Indestructibility and $C^{(n)}$-Supercompact Cardinals

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The candidate confirms that the work submitted is their own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Chapter 6 is comprised of joint work with John Howe and Rosario Mennuni, which has been published as 'On double-membership graphs of models of Anti-Foundation', by The Bulletin of Symbolic Logic [2].

The main results in this paper are: Proposition 3.3 (see 6.5.3); Theorem 3.4 (see 6.5.4); Proposition 3.5 (see 6.5.5); Proposition 4.3 (see 6.6.3); Corollary 4.5 (see 6.6.5); Lemma 5.5 (see 6.7.5); Corollary 5.8 (see 6.7.8); Corollary 5.11 (see 6.7.11); Theorem 5.14 (see 6.7.14); Corollary 5.15 (see 6.7.15); Theorem 5.16 (see 6.7.17).

Of these results the candidate was an essential contributor to Proposition 3.3; Theorem 3.4; Proposition 3.5; Proposition 4.3; Corollary 4.5; Lemma 5.5; Theorem 5.14; Corollary 5.15. The other results are credited among the other authors.

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Abstract

This thesis is split into two areas of interest. The first, a study of indestructibility results for two variants of supercompactness; the second, a discussion of double-membership graphs of models of Anti-Foundational set theory.

In Chapter 3 we will consider α-subcompact cardinals — which can be viewed as a weakened version of supercompact cardinals — and we will show that, by defining a suitable preparatory forcing, an α-subcompact cardinal (with $\alpha \in \mathrm{Reg}$) can be made indestructible by all $< \kappa$-directed closed forcing.

We will then turn our attention to stronger forms of supercompactness, namely $C^{(n)}$-supercompacts, and answer (in part) open questions about their indestructibility, by showing that, for a $C^{(2)}$-extendible, we can make its $C^{(2)}$-supercompactness indestructible by $< \kappa$-directed closed forcing.

We will then combine the concepts of α-subcompact cardinals and $C^{(n)}$-cardinals and show that, below an α-subcompact cardinal where $\alpha \in C^{(n)}$ with $n \geq 1$, there is a stationary set of partial extendibles below κ, determined by α-subcompactness embeddings for κ.

Lastly, we consider reducts of countable models of Anti-Foundational set theory to the double-membership relation D, where $D(x, y)$ if and only if $x \in y \in x$. We show that there are continuum many such graphs, and study their connected components. We describe their complete theories, and prove that each one has continuum-many countable models, some of which are not reducts of models of Anti-Foundation.

To disabled people in academia.

Contents

Chapter 1

Introduction

In 1978, Laver showed that, using a suitably defined preparatory forcing, it is possible to make the supercompactness of some κ indestructible by $< \kappa$-directed closed forcing. This can be seen as a starting point for the study of indestructibility results for large cardinals, a field which has seen many advances in the decades since Laver's result. It is natural to wonder which large cardinals have indestructibility results like this, and what variants of supercompactness may have similar techniques applied to them.

This work is comprised of two parts. The first, a study of indestructibility results for two large cardinals which may be considered variants of supercompactness, namely α-subcompact cardinals and $C^{(n)}$-supercompact cardinals. The second part, in Chapter 6, is a continuation of previous work in the field of Anti-Foundational set theory, from a joint paper .

The structure of the thesis is as follows. In Chapter 2 we provide some background concepts and results which will be relied upon later, specifically regarding the construction of ultrafilters and extenders, as well as a brief survey of supercompact cardinals and iterated forcing. We also discuss indestructibility results in general, and give an introduction to $C^{(n)}$-large cardinals, with specific results regarding $C^{(n)}$-extendibles and $C^{(n)}$-supercompacts which will be used in later chapters.

In Chapter 3 we give an introduction to *α-subcompact cardinals*, and show that, for $\alpha \in \mathrm{Reg}$, we may make the α-subcompactness of κ indestructible by all $< \kappa$-directed forcings in H_α. To do this we utilise the technique of lottery sums of minimal counterexamples, as seen in the work of Schlicht and Lücke ([30]) and Schlöder ([41]). This machinery is well-suited to large cardinals which are defined as the image of critical points of elementary embeddings.

In Chapter 4 we consider large cardinals which are strengthenings of supercompact cardinals, namely *$C^{(n)}$-supercompact cardinals*, introduced by Bagaria in 2012 ([6]). We show that, with the extra assumption that κ is $C^{(2)}$-extendible, we may make the $C^{(2)}$-supercompactness of κ indestructible by all $< \kappa$-directed closed forcing. For this we leverage the power of $C^{(2)}$-extendibles and utilise the correspondence between extenders and $C^{(n)}$-supercompactness elementary embeddings.

Chapter 5 contains a discussion of how we may combine the concepts of $C^{(n)}$-cardinals with α-subcompact cardinals, and how doing so results in many partial $C^{(n)}$-extendibles below any α-subcompact — so, in particular, since a supercompact cardinal κ is α-subcompact for all α, there are many partial $C^{(n)}$-extendibles below every κ, of increasing strength.

Finally, Chapter 6 contains joint work with John Howe and Rosario Mennuni, from a paper 'On double-membership graphs of models of Anti-Foundation' ([2]). In a previous paper ([1]) it is shown that, just as countable membership graphs of ZFC are isomorphic to the Random Graph, countable models of ZFA[1] (without multiple edges) are isomorphic to the *Random Loopy Graph*, the Fraïssé limit of finite graphs with loops. This structure has many desirable model-theoretic properties, and echoes the well-founded case in a pleasing way.

However, in this Chapter we show that considering the *double-membership* graphs — i.e reducts of the countable membership graph M to the binary relation

[1] ZFA is ZFC with the axiom of Foundation replaced by the *Anti-Foundation Axiom*, defined in § 6.3.

D where $D(x, y)$ iff $x \in y \wedge y \in x$ — gives a much less straightforward picture.

We show that there are infinitely many countable double-membership graphs of models of ZFA, and continuum-many countable models of each of their theories. We study the common theory of double-membership graphs, and show that it is incomplete. The completions of this theory are then characterised, by showing that double-membership graphs of two models of ZFA are elementarily equivalent precisely when the models satisfy the same consistency statements.

Further we show that, in ZFC \ Foundation, with no axiom of Anti Foundation, the double-membership graphs are almost arbitrary. Our final result gives that, for every (single-)double-membership graph, there is a countable elementarily equivalent structure which is not the (single-)double-membership graph of any model of ZFA.

Chapter 2

Preliminaries

This Chapter contains the main concepts and theorems which will be relied upon throughout the document. It is not intended to provide a thorough grounding in these topics, so references will be given to introductory texts for each Section.

2.1 Set Theory

In this Section we give a general overview of set theoretical concepts and key results in the field. We also fix our notation, all of which is standard. For any omitted or assumed definitions, we refer the reader to [27], [25], [33], [28] or to any other references given in specific Sections.

2.1.1 Notation

For a set x, $|x|$ denotes its *cardinality*, $\mathrm{TC}(x)$ its *transitive closure*, $\mathrm{rk}(x)$ its rank, and $\mathcal{P}(x)$ is its *power set*. For $\lambda \geq \kappa$, $\mathcal{P}_\kappa(\lambda)$ — or equivalently, $[\lambda]^{<\kappa}$ — consists of all subsets of λ of cardinality less than κ.

The class of all ordinals is denoted On, and Reg is the class of all regular cardinals.

For an ordinal α, H_α is the set of all sets x with $|\operatorname{TC}(x)| < \alpha$, i.e. the set of all sets of hereditary cardinality less than α. The *order type* of a well-ordered set X is denoted $\operatorname{ot}(X)$.

For $X \subseteq \mathrm{On}$, an limit ordinal $\gamma > 0$ is a *limit point* of X if $\sup(X \cap \gamma) = \gamma$. We denote the set of limit points of X by $\lim(X)$ and we denote by Lim the class of all limit ordinals.

If κ is a regular uncountable cardinal, a set $C \subset \kappa$ is *closed unbounded* (or *club*) in κ if C is unbounded in κ and contains all its limit points below κ. A set $S \subset \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for all club subsets of κ.

For an embedding j, its *critical point* is $\operatorname{cf}(j)$, its *range* is $\operatorname{range}(j)$ and its *domain* is $\operatorname{dom}(j)$. For any $X \subseteq \operatorname{dom}(j)$, we denote by $j \upharpoonright X$ the *restriction* of j to X. The *pointwise image* of some $x \subseteq \operatorname{dom}(j)$ under j is $j``(x) = \{j(y) : y \in x\}$.

A partial function f with domain X and range Y is denoted by $f \vdots X \to Y$.

Definition 2.1.2. For any cardinal α, a $\square_\alpha$-*sequence* (a 'square-alpha sequence') is $\langle C_\beta : \beta \in \alpha^+ \cap \mathrm{Lim}\rangle$ such that, for every $\beta \in \alpha^+ \cap \mathrm{Lim}$:

- C_β is a club subset of β;
- $\operatorname{ot}(C_\beta) \leq \alpha$;
- for any $\gamma \in \lim(C_\beta)$, $C_\gamma = C_\beta \cap \gamma$.

If there exists an α-sequence, then we say that $\square_\alpha$ holds.

The *Beth numbers* are defined by transfinite recursion as follows: $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ and $\beth_\lambda = \sup\{\beth_\alpha : \alpha < \lambda\}$ for λ a limit ordinal.

The Continuum Hypothesis (CH) states that $\beth_1 = \aleph_1$, i.e. $\aleph_1 = 2^{\aleph_0}$. The *Generalised Continuum Hypothesis* (or GCH), is the statement that, for all ordinals α, $\aleph_\alpha = \beth_\alpha$.

Until Chapter 6 we will be working in ZFC, making tacit use of choice throughout.

2.1.3 Elementarity

Throughout this thesis we will be dealing with many concepts involving elementarity: all the large cardinals discussed are defined in terms of elementary embeddings, and we deal in particular with *correctness* in Chapters 4 and 5, which stipulates the elementarity of substructures of V.

Definition 2.1.4. Let $\mathcal{M}_0 = \langle M_0, \dots \rangle$ and $\mathcal{M}_1 = \langle M_1, \dots \rangle$ be structures over the same language $\mathcal{L}$. An injective function $j : M_0 \to M_1$ is an *elementary embedding* of $\mathcal{M}_0$ into $\mathcal{M}_1$ (often written $j : \mathcal{M}_0 \prec \mathcal{M}_1$) if and only if, for all formulas $\varphi(v_1, \dots, v_n)$ of $\mathcal{L}$, and $x_1, \dots x_n \in \mathcal{M}_0$,

$$\mathcal{M}_0 \models \varphi[x_1, \dots, x_n] \iff \mathcal{M}_1 \models \varphi[j(x_1), \dots, j(x_n)]. \qquad (\dagger)$$

If, further, j is the identity on $\mathcal{M}_0$, then $\mathcal{M}_0$ is an *elementary substructure* of $\mathcal{M}_1$, written $\mathcal{M}_0 \prec \mathcal{M}_1$.

Where there is no confusion, we identify $\mathcal{M}_0$ and $\mathcal{M}_1$ with their underlying universes M_0 and M_1, respectively. Typically we omit mention of $\mathcal{L}$ if it is assumed clear — for our purposes here we will be dealing with non-trivial elementary embeddings $j : V \to M$, where V is the universe of sets and $M \subseteq V$ is a transitive proper class (hence $\mathcal{L} = \mathcal{L}_{\in} := \{\in\}$ is the language of set theory). Unless stated otherwise, all elementary embeddings will be non-trivial.

In later Chapters we will be concerned with Σ_n-*elementary embeddings*, and corresponding Σ_n-*elementary substructures*. Here recall that a formula is Σ_0 and Π_0 if all of its quantifiers are bounded. By induction, we define Σ_{n+1} formulas to be those which have the form $\exists x \varphi$, where φ is Π_n; and Π_{n+1} to be those of the form $\forall x \varphi$, where φ is Σ_n. If φ is both Σ_n and Π_n, then it is Δ_n.

Accordingly, a Σ_n (or Π_n) elementary embedding is $j : M_0 \to M_1$ with $\mathcal{M}_0$ and $\mathcal{M}_1$ as above, where ($\dagger$) holds for all formulas φ which are Σ_n (or Π_n). This is denoted

$j : \mathcal{M}_0 \prec_n \mathcal{M}_1$.

Sometimes we will need to be able to determine whether a substructure is elementary or not, for this we will use the *Tarski-Vaught criterion*, namely:

Theorem 2.1.5. (Tarski, Vaught, as seen in [24]) Let $\mathcal{L}$ be a first-order language and let $\mathcal{M}_0$ and $\mathcal{M}_1$ be $\mathcal{L}$-structures with $\mathcal{M}_0 \subseteq \mathcal{M}_1$. Then the following are equivalent:

- $\mathcal{M}_0$ is an elementary substructure of $\mathcal{M}_1$
- For every formula $\varphi(v_1, \dots v_n, y)$ of $\mathcal{L}$ and all $x_1, \dots x_n$ in $\mathcal{M}_0$, $\mathcal{M}_1 \models \exists y\, \varphi(x_1, \dots x_n, y) \rightarrow \mathcal{M}_1 \models \varphi(x_1, \dots x_n, z)$ for some element z of $\mathcal{M}_0$.

2.1.6 Ultrafilters

Ultrafilters and ultrapowers have been used extensively in set theory since the 1960's, as they provide equivalent formulations of many large cardinals. Their construction is briefly outlined here.

Definition 2.1.7. For a non-empty set X, $\mathcal{U} \subseteq \mathcal{P}(X)$ is an *ultrafilter* over X if

- It is a *filter*, namely:
  - $X \in \mathcal{U}$ and $\emptyset \notin \mathcal{U}$;
  - If $A \in \mathcal{U}$ and $B \in \mathcal{U}$ then $A \cap B \in \mathcal{U}$;
  - If $A, B \subseteq X$, $A \in \mathcal{U}$ and $A \subseteq B$, then $B \in \mathcal{U}$.
- Is is *maximal*, namely there is no filter F over X such that $\mathcal{U} \subseteq F$ but $F \neq \mathcal{U}$. Equivalently, for every $A \subseteq X$, $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Definition 2.1.8. An ultrafilter $\mathcal{U}$ over $\mathcal{P}(X)$ is *κ-complete* for some κ if, for any $\gamma < \kappa$ and $\{X_\alpha : \alpha < \gamma\} \subseteq \mathcal{U}$, $\bigcap_{\alpha<\gamma} X_\alpha \in \mathcal{U}$.

We will mainly be interested in ultrafilters over $\mathcal{P}_\kappa(\lambda)$ (also denoted $[\lambda]^\kappa$), since they offer a characterisation of supercompactness (see §2.1.15), and because we use them to define extenders (see §2.1.11).

Definition 2.1.9. An ultrafilter $\mathcal{U}$ over $\mathcal{P}_\kappa(\lambda)$ is *normal* if:

- It is *κ-complete*, i.e. for any collection $\{X_\alpha : \alpha < \gamma\}$ with $\gamma < \kappa$ and each $X_\alpha \in \mathcal{U}$, then $\bigcap_{\alpha<\gamma} X_\alpha \in \mathcal{U}$;
- It is *fine*, i.e. for any $\gamma < \lambda$, $\{X \in \mathcal{P}_\kappa(\lambda) : \gamma \in X\} \in \mathcal{U}$;
- It is closed under *diagonal intersection*, namely, for any $\{X_\alpha : \alpha < \lambda\}$ with $X_\alpha \in \mathcal{U}$ for all $\alpha < \lambda$, $\Delta_{\alpha<\lambda} X_\alpha = \{x \in \mathcal{P}_\kappa(\lambda) : x \in \bigcap_{i\in x} X_i\} \in \mathcal{U}$.

From an ultrafilter $\mathcal{U}$ over a set X we may derive its *ultraproduct* as follows. For each $i \in X$ let $\mathcal{M}_i = \langle M_i, \ldots\rangle$ be a structure over some fixed language $\mathcal{L}$, with universe M_i. Let $\prod_{i\in X} M_i$ be the Cartesian product of all the M_i, so elements are functions f with domain X such that $f(i) \in M_i$ for all $i \in X$. We construct an equivalence relation $=_\mathcal{U}$ on $\prod_{i\in X} M_i$ by:

$$f =_\mathcal{U} g \iff \{i \in X : f(i) = g(i)\} \in \mathcal{U}$$

and we denote by $[f]_\mathcal{U}$ the equivalence class of f (or, when clear, we simply denote it $[f]$, or even f). Let $\prod_{i\in X} M_i/\mathcal{U} = \left\{[f]_\mathcal{U} : f \in \prod_{i\in X} M_i\right\}$. For any n-ary predicate symbol in $\mathcal{L}$ interpreted in $\mathcal{M}_i$ by the n-ary relation $R_i \subseteq M_i^n$, we define its interpretation by:

$$\langle [f_1] \ldots, [f_n]\rangle \in R_\mathcal{U} \iff \left\{i \in X : \langle f_1(i) \ldots, f_n(i)\rangle \in R_i\right\} \in \mathcal{U}.$$

We interpret function and constant symbols the same way, so that we can define the ultraproduct of the $\mathcal{M}_i$ to be an $\mathcal{L}$-structure denoted by $\prod_{i\in X} \mathcal{M}_i/\mathcal{U}$, which has domain $\prod_{i\in X} M_i/\mathcal{U}$.

A fundamental result regarding ultraproducts and their first-order theory is Łos's Theorem:

Theorem 2.1.10. (Łos) For a formula $\varphi(v_1, \dots v_n)$ and $f_1, \dots, f_n \in \prod_{i\in X} M_i$,

$$\prod_{i\in X} \mathcal{M}_i/\mathcal{U} \vDash \varphi\left([f_1], \dots, [f_n]\right) \iff \left\{i \in X \ : \ \mathcal{M}_i \vDash \varphi\left(f_1(i), \dots, f_n(i)\right)\right\} \in \mathcal{U}.$$

If we construct an ultraproduct where all the $\mathcal{M}_i$ are equal to some $\mathcal{M} = \langle M, \dots\rangle$, this is called the *ultrapower* of $\mathcal{M}$ by $\mathcal{U}$, and is denoted ${}^X\mathcal{M}/\mathcal{U}$, or $\mathrm{Ult}(\mathcal{M}, \mathcal{U})$.

For $x \in M$ define $c_x : X \to \{x\}$ to be the constant function for x. Then the embedding $j_{\mathcal{U}} : \mathcal{M} \to \mathrm{Ult}(\mathcal{M}, \mathcal{U})$ defined by $j_{\mathcal{U}}(x) = [c_x]_{\mathcal{U}}$ is an elementary embedding. Provided $\mathrm{Ult}(V, \mathcal{U})$ is well-founded[1] we may then take the Mostowski collapse $\langle M, \in\rangle$ of $\mathrm{Ult}(V, \mathcal{U})$, given by the collapsing map $\pi : \mathrm{Ult}(V, \mathcal{U}) \to \langle M, \in\rangle$, where $\langle M, \in\rangle$ is a transitive class. Using this, we define the elementary embedding $j : V \to M$ by $j = \pi \circ j_{\mathcal{U}}$. It is common to abuse notation and identify j with $j_{\mathcal{U}}$.

2.1.11 Extenders

Given an elementary embedding $j : V \to M$ between inner models it is possible to derive an approximation for j using sequences of ultrafilters, called an *extender*. The formulation we will present is as seen in Cummings's [14] and Kanamori's [27] work, though their construction comes from Dodd and Jensen (see [15]), themselves simplifying work of Mitchell. This section is not intended to be a full introduction, and details omitted can be found in the above sources, and the Appendix of Tsaprounis's [43], as well as Schindler's [40].

First we give details on how to derive an extender from an elementary embedding.

[1]That $\mathrm{Ult}(V, \mathcal{U})$ is well-founded is actually equivalent to $\mathcal{U}$ being $\aleph_1$-complete (see §5 of [27].)

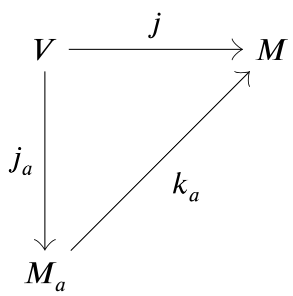
Definition 2.1.12. Suppose that $j : V \to M$ is an elementary embedding into a transitive inner model M, with $\mathrm{cp}(j) = \kappa$. Let $\kappa < \lambda$ and let $\zeta \geq \kappa$ be the least ordinal such that $\lambda \leq j(\zeta)$. For each $a \in [\lambda]^{<\omega}$, define E_a, an ultrafilter on $[\zeta]^{|a|}$ by:

$$X \in E_a \iff a \in j(X)$$

Now define $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$, which we call the *(ordinary)* (κ, λ)*-extender derived from* j.

The basic (and perhaps most common) presentation of such an extender carries the assumption that $\zeta = \kappa$, however we will later be exploiting this more generalised version of the definition. We say that E is *long* if $\lambda > j(\kappa)$, otherwise it is called *short*.

From each of the ultrafilters E_a we can construct ultrapowers, which, by the κ-completeness of each E_a, will be well-founded. Let $M_a \cong \mathrm{Ult}(V, E_a)$ be the transitive collapse of the ultrapower. For each a the following diagram commutes:



Here $j_a(x) = [c_x^a]_{E_a}$ for each $x \in V$, where $c_x^a : [\zeta]^{|a|} \to \{x\}$ is the constant function, and $k_a([f]_{E_a}) = j(f)(a)$ for each $f : [\zeta]^{|a|} \to V$. Standard arguments show that, for each $a \in [\lambda]^{<\omega}$, k_a is well-defined and elementary, and the diagram is commutative.

Each of the M_a are interrelated by functions defined using projections between the $[\zeta]^{|a|}$. For $a \subseteq b$ in $[\lambda]^{<\omega}$, these projection functions $\pi_{ba} : [\zeta]^{|b|} \to [\zeta]^{|a|}$ can be thought of as 'projecting down' according to the relation between $b = \{\gamma_1, \dots, \gamma_n\}$

(where $\gamma_1 < \ldots < \gamma_n$) and $a = \{\gamma_{i_1}, \ldots, \gamma_{i_m}\}$ (where $1 \leq i_1 < \ldots < i_m \leq n$). Specifically, $\pi_{ba}\left(\left\{\delta_1, \ldots, \delta_n\right\}\right) = \left\{\delta_{i_1}, \ldots, \delta_{i_m}\right\}$, where $\delta_1 < \ldots < \delta_n$.

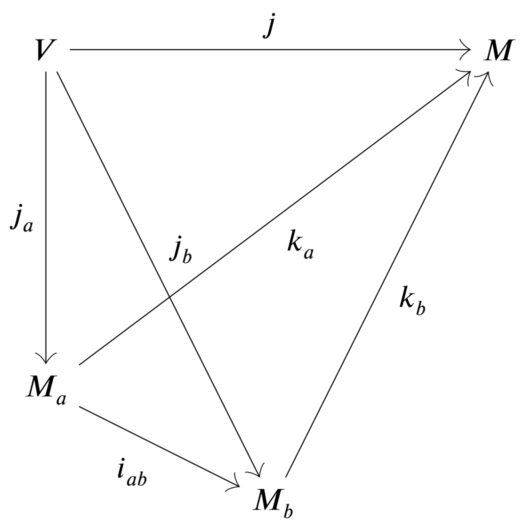
A key property satisfied by the ultrafilters E_a involves the projections π_{ba}, namely the *coherence property*: that for all $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$,

$$X \in E_a \iff \left\{x \in [\zeta]^{|b|} \,:\, \pi_{ba}(x) \in X\right\} \in E_b.$$

Using the π_{ab} we now define elementary embeddings $i_{ab} : M_a \to M_b$ by:

$$i_{ab}\left([f]_{E_a}\right) = \left[f \circ \pi_{ba}\right]_{E_b}$$

for all $f : [\zeta]^{|a|} \to V$ in V. These embeddings are also well-defined, elementary and commute — this is shown by applying the coherence property alongside standard arguments. We obtain the following commutative diagram:



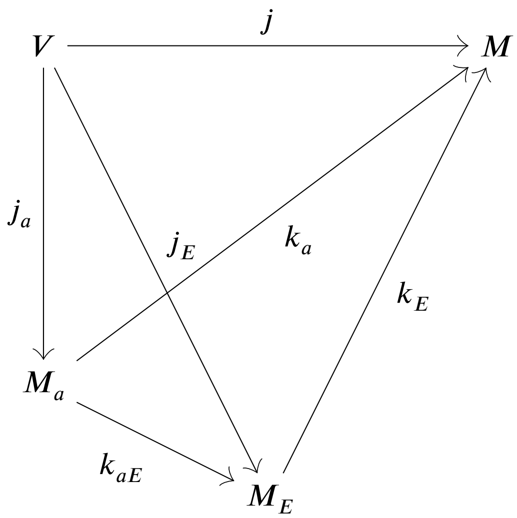
Now, with these systems of embeddings and ultrafilters, we define the directed system $\langle\langle M_a \,:\, a \in [\lambda]^{<\omega}\rangle; \langle i_{ab} \,:\, a \subseteq b \in [\lambda]^{<\omega}\rangle\rangle$. Its corresponding *direct limit* is $\widetilde{M_E}$, obtained by taking equivalence classes of a suitably defined equivalence relation. A little work gives us that $\widetilde{M_E}$ is well-founded (since it can be viewed as a substructure of M), and its members have the form $\left[\langle a, [f]_{E_a}\rangle\right]_E$ where $a \in [\lambda]^{<\omega}$

and $f : [\zeta]^{|a|} \to V$. Let M_E be the transitive collapse of $\widetilde{M}_E$. We will identify M_E with $\widetilde{M}_E$ throughout.

Now we define elementary embeddings between M_E, the M_a and M by:

- $j_E : V \to M_E$, with $j_E(x) = [a, [c^a_x]]$, for $a \in [\lambda]^{<\omega}$;
- $k_{aE} : M_a \to M_E$ with $k_{aE}([f]) = [a, [f]]$, for $a \in [\lambda]^{<\omega}$ and $f : [\zeta]^{|a|} \to V$;
- $k_E : M_E \to M$ with $k_E([a, [f]]) = j(f)(a)$ for $a \in [\lambda]^{<\omega}$ and $f : [\zeta]^{|a|} \to V$.

