Young-diagrammatic Methods for the Representation Theory of G_2

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Abstract

We study the G_2 fixed-point spaces $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ for an algebraically closed field $\Bbbk$ of characteristic $p > 2$, and dominant $\mathrm{GL}_7(\Bbbk)$-weights $\lambda \in X^+(7)$. Our primary focus is to develop a formula for the calculation of the dimension $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$. We obtain this formula by studying the fixed-point spaces $\nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2}$ for dominant $\mathrm{SO}_7(\Bbbk)$-weights $\mu \in X^+(T_{\mathrm{SO}_7(\Bbbk)})$, and then obtaining a good $\mathrm{SO}_7(\Bbbk)$-filtration of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$ when viewed as an $\mathrm{SO}_7(\Bbbk)$-module.

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I joined the University of York in 2014 as a fresh-faced undergraduate mature student. A common joke that I tell when playfully mocked about my age is that I'm an antediluvian, having arrived in York prior to the great flood of 2015. During my years at York, I've obtained my Master's degree, (almost) obtained my PhD, met my partner of nine years, and made many lifelong friends. It's a bittersweet feeling that this significant era of my life is coming to an end.

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I declare that this thesis is a presentation of original work and I am the sole author, except where indicated otherwise. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

— 1 —

Introduction

In this thesis we investigate the 0-th cohomology groups $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$, where $\Bbbk$ is an algebraically closed field of characterstic $p > 2$ and dominant $\mathrm{GL}_7(\Bbbk)$ weights $\lambda \in X^+(7)$. This study is in part motivated by the work of Brundan, who in his 1998 paper "Dense Orbits and Double Cosets" [3] showed that, for a spherical subgroup H of a connected reductive algebraic group G, the fixed-point space $\nabla_G(\mu)^H$ is at most one-dimensional for $\mu \in X^+(T_G)$. As G_2 is not a spherical subgroup of $\mathrm{GL}_7(\Bbbk)$, there is no such restriction on $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ for $\lambda \in X^+(7)$.

Chapter two provides all necessary preliminary definitions and results. Where it is not cumbersome, we state results in their most general form. However, for some key results we are concerned only with how the result relates to the theory of $\mathrm{GL}_n(\Bbbk)$, $\mathrm{SO}_7(\Bbbk)$, and G_2. Though we frequently reference Jantzen's book "Representations of Algebraic groups" [10], we present many results in the language of affine algebraic groups rather than that of group schemes.

Chapter three establishes a collection of tools which we frequently use throughout the thesis. We introduce key formulae of Pieri, Littlewood and Richardson, and Brauer which permit us to obtain good filtrations of certain tensor products of $\mathrm{GL}_7(\Bbbk)$-modules. We also develop a partition tool which allows us to restrict our study to dominant $\mathrm{GL}_7(\Bbbk)$-weights corresponding to partitions of at most 7 parts. The type **B** procedure of Koike and Terada, introduced in their 1987 paper "Young-diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n" [12], permits us to obtain a good $\mathrm{SO}_7(\Bbbk)$-filtration of the induced $\mathrm{GL}_7(\Bbbk)$-module $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$ for $\lambda \in X^+(7)$.

In chapter four we calculate the dimension $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ where λ is a partition of at most three parts. In chapter five we apply the main theorem of chapter four in order to calculate $\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2}$ for all dominant $\mathrm{SO}_7(\Bbbk)$-weights $\mu \in X^+(T_{\mathrm{SO}_7(\Bbbk)})$. Combining this with the type **B** procedure of Koike and Terada, we obtain a formula for $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ for $\lambda \in \Lambda^+(7)$. In chapter six we refine the dimension formula.

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Preliminaries

In this thesis we calculate the dimension of the 0-th cohomology group V^{G_2} for the exceptional group G_2 and certain G_2-modules V, which arise as restrictions of an important class of representations of $\mathrm{GL}_7(\Bbbk)$. The underlying theory presented in this chapter is introduced to build a framework which permits the study of these cohomology groups. For a more general coverage of linear algebraic groups, the texts used in writing this thesis are Humphreys [9] and Borel [2]. For a more general coverage of the representation theory of algebraic groups, in particular the representation theory of $\mathrm{GL}_n(\Bbbk)$, we refer to Green [6], Donkin [4], and Jantzen [10].

2.1 PARTITIONS, DIAGRAMS, AND TABLEAUX

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be an infinite sequence of non-negative integers. We call λ a *partition* if the terms of λ are weakly decreasing and only finitely many terms are non-zero. We call the non-zero terms λ_i the *parts* of λ. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n, 0, \ldots)$ such that $\lambda_n \neq 0$, we may ignore the zero terms and write $\lambda = (\lambda_1, \ldots, \lambda_n)$. The set of partitions of at most n parts is denoted as $\Lambda^+(n)$, and is a subset of the set $X(n)$ of integral sequences of at most n terms.

The *degree* of λ, denoted $|\lambda|$ is equal to the sum of its parts. The *length* of λ, denoted $l(\lambda)$, is the largest index n for which $\lambda_n \neq 0$; otherwise we say that $l(\lambda) = 0$ if $\lambda = (0)$. For a partition λ, the *transpose partition* of λ, denoted λ', is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, where λ'_i is equal to the number of parts of λ which are at least as large as i.

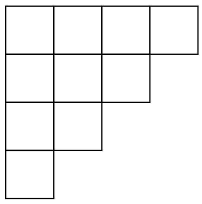
Let λ be a partition. We call a partition σ a *subpartition* of λ, denoted $\sigma \subseteq \lambda$, if $l(\sigma) \le l(\lambda)$ and for each index i we have $\sigma_i \le \lambda_i$.

Let λ be a partition. The *Young diagram* of shape λ, denoted Δ_λ, is a set of ordered pairs (i, j) of positive integers,

$$\Delta_\lambda = \{(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \,|\, 1 \le i \le l(\lambda),\, 1 \le j \le \lambda_i\}.$$

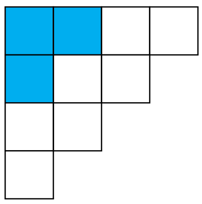
This diagram can be expressed pictorially as a collection of boxes, consisting of $l(\lambda)$ rows, such that the number of boxes on row i, counting from top to bottom, is equal to λ_i for $1 \le i \le l(\lambda)$.

Example 2.1.1. Let $\lambda = (4, 3, 2, 1)$. Then the Young diagram Δ_λ can be drawn as

$\Delta_\lambda =$  .

Let λ be a partition and let $\sigma \subseteq \lambda$. The *skew diagram* of shape λ/σ, denoted $\Delta_{\lambda/\sigma}$, is equal to the set of ordered pairs in the difference $\Delta_\lambda \setminus \Delta_\sigma$. We express this skew diagram by drawing the picture corresponding to Δ_λ and shading the picture corresponding to Δ_σ in the top left of the diagram.

Example 2.1.2. Let $\lambda = (4, 3, 2, 1)$ and let $\sigma = (2, 1)$. Then $\Delta_{\lambda/\sigma}$ is drawn as

$\Delta_{\lambda/\sigma} =$  .

We may view any Young diagram Δ_λ as the skew diagram $\Delta_{\lambda/(0)}$. From now on, we understand a diagram of shape λ/σ to mean a skew diagram of shape λ/σ and a diagram of shape λ to mean a skew diagram of shape $\lambda/(0)$. We call a diagram of shape λ/σ *empty* if $\lambda = \sigma$.

Let $\sigma \subseteq \lambda$ be two partitions and let $\Delta_{\lambda/\sigma}$ be the diagram of shape λ/σ. A *Young tableau* T with shape λ/σ is a map which assigns to each ordered pair (i, j) in $\Delta_{\lambda/\sigma}$ a positive integer $T(i, j)$. A Young tableau is expressible pictorially; for each $(i, j) \in \Delta_{\lambda/\sigma}$, inscribe the integer $T(i, j)$ into the box in the i-th row and j-th column of $\Delta_{\lambda/\sigma}$.

We call a tableau T *row-standard* if $T(i,j) \leq T(i,j+1)$ for every $(i,j),(i,j+1) \in \Delta_{\lambda/\sigma}$. We call T *column-standard* if $T(i,j) < T(i+1,j)$ for every $(i,j),(i+1,j) \in \Delta_{\lambda/\sigma}$. We call T a *standard tableau* if T is both row-standard and column-standard.

Let T be a Young tableau of shape λ/σ. The *weight* of T is a tuple $\alpha = (\alpha_1, \alpha_2, \dots)$ where α_k records the number of times k appears in the diagram $\Delta_{\lambda/\sigma}$. We call a tableau T with weight α an *α-tableau*. We call T a *(0)-tableau* if its corresponding diagram is empty.

The *word* of T, denoted $w(T)$, is the word obtained by reading the integers $T(i,j)$ from top to bottom and from right to left. Let $w(T) = a_1 a_2 \dots a_r$. For any $k \leq r$ we denote by $w_k(T)$ the subword containing the first k letters of $w(T)$. We call $w(T)$ a *lattice permutation* if for any index $k \leq r$, the number of instances of a positive number n in $w_k(T)$ is at least as many as the number of instances of $n+1$ in $w_k(T)$.

Definition 2.1.3. A *Littlewood-Richardson tableau* or an *LR-tableau* is a standard tableau whose word is a lattice permutation.

Let T be a tableau of shape λ/σ and let S be a tableau of shape μ/τ. We call S a *subtableau* of T if $\mu \subseteq \lambda$, $\tau \subseteq \sigma$, and for each $1 \leq i \leq l(\mu)$ we have

$$\{S(i,j) \,|\, \tau_i + 1 \leq j \leq \mu_i\} \subseteq \{T(i,j) \,|\, \sigma_i + 1 \leq j \leq \lambda_i\}.$$

Example 2.1.4. Continuing from our previous example, one possible tableau of shape $(4,3,2,1)/(2,1)$ is the following,

$$T = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}.$$

Note that T is both row standard (as rows weakly increase from left to right) and column standard (as columns strongly increase from top to bottom) and is thus a standard tableau. Counting the instances of 1, 2, and 3, we see that the weight of T is $(3,3,1)$. Finally, reading the T from right to left and top to bottom, we obtain the word $w(T) = 1121223$. Note that this word is a lattice permutation and thus T

is an LR-tableau.Now consider the following subtableau:

$$S = \begin{array}{|c|c|} \hline & \\ \hline & 1 \\ \hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Note that S is also a standard tableau whose weight is $(1, 2, 1)$. However, the word of this tableau is $w(S) = 1223$, which is not a lattice permutation. Therefore, S is not an LR-tableau.

2.2 ALGEBRAIC GEOMETRY

Algebraic groups are understood as affine varieties G endowed with a group structure, such that the defining group operations of G satisfy certain geometric properties.

Let $\Bbbk$ be an algebraically closed field. Denote by $(\mathbb{A}^n, \Bbbk[T_1, \ldots, T_n])$ the affine n-space over $\Bbbk$ together with the polynomial ring $\Bbbk[T_1, \ldots, T_n]$ in n independent variables. We interpret $\Bbbk[T_1, \ldots, T_n]$ as a subset of $\mathrm{Map}(\mathbb{A}^n, \Bbbk)$ by defining $T_i(x) = x_i$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{A}^n$. We call this polynomial ring the *coordinate ring* of $\mathbb{A}^n$. For a set of polynomials $S \subseteq \Bbbk[T_1, \ldots, T_n]$, we define $\mathcal{V}(S)$ to be the set of common zeros of S in $\mathbb{A}^n$, that is

$$\mathcal{V}(S) := \{x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

We call this set the *algebraic set* of S. It is easy to see that the algebraic set of the ideal $\mathfrak{I}$ generated by S coincides with the algebraic set of S. As the polynomial ring is a Noetherian ring, any ideal of the polynomial ring has a finite set of generators. Thus, the algebraic set of any ideal of $\Bbbk[T_1, \ldots, T_n]$ can be expressed as the algebraic set of a finite set of polynomials which generates that ideal.

Using algebraic sets we assign a topological structure to $\mathbb{A}^n$. The *Zariski topology* on $\mathbb{A}^n$ is defined by taking the closed subsets of $\mathbb{A}^n$ to be the algebraic sets of subsets of $\Bbbk[T_1, \ldots, T_n]$. This satisfies the axioms of a topology:

- both $\mathbb{A}^n$ and $\emptyset$ are closed, as they are the respective algebraic sets of the ideals $\{0\}$ and $\Bbbk[T_1, \ldots, T_n]$ respectively;
- The union of two closed sets is a closed set. Indeed, if $U = \mathcal{V}(S)$ and $V = \mathcal{V}(T)$, then $U \cup V = \mathcal{V}(ST)$, where $ST = \{fg \mid f \in S,\, g \in T\}$;

- If U_i is a family of closed sets indexed by a set I, then $\cap_{i \in I} U_i$ is closed. Suppose $U_i = \mathcal{V}(S_i)$. Then $\bigcap_{i \in I} U_i = \mathcal{V}(\bigcup_{i \in I} S_i)$.

Definition 2.2.1. An *affine variety* over $\Bbbk$ is a pair $(V, \Bbbk[V])$ where V is a Zariski-closed subset of $\mathbb{A}^n$, and $\Bbbk[V]$ is a finitely generated $\Bbbk$-subalgebra of the $\Bbbk$-algebra $\mathrm{Map}(V, \Bbbk)$, such that there is a bijection $\epsilon : V \simeq \mathrm{Hom}_{\Bbbk}(\Bbbk[V], \Bbbk)$. We call the image ϵ_v of $v \in V$ under ϵ the evaluation homomorphism at v, which is the map $f \mapsto f(v)$. We often use the shorthand V to denote the affine variety $(V, \Bbbk[V])$.

Let V be an affine variety. Then V has a corresponding coordinate ring inherited from $\Bbbk[T_1, \dots, T_n]$. We define the *vanishing ideal* $\mathcal{I}(V)$ of V to be the ideal

$$\mathcal{I}(V) = \{f \in \Bbbk[T_1, \dots, T_n] \mid f(x) = 0 \text{ for all } x \in V\}.$$

The coordinate ring of V is equal to the quotient ring $\Bbbk[V] = \Bbbk[T_1, \dots, T_n]/\mathcal{I}(V)$.

Example 2.2.2. Affine n-space has the structure of an affine variety with coordinate algebra $\Bbbk[\mathbb{A}^n] = \Bbbk[T_1, \dots, T_n]$. Singleton sets $\{x\}$ are affine varieties when viewed as the algebraic set of the n polynomials $T_1 - x_1, \dots, T_n - x_n$.

Given an ideal $\mathfrak{I}$ of $\Bbbk[V]$ we write $\sqrt{\mathfrak{I}}$ to denote its *radical* in $\Bbbk[V]$,

$$\sqrt{\mathfrak{I}} = \{f \in \Bbbk[V] \mid f^r \in \mathfrak{I} \text{ for some positive integer } r\}.$$

Clearly the radical $\sqrt{\mathfrak{I}}$ is an ideal of $\Bbbk[V]$ such that $\sqrt{\mathfrak{I}} \subseteq \mathcal{I}(\mathcal{V}(\mathfrak{I}))$. Hilbert's Nullstellensatz establishes a 1-1 correspondence between radical ideals of $\Bbbk[T_1, \dots, T_n]$ and algebraic sets of $\mathbb{A}^n$.

Theorem 2.2.3 (Hilbert's Nullstellensatz)**.** *Let $\mathfrak{I}$ be an ideal of $\Bbbk[T_1, \dots, T_n]$ and let $f \in \mathcal{I}(\mathcal{V}(\mathfrak{I}))$. Then $f^r \in \mathfrak{I}$ for some integer $r > 0$.*

Proof. See [9, Theorem 1.1] □

If V is an affine variety and W is a closed subset of V in the Zariski topology induced from $\mathbb{A}^n$, then by viewing W as a closed subset of the ambient affine space of V we see that W is also an affine variety with coordinate ring $\Bbbk[W] = \Bbbk[V]/\mathcal{I}(W)$. We call an affine variety V *irreducible* if there do not exist proper subvarieties V_1, V_2 such that

$$V = V_1 \cup V_2.$$

Equivalently, V is irreducible if the ideal $\mathcal{I}(V)$ is a prime ideal in $\Bbbk[\mathbb{A}^n]$.

We call an affine variety V *connected* if V is not expressible as the disjoint union of two disjoint non-empty subsets.

Definition 2.2.4. A topological space X is *Noetherian* if for any sequence $V_1 \supseteq V_2 \supseteq \ldots$ of closed subsets of X, there exists an integer r such that $V_r = V_{r+1} = \ldots$.

It is clear that $\mathbb{A}^n$ is a Noetherian topological space. Indeed, the inclusion reversing vanishing ideal map associates to any descending chain of closed subsets of $\mathbb{A}^n$ an ascending chain of ideals in the polynomial ring $\Bbbk[T_1, \ldots, T_n]$. As the polynomial ring is Noetherian, this ascending chain must terminate, which in turn means that the descending chain of algebraic sets must terminate.

Viewing an affine variety V as a subset of an affine space $\mathbb{A}^n$ for some sufficient n permits us to observe useful properties about its structure.

Proposition 2.2.5. *Let V be an affine variety in $\mathbb{A}^n$. Then V is expressible as a finite union $V = V_1 \cup \ldots \cup V_r$ of irreducible affine varieties, no one containing another.*

Proof. See [7, Proposition 1.5]. □

Let $V \subseteq \mathbb{A}^n$ be an affine variety and let P be the prime ideal in $\Bbbk[V]$ generated by a polynomial f. In the ring $P \times \Bbbk[V]$, we define an equivalence relation

$$(p, g) \sim (q, h) \iff ph - qg = 0.$$

We define the *localisation* of $\Bbbk[V]$ at f to be the quotient ring,

$$\Bbbk[V]_f = P \times \Bbbk[V]/\sim .$$

The affine variety V_f corresponding to $\Bbbk[V]_f$ is identified with the non-vanishing points of f in V; we call this variety the *principal open set* of f.

Example 2.2.6. The set $\mathrm{GL}_n(\Bbbk)$ is an affine variety corresponding to the localization $\Bbbk[\mathbb{A}^{n^2}]_{\det}$ of $\Bbbk[\mathbb{A}^{n^2}]$ at the determinant function det. Indeed, $\mathrm{GL}_n(\Bbbk)$ is the principal open set $\mathbb{A}^{n^2}_{\det}$ defined by the non-vanishing of the determinant on $\mathbb{A}^{n^2}$.

Definition 2.2.7. Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine varieties. A map $\phi : V \to W$ is a *morphism of affine varieties* if, for any $f \in \Bbbk[W]$, the map $f \circ \phi \in \mathrm{Map}(V, \Bbbk)$ is an element of $\Bbbk[V]$. We call the map $\phi^{\#} : \Bbbk[W] \to \Bbbk[V]$ mapping f to $f \circ \phi$ the *comorphism* of ϕ.

A morphism $\phi : V \to W$ of affine varieties is called an *isomorphism* if the comorphism $\phi^{\#} : \Bbbk[W] \to \Bbbk[V]$ is an isomorphism of $\Bbbk$-algebras. Note that a bijective morphism of affine varieties is not necessarily an isomorphism of affine varieties.

2.3 ALGEBRAIC GROUPS

With the necessary algebraic geometry in place, we move onto the theory of algebraic groups.

AFFINE ALGEBRAIC GROUPS

Definition 2.3.1. A group G is called an *affine algebraic group*, or simply an *algebraic group*, over $\Bbbk$ if G is an affine variety over $\Bbbk$ and the group multiplication map

$$\mu : G \times G \to G, \quad (g, h) \mapsto gh$$

and the group inversion map

$$\iota : G \to G, \quad g \mapsto g^{-1}$$

are morphisms of affine varieties.

Example 2.3.2. The two fundamental examples of algebraic groups over a field $\Bbbk$ are the additive group and the multiplicative group. The additive group $\mathbb{G}_a$ is the affine line $\mathbb{A}^1$ with coordinate ring $\Bbbk[T]$, equipped with the group operations of addition $(a, b) \mapsto a + b$ and inversion $a \mapsto -a$. The multiplicative group $\mathbb{G}_m$ is the principal open set $\mathbb{A}^1_T = \mathbb{A}^1 \setminus \{0\}$ with coordinate ring $\Bbbk[T, T^{-1}]$, equipped with group operations of multiplication $(g, h) \mapsto gh$ and inversion $g \mapsto g^{-1}$.

Let H be a closed subset of G which is also a subgroup of G. We call H an *algebraic subgroup* of G. Given algebraic groups K, L, the direct product $K \times L$ is an algebraic group, with coordinate ring $\Bbbk[K \times L] = \Bbbk[K] \otimes \Bbbk[L]$. As an algebraic group G has the structure of an affine variety, then it is expressible as a disjoint union of irreducible subsets, which we call the *connected components* of G. The connected component containing the group identity is called the *identity component* and is denoted by G^0. If $G = G^0$ then we call G a *connected group*.

Example 2.3.3. Over any algebraically closed field $\Bbbk$, the general linear group $\mathrm{GL}_n(\Bbbk)$ is a connected algebraic group.

Let G and H be algebraic groups. A *morphism of algebraic groups* is a group homomorphism $\phi : G \to H$ which is a morphism of algebraic varieties. Two examples of morphisms of algebraic groups are the following.

Definition 2.3.4. Let G be an algebraic group. A *character* of G is a morphism of algebraic groups $\lambda : G \to \mathbb{G}_m$. A *cocharacter* of G is a morphism of algebraic groups $\phi : \mathbb{G}_m \to G$.

The set of characters $X(G)$ of a group G forms an abelian group under the operation $(\lambda+\mu)(g) = \lambda(g)\mu(g)$ for $\lambda, \mu \in X(G)$, $g \in G$. We call $X(G)$ the *character group* of G. We denote by $Y(G)$ the set of cocharacters of G.

Definition 2.3.5. We say that an algebraic group G *acts* on an affine variety X over $\Bbbk$ if there exists a morphism of affine varieties $\phi : G \times X \to X$ satisfying the following conditions:

- $\phi(gh, x) = \phi(g, \phi(h, x))$ for all $g, h \in G$ and $x \in X$;
- $\phi(e_G, x) = x$ for all $x \in X$.

We often use the shorthand $g \cdot x$ for $\phi(g, x)$. Given an element $y \in X$, we define the *orbit* of y under the action of G, denoted $\mathrm{orb}_G(y)$, to be the set

$$\mathrm{orb}_G(y) = \{g \cdot y \mid g \in G\}.$$

Example 2.3.6. An algebraic group G acts on itself by left multiplication $g \cdot h = gh$ and by right inverse multiplication $g \cdot h = hg^{-1}$. We call these actions respectively the left regular and right regular maps of G. These actions induce actions of G on its coordinate ring $\Bbbk[G]$ which are given by

$$(\lambda_g f)(h) = (\lambda(g)f)(h) = f(g^{-1}h), \qquad (\rho(g)f)(h) = f(hg), \qquad g, h \in G,\ f \in \Bbbk[G].$$

We call these actions respectively *left and right translation of functions*.

By considering the action of G on its coordinate ring via translation of functions, we establish a correspondence between algebraic groups and closed subgroups of $\mathrm{GL}_n(\Bbbk)$. This is a key result in the study of algebraic groups, as it restricts our study to closed subgroups of $\mathrm{GL}_n(\Bbbk)$.

Theorem 2.3.7. *Let G be an algebraic group. Then G is isomorphic to a closed subgroup of* $\mathrm{GL}_n(\Bbbk)$ *for some* $n \in \mathbb{N}$.

Proof. See [9, Theorem 8.6] □

Tori and Borel subgroups

Henceforth, we assume without further mention that G is a connected algebraic group. The main focus of this thesis is the study of the connected algebraic groups $\mathrm{GL}_7(\Bbbk)$, $\mathrm{SO}_7(\Bbbk)$, and G_2. Some important subgroups which regularly occur in the representation theory of algebraic groups are tori and Borel subgroups.

Inside the general linear group $\mathrm{GL}_n(\Bbbk)$ there exists the closed connected subgroup $T_n(\Bbbk)$ of invertible diagonal matrices. This subgroup is isomorphic to the direct product of n copies of the multiplicative group $\mathbb{G}_m$ over $\Bbbk$.

Definition 2.3.8. A *torus* S of G is a closed subgroup of G which is isomorphic to $T_r(\Bbbk)$ for some $r > 0$. A torus is *maximal* if it is not a proper subgroup of another torus of G.

Note that as a torus S is isomorphic to some $T_r(\Bbbk)$ then S is an abelian subgroup of G. As with all algebraic groups, the character group $X(S)$ is an abelian group. Note also that the set of cocharacters $Y(S)$ of S has the form of an abelian group under the group operation $(\phi + \psi)(t) = \phi(t)\psi(t)$ for any $\phi, \psi \in Y(S)$ and $t \in \mathbb{G}_m$.

Let H be a subset of G. The *centralizer* of H, denoted $C_G(H)$ is the closed subgroup

$$C_G(H) = \{g \in G \,|\, ghg^{-1} = h \text{ for all } h \in H\}.$$

The *normalizer* of H, denoted $N_G(H)$ is the closed subgroup

$$N_G(H) = \{g \in G \,|\, gHg^{-1} = H\}.$$

The centralizer fixes each element of H by conjugation, whereas the normalizer contains elements which do not necessarily fix the elements of H.

Let S be a torus. Then the centraliser of S is a closed subgroup of $N_G(S)$. The quotient group $W(G,S) = N_G(S)/C_G(S)$ is a finite group which we call the *Weyl group* of G relative to S.

An endomorphism $x \in \mathrm{End}_{\Bbbk}(V)$ of a finite dimensional $\Bbbk$-space V is called *nilpotent* if $x^n = 0$ for some $n \in \mathbb{N}$, and *semisimple* if x is diagonalisable over $\Bbbk$. We call x *unipotent* if $x - 1$ is nilpotent.

Lemma 2.3.9 (Additive Jordan Decomposition). *Let $x \in \mathrm{End}_{\Bbbk}(V)$. Then there exist unique elements $x_s, x_n \in \mathrm{End}_{\Bbbk}(V)$ such that:*

- *x_s is semisimple;*
- *x_n is nilpotent;*
- *$x_s x_n = x_n x_s$;*
- *$x = x_s + x_n$.*

If $x \in \mathrm{GL}(V)$ then its eigenvalues are non-zero and hence x_s is invertible. Therefore, by defining $x_u = 1 + x_s^{-1} x_n$, we obtain a multiplicative Jordan decomposition for automorphisms of V. By viewing the right translation of functions $\rho(g)$ as an automorphism of the coordinate ring $\Bbbk[G]$, we obtain a result for the multiplicative decomposition of elements of an algebraic group G.

Theorem 2.3.10 (Multiplicative Jordan Decomposition). *Let g be an element of an algebraic group G. Then there exist unique elements $g_s, g_u \in G$ satisfying the following properties:*

- *$g = g_s g_u = g_u g_s$;*
- *$\rho(g_s) = \rho(g)_s$, $\rho(g_u) = \rho(g)_u$;*
- *if $\phi : G \to H$ is a morphism of algebraic groups then $\phi(g_s) = \phi(g)_s$ and $\phi(g_u) = \phi(g)_u$.*

We call g_s and g_u respectively the semisimple and unipotent parts of g.

A subgroup H of an algebraic group G is called *unipotent* if all of its elements are unipotent; that is $h_s = e_G$ for all $h \in H$.

For elements $g, h \in G$ we define the *commutator* of g and h to be $(g, h) = g^{-1}h^{-1}gh$. We define the *commutator* of G, denoted (G, G) to be the closed normal subgroup of G generated by the elements (g, h) for $g, h \in G$. We define the *derived series* $\mathcal{D}(G)$ of an algebraic group G inductively: we define $\mathcal{D}^0(G) = G$ and $\mathcal{D}^{i+1}(G) = (\mathcal{D}^i(G), \mathcal{D}^i(G))$ for $i \geq 0$. We call G *solvable* if the derived series of G terminates. That is, there exists an index i such that $\mathcal{D}^i(G)$ is the trivial subgroup $\{e_G\}$.

Example 2.3.11. A trivial example is that any abelian subgroup G is solvable, since $(G, G) = \{e_G\}$.

Definition 2.3.12. A *Borel subgroup* B of an algebraic group G is a maximal (with respect to inclusion), closed, connected, solvable subgroup of G.

Each Borel subgroup is expressible as the semi-direct product $T \ltimes U$ of a maximal torus T of G and a maximal connected unipotent subgroup U of G.

Example 2.3.13. Fix the maximal torus $T_n(\Bbbk) = \{\mathrm{diag}(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in \Bbbk^\times\}$ of $\mathrm{GL}_n(\Bbbk)$. One example of a Borel subgroup of $\mathrm{GL}_n(\Bbbk)$ is the subset B of lower triangular invertible $n \times n$ matrices over $\Bbbk$. This is expressible as the semi-direct product of T with the subgroup U of lower triangular unipotent matrices.

