On Homomorphisms Between Specht Modules in Even Characteristic

Adam Higgins

PhD

University of York
Mathematics

October 2023

Abstract

Abstract. Over fields of characteristic 2, Specht modules may decompose and there is no upper bound for the dimension of their endomorphism algebra. A classification of the (in)decomposable Specht modules and a closed formula for the dimension of their endomorphism algebra remain two important open problems in the area. More generally, the space of homomorphisms between two Specht modules is of interest in its own right. In this thesis, we develop a novel description for the homomorphism space between two Specht modules, which we then utilise to deduce new results. Most notably, we provide infinite families of Specht modules with one-dimensional endomorphism algebra in characteristic 2. We conclude by providing a dimension formula for the space of homomorphisms between hook Specht modules in characteristic 2, thereby generalising a result of Murphy who provided an analogous formula covering the endomorphism case [Mur].

Contents

Acknowledgements

I would like to offer my wholehearted thanks to my supervisors Dr. Haralampos Geranios and Prof. Michael Bate. You have gifted me with an experience and an opportunity that I will never cease to cherish. I am grateful beyond bound for the degree to which you have both put up with, and made great efforts to accommodate, my unconventional working habits. It goes without saying, but I will say it anyway; sincerely, thank you. I would also like to extend my thanks to the wider academic group at the University of York. From day one, so many of you have gone above and beyond to make me feel welcome. I will list a couples names, but I would also like to express acknowledge that this list is by no means exhaustive. To Andrew: I think we would both admit that we got off to a rocky start, but to me, that just puts even more weight behind the invaluable support that you have provided over the last few years. In you, I know that I have gained a friend for life. To Ambroise, thank you for keeping me running. To the rest of (extended) Millfield, thank you for making lockdown bearable. To Ross: Thank you for listening to my whinging and only occasionally giving me a hard time about it. Outside of York, I would like to thank Prof. Marc Lackenby and Zubin Siganporia for your continued support and advice over the last few years.

Moving on, I would like to express my deepest thanks to Alice, Ollie, and Brad. I never thought I would be lucky enough to have friends as close as you. I would struggle to express in writing the true extent of my thanks, but thankfully I'm sure you already know. To Scott: thank you for your boundless capacity for fun. To Nate and Joe: I think you both know how much your support has meant to me. You have been there at every last second, and I cannot express how grateful I am. To my (extended) family: I couldn't have done this without your support. Finally, the greatest of my thanks must go to Abbie: Thank you, from the bottom of my heart, for your unwavering support and empathy. You are, as always, my love.

I would like to thank and gratefully acknowledge The Royal Society for their support during my PhD studentship that has been funded by my supervisor's research grant RGF\R1\181015.

Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Many parts of this thesis, are taken directly from, or are heavily influenced by, joint work with the supervisor of this thesis: Dr. Haralampos Geranios. Much of this work has been compiled into a manuscript that has been peer-reviewed and is currently under review following revision. In particular, the abstract to this thesis is almost identical to the abstract from the aforementioned manuscript. A preprint is available on the ArXiV at [GH]. Note that in order to be included within this thesis, and in particular to ensure compliance with formatting requirements of this thesis, a handful of minor cosmetic edits had to be made to the content from this manuscript. The final chapter of this thesis contains a result that will form a component of a follow up paper, currently under preparation, baring the same name as this thesis. In order to ensure compatibility with this final chapter, a handful of minor modifications to terminology and notation were made to the content of the aforementioned manuscript.

In loving memory of Alia Vahedi

1 Introduction

For r a positive integer, denote by $\mathfrak{S}_r$ the symmetric group on r letters, with $\Bbbk\mathfrak{S}_r$ its group algebra over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$. When $p = 0$, the irreducible $\Bbbk\mathfrak{S}_r$-modules are parametrised by the partitions of r, where for each such partition λ, the corresponding irreducible $\Bbbk\mathfrak{S}_r$-module is given by the *Specht module* $\mathrm{Sp}_{\Bbbk}(\lambda)$. In positive characteristic however, Specht modules are not necessarily irreducible. That said until relatively recently, despite significant effort over an extended period of time, a classification describing the irreducible Specht modules for the symmetric groups remained open. In a 1977 paper, James stated a conjecture — which he credits to Carter — that would provide a criterion to determine the (ir)reducibility of a Specht module corresponding to a p-regular partition λ [J_2]. James stated the so-called *Carter conjecture* in terms of a certain combinatorial condition on the partition λ. In the same paper, James proved that the Carter conjecture is necessary in general [J_2, Theorem 2.10], which is to say, if a partition λ does not satisfy this combinatorial condition, then the corresponding Specht module is reducible. Furthermore, James went on to provide a sufficiency result in general characteristic in terms of a certain arithmetical condition on λ [J_2, Theorem 2.13]. When $p = 2$, this arithmetical condition had already previously been observed by James to be equivalent to the condition in the Carter conjecture [J_3, Lemma 3.14], thereby showing that the Carter conjecture holds for $p = 2$ [J_2, Corollary 2.14]. Shortly thereafter, in a 1979 paper, James and Murphy examined the determinant of the Gram matrix of a certain bilinear form on Specht modules, and ultimately resolved the remainder of the proof of Carter's conjecture in odd characteristic [JMur]. Later, in 1999, James and Mathas analysed the arithmetical condition in the case that $p = 2$, and showed that with the exception of $\lambda = (2, 2)$, the Specht module $\mathrm{Sp}_{\Bbbk}(\lambda)$ is irreducible in characteristic 2 if and only if either λ is 2-regular — that is to say, has distinct terms — and satisfies Carter's condition, or λ' is 2-regular and satisfies Carter's condition. At the same time, Mathas stated a conjecture with James for the odd characteristic case [Mat, Conjecture 5.47]. In a 2004 paper, Fayers, building on the work of Lyle [L], showed that the condition stated in the James-Mathas conjecture was necessary [F_1]. Following on shortly after this, Fayers showed that this condition was sufficient in the more general context of the *Iwahori-Hecke algebra* $\mathscr{H}_{\Bbbk,q}(\mathfrak{S}_r)$ of the symmetric group $\mathfrak{S}_r$ so long as $q \neq -1$. In particular, this applies in the case that $q = 1$, where we have that $\mathscr{H}_{\Bbbk,q}(\mathfrak{S}_r)$ is isomorphic to the group algebra $\Bbbk\mathfrak{S}_r$ [F_2].

Meanwhile, in a series of lectures delivered in 1977 – and then later published in 1978 – James showed that so long as $p \neq 2$, Specht modules have one-dimensional endomorphism algebras [J_1, 13.17], and are hence indecomposable [J_1, 13.18]. On the other hand, in the same set of lectures, James showed that $\mathrm{Sp}_{\Bbbk}(5, 1^2)$ decomposes over fields $\Bbbk$ of characteristic 2 [J_1, 23.10(iii)]. In the intervening years, this phenomenon of decomposable Specht modules has received extensive investigation. In 1980, Murphy examined Specht modules labelled by hook partitions, resulting in a complete classification of all such decomposable Specht modules in terms of a parity condition on the parameters a, b determining the hook partition $(a, 1^b)$ [Mur]. At the time, this result subsumed all known examples of decomposable Specht modules, and this remained the case for many years thereafter. Then, in 2012, Dodge and Fayers examined partitions of the form $(a, 3, 1^b)$, and produced novel examples of decomposable Specht modules [DF]. More recently, Donkin and Geranios [DG$_2$] examined partitions of the form $\lambda = (a, m-1, \dots, 2, 1^b)$ for parameters $a \geq m$, $b \geq 1$, and found precise decompositions when $a - m$ is even and b is odd. An interesting feature arising in these decompositions is that there is no upper bound for the number of indecomposable summands of the $\mathrm{Sp}_{\Bbbk}(\lambda)$, and so in turn for the dimension of its endomorphism algebra. The results of Donkin and Geranios apply equally well in the more general context of the Hecke algebras. Other results in this general context can be found in [BBS], [Spe]. Despite this extensive study, a classification of the (in)decomposable Specht modules remains to be found. Much of the research into indecomposable Specht modules shares a common approach, that is through studying the endomorphism algebra of a Specht module. More generally, the vector space of linear homomorphisms between a pair of Specht modules is of interest in its own right, in characteristic 2 and beyond. It is the initial purpose of this thesis to present a new general characterisation of the space of homomorphisms between two Specht modules in terms of certain composition relations. As a proof of concept, we use our new description to provide infinite families of Specht modules with one-dimensional endomorphism algebra.

In Chapter 2, we begin by covering some of the prerequisite background for many of the concepts present in this thesis. We start by reviewing some of the terminology from combinatorics and multilinear algebra. Then, we cover the basics of the polynomial representation theory of the general linear groups, along the lines of [G, §2]. In particular, we review some constructions of induced and Weyl modules due to Akin, Buchsbaum, and Weyman [ABW, §II], and James [J$_1$, §26], using results from [G, §4.8], [D$_1$, §2.7(5)] to tie them together. Then, following [G, §2], [D$_3$] closely, we provide some details on the structure of the Schur algebra and its connection to the general liner groups. Then, we move on and establish some background relating to the representation theory of the symmetric groups, where we follow the guide of [J$_1$]. Finally, we present the background on the Schur functor, which will be critical when establishing the connection between the representation theories of the general linear groups and symmetric groups. In doing so, we follow the presentation in [G, §6.1-§6.2], which we supplement with [D$_3$].

