Properties of Convolution Operators on $L^p(0,1)$

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Abstract

Convolution operators on $L^p(0,1)$ have many similarities with the classical Volterra operator V, but it is not known in general for which convolution kernels the resulting operator behaves like V.

It is shown that many convolution operators are cyclic, and the cyclic property is related to the invariant subspace lattice of the operator, and to the behaviour of the kernel as an element of the Volterra algebra.

The convolution operators induced by kernels satisfying a smoothness condition near the origin are shown to have asymptotic behaviour that matches that of powers of V, and a new class of convolution operators that are not nilpotent, but have kernels that are not polynomial generators for $L^1(0,1)$, are produced.

For kernels that are polynomial generators for $L^1(0,1)$, the corresponding convolution operators are shown to have the property that their commutant and the strongly-closed subalgebra of $\mathcal{B}(L^p(0,1))$ they generate are equal.

Contents

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Declaration

No part of this thesis has been submitted before for any qualification, nor has it been published elsewhere. The work is the author's own, except where otherwise attributed.

Chapter 1

Introduction

The classical Volterra operator, the indefinite integral, has been much studied over the years, and interesting discoveries about it are still being made. For example, in [12], Gallardo and Montes demonstrate that the Volterra operator is not supercyclic, thus solving a conjecture of Salas.

The Volterra operator can be regarded as a special example of a more general class: the convolution operators. In general, convolution operators have been much less well-studied. As all convolution operators are quasinilpotent, spectral theory does little to help study them, and other methods must be employed. For something of a survey of complex analytic methods with regards to convolution operators, see [11].

Given that the Volterra operator has been so well-studied, often questions about convolution operators take the form: "for which kernels does the corresponding convolution operator behave like V?" The answer can sometimes be surprisingly many; in [10], Eveson shows that for any absolutely continuous kernel k, the corresponding convolution operator V_k is not supercyclic.

This is the motivation for this thesis: for three properties known about the Volterra operator, I attempt to provide answers to the question "for which kernels does V_k behave like V?"

The first property looked at is that of the cyclicity of V_k. This is a property related to the supercyclicity of Gallardo and Montes' paper, but the Volterra operator is a cyclic operator. In Chapter 3, it is discovered that V_k is cyclic for a large class of k; in particular, V_k is cyclic if k is a polynomial generator for $L^1(0,1)$.

In Chapter 4 the behaviour of iterates of V_k in the opertor algebra is related to the behaviour of iterates of k in the convolution algebra. The main result here is a slightly more general version of a theorem by Eveson [9], which provides some new asymptotic results, based around a kernel that has been previously studied [2].

Finally, in Chapter 5 a result of Erdos [7] is extended to show that, if k a polynomial generator for $L^1(0,1)$, then the commutant of V_k is as large as it could be.

Throughout the theme is that "nice" properties of k translate into "nice" properties of V_k.

Chapter 2

Preliminaries

2.1 L^p-spaces

[15, 3] Let $X = \mathbb{R}$, $X = \mathbb{R}^+$ or $X = (0, 1)$, and let $\mathcal{M}(X)$ denote the set of real-valued measurable functions with domain X. Fix $p \in [1, \infty)$. For any $f \in \mathcal{M}(X)$ the *p-norm* of f is denoted $\|f\|_p$ and defined by

$$\|f\|_p = \left(\int_X |f(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}}.$$

For $p = \infty$, the ∞-*norm* of $f \in \mathcal{M}(X)$ is defined by

$$\|f\|_\infty = \inf\{a \in \mathbb{R} : |f(t)| \leqslant a \text{ for a.a. } t \in X\}$$

(this is also known as the *supremum norm*, as $\|f\|_\infty$ is also called the *essential supremum* of f).

Then for $p \in [1, \infty]$ the set $L^p(X)$ is defined by

$$L^p(X) = \{f \in \mathcal{M}(X) : \|f\|_p < \infty\}.$$

This is easily seen to be a vector space, and in fact is a normed space with the p-norm defining a norm on $L^p(X)$, where elements of $L^p(X)$ are regarded as equivalence classes of functions which differ only on a set of measure zero. It is customary to continue to refer to elements of $L^p(X)$ as "functions", and to treat them as such in most instances, remembering always that they are in fact equivalence classes. For each $p \in [1, \infty]$ the space $L^p(X)$ is complete in its norm, so that it is a Banach space.

2.2 Convolution

For any $f, g \in L^1(\mathbb{R})$, define their *convolution product* (or just "the *convolution* of f and g") as

$$h(t) = \int_{-\infty}^{\infty} f(t-s)g(s)\,\mathrm{d}s$$

for all $t \in \mathbb{R}$ [15, §8.13]. Clearly h is a measurable, real-valued function; in fact, $h \in L^1(\mathbb{R})$, and $\|h\|_1 \leqslant \|f\|_1 \|g\|_1$. The convolution of f and g is denoted $f * g$, which leads to the more familiar inequality

$$\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1 \,. \tag{2.1}$$

It is easily verified that convolution is commutative ($f * g = g * f$) associative ($(f * g) * h = f * (g * h)$), and distributes across addition ($(f + g) * h = (f * h) + (g * h)$) and scalar multiplication ($(\alpha f) * g = \alpha(f * g)$). For any $n \in \mathbb{N}$, define f^{*n} as f convolved with itself n-times: $f^{*n} = f * \cdots * f$.

This definition can be extended to $L^1(\mathbb{R}^+)$ by embedding $L^1(\mathbb{R}^+)$ into $L^1(\mathbb{R})$ by

$$(Ef)(t) = \begin{cases} f(t) & t \geqslant 0 \\ 0 & t < 0 \end{cases}$$

for each $f \in L^1(\mathbb{R}^+)$, which is clearly an isometric embedding. Then for $f, g \in L^1(\mathbb{R}^+)$ their convolution product can be defined by

$$(f * g)(t) = (Ef * Eg)(t)$$

for $t \in \mathbb{R}^+$. It is easily verified that $f * g \in L^1(\mathbb{R}^+)$, and $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$.

A simplified expression for $f * g$ can be obtained, by virtue of the fact that $(Ef)(t)$ is zero for all $t < 0$. Using this, and a change of variables, for any $f, g \in L^1(\mathbb{R}^+)$ and $t \in \mathbb{R}^+$,

$$\begin{aligned}(f * g)(t) &= \int_{-\infty}^{\infty} (Ef)(t-s)(Eg)(s)\,\mathrm{d}s \\ &= \int_{0}^{\infty} (Ef)(t-s)g(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{t} (Ef)(u)g(t-u)\,\mathrm{d}u \\ &= \int_{0}^{t} f(u)g(t-u)\,\mathrm{d}u\,.\end{aligned}$$

The definition can be further extended to $L^1(0,1)$ by a similar method. Define $E' : L^1(0,1) \to L^1(\mathbb{R})$ by

$$(E'f)(t) = \begin{cases} f(t) & t \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

which again defines an isometric embedding. The convolution product for $f, g \in L^1(0,1)$ is again defined by

$$(f * g)(t) = (E'f * E'g)(t)$$

for $t \in (0,1)$, and $f * g \in L^1(0,1)$ with $\|f * g\|_1 \leqslant \|f\|_1 \|g\|_1$. Again, the expression for $f * g$ can be simplified to

$$(f * g)(t) = \int_0^t f(s)g(t-s)\,\mathrm{d}s$$

for $t \in [0,1]$.

The addition of the convolution product gives $L^1(X)$ an algebra structure, and since the 1-norm interacts properly with the multiplication (as in (2.1)), $L^1(X)$ forms a commutative Banach algebra [3, §2.1].

The convolution algebra $L^1(0,1)$ is also called the *Volterra algebra*. It will occasionally be denoted $\mathcal{V}$ to emphasise the algebraic structure.

2.2.1 Integral Transforms

The *Fourier* and *Laplace transforms* are especially useful when working with convolutions, as for any $f, g \in L^1(\mathbb{R})$,

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g) \qquad \text{and} \qquad \mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$$

where $\mathcal{F}$ and $\mathcal{L}$ represent the Fourier and Laplace transforms respectively. [16, Theorem 7.19]

The transforms have the effect of turning convolution into pointwise multiplication, which can be much easier to handle. The invertibility of the transforms shows that

$$f * g = \mathcal{F}^{-1}\left(\mathcal{F}(f)\mathcal{F}(g)\right)$$

and similarly for the Laplace transform; this can mean that a complicated convolution product can be calculated explicitly by moving to the transform domain and back. See Chapter 4 to see this in use.

2.3 The Titchmarsh Convolution Theorem

For $f \in L^1(\mathbb{R})\backslash\mathbf{0}$ define the *minimum support* of f, denoted $\alpha(f)$, by

$$\alpha(f) = \sup\{a \in \mathbb{R} : f \text{ is a.e. zero on } (-\infty, a)\}.$$

If the set is empty, then $\alpha(f) = -\infty$. For $f = \mathbf{0}$, define $\alpha(f) = \infty$.

If A is a non-empty subset of $L^1(\mathbb{R})$ then it is convenient to define $\alpha(A) = \inf\{\alpha(f) : f \in A\}$.

This is a useful tool for dealing with convolutions, as if $\alpha(f) > 0$ and

$\alpha(g) > 0$ then

$$\begin{aligned}(f * g)(t) &= \int_{\alpha(g)}^{\infty} f(t-s)g(s)\,\mathrm{d}s \\ &= \int_{-\infty}^{t-\alpha(g)} f(u)g(t-u)\,\mathrm{d}u \\ &= \int_{\alpha(f)}^{t-\alpha(g)} f(u)g(t-u)\,\mathrm{d}u\end{aligned}$$

so that $(f * g)(t)$ is zero for $t - \alpha(g) < \alpha(f)$; that is, $\alpha(f * g) \geqslant \alpha(f) + \alpha(g)$. In fact there is a much stronger version of this result, which is often referred to as the Titchmarsh Convolution Theorem.

2.1 Theorem (Titchmarsh Convolution Theorem)**.** *For all $f, g \in L^1(\mathbb{R})$ with $\alpha(f) > -\infty$ and $\alpha(g) > -\infty$*

$$\alpha(f * g) = \alpha(f) + \alpha(g).$$

This is presented here without proof, but several interesting proofs exist, including [17, VI.5] and [3, Theorem 4.7.22].

The Titchmarsh Convolution Theorem is sometimes presented in the superficially more general form below:

2.2 Theorem (Titchmarsh Convolution Theorem (interval version))**.** *Let $f, g \in L^1(\mathbb{R})$. If f is a.e. zero outside of $[a, b]$, and g is a.e. zero outside of $[c, d]$ then $f * g$ is a.e. zero outside of $[a+c, b+d]$. Furthermore, if $[a, b]$ and $[c, d]$ are the smallest such intervals for f and g respectively, then $[a+c, b+d]$ is the smallest interval for $f * g$.*

However, the two forms are in fact equivalent.

Proof. Clearly Theorem 2.2 implies Theorem 2.1, by the definition of $\alpha(\cdot)$. To see the reverse implication notice that

$$\inf\{b : f \text{ is a.e. zero on } (b, \infty)\} = \sup\{a : Rf \text{ is a.e. zero on } (-\infty, a)\}$$

where Rf denotes the function $t \mapsto f(-t)$. Temporarily adopting the notation $\beta(f)$ for the former expression, it is clear that $\beta(f) = \alpha(Rf)$. But since

$$\begin{aligned}
(Rf * Rg)(t) &= \int_{-\infty}^{\infty} (Rf)(t-s)(Rg)(s)\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty} f(s-t)g(-s)\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty} f(-u-t)g(-u)\,\mathrm{d}u \qquad \text{(with } u = -s\text{)} \\
&= \int_{-\infty}^{\infty} f((-t)-u)g(u)\,\mathrm{d}u \\
&= (f * g)(-t) \\
&= (R(f * g))(t)
\end{aligned}$$

it follows immediately that $\beta(f * g) = \beta(f) + \beta(g)$, which gives the result. □

When dealing with $f \in L^1(\mathbb{R}^+)$, $\alpha(f)$ is defined in precisely the same way. By definition, therefore, $\alpha(f) \geqslant 0$ for all $f \in L^1(\mathbb{R}^+)$. Both forms of the Titchmarsh Convolution Theorem are applicable to $L^1(\mathbb{R}^+)$, but there is also a useful third form in this context.

2.3 Theorem (Titchmarsh Convolution Theorem (integral domain form))**.** *Let $f, g \in L^1(\mathbb{R}^+)$. If $f * g$ is a.e. zero then either f is a.e. zero or g is a.e. zero.*

In other words, the convolution algebra $L^1(\mathbb{R}^+)$ has no zero divisors.

This is clearly a consequence of the previous versions of the theorem, but in fact it is also equivalent. A couple of quick results are required first.

2.4 Lemma. *If $f, g \in L^1(\mathbb{R})$ and τ_a represents a left-shift by $a \in \mathbb{R}$ (that is, $(\tau_a f)(t) = f(t-a)$), then*

$$\tau_a f * g = \tau_a(f * g).$$

Proof. For any $t \in \mathbb{R}$,

$$\begin{aligned}
(\tau_a f * g)(t) &= \int_{-\infty}^{\infty} (\tau_a f)(s) g(t-s)\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty} f(s-a) g(t-s)\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty} f(u) g(t-(u+a))\,\mathrm{d}u \qquad \text{(where } u = s-a\text{)} \\
&= \int_{-\infty}^{\infty} f(u) g((t-a)-u)\,\mathrm{d}u \\
&= (f * g)(t-a) \\
&= \big(\tau_a (f * g)\big)(t). \qquad \square
\end{aligned}$$

2.5 Lemma. *For any $f \in L^1(\mathbb{R})$ and $a \in \mathbb{R}$, $\alpha(\tau_a f) = \alpha(f) - a$.*

Proof. If $\alpha(f) = \infty$ then $f = 0$ a.e., so $\tau_a f$ is a.e. zero, and hence $\alpha(\tau_a f) = \infty = \infty - a$. If $\alpha(f) = -\infty$ then clearly $\alpha(\tau_a f) = -\infty = -\infty - a$ also.

So assume that $\alpha(f)$ is finite. Then

$$\begin{aligned}
\alpha(\tau_a f) &= \sup_s \{(\tau_a f)(t) = 0 \text{ a.a. } t < s\} \\
&= \sup_s \{f(t-a) = 0 \text{ a.a. } t < s\} \\
&= \sup_s \{f(t) = 0 \text{ a.a. } t < s - a\} \\
&= \sup_s \{f(t) = 0 \text{ a.a. } t < s\} - a \\
&= \alpha(f) - a. \qquad \square
\end{aligned}$$

Proof that Theorems 2.1 and 2.3 are equivalent for $L^1(\mathbb{R}^+)$. Assume that Theorem 2.1 holds; then for any $f, g \in L^1(\mathbb{R}^+)$, $\alpha(f) \geqslant 0$ and $\alpha(g) \geqslant 0$, so $\alpha(f * g) = \alpha(f) + \alpha(g)$. So if $\alpha(f * g) = \infty$ then either $\alpha(f) = \infty$ or $\alpha(g) = \infty$; i.e. either f is a.e. zero or g is a.e. zero.

Now assume that Theorem 2.3 holds, and let $f, g \in L^1(\mathbb{R}^+)$. From the previous Lemma, if $\alpha(f) > 0$ or $\alpha(g) > 0$ then

$$\alpha(f * g) = \alpha(\tau_{\alpha(f)} f * \tau_{\alpha(g)} g) + \alpha(f) + \alpha(g),$$

so it is enough to show that $\alpha(f * g) = 0$ when $\alpha(f) = \alpha(g) = 0$.

