) &= \frac{f(q)g(p)\gamma(p) - f(p)g(q)\gamma(q)}{\gamma(p) - \gamma(q)}. \end{aligned} \tag{7.21}$$

And it can be verified that (7.21) satisfies (7.2) identically.

To summarise, suppose that a, b and c map D to $\mathbb{C}$ and satisfy the functional equation (7.2). We have shown that a, b and c must be either of the form (7.13) or of the form (7.21), moreover it is straightforward to verify that both (7.13) and (7.21) satisfy (7.2) identically.

7.5 Discussion of the constructed equations

From the solutions of the constraint equation (7.2) we will now reconstruct the lattice equations. Both solutions (7.13) and (7.21) contain three arbitrary functions f, g and γ. If we make the identifications

$$\begin{aligned} p \to (p_1, p_2, p_3) &= (f(p), g(p), \gamma(p)), \\ q \to (q_1, q_2, q_3) &= (f(q), g(q), \gamma(q)), \end{aligned} \tag{7.22}$$

then without loss of generality $(p_1, p_2, p_3), (q_1, q_2, q_3) \in \mathbb{C}^3$ may be taken as the lattice parameters. Substitution of (7.13) and (7.21) back into the ansatz (7.1) then yields the equations

$$\begin{aligned}\widehat{\widetilde{u}} + q_2\widetilde{u} + p_1\widehat{u} + p_1q_2u = p_3(&\widehat{\widetilde{u}} + q_2\widetilde{u} + p_2\widehat{u} + p_2q_2u) \\ &+ (\widehat{\widetilde{u}} + q_1\widetilde{u} + p_1\widehat{u} + p_1q_1u)/q_3\end{aligned} \tag{7.23}$$

and

$$p_3(\widehat{\widetilde{u}} + q_1\widetilde{u} + p_2\widehat{u} + p_2q_1u) = q_3(\widehat{\widetilde{u}} + q_2\widetilde{u} + p_1\widehat{u} + p_1q_2u) \tag{7.24}$$

respectively. Both of these equations lie outside any previous classification schemes [4, 6, 27, 28].

Let us now recast these equations into a different form by applying a point transformation to the lattice parameters,

$$(p_1, p_2, p_3) \to \left(-p_3\frac{\omega + (1+\omega)p_1}{\omega + (1+\omega)p_2}, -p_3\frac{\omega - p_1}{\omega - p_2}, \frac{-\omega(\omega - p_2)}{\omega + (1+\omega)p_2}\right), \tag{7.25}$$

here ω is a parameter of the transformation. For (7.23) we intend that ω satisfy $\omega^2 + \omega + 1 = 0$ and for (7.24) that $\omega = 1$. When this assumption is made the transformation (7.25) brings both (7.23) and (7.24) to the same equation,

$$(p_1 - \omega q_1)p_3q_3u + (p_2 - \omega q_2)\widehat{\widetilde{u}} - (p_2 - \omega q_1)q_3\widetilde{u} - (p_1 - \omega q_2)p_3\widehat{u} = 0, \tag{7.26}$$

except in this choice of ω. It can be verified that equation (7.26) is consistent in the sense of (1.12) if and only if $\omega^3 = 1$, which is equivalent to $(\omega - 1)(\omega^2 + \omega + 1) = 0$. The equations (7.23) and (7.24) are obviously easier to write down in the form (7.26). The principal benefit however is that from (7.26) it is obvious that one component of the lattice parameters, specifically p_3 and q_3, can be *removed* by the gauge transformation $u(n, m) \to p_3^n q_3^m u(n, m)$. The resulting equation

$$(p_1 - \omega q_1)u + (p_2 - \omega q_2)\widehat{\widetilde{u}} - (p_2 - \omega q_1)\widetilde{u} - (p_1 - \omega q_2)\widehat{u} = 0, \tag{7.27}$$

depends on fewer parameters, but is equivalent to (7.26) up to gauge transformations.

Observe also that (7.27) is invariant under translations of u, $u \mapsto u + \delta$, i.e., invariant under $\mathsf{F}(\infty)$. Effectively (7.27) corresponds to a particular choice of gauge which leaves (7.26) with a larger point symmetry group. To conclude this section we will demonstrate that this observation has a converse, something which will play an important role later. Let us *impose* translation invariance on (7.26) whilst demanding consistency is preserved. A direct calculation leads to the following condition

$$(p_2 - p_3p_1)(1 - q_3) = \omega(q_2 - q_3q_1)(1 - p_3), \tag{7.28}$$

which should hold for any choice of the lattice parameters. In particular, if we fix r and suppose

$$\omega\frac{r_2 - r_3r_1}{1 - r_3} = \chi \tag{7.29}$$

then it must be that

$$\frac{p_2 - p_3p_1}{1 - p_3} = \chi, \qquad \frac{q_2 - q_3q_1}{1 - q_3} = \chi. \tag{7.30}$$

Directly from (7.28) we then see that $\omega\chi = \chi$ so if $\omega \neq 1$ we must choose $\chi = 0$, otherwise χ is arbitrary. Effectively then, condition (7.28) fixes the gauge,

$$p_3 = \frac{p_2 - \chi}{p_1 - \chi}, \qquad q_3 = \frac{q_2 - \chi}{q_1 - \chi}. \tag{7.31}$$

After fixing p_3 and q_3 this way in (7.26), if we apply the point transformation

$$(p_1, p_2) \to \left(\frac{1}{p_1 - \chi}, \frac{1}{p_2 - \chi}\right), \tag{7.32}$$

then the equation (7.26) goes to (7.27).

So to summarise, up to the assumptions made in the solution of the functional equation (7.2), we have shown that all equations of the form (7.1) which are consistent in the sense of (1.12) can be brought to the form (7.27) by point-transformations of the lattice parameters and gauge transformations of the dependent variable u. Moreover, equations which additionally have symmetry under translations of u can be brought to the form (7.27), this time by only point-transformations of the lattice parameters.

7.6 The inhomogeneous case

In this section we extend the original ansatz (7.1) to equations of the form

$$a(p,q)u + b(p,q)\widetilde{u} + c(p,q)\widehat{u} + \widehat{\widetilde{u}} = d(p,q). \tag{7.33}$$

If we impose consistency in the sense of (1.12) on this more general ansatz the conditions which emerge are equivalent to the original condition (7.2) plus the following additional condition involving the coefficient d,

$$(b(q,r) - c(r,p))d(p,q) = (1 + b(r,p))d(q,r) - (1 + c(q,r))d(r,p). \tag{7.34}$$

The reappearance of the original condition (7.2) suggests that we can build on the results already established in order to solve this system of functional equations. In fact, without much further effort (but the calculation is a little tedious), we will be able to show that (almost) all consistent equations of the form (7.33) can all be transformed back to the homogeneous case (7.1).

Consider the additional functional equation (7.34). The substitution $(p,q,r) \to (r,p,q)$ in (7.34) leads to

$$(b(p,q) - c(q,r))d(r,p) = (1 + b(q,r))d(p,q) - (1 + c(p,q))d(q,r). \tag{7.35}$$

Elimination of $d(p,q)$ between (7.35) and (7.34), then employing (7.2) reveals the following derived functional equation

$$(a(r,p)+b(r,p)+c(r,p)+1)d(q,r) = (a(q,r)+b(q,r)+c(q,r)+1)d(r,p). \tag{7.36}$$

There are two cases, either

$$a(p,q) + b(p,q) + c(p,q) + 1 = 0 \tag{7.37}$$

or, without loss of generality, we may write

$$d(p,q) = (1 + a(p,q) + b(p,q) + c(p,q))d_*(p,q) \tag{7.38}$$

for some new function d_*. We observe from (7.36) that the resulting condition on d_* is simply that

$$d_*(q,r) = d_*(r,p), \tag{7.39}$$

which reveals that d_* depends neither on its first argument, or its second, in other words $d_*(p,q) = \delta$ for some constant δ. But substituting $d(p,q) = \delta(a(p,q)+b(p,q)+c(p,q)+1)$ in (7.33) yields (7.1) up to the point transformation $u \to u - \delta$. So, assuming (7.37) is false, the ansatz (7.33) reduces to the homogeneous case (7.1) up to this point-transformation.

It remains to consider the case when (7.37) is true. Observe that this further condition on the coefficients a, b and c arises exactly when we demand the translation invariance of (7.33). This condition does not involve d and is therefore equivalent to considering the same problem in the homogeneous case (7.1). But this is exactly the problem we solved at the end of section 7.5, so we may conclude that if (7.37) holds then the homogeneous part of (7.33) must - up to point transformations of the lattice parameters - be of the form (7.27). So, assuming (7.37), consistent equations of the form (7.33) can be brought to the form

$$\frac{p_1 - \omega q_1}{p_2 - \omega q_2} u - \frac{p_2 - \omega q_1}{p_2 - \omega q_2} \widetilde{u} - \frac{p_1 - \omega q_2}{p_2 - \omega q_2} \widehat{u} + \widehat{\widetilde{u}} = d(p,q), \tag{7.40}$$

The only remaining condition is on the function d, it reads

$$\begin{aligned} &\omega^2 (r_1 - r_2)(p_2 - \omega q_2) d(p,q) + \\ &\omega (p_1 - p_2)(q_2 - \omega r_2) d(q,r) + \\ &(q_1 - q_2)(r_2 - \omega p_2) d(r,p) = 0. \end{aligned} \tag{7.41}$$

This can be solved directly. First, when the components of the lattice parameters are related, $p_1 = p_2$ etc., this condition is satisfied trivially. In other words the equation

$$u - \widetilde{u} - \widehat{u} + \widehat{\widetilde{u}} = d(p,q) \tag{7.42}$$

is consistent whatever the choice of d. Otherwise if we suppose that r is a fixed constant and $r_1 \neq r_2$, then from (7.41) we conclude that

$$d(p,q) = \frac{(p_1 - p_2) f(q) + (q_1 - q_2) g(p)}{p_2 - \omega q_2} \tag{7.43}$$

for some functions f and g. Substituting this expression for d back into (7.41) we find the condition

$$\omega^2 \frac{\omega f(p) + g(p)}{p_1 - p_2} + \omega \frac{\omega f(q) + g(q)}{q_1 - q_2} + \frac{\omega f(r) + g(r)}{r_1 - r_2} = 0, \tag{7.44}$$

which shows that f and g cannot be chosen independently. By substituting $(q,r) \to (p,p)$ in this condition we find that $(\omega^2+\omega+1)(\omega f(p)+g(p)) = 0$ so there are two cases. If $\omega = 1$ then it is necessary that $f(p) + g(p) = 0$. If $\omega^2 + \omega + 1 = 0$ then from (7.44) we know that $(\omega f(p) + g(p))/(p_1 - p_2)$ is a constant, without loss of generality we may write this constant as $(\omega - 1)\chi$ for arbitrary χ, so both cases are covered by the single relation

$$\omega f(p) + g(p) = (p_1 - p_2)(\omega - 1)\chi. \tag{7.45}$$

This is necessary, but also sufficient, i.e., given (7.45) the functional equation (7.44) is satisfied by virtue of the fact that $\omega^3 = 1$. It remains to (without loss of generality) make the identification $p_3 = f(p) - \chi(p_1 - p_2)$ whence $g(p) = -\omega p_3 - \chi(p_1 - p_2)$. Substitution of f and g into (7.43) reveals

$$d(p,q) = \frac{(p_1 - p_2)q_3 - \omega(q_1 - q_2)p_3}{p_2 - \omega q_2}. \tag{7.46}$$

Then substitution of this into (7.40) followed by a rearrangement yields

$$\begin{aligned} 0 =&(p_1 - \omega q_1)u + (p_2 - \omega q_2)(\widehat{\widetilde{u}} - p_3 - q_3) \\ &-(p_2 - \omega q_1)(\widetilde{u} - p_3) - (p_1 - \omega q_2)(\widehat{u} - q_3). \end{aligned} \tag{7.47}$$

Writing the equation in this way it becomes clear that the gauge transformation $u(n,m) \to u(n,m) + np_3 + mq_3$ brings this equation to (7.27).

So finally we may conclude that equations of the form (7.33) which are consistent in the sense of (1.12) may (up to the assumptions made in the solution of (7.2)) be brought to the form (7.42) or the form (7.27) by point transformations of the parameters and gauge/point transformations of the variables.

7.7 Conclusion

The direct method we have applied here for a small classification problem (The linear case with 'lattice parameters') is unwieldy for larger problems. However the result is a new, albeit incremental, contribution to the classification of scalar equations which are consistent on the cube. Note that the main equation listed here turns up in chapter 8, section 8.3.

Chapter 8

Other Bäcklund transformations

We give new Bäcklund transformations (BTs) for the quadrilateral lattice equations listed in section 6.2. As opposed to the natural auto-BT we described in section 1.2, these BTs are of two other kinds. Specifically, it is found that some equations admit additional auto-BTs (with Bäcklund parameter), whilst some pairs of distinct equations admit a BT which connects them. (We use the term auto-BT in this chapter for precision because BTs relating distinct equations are also discussed.)

The BTs given here are similar in essence to the notion of consistency on the cube. However, rather than an equation being consistent with copies of itself (like described for the example in section 1.2), distinct equations are consistent with each other. The point of view we adopt, of giving BTs rather than listing equations and how they fit on the faces of a cube, lends more in the way of intuition to the results as well as providing a natural way to organise them. With reference to the classification results described in section 6.3, we note that the systems on the cube given here consist of equations which are not of 'type Q' on at least one pair of faces.

Eq	Bäcklund transformation	SP
$Q3^0$	$(pr - \frac{1}{pr})(uv + \widetilde{u}\widetilde{v}) - (r - \frac{1}{r})(u\widetilde{v} + \widetilde{u}v) = (p - \frac{1}{p})(1 + u\widetilde{u}v\widetilde{v})$	$A2$
$Q1^\delta$	$p(u+v)(\widetilde{u}+\widetilde{v}) - r(u-\widetilde{u})(v-\widetilde{v}) = \delta^2 pr(p+r)$	$A1^\delta$
$Q3^0$	$p(u\widetilde{v} + \widetilde{u}v) - uv - \widetilde{u}\widetilde{v} = r(1-p^2)$	$H3^1$
$Q1^1$	$(u-\widetilde{u})(v-\widetilde{v}) = -p(u+\widetilde{u}+v+\widetilde{v}+p+2r)$	$H2$

Table 8.1: Each equation, Eq, admits the given auto-BT with Bäcklund parameter r. The equation SP emerges as the superposition principle for solutions of Eq related by this BT. It turns out that the converse associations also hold (see main text).

8.1 Alternative auto-Bäcklund transformations

Table 8.1 lists auto-BTs for some particular equations from the list (6.11). One significant difference between these and the natural auto-BT (descirbed in section 1.2) is that the superposition principle associated with these alternative auto-BTs coincides with some *other* equation present in the list (6.11). This can be compared to the natural auto-BT, (described in section 1.2) for which the superposition principle esentially coincides with the equation itself.

To explain the BTs in table 8.1 we give an example here (the last entry in the table). Consider the following system of equations in the two variables $u = u(n,m)$ and $v = v(n,m)$,

$$\begin{aligned} (u-\widetilde{u})(v-\widetilde{v}) &= -p(u+\widetilde{u}+v+\widetilde{v}+p+2r), \\ (u-\widehat{u})(v-\widehat{v}) &= -q(u+\widehat{u}+v+\widehat{v}+q+2r) \end{aligned} \tag{8.1}$$

(the second equation here is implicit from the first and so is omitted from the table for brevity). With u fixed throughout the lattice (i.e., for all n,m), (8.1) constitutes an overdetermined system for v. This is resolved ($\widehat{\widetilde{v}} = \widetilde{\widehat{v}}$) if u is chosen to satisfy the equation $Q1^1$, moreover v which then emerges in the solution of (8.1) also satisfies $Q1^1$. If (8.1) holds we say that the solutions u and v of $Q1^1$ are related by the BT (8.1) and for convenience write

$$u \overset{r}{\sim} v. \tag{8.2}$$

The BT relation (8.2) is symmetric with Bäcklund parameter r.

Transformations (8.1) with different choices of the parameter r commute in the sense that a superposition principle exists. That is, given solutions $u = u(n,m)$, $\overline{u} = \overline{u}(n,m)$ and $\dot{u} = \dot{u}(n,m)$ of $Q1^1$ for which

$$u \overset{r}{\sim} \overline{u}, \qquad u \overset{s}{\sim} \dot{u}, \tag{8.3}$$

the function $\dot{\overline{u}}$ defined by the relation

$$(u - \dot{\overline{u}})(\overline{u} - \dot{u}) = (r-s)(u + \overline{u} + \dot{u} + \dot{\overline{u}} + r + s) \tag{8.4}$$

satisfies

$$\dot{u} \overset{r}{\sim} \dot{\overline{u}}, \qquad \overline{u} \overset{r}{\sim} \dot{\overline{u}}. \tag{8.5}$$

The relation (8.4) is the superposition principle for solutions of $Q1^1$ related by the BT (8.1). But up to a change in notation (8.4) is exactly the lattice equation $H2$ from the list (6.11).

Remarkably the preceeding facts are also true in the reverse sense. Specifically, the same consistent system, just considered from a different perspective, constitutes a new auto-BT for the equation $H2$ and the superposition principle which emerges coincides with $Q1^1$. In this sense the BT defines a kind of duality between these equations, each equation arising as the superposition principle for BTs that relate solutions of the other.