In Chapter 3, we apply the Schur functor to the constructions of the induced modules provided by James and by Akin, Buchsbaum, and Weyman. The first of these recovers a description of a Specht module known as *James' kernel intersection theorem* [J$_1$, Corollary 17.18]. On the other hand, the second of these produces a cokernel description of a Specht module dual to James' kernel intersection theorem §3.1.1. Next, in Proposition 3.1.3, we provide some new results on the g-functor, that is, the right-inverse of the Schur functor. Using these results, in Lemma 3.2.2, we produce the desired description for the space of homomorphisms between two Specht modules as a subspace of the homomorphism space between a signed permutation module and a permutation module, or between two permutations modules in the case of even characteristic. In order to make use of this description, we produce a description of a basis for the space of homomorphisms between two permutation modules in characteristic 2, and then we interpret this description in terms of this basis. For the remainder of the chapter, we aim to examine the same family of partitions studied by Donkin and Geranios [DG_2], that is to say, partitions of the form $(a, m-1, \ldots, 2, 1^b)$, but now in the parity case where $a - m \equiv b$. In order to do so, we once again return to the context of the polynomial representation theory of the general linear groups in order to produce a reduction technique. We proceed by developing certain combinatorial techniques and terminology. The chapter concludes by utilising this technology to show that the corresponding Specht modules have one-dimensional endomorphism algebra Theorem 3.4.25, thereby providing a novel large family of indecomposable Specht modules in characteristic 2.

Finally, in Chapter 4, we finish by analysing the special case of hook Specht modules in characteristic 2. First, in Proposition 4.2.1, we recover Murphy's result on the dimension of the endomorphism algebra of a hook Specht module [Mur]. Then, in Proposition 4.3.1, we generalise this approach to produce a new dimension formula for the space of homomorphisms between any two hook Specht modules in characteristic 2.

2 BACKGROUND

2.1. CONVENTIONS, TERMINOLOGY, AND NOTATION

We denote by $\mathbb{N}$ the set of non-negative integers, and by $\mathbb{Z}_{>0}$ the set of positive integers. Throughout, we fix $\Bbbk$ to be an algebraically closed field of characteristic $p \geq 0$.

2.1.1. **Algebraic Notation.** Let R be a ring. We denote by R–mod (resp. mod–R) the category of finite-dimensional left R-modules (resp. right R-modules). If $U, V \in R$–mod, then we say that a function $U \xrightarrow{h} V$ is *R-linear* if h is a homomorphism of R-modules. If S is a set, then by the *R-module defined on S*, we shall mean the free R-module $RS \coloneqq \bigoplus_{s \in S} Rs$ whose elements are given by all formal linear combinations $\sum_{s \in S} r_s s$ for subsets of the form $\{r_s \mid s \in S\} \subseteq R$ where only finitely many of the r_s are non-zero. If G is a group, then RG is the *group ring of G (over R)*. When the ring R is implicit, we refer to RG-modules simply as *G-modules*, and accordingly, we say that a function $U \xrightarrow{h} V$ on G-modules is *G-linear*, or is a G-homomorphism, to mean that h is RG-linear. By a $\Bbbk$-space, we shall mean a vector space over $\Bbbk$. Accordingly, we refer to $\Bbbk S$ as the *$\Bbbk$-space defined on S*.

Given a $\Bbbk$-algebra A, then an *$\mathbb{N}$-grading* on A is a $\Bbbk$-decomposition $A = \bigoplus_{r \in \mathbb{N}} A^{(r)}$ into $\Bbbk$-subspaces $\{A^{(r)} \mid r \in \mathbb{N}\}$ of A such that $A^{(r)} \cdot A^{(s)} \subseteq A^{(r+s)}$ for $r, s \in \mathbb{N}$. For each $r \in \mathbb{N}$, we refer to the $\Bbbk$-subspace $A^{(r)}$ of A as the *component of A in degree r*. Throughout this thesis, we impose the additional condition that $A^{(0)} = \Bbbk$.

2.1.2. **Integers, Sequences, and Matrices.** For $a, b \in \mathbb{Z}$, we write $a \equiv b$ to mean that a and b are congruent modulo 2. Given $d \in \mathbb{Z}_{>0}$, we identify the set $\mathbb{N}^d$ as the set of sequences of d elements of $\mathbb{N}$. Then, given $n \in \mathbb{N}$, we denote by $(n^d) \coloneqq (n, \ldots, n)$ the sequence formed from d consecutive copies of n, and accordingly, if n^d appears as a term in a sequence, it is taken to mean d consecutive copies of the term n. Given $d, d' \in \mathbb{Z}_{>0}$ with $\imath \in \mathbb{N}^d$, $\imath' \in \mathbb{N}^{d'}$, we denote by $\imath \mathbin{+\!\!+} \imath' \in \mathbb{N}^{d+d'}$ the *concatenation* of $\imath$ and $\imath'$, that is to say, the sequence $\imath \mathbin{+\!\!+} \imath' \coloneqq (\imath_1, \ldots, \imath_d, \imath'_1, \ldots, \imath'_{d'})$. For $n \in \mathbb{Z}_{>0}$, we write $[n] \coloneqq \{1, \ldots, n\}$, with $[0] \coloneqq \varnothing$. Given $a, b \in \mathbb{Z}_{>0}$ and a set S, we write $M_{a \times b}(S)$ for the set of $(a \times b)$-matrices with entries in S. If $A \in M_{a \times b}(S)$, then for $i, j \in [n]$, we denote by $a_{ij} \in S$ the $(i,j)^{\text{th}}$-entry of A. Given $A \in M_{a \times b}(S)$, we denote by $A' \in M_{b \times a}(S)$ the transpose matrix of A.

2.1.3. **Symmetric Groups.** For a set S, we denote by $\mathrm{Sym}(S)$ the *symmetric group* on the elements of S. We take the convention that the natural action of $\mathrm{Sym}(S)$ on the set S is from the **left**. Explicitly, for $s \in S$, $\sigma \in \mathrm{Sym}(S)$, we denote by $\sigma(s) \in S$ the image of s under σ. Accordingly, composition in $\mathrm{Sym}(S)$ is performed from **right to left**. For $r \in \mathbb{Z}_{>0}$, we write $\mathfrak{S}_r \coloneqq \mathrm{Sym}([r])$ for the *the symmetric group on r letters*. Recall that $\mathfrak{S}_r$ is generated by the transpositions $\{\sigma_k \mid 1 \leq k < r\}$, where for such k, σ_k denotes the transposition $k \leftrightarrow k+1$. For $r \in \mathbb{Z}_{>0}$, we denote by $\mathsf{triv}_r \coloneqq \Bbbk \cdot 1_{\text{triv}}$ the $\Bbbk\mathfrak{S}_r$-module that affords the trivial character of $\mathfrak{S}_r$, and by $\mathsf{sgn}_r \coloneqq \Bbbk \cdot 1_{\text{sgn}}$ the $\Bbbk\mathfrak{S}_r$-module that affords the sign character of $\mathfrak{S}_r$. Finally, for $S \subseteq \mathfrak{S}_r$, we define the elements:

$$[S] \coloneqq \sum_{\sigma \in S} \sigma \in \mathbb{Z}\mathfrak{S}_r, \qquad \{S\} \coloneqq \sum_{\sigma \in S} \mathtt{sgn}(\sigma)\sigma \in \mathbb{Z}\mathfrak{S}_r. \tag{2.1.1}$$

where $\mathtt{sgn}(\sigma) \in \mathbb{Z}$ denotes the *sign* (*parity*) of $\sigma \in \mathfrak{S}_r$.

2.2. Combinatorics

2.2.1. **Compositions and Partitions.** By a *composition*, we mean a sequence α in the set $\Lambda(n) := \mathbb{N}^n$ for some $n \in \mathbb{Z}_{>0}$. For such a composition α, we refer to the terms of a composition $\alpha \in \Lambda(n)$ as *the parts of* α. Now, for $\alpha \in \Lambda(n)$, we denote by $\deg(\alpha) := \sum_{v=1}^{n} \alpha_v$ the *degree* of α, and by $\operatorname{len}(\alpha)$ the *length* of α, that is to say $\operatorname{len}(\alpha) := \max\{v \in [n] \mid \alpha_v \neq 0\}$ if $\alpha \neq (0^n)$, and $\operatorname{len}(0^n) := 0$. Then, we say that $\alpha \in \Lambda(n)$ is a *composition of* r if $\deg(\alpha) = r$, and we denote by $\Lambda(n,r) \subseteq \Lambda(n)$ the subset of $\Lambda(n)$ consisting of the compositions of r with at most n non-zero parts. For $\alpha, \beta \in \Lambda(n,r)$, we write:

$$\operatorname{Tab}(\alpha,\beta) := \left\{ A \in M_{n\times n}(\mathbb{N}) \;\middle|\; \sum_j a_{ij} = \alpha_i,\ \sum_i a_{ij} = \beta_j \right\}. \tag{2.2.1}$$

Now, we say that a composition α is a *partition* if its parts are weakly decreasing, and we denote by $\Lambda^+(n) \subseteq \Lambda(n)$ the set of partitions with at most n non-zero parts. Then, for $r \in \mathbb{N}$, as with compositions, we say that a partition λ is a *partition* of r if $\deg(\lambda) = r$. Given a partition λ, we denote by λ' the *transpose* (*conjugate*) partition of λ. We denote by $\Lambda^+(n,r) := \Lambda^+(n) \cap \Lambda(n,r)$ the set of partitions of r into at most n non-zero parts. Note that if $n \geq r$, then $\Lambda^+(n,r)$ contains all partitions of r up to trailing zeros. In particular, if $n \geq r$ with $\lambda \in \Lambda^+(n,r)$, then we have that $\lambda' \in \Lambda^+(n,r)$.