We then obtain the following commutative diagram:



The structure M_E, and the embedding j_E, have some valuable properties, which will be used in later arguments regarding $C^{(n)}$-supercompact cardinals. Namely:

- $\mathrm{cp}(j_E) = \kappa$ and $j_E(\zeta) \geq \lambda$, and if $\lambda = j(\kappa)$ then $j_E(\kappa) = j(\kappa) = \lambda$;
- $M_E = \left\{ j_E(f)(a) : a \in [\lambda]^{<\omega},\ f : [\zeta]^{|a|} \to V,\ f \in V \right\}$.

The proof of these results can be found, for example, in Lemma 26.1 of [27].

Now, having approached extenders from one direction by deriving them from an elementary embedding, we show that we may also directly construct an extender from ultrafilters, and from this extender derive an elementary embedding $j_E : V \to M_E$, much like the one constructed above.

Definition 2.1.13. Let κ be a regular cardinal and let $\lambda > \kappa$. Then $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$ is a (κ, λ)-extender if, for some $\zeta \geq \kappa$:

1. (a) For all $a \in [\lambda]^{<\omega}$, E_a is a κ-complete ultrafilter on $[\zeta]^{|a|}$;

   (b) There is some $a \in [\lambda]^{<\omega}$ such that E_a is not κ^+-complete;

   (c) For all $\gamma < \zeta$ there is an $a \in [\lambda]^{<\omega}$ such that $\{x \in [\zeta]^{|a|} : \gamma \in x\} \in E_a$.

2. (Coherence) For all $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$,

$$X \in E_a \iff \left\{x \in [\zeta]^{|b|} : \pi_{ba}(x) \in X\right\} \in E_b.$$

3. (Normality) If there is an $a \in [\lambda]^{<\omega}$ and $f : [\zeta]^{|a|} \to V$ such that

$$\left\{x \in [\zeta]^{|a|} : f(x) \in \max(x)\right\} \in E_a$$

   then there is some b with $a \subseteq b$ such that

$$\left\{x \in [\zeta]^{|b|} : f \circ \pi_{ba}(x) \in x\right\} \in E_b$$

4. (Well-foundedness) For any $a_n \in [\lambda]^{<\omega}$ and $X_n \subseteq [\zeta]^{|a_n|}$ with $X_n \in E_{a_n}$ for all $n \in \omega$, there is an order-preserving function $f : \bigcup_{n\in\omega} a_n \to \zeta$ such that $f``a_n \in X_{a_n}$ for all $n \in \omega$. Equivalently, the direct limit $\widetilde{M_E}$ is well-founded.

Here we construct the direct limit $\widetilde{M_E}$ as before: by first constructing the ultrapowers for each E_a, the defining projection functions and elementary embeddings, then the directed system with direct limit $\widetilde{M_E}$. Then, using the well-foundedness of $\widetilde{M_E}$ we obtain, as before, the transitive collapse M_E, and the elementary embedding $j_E : V \to M_E$.

The same nice properties hold for M_E and j_E as before, namely:

- $\mathrm{cp}(j_E) = \kappa$ and ζ is the least ordinal such that $j_E(\zeta) \geq \lambda$;
- $M_E = \left\{ j_E(f)(a) : a \in [\lambda]^{<\omega},\ f : [\zeta]^{|a|} \to V,\ f \in V \right\}$.

Every element of $\widetilde{M_E}$ is of the form $x = \left[\langle a, [f]_{E_a}\rangle\right]_E$ for some $a \in [\lambda]^{<\omega}$ and $[f]_{E_a} \in M_a$, where $f : [\zeta]^{|a|} \to V$. For notational ease we will express this simply as $x = [a, [f]]$.

For elements of M_E, we can, using this characterisation, form equivalents for equality and membership as follows:

Lemma 2.1.14. For $a \in [\lambda]^{<\omega}$ and $f : [\zeta]^{|a|} \to V$ we have that:

$$[a, [f]] =_E [b, [g]] \iff j_E(f)(a) = j_E(g)(b)$$
$$[a, [f]] \in_E [b, [g]] \iff j_E(f)(a) \in j_E(g)(b).$$

So in fact we have that every extender is derived from an elementary embedding, and that from an extender we can derive an elementary embedding. Despite this correspondence we will be careful to specify whether the extenders we use are derived from an elementary embedding or vice-versa.

As with extenders derived from an elementary embedding, we say that the extender is *long* if $\lambda > j_E(\kappa)$, otherwise it is *short*.

2.1.15 Supercompact cardinals

Many large cardinals are constructed using elementary embeddings, often as the critical point of some non-trivial $j : V \prec M$, where M is a transitive class model of ZFC. Supercompact cardinals can be formulated in this way:

Definition 2.1.16. For $\lambda \geq \kappa$ we say that κ is *λ-supercompact* if there exists and elementary embedding $j : V \to M$ with M a transitive class such that ${}^{\lambda}M \subseteq M$, where the critical point of j is κ and $j(\kappa) > \lambda$. If κ is λ-supercompact for all $\lambda \geq \kappa$, then we say it is *supercompact*.

An equivalent definition, shown in [32], will be relevant to later discussions of α-subcompact cardinals, as, under this characterisation, the supercompact cardinal κ is no longer the critical point of the elementary embedding, but the image of the critical point.

Definition 2.1.17. (Magidor Characterisation) Let $\lambda \geq \kappa$. Then κ is λ-supercompact if and only if, for some $\gamma < \kappa$, there exists an elementary embedding $j : V_\gamma \to V_\lambda$ such that $j\,(\mathrm{cp}(j)) = \kappa$.

One final alternative characterisation which we will discuss is one in terms of normal ultrafilters over $\mathcal{P}_\kappa(\lambda)$, namely:

Theorem 2.1.18. (Solovay, Reinhardt) A cardinal κ is λ-supercompact for $\lambda \geq \kappa$ if and only if there exists a normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

2.1.19 Forcing

In this section we provide a survey of key forcing results and concepts which will be used later. It is not intended to be an introduction to the rich topic of forcing and its numerous applications across set theory. For a thorough treatment, see [28], [25] or [27], to name but a few.

Notation

We denote using blackboard bold capital letters any posets used in forcing constructions, e.g. $\mathbb{P}$ and $\mathbb{Q}$, and we suppress, unless necessary for clarity, discussion of the ordering $\leq_{\mathbb{P}}$, merely referring to it as $\leq$. Conditions are p and q in $\mathbb{P}$, and $p < q$ means that p is stronger than q, or extends it. The greatest element of a forcing poset $\mathbb{P}$ is denoted $\mathbb{1}$, or $\mathbb{1}_{\mathbb{P}}$ when it is important to differentiate it from some other $\mathbb{1}_{\mathbb{Q}}$.

We denote a $\mathbb{P}$-*name* by $\dot{p}$. The *canonical name* for x is recursively defined as $\check{x} = \{\langle \mathbb{1}, \check{y}\rangle : y \in x\}$. For a $\mathbb{P}$-name $\dot{x}$ and G an M generic filter, the *interpretation* of $\dot{x}$ is denoted $\dot{x}^G$, and is defined recursively as $\{\dot{y}^G : \exists p \in G, \langle p, \dot{y}\rangle \in \dot{x}\}$. This notation may be suppressed when it is clear from context.

A notion of forcing $\mathbb{P}$ has the κ-*chain condition* (or, the κ-c.c. for brevity) if it has no antichain of size κ. Here $A \subseteq \mathbb{P}$ is an *antichain* if every pair of $p, q \in A$ are incompatible, namely there is no r such that $r \leq p$ and $r \leq q$.

It is κ-*closed* if, for every decreasing sequence of conditions $D = \{d_\alpha : \alpha < \gamma\} \subseteq \mathbb{P}$ with $\gamma < \kappa$, there is some $p \in \mathbb{P}$ such that $p \leq d_\alpha$ for all $\alpha < \gamma$.

A forcing $\mathbb{P}$ is $< \kappa$-*directed closed* if, whenever $D \subseteq \mathbb{P}$ is directed, with $|D| < \kappa$, there is some $p \in \mathbb{P}$ such that $p \leq d$ for all $d \in D$. Here, D is *directed* if for any d_1, $d_2 \in D$ there is some $e \in D$ such that $e \leq d_1$ and $e \leq d_2$.

A weaker (but nonetheless useful) property is defined using the following game:

Definition 2.1.20. Let $\mathbb{P}$ be a notion of forcing, and let α be an ordinal. The game $G_\alpha(\mathbb{P})$ has two players, 'Odd' and 'Even', and is played with perfect information. The players take turns playing conditions from $\mathbb{P}$, for up to α many moves. Odd plays at odd stages and Even plays at even (and limit) stages. In move zero, Even plays $\mathbb{1}$. If p_β is the condition played at turn β, then the player who played p_β loses immediately unless $p_\beta \leq p_\gamma$ for all $\gamma < \beta$. If neither player loses at any stage before α, then Even wins.

A notion of forcing $\mathbb{P}$ is $< \kappa$-*strategically closed* if, for all $\alpha < \kappa$, player Even has a winning strategy for the game $G_\alpha(\mathbb{P})$. If player Even has a winning strategy for $G_\kappa(\mathbb{P})$, then $\mathbb{P}$ is κ-strategically closed.

Iterated Forcing

The forcings utilised in later chapters will mostly be long iterations of ‘well-behaved’ forcings. We summarise here briefly how such iterations are generally constructed, as well as specific types of iteration which will be used later. See [14] for any omitted detail.

First we construct a *two-step iteration*, using some forcing $\mathbb{P}$, and a $\mathbb{P}$-name for a forcing $\dot{\mathbb{Q}}$.

Definition 2.1.21. Let $\mathbb{P}$ be a notion of forcing. A $\mathbb{P}$-name $\dot{x}$ is *canonical* if there is no $\mathbb{P}$-name $\dot{y}$ such that $|\,\mathrm{TC}(\dot{y})| < |\,\mathrm{TC}(\dot{x})|$ and $\mathbb{1} \Vdash \dot{x} = \dot{y}$. If $\mathbb{Q}$ is a $\mathbb{P}$-name for a notion of forcing, then $\mathbb{P} * \dot{\mathbb{Q}}$ is the set of all pairs $(p, \dot{q})$, such that $p \in \mathbb{P}$, $\mathbb{1} \Vdash \dot{q} \in \dot{\mathbb{Q}}$ and $\dot{q}$ is canonical. It is ordered by $(p_0, \dot{q}_0) \leq (p_1, \dot{q}_1)$ if and only if $p_0 \leq p_1$ (with respect to the ordering on $\mathbb{P}$) and $p_0 \Vdash \dot{q}_0 \leq \dot{q}_1$.

Now we build iterations of arbitrary length as follows:

Definition 2.1.22. An iteration of forcing of length α is an object of the form

$$\left(\left\langle \mathbb{P}_\beta : \beta \leq \alpha \right\rangle, \left\langle \dot{\mathbb{Q}}_\beta : \beta < \alpha \right\rangle\right)$$

where, for every $\beta \leq \alpha$:

- $\mathbb{P}_\beta$ is a notion of forcing whose elements are β-sequences;
- If $p \in \mathbb{P}_\beta$ and $\gamma < \beta$ then $p \restriction \gamma \in \mathbb{P}_\gamma$;
- If $\beta < \alpha$ then $\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}_\beta$ is a notion of forcing;
- If $p \in \mathbb{P}_\beta$ and $\gamma < \beta$, then $p(\gamma)$ is a $\mathbb{P}_\gamma$-name for an element of $\dot{\mathbb{Q}}_\gamma$;
- If $\beta < \alpha$ then $\mathbb{P}_{\beta+1} \cong \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$, via the map $f : h \mapsto (h \restriction \beta, h(\beta))$;
- If $p, q \in \mathbb{P}_\beta$ then $p \leq_{\mathbb{P}_\beta} q$ if and only if $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} p(\gamma) \leq_{\dot{\mathbb{Q}}_\gamma} q(\gamma)$ for all $\gamma < \beta$;

- $\mathbb{1}_{\mathbb{P}_\beta}(\gamma) = \dot{\mathbb{1}}_{\mathbb{Q}_\gamma}$ for all $\gamma < \beta$;
- If $p \in \mathbb{P}_\beta$ and $\gamma < \beta$, with $p \leq_{\mathbb{P}_\gamma} p \restriction \gamma$, then $q^\frown p \restriction [\gamma, \beta) \in \mathbb{P}_\beta$.

For notational ease, we refer to the iteration $\left(\left\langle \mathbb{P}_\beta : \beta \leq \alpha \right\rangle, \left\langle \dot{\mathbb{Q}}_\beta : \beta < \alpha \right\rangle\right)$ as $\mathbb{P}_\alpha$, and write $\Vdash_\beta$ instead of $\Vdash_{\mathbb{P}_\beta}$.

Sometimes, however, it is useful to weaken the definition of an iteration of length α to merely be some pair $\left(\langle \mathbb{P}_\beta : \beta < \alpha \rangle, \langle \dot{\mathbb{Q}}_\beta : \beta < \alpha \rangle\right)$ where each $\mathbb{P}_\beta$ is a forcing with conditions are sequences of length β; $\dot{\mathbb{Q}}_\beta$ is a $\mathbb{P}_\beta$-name for a forcing notion for all $\beta < \alpha$; $\mathbb{P}_{\beta+1} \cong \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta$ and the restriction map from $\mathbb{P}_\beta$ to $\mathbb{P}_\gamma$ is a projection for all $\gamma < \beta$. This will allow us some flexibility with our definitions, while maintaining the important properties of our iterations.

When constructing iterated forcing notions one has some choice over what to do at limit stages, depending on the intended purpose of the iteration. In later Chapters we will only need two types of limit behaviour, namely, for a limit ordinal α and an iteration $\mathbb{P}_\alpha$:

Definition 2.1.23.
- The *inverse limit*, often denoted $\varprojlim \mathbb{P} \restriction \alpha$, is the set of sequences p of length α such that, for all $\beta < \alpha$, $p \restriction \beta \in \mathbb{P}_\beta$;
- The *direct limit*, often denoted $\varinjlim \mathbb{P} \restriction \alpha$, is the subset of the inverse limit where each p is such that $p(\beta) = \dot{\mathbb{1}}_{\mathbb{Q}_\alpha}$ for sufficiently large β.

For an iteration of length α, the *support* of some condition $p \in \mathbb{P}_\alpha$ is $\operatorname{supp}(p) = \{\beta < \alpha : p(\beta) \neq \dot{\mathbb{1}}_{\mathbb{Q}_\beta}\}$. So, for example, if a direct limit is taken at stage α, then $p \in \mathbb{P}_\alpha$ will have $\operatorname{supp}(p) = \{\beta < \gamma : \text{ for some } \gamma < \alpha\}$.

It turns out that to preserve chain condition properties with an iterated forcing notion, one should take many direct limits:

Lemma 2.1.24. (See [10]) Let α be a limit ordinal and $\mathbb{P}_\alpha$ an iteration of length α with a direct limit taken at stage α. Suppose $\kappa = \operatorname{cf}(\kappa) > \omega$, and, for all $\beta < \alpha$,

$\mathbb{P}_\beta$ is κ-c.c.. Also, if $\mathrm{cf}(\alpha) = \kappa$, then suppose that $\{\gamma < \alpha : \mathbb{P}_\gamma$ is a direct limit$\}$ is stationary in α. Then the iteration $\mathbb{P}_\alpha$ is κ-c.c..

In order to preserve closure properties, however, one should take many inverse limits:

Lemma 2.1.25. (See [10] and [14]) Let $\kappa = \mathrm{cf}(\kappa) > \omega$, and let X be any of the properties 'κ-closed', '$< \kappa$-directed closed' or 'κ-strategically closed'. Suppose that, for each $\beta < \alpha$, $\Vdash_\beta$'$\dot{\mathbb{Q}}_\beta$ is X'. Suppose also that all limits are either direct or inverse, with inverse limits at every limit stage of cofinality less than κ. Then $\mathbb{P}_\alpha$ has property X also.

This leads us to Easton's construction, which has as support a set of ordinals which is bounded at each regular cardinal.

Definition 2.1.26. An iteration with *Easton support* is an iteration where direct limits are taken at regular limit stages, and inverse limits are taken at all other limit stages.

We note that this type of iteration has also been referred to as a *reverse Easton support* iteration, to differentiate it from Easton's product construction, but it is now common to simply call this an Easton support iteration.

When working with iterated forcing later, we will perform a number of factorisation arguments, for which we will use the *Factor Lemma* (Lemma 21.8 in [25]). This result allows us to deduce that an iteration $\mathbb{P}_{\alpha+\beta}$ is sometimes equivalent to $\mathbb{P}_\alpha * \dot{\mathbb{P}}^{(\alpha)}_\beta$, where $\dot{\mathbb{P}}^{(\alpha)}_\beta$ is an iteration in $V^{\mathbb{P}_\alpha}$ of length β.

Theorem 2.1.27 (The Factor Lemma)**.** Let $\mathbb{P}_{\alpha+\beta}$ be an $\alpha+\beta$-length forcing iteration $\big(\langle \mathbb{P}_\xi : \xi \leq \alpha+\beta\rangle, \langle \dot{\mathbb{Q}}_\xi : \xi < \alpha+\beta\rangle\big)$ where each $\mathbb{P}_\xi$ is a direct or inverse limit. In $V^{\mathbb{P}_\alpha}$ let $\dot{\mathbb{P}}^{(\alpha)}_\beta$ be the forcing iteration of $\big(\langle \mathbb{P}_{\alpha+\xi} : \xi \leq \beta\rangle, \langle \dot{\mathbb{Q}}_{\alpha+\xi} : \xi < \beta\rangle\big)$, such that $\dot{\mathbb{P}}^{(\alpha)}_\xi$ is a direct or inverse limit, depending on whether $\mathbb{P}_{\alpha+\xi}$ is a direct or inverse limit.

Then if $\mathbb{P}_{\alpha+\xi}$ is an inverse limit for every limit ordinal $\xi \leq \beta$ with $\mathrm{cf}(\xi) \leq |\mathbb{P}_\alpha|$, then $\mathbb{P}_{\alpha+\beta}$ is isomorphic to $\mathbb{P}_\alpha * \dot{\mathbb{P}}^{(\alpha)}_\beta$.

In such an instance we say that we *factor* $\mathbb{P}_{\alpha+\beta}$ *as* $\mathbb{P}_\alpha * \dot{\mathbb{P}}^{(\alpha)}_\beta$.

2.2 Indestructibility

The concept of making a large cardinal *indestructible* by a certain class of forcings came to prominence in Laver's paper [29] where it was showed that a supercompact cardinal κ may be made indestructible by all $< \kappa$-directed closed forcings — that is, after some preparatory forcing, κ will remain supercompact after further forcing with any $< \kappa$-directed closed forcing.

In general, an indestructibility result has the form: for a large cardinal κ with large cardinal property $\mathcal{L}$, it is *indestructible by forcing of type* $\mathcal{A}$ if, for any forcing $\mathbb{Q}$ with property $\mathcal{A}$, κ maintains the large cardinal property after forcing with $\mathbb{Q}$. Typically (though not always) one has to first apply a preparatory forcing, as Laver did, and show that it preserves the large cardinal notion, and that after this forcing, any additional forcing in $\mathcal{A}$ cannot destroy the $\mathcal{L}$-largeness of κ.

The preparatory forcing used by Laver for this result is known as the *Laver Preparation*, which is defined relative to the *Laver function*, a partial function $f : \kappa \to V_\kappa$, with the property that, for all sets x and every $\lambda \geq |\mathrm{TC}(x)|$, there is a normal ultrafilter on $\mathcal{P}_\kappa\lambda$ such that $j_\lambda(f)(\kappa) = x$. Due to its anticipatory power, Laver functions are sometimes referred to as Laver $\diamondsuit$ (Laver diamond).

Given such a partial function, the Laver preparation $\mathbb{P}_\kappa$ is an Easton support iteration of length κ with an associated sequence of λ_i for $i < \kappa$, where here, at limit γ, we let $\lambda_\gamma = \sup_{\beta<\gamma} \lambda_\beta$. At successor stage $\gamma + 1 < \kappa$, we do trivial forcing unless, for all $\beta < \gamma$, $\lambda_\beta < \gamma$, and $f(\gamma) = \langle \mathbb{Q}, \lambda \rangle$ where λ is an ordinal and $\Vdash_\gamma$'$\mathbb{Q}$ is a γ-directed closed forcing'. In such a case, let $\mathbb{Q}_\gamma = \mathbb{Q}$ and $\lambda_\gamma = \lambda$. So the preparation is guided by the Laver function, and has long stages of trivial forcing, interspersed with suitably directed-closed forcing.

This work was among the first of many indestructibility results that followed in the next few years — see [20], where it is shown that a strongly compact cardinal κ may be made indestructible under κ^+-weakly closed forcing notions satisfying the Prikry condition. However much was still unknown in this area, including the answers to such questions as: can a strongly compact cardinal κ be made indestructible by the forcing $\mathrm{Add}(\kappa, 1)$ which adds a Cohen subset to κ?

In 2000 Hamkins published 'The Lottery Preparation' [22], which answered this question and more. It contained a new method of showing Laver's result without the use of Laver functions, and provided a plethora of other indestructibility results. For example the indestructibility of strong compactness is shown, as well as a level-by-level indestructibility result for supercompactness, provided one assumes some degree of GCH. In this paper is defined the Lottery Sum of a collection $\mathcal{A}$ of forcings, which has also been referred to as the *disjoint sum*, *side-by-side forcing*, or *choosing which partial ordering to force with generically*. The intuition behind the lottery sum is that it 'holds a lottery' among all the notions of forcing in $\mathcal{A}$, and the generic filter chooses a 'winning' poset $\mathbb{Q}$ and forces with it. More precisely:

Definition 2.2.1. The *Lottery Sum* of a collection $\mathcal{A}$ of forcing notions is the poset

$$\oplus\mathcal{A} := \{\langle\mathbb{Q}, p\rangle : \mathbb{Q} \in \mathcal{A} \wedge p \in \mathbb{Q}\} \cup \{\mathbb{1}\}$$

which is ordered with $\mathbb{1}$ larger than everything else, and $\langle\mathbb{Q}, p\rangle \leq \langle\mathbb{Q}', p'\rangle$ if and only if $\mathbb{Q} = \mathbb{Q}'$ and $p \leq_{\mathbb{Q}} p'$.

Definition 2.2.2. The Lottery Preparation for a cardinal κ is an Easton support iteration of length κ defined relative to a partial function $f \,\vdots\, \kappa \to \kappa$ as follows. The iteration has trivial forcing at stage $\gamma + 1$, for $\gamma < \kappa$, unless $\gamma \in \mathrm{dom}(f)$ and $f``(\gamma) \subseteq \gamma$. At non trivial stage γ we take the lottery sum of all forcing notions in

$H_{f(\gamma)^+}$ which are $< \gamma$-strategically closed.

The lottery preparation works best when defined relative to a fast function f, or a function with the appropriate version of the *Menas property*, in Hamkins's terminology.

Definition 2.2.3. Let κ have large cardinal property $\mathcal{L}$. Then $f \vdots \kappa \to \kappa$ is a *$\mathcal{L}$-largeness Menas function* if, for all $\lambda > \kappa$, there is a λ-$\mathcal{L}$-largeness embedding $j : M \to N$ such that $j(f)(\kappa) > \lambda$.

A function satisfying this definition is said to have the *Menas property*.

Such functions allow for $j(f)(\kappa)$ to be sufficiently large that the preparation has vast segments which are comprised of trivial forcing only, thus isolating the key property of the Laver function, without requiring its full anticipatory strength.

The lottery preparation has a number of desirable properties, shown in [22]:

Lemma 2.2.4. (Hamkins) If in the lottery preparation there is no non-trivial forcing until beyond stage γ, the preparation is $\leq \gamma$-strategically closed.

In order to show indestructibility, it is common to apply *the lifting criterion*, as seen in Proposition 9.1 of [14], which provides a way to show that an elementary embedding in V has been *lifted* to an elementary embedding in some extension after forcing.

Theorem 2.2.5. [The Lifting Criterion] Let M and N be transitive models of ZFC$^-$, let $\pi : M \to N$ be an elementary embedding, let $\mathbb{P} \in M$ be a notion of forcing with G generic over $\mathbb{P}$, and let H be $\pi(G)$-generic over N. Then there exists an elementary embedding $\pi^+ : M[G] \to N[H]$ — the *lifted embedding* of π — with $\pi^+(G) = H$ and $\pi^+ \restriction M = N$ if and only if $\pi(p) \in H$ for all $p \in G$.

Another invaluable tool for indestructibility results is Silver's concept of *master conditions*. These allow us to arrange the compatibility between generic filters required to apply the lifting criterion.

Definition 2.2.6. For an elementary embedding $j : M \to N$ and a forcing $\mathbb{P} \in M$, a master condition for j and $\mathbb{P}$ is a condition $q \in j(\mathbb{P})$ such that, for every dense $D \subseteq \mathbb{P}$, there is a condition $p \in D$ such that $q \leq j(p)$.

Note here that it is enough that $q \leq j(p)$ for all $p \in G$.

2.3 $C^{(n)}$-large cardinals

The notion of $C^{(n)}$ cardinals were introduced by Bagaria in [6], and offer a strengthening of large cardinal concepts with strong reflection properties. For a fixed natural number n, the closed unbounded (club) class $C^{(n)}$ is defined to be all ordinals α such that V_α is a Σ_n-elementary substructure of V, written $V_\alpha \prec_n V$. So such ordinals reflect true Σ_n statements with parameters in V_α.

The class $C^{(0)}$ consists of all ordinals, while $C^{(1)}$ consists of all uncountable cardinals α with $H_\alpha = V_\alpha$. Unfortunately, no such combinatorial characterisation of $C^{(n)}$, for $n \geq 2$ (yet) exists. We do, however, know that $C^{(n+1)} \subset C^{(n)}$ for all n, since the least ordinal in $C^{(n)}$ does not belong to $C^{(n+1)}$. Further, we have that the class $C^{(n)}$ is Π_n definable:

$$\begin{aligned}&\alpha \in C^{(n)} \iff \\ &\alpha \in C^{(n-1)} \wedge \forall \varphi(x) \in \Sigma_n, \forall b \in V_\alpha : \left(\models_n \varphi(b) \implies V_\alpha \models \varphi(b)\right).\end{aligned}$$

Here it is worth noting that the class $C^{(n)}$ cannot be Σ_n-definable, since then if α is the least $C^{(n)}$ ordinal, the sentence 'There is a $C^{(n)}$ ordinal' would be true in V_α, contradicting its minimality. We also have the following observation:

Fact 2.3.1.
- An ordinal α is Σ_n-correct in V if and only if it is Π_n-correct in V.
- If α is Σ_n-correct and φ is a Σ_{n+1} sentence with parameters in V_α that is true in V_α, then φ holds in V. Similarly, if ψ is a Π_{n+1} sentence with parameters in

V_α that holds in V, then it holds in V_α.