Assume then that $f, g \in L^1(\mathbb{R}^+)$ with $\alpha(f) = \alpha(g) = 0$. Then for any $\varepsilon > 0$,

$$\big((f\chi_{(0,\varepsilon)} * g\chi_{(0,\varepsilon)}\big)\,\chi_{(0,\varepsilon)} = (f * g)\chi_{(0,\varepsilon)}.$$

So if $\alpha(f * g) \geqslant 2\varepsilon$ then $f\chi_{(0,\varepsilon)} * g\chi_{(0,\varepsilon)}$ is a.e. zero, which means that either $\alpha(f) > \varepsilon$ or $\alpha(g) > 0$, which contradicts our assumption. So $\alpha(f * g) \leqslant 2\varepsilon$ for all $\varepsilon > 0$, and therefore $\alpha(f * g) = 0$. □

Remark. This form of the Titchmarsh Convolution Theorem is not applicable to $L^1(\mathbb{R})$, due to the presence of functions with minimum support $-\infty$ which are zero divisors. For example, let

$$f(t) = \frac{1 - \cos(t)}{t^2} \qquad \text{and} \qquad g(t) = \frac{-\cos(3t) + 2\cos(2t) - \cos(t)}{t^2}.$$

Taking the Fourier transforms of f and g gives

$$(\mathcal{F}f)(x) = \begin{cases} x + 1 & x \in (-1, 0) \\ 1 - x & x \in (0, 1) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (\mathcal{F}g)(x) = \begin{cases} x + 3 & x \in (-3, -2) \\ (-1) - x & x \in (-2, -1) \\ x - 1 & x \in (1, 2) \\ 3 - x & x \in (2, 3) \\ 0 & \text{otherwise} \end{cases}$$

respectively. Since $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$ (see §2.2.1), this shows that $\mathcal{F}(f * g)(x) = 0$ for all $x \in \mathbb{R}$, and so $f * g$ is a.e. zero.

2.3.1 The Standard Ideals of $L^1(\mathbb{R}^+)$

An *ideal* in a commutative algebra A is a set $U \subseteq A$ such that

1. U is a subspace of A
2. $au \in U$ for all $a \in A$ and all $u \in U$.

[16, Def'n 11.1] In the convolution algebras there are sets known as the *standard ideals*. These have the form

$$\{f \in L^1(0,1) : \alpha(f) \geqslant a\}$$

for $a \in \mathbb{R}^+$. These sets are denoted $L^1(a,\infty)$, and are regarded as subspaces of $L^1(\mathbb{R}^+)$. That these are indeed ideals follows immediately from the inequality

$$\alpha(f * g) \geqslant \alpha(f) + \alpha(g) \geqslant \alpha(g)$$

so that $f * g \in L^1(a,\infty)$ for any $f \in L^1(\mathbb{R}^+)$ and $g \in L^1(a,\infty)$.

These subspaces are closed in the 1-norm and form a nested sequence, as

$$L^1(a,\infty) \subseteq L^1(b,\infty) \qquad \text{iff} \qquad a \geqslant b.$$

2.4 The Unitisation of $\mathcal{V}$

The Volterra algebra is a non-unital algebra – an algebra without an element e such that $e * f = f$ for every $f \in \mathcal{V}$. However, it is sometimes useful to work with $\mathcal{V}$ as if it did have a unit.

Let $\mathcal{V}^1$ denote the unitisation of $\mathcal{V}$, defined by adjoining a unit e. Define a function $\tilde{\alpha}(\cdot)$ on $\mathcal{V}^1$ by

$$\tilde{\alpha}(\lambda e + f) = \begin{cases} 0 & \text{if } \lambda \neq 0 \\ \alpha(f) & \text{if } \lambda = 0 \end{cases}. \tag{2.2}$$

Remark. This definition formalises the notion of e, being essentially a delta-function and therefore having all of its mass concentrated at the origin, having a minimum support of 0.

It is immediate from the definition that $\tilde{\alpha}(f) = \alpha(f)$ for any $f \in \mathcal{V}$, so that $\tilde{\alpha}$ is an extension of α to $\mathcal{V}^1$. In fact the properties of α on $\mathcal{V}$ carry across to $\tilde{\alpha}$ on $\mathcal{V}^1$, as the following proposition shows.

2.6 Proposition. *For any $f, g \in \mathcal{V}^1$, $\tilde{\alpha}(f * g) = \tilde{\alpha}(f) + \tilde{\alpha}(g)$.*

Proof. Consider $\lambda e + f$ and $\mu e + g$ in $\mathcal{V}^1$, where $\lambda, \mu \in \mathbb{C}$ and $f, g \in \mathcal{V}$. There are three cases to consider:

- If $\lambda = \mu = 0$ then

$$\tilde{\alpha}\big((\lambda e+f) * (\mu e+g)\big) = \alpha(f * g) = \alpha(f) + \alpha(g) = \tilde{\alpha}(\lambda e+f) + \tilde{\alpha}(\mu e+g).$$

- If $\lambda \neq 0$ and $\mu \neq 0$ then

$$(\lambda e + f) * (\mu e + g) = \lambda\mu e + \lambda g + \mu f + f * g.$$

  Since the coefficient of e is non-zero, it follows immediately that

$$\tilde{\alpha}\big((\lambda e + f) * (\mu e + g)\big) = 0 = \tilde{\alpha}(\lambda e + f) + \tilde{\alpha}(\mu e + g).$$

- In the case that one of λ, μ is zero and the other non-zero, assume without loss of generality that $\mu = 0$ so that

$$(\lambda e + f) * (\mu e + g) = \lambda g + f * g.$$

  Clearly then $(\lambda e+f) * (\mu e+g)) = \alpha(\lambda g + f * g)$, which must be greater than or equal to $\alpha(g)$. If the inequality is strict then there exists an $a > \alpha(g)$ such that

$$(f * g)(t) = \lambda g(t) \tag{2.3}$$

  for all $t \in (0, a)$. However, if this were the case then restricting to the interval $(0, a)$ would give an element $g \in L^1(0, a)$ such that $V_f g = \lambda g$. This is a contradiction, since V_f has trivial spectrum and therefore no non-zero eigenvalues. Hence it must be that

$$\tilde{\alpha}\big((\lambda e + f) * (\mu e + g)\big) = \alpha(g) = \tilde{\alpha}(\lambda e + f) + \tilde{\alpha}(\mu e + g). \qquad \square$$

Given this result, we drop the tilde and define α on $\mathcal{V}^1$ as in (2.2), and call this the minimum support in analogy with $\mathcal{V}$.

This definition is useful for proving results on $\mathcal{V}$, as the following result shows.

2.7 Theorem. *If* $k \in L^1(0,1)$ *with* $\alpha(k) = 0$ *and* P *a non-zero polynomial with zero constant term, then*

$$\alpha(P(k)) = 0.$$

Proof. We proceed by induction on the degree of P. If P has degree 1 then $P(k)$ is a non-zero multiple of k, and hence $\alpha(P(k)) = \alpha(k)$.

Now suppose the result holds for polynomials of degree n, and that P is a polynomial of degree $n + 1$. Since $P(x)$ has zero constant term, we can re-write it as

$$P(x) = xQ(x)$$

where $Q(x)$ is a polynomial of degree n, which may or may not have zero constant term.

Thus $P(k)$ can be rewritten as $k * Q(k)$, and from the previous results we can now say that

$$\alpha(P(k)) = \alpha(k) + \alpha(Q(k))$$

regardless of whether Q has zero constant term.

If $Q(k)$ does have zero constant term then $\alpha(Q(k)) = 0$ by the inductive assumption; if, however, it has non-zero constant term then $\alpha(Q(k)) = 0$ by the definition of α on $\mathcal{V}^1$. In either case, $\alpha(Q(k)) = 0$, and so by the inductive principle the claim follows. □

2.5 $L^p(0,1)$

The spaces $L^p(0,1)$, $p, q \in [1,\infty]$ spaces have the interesting property that they are nested: for all $p, q \in [1,\infty]$ such that $p \leqslant q$,

$$\|f\|_p \leqslant \|f\|_q \tag{2.4}$$

for all $f \in \mathcal{M}(0,1)$, and so $L^q(0,1)$ can be embedded in $L^p(0,1)$. In particular, $L^p(0,1)$ can be embedded in $L^1(0,1)$ for every $p \in [1,\infty]$; this is done by taking the identity map on $\mathcal{M}(0,1)$ and restricting the domain to $L^p(0,1)$ and the co-domain to $L^1(0,1)$. That this map is a well-defined embedding follows from (2.4).

The range of this embedding is dense in $L^1(0,1)$, as can be seen by noting that the continuous functions on $(0,1)$, $\mathrm{C}(0,1)$, are dense in each $L^p(0,1)$.

The standard ideals form a kind of invariant in the $L^p(0,1)$ spaces, as the following lemma shows.

2.8 Lemma. *Let $a \in [0,1]$, and $1 \leqslant p < q < \infty$. Then*

$$L^p(a,1) \cap L^q(0,1) = L^q(a,1)$$

and

$$\mathrm{Cl}_p(L^q(a,1)) = L^p(a,1)$$

(where Cl_p denotes closure in the $L^p(0,1)$ norm).

Proof. First notice that $L^p(a,1) \cap L^q(0,1) \subseteq L^q(a,1)$, since all $f \in L^p(a,1)$ have $\alpha(f) \geqslant a$. But for any $f \in L^q(a,1)$, f is both in $L^q(0,1)$ and $L^p(a,1)$, so that $L^q(a,1) \subseteq L^p(a,1) \cap L^q(0,1)$. Hence $L^p(a,1) \cap L^q(0,1) = L^q(a,1)$.

Since the elements of $L^q(a,1)$ are a.e. zero on $(0,a)$ taking the $L^p(0,1)$-closure cannot introduce any non-zero elements on this interval, so $\mathrm{Cl}_p(L^q(a,1)) \subseteq L^p(a,1)$. Now let $f \in L^p(a,1)$. Since $L^q(0,1)$ is dense in $L^p(0,1)$, there exists a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subseteq L^q(0,1)$ such that $f_n \overset{L^p(0,1)}{\to} f$. So there exists a subsequence $\{g_n\}_{n\in\mathbb{N}}$ that tends to f pointwise almost everywhere. By considering $h_n = g_n\chi_{(a,1)}$ it follows that $\{h_n\}_{n\in\mathbb{N}} \subseteq L^q(a,1)$ and that $h_n \to f$ pointwise a.e.. So $h_n \overset{L^p(0,1)}{\to} f$, and therefore $f \in \mathrm{Cl}_p(L^q(a,1))$, so that $L^p(a,1) \subseteq \mathrm{Cl}_p(L^q(a,1))$. □

This will have applications to the invariant subspaces of convolution operators.

2.5.1 The dual of $L^p(0,1)$

There is another relationship between certain of the $L^p(0,1)$ spaces: for each $p \in [1,\infty]$, define the *Hölder conjugate* of p as

$$p\prime = \frac{p}{p-1}$$

(that is, $1/p + 1/p\prime = 1$).

This has an important consequence, in *Hölder's inequality* [15, 3.5]: for any $p \in [1,\infty]$ with $p\prime$ its Hölder conjugate and $f, g \in \mathcal{M}(0,1)$, then

$$\|fg\|_1 \leqslant \|f\|_p \, \|p\prime\| \, g.$$

For $p \in [1,\infty)$, the *dual-space* of $L^p(0,1)$ (that is, the space of bounded linear functionals from $L^p(0,1)$ to $\mathbb{R}$) is isometrically isomorphic to $L^{p'}(0,1)$. [15, 6.16] Similarly, for $p \in (1,\infty]$, the pre-dual of $L^p(0,1)$ is isometrically isomorphic to $L^{p'}(0,1)$.

For any $f \in L^p(0,1)$ and $g \in L^{p'}(0,1)$, the functional relationship between the spaces is denoted by angle brackets:

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,\mathrm{d}t\,.$$

This is guaranteed to be finite by Hölder's inequality.

For any $p \in (1,\infty]$, the pre-dual of $L^p(0,1)$ induces a topology on $L^p(0,1)$, called here the *weak$*$-topology* [16, §3.14]. For $p \in (1,\infty)$, $L^p(0,1)$ is a reflexive space, so that the bi-dual of $L^p(0,1)$ is $L^p(0,1)$ itself. In this case, the choice of pre-dual is unambiguous. However, in the case of $L^\infty(0,1)$, there are potentially other Banach spaces whose dual-space is isometrically isomorphic to $L^\infty(0,1)$. In the rest of this text, the term "weak$*$-topology of $L^p(0,1)$" is used to refer exclusively to the weak$*$-topology induced by the pre-dual $L^{p'}(0,1)$.

An important and useful tool when studying the norm- or weak$*$-topology on $L^p(0,1)$ are *annihilators* [16, §4.6]. For $p \in [1,\infty)$ and a set $A \subseteq L^p(0,1)$, the *annihilator* of A is a subset of $L^{p'}(0,1)$ denoted $A^\perp$ and defined by

$$A^\perp = \{g \in L^{p'}(0,1) : \langle f, g\rangle = 0 \text{ for all } f \in A\}.$$

Similarly, for $p \in (1,\infty]$ and $A \subseteq L^p(0,1)$, the *pre-annihilator* of A is denoted $A^\top$ and defined as

$$A^\top = \{g \in L^{p'}(0,1) : \langle g, f\rangle = 0 \text{ for all } f \in A\}.$$

For any A it is easy to show that $A^\perp$ is a weak$*$-closed subspace of $L^{p'}(0,1)$, and similarly $A^\top$ is a norm-closed subspace of $L^{p'}(0,1)$. In fact for any $A \in L^p(0,1)$, $p \in [1,\infty)$,

$$A^{\perp\top} = \mathrm{Cl}_p(\mathrm{span}(A))$$

and for $p \in (1,\infty]$,

$$A^{\top\perp} = \mathrm{Cl}_p^{\mathrm{w}*}(\mathrm{span}(A))$$

where span is the algebraic span of A, Cl_p denotes closure in the norm topology on $L^p(0,1)$, and $\mathrm{Cl}_p^{\mathrm{w}*}$ means closure in the weak$*$-topology on $L^p(0,1)$.

2.6 Convolution Operators

An operator A on the Banach space $L^p(0,1)$ is *bounded* if there exists a constant U such that

$$\|Af\|_p \leqslant U\,\|f\|_p$$

for all $f \in L^p(0,1)$; say that U is a *bound* for A. In this case, write $A \in \mathcal{B}(L^p(0,1))$, and define the *operator norm* of A as

$$|||A|||_p = \inf\{U : \|Af\|_p \leqslant U\,\|f\|_p\}.$$

Notice that the operator norm is dependant on p, and that in general even if A is defined on $L^p(0,1)$ for more than one p, A will not necessarily be bounded for both. (For more details on when boundedness can be inferred, see the Riesz–Thorin theorem in e.g. [6].)

For any $k \in L^1(0,1)$ and any $p \in [1,\infty]$, define the *convolution operator* $V_k : L^p(0,1) \to L^p(0,1)$ by

$$V_k f = k * f$$

for $f \in L^p(0,1)$. The function k is called the *kernel* of the operator V_k. That $V_k \in \mathcal{B}(L^p(0,1))$ follows from an application of the Riesz–Thorin Theorem: it is clear that $V_k \in \mathcal{B}(L^1(0,1))$ and $V_k \in \mathcal{B}(L^\infty(0,1))$, from Hölder's inequality. The Riesz–Thorin Interpolation Theorem [6, VI.10.11] then shows that $V_k \in \mathcal{B}(L^p(0,1))$ for all $p \in (1,\infty)$.