Finally, the following notation will be of particular use for this thesis. Given some $\alpha \in \Lambda(n,r)$ with $1 \leq i < j \leq n$ with $\alpha_j \neq 0$ and $1 \leq s \leq \alpha_j$, we write $\alpha^{(i,j,s)}$ for the element of $\Lambda(n,r)$ with terms defined by $\alpha_k^{(i,j,s)} := \alpha_k + s(\delta_{i,k} - \delta_{j,k})$, which is to say the sequence obtained from α by raising its i^{th}-part α_i by k, and lowering its j^{th}-part by k.

Example 2.2.2. Let $\alpha := (5,4,0,1)$. Then $\alpha^{(2,4,1)} = (5,5,0,0)$, and $\alpha^{(1,2,3)} = (8,1,0,1)$.

2.2.2. **Multi-indices.** Fix $n \in \mathbb{N}$ and $r \in \mathbb{Z}_{>0}$. We write $I(n,r) := [n]^r$ for the set of multi-indices $\imath = (\imath_1, \ldots, \imath_r)$ with terms in $[n]$. Note that the left-action of $\mathfrak{S}_r$ on $[r]$ induces a right-action of $\mathfrak{S}_r$ on $I(n,r)$ by *place-permutation* on indices. For the sake of clarity, for $\imath \in I(n,r)$, $\sigma \in \mathfrak{S}_r$, the sequence $\imath\sigma$ is determined by $(\imath\sigma)_j = \imath_{\sigma(j)}$ for $j \in [r]$.

Example 2.2.3. Let $a,b,c \in [n]$ and write $\imath := (a,b,c) \in I(n,3)$. Then, we have that $\imath(1\ 2\ 3) = (b,c,a)$ and $\imath(1\ 3\ 2) = (c,a,b)$.

Given $\imath, \jmath \in I(n,r)$, we write $\imath \sim \jmath$ to say that $\imath, \jmath$ share a $\mathfrak{S}_r$-orbit in $I(n,r)$. Now, for $\imath = (\imath_1, \ldots, \imath_r) \in I(n,r)$, by the *content* of $\imath$, we shall mean the sequence $\mathsf{c}(\imath) := (\mathsf{c}_1(\imath), \ldots, \mathsf{c}_n(\imath)) \in \mathbb{N}^n$ whose terms are given by $\mathsf{c}_k(\imath) := |\{j \in [r] \mid \imath_j = k\}|$. Clearly, given $\imath, \jmath \in I(n,r)$, we have that $\imath \sim \jmath$ if and only if $\mathsf{c}(\imath) = \mathsf{c}(\jmath)$. Viewing c as a function $I(n,r) \xrightarrow{\mathsf{c}} \mathbb{N}^n$, we have that the image of c is precisely the set $\Lambda(n,r)$. Thus, we may view the set $\Lambda(n,r)$ as a parametrisation of the $\mathfrak{S}_r$-orbits in $I(n,r)$, and accordingly, for $\imath \in I(n,r)$, $\alpha \in \Lambda(n,r)$, we write $\imath \in \alpha$ to mean that $\mathsf{c}(\imath) = \alpha$.

Finally, we denote by $\hat{I}(n,r) \subseteq I(n,r)$ the set consisting of those $\imath \in I(n,r)$ with distinct terms, and by $\hat{I}^+(n,r) \subseteq \hat{I}(n,r)$ the set consisting of those $\imath \in \hat{I}(n,r)$ whose terms are strictly increasing.

2.3. Multilinear Algebra

Unless otherwise stated, all tensor products are taken over the field $\Bbbk$.

Fix $n \in \mathbb{Z}_{>0}$. We write $E \coloneqq \Bbbk^{\oplus n}$ for the n-dimensional $\Bbbk$-space defined on the standard basis $\{e_u \mid u \in [n]\}$ of column-vectors.

2.3.1. **Tensor Algebra.** For $r \in \mathbb{Z}_{>0}$, we write $E^{\otimes r}$ for the r-fold tensor product of E, where $E^{\otimes 0} \coloneqq \Bbbk$. We observe that $E^{\otimes r}$ has a $\Bbbk$-basis given by:

$$\left\{ e_{\imath} \coloneqq e_{\imath_1} \otimes \cdots \otimes e_{\imath_r} \in E^{\otimes r} \;\middle|\; \imath \in I(n,r) \right\}.$$

Note that the right-action $I(n,r) \curvearrowleft \mathfrak{S}_r$ endows the $\Bbbk$-space $E^{\otimes r}$ with the structure of a right $\Bbbk\mathfrak{S}_r$-module. Explicitly, we have $e_{\imath} \cdot \sigma \coloneqq e_{\imath\sigma} = \bigotimes_{k=1}^{r} e_{\imath_{\sigma(k)}}$ for $\imath \in I(n,r)$, $\sigma \in \mathfrak{S}_r$.

Now, by the *tensor algebra* of E, we mean the $\Bbbk$-space $T(E) \coloneqq \bigoplus_{r \in \mathbb{N}} E^{\otimes r}$. The tensor algebra $T(E)$ has the structure of an $\mathbb{N}$-graded $\Bbbk$-algebra under the *concatenation product* $\Pi_{\otimes}$. Explicitly, for $r, s \in \mathbb{N}$, the component $\Pi_{\otimes}^{(r,s)} : E^{\otimes r} \otimes E^{\otimes s} \to E^{\otimes(r+s)}$ of $\Pi_{\otimes}$ is given by $e_{\imath} \otimes e_{\jmath} \mapsto e_{\imath + \jmath}$ for $\imath \in I(n,r)$, $\jmath \in I(n,s)$. In addition to its $\Bbbk$-algebra structure, the tensor algebra $T(E)$ carries the structure of a $\Bbbk$-coalgebra. For $r, s \in \mathbb{N}$, we denote by $\Delta_{\otimes}^{(r,s)} : E^{\otimes(r+s)} \to E^{\otimes r} \otimes E^{\otimes s}$ the appropriate component of the comultiplication $\Delta_{\otimes}$ of $T(E)$. Explicitly, we have $\Delta_{\otimes}^{(r,s)} : e_{\imath} \mapsto e_{\imath} \cdot \left[\mathrm{Sh}_{(r,s)}\right]$ for $\imath \in I(n,r)$, where $\mathrm{Sh}_{(r,s)} \leq \mathfrak{S}_{r+s}$ denotes the set of (r,s)-*shuffles* within $\mathfrak{S}_{r+s}$, that is:

$$\mathrm{Sh}_{(r,s)} \coloneqq \{\sigma \in \mathfrak{S}_{r+s} \mid \sigma(i) < \sigma(j) \text{ for } i < j \text{ with } j \leq r \text{ or } i \geq r+1\}. \tag{2.3.1}$$

2.3.2. **Symmetric Algebra.** We denote by I the two-sided ideal of $T(E)$ generated by elements of the form $x \otimes y - y \otimes x \in E^{\otimes 2}$ for $x, y \in E$. Note that I is an ideal in the graded sense. Moreover, for $r \in \mathbb{N}$, the component $I^{(r)} = I \cap E^{\otimes r}$ of I in degree r is $\Bbbk$-spanned by elements of the form $e_{\imath} \cdot (1 - \sigma_k)$ for $\imath \in I(n,r)$ and $1 \leq k < r$. Then, by the *symmetric algebra of* E, we mean the quotient $S(E) \coloneqq T(E)/I$. Note that $E \hookrightarrow S(E)$, where for $u \in [n]$, we have $e_u \mapsto \bar{e}_u \coloneqq e_u + I \in S(E)$. Now, since I is graded, $S(E)$ inherits an $\mathbb{N}$-grading from $T(E)$. For $r \in \mathbb{N}$, we denote by $S^r E \coloneqq S(E)^{(r)}$ the component of $S(E)$ in degree r, and we refer to $S^r E$ as the r^{th}*-symmetric power of* E. Accordingly, we denote by $\Pi_S^{(1^r)} : E^{\otimes r} \twoheadrightarrow S^r E$ the appropriate component of the quotient map $T(E) \twoheadrightarrow S(E)$.