Let us state this more precisely, but without changing the notation so that the two systems can be seen in a unified way. Observe first that the relations (8.3) imply (with two others) the following pair of equations,

$$\begin{aligned} (u - \widetilde{u})(\overline{u} - \widetilde{\overline{u}}) &= -p(u + \widetilde{u} + \overline{u} + \widetilde{\overline{u}} + p + 2r), \\ (u - \widetilde{u})(\dot{u} - \dot{\widetilde{u}}) &= -p(u + \widetilde{u} + \dot{u} + \dot{\widetilde{u}} + p + 2s). \end{aligned} \tag{8.6}$$

Now introduce new discrete variables l and k (forgetting for the moment n and m), so that $u = u(l,k)$, $\overline{u} = u(l+1,k)$, $\dot{u} = u(l,k+1)$ and $\dot{\overline{u}} =$

$u(l+1,k+1)$ for independent variables $l,k \in \mathbb{Z}$. Then it may be verified that the system (8.6) forms a BT, with Bäcklund parameter p, between solutions $u(l,k)$ and $\widetilde{u}(l,k)$ of the equation (8.4) (i.e., constitutes an auto-BT for $H2$). This BT commutes with its counterpart with Bäcklund parameter q (which relates solutions $u(l,k)$ and $\widehat{u}(l,k)$ of (8.4)). Finally, the superposition principle in this case is found to be

$$p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})=pq(q-p), \tag{8.7}$$

which is exactly the equation $Q1^1$.

It is neat to realise, but difficult to visualise, that both perspectives can be acommodated simultaneously by considering all four discrete variables k,l,m and n at once, in other words this system lies naturally on the four-dimensional lattice. All the BTs given in table 8.1 establish the same kind of duality between 'Eq' and 'SP'.

8.2 Bäcklund transformations between distinct equations

Table 8.2 lists BTs that connect particular pairs of equations from the list (6.11). To be precise about the meaning of the entries in this table we again give an example. Consider the system of equations

$$\begin{aligned} 2u\widetilde{u} &= v+\widetilde{v}+p, \\ 2u\widehat{u} &= v+\widehat{v}+q. \end{aligned} \tag{8.8}$$

This system is compatible in v if the variable u satisfies the equation $H1$. Given such u, it can be verified that v which emerges in the solution of (8.8) then satisfies the equation $H2$. Conversely, if v satisfies $H2$ then solving (8.8) yields u which satisfies $H1$. In this way the system (8.8) constitutes a BT between the equations $H1$ and $H2$, which corresponds to the fifth entry in table 8.2 (where we give only one equation from the pair (8.8), the other being implicit).

Eq in u	Bäcklund transformation	Eq in v
$Q3^0$	$uv+\widetilde{u}\widetilde{v}-p(u\widetilde{v}+\widetilde{u}v)=(p-\frac{1}{p})(u\widetilde{u}+\frac{\delta^2}{4}p)$	$Q3^\delta$
$Q1^1$	$(u-\widetilde{u})(v-\widetilde{v})=p(2u\widetilde{u}-v-\widetilde{v})+p^2(u+\widetilde{u}+p)$	$Q2$
$Q1^0$	$(u-\widetilde{u})(v-\widetilde{v})=p(u\widetilde{u}-\delta^2)$	$Q1^\delta$
$H3^0$	$pu\widetilde{u}-uv-\widetilde{u}\widetilde{v}=\delta$	$H3^\delta$
$H1$	$2u\widetilde{u}=v+\widetilde{v}+p$	$H2$
$A1^0$	$(u+\widetilde{u})(v+\widetilde{v})=p(u\widetilde{u}+\delta^2)$	$A1^\delta$
$A1^0$	$(u+\widetilde{u})(v-\widetilde{v})=p(u-\widetilde{u})$	$Q1^1$
$^\dagger A1^\delta$	$u+\widetilde{u}=2pv\widetilde{v}+\delta p^2$	$H3^\delta$
$^\dagger A1^0$	$(u+\widetilde{u})v\widetilde{v}=p(1-\frac{\delta}{2}u)(1-\frac{\delta}{2}\widetilde{u})$	$H3^\delta$

Table 8.2: BTs between distinct lattice equations. The BT between $Q1^0$ and $Q1^\delta$ was given originally by ABS in [6]. † indicates application of the point transformation $p \to p^2$, $q \to q^2$ to the lattice parameters. Transformations are stated up to composition with point symmetries of the equations in u and v.

The BT (8.8) can be explained as a non-symmetric degeneration of the natural auto-BT for the equation $H2$, which is defined by the system

$$\begin{aligned}(u-\widetilde{v})(\widetilde{u}-v) &= (p-r)(u+\widetilde{u}+v+\widetilde{v}+p+r),\\ (u-\widehat{v})(\widehat{u}-v) &= (q-r)(u+\widehat{u}+v+\widehat{v}+q+r)\end{aligned} \tag{8.9}$$

(r is the Bäcklund parameter). Now, the substitution $u \to \frac{1}{\epsilon^2}+\frac{2}{\epsilon}u$ in the equation $H2$ leads to the equation $H1$ in the limit $\epsilon \longrightarrow 0$. This substitution in the system (8.9) together with the particular choice $r = -\frac{1}{\epsilon^2}$ yields the system (8.8) in the limit $\epsilon \longrightarrow 0$. Note that it is not a priori obvious that the BT will be preserved in this limit, by which we mean that once the system (8.8) has been found, it remains to verify the result.

Each of the first six entries in table 8.2 can be explained as a non-symmetric degeneration of the natural auto-BT for the equation in v.

However the remaining three entries, which have been uncovered by a (non-exhaustive) computer algebra search, do not appear to fit this explanation.

8.3 Linearisation of the Hietarinta equation

Consider now the system (with 2-component lattice parameters)

$$\begin{aligned}(u+p_1)v &= (\widetilde{u}+p_2)\widetilde{v},\\ (u+q_1)v &= (\widehat{u}+q_2)\widehat{v}.\end{aligned} \tag{8.10}$$

The following equations emerge as compatibility constraints

$$(u+q_1)(\widetilde{u}+p_2)(\widehat{u}+p_1)(\widehat{\widetilde{u}}+q_2) = (u+p_1)(\widehat{u}+q_2)(\widetilde{u}+q_1)(\widehat{\widetilde{u}}+p_2), \tag{8.11}$$

$$(p_1-q_1)v + (p_2-q_2)\widehat{\widetilde{v}} = (p_2-q_1)\widetilde{v} + (p_1-q_2)\widehat{v}, \tag{8.12}$$

although they lie outside the list (6.11) they are both consistent on the cube. The equation (8.11) was given originally by Hietarinta in [26] and subsequently shown to be linearisable by Ramani et. al. in [69], whereas (8.12) is the covariant form ($\omega = 1$) of the linear equation (7.27) constructed in chapter 7. The system (8.10) constitutes a BT which relates (8.11) to (8.12) and therefore provides an alternative linearisation of the Hietarinta equation.

8.4 Comutativity

In sections 8.1, 8.2 and 8.3 we have given systems of equations which may be written generically in the form

$$\begin{aligned}f_p(u,\widetilde{u},v,\widetilde{v}) &= 0,\\ f_q(u,\widehat{u},v,\widehat{v}) &= 0,\end{aligned} \tag{8.13}$$

and that constitute a BT between a lattice equation in $u = u(n,m)$ and a possibly different lattice equation in $v = v(n,m)$, say

$$\mathcal{Q}_{pq}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = 0, \tag{8.14}$$

$$\mathcal{Q}^*_{pq}(v,\widetilde{v},\widehat{v},\widehat{\widetilde{v}}) = 0. \tag{8.15}$$

(Here we suppose that u and v are scalar fields and f, $\mathcal{Q}$ and $\mathcal{Q}^*$ are polynomials of degree 1 in which the coefficients are functions of the lattice parameters.) In this generic (scalar) case it can be deduced (by considering an initial value problem on the cube) that the multidimensional consistency of (8.14) implies the multidimensional consistency of (8.15). Furthermore, when (8.14) and (8.15) are consistent on the cube, the BT (8.13) commutes with the natural auto-BTs for these equations, the superposition principle being the equation

$$f_r(u, \overline{u}, v, \overline{v}) \;=\; 0. \tag{8.16}$$

Here u and $\overline{u}$ are solutions of (8.14) related by its natural auto-BT (with Bäcklund parameter r), similarly v and $\overline{v}$ are solutions of (8.15) related by its natural auto-BT (also with Bäcklund parameter r), and finally, u and $\overline{u}$ are related to v and $\overline{v}$ respectively by the BT (8.13).

Finally let us remark that lattice equations which are connected by a BT are not always consistent on the cube (by which we mean consistent with copies of themselves as described in section 1.2). Consider the following example (which involves 2-component lattice parameters),

$$\begin{aligned} p_1 u\widetilde{u} &= v + \widetilde{v} + p_2, \\ q_1 u\widetilde{u} &= v + \widetilde{v} + q_2. \end{aligned} \tag{8.17}$$

This system constitutes a BT between the equations

$$\begin{aligned} &p_1(u\widetilde{u} + \widehat{u}\widehat{\widetilde{u}}) - q_1(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}) = 2(p_2 - q_2), \\ &p_1^2(v + \widehat{v})(\widetilde{v} + \widehat{\widetilde{v}}) - q_1^2(v + \widetilde{v})(\widehat{v} + \widehat{\widetilde{v}}) = p_2^2 q_1^2 - q_2^2 p_1^2. \end{aligned} \tag{8.18}$$

The equations (8.18) are consistent on the cube if and only if the components of the lattice parameters are connected by the relations

$$a + bp_1^2 + cp_2 = 0, \qquad a + bq_1^2 + cq_2 = 0, \tag{8.19}$$

for some constants a, b and c not all equal to zero. (The solution of (8.19) yields the fifth and eighth entries in table 8.2.) On the other hand, when $\mathcal{Q}^* = \mathcal{Q}$ in the above, so that (8.14) admits the auto-BT (8.13), we have found no counterexamples to the conjecture that (8.14) is consistent on the cube.

8.5 Conclusion

Two types of BT have been presented. First there are new auto-BTs for some known integrable lattice equations (described in chapter 6). These turn out to define a rather elegant duality between distinct pairs of known equations. The duality originates in the consideration of commutativity for the new auto-BTs because the superposition principle which emerges coincides with some other known equation.

Second there are BTs which relate distinct equations. These provide transformations between some of the canonical forms listed in (6.11).

Importantly the BTs presented commute with the natural auto-BT for the equation(s) concerned. In fact there is a superposition principle relating solutions of the equations algebraically. This commutativity is of practical significance, for example allowing for soliton type solutions to be found for one equation from the soliton solution of an equation to which it is related (cf. [11]).

Chapter 9

The construction of solutions for Adler's equation

In this chapter we construct solutions of Adler's equation. We introduced Adler's equation in chapter 6 as the superposition principle for BTs of the Krichever-Novikov equation and also pointed out its significance vis-à-vis the ABS classification result (cf. also section 1.1). The methods used here are applicable more widely, for example to the equations in the list (6.11).

9.1 The symmetric biquadratic correspondence

We begin by reviewing some facts about the biquadratic algebraic correspondence, in particular the associated group and the connection to elliptic functions. The solution of such correspondences will play a pivotal role in the constructions we present here. If h is a non-degenerate symmetric biquadratic (cf. section 6.3), the correspondence $u \mapsto \widetilde{u}$ defined by the equation $h(u, \widetilde{u}) = 0$ is in general two-valued. It turns out that if two symmetric biquadratics, say f and g, have the same discriminant up to an overall scaling, then they define correspondences which commute. By commutativity here we mean that the elimination of $\widetilde{u}$ from the system $\{f(u,\widetilde{u})=0,\, g(\widetilde{u},\widehat{\widetilde{u}})=0\}$ yields the same polynomial equation in u and $\widehat{\widetilde{u}}$ as the elimination of $\widehat{u}$ from the system $\{g(u,\widehat{u})=0, f(\widehat{u},\widehat{\widetilde{u}})=0\}$ (which is equivalent to this resulting polynomial being symmetric [86]).

In section (6.1) we constructed a family of biquadratics with shared discriminant and we will focus now on that example. Let us recall this family of biquadratics (of Jacobi type) and their common discriminant

$$\begin{aligned} \mathcal{H}_{\mathfrak{p}}(u,\widetilde{u}) =& \frac{1}{2p}\left(u^2+\widetilde{u}^2-(1+u^2\widetilde{u}^2)p^2-2u\widetilde{u}P\right), \\ & \mathfrak{p}\in\Gamma=\left\{(p,P)\mid P^2=\mathcal{R}(p)\right\}, \\ & \mathcal{R}(u)=1-(k+1/k)u^2+u^4, \end{aligned} \tag{9.1}$$

which are parametrised by $\mathfrak{p}\in\Gamma$. This family of biquadratics is associated with a natural group structure on Γ. To construct a rational representation of this group is quite straightforward. We begin by solving the equation $\mathcal{H}_{\mathfrak{p}}(u,\widetilde{u})=0$ for $\widetilde{u}$, and then go back and solve it again, this time for u,

$$\widetilde{u}=\frac{uP+pU}{1-u^2p^2}, \qquad u=\frac{\widetilde{u}P-p\widetilde{U}}{1-\widetilde{u}^2p^2}, \tag{9.2}$$

where we have introduced U and $\widetilde{U}$ for which $(u,U),(\widetilde{u},\widetilde{U})\in\Gamma$. The expressions (9.2) for $\widetilde{u}$ and u are unique up to a choice of sign for U and $\widetilde{U}$, the particular choice in (9.2) is for convenience later. Now consider rearranging (9.2) for the pair $(\widetilde{u},\widetilde{U})$, if we also denote $(u,U)=\mathfrak{u}$ and $(\widetilde{u},\widetilde{U})=\widetilde{\mathfrak{u}}$ then we have written $\widetilde{\mathfrak{u}}$ as a function of $\mathfrak{u}$ and $\mathfrak{p}$, which we in turn use to define binary operation

$$\widetilde{\mathfrak{u}}=\mathfrak{u}\cdot\mathfrak{p}=\left(\frac{uP+pU}{1-u^2p^2},\frac{Uu(p^4-1)-Pp(u^4-1)}{(1-u^2p^2)(pU-uP)}\right). \tag{9.3}$$

It can be verified that (9.3) constitutes a product which turns Γ into an abelian group, the main non-trivial property is the associativity which is basically inherited from the commutativity property in the family of biquadratics (9.1). The identity in the group is the point $\mathfrak{e}=(0,1)$ (which is the result of an earlier choice we made in the construction (6.5)), the inverse of a point $\mathfrak{p} = (p, P)$ is the point $(-p, P) = \mathfrak{p}^{-1}$. The (single-valued) product we have defined by (9.3) relates back to the (two-valued) biquadratic correspondence by the observation

$$\widetilde{\mathfrak{u}} \in \{\mathfrak{u} \cdot \mathfrak{p}, \mathfrak{u} \cdot \mathfrak{p}^{-1}\} \quad \Rightarrow \quad \mathcal{H}_{\mathfrak{p}}(u, \widetilde{u}) = 0. \tag{9.4}$$

The other solution arising by the converse choice of sign for U in the first equation of (9.2). (Note also that $\mathfrak{s} = (\sqrt{(}k), 0) \in \Gamma$ and the involution $\mathfrak{p} \mapsto \mathfrak{s}^2 \cdot \mathfrak{p}^{-1} = (p, -P)$ doubles the number of points on the LHS of the above and thus explains the *four* permutations of sign choice in (9.2), but this is just a technical detail and has no consequence for us.)

The parametrisation of this group on Γ by elliptic functions is often referred to as the introduction of *uniformizing variables*. To do this we define a function $\mathfrak{f} = \mathfrak{f}(\xi)$ in terms of another function $f = f(\xi)$ by the relation $\mathfrak{f} = (f, f')$. The constraint

$$\mathfrak{f} \in \Gamma, \quad \mathfrak{f}(0) = \mathfrak{e}, \tag{9.5}$$

amounts to an initial-value ODE which defines f. This system turns out to have a meromorphic solution which is in fact an elliptic function [25]. Remarkably (cf. [25]) $\mathfrak{f}$ defined in this way satisfies

$$\mathfrak{f}(\xi + \eta) = \mathfrak{f}(\xi) \cdot \mathfrak{f}(\eta), \tag{9.6}$$

bringing composition in Γ down to addition on the torus. In the case of our particular example the function f in question is given by

$$f(\xi) = \sqrt{k}\ \mathrm{sn}(\xi/\sqrt{k}) \tag{9.7}$$

where sn is the Jacobi elliptic function with modulus k (defined for example in [25]).

Finally, we may use the construction described here to parametrise solutions of the biquadratic correspondence

$$\mathcal{H}_{\mathfrak{p}}(u, \widetilde{u}) = 0 \tag{9.8}$$

where now we consider $u = u(n)$ and $\widetilde{u} = u(n+1)$ as values of a dependant variable u as a function of the independent variable $n \in \mathbb{Z}$. If we define α by the equation $\mathfrak{f}(\alpha) = \mathfrak{p}$ then the (canonical) solution may be written simply as $u(n) = f(\xi_0 + n\alpha)$ for some arbitrary constant ξ_0. Given (9.4) other (non-canonical) solutions, like for example $u(n) = f(\xi_0 - n\alpha)$ or $u(n) = f(\xi_0 + (1 - (-1)^n)\alpha/2)$, are of course possible.