Theorem 2.3.14. *Let B be any Borel subgroup of G. Then all other Borel subgroups of G are conjugate to B.*

Proof. See [9, Theorem 21.3]. □

Corollary 2.3.15. *The maximal tori (respectively the maximal connected unipotent subgroups) of G are the maximal tori (respectively the maximal connected unipotent subgroups) of the Borel subgroups, and are conjugate.*

Proof. See [9, Corollary 21.3A]. □

As all maximal tori of G are conjugate, they must all have the same dimension, say n. We call n the *rank* of G. The Weyl group $W(G, T)$ of G relative to a maximal torus T is isomorphic to the Weyl group relative to any other maximal torus of G. Therefore, for a maximal torus T of G we call the $W = W(G, T)$ the *Weyl group* of G.

For a connected algebraic group G, the *radical* of G, denoted $R(G)$ is the maximal, connected, normal, solvable subgroup of G. The *unipotent radical* of G, denoted $R_u(G)$ is the maximal, connected, normal, unipotent subgroup of G.

Definition 2.3.16. We call G *semisimple* if $R(G)$ is trivial. We call G *reductive* if $R_u(G)$ is trivial.

Remark 2.3.17. Note that all semisimple groups are reductive. The general linear group $\mathrm{GL}_n(\Bbbk)$ is reductive but not semisimple.

Borel subgroups are an important point of study in algebraic groups as the structure of a connected algebraic group can be understood from its Borel subgroups.

Theorem 2.3.18. *Let G be reductive and let B be a Borel subgroup of G. Then G has a decomposition into the disjoint union of double B-cosets,*

$$G = \bigsqcup_{w \in W} BwB.$$

Proof. See [9, Theorem 28.3]. □

The Lie Algebra

Let G be a connected algebraic group with coordinate ring $\Bbbk[G]$. The set of *derivations* of $\Bbbk[G]$, denoted $\operatorname{Der}\Bbbk[G]$, is the set of linear maps $\delta : \Bbbk[G] \to \Bbbk[G]$ which satisfy the condition $\delta(fg) = \delta(f)g + f\delta(g)$ for $f, g \in \Bbbk[G]$. It is clear that sums of derivations are themselves derivations. We may view this space as a $\Bbbk$-algebra by endowing it with bracket operation $[-,-]$ which is defined by $[\delta, \gamma] = \delta\gamma - \gamma\delta$ for derivations δ, γ, where $\delta\gamma$ denotes the composition of functions $\delta \circ \gamma$. The set $\mathcal{L}(G)$ of all *left invariant derivations* (derivations $\delta \in \operatorname{Der}\Bbbk[G]$ satisfying $\delta\lambda_g = \lambda_g\delta$ for all $g \in G$, where λ_g denotes the left translation of functions by g) is a $\Bbbk$-subalgebra of $\operatorname{Der}\Bbbk[G]$ which we call the *Lie algebra* of G. This Lie algebra is isomorphic as a $\Bbbk$-vector space to the tangent space of G at the identity $\mathfrak{g} = T_e(G)$. For $g \in G$ the *inner automorphism* Inn_g is the map $\operatorname{Inn}_g(h) = ghg^{-1}$. The differential Ad_g of this inner automorphism is an automorphism of $\mathfrak{g}$. Thus the map $\operatorname{Ad} : G \to \operatorname{Aut}\mathfrak{g} \subset \operatorname{GL}(\mathfrak{g})$ is a morphism of algebraic groups which we call the *adjoint representation* of G. The group G acts on the tangent space $\mathfrak{g}$ naturally via the adjoint representation.

Root systems

The character group $X(\mathbb{G}_m)$ of the multiplicative group $\mathbb{G}_m$ is isomorphic as an algebraic group to $\mathbb{Z}$. Let S be a torus of G which is isomorphic to $T_r(\Bbbk)$ for some $r > 0$. Then the character group $X(S)$ of S is isomorphic to $\mathbb{Z}^r$.

Let S be a torus of the general linear group $\operatorname{GL}(V)$ of a $\Bbbk$-vector space V. We can write V as a direct sum of $\Bbbk$-subspaces

$$V = \bigoplus_{\alpha \in X(S)} V_\alpha,$$

where V_α is defined to be the subspace

$$V_\alpha = \{v \in V \mid h \cdot v = \alpha(h)v, \text{ for all } h \in S\}$$

for some $\alpha \in X(S)$. We call the characters α corresponding to non-zero subspaces V_α of V the *weights* of S in V.

In particular, consider an algebraic group G acting on its tangent space $\mathfrak{g}$ via the adjoint representation. Then there is a decomposition of $\mathfrak{g}$ into the direct sum of weight spaces,

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0$ denotes the 0-weight space. We call the non-zero weights of S in $\mathfrak{g}$ the *set of roots* of G relative to S. If we choose S to be a maximal torus then the roots of S in $\mathfrak{g}$ is called the *root system* of G, often denoted Φ.

Example 2.3.19. Let T be the maximal torus of $\mathrm{GL}_n(\Bbbk)$. For $1 \leq i \leq n$ the (i,i)-th coordinate map $\epsilon_i : T \to \mathbb{G}_m$ maps a matrix $M \in T$ to its (i,i)-th coordinate M_{ii}. Then the set of weights of T is given by

$$X(n) = X(T) = \{\lambda_1\epsilon_1 + \ldots + \lambda_n\epsilon_n \,|\, \lambda_1, \ldots, \lambda_n \in \mathbb{Z}\}.$$

The root system of $\mathrm{GL}_n(\Bbbk)$ relative to T is the set

$$\Phi = \{\epsilon_i - \epsilon_j \,|\, 1 \leq i \neq j \leq n\}.$$

We can construct root systems in a more abstract sense. Let E be a *Euclidean space* i.e. a finite dimensional $\mathbb{R}$-vector space with a positive definite, symmetric bilinear form $(-,-)$. For $\alpha \in E$ we define the orthogonal hyperplane of α, denoted H_α, by

$$H_\alpha := \{\beta \in E \,|\, (\beta, \alpha) = 0\}.$$

Given an orthogonal hyperplane H_α we denote by σ_α the reflection in this hyperplane

$$\sigma_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha, \qquad \beta \in E.$$

Definition 2.3.20. Let E be a Euclidean space. Then an *abstract root system* of E is a subset Ψ of E satistfying

- Ψ is finite, spans E, and does not contain 0;
- $\mathbb{R}\alpha \cap \Psi = \{\pm\alpha\}$ for all $\alpha \in \Psi$;
- If $\alpha \in \Psi$ then the reflection σ_α leaves Ψ stable;
- If $\alpha, \beta \in \Psi$, then $2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer.

The *abstract Weyl group* $W(\Psi)$ is the finite group generated by all reflections σ_α for $\alpha \in \Psi$. The *rank* of Ψ is the dimension of E.

When G is a semisimple group, then the root system of G is in fact an abstract root system.

Theorem 2.3.21. *Let G be semisimple with maximal torus T_G and root system Φ, and let $E = \mathbb{R} \otimes_{\mathbb{Z}} X(T_G)$. Then Φ is an abstract root system whose rank is* $\operatorname{rank} G$, *and whose abstract Weyl group is isomorphic to W.*

Proof. See [9, Theorem 27.1]. □

Definition 2.3.22. A subset $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subseteq \Phi$ is called a *base* of Φ if:

- Δ is a basis for E as an $\mathbb{R}$-space;
- For each $\beta \in \Phi$ there is an expression

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

  where the c_α are integers which are either all non-negative or all non-positive.

The set of roots for which the constants c_α are all non-negative are called the *positive roots*, denoted Φ^+, and the set of roots for which the constants c_α are all non-positive are called the *negative roots*, denoted Φ^-. We have that $\Phi = \Phi^+ \sqcup \Phi^-$ and $\Phi^- = -\Phi^+$.

Weyl groups and their associated root systems are often diagrammatically classified via directed graphs known as Dynkin diagrams. These diagrams are ubiquitous across many areas of study in abstract algebra, and are useful tools in the classification of certain objects. Some algebraic groups and their corresponding Lie algebras are known primarily by the label attached to the Dynkin diagram of their Weyl group in this classification.

Note that since the composition of morphisms of algebraic groups is a morphism of algebraic groups, the composition of a cocharacter with a character $\lambda \circ \phi$ is a morphism $\mathbb{G}_m \to \mathbb{G}_m$. The group of endomorphisms of $\mathbb{G}_m$ is isomorphic as an algebraic group to the additive group of $\mathbb{Z}$. Thus, for any character λ and cocharacter ϕ there exists a unique integer $\langle \lambda, \phi \rangle$ such that $\lambda \circ \phi$ is the map $t \mapsto t^{\langle \lambda, \phi \rangle}$.

Definition 2.3.23. Let $\alpha \in \Phi$. We define the *coroot* of α, often denoted to be $\alpha^\vee$, to be the unique cocharacted satisfying

$$\langle \beta, \alpha^\vee \rangle = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}, \text{ for all } \beta \in \Phi.$$

For a root system Φ, define a *weight* to be an element $\lambda \in E$ such that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$. We denote by Λ the set of these weights. If Δ is a base for Φ, then we call a weight λ *dominant* if $\langle \lambda, \alpha^\vee \rangle$ is non-negative for each $\alpha \in \Delta$. Denote by $\Lambda^+ \subset \Lambda$ the set of all dominant weights. The *fundamental dominant weights* with respect to Δ are the roots λ_i which satisfy the property

$$\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}.$$

In fact, any weight $\lambda \in \Lambda$ is expressible as an integral combination $\lambda = \sum_i m_i \lambda_i$. Thus the set of weights Λ forms a lattice. The weight lattice Λ is stable under the action of the Weyl group W. That is, $w(\lambda) \in \Lambda$ for any $\lambda \in \Lambda$ and $w \in W$. There exists a unique longest element $w_0 \in W$ which acts on Λ by sending λ to $-\lambda$ for all $\lambda \in \Lambda$.

Since weights have expressions as integral combinations of fundamental dominant weights, we may establish a partial order on the set of weights. Let λ, μ be weights. We say that $\lambda \geq \mu$ if $\lambda - \mu$ is expressible as an non-negative integral combination of fundamental dominant weights. We call this partial ordering the *dominance order* on Λ.

For each root $\alpha \in \Phi$ there exists a *root morphism* $x_\alpha \colon \mathbb{G}_a \to G$ with the property $t x_\alpha(r) t^{-1} = x_\alpha(\alpha(t) r)$ for all $r \in \mathbb{G}_a$, $t \in T_G$. The image of this morphism is a closed subgroup U_α of G called the *root subgroup* of G corresponding to α. The Lie algebra of this root subgroup is isomorphic to the weight space $\mathfrak{g}_\alpha$. Denote by U^+ the closed subgroup generated by all root subgroups U_α corresponding to positive roots α and denote by U the closed subgroup generated by all root subgroups U_α corresponding to negative roots. The subgroups $B = T_G \ltimes U$ and $B^+ = T_G \ltimes U^+$ are Borel subgroups of G, and $B \cap B^+ = T_G$.

The groups $\mathrm{GL}_7(\Bbbk)$, $\mathrm{SO}_7(\Bbbk)$, and G_2

As we are interested in the representation theory of $\mathrm{GL}_7(\Bbbk)$, $\mathrm{SO}_7(\Bbbk)$, and G_2, as an extended example of the above theory of algebraic groups, we fix our choice of maximal tori, root systems, and bases for these groups. In the context of this thesis we work over an algebraically closed field of characteristic $p > 2$. We follow the notation of [13, Section 2.1] for the following work.

The general linear group $\mathrm{GL}_7(\Bbbk) = \mathbb{A}^{7^2}_{\det}$ has maximal torus

$$T_7 = \{\mathrm{diag}(t_1, \ldots, t_7) \mid t_1, \ldots, t_7 \in \Bbbk^\times\}.$$

The generators of the free abelian group $X(7) = X(T_7)$ are the maps $\epsilon_i : T_7 \to \mathbb{G}_m$ which map a matrix $\mathrm{diag}(t_1, \ldots, t_7)$ to its (i,i)-th coordinate t_i, where $1 \leq i \leq 7$. The character group is then

$$X(7) = \{\lambda_1\epsilon_1 + \ldots + \lambda_7\epsilon_7 \,|\, \lambda_1, \ldots, \lambda_7 \in \mathbb{Z}\}.$$

The root system of $\mathrm{GL}_7(\mathbb{k})$ is given by

$$\Phi_7 = \{\epsilon_i - \epsilon_j \,|\, 1 \leq i \neq j \leq 7\}.$$

We choose the base

$$\Delta_{\mathrm{GL}_7(\mathbb{k})} = \{\epsilon_i - \epsilon_{i+1} \,|\, 1 \leq i \leq 6\},$$

which gives the positive root system

$$\Phi_7^+ = \{\epsilon_i - \epsilon_j \,|\, 1 \leq i < j \leq 7\}.$$

The set of dominant weights is the set

$$X^+(7) = \{\lambda_1\epsilon_1 + \ldots + \lambda_7\epsilon_7 \,|\, \lambda_1 \geq \ldots \geq \lambda_7\}.$$

Note that the set of dominant $\mathrm{GL}_7(\mathbb{k})$-weights contains the set of partitions of at most seven parts,

$$\Lambda^+(7) = \{\lambda_1\epsilon_1 + \ldots + \lambda_7\epsilon_7 \,|\, \lambda_1 \geq \ldots \geq \lambda_7 \geq 0\}.$$

The special orthogonal group $\mathrm{SO}_7(\mathbb{k})$ is defined as

$$\mathrm{SO}_7(\mathbb{k}) = \{M \in \mathrm{SL}_7(\mathbb{k}) \,|\, M'JM = J\},$$

where J is the 7×7 matrix with entries 1 on the anti-diagonal, and 0 elsewhere, and M' denotes the transpose matrix of M. We choose the maximal torus

$$T_{\mathrm{SO}_7(\mathbb{k})} = \{\mathrm{diag}(t, u, v, 1, v^{-1}, u^{-1}, t^{-1}) \,|\, t, u, v \in \mathbb{k}^\times\}.$$

We denote again by ϵ_1, ϵ_2, ϵ_3 the restrictions of the generators $\epsilon_1, \epsilon_2, \epsilon_3$ of $X(7)$ to $T_{\mathrm{SO}_7(\mathbb{k})}$. Then the character group $X(T_{\mathrm{SO}_7(\mathbb{k})})$ is written as

$$X(T_{\mathrm{SO}_7(\mathbb{k})}) = \{\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \lambda_3\epsilon_3 \,|\, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}\}.$$

The root system of $\mathrm{SO}_7(\mathbb{k})$ is given by

$$\Phi_{\mathrm{SO}_7(\mathbb{k})} = \{\pm\epsilon_1 \pm \epsilon_2, \pm\epsilon_1 \pm \epsilon_3, \pm\epsilon_2 \pm \epsilon_3,\} \cup \{\pm\epsilon_1, \pm\epsilon_2, \pm\epsilon_3,\}.$$

Within this root system we choose the base

$$\Delta_{\mathrm{SO}_7(\Bbbk)} = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3\},$$

which fixes our positive root system,

$$\Phi^+_{\mathrm{SO}_7(\Bbbk)} = \{\epsilon_1 \pm \epsilon_2, \epsilon_1 \pm \epsilon_3, \epsilon_2 \pm \epsilon_3\} \cup \{\epsilon_1, \epsilon_2, \epsilon_3\}.$$

The set of dominant weights is defined to be the weights λ such that $\langle \lambda, \alpha^\vee \rangle$ is non-negative for all simple roots α. This is the set

$$X^+(T_{\mathrm{SO}_7(\Bbbk)}) = \{\lambda_1\epsilon_1 + \lambda_2\epsilon_2 + \lambda_3\epsilon_3 \,|\, \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0\}.$$

The algebraic group of type G_2 is one exampe of an exceptional algebraic group named for its label in the Dynkin diagram classification. From this information we know that the group has rank 2 and is not a group of one of the classical families (SL_n, SO_n, SP_n). We view G_2 as the automorphism group of the octonion algebra $\mathbb{O}_\Bbbk$. Let $M_2(\Bbbk)$ denote the set of 2×2 matrices over $\Bbbk$. Let $\mathbb{O}_\Bbbk = M_2(\Bbbk) \oplus M_2(\Bbbk)$. The elements of $\mathbb{O}_\Bbbk$ are represented by ordered pairs (a, b) for $a, b \in M_2(\Bbbk)$. We view $\mathbb{O}_\Bbbk$ as a $\Bbbk$-algebra with the following algebraic structures:

- There exists an element $1 = (\mathbb{I}_2, 0)$, where $\mathbb{I}_2$ denotes the 2×2 identity matrix.
- There exists a quadratic form $n_{\mathbb{O}_\Bbbk} : \mathbb{O}_\Bbbk \to \Bbbk$ given by,

$$n_{\mathbb{O}_\Bbbk}((a, b)) = \det a - \det b.$$

- There exists an involution $*$ given by

$$(a, b)^* = (a', -b),$$

  for $(a, b) \in \mathbb{O}_\Bbbk$, where a' is the transpose of the matrix a.
- There exists a multiplication $\mathbb{O}_\Bbbk \times \mathbb{O}_\Bbbk \to \mathbb{O}_\Bbbk$ given by

$$(a, b)(c, d) = (ac + d'b, da + bc').$$

As an 8-dimensional $\Bbbk$-space, the octonion algebra $\mathbb{O}_\Bbbk$ can be viewed as the $\Bbbk$-span of 8 basis vectors $\{e_0, e_1, \ldots, e_7\}$, where e_0 is chosen to be the identity element of $\mathbb{O}_\Bbbk$ and the basis elements $e_1, \ldots, e_7$ are chosen such that

$$e_i^2 = -e_0, \text{ for } 1 \leq i \leq 7.$$

The automorphism group G_2 of $\mathbb{O}_{\Bbbk}$ is defined as

$$G_2 = \{g \in \mathrm{GL}(\mathbb{O}_{\Bbbk}) \mid g1 = g,\ g(x)^* = g(x^*),\ g(xy) = g(x)g(y), \text{ for } x, y \in \mathbb{O}_{\Bbbk}\}.$$

As the action of G_2 preserves multiplication and involution in the octonion algebra, the group necessarily stabilises the one-dimensional subspace of $\mathbb{O}_{\Bbbk}$ spanned by e_0 and thus also stabilises the seven-dimensional subspace spanned by $\{e_1, \ldots, e_7\}$. Through this lens, it is possible to identify G_2 as a algebraic subgroup of $\mathrm{SO}_7(\Bbbk)$.

We choose the maximal torus

$$T_{G_2} = \{\mathrm{diag}(s, s^{-1}t, s^2t^{-1}, 1, s^{-2}t, st^{-1}, s^{-1}) \mid s, t \in \Bbbk^*\}.$$

The character group of T_{G_2} is expressible as the a free abelian group with two generators,

$$X(T_{G_2}) = \{\lambda_1\gamma_1 + \lambda_2\gamma_2 \mid \lambda_1, \lambda_2 \in \mathbb{Z}\}.$$

Let $\alpha = 2\gamma_1 - 3\gamma_2$ and let $\beta = -\gamma_1 + 2\gamma_2$. Inside $X(T_{G_2})$ we have the root system

$$\Phi_{G_2} = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}.$$

Choosing the base $\Delta_{G_2} = \{\alpha, \beta\}$ the fundamental dominant weights are given by $\omega_1 = 2\alpha + 3\beta$, $\omega_2 = \alpha + 2\beta$. Then the set of dominant weights is given by

$$X^+(T_{G_2}) = \{(2x+y)\alpha + (3x+2y)\beta \mid x, y \in \mathbb{Z}_{\geq 0}\}.$$

The Weyl group of G_2 is given by

$$W_{G_2} = \langle s_\alpha, s_\beta \mid (s_\alpha s_\beta)^6 = e\rangle.$$

2.4 REPRESENTATION THEORY

We are interested in the representation theory of connected algebraic groups. To be more precise, we are interested in the dimension of the G_2 fixed-point space of induced $\mathrm{GL}_7(\Bbbk)$-modules $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$. We achieve this by calculating the filtration multiplicity of the trivial G_2-module $\nabla_{G_2}(0,0)$ in the G_2-filtration of the induced $\mathrm{GL}_7(\Bbbk)$-module $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$.

MODULES

Let G be a connected algebraic group over $\Bbbk$ and let V be a $\Bbbk$-vector space. A *representation* of G on V is morphism of algebraic groups $G \to \mathrm{GL}(V)$. We call V a *rational G-module*, or simply a *G-module*. Given two G-modules V and W, a *G-module homomorphism* is a G-equivariant map $\xi : V \to W$. That is, for each $v \in V$ and $g \in G$ we have $\xi(g \cdot v) = g \cdot \xi(v)$ where the left hand action is the action of G on V and the right hand action is the action of G on W.

Example 2.4.1. The $\Bbbk$-vector space $E \simeq \Bbbk^n$ of column vectors is a $\mathrm{GL}_n(\Bbbk)$-module via left multiplication. We call E the natural $\mathrm{GL}_n(\Bbbk)$-module.

Given two G-modules V and W, the direct sum $V \oplus W$ is a G-module whose representation is induced from the representations on V and W. That is, for $g \in G$, $v \in V$, and $w \in W$ we have $g \cdot (v, w) = (g \cdot v, g \cdot w)$ where the respective actions in the brackets are the actions on V and W respectively. Naturally, V and W can be viewed as G-submodules of $V \oplus W$. The tensor product of two G-modules V and W is also a G-module in the natural way induced from V and W. Symmetric powers and exterior powers of V are G-modules by the action induced from the G-representation on the tensor algebra of V.

Example 2.4.2. Let E be the natural $\mathrm{GL}_n(\Bbbk)$-module. The tensor algebra $T(E)$ is a $\mathrm{GL}_n(\Bbbk)$-module via the action induced from E. In fact, $T(E)$ is a graded $\Bbbk$-algebra,

$$T(E) = \bigoplus_{i \geq 0} T^i(E),$$

where by convention $T^0(E) = \Bbbk$ and for $i > 0$ we have

$$T^i(E) = \underbrace{E \otimes \ldots \otimes E}_{i \text{ times}}.$$

Then for $i \geq 0$, the i-th tensor power $T^i(E)$ is a $\mathrm{GL}_n(\Bbbk)$-submodule of $T(E)$.

Example 2.4.3. Denote by $S\,E$ the symmetric algebra of E. This is the quotient of $T(E)$ by the ideal generated by all elements of the form $e \otimes f - f \otimes e$, for $e, f \in T(E)$. Denote by $\bigwedge E$ the exterior algebra of E, which is the quotient of $T(E)$ by the ideal generated by all elements $e \otimes e$, for $e \in T(E)$. The symmetric and exterior algebras of E are both $\mathrm{GL}_n(\Bbbk)$-quotients of the $\mathrm{GL}_n(\Bbbk)$-module $T(E)$. They are both graded,

$$S\,E = \bigoplus_{i \geq 0} S^i\,E, \qquad \bigwedge E = \bigoplus_{i \geq 0} \bigwedge^i E.$$

For $i \geq 0$ the symmetric power $S^i\, E$ and the exterior power $\bigwedge^i E$ are respectively $\mathrm{GL}_n(\Bbbk)$-submodules of $S\,E$ and $\bigwedge E$.

Example 2.4.4. We have a 1-dimensional $\mathrm{GL}_n(\Bbbk)$-module

$$\det_n := \bigwedge^n E = \Bbbk(x_1 \wedge x_2 \wedge \ldots \wedge x_n),$$

Where $x_1, \ldots, x_n$ is a basis for the natural $\mathrm{GL}_n(\Bbbk)$-module E. We call this $\mathrm{GL}_n(\Bbbk)$-module the determinant module of $\mathrm{GL}_n(\Bbbk)$, as the natural action of $\mathrm{GL}_n(\Bbbk)$ on $\det_n$ is given by

$$g \cdot v = \det(g)v,$$

for all $v \in \det_n$.

Definition 2.4.5. A non-zero G-module V is called *simple* if the only G-submodules of V are $\{0\}$ and V. We call V *semisimple* if V is expressible as the direct sum of simple G-modules.

Definition 2.4.6. Let V be a G-module. Then the *socle* $\mathrm{soc}_G(V)$ of V is defined to be the sum of all simple G-submodules of V.

Simple modules are a key area of study in the representation theory of connected algebraic groups. Let T_G be the maximal torus of G; then each dominant weight $\lambda \in X^+(T_G)$ has a unique simple module $L_G(\lambda)$ of highest weight λ.

If V is a G-module, then the fixed-point set V^G is a G-submodule of V on which G acts trivially.

$$V^G = \{v \in V \mid g \cdot v = v \text{ for all } g \in G\}.$$

If H is a closed subgroup of G then we may interpret a G-module V as a H-module by considering the restriction to H of the morphism $G \to \mathrm{GL}(V)$. We denote this restriction by $\mathrm{res}^G_H V$. For an H-module W we may induce a representation of G on W as follows,

$$\mathrm{ind}^G_H W = \{\phi \in \mathrm{Mor}(G, W) \mid \phi(gh) = h^{-1}\phi(g), \text{ for } g \in G, h \in H\}.$$

Proposition 2.4.7. *Let G be an algebraic group and let H be a closed subgroup. Let V be an H-module. Then for each G-module W there exists an bijective correspondence of $\Bbbk$-spaces,*

$$\mathrm{Hom}_G(W, \mathrm{ind}^G_H V) \cong \mathrm{Hom}_H(\mathrm{res}^G_H W, V).$$

Proof. See [10, Part I 3.4]. □

Given an algebraic group G with maximal torus T_G, and a G-module V, define the *formal character* of V as

$$\operatorname{ch}_G V = \sum_{\alpha \in X(T_G)} c_\alpha X^\alpha,$$

where c_α is the multiplicity of the weight space V^α and X^α is the formal polynomial $X_1^{\alpha_1} \dots X_n^{\alpha_n}$ where n is the rank of G.

If H is a closed subgroup of G then denote by $\operatorname{ch}_H^G V$ the character of $\operatorname{res}_H^G V$.

THE CATEGORY OF G-MODULES

We now concern ourselves with the category of G-modules. This is a category with many nice properties which aid in our study of representations of G.

Definition 2.4.8. A G-module I is called an *injective* if, for any G-modules V, W with injective morphism $f : V \to W$ and arbitrary morphism $g : V \to I$, there exists a morphism $h : W \to I$ satisfying $h \circ f = g$.

The category of G-modules is an abelian category and so has some useful properties:

- for each G-module V there exists a functor $\operatorname{Hom}_G(V, -)$ to the category of abelian groups;
- for any finite collection $V_1, \dots, V_n$ of G-modules there exists a biproduct, which is a G-module V together with embedding morphisms $V_i \to V$ and projection morphisms $V \to V_i$;
- we may define exact sequences of modules and exact functors.

In particular, the Hom-set functor to the category of abelian groups is a left-exact functor; this means that it is a functor that maps kernels of morphisms in the category of G-modules to kernels of morphisms in the category of abelian groups. Furthermore, the category of G-modules has enough injectives, meaning that for any G-module V there exists an injective G-module I and an injective morphism $f : V \to I$.

Let V_1, V_2, V be G-modules such that

$$0 \to V_1 \to V \to V_2 \to 0$$

is a short exact sequence of G-modules. Then considering the characters of these modules we have

$$\mathrm{ch}_G V = \mathrm{ch}_G V_1 + \mathrm{ch}_G V_2.$$

Given that the category of G-modules is an abelian category with enough injectives, for each G-module V we can define an *injective resolution* of V; this is a long exact sequence of G-modules,

$$0 \to V \to I_0 \to I_1 \to I_2 \to \dots,$$

where the I_i are injective G-modules. Given a left exact functor $\mathcal{F}$ between abelian categories, we obtain a cochain complex,

$$0 \to \mathcal{F}I_0 \to \mathcal{F}I_1 \to \dots.$$

We define the right derived functors $R^i\mathcal{F}$ to be the i-th cohomology of the chain complex. In this thesis we regularly apply the G fixed-point functor $(-)^G$ and the G-equivariant homomorphism functor $\mathrm{Hom}_G(W, -)$ for a fixed G-module W.

2.5 SIMPLE MODULES AND INDUCTION

For an algebraic group G with maximal torus T_G, we may view any G-module V as a T_G-module. As a T_G-module, there is a decomposition of V as a direct sum of T_G-modules on which the action of T_G is given by weights. The non-zero subspaces of V are the weight spaces of V:

$$V = \bigoplus_{\mu \in X(T_G)} V_\mu.$$

Now fix a root system Φ with base Δ, and let U be the unipotent subgroup generated by the root subgroups corresponding to negative roots. Set $B = T_G \ltimes U$ to be the Borel subgroup of G normalised by T_G and containing all negative root subgroups. We construct a unique 1-dimensional B-module isomorphic as a T-module to the 1-dimensional T-module of weight λ as follows: consider $\Bbbk$ as a B-module on which U acts trivially and T acts with weight λ. We denote the B-module $\Bbbk$ in this context by $\Bbbk_\lambda$.