Now, the symmetric algebra $S(E)$ has the structure of an $\mathbb{N}$-graded commutative $\Bbbk$-algebra whose product Π_S is inherited from the concatenation product $\Pi_{\otimes}$ of $T(E)$. For $r, s \in \mathbb{N}$, we write $\Pi_S^{(r,s)} : S^r E \otimes S^s E \to S^{r+s} E$ for the appropriate component of Π_S. Accordingly, we write $x \cdot y \coloneqq \Pi_S^{(r,s)}(x \otimes y)$ for $x \in S^r E$, $y \in S^s E$. Additionally, for $\imath \in I(n,r)$, we write $\bar{e}_{\imath} \coloneqq \prod_{u=1}^{r} \bar{e}_{\imath_u} \in S^r E$. Now, for $\imath, \jmath \in I(n,r)$, note that $\bar{e}_{\imath} = \bar{e}_{\jmath}$ if and only if $\imath \sim \jmath$. Hence, for $\alpha \in \Lambda(n,r)$, we may write $\bar{e}^{\alpha} \coloneqq \bar{e}_{\imath} \in S^r E$ for any choice of $\imath \in \alpha$. Clearly, for each $r \in \mathbb{N}$, we have that the set $\{\bar{e}^{\alpha} \mid \alpha \in \Lambda(n,r)\}$ forms a $\Bbbk$-basis for $S^r E$. Then, for $r, s \in \mathbb{N}$, we have $\bar{e}^{\alpha} \cdot \bar{e}^{\beta} = \bar{e}^{\alpha+\beta}$ for $\alpha \in \Lambda(n,r)$, $\beta \in \Lambda(n,s)$. Alongside the product Π_S, the symmetric algebra $S(E)$ inherits a comultiplication Δ_S descended from the comultiplication $\Delta_{\otimes}$ of $T(E)$. Note that, for $r, s \in \mathbb{N}$, the component $\Delta_S^{(r,s)} : S^{r+s} E \to S^r E \otimes S^s E$ of Δ_S is given by $\Delta_S^{(r,s)} : \bar{e}^{\gamma} \mapsto \sum_{\alpha,\beta} \binom{\alpha+\beta}{\alpha} \bar{e}^{\alpha} \otimes \bar{e}^{\beta}$, where the sum is over $\alpha \in \Lambda(n,r)$, $\beta \in \Lambda(n,s)$ with $\alpha + \beta = \gamma$, and for such α, β, we denote by $\binom{\alpha+\beta}{\alpha}$ the product of binomial coefficients given by $\binom{\alpha+\beta}{\alpha} \coloneqq \prod_{k \in [n]} \binom{\alpha_k + \beta_k}{\alpha_k}$.

2.3.3. **Exterior Algebra.** Now, we denote by J the two-sided ideal of $T(E)$ generated by elements of the form $x \otimes x \in E^{\otimes 2}$ for $x \in E$, and, once again, we observe that J is an ideal in the graded sense. Note that, for $r \in \mathbb{N}$, in this case we have that the component $J^{(r)} = J \cap E^{\otimes r}$ of J in degree r contains all elements of the form $e_{\imath} \cdot (1 - \mathtt{sgn}(\sigma)\sigma)$ for $\imath \in I(n,r)$ and $\sigma \in \mathfrak{S}_r$. Then, by the *exterior algebra of* E, we mean the quotient $\Lambda(E) \coloneqq T(E)/J$. Once again, we have that $E \hookrightarrow \Lambda(E)$, where now, for $u \in [n]$, we have that $e_u \mapsto \hat{e}_u \coloneqq e_u + J \in \Lambda(E)$. In this case, we write $\Lambda(E) = \bigoplus_{r\in\mathbb{N}} \Lambda^r E$ for the $\mathbb{N}$-grading on $\Lambda(E)$ inherited from that of $T(E)$, and for $r \in \mathbb{N}$, we refer to the component $\Lambda^r E$ as the r^{th}-*exterior power* of E. Accordingly, we denote by $\Pi_\Lambda^{(1^r)} : E^{\otimes r} \twoheadrightarrow \Lambda^r E$ the appropriate component of the quotient map $T(E) \twoheadrightarrow \Lambda(E)$.

Now, in this case, the product Π_Λ on $\Lambda(E)$ that is inherited from that of $T(E)$ gives $\Lambda(E)$ the structure of an $\mathbb{N}$-graded anti-commutative $\Bbbk$-algebra. For $r, s \in \mathbb{N}$, we write $\Pi_\Lambda^{(r,s)} : \Lambda^r E \otimes \Lambda^s E \to \Lambda^{r+s} E$ for the appropriate component of Π_Λ, and here, we write $x \wedge y \coloneqq \Pi_\Lambda^{(r,s)}(x \otimes y)$ for $x \in \Lambda^r E$, $y \in \Lambda^s E$. Then, for $\imath \in I(n,r)$, we write $\hat{e}_\imath \coloneqq \bigwedge_{u=1}^r \hat{e}_{\imath_u} \in \Lambda^r E$. Note that if $\imath, \jmath \in I(n,r)$ with $\imath \sim \jmath$, then $\hat{e}_\jmath = \pm \hat{e}_\imath$, and in particular, we see that $\hat{e}_\imath = 0$ if $\imath$ has a repeated term. Then, for $r \in \mathbb{N}$, we see that $\Lambda^r E$ has a $\Bbbk$-basis given by $\{\hat{e}_\imath \mid \imath \in \hat{I}^+(n,r)\}$. Note that, for $r, s \in \mathbb{N}$ with $\imath \in \hat{I}(n,r)$, $\jmath \in \hat{I}(n,s)$, then $\hat{e}_\imath \wedge \hat{e}_\jmath = \hat{e}_{\imath + \jmath}$. In particular, if $\imath, \jmath$ share a term, then $\hat{e}_\imath \wedge \hat{e}_\jmath = 0$. Meanwhile, the comultiplication Δ_Λ on $\Lambda(E)$, inherited from $\Delta_\otimes$ of $T(E)$, is given component-wise by $\Delta_\Lambda^{(r,s)} : \Lambda^{r+s} E \to \Lambda^r E \otimes \Lambda^s E$ for $r, s \in \mathbb{N}$, where for $\imath \in \hat{I}^+(n, r+s)$, we have $\Delta_\Lambda^{(r,s)} : \hat{e}_\imath \to (\hat{e}_\jmath \otimes \hat{e}_\Bbbk) \cdot \{\mathrm{Sh}_{(r,s)}\}$, where $\jmath \in \hat{I}^+(n,r)$, $\Bbbk \in \hat{I}^+(n,s)$ are uniquely determined by $\imath = \jmath + \Bbbk$.

2.3.4. **Divided Power Algebra.** Here, we introduce the *divided power algebra* $D(E)$ of E. In doing so, there are a multitude of approaches that one may choose to take. For the sake of expediency, we choose to define (as a $\Bbbk$-space) the divided power algebra $D(E)$ as the $\Bbbk$-linear dual $S(E^*)^* = \mathrm{Hom}_\Bbbk(S(E^*), \Bbbk)$ of the symmetric algebra of the $\Bbbk$-linear dual E^* of E. We have the $\mathbb{N}$-grading $D(E) = \bigoplus_{r\in\mathbb{N}} D^r E$ where $D^r E \cong (S^r(E^*))^*$. We identify the r^{th}-divided power $D^r E$ of E by the $\Bbbk$-basis $\{e^{(\alpha)} = \prod_{k=1}^n e_k^{(\alpha_k)} \mid \alpha \in \Lambda(n,r)\}$. We denote by $\Pi_D : D(E) \to D(E)$ the product on $D(E)$. For $r, s \in \mathbb{N}$, the component $\Pi_D^{(r,s)} : D^r E \otimes D^s E \to D^{r+s} E$ is given by $\Pi_D^{(r,s)} : e^{(\alpha)} \otimes e^{(\beta)} \mapsto \binom{\alpha+\beta}{\alpha} e^{(\alpha+\beta)}$ for $\alpha \in \Lambda(n,r)$, $\beta \in \Lambda(n,s)$. For $x \in D(E)$ and $t \in \mathbb{Z}_{>0}$, we denote by $x^{(t)} \in D(E)$ the image of x under the t-fold product $D(E)^{\otimes t} \to D(E)$, with $x^{(0)} \coloneqq 1 \in \Bbbk = D^0 E$.

Remark 2.3.2. For convenience, we list the following properties of Π_D:

- For $x \in D(E)$, we have $x^{(0)} = 1$ and $x^{(1)} = x$.
- For $x, y \in D(E)$, $t \in \mathbb{N}$, we have $(x+y)^{(t)} = \sum_{r+s=t} x^{(r)} \cdot y^{(s)}$.
- For $r, s \in \mathbb{N}$, $i \in [n]$, we have $e_i^{(r)} \cdot e_i^{(s)} = \binom{r+s}{r} e_i^{(r+s)}$,
- For $r, s \in \mathbb{N}$, $i, j \in [n]$ with $i \neq j$, we have $e_i^{(r)} \cdot e_j^{(s)} = e_j^{(s)} \cdot e_i^{(r)}$.
- For $r, s \in \mathbb{N}$, $\alpha \in \Lambda(n,r)$, $\beta \in \Lambda(n,s)$, we have $e^{(\alpha)} \cdot e^{(\beta)} = \binom{\alpha+\beta}{\alpha} e^{(\alpha+\beta)}$.