This result leads to a useful version of the Hausdorff–Young inequality: if $f, g \in \mathcal{M}(0,1)$ and $p, q \in [1,\infty]$ then

$$\|f * g\|_r \leqslant \|f\|_p \|g\|_q$$

for r such that

$$1/p + 1/q = 1 + 1/r. \tag{2.5}$$

Remark. It should be noted that if $f \in L^p(0,1)$ and $g \in L^{p'}(0,1)$ then $f * g$ is continuous. Thus $1/r \leqslant 0$ in (2.5) should be interpreted as meaning that $f * g$ is continuous.

The Hausdorff–Young inequality is often used when dealing with multiple L^p-spaces. It can produce surprisingly powerful results, including the following lemma.

2.9 Lemma. *If $k\chi_{(0,\delta)} \in L^p(0,1)$ for some $\delta > 0$, some $p > 1$, then there exists an $N \in \mathbb{N}$ such that k^{*n} is continuous for all $n > N$.*

Proof. If $f\chi_{(0,\delta)} \in L^p(0,1)$ then

$$f * f = f\chi_{(0,\delta)} * f\chi_{(0,\delta)} + 2f\chi_{(0,\delta)} * f\chi_{(\delta,1)} + f\chi_{(\delta,1)} * f\chi_{(\delta,1)}$$

so that

$$\begin{aligned} \|(f * f)\chi_{(0,2\delta)}\|_p &\leqslant \left\|\left(f\chi_{(0,\delta)} * f\chi_{(0,\delta)}\right)\chi_{(0,2\delta)}\right\|_p + 2\left\|\left(f\chi_{(0,\delta)} * f\chi_{(\delta,1)}\right)\chi_{(0,2\delta)}\right\|_p \\ &\leqslant \|f\chi_{(0,\delta)}\|_p \|f\chi_{(0,\delta)}\|_1 + 2\|f\chi_{(0,\delta)}\|_p \|f\chi_{(\delta,1)}\|_1 \end{aligned}$$

by the inequality. So $f^{*2}\chi_{(0,2\delta)} \in L^p(0,1)$; this process can be repeated, so that $f^{*2^n} \in L^p(0,1)$ for $2^n\delta > 1$.

Assume then without loss of generality that $f \in L^p(0,1)$. If $p \geqslant 2$ then shows that $f * f$ is continuous. If $p < 2$ then the inequality shows that $f * f \in L^r(0,1)$, for $2/p = 1 + 1/r$. Since $p \in (1,2)$, so $r = p/(2-p) > p$. In fact

$$f^{*2^n} \in L^r(0,1)$$

for $r = p/(2^n(1-p) - p)$, so that $f^{*2^n} \in L^2(0,1)$ when $2^n > p/(p-1)$.

Since a continuous function convolved with an $L^1(0,1)$ function is itself continuous, this shows that there exists an $N \in \mathbb{N}$ such that f^{*n} is continuous for all $n > N$. □

The Hausdorff–Young inequality can be applied to $V_k \in \mathcal{B}(L^p(0,1))$ to get an explicit bound for $|||V_k|||_p$:

$$\|V_k f\|_p = \|k * f\|_p \leqslant \|k\|_1 \|f\|_p$$

for all $f \in L^p(0,1)$, so that $|||V_k|||_p \leqslant \|k\|_1$.

With this definition, it is clear that the classical *Volterra operator* (or *indefinite integration operator*) V is the convolution operator with kernel $\mathbf{1}$, where $\mathbf{1}(t) = 1$ for all $t \in (0,1)$.

For $n \in \mathbb{N}$, $V^n = V_{\mathbf{1}^{*n}}$, and can be explicitly written out as

$$(V^n f)(t) = \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} f(s)\,\mathrm{d}s\,.$$

For $n \in \mathbb{R}$ with $n > 0$ this expression still makes sense; in this case it is known as the *Riemann–Liouville fractional integration operator*. Therefore, for $n \in \mathbb{R}$ such that $n > 0$, define $\mathbf{1}^{*n}$ by

$$\mathbf{1}^{*n}(t) = \frac{t^{n-1}}{\Gamma(n)}$$

(which is consistent for the existing expression for $\mathbf{1}^{*n}$ for $n \in \mathbb{N}$), so that $V^n = V_{\mathbf{1}^{*n}}$.

2.7 Asymptotics

Let $(\alpha_n)_{n\in I}$, $(\beta_n)_{n\in I}$ be real or complex nets, indexed by a directed set I. Then (α_n) and (β_n) are *asymptotically equal* as $n \to \infty$ (written $\alpha_n \sim \beta_n$ as $n \to \infty$) [4] if

$$\lim_n \frac{\alpha_n}{\beta_n} = 1.$$

This definition can be extended to nets in a normed vector space over $\mathbb{R}$ or $\mathbb{C}$. [9, §2] If $(u_n)_{n\in I}$, $(v_n)_{n\in I}$ are nets in a normed vector space say that $u_n \sim v_n$ if

$$\lim_n \frac{\|u_n - v_n\|}{\|u_n\|} = 0.$$

It is easy to check that this definition is reflexive, symmetric and transitive, so that asymptotic equality is an equivalence relation.

Although the same symbol is used for both scalar and vector asymptotic equality, it is unambiguous when considering particular nets.

Remark. It is easy to check that if $u_n \sim v_n$ then $\|u_n\|_1 \sim \|v_n\|_1$:

$$0 = \lim_n \frac{\|u_n - v_n\|_1}{\|u_n\|_1} \geqslant \lim_n \frac{|\|u_n\|_1 - \|v_n\|_1|}{\|u_n\|_1} = \lim_n \left|1 - \frac{\|v_n\|_1}{\|u_n\|_1}\right| \geqslant 0.$$

Chapter 3

Cyclicity of Convolution Operators

In [12], Gallardo and Montes show that the Volterra operator is not supercyclic – that there is no $f \in L^2(0,1)$ such that

$$\{\lambda V^n f : n \in \mathbb{N}, \lambda \in \mathbb{R}\}$$

is dense in $L^2(0,1)$.

A related concept is that of a *cyclic operator*. An operator $A \in \mathcal{B}(L^p(0,1))$ is *cyclic* if there exists an $f \in L^p(0,1)$ (a *cyclic vector*) such that

$$\text{span}\{A^n f : n \in \mathbb{N}\}$$

is dense in $L^p(0,1)$, where span denotes the algebraic span. This is clearly a weaker condition than supercyclicity (and than hypercyclicity, which is defined similarly and is stronger than both).

That V is a cyclic operator is a consequence of the Weierstrass approximation theorem, but in fact there is a much stronger result: for $V \in \mathcal{B}(L^p(0,1))$ and $f \in L^p(0,1)$ with $\alpha(f) = 0$, then f is a cyclic vector for V. This is strongly related to another concept: that of a unicellular operator.

3.1 Criteria for Unicellularity

Every bounded linear operator A acting on a Banach space X has a set of closed subspaces of X on which the operator is invariant; that is, for $U \subseteq X$, $AU \subseteq U$. These spaces form a lattice, with the meet defined as the intersection of subspaces, and the join as the span of their union.

Clearly the whole space X and the zero subspace are always invariant under linear transforms; if these are the only invariant subspaces then the lattice is called *trivial*.

Usually spectral theory is used to analyse the invariant subspace lattices of operators, but this is not applicable for quasi-nilpotent operators (that is, operators with trivial spectrum). In fact little can be said about the invariant subspaces of general quasi-nilpotent operators ([5]), but in the case of convolution operators there is more that can be obtained.

3.1 Lemma. *Let $k \in L^1(0,1)$ and V_k be the convolution operator with kernel k acting on $L^p(0,1)$, for $p \in [1,\infty]$. Then $L^p(a,1)$ is a closed subspace of $L^p(0,1)$ invariant under V_k, for all $a \in [0,1]$.*

Proof. Fix an $a \in [0,1]$. For any $f \in L^p(a,1)$, the Titchmarsh Convolution Theorem states that

$$\alpha(V_k f) = \alpha(k * f) = \alpha(k) + \alpha(f)$$

and since $\alpha(k) \geqslant 0$ and $\alpha(f) \geqslant a$,

$$\alpha(V_k f) \geqslant a.$$

In particular, $V_k f$ is a.e. zero outside of the interval $(a,1)$, and so $V_k f \in L^p(a,1)$. So $L^p(a,1)$ is an invariant subspace for V_k.

To show that $L^p(a,1)$ is closed, let f be the limit in the $L^p(0,1)$ norm of a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ such that $f_n \in L^p(a,1)$ for all $n \in \mathbb{N}$. It is a theorem [15, Thm 3.12] that there exists a convergent subsequence $\{g_n\}_{n\in\mathbb{N}}$ which converges to f pointwise. However, each of the f_n (and hence each g_n) is a.e. zero in the interval $(0,a)$, and so f must be a.e. zero on $(0,a)$. Hence

$f \in L^p(a,1)$, and so the subspace is closed. $\square$

This chain of nested subspaces is a particularly interesting feature of some invariant subspace lattices.

3.2 Definition. Fix $p \in [1,\infty)$, and let $X = \mathbb{R}$, $X = \mathbb{R}^+$ or $X = (0,1)$. For a bounded linear operator A on the Banach space $L^p(X)$, define the *invariant subspace lattice* of A as

$$L_A = \{U \subseteq L^p(X) : AU \subseteq U \text{ and } U \text{ norm-closed}\}.$$

In a similar way, if $p \in (1,\infty]$, define the *weak∗ invariant subspace lattice* of A as

$$L_A^* = \{U \subseteq L^p(X) : AU \subseteq U \text{ and } U \text{ closed in the weak∗ topology}\}$$

where the topology is the weak∗ topology induced by the predual $L^{p'}(X)$.

An operator $A \in \mathcal{B}(L^p(X))$ is called *unicellular* if the lattice L_A is linearly ordered by inclusion. Similarly, A is *weak∗-unicellular* if L_A^* is linearly ordered by inclusion.

Remark. The weak∗ invariant subspace lattice is only defined for $A \in \mathcal{B}(L^p(X))$ for $p > 1$, since $L^\infty(X)$ is not a predual for $L^1(X)$. In contrast the norm-based invariant subspace lattice is well-defined for $L^\infty(X)$; however, many of the properties explored later on in this chapter do not hold for the invariant subspace lattice on $L^\infty(X)$. See, for example, [5].

As reflexive spaces, the closed subspaces of the norm and weak∗ topologies of $L^p(0,1)$ coincide when $p \in (1,\infty)$. In that case, then, $L_A = L_A^*$.

In the case of convolution operators, then, if all of the elements in the lattice are of the form $L^p(a,1)$ then the operator is unicellular. In fact this is both a sufficient and necessary condition, and there is a simple characterisation of unicellularity for a convolution operator, based on the orbits of elements in $L^p(0,1)$.

3.3 Theorem. *Fix $p \in [1, \infty)$, and let $k \in L^1(0,1)$. Let V_k be the convolution operator with kernel k acting on $L^p(0,1)$ by convolution with k. Then the following are equivalent:*

1. *For all $f \in L^p(0,1)$, the norm-closure of* $\operatorname{span}\{V_k^n f : n \in \mathbb{N}\}$ *is $L^p(\alpha(f), 1)$.*

2. *V_k has the simplest closed invariant subspace lattice $\{L^p(a,1) : a \in [0,1]\}$.*

3. *V_k is unicellular on $L^p(0,1)$.*

In addition, for $p \in (1, \infty]$ the following are equivalent

1. *For all $f \in L^p(0,1)$, the weak$*$-closure of* $\operatorname{span}\{V_k^n f : n \in \mathbb{N}\}$ *is $L^p(\alpha(f), 1)$.*

2. *V_k has the simplest weak$*$-closed invariant subspace lattice $\{L^p(a,1) : a \in [0,1]\}$.*

3. *V_k is weak$*$-unicellular on $L^p(0,1)$.*

Before proving this, we prove the following Lemmas.

3.4 Lemma. *If $\alpha(k) > 0$ then V_k is nilpotent. In particular, if any of the three conditions in Theorem 3.3 hold, then $\alpha(k) = 0$.*

Proof. Assume that $\alpha(k) > 0$. Now $V_k^n f = k^{*n} * f$, and by the Titchmarsh Convolution Theorem,

$$\alpha(k^{*n}) = \min\{1, \alpha(k) + \alpha(k^{*(n-1)})\} = \min\{1, n\alpha(k)\}.$$

So if $n > 1/(\alpha(k))$ then k^{*n} is a.e. zero. Hence, by the Titchmarsh Convolution Theorem again, $V_k^n f$ is a.e. zero for all $f \in L^p(0,1)$; that is, V_k is nilpotent.

Any nilpotent operator has many finite-dimensional closed invariant subspaces – the orbit of any vector will have finitely many distinct elements, and therefore the closure of the span of the orbit will be finite dimensional – so if

V_k is nilpotent then it cannot be unicellular (condition 3), or have the simplest invariant subspace lattice (condition 2). The Titchmarsh Convolution Theorem gives

$$\alpha(k * f) = \alpha(k) + \alpha(f)$$

so the closed span of the orbit of f under V_k cannot contain functions with minimum support less than $\alpha(k) + \alpha(f)$ (condition 1). So if any of the three conditions in Theorem 3.3 hold then, by contradiction, $\alpha(k) = 0$. □

3.5 Lemma. *For $p \in [1, \infty)$, if U is a norm-closed subspace of $L^p(0,1)$ such that*

$$L^p(a + \varepsilon, 1) \subseteq U$$

for all $\varepsilon > 0$, then $L^p(a, 1) \subseteq U$.

Similarly, for $p \in (1, \infty]$, if U is a weak$*$-closed subspace of $L^p(0,1)$ such that

$$L^p(a + \varepsilon, 1) \subseteq U$$

for all $\varepsilon > 0$, then $L^p(a, 1) \subseteq U$.

Proof. For $p \in [1, \infty)$ and $U \subseteq L^p(0,1)$ a norm-closed subspace of $L^p(0,1)$ such that

$$L^p(a + \varepsilon, 1) \subseteq U$$

for all $\varepsilon > 0$, consider an arbitrary $f \in L^p(a, 1)$. It is enough to show that $f \in U$.

Define $f_\varepsilon \coloneqq f\chi_{(a+\varepsilon,1)}$, so that $f_\varepsilon \in L^p(a + \varepsilon, 1)$ and therefore $f_\varepsilon \in U$ for all $\varepsilon > 0$.

Now

$$\|f - f_\varepsilon\|_p^p = \int_0^1 |f(t) - f_\varepsilon(t)|^p \, \mathrm{d}t = \int_a^{a+\varepsilon} |f(t)|^p \, \mathrm{d}t$$

which tends to zero as $\varepsilon \to 0$, so that $f_\varepsilon \to f$ in the p-norm. Since U is norm-closed it must be that $f \in U$.

Similarly for $p \in (1, \infty]$ and $U \subseteq L^p(0,1)$ a weak$*$-closed subspace of $L^p(0,1)$ such that

$$L^p(a + \varepsilon, 1) \subseteq U$$

for all $\varepsilon > 0$, consider an arbitrary $f \in L^p(a,1)$. It is enough to show that $f \in U$.

Again, define $f_\varepsilon := f\chi_{(a+\varepsilon,1)}$, so that $f_\varepsilon \in L^p(a+\varepsilon,1) \subseteq U$.