Definition 2.5.1. The *induced G-module* of highest weight λ, which we denote $\nabla_G(\lambda)$, is given by

$$\mathrm{ind}_B^G \Bbbk_\lambda = \{f \in \Bbbk[G] \mid f(gb) = \lambda(b^{-1})f(g) \text{ for all } g \in G, b \in B\}.$$

Example 2.5.2. Let $G = \mathrm{GL}_n(\Bbbk)$, Let E be the natural $\mathrm{GL}_n(\Bbbk)$-module, and let $r \geq 0$. Then

$$\nabla_{\mathrm{GL}_7(\Bbbk)}(r) \cong S^r(E),$$

$$\nabla_{\mathrm{GL}_7(\Bbbk)}(1^r) \cong \bigwedge^r E.$$

To each dominant weight $\lambda \in X^+(T_G)$ we have an induced G-module $\nabla_G(\lambda)$ and a simple G-module $L_G(\lambda)$. There is in fact a relation between these two modules.

Proposition 2.5.3. *If* $\nabla_G(\lambda) \neq 0$ *then* $\mathrm{soc}_G \nabla_G(\lambda)$ *is simple.*

Proof. See [10, Corollary I 2.3]. □

For any $\lambda \in X(T_G)$ with $\nabla_G(\lambda) \neq 0$, denote by $L_G(\lambda)$ the simple socle of $\nabla_G(\lambda)$. Over fields of characteristic zero, the induced modules $\nabla_G(\lambda)$ are isomorphic as G-modules to the simple modules $L_G(\lambda)$. This is not the case in general over fields of positive characteristic, yet we still find these simple modules as the socles of their respective induced modules.

Proposition 2.5.4. *Any simple G-module is isomorphic to exactly one $L_G(\lambda)$ with $\lambda \in X^+(T_G)$.*

Proof. See [10, Proposition I 2.4a]. □

With this we see that the simple G-modules coincide exactly with the simple socles of induced modules. We now wish to know exactly which weights of T have a corresponding simple module.

Proposition 2.5.5. *The induced G-module $\nabla_G(\lambda)$ is non-zero exactly when λ is a dominant weight.*

Proof. see [10, Proposition I 2.5] □

In order to obtain useful results for induced modules $\nabla_G(\lambda)$ for $\lambda \in T_G$, we must define some actions of the Weyl group W of G on $\nabla_G(\lambda)$. We may define the natural action of W on $X(T_G)$ as $w : \lambda \to w(\lambda)$ for all $w \in W, \lambda \in X(T_G)$. For a fixed positive root system Φ^+, we define $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. Weyl's character formula gives us an alternate method of calculating the character $\chi_G(\lambda)$ of the induced module $\nabla_G(\lambda)$.

Theorem 2.5.6 (Weyl's Character Formula)**.** *For $\lambda \in X^+(T_G)$ we have*

$$\chi_G(\lambda) = \frac{\sum_{w\in W} \operatorname{sgn}(w) X^{w(\lambda+\rho)}}{\sum_{w\in W} \operatorname{sgn}(w) X^{w(\rho)}},$$

where X^ξ is the formal polynomial corresponding to the weight ξ and $\operatorname{sgn}(w)$ is defined to be +1 if w is the composition of an even number of simple reflections, or -1 otherwise.

Proof. See [4, Proposition 2.2.7]. □

From Weyl's character formula we derive a formula for the dimension of an induced module $\nabla_G(\lambda)$.

Theorem 2.5.7 (Weyl's Dimension Formula)**.** *For $\lambda \in X^+(T_G)$ we have*

$$\dim(\nabla_G(\lambda)) = \frac{\prod_{\alpha\in\Delta}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Delta}(\rho,\alpha)}.$$

Proof. See [8, Corollary 24.3]. □

We wish to restrict our study of G-modules to those which have a nice construction from induced modules. We say that a G-module V admits a *good filtration* if there exists an ascending chain of G-submodules of V

$$0 = V_0 \subset V_1 \subset \ldots \subset V_k = V,$$

such that the quotient of subsequent submodules V_i/V_{i-1} is isomorphic to some $\nabla_G(\lambda^i)$ for some $\lambda^i \in X^+(T_G)$. We call these quotients the *sections* of V. For $\lambda \in X^+(T_G)$ we denote by $(V : \nabla_G(\lambda))$ the *filtration multiplicity* of $\nabla_G(\lambda)$ as a section of V; that is, the number of sections of V which are isomorphic to $\nabla_G(\lambda)$.

Theorem 2.5.8. *Let V be a rational G-module which admits a good filtration. Then for $\lambda \in X^+(T_G)$, the filtration multiplicity $(V : \nabla_G(\lambda))$ is independent of the choice of good filtration.*

Proof. See [4, Proposition 12.1.1]. □

We may choose a good filtration such that $\nabla_G(\lambda)$ appears higher in the filtration than $\nabla_G(\mu)$ whenever $\lambda \geq \mu$ with respect to the dominance order. We may thus express a good filtration for V as

$$V = \nabla_G(\mu_1)|\nabla_G(\mu_2)|\ldots|\nabla_G(\mu_n),$$

where $\nabla_G(\xi)$ appears higher in the filtration order than $\nabla_G(\nu)$ if $\xi \geq \nu$.

Given a G-module V with a good filtration, we would like to express the character of V in terms of the characters of its filtration sections.

Lemma 2.5.9. *Let V be a G-module with a good G-filtration. Then the character of V is given by,*

$$\operatorname{ch} V = \sum_{\lambda \in X^+(T_G)} m_\lambda \chi_G(\lambda),$$

where $m_\lambda = (V : \nabla_G(\lambda))$ for $\lambda \in X^+(T_G)$.

Proof. If the G-module V has a good G-filtration, then there exists a sequence of G-submodules

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots \subseteq V_k = V,$$

where the sections V_i/V_{i-1} are isomorphic to some induced G-module $\nabla_G(\lambda^i)$ for $1 \leq i \leq k$ and $\lambda^i \in X^+(T_G)$. We proceed by induction on the length of the filtration of V.

If $k = 1$ then the statement is clearly true, as $\operatorname{ch}_G V = \chi_G(\lambda^1)$ for some $\lambda^1 \in X^+(T_G)$. Suppose $k > 1$. There exists a short exact sequence of G-modules

$$0 \to V_1 \to V \to V/V_1 \to 0.$$

We note that as a G-module, $V_1 \cong \nabla_G(\lambda^1)$ for some dominant weight $\lambda^1 \in X^+(T_G)$. Note also that V/V_1 admits a good filtration and that the corresponding sequence of G-submodules has length $k-1$. We have that

$$\operatorname{ch}_G(V) = \chi_G(\lambda^1) + \operatorname{ch}_G V/V_1.$$

By the induction hypothesis we have that

$$\operatorname{ch} V/V_1 = \sum_{\lambda \in X^+(T_G)} \bar{m}_\lambda \chi_G(\lambda),$$

where $\bar{m}_\lambda = (V/V_1 : \nabla_G(\lambda))$ for $\lambda \in X^+(T_G)$. Therefore, the character of V is given by

$$\operatorname{ch} V = \sum_{\lambda \in X^+(T_G)} m_\lambda \chi_G(\lambda),$$

where $m_\lambda = \bar{m}_\lambda + 1$ if $\lambda = \lambda^1$; otherwise $m_\lambda = \bar{m}_\lambda$. $\square$

In the monograph [4], Donkin studies the tensor products $V \otimes W$ of rational G-modules V, W which both have good filtrations. Donkin shows that if the characteristic of $\Bbbk$ is not 2 and G has no components of type E_7 or E_8, then $V \otimes W$ admits a good filtration. Mathieu completes this study in [14], permitting the following result.

Theorem 2.5.10. *Let V, W be rational G-modules which admit good filtrations. Then the G-module $V \otimes W$ admits a good filtration as a G-module.*

Proof. Due to [4] and [14] as stated above. □

Good pairs

Let G be a connected reductive algebraic group and let H be a connected reductive algebraic subgroup of G. We call the pair (G, H) a *good pair* if, for any G-module V with a good filtration, the H-restriction of V has a good filtration as an H-module.

Let T_H be a maximal torus for H. It follows from Lemma 2.5.9 that if (G, H) is a good pair and $\lambda \in X^+(T_G)$, then as an H-module, $\nabla_G(\lambda)$ has character $\mathrm{ch}_H \nabla_G(\lambda)$ of the form

$$\mathrm{ch}_H \nabla_G(\lambda) = \sum_{\mu \in X(T_H)} m_\mu \chi_H(\mu),$$

where $m_\mu = (\nabla_G(\lambda) : \nabla_H(\mu))$ for $\mu \in X^+(T_H)$. Denote by $\chi_H^G(\lambda)$ the character of $\nabla_G(\lambda)$ as an H-module.

Proposition 2.5.11. *Let $\Bbbk$ be an algebraically closed field of characterstic $p > 2$.*

(i) The pair $(\mathrm{GL}_n(\Bbbk), \mathrm{SO}_n(\Bbbk))$ is a good pair.

(ii) the pair $(\mathrm{SO}_7(\Bbbk), G_2)$ is a good pair.

Proof. See [3, Proposition 3.3 (iii) and (iv)]. □

It is clear that if (G, H) and (H, K) are good pairs, then (G, K) is a good pair.

Corollary 2.5.12. *Let $\Bbbk$ be an algebraically closed field of characterstic $p > 2$. Then $(\mathrm{GL}_7(\Bbbk), G_2)$ is a good pair.*

Proof. This follows immediately from Proposition 2.5.11 and by the fact the good pair property is transitive. □

Example 2.5.13. If the characteristic of $\Bbbk$ is 2 then $(\mathrm{GL}_n(\Bbbk), \mathrm{SO}_n(\Bbbk))$ is not a good pair for $n \geq 3$. This is due to [3, Proposition 3.4 (i)].

2.6 INVARIANT SPACES OF MODULES

Let G be a connected reductive algebraic group with Borel subgroup B containing the maximal torus T_G, and let H be a connected reductive subgroup of G. We call H a *spherical subgroup* of G if H has an open dense orbit on the variety G/B, or equivalently B has an open dense orbit on G/H. Let $V \simeq \Bbbk^n$ be the natural G-module. We call a spherical subgroup H of G *irreducible* if V restricted to H is irreducible as an H-module.

Proposition 2.6.1. *If $\Bbbk$ has characteristic $p > 2$ then $\mathrm{SO}_7(\Bbbk)$ is an irreducible spherical subgroup of $\mathrm{GL}_7(\Bbbk)$ and G_2 is an irreducible spherical subgroup of $\mathrm{SO}_7(\Bbbk)$. However, G_2 is not an irreducible spherical subgroup of $\mathrm{GL}_7(\Bbbk)$. If $\operatorname{char} \Bbbk = 2$ then $\mathrm{SO}_7(\Bbbk)$ is not a spherical subgroup of $\mathrm{GL}_7(\Bbbk)$.*

Proof. See [11, Lemma 5.5]. □

Theorem 2.6.2. *A connected reductive subgroup H of G is a spherical subgroup of G if and only if $\nabla_G(\lambda)^H$ is at most one-dimensional for all $\lambda \in X^+(T_G)$.*

Proof. See [3, Theorem 1.2]. □

— 3 —

Methodology and Tools

As G_2 is not a spherical subgroup of $\mathrm{GL}_n(\Bbbk)$ then by Theorem 2.6.2 we know that there exist dominant weights $\lambda \in X^+(7)$ such that $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} > 1$. We now establish our tools for studying the invariant space $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$. Throughout this chapter, unless explicitly stated, let G be a connected reductive group, H a connected reductive subgroup of G, and T_G a fixed maximal torus of G.

3.1 TOOLS FOR FILTRATIONS

Lemma 3.1.1. *Let $\lambda, \mu \in X^+(T_G)$. Then*

(i)

$$\mathrm{Ext}^i_G(\Bbbk, \nabla_G(\lambda) \otimes \nabla_G(\mu)) = \begin{cases} \Bbbk, & \text{if } i = 0 \text{ and } \lambda = -w_0\mu; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) $\mathrm{Ext}^i_G(\Bbbk, \nabla_G(\lambda)) = 0$ *for* $i > 0$.

Proof. Part (i) is due to [10, II Proposition 4.13]. Setting $\mu = 0$ we obtain part (ii) from part (i). □

Lemma 3.1.2 (Filtration Tool)**.** *Let V be a G-module with a good filtration. Then*

$$\mathrm{Ext}^1_G(\Bbbk, V) = 0$$

Proof. Suppose V has the good filtration,

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k = V,$$

with sections $\nabla_G(\lambda^i)$ for some $\lambda^i \in X^+(T_G)$. We proceed by induction on k.

If $k = 1$ then the statement follows from 3.1.1. Suppose $k > 1$. There exists a short exact sequence of G-modules

$$0 \to V_1 \to V \to V/V_1 \to 0.$$

We note that as a G-module, $V_1 \cong \nabla_G(\lambda^1)$ for some dominant weight $\lambda^1 \in X^+(T_G)$. Note also that V/V_1 admits a good filtration and that the corresponding sequence of G-submodules has length $k-1$. We apply $\mathrm{Hom}_G(\Bbbk, -)$ to 3.1 and obtain the following fragment of an exact sequence,

$$\mathrm{Ext}^1_G(\Bbbk, V_1) \to \mathrm{Ext}^1_G(\Bbbk, V) \to \mathrm{Ext}^1_G(\Bbbk, V/V_1).$$

by Lemma 3.1.1 part (ii) we have $\mathrm{Ext}^1_G(\Bbbk, V_1) = 0$, and by our induction hypothesis we have $\mathrm{Ext}^i_G(\Bbbk, V/V_1) = 0$. Therefore, we deduce that $\mathrm{Ext}^1_G(\Bbbk, V) = 0$. □

As we are interested in calculating $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ for $\lambda \in X^+(7)$, we require a tool for calculating this dimension.

Lemma 3.1.3 (Dimension Tool). *Let (G, H) be a good pair and let V be a G-module with a good G-filtration. Then*

$$\mathrm{dim} V^H = \sum_{\lambda \in X^+(T_G)} m_\lambda \mathrm{dim} \nabla_G(\lambda)^H,$$

where $m_\lambda := (V : \nabla_G(\lambda))$ for $\lambda \in X^+(T_G)$.

Proof. Suppose V has the good filtration,

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k = V,$$

with sections $\nabla_G(\lambda^i)$ for some $\lambda^i \in X^+(T_G)$. We proceed by induction on k.

If $k = 1$ then $\dim V^H = \dim \nabla_G(\lambda^1)^H$. Suppose that $k > 1$. There exists a short exact sequence of G-modules

$$0 \to V_1 \to V \to V/V_1 \to 0.$$

Note that $V_1 \cong \nabla_G(\lambda^1)$ and that V/V_1 admits a good filtration and that the corresponding sequence of G-submodules has length $k-1$. We apply $\mathrm{Hom}_H(\Bbbk, -)$ and obtain the sequence,

$$0 \to \mathrm{Hom}_H(\Bbbk, V_1) \to \mathrm{Hom}_H(\Bbbk, V) \to \mathrm{Hom}_H(\Bbbk, V/V_1) \to \mathrm{Ext}^1_H(\Bbbk, V_1) \to \dots .$$

Note that (G, H) is a good pair. Thus, V_1 admits a good filtration as an H-module. It follows by Lemma 3.1.2 that we have $\mathrm{Ext}^1_H(\Bbbk, V_1) = 0$. Therefore, from the above exact sequence we obtain the following short exact sequence,

$$0 \to \mathrm{Hom}_H(\Bbbk, V_1) \to \mathrm{Hom}_H(\Bbbk, V) \to \mathrm{Hom}_H(\Bbbk, V/V_1) \to 0.$$

It follows that

$$\dim V^H = \dim V_1^H + \dim (V/V_1)^H = \dim \nabla_G(\lambda^1)^H + \dim (V/V_1)^H.$$

By the induction hypothesis we have

$$\dim(V/V_1)^H = \sum_{\lambda \in X^+(T_G)} \bar{m}_\lambda \dim \nabla_G(\lambda)^H,$$

where $\bar{m}_\lambda = (V/V_1 : \nabla_G(\lambda))$ for $\lambda \in X^+(T_G)$. It follows that

$$\dim V^H = \sum_{\lambda \in X^+(T_G)} m_\lambda \dim \nabla_G(\lambda)^H,$$

where $m_\lambda = \bar{m}_\lambda + 1$ if $\lambda = \lambda^1$; otherwise $m_\lambda = \bar{m}_\lambda$. □

Let V and W be G-modules with good filtrations. We require methods of describing a good filtration of the tensor product $V \otimes W$. To this end, we require a set of tools for calculating these filtrations. Pieri's formula gives a formula for the $\mathrm{GL}_7(\Bbbk)$-filtration of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \otimes S^r E$, where E is the natural $\mathrm{GL}_7(\Bbbk)$-module.

Lemma 3.1.4 (Pieri's Formula). *Let $\lambda \in \Lambda^+(7)$ and let $r > 0$. Then the G-module $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \otimes S^r E$ has a good filtration with sections $\nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)$, where $\mu \in \Lambda^+(7)$ are partitions such that $|\mu| = |\lambda| + r$ and $\mu_{i+1} \leq \lambda_i$ for $1 \leq i \leq l(\lambda)$.*

Proof. See [1, IV Corollary 2.6]. □

The Littlewood-Richardson rule is a more general form of Pieri's rule, allowing us to calculate the $\mathrm{GL}_7(\Bbbk)$-filtration of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)$, for $\lambda, \mu \in \Lambda^+(7)$. Recall that a Littlewood-Richardson tableau is a standard tableau whose word is a lattice permutation.

Theorem 3.1.5 (Littlewood-Richardson Rule). *Let $\lambda, \mu \in \Lambda^+(T_G)$. Then the G-module $\nabla_G(\lambda) \otimes \nabla_G(\mu)$ has a good filtration with sections $\nabla_G(\nu)$, where $\nu \in \Lambda^+(T_G)$, and the filtration multiplicities are given by Littlewood-Richardson coefficients $(\nabla_G(\lambda) \otimes \nabla_G(\mu) : \nabla_G(\nu)) = c^\nu_{\lambda,\mu}$.*

Furthermore, the Littlewood-Richardson coefficients $c^{\nu}_{\lambda,\mu}$ *are given by*

$$c^{\nu}_{\lambda,\mu} = \#\{T \mid T \textit{ is an LR-tableau with shape } \nu/\lambda \textit{ and weight } \mu\}.$$

Proof. See [5, Chapter 5]. □

Finally, if V is a G_2-module with a good G_2-filtration, we require a tool for calculating the character of the module $\nabla_{G_2}(\xi) \otimes V$, where $\xi \in X^+(T_{G_2})$.

We recall some terminology. For a fixed maximal torus T_G of G, let $X(T_G)$ be the character group of T_G and let W be the corresponding Weyl group. Let Φ^+ be the set of positive roots with respect to a base Δ, and let the dominant weight ρ be

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

The dot action of W on E is given by

$$w \cdot \lambda = w(\lambda + \rho) - \rho. \tag{3.1}$$

Lemma 3.1.6 (Brauer's Formula)**.** *Let* $\lambda \in X^+(T_G)$ *and let* V *be a* G*-module with formal character* $\operatorname{ch} V = \sum_{\mu \in X(T_G)} a_\mu X^\mu$.

(i)

$$\chi_G(\lambda)\operatorname{ch} V = \sum_{\mu \in X(T_G)} a_\mu \chi_G(\lambda + \mu).$$

(ii) For $\sigma \in X(T_G)$ *we have* $\chi_G(w \cdot \sigma) = \operatorname{sgn}(w)\chi_G(\sigma)$ *for* $w \in W$.

(iii) If $(\sigma, \alpha) = 0$ *for some simple root* $\alpha \in \Delta$*, then* $\chi_G(\sigma) = 0$.

Proof. See [4, Chapter 2]. □

Note that in the formula in part (i) that we may obtain characters $\chi_G(\lambda + \mu)$ such that $\lambda + \mu$ is not a dominant weight. Parts (ii) and (iii) give us an expression

$$\chi_G(\lambda + \mu) = \operatorname{sgn}(w)\chi_G(\xi),$$

where $\xi \in X^+(T_G)$ and w is some element of W such that $w \cdot \xi = \lambda + \mu$.

3.2 TOOLS FOR CALCULATING INVARIANTS

Now, we wish to be able to calculate the dimension of $(\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu))^{G_2}$, for $\lambda, \mu \in X^+(7)$. The comparison tool allows us to calculate this dimension by comparing the good G_2-filtrations of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$ and $\nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)$.

Lemma 3.2.1 (Comparison Tool)**.** *Let $\lambda, \mu \in X^+(7)$. Then*

$$\dim(\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu))^{G_2} = \sum_{\xi \in X^+(T_{G_2})} m_{\lambda,\xi} m_{\mu,\xi},$$

where $m_{\lambda,\xi} = (\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) : \nabla_{G_2}(\xi))$ and $m_{\mu,\xi} = (\nabla_{\mathrm{GL}_7(\Bbbk)}(\mu) : \nabla_{G_2}(\xi))$.

Proof. Since $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$ and $\nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)$ have good G_2-filtrations, we have that the tensor product $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)$ has a good G_2-filtration with sections $\nabla_{G_2}(\xi) \otimes \nabla_{G_2}(\zeta)$, where $\xi, \zeta \in X^+(T_{G_2})$ such that $m_{\lambda,\xi} \neq 0$ and $m_{\lambda,\zeta} \neq 0$. By Lemma 3.1.1 we have

$$\dim \mathrm{Hom}_{G_2}(\Bbbk, \nabla_{G_2}(\xi) \otimes \nabla_{G_2}(\zeta)) = \begin{cases} 1, & \text{if } \xi = \zeta; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we deduce the desired formula. $\square$

A key result is the partition tool, which permits us to restrict our study to $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$, for $\lambda \in \Lambda^+(7)$. In fact, it allows us to focus on partitions of at most six parts. Recall that we denote by $\det_n$ the determinant module $\nabla_{\mathrm{GL}_n(\Bbbk)}(1^n)$ of $\mathrm{GL}_n(\Bbbk)$.

Lemma 3.2.2 (Partition Tool)**.** *Let $\lambda \in X^+(n)$. Then we have the following isomorphisms of $\mathrm{GL}_n(\Bbbk)$-modules.*

(i) For $r \geq 0$ we have

$$\nabla_{\mathrm{GL}_n(\Bbbk)}(\lambda) \otimes \det_n^{\otimes r} \cong \nabla_{\mathrm{GL}_n(\Bbbk)}(\lambda_1 + r, \dots, \lambda_n + r).$$

(ii) If $\lambda_n < 0$ then

$$\nabla_{\mathrm{GL}_n(\Bbbk)}(\lambda) \otimes \det_n^{\otimes -\lambda_n} \cong \nabla_{\mathrm{GL}_n(\Bbbk)}(\lambda_1 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n).$$

(iii) If $\lambda_n > 0$ then

$$\nabla_{\mathrm{GL}_n(\Bbbk)}(\lambda) \cong \nabla_{\mathrm{GL}_n(\Bbbk)}(\lambda_1 - \lambda_n, \dots, \lambda_{n-1} - \lambda_n) \otimes \det_n^{\otimes \lambda_n}.$$

Proof. Part (i) follows from the tensor identity for algebraic groups [10, I Proposition 2.6]. Parts (ii) and (iii) follow immediately from part (i). □

Corollary 3.2.3. *Let $\lambda \in X^+(7)$. Then there exists a partition $\mu \in \Lambda^+(7)$ of at most six parts such that there is an isomorphism of G_2-modules*

$$\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \cong \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu).$$

Proof. Assume that $\lambda_7 < 0$. Then by Lemma 3.2.2 part (ii) we have an isomorphism of $\mathrm{GL}_7(\Bbbk)$-modules

$$\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \otimes \det_7^{\otimes -\lambda_7} \cong \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda_1 - \lambda_7, \dots, \lambda_6 - \lambda_7).$$

As a G_2-module, the determinant module $\det_7$ is isomorphic to the trivial G_2-representation $\nabla_{G_2}(0)$. Therefore, restricting to G_2 we have an isomorphism of G_2-modules

$$\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \cong \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda_1 - \lambda_7, \dots, \lambda_6 - \lambda_7).$$

Assume now that $\lambda_7 > 0$. Then by Lemma 3.2.2 part (iii) we have an isomorphism of $\mathrm{GL}_7(\Bbbk)$-modules

$$\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \cong \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda_1 - \lambda_7, \dots, \lambda_6 - \lambda_7) \otimes \det_7^{\otimes \lambda_7}.$$

It follows in the same way as the previous case that we obtain an isomorphism of G_2-modules

$$\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \cong \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda_1 - \lambda_7, \dots, \lambda_6 - \lambda_7).$$

□

3.3 THE TYPE **B** PROCEDURE OF KOIKE AND TERADA

The final tool that we use is the type **B** procedure of Koike and Terada [12], which provides a tool for calculating $\mathrm{SO}_7(\Bbbk)$-filtrations of induced $\mathrm{GL}_7(\Bbbk)$-modules). The type **B** procedure defines a surjective map

$$\pi : \Lambda^+(7) \to \Lambda^+(3) \cup \{\emptyset\},$$

$$\mu \mapsto \mu^{\#},$$

which is defined by the following algorithm.

(i) If $l(\mu) \leq 3$ then $\mu^{\#} = \mu$ and $\mathrm{sgn}(\mu^{\#}) = +1$.

(ii) Suppose $l(\mu) > 3$. Then consider its transpose partition $\mu' = (\mu'_1, \dots, \mu'_k)$. Define the tuple $t = (t_1, \dots, t_k)$ as follows.

- If $\mu'_i - (i-1) \leq 3$ then set $t_i = \mu'_i - (i-1)$.
- otherwise set $t_i = 7 - (\mu'_i - (i-1))$.

If $t_i = t_j$ for any $i \neq j$ then set $\mu^{\#} = \emptyset$. Otherwise, let $\tilde{t}$ be the tuple obtained by reordering the parts of t such that $\tilde{t}_i > \tilde{t}_{i+1}$ for all $i < k$. Denote by σ_t the permutation which maps t to $\tilde{t}$ by the natural action of the symmetric group S_k on t.

(iii) If $\mu^{\#} \neq \emptyset$ then set $\xi = (\xi_1, \dots, \xi_k)$ where $\xi_i = \tilde{t}_i + (i-1)$.

(iv) Finally, if $\mu^{\#} = \emptyset$ then set $\chi_{\mathrm{SO}_7(\Bbbk)}(\emptyset) = 0$ and $\mathrm{sgn}(\emptyset) = 1$. Otherwise, set $\mu^{\#} = \xi'$ and set $\mathrm{sgn}(\mu^{\#}) = +1$ if the transposition σ_t is even or -1 otherwise.

Theorem 3.3.1. *Let $\lambda \in \Lambda^+(7)$. Then*

$$\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\lambda) = \sum_{2\nu, \mu \subseteq \lambda} c^{\lambda}_{2\nu,\mu} \mathrm{sgn}(\mu^{\#}) \chi_{\mathrm{SO}_7(\Bbbk)}(\mu^{\#}),$$

where μ is a sub-partition of λ, 2ν is an even sub-partition (a partition whose parts are even) of λ, the numbers $c^{\lambda}_{2\nu,\mu}$ are Littlewood-Richardson coefficients, and $\mu^{\#}$ is the image of μ under the map π defined above.

Proof. We obtain this result as a consequence of [12, Proposition 2.5.1 (1)]. □

Using the preceding theorem, we may explicitly calculate the $\mathrm{SO}_7(\Bbbk)$-filtration of the restriction of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$, for $\lambda \in \Lambda^+(7)$. Combining this result with the corollary to the partition tool in Lemma 3.2.2, we may completely study $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ for $\lambda \in X^+(7)$ by studying $\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2}$ for $\mu \in X^+(T_{\mathrm{SO}_7(\Bbbk)})$.

3.4 ROADMAP FOR THE THESIS

Equipped with the full list of tools described in this chapter, we have a firm plan of how to tackle the problem of calculating $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ for $\lambda \in X^+(7)$.

1. Calculate $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ for all partitions λ of at most three parts.

2. With this calculation, employ the type **B** procedure of Theorem 3.3.1 to calculate
$$\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2}$$
for all $\mu \in X^+(T_{\mathrm{SO}_7(\Bbbk)})$.
3. For partitions λ of at most 6 parts, calculate the filtration multiplicities $(\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) : \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu))$ of the $\mathrm{SO}_7(\Bbbk)$-filtration of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$ for all sections $\nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)$ with non-zero G_2-invariants.
4. Observe that by Lemma 3.2.2, this is sufficient to obtain an answer for any $\lambda \in X^+(7)$.