Meanwhile, we denote by $\Delta_D : D(E) \to D(E) \otimes D(E)$ the comultiplication on $D(E)$. For $r, s \in \mathbb{N}$, the component $\Delta_D^{(r,s)} : D^{r+s} E \to D^r E \otimes D^s E$ of Δ_D is given by $\Delta_D^{(r,s)} : e^{(\gamma)} \mapsto \sum_{\alpha,\beta} e^{(\alpha)} \otimes e^{(\beta)}$ for $\gamma \in \Lambda(n, r+s)$, where the sum is over all $\alpha \in \Lambda(n,r)$, $\beta \in \Lambda(n,s)$ with $\alpha + \beta = \gamma$.

2.4. POLYNOMIAL REPRESENTATIONS OF THE GENERAL LINEAR GROUPS

Here, we review the prerequisite background related to the polynomial representation theory of the general linear groups. See [G], [D$_3$], [Mar] for more details.

We fix an integer $n \geq 1$ with $G := \mathrm{GL}_n(\Bbbk)$ the group of invertible $(n\times n)$-matrices with entries in $\Bbbk$. We identify G as an affine variety embedded within the affine space $\mathbb{A}^{n^2}$. Accordingly, we identify the coordinate algebra of G as $\Bbbk[G] = \Bbbk[c_{11}, \ldots, c_{nn}, \det^{-1}]$. Recall that the group multiplication $G \times G \xrightarrow{m} G$ induces the comultiplication map $\Bbbk[G] \xrightarrow{\Delta_G} \Bbbk[G] \otimes \Bbbk[G]$ determined by $\Delta_G(h)(g, g') = h(gg')$ for $g, g' \in G$, $h \in \Bbbk[G]$. Here, we have applied the standard identification $\Bbbk[G \times G] \cong \Bbbk[G] \otimes \Bbbk[G]$.

Remark 2.4.1. Note that $\Bbbk[G]$ has the structure of a (G, G)-bimodule. Indeed, for $h \in \Bbbk[G]$, $g \in G$, $x \in G$, we have that $(g \cdot h)(x) := h(xg) = \Delta_G(h)(x, g)$. On the other hand, in a similar fashion, we have that $(h \cdot g)(x) = \Delta_G(h)(g, x)$ for such h, g, x.

2.4.1. **Rational, Polynomial, and Homogenous Modules.** Given a finite dimensional G-module V and some $\Bbbk$-basis $\mathcal{V}$, the *coefficient functions* of V (with respect to $\mathcal{V}$) are the functions $f_{vv'} : G \to \Bbbk$ determined by $g \cdot v' = \sum_{v\in\mathcal{V}} f_{vv'}(g)v$ for $g \in G$, $v' \in \mathcal{V}$. We write $\mathrm{cf}(V)$ for the *coefficient space* of V, that is, the $\Bbbk$-space defined on the set $\{f_{vv'} \mid v, v' \in \mathcal{V}\}$ of coefficient functions. The coefficient space $\mathrm{cf}(V)$ is independent of the choice of $\Bbbk$-basis $\mathcal{V}$. We say that V is *rational* if $\mathrm{cf}(V) \subseteq \Bbbk[G]$, and *polynomial* if $\mathrm{cf}(V) \subseteq A_\Bbbk(n) := \Bbbk[c_{11}, \ldots, c_{nn}]$. The polynomial G-modules form an Abelian category, which we denote by $M_\Bbbk(n)$. Note that $M_\Bbbk(n)$ is closed under tensor products.

Example 2.4.2. Let $V = \Bbbk \cdot v$ be the 1-dimensional G-representation determined by $g\cdot v = \det(g)^{-1}v$. Then $\mathcal{V} := \{v\}$ is a $\Bbbk$-basis for V and we have $f_{vv} : g \mapsto \det(g)^{-1}$. Since $\det^{-1} \in \Bbbk[G]$, we see that V is rational. However, V is not polynomial since $f_{vv} \notin A_\Bbbk(n)$.

Now, the polynomial algebra $A_\Bbbk(n) = \Bbbk[c_{11}, \ldots, c_{nn}]$ has an $\mathbb{N}$-grading of the form $A_\Bbbk(n) = \bigoplus_{r\in\mathbb{N}} A_\Bbbk(n, r)$, where for $r \in \mathbb{N}$, the $\Bbbk$-space $A_\Bbbk(n, r)$ is the $\Bbbk$-span of the homogenous degree r monomials $\{c_{\imath\jmath} := \prod_k c_{\imath_k\jmath_k} \mid \imath, \jmath \in I(n, r)\}$ in the c_{ij}. Then, for $r \in \mathbb{N}$, we say that a non-zero polynomial G-module V is *homogenous of degree* r if $\mathrm{cf}(V) \subseteq A_\Bbbk(n, r)$. The homogenous degree r polynomial G-modules form a subcategory of $M_\Bbbk(n)$, which we denote by $M_\Bbbk(n, r)$.

Examples 2.4.3.

(i) Recall that we identify $E = \Bbbk^{\oplus n}$ as the $\Bbbk$-space defined by the standard basis $\{e_i \mid i \in [n]\}$. Then, for $g \in G$, we have $g \cdot e_j = \sum_{i\in[n]} g_{ij}e_i$ for $j \in [n]$. Hence, $\mathrm{cf}(E) = A_\Bbbk(n, 1)$, and so $E \in M_\Bbbk(n, 1)$.

(ii) More generally, for $r \in \mathbb{N}$, we have $g \cdot e_\jmath = \sum_{\imath\in I(n,r)} g_{\imath\jmath}e_\imath$ for $g \in G$, $\jmath \in I(n, r)$, where $g_{\imath\jmath} := c_{\imath\jmath}(g)$. Hence $\mathrm{cf}(E^{\otimes r}) = A_\Bbbk(n, r)$, and so $E^{\otimes r} \in M_\Bbbk(n, r)$.

(iii) Denote by $\mathsf{det} = \Bbbk \cdot 1_{\det}$ the 1-dimensional G-module afforded by the determinant function: $g\cdot 1_{\det} = \det(g)1_{\det}$ for $g \in G$. Then, as functions, we identify det with the polynomial $\sum_{\sigma\in\mathfrak{S}_n} \mathsf{sgn}(\sigma)c_{\boldsymbol{n}\boldsymbol{n}\sigma} \in A_\Bbbk(n, n)$, where here $\boldsymbol{n} = (1, 2, \ldots, n) \in I(n, n)$, and so $\mathsf{det} \in M_\Bbbk(n, n)$.

Note that given $V \in M_{\Bbbk}(n)$, the $\mathbb{N}$-grading on $A_{\Bbbk}(n)$ induces a G-module decomposition of V of the form $V = \bigoplus_{r\in\mathbb{N}} V^{(r)}$ where each $V^{(r)} \in M_{\Bbbk}(n,r)$ [G, Theorem (2.2c)]. Hence, in particular, indecomposable polynomial G-modules are necessarily homogenous. Since polynomial G-modules are by definition finite-dimensional, for a given non-zero $V \in M_{\Bbbk}(n)$, we have that $V^{(r)} \neq 0$ for only finitely many $r \in \mathbb{N}$. For $r \in \mathbb{N}$, we refer to the submodule $V^{(r)}$ as the *component of V in degree r*. Note that, given $V, W \in M_{\Bbbk}(n)$ and a G-homomorphism $V \xrightarrow{h} W$, then for $r \in \mathbb{N}$, the restriction of h to $V^{(r)}$ gives a G-homomorphism $V^{(r)} \xrightarrow{h^{(r)}} W^{(r)}$, which we call the *component of h in degree r*. Observe that if $r, s \in \mathbb{N}$ with $V \in M_{\Bbbk}(n,r)$, $W \in M_{\Bbbk}(n,s)$, then the tensor product $V \otimes W$ satisfies $V \otimes W \in M_{\Bbbk}(n, r+s)$.

Remark 2.4.4. Fix $r \in \mathbb{N}$. Then, recall from Examples 2.4.3(ii) that the r-fold tensor power $E^{\otimes r}$ is homogenous of degree r with $g \cdot e_{\mathscr{j}} = \sum_{\mathscr{i}\mathscr{j}} g_{\mathscr{i}\mathscr{j}} e_{\mathscr{i}}$ for $g \in G$, $\mathscr{j} \in I(n,r)$. Now, for $\sigma \in \mathfrak{S}_r$, $g \in G$, $\mathscr{j} \in I(n,r)$, note that:

$$\begin{aligned}(g \cdot e_{\mathscr{j}}) \cdot \sigma &= \sum_{\mathscr{i}\in I(n,r)} (g_{\mathscr{i}\mathscr{j}} e_{\mathscr{i}}) \cdot \sigma = \sum_{\mathscr{i}\in I(n,r)} g_{\mathscr{i}\mathscr{j}} e_{\mathscr{i}\sigma} \\ &= \sum_{\mathscr{i}\in I(n,r)} g_{(\mathscr{i}\sigma^{-1})\mathscr{j}} e_{\mathscr{i}} = \sum_{\mathscr{i}\in I(n,r)} g_{\mathscr{i}\mathscr{j}\sigma} e_{\mathscr{i}} = g \cdot (e_{\mathscr{j}}\sigma)\end{aligned} \tag{2.4.5}$$

From (2.4.5), we deduce that actions of G and $\mathfrak{S}_r$ on $E^{\otimes r}$ commute.