When considering a large cardinal κ defined as the critical point of an elementary embedding $j : V \to M$ it is natural to ask what kind of properties $V_{j(\kappa)}$ might hold. Does it, for example, reflect some properties of V? To consider such questions Bagaria introduced $C^{(n)}$-large cardinals in [6], which are cardinals κ typically defined as being the critical point of some large cardinal elementary embedding j, with the additional requirement that $j(\kappa) \in C^{(n)}$. For example, a $C^{(n)}$-measurable is defined to be the critical point of an elementary embedding $j : V \to M$, with M a transitive class, such that $j(\kappa) \in C^{(n)}$. Similarly, if one takes the definition of a strong, superstrong, supercompact or extendible elementary embedding and demands that $j(\kappa)$ be Σ_n-correct, this gives the definition of their $C^{(n)}$ analogue.

For some $C^{(n)}$ versions of large cardinals, the result is not so interesting, for example every measurable cardinal is $C^{(n)}$-measurable for all n, and the same is true for strong cardinals. For other large cardinals the $C^{(n)}$ analogues form a true hierarchy. For example, while every superstrong cardinal is $C^{(1)}$-superstrong, we have that the least $C^{(n)}$-superstrong — if it exists — is not $C^{(n+1)}$-superstrong.

2.3.2 $C^{(n)}$-extendible cardinals

The subject of $C^{(n)}$-extendible cardinals is a rich one, and many papers have been published discussing their properties and interrelation with other interesting large cardinal notions.

Definition 2.3.3. A cardinal κ is λ-$C^{(n)}$-extendible if there is an elementary embedding $j : V_\lambda \to V_\mu$ for some $\mu > \lambda$, with critical point κ, such that $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. If κ is λ-$C^{(n)}$-extendible for all $\lambda > \kappa$ then we say that κ is $C^{(n)}$-extendible.

An interesting result of Bagaria in [6] is that there is a strong level-by-level relationship between $C^{(n)}$-extendibility and Vopěnka's Principle (VP).

Definition 2.3.4. VP(Π_n) holds if, for every proper class C of Π_n-definable structures of the same type, there are distinct A and B in C where A is elementarily embeddable into B. If VP(Π_n) holds for all n then we say VP holds.

Bagaria showed that, for $n \geq 1$, VP(Π_{n+1}) holds if and only if there exists a $C^{(n)}$-extendible cardinal. Further, full VP holds if and only if, for all n, there exists a $C^{(n)}$-extendible cardinal.

In fact, there are many pleasing results about $C^{(n)}$-extendibles, including the following in [6].

Lemma 2.3.5. (Bagaria) A cardinal κ is $C^{(1)}$-extendible if and only if it is extendible.

Theorem 2.3.6. (Bagaria) For $n \geq 0$, if κ is $C^{(n)}$-extendible, then $\kappa \in C^{(n+2)}$

Theorem 2.3.7. (Bagaria) For any $\alpha > \kappa$, the relation 'κ is α-$C^{(n)}$-extendible' is Σ_{n+1}, and so 'κ is $C^{(n)}$-extendible' is Π_{n+2}.

2.3.8 $C^{(n)}$-supercompact cardinals

In general, $C^{(n)}$-supercompacts seem more elusive than their extendible counterparts, and while something is known about the interaction between $C^{(n)}$-extendibles and forcing, there are only a few results about how $C^{(n)}$-supercompact cardinals behave under forcing.

The definition of a $C^{(n)}$-supercompact follows the usual paradigm for $C^{(n)}$-versions of large cardinals:

Definition 2.3.9. A cardinal κ is λ-$C^{(n)}$-supercompact for some $\lambda \geq \kappa$ if there exists an elementary embedding $j : V \to M$ with M a transitive class, such that ${}^{\lambda}M \subseteq M$, $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$. If κ is λ-$C^{(n)}$-supercompact for all $\lambda \geq \kappa$ then we say that it is $C^{(n)}$-supercompact.

A key property of $C^{(n)}$-supercompact cardinals, which makes them harder to investigate than supercompact cardinals, is that they cannot be characterised by normal ultrafilters of $\mathcal{P}_\kappa(\lambda)$, since an ultrapower embedding j arising from such an ultrafilter would have that $2^{\lambda^{<\kappa}} < j(\kappa) < \left(2^{\lambda^{<\kappa}}\right)^+$, and so cannot be a cardinal.

However, in [6], Bagaria shows that we can formulate λ-$C^{(n)}$-supercompactness in terms of *Martin-Steel extenders* with sufficiently rich transitive sets as their supports, as seen in [34]. Such extenders are variants of the extenders seen in §2.1.11, but have as a parameter some set Y instead of λ.

A Martin-Steel (κ, Y)-extender over V_γ with critical point κ and support Y is a sequence $E = \langle E_a : a \in [Y]^{<\omega} \rangle$ such that:

- Each E_a is a κ-complete ultrafilter over ${}^a(V_\gamma)$, and at least one E_a is not κ^+-complete;
- For every $a \in [Y]^{<\omega}$, $\{f : a \to \mathrm{range}(f) : f \text{ is an } \in\text{-isomorphism}\} \in E_a$;
- (Coherence) If $a, b \in [Y]^{<\omega}$ and $a \subseteq b$, then $X \in E_a$ if and only if $\{f \in^b (V_\gamma) : f \restriction a \in X\} \in E_b$;
- (Normality) If $F :^a (V_\gamma) \to V$ is such that

  $$\left\{f : F(f) \in \bigcup \mathrm{range}(f)\right\} \in E_a$$

  then there is a $x \in Y$ such that

  $$\left\{f \in (V_\gamma)^{a \cup \{x\}} : F(f \restriction a) = f(x)\right\} \in E_{a \cup \{x\}};$$

- The ultrapower $M_E \cong \mathrm{Ult}(V, E)$ is well-founded.

Provided that Y is closed under j_E and under λ-sequences, the ultrapower embedding $j_E : V \to M_E \cong \mathrm{Ult}(V, E)$ is a λ-$C^{(n)}$-supercompactness embedding. Similarly, from a λ-$C^{(n)}$-supercompactness elementary embedding $j : V \to M$ one can construct a (κ, Y)-extender as above, and so we have the following:

Theorem 2.3.10. (Bagaria) For $n \geq 1$, κ is λ-$C^{(n)}$-supercompact if and only if:

$$\exists \mu, \exists E, \exists Y, \exists \gamma \big(\mu \in C^{(n)} \wedge \lambda, E, Y \in V_\mu \wedge Y \text{ is transitive} \wedge [Y]^{\leq \lambda} \subseteq Y \wedge$$
$$V_\mu \models E \text{ is a } (\kappa, Y)\text{-extender over } V_\gamma \text{ with critical point } \kappa \text{ and support } Y \wedge$$
$$j_E``(Y) \subseteq Y \wedge j_E(\kappa) > \lambda \wedge j_E(\kappa) \in C^{(n)} \big)$$

This characterisation, apart from providing another way to show whether $C^{(n)}$-supercompactness holds, also gives that 'κ is λ-$C^{(n)}$-supercompact' is Σ_{n+1} expressible, and so 'κ is $C^{(n)}$-supercompact' is Σ_{n+2} expressible.

Further results about $C^{(n)}$-supercompact cardinals are given in [23], where the authors show that, under GCH, a supercompact cardinal κ may be forced to be supercompact but not $C^{(1)}$-supercompact — indeed in this extension $V^{\mathbb{P}}$ there can be no elementary embedding $j : V^{\mathbb{P}} \to M$ with critical point κ such that ${}^{\omega}M \subseteq M$ and where $j(\kappa)$ is a limit cardinal. The authors also show a strong identity crisis is possible; that after forcing with a Magidor iteration of Prikry forcings a $C^{(n)}$-supercompact can be made to be the least (ω_1-) strongly compact. Indeed, in the extension, the minimal ω_1-strongly compact is the minimal supercompact which is equal to the minimal $C^{(n)}$ supercompact, all of which are smaller than the minimal extendible.

These results lead us to conclude that the situation for $C^{(n)}$-supercompacts is much more complex than that of $C^{(n)}$-extendibles, where $C^{(1)}$-extendibility is equivalent to extendibility, and the classes of $C^{(n)}$-extendibles form a strict hierarchy.

Chapter 3

Indestructibility of α-Subcompact Cardinals

3.1 α-Subcompact cardinals

Subcompact cardinals were first introduced by Jensen [26], who showed that if κ is subcompact then $\square_\kappa$ fails. The definition was later stratified by Brooke-Taylor and Friedman in [11], where they defined an *α-subcompact cardinal* κ, which generalises the original definition (equivalent to the case when $\alpha = \kappa^+$) and provides a strict hierarchy in consistency strength all the way up to supercompact.

Definition 3.1.1. For $\alpha > \kappa$, κ is *α-subcompact* if, for all $A \subseteq H_\alpha$ there exist $\bar{\kappa} < \bar{\alpha} < \kappa$, $\bar{A} \subseteq H_{\bar{\alpha}}$ and an elementary embedding

$$\pi : \left(H_{\bar{\alpha}}, \in, \bar{A}\right) \to \left(H_\alpha, \in, A\right)$$

with critical point $\bar{\kappa}$ such that $\pi(\bar{\kappa}) = \kappa$.

An immediate observation is that, if κ is β-subcompact for $\beta > \alpha$, then it is also α-subcompact.

The definition of an α-subcompact cardinal invites comparison to the Magidor characterisation of supercompactness (Definition 2.1.17), the only difference being that the embeddings are between $H_{\bar{\alpha}}$ and H_α, and we have added a predicate A to the embedding. Indeed, a key result in [11] is that levels of subcompactness are interleaved with levels of partial supercompactness in strength, giving the following:

Theorem 3.1.2. (Brooke-Taylor and Friedman)

1. If κ is $2^{<\alpha}$-supercompact, then κ is α-subcompact.
2. If κ is $\left(2^{\lambda^{<\kappa}}\right)^+$-subcompact, then it is α-supercompact

In particular, κ is supercompact if and only if κ is α-subcompact for all α.

For the reader's convenience we provide the proof of this result.

Proof. 1. Suppose κ is $2^{<\alpha}$-supercompact, witnessed by $j : V \to M$ with critical point κ, such that $j(\kappa) > 2^{<\alpha}$, and ${}^{<\alpha}M \subseteq M$. For a given $A \subseteq H_\alpha$, the restricted embedding

$$j \upharpoonright H_\alpha : \left(H_\alpha, \in, A\right) \to \left(H^M_{j(\alpha)}, \in, j(A)\right)$$

is elementary. Since $|H_\alpha| = 2^{<\alpha}$, and since M is closed under $2^{<\alpha}$ sequences, $j \upharpoonright H_\alpha \in M$. So α, A and $j \upharpoonright H_\alpha$ respectively witness the existential quantifications below:

$$M \vDash \exists\bar{\alpha} < j(\kappa), \exists\bar{A} \subseteq H_{\bar{\alpha}}, \exists\pi : \left(H_{\bar{\alpha}}, \in, \bar{A}\right) \prec \left(H_{j(\alpha)}, \in, j(A)\right), \\ \pi\left(\mathrm{cp}(\pi)\right) = j(\kappa).$$

And so, by elementarity of j, we have:

$$V \vDash \exists\bar{\alpha} < \kappa, \exists\bar{A} \subseteq H_{\bar{\alpha}}, \exists\pi : \left(H_{\bar{\alpha}}, \in, \bar{A}\right) \prec \left(H_\alpha, \in, A\right), \\ \pi\left(\mathrm{cp}(\pi)\right) = \kappa,$$

thus giving an α-subcompactness elementary embedding for κ as required.

2. Suppose κ is $\left(2^{\lambda^{<\kappa}}\right)^+$-subcompact, witnessed by

$$\pi : \left(H_{\bar{\alpha}}, \in, \{\bar{\kappa}, \bar{\lambda}\}\right) \prec \left(H_{\left(2^{\lambda^{<\kappa}}\right)^+}, \in, \{\kappa, \lambda\}\right)$$

for the predicate $A = \{\kappa, \lambda\}$, with critical point $\bar{\kappa}$. By elementarity $\bar{\alpha} = \left(2^{\bar{\lambda}^{<\bar{\kappa}}}\right)^+$, and since $\bar{\alpha} < \kappa$, we have that $\bar{\lambda} < \kappa$.

Define an ultrafilter $\mathcal{U}$ on $\mathcal{P}_{\bar{\kappa}}(\bar{\lambda})$ by:

$$X \in \mathcal{U} \iff X \subseteq \mathcal{P}_{\bar{\kappa}}(\bar{\lambda}) \wedge \left\{\pi(\gamma) : \gamma \in \bar{\lambda}\right\} \in \pi(X)$$

We claim that $\mathcal{U}$ is normal. First, $\mathcal{U}$ is $\bar{\kappa}$-complete since, if for some $\beta < \bar{\kappa}$, $\langle X_\theta : \theta < \beta\rangle$ is a family of sets in $\mathcal{U}$, then $\{\pi(\gamma) : \gamma \in \bar{\lambda}\} \in \pi(X_\theta)$ for all $\theta < \beta$, and so $\{\pi(\gamma) : \gamma \in \bar{\lambda}\} \in \pi\left(\bigcap_{\theta<\beta} X_\theta\right)$, so the intersection is in $\mathcal{U}$.

It is also fine, since, for $\theta < \bar{\lambda}$, $\{X \in \mathcal{P}_{\bar{\kappa}}(\bar{\lambda}) : \theta \in X\}$ is in $\mathcal{U}$ if and only if

$$\left\{\pi(\gamma) : \gamma \in \bar{\lambda}\right\} \in \pi\left\{X \in \mathcal{P}_{\bar{\kappa}}(\bar{\lambda}) : \theta \in X\right\} = \left\{X \in \mathcal{P}_{\kappa}(\lambda) : \pi(\theta) \in X\right\}.$$

This holds since $\theta < \bar{\lambda}$, so $\pi(\theta) \in \{\pi(\gamma) : \gamma < \bar{\lambda}\}$.

It is also closed under diagonal intersection. To see this suppose $\langle X_\theta : \theta < \bar{\kappa}\rangle$ is a family of sets in $\mathcal{U}$. Then $\Delta_{\theta<\bar{\lambda}} X_\theta = \left\{x \in \mathcal{P}_{\bar{\kappa}}(\bar{\lambda}) : x \in \bigcap_{i\in x} X_i\right\}$ is in $\mathcal{U}$ if and only if $\{\pi(\gamma) : \gamma \in \bar{\lambda}\} \in \pi\left(\Delta_{\theta<\bar{\lambda}} X_\theta\right)$. By the elementarity of π, this holds if and only if:

$$\left\{\pi(\gamma) : \gamma \in \bar{\lambda}\right\} \in \left\{x \in \mathcal{P}_{\kappa}(\lambda) : x \in \bigcap_{i\in x} \pi\left(\langle X_\theta\rangle\right)_i\right\}.$$

So we wish to show that $\{\pi(\gamma) : \gamma \in \bar{\lambda}\} \in \bigcap_{i\in\{\pi(\gamma):\gamma\in\bar{\lambda}\}} \pi\left(\langle X_\theta\rangle\right)_i$. Since $X_\gamma \in \mathcal{U}$ for every $\gamma \in \bar{\lambda}$, $\{\pi(\gamma) : \gamma \in \bar{\lambda}\} \in \pi(X_\gamma)$ for each γ, and $\pi\left(X_\gamma\right) = \pi\left(\langle X_\theta\rangle_\gamma\right) = \pi\left(\langle X_\theta\rangle\right)_{\pi(\gamma)}$ so the condition holds.

Now, $\mathcal{U} \in H_{(2^{\bar{\lambda}^{<\bar{\kappa}}})^+}$ and:

$$H_{(2^{\bar{\lambda}^{<\bar{\kappa}}})^+} \vDash \mathcal{U} \text{ is a normal ultrafilter on } \mathcal{P}_{\bar{\kappa}}(\bar{\lambda}).$$

So, by elementarity of π:

$$H_{(2^{\lambda^{<\kappa}})^+} \vDash \mathcal{U} \text{ is a normal ultrafilter on } \mathcal{P}_{\beta}(\lambda),$$

and $H_{(2^{\lambda^{<\kappa}})^+}$ is correct for this statement. So κ is λ-supercompact, as required.

□

Another key property of subcompactness embeddings is that an α-subcompactness embedding may have as parameters finitely many subsets A_i, since we can encode them all into one subset, for example by using pairs (i, x) to denote that $x \in A_i$.

Since we will be extensively dealing with H_α and its interaction with forcing, we give two crucial lemmas, found across the literature. We provide proofs for completeness.

Lemma 3.1.3. (Folklore) If $\mathbb{P}$ is a forcing notion which doesn't collapse α and $\dot{x} \in H_\alpha$ then $\forall p \in \mathbb{P}$, $p \Vdash \dot{x} \in H_\alpha$ i.e. $\mathbb{1} \Vdash \dot{x} \in H_\alpha$.

Proof. We induct upon $\text{rk}(\dot{x})$ for $\dot{x} \in H_\alpha$. For $\text{rk}(\dot{x}) = 0$ the result is clear. So suppose that the theorem holds for all $\dot{y} \in H_\alpha$ with $\text{rk}(\dot{y}) < \text{rk}(\dot{x})$. Express $\dot{x}$ as $\{(\dot{y}_i, p_i) : i \in I\}$ where I is some indexing set, and then note that, by the induction hypothesis, each $\dot{y}_i$ has that $\mathbb{1} \Vdash \dot{y}_i \in H_\alpha$. Since $\dot{x} \in H_\alpha$, $|\dot{x}| < \alpha$, and, since the forcing doesn't collapse α, $\mathbb{1} \Vdash |\dot{x}| < \alpha$. So $\mathbb{1} \Vdash \dot{x} \in H_\alpha$ as required. □

Lemma 3.1.4. (Folklore, see e.g. 3.6 of [21]) Let α be a regular cardinal, let $\mathbb{P} \in H_\alpha$ be a notion of forcing. Then $\forall p \in \mathbb{P}$, if $p \Vdash \dot{x} \in H_\alpha$, then $\exists \dot{y} \in H_\alpha$ such that $p \Vdash \dot{x} = \dot{y}$.

Proof. To show this we will first prove that for all $x \in H_\alpha$, there is a $\lambda^y < \alpha$ and

$\big(x_\delta\big)_{\delta\leq\lambda^y}$ with $x_\delta \in H_\alpha$ such that $\forall\delta \leq \lambda$, $x_\delta \subseteq \{x_\beta : \beta \leq \lambda\}$ and $x = x_\lambda$. To show that this is the case we will induct upon $x \in H_\alpha$: the case when $x = \emptyset$ is clear.

Now suppose that the above is true for all $y \in x$, and for each such y let $\lambda^y < \alpha$ and $(x^y_\delta)_{\delta\leq\lambda^y}$ witness this. Then let $\lambda = \Sigma_{y\in x}\lambda^y$. Now since $|x| < \alpha$ and α is regular, we must have that $\lambda < \alpha$. Now define the witnessing sequence for x to be the concatenation of all the $(x^y_\delta)_{\delta\leq\lambda^y}$, and set $x_\lambda = x$. Then this sequence will be a witness to the property for x, since every $y \in x$ appears at some point of the sequence.

Now since $\mathbb{P} \in H_\alpha$, $\mathbb{P}$ has the α-c.c. and so doesn't collapse α. Let $p \in \mathbb{P}$ be such that $p \Vdash \dot{x} \in H_\alpha$. Then there are names $\dot{\lambda}$ and $(\dot{x}_\delta)_{\delta\leq\dot{\lambda}}$ for λ and $(x^y_\delta)_{\delta\leq\lambda^y}$ as above. Further, there is an ordinal $\lambda < \alpha$ such that $p \Vdash \dot{\lambda} \leq \check{\lambda}$. Since in $V[G]$ we may set $x_\delta = \emptyset$ for all $\dot{\lambda}^G < \delta < \check{\lambda}^G$ we can assume that $\dot{\lambda} = \check{\lambda}$. Now define inductively:

$$\ddot{x}_\delta = \big\{(\ddot{x}_\beta, q) : \beta < \delta \wedge q \leq p \wedge q \Vdash \ddot{x}_\beta \in \dot{x}_\delta\big\}$$

and let $\ddot{x} = \ddot{x}_\lambda$. By induction, all the $\ddot{x}_\delta$ are in H_α. Finally we show that, for all $\delta < \lambda$, $p \Vdash \dot{x}_\delta = \ddot{x}_\delta$, so in particular $p \Vdash \dot{x} = \ddot{x}$.

To see this, note that the base case, when $\delta = 0$, is immediate. So suppose that the statement holds for all $\beta < \delta$, i.e. $p \Vdash \dot{x}_\beta = \ddot{x}_\beta$. Then, for any generic G containing p:

$$\begin{aligned}
\ddot{x}^G_\delta &= \big\{(\ddot{x}_\beta, q) : \beta < \delta \wedge q \leq p \wedge q \Vdash \ddot{x}_\beta \in \dot{x}_\delta\big\} \\
&= \big\{(\dot{x}_\beta, q) : \beta < \delta \wedge q \leq p \wedge q \Vdash \ddot{x}_\beta \in \dot{x}_\delta\big\} && \text{(by induction)} \\
&= \dot{x}^G_\delta && \text{(by the Forcing Theorem)}
\end{aligned}$$

□

3.2 Minimal Counterexamples

In order to show that α-subcompact cardinals may be made indestructible, we first introduce the concept of *minimal counterexamples*, which will aid us in construction of an appropriate preparatory forcing.

Since Hamkins's [22], much use has been made of the Lottery Preparation to show indestructibility results. However, since its definition relies of the existence of suitably fast-growing functions — e.g. Laver functions ot those with the Menas property — it has been asked whether it is possible to show indestructibility results without use of such functions.

In [4], Apter uses an iteration of lottery sums of forcings of 'small' rank to re-prove, and improve upon, various indestructibility results. For example, a proof is given that every supercompact cardinal κ can simultaneously be made indestructible by κ-directed closed forcing, using an iteration which makes no mention of either a Laver function or a fast function (though the proof does require a function with the Menas property).

Building upon this work, Apter later published [5], extending the ideas of [4] to other uses of Laver functions, in particular to show the consistency of the Proper Forcing Axiom (PFA) and the Semiproper Forcing Axiom (SPFA) as seen in [31]. The iterations used are iterations of lottery sums of *counterexamples of minimal rank*, i.e. forcings which destroy the property at hand, which are of minimal rank among such counterexamples.

Based on a suggestion of Schlicht, the Master's Thesis of Schlöder [41] detailed a new method using iterations of lottery sums of *counterexamples of minimal hereditary cardinality*. There they were used mainly in the context of PFA, but their usage in other proofs, including indestructibility arguments, was suggested. To this end, in the lecture notes of Lücke and Schlicht ([30]) the authors were able to re-prove Laver's original indestructibility result using these counterexamples of minimal hereditary cardinality.

In this Chapter we extend the results obtained using this method, and show that an α-subcompact cardinal κ where $\alpha \in \mathrm{Reg}$ may be made indestructible by $< \kappa$-directed closed forcings $\mathbb{Q} \in H_\alpha$ using an Easton support iteration of length κ involving lottery sums of counterexamples to subcompactness of minimal hereditary cardinality.

The key idea behind this method is to build a preparatory forcing, with non-trivial stages which are lottery sums of forcings minimal among those that break a minimal amount of subcompactness. Having forced with many minimal counterexamples, any α-subcompact cardinal left standing will be indestructible. This idea has also been referred to as a 'trial by fire', since whatever remains after our many forcings will be very robust after an onslaught of small counterexample forcings (see [3]).

Let us now make this idea precise for the case of subcompactness.

Definition 3.2.1. A *counterexample for the θ-subcompactness of κ* (often simply written as a *counterexample*) is a triple $(\mathbb{Q}, \theta, \kappa)$, where:

- $\mathbb{Q}$ is a $< \kappa$-directed closed forcing;
- κ is θ-subcompact;
- $\Vdash_{\mathbb{Q}} \kappa$ is not θ-subcompact.

We say that $(\mathbb{Q}, \theta, \kappa)$ is a *minimal counterexample* if (θ, η) is lexicographically least among counterexamples, where $\eta = |\operatorname{TC}(\mathbb{Q})|$.

Our preparatory forcing is then defined as follows:

Definition 3.2.2. Fix a cardinal κ and $\alpha > \kappa$. Define inductively an Easton support iteration $\langle \mathbb{P}^\kappa_\gamma, \dot{\mathbb{Q}}^\kappa_\gamma \rangle_{\gamma<\kappa}$ and a sequence $(\theta^\kappa_\gamma, \eta^\kappa_\gamma)_{\gamma<\kappa}$ as follows: suppose that $\mathbb{P}^\kappa_\delta$ has been defined and that $\theta^\kappa_\gamma, \eta^\kappa_\gamma$ have been defined for each $\gamma < \delta$.

(i) If $\delta > \theta^\kappa_\gamma, \eta^\kappa_\gamma$ for all $\gamma < \delta$ then let $\dot{\mathbb{Q}}^\kappa_\delta$ denote a $\mathbb{P}^\kappa_\delta$-name for the lottery sum of all forcings $\mathbb{Q}$ with $|\operatorname{TC}(\mathbb{Q})| < \kappa$ such that $(\mathbb{Q}, \theta, \delta)$ is a minimal counterexample for some $\theta \leq \kappa$.

Let $\eta^\kappa_\delta = |\operatorname{TC}(\mathbb{Q})|$ and $\theta^\kappa_\delta = \theta$ for such $\mathbb{Q}$ and θ.

(ii) Otherwise let $\dot{\mathbb{Q}}^\kappa_\delta$ denote a $\mathbb{P}^\kappa_\delta$-name for the trivial forcing and let $\theta^\kappa_\delta = 1 = \eta^\kappa_\delta$.

We now show a number of results about the iteration defined above, which will prove useful in our indestructibility argument. The proofs of the preliminary results (Lemmas 3.2.3 - 3.2.6) are the same as those in Schlöder's [41] — though we take minimal counterexamples to different large cardinal properties — and are shown here for completeness.

Firstly we have that the stages of the iteration are not large.