— 4 —

Partitions of at most three parts

We begin by directly calculating $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ where λ is a partition of at most three parts. We achieve this by calculating the filtration multiplicity of the trivial module $\nabla_{G_2}(0,0)$ in the good G_2-filtration of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$.

4.1 PARTITIONS OF ONE PART

Let $\lambda = (r)$. Then we have $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) = S^r E$, the r-th symmetric power of the natural $\mathrm{GL}_7(\Bbbk)$-module E.

Lemma 4.1.1. *Let $r \in \mathbb{N}$ with $r \geq 2$. Then there exists an exact sequence of G_2-modules*

$$0 \to S^{r-2}E \to S^r E \to \nabla_{G_2}(r,0) \to 0.$$

Proof. By the formula of Koike and Terada given in Theorem 3.3.1, we have an exact sequence of $\mathrm{SO}_7(\Bbbk)$-modules,

$$0 \to S^{r-2}\, E \to S^r\, E \to \nabla_{\mathrm{SO}_7(\Bbbk)}(r) \to 0.$$

Restricting to the closed subgroup G_2 of $\mathrm{SO}_7(\Bbbk)$ we have a short exact sequence of G_2-modules. Since $(\mathrm{SO}_7(\Bbbk), G_2)$ is a good pair over fields of odd characteristic by Proposition 2.5.11(ii), we have that $\nabla_{\mathrm{SO}_7(\Bbbk)}(r)$ has highest T_{G_2}-weight $(r,0)$ and thus contains the induced G_2-module $\nabla_{G_2}(r,0)$ as a section of its G_2-filtration. By comparing the dimensions of $\nabla_{\mathrm{SO}_7(\Bbbk)}(r)$ and $\nabla_{G_2}(r,0)$ using Weyl's dimension formula from Lemma 2.5.7, we see that $\nabla_{G_2}(r,0)$ is found as a section of a G_2-module of the same dimension. We conclude that as G_2-modules, $\nabla_{\mathrm{SO}_7(\Bbbk)}(r) \simeq \nabla_{G_2}(r,0)$, and we obtain the short exact sequence of G_2-modules,

$$0 \to S^{r-2} E \to S^r E \to \nabla_{G_2}(r,0) \to 0.$$

$\square$

Lemma 4.1.2. *We have*

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r)^{G_2} = \begin{cases} 1, & \textit{if } r \textit{ is even;} \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We proceed by induction on r. The base case of $r = 0$ is immediate, as $S^0 E \simeq \Bbbk$. As $\mathrm{SO}_7(\Bbbk)$ is an irreducible spherical subgroup of $\mathrm{GL}_7(\Bbbk)$ and G_2 is an irreducible spherical subgroup of $\mathrm{SO}_7(\Bbbk)$ in odd characteristic by Proposition 2.6.1, the natural $\mathrm{GL}_7(\Bbbk)$-module $E \simeq \nabla_{\mathrm{SO}_7(\Bbbk)}(1)$ is irreducible as a G_2-module and hence $\dim E^{G_2} = 0$.

Suppose $r \geq 2$. By Lemma 4.1.1 we have a short exact sequence of G_2-modules

$$0 \to S^{r-2} E \to S^r E \to \nabla_{G_2}(r,0) \to 0.$$

Recall that $(\mathrm{GL}_7(\Bbbk), G_2)$ is a good pair by Corollary 2.5.12. Therefore, $\nabla_{\mathrm{GL}_7(\Bbbk)}(r)$ admits a good filtration as a G_2-module. Then we apply the functor $\mathrm{Hom}_{G_2}(\Bbbk, -)$ to the above exact sequence to obtain

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r)^{G_2} = \dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r-2)^{G_2} + \dim \nabla_{G_2}(r,0)^{G_2}.$$

Note by Proposition 2.5.4 that $\dim \nabla_{G_2}(r,0)^{G_2} = 0$. Therefore, we have

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r)^{G_2} = \dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r-2)^{G_2},$$

where by the induction hypothesis we have

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r-2)^{G_2} = \begin{cases} 1, & \text{if } r \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$$

The result follows. $\square$

4.2 PARTITIONS OF TWO PARTS

To calculate $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s)^{G_2}$ for a partition $(r,s) \in \Lambda^+(7)$, we view $\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s)$ as a $\mathrm{GL}_7(\Bbbk)$-filtration section of $S^rE \otimes S^sE$ and apply Lemma 4.1.2 to calculate the dimension of $(S^rE \otimes S^sE)/\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s)$. Lemma 3.1.4 gives us the good $\mathrm{GL}_7(\Bbbk)$-filtration of $S^rE \otimes S^sE$.

Lemma 4.2.1. *Let* $(r,s) \in \Lambda^+(7)$. *Then*

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s)^{G_2} = \begin{cases} 1, & \text{if } r \equiv s \equiv 0 \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Applying Lemma 3.1.4, we obtain an exact sequence of $\mathrm{GL}_7(\Bbbk)$-modules,

$$0 \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \to S^rE \otimes S^sE \to X \to 0,$$

where X has a good $\mathrm{GL}_7(\Bbbk)$-filtration with sections

$$\nabla_{\mathrm{GL}_7(\Bbbk)}(r+s)|\nabla_{\mathrm{GL}_7(\Bbbk)}(r+s-1,1)|\cdots|\nabla_{\mathrm{GL}_7(\Bbbk)}(r+1,s-1).$$

The fixed point functor $(-)^{G_2}$ is left exact, so applying the fixed point functor to the above exact sequence yields the exact sequence,

$$0 \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s)^{G_2} \to (\nabla_{\mathrm{GL}_7(\Bbbk)}(r) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(s))^{G_2} \to X^{G_2} \to \dots.$$

By Lemma 3.1.2, we obtain the short exact sequence

$$0 \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s)^{G_2} \to (\nabla_{\mathrm{GL}_7(\Bbbk)}(r) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(s))^{G_2} \to X^{G_2} \to 0.$$

We proceed by induction on s.

The case where $s = 0$ is given in Lemma 4.1.2. If $s = 1$ then by Lemma 3.1.4 we have a short exact sequence of $\mathrm{GL}_7(\Bbbk)$-modules

$$0 \to \nabla_{GL_7(k)}(r,1) \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r) \otimes E \to \nabla_{GL_7(k)}(r+1) \to 0.$$

Now, by Lemma 4.1.2 we have that

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r+1)^{G_2} = \begin{cases} 1, & \text{if } r \text{ is odd;} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore by Lemma 3.1.3 we deduce that $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r,1)^{G_2} = 0$ for any $r > 0$.

Now let $s \leq r$ such that $\left\lfloor \frac{s}{2} \right\rfloor = d$ and suppose for $t < s$ that

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r,t)^{G_2} = \begin{cases} 1, & \text{if } r \equiv t \equiv 0 \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 3.2.1 we have

$$\dim \left(\nabla_{\mathrm{GL}_7(\Bbbk)}(r) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(s)\right)^{G_2} = \begin{cases} d+1, & \text{if } r \equiv s \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

Note also that X has a good $\mathrm{GL}_7(\Bbbk)$-filtration by Pieri's formula. Note that by Lemma 3.1.4 that $\nabla_{\mathrm{GL}_7(\Bbbk)}(r) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(s)$ has a good $\mathrm{GL}_7(\Bbbk)$-filtration with sections $\nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)$, where $\mu = (\mu_1, \mu_2)$ is a partition such that $|\mu| = r + s$ and $\mu_2 \leq s$. It then follows that X has a good filtration with sections $\nabla_{\mathrm{GL}_7(\Bbbk)}(r+s-t, t)$, where $0 \leq t < s$. In particular, if $r \equiv s \equiv 1 \pmod 2$ then X has the filtration sections $\nabla_{\mathrm{GL}_7(\Bbbk)}(r+1+2a, s-1-2a)$ where both parts are even, for $0 \leq a \leq d$; if $r \equiv s \equiv 0 \pmod 2$ then X has filtration sections $\nabla_{\mathrm{GL}_7(\Bbbk)}(r+2a, s-2a)$ where both parts are even, for $1 \leq a \leq d$. If $r \not\equiv s \pmod 2$ then X has no filtration sections of the form $\nabla_{\mathrm{GL}_7(\Bbbk)}(a,b)$ where both a and b are even. We conclude by the induction hypothesis that

$$\dim X^{G_2} = \begin{cases} d+1, & \text{if } r \equiv s \equiv 1 \pmod 2; \\ d, & \text{if } r \equiv s \equiv 0 \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore we have

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s)^{G_2} = \begin{cases} 1, & \text{if } r \equiv s \equiv 0 \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

$\square$

4.3 PARTITIONS OF THREE PARTS

Let $(r,s,t) \in \Lambda^+(7)$. We may view $\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s,t)$ as a filtration section of

$$\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes S^t(E).$$

By Lemma 3.1.4 we obtain a short exact sequence of $\mathrm{GL}_7(\Bbbk)$-modules,

$$0 \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s,t) \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(t) \to X \to 0.$$

The module X has a good $\mathrm{GL}_7(\Bbbk)$-filtration with sections $\nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)$ corresponding to dominant weights μ of degree $r+s+t$ and length at most 3, such that $s \leq \mu_2 \leq r$ and $\mu_3 < t$.

For legibility, throughout this section we use the notation $\chi(\mu)$ for the character $\chi_{G_2}^{\mathrm{GL}_7(\Bbbk)}(\mu)$.

By the exactness of the fixed-point functor by Lemma 3.1.2, we obtain the following short exact sequence,

$$0 \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s,t)^{G_2} \to (\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(t))^{G_2} \to X^{G_2} \to 0.$$

By Lemma 3.2.1 and Lemma 4.1.2 we know that the dimension of the G_2-invariant space $(\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(t))^{G_2}$ is equal to the number of filtration sections in the G_2-filtration of $\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s)$ of the form $\nabla_{G_2}(a,0)$ where $a \equiv t \pmod 2$ and $a \leq t$. It is important that we count only weights where $a \leq t$, as the symmetric power S^tE has highest T_{G_2}-weight $(t,0)$.

Lemma 4.3.1. *Let $s \in \mathbb{N}$.*

1. *If $r \geq s+2$ then*

$$\chi(r,s) = \chi_{G_2}(r,0)\chi(s) - \chi_{G_2}(r+1,0)\chi(s-1) + \chi(r-2,s).$$

2. *If $s \geq 2$ then*

$$\chi(r,s) = \chi(r)\chi_{G_2}(s,0) - \chi(r+1)\chi_{G_2}(s-1,0) + \chi(r,s-2).$$

Proof. By Lemma 3.1.4 we obtain a short exact sequence of $\mathrm{GL}_7(\Bbbk)$-modules,

$$0 \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(s) \to X \to 0,$$

where X has the good $\mathrm{GL}_7(\Bbbk)$-filtration with sections $\nabla_{\mathrm{GL}_7(\Bbbk)}(r+s)|\nabla_{\mathrm{GL}_7(\Bbbk)}(r+s-1,1)|\cdots|\nabla_{\mathrm{GL}_7(\Bbbk)}(r+1,s-1)$. A second application of Pieri's formula yields the exact sequence,

$$0 \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r+1,s-1) \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r+1) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(s-1) \to Y \to 0,$$

where Y has the good $\mathrm{GL}_7(\Bbbk)$-filtration $\nabla_{\mathrm{GL}_7(\Bbbk)}(r+s)|\cdots|\nabla_{\mathrm{GL}_7(\Bbbk)}(r+2,s-2)$. We take the formal character of all terms to obtain

$$\chi(r)\chi(s) = \chi(r,s) + \chi(r+1)\chi(s-1).$$

Rearranging this sum we obtain a formula for $\chi(r,s)$,

$$\chi(r,s) = \chi(r)\chi(s) - \chi(r+1)\chi(s-1).$$

Suppose $r \geq s+2$. By Lemma 4.2.1, the above equation has the expression,

$$\begin{aligned}\chi(r,s) =& \chi_{G_2}(r,0)\chi(s) - \chi_{G_2}(r+1,0)\chi(s-1)\\ &+ \chi(r-2)\chi(s) - \chi(r-1)\chi(s-1)\\ =& \chi_{G_2}(r,0)\chi(s) - \chi_{G_2}(r+1,0)\chi(s-1) + \chi(r-2,s).\end{aligned}$$

If $s \geq 3$ then we have the expression

$$\begin{aligned}\chi(r,s) =& \chi(r)\chi_{G_2}(s,0) - \chi(r+1)\chi_{G_2}(s-1,0)\\ &+ \chi(r)\chi(s-2) - \chi(r+1)\chi(s-3)\\ =& \chi(r)\chi_{G_2}(s,0) - \chi(r+1)\chi_{G_2}(s-1,0) + \chi(r,s-2).\end{aligned}$$

If $s = 2$ then

$$\chi(r,s) = \chi(r)\chi_{G_2}(2,0) - \chi(r+1)\chi_{G_2}(1,0) + \chi(r).$$

□

Let W_G be the Weyl group of an algebraic group G. Recall the dot action of W_G on $X(T_G)$ as given in equation 3.1. Let W be the Weyl group of G_2. Taking ρ to be half the sum of positive T_{G_2}-weights, we obtain the dot action of W on $X(T_{G_2})$:

$$w \cdot (a,b) = w(a+1,b+1) - (1,1).$$

In order to calculate the filtration multiplicity of G_2-modules $\nabla_{G_2}(a,0)$ in the G_2-filtration of $\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s)$, we must count the number of filtration sections $\nabla_{G_2}(\sigma)$, where σ is a T_{G_2}-weight in the W-orbit of $(a,0)$ under the dot action.

Recall that we may use Lemma 3.1.6 to obtain the sections for a good $\mathrm{GL}_7(\Bbbk)$-filtration of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)$ for $\lambda, \mu \in X^+(7)$.

By Lemma 3.1.6 we have

$$\chi_{G_2}(r)\chi(s) = \sum_{\xi \in X(T_{G_2})} a_\xi \chi_{G_2}((r) + \xi),$$

where $\chi(s) = \sum_{\lambda \in X(T_{G_2})} a_\lambda X^\lambda$. Recall that $(r) + \xi$ is not necessarily a dominant T_{G_2}-weight for all $\xi \in X(T_{G_2})$. However, from Lemma 3.1.6 parts (ii) and (iii) we can express $\chi_{G_2}((r,0) + \xi)$ as $\pm\chi_{G_2}(\zeta)$ for some dominant T_{G_2}-weight ζ, or 0. This requires us to know the orbit of the dominant T_{G_2}-weight $(a,0)$ under the dot action of W. The orbit of this weight is given in the following table.

w	$w \cdot (a,0)$
e	$(a,0)$
s_α	$(-a-2, a+1)$
s_β	$(a+3, -2)$
$s_\alpha s_\beta$	$(-a-5, a+2)$
$s_\beta s_\alpha$	$(2a+4, -a-3)$
$s_\alpha s_\beta s_\alpha$	$(-2a-6, a+2)$
$s_\beta s_\alpha s_\beta$	$(2a+4, -a-4)$
$(s_\alpha s_\beta)^2$	$(-2a-6, a+1)$
$(s_\beta s_\alpha)^2$	$(a+3, -a-4)$
$s_\alpha(s_\beta s_\alpha)^2$	$(-a-5, 0)$
$s_\beta(s_\alpha s_\beta)^2$	$(a, -a-3)$
$(s_\alpha s_\beta)^3$	$(-a-2, -2)$

From this we can identify those weights in the character $\chi_{G_2}(r,0)\chi(s)$ which lie in the W-orbit of a given weight $(a,0)$.

Proposition 4.3.2. *Let $r \geq s+2$. In the sum $\chi_{G_2}(r,0)\chi(s)$, the weights in the W_{G_2}-orbit of the dominant weight $(a,0)$ are*

$$\begin{cases} \{(a,0),(a+3,-2)\}, & \textit{if } r-s < a < r+s, \\ \{(a,0)\}, & \textit{if } a = r \pm s \end{cases}$$

Proof. The natural $\mathrm{GL}_7(\Bbbk)$-module E has $\Bbbk$-basis $x_1, \ldots, x_7$ with respective T_{G_2}-weights

$$(1,0),\ (-1,1),\ (2,-1),\ (0,0),\ (-2,1),\ (1,-1),\ (-1,0).$$

It follows that $\nabla_{\mathrm{GL}_7(\Bbbk)}(s)$ has basis elements $\Pi_{n=1}^7 x_n^{i_n}$, where $\sum_{n=1}^7 i_n = s$. These basis elements have T_{G_2}-weights (u,v), where $-2s \leq u \leq 2s$ and $-s \leq v \leq s$.

For $v = 0$, the first coordinate u ranges from $-s$ to s. Then by Lemma 3.1.6, in the character sum $\chi_{G_2}(r)\chi(s)$ we find the characters $\chi_{G_2}(a,0)$ corresponding to T_{G_2}-weights $(a,0)$ for $r-s \le a \le r+s$.

Now suppose $v < 0$ is fixed. Then we have $-2v - s \le u \le -v + s$. In particular when $v = -2$ we have that $4 - s \le u \le 2 + s$. Thus in the G_2 character sum representing $\chi_{G_2}(r)\chi(s)$ we find characters $\chi_{G_2}(a+3,-2)$ corresponding to T_{G_2}-weights $(a+3,-2)$ for $r-s+1 \le a \le r+s-1$. For all other b with $-(a+b) = v < 0$ we have $2a+2b+r-2 \le r+u$, and so we cannot have another weight in the W-orbit of $(a,0)$ with negative second coordinate.

Now suppose $v > 0$ is fixed. We have $-v - s \le u \le -2v + s$. In particular for $v = a+b$ for some b, we have $r-s-a-b \le r+u$. Since $r \ge s+2$, we see that weights in the W-orbit with positive second coordinate do not appear in $\chi_{G_2}(r)\chi(s)$. $\square$

Note that by Lemma 3.1.6, we have $\chi_{G_2}(a+3,-2) = -\chi_{G_2}(a,0)$. We find the total multiplicity of $\chi_{G_2}(r+a,0)$ as the difference between the multipicities of $X^{(a,0)}$ and $X^{(a+3,-2)}$ in the formal character $\chi(s)$. henceforth, we denote by m_i the binomial coefficient $\binom{i+1}{2} = \frac{i(i+1)}{2}$.

Lemma 4.3.3. *Let $s \in \mathbb{N}$. For $-s \le u \le s$, let δ_u denote the difference between the multiplicities of $X^{(u,0)} and X^{(u+3,-2)}$ in the formal T_{G_2}-character $\chi(s)$.*

1. *Let $s = 2k$. Then*

$$\delta_u = \begin{cases} m_{k+1}, & \text{if } u = 0; \\ m_{i+1}, & \text{if } u = \pm(s-2i) \text{ or } \pm(s-2i-1) \text{ for some } i < k. \end{cases}$$

2. *Let $s = 2k+1$. Then*

$$\delta_u = m_{i+1}, \text{ if } u = \pm(s-2i) \text{ or } \pm(s-2i-1) \text{ for some } i \le k.$$

Proof. Recall that the natural $\mathrm{GL}_7(\Bbbk)$-module E has basis $x_1, \ldots, x_7$. Recall also that these basis elements the the respective T_{G_2}-weights

$$(1,0), (-1,1), (2,-1), (0,0), (-2,1), (1,-1), (-1,0).$$

Now, let $x = \prod_{n=1}^{7} x_n^{i_n}$, where $\sum_{n=1}^{7} i_n = s$. Then $x \in \nabla_{\mathrm{GL}_7(\Bbbk)}(s)$. Suppose x has T_{G_2}-weight (a,b). To this element we can associate the tuple $\underline{i} = (i_1, \ldots, i_7)$ where

$\sum_{k=1}^{7} i_k = s$. The weight of this tuple can be expressed in the form

$$\begin{aligned} &i_1(1,0) + i_2(-1,1) + i_3(2,-1) + i_4(0,0) + i_5(-2,1) + i_6(1,-1) + i_7(-1,0) \\ &= (i_1 - i_2 + 2(i_3 - i_5) + i_6 - i_7, i_2 - i_3 + i_5 - i_6). \end{aligned}$$

We prove the result for non-negative u since by symmetry we obtain the multiplicities for negative u. Let $I_{(u,0)}$ and $I_{(u+3,-2)}$ be the sets of tuples corresponding to elements of T_{G_2}-weight $(u,0)$ and $(u+3,-2)$ respectively. For $u = s$ exactly one element, namely x_1^s, corresponds to the weight $(u,0)$. For $0 \leq u < s$ we proceed in two steps. First we provide an injective map ϕ from $I_{(u+3,-2)}$ into $I_{(u,0)}$. We then find the cardinality of $I_{(u,0)} \setminus \operatorname{Im}\phi$.

Suppose $\underline{i}$ is a tuple corresponding to the weight $(u+3,-2)$. Since $v = -2$, we must have that either i_3 or i_6 is positive. We define the map $\phi : I_{(u+3,-2)} \to I_{(u,0)}$ pointwise. If $i_3 > 0$ then we denote by $\underline{i}' = (i'_1, \ldots, i'_7)$ the image of $\underline{i}$ under ϕ,

$$\underline{i}' = \phi((i_1, \ldots, i_7)) = (i_1, i_2 + 1, i_3 - 1, i_4, i_5, i_6, i_7),$$

which has T_{G_2}-weight

$$\begin{aligned} &i_1(1,0) + (i_2+1)(-1,1) + (i_3-1)(2,-1) + i_4(0,0) + i_5(-2,1) + i_6(1,-1) + i_7(-1,0) \\ &= (i_1 - (i_2+1) + 2((i_3-1) - i_5) + i_6 - i_7, i_2 + 1 - (i_3 - 1) + i_5 - i_6) \\ &= (i_1 - i_2 + 2(i_3 - i_5) + i_6 - i_7, i_2 - i_3 + i_5 - i_6) + (-3, 2) \\ &= (u, 0). \end{aligned}$$

If $i_3 = 0$ then we have $i_6 > 0$. Then set $\underline{i}'$ to be

$$\underline{i}' = \phi((i_1, \ldots, i_7)) = (i_1, 0, i_2, i_4, i_6 - 1, i_5 + 1, i_7),$$

which has T_{G_2}-weight

$$\begin{aligned} &i_1(1,0) + 0(-1,1) + (i_2)(2,-1) + i_4(0,0) + i_5(-2,1) + i_6(1,-1) + i_7(-1,0) \\ &= (i_1 - 0 + 2(i_2 - i_5) + i_6 - i_7, 0 - i_2 + i_5 - i_6) \\ &= (i_1 - i_2 + 2(i_3 - i_5) + i_6 - i_7, i_2 - i_3 + i_5 - i_6) + (-3, 2) \\ &= (u, 0). \end{aligned}$$

Note that in either case, the weight of $\underline{i}'$ is

$$(u + 3 - 3, -2 + 2) = (u, 0).$$

Similarly, we can define a reverse map ψ from a subset of $I_{(u,0)}$ to $I_{(u+3,-2)}$. Suppose $\underline{j} = (j_1, \dots, j_7)$ is a tuple with weight $(u, 0)$ such that $j_2 > 0$. Then we define $\underline{j}'$ to be the image of $\underline{j}$ under ψ,

$$\underline{j}' = \psi(j_1, \dots, j_7) = (j_1, j_2 - 1, j_3 + 1, j_4, j_5, j_6, j_7).$$

If $j_2 = 0$ and $j_6 > 0$, we define $\underline{j}'$ to be

$$\underline{j}' = \psi(j_1, \dots, j_7) = (j_1, j_3, 0, j_4, j_6 - 1, j_5 + 1, j_7).$$

Note that in both cases $\underline{j}'$ has weight

$$(u + 3, -2).$$

We have defined a bijection ϕ from $I_{(u+3,-2)}$ to a subset of $I_{(u,0)}$ with inverse ψ. Thus, we can find the cardinality of $J = I_{(u,0)} \setminus \operatorname{Im}\phi$. A tuple $\underline{i}$ with weight $(u, 0)$ can be paired with a tuple with weight $(u + 3, -2)$ if $i_2 > 0$ or $i_6 > 0$. The elements of J must be tuples $\underline{j}$ where $j_2, j_6 = 0$. Let $\underline{j} \in J$. Using the equation $j_2 - j_3 + j_5 - j_6 = 0$, we must have $j_3 = j_5$. We now have that $j_1 - j_7 = u$, thus $j_1 = j_7 + u$. Since we have $\sum_{n=1}^{7} j_n = s$, then

$$2j_7 + 2j_3 + j_4 + u = s.$$

Our choices of j_3 and j_7 will determine the value of j_4. We may choose j_3 from $0, \dots, \lfloor \frac{s-u}{2} \rfloor$. For fixed j_3 we have $\lfloor \frac{s-u}{2} \rfloor - j_3$ choices or j_7. Therefore the cardinality of J is expressible as the sum

$$\sum_{r=0}^{\lfloor \frac{s-u}{2} \rfloor} (r + 1),$$

which is the binomial coefficient $m_{\lfloor \frac{s-u}{2} \rfloor + 1}$. □

Corollary 4.3.4. *Let $r \geq s + 2$, and let $s = 2l$ or $s = 2l + 1$. Let γ_a denote the multiplicity of $\chi_{G_2}(a, 0)$ in the sum $\chi(r, s) - \chi(r - 2, s)$. Then we have*

$$\gamma_a = \begin{cases} i + 1, & \text{if } a = r + 2i - s,\ i < l; \\ i + 1, & \text{if } a = r + 2i + 1 - s,\ i < l; \\ l + 1, & \text{if } a = r, \text{ or } a = r - 1 \text{ and } s \text{ is odd.} \end{cases}$$

Lemma 4.3.5. *Let $s = 2l$ or $s = 2l + 1$ with $l \geq 1$. Then the partial sum P of characters of the form $\chi_{G_2}(x, 0)$ in the character sum $\chi(s, s) - \chi(s, s - 2)$ is given by*

$$P = \chi_{G_2}(s, 0) - \sum_{j=1}^{l-1} j(\chi_{G_2}(2j, 0) + \chi_{G_2}(2j + 1, 0)),$$

if $s = 2l$, *or*

$$P = \chi_{G_2}(s,0) - \Big(\chi_{G_2}(2l,0) + \sum_{j=1}^{l-1} j(\chi_{G_2}(2j,0) + \chi_{G_2}(2j+1,0))\Big),$$

if $s = 2l + 1$.

Proof. We have by 4.3.1 that

$$\chi(s,s) - \chi(s,s-2) = \chi(s)\chi_{G_2}(s,0) - \chi(s+1)\chi_{G_2}(s-1,0).$$

Considering $\chi(s)\chi_{G_2}(s,0)$, for $0 \leq a \leq 2s$, by Lemma 4.3.3 we have that the difference in multiplicities of $X^{(a,0)}$ and $X^{(a+3,-2)}$ is a binomial coefficient. When s is even this number is $m_{l+1-j} = \binom{l+2-j}{2}$ when a is expressible as $s \pm 2j$ $(j \geq 0)$ or $s \pm (2j-1)$ $(j > 0)$. When s is odd this number is m_{l+1-j} when a is expressible as $s \pm 2j$ or $s \pm (2j+1)$ $(j \geq 0)$.

Now consider the character $\chi(s+1)\chi_{G_2}(s-1,0)$. Again for $0 \leq a \leq 2s$ the difference in multiplicities of the weights $X^{(a,0)}$ and $X^{(a+3,-2)}$ is a binomial coefficient. For $0 \leq a < s$ this binomial coefficient is m_{l+2-j} where the multiplicity in $\chi(s)\chi_{G_2}(s,0)$ would be m_{l+1-j}.

In the sum $\chi(s+1)\chi_{G_2}(s-1,0)$ we also have weights $X^{s_\beta s_\alpha \cdot (a,0)} = X^{(2a+4,-a-3)}$ for $0 \leq a \leq s-2$. Using the same tuple notation that we use in proof of Lemma 4.3.3, these weights correspond to the tuples $(0,0,0,0,0,a+3,s-2-a)$ and all have multiplicity 1. We also find weights $X^{s_\alpha \cdot (a,0)} = X^{(-a-2,a+1)}$ for $0 \leq a \leq s$. For $a = s$ the multiplicity of this weight is 1 corresponding to the tuple $(0,1,0,0,a,0,s-a)$. For $a = s-1$ the multiplicity of this weight is 2 corresponding to the tuples $(0,1,0,0,a,0,s-a)$ and $(0,0,0,1,a+1,0,s-a-1)$. For all other $0 \leq a \leq s-2$ the multiplicity of this weight is 3 corresponding to the tuples $(0,1,0,0,a,0,s-a)$, $(0,0,0,1,a+1,0,s-a-1)$, and $(0,0,0,0,a+2,1,s-a-2)$.