2.4.2. **Symmetric and Exterior Powers.** Firstly, recall that the $\Bbbk$-space $S^r E$ is the quotient of $E^{\otimes r}$ by the subspace $I^{(r)}$, where $I^{(r)}$ is $\Bbbk$-spanned by elements of the form $e_{\mathscr{i}} \cdot (1-\sigma)$ for $\mathscr{i} \in I(n,r)$ and $\sigma \in \mathfrak{S}_r$. Now, $I^{(r)}$ is a G-submodule of $E^{\otimes r}$ since the G-action and $\mathfrak{S}_r$-action on $E^{\otimes r}$ commute, and so the G-module structure on $E^{\otimes r}$ descends to a G-module structure on $S^r E$. Moreover, for $r, s \in \mathbb{N}$, it is clear to see that the components $\Pi_S^{(r,s)} : S^r E \otimes S^s E \to S^{r+s} E$ and $\Delta_S^{(r,s)} : S^{r+s} E \to S^r E \otimes S^s E$ are G-module homomorphisms, where the tensor product $S^r E \otimes S^s E$ is endowed with a G-module structure in the standard way. Note that, for $r \in \mathbb{N}$ the symmetric power $S^r E$ is homogenous of degree r under this G-module structure.

Now, fix $\mathscr{r} = (\mathscr{r}_1, \dots, \mathscr{r}_k) \in \mathbb{N}^k$ with $k > 1$ and write $\mathscr{r}' = (\mathscr{r}_1, \dots, \mathscr{r}_{k-1}) \in \mathbb{N}^{k-1}$ with $r \coloneqq \deg(\mathscr{r}) \in \mathbb{N}$ and $r' \coloneqq \deg(\mathscr{r}') \in \mathbb{N}$. We endow the $\Bbbk$-space $S^{\mathscr{r}} E \coloneqq \bigotimes_{i=1}^k S^{\mathscr{r}_i} E$ with a G-module structure through the diagonal action. Then, we write $\Pi_S^{\mathscr{r}} : S^{\mathscr{r}} E \to S^r E$ for the *$\mathscr{r}$-fold product*, which we define recursively as the composition:

$$S^{\mathscr{r}} E = S^{\mathscr{r}'} E \otimes S^{\mathscr{r}_k} E \xrightarrow{\Pi_S^{\mathscr{r}'} \otimes \mathbb{1}} S^{r'} E \otimes S^{\mathscr{r}_k} \xrightarrow{\Pi_S^{(r', \mathscr{r}_k)}} S^r E. \tag{2.4.6}$$

On the other hand, we write $\Delta_S^{\mathscr{r}} E : S^r E \to S^{\mathscr{r}} E$ for the *$\mathscr{r}$-fold coproduct*, which we once again define recursively as the composition:

$$S^r E \xrightarrow{\Delta_S^{(r', \mathscr{r}_k)}} S^{r'} E \otimes S^{\mathscr{r}_k} \xrightarrow{\Delta_S^{\mathscr{r}'} \otimes \mathbb{1}} S^{\mathscr{r}'} E \otimes S^{\mathscr{r}_k} E = S^{\mathscr{r}} E. \tag{2.4.7}$$

Then, the G-module $S^{\mathscr{r}} E$ is homogenous of degree r, and the $\mathscr{r}$-fold product $\Pi_S^{\mathscr{r}}$ and $\mathscr{r}$-fold coproduct $\Delta_S^{\mathscr{r}}$ are G-module homomorphisms.

Finally, we note that the constructions of this section performed for the symmetric powers transfer analogously to the exterior powers. In this case, however, note that

$\Lambda^r E = 0$ whenever $r > n$. So here, we restrict attention to sequences $\mathscr{r} \in \mathbb{N}^k$ of degree at most n. Then, for such $\mathscr{r}$ with $r := \deg(\mathscr{r})$, we write $\Lambda^{\mathscr{r}} E := \bigotimes_{i=1}^{k} \Lambda^{\mathscr{r}_i} E$ with the $\mathscr{r}$-fold product $\Pi^{\mathscr{r}}_{\Lambda} : \Lambda^{\mathscr{r}} E \to \Lambda^r E$ and $\mathscr{r}$-fold coproduct $\Delta^{\mathscr{r}}_{\Lambda} : \Lambda^r E \to \Lambda^{\mathscr{r}} E$ defined similarly. Then $\Lambda^{\mathscr{r}} E$ is homogenous of degree r, and the maps $\Pi^{\mathscr{r}}_{\Lambda}$, $\Delta^{\mathscr{r}}_{\Lambda}$ are G-module homomorphisms.

2.4.3. **Contravariant Duality.** Recall that if V is a G-module, then the $\Bbbk$-linear dual $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ has the structure of a G-module via $(g \cdot h)(v) = h(g^{-1} \cdot v)$ for $g \in G$, $h \in H$, and $v \in V$.

Remark 2.4.8. Recall that in Examples 2.4.3(iii), we saw that the 1-dimensional G-module $\mathsf{det} = \Bbbk \cdot 1_{\mathrm{det}}$ afforded by the determinant function is homogenous of degree n. Then, the $\Bbbk$-linear dual det^* of det is a one-dimensional G-module with $g \cdot 1^*_{\mathrm{det}} = \det(g)^{-1} 1^*_{\mathrm{det}}$ for $g \in G$. The reader may observe that det^* is precisely the module that was observed not to be polynomial in Example 2.4.2.

Now, following the discussion in Remark 2.4.8, we see that the polynomial category $M_{\Bbbk}(n)$ is not closed under the (standard) duality. Instead, the polynomial category carries an alternate notion of duality called *contravariant duality*.

Given $g \in G$, we denote by $g^t \in G$ the transpose element of g. Then for $V \in M_{\Bbbk}(n)$, by the *contravariant dual* V° of V, we shall mean the $\Bbbk$-space $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ endowed with a G-module structure via $(g \cdot h)(v) := h(g^t \cdot v)$ for $g \in G$, $h \in \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$, and $v \in V$. Now, since $c_{ij}(g^t) = c_{ji}(g)$ for $g \in G$, $i, j \in [n]$, it is clear that $V^\circ \in M_{\Bbbk}(n)$. Moreover, if $V \in M_{\Bbbk}(n, r)$, then we also have that $V^\circ \in M_{\Bbbk}(n, r)$. If $V \xrightarrow{h} W$ is a G-module homomorphism for $V, W \in M_{\Bbbk}(n, r)$, then the dual map $h^\circ := h^*$ provides a G-module homomorphism $W^\circ \xrightarrow{h^\circ} V^\circ$.

Remark 2.4.9. Let us consider the contravariant dual E° of the natural module E. Here, E° has a $\Bbbk$-basis $\{e_i^* \mid i \in [n]\}$ dual to the standard basis of E. Then, for $g \in G$, $j, k \in [n]$, we have that $(g \cdot e_k^*)(e_j) = e_k^*(g^t \cdot e_j) = e_k^* \left(\sum_i g^t_{ij} e_i\right) = g^t_{kj} = g_{jk}$, and so $g \cdot e_k^* = \sum_j g_{jk} e_j^*$ for $g \in G$, $k \in [n]$. Hence, we deduce that the $\Bbbk$-linear map $E^\circ \to E$ with $e_k^* \mapsto e_k$ is a G-module isomorphism.

For an alternate perspective on contravariant duality, and in particular Remark 2.4.9, the reader may wish to consult [G, §2.7].

Remark 2.4.10. Let $r \in \mathbb{N}$. Recall that, as a $\Bbbk$-space, the r^{th}-divided power $D^r E$ is given by the dual of $S^r(E^*)^*$ of the r^{th}-symmetric power $S^r(E^*)$ of the dual E^* of E. Accordingly, we endow the r^{th}-divided power $D^r E$ with a G-module structure by identifying it with $S^r(E^\circ)^\circ$. Note that since E is self-dual under the contravariant dual, we have that $D^r E \cong S^r(E^\circ)^\circ \cong (S^r E)^\circ$. Note that $D^r E$ is homogenous of degree r. Once again, for $\mathscr{r} \in \mathbb{N}^k$ with $r := \deg(\mathscr{r}) \in \mathbb{N}$, we write $D^{\mathscr{r}} E := \bigotimes_{i=1}^{k} D^{\mathscr{r}_i} E$ with the $\mathscr{r}$-fold product $\Pi^{\mathscr{r}}_D : D^{\mathscr{r}} E \to D^r E$ and $\mathscr{r}$-fold coproduct $\Delta^{\mathscr{r}}_D : D^r E \to D^{\mathscr{r}} E$ defined as in §2.4.2. Finally, after endowing the tensor product $D^{\mathscr{r}} E$ with a G-module structure in the standard way, we see that $D^{\mathscr{r}} E$ is homogeneous of degree r and the maps $\Pi^{\mathscr{r}}_D$, $\Delta^{\mathscr{r}}_D$ are G-module homomorphisms.