Lemma 3.2.3. For $\beta < \kappa$, $|\mathbb{P}^\kappa_\beta| < \kappa$.

Proof. Proceed by induction on β: for $\beta = 0$, $\mathbb{P}^\kappa_0$ is trivial.

If $\nu = \beta + 1$ and we are in case (i), $\mathbb{P}^\kappa_\beta$ forces that $\mathbb{Q}$ is a union of forcing notions with hereditary cardinality $\gamma = \eta^\kappa_\nu < \kappa$, so

$$\Vdash_{\mathbb{P}^\kappa_\beta} |\mathbb{Q}^\kappa_\nu| \leq |H_{\gamma^+}| \leq 2^\gamma.$$

Now, by induction $|\mathbb{P}^\kappa_\beta| < \kappa$, so

$$\Vdash_{\mathbb{P}^\kappa_\beta} \check{2^\gamma} \leq \check{2^\delta} \leq \widecheck{\left(2^\delta\right)^V}$$

for some δ with $\max(\gamma, |\mathbb{P}^\kappa_\beta|) < \delta < \kappa$. Since κ is inaccessible $|\mathbb{P}^\kappa_\nu| \leq 2^\gamma \leq 2^\delta < \kappa$. If we are in case (ii) then we perform trivial forcing so the same argument holds.

If $\nu < \kappa$ is a limit and the result holds below ν, then since κ is regular, there exists a $\delta < \kappa$ such that $\delta > |\mathbb{P}^\kappa_\beta|$ for all $\beta < \nu$. Since the function $p \mapsto (p \restriction \beta)_{\beta<\nu}$ is injective, $|\mathbb{P}^\kappa_\nu| \leq \prod_{\beta<\nu} |\mathbb{P}^\kappa_\beta|$.

So $|\mathbb{P}^\kappa_\nu| \leq \prod_{\beta<\nu} \delta = \delta^\nu < \kappa$ as required. □

Not only are the stages of the iteration not too large, but in fact each stage below κ is in H_κ.

Lemma 3.2.4. For $\beta < \kappa$, $\mathbb{P}^\kappa_\beta \in H_\kappa$.

Proof. Proceed by induction on β; again $\mathbb{P}^\kappa_0$ is trivial. For successor $\beta = \gamma + 1$ we have, as in Lemma 3.2.3, that $\Vdash_{\mathbb{P}^\kappa_\gamma} \dot{\mathbb{Q}}^\kappa_\gamma \in H_\kappa$. Of course, if the forcing at stage γ is trivial we are done, so let us assume that $\dot{\mathbb{Q}}^\kappa_\gamma$ is a $\mathbb{P}^\kappa_\gamma$ name for the lottery sum of minimal counterexamples. By induction and by an application of Lemma 3.1.4 there is a name for $\mathbb{Q}^\kappa_\gamma$ in H_κ, and so $\mathbb{P}^\kappa_\beta = \mathbb{P}^\kappa_\gamma * \dot{\mathbb{Q}}^\kappa_\gamma$. Every canonical name (as in Definition 2.1.21) for an element of $\mathbb{Q}^\kappa_\gamma$ is equivalent to a subset of $\mathrm{dom}(\dot{\mathbb{Q}}^\kappa_\gamma) \times \mathbb{P}^\kappa_\gamma$, and κ is inaccessible, so $\mathbb{P}^\kappa_\beta \in H_\kappa$.

For limit $\beta < \kappa$ we take either an inverse or a direct limit of forcings which are all in H_κ, and so the result follows from the fact that κ is inaccessible. □

Now, we have that the full iteration is not too large either.

Lemma 3.2.5. $|\mathbb{P}^\kappa_\kappa| \leq \kappa$.

Proof. By the regularity of κ we take a direct limit at stage κ. For $p \in \mathbb{P}^\kappa_\kappa$ let $\beta_p = \sup(\mathrm{supp}(p))$. For $\gamma < \kappa$ let $B_\gamma = \{p \in \mathbb{P}^\kappa_\kappa : \beta_p = \gamma\}$. Now the function $f : B_\gamma \to \mathbb{P}^\kappa_\gamma$ with $f(p) = p \upharpoonright \beta$ is an injection, so $|B_\gamma| < \kappa$ for all $\gamma < \kappa$, by Lemma 3.2.3. Further $\mathbb{P}^\kappa_\kappa = \bigcup_{\gamma<\kappa} B_\gamma$ and so $|\mathbb{P}^\kappa_\kappa| \leq \kappa$. □

Indeed, we can consider the iteration to be a subset of H_κ, by utilising the direct limit at stage κ.

Lemma 3.2.6. We may, without loss of generality, assume that $\mathbb{P}^\kappa_\kappa \subseteq H_\kappa$.

Proof. Since κ is a regular cardinal, we take a direct limit at stage κ, and so each $p \in \mathbb{P}^{\kappa}_{\kappa}$ looks like $p^\frown \mathbb{1}^{(\kappa)}$. So we may 'forget' the trailing $\mathbb{1}$'s (though implicitly add them back on when needed) and consider p a member of $\mathbb{P}^{\kappa}_{\beta}$ where $\beta = \sup(\mathrm{supp}(p))$. Then use Lemma 3.2.4 which gives us that $\mathbb{P}^{\kappa}_{\beta} \subseteq H_{\kappa}$. □

The following lemma is an adaptation of Lemma 3.2 of [22], which allows us to show strategic closure of the tail of our iteration.

Lemma 3.2.7. If in $\mathbb{P}^{\kappa}_{\kappa}$ there is no nontrivial forcing until beyond stage δ then the iteration is δ-strategically closed.

Proof. To see that this is true, recall that, at each stage $\beta > \delta$ the forcings in the lottery sum are chosen so that they are β-directed closed. So, in particular, they are β-strategically closed. Let σ_β be a strategy for the game of length δ on the forcing at stage β. A partial play in the game on $\mathbb{P}^{\kappa}_{\kappa}$ is a descending sequence $\langle p^\gamma : \gamma < \delta' \rangle$ for some $\delta' < \delta$, where $p^\gamma = \langle \check{\mathbb{1}} : \alpha \leq \delta \rangle^\frown \langle \dot{p}^\gamma_\alpha : \delta < \alpha < \kappa \rangle$. Apply the strategies σ_β for each $\beta > \delta$ to obtain the strategy

$$\sigma\left(\langle p^\gamma : \gamma < \delta' \rangle\right) = \langle \check{\mathbb{1}} : \beta \leq \delta \rangle^\frown \langle \dot{q}_\beta : \delta < \beta < \kappa \rangle$$

where $\dot{q}_\beta$ is the name for the condition obtained by applying σ_β to $\langle \dot{p}^\gamma_\beta : \beta < \delta' \rangle$. Each of the strategies σ_β can successfully navigate all limits up to δ and thus so can σ. So σ gives a strategy for the game of length δ as required. □

We now fix some non-standard terminology which will be used later.

Definition 3.2.8. We say that $\gamma < \kappa$ is a *closure point* of $\mathbb{P}^{\kappa}_{\kappa}$ if $\mathbb{P}^{\kappa}_{\gamma} = \mathbb{P}^{\gamma}_{\gamma}$.

So, if γ is a closure point of $\mathbb{P}^{\kappa}_{\kappa}$ then the iteration for κ up to stage γ is the same as the iteration $\mathbb{P}^{\gamma}_{\gamma}$ defined for γ itself.

Note that the set of closure points of $\mathbb{P}^{\kappa}_{\kappa}$ are club in κ and, in particular, κ itself is a closure point. Note here that the closure points need not be inaccessible, but, by standard arguments about clubs closed under functions, the previous lemmata will also hold for them.

In showing that some γ is a closure point of $\mathbb{P}^{\kappa}_{\kappa}$ it is not sufficient to simply apply the Factor Lemma (2.1.27), as we require not only that the longer iteration may be 'cut' into its first γ stages $\mathbb{P}^{\kappa}_{\gamma}$ followed by the rest of the iteration $\mathbb{P}^{\kappa}_{[\gamma,\kappa)}$, but that the first γ stages are *exactly* the stages of the corresponding lottery sum of minimal counterexamples iteration for γ, namely $\mathbb{P}^{\gamma}_{\gamma}$.

3.3 Indestructibility Result

Having laid the groundwork we are now ready to state the main Theorem of this Chapter:

Theorem 3.3.1. Let κ be α-subcompact for some regular cardinal $\alpha > \kappa$. Then, after preparatory forcing with $\mathbb{P}^{\kappa}_{\kappa}$, the α-subcompactness of κ will be indestructible under any $< \kappa$-directed closed forcing $\mathbb{Q} \in H_\alpha$.

Proof. Suppose the theorem does not hold, so there is some minimal counterexample $(\mathbb{Q}, \Theta, \kappa)$ in $V[G_\kappa]$, for some $\Theta \leq \alpha$, where G_κ is $\mathbb{P}^{\kappa}_{\kappa}$-generic over V, and $\mathbb{Q} \in H_\alpha$. Note that, if $\mathbb{Q}$ breaks the Θ-subcompactness of κ for $\Theta < \alpha$ then it also breaks its α-subcompactness. We will show that this gives a contradiction by showing that κ is in fact Θ-subcompact in $V[G_\kappa * \dot{g}]$, where g is a $\mathbb{Q}$-generic over $V[G_\kappa]$.

So we must show that, for all $A \subseteq H_\Theta$ in $V[G_\kappa * \dot{g}]$ there is a Θ-subcompactness elementary embedding

$$\pi : \left(H^{V[G_\kappa * \dot{g}]}_{\bar{\Theta}}, \in, \bar{A} \right) \to \left(H^{V[G_\kappa * \dot{g}]}_{\Theta}, \in, A \right)$$

with critical point $\bar{\kappa}$, $\pi(\bar{\kappa}) = \kappa$ for some $\bar{\kappa} < \bar{\Theta} < \kappa$ and $\bar{A} \subseteq H^{V[G_\kappa * \dot{g}]}_{\bar{\Theta}}$.

The general structure to do this will be by lifting an α-subcompactness embedding for the $\mathbb{P}^{\kappa}_{\kappa} * \dot{\mathbb{Q}}$ name in V that A interprets, and then restricting the lifted α-subcompactness embedding to get a Θ-subcompactness embedding in the extension. Note here that we will provide an α-subcompactness embedding for each $A \subseteq H_\Theta$ in $V[G_\kappa * \dot{g}]$, not for all subsets of H_α in the extension (unless of course $\Theta = \alpha$).

So let $A \subseteq H^{V[G_\kappa * \dot{g}]}_{\Theta}$. By Lemma 3.2.6 we can assume that $\mathbb{P}^{\kappa}_{\kappa} \subseteq H_\kappa$, so we have that $\mathbb{P}^{\kappa}_{\kappa} \in H_\alpha$. Now, since also α is regular,we apply Lemma 3.1.4[1] to conclude that $A = \dot{B}^{G_\kappa * \dot{g}}$ for some $\dot{B} \subseteq H_\alpha$ in V.

[1]The repeated use of this Lemma is the key area where we utilise the assumption that $\alpha \in \mathrm{Reg}$.

Since κ is α-subcompact in V, there exist $\bar{\kappa} < \bar{\alpha} < \kappa$, $\bar{B} \subseteq H_{\bar{\alpha}}$ and an elementary embedding $\pi : (H_{\bar{\alpha}}, \in, \bar{B}) \to (H_\alpha, \in, B)$ with critical point $\bar{\kappa}$ such that $\pi(\bar{\kappa}) = \kappa$.

Now let $\mathbb{P}^{\bar{\kappa}}_{\bar{\kappa}}$ be the $\bar{\kappa}$-length lottery sum of minimal counterexamples iteration for $\bar{\kappa}$, and let $G_{\bar{\kappa}}$ be a $\mathbb{P}^{\bar{\kappa}}_{\bar{\kappa}}$-generic over V. Let $\mathbb{P}^\alpha_\alpha$ be the α-length lottery sum of minimal counterexamples iteration for α. Below a condition which opts for $\mathbb{Q}$ in the stage κ lottery sum of minimal counterexamples, we may factor $\mathbb{P}^\alpha_\alpha$ as $\mathbb{P}^\alpha_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{P}}^\alpha_{(\kappa+1,\alpha)}$.

We will also add in some extra predicates to the subcompactness embedding, namely $\{\Theta\}$ and a $\mathbb{P}^\kappa_\kappa$-name $\dot{\mathbb{Q}} \in H_\alpha$ that $\mathbb{Q}$ interprets. Thus the embedding looks like:

$$\pi : \left(H_{\bar{\alpha}}, \in, \dot{\bar{B}}, \dot{\bar{\mathbb{Q}}}, \{\bar{\Theta}\}\right) \to \left(H_\alpha, \in, \dot{B}, \dot{\mathbb{Q}}, \{\Theta\}\right).$$

Here $\dot{\bar{\mathbb{Q}}} \in H_{\bar{\alpha}}$ is a name for a $< \bar{\kappa}$-directed closed forcing whose interpretation is some $\bar{\mathbb{Q}} \in H^{V[G_{\bar{\kappa}} * \dot{\bar{g}}]}_{\bar{\alpha}}$, where $\dot{\bar{g}}$ is a $\dot{\bar{\mathbb{Q}}}$-generic over $V[G_{\bar{\kappa}}]$. By elementarity this $\bar{\mathbb{Q}}$ breaks the $\bar{\Theta}$-subcompactness of $\bar{\kappa}$ and has $|\operatorname{TC}\bar{\mathbb{Q}}| < \bar{\alpha}$.

We now wish to perform lifting arguments to provide an α-subcompactness embedding for A in $V[G_\kappa * \dot{g}]$. But to do this we first need to prove some preliminary results about closure points.

In the following Lemmas we will show that we can factor $\mathbb{P}^\alpha_\alpha$ as $\mathbb{P}^\kappa_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{P}}^\alpha_{(\kappa+1,\alpha)}$, and that we can factor $\mathbb{P}^\kappa_\kappa$ as $\mathbb{P}^{\bar{\kappa}}_{\bar{\kappa}} * \dot{\mathbb{P}}_{[\bar{\kappa},\kappa)}$.

To begin we show:

Claim 3.3.2. $\mathbb{P}^\kappa_{\bar{\kappa}} = \mathbb{P}^{\bar{\kappa}}_{\bar{\kappa}}$, i.e. $\bar{\kappa}$ is a closure point of $\mathbb{P}^\kappa_\kappa$.

Proof. To see this first note that $\pi(\mathbb{P}^{\bar{\kappa}}_{\bar{\kappa}}) = \mathbb{P}^\kappa_\kappa$ and that the critical point of π is $\bar{\kappa}$, so that $\pi \restriction H_{\bar{\kappa}}$ is the identity function. Further, for $\gamma < \bar{\kappa}$, each $\mathbb{P}^{\bar{\kappa}}_\gamma$ is a member of $H_{\bar{\kappa}}$ by Lemma 3.2.4, and by elementarity $\bar{\kappa}$ is inaccessible. So $\pi(\mathbb{P}^{\bar{\kappa}}_\gamma) = \mathbb{P}^\kappa_\gamma$, but $\pi(\mathbb{P}^{\bar{\kappa}}_\gamma) = \mathbb{P}^{\bar{\kappa}}_\gamma$, so $\mathbb{P}^\kappa_\gamma = \mathbb{P}^{\bar{\kappa}}_\gamma$, for every $\gamma < \bar{\kappa}$.

At stage $\bar{\kappa}$ we take a direct limit, since $\bar{\kappa} \in \mathrm{Reg}$, so conditions are bounded below

$\bar{\kappa}$, and so $\pi(p) = p$ for all p in the stage κ forcing. So the κ length iteration and the $\bar{\kappa}$-length iteration are identical up to stage $\bar{\kappa}$. □

We also have that:

Claim 3.3.3. $\mathbb{P}^{\alpha}_{\bar{\kappa}} = \mathbb{P}^{\kappa}_{\bar{\kappa}}$, i.e. the iterations $\mathbb{P}^{\alpha}_{\alpha}$ and $\mathbb{P}^{\kappa}_{\kappa}$ up to stage $\bar{\kappa}$ are identical.

Proof. We proceed by induction, showing that $\mathbb{P}^{\alpha}_{\delta} = \mathbb{P}^{\bar{\kappa}}_{\delta}$ for all $\delta < \bar{\kappa}$. The base case is $\mathbb{P}^{\alpha}_{0} = \mathbb{P}^{\kappa}_{0}$, since both are trivial. The result holds for limit stages since if $\mathbb{P}^{\alpha}_{\delta} = \mathbb{P}^{\kappa}_{\delta}$ for limit δ then both iterations take either a direct or inverse limit at stage δ depending on the regularity of δ.

For successor stages suppose by induction that $\mathbb{P}^{\alpha}_{\delta} = \mathbb{P}^{\kappa}_{\delta}$. The forcing done at successor stages is either trivial forcing or a lottery sum of minimal counterexamples. We must now show that $\dot{\mathbb{Q}}^{\alpha}_{\delta} = \dot{\mathbb{Q}}^{\kappa}_{\delta}$, and for this we examine each possible case.

If $\dot{\mathbb{Q}}^{\alpha}_{\delta}$ is a $\mathbb{P}^{\alpha}_{\delta}$-name for trivial forcing, then so must $\dot{\mathbb{Q}}^{\kappa}_{\delta}$ be the $\mathbb{P}^{\kappa}_{\delta}$-name for trivial forcing. So let us consider when $\dot{\mathbb{Q}}^{\alpha}_{\delta}$ is not trivial. It is a $\mathbb{P}^{\alpha}_{\delta}$-name for the lottery sum of forcings $\mathbb{R}$ such that $(\mathbb{R}, \gamma^{\alpha}_{\delta}, \delta)$ is a minimal counterexample with $\gamma^{\alpha}_{\delta} \leq \alpha$ and $\eta^{\alpha}_{\delta} < \alpha$, where $\eta^{\alpha}_{\delta} = |\operatorname{TC}(\mathbb{R})|$. If further $\dot{\mathbb{Q}}^{\kappa}_{\delta}$ is non-trivial, then actually it must be equal to $\dot{\mathbb{Q}}^{\alpha}_{\delta}$, since, by definition, they are both lottery sums of *minimal* counterexamples, so the length of the iteration changes nothing.

Our final possibility is if $\dot{\mathbb{Q}}^{\kappa}_{\delta}$ is trivial but $\dot{\mathbb{Q}}^{\alpha}_{\delta}$ is not. By the definition of our iteration, there are two ways in which $\dot{\mathbb{Q}}^{\kappa}_{\delta}$ may be trivial:

Case 1. There is some $\beta < \delta$ such that either $\delta \leq \gamma^{\kappa}_{\beta}$ or $\delta \leq \eta^{\kappa}_{\beta}$.

Case 2. There does not exist a counterexample $(\mathbb{R}, \gamma^{\kappa}_{\delta}, \delta)$ with $\gamma^{\kappa}_{\delta} \leq \kappa$ and $|\operatorname{TC}(\mathbb{R})| = \eta^{\kappa}_{\delta} < \kappa$.

Now Case 1 cannot hold: by induction, the stages up to β will be the same for both iterations, and so since $\dot{\mathbb{Q}}^{\alpha}_{\delta}$ is non-trivial, so must $\dot{\mathbb{Q}}^{\kappa}_{\delta}$ be.

Also, Case 2 cannot hold: we show that if there exists a counterexample $(\mathbb{R}, \gamma, \delta)$ for some $\gamma < \bar{\kappa}$ with $|\operatorname{TC}(\mathbb{R})| \in [\kappa, \alpha)$ then there exists a counterexample $(\mathbb{S}, \gamma, \delta)$ with $|\operatorname{TC}(\mathbb{S})| < \kappa$.

The first thing to note is that the counterexample forcing $\mathbb{R}$ described above is in H_α and so, in V, there are $\bar{\kappa}_{\mathbb{R}} < \bar{\alpha}_{\mathbb{R}} < \kappa$, $\bar{\mathbb{R}} \subseteq H_{\bar{\alpha}_{\mathbb{R}}}$ and a α-subcompactness elementary embedding

$$j : \left(H_{\bar{\alpha}_{\mathbb{R}}}, \in, \bar{\mathbb{R}}, \bar{\gamma}, \bar{\delta}\right) \to \left(H_\alpha, \in, \mathbb{R}, \gamma, \delta\right)$$

with critical point $\bar{\kappa}_{\mathbb{R}}$ such that $j(\bar{\kappa}_{\mathbb{R}}) = \kappa$. Now, in H_α, $\mathbb{R}$ is $<\delta$-directed closed and breaks the γ-subcompactness of δ for $\delta < \gamma < \bar{\kappa}$. So, by elementarity of j, $\bar{\mathbb{R}}$ is $<\bar{\delta}$-directed closed and breaks the $\bar{\gamma}$-subcompactness of $\bar{\delta}$.

Observe that $\bar{\delta} < \bar{\gamma} < \bar{\kappa}_{\mathbb{R}}$ since $\delta < \gamma < \kappa$ and j is elementary. Now, since the critical point of j is $\bar{\kappa}_{\mathbb{R}}$, we have that $\bar{\delta} = \delta$ and $\bar{\gamma} = \gamma$, and so $\delta < \gamma < \bar{\kappa}_{\mathbb{R}}$ and $\bar{\mathbb{R}}$ in fact breaks the γ-subcompactness of δ, and is $<\delta$-directed closed, but has $|\operatorname{TC}(\bar{\mathbb{R}})| \in [\bar{\kappa}_{\mathbb{R}}, \bar{\alpha}_{\mathbb{R}})$, thus in particular $|\operatorname{TC}(\bar{\mathbb{R}})| < \kappa$. □

So combining Claims 3.3.2 and 3.3.3 we get that:

Lemma 3.3.4. $\bar{\kappa}$ is a closure point of $\mathbb{P}^\alpha_\alpha$ i.e. $\mathbb{P}^\alpha_{\bar{\kappa}} = \mathbb{P}^{\bar{\kappa}}_{\bar{\kappa}}$.

Now, we show our final closure point Lemma; that κ itself is a closure point of the long iteration $\mathbb{P}^\alpha_\alpha$. This will allow us to factor $\mathbb{P}^\alpha_\alpha$ as $\mathbb{P}^\kappa_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{P}}^\alpha_{[\kappa+1,\alpha)}$.

Lemma 3.3.5. κ is a closure point of $\mathbb{P}^\alpha_\alpha$, i.e. $\mathbb{P}^\alpha_\kappa = \mathbb{P}^\kappa_\kappa$.

Proof. By Lemma 3.3.4 the iterations agree up to stage $\bar{\kappa}$. Since we choose $\mathbb{Q}$ in the stage κ lottery sum, which is a minimal counterexample to the Θ-subcompactness of κ, by elementarity we choose some $\bar{\mathbb{Q}}$ in the stage $\bar{\kappa}$ forcing of $\mathbb{P}^\alpha_\alpha$, which is a minimal counterexample to the $\bar{\Theta}$-subcompactness of $\bar{\kappa}$, for some $\bar{\Theta} \leq \bar{\alpha}$. Since we only use properties of the iteration $\mathbb{P}^\kappa_{\bar{\kappa}}$ to determine whether the forcing done at the

$\bar{\kappa}$th stage of $\mathbb{P}^{\kappa}_{\kappa}$ is trivial or not, we also choose the same forcing $\bar{\mathbb{Q}}$ in the stage $\bar{\kappa}$ forcing of $\mathbb{P}^{\kappa}_{\kappa}$.

Thus, by definition of our iterations, they have trivial stages up to at least stage $\bar{\Theta}$. So we must show that, from stage $\bar{\Theta}$ to stage κ the two iterations are the same. For this we argue inductively as in Claim 3.3.3, and it suffices to show that we cannot have $\dot{\mathbb{Q}}^{\kappa}_{\delta}$ be trivial, while $\dot{\mathbb{Q}}^{\alpha}_{\delta}$ is non-trivial, for successor stages where $\delta \in [\bar{\Theta}, \kappa)$.

Since, inductively, all the stages up to δ are the same for both iterations, we need to show that, if there is a counterexample $(\mathbb{R}, \gamma, \delta)$ to the γ-subcompactness of δ, with $\bar{\Theta} \leq \delta < \gamma < \kappa$ with $|\operatorname{TC}(\mathbb{R})| \in [\kappa, \alpha)$, then there is a counterexample $(\mathbb{S}, \gamma, \delta)$ with $|\operatorname{TC}(\mathbb{S})| < \kappa$.

As before, we have that $\mathbb{R} \in H_{\alpha}$ so we obtain an α-subcompactness elementary embedding $j : (H_{\bar{\alpha}_{\mathbb{R}}}, \in, \bar{\mathbb{R}}, \bar{\gamma}, \bar{\delta}) \to (H_{\alpha}, \in, \mathbb{R}, \gamma, \delta)$ with critical point $\bar{\kappa}_{\mathbb{R}}$, where $\bar{\alpha}_{\mathbb{R}} \leq \bar{\delta} < \bar{\gamma} < \bar{\kappa}_{\mathbb{R}}$ and $\bar{\mathbb{R}}$ breaks the $\bar{\gamma}$-subcompactness of $\bar{\delta}$. By elementarity, $|\operatorname{TC}\bar{\mathbb{R}}| \in [\bar{\kappa}_{\mathbb{R}}, \bar{\alpha}_{\mathbb{R}})$.

Since $\bar{\delta} < \bar{\gamma} < \bar{\kappa}_{\mathbb{R}}$, we have that $j(\bar{\delta}) = \bar{\delta}$ and $j(\bar{\gamma}) = \bar{\gamma}$, meaning that $\bar{\mathbb{R}}$ breaks the γ-subcompactness of δ and has $|\operatorname{TC}\bar{\mathbb{R}}| < \bar{\alpha}_{\mathbb{R}} < \kappa$ as desired. □

From now we will implicitly use these closure results without comment. We now progress by showing that we may lift the α-subcompactness embedding π in V to an α-subcompactness embedding:

$$\pi^{+} : \left(H_{\bar{\alpha}}[G_{\bar{\kappa}}], \in, \dot{\bar{B}}^{G_{\bar{\kappa}}}, \dot{\bar{\mathbb{R}}}^{G_{\bar{\kappa}}}, \dot{\bar{\Theta}}^{G_{\bar{\kappa}}}\right) \to \left(H_{\alpha}[G_{\kappa}], \in, \dot{B}^{G_{\kappa}}, \dot{\mathbb{R}}^{G_{\kappa}}, \dot{\Theta}^{G_{\kappa}}\right).$$

For notational ease we will often suppress the extra parameters, and refer the the embedding simply as $\pi^{+} : H_{\bar{\alpha}}[G_{\bar{\kappa}}] \to H_{\alpha}[G_{\kappa}]$.