Comparing multiplicities we have for both s odd and s even that the multiplicity of $\chi_{G_2}(a,0)$ is 0 for $a > s$, and the multiplicity of $\chi_{G_2}(s,0)$ is 1. For s even we have that the net multiplicity of $\chi_{G_2}(2j,0)$ and $\chi_{G_2}(2j+1,0)$ is $-j$ for $0 \leq j < k$.

For odd s we have by comparing multiplicities that the net multiplicity of $\chi_{G_2}(2l,0)$ is $-l$. For $0 \leq j < l$ the net multiplicities of $\chi_{G_2}(2j,0)$ and $\chi_{G_2}(2j+1,0)$ are $-j$. $\square$

In order to show that $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s,t)^{G_2} = 0$ when $r \not\equiv s \pmod 2$, we must calculate $\chi(s+1,s)$. We need not calculate the full character explicitly, as a consequence of Lemma 3.2.1.

Lemma 4.3.6. *Let $s = 2l$ (s even) or $s = 2l+1$ (s odd), with $l \in \mathbb{N}$. Then considering $\nabla_{\mathrm{GL}_7(\Bbbk)}(s, s-1)$ as a G_2-module, the character $\chi(s, s-1)$ is expressible as a sum. Let P be the partial sum over all terms $\chi_{G_2}(\mu)$, for those $\mu \in X^+(T_{G_2})$ in the W-orbit of $(a, 0)$ for some $a \geq 0$. Then P has the form*

$$P = \sum_{j=1}^{l} j(\chi_{G_2}(2j, 0) + \chi_{G_2}(2j-1, 0))$$

if s is even, or

$$P = (l+1)\chi_{G_2}(2l+1, 0) + \sum_{j=1}^{l} j(\chi_{G_2}(2j, 0) + \chi_{G_2}(2j-1, 0))$$

if s is odd.

Proof. Let $s = 2$, then by Lemma 3.1.4 we have following exact sequence of $\mathrm{GL}_7(\Bbbk)$-modules,

$$0 \to \nabla_{\mathrm{GL}_7(\Bbbk)}(2, 1) \to S^2E \otimes E \to S^3E \to 0. \tag{4.1}$$

We have by Lemma 4.1.1 that S^3E has G_2-filtration with sections $\nabla_{G_2}(3, 0)|\nabla_{G_2}(1, 0)$. Note also that S^2E has G_2-filtration with sections $\nabla_{G_2}(2, 0)|\nabla_{G_2}(0)$. The formal character of the natural $\mathrm{GL}_7(\Bbbk)$-module E is given by

$$\mathrm{ch}_{G_2} E = X^{(1,0)} + X^{(-1,1)} + X^{(2,-1)} + X^{(0)} + X^{(-2,1)} + X^{(1,-1)} + X^{(-1,0)}.$$

By Lemma 3.1.6 we obtain the T_{G_2}-character of $\nabla_{\mathrm{GL}_7(\Bbbk)}(2) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(1)$ as follows,

$$\begin{aligned}\chi(2)\chi(1) &= (\chi_{G_2}(2, 0) + \chi_{G_2}(0, 0))\mathrm{ch}_{G_2} E \\ &= \chi_{G_2}(3, 0) + \chi_{G_2}(2, 0) + 2\chi_{G_2}(1, 0).\end{aligned}$$

By (4.1), the $\mathrm{GL}_7(\Bbbk)$-character of $\nabla_{\mathrm{GL}_7(\Bbbk)}(2, 1)$ is given by $\chi(2)\chi(1) - \chi(3)$. Therefore we conclude that $\nabla_{\mathrm{GL}_7(\Bbbk)}(2, 1)$ has a G_2-filtration containing the sections $\nabla_{G_2}(2, 0)|\nabla_{G_2}(1, 0)$.

Now suppose $s > 2$. By Lemma 4.3.3 we may write $\chi(s, s-1)$ as the sum

$$\begin{aligned}&\chi_{G_2}(s, 0)\chi(s-1) + \chi(s-2)\chi(s-1) \\ &- \chi_{G_2}(s+1, 0)\chi(s-2) - \chi(s-1)\chi(s-2),\end{aligned}$$

which simplifies to

$$\chi_{G_2}(s, 0)\chi(s-1) - \chi_{G_2}(s+1, 0)\chi(s-2).$$

Suppose $s = 2k$. Then by Lemma 4.3.3 we have the filtration multiplicity

$$(\nabla_{G_2}(s,0) \otimes \nabla_{G_2}^{\mathrm{GL}_7(\Bbbk)}(s-1) : \nabla_{G_2}(a,0))$$

is a binomial coefficient for $1 \leq a \leq 2s-1$. The sum of characters of the sections $\nabla_{G_2}(a,0)$ is given by

$$m_l(\chi_{G_2}(s+1,0) + \chi_{G_2}(s,0) + \chi_{G_2}(s-1,0)$$
$$+ \sum_{i=0}^{l-2} m_{i+1}(\chi_{G_2}(2s-2i-1,0) + \chi_{G_2}(2s-2(i-1),0) + \chi_{G_2}(2i+1,0) + \chi_{G_2}(2(i+1),0)).$$

Similarly we have that the filtration multiplicity $(\nabla_{G_2}(s+1) \otimes \nabla_{G_2}^{\mathrm{GL}_7(\Bbbk)}(s-2) : \nabla_{G_2}(a,0))$ is a binomial coefficient for $3 \leq c \leq 2s-1$. The sum of characters of the sections $\nabla_{G_2}(a,0)$ is given by

$$m_l\chi_{G_2}(s+1,0) + \sum_{i=0}^{l-2} m_{i+1}(\chi_{G_2}(2(i+1)+1,0) + \chi_{G_2}(2(i+2),0)$$
$$+ \chi_{G_2}(2s-2i-1,0) + \chi_{G_2}(2s-2(i+1),0)).$$

Comparing multiplicities we get the final sum

$$\sum_{j=1}^{k} j(\chi_{G_2}(2j,0) + \chi_{G_2}(2j-1,0)).$$

The case for s odd is similar. □

Now that we have the filtration for $\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s)$ as a G_2-module, we may finally calculate the dimension of G_2-invariants of $\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s,t)$.

Theorem 4.3.7. *Let $\lambda = (r,s,t)$ be a partition. Then*

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \begin{cases} \lfloor \frac{t}{2} \rfloor + 1, & \text{if } r \equiv s \equiv t \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The cases where $s = 0$ and $t = 0$ are due to Lemma 4.1.2 and Lemma 4.2.1 respectively. Suppose $t = 1$. By Lemma 3.1.4 we have a short exact sequence of $\mathrm{GL}_7(\Bbbk)$-modules

$$0 \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s,1) \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes E \to X \to 0,$$

where $X \simeq \nabla_{\mathrm{GL}_7(\Bbbk)}(r+1,s)$ if $r = s$, otherwise X has a good $\mathrm{GL}_7(\Bbbk)$-filtration with sections $\nabla_{\mathrm{GL}_7(\Bbbk)}(r+1,s)|\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s+1)$.

If $r \not\equiv s \pmod 2$ and $r \neq s+1$ then by repeated use of Lemma 4.3.1, we have the following,

$$\chi(r,s) = \chi(s+1,s) + \sum_{j=0}^{\lfloor \frac{r-s}{2} \rfloor} (\chi_{G_2}(r-2j,0)\chi(s) - \chi_{G_2}(r-2j+1,0)\chi(s-1)).$$

By Corollary 4.3.4 we have that the multiplicity of $\chi_{G_2}(1,0)$ in the partial sum

$$\sum_{j=0}^{\lfloor \frac{r-s}{2} \rfloor} (\chi_{G_2}(r-2j,0)\chi(s) - \chi_{G_2}(r-2j+1,0)\chi(s-1)),$$

is 0. By Lemma 4.3.6 we have that the term $\chi_{G_2}(1,0)$ appears in the sum of $\chi(s+1,s)$ with multiplicity 1. Using Lemma 3.2.1 we conclude that

$$\dim{(\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes E)}^{G_2} = 1,$$

when $r \not\equiv s \pmod 2$. If $r \equiv s \pmod 2$ then we have $\dim X^{G_2} = 0$. By repeated use of Lemma 4.3.1, we have that the character sum for $\chi(r,s)$ contains the term $\chi_{G_2}(1,0)$ with multiplicity 1 if and only if $r \equiv 1 \pmod 2$. Using Lemma 3.2.1 we have

$$\dim{(\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes E)}^{G_2} = \begin{cases} 1, & \text{if } r \equiv s \equiv 1 \pmod 2; \\ 0, & \text{if } r \equiv s \equiv 0 \pmod 2. \end{cases}$$

Comparing dimensions of these modules we have that

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s,1)^{G_2} = \dim{(\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes E)}^{G_2} - \dim X^{G_2}.$$

Therefore, we conclude that

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s,1)^{G_2} = \begin{cases} 1, & \text{if } r \equiv s \equiv 1 \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

We now proceed by induction on t. Let $k = \left\lfloor \frac{t}{2} \right\rfloor$. For $t \geq 2$, by Lemma 3.1.4 we have a short exact sequence of $\mathrm{GL}_7(\Bbbk)$-modules

$$0 \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s,t) \to \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(t) \to X \to 0,$$

where X has a good $\mathrm{GL}_7(\Bbbk)$-filtration with sections $\nabla_{\mathrm{GL}_7(\Bbbk)}(r+a, s+b, t-(a+b))$ with filtration multiplicity 1, for $0 \leq b \leq (r-s)$ and $1 \leq a+b \leq t$.

If $r \geq s+2$ then $\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(t)$ contains the $\mathrm{GL}_7(\Bbbk)$ filtration for $\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s+2) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(t-2)$. In particular we have

$$\dim\left(\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(t)\right)^{G_2} = \frac{k(k+1)}{2} + \dim\left(\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s+2) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(t-2)\right)^{G_2}.$$

Let $j = \left\lfloor \frac{r-s}{2} \right\rfloor$, then we have

$$\dim\left(\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(t)\right)^{G_2} = \sum_{i=0}^{j} \frac{(k-i)(k-i+1)}{2}.$$

If $r \not\equiv s, \pmod 2$ then by the induction hypothesis we have

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r+2a+1, s, t-(2a+1))^{G_2} = \begin{cases} 1 + \left\lfloor \frac{t-1-2a}{2} \right\rfloor, & \text{if } r \equiv t \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, we have

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r+2a, s+1, t-(2a+1))^{G_2} = \begin{cases} 1 + \left\lfloor \frac{t-1-2a}{2} \right\rfloor, & \text{if } r \not\equiv t \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

If $r > s+2$ then we also observe X has $\mathrm{GL}_7(\Bbbk)$-filtration sections $\nabla_{\mathrm{GL}_7(\Bbbk)}(r+2a+1, s+2, t-(2a+3))$ and $\nabla_{\mathrm{GL}_7(\Bbbk)}(r+2a, s+3, t-(2a+3))$, such that

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r+2a+1, s+2, t-(2a+3))^{G_2} = \begin{cases} 1 + \left\lfloor \frac{t-(2a+3)}{2} \right\rfloor, & \text{if } r \equiv t \pmod 2; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r+2a, s+3, t-(2a+3))^{G_2} = \begin{cases} 1 + \left\lfloor \frac{t-(2a+3)}{2} \right\rfloor, & \text{if } r \not\equiv t \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that when $r \not\equiv s \pmod 2$ we have

$$\dim X^{G_2} = \sum_{i=0}^{j} \frac{(k-i)(k-i+1)}{2}.$$

Comparing dimensions of invariant spaces we have that

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s,t)^{G_2} = 0,$$

whenever $r \not\equiv s \pmod 2$.

Suppose $r \equiv s \pmod 2$. If $r \equiv t \pmod 2$ then X has $\mathrm{GL}_7(\Bbbk)$-filtration sections of the form

$$\nabla_{\mathrm{GL}_7(\Bbbk)}(r+2a, s+2b, t-2(a+b)).$$

By the induction hypothesis we have that

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r+2a, s+2b, t-2(a+b))^{G_2} = \left\lfloor \frac{t}{2} \right\rfloor - (a+b).$$

If $r \not\equiv t \pmod 2$ then X has $\mathrm{GL}_7(\Bbbk)$-filtration sections of the form

$$\nabla_{\mathrm{GL}_7(\Bbbk)}(r+2a+1, s+2b+1, t-2(a+b+1)),$$

if $r \neq s$. By the induction hypothesis we have that

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r+2a+1, s+2b+1, t-2(a+b+1))^{G_2} = \left\lfloor \frac{t}{2} \right\rfloor - (a+b+1).$$

Comparing $\dim \left(\nabla_{\mathrm{GL}_7(\Bbbk)}(r,s) \otimes \nabla_{\mathrm{GL}_7(\Bbbk)}(t)\right)^{G_2}$ against $\dim X^{G_2}$, it follows from the induction hypothesis that

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(r,s,t)^{G_2} = \begin{cases} \left\lfloor \frac{t}{2} \right\rfloor + 1, & \text{if } r \equiv s \equiv t \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

□

— 5 —

Classification for dominant $\mathrm{SO}_7(\Bbbk)$ and $\mathrm{GL}_7(\Bbbk)$-weights

Equipped with Theorem 4.3.7, we utilise the type **B** procedure of Koike and Terada given in Theorem 3.3.1 in order to obtain a complete calculation of $\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2}$ for dominant weights $\mu \in X^+(T_{\mathrm{SO}_7(\Bbbk)})$.

5.1 DOMINANT $\mathrm{SO}_7(\Bbbk)$-WEIGHTS

Lemma 5.1.1. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition of at most three parts. Then*

(i)

$$\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\lambda) = \chi_{\mathrm{SO}_7(\Bbbk)}(\lambda) + \sum_{\substack{\mu, 2\nu \subseteq \lambda \\ \mu \neq \lambda}} c^{\lambda}_{2\nu,\mu} \chi_{\mathrm{SO}_7(\Bbbk)}(\mu),$$

where the coefficients $c^{\lambda}_{2\nu,\mu}$ are the Littlewood-Richardson coefficients.

(ii)

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\lambda)^{G_2} + \sum_{\mu \neq \lambda} m_{\mu} \dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2},$$

where the coefficients m_μ are given by

$$m_\mu = \sum_{2\nu \subseteq \lambda} c^{\lambda}_{2\nu,\mu}.$$

Proof. For part (i) note that by Lemma 3.3.1, we have that the $\mathrm{SO}_7(\Bbbk)$-character of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$ is given by

$$\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\lambda) = \sum_{\mu, 2\nu \subseteq \lambda} c^{\lambda}_{2\nu,\mu} \chi_{\mathrm{SO}_7(\Bbbk)}(\mu).$$

For $\nu = 0$ we have $c^{\lambda}_{0,\lambda} = 1$ and $c^{\lambda}_{0,\mu} = 0$ if $\lambda \neq \mu$. Thus, we obtain the required formula,

$$\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\lambda) = \chi_{\mathrm{SO}_7(\Bbbk)}(\lambda) + \sum_{\substack{\mu, 2\nu \subseteq \lambda \\ \mu \neq \lambda}} c^{\lambda}_{2\nu,\mu} \chi_{\mathrm{SO}_7(\Bbbk)}(\mu).$$

For part (ii), recall from Proposition 2.5.11 that $(\mathrm{GL}_7(\Bbbk), \mathrm{SO}_7(\Bbbk))$ is a good pair when $\mathrm{char}\Bbbk \neq 2$. In particular, the $\mathrm{GL}_7(\Bbbk)$-module $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$ has a good $\mathrm{SO}_7(\Bbbk)$-filtration. Moreover, recall from Proposition 2.5.11 that $(\mathrm{SO}_7(\Bbbk), G_2)$ is a good pair when $\mathrm{char}\Bbbk \neq 2$. Therefore, by Lemma 3.1.3, we have

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{\mu \in X^+(T_{\mathrm{SO}_7(\Bbbk)})} m_\mu \dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2},$$

where m_μ is the filtration multiplicity,

$$m_\mu = (\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) : \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)),$$

for $\mu \in X^+(T_{\mathrm{SO}_7(\Bbbk)})$.

It follows from Lemma 3.1.3 that the filtration multiplicities m_μ are in fact the coefficients of $\chi_{\mathrm{SO}_7(\Bbbk)}(\mu)$ in the $\mathrm{SO}_7(\Bbbk)$-character $\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\lambda)$. Hence by part (i) we conclude that

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\lambda)^{G_2} + \sum_{\mu \neq \lambda} m_\mu \dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2},$$

where the coefficients m_μ are

$$m_\mu = \sum_{2\nu \subseteq \lambda} c^{\lambda}_{2\nu,\mu}.$$

□

With this last tool, we have everything we require to calculate $\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2}$ for any $\mu \in X^+(T_{\mathrm{SO}_7(\Bbbk)})$. Recall from Theorem 2.6.2 that, as G_2 is a spherical subgroup of $\mathrm{SO}_7(\Bbbk)$, then $\nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2}$ is at most one-dimensional for all $\mu \in X^+(T_{\mathrm{SO}_7(\Bbbk)})$.

Theorem 5.1.2. *Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a partition of at most three parts. Then,*

$$\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2} = \begin{cases} 1, & \text{if } \mu_1 = \mu_2 = \mu_3; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $\mu = (0)$ then $\nabla_{\mathrm{SO}_7(\Bbbk)}(0) \cong \Bbbk$ and the result is immediate.

Suppose that μ is a non-zero partition such that $\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2} \neq 0$. By Lemma 5.1.1(ii) we have that $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)^{G_2} \neq 0$. Then by Theorem 4.3.7, we have that $\mu_1 \equiv \mu_2 \equiv \mu_3 \pmod 2$.

If $\mu_3 = 0$ then $\mu = (\mu_1, \mu_2)$ is a partition with even parts and so by Lemma 4.2.1 we have that $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)^{G_2} = 1$. Now, for $2\nu = \mu$ we have that the Littlewood-Richardson constant $c^{\mu}_{\mu,0}$ is equal to 1. Therefore, by Lemma 5.1.1(i) we have that the trivial $\mathrm{SO}_7(\Bbbk)$-module $\nabla_{\mathrm{SO}_7(\Bbbk)}(0)$ appears as a section in the $\mathrm{SO}_7(\Bbbk)$-filtration of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)$ with multiplicity 1. Therefore, by Lemma 5.1.1(i), we conclude that

$$\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu_1, \mu_2)^{G_2} = 0,$$

if $\mu_1 \neq 0$.

Suppose $\mu_3 \neq 0$. If $\mu_3 = 1$ then we have that μ_1 and μ_2 are both odd, and by Theorem 4.3.7 we have

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)^{G_2} = 1.$$

If $\mu = (1,1,1)$ then by Lemma 5.1.1(i) we have

$$\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(1,1,1) = \chi_{\mathrm{SO}_7(\Bbbk)}(1,1,1).$$

It follows from Lemma 5.1.1(ii) that

$$\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(1,1,1)^{G_2} = 1.$$

Suppose $\mu_1 \neq 1$ then $2\nu = (\mu_1 - 1, \mu_2 - 1)$ is an even non-zero partition and for $\kappa = (1,1,1)$ we have

$$c^{\mu}_{2\nu,\kappa} = 1.$$

Then $\nabla_{\mathrm{SO}_7(\Bbbk)}(1,1,1)$ is a filtration section of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)$ with multiplicity 1. It follows from Lemma 5.1.1(i), that

$$\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu_1, \mu_2, 1)^{G_2} = 0,$$

if $\mu_1 \neq 1$.

We proceed by induction on μ_3. By Theorem 4.3.7 we have

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)^{G_2} = \left\lfloor \frac{\mu_3}{2} \right\rfloor.$$

First suppose that $\mu = (\mu_1, \mu_2, \mu_3)$ has 3 equal parts.

Suppose μ_3 is even. Then it is easy to see that for any $2\nu = (2c, 2c, 2c)$ with $c \leq \frac{\mu_3}{2}$ and $\kappa = \mu - (2c, 2c, 2c)$ that $c^{\mu}_{2\nu,\kappa} = 1$. Therefore, for $\kappa = \mu - (2c, 2c, 2c)$, the character $\chi_{\mathrm{SO}_7(\Bbbk)}(\kappa)$ appears as a summand of $\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\mu)$ for $0 \leq c \leq \frac{\mu_3}{2}$. By the induction hypothesis, we have

$$\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\kappa)^{G_2} = 1.$$

Then by Lemma 5.1.1(ii) we have

$$\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu_3, \mu_3, \mu_3)^{G_2} = 1.$$

Now suppose that μ_3 is odd. Then by a similar argument, for $0 \leq c \leq \frac{\mu_3 - 1}{2}$, for $2\nu = (2c, 2c, 2c)$ and $\kappa = \mu - 2\nu$, we have that $c^{\mu}_{2\nu,\kappa} = 1$. Therefore, for $\kappa = \mu - (2c, 2c, 2c)$, we have by Lemma 5.1.1(i) that the character $\chi_{\mathrm{SO}_7(\Bbbk)}(\kappa)$ appears as a summand of $\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\mu)$ for $0 \leq c \leq \frac{\mu_3}{2}$. By the induction hypothesis, we have

$$\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\kappa)^{G_2} = 1.$$

Then by Lemma 5.1.1(ii) we have

$$\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu_3, \mu_3, \mu_3)^{G_2} = 1. \tag{5.1}$$

Now suppose μ is a partition of even parts that does not have three equal parts. Then consider the subpartitions $2\nu = (\mu_1 - 2c, \mu_2 - 2c, \mu_3 - 2c)$ and $\kappa = (2c, 2c, 2c)$ for $0 \leq c \leq \frac{\mu_3}{2}$. Then again we have that $c^{\mu}_{2\nu,\kappa} = 1$. Then by Lemma 5.1.1(i) we have that $\chi_{\mathrm{SO}_7(\Bbbk)}(\kappa)$ appears as a summand in the $\mathrm{SO}_7(\Bbbk)$-character formula for $\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\mu)$. Moreover, by 5.1 we have that $\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\kappa)^{G_2} = 1$. Then by Lemma 5.1.1(ii) we have that

$$\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2} = 0.$$

If μ is a partition of odd parts that does not have three equal parts then set $2\nu = (\mu_1 - 2c - 1, \mu_2 - 2c - 1, \mu_3 - 2c - 1)$ and $\kappa = (2c + 1, 2c + 1, 2c + 1)$ for $0 \leq c \leq \frac{\mu_3 - 1}{2}$. Following a similar argument as previously stated, we have that $c^{\mu}_{2\nu,\kappa} = 1$. Then by Lemma 5.1.1(i) we have that $\chi_{\mathrm{SO}_7(\Bbbk)}(\kappa)$ appears as a summand in the $\mathrm{SO}_7(\Bbbk)$-character formula for $\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\mu)$. By a previous step we have that $\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\kappa)^{G_2} = 1$. It follows from Lemma 5.1.1(ii) that

$$\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu)^{G_2} = 0.$$

□

One final result gives us a way to calculate $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ for any partition $\lambda \in \Lambda^+(7)$.

Corollary 5.1.3. *Let $\lambda \in \Lambda^+(7)$ and write*

$$\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\lambda) = \sum_{\mu \in \lambda^+(3)} m_\mu \chi_{\mathrm{SO}_7(\Bbbk)}(\mu),$$

where $m_\mu = (\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda) : \nabla_{\mathrm{SO}_7(\Bbbk)}(\mu))$. Then

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{r \geq 0} m_{(r,r,r)}.$$

Proof. This follows immediately from Theorem 5.1.2 and Lemma 3.1.3. □

5.2 DOMINANT $\mathrm{GL}_7(\Bbbk)$-WEIGHTS

The next step in calculating $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ for $\lambda \in X^+(T_{\mathrm{GL}_7(\Bbbk)})$ is to calculate the filtration multiplicities $m_{(r,r,r)}$ given in Corollary 5.1.3. Recall that the type **B** procedure of Koike and Terada gives us a surjection

$$\pi : \Lambda^+(7) \to \Lambda^+(3) \cup \{\emptyset\}.$$

We define $\mu^\#$ to be the image of μ under π, and $\mathrm{sgn}(\mu^\#)$ to be the signature of $\mu^\#$ as given in the type **B** procedure.

Lemma 5.2.1. *Let $\lambda \in \Lambda^+(7)$. Then the coefficients $m_{(r,r,r)}$ of the dimension sum of* $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ *are given by*

$$m_{(r,r,r)} = \sum_{\substack{2\nu, \mu \subseteq \lambda \\ \mu^\# = (r,r,r) \\ \mathrm{sgn}(\mu^\#) = +1}} c^\lambda_{2\nu,\mu} - \sum_{\substack{2\nu, \mu \subseteq \lambda \\ \mu^\# = (r,r,r) \\ \mathrm{sgn}(\mu^\#) = -1}} c^\lambda_{2\nu,\mu},$$

where the coefficients $c^\lambda_{2\nu,\mu}$ are Littlewood-Richardson coefficients.

Proof. Recall from Theorem 3.3.1 that the $\mathrm{SO}_7(\Bbbk)$-character $\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\lambda)$ has a sum of the form

$$\chi^{\mathrm{GL}_7(\Bbbk)}_{\mathrm{SO}_7(\Bbbk)}(\lambda) = \sum_{2\nu, \xi \subseteq \lambda} c^\lambda_{2\nu,\mu} \mathrm{sgn}(\mu^\#) \chi_{\mathrm{SO}_7(\Bbbk)}(\mu^\#).$$

From Lemma 2.5.9 we have that, for $\xi \in X^+(T_{\mathrm{SO}_7(\Bbbk)})$, the filtration multiplicities m_ξ are in fact the coefficients of $\chi_{\mathrm{SO}_7(\Bbbk)}(\xi)$ in the $\mathrm{SO}_7(\Bbbk)$-character $\chi_{\mathrm{SO}_7(\Bbbk)}^{\mathrm{GL}_7(\Bbbk)}(\lambda)$. By Theorem 3.3.1, for $\xi \in X^+(T_{\mathrm{SO}_7(\Bbbk)})$ we have that

$$m_\xi = \sum_{\substack{2\nu,\mu\subseteq\lambda \\ \mu^\#=\xi \\ \mathrm{sgn}\mu^\#=+1}} c^\lambda_{2\nu,\mu} - \sum_{\substack{2\nu,\mu\subseteq\lambda \\ \mu^\#=\xi \\ \mathrm{sgn}\mu^\#=-1}} c^\lambda_{2\nu,\mu}.$$

By Theorem 5.1.2 we have that $\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(\xi)^{G_2} \neq 0$ if and only if $\xi = (r,r,r)$ for some $r \geq 0$. Therefore we have

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{r\geq 0} \sum_{\substack{2\nu,\mu\subseteq\lambda \\ \mu^\#=(r,r,r)}} \mathrm{sgn}(\mu^\#) c^\lambda_{2\nu,\mu} \dim \nabla_{\mathrm{SO}_7(\Bbbk)}(r,r,r)^{G_2}.$$

By Theorem 5.1.2 we have that $\dim \nabla_{\mathrm{SO}_7(\Bbbk)}(r,r,r)^{G_2} = 1$ for $r \geq 0$. Thus we deduce

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{r\geq 0} \left(\sum_{\substack{2\nu,\mu\subseteq\lambda \\ \mu^\#=(r,r,r) \\ \mathrm{sgn}(\mu^\#)=+1}} c^\lambda_{2\nu,\mu} - \sum_{\substack{2\nu,\mu\subseteq\lambda \\ \mu^\#=(r,r,r) \\ \mathrm{sgn}(\mu^\#)=-1}} c^\lambda_{2\nu,\mu} \right).$$

□

We now know that, for any $\lambda \in \Lambda^+(7)$ the dimension of the G_2-invariant space $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$ is equal to the sum of the multiplicities of the induced $\mathrm{SO}_7(\Bbbk)$-modules $\nabla_{\mathrm{SO}_7(\Bbbk)}(r,r,r)$ as sections of the $\mathrm{SO}_7(\Bbbk)$-filtration of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$. Moreover, we obtain these multiplicities as the difference of sums of Littlewood-Richardson coefficients. Our next result is to calculate $\lambda^\#$ and $\mathrm{sgn}(\lambda)$ for a partition $\lambda \in \Lambda^+(7)$.

Lemma 5.2.2. *Let $\lambda = (\lambda_1, \dots, \lambda_7) \in \lambda^+(7)$.*

(i) Suppose $l(\lambda) = 3$. Then $\lambda^\# = \lambda$ and $\mathrm{sgn}(\lambda^\#) = +1$.