9.2 A seed solution

Let us recall the Jacobi form of Adler's equation given in section 6.1

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{q}}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) :=$$
$$p(u\widetilde{u} + \widehat{u}\widehat{\widetilde{u}}) - q(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}) - \frac{pQ - qP}{1 - p^2q^2}(u\widehat{\widetilde{u}} + \widetilde{u}\widehat{u} - pq(1 + u\widetilde{u}\widehat{u}\widehat{\widetilde{u}})) = 0, \tag{9.9}$$

where $\mathfrak{p}, \mathfrak{q} \in \Gamma$ (where Γ is defined in (9.1)) are the lattice parameters. Using the facts established in the previous section it is clear that the function

$$u(n, m) = f(\xi_0 + n\alpha + m\beta), \tag{9.10}$$

satisfies both equations

$$\mathcal{H}_{\mathfrak{p}}(u, \widetilde{u}) = 0, \qquad \mathcal{H}_{\mathfrak{q}}(u, \widehat{u}) = 0 \tag{9.11}$$

simultaneously. Here α and β are defined by the relations $\mathfrak{f}(\alpha) = \mathfrak{p}$ and $\mathfrak{f}(\beta) = \mathfrak{q}$, ξ_0 is an arbitrary constant, and as usual $\widetilde{u} = u(n + 1, m)$, $\widehat{u} = u(n, m+1)$ with $n, m \in \mathbb{Z}$. Now consider the identity (6.9) previously established for Adler's equation, this suggests, and indeed it is straightforward to verify, that (9.10) is a solution of Adler's equation (9.9). It turns out, however, that this solution is not suitable as a *seed* for constructing soliton solutions using the BT. To see this we consider the action of the BT for Adler's equation on the solution (9.10), this is the solution $v = v(n, m)$ defined by the equations

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{r}}(u, \widetilde{u}, v, \widetilde{v}) = 0, \qquad \mathcal{Q}_{\mathfrak{q},\mathfrak{r}}(u, \widehat{u}, v, \widehat{v}) = 0. \tag{9.12}$$

These equations for v are reducible and have only two solutions, which we label $\overline{u}$ and $\underline{u}$, given by

$$\begin{aligned}\overline{u} &= f(\xi_0 + n\alpha + m\beta + \gamma),\\ \underline{u} &= f(\xi_0 + n\alpha + m\beta - \gamma),\end{aligned} \tag{9.13}$$

where $\mathfrak{f}(\gamma) = \mathfrak{r}$. In other words the solution is trivially extended into a third lattice direction associated with the BT (9.12). This kind of augmented solution will be of use later, we call it a *covariant extension*. However this situation is a little disappointing, the application of the BT has not produced a functionally distinct solution from the seed (9.10). As we will see in section 9.3 there is nothing fundamentally wrong with the BT, we have just found a *non-germinating* seed.

The method we employ to find a germinating seed is to look for solutions $u = u(n, m)$ which are related to themselves by the BT,

$$u \overset{\mathfrak{t}}{\sim} u, \tag{9.14}$$

for some parameter $\mathfrak{t} \in \Gamma$ (an idea which can be attributed to Weiss [81, 82]), we will refer to such solutions as 1-cycles (or fixed-points) of the BT. The relation (9.14) means that

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u, \widetilde{u}, u, \widetilde{u}) = 0, \qquad \mathcal{Q}_{\mathfrak{q},\mathfrak{t}}(u, \widehat{u}, u, \widehat{u}) = 0. \tag{9.15}$$

It can be verified that the BT for Adler's equation is a *strong* BT, i.e., the equation is necessary, not just sufficient for compatibility of the system. It follows that if there exists u satisfying (9.14) then u is also a solution of Adler's equation (9.9). So the existence of solutions for (9.9) which are 1-cycles of the BT is equivalent to the commutativity of the mappings in this system. Considering the definition (9.9) it is clear that (9.15) are symmetric biquadratic correspondences. It turns out that these biquadratics commute. To see this it is sufficient to compute the discriminant

$$\begin{aligned}\Delta[\mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u, \widetilde{u}, u, \widetilde{u}), \widetilde{u}] &= c\mathcal{H}_{\mathfrak{t}}(u, u)\\ &= c\frac{t}{2}(1 - [2(1-T)/t^2]u^2 + u^4)\end{aligned} \tag{9.16}$$

where $c = 8pt(pT - tP)/(1 - p^2t^2)$, because this polynomial depends on $\mathfrak{p}$ only in the overall factor.

We will now proceed to solve these simultaneous biquadratics using the method described in section 9.1. An inspection of (9.16) actually reveals that the biquadratics (9.15) are members of a commuting family which is once again of Jacobi type, just with a different modulus than in (9.1). If we define the new curve

$$\Gamma_* = \{(u, U) \mid U^2 = 1 - [2(1 - T)/t^2]u^2 + u^4\} \tag{9.17}$$

then it may be verified that

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{t}}(u, \widetilde{u}, u, \widetilde{u}) = -2p_* t \mathcal{H}_{\mathfrak{p}_*}(u, \widetilde{u}) \tag{9.18}$$

(usefully $\mathcal{H}$ itself does not depend on the modulus k) where $\mathfrak{p}_* = (p_*, P_*) \in \Gamma_*$ is defined by the relation

$$p_*^2 = p\frac{pT - tP}{1 - p^2t^2}, \qquad P_* = \frac{1}{t}\left(p - \frac{pT - tP}{1 - p^2t^2}\right) \tag{9.19}$$

For later convenience we use the symbolic notation $\delta(\mathfrak{p}, \mathfrak{p}_*)$ to represent that $\mathfrak{p}$ and $\mathfrak{p}_*$ are related by (9.19). (Note that this relation defines $\mathfrak{p}_*$ as a two-valued function of $\mathfrak{p}$ and in fact $\mathfrak{p}$ is also a two-valued function of $\mathfrak{p}_*$, moreover $\delta(\mathfrak{p}, \mathfrak{p}_*) \Leftrightarrow \delta(\mathfrak{p}^{-1} \cdot \mathfrak{t}, \mathfrak{p}_*) \Leftrightarrow \delta(\mathfrak{p}, \mathfrak{p}_*^{-1})$.) The result is that the system (9.15) which defines solutions of (9.9) satisfying (9.14) is equivalent to

$$\mathcal{H}_{\mathfrak{p}_*}(u, \widetilde{u}) = 0, \qquad \mathcal{H}_{\mathfrak{q}_*}(u, \widehat{u}) = 0, \tag{9.20}$$

where also the relation $\delta(\mathfrak{q}, \mathfrak{q}_*)$ holds. It remains to parametrise the solution, to do so we define (cf. section 9.1) $\mathfrak{f}_* = (f_*, f_*') \in \Gamma_*$, $\mathfrak{f}_*(0) = \mathfrak{e}_* = (0, 1)$ ($\mathfrak{e}$ does not depend on the modulus k) and the (canonical) solution of (9.20) is simply

$$u(n, m) = f_*(\xi_0 + n\alpha_* + m\beta_*), \tag{9.21}$$

where ξ_0 is an arbitrary constant and α_*, β_* are defined by $\mathfrak{f}_*(\alpha_*) = \mathfrak{p}_*$ and $\mathfrak{f}_*(\beta_*) = \mathfrak{q}_*$. The function (9.21) can be verified as a solution of (9.9) directly, we will refer to it as the *canonical seed solution*. It depends on the parameter $\mathfrak{t}$ chosen in (9.14), in the limit $\mathfrak{t} \longrightarrow \mathfrak{e}$ we find $\Gamma_* \longrightarrow \Gamma$, $\mathfrak{p}_* \longrightarrow \mathfrak{p}^{\pm 1}$, $\mathfrak{q}_* \longrightarrow \mathfrak{q}^{\pm 1}$ and $\mathfrak{f}_* \longrightarrow \mathfrak{f}$, and in this sense these constitute $\mathfrak{t}$-dependent deformations of the curve, lattice parameters and elliptic function.

9.3 One-soliton solution

We will now show that the canonical seed solution (9.21) germinates by applying the BT to it, i.e., by computing (what we shall call) the one-soliton solution. We need to solve the simultaneous ordinary difference equations in v

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u, \widetilde{u}, v, \widetilde{v}) = 0, \quad \mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u, \widehat{u}, v, \widehat{v}) = 0, \tag{9.22}$$

which define the BT, with Bäcklund parameter $\mathfrak{l}$, from u which we take to be (9.21) to a new solution v. The seed itself can be invariantly extended in the lattice direction associated with this BT, we complement the defining equations for the seed (9.20) with the compatible equation

$$\mathcal{H}_{\mathfrak{l}_*}(u, \overline{u}) = 0 \tag{9.23}$$

where $\mathfrak{l}_*$ is related to $\mathfrak{l}$ by $\delta(\mathfrak{l}, \mathfrak{l}_*)$ (cf. (9.19)). In particular then,

$$\begin{aligned} \overline{u}(n, m) &= f_*(\xi_0 + n\alpha_* + m\beta_* + \lambda_*), \\ \underline{u}(n, m) &= f_*(\xi_0 + n\alpha_* + m\beta_* - \lambda_*), \end{aligned} \tag{9.24}$$

where λ_* is defined by the equation $\mathfrak{f}_*(\lambda_*) = \mathfrak{l}_*$. The problem of solving the system (9.22) can be simplified because the invariantly extended seed (9.24) provides two particular solutions, i.e.,

$$\begin{aligned} \mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u, \widetilde{u}, \overline{u}, \widetilde{\overline{u}}) = 0, \quad \mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u, \widehat{u}, \overline{u}, \widehat{\overline{u}}) = 0, \\ \mathcal{Q}_{\mathfrak{p},\mathfrak{l}}(u, \widetilde{u}, \underline{u}, \widetilde{\underline{u}}) = 0, \quad \mathcal{Q}_{\mathfrak{q},\mathfrak{l}}(u, \widehat{u}, \underline{u}, \widehat{\underline{u}}) = 0. \end{aligned} \tag{9.25}$$

(Compare with the non-germinating seed for which these were the only solutions.) From the multilinearity of (9.9) the equations (9.22) are discrete Riccati equations for v. The key observation is that since these equations share two solutions (9.25) they can be simultaneously reduced to homogeneous linear equations for a new variable $\rho = \rho(n,m)$ by the substitution

$$v = \frac{1}{1-\rho}\,\underline{u} - \frac{\rho}{1-\rho}\,\overline{u}. \tag{9.26}$$

After some manipulation, the system for ρ found by substituting (9.26) into (9.22) can be written as

$$\begin{aligned} \widetilde{\rho} &= \left(\frac{p_*l - l_*p}{p_*l + l_*p}\right)\left(\frac{1 - l_*\overline{p}_*u\widetilde{u}}{1 + l_*\underline{p}_*u\widetilde{u}}\right)\rho, \\ \widehat{\rho} &= \left(\frac{q_*l - l_*q}{q_*l + l_*q}\right)\left(\frac{1 - l_*\overline{q}_*u\widehat{u}}{1 + l_*\underline{q}_*u\widehat{u}}\right)\rho, \end{aligned} \tag{9.27}$$

where we mildly abuse notation by introducing the modified parameters

$$\begin{aligned} \overline{\mathfrak{p}}_* &= \mathfrak{p}_* \cdot \mathfrak{l}_*, \quad \underline{\mathfrak{p}}_* = \mathfrak{p}_* \cdot \mathfrak{l}_*^{-1}, \\ \overline{\mathfrak{q}}_* &= \mathfrak{p}_* \cdot \mathfrak{l}_*, \quad \underline{\mathfrak{q}}_* = \mathfrak{p}_* \cdot \mathfrak{l}_*^{-1}, \end{aligned} \tag{9.28}$$

(p_* and q_* do not depend on lattice shifts). We take (9.27) as the defining equations for ρ, which we refer to as the plane-wave factor. (The compatibility of this system for ρ can be verified directly.)

The one-soliton solution for (9.9) is therefore the function $v = v(n,m)$ defined in (9.26) in terms of $\overline{u}, \underline{u}$ (9.24) and the plane-wave factor ρ (9.27) which itself depends on the function u (9.21).

9.4 2-cycles of the Bäcklund transformation

In section (9.2) we found a seed solution for Adler's equation which was actually the simplest case of the more general problem to find the N-*cycles* of the BT. These are solutions $u_1 \dots u_N$ for which

$$u_1 \overset{\mathfrak{t}_1}{\sim} u_2, \qquad u_2 \overset{\mathfrak{t}_2}{\sim} u_3, \qquad \dots \qquad u_N \overset{\mathfrak{t}_N}{\sim} u_1 \tag{9.29}$$

for some parameters $\mathfrak{t}_1 \dots \mathfrak{t}_N \in \Gamma$. That such solutions of (9.9) exist is not a-priori obvious, however their existence is equivalent to the commutativity of two rank-N (i.e., N equations in N unknowns) 2-valued mappings.

In this section we consider this problem in the case $N = 2$, that is to find u for which there exists v such that

$$u \overset{\mathfrak{t}_1}{\sim} v, \qquad v \overset{\mathfrak{t}_2}{\sim} u, \tag{9.30}$$

for some fixed parameters $\mathfrak{t}_1, \mathfrak{t}_2 \in \Gamma$. If $\mathfrak{t}_1 = \mathfrak{t}_2$ this includes *all* solutions of (9.9) because the BT relation is symmetric. If $\mathfrak{t}_1 \neq \mathfrak{t}_2$ we will refer to any such u as a *2-cycle* of the BT. Clearly the solution v is also a 2-cycle of the BT. By definition, the equations implied by (9.30) are

$$\mathcal{Q}_{\mathfrak{p},\mathfrak{t}_1}(u, \widetilde{u}, v, \widetilde{v}) = 0, \qquad \mathcal{Q}_{\mathfrak{p},\mathfrak{t}_2}(u, \widetilde{u}, v, \widetilde{v}) = 0, \tag{9.31}$$

$$\mathcal{Q}_{\mathfrak{q},\mathfrak{t}_1}(u, \widehat{u}, v, \widehat{v}) = 0, \qquad \mathcal{Q}_{\mathfrak{q},\mathfrak{t}_2}(u, \widehat{u}, v, \widehat{v}) = 0. \tag{9.32}$$

We will treat the systems (9.31) and (9.32) as mappings $(u, v) \overset{\mathfrak{p}}{\mapsto} (\widetilde{u}, \widetilde{v})$ and $(u, v) \overset{\mathfrak{q}}{\mapsto} (\widehat{u}, \widehat{v})$ respectively. As such they are 2-valued, that is if we fix points (u, v) then there are two possible values of the pair $(\widetilde{u}, \widetilde{v})$ which satisfy (9.31), and two possible values of the pair $(\widehat{u}, \widehat{v})$ which satisfy (9.32). In order to construct the general simultaneous solution of these mappings we begin in the following subsection by solving the first of them, the system (9.31).

The explicit solution of the rank-2, 2-valued mapping and the deformed elliptic curve

Consider the mapping $(u, v) \overset{\mathfrak{p}}{\mapsto} (\widetilde{u}, \widetilde{v})$ defined by the system (9.31). This involves shifts in the discrete variable n only, hence throughout this section, in which we solve (9.31), we will restrict our attention to the variable n alone, that is we consider $(u, v) = (u(n), v(n))$. In the first step toward the solution of this mapping we use the identity (6.9), which shows that the defining equations (9.31) imply that

$$\begin{aligned} \mathcal{H}_{\mathfrak{p}}(u, \widetilde{u})\mathcal{H}_{\mathfrak{p}}(v, \widetilde{v}) &= \mathcal{H}_{\mathfrak{t}_1}(u, v)\mathcal{H}_{\mathfrak{t}_1}(\widetilde{u}, \widetilde{v}), \\ \mathcal{H}_{\mathfrak{p}}(u, \widetilde{u})\mathcal{H}_{\mathfrak{p}}(v, \widetilde{v}) &= \mathcal{H}_{\mathfrak{t}_2}(u, v)\mathcal{H}_{\mathfrak{t}_2}(\widetilde{u}, \widetilde{v}). \end{aligned} \tag{9.33}$$

Elimination of the common LHS from the derived system (9.33) leads naturally to the following

$$J\widetilde{J} = 1, \qquad J = \frac{\mathcal{H}_{\mathfrak{t}_1}(u,v)}{\mathcal{H}_{\mathfrak{t}_2}(u,v)}. \tag{9.34}$$

The dynamical equation for the new variable J is trivial and provides a first integral of the derived mapping defined by (9.33). In the remainder of this section we solve the essentially technical problem of using this to find the explicit solution of (9.31).

Fixing $J(0)$ from the initial data $(u(0), v(0))$, the second equation of (9.34) amounts to a mildly non-autonomous biquadratic constraint on u and v which is actually of Jacobi type:

$$\mathcal{H}_{\mathfrak{t}_2}(u,v)J - \mathcal{H}_{\mathfrak{t}_1}(u,v) \quad = \quad t_* \frac{t_2 - t_1 J}{t_1 t_2}\, \mathcal{H}_{\mathfrak{t}_*}(u,v), \tag{9.35}$$

the new parameter $\mathfrak{t}_* = (t_*, T_*)$ is defined by the equations

$$t_*^2 = t_1 t_2 \frac{t_1 - t_2 J}{t_2 - t_1 J}, \quad T_* = \frac{t_2 T_1 - t_1 T_2 J}{t_2 - t_1 J} \tag{9.36}$$

and lies on a new curve, $\mathfrak{t}_* \in \Gamma_*$,

$$\Gamma_* = \{(x, X) : X^2 = x^4 + 1 - (k_* + 1/k_*)x^2\}, \tag{9.37}$$

$$k_* + \frac{1}{k_*} = \frac{t_1 t_2 \,(k + 1/k)\,(J + 1/J) + 2\left(T_1 T_2 - 1 - t_1^2 t_2^2\right)}{t_1 t_2 (J + 1/J) - t_1^2 - t_2^2}, \tag{9.38}$$

which is again of Jacobi type but with a new elliptic modulus k_*. The parameter $\mathfrak{t}_* \in \Gamma_*$ defined by (9.36) depends on J so it is non-autonomous, the curve itself depends on J only through the combination $J + 1/J$ which from (9.34) is clearly autonomous, it follows that $\widetilde{\Gamma}_* = \Gamma_*$.