2.4.4. **Weight Spaces.** Now, we denote by $T \leq G$ the maximal torus of G consisting of the diagonal matrices in G. Recall that the *character group* $X(T) \coloneqq \mathrm{Hom}(T, \Bbbk^\times)$ of T is identified with $\mathbb{Z}^n$, where a sequence $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ is identified with $\mathrm{diag}(t_1, \dots, t_n) \overset{\alpha}{\mapsto} \prod_{k=1}^n t_k^{\alpha_k}$. Now, let V be a rational G-module. Then, for $\alpha \in \mathbb{Z}^n$, we write V^α for the α-*weight space* of V, that is to say, the $\Bbbk$-subspace of V given by $V^\alpha = \{v \in V \mid t \cdot v = \alpha(t)v \text{ for } t \in T\}$. Note that if V is polynomial, then the weight-space V^α may only be non-zero if $\alpha \in \Lambda(n) = \mathbb{N}^n$, whilst if V is homogenous of degree r, then V^α may only be non-zero if $\alpha \in \Lambda(n,r)$. Recall that for $V \in M_\Bbbk(n,r)$, we have the $\Bbbk$-linear *weight-space decomposition* $V = \bigoplus_{\alpha \in \Lambda(n,r)} V^\alpha$ [Spr, Theorem 3.2.3].

2.4.5. **Induced Modules.** Now, we denote by $B \leq G$ the subgroup consisting of the the lower-triangular matrices in G, with $U \leq B$ the subgroup of lower-uni-triangular matrices. Recall that U is a unipotent group and that we have the semi-direct product decomposition $B = T \ltimes U$. Then, for $\alpha \in \Lambda(n)$, we denote by $\Bbbk_\alpha$ the one-dimensional rational T-module on which $t \in T$ acts by multiplication by $\alpha(t)$. Note that since $B = T \ltimes U$, we may endow $\Bbbk_\alpha$ with the structure of a rational B-module by letting U act trivially. Then, for $\lambda \in \Lambda^+(n,r)$, by the *induced module* (associated to λ), we mean the G-module $\nabla_\Bbbk(\lambda) \coloneqq \mathrm{ind}_B^G \Bbbk_\lambda$ [Jan, §II.2.1] (note that here, the author writes $H^0(\lambda)$ in place of $\nabla_\Bbbk(\lambda)$). For details on the induction functor ind_B^G, the reader may consult [Jan, §I.3.3].

Note that in characteristic zero, the induced modules labelled by $\Lambda^+(n,r)$ are precisely the irreducible modules in $M_\Bbbk(n,r)$:

Theorem 2.4.11 ([G, (4.7b)])**.** *Suppose that the field $\Bbbk$ is of characteristic zero. Then, the induced modules labelled by $\Lambda^+(n,r)$ form a complete set of representatives of the isomorphism classes of irreducible homogenous degree r polynomial G-modules.*

Remark 2.4.12. Note that in [G, §4], Green constructs the module $D_{\lambda,\Bbbk}$ which is isomorphic to our induced module $\nabla_\Bbbk(\lambda)$. See [G, §4.8], [J$_1$, §27] for details.

Now, we review a construction of $\nabla_\Bbbk(\lambda)$ by Akin, Buchsbaum, and Weyman, which we refer to as the *ABW-construction* of $\nabla_\Bbbk(\lambda)$. In [ABW, §II.1], the authors associate to a partition λ with $\lambda_1 \leq n$ a G-module denoted $L_\lambda(E)$, which they call the *Schur functor of* E, and sometimes is referred to as the *Schur module associated to* λ. Further, in [ABW, §II.2], the authors provide a description of $L_\lambda(E)$ by generators and relations. More precisely, in [ABW, Theorem II.2.16], the authors identify $L_\lambda(E)$ with the cokernel of a G-homomorphism between a pair of (direct sums of) tensor products of exterior powers of E. By [D$_1$, §2.7(5)], we have that $L_\lambda(E)$ is isomorphic to an induced module, namely $L_\lambda(E) \cong \nabla_\Bbbk(\lambda')$ for partitions λ with $\lambda_1 \leq n$ (note that $Y(\lambda)$ is used in place of $\nabla_\Bbbk(\lambda)$ in [D$_1$]). The cokernel construction by Akin, Buchsbaum, and Weyman is as follows. Fix a partition λ with $\lambda_1 \leq n$, and write $\ell \coloneqq \mathrm{len}(\lambda)$. Then, for $1 \leq i < j \leq \ell$, $1 \leq s \leq \lambda_j$, we denote by $\phi_\lambda^{(i,j,s)} : \Lambda^{\lambda(i,j,s)} E \to \Lambda^\lambda E$ the G-homomorphism given by the composition:

$$\begin{aligned} &\Lambda^{\lambda(i,j,s)} E \xrightarrow{\mathbb{1} \otimes \cdots \otimes \Delta_\Lambda^{(\lambda_i, s)} \otimes \cdots \otimes \mathbb{1}} \Lambda^{\lambda_1} E \otimes \cdots \otimes \Lambda^{\lambda_i} E \otimes \Lambda^s E \otimes \cdots \otimes \Lambda^{\lambda_j - s} E \otimes \cdots \otimes \Lambda^{\lambda_\ell} E \\ &\xrightarrow{\sigma_\Lambda} \Lambda^{\lambda_1} E \otimes \cdots \otimes \Lambda^{\lambda_i} E \otimes \cdots \otimes \Lambda^s E \otimes \Lambda^{\lambda_j - s} E \otimes \cdots \otimes \Lambda^{\lambda_\ell} E \xrightarrow{\mathbb{1} \otimes \cdots \otimes \Pi_\Lambda^{(s, \lambda_j - s)} \otimes \cdots \otimes \mathbb{1}} \Lambda^\lambda E, \end{aligned} \tag{2.4.13}$$

where σ_Λ denotes the isomorphism that permutes the appropriate tensor factors, and each $\mathbb{1}$ denotes the identity map on the appropriate tensor factor. Now, set:

$$\begin{aligned} \phi_\lambda^{(i,i+1)} &:= \sum_{s=1}^{\lambda_{i+1}} \phi_\lambda^{(i,i+1,s)} : \sum_{s=1}^{\lambda_{i+1}} \Lambda^{\lambda^{(i,i+1,s)}} E \to \Lambda^\lambda E, \\ \phi_\lambda &:= \sum_{i=1}^{\ell-1} \phi_\lambda^{(i,i+1)} : \sum_{i=1}^{\ell-1} \sum_{s=1}^{\lambda_{i+1}} \Lambda^{\lambda^{(i,i+1,s)}} E \to \Lambda^\lambda E. \end{aligned} \tag{2.4.14}$$

Then, for $\lambda \in \Lambda^+(n)$, we have that $\operatorname{coker} \phi_{\lambda'} \cong L_{\lambda'}(E)$ [ABW, Theorem II.2.16], and hence $\operatorname{coker} \phi_{\lambda'} \cong \nabla_{\mathbb{k}}(\lambda)$ [D_1, §2.7(5)].

Now, we review an alternative description of $\nabla_{\mathbb{k}}(\lambda)$ due to James [J_1, §26], which we refer to as the *James-construction* of $\nabla_{\mathbb{k}}(\lambda)$. James' construction is as follows. Once again, let $\lambda \in \Lambda^+(n)$ with $\ell := \operatorname{len}(\lambda)$. Then, for $1 \leq i < j \leq \ell$, $1 \leq t \leq \lambda_j$, we construct the G-homomorphism $\psi_\lambda^{(i,j,t)} : S^\lambda E \to S^{\lambda^{(i,j,t)}} E$ as the composition:

$$\begin{aligned} &S^\lambda E \xrightarrow{\mathbb{1}\otimes\cdots\otimes\Delta_S^{(t,\lambda_j-t)}\otimes\cdots\otimes\mathbb{1}} S^{\lambda_1}E \otimes \cdots \otimes S^{\lambda_i}E \otimes \cdots \otimes S^t E \otimes S^{\lambda_j - t} E \otimes \cdots \otimes S^{\lambda_\ell} E \xrightarrow{\sigma_S} \\ &S^{\lambda_1}E \otimes \cdots \otimes S^{\lambda_i}E \otimes S^t E \otimes \cdots \otimes S^{\lambda_j - t} E \otimes \cdots \otimes S^{\lambda_\ell} E \xrightarrow{\mathbb{1}\otimes\cdots\otimes\Pi_S^{(\lambda_i,t)}\otimes\cdots\otimes\mathbb{1}} S^{\lambda^{(i,j,t)}} E, \end{aligned} \tag{2.4.15}$$

where σ_S denotes the isomorphism that permutes the appropriate tensor factors, and each $\mathbb{1}$ refers to the identity map on the appropriate tensor factor. Now, set:

$$\begin{aligned} \psi_\lambda^{(i,i+1)} &:= \sum_{t=1}^{\lambda_{i+1}} \psi_\lambda^{(i,i+1,t)} : S^\lambda E \to \sum_{t=1}^{\lambda_{i+1}} S^{\lambda^{(i,i+1,t)}} E, \\ \psi_\lambda &:= \sum_{i=1}^{\ell-1} \psi_\lambda^{(i,i+1)} : S^\lambda E \to \sum_{i=1}^{\ell-1} \sum_{t=1}^{\lambda_{i+1}} S^{\lambda^{(i,i+1,t)}} E. \end{aligned} \tag{2.4.16}$$

Then, for $\lambda \in \Lambda^+(n)$, we have that $\nabla_{\mathbb{k}}(\lambda) \cong \ker \psi_\lambda$ [J_1, Theorem 26.5].