(ii) Suppose $l(\lambda) = 4$. If $\lambda_4 = 1$, then $\lambda^\# = (\lambda_1, \lambda_2, \lambda_3)$ and $\mathrm{sgn}(\lambda^\#) = +1$. *For all other partitions with four parts we have $\lambda^\# = \emptyset$.*

(iii) Suppose $l(\lambda) = 5$. If $\lambda_5 = 1$ then $\lambda^\# = (\lambda_1, \lambda_2)$ with $\mathrm{sgn}(\lambda^\#) = +1$ *if and only if $\lambda = (\lambda_1, \lambda_2, 1^3)$. If $\lambda_5 = 2$ then $\lambda^\# = (\lambda_1, \lambda_2, \lambda_3)$ and* $\mathrm{sgn}(\lambda^\#) = -1$ *if and only if $\lambda = (\lambda_1, \lambda_2, \lambda_3, 2, 1)$ or $\lambda = (\lambda_1, \lambda_2, \lambda_3, 2, 2)$. For all other partitions with five parts we have $\lambda^\# = \emptyset$.*

(iv) Suppose $l(\lambda) = 6$. *Then*

(a) $\lambda^{\#} = (\lambda_1)$ *with* $\mathrm{sgn}(\lambda^{\#}) = +1$ *if and only if* $\lambda = (\lambda_1, 1^5)$.

(b) $\lambda^{\#} = (\lambda_1, \lambda_2)$ *with* $\mathrm{sgn}(\lambda^{\#}) = -1$ *if and only if* $\lambda = (\lambda_1, \lambda_2, 2, 1^3)$ *or* $\lambda = (\lambda_1, \lambda_2, 2^4)$.

(c) $\lambda^{\#} = (\lambda_1, \lambda_2, 1)$ *with* $\mathrm{sgn}(\lambda^{\#}) = -1$ *if and only if* $\lambda = (\lambda_1, \lambda_2, 2, 2, 1, 1)$ *or* $\lambda = (\lambda_1, \lambda_2, 2^3, 1)$

(d) $\lambda^{\#} = (\lambda_1, \lambda_2, \lambda_3)$ *with* $\mathrm{sgn}(\lambda^{\#}) = +1$ *if and only if* $\lambda = (\lambda_1, \lambda_2, \lambda_3, 3, 1, 1)$ *or* $\lambda = (\lambda_1, \lambda_2, \lambda_3, 3, 2, 1)$.

(e) $\lambda^{\#} = (\lambda_1, \lambda_2, \lambda_3)$ *with* $\mathrm{sgn}(\lambda^{\#}) = -1$ *if and only if* $\lambda = (\lambda_1, \lambda_2, \lambda_3, 3, 3, 2)$ *or* $\lambda = (\lambda_1, \lambda_2, \lambda_3, 3^3)$.

(f) For all other partitions of six parts we have $\lambda^{\#} = \emptyset$.

Proof. (i) This immediately follows from the fact that if $l(\lambda) \leq 3$ then by the type **B** procedure 3.3, we have $\lambda^{\#} = \lambda$ and $\mathrm{sgn}(\lambda^{\#}) = +1$.

(ii) Let $l(\lambda) = 4$. We follow the type **B** procedure of Koike and Terada 3.3. Set λ' to be

$$\lambda' = (4^{\lambda_4}, 3^{\lambda_3 - \lambda_4}, 2^{\lambda_2 - \lambda_3}, 1^{\lambda_1 - \lambda_2}),$$

the transpose partition of λ, from which we calculate the tuple t as in step two of the type **B** procedure 3.3. Note that if $\lambda_4 \geq 2$ then we have $t_1 = 7 - (4 - 0) = 3 = 4 - (2 - 1) = t_2$. Then we set $\lambda^{\#} = \emptyset$, set $\mathrm{sgn}(\lambda^{\#}) = +1$, and terminate the algorithm.

Now we suppose $\lambda_4 = 1$. Then $\lambda' = (4, 3^{\lambda_3 - 1}, 2^{\lambda_2 - \lambda_3}, 1^{\lambda_1 - \lambda_2})$. The tuple t can be written $t = (3, t_2, t_3, \ldots, t_{\lambda_1})$. Now note that since $t_2 \leq 3 - 1 = 2$, we have $t_1 > t_2$. Note also that for $i \geq 2$ we have $3 \geq \lambda'_i \geq \lambda'_{i+1}$. It follows therefore that $3 - (i - 1) \geq t_i = \lambda'_i - (i - 1) > \lambda'_i - i \geq t_{i+1}$. We therefore conclude that the tuple $t = (3, t_2, \ldots, t_{\lambda_1})$ already satisfies $t_i > t_{i+1}$ for all $1 \leq i < \lambda_1$. Thus, the permutation $\sigma \in S_{\lambda_1}$ which rearranges t into descending order is the identity permutation $\sigma = 1_{\lambda_1}$ and so $\mathrm{sgn}(\lambda^{\#}) = +1$. Now let $\xi = (\xi_1, \ldots, \xi_{\lambda_1})$ be the partition such that $\xi_i = t_i + (i - 1)$ for $1 \leq i \leq \lambda_1$. Then ξ has the form

$$\xi = (3^{\lambda_3}, 2^{\lambda_2 - \lambda_3}, 1^{\lambda_1 - \lambda_2}).$$

Taking the transpose of ξ we obtain $\lambda^{\#} = (\lambda_1, \lambda_2, \lambda_3)$.

(iii) Let $l(\lambda) = 5$. We follow the type **B** procedure 3.3. Set λ' to be

$$\lambda' = (5^{\lambda_5}, 4^{\lambda_4-\lambda_5}, 3^{\lambda_3-\lambda_4}, 2^{\lambda_2-\lambda_3}, 1^{\lambda_1-\lambda_1}).$$

note that if $\lambda_5 \geq 3$ then we have $t_2 = 7-(5-1) = 3 = 5-2 = t_3$. Then we set $\lambda^\# = \emptyset$, set $\operatorname{sgn}(\lambda^\#) = +1$, and terminate the algorithm.

Suppose $\lambda_5 = 2$. If $\lambda_4 \geq 3$ then we have $t_1 = 7-(5-0) = 2 = 4-(3-1) = t_3$. Then we set $\lambda^\# = \emptyset$, set $\operatorname{sgn}(\lambda^\#) = +1$, and terminate the algorithm. Suppose $\lambda_4 = 2$. Then the tuple t has the form $t = (2, 3, t_3, \dots, t_{\lambda_1})$, where $t_3 \leq 3-(2-1) = 1 < t_1$ and $t_i > t_{i+1}$ for $3 \leq i < \lambda_1$ by the same argument as above. Note that the tuple of decreasing order $\tilde{t}$ is written as $\tilde{t} = (3, 2, t_3, \dots, t_{\lambda_1})$ and the permutation which reorders t to obtain $\tilde{t}$ is the transposition (12). Therefore, we have $\operatorname{sgn}(\lambda^\#) = +1$. The partition ξ obtained from $\tilde{t}$ has the form

$$\xi = (3^{\lambda_3}, 2^{\lambda_2-\lambda_3}, 1^{\lambda_1-\lambda_2}).$$

Taking the transpose of ξ we obtain $\lambda^\# = (\lambda_1, \lambda_2, \lambda_3)$.

Now suppose $\lambda_5 = 1$. If $\lambda_4 \geq 3$ then for the same reason as above, we set $\lambda^\# = \emptyset$, set $\operatorname{sgn}(\lambda^\#) = +1$, and terminate the algorithm. Suppose $\lambda_4 = 2$. Then the tuple t has the form $t = (2, 3, t_3, \dots, t_{\lambda_1})$, where $t_3 \leq 3-(2-1) = 1 < t_1$ and $t_i > t_{i+1}$ for $3 \leq i < \lambda_1$ by the same argument as above. It follows by the same argument as above that $\lambda^\# = (\lambda_1, \lambda_2, \lambda_3)$ and $\operatorname{sgn}(\lambda^\#) = -1$. Now suppose that $\lambda_4 = 1$. If $\lambda_3 \geq 2$ then we have $t_1 = 7-(5-0) = 2 = 3-(2-1) = t_2$. We set $\lambda^\# = \emptyset$, set $\operatorname{sgn}(\lambda^\#) = +1$, and terminate the algorithm. Suppose $\lambda_3 = 1$. Then $t = (2, t_2, t_3, \dots, t_{\lambda_1})$, such that $t_2 \leq 2-1 = 1 < t_1$, and $t_i > t_{i+1}$ for $2 \leq i < \lambda_1$. Then the permutation which orders t is the identity permutation and $\operatorname{sgn}(\lambda^\#) = +1$. The partition ξ obtained from t has the form

$$\xi = (2^{\lambda_2}, 1^{\lambda_1}).$$

Taking the transpose of ξ we obtain $\lambda^\# = (\lambda_1, \lambda_2)$.

(iv) Let $l(\lambda) = 6$. We follow the type **B** procedure 3.3. Set λ' to be

$$\lambda' = (6^{\lambda_6}, 5^{\lambda_5-\lambda_6}, 4^{\lambda_4-\lambda_5}, 3^{\lambda_3-\lambda_4}, 2^{\lambda_2-\lambda_3}, 1^{\lambda_1-\lambda_1}).$$

note that if $\lambda_6 \geq 4$ then we have $t_3 = 7-(3-2) = 3 = 6-3 = t_3$. Then we set $\lambda^\# = \emptyset$, set $\operatorname{sgn}(\lambda^\#) = +1$, and terminate the algorithm.

(e) Suppose $\lambda_6 = 3$. If $\lambda_4 \geq 4$ then we have $t_4 = 4-(4-1) = 1 = t_1$ or $t_4 = 5-(4-1) = 2 = t_2$. In either case we set $\lambda^{\#} = \emptyset$, set $\mathrm{sgn}(\lambda^{\#}) = +1$, and terminate the algorithm.

Suppose $\lambda_4 = 3$. Then t has the form $t = (1,2,3,t_4,\dots,t_{\lambda_1})$, such that $t_4 \leq 3-(4-1) = 0$ and $t_i > t_{i+1}$ for $4 \leq i < \lambda_1$. The permutation which reorders t into the ordered tuple $\tilde{t} = (3,2,1,t_4,\dots t_{\lambda_1})$ is the transposition (13). The partition ξ obtained from $\tilde{t}$ has the form

$$\xi = (3^{\lambda_3}, 2^{\lambda_2-\lambda_3}, 1^{\lambda_1-\lambda_2}).$$

Taking the transpose of ξ we obtain $\lambda^{\#} = (\lambda_1,\lambda_2,\lambda_3)$.

Now suppose $\lambda_6 = 2$. By the same logic as above, we cannot have $\lambda_4 \geq 4$. Similarly, if $\lambda_5 = 2$ and $\lambda_4 = 3$ then we have $t_3 = 4-(3-1) = 2 = t_2$. Then we set $\lambda^{\#} = \emptyset$, set $\mathrm{sgn}(\lambda^{\#}) = +1$, and terminate the algorithm. Suppose $\lambda_5 = 3$. Then the tuple t has the form $t = (1,2,3,t_4,\dots,t_{\lambda_1})$. The rest of the algorithm follows as written above, and so we conclude that $\lambda^{\#} = (\lambda_1,\lambda_2,\lambda_3)$, set $\mathrm{sgn}(\lambda^{\#}) = -1$.

(b) Suppose $\lambda_4 = 2$. If $\lambda_3 \geq 3$ then $t_3 = 3-(3-1) = 1 = t_1$ and so we set $\lambda^{\#} = \emptyset$, set $\mathrm{sgn}(\lambda^{\#}) = +1$, and terminate the algorithm. Suppose $\lambda_3 = 2$. Then t has the form $t = (1,2,t_3,\dots,t_{\lambda_1})$ where $t_3 \leq 2-(3-1) = 0$. The permutation which orders t is the transposition (12) and so we set $\mathrm{sgn}(\lambda^{\#}) = -1$. The transpose partition ξ has form

$$\xi = (2^{\lambda_2}, 1^{\lambda_1-\lambda_2}).$$

Taking the transpose of ξ we obtain $\lambda^{\#} = (\lambda_1,\lambda_2)$.

(d)Now suppose $\lambda_6 = 1$. We cannot have $\lambda_4 \geq 4$ for the same reason as established previously. We cannot have $\lambda_5 = 3$, otherwise $t_2 = 3 = t_3$. So we suppose $\lambda_4 = 3$ and $\lambda_5 = 2$. Then tuple t has form $t = (1,3,2,t_4,\dots,t_{\lambda_1})$ such that $t_4 \leq 3-(4-1) = 0$ and $t_i > t_{i+1}$ for $4 \leq i < \lambda_1$. The permutation which orders t is the 3-cycle (123) and so we set $\mathrm{sgn}(\lambda^{\#}) = +1$. The partition ξ has form

$$\xi = (3^{\lambda_3}, 2^{\lambda_2-\lambda_3}, 1^{\lambda_1-\lambda_2}),$$

and so we obtain $\lambda^{\#} = (\lambda_1,\lambda_2,\lambda_3)$.

(d) Similarly, if $\lambda_4 = 3$ and $\lambda_5 = 1$ then the tuple t again has form $t = (1,3,2,t_4,\dots,t_{\lambda_1})$. By the same argument as above we obtain $\lambda^{\#} = (\lambda_1,\lambda_2,\lambda_3)$ and $\mathrm{sgn}(\lambda^{\#}) = +1$.

(c) Set $\lambda_4 = 2$. For $\lambda_5 = 1$ or 2 we have $t_2 = 3$. If $\lambda_3 \geq 3$ then we have $t_3 = 3 - (3-1) = 1 = t_1$, so we set $\lambda^\# = \emptyset$, set $\mathrm{sgn}(\lambda^\#) = +1$, and terminate the algorithm. Suppose $\lambda_3 = 2$, then we obtain the tuple $t = (1, 3, t_3, \ldots, t_{\lambda_n})$ where $t_3 \leq 2 - (3-1) = 0 < t_1$. The transposition reordering t is the transposition (12) and so $\mathrm{sgn}(\lambda^\#) = -1$. The partition ξ corresponding to $\tilde{t}$ has the form

$$\xi = (3, 2^{\lambda_2 - 1}, 1^{\lambda_1 - \lambda_2}).$$

Taking the transpose of ξ we obtain $\lambda^\# = (\lambda_1, \lambda_2)$.

(b) Now suppose $\lambda_4 = 1$. We cannot have $\lambda_3 \geq 3$ for the same reason as stated above. If $\lambda_3 = 2$ then $t = (1, 2, t_3, \ldots)$ such that $t_3 \leq 0 < t_1$. It follows that the transposition reording t is the transposition (12) and so $\mathrm{sgn}(\lambda^\#) = -1$. The partition corresponding to $\tilde{t}$ has the form

$$\xi = (2^{\lambda_2}, 1^{\lambda_1 - \lambda_2}).$$

Taking the transpose of ξ we obtain $\lambda^\# = (\lambda_1, \lambda_2)$.

(a) Finally, suppose $\lambda_3 = 1$. If $\lambda_2 \geq 2$ then $t_2 = 2 - (2-1) = 1 = t_1$. We then set $\lambda^\# = \emptyset$, set $\mathrm{sgn}(\lambda^\#) = +1$, and terminate the algorithm. Suppose $\lambda_2 = 1$. then t has the form $t = (1)$ if $\lambda_1 = 1$ or $t = (1, 0, \ldots, 2 - \lambda_1)$. In both cases, t is a tuple in descending order, thus we conclude that $\mathrm{sgn}(\lambda^\#) = +1$. The partition ξ corresponding to t has the form

$$\xi = (1^{\lambda_1}).$$

Taking the transpose of ξ we obtain $\lambda^\# = (\lambda_1)$.

□

As we are concerned only with $SO_7(\Bbbk)$-filtration sections $\nabla_{SO_7(\Bbbk)}(r, r, r)$, the following result follows immediately from Lemma 5.2.2.

Corollary 5.2.3. *Let* $\lambda \in \Lambda^+(7)$.

(i) Suppose $l(\lambda) \leq 6$. *We have* $\lambda^\# = (r, r, r)$ *for some* $r \geq 0$, *with* $\mathrm{sgn}(\lambda^\#) = +1$ *if and only if* λ *is one of the following partitions,*

$$\{(r, r, r),\ (r, r, r, 1),\ (r, r, r, 3, 2, 1)\ (r, r, r, 3, 1, 1)\}.$$

(ii) Suppose $l(\lambda) \leq 6$. *We have* $\lambda^\# = (r, r, r)$ *for some* $r \geq 0$, *with* $\mathrm{sgn}(\lambda^\#) = -1$ *if and only if* λ *is one of the following partitions,*

$$\{(r, r, r, 2, 2),\ (r, r, r, 2, 1),\ (r, r, r, 3, 3, 2)\ (r, r, r, 3, 3, 3)\}.$$

Proof. The result follows immediately from Lemma 5.2.2 by setting $\lambda_1 = \lambda_2 = \lambda_3 = r$. $\square$

We combine the results of Corollary 5.2.3 and Lemma 3.2.2(iii) to obtain the following result.

Lemma 5.2.4. *Let $\lambda \in \Lambda^+(7)$ such that $l(\lambda) = 7$. Then we have $\lambda^\# = (r, r, r)$ for some $r \geq 0$ if and only if λ can be written as $\lambda = (s^7) + \bar{\lambda}$, where $s > 0$ and $\bar{\lambda}$ is one of the eight partitions in Corollary 5.2.3.*

Proof. Note that as an $\mathrm{SO}_7(\Bbbk)$-module, the determinant module $\det_7$ is isomorphic to the trivial module $\nabla_{\mathrm{SO}_7(\Bbbk)}(0)$. Then by Lemma 3.2.2(iii) we have that $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)$ is isomorphic as an $\mathrm{SO}_7(\Bbbk)$-module to the induced module $\nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)$ corresponding to a partition $\mu = (\lambda_1 - \lambda_7, \ldots, \lambda_6 - \lambda_7)$ of at most 6 parts. Setting $s = \lambda_7$, the result follows from Corollary 5.2.3. $\square$

Finally, we state the final theorem of this section, which provides a complete calculation of $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ for $\lambda \in X^+(7)$.

Theorem 5.2.5. *Let $\lambda \in \Lambda^+(7)$. For $\xi, \zeta \subseteq \lambda$ let $c^\lambda_{\xi,\zeta}$ denote the Littlewood-Richardson coefficients.*

(i) If $l(\lambda) = 3$ then

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \begin{cases} \lfloor \frac{\lambda_3}{2} \rfloor + 1, & \text{if } \lambda_1 \equiv \lambda_2 \equiv \lambda_3 \pmod 2; \\ 0, & \text{otherwise.} \end{cases}$$

(ii) If $l(\lambda) = 4$ then

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{r \geq 0} \sum_{2\nu \subseteq \lambda} c^\lambda_{2\nu,(r,r,r)} + \sum_{r \geq 1} \sum_{2\nu \subseteq \lambda} c^\lambda_{2\nu,(r,r,r,1)}.$$

(iii) If $l(\lambda) = 5$ then

$$\begin{aligned} \dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = & \sum_{r \geq 0} \sum_{2\nu \subseteq \lambda} c^\lambda_{2\nu,(r,r,r)} + \sum_{r \geq 1} \sum_{2\nu \subseteq \lambda} c^\lambda_{2\nu,(r,r,r,1)} \\ & - \sum_{r \geq 2} \sum_{2\nu \subseteq \lambda} \left(c^\lambda_{2\nu,(r,r,r,2,1)} + c^\lambda_{2\nu,(r,r,r,2,2)} \right). \end{aligned}$$

(iv) If $l(\lambda) = 6$ *then*

$$\begin{aligned}\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = &\sum_{r\geq 0}\sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r)} + \sum_{r\geq 1}\sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r,1)} \\ &- \sum_{r\geq 2}\sum_{2\nu\subseteq\lambda}\left(c^{\lambda}_{2\nu,(r,r,r,2,1)} + c^{\lambda}_{2\nu,(r,r,r,2,2)}\right) \\ &+ \sum_{r\geq 3}\sum_{2\nu\subseteq\lambda}\left(c^{\lambda}_{2\nu,(r,r,r,3,1,1)} + c^{\lambda}_{2\nu,(r,r,r,3,2,1)}\right) \\ &- \sum_{r\geq 3}\sum_{2\nu\subseteq\lambda}\left(c^{\lambda}_{2\nu,(r,r,r,3,3,2)} + c^{\lambda}_{2\nu,(r,r,r,3,3,3)}\right).\end{aligned}$$

(v) If $l(\lambda) = 7$ *then*

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)^{G_2},$$

where μ *is the six part partition*

$$\mu = (\lambda_1 - \lambda_7, \ldots, \lambda_6 - \lambda_7).$$

Proof. Part (i) was proved in Theorem 4.3.7.

Suppose $l(\lambda) = 4$. By Corollary 5.2.3, the only four part partitions μ for which $\mu^{\#} = (r, r, r)$ for some $r \geq 0$ are the partitions (r, r, r) and $(r, r, r, 1)$. We have $\mathrm{sgn}(\mu^{\#}) = +1$ for both these partitions. It follows by Lemma 5.2.1 that

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{r\geq 0} m_{(r,r,r)},$$

where the coefficients $m_{(r,r,r)}$ are given by

$$m_{(r,r,r)} = \begin{cases} \sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r)}, & \text{if } r = 0; \\ \sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r)} + \sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r,1)}, & \text{otherwise.}\end{cases}$$

Suppose $l(\lambda) = 5$. Again, by Corollary 5.2.3 λ has subpartitions (r, r, r) and $(r, r, r, 1)$ with corresponding signature $\mathrm{sgn}(\mu^{\#}) = +1$. We also have that λ has subpartitions $(r, r, r, 2, 1)$ and $(r, r, r, 2, 2)$ with signature $\mathrm{sgn}(\mu^{\#})$. Then statement (iii) follows again by Lemma 5.2.1.

Suppose $l(\lambda) = 6$. By Corollary 5.2.3, λ contains subpartitions μ of each type such that $\mu^{\#} = (r, r, r)$. In particular, the subpartitions (r, r, r), $(r, r, r, 1)$, $(r, r, r, 3, 1, 1)$, and $(r, r, r, 3, 2, 1)$ all have signature $\mathrm{sgn}(\mu^{\#}) = +1$. The subpartitions $(r, r, r, 2, 1)$, $(r, r, r, 2, 2)$, $(r, r, r, 3, 3, 2)$, and $(r, r, r, 3, 3, 3)$ all has signature $\mathrm{sgn}(\mu^{\#}) = -1$. The statement (iv) follows by Lemma 5.2.1.

Suppose $l(\lambda) = 7$. Then statement (v) follows from Lemma 5.2.4. □

— 6 —

Reduction of the dimension formula

In this final chapter, we provide a reduction of the formula presented in Theorem 5.2.5 by making comparisons of tableaux with weights ξ and ζ, where $\xi^{\#} = \zeta^{\#}$ and $\mathrm{sgn}(\xi^{\#}) = -\mathrm{sgn}(\zeta^{\#})$. This permits a reformulation of the dimension formula for $\nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ as a net sum taken over a subset of even subpartitions of λ.

6.1 A MOTIVATING EXAMPLE

We begin this section with an example concerning tableaux of four parts. We recall the definition of a subtableau. A tableau S with shape μ/τ is a subtableau of a tableau T with shape λ/σ if the following hold:

- $\mu \subseteq \lambda$,
- $\tau \subseteq \sigma$,
- For $1 \leq i \leq l(\mu)$ we have

$$\{S(i,j) \,|\, \tau_i + 1 \leq j \leq \mu_i\} \subseteq \{T(i,j) \,|\, \sigma_i + 1 \leq j \leq \lambda_i\}.$$

Definition 6.1.1. Let T be a tableau with shape λ/σ and let S be a subtableau of T with shape μ/τ. The *remainder* of T by S, denoted T/S is the subtableau obtained by deleting the subtableau S from T. The shape of this tableau is ξ/ζ, where $\xi = \lambda - \mu$ and $\zeta = \sigma - \tau$.

Example 6.1.2. Let T be a tableau with shape $(3^3,1)/(0)$ and weight $(3^3,1)$,

$$T = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}.$$

Then two examples of a subtableau of T are the tableau R with shape $(1^4)/(0)$ and weight (1^4), and the tableau T/R with shape $(2^3)/(0)$ and weight (2^3),

$$R = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}, \qquad T/R = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}.$$

Let $\lambda \in \Lambda^+(7)$. We wish to devise a method of constructing a tableau T from its subtableaux.

Definition 6.1.3. Let R, S be standard Young tableaux, of respective shapes λ/σ and μ/τ, where $\lambda, \mu \in \Lambda^+(7)$, $\sigma \subseteq \lambda$, and $\tau \subseteq \mu$. The *standardised concatenation* of R and S, denoted $R|_s S$, is the row-standard tableau with shape $(\lambda+\mu)/(\sigma+\tau)$ whose row-wise content is the ordered collective row-wise content of R and S.

Remark 6.1.4. As this operation preserves the row-wise content of R and S, the operation is commutative, so $R|_s S = S|_s R$. The operation is also associative, thus $Q|_s R|_s S$ is an unambiguous expression for any standard tableaux Q, R, S.

Example 6.1.5. Continuing from the previous example, it is easy to see that we may express T as the standardised concatenation of R and T/R,

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}|_s \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline \end{array}.$$

Our method of reducing the dimension formula involves making comparisons between Littlewood-Richardson coefficients $c^{\lambda}_{2\mu,\xi}$ and $c^{\lambda}_{2\nu,\zeta}$, where $\xi^{\#} = \zeta^{\#}$ and $\mathrm{sgn}(\xi^{\#}) = -\mathrm{sgn}(\zeta^{\#})$. As the Littlewood-Richardson constant $c^{\lambda}_{2\mu,\xi}$ can be calculated as the number of LR-tableaux with shape $\lambda/2\mu$ and weight ξ, we can make a comparison between the coefficients $c^{\lambda}_{2\mu,\xi}$ and $c^{\lambda}_{2\nu,\zeta}$ by establishing a correspondence between LR-tableaux with shape $\lambda/2\mu$ and weight ξ, and LR-tableaux with shape $\lambda/2\nu$ and weight ζ.

Note that we cannot establish such a correspondence between LR-tableaux of weights (r,r,r) and $(r,r,r,1)$. Suppose T is an LR-tableau with shape $\lambda/2\mu$ and weight (r,r,r) and suppose S is an LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,1)$. Comparing degrees of all partitions we have

$$3r + 2m = |(r,r,r)| + |2\mu| = |\lambda| = |(r,r,r,1)| + |2\nu| = 3r + 1 + 2n,$$

where $2m = |2\mu|$ and $2n = |2\nu|$. There do not exist integers m, n such that $2m = 2n + 1$.

6.2 REDUCTION FOR PARTITIONS OF FIVE PARTS

Note that given a tableau with shape $\lambda/2\nu$ and weight $(r,r,r,2,1)$ for $r \geq 2$, the degree of $(r,r,r,2,1)$ is given by,

$$|(r,r,r,2,1)| = 3r + 3.$$

In particular, the degree of the weight $(r,r,r,2,1)$ has the same parity as the degree of the weight $(r,r,r,1)$.

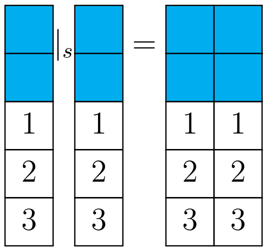
Similarly, we have that the degree of the weight $(r,r,r,2,2)$ has the same parity as the degree of the weight (r,r,r). This is why, below, we choose to match LR-tableaux with weight $(r,r,r,2,1)$ (respectively $(r,r,r,2,2)$) and tableaux with weight $(r,r,r,1)$ (respectively (r,r,r)).

MATCHING TABLEAUX OF SIMILAR WEIGHTS

Note that there exists exactly one LR-tableau with weight (2^5) (up to equivalence by adding some empty diagram $\Delta_{2\mu/2\mu}$), obtained by taking the concatenation of two copies of the tableau with weight (1^5) given in Appendix A.1.3.

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array} \,\|_s\, \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array} = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline 4&4\\\hline 5&5\\\hline\end{array}$$

This tableau has shape $(2^5)/(0)$. Note that there exist ten standard tableau of weight (1^3) as given in Appendix A.1.1, which yields 100 combinations of two such tableaux to produce a standard tableau of weight (2^3). There is exactly one combination of two tableaux of weight (1^3) which has the shape $(2^5)/2\nu$ for some $2\nu \subseteq (2^5)$. The combination of two LR-tableaux weight shape $(1^5)/(1^2)$ and weight (1^3) gives us an LR-tableau with shape $(2^5)/(2^2)$ and weight (2^3).



Note that this tableau can be drawn within the same shape as the tableau with weight (2^5). Note also that the even subpartition (2^2) is obtained by the addition of an even subpartition to (0). It immediately follows that we may match five row tableaux with weight $(r, r, r, 2, 2)$ to five row tableaux with weight (r, r, r). If T is a five row tableau with weight $(r, r, r, 2, 2)$, then T necessarily contains a subtableau R with shape $(2^5)/(0)$ and weight (2^5). We have established a correspondence between R and a tableau S with shape $(2^5)/(2^2)$ and weight (2^3). The concatenation $S|_s T/R$ of S with the remainder of T by R yields a unique tableau U with weight (r, r, r). We say that U *corresponds* to T.