Now for any $g \in L^{p'}(0,1)$,

$$\langle g, f - f_\varepsilon \rangle = \int_0^1 g(t)(f - f_\varepsilon)(t)\,\mathrm{d}t = \int_a^{a+\varepsilon} g(t)f(t)\,\mathrm{d}t$$

which tends to zero as $\varepsilon \to 0$, so that $f_\varepsilon \to f$ in the weak$*$ topology on $L^p(0,1)$. Since U is weak$*$-closed, it follows immediately that $f \in U$. □

Proof of Theorem 3.3. Fix $p \in [1,\infty]$. The proof for the norm-topology on $L^p(0,1)$ for $p \in [1,\infty)$ is the same as the proof for the weak$*$-topology for $p \in (1,\infty)$, with minor differences denoted by (norm/weak$*$).

($1 \implies 2$) Let U be a (norm-/weak$*$-) closed subspace of $L^p(0,1)$, invariant under V_k. Clearly $U \subseteq L^p(\alpha(U),1)$; it is therefore enough to show that $L^p(\alpha(U),1) \subseteq U$.

Now for all $f \in U$,

$$\{V_k^n f : n \in \mathbb{N}\} \subseteq U$$

and since U is (norm-/weak$*$-)closed and using the assumption, this gives that $L^p(\alpha(f),1) \subseteq U$ for all $f \in U$.

Now for any $\varepsilon > 0$ there exists an $f \in U$ with $\alpha(f) \leqslant \alpha(U) + \varepsilon$, so by the previous observation

$$L^p(\alpha(U)+\varepsilon,1) \subseteq L^p(\alpha(f),1) \subseteq U$$

for all $\varepsilon > 0$, and applying Lemma 3.5 gives that $L^p(\alpha(U),1) \subseteq U$.

($2 \implies 1$) Assume that V_k has the simple invariant subspace lattice, and let $f \in L^p(0,1)$. Clearly the (norm/weak$*$) closure of $A := \operatorname{span}\{V_k^n f : n \in \mathbb{N}\}$ is a closed subspace of $L^p(0,1)$, and is invariant under V_k, and must therefore be an $L^p(a,1)$ for some $a \in [0,1]$. It is enough to show that $a = \alpha(f)$.

Clearly $k * f \in A$, and $\alpha(k * f) = \alpha(k) + \alpha(f) = \alpha(f)$ since $\alpha(k) = 0$ (from Lemma 3.4). So $a \leqslant \alpha(f)$; however, from the Titchmarsh Convolution

Theorem it is clear that $\alpha(g) \geqslant \alpha(f)$ for all $g \in A$, and so no g in the (norm/weak∗) closure of A can have minimum support less than $\alpha(f)$. So $a \geqslant \alpha(f)$.

($2 \implies 3$) If V_k has the simple invariant subspace lattice $\{L^p(a,1) : a \in [0,1]\}$ then clearly the spaces are linearly ordered by inclusion, and so V_k is (weak∗-) unicellular.

($3 \implies 2$) Lemma 3.1 shows that the subspaces $\{L^p(a,1) : a \in [0,1]\}$ of $L^p(0,1)$ are invariant under V_k. It remains to show that these are the only elements of L_{V_k}.

Let U be a (norm/weak∗)-closed subspace of $L^p(0,1)$ invariant under V_k. Clearly $U \subseteq L^p(\alpha(U),1)$. For any $\varepsilon > 0$ there exists an $f \in U$ such that $\alpha(f) \leqslant \alpha(U) + \varepsilon/2$; hence $U \nsubseteq L^p(\alpha(U)+\varepsilon,1)$, and since V_k is unicellular, it must be that $L^p(\alpha(U)+\varepsilon,1) \subseteq U$ for all $\varepsilon > 0$. Hence, by Lemma 3.5, $L^p(\alpha(U),1) \subseteq U$. □

3.2 Unicellularity on different spaces

One of the unusual properties of convolution operators is that they are bounded operators on $L^p(0,1)$ for each $p \in [1,\infty]$. This leads to a natural question about the invariant subspace lattices of convolution operators: does the invariant subspace lattice of $V_k \in \mathcal{B}(L^p(0,1))$ have any relation to the invariant subspace lattice of $V_k \in \mathcal{B}(L^q(0,1))$?

As the following lemma shows, there is a relationship between the invariant subspace lattices on different $L^p(0,1)$ spaces; however, the relationship may not be simple.

3.6 Lemma. *Let $1 \leqslant p < q < \infty$, $k \in L^1(0,1)$. Let U_p be a closed invariant subspace of $L^p(0,1)$ for the operator V_k, and U_q be a closed invariant subspace of $L^q(0,1)$ for V_k Then*

$$U_p \cap L^q(0,1)$$

is a closed subspace of $L^q(0,1)$ invariant under V_k. Similarly,

$$\mathrm{Cl}_p(U_q)$$

is a closed subspace of $L^p(0,1)$ invariant under V_k.

Proof. Let $f \in U_p \cap L^q(0,1)$. Then $V_k f \in U_p$, since U_p is an invariant subspace for V_k; but $V_k f \in L^q(0,1)$ as well, by the Hausdorff–Young inequality. So $V_k f \in U_p \cap L^q(0,1)$, and so this is an invariant subspace for V_k.

To check that $U_p \cap L^q(0,1)$ is closed, take any $L^q(0,1)$-Cauchy sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq U_p \cap L^q(0,1)$. Then g_n tends to a limit, say f, such that $f \in L^q(0,1)$. Now

$$\|f - g_n\|_p \leqslant \|f - g_n\|_q$$

so $\{g_n\}$ is also an $L^p(0,1)$-Cauchy sequence, entirely contained in U_p. Since U_p is closed in the $L^p(0,1)$-norm, it follows that $f \in U_p$, and hence $f \in U_p \cap L^q(0,1)$. So $U_p \cap L^q(0,1)$ is a closed subspace of $L^q(0,1)$, invariant under V_k.

Clearly $\mathrm{Cl}_p(U_q)$ is closed, and since it is the L^p-closure of a subspace of $L^q(0,1)$, it must itself be a subspace of $L^p(0,1)$. We need only check it is invariant under V_k. If $f \in \mathrm{Cl}_p(U_q)$ then there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq U_q$ such that $g_n \overset{L^p(0,1)}{\to} f$. Fix an $\varepsilon > 0$, and an $n \in \mathbb{N}$ such that $\|f - g_n\|_p \leqslant \varepsilon$. Then

$$\|V_k f - V_k g_n\|_p = \|V_k(f - g_n)\|_p \leqslant \|k\|_1 \|f - g_n\|_p \leqslant \|k\|_1 \varepsilon.$$

Since $V_k g_n \in U_q$ for all n, there exist elements of U_q arbitrarily close to $V_k f$. Hence $V_k f \in \mathrm{Cl}_p(U_q)$, so that $\mathrm{Cl}_p(U_q)$ is a closed subspace of $L^p(0,1)$, invariant under V_k. □

In some respects the two operations above can be regarded as inverses, but it is not known if $\mathrm{Cl}_p(U_p \cap L^q(0,1)) = U_p$ in general.

In the case where $q = p\prime$ the Hölder conjugate of p, there is an additional link between the two spaces; this relationship can be exploited to show that there is a relationship between invariant subspace lattices on $L^p(0,1)$ and

$L^{p'}(0,1)$.

3.7 Lemma. *Let $k \in L^1(0,1)$ and V_k be the convolution operator with kernel k, and define $R : L^1(0,1) \to L^1(0,1)$ by $(Rf)(t) = f(1-t)$.*

For $p \in [1,\infty)$ and $p\prime$ the Hölder conjugate of p, and $A \subseteq L^p(0,1)$ a (not necessarily closed) subspace of $L^p(0,1)$ invariant under V_k, then

$$R(A^{\perp}) \coloneqq \{Rg : g \in A^{\perp}\}$$

is a weak∗-closed subspace of $L^{p'}(0,1)$ invariant under $V_k \in \mathcal{B}(L^{p'}(0,1))$.

Similarly, if $p \in (1,\infty]$ and $A \subseteq L^p(0,1)$ a subspace of $L^p(0,1)$ invariant under V_k, then $R(A^{\top})$ is a norm-closed subspace of $L^{p'}(0,1)$ invariant under V_k.

Proof. It is immediate from the properties of annihilators that for $p \in [1,\infty)$ $A^{\perp}$ is a weak∗-closed subspace of $L^{p'}(0,1)$, and therefore $R(A^{\perp})$ is also. Similarly for $p \in (1,\infty]$, $A^{\top}$ is a norm-closed subspace of $L^{p'}(0,1)$, so $R(A^{\top})$ is. It remains to show that these sets are invariant under V_k.

Fix $p \in [1,\infty)$, and let $g \in A^{\perp}$. Then $Rg \in R(A^{\perp})$, and it is enough to show that $h \coloneqq R(k * (Rg)) \in A^{\perp}$ to show that $R(A^{\perp})$ is invariant under V_k.

Similarly, if $p \in (1,\infty]$ and $g \in A^{\top}$, showing that $h \coloneqq R(k * (Rg))$ is in $A^{\top}$ is enough.

The rest of the proof works for both versions of the statement of the Lemma, with some minor differences. It is assumed that $g \in A^{\perp}$, except where indicted with "resp.".

For all $f \in A$, $k * f \in A$, so that $\langle k * f, g\rangle = 0$ (resp. $\langle g, k * f\rangle = 0$). Consider $(k * f) * \tilde{g}$, where $\tilde{g} = Rg$. Now $(k * f) * \tilde{g} = f * (k * \tilde{g})$, by the basic properties of convolution, and since $f \in L^p(0,1)$ and $(k * \tilde{g}) \in L^{p'}(0,1)$, this is a continuous function. Hence $(f * k * \tilde{g})(1)$ is well-defined, and

$$\begin{aligned}(k * f * \tilde{g})(1) &= \int_0^1 (k * f)(t)\tilde{g}(1-t)\,\mathrm{d}t \\ &= \int_0^1 (k * f)(t)g(t)\,\mathrm{d}t\end{aligned}$$

which is $\langle f * k, g\rangle = 0$ (resp. $\langle g, f * k\rangle = 0$), since $g \in A^{\perp}$ (resp. $g \in A^{\top}$).

Then $\langle f, h\rangle = \int_0^1 f(t)h(t)\,\mathrm{d}t$ (resp. $\langle h, f\rangle = \int_0^1 f(t)h(t)\,\mathrm{d}t$), and

$$\int_0^1 f(t)h(t)\,\mathrm{d}t \;=\; \int_0^1 f(t)(k * \tilde{g})(1-t)\,\mathrm{d}t \;=\; (f * k * \tilde{g})(1) = 0.$$

Since $f \in A$ was chosen arbitrarily, this shows that $h \in A^{\perp}$ (resp. $h \in A^{\top}$. $\square$

This result will be useful in later results, and it can be immediately expanded into a unicellularity result.

3.8 Theorem. *Let $k \in L^1(0,1)$ and V_k the convolution operator with kernel k. Then for all $p \in [1,\infty)$ V_k is unicellular on $L^p(0,1)$ if and only if V_k is weak$*$-unicellular on $L^{p'}(0,1)$.*

Proof. Fix $p \in [1,\infty)$ and assume that V_k is weak$*$-unicellular on $L^{p'}(0,1)$. Let $f \in L^p(0,1)$, and define $A = \operatorname{span}\{V_k^n f : n \in \mathbb{N}\}$. If the norm-closure of A is $L^p(\alpha(f),1)$ then Theorem 3.3 shows that V_k is unicellular on $L^p(0,1)$.

The previous result shows that $R(A^{\perp})$ is a weak$*$-closed subspace of $L^{p'}(0,1)$ invariant under V_k, so by the assumption that V_k is weak$*$-unicellular on $L^{p'}(0,1)$, $R(A^{\perp}) = L^{p'}(1-b,1)$ for some $b \in [0,1]$. So $A^{\perp} = L^{p'}(0,b)$.

It is a simple application of the Titchmarsh Convolution Theorem to see that $A \subseteq L^p(\alpha(f),1)$, and therefore $A^{\perp} \supseteq L^{p'}(0,\alpha(f))$. So $b \geqslant \alpha(f)$.

Consider $\chi_{[0,\varepsilon]}$ for $\varepsilon \in [0,b]$, and let $g \in A$ such that $\alpha(g) = \alpha(f)$ (e.g. $g = k * f$). Then, since $\chi_{[0,\varepsilon]} \in L^{p'}(0,b) = A^{\perp}$, $\langle g, \chi_{[0,\varepsilon]}\rangle = 0$. But

$$\begin{aligned}\langle g, \chi_{[0,\varepsilon]}\rangle &= \int_0^1 g(t)\chi_{[0,\varepsilon]}(t)\,\mathrm{d}t\\ &= \int_0^{\varepsilon} g(t)\,\mathrm{d}t\end{aligned}$$

and so $g(t) = 0$ for almost all $t \in (0,b)$. But that means that $\alpha(g) \geqslant b$. Thus $b = \alpha(f)$, and so

$$A^{\perp\top} = L^{p'}(0,\alpha(f))^{\top} = L^p(\alpha(f),1)$$

as required.

Now fix $p \in (1,\infty]$, $f \in L^p(0,1)$, and set $A = \operatorname{span}\{V_k^n f : n \in \mathbb{N}\}$. Assume that V_k is unicellular on $L^{p'}(0,1)$. The proof that $A^\top = L^{p'}(0,\alpha(f))$ is precisely the same as before, just replacing $A^\perp$ with $A^\top$. Then $A^{\top\perp} = L^p(\alpha(f),1)$, and Theorem 3.3 gives that V_k is weak∗-unicellular on $L^p(0,1)$. □

3.3 Generators for $L^1(0,1)$

In [5] Donoghue proves that the Volterra operator is unicellular on $L^2(0,1)$. The proof relies on the Weierstrass approximation theorem: that the polynomials are dense in the continuous functions on $[0,1]$, and so in $L^p(0,1)$ for all $p \in [1,\infty]$.

This can be regarded as a special case of a more general concept.

3.9 Definition. Let $k \in L^1(0,1)$. k is called a *polynomial generator* of $L^1(0,1)$ if

$$\operatorname{span}\{k^{*n} : n \in \mathbb{N}\}$$

is dense in $L^1(0,1)$.

That the constant function $\mathbf{1}$ is a polynomial generator of $L^1(0,1)$ is just a restatement of the Weierstrass approximation theorem. With this definition in place, the result in [5] can be extended.

3.10 Theorem. *Let $k \in L^1(0,1)$ and V_k the convolution operator with kernel k. If k is a polynomial generator for $L^1(0,1)$ then*

- *V_k is unicellular on $L^p(0,1)$, for all $p \in [1,\infty)$, and*
- *V_k is weak∗-unicellular on $L^p(0,1)$, for all $p \in (1,\infty]$.*

Proof. The proof of this theorem is similar to that of Theorem 3.8; fix $p \in [1,\infty)$ and let p' be its Hölder conjugate. Let $f \in L^p(0,1)$ and set $A = \{V_k^n f : n \in \mathbb{N}\}$. Clearly $A \subseteq L^p(\alpha(f),1)$, so $A^\perp \supseteq L^{p'}(0,\alpha(f))$. The aim is to show that $A^\perp = L^{p'}(0,\alpha(f))$, which is enough to show that V_k is unicellular on $L^p(0,1)$.