By the Lifting Criterion (Theorem 2.2.5) it suffices to show that $\pi(p) \in G_{\kappa}$ for all $p \in G_{\bar{\kappa}}$. For this, note that such a condition p is a member of $\mathbb{P}^{\bar{\kappa}}_{\bar{\kappa}}$, and we take a direct limit at stage $\bar{\kappa}$, so there is some $\beta < \bar{\kappa}$ such that $p(\delta) = \mathbb{1}$ for all $\delta \geq \beta$. Then

$$H_{\bar{\alpha}}[G_{\bar{\kappa}}] \vDash \forall \gamma < \beta : \ p(\gamma) = (p \restriction_\beta)(\gamma)$$

and so for all $\gamma < \pi(\beta)$, $\pi(p)(\gamma) = \left(\pi(p \restriction_\beta)\right)(\gamma)$. Now, by Lemma 3.2.4 we have that $\mathbb{P}^\kappa_\gamma \in H_{\bar{\kappa}}$; also since the critical point of π is $\bar{\kappa}$, $\pi \restriction H_{\bar{\kappa}}$ is the identity function. So $\pi(p)$ is p followed by a sequence of length κ, but in fact this sequence must be a sequence of $\mathbb{1}$s only: $\pi(\beta) = \beta$ since $\beta < \bar{\kappa}$, and $\pi(p)$ is a sequence of length $\pi(\bar{\kappa}) = \kappa$ with maximum support $\pi(\beta) = \beta$. Thus $\pi(p) = p^\frown \mathbb{1}^{(\kappa)} \in G_\kappa$. So, as desired we may apply the lifting criterion to obtain the lifted embedding π^+.

Now that we have π^+, we wish to lift again. To do this we will apply Silver's Master Condition argument (Definition 2.2.6). Since g picks $\mathbb{Q}$ in the stage κ lottery sum, we have by elementarity that there is $\bar{\mathbb{Q}} \subseteq H_{\bar{\alpha}}$ with $\pi^+(\bar{\mathbb{Q}}) = \mathbb{Q}$ which is chosen in the stage $\bar{\kappa}$ lottery by some generic $\bar{g}$. To see that we may indeed lift the embedding π^+ note that $\pi^{+}``(\bar{g}) \subseteq \mathbb{Q}$ is directed and has cardinality at most $\bar{\alpha} < \kappa$. So by the $< \kappa$-directed closure of $\mathbb{Q}$ there is some $q \in \mathbb{Q}$ with $q \leq \pi^+(p)$ for all $p \in \bar{g}$. So we specify that q is a member of our generic g, and so $\pi^{+}``(\bar{g}) \subseteq g$. This means that the Lifting Criterion is satisfied, giving us a lifted elementary embedding

$$\pi^{++} : \left(H_{\bar{\alpha}}[G_{\bar{\kappa}} * \dot{\bar{g}}], \in, \dot{\bar{B}}^{G_{\bar{\kappa}} * \bar{g}}, \bar{\mathbb{Q}}, \bar{\Theta}\right) \to \left(H_\alpha[G_\kappa * \dot{g}], \in, \dot{B}^{G_\kappa * \dot{g}}, \mathbb{Q}, \Theta\right).$$

But recall that $\dot{B}^{G_\kappa * \dot{g}} = A$ and so π^{++} is an α-subcompactness embedding for κ with predicate A in the extension $V[G_\kappa * \dot{g}]$. Now we restrict this embedding in $V[G_\kappa * \dot{g}]$ to give a Θ-subcompactness embedding

$$\pi^* : \left(H_{\bar{\Theta}}[G_{\bar{\kappa}} * \dot{\bar{g}}], \in, \bar{A}, \bar{\Theta}\right) \to \left(H_\Theta[G_\kappa * \dot{g}], \in, A, \Theta\right)$$

where $\bar{A} = \dot{\bar{B}}^{G_{\bar{\kappa}} * \dot{\bar{g}}}$.

All that remains to do now is to show that the domain of this embedding, $H_{\bar{\Theta}}[G_{\bar{\kappa}} *$

$\dot{g}]$, is equal to $H_{\bar{\Theta}}^{V[G_\kappa * \dot{g}]}$ and that its range, $H_\Theta[G_\kappa * \dot{g}]$, is equal to $H_\Theta^{V[G_\kappa * \dot{g}]}$.

We do this by showing the following three equalities:

$$H_{\bar{\Theta}}[G_{\bar{\kappa}} * \dot{\bar{g}}] = H_{\bar{\Theta}}^{V[G_{\bar{\kappa}} * \dot{\bar{g}}]} \tag{3.3.1}$$

$$H_{\bar{\Theta}}^{V[G_\kappa * \dot{g}]} = H_{\bar{\Theta}}^{V[G_{\bar{\kappa}} * \dot{\bar{g}}]} \tag{3.3.2}$$

$$H_\Theta[G_\kappa * \dot{g}] = H_\Theta^{V[G_\kappa * \dot{g}]} \tag{3.3.3}$$

Equalities 3.3.1 and 3.3.3 can be seen by an application of Lemmas 3.1.4 and 3.1.3. To see that Equality 3.3.2 holds we need to show that $H_{\bar{\Theta}}$ has not been altered by the forcing iteration from stage $\bar{\kappa} + 1$ to stage $\kappa + 1$.

By standard encoding arguments, a forcing adds no new elements of $H_{\bar{\Theta}}$ if and only if it adds no bounded subsets of $\bar{\Theta}$. Now, a $\bar{\Theta}$-strategically closed forcing will add no new bounded subsets of $\bar{\Theta}$. So we claim that $\mathbb{P}^\alpha_{(\bar{\kappa},\kappa+1)} \cong \mathbb{P}^\kappa_{(\bar{\kappa},\kappa)} * \dot{\mathbb{Q}}$ is $< \bar{\Theta}$-strategically closed.

Factor $\mathbb{P}^\kappa_\kappa$ as $\mathbb{P}^{\bar{\kappa}}_{\bar{\kappa}} * \bar{\mathbb{Q}} * \mathbb{P}^\kappa_{(\bar{\kappa}+1,\kappa)}$, then note that between stage $\bar{\kappa}+1$ and stage $\bar{\Theta}$ there can only be trivial forcing by the definition of the iteration. Thus, by Lemma 3.2.7 the tail of the forcing $\mathbb{P}^\kappa_{(\bar{\kappa}+1,\kappa)}$ is $\bar{\Theta}$-strategically closed. Also $\mathbb{Q}$ is $< \kappa$-directed closed in $V[G_\kappa]$, so the iteration $\mathbb{P}^\alpha_{(\bar{\kappa}+1,\kappa+1)} \cong \mathbb{P}^\alpha_{(\bar{\kappa}+1,\kappa)} * \dot{\mathbb{Q}}$ between stage $\bar{\kappa} + 1$ and $\kappa + 1$ will be $\bar{\Theta}$-strategically closed by Lemma 2.1.25.

So we have not altered $H_{\bar{\Theta}}$ with our iteration, and so Equality 3.3.2 holds.

So in fact we have

$$\pi^* : \left(H_{\bar{\Theta}}^{V[G_\kappa * \dot{g}]}, \in, \bar{A}, \bar{\Theta}\right) \to \left(H_\Theta^{V[G_\kappa * \dot{g}]}, \in, A, \Theta\right)$$

which is a Θ-subcompactness elementary embedding in $V[G_\kappa * \dot{g}]$ for κ with predicate $A \subseteq H_\alpha$ as required.

So we have a Θ-subcompactness elementary embedding π^* in $V[G_\kappa * \dot{g}]$ for any subset A in H_Θ, and so κ is Θ-subcompact in the extension, which contradicts our assumption that $\mathbb{Q}$ breaks the Θ-subcompactness of κ. Thus κ is α-subcompact after forcing with $\mathbb{P}^\kappa_\kappa * \dot{\mathbb{Q}}$. □

This leaves us with some questions.

Question 3.3.6. Is it possible to show indestructibility for a α-subcompact cardinal where $\alpha \notin \text{Reg}$?

Due to the fact that α-subcompactness interleaves with partial supercompactness, the requirement that $\alpha \in \text{Reg}$ is not so egregious — many large cardinals will automatically be α-subcompact with regular α, but for finer detail it would be good to do without the regularity condition. Our main use of this assumption was in usage of Lemma 3.1.4, so it is possible that utilising some other techniques to ensure membership in H_α will allow us to do without the regularity of α.

Question 3.3.7. Can the techniques here be used to show indestructibility of other large cardinals where other methods have proved unsuccessful?

Perhaps this approach would suit other large cardinals which are defined as the image of the critical point of an elementary embedding, which makes defining an appropriate Menas function challenging.

Chapter 4

Indestructibility for $C^{(n)}$-Supercompact Cardinals

The preservation and indestructibility of $C^{(n)}$-supercompact cardinals has been an open topic since their definition in 2012. Some progress has been made in this direction: in [44], Tsaprounis showed that $C^{(n)}$-supercompacts (and $C^{(n)}$-extendibles and more) are preserved by small[1] forcing, and that λ-$C^{(n)}$-supercompactness is preserved by $\leq \lambda^{<\kappa}$-distributive forcing.

However we are still some distance from answering the open question in [23], namely:

> *Let κ be a $C^{(n)}$-supercompact cardinal. What kind of forcings preserve the $C^{(n)}$-supercompactness of κ? For instance, is it possible to add many Cohen subsets to κ, while preserving $C^{(n)}$-supercompactness?*

In this chapter we show that, provided we begin with a $C^{(2)}$-extendible cardinal, we may make its $C^{(2)}$-supercompactness indestructible by all $< \kappa$-directed closed forcing. While this does not entirely answer the above question, this is the first such

[1]Here, 'small' forcing for a $C^{(n)}$-supercompact κ means forcing of cardinality $< \kappa$.

indestructibility result for $C^{(n)}$-supercompacts, and it is possible the techniques used here could be used to show a similar result without the additional assumption of $C^{(2)}$-extendibility.

Note that in this argument we will not show that, in the forcing extension, κ is $C^{(2)}$-extendible, we merely leverage the properties of a $C^{(2)}$-extendible in V to allow us to define the preparatory forcing and lift the embedding. In fact, showing any indestructibility arguments for $C^{(n)}$-extendibles is impossible, by the following theorem of Bagaria, Hamkins, Tsaprounis and Usuba [7].

Theorem 4.0.1. (Bagaria, Hamkins, Tsaprounis) Σ_3-extendible cardinals κ are never indestructible by $< \kappa$-directed closed forcing.

In fact, they are *superdestructible*, meaning that, if κ is Σ_n-extendible with target θ or higher in V, then it is not Σ_3-extendible with target θ or higher after any non-trivial κ-strategically closed forcing $\mathbb{Q} \in V_\theta$.

Here, we say that κ is Σ_n*-extendible* with target θ if there is a Σ_n-elementary embedding $j : V_\kappa \prec_n V_\theta$ with critical point κ.

This theorem implies that a wide class of cardinals including superstrong cardinals, huge cardinals, $C^{(n)}$-extendible (and extendible) cardinals, as well as $C^{(n)}$ cardinals when $n \geq 3$, are superdestructible.

So for $C^{(n)}$-extendibles we cannot show any indestructibility results, and any large cardinal which is, by definition, Σ_3-extendible, will also be superdestructible. So we must be careful when discussing preservation and indestructibility of $C^{(n)}$-supercompacts (and indeed any large cardinal property) to avoid violating superdestructibility.

A promising result, which suggests that indestructibility for $C^{(n)}$-supercompacts may not be a lost case is that, though some $C^{(n)}$-supercompacts are in $C^{(n)}$, not all are. By [23] the first $C^{(n)}$-supercompact can be the first supercompact, which cannot be more than Σ_2-correct since 'there is a supercompact' is Σ_3, so this would contradict minimality. So it does not follow, by definition of $C^{(n)}$-supercompactness, that every $C^{(n)}$-supercompact is $C^{(n)}$. Our indestructibility result later will only claim to hold for $C^{(2)}$-supercompactness, mainly because we leverage key properties of Σ_2 formulas, but also because the Superdestructibilty Theorem seems to suggest that the case when $n \geq 3$ is harder, or perhaps impossible.

4.1 Background

Since $C^{(n)}$-supercompactness elementary embeddings require a degree of closure of the target, we will be using the following Lemma, which tells us that λ^+-c.c. forcing preserves the closure property we need.

Lemma 4.1.1. [Folklore] Let M be a transitive inner model with $\mathrm{Ord} \subseteq M$, $\mathbb{P} \in M$ a λ^+-c.c. forcing and G a $\mathbb{P}$-generic over M. Then, in $V[G]$, if $V \models M^\lambda \subseteq M$ then $M[G]^\lambda \subseteq M[G]$.

When dealing with Σ_n-correctness we will make use of the following folkloric result.

Lemma 4.1.2. Let M be a set model of $\mathsf{ZF} - \mathcal{P}$. Then the satisfaction relation $\models$ in M is Σ_1 definable.

The following definition, due to Tsaprounis, is one which we will exploit later.

Definition 4.1.3. A cardinal κ is jointly λ-$C^{(n)}$-supercompact and superstrong if there exists an elementary embedding $j : V \to M$, which is simultaneously a λ-$C^{(n)}$-supercompactness embedding, and witnesses the superstrongness of κ — meaning that M is transitive, ${}^\lambda M \subseteq M$, $j(\kappa) > \lambda$, $j(\kappa) \in C^{(n)}$ and $V_{j(\kappa)} \subseteq M$.

Such large cardinals will be of use to us soon, as it turns out that they are equivalent to $C^{(n)}$-extendibles. We present Tsaprounis's proof in [44] to show this equivalence.

We will be relying upon this result in the indestructibility proof, and will modify some of the arguments here to suit our purposes later.

We begin with one direction of the equivalence:

Theorem 4.1.4. (Tsaprounis) Suppose κ is $\lambda + 1$-$C^{(n)}$-extendible for some $n \geq 1$ and some $\lambda = \beth_\lambda$ with $\mathrm{cf}(\lambda) > \kappa$, witnessed by $j : V \to M$. Then κ is jointly λ-$C^{(n)}$-supercompact and superstrong. Moreover this is witnessed by j_E, the extender embedding arising from the $(\kappa, j(\lambda))$-extender E derived from j.

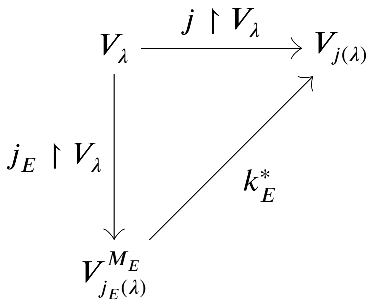
Proof. Fix $n \geq 1$ and $\lambda = \beth_\lambda$ with $\mathrm{cf}(\lambda) > \kappa$. Let $j : V_{\lambda+1} \to V_{j(\lambda)+1}$ be a $\lambda + 1$-$C^{(n)}$-extendibility elementary embedding for κ. Let $E = \langle E_a : a \in [j(\lambda)]^{<\omega} \rangle$ be the ordinary $(\kappa, j(\lambda))$-extender derived from j, where each E_a is a κ-complete ultrafilter on $[\lambda]^{|a|}$ with $X \in E_a$ if and only if $a \in j(X)$.

We may make this definition, despite the fact that j is an embedding between sets, not inner models, since, for any $m \in \omega$, $\mathcal{P}([\lambda]^m) \subseteq V_{j(\lambda)+1}$. Further $E \in V_{j(\lambda)+1}$ and $V_{j(\lambda)+1}$ is able to correctly verify that E is a $(\kappa, j(\lambda))$-extender, since it contains the E_a as well as all the projection functions.

Using the extender E we derive, as in Definition 2.1.12, the extender embedding $j_E : V \to M_E$ with $\mathrm{cp}(j) = \kappa$. Now we construct a factor embedding

$$k^*_E : V^{M_E}_{j_E(\lambda)} \to V_{j(\lambda)} \quad \text{with} \quad k^*_E([a, f]) = j(f)(a)$$

for all $[a, f] \in V^{M_E}_{j_E(\lambda)}$, where $a \in [j(\lambda)]^{<\omega}$ and $f : [\lambda]^{|a|} \to V_\lambda$. Note that this is a restricted version of the k_E used in the discussion after Definition 2.1.12, but standard arguments show that k^*_E is a well-defined $\in$-embedding, and is injective, and we have a commutative diagram:



where $j \restriction V_\lambda = k^*_E \circ (j_E \restriction V_\lambda)$.

In fact, the embedding k^*_E is surjective, and so — since its domain and range are transitive sets — it is the identity function. For surjectivity, we note that, since $\lambda = \beth_\lambda$, we can fix some bijection $g : [\lambda]^1 \to V_\lambda$, which is itself a member of $V_{\lambda+1}$. So, by elementarity, $j(g) : [j(\lambda)]^1 \to V_{j(\lambda)}$ is also a bijection with $j(g) \in V_{j(\lambda)+1}$. For a given $x \in V_{j(\kappa)}$ there is then some $\gamma < j(\kappa)$ such that $j(g)(\{\gamma\}) = x$. But then, by the definition of k^*_E, $x = k^*_E([\{\gamma\}, [g]])$, so k^*_E is surjective.

So we have that $V^{M_E}_{j_E(\lambda)} = V_{j(\lambda)}$, meaning $V_{j(\lambda)} \subseteq M_E$, and so j_E is superstrong. This gives that, for all $\alpha < \lambda$, $j_E(\alpha) = j(\lambda)$, so, in particular, $j_E(\kappa) = j(\kappa)$. Since (computed in V), $\mathrm{cf}(j(\lambda)) > \lambda$, we have that $j_E``(\lambda) = j``(\lambda) \in V_{j(\lambda)}$, so $j_E``(\lambda) \in M_E$.

It now suffices to show the remaining condition for λ-$C^{(n)}$-supercompactness holds — namely that ${}^\lambda M_E \subseteq M_E$. To see this first recall that M_E can be formulated as

$$M_E = \left\{ j_E(f)(a) : a \in [j(\lambda)]^{<\omega},\ f : [\lambda]^{|a|} \to V,\ f \in V \right\}.$$

So any subset of M_E of cardinality λ looks like $\{j_E(f_i)(a_i) : i < \lambda\}$ where $a_i \in [j(\lambda)]^{<\omega}$ and $f_i : [\lambda]^{|a|} \to V$, $f_i \in V$ for all $i < \lambda$.

Now, using that $\mathrm{cf}\,(j(\lambda)) > \lambda$, we have $\langle a_i : i < \lambda \rangle \in V_{j(\lambda)} \subseteq M_E$. Since $j_E``\lambda \in M_E$, the restriction $j_E \restriction \lambda : \lambda \to j_E``(\lambda)$ is in M_E (where here the restriction is viewed as an order-type function in M_E).

Define $G : j_E(\lambda) \to M_E$ by $G = j_E\left(\langle f_i : i < \lambda\rangle\right)$, where $G \in M_E$. Further define in M_E the function $F = G \circ \left(j_E \restriction \lambda\right)$, with domain λ and range M_E. But, we actually have that $F = \langle j_E(f_i) : i < \lambda\rangle$, since, for $i < \lambda$:

$$\begin{aligned}
F(i) &= G\left(j_E(i)\right) && \text{(by definition of } F\text{)}\\
&= j_E\left(\langle f_i : i < \lambda\rangle\right)\left(j_E(i)\right) && \text{(by definition of } G\text{)}\\
&= j_E\left(\langle f_i : i < \lambda\rangle(i)\right) && \text{(by elementarity of } j_E\text{)}\\
&= j_E(f_i).
\end{aligned}$$

This, together with the fact that $\langle a_i : i < \lambda\rangle \in M_E$, gives that $\left\{j_E(f_i)(a_i) : i < \lambda\right\} \in M_E$ as required. □

Now we consider the other direction of the equivalence, namely:

Theorem 4.1.5. (Tsaprounis) If κ is jointly $C^{(n)}$-supercompact and superstrong, then it is $C^{(n)}$-extendible (for $n \geq 0$).

Proof. We split into two cases, firstly, when $n \geq 1$, and the second when $n = 0$. For $n \geq 1$, fix some $\lambda > \kappa$ with $\lambda \in C^{(n+2)}$ and let us suppose that the joint λ-$C^{(n)}$-supercompactness and superstrongness of κ is witnessed by $j : V \to M$. So $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, $j(\kappa) \in C^{(n)}$, ${}^{\lambda}M \subseteq M$ and $V_{j(\kappa)} \subseteq M$. Now we apply the following result:

Lemma 4.1.6. (Tsaprounis) If κ is $C^{(n)}$-supercompact and superstrong, then $\kappa \in C^{(n+2)}$ (for any $n \geq 0$).

The proof of the Lemma is by an induction on n in the meta-theory. Full detail can be found in Lemma 2.29 of [43].

Using this Lemma, we have that $\kappa \in C^{(n+2)}$, and so, by elementarity $M \models j(\kappa) \in C^{(n+2)}$. Also, by definition of λ, $M \models j(\lambda) \in C^{(n+2)}$. Since M is closed under λ sequences, the restricted embedding $j \restriction V_\lambda \to V^M_{j(\lambda)}$ is a member of M. It is an elementary embedding, and has that $j \restriction V_\lambda(\kappa) > \lambda$, so in fact it witnesses, in M, that κ is $< \lambda$-$C^{(n)}$-extendible — namely, for all $\gamma < \lambda$, κ is γ-$C^{(n)}$-extendible.

Now, by the Σ_n-correctness of $j(\kappa)$, $V_{j(\kappa)} \models \lambda \in C^{(n+1)}$ (recalling here Fact 2.3.1). So, since $V_{j(\kappa)} \subseteq M$ by superstrongness, and $M \models j(\kappa) \in C^{(n+2)}$, we have that $M \models \lambda \in C^{(n+1)}$. So in fact the $< \lambda$-$C^{(n)}$-extendibility of κ in M can be verified in V_λ, i.e. $M \models V_\lambda \models \kappa$ is $C^{(n)}$-extendible.

But $V_\lambda \subseteq M$ so in fact $V_\lambda \models \kappa$ is $C^{(n)}$-extendible. Since $\lambda \in C^{(n+2)}$, and, by Theorem 2.3.6, 'κ is $C^{(n)}$-extendible' is $C^{(n+2)}$, so κ is in fact $C^{(n)}$-extendible.

For the case when $n = 0$, one follows the same argument, but with a λ chosen to be in $C^{(3)}$, and use that 'κ is extendible' is Π_3. Then, analogously to above, one verifies that $M \models \lambda \in C^{(2)}$ and argues the same. □

Now combining Theorems 4.1.4 and 4.1.5 we obtain the equivalence:

Theorem 4.1.7. (Tsaprounis) A cardinal κ is $C^{(n)}$-extendible if and only if it is jointly $C^{(n)}$-supercompact and superstrong.

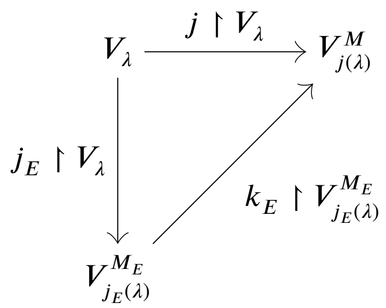
The following theorem, from [45], will also be used extensively in later $C^{(n)}$-supercompactness arguments. Its form here is a summarisation of Corollary 2.32 of that paper, and the discussion afterwards.

Theorem 4.1.8. (Tsaprounis) Suppose that $j : V \to M$ is a θ-$C^{(n)}$-supercompactness elementary embedding for κ, for some $\theta > \kappa$. Let $\lambda = \beth_\lambda \geq \theta$ and let E be the $(\kappa, j(\lambda))$-extender derived from j. Then the extender embedding $j_E : V \to M_E$ derived from E is a θ-$C^{(n)}$-supercompactness elementary embedding for κ with $j_E(\kappa) = j(\kappa)$.

Further, if j is also superstrong (i.e. κ is θ-$C^{(n)}$-extendible, by Theorem 4.1.7) then j_E will also be superstrong, so witnesses that κ is θ-$C^{(n)}$-extendible.

Proof. Fix $\theta > \kappa$, $j : V \to M$ a θ-$C^{(n)}$-supercompactness elementary embedding, and pick $\lambda = \beth_\lambda$ with $\mathrm{cf}(\lambda) > \kappa$. Let E be the $(\kappa, j(\lambda))$-extender derived from j. So $E = \langle E_a : a \in [j(\lambda)]^{<\omega} \rangle$, where each E_a is a κ-complete ultrafilter on $[\lambda]^{|a|}$ with $X \in E_a$ if and only if $a \in j(X)$. Let $j_E : V \to M_E$ be the derived extender embedding, which has $\mathrm{cp}(j_E) = \kappa$.

Now, as in the discussion after Definition 2.1.12, we define $k_E : M_E \to M$ by $k_E([a, f]) = j(f)(a)$ for all $[a, f] \in M_E$. We consider its restriction to $V^{M_E}_{j_E(\lambda)}$ and obtain the following commutative diagram:



As in the proof of Theorem 4.1.4, we have that $k_E \restriction V^{M_E}_{j_E(\lambda)} : V^{M_E}_{j_E(\lambda)} \to V^M_{j(\lambda)}$ is actually the identity function, and so $V^{M_E}_{j_E(\lambda)} = V^M_{j(\lambda)}$. So, for all $\beta < \lambda$ — in particular, for κ — we have that $j_E(\beta) = j(\beta)$. Indeed, since $\mathrm{cf}\,(j(\lambda)) > j(\kappa) > \theta$, we have that $j_E``(\theta) = j``(\theta) \in V^M_{j(\lambda)}$, and $j_E``(\theta) \in M_E$.

Here, if κ is also superstrong, i.e. $V_{j(\kappa)} \subseteq M$, then $V_{j(\kappa)} \subseteq V^M_{j(\lambda)}$, so $V_{j(\kappa)} \subseteq V^{M_E}_{j_E(\lambda)} \subseteq M_E$, but $j_E(\kappa) = j(\kappa)$, so in fact $V_{j_E(\kappa)} \subseteq M_E$, so j_E is a superstrong embedding too.

The last condition for θ-$C^{(n)}$-supercompactness is closure in M_E under θ-sequences. For this we apply the exact same argument as was used in Theorem 4.1.4.