Note that there exist (up to equivalence) five LR-tableaux with weight $(2^4, 1)$, obtained by concatenating the tableau with weight (1^5) given in Appendix A.1.3 with one of the five tableaux with weight (1^4) given in Appendix A.1.2.

These five tableaux with weight $(2^4, 1)$ have shape λ/μ, where μ is not necessarily an even subpartition of λ. In particular, the partition μ is either (0), or has the form (1^j) for some $j \geq 1$. As we are counting Littlewood-Richardson coefficients over even subpartitions of λ, we will associate each of these tableaux to a shape $\rho/2\nu$, where $\lambda \subseteq \rho$, and $\mu \subseteq 2\nu \subseteq \rho$. The purpose of this association is to devise a method of matching an LR-tableau with weight $(2^4, 1)$ and shape $\rho/2\nu$ to an LR-tableau with weight $(2^3, 1)$ and shape $\rho/2\sigma$ such that $2\sigma \subseteq \rho$ is obtained by adding an even subpartition to 2ν.

Definition 6.2.1. Let T be an LR-tableau with shape λ/μ and weight σ, where $\lambda \in \Lambda^+(7)$ and $\mu \subseteq \lambda$. We define a *corresponding tableau* with shape $\rho/2\nu$ by the following procedure.

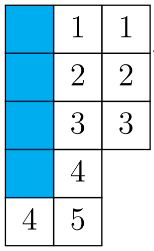
(i) Let $\xi = \mu$. Let $\zeta = \lambda$

(ii) Let $l = l(\mu)$. If μ_l is odd then add (1^l) to ξ and add (1^l) to ζ.

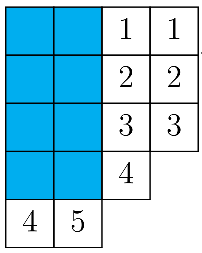
(iii) For $i < l$, if $\xi_i \not\equiv \xi_{i+1} \pmod 2$ then add (1^i) to ξ and add (1^i) to ζ.

(iv) Set $\rho = \zeta$ and set $2\nu = \xi$.

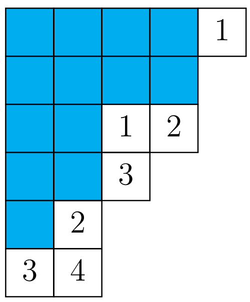
Example 6.2.2. Consider the following tableau T with shape $(3^3, 2^2)/(1^4)$ and weight $(2^4, 1)$,

$T =$  .

Here $\lambda = (3^3, 2^2)$ and $\mu = (1^4)$. We set $\xi = \mu$ and $\zeta = \lambda$. Note that $\xi_4 = 1$, thus we add (1^4) to ξ and ζ. Note for $1 \le i \le 3$ that $\xi_i \equiv \xi_{i+1} \pmod 2$. Thus, we set $\rho = \zeta$ and $2\nu = \xi$. Then the tableau R with shape $\rho/2\nu$ which corresponds to T is given by

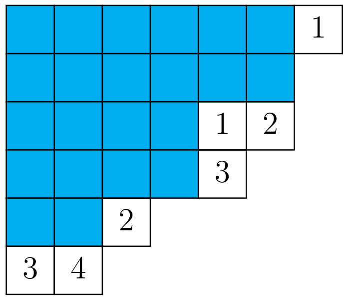
$R =$  .

Example 6.2.3. Consider the tableau T with shape $(5, 4, 4, 3, 2, 2)/(4, 4, 2, 2, 1)$ and weight $(2^3, 1)$,

$T =$ 

here $\lambda = (6, 4, 4, 3, 2, 2)$ and $\mu = (4, 4, 2, 2, 1)$. We set $\xi = \mu$ and $\zeta = \lambda$. Note that $\xi_5 = 1$, thus we add (1^5) to ξ and ζ. Note also that $\xi_4 \not\equiv \xi_5 \pmod 2$, thus we add (1^4) to ξ and ζ. Note that $\xi_3 = \xi_4$ and $\xi_1 = \xi_2 = \xi_3 + 2$, so we make no additional changes to ξ and ζ. We set $\rho = \zeta$ and $2\nu = \xi$. Then the tableau R with shape $\rho/2\nu$

which corresponds to T is given by

$R =$ 

Lemma 6.2.4. *Let λ be a partition of five parts.*

(i) Let T be an LR-tableau with shape $\lambda/2\mu$ and weight $(r, r, r, 2, 2)$ for $r \geq 2$. Then there exists an LR-tableau with shape $\lambda/2\nu$ and weight (r, r, r), where $2\nu = 2\mu + (2, 2)$

(ii) Suppose $\lambda_1 \equiv \lambda_2 \pmod 2$ or $\lambda_1 \equiv \lambda_3 \pmod 2$. Let T be an LR-tableau with shape $\lambda/2\mu$ and weight $(r, r, r, 2, 1)$ for $r \geq 2$. Then there exists an LR-tableau with shape $\lambda/2\nu$ and weight $(r, r, r, 1)$, where $2\nu = 2\mu + (2)$.

(iii) Suppose $(\lambda_1 - 1) \equiv \lambda_2 \equiv \lambda_3 \pmod 2$. Let T be an LR-tableau with shape $\lambda/2\mu$ and weight $(r, r, r, 2, 1)$, where $r \geq 2$ and $2\mu = 2\xi + (2)$ for some $2\xi \subseteq \lambda$. Then there exists an LR-tableau with shape $\lambda/2\nu$ and weight $(r, r, r, 1)$, where $2\nu = 2\xi + (2, 2)$.

Proof. (i) Let T be a tableau with shape $\lambda/2\mu$ and weight $(r, r, r, 2, 2)$. Then T contains, as a subtableau, the tableau R with shape $(2^5)/(0)$. By removing a pair of cells from both rows 1 and 2, and by replacing the content i of each other cell with $i - 1$, we find a tableau S with shape $(2^5)/(2^2)$ and weight (2^3). Concatenating with the remainder T/R, we have that $S|_s T/R$ is a tableau with shape $2\lambda/2\nu$ and weight (r, r, r), where $2\nu = 2\mu + (2, 2)$.

(ii) Now let T be an LR-tableau with shape $\lambda/2\mu$ and weight $(r, r, r, 2, 1)$, such that $\lambda_1 \equiv \lambda_2 \pmod 2$ or $\lambda_1 \equiv \lambda_3 \pmod 2$. Then necessarily T contains, as a subtableau, a tableau R with shape $\rho/2\sigma$ and weight $(2^4, 1)$, such that $\rho \subseteq \lambda$, $2\sigma \subseteq 2\mu$, and R contains a pair of cells on row 1 with content 1. By removing the pair of cells from row 1 and, on each subsequent row, replacing any cell with content i with a cell of content $i - 1$, we obtain a tableau S with shape $\rho/2\tau$ and weight $(2^3, 1)$, where $2\tau = 2\sigma + (2)$. The standardised concatenation $S|_s T/R$ yields a tableau with shape $\lambda/2\nu$ and weight $(r, r, r, 1)$, such that $2\nu = 2\mu + (2)$.

(iii) Now, let T be an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,2,1)$ such that $(\lambda_1 - 1) \equiv \lambda_2 \equiv \lambda_3 \pmod 2$. Then necessarily, T contains a subtableau R with shape $\rho/2\sigma$ and weight $(2^4, 1)$, where $\rho \subseteq \lambda$, $2\sigma \subseteq 2\mu$, and $\rho_1 = \rho_2 + 1$. This is because R must contain a subtableau with shape $(1^5)/(0)$ and weight (1^5). There must be an odd number of cells with content 1 on row 1. Indeed, if we have a tableau with shape $\rho/2\sigma$ and weight $(2^4, 1)$ containing a pair of cells with content 1 on row 1, this would result in the remainder tableau T/R having shape $(\lambda - \rho)/(2\mu - 2\sigma)$, where $2\mu - 2\sigma$ is partition whose first part is odd.

Therefore, we take R such that there exists one cell on row 1 with content 1. As there does not exist a pair of cells which can be deleted from row 1, we instead delete a pair of cells from row 2. On each row below row 2, we replace any cell with content i with a cell of content $i-1$. This results in an LR-tableau S with shape $\rho/2\tau$ and weight $(2,2,2,1)$, where $2\tau = 2\sigma + (0,2)$. Taking the concatenation of S with the remainder tableau T/R, we have that $S|_s T/R$ is a tableau with shape $\lambda/2\nu$ and weight $(r,r,r,1)$, such that $2\nu = 2\mu + (0,2)$.

Now we prove that indeed these tableaux are LR-tableaux. By definition, the standardised concatenation $S|_s T/R$ is row-standard.

The algorithm for matching the above tableaux acts by relabelling content on two cells of each affected row and then row-standardising the result. Thus, if a cell with new content $i-1$ (respectively $i-2$) in part (i) is directly above a non-empty cell with content m, we have that $m > i > i-1$ (respectively, after row-standardising we have $m > i-2$). If a cell of new content $i-1$ (respectively $i-2$) is directly underneath a non-empty cell of content n, then either the cell is invariant under the algorithm and its content n is at most $i-2$ (respectively $i-3$), or the cell has new content $i-2$ (respectively $i-3$) by the algorithm. Thus we have $n < i-1$ (respectively, $n < i-2$) and therefore, the new tableau is column-standard.

Finally, note that an equivalent formulation of the lattice word property is that the total number of instances of a letter i in the first j rows of a tableau must be at least as large as the total number of instances of the letter $i+1$ in the first $j+1$ rows. It is easy to see that, since the we remove a cell of content i from row j exactly when we remove a cell of content $i+1$ from row $j+1$, that the word of the new tableau is a lattice permutation.

We conclude in all three cases that $S|_s T/R$ is an LR-tableau. $\square$

The reverse correspondence follows from observing that if S is a tableau with shape $\lambda/2\nu$ for some 2ν described in Lemma 6.2.4, then necessarily S must be a tableau described in that Lemma.

Lemma 6.2.5. *Let λ be a partition of five parts.*

(i) Let T be an LR-tableau with shape $\lambda/2\nu$ and weight (r,r,r) for $r \geq 2$ and $2\nu = 2\mu + (2,2)$ for some $2\mu \subseteq \lambda$. Then there exists an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,2,2)$.

(ii) Suppose $\lambda_1 \equiv \lambda_2 \pmod 2$ or $\lambda_1 \equiv \lambda_3 \pmod 2$. Let T be an LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,1)$ for $r \geq 2$, and $2\nu = 2\mu + (2)$. Then there exists an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,2,1)$.

(iii) Suppose $(\lambda_1 - 1) \equiv \lambda_2 \equiv \lambda_3 \pmod 2$. Let T be an LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,1)$, where $r \geq 2$ and $2\nu = 2\xi + (2,2)$ for some $2\xi \subseteq \lambda$. Then there exists an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,2,1)$, where $2\mu = 2\xi + (2)$.

Proof. (i) Suppose T is an LR-tableau with shape $\lambda/2\nu$ and weight (r,r,r) for $r \geq 2$ and $2\nu = 2\mu + (2,2)$ for some $2\mu \subseteq \lambda$. Then T contains a subtableau R with shape $(2^5)/(2^2)$ and weight (2^3). As there is a unique tableau of this form, and it corresponds to the tableau with shape $(2^5)/(0)$ and weight (2^5) by the algorithm of Lemma 6.2.4, by reversing the algorithm we get the reverse correspondence.

(ii) Note that for five row tableau, the tableaux with weight (1^4) given in Appendix A.1.2 have shape $\lambda/(1^k)$ for some $k \leq 4$. If T is a tableau with shape $\lambda/2\nu$ such that $2\nu = 2\mu + (2)$, then 2ν must be the even partition corresponding to a tableau R with shape ρ/μ and weight $(2^3, 1)$, such that $\rho \subseteq \lambda$ and $\mu \subseteq \lambda$ such that $\mu_1 = \mu_2 + 1$. We see by combining tableaux with weight (1^3) from Appendix A.1.1 and tableaux with weight (1^4) from Appendix A.1.2, that there exist four such combinations, each of which corresponds to a tableau with weight $(2^4, 1)$ via Lemma 6.2.4. Therefore, by reversing the algorithm described in Lemma 6.2.4, we establish the reverse correspondence.

(iii) Finally, suppose that $(\lambda_1 - 1) \equiv \lambda_2 \equiv \lambda_3 \pmod 2$ and let T be an LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,1)$. Then necessarily, T contains the subtableau R with shape $(3, 2^4)/(2,2)$, which corresponds to a tableau with weight $(2^4, 1)$ via the algorithm of Lemma 6.2.4. By reversing the algorithm we obtain the reverse correspondence. □

With this matching of different tableaux, we may calculate the net sum of tableaux by counting only those tableau with shape $\lambda/2\nu$ and weight (r,r,r) or $(r,r,r,1)$, such that 2ν does not correspond to a partition 2μ by the above procedure.

Lemma 6.2.6. *Let λ be a partition of five parts. Set the sets $2L$, $2M_1$, $2M_2$, $2X_1$, and $2X_2$ to be,*

$$2L = \{2\nu \,|\, 2\nu \subseteq \lambda\},$$

$$2X_1 = \{2\nu \subseteq \lambda \,|\, 2\nu = (2) + 2\mu \text{ for some } 2\mu \subseteq \lambda\},$$

$$2X_2 = \{2\nu \subseteq \lambda \,|\, 2\nu = (2,2) + 2\mu \text{ for some } 2\mu \subseteq \lambda\},$$

$$2M_2 = 2L \setminus 2X_2.$$

$$2M_1 = \begin{cases} 2L \setminus (2X_1), & \text{if } \lambda_1 \equiv \lambda_2 \pmod 2 \text{ or } \lambda_1 \equiv \lambda_3 \pmod 2; \\ 2L \setminus 2X_2, & \text{otherwise.} \end{cases}$$

Then,

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{\substack{r \geq 1 \\ 2\nu \in 2M_1}} c^{\lambda}_{2\nu,(r,r,r,1)} + \sum_{\substack{r \geq 0 \\ 2\nu \in 2M_2}} c^{\lambda}_{2\nu,(r,r,r)},$$

where the coefficients $c^{\lambda}_{2\nu,\mu}$ are Littlewood-Richardson coefficients.

Proof. We have by Theorem 5.2.5 part (iii) that

$$\begin{aligned} \dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = & \sum_{r \geq 0} \sum_{2\nu \subseteq \lambda} c^{\lambda}_{2\nu,(r,r,r)} + \sum_{r \geq 1} \sum_{2\nu \subseteq \lambda} c^{\lambda}_{2\nu,(r,r,r,1)} \\ & - \sum_{r \geq 2} \sum_{2\nu \subseteq \lambda} \left(c^{\lambda}_{2\nu,(r,r,r,2,1)} + c^{\lambda}_{2\nu,(r,r,r,2,2)} \right). \end{aligned}$$

By Lemma 6.2.4 we have that if T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,2,1)$, then there exists a corresponding LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,1)$, where $2\nu = 2\mu + (2)$ or $2\nu = 2\mu + (0,2)$. Similarly, by 6.2.5 we see that by reversing the algorithm, we have that if S is an LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,1)$ where $2\nu = 2\xi + (2)$ or $2\xi + (2,2)$ for some $2\xi \subseteq \lambda$, then there exists a corresponding LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,2,1)$, where $2\mu = 2\xi$ or $2\mu = 2\xi + (2)$ respectively. Therefore, the LR-tableaux with weight $(r,r,r,1)$ which do not have a corresponding LR-tableau with weight $(r,r,r,2,1)$, are those which do not correspond to an even subpartition 2ν of the form $2\mu + (2)$ or $2\mu + (2,2)$.

By Lemma 6.2.4 we have that if T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,2,2)$, then there exists a corresponding LR-tableau with shape $\lambda/2\nu$ and weight (r,r,r), where $2\nu = 2\mu + (2,2)$. Similarly, by reversing the algorithm in Lemma 6.2.4 we have that if S is an LR-tableau with shape $\lambda/2\nu$ and weight (r,r,r) where $2\nu = 2\xi + (2,2)$ for some $2\xi \subseteq \lambda$, then there exists a corresponding LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,2,2)$, where $2\mu = 2\xi$. Therefore, the LR-tableaux with weight (r,r,r) which do not have a corresponding LR-tableau with weight $(r,r,r,2,2)$, are those which do not correspond to an even subpartition 2ν of the form $2\xi + (2,2)$. □

6.3 REDUCTION FOR PARTITIONS OF SIX PARTS

Note that given a tableau with shape $\lambda/2\nu$ and weight $(r,r,r,3,3,2)$ for $r \geq 3$, the degree of $(r,r,r,3,3,2)$ is given by

$$|(r,r,r,3,3,2)| = 3r + 8.$$

Note that for a tableau with shape $\lambda/2\xi$ and weight $(r,r,r,3,2,1)$ for $r \geq 3$, the degree of $(r,r,r,3,2,1)$ is given by

$$|(r,r,r,3,2,1)| = 3r + 6.$$

In particular, the degrees of the above two weights have the same parity as the degrees of the weights (r,r,r) and $(r,r,r,2,2)$.

Similarly, we have that the degrees of the weights $(r,r,r,3,3,3)$ and $(r,r,r,3,1,1)$ have the same parity as the degrees of the weights (r,r,r) and $(r,r,r,2,1)$. By considering tableaux whose weights have degrees of the of the same parity, we devise a method of matching tableaux in order to calculate $\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2}$ as a reduced sum.

Lemma 6.3.1. *Let λ be a partition of six parts.*

(i) If T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,3,3,3)$ then there exists an LR-tableau with weight $(r,r,r,3,1,1)$ and shape $\lambda/2\nu$ where $2\nu = 2\mu + (2,2)$.

(ii) If T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,3,3,2)$, then there exists an LR-tableau with weight $(r,r,r,3,2,1)$ and shape $\lambda/2\nu$, where $2\nu = 2\mu + (2)$.

(iii) If T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,3,3,2)$, such that $\lambda_1 - 1 \equiv \lambda_2 \,(\mathrm{mod}\, 2)$ and $2\mu = 2\zeta + (2)$ then there exists an LR-tableau with weight $(r,r,r,3,2,1)$ and shape $\lambda/2\nu$, where $2\nu = 2\zeta + (2,2)$.

Proof. (i) Suppose that T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,3,3,3)$. Then necessarily, T contains a subtableau R with shape $(2^6)/(0)$ and weight (2^6). We obtain the tableau S with shape $(2^6)/(2^2)$ and weight (2^4) as follows. We delete the top two rows of this tableau and, on all other rows, relabel any cell with content i to have content $i-2$. The concatenation $S|_sT/R$ is an LR-tableau with shape $2\lambda/2\nu$ and weight $(r,r,r,3,1,1)$, where $2\nu = \xi + (2,2)$.

(ii) Now suppose that T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,3,3,2)$, such that $\lambda_1 - 1 \not\equiv \lambda_2 \,(\mathrm{mod}\, 1)$. Then necessarily T contains a subtableau R which has shape $\rho/2\sigma$ and weight $(2^5,1)$, such that there exists a pair of cells with content 1 on the first row of R. Then by deleting the two cells from row 1 and, on each subsequent row, relabelling a cell with content i to have content $i-1$, we obtain a tableau S with shape $\rho/2\nu$ where $2\tau = 2\sigma + (2)$. The concatenation $S|_sT/R$ has weight $(r,r,r,3,2,1)$ and shape $\lambda/2\nu$, where $2\nu = 2\mu + (2)$.

(iii) Now suppose that T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,3,3,2)$, such that $\lambda_1 - 1 \equiv \lambda_2 \,(\mathrm{mod}\, 1)$ and $2\mu = 2\zeta + (2)$. Then T contains a subtableau R with shape $\rho/2\sigma$ and weight $(2^5,1)$. This is because R must contain a subtableau with shape $(1^6)/(0)$ and weight (1^6). There must be an odd number of cells with content 1 on row 1. Indeed, if we have a tableau with shape $\rho/2\sigma$ and weight $(2^5,1)$ containing a pair of cells with content 1 on row 1, this would result in the remainder tableau T/R having shape $(\lambda-\rho)/(2\mu-2\sigma)$, where $2\mu - 2\sigma$ is partition whose first part is odd.

Therefore, we take R such that there exists one cell on row 1 with content 1. As there does not exist a pair of cells which can be deleted from row 1, we instead delete a pair of cells from row 2. On each row below row 2, we replace any cell with content i with a cell of content $i-1$. This results in an LR-tableau S with shape $\rho/2\tau$ and weight $(2,2,2,2,1)$, where $2\tau = 2\sigma + (0,2)$. Taking the concatenation of S with the remainder tableau T/R, we have that $S|_sT/R$ is a tableau with shape $\lambda/2\nu$ and weight $(r,r,r,3,2,1)$, such that $2\nu = 2\mu + (0,2)$.

Now we prove that indeed these tableaux are LR-tableaux. By definition, the standardised concatenation $S|_sT/R$ is row-standard.

The algorithm for matching the above tableaux acts by relabelling content on two cells of each affected row and then row-standardising the result. Thus, if a cell with new content $i-1$ (respectively $i-2$) in part (i) is directly above a non-empty cell with content m, we have that $m > i > i-1$ (respectively, after row-standardising we have $m > i-2$). If a cell of new content $i-1$ (respectively $i-2$) is directly underneath a non-empty cell of content n, then either the cell is invariant under the algorithm and its content n is at most $i-2$ (respectively $i-3$), or the cell has new content $i-2$ (respectively $i-3$) by the algorithm. Thus we have $n < i-1$ (respectively, $n < i-2$) and therefore, the new tableau is column-standard.

Finally, note that an equivalent formulation of the lattice word property is that the total number of instances of a letter i in the first j rows of a tableau must be at least as large as the total number of instances of the letter $i+1$ in the first $j+1$ rows. It is easy to see that, since the we remove a cell of content i from row j exactly when we remove a cell of content $i+1$ from row $j+1$, that the word of the new tableau is a lattice permutation.

We conclude in all three cases that $S|_s T/R$ is an LR-tableau. □

We also obtain a reverse correspondence by noticing that the tableaux obtained via the algorithm of Lemma 6.3.1 are the only tableaux of those respective shapes and weight $(r,r,r,3,2,1)$ (respectively $(r,r,r,3,1,1)$.

Lemma 6.3.2. *Let λ be a partition of six parts.*

(i) If T is an LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,3,1,1)$, where $2\nu = 2\mu + (2,2)$, then there exists an LR-tableau with weight $(r,r,r,3,1,1)$ and shape $\lambda/2\mu$.

(ii) Let T be an LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,3,2,1)$, where $2\nu = 2\mu + (2)$. Then there exists an LR-tableau with weight $(r,r,r,3,3,2)$ and shape $\lambda/2\mu$.

(iii) Let T be an LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,3,2,1)$, such that $\lambda_1 - 1 \equiv \lambda_2 \pmod 1$ and $2\nu = 2\zeta + (2,2)$. Then there exists an LR-tableau with weight $(r,r,r,3,3,2)$ and shape $\lambda/2\mu$, where $2\mu = 2\zeta + (2)$.

Proof. Note that in case (i), If T has shape $\lambda/2\nu$ and weight $(r,r,r,3,1,1)$, such that $2\nu = 2\mu + (2,2)$ then T necessarily contains as a subtableau the tableau R with shape $(2^6)/(2^2)$ and weight (2^4). By reversing the algorithm given in part (i) of Lemma 6.3.1, we obtain the reverse correspondence.

In case (ii), we note that T must contain a subtableau R with shape $\rho/2\tau$ and weight $(2^4, 1)$, such that $2\tau_1 > 2\tau_2$. By constructing tableaux using Appendix A.2.4 and Appendix A.2.3, we see that there are six such tableaux, and each tableau corresponds to a tableau with weight $(2^5, 1)$ by the algorithm in Lemma 6.3.1 part (ii). By reversing this algorithm we obtain the reverse correspondence.

Finally, in case (iii) there is exactly one such tableau, which has shape $(3, 2^5)/(2, 2)$ and corresponds to the tableau with shape $(3, 2^5)/(2)$ and weight $(2^5, 1)$ by part (iii) of Lemma 6.3.1. Therefore, reversing the algorithm gives the reverse correspondence. □

Lemma 6.3.3. *Let the sets $2L$, $2O_1$, $2O_2$, $2Z_1$, and $2Z_2$ be as follows,*

$$2L = \{2\nu \,|\, 2\nu \subseteq \lambda\},$$

$$2Z_1 = \{2\nu \,|\, 2\nu \subseteq \lambda \textit{ and } 2\nu = (2,2) + 2\mu \textit{ for some } 2\mu \subseteq \lambda\},$$

$$2Z_2 = \{2\nu \,|\, 2\nu \subseteq \lambda \textit{ and } 2\nu = (2) + 2\mu \textit{ for some } 2\mu \subseteq \lambda\},$$

$$2O_1 = 2L \setminus 2Z_1,$$

$$2O_2 = \begin{cases} 2L \setminus (2Z_1 \cup 2Z_2), & \textit{if } \lambda_1 - 1 \equiv \lambda_2 \pmod 2; \\ 2L \setminus 2Z_2, & \textit{otherwise.} \end{cases}$$

(i) If T is an LR tableaux with weight $(r, r, r, 3, 1, 1)$ and shape $\lambda/2\mu$ such that T does not correspond to an LR tableau with weight $(r, r, r, 3, 3, 3)$ as described in Lemma 6.3.1 part (i), Then $2\mu \in 2O_1$.

(ii) If T is an LR tableaux with weight $(r, r, r, 3, 2, 1)$ and shape $\lambda/2\mu$ such that T does not correspond to an LR tableau with weight $(r, r, r, 3, 3, 2)$ as described in Lemma 6.3.1 parts (ii) and (iii), Then $2\mu \in 2O_2$.

Proof. By Lemma 6.3.1 we have that if T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r, r, r, 3, 3, 3)$, then there exists a corresponding LR-tableau with shape $\lambda/2\nu$ and weight $(r, r, r, 3, 1, 1)$, where $2\nu = 2\mu + (2, 2)$. Similarly, by Lemma 6.3.2 we see that by reversing the algorithm, we have that if S is an LR-tableau with shape $\lambda/2\nu$ and weight $(r, r, r, 3, 1, 1)$ where $2\nu = 2\mu + (2, 2)$ for some $2\mu \subseteq \lambda$, then there exists a corresponding LR-tableau with shape $\lambda/2\mu$ and weight $(r, r, r, 3, 3, 3)$. Therefore, the LR-tableaux with weight $(r, r, r, 3, 1, 1)$ which do not have a corresponding LR-tableau with weight $(r, r, r, 3, 3, 3)$, are those which do not correspond to an even subpartition 2ν of the form $2\mu + (2, 2)$

By Lemma 6.3.1 we have that if T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,3,3,2)$, then there exists a corresponding LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,3,2,1)$, where $2\nu = 2\mu + (2)$ or $2\nu = 2\mu + (0,2)$. Similarly, by reversing the algorithm in Lemma 6.3.1 we have that if S is an LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,3,2,1)$ where $2\nu = 2\mu + (2)$ or $2\nu = 2\mu + (0,2)$ for some $2\mu \subseteq \lambda$, then there exists a corresponding LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,3,3,2)$. Therefore, the LR-tableaux with weight $(r,r,r,3,2,1)$ which do not have a corresponding LR-tableau with weight $(r,r,r,3,3,2)$, are those which do not correspond to an even subpartition 2ν of the form $2\mu + (2)$ or $2\mu + (0,2)$. $\square$

Given the set of tableaux with weight $(r,r,r,3,1,1)$ (respectively with weight $(r,r,r,3,2,1)$) which do not correspond to tableaux with weight $(r,r,r,3,3,3)$ (respectively with weight $(r,r,r,3,3,2)$), we establish a one-to-one correspondence between these tableaux and a subset of tableaux with weight $(r,r,r,2,1)$ (respectively with weight $(r,r,r,2,2)$).

Lemma 6.3.4. *Let λ be a partition of six parts.*

(i) Suppose T is an LR tableau with weight $(r,r,r,3,1,1)$ (respectively with weight $(r,r,r,3,2,1)$) and shape $\lambda/2\mu$, where $2\mu \in 2O_1$ (respectively where $2\mu \in 2O_2$). Then there exists an LR tableau with weight $(r,r,r,2,1)$ (respectively with weight $(r,r,r,2,2)$) and shape $2\lambda/2\nu$ where $2\nu = 2\mu + (2)$.

(ii) Suppose T is an LR tableau with weight $(r,r,r,3,1,1)$ and shape $\lambda/2\mu$, where $2\mu = 2\zeta + (2) \in 2O_1$ such that $\lambda_1 - 1 \equiv \lambda_2 \pmod 2$. Then there exists an LR tableau with weight $(r,r,r,2,1)$ $2\lambda/2\nu$ where $2\nu = 2\zeta + (2,2)$.