Let $g \in A^{\perp}$ and define $\tilde{g} = Rg$. For $n \in \mathbb{N}$, consider

$$(V_k^n f) * \tilde{g} = k^{*n} * f * \tilde{g}.$$

Since $k^{*n} * f \in L^p(0,1)$ and $\tilde{g} \in L^{p'}(0,1)$, the resulting function is continuous, so $(k^{*n} * f * \tilde{g})(1)$ is well-defined. But

$$\begin{aligned}(k^{*n} * f * \tilde{g})(1) &= \int_0^1 (k^{*n} * f)(t)\tilde{g}(1-t)\,\mathrm{d}t \\ &= \int_0^1 (k^{*n} * f)(t)g(t)\,\mathrm{d}t \\ &= \langle k^{*n} * f, g\rangle = 0\end{aligned}$$

since $V_k^n f \in A$ and $g \in A^{\perp}$.

Now let $h = R(f * \tilde{g}) \in L^{\infty}(0,1)$. Then

$$\langle k^{*n}, h\rangle = (k^{*n} * f * \tilde{g})(1) = 0$$

for all $n \in \mathbb{N}$, and so $h \in (\{k^{*n} : n \in \mathbb{N}\})^{\perp}$. However, since k is a polynomial generator for $L^1(0,1)$, this set has trivial annihilator, so $(f * \tilde{g})(t) = 0$ for all $t \in [0,1]$, and therefore, by the Titchmarsh Convolution Theorem,

$$1 \leqslant \alpha(f * \tilde{g}) = \alpha(f) = \alpha(\tilde{g})$$

so that $\alpha(\tilde{g}) \geqslant 1 - \alpha(f)$. That is to say, $\tilde{g}(t) = 0$ for a.a. $t \in (0, 1-\alpha(f)$; which means in turn that $g(t) = 0$ for a.a. $t \in (\alpha(f), 1)$; or equivalently, $g \in L^{p'}(0, \alpha(f))$.

The proof of the second statement is similar: let $p \in (1,\infty]$, let $f \in L^p(0,1)$ and define $A = \{V_k^n f : n \in \mathbb{N}\}$, as before. The aim is to show that $A^{\top} = L^{p'}(0,\alpha(f))$, which is enough to show that V_k is weak$*$-unicellular on $L^p(0,1)$. The rest of the proof is the same, replacing $A^{\perp}$ with $A^{\top}$. □

So if k is a polynomial generator for $L^1(0,1)$, then V_k is unicellular on every $L^p(0,1)$ space. What about the converse? There is evidence to suggest that if V_k is unicellular on $L^p(0,1)$ for some $p \in [1,\infty)$ (or weak$*$-unicellular on $L^p(0,1)$ for $p \in (1,\infty])$ then k is a polynomial generator for $L^1(0,1)$.

3.11 Theorem. *If V_k is unicellular on $L^1(0,1)$, then k is a polynomial generator for $L^1(0,1)$.*

Proof. Since V_k is unicellular on $L^1(0,1)$, the norm-closure of

$$\operatorname{span}\{V_k^n f : n \in \mathbb{N}\}$$

is $L^1(\alpha(f),1)$ for all $f \in L^1(0,1)$ (Theorem 3.3). In particular, since $\alpha(k) = 0$ and $k \in L^1(0,1)$,

$$\operatorname{span}\{V_k^n k : n \in \mathbb{N}\} = \operatorname{span}\{k^{*n} : n \in \mathbb{N}\}$$

is dense in $L^1(0,1)$. So k is a polynomial generator for $L^1(0,1)$. □

3.12 Corollary. *If V_k is weak$*$-unicellular on $L^\infty(0,1)$ then k is a polynomial generator for $L^1(0,1)$.*

Proof. If V_k is weak$*$-unicellular on $L^\infty(0,1)$ then V_k is unicellular on $L^1(0,1)$ (Theorem 3.8) so the previous result applies. □

If V_k is unicellular on $L^p(0,1)$ for some $p \in (1,\infty)$ then the same proof can be applied, as long as $k \in L^p(0,1)$. In fact this can be extended to a much larger class of kernels, using the same principles.

3.13 Theorem. *Fix $p \in (1,\infty)$ and let $k \in L^1(0,1)$ such that V_k is unicellular on $L^p(0,1)$. Suppose $h \in L^p(0,1)$ and there is a sequence of polynomials $\{P_n\}_{n\in\mathbb{N}}$ such that*

$$\|P_n(k) - h\|_1 \to 0$$

as $n \to \infty$. Then

$$L^1(\alpha(h),1) \subseteq \operatorname{Cl}_1(\operatorname{span}\{k^{*n} : n \in \mathbb{N}\}).$$

If $\alpha(h) = 0$, then k is a polynomial generator for $L^1(0,1)$.

Proof. Let $h \in L^p(0,1)$, and let $A = \operatorname{span}\{V_k^n h : n \in \mathbb{N}\}$. Since V_k is unicellular on $L^p(0,1)$, it has the cyclic property (Theorem 3.3), so that

$$\operatorname{Cl}_p A = L^p(\alpha(h), 1).$$

Since $L^p(0,1)$ is dense in $L^1(0,1)$, taking the L^1 closure shows that

$$\operatorname{Cl}_1 A = L^1(\alpha(h), 1).$$

Since $\operatorname{span}\{V_k^n h : n \in \mathbb{N}\}$ is dense in $L^1(\alpha(h),1)$ it follows that for any $f \in L^1(\alpha(h),1)$ and $\varepsilon > 0$, there exists a polynomial Q such that,

$$\|Q(k) * h - f\|_1 < \varepsilon/2.$$

In addition, h is the limit of a sequence of polynomials, so for fixed ε and Q there exists an $n \in \mathbb{N}$ such that

$$\|P_n(k) - h\|_1 < \frac{\varepsilon}{2\,\|Q(k)\|_1}.$$

Hence

$$\begin{aligned}
\|Q(k) * P_n(k) - f\|_1 &= \|Q(k) * (P_n(k) - h) + Q(k) * h - f\|_1 \\
&\leqslant \|Q(k)\|_1 \|P_n(k) - h\|_1 + \|Q(k) * h - f\|_1 \\
&\leqslant \frac{\varepsilon\,\|Q(k)\|_1}{2\,\|Q(k)\|_1} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

So for any $f \in L^1(a,1)$ there exist convolution polynomials $(Q(k) * P_n(k))$ arbitrarially close to f; hence

$$L^1(\alpha(h),1) \subseteq \operatorname{Cl}_1 \operatorname{span}\{k^{*n} : n \in \mathbb{N}\}.$$

If $\alpha(h) = 0$ it follows immediately that k is a polynomial generator for $L^1(0,1)$. □

3.4 Conclusion

From these explorations it is clear that there is a definite link between unicellularity of the operator V_k on $L^p(0,1)$ and k being a polynomial generator for $L^1(0,1)$. If $\alpha(k) > 0$ then neither of these can be true, as V_k is nilpotent. But aside from this trivial example, are there any examples of kernels which are *not* polynomial generators of $L^1(0,1)$?

3.14 Theorem. *Let $k \in L^1(0,1)$ be defined by*

$$k(t) = \frac{\mathrm{e}^{-1/t}}{\sqrt{\pi} t^{3/2}}.$$

Then k is not a polynomial generator for $L^1(0,1)$.

Proof. If $\langle k \rangle := \operatorname{span}\{k^{*n} : n \in \mathbb{N}\}$ is dense in $L^1(0,1)$ then the image of $\langle k \rangle$ under a bounded operator with dense range will be dense in the target space. In particular, for a fixed $\varepsilon \in (0,1)$, the projection operator $P : L^1(0,1) \to L^1(\varepsilon,1)$ is bounded and surjective. So if $P\langle k \rangle$ is *not* dense in $L^1(\varepsilon,1)$ then k is not a polynomial generator for $L^1(0,1)$.

For simplicity, define

$$e_n(t) = \frac{\mathrm{e}^{-n^2/t}}{t^{3/2}}$$

for all $n \in \mathbb{N}$, $t \in (\varepsilon,1)$, and notice that $\operatorname{span}\{e_n : n \in \mathbb{N}\} = P\langle k \rangle$. This follows from the formula for k^{*n} obtained in Lemma 4.15.

Let M be the multiplication operator on $L^1(\varepsilon,1)$ defined by

$$(Mf)(t) = f(t)t^{3/2}.$$

This is bounded, since for any $f \in L^1(\varepsilon,1)$,

$$\begin{aligned} \|Mf\|_1 &= \int_\varepsilon^1 \left|f(t)t^{3/2}\right| \,\mathrm{d}t \\ &\leqslant \int_\varepsilon^1 |f(t)| \,\mathrm{d}t = \|f\|_1 \end{aligned}$$

since $t^{3/2} \leqslant 1$ for $t \in (\varepsilon,1)$. Clearly $(M^{-1}f)(t) = f(t)t^{-3/2}$ defines an inverse for M; this is also bounded: for any $f \in L^1(\varepsilon, 1)$,

$$\begin{aligned}\left\|M^{-1}f\right\|_1 &= \int_\varepsilon^1 \left|f(t)t^{-3/2}\right| \,\mathrm{d}t \\ &\leqslant \int_\varepsilon^1 |f(t)| \left|\varepsilon^{-3/2}\right| \,\mathrm{d}t \; = \varepsilon^{-3/2} \|f\|_1 .\end{aligned}$$

Now define the composition operator $T : L^1(\varepsilon, 1) \to L^1(\mathrm{e}^{-1/\varepsilon}, \mathrm{e}^{-1})$ by

$$(Tf)(t) = f\left(-\frac{1}{\log t}\right).$$

T is bounded, since for $f \in L^1(\varepsilon, 1)$

$$\begin{aligned}\|Tf\|_1 &= \int_{\mathrm{e}^{-1/\varepsilon}}^{\mathrm{e}^{-1}} \left|f\left(-\frac{1}{\log t}\right)\right| \,\mathrm{d}t \\ &= \int_\varepsilon^1 |f(u)| \frac{\mathrm{e}^{-1/u}}{u^2} \,\mathrm{d}u \\ &\leqslant \int_\varepsilon^1 |f(u)| \frac{\mathrm{e}^{-1}}{\varepsilon^2} \,\mathrm{d}u \\ &= \frac{\mathrm{e}^{-1}}{\varepsilon^2} \|f\|_1 .\end{aligned}$$

The map $T^{-1} : L^1(\mathrm{e}^{-1/\varepsilon}, \mathrm{e}^{-1}) \to L^1(\varepsilon, 1)$ defined by $(T^{-1}f)(t) = f(\mathrm{e}^{-1/t})$ is an inverse for T, and is also bounded:

$$\begin{aligned}\left\|T^{-1}f\right\|_1 &= \int_\varepsilon^1 \left|f(\mathrm{e}^{-1/t})\right| \,\mathrm{d}t \\ &= \int_{\mathrm{e}^{-1/\varepsilon}}^{\mathrm{e}^{-1}} |f(u)| \frac{1}{u \log^2 u} \,\mathrm{d}u \\ &\leqslant \int_{\mathrm{e}^{-1/\varepsilon}}^{\mathrm{e}^{-1}} |f(u)| \,\mathrm{e}^{1/\varepsilon} \,\mathrm{d}u \\ &= \mathrm{e}^{1/\varepsilon} \|f\|_1 .\end{aligned}$$

Now since TM is a bounded, invertible map with bounded inverse, it follows immediately that $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ is dense in $L^1(\varepsilon, 1)$ if and only if $\operatorname{span}\{TMe_n : n \in \mathbb{N}\}$ is dense in $L^1(\mathrm{e}^{-1/\varepsilon}, \mathrm{e}^{-1})$.

The Müntz–Szász Theorem [15, §15.25] for an interval $[a, b]$ not containing the origin states that for an increasing sequence of positive real numbers λ_i the set of functions

$$\operatorname{span}\{t^{\lambda_i} : i \in \mathbb{N}\}$$

is dense in $L^1(a, b)$ if and only if

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty.$$

However, $(TMe_n)(t) = t^{n^2}$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, so the Müntz–Szász Theorem shows that $\operatorname{span}\{TMe_n : n \in \mathbb{N}\}$ is not dense in $L^1(\mathrm{e}^{-1/\varepsilon}, \mathrm{e}^{-1})$. Therefore $\operatorname{span}\{e_n : n \in \mathbb{N}\}$ is not dense in $L^1(\varepsilon, 1)$, and hence $\operatorname{span}\{k^{*n} : n \in \mathbb{N}\}$ is not dense in $L^1(0, 1)$. □

Chapter 4

Asymptotics of Iterated Convolution Operators

We are going to focus on two different normed spaces when discussing the asymptotics of convolution operators on $L^p(0,1)$, for $p \in [1,\infty]$: the space $\mathcal{B}(L^p(0,1))$ of bounded operators of $L^p(0,1)$; and the space $L^1(0,1)$ regarded as an algebra of kernels, with convolution as the multiplicative operation.

There is a natural embedding of $L^1(0,1)$ into $\mathcal{B}(L^p(0,1))$, defined by $\phi : k \mapsto V_k$. This mapping is injective and its range is the set of convolution operators. That this is a algebra morphism is easily verified; the map however is not isometric. That it is continuous follows from the inequality:

$$\|(\phi k)f\|_p = \|V_k f\|_p = \|k * f\|_p \leqslant \|k\|_1 \|f\|_p$$

for all $f \in L^p(0,1)$, so that $|||\phi k|||_p \leqslant \|k\|_1$ for all $k \in L^1(0,1)$, and therefore the norm of ϕ is bounded by 1.

In some cases, asymptotic equality in the Volterra algebra implies asymptotic equality in the bounded operators, as shown in the next theorem. The choice of comparison kernels is motivated by the approximation methods Eveson uses in [9].

4.1 Definition. For any $\mu \in \mathbb{R}$, let e_μ denote the function defined by

$$\mathrm{e}_\mu(t) = \mathrm{e}^{\mu t}$$

for all $t \in (0,1)$.

4.2 Theorem. *Let $k \in L^1(0,1)$ be a kernel such that*

$$k^{*n} \sim \alpha_n \mathrm{e}_{\beta_n}$$

for some real nets $(\alpha_n)_{n\in I}, (\beta_n)_{n\in I}$, where $\lim_n \beta_n = \infty$*. Then as elements of $\mathcal{B}(L^p(0,1))$ for any $p \in [1,\infty]$,*

$$V_k^n \sim \alpha_n V_{\mathrm{e}_{\beta_n}}.$$

Proof. Using the standard estimate for the numerator,

$$\begin{aligned}\frac{\left\|\left|V_k^n - \alpha_n V_{\mathrm{e}_{\beta_n}}\right|\right\|_p}{\left\|\left|\alpha_n V_{\mathrm{e}_{\beta_n}}\right|\right\|_p} &\leqslant \frac{\left\|k^{*n} - \alpha_n \mathrm{e}_{\beta_n}\right\|_1}{\left\|\left|\alpha_n V_{\mathrm{e}_{\beta_n}}\right|\right\|_p} \\ &= \frac{\left\|k^{*n} - \alpha_n \mathrm{e}_{\beta_n}\right\|_1}{\left\|\alpha_n \mathrm{e}_{\beta_n}\right\|_1} \frac{\left\|\alpha_n \mathrm{e}_{\beta_n}\right\|_1}{\left\|\left|\alpha_n V_{\mathrm{e}_{\beta_n}}\right|\right\|_p} \\ &= \frac{\left\|k^{*n} - \alpha_n \mathrm{e}_{\beta_n}\right\|_1}{\left\|\alpha_n \mathrm{e}_{\beta_n}\right\|_1} \frac{(\mathrm{e}^{\beta_n}-1)/\beta_n}{\left\|\left|V_{\mathrm{e}_{\beta_n}}\right|\right\|_p}.\end{aligned}$$

Eveson [9] provides an asymptotic formula for the norm of V_{e_μ}, given by

$$\left\|\left|V_{\mathrm{e}_\mu}\right|\right\|_p \sim \frac{C_p \mathrm{e}^\mu}{\mu}$$

as $\mu \to \infty$, where C_p is a constant depending only on p. In particular, $\left|\beta_n \left\|\left|V_{\mathrm{e}_{\beta_n}}\right|\right\|_p / C_p \mathrm{e}^{\beta_n}\right| \to 1$ as $n \to \infty$.