It is worth noting here that the assumption that $\mathrm{cf}(\lambda) > \kappa$, hence $\mathrm{cf}\,(j(\lambda)) > j(\kappa) > \lambda$, is what allows the argument to work. □

We now turn our attention to the function which will guide our preparatory forcing. The following is a combination of Theorem 4.2 of [44] and Theorem 4.2 of [43], the latter of which was also independently shown by Corazza in [13].

Theorem 4.1.9 (Tsaprounis)**.** Every $C^{(n)}$-extendible cardinal (for $n \geq 1$) carries a *$C^{(n)}$-extendibility Laver function*, namely $f : \kappa \to V_\kappa$ such that, for every $\lambda \geq \kappa$ and any $x \in H_{\lambda^+}$, there is a jointly λ-$C^{(n)}$-supercompact and superstrong elementary embedding $j : V \to M$, such that $j(f)(\kappa) = x$.

For the preparatory forcing in the indestructibility argument we will utilise a weaker version of this Laver function, namely a *$C^{(n)}$-extendibility Menas function*. Using Hamkins's techniques in [22], we are able to isolate the important aspects of the Laver function — namely its 'fastness' — and use it to insist that there are many trivial forcings between each non-trivial stage of our iterated forcing. In this fashion, we no longer need the full anticipatory power of a Laver function, as the ability of the Menas function to create long periods of trivial forcing is enough.

Definition 4.1.10. For a $C^{(n)}$-extendible cardinal κ, a *$C^{(n)}$-extendibility Menas function* is $f : \kappa \to \kappa$ such that, for all $\lambda > \kappa$, there is a jointly λ-$C^{(n)}$-supercompact and superstrong elementary embedding $j : V \to M$, such that $j(f)(\kappa) > \lambda$.

The existence of a $C^{(n)}$-extendibility Laver function immediately gives us a $C^{(n)}$-extendibility Menas function.

4.2 Indestructibility Result

As is pointed out in [23], there are many issues with showing indestructibility results for $C^{(n)}$-supercompacts. Chief among them are concerns with *lifting embeddings* and with *defining extenders appropriately*.

In order to lift elementary embeddings for $C^{(n)}$-supercompactness one needs to show definability of the lifted embedding, i.e. if we wish to lift $j : V \to M$ to some $j^+ : V[G] \to M[G * H]$, we need to show that the $M[G]$-generic filter H is in

$V[G]$. Many lifting arguments in the literature rely upon their $j(\kappa)$ being 'small' in V, together with distributivity or closure properties of the forcing used. However with $C^{(n)}$-supercompacts, we must work a little harder, and we will show definability in a somewhat roundabout way — by lifting an embedding, using this embedding and a ground model condition to derive an extender, and from that extender defining a new elementary embedding in the extension.

We begin our proof of indestructibility of a $C^{(2)}$-supercompact with a stronger assumption, that of $C^{(2)}$-extendibility. The properties of $C^{(2)}$-extendibles, and the existence of a suitable Menas function, will be used to show that we can make $C^{(2)}$-supercompactness (but not $C^{(2)}$-extendibility) indestructible.

Theorem 4.2.1. Let κ be $C^{(2)}$-extendible, then after forcing with the Lottery Preparation defined relative to a $C^{(2)}$-extendibility Menas function f, the $C^{(2)}$-supercompactness of κ is indestructible by $<\kappa$-directed closed forcing.

Proof. Let $\theta \geq \kappa$ and suppose that G_κ is $\mathbb{P}_\kappa$-generic over V and g is $\mathbb{Q}$-generic over $V[G_\kappa]$, for $\mathbb{Q}$ a $<\kappa$-directed closed forcing in $V[G_\kappa]$. Let $\lambda > \max(|\mathbb{Q}|, 2^{\theta^{<\kappa}})$ be such that $\lambda = \beth_\lambda$ and $\mathrm{cf}(\lambda) > \kappa$.

Let $j : V \to M$ be a jointly λ-$C^{(2)}$-supercompact and superstrong elementary embedding witnessing the Menas property of f; so in particular $j(f)(\kappa) > \lambda$. Note that, below a condition which chooses $\mathbb{Q}$ in the stage κ lottery, $j(\mathbb{P}_\kappa)$ factors as $\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathbb{P}}_{\text{tail}}$ in M, and $\mathbb{P}_{\text{tail}}$ is $<\lambda$-strategically closed in $M[G_\kappa * \dot{g}]$ by Lemma 2.2.4.

Now $V \models j(\kappa) \in C^{(1)}$ and, by the superstrongness of j, we have that $j(\kappa)$ is regular (hence inaccessible).

We now lift the embedding $j : V \to M$ to $j^+ : V[G_\kappa] \to M[H]$ where H is $j(\mathbb{P}_\kappa)$-generic over M, using the Lifting Criterion (Theorem 2.2.5), which holds since we take a direct limit at stage κ and so $j(p) = p^\frown \mathbb{1}^{j(\kappa)} \in G_\kappa * \dot{g} * \dot{G}_{\text{tail}}$ for all $p \in G_\kappa$. Now apply Lemma 4.1.1 to conclude that ${}^\lambda M[H] \subseteq M[H]$ — for this recall that $|\mathbb{P}_\kappa| = \kappa$ and so it has the λ^+-c.c..

Now, since $|\mathbb{P}_\kappa| < j(\kappa)$, forcing with $\mathbb{P}_\kappa$ cannot destroy the inaccessibility (hence Σ_1-correctness) of $j(\kappa)$. So in $V[G_\kappa]$, $j^+(\kappa) = j(\kappa) \in C^{(1)}$.

Now we note that any Σ_2 formula φ with parameters in $V_{j(\kappa)}$ may be reformulated as $\exists\alpha < j(\kappa) : V_\alpha \models \varphi$. By Lemma 4.1.2, for set models of $\mathsf{ZF} - \mathcal{P}$, satisfaction is Σ_1 definable. Now, $V[G_\kappa] \models j(\kappa) \in C^{(1)}$, and any Σ_2 formula with parameters in $V[G_\kappa]_{j(\kappa)}$ is true in $V[G_\kappa]$ if and only if it's true in $V[G_\kappa]_{j(\kappa)}$. Thus $j(\kappa) \in C^{(2)}$ in $V[G_\kappa]$. For clarity:

$$
\begin{aligned}
&V[G_\kappa]_{j(\kappa)} \models \varphi \text{ iff} \\
&V[G_\kappa]_{j(\kappa)} \models \exists\alpha < j(\kappa) : V[G_\kappa]_\alpha \models \varphi \text{ iff} \\
&V[G_\kappa] \models \exists\alpha < j(\kappa) : V[G_\kappa]_\alpha \models \varphi \text{ iff} \\
&V[G_\kappa] \models \varphi.
\end{aligned}
$$

We now must lift the embedding again, and show that, after forcing with $\mathbb{Q}$, κ is still λ-$C^{(2)}$-supercompact. Note here that we do not — and cannot — claim that the lifted embedding is also superstrong like j, since this would violate the Superdestructibility Theorem 4.0.1.

For the second lift note that $\mathbb{Q}$ is $<\kappa$-directed closed, hence $j^+(\mathbb{Q})$ is $<j^+(\kappa)$-directed closed, so in particular it is $<\lambda$-directed closed. Now $|\dot{g}| < \lambda$, so $|j^{+}``(\dot{g})| \leq \lambda$, and since ${}^\lambda M[H] \subseteq M[H]$, there is some master condition $q \in j^+(\mathbb{Q})$ such that $q \leq j^+(p)$ for all $p \in g$. Since $q \in M[H]$ there is a $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$-name $\dot{q}$ in M, hence in V, which q interprets.

Using this master condition (and applying the lifting criterion) we lift again to an elementary embedding $j^{++} : V[G_\kappa * \dot{g}] \to M[H * \dot{h}]$, where h is $j^+(\mathbb{Q})$-generic over $M[H]$ and contains the master condition q. This elementary embedding has that $j^{++}(\kappa) = j^+(\kappa) = j(\kappa) > \lambda$, and we can again apply Lemma 4.1.1 to conclude that ${}^\lambda M\left[j(G_\kappa) * j^+\dot{(g)}\right] \subseteq M\left[j(G_\kappa) * j^+\dot{(g)}\right]$ since $|\mathbb{Q}| < \lambda$ so it clearly has the λ^+-c.c.

As with the first lift we have that $j^{++}(\kappa) = j(\kappa)$ is Σ_2-correct in $V[G_\kappa * \dot{g}]$, since forcing with $\mathbb{Q}$ cannot break the inaccessibility, hence Σ_1-correctness of $j^{++}(\kappa)$, and Σ_2 formulas are reflected down below $j^{++}(\kappa)$, with satisfaction in $V[G_\kappa * \dot{g}]_{j^{++}(\kappa)}$ being Σ_1 definable.

We can now, in $V[G_\kappa * \dot{g}]$, define a $(\kappa, j^{++}(\lambda))$-extender $E = \langle E_a : a \in [j^{++}(\lambda)]^{<\omega}\rangle$ by:

$$X \in E_a \leftrightarrow X \subseteq [\lambda]^{|a|} \wedge q \Vdash a \in j^{++}(X)$$

where q is the master condition above.

Since the definition of each E_a is different to the usual $\big($namely $X \in E_a$ if and only if $q \Vdash a \in j^{++}(X)$ instead of $a \in j^{++}(X)\big)$, we will show that this definition does indeed give rise to an extender. We verify each condition of Definition 2.1.13:

1. (a) First we show that each E_a is a κ-complete ultrafilter on $[\lambda]^{|a|}$.
   First, we have that $[\lambda]^{|a|} \in E_a$ since $a \in [j^{++}(\lambda)]^{<\omega}$, so $\mathbb{1} \Vdash a \in j^{++}([\lambda]^{|a|})$. Clearly q does not force that $a \in j^{++}(\emptyset) = \emptyset$. If A and B are in E_a then $q \Vdash a \in j^{++}(A)$ and $q \Vdash a \in j^{++}(B)$, so

   $$q \Vdash a \in j^{++}(A) \cap j^{++}(B) = j^{++}(A \cap B).$$

   So $A \cap B \in E_a$. If $A, B \subseteq [\lambda]^{|a|}$, $A \in E_a$ and $A \subseteq B$, then $q \Vdash a \in j^{++}(A) \subseteq j^{++}(B)$, so $q \Vdash a \in j^{++}(B)$, so $B \in E_a$.
   Since either $q \Vdash a \in j^{++}(X)$ or $q \Vdash a \in j^{++}\left([\lambda]^{|a|}\right) \setminus j^{++}(X)$ for any $X \subseteq [\lambda]^{|a|}$, either $X \in E_a$ or $[\lambda]^{|a|} \setminus X \in E_a$.
   To see that E_a is κ-complete, let $\gamma < \kappa$ and suppose $W = \{X_\delta : \delta < \gamma\} \subseteq E_a$. Then $q \Vdash \forall X_\delta \in W : a \in j^{++}(X_\delta)$, and, since the critical point of j^{++} is κ, $j^{++}(W) = j^{++}``(W)$. So in fact $q \Vdash a \in j^{++}(X_\delta)$ for each $\delta < \gamma$, and so $q \Vdash a \in \bigcap_{\delta<\gamma} j^{++}(X_\delta)$, so the intersection is in E_a also.

(b) We need now some $a \in [j^{++}(\lambda)]^{<\omega}$ such that E_a is not a κ^+-complete ultrafilter on $[\lambda]^{|a|}$. Let us pick $a = \{\kappa\}$ and define, for each $\gamma < \kappa$,

$$X_\gamma = \{\{\delta\} : \gamma < \delta < \lambda\} \subseteq [\lambda]^{|a|} = [\lambda]^1.$$

For all $\gamma < \kappa$, we have that $a = \{\kappa\} \in j^{++}(X_\gamma)$ since $j^{++}(X_\gamma) = \{\{\delta\} : \gamma < \delta < j^{++}(\lambda)\}$ (recall that the critical point of j^{++} is κ and $\gamma < \kappa$). But $\bigcap_{\gamma<\kappa} X_\gamma = \emptyset$ so it is not in E_a, as required.

(c) Now we wish to show that for all $\gamma < \lambda$ there is some $a \in [j^{++}(\lambda)]^{<\omega}$ such that $\{x \in [\lambda]^{|a|} : \gamma \in x\} \in E_a$. Note that, for $\gamma < \lambda$, $j^{++}(\gamma) < j^{++}(\lambda)$. So, for a given γ, let $a = \{j^{++}(\gamma)\} \in [j^{++}(\lambda)]^{<\omega}$. Then $X = \{x \in [\lambda]^1 : \gamma \in x\} \in E_a$ if and only if $q \Vdash a \in j^{++}(X)$, i.e.

$$q \Vdash \{j^{++}(\gamma)\} \in \{x \in [j^{++}(\lambda)]^1 : j^{++}(\gamma) \in x\},$$

so we are done.

2. Now we show Coherence: that for all $a, b \in [j^{++}(\lambda)]^{<\omega}$ with $a \subseteq b$, $X \in E_a$ if and only if $W = \{x \in [\lambda]^{|b|} : \pi_{ba}(x) \in X\} \in E_b$, where $\pi_{ba} : [\lambda]^{|b|} \to [\lambda]^{|a|}$ is as defined in §2.1.11. To see this first note that $j^{++}(\pi_{ba})(b) = a$, so $X \in E_a$ if and only if $q \Vdash a \in j^{++}(X)$, i.e.

$$q \Vdash j^{++}(\pi_{ba})(b) \in j^{++}(X).$$

This is true if and only if

$$q \Vdash b \in \left\{x \in [j^{++}(\lambda)]^{|b|} : j^{++}(\pi_{ba})(x) \in j^{++}(X)\right\},$$

which is precisely $q \Vdash b \in j^{++}(W)$, so $a \in E_a$ if and only if $W \in E_b$.

3. For Normality, let $a \in [j^{++}(\lambda)]^{<\omega}$ and $f \in {}^{[\lambda]^{|a|}}V[G_\kappa * \dot{g}] \cap V[G_\kappa * \dot{g}]$ be such that $\{x \in [\lambda]^{|a|} : f(x) \in \max(x)\} \in E_a$. So there is some name

$\dot{f}$ for f such that $\mathbb{1} \Vdash a \in j^{++}\left(\{x \in [\lambda]^{|a|} : \dot{f}(x) \in \max(x)\}\right)$, so $\mathbb{1} \Vdash j^{++}(\dot{f})(a) \in \max(a) \in \lambda$. Since q is a common extension for q and $\mathbb{1}$, we have that $q \Vdash j^{++}(\dot{f})(a) \in \lambda$.

Now let $b = a \cup \{j^{++}(\dot{f})(a)\}$ and note that $\{x \in [\lambda]^{|b|} : \dot{f} \circ \pi_{ba}(x) \in x\} \in E_b$ if and only if $q \Vdash b \in j^{++}\left(\{x \in [\lambda]^{|b|} : \dot{f} \circ \pi_{ba}(x) \in x\}\right)$, i.e. if $q \Vdash j^{++}\left(\dot{f} \circ \pi_{ba}(b)\right) \in b$, which holds by definition of b.

4. For Well-Foundedness we verify the equivalent condition (as seen in [27]), namely that the direct limit $\widetilde{M_E}$ is well-founded. This holds since, if in $\widetilde{M_E}$ there is an $\in_E$-descending chain $x_n = [a_n, [f_n]]$ with $x_{n+1} \in x_n$ for all $n \in \omega$, then there is, by the equivalence in Lemma 2.1.14, an $\in$-descending chain $j(f_n)(a_n)$ in M, which contradicts that M itself is well-founded.

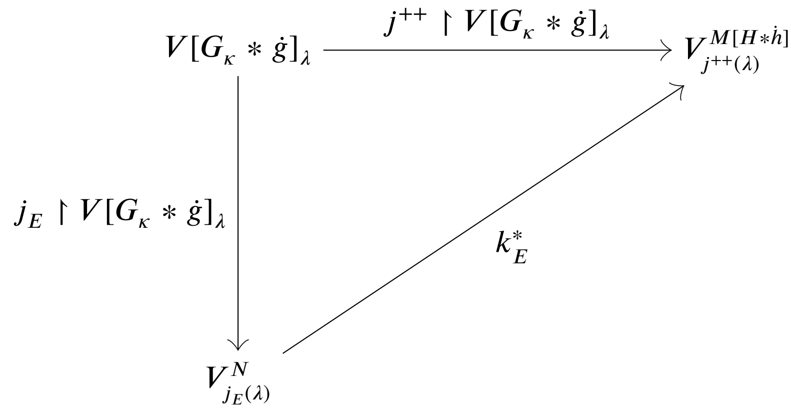
Now, from the extender E we derive the extender embedding $j_E : V[G_\kappa * \dot{g}] \to N$ for $N \cong \text{Ult}\left(V[G_\kappa * \dot{g}], E\right)$, which we show is a θ-$C^{(2)}$-supercompactness embedding with $j_E(\kappa) = j^{++}(\kappa)$.

To see this we follow the same proof structure as that of Theorem 4.1.8, the only difference will be that, instead of defining the ultrafilters E_a by $X \in E_a \iff a \in j^{++}(X)$, we have the requirement that $X \in E_a \iff \dot{q} \Vdash a \in j^{++}(X)$. For clarity we will repeat the full argument here.

We define a restricted factor embedding

$$k_E \restriction V^N_{j_E(\lambda)} : V^N_{j_E(\lambda)} \to V^{M[H*\dot{h}]}_{j^{++}(\lambda)}$$

by $k_E \restriction V^N_{j_E(\lambda)}([a, [f]]) = j^{++}(f)(a)$, where $a \in [j^{++}(\lambda)]^{<\omega}$ and $f : [\lambda]^{|a|} \to V[G_\kappa * \dot{g}]_\lambda$. For brevity, we refer to $k_E \restriction V^N_{j_E(\lambda)}$ as k^*_E. We then obtain the following commutative diagram:



Claim 4.2.2. The restricted embedding k^*_E is the identity function.

Proof. Injectivity follows since, if $j^{++}(f)(a) = j^{++}(g)(b)$, then this is precisely equivalent to $[a, [f]] = [b, [g]]$ as seen in Lemma 2.1.14. Similarly, this equivalence gives that k^*_E is well-defined. For surjectivity, since $\lambda = \beth_\lambda$, we may fix a bijection $g : [\lambda] \to V_\lambda$ in $V[G_\kappa * \dot{g}]$. Now consider $j^{++}(g) : \left[j^{++}(\lambda)\right]^1 \to V^{M[H*\dot{h}]}_{j^{++}(\lambda)}$, which is, by elementarity, also a bijection. So for any $x \in V^{M[H*\dot{h}]}_{j^{++}(\lambda)}$ there is some $\gamma < j^{++}(\lambda)$ such that $x = j^{++}(g)(\{\gamma\})$. But, by the definition of k^*_E, this is the same as $x = k^*_E([\{\gamma\}, [g]])$. Since both the domain and range of k^*_E are transitive sets, it is the identity function. □

So, this means that $V^N_{j_E(\lambda)} = V^{M[H*\dot{h}]}_{j^{++}(\lambda)}$. Note here that, while this implies that $V^{M[H*\dot{h}]}_{j^{++}(\lambda)} \subseteq N$, it does *not* imply that $V^{V[G*\dot{g}]}_{j^{++}(\lambda)} \subseteq N$, since j^{++} is not superstrong. So we have not violated the Superdestructibility Theorem (4.0.1).

That k^*_E is the identity also gives that, for every ordinal $\gamma \leq \lambda$, $j_E(\gamma) = j^{++}(\gamma)$, so, in particular, $j_E(\kappa) = j^{++}(\kappa)$. Thus $j_E(\kappa) \in C^{(2)}$. Now, since $\mathrm{cf}(\lambda) > \kappa$ and $j^{++}(\kappa) > \theta$, we have that $\mathrm{cf}\,(j^{++}(\lambda)) > \theta$ (when computed in $V[G_\kappa * \dot{g}]$), and so $j_E``(\theta) = j^{++}``(\theta) \in V^{M[H*\dot{h}]}_{j^{++}(\lambda)}$. Since $V^N_{j_E(\lambda)} = V^{M[H*\dot{h}]}_{j^{++}(\lambda)}$, it follows that $j_E``(\theta) \in N$.

All that remains to show that j_E is a θ-$C^{(2)}$-supercompactness embedding is that ${}^\theta N \subseteq N$. This argument also follows the structure of Theorem 4.1.8.

We use the fact that, for our extender E derived from j^{++}, with associated embedding j_E, the transitive collapse of the direct limit can be expressed as follows:

$$N = \left\{ j_E(f)(a) : a \in \left[j^{++}(\lambda)\right]^{<\omega}, f : [\lambda]^{|a|} \to V\left[G_\kappa * \dot{g}\right], f \in V\left[G_\kappa * \dot{g}\right] \right\}.$$

So a subset of N of cardinality θ can be expressed as $\{j_E(f_i)(a_i) : i < \theta\}$, where a_i and f_i are as above. Since $\mathrm{cf}(j^{++}(\lambda)) > \theta$ we have that $\langle a_i : i < \theta \rangle \in V[G_\kappa * \dot{g}]_{j^{++}(\lambda)} \subseteq N$. So we will show that the sequence $\langle j_E(f_i) : i < \theta \rangle$ is a member of N, so that N can compute our sequence $\langle j_E(f_i)(a_i) : i < \theta \rangle$ using pointwise evaluation of the a_i under the $j_E(f_i)$.

To see this, note that $j_E``(\theta) \in N$ means that $j_E \restriction \theta \in N$, where here we consider the restriction $j_E \restriction \theta : \theta \to j_E``(\theta)$ as an order-type function in N.

As in Theorem 4.1.8 we can define $G : j_E(\theta) \to N$, where $G = j_E\left(\langle f_i : i < \theta \rangle\right)$. Again, this G is in N. Define in N the function $F = G \circ j_E \restriction \theta : \theta \to N$, which is equivalent to $\langle j_E(f_i) : i < \theta \rangle$ since for every $i < \theta$,

$$F(i) = G\left(j_E(i)\right) = j_E\left(\langle f_i : i < \theta \rangle\right)\left(j_E(i)\right) = j_E\left(\langle f_i : i < \theta \rangle(i)\right) = j_E(f_i).$$

So, as required, we have that $F = \langle j_E(f_i) : i < \theta \rangle$ is a member of N, being a composition of functions in N. So $\langle j_E(f_i) : i < \theta \rangle \in N$, and so N is closed under θ-sequences.

Hence, j_E witnesses that κ is θ-$C^{(2)}$-supercompact in $V[G_\kappa * \dot{g}]$ and so, since this holds for any θ, we have that κ is indestructibly $C^{(2)}$-supercompact after any $< \kappa$-directed closed forcing. □

Note that, an immediate corollary of this result is:

Corollary 4.2.3. Let κ be $C^{(1)}$-extendible (namely, extendible). Then we can make the $C^{(1)}$-supercompactness of κ indestructible by all $< \kappa$-directed closed forcing.

To see this, one argues as above, and simply omits the argument about why the image of the lifted embeddings is in $C^{(2)}$, simply relying on the fact that '$j(\kappa)$ is inaccessible' is preserved.

Of course, this still leaves us with some questions to consider.

Question 4.2.4. Is it possible to make a $C^{(2)}$-supercompact cardinal which is not $C^{(2)}$-extendible indestructible by all $< \kappa$-directed closed forcing?

The extra assumption of $C^{(2)}$-extendibility in our proof is mainly used to affirm the existence of a $C^{(2)}$-extendibility Menas function, and to give that $j(\kappa) \in \text{Reg}$, which helps with some preservation arguments. Due to the equivalence, shown my Tsaprounis in [44], between $C^{(n)}$-extendible cardinals and jointly $C^{(n)}$-supercompact and superstrong cardinals, it is difficult to add much extra strength to a $C^{(n)}$-supercompactness embedding without accidentally creating a $C^{(n)}$-extendible one. One possible avenue is to weaken the assumption that j is a superstrong embedding to merely $j(\kappa)$ being a regular cardinal. This allows for the lifting arguments to work, but we would have to show the existence of a Menas function for $C^{(2)}$-supercompactness, where each witnessing embedding j has that $j(\kappa) \in \text{Reg}$. Much like the existence of a $C^{(n)}$-supercompactness embedding, this is as yet unknown.

Question 4.2.5. What can be said about the indestructibility of a $C^{(n)}$-supercompact when $n \geq 3$?

With this question we have to carefully consider the Superdestructibility Theorem (4.0.1), and whether the indestructibility of a $C^{(3)}$-supercompact would violate this result. Further, the proof we have utilised relies upon reflection properties of Σ_2 formulas, so for $n > 2$ it isn't clear how to use our methods, if they are even applicable.

Chapter 5

Combining Subcompactness with $C^{(n)}$

When considering $C^{(n)}$-large cardinals it is natural to ask whether any given large cardinal may be given a $C^{(n)}$ counterpart. For cardinals κ defined as the critical point of an elementary embedding j this is simple — we just require that the image of the critical point $j(\kappa)$ is in $C^{(n)}$. However, for other large cardinals, such as α-subcompact cardinals, it is not so clear how a $C^{(n)}$ version might be defined. One option is to stick with the paradigm of other $C^{(n)}$-large cardinals, and, for an α-subcompact κ, require that the image of the critical point (namely κ) is in $C^{(n)}$. This would of course give a strengthening of α-subcompactness, but for our purposes doesn't appear to offer much interest.

A potentially more interesting option is to instead require that α is $C^{(n)}$, so that the witnessing embeddings have domain H_α where $\alpha \in C^{(n)}$.

Note here that, if κ is β-subcompact then it is α-subcompact for any $\beta > \alpha > \kappa$ with $\alpha \in C^{(n)}$. Moreover, if κ is supercompact (hence α-subcompact for all α), then for all $n \in \omega$ and for all $\alpha \in C^{(n)}$, κ is α-subcompact.

Given such an α-subcompact cardinal, for $n \geq 1$, then we have the following:

Theorem 5.0.1. Let κ be α-subcompact, where $\alpha \in C^{(n)}$ and $n \geq 1$. Let $\pi_A : (H_{\bar{\alpha}}, \in, \bar{A}) \to (H_\alpha, \in, A)$ be an α-subcompactness elementary embedding for κ with critical point $\bar{\kappa}$, such that $\pi_A(\bar{\kappa}) = \kappa$ (for some $A \subseteq H_\alpha$). Then $\bar{\kappa}$ is $\bar{\alpha}$-extendible, witnessed by π_A.