The biquadratic constraint $\mathcal{H}_{\mathfrak{t}_*}(u,v) = 0$ can be used to eliminate v from (9.31), solving for v we find

$$v = \frac{uT_* + t_* U}{1 - t_*^2 u^2}. \tag{9.39}$$

this choice of sign for U is without loss of generality because choosing the other sign leads to (9.39) with $t_* \to -t_*$, under which the equations (9.36) defining $\mathfrak{t}_*$ are invariant. Note that the relation (9.39) applied at $n = 0$, taken with the definition of $\mathfrak{t}_*$ (9.36), fixes t_* uniquely at the origin in terms of $\mathfrak{u}(0) \in \Gamma_*$ and $v(0)$ which we take as the initial conditions.

On substituting for v using (9.39), the system (9.31) reduces to two relations between $\mathfrak{u}$ and $\widetilde{\mathfrak{u}}$. It turns out that these relations are compatible only if

$$t_*\widetilde{t}_* + t_1t_2 = 0. \tag{9.40}$$

It can be confirmed that this constraint is compatible with the definition of $\mathfrak{t}_*$ (9.36), in fact (9.40) refines this definition by fixing $\mathfrak{t}_*$ uniquely at each iteration in terms of its previous value. Note also that $\widetilde{\widetilde{\mathfrak{t}}}_* = \mathfrak{t}_*$ so the value of the parameter oscillates.

When (9.40) holds the substitution of (9.39) reduces (9.31) to a single equation on the curve Γ_*,

$$\widetilde{\mathfrak{u}} = \mathfrak{p}_* \cdot \mathfrak{u}, \tag{9.41}$$

where the new (non-autonomous) parameter $\mathfrak{p}_* = (p_*, P_*) \in \Gamma_*$ is defined by the equations

$$\begin{aligned} &\frac{p_*^2 - pp_{12}}{p_*(p - p_{12})} = \frac{t_*^2 + t_1t_2}{t_*(t_1 + t_2)}, \\ &P_* = \frac{1}{t_{12}}(p - p_{12}) + \frac{p_*^2 - pp_{12}}{p - p_{12}}\left(\frac{T_1 - T_2}{t_1 - t_2} - \frac{t_1 - t_2}{p_1 - p_2}(p - p_{12})p_1p_2\right). \end{aligned} \tag{9.42}$$

We have used the notation:

$$\mathfrak{p}_1 = \mathfrak{p} \cdot \mathfrak{t}_1^{-1}, \quad \mathfrak{p}_2 = \mathfrak{p} \cdot \mathfrak{t}_2^{-1}, \quad \mathfrak{t}_{12} = \mathfrak{t}_1 \cdot \mathfrak{t}_2, \quad \mathfrak{p}_{12} = \mathfrak{p} \cdot \mathfrak{t}_{12}^{-1}$$

where $\mathfrak{p}_1 = (p_1, P_1)$ etc. The equations (9.42) for $\mathfrak{p}_*$ have two solutions, so there is some choice in the parameter $\mathfrak{p}_*$ at each iteration of (9.41). In fact

$$\widetilde{\mathfrak{p}}_* \in \{\mathfrak{p}_*^{-1}, \mathfrak{t}_*^{-1} \cdot \widetilde{\mathfrak{t}}_* \cdot \mathfrak{p}_*\}, \tag{9.43}$$

which can be verified directly. The existence of this choice in the value of $\mathfrak{p}_*$ is a consequence of the underlying mapping defined by (9.31) being 2-valued. Apart from the book-keeping involved in this detail, the solution of the dynamical equation defined by (9.41) and (9.43) is trivial.

We conclude this section by giving a concrete example. Let us define the *canonical* solution by choosing from (9.43) $\widetilde{\mathfrak{p}}_* = \mathfrak{t}_*^{-1} \cdot \widetilde{\mathfrak{t}}_* \cdot \mathfrak{p}_*$, which fixes $\mathfrak{p}_*$ at each iteration in terms of its value at the origin, $\mathfrak{p}_*(0)$, moreover $\widetilde{\widetilde{\mathfrak{p}}}_* = \mathfrak{p}_*$ so the value of $\mathfrak{p}_*$ oscillates. Now from (9.41) we find the solution directly,

$$\mathfrak{u}(n) = \begin{cases} \mathfrak{p}_*(0)^{n/2} \cdot \mathfrak{p}_*(1)^{n/2} \cdot \mathfrak{u}(0), & n \text{ even}, \\ \mathfrak{p}_*(0)^{(n+1)/2} \cdot \mathfrak{p}_*(1)^{(n-1)/2} \cdot \mathfrak{u}(0), & n \text{ odd}, \end{cases} \tag{9.44}$$

where $\mathfrak{p}_*(1) = \mathfrak{t}_*(0)^{-1} \cdot \mathfrak{t}_*(1) \cdot \mathfrak{p}_*(0)$. The canonical solution of the system (9.31) itself is actually the pair $(u(n), v(n))$, but $v(n)$ can also be found from $\mathfrak{u}(n) = (u(n), U(n))$ given in (9.44) by using the relation (9.39). To parametrise in elliptic functions we let $\mathfrak{f}_* = (f_*, f_*') \in \Gamma_*$ and $\mathfrak{f}_*(0) = \mathfrak{e}_* = (0, 1)$. We may then write the solution in the following way

$$u(n) = \begin{cases} f_*(\xi_0 + n(\alpha_* + \alpha_{**})/2), & n \text{ even}, \\ f_*(\xi_0 + n(\alpha_* + \alpha_{**})/2 + (\alpha_* - \alpha_{**})/2), & n \text{ odd}, \end{cases} \tag{9.45}$$

where α_* and α_{**} are defined by the relations $\mathfrak{f}_*(\alpha_*) = \mathfrak{p}_*(0)$ and $\mathfrak{f}_*(\alpha_*) = \mathfrak{p}_*(1)$.

The solution on the lattice

So let us review. As we have established, any 2-cycle of the BT for Adler's equation satisfies the coupled systems (9.31) and (9.32). In the previous section we have solved the first system (9.31) by reducing it to a single equation on the new curve Γ_*. The second system, defined by (9.32), differs from the first only in the change of parameter $\mathfrak{p} \to \mathfrak{q}$ (and that it involves shifts in the other direction, the discrete variable m and not n). In particular the (mildly non-autonomous) biquadratic constraint is the same: $\mathcal{H}_{\mathfrak{t}_*}(u, v) = 0$, which therefore holds throughout the lattice. We can use the same substitution (9.39) to eliminate v from (9.32) which, provided $\widehat{t}_* t_* + t_1 t_2 = 0$ (note from (9.40) we see that $\widehat{\mathfrak{t}}_* = \widetilde{\mathfrak{t}}_*$ so that $\mathfrak{t}_*$ is a function of $n + m$ only) then reduces to the single equation

$$\widehat{\mathfrak{u}} = \mathfrak{q}_* \cdot \mathfrak{u}, \tag{9.46}$$

on the curve Γ_*. The new parameter $\mathfrak{q}_*$ is defined by the relations (9.42) with the change $\mathfrak{p} \to \mathfrak{q}$ and $\mathfrak{p}_* \to \mathfrak{q}_*$, and it satisfies the dynamical equation

$$\widehat{\mathfrak{q}}_* \in \{\mathfrak{q}_*^{-1}, \mathfrak{t}_*^{-1} \cdot \widehat{\mathfrak{t}}_* \cdot \mathfrak{q}_*\}. \tag{9.47}$$

So we have the solutions of both correspondences (9.31) and (9.32), they are defined in terms of the mappings (9.41) and (9.46).

To find the full solution it remains to couple mappings (9.41) and (9.46), while taking into account the correspondences (9.47) and (9.43). These mappings are compatible, i.e., $\widehat{\widetilde{\mathfrak{u}}} = \widetilde{\widehat{\mathfrak{u}}}$, if and only if

$$\widehat{\mathfrak{p}}_* \cdot \mathfrak{q}_* = \widetilde{\mathfrak{q}}_* \cdot \mathfrak{p}_* \tag{9.48}$$

throughout the lattice. Now, from the definition of $\mathfrak{p}_*$ together with the observation that $\widetilde{\mathfrak{t}}_* = \widehat{\mathfrak{t}}_*$, we see that $\widehat{\mathfrak{p}}_*$ is subject (in principle) to the same choice as $\widetilde{\mathfrak{p}}_*$ in (9.43). Similarly $\widetilde{\mathfrak{q}}_*$ is subject to the same choice as $\widehat{\mathfrak{q}}_*$ in (9.47). However the condition (9.48) constrains these choices, specifically we must choose that

$$\widehat{\mathfrak{p}}_* = \mathfrak{t}_*^{-1} \cdot \widehat{\mathfrak{t}}_* \cdot \mathfrak{p}_*, \qquad \widetilde{\mathfrak{q}}_* = \mathfrak{t}_*^{-1} \cdot \widetilde{\mathfrak{t}}_* \cdot \mathfrak{q}_*. \tag{9.49}$$

So in fact the dynamics of $\mathfrak{p}_*$ in the $\widehat{\ }$ direction and $\mathfrak{q}_*$ in the $\widetilde{\ }$ direction are single-valued. When the dynamics of $\mathfrak{p}_*$ and $\mathfrak{q}_*$ satisfy (9.49) the equations (9.41) and (9.46) can be coupled and the full solution on the lattice, $u(n,m)$, follows from their general simultaneous solution $\mathfrak{u}(n,m)$.

It is natural to define the *canonical 2-cycle of the BT* by fixing the choices (9.43) and (9.47) so that

$$\widetilde{\mathfrak{p}}_* = \mathfrak{p}_* \cdot \widetilde{\mathfrak{t}}_* \cdot \mathfrak{t}_*^{-1}, \qquad \widehat{\mathfrak{q}}_* = \mathfrak{q}_* \cdot \widehat{\mathfrak{t}}_* \cdot \mathfrak{t}_*^{-1}. \tag{9.50}$$

Given (9.49) this means that $\widehat{\mathfrak{p}}_* = \widetilde{\mathfrak{p}}_*$ and $\widehat{\mathfrak{q}}_* = \widetilde{\mathfrak{q}}_*$ throughout the lattice, so both parameters are a function of $n+m$ only, moreover they oscillate, $\widetilde{\widetilde{\mathfrak{p}}}_* = \mathfrak{p}_*$ etc. This canonical solution written explicitly is

$$\mathfrak{u}(n,m) = \begin{cases} \mathfrak{t}_*(0)^{-(n+m)/2} \cdot \mathfrak{t}_*(1)^{(n+m)/2} \cdot \mathfrak{p}_*(0)^n \cdot \mathfrak{q}_*(0)^m \cdot \mathfrak{u}(0), \\ \hfill n+m \text{ even}, \\ \mathfrak{t}_*(0)^{-(n+m-1)/2} \cdot \mathfrak{t}_*(1)^{(n+m-1)/2} \cdot \mathfrak{p}_*(0)^n \cdot \mathfrak{q}_*(0)^m \cdot \mathfrak{u}(0), \\ \hfill n+m \text{ odd}. \end{cases} \tag{9.51}$$

The parametrisation in terms of elliptic functions follows in the same way as for (9.44). We finish this section with a number of remarks regarding the obtained results.

Remark 1

Choosing initial data so that J defined in (9.34) is equal to 1 at the origin has the consequence that $J = 1$ throughout the lattice. Fixing $J = 1$ in (9.34) and using this to eliminate v from the systems (9.31) and (9.32) which define the 2-cycle of the BT, yields the reduced system

$$\begin{aligned} \mathcal{Q}_{\mathfrak{p},\mathfrak{t}_1\cdot\mathfrak{t}_2}(u,\widetilde{u},u,\widetilde{u}) = 0, \\ \mathcal{Q}_{\mathfrak{q},\mathfrak{t}_1\cdot\mathfrak{t}_2}(u,\widehat{u},u,\widehat{u}) = 0. \end{aligned} \tag{9.52}$$

But this is exactly the system that defines the 1-cycle of the BT:

$$u \overset{\mathfrak{t}_1\cdot\mathfrak{t}_2}{\sim} u, \tag{9.53}$$

where the Bäcklund parameter associated to this solution is the point $\mathfrak{t}_1 \cdot \mathfrak{t}_2 \in \Gamma$. So the solution found in this section as a 2-cycle of the BT is a generalisation of the solution found in section 4.1 (as a 1-cycle of the BT) because it reduces to that solution if we choose the initial data so that $J = 1$.

Remark 2

In the limit $\mathfrak{t}_2 \longrightarrow \mathfrak{t}_1^{-1}$ we find that $\Gamma_* \longrightarrow \Gamma$ and $\mathfrak{p}_*, \mathfrak{q}_* \longrightarrow \mathfrak{p}^{\pm 1}, \mathfrak{q}^{\pm 1}$. In this sense the new curve and parameters are deformations of the original curve and lattice parameters associated to the equation (9.9). In the same limit the solution presented here goes to the non-germinating seed solution (9.10).

Remark 3

Given that the 2-cycle of the BT is defined by the relations $u \overset{\mathfrak{t}_1}{\sim} v$, $u \overset{\mathfrak{t}_2}{\sim} v$, we can naturally construct a new solution $\overline{u}$ by superposition,

$$\mathcal{Q}_{\mathfrak{t}_1,\mathfrak{t}_2}(u, v, v, \overline{u}) = 0, \tag{9.54}$$

so that $v \overset{\mathfrak{t}_2}{\sim} \overline{u}$ and $v \overset{\mathfrak{t}_1}{\sim} \overline{u}$, and hence $\overline{u}$ is another 2-cycle of the BT. Clearly by iteration of this procedure we can construct a sequence of such solutions. Note however that solutions related in this way are associated with the same deformed curve Γ_*.

Remark 4

Let us restrict our attention to the mapping defined by (9.31) in the special case that $\mathfrak{p} = \mathfrak{t}_1$, that is we choose one lattice parameter of the equation to coincide with one of the Bäcklund parameters. In this case the first equation of (9.31) reduces to the trivial equation $(\widetilde{u} - v)(\widetilde{v} - u) = 0$. Choosing the solution $v = \widetilde{u}$ brings the second equation of (9.31) to

$$\mathcal{Q}_{\mathfrak{t}_1,\mathfrak{t}_2}(u, \widetilde{u}, \widetilde{u}, \widetilde{\widetilde{u}}) = 0. \tag{9.55}$$

This scalar second-order ordinary difference equation is the two-step periodic "staircase" reduction of Adler's equation considered first by Joshi et. al. [32]. Note that, apart from notational differences, the equation (9.55) coincides with the superposition formula for 2-cycles of the BT, (9.54). The first integral of the mapping defined by (9.31) given in (9.34) is unchanged in the case $\mathfrak{p} = \mathfrak{t}_1$ because it is independent of the parameter $\mathfrak{p}$. This integral was first given for (9.55) in [32].

Now, it is of some interest to consider the solution of (9.55) in its own right. The solution method of section 4 applied in this case leads to (9.41) being simplified to the (single-valued) mapping $\widetilde{\mathfrak{u}} = \mathfrak{t}_* \cdot \mathfrak{u}$. The solution itself can be written explicitly as

$$\mathfrak{u}(n) = \begin{cases} \mathfrak{t}_*(0)^{n/2} \cdot \mathfrak{t}_*(1)^{n/2} \cdot \mathfrak{u}(0), & n \text{ even}, \\ \mathfrak{t}_*(0)^{(n+1)/2} \cdot \mathfrak{t}_*(1)^{(n-1)/2} \cdot \mathfrak{u}(0), & n \text{ odd}. \end{cases} \tag{9.56}$$

Note that the parameter $\mathfrak{t}_*$ that was defined originally in terms of $(u(0), v(0))$ is now defined in terms of $(u(0), u(1))$ because we have chosen $v = \widetilde{u}$, so that in particular, $v(0) = u(1)$.

9.5 Conclusion

We have given three distinct solutions of Adler's equation in Jacobi form. The most elementary is the 1-cycle of the BT. Interestingly the construction involves a deformation of the elliptic curve associated to the equation, and in fact this kind of situation has a precedent in different area [39]. This deformation is characterised more precisely in chapter 10.

We also applied the BT to the 1-cycle solution to yield the one-soliton solution, this was computed explicitly up to the integration of the 'plane-wave' factor defined by compatible first-order homogeneous linear ordinary difference equations. (We remark that Adler's equation is distinguished from its degenerate sub-cases (6.11) by this situation, in all those cases the system which defines the plane-waves is additionally *autonomous*.)

In the third instance we constructed solutions which arise as a 2-cycle of the BT, which were shown to generalise the 1-cycle solution. Like the 1-cycle, the 2-cycle of the BT is in terms of a deformation of the elliptic curve associated with the equation itself. The new features in this case are that the deformation of the curve depends on the choice of initial data, and the shifts on the deformed curve are themselves non-autonomous (in fact they oscillate with period 2).

Chapter 10

Perspectives

10.1 N-Cycles of the Bäcklund transformation

A theme which has emerged in both parts of the thesis is the notion of N-Cycles of the BT. This is an idea originally due to Weiss [81] and which he connected, in the continuous case, to finite-gap solitons. The connection between periodic solitons on the lattice and N-Cycles of the BT is manifest because the N-Cycle just arises as a periodic reduction of the higher-dimensional lattice. In this situation the integrability of the reduced system is also clear because the equations defining it are overdetermined (lattice equation on a cube for reductins of SKP 5, and commuting biquadratic correspondences for Adler's equation in Chapter 9). The obvious open question here, which is to the author's knowledge unanswered, is the discrete analog of the original construction due to Krichever-Novikov [37, 38] of finite-gap solutions of the KP equation. I.e. by what means does Adler's equation arise from a reduction of a lattice KP type system?