So

$$\frac{\left\|\left|V_k^n - \alpha_n V_{\mathrm{e}_{\beta_n}}\right|\right\|_p}{\left\|\left|\alpha_n V_{\mathrm{e}_{\beta_n}}\right|\right\|_p} \leqslant \frac{\left\|k^{*n} - \alpha_n \mathrm{e}_{\beta_n}\right\|_1}{\left\|\alpha_n \mathrm{e}_{\beta_n}\right\|_1} \frac{(\mathrm{e}^{\beta_n}-1)/\beta_n}{C_p \mathrm{e}^{\beta_n}/\beta_n} \frac{C_p \mathrm{e}^{\beta_n}/\beta_n}{\left\|\left|V_{\mathrm{e}_{\beta_n}}\right|\right\|_p}$$

(as long as $\beta_n > 0$, which is true for sufficiently large n). The first term tends to zero by the assumptions on k, and the middle term tends to $1/C_p$, since $\beta_n \to \infty$ as $n \to \infty$. Finally, the last term tends to 1 as $n \to \infty$, from the definition of C_p

Hence

$$\frac{\left\| V_k^n - \alpha_n V_{\mathrm{e}_{\beta_n}} \right\|_p}{\left\| \alpha_n V_{\mathrm{e}_{\beta_n}} \right\|_p} \to 0$$

as $n \to \infty$, as required. □

4.3 Corollary. *Let $k \in L^1(0,1)$. If $k^{*n} \sim \alpha_n 1^{*\beta_n}$ for some α_n, β_n, where $\beta_n \to \infty$, then*

$$V_k^n \sim \alpha_n V^{\beta_n}.$$

Proof. Since $\beta_n \to \infty$ choose N such that $\beta_n > 1$ for all $n \geqslant N$. For all $t \in (0,1)$, $\ln t \leqslant t - 1$, from which it follows that

$$t^{\beta_n - 1} \leqslant \mathrm{e}^{1-\beta_n} \mathrm{e}^{(\beta_n - 1)t} \tag{$*$}$$

for all $n \geqslant N$, and consequently

$$\mathbf{1}^{*\beta_n} \leqslant \frac{\mathrm{e}^{1-\beta_n}}{\Gamma(\beta_n - 1)} \mathrm{e}_{\beta_n - 1}.$$

Call the right-hand-side f_n. Then for $n \geqslant N$,

$$\begin{aligned}
\left\| \mathbf{1}^{*\beta_n} - f_n \right\|_1 &= \int_0^1 \left| \frac{t^{\beta_n - 1}}{\Gamma(\beta_n - 1)} - \frac{\mathrm{e}^{1-\beta_n}}{\Gamma(\beta_n - 1)} \mathrm{e}^{(\beta_n - 1)t} \right| \mathrm{d}t \\
&= \frac{1}{\Gamma(\beta_n - 1)} \int_0^1 \mathrm{e}^{1-\beta_n} \mathrm{e}^{(\beta_n - 1)t} - t^{\beta_n - 1} \, \mathrm{d}t \qquad \text{by } (*) \\
&= \frac{1}{\Gamma(\beta_n - 1)} \left(\frac{1 - \mathrm{e}^{1-\beta_n}}{\beta_n - 1} - \frac{1}{\beta_n} \right) \\
&= \frac{\frac{1}{\beta_n} - \mathrm{e}^{1-\beta_n}}{\Gamma(\beta_n)}.
\end{aligned}$$

So

$$\frac{\left\| \mathbf{1}^{*\beta_n} - f_n \right\|_1}{\left\| \mathbf{1}^{*\beta_n} \right\|_1} = \frac{\frac{\frac{1}{\beta_n} - \mathrm{e}^{1-\beta_n}}{\Gamma(\beta_n)}}{1/\Gamma(\beta_n)} = \frac{1}{\beta_n} - \mathrm{e}^{1-\beta_n}$$

which tends to zero as $\beta_n \to \infty$. Hence

$$k^{*n} \sim \alpha_n \mathbf{1}^{*\beta_n} \sim \frac{\alpha_n \mathrm{e}^{1-\beta_n}}{\Gamma(\beta_n - 1)} \mathrm{e}_{\beta_n - 1},$$

and the result follows from Theorem 4.2. □

4.4 Lemma. *For any $f, g \in L^1(0,1)$ and any $\mu \in \mathbb{R}$,*

$$(f\mathrm{e}_\mu) * (g\mathrm{e}_\mu) = (f * g)\mathrm{e}_\mu.$$

Proof. Fix $f, g \in L^1(0,1)$ and $\mu \in \mathbb{R}$. Then for any $t \in (0,1)$,

$$\begin{aligned}(f\mathrm{e}_\mu * g\mathrm{e}_\mu)(t) &= \int_0^t f(t-s)\mathrm{e}^{\mu(t-s)} g(s)\mathrm{e}^{\mu s}\,\mathrm{d}s \\ &= \int_0^t f(t-s)g(s)\mathrm{e}^{\mu t}\,\mathrm{d}s \\ &= \mathrm{e}^{\mu t}\int_0^t f(t-s)g(s)\,\mathrm{d}s \\ &= ((f * g)\mathrm{e}_\mu)(t).\end{aligned}$$

□

4.5 Corollary. *Let $f \in L^1(0,1)$ and $\mu \in \mathbb{R}$. Then*

$$(f\mathrm{e}_\mu)^{*n} = f^{*n}\mathrm{e}_\mu$$

for all $n \in \mathbb{N}$.

4.6 Lemma. *Let V be a normed vector algebra and $T \in \mathcal{B}(V)$ an invertible map with bounded inverse. Then if $\{u_n\}, \{v_n\}$ are sequences in V such that $u_n \sim v_n$, then*

$$Tu_n \sim Tv_n.$$

Proof. Since T is bounded, $\|T(u_n - v_n)\| \leqslant \|T\|\,\|u_n - v_n\|$, and since T is invertible, so also

$$\|u_n\| = \left\|T^{-1}Tu_n\right\| \leqslant \left\|T^{-1}\right\|\,\|Tu_n\|.$$

Hence

$$\frac{\|Tu_n - Tv_n\|}{\|Tu_n\|} \leqslant \frac{\|T\|\,\|u_n - v_n\|}{\|u_n\| / \|T^{-1}\|} = \|T\|\,\left\|T^{-1}\right\|\frac{\|u_n - v_n\|}{\|u_n\|}.$$

Hence $\lim_n \frac{\|Tu_n - Tv_n\|}{\|Tu_n\|} = 0$, so $Tu_n \sim Tv_n$. □

4.7 Proposition. *Let* $V_{k_1}^n \sim V_{k_2}^n$ *on* $L^p(0,1)$*. Then, for any* $\mu \in \mathbb{R}$*,*

$$V_{k_1 \mathrm{e}_\mu}^n \sim V_{k_2 \mathrm{e}_\mu}^n.$$

Proof. Let M_h denote the multiplication operator with symbol h. Then, for any $f \in L^p(0,1)$, $t \in (0,1)$,

$$\begin{aligned}
[M_{\mathrm{e}_\mu} V_k M_{\mathrm{e}_\mu}^{-1} f](t) &= [\mathrm{e}_\mu (k * f\mathrm{e}_{-\mu})](t) \\
&= \mathrm{e}^{\mu t} \int_0^t k(t-s) f(s) \mathrm{e}^{-\mu s} \, \mathrm{d}s \\
&= \int_0^t k(t-s) f(s) \mathrm{e}^{\mu t} \mathrm{e}^{-\mu s} \, \mathrm{d}s \\
&= \int_0^t k(t-s) \mathrm{e}^{\mu(t-s)} f(s) \, \mathrm{d}s \\
&= [(k\mathrm{e}_\mu) * f](t) \\
&= [V_{k\mathrm{e}_\mu} f](t)
\end{aligned}$$

It is then immediate that $V_{k_1 \mathrm{e}_\mu}^n = M_{\mathrm{e}_\mu} V_{k_1}^n M_{\mathrm{e}_\mu}^{-1}$, and similarly for $V_{k_2 \mathrm{e}_\mu}$. The result follows from Lemma 4.6. □

4.8 Lemma. *For all* $\alpha > 0$ *and* $n \in \mathbb{N}$*,*

$$\frac{\|\mathbf{1}^{*n+\alpha}\|}{\|\mathbf{1}^{*n}\|} \leqslant \frac{1}{n^\alpha}.$$

Proof. Fix $\alpha > 0$. Then for any $n \in \mathbb{N}$,

$$\frac{\|\mathbf{1}^{*n+\alpha}\|}{\|\mathbf{1}^{*n}\|} = \frac{1/\Gamma(n+\alpha+1)}{1/\Gamma(n+1)} = \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}.$$

Since Γ is log-convex and $\log(\Gamma(x))$ is increasing for $x > 2$, so

$$\log(\Gamma(n+1+\alpha)) \geqslant \log(\Gamma(n+1)) + \alpha \frac{\mathrm{d}}{\mathrm{d}x} (\log(\Gamma(x)))|_{x=n+1}.$$

The derivative of $\log\Gamma$ at a positive integer is given by

$$(\log\Gamma)'(n+1) = \frac{\Gamma'(n+1)}{\Gamma(n+1)} = -\gamma + \sum_{k=1}^{n} \frac{1}{k}$$

where γ is Euler's constant. It follows from the integral test [1, Theorem 12-23] that $\sum_{k=1}^{n} \frac{1}{k} \geqslant \log(n) + \gamma$, so combining this and the above expressions gives

$$\log(\Gamma(n+1+\alpha)) \geqslant \log(\Gamma(n+1)) + \alpha\log(n).$$

Hence $\Gamma(n+1+\alpha) \geqslant n^{\alpha}\Gamma(n+1)$ for all $n \in \mathbb{N}$. So

$$\frac{\|\mathbf{1}^{*n+\alpha}\|}{\|\mathbf{1}^{*n}\|} = \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} \leqslant \frac{\Gamma(n+1)}{n^{\alpha}\Gamma(n+1)} = \frac{1}{n^{\alpha}}$$

as required. □

4.1 Asymptotics of Perturbations

The asymptotic behaviour of the norms of iterates of the classical Volterra operator acting on $L^2(0,1)$ have been established by various means (see for example [13]). In [8] and [9], Eveson gives more details, calculating the norms of iterates of more general convolution operators in $L^2(0,1)$ and $L^p(0,1)$ respectively.

In this chapter, Eveson's work is built upon to give a result which is applicable to a larger class of kernels, at the cost of slightly less sharp estimates of the known results.

The main result of this chapter is the following theorem. This is phrased in terms of functions in $L^1(0,1)$; this is because such statements can be used to show results about operators on $L^p(0,1)$, for any $p \in [1,\infty]$, using Corollary 4.3.

4.9 Theorem. *If $h, r \in L^1(0,1)$, p_n a sequence in $\mathbb{N}$ and $\alpha > 0$ are such that*

i. $h^{*n} \sim a_n \mathrm{e}_{\mu}^{*p_n}$;

ii. $\left|h^{*n}(t)\right| \leqslant c\left|a_n \mathrm{e}_\mu^{*p_n}(t)\right|$ *for some* c*, all* $t \in (0,1)$ *and all sufficiently large* n*;*

iii. $|r| \leqslant \left|h * \mathrm{e}_\mu^{*\alpha}\right|$*;*

iv. $p_n^\alpha / n \to \infty$ *as* $n \to \infty$

for some $\mu \in \mathbb{R}$ *and* a_n *a real sequence, then*

$$(h+r)^{*n} \sim h^{*n}.$$

Remark. The Theorem states, then, that for "well-behaved" h and "small" r, perturbation by r makes no difference to the asymptotics of convolution powers of h.

This extends the work of Eveson [9, §4] by providing new criteria for the size of the perturbation, as well as allowing more general kernels to be perturbed. See Applications (page 52) for details.

Proof. We prove that $(h+r)^{*n} \sim a_n \mathrm{e}_\mu^{*p_n}$, and the result follows by transitivity of asymptotic equality.

First notice that

$$(h+r)^{*n} = h^{*n} + r^{*n} + \sum_{j=1}^{n-1} \binom{n}{j} h^{*n-j} * r^{*j}$$

so that

$$\begin{aligned} \left\|(h+r)^{*n} - a_n \mathrm{e}_\mu^{*p_n}\right\| &= \left\| r^{*n} + h^{*n} - a_n \mathrm{e}_\mu^{*p_n} + \sum_{j=1}^{n-1} \binom{n}{j} h^{*n-j} * r^{*j} \right\| \\ &\leqslant \left\|r^{*n}\right\| + \left\|h^{*n} - a_n \mathrm{e}_\mu^{*p_n}\right\| + \sum_{j=1}^{n-1} \binom{n}{j} \left\|h^{*n-j} * r^{*j}\right\|. \end{aligned}$$

Now using $|r| \leqslant \left|h * \mathrm{e}_\mu^{*\alpha}\right|$, this gives

$$\left\|(h+r)^{*n} - a_n \mathrm{e}_\mu^{*p_n}\right\| \leqslant \left\|h^{*n} - a_n \mathrm{e}_\mu^{*p_n}\right\| + \left\|h^{*n} * \mathrm{e}_\mu^{*\alpha n}\right\| + \sum_{j=1}^{n-1} \binom{n}{j} \left\|h^{*n-j} * h^{*j} * \mathrm{e}_\mu^{*\alpha j}\right\|$$

$$= \left\|h^{*n} - a_n \mathrm{e}_\mu^{*p_n}\right\| + \sum_{j=1}^{n} \binom{n}{j} \left\|h^{*n} * \mathrm{e}_\mu^{*\alpha j}\right\|.$$

Further using the estimate $|h^{*n}| \leqslant c\left|a_n \mathrm{e}_\mu^{*p_n}\right|$, this gives

$$\left\|(h+r)^{*n} - a_n \mathrm{e}_\mu^{*p_n}\right\| \leqslant \left\|h^{*n} - a_n \mathrm{e}_\mu^{*p_n}\right\| + \sum_{j=1}^{n} \binom{n}{j} c \left\|a_n \mathrm{e}_\mu^{*p_n} * \mathrm{e}_\mu^{*\alpha j}\right\|$$

$$= \left\|h^{*n} - a_n \mathrm{e}_\mu^{*p_n}\right\| + c\,|a_n| \sum_{j=1}^{n} \binom{n}{j} \left\|\mathrm{e}_\mu^{*p_n+\alpha j}\right\|.$$

So

$$\frac{\left\|(h+r)^{*n} - a_n \mathrm{e}_\mu^{*p_n}\right\|}{\left\|a_n \mathrm{e}_\mu^{*p_n}\right\|} \leqslant \frac{\left\|h^{*n} - a_n \mathrm{e}_\mu^{*p_n}\right\|}{\left\|a_n \mathrm{e}_\mu^{*p_n}\right\|} + c \sum_{j=1}^{n} \binom{n}{j} \frac{\left\|\mathrm{e}_\mu^{*p_n+\alpha j}\right\|}{\left\|\mathrm{e}_\mu^{*p_n}\right\|}.$$

The centre term tends to zero as $n \to \infty$, because $h^{*n} \sim a_n \mathrm{e}_\mu^{*p_n}$ as $n \to \infty$, and so it only remains to show that the final term tends to zero as $n \to \infty$.