Proof. First note that, since $\alpha \in C^{(n)}$ and $n \geq 1$, $H_\alpha = V_\alpha$. Note also that $\pi_A(\bar{\kappa}) = \kappa > \bar{\alpha}$. So to show that π_A is indeed an $\bar{\alpha}$-extendibility elementary embedding for $\bar{\kappa}$, it suffices to show that $V_{\bar{\alpha}} = H_{\bar{\alpha}}$.

To verify this condition we characterise $V_\alpha = H_\alpha$ by:

$$H_\alpha \vDash \forall x\, \exists y\, (y = \mathcal{P}(x))\,.$$

Now, by elementarity of π_A, we have that $H_{\bar{\alpha}} \vDash \forall x\, \exists y\, (y = \mathcal{P}(x))$, and so $V_{\bar{\alpha}} = H_{\bar{\alpha}}$ also. □

So we have deduced, from the existence of an α-subcompact cardinal, the existence of an $\bar{\alpha}$-extendible $\bar{\kappa} < \kappa$, where $\bar{\alpha} < \kappa$. But there are many more such critical points below κ, as we see in the following theorem.

Theorem 5.0.2. Let κ be an α-subcompact cardinal for any $\alpha > \kappa$. Then there is a stationary set of $\bar{\kappa}_i$ below κ, where each $\bar{\kappa}_i$ is the critical point of a α-subcompactness embedding for κ.

Proof. Suppose κ is α-subcompact and let C be club in κ. We show that there is a critical point of an α-subcompactness embedding contained in C.

Now, consider an α-subcompactness embedding with predicate C, namely $\pi_C : (H_{\bar{\alpha}}, \in, \bar{C}) \to (H_\alpha, \in, C)$, which has critical point some $\bar{\kappa}$, with $\pi_C(\bar{\kappa}) = \kappa$.

Note that, by elementarity, $\bar{C}$ is a club subset of $\bar{\kappa}$. But, since the critical point of π_C is $\bar{\kappa}$, in fact $\bar{C} = C \cap \bar{\kappa}$. Since C is club, this means that $\bar{\kappa} \in C$, as required. □

Now we combine these two theorems to deduce the existence of many partially extendible cardinals below an α-subcompact.

Theorem 5.0.3. Let κ be α-subcompact, for $\alpha \in C^{(n)}$ with $n \geq 1$. Then there is a stationary set of $\bar{\kappa}_i < \kappa$ where each $\bar{\kappa}_i$ is $\bar{\alpha}_i$-extendible, for $\bar{\alpha}_i < \kappa$ which is given by an α-subcompactness embedding.

With this shown, we can direct our attention to partially supercompact cardinals, recalling the relationship between partial supercompactness and α-subcompactness.

Theorem 5.0.4. Let κ be $2^{<\alpha}$-supercompact for some $\alpha \in C^{(n)}$ with $n \geq 1$. Then there is a stationary set of $\bar{\kappa}_i < \kappa$, where each $\bar{\kappa}_i$ is $\bar{\alpha}_i$-extendible, and $\bar{\alpha}_i < \kappa$ is given by an α-subcompactness elementary embedding for κ.

Proof. The result follows by Theorem 3.1.2, which states that any $2^{<\alpha}$-supercompact is α-subcompact, and then applying Theorem 5.0.3. □

So, taking this idea to its natural conclusion, we consider the case of full supercompactness.

Corollary 5.0.5. Suppose κ is supercompact. Then there are unboundedly many $\alpha \in C^{(n)}$ for each $n \geq 1$, and for each such α there exists a stationary set of $\bar{\alpha}_i$-extendible cardinals $\bar{\kappa}_i < \kappa$, where $\bar{\alpha}_i < \kappa$ is given by an α-subcompactness embedding for κ.

One thing to note here is that, if we were to make a supercompact cardinal κ indestructible by $< \kappa$-directed closed forcing, then, since the existence of many partial extendibles below κ follows by definition, we would make this property indestructible too. However, we would not have that the same $\bar{\kappa}_i$ are $\bar{\alpha}_i$-extendible in the forcing extension, since this would imply that $\bar{\kappa}_i$ is indestructibly $\bar{\alpha}_i$-extendible, which clearly violates the Superdestructibility Theorem 4.0.1.

Our final remark regarding this topic is that, should κ itself be a member of $C^{(n)}$, then in fact all the $\bar{\alpha}_i$-extendible embeddings below κ are actually $\bar{\alpha}_i$-$C^{(n)}$-extendibility embeddings.

Theorem 5.0.6. Let κ be α-subcompact for any α, and suppose $\kappa \in C^{(n)}$ for $n \geq 0$. Then there is a stationary set of $\bar{\kappa}_i < \kappa$ where each $\bar{\kappa}_i$ is an $\bar{\alpha}_i$-$C^{(n)}$-extendible cardinal, where $\bar{\alpha}_i < \kappa$ is given by an α-subcompactness embedding for κ.

Proof. This follows by Theorem 5.0.3, and by noting that, for a witnessing $\bar{\alpha}_i$-extendibility elementary embedding π_i with critical point $\bar{\kappa}_i < \kappa$, we have that $\pi(\bar{\kappa}) = \kappa \in C^{(n)}$, thus fulfilling the extra requirement for an $\bar{\alpha}_i$-extendibility embedding to be an $\bar{\alpha}_i$-$C^{(n)}$-extendibility embedding. □

The analogue for fully supercompact κ follows as before:

Corollary 5.0.7. Suppose κ is supercompact and $\kappa \in C^{(n)}$. Then there are unboundedly many $\alpha \in C^{(m)}$ with $m \geq 1$, and for each such α there exists a stationary set of $\bar{\alpha}_i$-$C^{(n)}$-extendible cardinals $\bar{\kappa}_i < \kappa$, where $\bar{\alpha}_i < \kappa$ is given by an α-subcompactness embedding for κ.

In particular, since a supercompact cardinal is $C^{(2)}$, there are, below any supercompact κ, unboundedly many $\alpha \in C^{(2)}$, and for each such α a stationary set of partial $C^{(2)}$-extendibles (and the same holds for $n = 1$).

Further, if we have a $C^{(n)}$-extendible cardinal κ, then it is automatically Σ_{n+2}-correct, and so the result gives us a stationary set of $\bar{\alpha}_i$-$C^{(n+2)}$-extendibles below κ, for unboundedly many α, as above.

Chapter 6

Anti Foundation

This Chapter is comprised of joint work with John Howe and Rosario Mennuni, regarding reducts of countable models of Anti-Foundational set theory. The work came about after I gave a seminar about my Master's thesis, which culminated in a paper 'Undirecting membership in models of Anti-Foundation' ([1]), co-authored with my supervisor Peter Cameron. The joint paper with Howe and Mennuni has now been published in the Bulletin of Symbolic Logic ('On double-membership graphs of models of Anti-Foundation', [2]) and extends the results of the original paper in new and perhaps unexpected ways.

In it we answer some questions about graphs that are reducts of countable models of Anti-Foundation, obtained by considering the binary relation of double-membership $x \in y \in x$. Since its content is quite different from the rest of the thesis, some additional preliminaries are required.

6.1 Background

We will give a brief survey of relevant definitions and results in the field of model theory, more substantial detail can be found in e.g. [24], [33].

Let $\mathcal{L}$ be a first-order language, and T a theory in $\mathcal{L}$. Then T is *complete* if it has models, and any two of its models are elementarily equivalent.

We say that a structure X is *ultrahomogeneous* if every isomorphism between finitely generated substructures of X extends to an automorphism of X.

Let λ be a cardinal. A complete theory with exactly one model of cardinality λ up to isomorphism is called *λ-categorical*. A structure X is λ-categorical if the theory of X is λ-categorical.

For a theory T, an *n-type* is a set $\Phi(\bar{x})$, where $\bar{x} = (x_0, \dots, x_{n-1})$ such that, for some model M of T, and some n-tuple $\bar{a} = (a_0, \dots, a_{n-1})$ with $a_i \in M$, $M \vDash \psi(\bar{a})$ for all $\psi \in \Phi$. In this case we say that the model M *realises* the n-type Φ, and that $\bar{a}$ realises Φ in M.

We say that M *omits* Φ if no tuple in M realises Φ.

Definition 6.1.1. The *Random Graph R* is the Fraïssé limit of the class of finite graphs. If a graph G with countable vertices has edges placed between vertices x and y with some probability $p \in (0, 1)$, then, with probability 1, G is isomorphic to R.

6.2 Introduction

Ultimately, models of a set theory are digraphs, where a directed edge between two points denotes membership. To such a model, one can associate various graphs, such as the *membership graph*, obtained by symmetrising the binary relation $\in$, or the *double-membership graph*, which has an edge between x and y when $x \in y$ and $y \in x$ hold simultaneously. We also consider the structure equipped with the two previous graph relations, which we call the *single-double-membership graph*. In [1] this kind of object is investigated in the non-well-founded case. We continue this line of study, and answer some questions regarding such graphs that were left open in the aforementioned work.

It is well-known that every membership graph of a countable model of ZFC is isomorphic to the Random Graph (see e.g. [12]). The usual proof of this fact goes through for set theories much weaker than ZFC, but uses the Axiom of Foundation in a crucial way, hence the interest in (double-)membership graphs of non-well-founded set theories.

In 1917 Mirimanoff [35, 36] discussed the distinction between non-well-founded sets and their well-founded counterparts, and even presented a notion of isomorphism between possibly non-well-founded sets. Throughout the years they have appeared — implicitly and explicitly — in myriad places, and various formulations of axioms allowing such sets to exist have been developed and utilised. A uniform treatment of many of these axioms can be found in [38], along with historical notes.

Perhaps the most famous non-well-founded set theory is obtained from ZFC by replacing the Axiom of Foundation with the *Anti-Foundation Axiom* AFA, and is called ZFA (not to be confused with another ZFA, a set theory with *Atoms*). This axiom provides the universe with a rich class of non-well-founded sets, the structure of which reflects that of the well-founded sets: in models of ZFA there are, for example, unique a and b such that $a = \{b, \emptyset\}$ and $b = \{a, \{\emptyset\}\}$, and a unique $c = \{c, \emptyset, \{\emptyset\}\}$, pictured in Figure 6.1.

By facilitating the modelling of circular behaviours, ZFA has found applications in computer science and category theory for the study of streams, communicating systems and final coalgebras, and in philosophy, for the study of paradoxes involving circularity and natural language semantics. We refer the interested reader to [38, 9, 8].

On many accounts, models of ZFC and of ZFA are closely related, and the two set theories behave very similarly, even under forcing extensions: see for instance [46, 17]. Now, when we symmetrise the membership relation, we have two choices: we can either forget which edges were symmetric in the first place — that is, consider the membership graph — or remember this information — that is, consider the single-double-membership graph. In the first case, we find ourselves in yet another situation where the behaviour of ZFA parallels closely that of ZFC.

Namely, in [1] it was proven that all membership graphs of countable models of ZFA are isomorphic to the 'Random Loopy Graph': the Fraïssé limit of finite graphs with self-edges. Much like the Random Graph, it can be obtained by adding edges between (not necessarily distinct) vertices of a countable graph with some probability $p \in (0, 1)$. This structure is easily seen to be $\aleph_0$-categorical, ultrahomogeneous, and supersimple of SU-rank 1. If instead we take the second option, the situation changes drastically, and already double-membership graphs of models of ZFA are, in a number of senses, much more complicated. For instance, [1, Theorem 3] shows that they are not $\aleph_0$-categorical, and here we show further results in this direction.

In this Chapter we aim to answer some of the open questions in [1], namely

1. Is it true that the first-order theory of the membership graph of a countable model of ZFA has infinitely many countable models?
2. Is it true that there are infinitely many non-isomorphic graphs which are membership graphs of countable models of ZFA?
3. Can more be said about infinite connected components of the double-edge graph?
4. What about models of ZFA where the Axiom of Infinity is replaced with its negation?

5. Is it true that, if two countable multigraphs are elementarily equivalent, and one is the membership graph of a model of ZFA, then so is the other?

The structure of the Chapter is as follows. After a brief introduction to Anti-Foundation in Section 6.3, and after setting up the context in Section 6.4, we answer [1, Question 3] in Section 6.5 by characterising the connected components of double-membership graphs of models of ZFA. In the same section, we show that if we do not assume Anti-Foundation, but merely drop Foundation, then double-membership graphs can be almost arbitrary. Section 6.6 answers [1, Questions 1 and 2] by proving the following theorem.

Theorem (Corollary 6.6.5)**.** There are, up to isomorphism, continuum-many countable (single-)double-membership graphs of models of ZFA, and continuum-many countable models of each of their theories.

In Section 6.7 we study the common theory of double-membership graphs, which we show to be incomplete. Then, by using methods more commonly encountered in finite model theory, we characterise the completions of said theory in terms of consistent collections of consistency statements.

Theorem (Theorem 6.7.14)**.** The double-membership graphs of two models M and N of ZFA are elementarily equivalent precisely when M and N satisfy the same consistency statements.

We also show that all of these completions are wild in the sense of neostability theory, since each of their models interprets (with parameters) arbitrarily large finite fragments of ZFC. Our final result, below — obtained with similar techniques — answers [1, Question 5] negatively. The analogous statement for double-membership graphs holds as well.

Theorem (Corollary 6.7.18)**.** For every single-double-membership graph of a model of ZFA, there is a countable elementarily equivalent structure that is not the single-double-membership graph of any model of ZFA.

6.3 The Anti-Foundation Axiom

There are a number of equivalent formulations of AFA. Expressed in terms of *f-inductive functions*, or of *homomorphism onto transitive structures*, it first appeared in [19], under the name of axiom X_1. It gained its current name in [38], where it was defined via *decorations*. The form that we shall be using is known in the literature (e.g. [9, p. 71]) as the *Solution Lemma*. For the equivalence with other formulations, see e.g. [38, p. 16].

Definition 6.3.1. Let X be a set of 'indeterminates', and A a set of sets. A *flat system of equations* is a set of equations of the form $x = S_x$, where S_x is a subset of $X \cup A$ for each $x \in X$. A *solution* f to the flat system is a function taking elements of X to sets, such that after replacing each $x \in X$ with $f(x)$ inside the system, all of its equations become true.

The *Anti-Foundation Axiom* (AFA) is the statement that every flat system of equations has a unique solution.

Example 6.3.2. Consider the flat system with $X = \{x, y\}$, $A = \{\emptyset, \{\emptyset\}\}$ and the following equations.

$$x = \{y, \emptyset\}$$
$$y = \{x, \{\emptyset\}\}$$

The image of its unique solution $x \mapsto a, y \mapsto b$ is pictured in Figure 6.1.

Note that solutions of systems need not be injective, and in fact uniqueness sometimes prevents injectivity. For instance, if $x \mapsto a$ is the solution of the flat system consisting of the single equation $x = \{x\}$, then $x \mapsto a, y \mapsto a$ solves the system with equations $x = \{y\}$ and $y = \{x\}$, whose unique solution is therefore not injective.

Figure 6.1: On the left, a picture of the unique sets a and b such that $a = \{b, \emptyset\}$ and $b = \{a, \{\emptyset\}\}$. On the right, a picture of the unique set c such that $c = \{c, \emptyset, \{\emptyset\}\}$. The arrows denote membership.

Fact 6.3.3. ZFC without the Axiom of Foundation proves the equiconsistency of ZFC and ZFA.

Proof. In one direction, from a model of ZFA one obtains one of ZFC by restricting to the well-founded sets. In the other direction, see [19, Theorem 4.2] for a class theory version, or [38, Chapter 3] for the ZFC statement. □

Remark 6.3.4. There exists a weak form of AFA that only postulates the existence of solutions to flat systems, but not necessarily their uniqueness, known as axiom X in [19] or AFA_1 in [38]. Below, and in [1], uniqueness is never used, hence all the results go through for models of ZFC with Foundation replaced by AFA_1. For brevity, we still state everything for ZFA.

6.4 Set-Up

Since Anti-Foundation allows for sets that are members of themselves, in what follows we will need to deal with graphs where there might be an edge between a point and itself. These are called *loopy graphs* in [1] but, for the sake of concision, we depart from common usage by adopting the following convention.

Notation. By *graph* we mean a first-order structure with a single relation that is binary and symmetric (it is not required to be irreflexive).

Since we are interested in studying (reducts of) models of ZFA, we need to assume they exist in the first place, since otherwise the answers to the questions we are studying are trivial. Therefore, in this chapter we work in a set theory that is slightly stronger than usual.

Assumption 6.4.1. The ambient metatheory is ZFC + Con(ZFC).

Definition 6.4.2. Let $L = \{\in\}$, where $\in$ is a binary relation symbol, and M an L-structure. Let S and D be the definable relations

$$S(x, y) := x \in y \vee y \in x$$

$$D(x, y) := x \in y \wedge y \in x$$

The *single-double-membership graph*, or *SD-graph*, M_0 of M is the reduct of M to $L_0 := \{S, D\}$. The *double-membership graph*, or *D-graph*, M_1 of M is the reduct of M to $L_1 := \{D\}$.

So, given an L-structure M, i.e. a digraph (possibly with loops) where the edge relation is $\in$, we have that $M_0 \models S(x, y)$ if and only if in M there is at least one $\in$-edge between x and y. Similarly $M_0 \models D(x, y)$ means that in M we have both $\in$-edges between x and y. The idea is that, if M is a model of some set theory, then M_0 is a symmetrisation of M that keeps track of double-membership as well as single-membership, and M_1 only keeps track of double-membership.

In [1], M_0 is called the *membership graph (keeping double-edges)* of M and M_1 is called the *double-edge graph* of M. Note that, strictly speaking, SD-graphs are not graphs, according to our terminology.

For the majority of the chapter we are concerned with D-graphs, since most of the results we obtain for them imply the analogous versions for SD-graphs. This situation will reverse in Theorem 6.7.17.

Definition 6.4.3. Let $M \models$ ZFA. We say that $A \subseteq M$ is an *M-set* iff there is $a \in M$ such that $A = \{b \in M : M \models b \in a\}$.

So an M-set A is a definable subset of M that is the extension of a set in the sense of M, namely the $a \in M$ in the definition. We will occasionally abuse notation and refer to an M-set A when we actually mean the corresponding $a \in M$.

6.5 Connected Components

Let $M \models \mathsf{ZFA}$. It was proven in [1, Theorem 4] that, for every finite connected graph G, the D-graph M_1 has infinitely many connected components isomorphic to G. It was asked in [1, Question 3] if more can be said about the infinite connected components of M_1. In this section we characterise them in terms of the graphs inside M.

Let G be a graph in the sense of $M \models \mathsf{ZFA}$, i.e. a graph whose domain and edge relation are M-sets, the latter as, say, a set of Kuratowski pairs. If G is such a graph and $M \models$ 'G is connected', then G need not necessarily be connected. This is due to the fact that M may have non-standard natural numbers, hence relations may have non-standard transitive closures. We therefore introduce the following notion.

Definition 6.5.1. Let $a \in M \models \mathsf{ZFA}$. Let $b \in M$ be such that

$$M \models \text{`}b \text{ is the transitive closure of } \{a\} \text{ under } D\text{'}$$

The *region of a in M* is $\{c \in M : M \models c \in b\}$. If $A \subseteq M$, we say that A is a *region of M* iff it is the region of some $a \in M$.

Remark 6.5.2. For each $a \in M$, the region of a in M is an M-set.

For $a \in M$, if A is the region of a and B is the transitive closure of $\{a\}$ under D computed in the metatheory, i.e. the connected component of a in M_1, then $B \subseteq A$. In particular, regions of M are unions of connected components of M_1. If M contains non-standard natural numbers and the diameter of B is infinite then the inclusion $B \subseteq A$ may be strict, and B may not even be an M-set. From now on, the words 'connected component' will only be used in the sense of the metatheory.

Most of the appeals to AFA in the rest of the chapter will be applications of the following proposition. In fact, after proving it, we will only deal directly with flat systems twice more.

Proposition 6.5.3. Let M_1 be the D-graph of $M \models \mathsf{ZFA}$, and let G be a graph in M. Then there is $H \subseteq M_1$ such that

1. $(H, D^{M_1} \upharpoonright H)$ is isomorphic to G,
2. H is a union of regions of M, and
3. H is an M-set.

Proof. Work in M until further notice. Let G be a graph in M, say in the language $\{R\}$. Let κ be its cardinality, and assume up to a suitable isomorphism that $\operatorname{dom} G = \kappa$. In particular, note that every element of $\operatorname{dom} G$ is a well-founded set. Consider the flat system

$$\left\{x_i = \{i, x_j \,:\, j \in \kappa, G \models R(i,j)\} \,:\, i \in \kappa\right\}$$

Let $s :\ x_i \mapsto a_i$ be a solution to the system. If $i \neq j$, then $i \in a_i \setminus a_j$, and therefore s is injective. Observe that

(i) since R is symmetric, we have $a_i \in a_j \in a_i \iff G \models R(i,j)$, and

(ii) for all $b \in M$ and all $i \in \kappa$, we have $b \in a_i \in b$ if and only if there is $j < \kappa$ such that $b = a_j$ and $G \models R(i,j)$.

Now work in the ambient metatheory. Consider the M-set

$$H := \{a_i \,:\, M \models i \in \kappa\} = \{b \in M \,:\, M \models b \in \operatorname{Im}(s)\} \subseteq M_1$$

By (i) above, $(H, D^{M_1} \upharpoonright H)$ is isomorphic to G and, by (ii) above, H is a union of regions of M. □

We can now generalise [1, Theorem 4], answering [1, Question 3]. The words 'up to isomorphism' are to be interpreted in the sense of the metatheory, i.e. the isomorphism need not be in M.

Theorem 6.5.4. Let $M \models \mathsf{ZFA}$. Up to isomorphism, the connected components of M_1 are exactly the connected components (in the sense of the metatheory) of graphs in the sense of M. In particular, there are infinitely many copies of each of them.

Proof. Let C be a connected component of a graph G in M. By Proposition 6.5.3 there is an isomorphic copy H of G that is a union of regions of M, hence, in particular, of connected components of M_1. Clearly, one of the connected components of H is isomorphic to C.

In the other direction, let $a \in M_1$ and consider its connected component. Inside M, let G be the region of a. Using Remark 6.5.2 it is easy to see that $(G, D \upharpoonright G)$ is a graph in M, and one of its connected components is isomorphic to the connected component of a in M_1.

For the last part of the conclusion take, inside M, disjoint unions of copies of a given graph. □

If one does not assume some form of AFA and, for instance, merely drops Foundation, then double-membership graphs can be essentially arbitrary, as the following proposition shows.

Proposition 6.5.5. Let $M \models \mathsf{ZFC}$ and let G be a graph in M. There is a model N of ZFC without Foundation such that N_1 is isomorphic to the union of G with infinitely many isolated vertices, i.e. points without any edges or self-loops.

Note that the isolated vertices are necessary, as N will always contain well-founded sets.

Proof. Let G be a graph in M, say in the language $\{R\}$. Assume without loss of generality that G has no isolated vertices, and that $\operatorname{dom} G$ equals its cardinality κ. For each $i \in \kappa$ choose $a_i \subseteq \kappa$ that has foundational rank κ in M, e.g. let $a_i := \kappa \setminus \{i\}$. Let $b_j := \{a_i : G \models R(i, j)\}$ and note that, since no vertex of G is isolated, b_j is non-empty, thus has rank $\kappa+1$. Define $\pi : M \to M$ to be the permutation swapping each a_i with the corresponding b_i and fixing the rest of M. Let N be the structure with the same domain as M, but with membership relation defined as

$$N \models x \in y \iff M \models x \in \pi(y)$$

By [39, Section 3][1], N is a model of ZFC without Foundation. To check that N_1 is as required, first observe that

$$N \models a_i \in a_j \iff M \models a_i \in \pi(a_j) = b_j \iff G \models R(i, j)$$

so $\{a_i : M \models i \in \kappa\}$, equipped with the restriction of D^{N_1}, is isomorphic to G. To show that there are no other D-edges in N_1, assume that $N_1 \models D(x, y)$, and consider the following three cases (which are exhaustive since D is symmetric).

(i) x and y are both fixed points of π. This contradicts Foundation in M.

(ii) $y = a_i$ for some i, so $N \models x \in a_i$, hence $M \models x \in \pi(a_i) = b_i$. Then $x = a_j$ for some j by construction.

(iii) $y = b_i$ for some i. From $N \models x \in b_i$ we get $M \models x \in a_i \subseteq \kappa$, thus x has rank strictly less than κ. Therefore, x is not equal to any a_j or b_j, hence

[1] Strictly speaking, [39] works in class theory. The exact statement we use is that of [28, Chapter IV, Exercise 18].

$\pi(x) = x$. Again by rank considerations, it follows that $M \vDash b_i \notin x = \pi(x)$, so $N \vDash b_i \notin x$, a contradiction. □

6.6 Continuum-Many Countable Models

We now turn our attention to answering [1, Questions 1 and 2]. Namely, we compute, via a type-counting argument, the number of non-isomorphic D-graphs of countable models of ZFA, and the number of countable models of their complete theories. The analogous results for SD-graphs also hold.

Definition 6.6.1. Let $n \in \omega \setminus \{0\}$. Define the L_1-formula

$$\varphi_n(x) := \neg D(x,x) \wedge \exists z_0, \ldots, z_{n-1}\Big(\big(\bigwedge_{0 \le i < j < n} z_i \neq z_j \big) \wedge \big(\bigwedge_{0 \le i < n} D(z_i, x) \big) \wedge \big(\forall z\, D(z,x) \to \bigvee_{0 \le i < n} z = z_i \big) \Big)$$

For A a subset of $\omega \setminus \{0\}$, define the set of L_1-formulas

$$\beta_A(y) := \{\neg D(y,y)\} \cup \{\exists x_n\ \varphi_n(x_n) \wedge D(y, x_n) : n \in A\} \cup \{\neg(\exists x_n\ \varphi_n(x_n) \wedge D(y, x_n)) : n \in \omega \setminus (\{0\} \cup A)\}$$

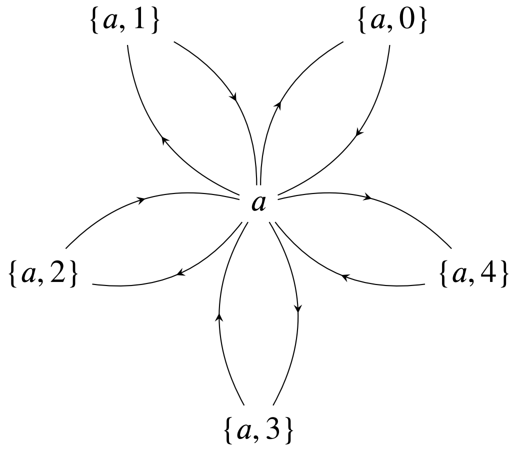
We say that $a \in M_1$ is an *n-flower* iff $M_1 \vDash \varphi_n(a)$. We say that $b \in M_1$ is an *A-bouquet* iff for all $\psi(y) \in \beta_A(y)$ we have $M_1 \vDash \psi(b)$. Figure 6.2 gives a visualisation of a 5-flower, which explains the reason for its name.