10.2 Connecting Parts I and II

It is the impression of the author that Adler's equation is a very natural object associated to a family of commuting biquadratic correspondences. Superfical similarities with the ideas presented in part I of the thesis conneting the Möbius group naturally with the lattice Schwarzian KdV equation. It therefore seems natural to ask what kind of lattice systems emerge naturally if we study the set of biquadratic correspondences.

We finish this section, and hence the thesis, with a further consideration of the biquadratic correspondence, drawing analogies where possible with the Möbius group, in particular the facts of chapter 2. Of course technically the biquadratics are much more difficult to study, in particular they do not form a group (although the family with shared discriminant constitute a 2-group [16]).

10.3 The generic non-degenerate symmetric biquadratic

In chapter 6, section 9.1, we constructed a family of biquadratics h with common discriminant $\mathcal{R}(u) = 1 - (k + 1/k)u^2 + u^4$. If we had supposed that $h(\sqrt{k}, p) = 0$ rather than $h(0, p) = 0$ (6.5) then we would have found (up to an overall scaling) that

$$h(u,\widetilde{u}) = \frac{1}{2\sqrt{k}}(p^2+u^2+\widetilde{u}^2+p^2u^2\widetilde{u}^2) - \frac{\sqrt{k}}{2}(1+p^2u^2+u^2\widetilde{u}^2+\widetilde{u}^2p^2) + (k-\frac{1}{k})pu\widetilde{u}. \tag{10.1}$$

which is just a re-parametrisation of the same set of biquadratics. Note that a direct calculation confirms that in fact $\Delta[h(u,\widetilde{u}),\widetilde{u}] = \mathcal{R}(u)\mathcal{R}(p)$. The biquadratic (10.1) is symmetric in u and $\widetilde{u}$ by construction, but the symmetry also in the parameter p might not have been expected. We refer to (10.1) as a symmetric *triquadratic*.

In this chapter we consider the symmetric triquadratic associated with discriminant $R(u) = (u - e)(u - a)(u - b)(u - c)$ where we suppose e, a, b

and c are distinct, we will refer to these as *branch points*. Our aim is to give a characterisation of the deformation introduced in the construction of the seed solution for Adler's equation (section 9.2). This characterisation only emerges when we move away from a particular curve (like the Jacobi or Weierstrass forms) to this more parameter-rich but equivalent (by Möbius conjugation) case. However, along the way we will draw analogies between dynamical aspects of the biquadratic correspondence and the Möbius group. This is natural becuase the parametrisation of the biquadratic considered here is similar in spirit to the parametrisation of Möbius transformations based on fixed-points described in chapter 3 theorem 3. In fact a direct association can be made between M and the set of biquadratics which have discriminant R that is a quadratic squared, so the biquadratic correspondence is of the form

$$(\mathsf{m}(u) - \widetilde{u})(\mathsf{m}(\widetilde{u}) - u) = 0. \tag{10.2}$$

for some $\mathsf{m} \in \mathsf{M}$. The roots of R in this case are the fixed-points of m, of which there are at most two.

Construction

The triquadratic, associated to the four *distinct* branch points $\{e, a, b, c\}$, can be obtained in the following way. Start with the expression

$$G = \frac{r_1(y)r_2(e) - r_2(y)r_1(e)}{2(y - e)} \tag{10.3}$$

where for compactness we have introduced the polynomials

$$\begin{aligned} r_1(y) &= (y - u)(y - v)(y - w), \\ r_2(y) &= (y - a)(y - b)(y - c). \end{aligned} \tag{10.4}$$

G is actually a polynomial of degree two in y (and symmetric in y and e). The triquadratic under consideration may be defined in terms of the discriminant of G with respect to y, specifically we define

$$H(u, v, w; e, a, b, c) := \frac{1}{r_2(e)}\Delta[G, y]. \tag{10.5}$$

This expression is clearly quadratic and symmetric in u, v, w. A direct calculation confirms that

$$\Delta[H(u, v, w; e, a, b, c), v] = R(u)R(w), \tag{10.6}$$

where R is the polynomial

$$R(u) = (u - e)(u - a)(u - b)(u - c). \tag{10.7}$$

For fixed $\{e, a, b, c\}$ the equation

$$H(u, v, w; e, a, b, c) = 0 \tag{10.8}$$

defines w as a (generally two-valued) function of u and v. This can be interpreted as a product which turns $\widehat{\mathbb{C}}$ into an abelian two-group (cf. [16]). The commutativity of the product is clear from the $u \leftrightarrow v$ symmetry. The associativity is non-trivial but can be verified, it is equivalent to the commutativity property described below.

One interesting observation about H (which we do not exploit here) is the remarkable symmetry,

$$H(u, v, w; e, a, b, c) = 0 \quad \Leftrightarrow \quad H(a, b, c; e, u, v, w) = 0, \tag{10.9}$$

which is apparent from a consideration of our construction.

We also observe that for any Möbius transformation $\mathsf{m} \in \mathsf{M}$,

$$\begin{aligned} &H(\mathsf{m}(u), \mathsf{m}(v), \mathsf{m}(w); \mathsf{m}(e), \mathsf{m}(a), \mathsf{m}(b), \mathsf{m}(c)) = 0 \\ &\qquad\qquad \Leftrightarrow \quad H(u, v, w; e, a, b, c) = 0. \end{aligned} \tag{10.10}$$

I.e., the triquadratic is naturally Möbius invariant. Möbius changes of variables preserve the form of the biquadratic correspondence, this Möbius invariance just reflects the fact that our co-ordinates are homogeneous.

Behaviour at the branch-points

Let us for the moment view the triquadratic correspondence (10.8) as a family of biquadratic correspondences parametrized by the variable w. As such, the biquadratics associated to the particular choice of $w \in \{e, a, b, c\}$ are special. In fact the vanishing of the discriminant (10.6) means that for these values of w, (10.8) defines a one-to-one correspondence between u and v, in other words u and v are related by a Möbius transformation.

It is straightforward to calculate which Möbius transformations these are, they can be written in terms of just two, $\mathsf{i}, \mathsf{j} \in \mathsf{M}$,

$$\begin{aligned} w = e &\quad\Rightarrow\quad v = u, \\ w = a &\quad\Rightarrow\quad v = \mathsf{i}(u), \\ w = b &\quad\Rightarrow\quad v = \mathsf{j}(u), \\ w = c &\quad\Rightarrow\quad v = \mathsf{i}(\mathsf{j}(u)), \end{aligned} \tag{10.11}$$

where i and j are the *unique* (cf. theorem 6 page 25) commuting Möbius involutions for which

$$\begin{aligned} \mathsf{i}(e) = a, \quad \mathsf{i}(b) = c, \\ \mathsf{j}(e) = b, \quad \mathsf{j}(a) = c. \end{aligned} \tag{10.12}$$

I.e., they correspond to kleinian permutations of the points $\{e, a, b, c\}$. It is clear that $\langle \mathsf{i}, \mathsf{j} \rangle < \mathsf{S}(e, a, b, c)$, this group is discussed further in appendix A.2.

So, to each non-degenerate triquadratic we may associate a unique group of Möbius involutions $\langle \mathsf{i}, \mathsf{j} \rangle$. This associated group will be important to us here.

The commutativity property

It can be verified that the composition $u \xrightarrow{p} \widetilde{u}$, $\widetilde{u} \xrightarrow{q} \widehat{\widetilde{u}}$ defined by

$$H(u, \widetilde{u}, p; e, a, b, c) = 0, \qquad H(\widetilde{u}, \widehat{\widetilde{u}}, q; e, a, b, c) = 0, \tag{10.13}$$

is equivalent to the composition $u \xrightarrow{q} \widehat{u}$, $\widehat{u} \xrightarrow{p} \widetilde{\widehat{u}}$ defined by

$$H(u, \widehat{u}, q; e, a, b, c) = 0, \qquad H(\widehat{u}, \widetilde{\widehat{u}}, p; e, a, b, c) = 0. \tag{10.14}$$

By which we mean that the set of possible values of $\widetilde{\widehat{u}}$ coincides with the set of possible values of $\widehat{\widetilde{u}}$, this set containing at most four points.

This fact in the case that $q \in \{e, a, b, c\}$ (so that one of the mappings is single-valued and in fact belongs to $\langle \mathsf{i}, \mathsf{j} \rangle$) can be stated as

$$H(u, v, w; e, a, b, c) = 0 \;\Leftrightarrow\; H(u, \mathsf{m}(v), \mathsf{m}(w); e, a, b, c) = 0, \quad \mathsf{m} \in \langle \mathsf{i}, \mathsf{j} \rangle. \tag{10.15}$$

In other words, it reveals symmetry of the associated biquadratic. If we regard the (un-ordered) triplet $\{u, v, w\}$ as a solution of (10.8), then clearly $\{u, \mathsf{i}(v), \mathsf{i}(w)\}$ is also a solution. One may ask how many other solutions may be constructed by exploiting this symmetry. In fact the number is sixteen as may be seen from table 10.1.

Triplet	multiplicity
$\{u, v, w\}$	1
$\{u, \mathsf{i}(v), \mathsf{i}(w)\}$	3
$\{u, \mathsf{j}(v), \mathsf{j}(w)\}$	3
$\{u, \mathsf{i}(\mathsf{j}(v)), \mathsf{i}(\mathsf{j}(w))\}$	3
$\{\mathsf{i}(u), \mathsf{j}(v), \mathsf{i}(\mathsf{j}(w))\}$	6

Table 10.1: Counting the elements in the symmetry group of the triquadratic equation, the multiplicity arises by permutations of u, v, w. We observe that this group is abelian and each element has order two.

The cyclic biquadratic

Let us now examine further the object we introduced for the construction of H, the polynomial G (10.3) which we now treat as a biquadratic correspondence $G = G(y, e)$. In fact this is a very particular biquadratic. Specifically it can be shown that,

$$G(x, y) = 0, \quad G(x, z) = 0, \qquad \Rightarrow \qquad (y - z)G(y, z) = 0. \tag{10.16}$$

The dynamics defined by iteration of such a correspondence are cyclic as in the following diagram:

$$\longleftarrow x \longleftrightarrow y \longleftrightarrow z \longleftrightarrow x \longleftrightarrow y \longrightarrow$$

In this way, the correspondence partitions $\widehat{\mathbb{C}}$ into distinct orbits, each containing at most three points. Note that two such orbits are the sets $\{u, v, w\}$ and $\{a, b, c\}$. This construction, and the observation that a biquadratic is fixed by its action on a set of five distinct points, demonstrates there is a bijective correspondence from each cyclic biquadratic to a distinct pair of its orbits. Because of these facts it is natural to associate such cyclic biquadratics to a generalisation of the Möbius involution, cf. section 2.3.

Deformation of triquadratics

For convenience in this section we will write

$$H(u, v, w) = H(u, v, w; e, a, b, c)$$

and also define a deformed triquadratic

$$H_*(u, v, w) = H(u, v, w; e_*, a_*, b_*, c_*) \tag{10.17}$$

and a deformed polynomial

$$R_*(u) = (u - e_*)(u - a_*)(u - b_*)(u - c_*) \tag{10.18}$$

where e_* is a new parameter and a_*, b_*, c_* are fixed by the relations

$$\begin{aligned} a_* &= \mathsf{i}(e_*), \\ b_* &= \mathsf{j}(e_*), \\ c_* &= \mathsf{i}(\mathsf{j}(e_*)), \end{aligned} \tag{10.19}$$

where i, j are defined in 10.12. This is a deformation in the sense that $e_* \to e$ implies $H_* \to H$ and $r_* \to r$. By construction, the set of all such deformed triquadratics is exactly the set of triquadratics which share the same associated Möbius group $\langle \mathsf{i}, \mathsf{j} \rangle$.

Now, to the deformation parameter e_* we may associate a new parameter t, fixed uniquely by the equation

$$H(e_*, e_*, t) = 0, \qquad t \neq e. \tag{10.20}$$

In fact t defined by (10.20) may be written explicitly as

$$t = e - 4\frac{R_*(e)}{R'_*(e)}. \tag{10.21}$$

At this point let us observe that

$$R_*(u) = \gamma H(u, u, t) \tag{10.22}$$

for some constant γ. This can be seen by observing that the RHS is a polynomial of degree four which vanishes on e_*, and by (10.15) it also vanishes on $\mathsf{i}(e_*)$, $\mathsf{j}(e_*)$ and $\mathsf{i}(\mathsf{j}(e_*))$. But by (10.19) these are just the roots of R_*.

We claim that there exists a Möbius transformation, m, such that

$$\begin{aligned} \mathsf{m}(e_*) &= t, \\ \mathsf{m}(a_*) &= a, \\ \mathsf{m}(b_*) &= b, \\ \mathsf{m}(c_*) &= c. \end{aligned} \tag{10.23}$$

For the existence of m it is sufficient that the cross-ratio for these two sets coincide

$$\frac{(t-a)(b-c)}{(t-b)(a-c)} = \frac{(e_*-a_*)(b_*-c_*)}{(e_*-b_*)(a_*-c_*)}, \tag{10.24}$$

and a calculation reveals this is true by virtue of (10.20).

This means that the deformed triquadratic is Möbius conjugate to a triquadratic with branch points $\{t, a, b, c\}$. Specifically, using (10.10) together with (10.23) we see that

$$H_*(u, v, w) = 0 \;\Leftrightarrow\; H(\mathsf{m}(u), \mathsf{m}(v), \mathsf{m}(w); t, a, b, c) = 0. \tag{10.25}$$

The converse also holds as can be seen by applying the deformation again to H_* and choosing $e_{**} = e$ so that $H_{**} = H$ etc. Specifically, we may define a parameter t_* by the relation

$$H_*(e, e, t_*) = 0, \qquad t_* \neq e_*, \tag{10.26}$$

which is equivalent to

$$t_* = e_* - 4\frac{R(e_*)}{R'(e_*)}. \tag{10.27}$$

And according to our previous claim there exists a Möbius transformation, say m_*, such that

$$\begin{aligned} \mathsf{m}_*(e) &= t_*, \\ \mathsf{m}_*(a) &= a_*, \\ \mathsf{m}_*(b) &= b_*, \\ \mathsf{m}_*(c) &= c_*. \end{aligned} \tag{10.28}$$

Of course comparing (10.28) and (10.23) we see that $\mathsf{m}_* = \mathsf{m}^{-1}$ (because a Möbius transformation is fixed uniquely by its action on a set of three points). This means that the original triquadratic is Möbius conjugate to a triquadratic with branch points $\{t_*, a_*, b_*, c_*\}$,

$$H(u, v, w) = 0 \;\Leftrightarrow\; H(\mathsf{m}_*(u), \mathsf{m}_*(v), \mathsf{m}_*(w); t_*, a_*, b_*, c_*) = 0. \tag{10.29}$$

Connection to the 'seed' biquadratic

Now we make a connection between the deformation defined in the previous section and the deformation which arose naturally in the construction of the seed solution of Adler's equation (section 9.2).

Given a non-degenerate family of biquadratics we may, by the construction of section 6.1, construct an associated form of Adlers equation. Then, by the construction in section 9.2 we can construct its seed solution which is defined as the solution of a new commuting family of biquadratics which are a parameter-deformation of the original family. If we construct a form of Adlers equation starting from the family of biqudratics $H(u, \widetilde{u}, p)$, the resulting seed solution (the 1-cycle of the BT with Bäcklund parameter $(t, \sqrt{R(t)})$) for this form of Adlers equation is defined by commuting biquadratics with shared discriminant equal to $H(u, u, t)$ (up to a constant factor) as is shown by equation (9.16). Now, the family $H_*(u, \widetilde{u}, p)$ have disciminant R_*, so from (10.22) we see that the seed solution is defined by commuting biquadratics in the family $H_*(u, \widetilde{u}, p)$.

Thus we have characterised this 'seed deformation' in terms of the associated Kleinian Möbius group $\langle \mathsf{i}, \mathsf{j} \rangle$ and shown that, keeping three of the branch-points fixed by Möbius conjugation, the fourth deforms and actually *coincides* with the natural deformation parameter t, the projection of the Bäcklund parameter (t, T) associated with the seed solution.

Appendix A

The cross-ratio

Here we discuss some elementary properties of the cross-ratio, ϕ:

$$\phi : \widehat{\mathbb{C}}^4 \to \widehat{\mathbb{C}}, \qquad \phi(w,x,y,z) = \frac{(w-x)(y-z)}{(w-y)(x-z)}. \tag{A.1}$$

We have already seen (2.4) that the cross-ratio of four points is preserved by the action of any Möbius transformation, i.e., given four distinct points $w, x, y, z \in \widehat{\mathbb{C}}$,

$$\phi(\mathsf{m}(w), \mathsf{m}(x), \mathsf{m}(y), \mathsf{m}(z)) = \phi(w,x,y,z) \qquad \forall\ \mathsf{m} \in \mathsf{M}. \tag{A.2}$$

We can reveal other properties of the cross-ratio by considering the following concrete example of theorem 1.