Notice that

$$\frac{\left\|\mathrm{e}_\mu^{*(p_n+\alpha j)}\right\|}{\left\|\mathrm{e}_\mu^{*p_n}\right\|} \leqslant \frac{\max\{1, \mathrm{e}^\mu\}}{\min\{1, \mathrm{e}^\mu\}} \frac{\left\|\mathbf{1}^{*(p_n+\alpha j)}\right\|}{\left\|\mathbf{1}^{*p_n}\right\|} \leqslant \frac{\max\{1, \mathrm{e}^\mu\}}{\min\{1, \mathrm{e}^\mu\}} \frac{1}{p_n^{\alpha j}}$$

and that

$$\sum_{j=1}^{n} \binom{n}{j} \frac{1}{p_n^{\alpha j}} = \left(1 + \frac{1}{p_n^\alpha}\right)^n - 1$$

so that

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{j=1}^{n}\binom{n}{j}\frac{1}{p_n^{\alpha j}} &= \lim_{n\to\infty}\left(1+\frac{1}{p_n^{\alpha}}\right)^{n}-1\\
&= \lim_{n\to\infty}\left(\left(1+\frac{1}{p_n^{\alpha}}\right)^{p_n^{\alpha}}\right)^{n/p_n^{\alpha}}-1\\
&= \left(\lim_{n\to\infty}\left(1+\frac{1}{p_n^{\alpha}}\right)^{p_n^{\alpha}}\right)^{\lim_{n\to\infty} n/p_n^{\alpha}}-1\\
&= \mathrm{e}^0-1=0
\end{aligned}
$$

using a well-known expression for e and the assumption that $p_n^{\alpha}/n \to \infty$ as $n \to \infty$. Hence $\sum_{j=1}^{n}\binom{n}{j}\frac{\left\|\mathrm{e}_{\mu}^{*(p_n+\alpha j)}\right\|}{\left\|\mathrm{e}_{\mu}^{*p_n}\right\|} \to 0$ as $n \to \infty$, and the result follows. □

While this is a useful result, it relies on global bounds for both the kernel to be disturbed, and the perturbation. The following lemma can be used to relax this requirement.

4.10 Lemma. *Let $k, r \in L^1(0,1)$. If $k^{*n} \sim a_n \mathrm{e}_{\mu}^{*p_n}$ for some $\mu \in \mathbb{R}$ and a_n a real sequence, and $|k^{*n}(t)| \leqslant c\left|a_n \mathrm{e}_{\mu}^{*p_n}(t)\right|$ for some $c > 0$, all $t \in (0,1)$ and all sufficiently large n, $p_n \to \infty$ as $n \to \infty$, and r is nilpotent, then*

$$(k+r)^{*n} \sim k^{*n}.$$

Proof. First fix $N \in \mathbb{N}$ such that $r^{*n} = 0$ for all $n > N$ – that is, $N \geqslant 1/\alpha(r)$. Use the convolution binomial theorem to show that

$$(k+r)^{*n} = k^{*n} + r^{*n} + \sum_{j=1}^{n-1}\binom{n}{j} r^{*j} * k^{*n-j}$$

but since r is nilpotent, if $n > N$ this becomes

$$(k+r)^{*n} = k^{*n} + \sum_{j=1}^{N}\binom{n}{j} r^{*j} * k^{*n-j}$$

so that

$$\frac{\left\|(k+r)^{*n}-k^{*n}\right\|_1}{\left\|k^{*n}\right\|_1} \leqslant \sum_{j=1}^{N} \binom{n}{j} \frac{\left\|r^{*j} * k^{*n-j}\right\|_1}{\left\|k^{*n}\right\|_1}.$$

Now $\left\|r^{*j} * k^{*n-j}\right\|_1 \leqslant \left\|r^{*j}\right\|_1 \left\|k^{*n-j}\chi_{(0,1-j\alpha(r))}\right\|_1$, because $r^{*j}(t) = 0$ for all $t \in (0, j\alpha(r))$.

But

$$\begin{aligned}
\frac{\int_0^{1-j\alpha(r)} \left|k^{*n-j}(t)\right| \,\mathrm{d}t}{\left\|k^{*n}\right\|_1} &\leqslant \frac{\int_0^{1-j\alpha(r)} c\left|a_n \mathrm{e}_\mu^{*p_{n-j}}(t)\right| \,\mathrm{d}t}{\left\|k^{*n}\right\|_1} \\
&= \frac{\int_0^{1-j\alpha(r)} c\left|a_n \mathrm{e}_\mu^{*p_{n-j}}(t)\right| \,\mathrm{d}t}{\left\|a_n \mathrm{e}_\mu^{*p_n}\right\|_1} \quad \frac{\left\|a_n \mathrm{e}_\mu^{*p_n}\right\|_1}{\left\|k^{*n}\right\|_1} \\
&\leqslant c' \frac{\left\|a_n \mathrm{e}_\mu^{*p_n}\right\|_1}{\left\|k^{*n}\right\|_1} \quad \frac{\int_0^{1-j\alpha(r)} \left|\mathbf{1}^{*p_{n-j}}(t)\right| \,\mathrm{d}t}{\left\|\mathbf{1}^{*p_n}\right\|_1} \\
&= c' \frac{\left\|a_n \mathrm{e}_\mu^{*p_n}\right\|_1}{\left\|k^{*n}\right\|_1} \quad \frac{\left[\frac{t^{p_{n-j}}}{\Gamma(p_{n-j}+1)}\right]_{t=0}^{1-j\alpha(r)}}{\left[\frac{t^{p_n}}{\Gamma(p_n+1)}\right]_{t=0}^{1}} \\
&= c' \frac{\left\|a_n \mathrm{e}_\mu^{*p_n}\right\|_1}{\left\|k^{*n}\right\|_1} \quad \frac{(1-j\alpha(r))^{p_{n-j}}/\Gamma(p_{n-j}+1)}{1/\Gamma(p_n+1)} \\
&= c' \frac{\left\|a_n \mathrm{e}_\mu^{*p_n}\right\|_1}{\left\|k^{*n}\right\|_1} \quad \frac{\Gamma(p_n+1)}{\Gamma(p_{n-j}+1)} \quad (1-j\alpha(r))^{p_{n-j}}.
\end{aligned}$$

The leftmost term tends to c', because $k^{*n} \sim a_n \mathrm{e}_\mu^{*p_n}$, and so the rightmost term dominates, and the entire expression tends to zero as $n \to \infty$. Hence

$$\binom{n}{j} \frac{\left\|r^{*j} * k^{*n-j}\right\|_1}{\left\|k^{*n}\right\|_1} \to 0$$

as $n \to \infty$, for all $j = 1, \ldots, N$, and so

$$\frac{\left\|(k+r)^{*n}-k^{*n}\right\|_1}{\left\|k^{*n}\right\|_1} \to 0$$

as $n \to \infty$. □

4.11 Corollary. *Let $f, g \in L^1(0,1)$. If $f^{*n} \sim a_n \mathrm{e}_\mu^{*p_n}$, for some $\mu \in \mathbb{R}$ and a_n a real sequence, and $\left|k^{*n}(t)\right| \leqslant c\left|a_n \mathrm{e}_\mu^{*p_n}(t)\right|$ for some c, all $t \in (0,1)$ and* *all sufficiently large n,* $p_n \to \infty$ *as* $n \to \infty$*, and* $g(t) = f(t)$ *for* $t \in (0, \varepsilon)$ *for some* $\varepsilon > 0$*, then*

$$g^{*n} \sim f^{*n}.$$

Proof. Let $h = g - f$; then h is nilpotent, so from Lemma 4.10

$$(f + h)^{*n} \sim f^{*n}$$

and since $f + h = g$, so $g^{*n} \sim f^{*n}$. □

For a large class of kernels, then, their asymptotic behaviour is determined entirely by the values taken by the function around the origin.

4.12 Theorem. *If* $h, r \in L^1(0,1)$*,* p_n *a sequence in* $\mathbb{N}$ *and* $\alpha > 0$ *are such that*

i. $h^{*n} \sim a_n \mathrm{e}_\mu^{*p_n}$*;*

ii. $\left|h^{*n}\right| \leqslant c \left|a_n \mathrm{e}_\mu^{*p_n}\right|$ *for some c and all sufficiently large n;*

iii. $|r(t)| \leqslant \left|(h * \mathrm{e}_\mu^{*\alpha})(t)\right|$ *for all* $t \in (0, \varepsilon)$*, for some* $\varepsilon > 0$*;*

iv. $p_n^\alpha / n \to \infty$ *as* $n \to \infty$

for some $\mu \in \mathbb{R}$ *and* a_n *a real sequence, then*

$$(h + r)^{*n} \sim h^{*n}.$$

4.2 Applications

Theorem 4.12 can be used to show that, for a large class of kernels, the asymptotic behaviour of $\left\|\left|V_k\right|\right\|_p$ depends only on the values that k takes near the origin.

4.13 Lemma. *Let* $f \in L^1(0,1)$ *be such that* $f(0) \neq 0$ *and*

$$f(t) = f(0) + f'(0)t + \mathrm{O}(t^{1+\varepsilon})$$

around the origin, for some $\varepsilon > 0$*. Let* $\mu = f'(0)/f(0)$*. Then*

$$f^{*n} \sim (f(0)\mathrm{e}_\mu)^{*n}$$

and

$$V_f^n \sim V_{f(0)\mathrm{e}_\mu}^n$$

as operators on any $L^p(0,1)$ *space.*

Proof. Let $q = f - f(0)\mathrm{e}_\mu$, and notice that $q(0) = q'(0) = 0$. Hence $q(t) = \mathrm{O}(t^{1+\varepsilon})$; that is,

$$|q| \leqslant c\mathbf{1}^{*2+\varepsilon}$$

for some constant c, and in particular

$$|q| \leqslant c' \left|(f(0)\mathrm{e}_\mu)^{*2+\varepsilon}\right|.$$

So, by Theorem 4.12

$$(f(0)\mathrm{e}_\mu + q)^{*n} \sim (f(0)\mathrm{e}_\mu)^{*n}$$

as required.

The operator statement is immediate from the first part and Corollary 4.3.

□

The next result is similar in character, but holds for a larger class of kernels. Note that this Lemma is modelled after [9, 4.3], but does not provide as sharp a result.

4.14 Lemma. *Let* $f \in L^1(0,1)$ *be such that* $f(0) \neq 0$ *and* $f(t) = f(0) + f'(0)t + \mathrm{O}(t^{1+\varepsilon})$ *at the origin, for some* $\varepsilon > 0$*. Let* $\mu = f'(0)/f(0)$*. Then*

$$(\mathbf{1}^{*a}f)^{*n} \sim (f(0)\mathrm{e}_\mu^{*a})^{*n}$$

and

$$V_{\mathbf{1}^{*a}f}^n \sim (f(0)V_{\mathrm{e}_\mu^{*a}})^n$$

as operators on any $L^p(0,1)$ *space, for any* $a > 0$*.*

Proof. As before, let $q = \mathbf{1}^{*a} f - \mathbf{1}^{*a} f(0) \mathrm{e}_\mu$. Differentiating shows that $q'(0) = q(0) = 0$, so that $q(t) = \mathrm{O}(t^2)$, or $|q| \leqslant c\mathbf{1}^{*3}$ for some constant c.

Letting $h = f(0)\mathrm{e}_\mu^{*a}$, so $|q| \leqslant ch * \mathbf{1}^{*2}$. This therefore satisfies the assumptions of Theorem 4.12, and we conclude that

$$(h + q)^{*n} \sim h^{*n}$$

or, rearranging, that

$$(\mathbf{1}^{*a} f)^{*n} \sim (f(0)\mathrm{e}_\mu^{*a})^{*n}$$

as required.

The operator statement is immediate from the first part and Corollary 4.3. □

The results above are already known from the literature [9], but the following results, culminating in Theorem 4.18, provide a new class of examples.

4.15 Lemma. *Let $k \in L^1(0, 1)$ be defined by*

$$k(t) = \frac{\mathrm{e}^{-1/t}}{\sqrt{\pi} t^{3/2}}.$$

Then

$$k^{*n}(t) = \frac{n\mathrm{e}^{-n^2/t}}{\sqrt{\pi} t^{3/2}}$$

and furthermore

$$k^{*n} \sim \frac{n\mathrm{e}^{-n^2}\Gamma(n^2 + \frac{1}{2})}{\sqrt{\pi}} \mathbf{1}^{*n^2 - \frac{1}{2}}$$

as $n \to \infty$.

In order to prove this, the following technical lemma is required.

4.16 Lemma. *Let $(k_n)_{n \in \mathbb{N}} \subset L^1(0, 1)$ be defined by*

$$k_n(t) = \mathrm{e}^{-n/t}$$

for all $n \in \mathbb{N}$. Then

$$k_n \sim \mathrm{e}^{-n}\Gamma(n + 1)\mathbf{1}^{*n+1}$$

as $n \to \infty$.

Proof. We wish to show that

$$\frac{\left\|e^{-n/t} - e^{-n}t^n\right\|}{\left\|e^{-n}t^n\right\|} = \frac{\int_0^1 \left|e^{-n/t} - e^{-n}t^n\right| \, dt}{\int_0^1 \left|e^{-n}t^n\right| \, dt} \to 0$$

as $n \to \infty$; this is simplified by the integrands of both expressions being single-signed. Notice that $t\log(t) + 1 \geqslant t$ for all $t \in (0, 1)$; from this it follows that

$$\begin{aligned} \log(t) - 1 &\geqslant -\frac{1}{t} \\ te^{-1} &\geqslant e^{-1/t} \\ e^{-n}t^n &\geqslant e^{-n/t} \end{aligned}$$

for any $n \in \mathbb{N}$, and therefore

$$e^{-n}t^n - e^{-n/t} \geqslant 0$$

for all $n \in \mathbb{N}$ and all $t \in (0, 1)$.

So

$$\begin{aligned} \frac{\left\|e^{-n/t} - e^{-n}t^n\right\|}{\left\|e^{-n}t^n\right\|} &= \frac{\int_0^1 e^{-n}t^n - e^{-n/t} \, dt}{\int_0^1 e^{-n}t^n \, dt} \\ &= 1 - \frac{\int_0^1 e^{-n/t} \, dt}{\int_0^1 e^{-n}t^n \, dt}. \end{aligned}$$

Taking these integrals individually, it is easy to show that

$$\int_0^1 e^{-n}t^n \, dt = \frac{e^{-n}}{n+1}$$

while the other integral can be transformed using the substitution $s = \frac{1}{t}$

$$\int_0^1 e^{-n/t} \, dt = \int_1^\infty \frac{1}{s^2} e^{-ns} \, ds$$

and further substituting $u = s - 1$, this becomes

$$\int_0^\infty \frac{\mathrm{e}^{-n(u+1)}}{(u+1)^2}\,\mathrm{d}u = \mathrm{e}^{-n}\int_0^\infty \frac{\mathrm{e}^{-nu}}{(u+1)^2}\,\mathrm{d}s = \mathrm{e}^{-n}(\mathcal{L}f)(n)$$

where $f(u) = 1/(u+1)^2$.