So a is an n-flower if and only if, in the D-graph, it is a point of degree n without a self-loop, while b is an A-bouquet iff it has no self-loop, it has D-edges to at least one n-flower for every $n \in A$, and it has no D-edges to any n-flower if $n \notin A$.

Lemma 6.6.2. Let A_0 be a finite subset of $\omega \setminus \{0\}$ and let $M \vDash$ ZFA. Then M_1 contains an A_0-bouquet.

Proof. It suffices to find a certain finite graph as a connected component of M_1, so this follows from Proposition 6.5.3 (or directly from [1, Theorem 4]). □

Figure 6.2: The set $a = \{\{a, i\} : i < 5\}$ is a 5-flower. The reason for the name 'n-flower' can be seen in this figure.



If M is a structure, denote by $\mathrm{Th}(M)$ its theory.

Proposition 6.6.3. Let $M \vDash \mathsf{ZFA}$. Then in $\mathrm{Th}(M_1)$ the $2^{\aleph_0}$ sets of formulas β_A, for $A \subseteq \omega \setminus \{0\}$, are each consistent, and pairwise contradictory. In particular, the same is true in $\mathrm{Th}(M)$.

Proof. If A, B are distinct subsets of $\omega \setminus \{0\}$ and, without loss of generality, there is an $n \in A \setminus B$, then β_A contradicts β_B because $\beta_A(y) \vdash \exists x_n\ (\varphi_n(x_n) \wedge D(y, x_n))$ and $\beta_B(y) \vdash \neg\exists x_n\ (\varphi_n(x_n) \wedge D(y, x_n))$.

To show that each β_A is consistent it is enough, by compactness, to show that if A_0 is a finite subset of A and A_1 is a finite subset of $\omega \setminus (\{0\} \cup A)$ then there is some $b \in M$ with a D-edge to an n-flower for every $n \in A_0$ and no D-edges to n-flowers whenever $n \in A_1$. Any A_0-bouquet will satisfy these requirements and, by Lemma 6.6.2, an A_0-bouquet exists inside M_1.

For the last part, note that all the theories at hand are complete (in different languages), and whether or not an intersection of definable sets is empty does not change after adding more definable sets. □

To conclude, we need the following standard fact from model theory.

Fact 6.6.4. Every partial type over $\emptyset$ of a countable theory can be realised in a countable model.

Corollary 6.6.5. Let M be a model of ZFA. There are $2^{\aleph_0}$ countable models of ZFA such that their D-graphs (resp. SD-graphs) are elementarily equivalent to M_1 (resp. M_0) and are pairwise non-isomorphic.

Proof. Consider the pairwise contradictory partial types β_A. By Fact 6.6.4, Th(M) has $2^{\aleph_0}$ distinct countable models, as each of them can only realise countably many of the β_A. The reducts to L_1 (resp. L_0) of models realising different subsets of $\{\beta_A : A \subseteq \omega \setminus \{0\}\}$ are still non-isomorphic, since the β_A are partial types in the language L_1. □

The previous Corollary answers affirmatively [1, Questions 1 and 2].

Remark 6.6.6. For the results in this section to hold, it is not necessary that M satisfies the whole of ZFA. It is enough to be able to prove Lemma 6.6.2 for M, and it is easy to see than one can provide a direct proof whenever in M it is possible to define infinitely many different well-founded sets, e.g. von Neumann natural numbers, and to ensure existence of solutions to flat systems of equations. This can be done as long as M satisfies Extensionality, Empty Set, Pairing, and AFA_1[2]. If we replace, in Definition 6.3.1, '$x = S_x$' with 'x and S_x have the same elements', then we can even drop Extensionality.

6.7 Common Theory

The main aim of this section is to study the common theory of the class of D-graphs of ZFA. We show in Corollary 6.7.11 that it is incomplete, and in Corollary 6.7.15

[2]Stated using a sensible coding of flat systems, which can be carried out using Pairing.

characterise its completions in terms of collections of consistency statements. Furthermore, we show that each of these completions is untame in the sense of neostability theory (Corollary 6.7.8) and has a countable model that is not a D-graph, and that the same holds for SD-graphs (Corollary 6.7.18), therefore solving negatively [1, Question 5].

Definition 6.7.1. Let K_1 be the class of D-graphs of models of ZFA. Let $\mathrm{Th}(K_1)$ be its common L_1-theory.

Definition 6.7.2. Let φ be an L_1-sentence. We define an L_1-sentence $\mu(\varphi)$ as follows. Let x be a variable not appearing in φ. Let $\chi(x)$ be obtained from φ by relativising $\exists y$ and $\forall y$ to $D(x, y)$. Let $\mu(\varphi)$ be the formula $\exists x\ (\neg D(x, x) \wedge \chi(x))$.

In other words, $\mu(\varphi)$ can be thought of as saying that there is a point whose set of neighbours is a model of φ.

Remark 6.7.3. Suppose φ is a 'standard' sentence, i.e. one that is a formula in the sense of the metatheory, say in the finite language L'. Let $M \vDash$ ZFA, and let N be an L'-structure in M. Then, whether $N \vDash \varphi$ or not is absolute between M and the metatheory. Every formula we mention is of this kind, and this fact will be used tacitly from now on.

Definition 6.7.4. Let Φ be the set of L_1-sentences that imply $\forall x, y\ (D(x, y) \rightarrow D(y, x))$.

Lemma 6.7.5. For every L_1-sentence $\varphi \in \Phi$ and every $M \vDash$ ZFA we have

$$M \vDash \mathrm{Con}(\varphi) \iff M_1 \vDash \mu(\varphi)$$

Moreover, if this is the case, then there is $H \subseteq M_1$ such that

1. $(H, D^{M_1} \restriction H)$ satisfies φ,

2. H is a union of regions of M, and

3. H is an M-set.

Proof. Note that the class of graphs in M is closed under the operations of removing a point or adding one and connecting it to everything. Now apply Proposition 6.5.3. □

define $L_{\mathsf{NBG}} := \{E\}$, where E is a binary relational symbol. We think of L_1 as 'the language of graphs' and of L_{NBG} as 'the language of digraphs', specifically, digraphs that are models of a certain class theory (see below), hence the notation. It is well-known that every digraph is interpretable in a graph, and that such an interpretation may be chosen to be uniform, in the sense below. See e.g. [24, Theorem 5.5.1].

Fact 6.7.6. Every L_{NBG}-structure N is interpretable in a graph N'. Moreover, for every L_{NBG}-sentence θ there is an L_1-sentence θ' such that

1. θ is consistent if and only if θ' is, and
2. for every L_{NBG}-structure N we have $N \models \theta \iff N' \models \theta'$.

Corollary 6.7.7. For every L_{NBG}-sentence θ, let θ' be as in Fact 6.7.6. For all $M \models$ ZFA

$$M \models \mathrm{Con}(\theta) \iff M_1 \models \mu(\theta')$$

Proof. Apply Lemma 6.7.5 to $\varphi := \theta'$. □

Corollary 6.7.8. Let $M \models \mathsf{ZFA}$. Then every model of $\mathrm{Th}(M_1)$ interprets with parameters arbitrarily large finite fragments of ZFC. In particular $\mathrm{Th}(M_1)$ has SOP, TP_2, and IP_k for all k.

Proof. If θ is the conjunction of a finite fragment of ZFC, it is well-known that $\mathsf{ZFA} \vdash \mathrm{Con}(\theta)$. Since a model of θ is a digraph, we can apply Corollary 6.7.7. If a witnesses the outermost existential quantifier in $\mu(\theta')$, then θ is interpretable with parameter a. □

We now want to use Corollary 6.7.7 to show that the common theory $\mathrm{Th}(K_1)$ of the class of D-graphs of models of ZFA is incomplete. Naively, this could be done by choosing θ to be a finite axiomatisation of some theory equiconsistent with ZFA, and then invoking the Second Incompleteness Theorem. For instance, one could choose von Neumann-Bernays-Gödel class theory NBG, axiomatised in the language L_{NBG}[3], as this is known to be equiconsistent with ZFC (see [18]), hence with ZFA. The problem with this argument is that, in order for it to work, we need a further set-theoretical assumption in our metatheory, namely Con(ZFC + Con(ZFC)). This can be avoided by using another sentence whose consistency is independent of ZFA, provably in ZFC + Con(ZFC) alone. We would like to thank Michael Rathjen for pointing out to us the existence of such a sentence.

Let NBG^- denote NBG without the axiom of Infinity. We will use special cases of a classical theorem of Rosser and of a related result. For proofs of these, together with their more general statements, we refer the reader to [42, Chapter 7, Application 2.1 and Corollary 2.6].

Fact 6.7.9 (Rosser's Theorem)**.** There is a Π^0_1 arithmetical statement ψ that is independent of ZFA.

Fact 6.7.10. Let ψ be a Π^0_1 arithmetical statement. There is another arithmetical statement $\widetilde{\psi}$ such that $\mathsf{ZFA} \vdash \psi \leftrightarrow \mathrm{Con}(\mathsf{NBG}^- + \widetilde{\psi})$.

Corollary 6.7.11. $\mathrm{Th}(K_1)$ is not complete.

Proof. Let ψ be given by Rosser's Theorem, and let $\widetilde{\psi}$ be given by Fact 6.7.10 applied to ψ. Apply Corollary 6.7.7 to $\theta := \mathsf{NBG}^- + \widetilde{\psi}$. □

It is therefore natural to study the completions of $\mathrm{Th}(K_1)$, and it follows easily from K_1 being pseudoelementary that all of these are the theory of some actual D-graph M_1. We provide a proof for completeness.

[3]The reader may have encountered an axiomatisation using two sorts; this can be avoided by declaring sets to be those classes that are elements of some other class.

Proposition 6.7.12. Let T be an L-theory, and let K be the class of its models. Let $L_1 \subseteq L$, and for $M \in K$ denote $M_1 := M \restriction L_1$. Let $K_1 := \{M_1 : M \in K\}$ and $N \models \mathrm{Th}(K_1)$. Then there is $M \in K$ such that $M_1 \equiv N$.

Proof. We are asking whether there is any $M \models T \cup \mathrm{Th}(N)$, so it is enough to show that the latter theory is consistent. If not, there is an L_1-formula $\varphi \in \mathrm{Th}(N)$ such that $T \vdash \neg\varphi$. In particular, since $\neg\varphi \in L_1$, we have that $\mathrm{Th}(K_1) \vdash \neg\varphi$, and this contradicts that $N \models \mathrm{Th}(K_1)$. □

In order to characterise the completions of $\mathrm{Th}(K_1)$, we will use techniques from finite model theory, namely Ehrenfeucht-Fraïssé games and k-equivalence. For background on these concepts, see [16].

Lemma 6.7.13. Let $G = G_0 \sqcup G_1$ be a graph with no edges between G_0 and G_1, and let $H = H_0 \sqcup H_1$ be a graph with no edges between H_0 and H_1. If $(G_0, a_1, \dots, a_{m-1}) \equiv_k (H_0, b_1, \dots, b_{m-1})$ and $(G_1, a_m) \equiv_k (H_1, b_m)$, then $(G, a_1, \dots, a_m) \equiv_k (H, b_1, \dots, b_m)$.

Proof. This is standard, see e.g. [16, Proposition 2.3.10]. □

Theorem 6.7.14. Let M and N be models of ZFA. The following are equivalent.

1. $M_1 \equiv N_1$.
2. M_1 and N_1 satisfy the same sentences of the form $\mu(\varphi)$, as φ ranges in Φ.
3. M and N satisfy the same consistency statements.

Proof. For statements about graphs, the equivalence of 2 and 3 follows from Lemma 6.7.5. For statements in other languages, it is enough to interpret them in graphs using [24, Theorem 5.5.1].

For the equivalence of 1 and 2, we show that for every $n \in \omega$ the Ehrenfeucht-Fraïssé game between M_1 and N_1 of length n is won by the Duplicator, by describing a winning strategy. The idea behind the strategy is the following. Recall that, for every finite relational language and every k, there is only a finite number of $\equiv_k$-classes, each characterised by a single sentence (see e.g. [16, Corollary 2.2.9]). After the Spoiler plays a point a, the Duplicator replicates the $\equiv_k$-class of the region of a using Lemma 6.7.5.

Fix the length n of the game and denote by $a_1, \dots, a_m \in M_1$ and $b_1, \dots, b_m \in N_1$ the points chosen at the end of turn m. The Duplicator defines, by simultaneous induction on m, sets $G_0^m \subseteq M_1$ and $H_0^m \subseteq N_1$, and makes sure that they satisfy the following conditions.

(C1) $a_1, \dots, a_m \in G_0^m$ and $b_1, \dots, b_m \in H_0^m$.

(C2) G_0^m and H_0^m are unions of regions of M and N respectively.

(C3) G_0^m and H_0^m are respectively an M-set and an N-set.

(C4) When G_0^m and H_0^m are equipped with the L_1-structures induced by M and N respectively, we have $(G_0^m, a_1, \dots, a_m) \equiv_{n-m} (H_0^m, b_1, \dots, b_m)$.

Before the game starts ('after turn 0') we set $G_0^0 = H_0^0 = \emptyset$ and all conditions trivially hold. Assume inductively that they hold after turn $m-1$. We deal with the case where the Spoiler plays $a_m \in M_1$; the case where the Spoiler plays $b_m \in N_1$ is symmetrical.

Let G_1^m be the region of a_m in M. If $G_1^m \subseteq G_0^{m-1}$ then, since by inductive hypothesis condition (C4) held after turn $m-1$, the Duplicator can find $b_m \in H_0^{m-1}$ such that $(G_0^{m-1}, a_0, \dots, a_m) \equiv_{n-m} (H_0^{m-1}, b_0, \dots, b_m)$. It is then clear that all conditions hold after setting $G_0^m = G_0^{m-1}$ and $H_0^m = H_0^{m-1}$.

Otherwise, by (C2), we have $G_1^m \cap G_0^{m-1} = \emptyset$. Let φ characterise the $\equiv_{n-m+1}$-class of G_1^m. Note that, if $n - m + 1 \geq 2$, then $\varphi \in \Phi$ automatically. Otherwise, replace φ with $\varphi \wedge \forall x \forall y\ (D(x, y) \to D(y, x))$. By Remark 6.5.2, G_1^m is an M-set, hence

$M \vDash \mathrm{Con}(\varphi)$. By Lemma 6.7.5 and assumption, there is a union H_1^m of regions of N which is an N-set and such that $G_1^m \equiv_{n-m+1} H_1^m$. By inductive hypothesis, H_0^{m-1} is also an N-set by (C3). Therefore, up to writing a suitable flat system in N, we may replace H_1^m with an isomorphic copy that is still a union of regions and an N-set, but with $H_1^m \cap H_0^{m-1} = \emptyset$.

Let $b_m \in H_1^m$ be the choice given by a winning strategy for the Duplicator in the game of length $n - m + 1$ between G_1^m and H_1^m after the Spoiler plays $a_m \in G_1^m$ as its first move. Set $G_0^m = G_0^{m-1} \cup G_1^m$ and $H_0^m = H_0^{m-1} \cup H_1^m$. Note that $G_0^{m-1}, G_1^m, H_0^{m-1}, H_1^m$ are all unions of regions and M-sets or N-sets, hence (C2) and (C3) hold (and (C1) is clear). Moreover both unions are disjoint, so the hypotheses of Lemma 6.7.13 are satisfied and $(G_0^m, a_1, \dots, a_m) \equiv_{n-m} (H_0^m, b_1, \dots, b_m)$, i.e. (C4) holds.

To show that this strategy is winning, note that the outcome of the game only depends on the induced structures on $a_1, \dots, a_n$ and $b_1, \dots, b_n$ at the end of the final turn. These do not depend on what is outside G_0^n and H_0^n since they are unions of regions, hence unions of connected components. As (C4) holds at the end of turn n, the structures induced on $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are isomorphic. □

Corollary 6.7.15. Let $N \vDash \mathrm{Th}(K_1)$. Then $\mathrm{Th}(N)$ is axiomatised by

$$\mathrm{Th}(K_1) \cup \{\mu(\varphi) : \varphi \in \Phi, N \vDash \mu(\varphi)\} \cup \{\neg\mu(\varphi) : \varphi \in \Phi, N \vDash \neg\mu(\varphi)\}$$

Proof. Let N' satisfy the axiomatisation above. Since N and N' are models of $\mathrm{Th}(K_1)$ we may, by Proposition 6.7.12, replace them with D-graphs $M_1 \equiv N$ and $M_1' \equiv N'$ of models of ZFA. By Theorem 6.7.14 $M_1 \equiv M_1'$. □

By the previous corollary, combined with Lemma 6.7.5, theories of double-membership graphs correspond bijectively to consistent (with ZFA, equivalently with ZFC) collections of consistency statements.

The reader familiar with finite model theory may have noticed similarities between the proof of Theorem 6.7.14 and certain proofs of the theorems of Hanf and Gaifman (see [16, Theorems 2.4.1 and 2.5.1]). In fact one could deduce a statement similar to Theorem 6.7.14 directly from Gaifman's Theorem:

Theorem 6.7.16 (Gaifman's Theorem)**.** Every first-order sentence is logically equivalent to a local sentence.[4]

This would characterise the completions of $\mathrm{Th}(K_1)$ in terms of *local formulas*, of which the $\mu(\varphi)$ form a subclass, yielding a less specific result than Corollary 6.7.15. Moreover, we believe that the correspondence with collections of consistency statements provides a conceptually clearer picture.

Similar ideas can be used to study [1, Question 5], which asks whether a countable structure elementarily equivalent to the SD-graph M_0 of some $M \vDash \mathsf{ZFA}$ must itself be the SD-graph of some model of ZFA. We provide a negative solution in Corollary 6.7.18. Again, Gaifman's Theorem could be used directly to deduce its second part.

Theorem 6.7.17. Let $M \vDash \mathsf{ZFA}$. There is a countable $N \equiv M_0$ such that $N \restriction L_1$ has no connected component of infinite diameter.

Before the proof, we show how this solves [1, Question 5].

Corollary 6.7.18. For every $M \vDash \mathsf{ZFA}$ there are a countable $N \equiv M_0$ that is not the SD-graph of any model of ZFA and a countable $N' \equiv M_1$ that is not the D-graph of any model of ZFA.

Proof. Let N be given by Theorem 6.7.17 and $N' := N \restriction L_1$. Now observe that, as follows easily from Proposition 6.5.3, any reduct to L_1 of a model of ZFA has a connected component of infinite diameter. □

Note that this proves slightly more: a negative solution to the question would only have required to find a single pair (M_0, N) satisfying the conclusion of the corollary.

[4] For definitions of this terminology see [16].

Proof of Theorem 6.7.17. Up to passing to a countable elementary substructure, we may assume that M itself is countable. Let N be obtained from M_0 by removing all points whose connected component in M_1 has infinite diameter. We show that $M_0 \equiv N$ by exhibiting, for every n, a sequence $(I_j)_{j \le n}$ of non-empty sets of partial isomorphisms between M_0 and N with the back-and-forth property (see [16, definition 2.3.1 and Corollary 2.3.4]). The idea is to adapt the proof of [37, Lemma 2.2.7] (essentially Hanf's Theorem) by considering the Gaifman balls with respect to L_1, while requiring the partial isomorphisms to preserve the richer language L_0.

On an L_0-structure A, consider the distance $d : A \to \omega \cup \{\infty\}$ given by the graph distance in the reduct $A \restriction L_1$ (where $d(a, b) = \infty$ iff a, b lie in distinct connected components). If $a_1, \dots, a_k \in A$ and $r \in \omega$, denote by $\operatorname{dom}(B(r, a_1, \dots, a_k))$ the union of the balls of radius r (with respect to d) centred on $a_1, \dots, a_k$. Equip $\operatorname{dom}(B(r, a_1, \dots, a_k))$ with the L_0-structure induced by A, then expand to an $L_0 \cup \{c_1, \dots, c_k\}$-structure $B(r, a_1, \dots, a_k)$ by interpreting each constant symbol c_i with the corresponding a_i. We stress that, even though $B(r, a_1, \dots, a_k)$ carries an $L_0 \cup \{c_1, \dots, c_k\}$-structure, and we consider isomorphisms with respect to this structure, the balls giving its domain are defined with respect to the distance induced by L_1 alone.

Set $r_j := (3^j - 1)/2$ and fix n. Define $I_n := \{\emptyset\}$, where $\emptyset$ is thought of as the empty partial map $M_0 \to N$. For $j < n$, let I_j be the following set of partial maps $M_0 \to N$:

$$I_j := \{a_1, \dots, a_k \mapsto b_1, \dots, b_k : k \le n - j, B(r_j, a_1, \dots, a_k) \cong B(r_j, b_1, \dots, b_k)\}$$

We have to show that for every map $a_1, \dots, a_k \mapsto b_1, \dots, b_k$ in I_{j+1} and every $a \in M_0$ [resp. every $b \in N$] there is $b \in N$ [resp. $a \in M_0$] such that $a_1, \dots, a_k, a \mapsto b_1, \dots, b_k, b$ is in I_j.

Denote by ι an isomorphism $B(r_{j+1}, a_1, \dots, a_k) \to B(r_{j+1}, b_1, \dots, b_k)$ and let $a \in M_0$. If a is chosen in $B(2 \cdot r_j + 1, a_1, \dots, a_k)$, then by the triangle inequality and the fact that $2 \cdot r_j + 1 + r_j = r_{j+1}$ we have $B(r_j, a) \subseteq B(r_{j+1}, a_1, \dots, a_k)$, and we can just set $b := \iota(a)$.

Otherwise, again by the triangle inequality, $B(r_j, a)$ and $B(r_j, a_1, \dots, a_k)$ are disjoint and there is no D-edge between them. Note, moreover, that they are M-sets. This allows us to write a suitable flat system, which will yield the desired b.

Working inside M, for every $d \in B(r_j, a)$ choose a well-founded set h_d such that for all $d, d_0, d_1 \in B(r_j, a)$ we have

(H1) $h_{d_0} \notin h_{d_1}$,

(H2) if $d_0 \neq d_1$ then $h_{d_0} \neq h_{d_1}$,

(H3) $h_d \notin B(r_j, b_1, \dots, b_k)$,

(H4) $h_d \notin \bigcup B(r_j, b_1, \dots, b_k)$, and

(H5) $h_d \notin \bigcup \bigcup B(r_j, b_1, \dots, b_k)$.

Let $\{x_d : d \in B(r_j, a)\}$ be a set of indeterminates. Define

$$P_d := \{x_e : e \in B(r_j, a), M \models e \in d\}$$
$$Q_d := \{\iota(f) : f \in B(r_j, a_1, \dots, a_k), M \models S(d, f)\}$$

and consider the flat system

$$\{x_d = \{h_d\} \cup P_d \cup Q_d : d \in B(r_j, a)\} \tag{$*$}$$

Intuitively, the terms P_d ensure that the image of a solution is an isomorphic copy of $B(r_j, a)$, while the terms Q_d create the appropriate S-edges between the image and $B(r_j, b_1, \dots, b_k)$ (note that we do not need any D-edges because there are none between $B(r_j, a)$ and $B(r_j, a_1, \dots, a_k)$). The $\{h_d\}$ are needed for bookkeeping reasons, in order to avoid pathologies. We now spell out the details; keep in mind that each P_d consists of indeterminates, and each Q_d is a subset of $B(r_j, b_1, \dots, b_k)$.

Let s be a solution of $(*)$, guaranteed to exist by AFA. By (H1) and the fact that each member of $\mathrm{Im}(s)$ contains some h_d, we have $\{h_d : d \in B(r_j, a)\} \cap \mathrm{Im}(s) = \emptyset$. Using this together with (H2) and (H3) we have $h_d \in s(x_e) \iff d = e$, hence s is injective.

Let $s' := d \mapsto s(x_d)$ and $b := s'(a)$. By (H4) we have that $\mathrm{Im}(s)$ does not intersect $B(r_j, b_1, \dots, b_k)$, and we already showed that it does not meet $\{h_d : d \in B(r_j, a)\}$. By looking at $(*)$ and at the definition of the terms P_d, we have that $\mathrm{Im}(s) = B(r_j, b)$ and that s' is an isomorphism $B(r_j, a) \to B(r_j, b)$.

Note that the only D-edges involving points of $\mathrm{Im}(s)$ can come from the terms P_d: the h_d are well-founded, and there are no $g \in \mathrm{Im}(s)$ and $\ell \in B(r_j, b_1, \dots, b_k)$ such that $g \in \ell$, since g contains some h_d but this cannot be the case for any element of ℓ because of (H5). Hence $\mathrm{Im}(s)$ is a connected component of M_1 and it has diameter not exceeding $2 \cdot r_j$, so is included in N.

Set $\iota' := s' \cup (\iota \restriction B(r_j, a_1, \dots, a_k))$. This map is injective because it is the union of two injective maps whose images $B(r_j, b)$ and $B(r_j, b_1, \dots, b_k)$ are, as shown above, disjoint. Moreover, there are no D-edges between $B(r_j, b)$ and $B(r_j, b_1, \dots, b_k)$, since the former is a connected component of M_1. By inspecting the terms Q_d, we conclude that ι' is an isomorphism $B(r_j, a_1, \dots, a_k, a) \to B(r_j, b_1, \dots, b_k, b)$, and this settles the 'forth' case.

The proof of the 'back' case, where we are given $b \in N$ and need to find $a \in M_0$, is analogous (and shorter, as we do not need to ensure that the new points are in N): we can consider statements such as $e \in d$ when $e, d \in N$ since the domain of the L_0-structure N is a subset of M. $\square$

Problems We leave the reader with some open problems.

1. Axiomatise the theory of D-graphs of models of ZFA.

2. Axiomatise the theory of SD-graphs of models of ZFA.

3. Characterise the completions of the theory of SD-graphs of models of ZFA.

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