Consider three distinct points $w, x, y \in \widehat{\mathbb{C}}$, by theorem 1 there exists a unique $\mathsf{m} \in \mathsf{M}$ for which

$$\begin{aligned} \mathsf{m}(w) &= 1, \\ \mathsf{m}(x) &= \infty, \\ \mathsf{m}(y) &= 0. \end{aligned} \tag{A.3}$$

Now, it is clear from the definition (A.1) that $\phi(1, \infty, 0, u) = u$, so from (A.2) we see that

$$\mathsf{m}(z) = \phi(w,x,y,z). \tag{A.4}$$

Note that *any* $\mathsf{m} \in \mathsf{M}$ can be written in the form (A.4) for some w, x and y, these are just the points which m sends to $1, \infty$ and 0. In this way we can view the cross-ratio as a particular parameterization of the Möbius transformations,

$$\mathsf{M} = \left\{ \, u \mapsto \phi(w,x,y,u) \;\middle|\; w,x,y \in \widehat{\mathbb{C}}, w \neq x \neq y \neq w \, \right\}. \tag{A.5}$$

A.1 Discrete symmetries

Consider now two particular Möbius transformations, $\mathsf{f}, \mathsf{g} \in \mathsf{M}$, for which

$$\begin{aligned} \mathsf{f}(0) &= 1, & \mathsf{g}(1) &= 1, \\ \mathsf{f}(\infty) &= \infty, & \mathsf{g}(0) &= \infty, \\ \mathsf{f}(1) &= 0, & \mathsf{g}(\infty) &= 0. \end{aligned} \tag{A.6}$$

A calculation shows that f and g are just

$$\mathsf{f}(u) = 1 - u, \qquad \mathsf{g}(u) = 1/u. \tag{A.7}$$

It is clear from theorem 1 that the group of Möbius transformations which permute the set $\{1, \infty, 0\}$, that is $\mathsf{S}(1, \infty, 0)$, is isomorphic to the permutation group of $\{1, \infty, 0\}$. i.e., S_3. In fact f and g constructed above generate this group,

$$\langle \mathsf{f}, \mathsf{g} \rangle = \mathsf{S}(1, \infty, 0). \tag{A.8}$$

Now, this group of Möbius transformations turns out to be connected to the discrete symmetries of the cross-ratio. Consider again the transformation m defined by (A.3). A permutation of the points $\{w, x, y\}$ in (A.3) is equivalent to a permutation of $\{1, \infty, 0\}$. This permutation can also be achieved by a composition of m with an element of the group (A.8). When this observation is applied to the equivalent definition of m, (A.4), it reveals transformations of the cross-ratio. Writing for convenience $\phi(w, x, y, z) = \varphi$ we find

$$\begin{aligned} \phi(w,x,y,z) &= \mathsf{e}(\varphi) &&= \varphi, \\ \phi(y,x,w,z) &= \mathsf{f}(\varphi) &&= 1-\varphi, \\ \phi(w,y,x,z) &= \mathsf{g}(\varphi) &&= 1/\varphi, \\ \phi(x,y,w,z) &= \mathsf{f}(\mathsf{g}(\varphi)) &&= 1-1/\varphi, \\ \phi(y,w,x,z) &= \mathsf{g}(\mathsf{f}(\varphi)) &&= 1/(1-\varphi), \\ \phi(x,w,y,z) &= \mathsf{f}(\mathsf{g}(\mathsf{f}(\varphi))) &&= \varphi/(\varphi-1). \end{aligned} \tag{A.9}$$

Our intention is that these now be taken at face-value as a table of discrete symmetries of the cross-ratio, i.e., a permutation of the points results in the listed point-transformation of their cross-ratio. As such this list is incomplete. In fact the relations (A.9) can be complemented by the following symmetries

$$\begin{aligned} \phi(w,x,y,z) &= \varphi, \\ \phi(x,w,z,y) &= \varphi, \\ \phi(y,z,w,x) &= \varphi, \\ \phi(z,y,x,w) &= \varphi, \end{aligned} \tag{A.10}$$

which can be seen directly from the definition (A.1). The correspondence defined by (A.9) and (A.10) between permutations of the arguments of ϕ and Möbius transformations in the group $\langle \mathsf{f}, \mathsf{g} \rangle$ is a surjective group homomorphism. The permutations listed in (A.10) constitute its kernel, which is the Klein 4-group. In fact *every* permutation of the arguments of ϕ can be expressed as the composition of unique pair of permutations, one from (A.9) and the other from (A.10). This is the sense in which the list of transformations given here is complete.

A.2 The stabilizer of four points

In the main text, chapter 4, we have given a fairly comprehensive description of the *stabilizer* in M of a set of one and two points. In section A.1 we have noted that the stabilizer in M of a set of three points, $\mathsf{S}(w,x,y)$, is isomorphic to S_3, a fact which is obvious from theorem 1. Now, it turns out that the discrete symmetries of the cross-ratio (section A.1) provide the exact tool necessary to describe the stabilizer in M of a set of *four* distinct points in $\widehat{\mathbb{C}}$, i.e., the group $\mathsf{S}(w,x,y,z)$.

Given two sets of distinct points $\{w,x,y,z\}$ and $\{\widetilde{w},\widetilde{x},\widetilde{y},\widetilde{z}\}$ we know by theorem 1 that there exists $\mathsf{m} \in \mathsf{M}$ that sends one set to the other if and only if the cross-ratio of the points coincide for the two sets,

$$\phi(\widetilde{w},\widetilde{x},\widetilde{y},\widetilde{z}) = \phi(w,x,y,z). \tag{A.11}$$

Here we are interested in the case where the second set of points is a permutation of the first, i.e., when $\{\widetilde{w}, \widetilde{x}, \widetilde{y}, \widetilde{z}\} = \{w, x, y, z\}$.

The relations (A.10) reveal that, whatever the values of the points $\{w, x, y, z\}$, to each Kleinian permutation of them there exists a Möbius transformation who's action on these points is that permutation. In fact, by theorem 6 we may construct unique involutions $\mathsf{i}, \mathsf{j} \in \mathsf{M}$ for which

$$\begin{aligned} \mathsf{i}(w) &= x, & \mathsf{j}(w) &= y, \\ \mathsf{i}(y) &= z, & \mathsf{j}(x) &= z. \end{aligned} \tag{A.12}$$

It is clear that $\langle \mathsf{i}, \mathsf{j} \rangle < \mathsf{S}(w, x, y, z)$ exhausts the Möbius transformations corresponding to the Kleinian permutations. There may be Möbius transformations in $\mathsf{S}(w, x, y, z)$ corresponding to permutations present in the other list (A.9), however their existance puts a constraint on the value taken by the cross-ratio $\phi(w, x, y, z) = \varphi$. In general $\varphi \in \widehat{\mathbb{C}} \setminus \{1, \infty, 0\}$ (provided the points are distinct), we observe from (A.9) that

$$\begin{aligned} \varphi = 2 \quad &\Leftrightarrow \quad \exists\, \mathsf{h} \in \mathsf{F}(w, x) \text{ s.}t.\ \mathsf{h}(y) = z,\ \mathsf{h}(z) = y, \\ \varphi = 1/2 \quad &\Leftrightarrow \quad \exists\, \mathsf{h} \in \mathsf{F}(w, y) \text{ s.}t.\ \mathsf{h}(x) = z,\ \mathsf{h}(z) = x, \\ \varphi = -1 \quad &\Leftrightarrow \quad \exists\, \mathsf{h} \in \mathsf{F}(w, z) \text{ s.}t.\ \mathsf{h}(x) = y,\ \mathsf{h}(y) = x, \\ \varphi + 1/\varphi = 1 \quad &\Leftrightarrow \quad \exists\, \mathsf{h} \in \mathsf{S}(w) \text{ s.}t.\ \mathsf{h}(x) = y,\ \mathsf{h}(y) = z,\ \mathsf{h}(z) = x. \end{aligned} \tag{A.13}$$

Given that these conditions on φ mutually exclude each other, we have in general that

$$\mathsf{S}(w, x, y, z) = \langle \mathsf{i}, \mathsf{j}, \mathsf{h} \rangle, \tag{A.14}$$

where if φ satisfies one of the conditions (A.13) then h is the corresponding Möbius transformation and otherwise we take $\mathsf{h} = \mathsf{e}$.

Appendix B

The trivial Toeplitz extension

Given an equation which is consistent around the cube (for simplicity we consider only scalar equations here), we outline a method for the construction of a higher rank system which is a trivial extension of the underlying equation and is also consistent around the cube. It turns out to be quite revealing to consider the interaction between this construction and the continuum limit.

B.1 The construction

This construction starts from a scalar lattice equation which we trivially extend to a diagonal rank-N system

$$\mathcal{Q}(u_n, \widetilde{u}_n, \widehat{u}_n, \widehat{\widetilde{u}}_n) = 0, \qquad n = 1 \ldots N, \tag{B.1}$$

where $u_n = u_n(l, m)$, $\widetilde{u}_n = u_n(l + 1, m)$, $\widehat{u}_n = u_n(l, m + 1)$ etc. for $n = 1 \ldots N$ are N dependent variables which depend on $l, m \in \mathbb{Z}$ and $\mathcal{Q}$ is a polynomial of degree one with coefficients which do not depend on l

and m (i.e., the system is autonomous). If we define the operator T

$$\begin{aligned} T(u_n) &= u_{n+1}, \qquad n = 1\dots N-1, \\ T(u_N) &= u_1. \end{aligned} \tag{B.2}$$

and apply the transformation

$$u_n(l,m) \to T^{l+m}(u_n(l,m)), \qquad n = 1\dots N, \tag{B.3}$$

to (B.1), we construct a new system which is no longer diagonal, but which is still autonomous

$$\mathcal{Q}(u_n, T(\widetilde{u}_n), T(\widehat{u}_n), T^2(\widehat{\widetilde{u}}_n)) = 0, \qquad n = 1\dots N. \tag{B.4}$$

This new system is trivial in the sense that application of the transformation inverse to (B.3) diagonalizes it. However it is clear that if the equation defined by $\mathcal{Q}$ were multidimensionally consistent, then the extended system (B.4) would be too. Note that this extended system is invariant under the transformation $u_n \to T(u_n)$, we refer to such systems as Toeplitz.

Consider, for example, the lattice Schwarzian KdV equation

$$p(u - \widehat{u})(\widetilde{u} - \widehat{\widetilde{u}}) = q(u - \widetilde{u})(\widehat{u} - \widehat{\widetilde{u}}) \tag{B.5}$$

extended to rank 2. The resulting rank-2 system is simply

$$\begin{aligned} p(u_1 - \widehat{u}_2)(\widetilde{u}_2 - \widehat{\widetilde{u}}_1) &= q(u_1 - \widetilde{u}_2)(\widehat{u}_2 - \widehat{\widetilde{u}}_1), \\ p(u_2 - \widehat{u}_1)(\widetilde{u}_1 - \widehat{\widetilde{u}}_2) &= q(u_2 - \widetilde{u}_1)(\widehat{u}_1 - \widehat{\widetilde{u}}_2), \end{aligned} \tag{B.6}$$

which is also multidimensionally consistent. The main point we want to make here is that in the exploration of rank-2 multidimensionally consistent equations, these trivial extensions of known scalar equations will emerge.

If we eliminate all but one of the variables in (B.4), i.e., write it as a scalar equation on a larger stencil, the result will not depend on the particular variable we choose because the system is Toeplitz. Moreover this higher order equation will be compatible with the underlying equation,

$$\mathcal{Q}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) = 0, \tag{B.7}$$

which is to say that any solution of (B.7) will also a solution of its higher order scalar counterpart. In the case of example (B.6) the elimination of one variable leads to the associated scalar equation

$$\begin{vmatrix} u^2 & 2u & 1 \\ \underset{\wedge}{\widetilde{u}}\,\underset{\sim}{\widehat{u}} & \underset{\wedge}{\widetilde{u}} + \underset{\sim}{\widehat{u}} & 1 \\ \underset{\sim}{\underset{\wedge}{u}}\,\widehat{\widetilde{u}} & \underset{\sim}{\underset{\wedge}{u}} + \widehat{\widetilde{u}} & 1 \end{vmatrix} = 0, \tag{B.8}$$

which it can be verified is compatible with (B.5). Actually, in general this is the same higher-order equation as would result from elimination of $\widetilde{u}, \widehat{u}, \underset{\sim}{u}$ and $\underset{\wedge}{u}$ between the equations

$$\begin{aligned} &\mathcal{Q}(\underset{\sim}{\underset{\wedge}{u}}, \underset{\wedge}{u}, \underset{\sim}{u}, u) = 0, \qquad &&\mathcal{Q}(\underset{\wedge}{u}, \underset{\wedge}{\widetilde{u}}, u, \widetilde{u}) = 0, \\ &\mathcal{Q}(\underset{\sim}{u}, u, \underset{\sim}{\widehat{u}}, \widehat{u}) = 0, \qquad &&\mathcal{Q}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) = 0. \end{aligned} \tag{B.9}$$

B.2 The continuum limit

Now, the continuum limit which takes (B.5) to the Schwarzian KdV equation can be achieved most straightforwardly by introducing the following operator,

$$C_p = e^{\sqrt{2p}(\partial_x + \frac{p}{6}\partial_y)}. \tag{B.10}$$

Supposing that u depends on continuous variables x and y, $u = u(x, y)$, and that $\widetilde{u} = C_p u$, $\widehat{u} = C_q u$ etc., then in the limit $(p, q) \longrightarrow (0, 0)$ the lattice equation (B.5) goes to

$$u_y - u_{xxx} + \frac{3}{2}\frac{u_{xx}^2}{u_x} = 0, \tag{B.11}$$

which is the Schwarzian KdV equation. Writing also that $\underset{\sim}{u} = C_p^{-1} u$ etc., we may calculate the continuum limit of the higher order equation (B.8), we find the continuous equation

$$\left(\frac{u_y}{u_x} - \frac{u_{xxx}}{u_x} + \frac{3}{2}\frac{u_{xx}^2}{u_x^2} \right)_x = 0, \tag{B.12}$$

i.e., just the x derivative of (B.11) after dividing it by u_x. This picture for the PDEs (B.11) and (B.12) is analogous to the relationship between the lattice equation (B.5) and its higher order scalar counterpart (B.8).

B.3 Adler's equation

A similar story is true for the other lattice equations in the list (6.11) and their continuous counterparts. In the most general case, that is for Adler's equation, the associated second-order scalar equation may be written in the form

$$\xi + u^2 - (1 + u^2\xi)r^2 + 2u\eta R = 0, \tag{B.13}$$

where we have introduced the new variables

$$\xi = \frac{\widehat{\widetilde{u}}\,\underset{\underset{\sim}{\wedge}}{u}\,\underset{\wedge}{\widetilde{u}} + \widehat{\widetilde{u}}\,\underset{\underset{\sim}{\wedge}}{u}\,\underset{\sim}{\widehat{u}} - \widehat{\widetilde{u}}\,\underset{\wedge}{\widetilde{u}}\,\underset{\sim}{\widehat{u}} - \underset{\underset{\sim}{\wedge}}{u}\,\underset{\wedge}{\widetilde{u}}\,\underset{\sim}{\widehat{u}}}{\widehat{\widetilde{u}} + \underset{\underset{\sim}{\wedge}}{u} - \underset{\wedge}{\widetilde{u}} - \underset{\sim}{\widehat{u}}}, \qquad \eta = \frac{\widehat{\widetilde{u}}\,\underset{\underset{\sim}{\wedge}}{u} - \underset{\wedge}{\widetilde{u}}\,\underset{\sim}{\widehat{u}}}{\widehat{\widetilde{u}} + \underset{\underset{\sim}{\wedge}}{u} - \underset{\wedge}{\widetilde{u}} - \underset{\sim}{\widehat{u}}}, \tag{B.14}$$

and $(r, R) = \mathfrak{r}$ is related to $(p, P) = \mathfrak{p}$ and $(q, Q) = \mathfrak{q}$, the lattice parameters of Adler's equation in Jacobi form (6.8) which lie on the Jacobi elliptic curve as described in section 9.1 by the relation $(r, R) = \mathfrak{r} = \mathfrak{p} \cdot \mathfrak{q}^{-1}$. The continuous equation associated to (B.13) is simply

$$\left(\frac{u_y}{u_x} - \frac{u_{xxx}}{u_x} + \frac{3}{2u_x^2}\left(u_{xx}^2 - u^4 - 1 + \left(k + \frac{1}{k}\right)u^2\right)\right)_x = 0, \tag{B.15}$$

just the derivative of the KN equation (6.1). which is basically the same situation as we found in the case of the Schwarzian KdV in the previous section. The remarkable similarity between (B.13) and the Jacobi biquadratic (6.6) does not (yet) have an obvious explanation.

B.4 Conclusion

The higher order compatible discrete equations described here exist for any lattice equation, regardless of whether it is integrable. This is analogous to the observation that any PDE can be differentiated to yield a higher order compatible PDE. Both of these notions are trivial extensions of an underlying equation, what we have demonstrated here is that the trivial extension commutes with the continuum limit for the known integrable examples.

Appendix C

A reduction of the Hirota-Miwa equation

Here we will give a new connection between what we refer to as the Hirota-Miwa [30, 47] equation

$$\overline{u}\widehat{\widetilde{u}} = \widehat{u}\widetilde{\overline{u}} - \widetilde{u}\widehat{\overline{u}} \tag{C.1}$$

(a.k.a the discrete KP equation [62]) and a hierarchy of quadrilateral equations, the lowest members of which are the lattice modified KdV and Boussinesq equations. We also construct a BT connecting this hierarchy to another one, this time with the lattice Schwarzian KdV and Boussinesq equations as the lowest two members.