Remark 6.3.5. Note that by definition there exist no partitions $2\zeta + (2) \in O_2$.

Proof. Proof of Lemma 6.3.4 For both types of tableaux we obtain a subtableau R with weight $(2^4, 1^2)$ and obtain a corresponding tableau S with weight $(2^3, 1)$.

Let T have weight $(r,r,r,3,2,1)$ or $(r,r,r,3,1,1)$. If $\lambda_1 \equiv \lambda_2 \pmod 2$ then T contains a subtableau R with shape $\rho/2\sigma$ and weight $(2^4, 1^2)$, such that R contains a pair of cells on row 1 with content 1. We delete this pair of cells and, for each lower row, we relabel a cell with content i such that it has content $i-1$. The resulting tableau S is an LR-tableau with weight $(2^3, 1^2)$. The standardised concatenation $S|_s T/R$ yields an LR-tableau with weight $(r,r,r,2,2)$ if T has weight $(r,r,r,3,2,1)$, otherwise $S|_s T/R$ has weight $(r,r,r,2,1)$.

By definition the tableau $S|_sT/R$ is row-standard. The tableau is also column-standard by the argument that the content of cells on row i is relabeled exactly when an equal number of cells on rows $i-1$ and $i+1$ either have their content relabeled in an equivalent way, or a pair of cells is removed from row $i-1$. By this same logic, the word of $S|_sT/R$ is a lattice permutation. $\square$

We also have a reverse correspondence which establishes a one-to-one correspondence.

Lemma 6.3.6. *Suppose that T is an LR-tableau with shape $\lambda/2\nu$ and weight $(r,r,r,2,1)$ (respectively $(r,r,r,2,2)$), such that $2\nu = 2\zeta+(2)$ (or also $2\nu = 2\zeta+(2,2)$ if $\lambda_1 - 1 \equiv \lambda_2 \pmod 2$). Then there exists an LR-tableau with shape $\lambda/2\mu$ and weight $(r,r,r,3,1,1)$ (respectively $(r,r,r,3,2,1)$), such that $2\mu = 2\zeta$ (or also $2\mu = 2\zeta + (2)$ if $\lambda_1 - 1 \equiv \lambda_2 \pmod 2$).*

Proof. Note that using Appendix A.2.1 and Appendix A.2.3 that there exist exactly 15 tableaux with shape $\rho/2\tau$ and weight $(2^3,1^2)$ such that $\tau = \sigma+(2)$ or $\tau = \sigma+(2,2)$. If T is a tableau with shape $\lambda/2\nu$ and weight $(r,r,r,2,1)$ or $(r,r,r,2,2)$ such that $2\nu = 2\zeta+(2)$ or $2\nu = 2\zeta+(2,2)$, then T necessarily contains one of these subtableaux. By Lemma 6.3.4 each of these subtableaux corresponds to a tableau with weight $(2^4,1^2)$ via the algorithm defined in the lemma. Therefore, by reversing the algorithm, we obtain the reverse correspondence. $\square$

Lemma 6.3.7. *Let λ be a partition of six parts. Let the sets $2L$, $2N_1$, $2N_2$, $2Y_1$, $2Y_2$, be as follows,*

$$2L = (2\nu \,|\, 2\nu \subseteq \lambda\},$$

$$\begin{aligned} 2Y_1 = \{2\nu \,|\, 2\nu \subseteq \lambda \textit{ and } 2\nu = (2) + \zeta \\ \textit{for some } 2\zeta \subseteq \lambda\}, \end{aligned}$$

$$\begin{aligned} 2Y_2 = \{2\nu \,|\, 2\nu \subseteq \lambda \textit{ and } 2\nu = (2,2) + 2\zeta \\ \textit{for some } 2\zeta \subseteq \lambda\}, \end{aligned}$$

$$2N = \begin{cases} 2L \setminus (2Y_1 \cup 2Y_2), & \textit{if } \lambda_1 - 1 \equiv \lambda_2 \pmod 2; \\ 2L \setminus 2Y_1, & \textit{otherwise.} \end{cases}$$

If T is an LR-tableau with weight $(r, r, r, 2, 1)$ (respectively with weight $(r, r, r, 2, 2)$) and shape $\lambda/2\nu$ such that T does not correspond to an LR-tableau with weight $(r, r, r, 3, 1, 1)$ (respectively with weight $(r, r, r, 3, 2, 1)$) then $2\nu \in 2N$.

Proof. By Lemma 6.3.4 we have that if T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r, r, r, 3, 1, 1)$, then there exists a corresponding LR-tableau with shape $\lambda/2\nu$ and weight $(r, r, r, 2, 1)$, where $2\nu = 2\mu + (2)$ or $2\nu = 2\mu + (0, 2)$. Similarly, by 6.3.6 we see that by reversing the algorithm, we have that if S is an LR-tableau with shape $\lambda/2\nu$ and weight $(r, r, r, 2, 1)$ where $2\nu = 2\zeta + (2, 2)$ or $2\nu = 2\zeta + (2)$ for some $2\zeta \subseteq \lambda$, then there exists a corresponding LR-tableau with shape $\lambda/2\mu$ and weight $(r, r, r, 3, 1, 1)$, where $2\mu = 2\zeta$ or $2\mu = 2\zeta + (2)$. Therefore, the LR-tableaux with weight $(r, r, r, 2, 1)$ which do not have a corresponding LR-tableau with weight $(r, r, r, 3, 1, 1)$, are those which do not correspond to an even subpartition 2ν of the form $2\zeta + (2)$ or $2\zeta + (2, 2)$

By Lemma 6.3.4 we have that if T is an LR-tableau with shape $\lambda/2\mu$ and weight $(r, r, r, 3, 2, 1)$, then there exists a corresponding LR-tableau with shape $\lambda/2\nu$ and weight $(r, r, r, 2, 2,)$, where $2\nu = 2\mu + (2)$. Similarly, by reversing the algorithm as in Lemma 6.3.6 we have that if S is an LR-tableau with shape $\lambda/2\nu$ and weight $(r, r, r, 2, 2)$ where $2\nu = 2\mu + (2)$ for some $2\mu \subseteq \lambda$, then there exists a corresponding LR-tableau with shape $\lambda/2\mu$ and weight $(r, r, r, 3, 2, 1)$. Therefore, the LR-tableaux with weight $(r, r, r, 2, 2)$ which do not have a corresponding LR-tableau with weight $(r, r, r, 3, 2, 1)$, are those which do not correspond to an even subpartition 2ν of the form $2\mu + (2)$. □

Lemma 6.3.8. *Let λ be a partition of six parts.*

(i) If T is an LR-tableau with weight $(r, r, r, 2, 1)$ and shape $\lambda/2\mu$ for some $2\mu \in 2N$ and $r \geq 2$, then T contains a subtableau R with shape $\rho/2\xi$ and weight $(2^4, 1)$ such that $2\xi = 2\zeta + (2^j)$ for $0 \leq j \leq 2$.

(ii) If T is an LR-tableau with weight $(r, r, r, 2, 2)$ and shape $\lambda/2\mu$ for some $2\mu \in 2N$ and $r \geq 2$, then T contains a subtableau R with shape $\rho/2\xi$ and weight (2^5) such that $2\xi = 2\zeta + (2^j)$ for $0 \leq j \leq 2$.

Proof. If T is an LR-tableau with weight $(r, r, r, 2, 1)$ then T necessarily contains a subtableau R with shape $\rho/2\xi$ and weight $(2^4, 1)$, for some $\rho \subseteq \lambda$ and $2\xi \subseteq 2\mu$. Note that R is constructed from a tableau with weight (1^5) given in Appendix A.2.3 and a tableau with weight (1^4) given in Appendix A.2.2. The tableau with weight

(1^5) has shape $(2^m, 1^{6-m})/(1^j)$ for some $0 \leq m, j \leq 6$ (note that if $j = 6$ then we remove the empty column with shape $(1^6)/(1^6)$ and obtain a tableau in Appendix A.2.3), and the tableau with weight (1^4) has shape $(3^m, 2^n, 1^{6-m-n})/(2^j, 1^k)$ for some $0 \leq m, n, k \leq 6$ and $0 \leq j \leq 2$ with $m + n \leq 6$ (note again that if $j + k = 6$ then we remove the empty column with shape $(1^6)/(1^6)$ and obtain a tableau in Appendix A.2.2). By taking the concatenation and calculating the corresponding even subpartition 2μ, we see that 2μ has the form $2\mu = 2\zeta + (2^j)$ for some $0 \leq j \leq 2$.

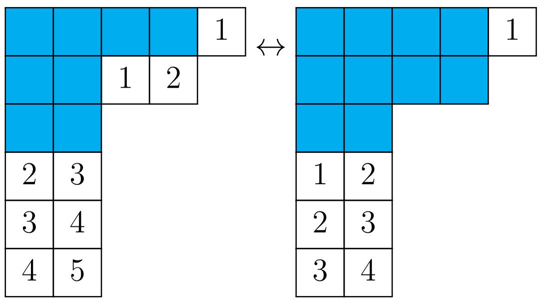
If T is an LR-tableau with weight $(r, r, r, 2, 2)$ then T necessarily contains a subtableau R with shape $\rho/2\xi$ and weight (2^5), for some $\rho \subseteq \lambda$ and $2\xi \subseteq 2\mu$. Note from Appendix A.2.3 that there are 25 such ways to construct this subtableau, each of which has shape $\rho/(4^j, 2^k)$ for some $0 \leq j \leq 2$ and $0 \leq k \leq 5 - j$. □

We now establish the correspondence between tableaux with weight $(2^4, 1)$ (respectively (2^5)) and a subset of tableaux with weight $(2^3, 1)$ (respectively (2^3)).

Lemma 6.3.9. *(i) Let T be an LR-tableau with shape $\rho/2\mu$ and weight $(2^4, 1)$, where $2\mu = 2\xi + (2^j)$ for some $0 \leq j \leq 2$. Then There exists an LR-tableau with shape $\rho/2\nu$ and weight $(2^3, 1)$, such that $2\nu = 2\xi + (2^{j+1})$.*

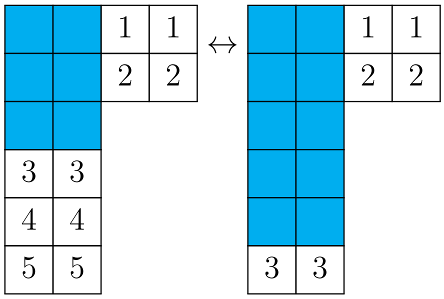
(ii) Let T be an LR-tableau with shape $\rho/2\mu$ and weight (2^5), where $2\mu = 2\xi + (2^j)$ for some $0 \leq j \leq 3$. Then There exists an LR-tableau with shape $\rho/2\nu$ and weight (2^3), such that $2\nu = 2\xi + (2^{j+2})$.

Example 6.3.10.



Example 6.3.11. Note that by this matching, we need not take a pair of cells from rows 1 and 2, even if both rows contain a pair of cells. As an example, take the LR-tableau with shape $(4^2, 2^4)/(2^3)$ and weight (2^5). By the algorithm we establish in the following proof, this tableau matches to the LR-tableau with shape $(4^2, 2^4)/(2^5)$

and weight (2^3).



Proof. Let T be an LR-tableau with shape $\rho/2\mu$ and weight $(2^4, 1)$, where $2\mu = 2\xi + (2^j)$ for some $0 \le j \le 2$. Then T contains a pair of non-empty cells on row $j+1$. We delete this pair of cells, and for each row beneath row $j+1$, we relabel each cell with content i so that the cell has content $i-1$. This yields an LR-tableau with shape $\rho/2\nu$ and weight $(2^3, 1)$, where $2\nu = 2\xi + (2^{j+1})$.

Let T be an LR-tableau with shape $\rho/2\mu$ and weight (2^5), where $2\mu = 2\xi + (2^j)$ for some $0 \le j \le 2$ chosen such that T has a pair of non-empty cells on rows $j+1$ and $j+2$. We delete the pairs of cells on rows $j+1$ and $j+1$, and for each row beneath row $j+2$, we relabel each cell with content i so that the cell has content $i-2$. This yields an LR-tableau with shape $\rho/2\nu$ and weight (2^3), where $2\nu = 2\xi + (2^{j+2})$. $\square$

We also establish the reverse correspondence to establish a one-to-one correspondence.

Lemma 6.3.12. *(i) Let T be an LR-tableau with shape $\rho/2\nu$ and weight $(2^3, 1)$, where $2\nu = 2\xi + (2^{j+1})$ for some $0 \le j \le 2$. Then There exists an LR-tableau with shape $\rho/2\mu$ and weight $(2^4, 1)$, such that $2\mu = 2\xi + (2^j)$.*

(ii) Let T be an LR-tableau with shape $\rho/2\nu$ and weight (2^3), where $2\nu = 2\xi + (2^{j+2})$ for some $0 \le j \le 2$. Then There exists an LR-tableau with shape $\rho/2\mu$ and weight (2^5), such that $2\mu = 2\xi + (2^j)$.

Proof. Let T be an LR-tableau with shape $\rho/2\nu$ and weight $(2^3, 1)$, where $2\nu = 2\xi + (2^{j+1})$. Then T contains, as subtableaux, an LR-tableau S of shape $\kappa/(1^{j+1})$ and LR-tableau U of shape $\zeta/(1^n)$ and . We note that these tableaux are contained, as subtableaux, in the tableau corresponding, via the algorithm given in Lemma 6.3.9 part (i), to the tableau R with shape $\rho/2\mu$ and weight $(2^4, 1)$. Therefore, we conclude that T corresponds to R via the algorithm given in Lemma 6.3.9 part (i), and we establish the reverse correspondence by reversing the algorithm. $\square$

Lemma 6.3.13. *Let λ be a partition of six parts. Let the sets $2L$, $2M_1$, $2M_2$, $2X_1$, $2X_2$ be as follows,*

$$2L = (2\nu \,|\, 2\nu \subseteq \lambda\},$$

$$\begin{aligned} 2X_1 = \{2\nu, \,|\, 2\nu \subseteq \lambda \text{ and } 2\nu = (2j+1)2\mu \\ \text{for some } 2\mu + (2^j) \in N\}, \end{aligned}$$

$$\begin{aligned} 2X_2 = \{2\nu, \,|\, 2\nu \subseteq \lambda \text{ and } 2\nu = (2j+2) + 2\mu \\ \text{for some } 2\mu + (2^j) \in N\}, \end{aligned}$$

$$2M_1 = 2L \setminus 2X_1,$$

$$2M_2 = 2L \setminus 2X_2,$$

(i) If T is an LR-tableau with weight $(r, r, r, 1)$ and shape $\lambda/2\nu$ such that T does not correspond to an LR-tableau with weight $(r, r, r, 2, 1)$ as described in Lemma 6.3.8 part (i), Then $2\nu \in 2M_1$.

(ii) If T is an LR-tableau with weight (r, r, r) and shape $\lambda/2\nu$ such that T does not correspond to an LR tableau with weight $(r, r, r, 2, 2)$ as described in Lemma 6.3.8 part (ii), Then $2\nu \in 2M_2$.

Proof. If T is an LR-tableau with shape $\lambda/2\nu$ and weight $(r, r, r, 1)$ such that T has a corresponding tableau with weight $(r, r, r, 2, 1)$, then by Lemma 6.3.8, Lemma 6.3.9, and Lemma 6.3.12, we have that T contains a subtableau R with shape $\rho/2\nu$ and weight $(2^3, 1)$, where $2\nu = 2\xi + (2^{j+1})$. Therefore, we conclude that $2\nu \in X_1$.

If T is an LR-tableau with shape $\lambda/2\nu$ and weight (r, r, r) such that T has a corresponding tableau with weight $(r, r, r, 2, 2)$, then by Lemma 6.3.8, Lemma 6.3.9, and Lemma 6.3.12, we have that T contains a subtableau R with shape $\rho/2\nu$ and weight (2^3), where $2\nu = 2\xi + (2^{j+2})$. Therefore, we conclude that $2\nu \in X_2$.

By Lemma 6.3.12 and Lemma 6.3.8 we have that $X_1 \cap M_1 = \emptyset$ and $X_2 \cap N_2 = \emptyset$. □

Lemma 6.3.14. *Let λ be a six part partition. Let M_1, M_2 be as described in Lemma 6.3.13. Then*

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{\substack{r \geq 1 \\ 2\nu \in 2M_1}} c^{\lambda}_{2\nu,(r,r,r,1)} + \sum_{\substack{r \geq 0 \\ 2\nu \in 2M_2}} c^{\lambda}_{2\nu,(r,r,r)},$$

where the coefficients $c^{\lambda}_{2\nu,\mu}$ are Littlewood-Richardson coefficients.

Proof. We have by Theorem 5.2.5 part (iv) that,

$$\begin{aligned}\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} =& \sum_{r\geq 0}\sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r)} + \sum_{r\geq 1}\sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r,1)} \\ &- \sum_{r\geq 2}\sum_{2\nu\subseteq\lambda}\left(c^{\lambda}_{2\nu,(r,r,r,2,1)} + c^{\lambda}_{2\nu,(r,r,r,2,2)}\right) \\ &+ \sum_{r\geq 3}\sum_{2\nu\subseteq\lambda}\left(c^{\lambda}_{2\nu,(r,r,r,3,1,1)} + c^{\lambda}_{2\nu,(r,r,r,3,2,1)}\right) \\ &- \sum_{r\geq 3}\sum_{2\nu\subseteq\lambda}\left(c^{\lambda}_{2\nu,(r,r,r,3,3,2)} + c^{\lambda}_{2\nu,(r,r,r,3,3,3)}\right).\end{aligned}$$

By Lemma 6.3.3, we obtain the following reduction of the formula by noting the correspondence between tableaux of weight $(r^3, 3^3)$ (respectively $(r^3, 3, 3, 2)$) and a subset of tableaux of weight $(r^3, 3, 1, 1)$ (respectively $(r^3, 3, 2, 1)$,

$$\begin{aligned}\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} =& \sum_{r\geq 0}\sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r)} + \sum_{r\geq 1}\sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r,1)} \\ &- \sum_{r\geq 2}\sum_{2\nu\subseteq\lambda}\left(c^{\lambda}_{2\nu,(r,r,r,2,1)} + c^{\lambda}_{2\nu,(r,r,r,2,2)}\right) \\ &+ \sum_{r\geq 3}\sum_{2\nu\in 2O_1}\left(c^{\lambda}_{2\nu,(r,r,r,3,1,1)} + c^{\lambda}_{2\nu,(r,r,r,3,2,1)}\right).\end{aligned}$$

By Lemma 6.3.7, we obtain a further reduction of the formula by noting the correspondence between tableaux of weight $(r^3, 3, 1, 1)$ (respectively $(r^3, 3, 2, 1)$) and a subset of tableaux of weight $(r^3, 2, 1)$ (respectively $(r^3, 2, 2)$,

$$\begin{aligned}\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} =& \sum_{r\geq 0}\sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r)} + \sum_{r\geq 1}\sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r,1)} \\ &- \sum_{r\geq 2}\sum_{2\nu\in N}\left(c^{\lambda}_{2\nu,(r,r,r,2,1)} + c^{\lambda}_{2\nu,(r,r,r,2,2)}.\right)\end{aligned}$$

By Lemma 6.3.9 and Lemma 6.3.13 we obtain the final reduction,

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{\substack{r\geq 1\\ 2\nu\in 2M_1}} c^{\lambda}_{2\nu,(r,r,r,1)} + \sum_{\substack{r\geq 0\\ 2\nu\in 2M_2}} c^{\lambda}_{2\nu,(r,r,r)}.$$

□

Theorem 6.3.15. *Let $\lambda \in \Lambda^+(7)$. For $\xi, \zeta \subseteq \lambda$ let $c^{\lambda}_{\xi,\zeta}$ denote the Littlewood-Richardson constants.*

(i) If $l(\lambda) = 3$ then

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \begin{cases} 1, & \text{if } \lambda_1 \equiv \lambda_2 \equiv \lambda_3 \pmod 2; \\ 0, & \text{otherwise.}\end{cases}$$

(ii) If $l(\lambda) = 4$ *then*

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{r\geq 0} \sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r)} + \sum_{r\geq 1} \sum_{2\nu\subseteq\lambda} c^{\lambda}_{2\nu,(r,r,r,1)}.$$

(iii) If $l(\lambda) = 5$ *then let* M_1*,* M_2 *be as defined in Lemma 6.2.6. Then*

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{\substack{r\geq 1\\ 2\nu\in 2M_1}} c^{\lambda}_{2\nu,(r,r,r,1)} + \sum_{\substack{r\geq 0\\ 2\nu\in 2M_2}} c^{\lambda}_{2\nu,(r,r,r)}$$

(iv) If $l(\lambda) = 6$ *then let* M_1*,* M_2 *be as defined in Lemma 6.3.14. Then*

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \sum_{\substack{r\geq 1\\ 2\nu\in 2M_1}} c^{\lambda}_{2\nu,(r,r,r,1)} + \sum_{\substack{r\geq 0\\ 2\nu\in 2M_2}} c^{\lambda}_{2\nu,(r,r,r)}.$$

(v) If $l(\lambda) = 7$ *then*

$$\dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\lambda)^{G_2} = \dim \nabla_{\mathrm{GL}_7(\Bbbk)}(\mu)^{G_2},$$

where μ *is the six part partition*

$$\mu = (\lambda_1 - \lambda_7, \dots, \lambda_6 - \lambda_7).$$

Proof. Parts (i), (ii) and (v) are due to Theorem 5.2.5. Part (iii) is due to Lemma 6.2.6 and part (iv) is due to Lemma 6.3.14. $\square$

— A —

Appendix A: Tableaux calculations

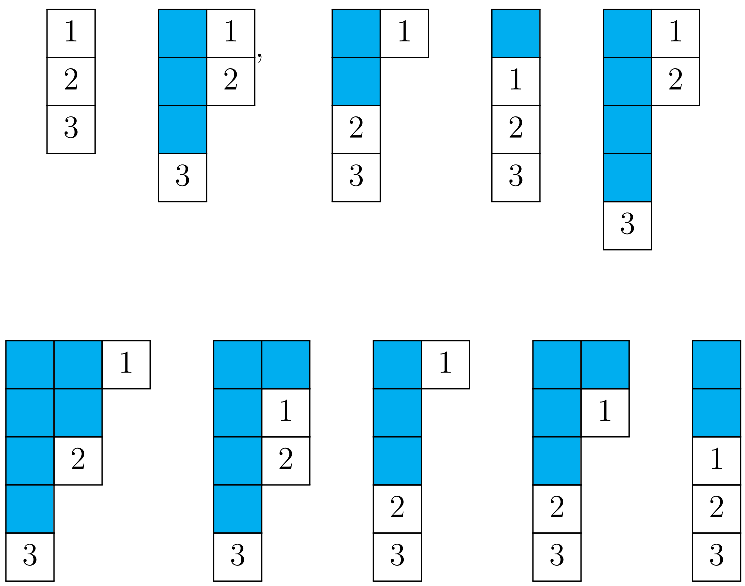
In this appendix we provide a list of tableaux from which we construct the tableaux with weight (1^4), $(2^4, 1)$, (2^5), and $(2^5, 1)$, which we use in the proofs of chapter six.

Generally speaking, given a diagram of n rows and a weight (1^k) for $k \leq n$, the number of distinct ways to draw a tableau which has at most n rows and weight (1^k) is the binomial coefficient $\binom{n}{k}$.

A.1 FIVE ROW TABLEAUX

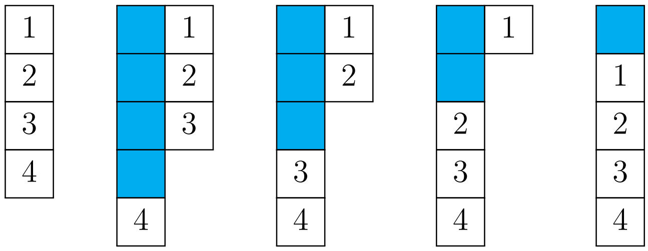
A.1.1 TABLEAUX OF WEIGHT (1^3)

There exist ten standard tableaux of up to five rows and weight (1^3).



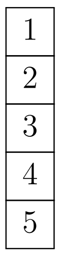
A.1.2 TABLEAUX OF WEIGHT (1^4)

There exist five standard tableaux of five rows and weight (1^4).



A.1.3 TABLEAUX OF WEIGHT (1^5)

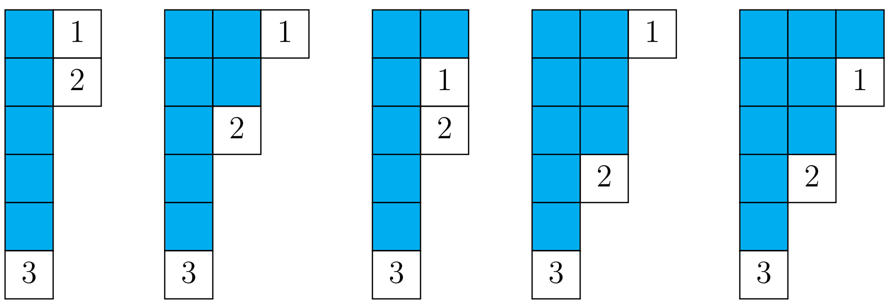
There exists a single standard tableau of five rows and weight (1^5).

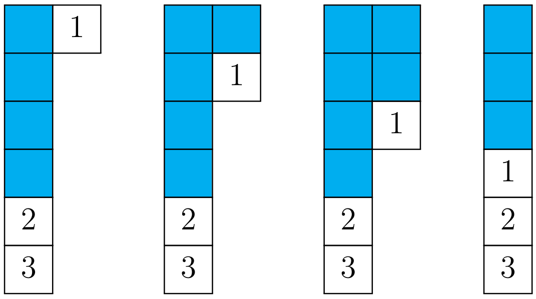


A.2 SIX ROW TABLEAUX

A.2.1 TABLEAUX OF WEIGHT (1^3)

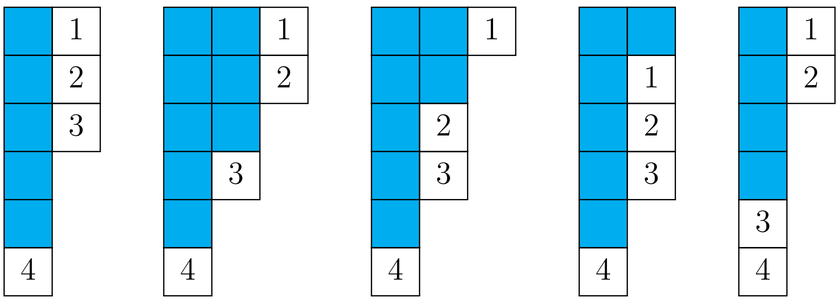
There exist twenty standard tableaux of six rows and weight (1^3); the ten tableaux of at most five rows listed above, and the following ten tableaux.

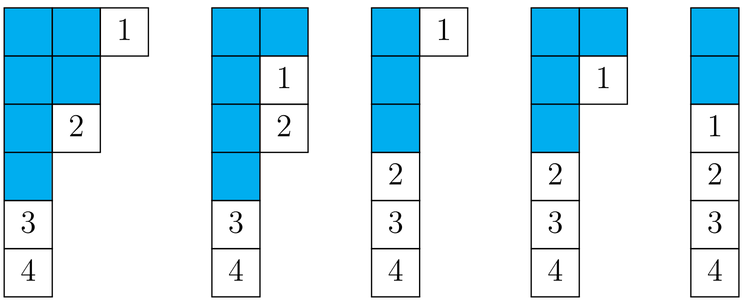




A.2.2 Tableaux of weight (1^4)

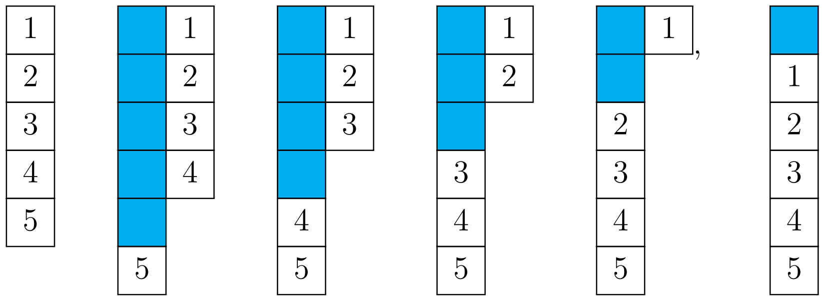
There exist fifteen standard tableaux of six rows and weight (1^4); the five tableaux of at most five rows listed above, and the following ten tableaux.





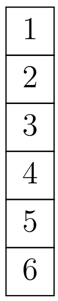
A.2.3 tableaux of weight (1^5)

There exist five standard tableaux of six rows with weight (1^5)



A.2.4 Tableaux of weight (1^6)

There exists a single standard six row tableau of weight (1^6).



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