The asymptotics of $f(u)$ as $u \to 0$ are simply computed using a binomial expansion, so that $f(u) = 1 - 2u + \mathrm{O}(u^2)$ as $u \to 0$. Then Watson's Lemma shows that

$$(\mathcal{L}f)(n) = \frac{1}{n} - \frac{2}{n^2} + \mathrm{O}\left(\frac{1}{n^3}\right).$$

as $n \to \infty$. So, taking just the first term of the expansion,

$$\begin{aligned}\frac{\int_0^1 \mathrm{e}^{-n/t}\,\mathrm{d}t}{\int_0^1 \mathrm{e}^{-n}t^n\,\mathrm{d}t} &\sim \frac{\mathrm{e}^{-n}/n}{\mathrm{e}^{-n}/(n+1)} \\ &= \frac{n+1}{n}\end{aligned}$$

as $n \to \infty$, and so

$$\frac{\left\|\mathrm{e}^{-n/t} - \mathrm{e}^{-n}t^n\right\|}{\left\|\mathrm{e}^{-n}t^n\right\|} \sim 1 - \frac{n+1}{n}$$

which tends to zero as $n \to \infty$. □

Proof of Lemma 4.15. A standard integral transform [14, Table 5.30, p41] shows that

$$\mathcal{L}(k)(s) = \mathrm{e}^{-2\sqrt{s}}$$

and therefore

$$\mathcal{L}(k^{*n})(s) = \mathrm{e}^{-2n\sqrt{s}} = \mathrm{e}^{-2\sqrt{(n^2 s)}} = \mathcal{L}(k)(n^2 s).$$

Substituting this into the definition of $\mathcal{L}(k)$ and using the substitution $u =$

n^2t gives

$$\begin{aligned}\mathcal{L}(k)(n^2s) &= \int_0^\infty \frac{\mathrm{e}^{-1/t}}{\sqrt{\pi}t^{3/2}}\mathrm{e}^{-n^2st}\,\mathrm{d}t\\ &= \int_0^\infty \frac{\mathrm{e}^{-n^2/u}}{\sqrt{\pi}u^{3/2}/n^3}\mathrm{e}^{-su}\frac{1}{n^2}\,\mathrm{d}u\\ &= \int_0^\infty \frac{n\mathrm{e}^{-n^2/u}}{\sqrt{\pi}u^{3/2}}\mathrm{e}^{-su}\,\mathrm{d}u\,.\end{aligned}$$

By the uniqueness of the Laplace transform this shows that k^{*n} has the form

$$k^{*n}(t) = \frac{n\mathrm{e}^{-n^2/t}}{\sqrt{\pi}t^{3/2}}.$$

The result now follows from Lemma 4.16, by using the multiplicative operator $M : f(t) \mapsto f(t)/t^{3/2}$ and Lemma 4.6, and noticing that $\Gamma(n+1)\mathbf{1}^{*n+1}(t)/t^{3/2} = \Gamma(n+\frac{1}{2})\mathbf{1}^{*n-\frac{1}{2}}$. □

4.17 Corollary. *For k defined as in Lemma 4.15 and $h \in L^1(0,1)$ defined by $h(t) = t^2k(t)$,*

$$h = (\mathbf{1} + \tfrac{1}{2}\mathbf{1}^{*3/2}) * k.$$

Proof. The previous proof establishes that

$$\mathcal{L}(k)(s) = \int_0^\infty \frac{\mathrm{e}^{-1/t}}{\sqrt{\pi}t^{3/2}}\mathrm{e}^{-st}\,\mathrm{d}t\ = \mathrm{e}^{-2\sqrt{s}}$$

so by differentiating under the integral this gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-2\sqrt{s}} = \int_0^\infty \frac{\partial}{\partial s}\left(\frac{\mathrm{e}^{-1/t}}{\sqrt{\pi}t^{3/2}}\mathrm{e}^{-st}\right)\,\mathrm{d}t\ = -\int_0^\infty \frac{\mathrm{e}^{-1/t}}{\sqrt{\pi}t^{1/2}}\mathrm{e}^{-st}\,\mathrm{d}t$$

and for general $n \in \mathbb{N}$,

$$\frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathrm{e}^{-2\sqrt{s}} = \int_0^\infty (-1)^n\frac{\mathrm{e}^{-1/t}t^n}{\sqrt{\pi}t^{3/2}}\mathrm{e}^{-st}\,\mathrm{d}t\,.$$

In particular, for $n = 2$, this shows that

$$\begin{aligned}\mathcal{L}(h)(s) = \int_0^\infty \frac{\mathrm{e}^{-1/t}t^2}{\sqrt{\pi}t^{3/2}}\mathrm{e}^{-st}\,\mathrm{d}t &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathrm{e}^{-2\sqrt{s}} \\ &= \mathrm{e}^{-2\sqrt{s}}\left(\frac{1}{2s^{3/2}} + \frac{1}{s}\right) \\ &= \mathcal{L}(k)(s)\left(\mathcal{L}(\tfrac{1}{2}\mathbf{1}^{*3/2} + \mathbf{1})(s)\right)\end{aligned}$$

and therefore, by uniqueness of Laplace transforms, that

$$h = k * (\mathbf{1} + \tfrac{1}{2}\mathbf{1}^{*3/2}). \qquad \square$$

The preceeding Corollary is used to give a class of examples of kernels with similar characteristics to k.

4.18 Theorem. *Let $k \in L^1(0,1)$ be as defined in Lemma 4.15, and let $f \in L^1(0,1)$ be linear + $\mathrm{O}(t^2)$ at the origin, with $f(0) \neq 0$. Then*

$$(kf)^{*n} \sim (kf(0)\mathrm{e}_\mu)^{*n}$$

as $n \to \infty$, where $\mu = f'(0)/f(0)$.

Proof. Given that f is linear + $\mathrm{O}(t^2)$ at the origin, there must exist an $\varepsilon > 0$ such that $f(t) = f(0)\mathrm{e}^{\mu t} + ct^2$ on $(0,\varepsilon)$, for some $c \in \mathbb{R}$. Corollary 4.11 shows that

$$(kf)^{*n} \sim (kf(0)\mathrm{e}_\mu + r)^{*n}$$

where $r(t) = ck(t)t^2$; it only remains to check the conditions of Theorem 4.9.

First notice from Lemma 4.15 that $k^{*n} \sim a_n\mathbf{1}^{*p_n}$, where $p_n = n^2 - \frac{1}{2}$. The proof of the Lemma also provides the second condition, that $k^{*n} \leqslant a_n\mathbf{1}^{*p_n}$.

From Corollary 4.17 it is seen that $r = k * (\mathbf{1} + \frac{1}{2}\mathbf{1}^{*3/2})$; in particular

$$r \leqslant k * (\delta\mathbf{1})$$

for some $\delta > 1 + 1/\sqrt{\pi}$. This fixes $\alpha = 1$ (in the statement of the Theorem), and so $\beta = 1$ (recalling Lemma 4.8 on the unit kernel). The final condition can then be checked: $p_n^\beta/n = (n^2 - \frac{1}{2})/n \to \infty$.

So by Theorem 4.9, $(k + r)^{*n} \sim k^{*n}$. $\square$

Chapter 5

The Commutant of a Convolution Operator

In [7], Erdos shows that any bounded operator on $L^2(0,1)$ that commutes with the classical Volterra operator is in the strongly closed algebra generated by V; that is, with $A \in \mathcal{B}(L^2(0,1))$ such that $AV = VA$, for any any $\varepsilon > 0$ and $f \in L^2(0,1)$ there exists a polynomial P such that $P(0) = 0$ and

$$\|(P(V) - A)f\|_2 < \varepsilon.$$

The proof used is easily generalised to $V_k \in \mathcal{B}(L^2(0,1))$, as long as k is a polynomial generator for $L^1(0,1)$. However, the methods used rely on the Hilbert-space nature of $L^2(0,1)$, so the question of whether the commutant of $V_k \in \mathcal{B}(L^p(0,1))$ is contained in the strongly closed algebra generated by V_k requires different techniques to answer.

Before tackling that, some results about the strongly-closed algebra generated by V_k are required. Since convolution operators commute with one-another, the most basic result in this class is the observation that V_k generates all convolution operators.

5.1 Lemma. *Let k be a polynomial generator for $L^1(0,1)$, and fix $p \in [1,\infty]$; let $V_k \in \mathcal{B}(L^p(0,1))$ be defined in the usual way. Then for any $g \in L^1(0,1)$, V_g is in the norm-closure of the algebra generated by V_k.*

Proof. Fix $g \in L^1(0,1)$ and let $\varepsilon > 0$. Since k is a polynomial generator for $L^1(0,1)$, there exists a convolution polynomial P such that

$$\|g - P(k)\|_1 < \varepsilon$$

and therefore

$$\|\|V_g - P(V_k)\|\|_p = \left\|\left\|V_g - V_{P(k)}\right\|\right\|_p = \left\|\left\|V_{g-P(k)}\right\|\right\|_p \leqslant \|g - P(k)\|_1 < \varepsilon$$

and therefore V_g is in the norm-closure of $\langle V_k \rangle$, as required. □

While the Volterra algebra does not contain an identity element, it does contain an *approximate identity* – a net $(e_i)_{i \in I}$ such that $\|e_i * f - f\|_1 \to 0$ for all $f \in L^1(0,1)$. If k is a polynomial generator for $L^1(0,1)$, then the subalgebra of $L^1(0,1)$ generated by k also contains an approximate identity, as the following result shows.

5.2 Lemma. *Let k be a polynomial generator for $L^1(0,1)$, and fix $p \in [1, \infty]$. Then* $\operatorname{span}\{k^{*n} : n \in \mathbb{N}\}$ *contains a bounded approximate identity for $L^p(0,1)$.*

Proof. To show the existence of an approximate identity, it is enough [3, Prop. 2.9.14] to show that for any $\varepsilon > 0$ and an arbitrary finite set of functions $F \subseteq L^p(0,1)$ there exists a polynomial P such that

$$\|P(k) * f - f\|_p < \varepsilon$$

for all $f \in F$. If, in addition, there exists $A > 0$ such that $\|P(k)\|_1 \leqslant A$ for each P so chosen, then the approximate identity is bounded.

Choose a sequence $(e_n)_{n \in \mathbb{N}} \subseteq L^1(0,1)$ that is an approximate identity for $L^1(0,1)$, bounded by $A > 0$. Choose $\delta > 0$.

Fix a finite set $F \subseteq L^p(0,1)$; then there exists $e_N \in L^1(0,1)$ such that

$$\|e_N * f - f\|_p < \varepsilon/2$$

for all $f \in F$.

Now since k is a polynomial generator for $L^1(0,1)$ there exists a polynomial P such that

$$\|P(k) - e_N\|_1 < \min\left\{\delta, \frac{\varepsilon}{2\max\{\|f\|_p : f \in F\}}\right\},$$

so that

$$\|P(k)\|_1 \leqslant \|P(k) - e_N\|_1 + \|e_N\|_1 < \delta + A.$$

Now

$$\begin{aligned}
\|P(k) * f - f\|_p &= \|P(k) * f - e_N * f + e_N * f - f\|_p \\
&\leqslant \|(P(k) - e_N) * f\|_p + \|e_N * f - f\|_p \\
&\leqslant \|P(k) - e_N\|_1 \|f\|_p + \|e_N * f - f\|_p \\
&\leqslant \frac{\varepsilon \|f\|_p}{2\max\{\|f\|_p : f \in F\}} + \frac{\varepsilon}{2} \leqslant \varepsilon
\end{aligned}$$

for all $f \in F$, and $\|P(k)\|_1 < A + \delta$, as required. □

Remark. Since δ was chosen arbitrarily in the proof above, this shows that not only does the algebra generated by k contain a bounded approximate identity, but that the bound can be chosen to be arbitrarily close to 1.

5.3 Theorem. *Let $k \in L^1(0,1)$ be a polynomial generator for $L^1(0,1)$, fix $p \in [1,\infty]$, and let $V_k \in \mathcal{B}(L^p(0,1))$. Then the commutant of V_k is equal to the strongly closed algebra generated by V_k.*

Proof. Let $\mathcal{V}_k$ be the strongly-closed algebra generated by V_k, and let $A \in \mathcal{B}(L^p(0,1))$ commute with V_k. Then A commutes with all $B \in \mathcal{V}_k$, since $B = \lim P_n(V_k)$ for some sequence of polynomials, and multiplication is continuous in the strong operator topology. So

$$AB = A \lim P_n(V_k) = \lim\left(AP_n(V_k)\right) = \lim\left(P_n(V_k)A\right) = \lim\left(P_n(V_k)\right)A = BA.$$

If $A = V_k$ (which trivially commutes with itself), this shows that B commutes with V_k, for any $B \in \mathcal{V}_k$. So $\mathcal{V}_k$ is a subset of the commutant of V_k.

Now fix $A \in \mathcal{B}(L^p(0,1))$, commuting with V_k. We aim to show that $A \in \mathcal{V}_k$.

For any $g \in L^1(0,1)$, Lemma 5.1 shows that $V_g \in \mathcal{V}_k$, and the above calculation shows that A commutes with V_g. So for any $f \in L^p(0,1)$,

$$AV_k f = AV_f k = V_f Ak = V_{Ak} f$$

and so $AV_k = V_{Ak} \in \mathcal{V}_k$.

To show that $A \in \mathcal{V}_k$, it is enough to show that, for any $\varepsilon > 0$ and any finite set of functions $F \subseteq L^p(0,1)$ there exists a polynomial S such that

$$\|Af - S(V_k)f\|_p \leqslant \varepsilon$$

for all $f \in F$.

Fix $\varepsilon > 0$ and let P be a polynomial such that

$$\|Af - P(V_k)Af\|_p < \varepsilon/2$$

for all f (which is guaranteed to exist by Lemma 5.2).

Let T be the polynomial defined by $T(x) = P(x)/x$. This is a polynomial, because P has zero constant term. Now $T(V_k)$ is a bounded operator, given by

$$T(V_k) = a_1 I + \sum_{n=2}^{N} a_n V_k^{n-1}$$

where $P(x) = \sum_{n=1}^{N} a_n x^n$.

Now let Q be a polynomial with $Q(0) = 0$ such that

$$\|(V_k A - Q(V_k))f\|_p < \varepsilon / \|T(V_k)\|_p$$

for all f, which exists because $V_k A \in \mathcal{V}_k$. Define the polynomial S by $S(x) =$

$T(x)Q(x)$. This has $S(0) = 0$, so that $S(V_k) \in \mathcal{V}_k$. Then

$$\begin{aligned}
\|P(V_k)Af - S(V_k)f\|_p &= \left\| \sum_{n=1}^{N} a_n V_k^n Af - \left(a_1 Q(V_k)f + \sum_{n=2}^{N} a_n V_k^{n-1} Q(V_k)f \right) \right\|_p \\
&= \left\| a_1(V_k A - Q(V_k))f + \sum_{n=1}^{N} a_n V_k^{n-1}(V_k A - Q(V_k))f \right\|_p \\
&\leqslant \left|\!\left|\!\left| a_1 I + \sum_{n=1}^{N} a_n V_k^{n-1} \right|\!\right|\!\right|_p \|(V_k A - Q(V_k))f\|_p \\
&= |\!|\!| T(V_k) |\!|\!|_p \, \|(V_k A - Q(V_k))f\|_p \\
&\leqslant |\!|\!| T(V_k) |\!|\!|_p \frac{\varepsilon}{2\, |\!|\!| T(V_k) |\!|\!|_p} = \varepsilon/2
\end{aligned}$$

and therefore

$$\|Af - S(V_k)f\|_p \leqslant \|Af - P(V_k)f\|_p + \|P(V_k)f - S(V_k)f\|_p \leqslant \varepsilon$$

for all $f \in F$. Hence $A \in \mathcal{V}_k$, and therefore the commutant of V_k is a subset of $\mathcal{V}_k$. $\square$

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