C.1 An integrable deformation of the Hirota-Miwa equation

Our interest will actually be in an equation related by a gauge transformation to (C.1), this gauge transformation singles out one particular direction on the lattice. Specifically, if we set $u(l,m,n) \to p^{ln}q^{mn}u(l,m,n)$ in (C.1), then the resulting equation is simply

$$\overline{u}\widehat{\widetilde{u}} = p\widehat{u}\widetilde{\overline{u}} - q\widetilde{u}\widehat{\overline{u}}. \tag{C.2}$$

This singling out of one direction means we should be quite explicit now about the notion of integrability for (C.2), but it is basically inherited from the multidimensional consistency of (C.1). What we will focus on is the notion of BT for (C.2), to which end we write the following system of equations in two variables $u = u(l, m, n)$ and $v = v(l, m, n)$

$$\begin{aligned} \overline{u}\widetilde{v} &= p\widetilde{\overline{u}}v - r\widetilde{u}\overline{v}, \\ \overline{u}\widehat{v} &= q\overline{\widehat{u}}v - r\widehat{u}\overline{v}. \end{aligned} \tag{C.3}$$

which, it can be verified, constitutes an auto-BT for (C.2). Now, there is a subtlety because this is a BT in the *weak* sense. Specifically, (C.3) actually constitutes an auto-BT, in the strong sense, for the equation

$$\frac{p\widehat{u}\widetilde{\overline{u}} - q\widetilde{u}\overline{\widehat{u}}}{\overline{u}\widehat{\widetilde{u}}} = \left[\frac{p\widehat{u}\widetilde{\overline{u}} - q\widetilde{u}\overline{\widehat{u}}}{\overline{u}\widehat{\widetilde{u}}}\right]^{-} \tag{C.4}$$

(this equation, not (C.2), arises as the compatibility condition) and clearly solutions of (C.2) form a subset of the solutions to this equation. It is this subtlety which we will exploit here, in that our interest now lies in the more general equation (C.4). This equation is clearly not covariant due to the singled-out lattice direction, it lies on a 10 point stencil on the three dimensional lattice.

C.2 Reduction to a modified hierarchy

We will consider a simple reduction of the equation (C.4) which brings it to a quadrilateral lattice system of N equations in N variables for $N \in \mathbb{N}$, i.e., a system of rank N. We suppose (C.4) holds, but simultaneously that $u(l, m, n) = u(l, m, n + N + 1)$ *and* that $u(l, m, 0) = 1$. Note that we mean for both of these conditions to hold everywhere, so that in particular $u(l, m, k(N + 1)) = 1$ for any integer k. In the case $N = 1$ the equation for $u = u(l, m, 1)$ is readily found to be

$$p(u\widehat{u} - \widetilde{u}\widehat{\widetilde{u}}) = q(u\widetilde{u} - \widehat{u}\widehat{\widetilde{u}}). \tag{C.5}$$

This is the lattice modified KdV equation given originally in ???. In the case $N = 2$ the rank-2 system for $(u_1, u_2) = (u(l, m, 1), u(l, m, 2))$ is

$$\begin{aligned} p(u_1\widehat{u}_1\widetilde{u}_2 - u_2\widetilde{u}_1\widehat{\widetilde{u}}_1) &= q(u_1\widetilde{u}_1\widehat{u}_2 - u_2\widehat{u}_1\widehat{\widetilde{u}}_1), \\ p(u_2\widehat{u}_2\widehat{\widetilde{u}}_1 - \widehat{u}_1\widetilde{u}_2\widehat{\widetilde{u}}_2) &= q(u_2\widetilde{u}_2\widehat{\widetilde{u}}_1 - \widetilde{u}_1\widehat{u}_2\widehat{\widetilde{u}}_2), \end{aligned} \tag{C.6}$$

which is the lattice modified Boussinesq equation which was given originally as a second order scalar equation in [53], the rank-2 version (C.6) is attributable to Nijhoff in [60]. In fact a rearrangement reveals that, in the general case, the system for $u_1 \dots u_N$ may be written in the form

$$p\left[\frac{u_{n+1}\widehat{u}_{n-1}}{u_n\widehat{u}_n} - \frac{\widetilde{u}_{n+1}\widehat{\widetilde{u}}_{n-1}}{\widetilde{u}_n\widehat{\widetilde{u}}_n}\right] = q\left[\frac{u_{n+1}\widetilde{u}_{n-1}}{u_n\widetilde{u}_n} - \frac{\widehat{u}_{n+1}\widehat{\widetilde{u}}_{n-1}}{\widehat{u}_n\widehat{\widetilde{u}}_n}\right], \tag{C.7}$$

for $n = 1 \dots N$ with the convention that $u_0 = u_{N+1} = 1$. This system is multidimensionally consistent in the usual sense. We also observe that the transformation

$$(u_1, u_2 \dots u_{N-1}, u_N) \to (u_2/u_1, u_3/u_1 \dots u_N/u_1, 1/u_1) \tag{C.8}$$

leaves (C.7) unchanged, which is basically a result of the *almost* cyclic boundary condition we imposed on (C.4). This symmetry generates a cyclic group which gives us the possibility to construct related multidimensionally consistent equations by the use of a gauge transformation.

C.3 Connection to a Schwarzian hierarchy

Now, the lattice modified KdV equation (C.5) is related to the lattice Schwarzian KdV equation written in the form

$$p^2(v - \widetilde{v})(\widehat{v} - \widehat{\widetilde{v}}) = q^2(v - \widehat{v})(\widetilde{v} - \widehat{\widetilde{v}}), \tag{C.9}$$

by the Bäcklund transformation

$$\widetilde{v} - v = \frac{1}{p}u\widetilde{u}, \qquad \widehat{v} - v = \frac{1}{q}u\widehat{u}. \tag{C.10}$$

This BT was given originally in [59]. Here we generalise this BT to the following system,

$$\widetilde{v}_n - v_n = \frac{1}{p}\frac{u_{n+1}\widetilde{u}_{n-1}}{u_n\widetilde{u}_n}, \qquad \widehat{v}_n - v_n = \frac{1}{q}\frac{u_{n+1}\widehat{u}_{n-1}}{u_n\widehat{u}_n}, \tag{C.11}$$

where $n = 1 \dots N$ and again we adopt the convention that $u_0 = u_{N+1} = 1$. It can be seen immediately that (C.11) is compatible in v provided u satisfies (C.7). And it does not take long to verify that (C.11) is compatible in u provided v satisfies the system

$$\begin{aligned} \frac{\widehat{v}_{n+1} - v_{n+1}}{\widetilde{v}_{n+1} - v_{n+1}} &= \frac{\widehat{\widetilde{v}}_n - \widetilde{v}_n}{\widehat{\widetilde{v}}_n - \widehat{v}_n}, \qquad n = 1 \dots N-1, \\ \frac{p^{N+1}}{q^{N+1}} \prod_{n=1\dots N} \frac{\widetilde{v}_n - v_n}{\widehat{v}_n - v_n} &= \frac{\widehat{\widetilde{v}}_N - \widetilde{v}_N}{\widehat{\widetilde{v}}_N - \widehat{v}_N}. \end{aligned} \tag{C.12}$$

In fact (C.11) constitutes a BT between (C.7) and (C.12). Note that as a result we can be sure that the new system (C.12), like (C.7), is multidimensionally consistent. In the case $N = 1$ (C.12) reduces to (C.9), and in the case $N = 2$ (C.12) can be written as

$$\begin{aligned} p^3(v_2 - \widetilde{v}_2)(\widehat{v}_2 - \widehat{\widetilde{v}}_2)(v_1 - \widetilde{v}_1) &= q^3(v_2 - \widehat{v}_2)(\widetilde{v}_2 - \widehat{\widetilde{v}}_2)(v_1 - \widehat{v}_1), \\ p^3(v_1 - \widetilde{v}_1)(\widehat{v}_1 - \widehat{\widetilde{v}}_1)(\widehat{v}_2 - \widehat{\widetilde{v}}_2) &= q^3(v_1 - \widehat{v}_1)(\widetilde{v}_1 - \widehat{\widetilde{v}}_1)(\widetilde{v}_2 - \widehat{\widetilde{v}}_2). \end{aligned} \tag{C.13}$$

The equation (C.13) is a rank-2 (i.e., 2-variable) version of the lattice Schwarzian Boussinesq equation which was given originally as a second order scalar equation in [56] (a second-order scalar equation can be recovered from (C.6) or (C.13) by elimination of one of the variables from the two-component system, by second order here we mean a lattice equation on a square nine point stencil).

C.4 Conclusion

We have established a method by which the lattice modified KdV and Boussinesq equations can be found as reductions of an integrable deformation of the Hirota-Miwa equation. The construction naturally yields a generalisation of a known BT between the lattice modified and Schwarzian KdV equations. As a result we are able to find a more convenient rank-2 version of the lattice Schwarzian Boussinesq equation.

The reduction we apply to the Hirota-Miwa equation is reminiscent of the notion introduced in the main text because shifts on the lattice are related to BTs. So, a periodic reduction is similar to an N-cycle of the BT. However, the actual reduction here is more contrived in that it is not purely cyclic.

Appendix D

3-Cycles of the BT for the lattice Schwarzian and modified KdV equations

It turns out that the lattice Schwarzian KdV equation has only the constant solution as a 1-cycle of its BT, and that this solution does not germinate. (In order to find a germinating seed for this equation we can consider the 1-cycle of a new BT formed by the composition of the natural BT and a point-symmetry.) Similarly the 2-cycle does not produce a germinating seed. However the 3-cycle does define non-trivial solutions of this equation.

In this note we solve the rank-3 nonlinear mappings associated with 3-cycles of the BT for the lattice SKdV and modified KdV equations (the situation is similar for the lattice modified KdV equation which is given below, it is gauge-related to $H3^0$ from the list (6.11)). The solution method relies on a special (associativity) property which is not a consequence of cubic consistency, for example it is not shared by other equations in the list (6.11).

D.1 The defining system

To find the 3-cycles of the BT for an arbitrary lattice equation defined by the polynomial $\mathcal{Q}$ we need to solve the rank-3 system

$$\begin{aligned} \mathcal{Q}_{p,t_0}(u_0,\widetilde{u}_0,u_1,\widetilde{u}_1) &= 0, \\ \mathcal{Q}_{p,t_1}(u_1,\widetilde{u}_1,u_2,\widetilde{u}_2) &= 0, \\ \mathcal{Q}_{p,t_2}(u_2,\widetilde{u}_2,u_0,\widetilde{u}_0) &= 0. \end{aligned} \tag{D.1}$$

For convenience we denote this system by the equation

$$\mathcal{S}_p(u,\widetilde{u}) = 0 \tag{D.2}$$

where $u = (u_0, u_1, u_2)$ and 0 is to be interpreted as a 3-vector. We view the system (D.2) as a mapping $u \overset{p}{\mapsto} \widetilde{u}$, as such it is two-valued, which is to say fixing u and p, (D.2) has two solutions $\widetilde{u}$, this being a consequence of the multilinearity of $\mathcal{Q}$.

It is the coupling of (D.2) with its counterpart, the mapping $u \overset{q}{\mapsto} \widehat{u}$ defined by the system

$$\mathcal{S}_q(u,\widehat{u}) = 0 \tag{D.3}$$

which yields the full solution of $\mathcal{Q}$ on the (p,q) lattice. Here we will assume (D.2) and (D.3) commute (there are no known counter-examples), and thus

$$\mathcal{Q}_{pq}(u_i,\widetilde{u}_i,\widehat{u}_i,\widehat{\widetilde{u}}_i) = 0, \quad i \in \{0,1,2\}$$

as a result of (strong) cubic consistency.

We will consider the system $\mathcal{S}$ defined here in the case of $\mathcal{Q}$ being the lattice Schwarzian and modified KdV equations.

D.2 The lattice Schwarzian KdV equation

The lattice SKdV equation,

$$\mathcal{Q}_{p,q}(z,\widetilde{z},\widehat{z},\widehat{\widetilde{z}}) = p(z-\widetilde{z})(\widehat{z}-\widehat{\widetilde{z}}) - q(z-\widehat{z})(\widetilde{z}-\widehat{\widetilde{z}}),$$

yields $\mathcal{S}$ with the following special property. The composition of two mappings, i.e. the mapping $u \mapsto \widehat{\widetilde{u}}$ defined by the equations

$$\mathcal{S}_p(u, \widetilde{u}) = 0, \qquad \mathcal{S}_q(\widetilde{u}, \widehat{\widetilde{u}}) = 0, \tag{D.4}$$

is also of the same form, which is to say

$$\mathcal{S}_r(u, \widehat{\widetilde{u}}) = 0 \tag{D.5}$$

for some r. In fact it can be verified that p, q and r are related by the symmetric triquadratic equation

$$\chi(p, q, r) = p^2 + q^2 + r^2 - 2(pq + qr + rp) - cpqr = 0 \tag{D.6}$$

where the constant c is defined by the equation $\chi(t_0, t_1, t_2) = 0$.

It follows that the dynamics defined by the rank-3 system $\mathcal{S}$ reduce to dynamics defined by χ. Specifically, the solution $u(n)$ of (D.2) can be defined by the equation

$$\mathcal{S}_{\pi(n)}(u(0), u(n)) = 0$$

where it remains to find π satisfying

$$\pi(1) = p, \quad \chi(\pi, \widetilde{\pi}, p) = 0. \tag{D.7}$$

The curve associated to χ is found by computing the discriminant,

$$\Delta[\chi(p, q, r), r] = (4 + cp)p\ (4 + cq)q.$$

Furthermore the 2-valued product defined by χ has (strong) identity at 0, i.e.,

$$\chi(p, e, p) = 0\ \forall p\ \Rightarrow e = 0,$$

and we observe that p is also its own (strong) inverse. This motivates our definition of the uniformizing transformation f,

$$(f')^2 = (4/c + f)f,\ f(0) = 0 \quad \Longrightarrow \quad f(\alpha) = \frac{2}{c}\left(\cosh(\alpha) - 1\right),$$

whereby

$$\chi(\ f(\alpha), f(\beta), f(\alpha \pm \beta)\ ) = 0 \quad \forall\ \alpha, \beta,$$

bringing the 2-valued product defined by χ to addition. The (canonical) solution of (D.7) can then be written as

$$\pi(n) = f(n\alpha)$$

where α is defined by the equation $f(\alpha) = p$.

For completeness we give two functions of u conserved by $\mathcal{S}$,

$$\begin{aligned} \xi &= \frac{t_0}{u_0 - u_1} + \frac{t_1}{u_1 - u_2} + \frac{t_2}{u_2 - u_0}, \\ \eta &= \frac{t_0 u_0 u_1}{u_0 - u_1} + \frac{t_1 u_1 u_2}{u_1 - u_2} + \frac{t_2 u_2 u_0}{u_2 - u_0}. \end{aligned}$$

That is if $\mathcal{S}_p(u, \widetilde{u}) = 0$, then $\widetilde{\xi} = \xi$ and $\widetilde{\eta} = \eta$, note that neither function depends on the parameter p.

ξ can be found by writing the equations of $\mathcal{S}$ in three-leg form (which importantly is rational for this equation), η follows from ξ by the transformation $(u_0, u_1, u_2) \mapsto (1/u_0, 1/u_1, 1/u_2)$ under which $\mathcal{S}$ is invariant.

D.3 The lattice modified KdV equation

The lattice mKdV equation

$$\mathcal{Q}_{p,q}(v, \widetilde{v}, \widehat{v}, \widehat{\widetilde{v}}) = p(v\widetilde{v} - \widehat{v}\widehat{\widetilde{v}}) - q(v\widehat{v} - \widetilde{v}\widehat{\widetilde{v}})$$

yields $\mathcal{S}$ with a similar associativity property, however the property is slightly weaker which seems to have its origin in the fact that this equation does not have $D4$ symmetry. It is *necessary* to write (D.1) as a single-valued mapping in order arrive to an explicit form for its general solution, this also naturally introduces the curve.

It is a straightforward but tedious calculation to write (D.1) as a single-valued mapping, we find that

$$\begin{aligned}
\widetilde{u}_0 &= \frac{(t_0t_1u_0/u_2 + t_1t_2u_1/u_0 + t_2t_0u_1/u_2 - \xi/2)\,p + t_0t_1t_2P}{(t_1u_0/u_1u_2 + t_0/u_1 + t_2/u_2)\,p^2 + t_0t_1t_2/u_0},\\
\widetilde{u}_1 &= \frac{(t_1t_2u_1/u_0 + t_2t_0u_2/u_1 + t_0t_1u_2/u_0 - \xi/2)\,p + t_0t_1t_2P}{(t_2u_1/u_2u_0 + t_1/u_2 + t_0/u_0)\,p^2 + t_0t_1t_2/u_1},\\
\widetilde{u}_2 &= \frac{(t_2t_0u_2/u_1 + t_0t_1u_0/u_2 + t_1t_2u_0/u_1 - \xi/2)\,p + t_0t_1t_2P}{(t_0u_2/u_0u_1 + t_2/u_0 + t_1/u_1)\,p^2 + t_0t_1t_2/u_2}
\end{aligned}$$

where $P^2 = \mathcal{R}(p)$,

$$\mathcal{R}(p) = ap^4 + bp^2 + 1,$$

$$a = \frac{\xi}{t_0^2t_1^2t_2^2} + \frac{1}{t_0^2t_1^2} + \frac{1}{t_1^2t_2^2} + \frac{1}{t_2^2t_0^2}, \quad b = \frac{\xi^2}{4t_0^2t_1^2t_2^2} - \frac{1}{t_0^2} - \frac{1}{t_1^2} - \frac{1}{t_2^2}$$

and

$$\xi = \left(\frac{u_0}{u_1} + \frac{u_1}{u_0}\right) t_1t_2 + \left(\frac{u_1}{u_2} + \frac{u_2}{u_1}\right) t_2t_0 + \left(\frac{u_2}{u_0} + \frac{u_0}{u_2}\right) t_0t_1,$$

it can be verified that ξ is conserved by $\mathcal{S}$. For convenience we will write this single-valued mapping as σ, i.e.,

$$\widetilde{u} = \sigma_{\mathfrak{p}}(u) \tag{D.8}$$

where $\mathfrak{p} = (p, P)$ denotes the parameter which we now consider to be a point on the curve defined by $\mathcal{R}$. The effect of taking $P \to -P$ is to flip between the two images of u under $\mathcal{S}$.

Now, the associativity property can be stated as follows. There exists $\mathfrak{r}$ such that

$$\sigma_{\mathfrak{q}}(\sigma_{\mathfrak{p}}(u)) = \sigma_{\mathfrak{r}}(u),$$

in fact a direct calculation confirms that

$$\mathfrak{r} = (r, R) = \left(\frac{pQ + qP}{1 - ap^2q^2},\ \frac{Pp(aq^4 - 1) - Qq(ap^4 - 1)}{(1 - ap^2q^2)(qP - pQ)} \right).$$

This is just the rational representation of the group operation on the curve defined by $\mathcal{R}$ with identity at the point $(0, 1)$. It follows directly that the solution of (D.8) is given by

$$u(n) = \sigma_{\mathfrak{p}^n}(u(0)).$$

We can find the uniformizing transformation

$$(f')^2 = \mathcal{R}(f),\ f(0) = 0 \quad \Longrightarrow \quad f(\alpha) = a^{\frac{1}{4}} k^{\frac{1}{2}} \mathrm{sn}(\alpha, k)$$

where sn is the Jacobi elliptic function with modulus k which can be chosen as any solution of the equation

$$k + \frac{1}{k} = \frac{-b}{a^{\frac{1}{2}}}.$$

Defining α by the equation $f(\alpha) = p$ we have that

$$\mathfrak{p}^n = \big(f(n\alpha), f'(n\alpha)\big)$$

i.e., we can write the (canonical) period 2 BT orbit solution of the lattice mKdV in totally explicit form.

At a glance it seems more natural to give a 2-valued description here as we did for the SKdV equation, however this fails, and we are now in a position to explain the reason for this failure. Fixing u, p and q and writing

$$\mathcal{S}_p(u, \widetilde{u}) = 0, \quad \mathcal{S}_q(\widetilde{u}, \widehat{\widetilde{u}}) = 0, \tag{D.9}$$

defines four values of $\widehat{\widetilde{u}}$ which are given exactly as

$$\widehat{\widetilde{u}} = \sigma_{\mathfrak{r}}(u), \quad \mathfrak{r} \in \{\mathfrak{p}\cdot\mathfrak{q}, \mathfrak{p}^{-1}\cdot\mathfrak{q}^{-1}, \mathfrak{p}\cdot\mathfrak{q}^{-1}\cdot\mathfrak{s}^2, \mathfrak{p}^{-1}\cdot\mathfrak{q}\cdot\mathfrak{s}^2\} = \rho,$$

where $\mathfrak{s} = (\sqrt{k}, 0)$ so that $\mathfrak{p}^{-1}\cdot\mathfrak{s}^2 = (p, -P)$. On the other hand $\mathfrak{r} = (r, R) \in \rho \implies \mathfrak{r}^{-1}\cdot\mathfrak{s}^2 = (r, -R) \notin \rho$, so for any $(r, R) \in \rho$ only one of the two values of $\widehat{\widetilde{u}}$ defined by

$$\mathcal{S}_r(u, \widehat{\widetilde{u}}) = 0$$

are in the set defined by the composition (D.9).

For completeness we also mention the other conserved quantity

$$\eta = u_0 u_1 t_0 + u_1 u_2 t_1 + u_2 u_0 t_2.$$

List of references

[1] Ablowitz M J and Ladik F J 1976 A nonlinear difference scheme and inverse scattering *Stud. Appl. Math.* **55** 213-29;

[2] Adler V E 1994 Nonlinear superposition principle for the Jordan NLS equation *Phys. Lett. A* **190** 53-8

[3] Adler V E 1998 Bäcklund Transformation for the Krichever-Novikov Equation *Int. Math. Res. Not.* **1** 1-4

[4] Adler V E, Bobenko A I and Suris Yu B 2002 Classification of Integrable Equations on Quad-Graphs. The Consistency Approach *Commun. Math. Phys.* **233** 513-43

[5] Adler V E and Suris Yu B 2004 $Q4$: Integrable Master Equation Related to an Elliptic Curve *Int. Math. Res. Not.* **47** 2523-53

[6] Adler V E, Bobenko A I and Suris Yu B 2007 Discrete nonlinear hyperbolic equations. Classification of integrable cases *Funct. Anal. Appl.* (to appear) arXiv:0705.1663v1 [nlin.SI]

[7] Anderson R L and Ibragimov N H 1979 Lie-Bäcklund transformations in applications *siam, Philadelphia*

[8] Atkinson J, Hietarinta J and Nijhoff F W 2006 Seed and soliton solutions for Adler's lattice equation *J. Phys A: Math. Theor.* **40** F1-F8

[9] Atkinson J and Nijhoff F W 2007 Solutions of Adler's lattice equation associated with 2-cycles of the Bäcklund transformation *J. Nonl. Math. Phys. (suppl. proceedings of NEEDS 2007)* (accepted) arXiv:0710.2643v1[nlin.SI]

[10] Atkinson J 2008 Bäcklund transformations for integrable lattice equations *J. Phys. A: Math. Theor.* **41** 135202 8pp

[11] Atkinson J, Hietarinta J and Nijhoff F 2008 Soliton solutions for $Q3$ *J. Phys. A: Math. Theor.* **41** 142001 11pp

[12] Bäcklund A V 1883 Om ytor med konstant negativ krökning *Lunds Universitets Årsskrift* **19** 1-48

[13] Bianchi L 1879 Ricerche sulle superficie a curvatura constante e sulle elicoidi. tesi di Abilitazione *Ann. Scuola Norm. Sup. Pisa (1)* **2** 285-304

[14] Bogdanov L V and Konopelchenko B G 1999 Mobius invariant integrable lattice equations associated with KP and 2DTL hierarchies *Phys. Lett. A* **256** 39-46

[15] Bobenko A I and Suris Yu B 2002 Integrable systems on quad-graphs *Intl. Math. Res. Notices* **11** 573-611

[16] Buchstaber V M and Veselov A P 1996 Integrable Correspondences and Algebraic Representations of Multivalued Groups *Intl. Math. Res. Notices* **8** 381-400

[17] Bullett S 1988 Dynamics of quadratic correspondences *Nonlinearity* **1** 27-50

[18] Capel H W, Nijhoff F W and Papageorgiou V G 1991 Complete integrability of Lagrangian mappings and lattices of KdV type *Phys. Lett. A* **155** 377-87

[19] Date E, Jimbo M and Miwa T 1982 Method for Generating Discrete Soliton Equations I-V *J. Phys. Soc. Japan* **51** 4116-31, **52** 388-93, 761-71

[20] Dorfman Y and Nijhoff F W 1991 On a (2+1) -dimensional version of the Krichever-Novikov equation *Phys. Lett. A* **157** 107-12

[21] Eisenhart L P 1962 Transformations of Surfaces *Chelsea, New York*

[22] Flashka H 1974 The Toda Lattice II. Inverse scattering solution *Prog. Theor. Phys.* **51**(3) 703-16

[23] Fokas A S and Ablowitz M J 1981 Linearization of the Korteweg-deVries and Painlevé II Equations *Phys. Rev. Lett.* **47** 1096-100

[24] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 Method for solving the Korteweg-de Vries equation *Phys. Rev. Lett* **19** 1095-7

[25] Hancock H 1910 Lectures on the theory of elliptic functions *John Wiley & Sons New York, Chapman & Hall Ltd. London*

[26] Hietarinta J 2004 A new two-dimensional lattice model that is 'consistent around a cube' *J. Phys. A: Math. Gen.* **37** L67-73

[27] Hietarinta J 2005 Searching for CAC-maps *J. Nonlinear Math. Phys.* **12** Suppl. 2 223-30

[28] Hietarinta J and Viallet C 2007 Searching for integrable lattice maps using factorization *J. Phys. A* **40** 12629-43

[29] Hirota R 1977 Nonlinear Partial Difference Equations I-III *J. Phys. Soc. Japan* **43** 1424-33, 2074-89

[30] Hirota R 1981 Discrete Analogue of a Generalized Toda Equation *J. Phys. Soc. Jpn.* **50** 3785-91

[31] Igonin S and Martini R 2002 Prolongation structure of the Krichever-Novikov equation *J. Phys. A: Math. Gen.* **35** 9801-10

[32] Joshi N, Grammaticos B, Tamizhmani T and Ramani A 2006 From Integrable Lattices to Non-QRT Mappings *Lett. Math. Phy.* **78**(1) 27-37

[33] Kac M and van Moerbeke P 1975 On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices *Adv. Math.* **16** 160-9

[34] van der Kamp P H, Rojas O and Quispel G R W 2007 Closed-form expressions for integral of MKdV and sine-Gordon maps *J. Phys. A: Math Theor.* **40** 12789-98

[35] Konopelchenko B G 1982 Elementary Bäcklund transformations, Nonlinear superposition principle and solutions of the integrable equations *Phys. Lett. A* **87**(9) 445-48

[36] Konopelchenko B G and Schief W K 2002 Menelaus'theorem, Clifford configurations and inversive geometry of the Schwarzian KP hierarchy*J. Phys. A: Math. Gen.* **35** 6125-44

[37] Krichever I M and Novikov S P 1979 Holomorphic Fiberings and Nonlinear Equations *Sov. Math. Dokl.* **20** 650-4

[38] Krichever I M and Novikov S P 1980 Holomorphic Bundles over Algebraic Curves and Nonlinear Equations *Russ. Math. Surv.* **35** 53-79

[39] Krichever I M 1981 Baxter's Equations and Algebraic Geometry *Funct. Anal. Appl.* **15** 92-103

[40] Krichever I M 2000 Elliptic Analog of the Toda Lattice *Intl. Math. Res. Notices* **8** 383-412

[41] Lamb G L 1974 Bäcklund transformations for certain nonlinear evolution equations *J. Math. Phys* **15** 2157-65

[42] Levi D and Benguria R 1980 Bäcklund transformations and nonlinear differential-difference equations *Proc. Natl. Acad. Sci. USA* **77** 5025-7

[43] Levi D 1981 Nonlinear differential difference equations as Backlund transformations *J. Phys. A: Math. Gen.* **14** 1083-98

[44] Levi D, Pilloni L and Santini P M 1981 Integrable three-dimensional lattices *J. Phys. A: Math. Gen.* **14** 1567-75

[45] Miura R M 1967 Korteweg-de Vries Equation and Generalizations. I. A Remarkable Explicit Nonlinear Transformation *J. Math. Phy.* **9**(8) 1202-04

[46] Miura R ed. 1976 Bäcklund Transformations, the Inverse Scattering Method, Solitons and Their Applications *Springer Berlin, Lecture Notes in Mathematics* **515**

[47] Miwa T 1982 On Hirota's difference equations *Proc. Jap. Acad. A* **58**(1) 9-12

[48] Miwa T, Jimbo M and Date E 2000 Differential equations, symmetries and infinite dimensional algebras *Cambridge University Press*

[49] Mokhov O I 1992 Canonical Hamiltonian representation of the Krichever-Novikov equation *Mathematical notes* **50**(3) 939-45

[50] Nijhoff F W, Quispel G R W and Capel H W 1983 Direct Linearization of Nonlinear Difference-Difference Equations *Phys Lett A* **97**(4) 125-8

[51] Nijhoff F W, Capel H W, Wiersma G L and Quispel G R W 1984 Bäcklund Transformations and Three-Dimensional Lattice Equations *Phys. Lett. A* **105**(6)267-72

[52] Nijhoff F W and Capel H W 1990 The direct linearisation approach to hierarchies of integrable PDEs in 2 + 1 dimensions: I. Lattice equations and the differential-difference hierarchies *Inverse Problems* **6** 567-90

[53] Nijhoff F W, Papageorgiou V G, Capel H W, Quispel G R W 1992 The lattice Gel'fand-Dikii hierarchy *Inverse Problems* **8(4)** 597-625

[54] Nijhoff F W and Capel H W 1995 The Discrete Korteweg-de Vries Equation *Act. App. Math* **39** 133-58

[55] Nijhoff F W and Pang G D 1996 Discrete-time Calogero-Moser Model and Lattice KP Equations, Eds. D. Levi, L. Vinet and P. Winternitz, in: Symmetries and Integrability of Difference Equations *CRM Lecture Notes and Proceedings Series* **9** 253-64

[56] Nijhoff F W 1996 On some "Schwarzian Equations" and their Discrete Analogues Eds. Fokas A S and Gel'fand I M in: Algebraic Aspects of Integrable Systems: In memory of Irene Dorfman *Birkhäuser Verlag* 237-60

[57] Nijhoff F W and Enolskii V Z 1999 Integrable Mappings of KdV type and hyperelliptic addition theorems in Symmetries and integrability of Difference Equations, eds. Clarkson P A and Nijhoff F W *Cambridge Univ. Press* 64-78

[58] Nijhoff F W and Walker A J 2001 The Discrete and Continuous Painlevé VI Hierarchy and the Garnier Systems Glasgow Math. J. **43A** 109-23

[59] Nijhoff F W, Ramani A, Grammaticos B and Ohta Y 2001 On Discrete Painlevé Equations associated with the Lattice KdV Systems and the Painlevé VI Equation *Studies in Applied Mathematics* **106** 261-314

[60] Nijhoff F W 1999 Discrete Painlevé Equations and Symmetry Reduction on the Lattice, Eds. Bobenko A I and Seiler R in: Discrete Integrable Geometry and Physics *Oxford Univ. Press* 209-34

[61] Nijhoff F W 2002 Lax pair for the Adler (lattice Krichever-Novikov) system *Phys. Lett. A* **297** 49-58

[62] Nimmo J J C 1997 Darboux transformations and the discrete KP equation *J. Phys. A: Math. Gen.* **30** 8693-704

[63] Novikov D P and Romanovkii R K 1997 On a metod for constructing algebro-geometric solutions to the zero curvature equation *Theoret. Math. Phys.* **110**(1) 47-56

[64] Novikov D P 1999 Algebraic-Geometric Solutions of the Krichever-Novikov equation *Theoret. Math. Phys.* **121**(3) 1567-73

[65] Papageorgiou V G, Nijhoff F W and Capel H W 1990 Integrable mappings and nonlinear integrable lattice equations *Phys. Lett. A* **147** 106-14

[66] Quispel G R W, Nijhoff F W, Capel H W and van der Linden J 1984 Linear Integral Equations and Nonlinear Difference-Difference Equations *Physica A* **125** 344-80

[67] Rogers C and Schief W K 2002 Bäcklund and Darboux Transformations - Geometry and Modern Applications in Soliton Theory *Cambridge University Press*

[68] Rose J S 1978 A course on group theory *Cambridge University Press (1978), Dover (1994)*

[69] Ramani A, Joshi N, Grammaticos B and Tamizhmani T 2006 Deconstructing an integrable lattice equation *J. Phys. A: Math. Gen.* **39** L145-9

[70] Ruijsenaars S N M 1990 Relativistic Toda systems *Comm. Math. Phys.* **133** (2) 217-47

[71] Schief W K 2003 Lattice Geometry of the Discrete Darboux, KP, BKP and CKP Equations. Menelaus' and Carnot's Theorems *Journal of Nonl. Math. Phys.* **10**(2) 194-208

[72] Svinolupov S I, Sokolov V V and Yamilov R I 1983 Bäcklund Transformations for Integrable Evolution Equations *Dokl. Akad. Nauk SSSR* **271** 802-5 English translation in *Sov. Math. Dokl.* **28** 165-8

[73] Svinolupov S I, Sokolov V V 1982 Evolution equations with nontrivial conservative laws *Funct. Anal. Appl.* **16** 317-9

[74] Toda M 1975 Studies of a non-linear lattice *Phys. Rep.* **18**(1) 1-123

[75] Tongas A and Nijhoff F W 2005 The Boussinesq integrable system: Compatible lattice and continuum structure *Glasgow Math J* **47A** 205-19

[76] Tsarev S P and Wolf T 2008 Classification of 3-dimensional integrable scalar discrete equations *Lett. in Math. Phys.* **84**(1) 31-9

[77] Veselov A P 1991 Integrable maps *Russ. Math. Surv.* **46**:5 1-51

[78] Wiersma G L and Capel H W 1987 Lattice equations, hierarchies and Hamiltonian structures *Physica A* **142** 199-244

[79] Weiss J 1982 The Painlevé property for partial differential equations *J. Math. Phys.* **24**(3) 552-26

[80] Weiss J 1983 The Painlevé property for partial differential equations. II: Bäcklund transformation, Lax pairs, and the Schwarzian derivative *J. Math. Phys.* **24**(6) 1405-13

[81] Weiss J 1986 Periodic fixed points of Bäcklund transformations and the Korteweg-de Vries equation *J. Math. Phys.* **27** 2647-56

[82] Weiss J 1987 Periodic fixed points of Bäcklund transformations *J. Math. Phys.* **28** 2025-39

[83] Wahlquist H D and Estabrook F B 1973 Bäcklund Transformation for Solutions of the Korteweg-de Vries Equation *Phys. Rev. Lett* **31**(23) 1386-90

[84] Wahlquist H D and Estabrook F B 1975 Prolongation structures of nonlinear equations *J. Math. Phys.* **16**(1) 1-7

[85] Zakharov V E and Shabat A B 1971 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media *Zh. Eksp. Teor. Fiz.* **61**(1) 118-34 [1972 *Sov. Phys. JETP* **34**(1) 62-9]

[86] Zefirov I V 1997 Commuting Algebraic Correspondences and Groups *Mathematical Notes* **61**(5) 553-60