Remark 2.4.17. Although James refers to the module $\ker \psi_\lambda$ as the *Weyl module*, it is not to be confused with the usual Weyl module $\Delta_{\mathbb{k}}(\lambda)$ that we introduce in the following section. See [G, §4.8] for details, and in particular [G, Theorem (4.8f)].

Examples 2.4.18. Fix $r \in \mathbb{Z}_{>0}$. Then:

(i) Let $\lambda = (1^r)$. Then, $\lambda' = (r)$ and so $\phi_{\lambda'} = 0$ since $\operatorname{len}(\lambda') = 1$. Hence, according to (2.4.14), we have $\nabla_{\mathbb{k}}(1^r) \cong \operatorname{codom} \phi_{\lambda'} = \Lambda^r E$.

(ii) On the other hand, let $\lambda = (r)$. Then, we have that $\psi_\lambda = 0$ since $\operatorname{len}(\lambda) = 1$. Hence, according to (2.4.16), we have $\nabla_{\mathbb{k}}(r) \cong \operatorname{dom} \psi_\lambda = S^r E$.

Remark 2.4.19. The proceeding Example shows the necessity of the condition that $\mathbb{k}$ has characteristic zero in Theorem 2.4.11.

Example 2.4.20. Suppose that $\mathbb{k}$ has characteristic 2, and let $r = 2$ with $\lambda = (2)$. Then, as in Examples 2.4.18(ii), we have that $\nabla_{\mathbb{k}}(2) = S^2 E$. Then, consider the $\mathbb{k}$-subspace U of $S^2 E$ spanned by elements of the form $\bar{e}_k^2$ for $k \in [n]$. Then, for $g \in G$, $k \in [n]$, we have that:

$$g \cdot \bar{e}_k^2 = \left(\sum_i g_{ik} \bar{e}_i \right)^2 = \sum_i g_{ik}^2 \bar{e}_i^2 + 2 \sum_{i<j} g_{ik} g_{jk} (\bar{e}_i \cdot \bar{e}_j) = \sum_i g_{ik}^2 \bar{e}_i^2 \in U,$$

and so U is a proper submodule of $\nabla_{\Bbbk}(2)$. Hence, $\nabla_{\Bbbk}(2)$ is reducible in characteristic 2.

2.4.6. **Weyl Modules.** For $\lambda \in \Lambda^+(n)$, we denote by $\Delta_{\Bbbk}(\lambda)$ the *Weyl module* (associated to λ). Note that for our purposes, we may identity the Weyl module $\Delta_{\Bbbk}(\lambda)$ as the contravariant dual $\nabla_{\Bbbk}(\lambda)^{\circ}$ of the induced module $\nabla_{\Bbbk}(\lambda)$ [G, §5.1], [Jan, §II.8.17]. For a definition of the Weyl module $\Delta_{\Bbbk}(\lambda)$ in a more general setting, consult [Jan, §II.2.13(1)].

Firstly, in light of the examples given in Examples 2.4.18 and the isomorphism $\Delta_{\Bbbk}(\lambda) \cong \nabla_{\Bbbk}(\lambda)^{\circ}$, we have the following:

Examples 2.4.21. Let $r \in \mathbb{Z}_{>0}$. Then:

(i) Let $\lambda = (1^r)$. Then, $\nabla_{\Bbbk}(1^r) \cong \Lambda^r E$ by Examples 2.4.18(i), and so it follows that $\Delta_{\Bbbk}(1^r) \cong \nabla_{\Bbbk}(1^r)^{\circ} \cong (\Lambda^r E)^{\circ} \cong \Lambda^r E$ since the r^{th}-exterior power $\Lambda^r E$ is self-dual under contravariant duality.

(ii) On the other hand, let $\lambda = (r)$. Then, $\nabla_{\Bbbk}(r) \cong S^r E$ by Examples 2.4.18(ii), and so $\Delta_{\Bbbk}(r) \cong \nabla_{\Bbbk}(r)^{\circ} \cong (S^r E)^{\circ} \cong D^r E$ since the r^{th}-divided power $D^r E$ is dual to the r^{th}-symmetric power $S^r E$ under contravariant duality.

Next, we have the analogue of Theorem 2.4.11 for Weyl modules:

Theorem 2.4.22. *Suppose that the field $\Bbbk$ is of characteristic zero. Then, the Weyl modules labelled by $\Lambda^+(n,r)$ form a complete set of representatives of the isomorphism classes of irreducible homogenous degree r polynomial G-modules.*

Remark 2.4.23. Once again, the condition that $\Bbbk$ has characteristic zero in Theorem 2.4.22 is necessary, as the proceeding Example shows.

Example 2.4.24. Suppose that $\Bbbk$ has characteristic 2 and let $\lambda = (2)$. Then, according to Examples 2.4.21(ii), we have that $\Delta_{\Bbbk}(2) \cong D^2 E$. Now, let $g \in G$, $j, l \in [n]$ with $j \neq l$. Then:

$$\begin{aligned} g \cdot (e_j^{(1)} \cdot e_l^{(1)}) &= \left(\sum_i g_{ij} e_i^{(1)} \right) \cdot \left(\sum_k g_{kl} e_k^{(1)} \right) \\ &= \sum_{i<k} (g_{ij} g_{kl} + g_{il} g_{kj}) \left(e_i^{(1)} \cdot e_k^{(1)} \right) + \sum_i g_{ij} g_{il} \left(e_i^{(1)} \cdot e_i^{(1)} \right) \qquad (2.4.25) \\ &= \sum_{i<k} (g_{ij} g_{kl} + g_{il} g_{kj}) \left(e_i^{(1)} \cdot e_k^{(1)} \right) + 2 \sum_i g_{ij} g_{il} e_i^{(2)} \in D^2 E. \end{aligned}$$

Hence, by (2.4.25), we have that the $\Bbbk$-span W of elements of the form $e_j^{(1)} \cdot e_l^{(1)} \in D^2 E$ for $j, l \in [n]$ with $k \neq l$ forms a proper G-submodule of $D^2 E$. Hence, in characteristic 2, the Weyl module $\Delta_{\Bbbk}(2)$ is reducible.

Remark 2.4.26. Here, we observe that Example 2.4.24 may be obtained from Example 2.4.20 via contravariant duality. Indeed, suppose that $\Bbbk$ has characteristic 2. Then, recall the (proper) G-module inclusion $U \hookrightarrow S^2 E \cong \nabla_{\Bbbk}(2)$ given in Example 2.4.20. By taking contravariant duals of the quotient map $S^2E \overset{\pi}{\twoheadrightarrow} S^2E/U =: \bar{U} \neq 0$, we receive a (proper) G-module embedding $\pi^\circ : \bar{U}^\circ \hookrightarrow (S^2E)^\circ \cong D^2E$. It is clear to see that the image of $\bar{U}^\circ$ in D^2E is precisely the (proper) G-submodule $W \leq D^2E \cong \Delta_{\Bbbk}(2)$ described in Example 2.4.24.

In the remainder of this section, we review a construction of the Weyl module $\Delta_{\Bbbk}(\lambda)$ by Akin, Buchsbaum, and Weyman [ABW, §II.3]. Let $\lambda \in \Lambda^+(n)$ and write $\ell := \mathrm{len}(\lambda)$. Then, in a similar manner to that of (2.4.14), for $1 \leq i < j \leq \ell$ and $1 \leq t \leq \lambda_j$, denote by $\theta_\lambda^{(i,j,t)} : D^{\lambda(i,j,t)} \to D^\lambda E$ the G-module homomorphism given by the composition:

$$\begin{aligned} &D^{\lambda(i,j,t)}E \xrightarrow{\mathbb{1}\otimes\cdots\otimes\Delta_D^{(\lambda_i,t)}\otimes\cdots\otimes\mathbb{1}} D^{\lambda_1}E\otimes\cdots\otimes D^{\lambda_i}E\otimes D^tE\otimes\cdots\otimes D^{\lambda_j-t}E\otimes\cdots\otimes D^{\lambda_\ell}E \\ &\xrightarrow{\sigma_D} D^{\lambda_1}E\otimes\cdots\otimes D^{\lambda_i}E\otimes\cdots\otimes D^tE\otimes D^{\lambda_j-t}E\otimes\cdots\otimes D^{\lambda_\ell}E \xrightarrow{\mathbb{1}\otimes\cdots\otimes\Pi_D^{(t,\lambda_j-t)}\otimes\cdots\otimes\mathbb{1}} D^\lambda E, \end{aligned} \tag{2.4.27}$$

where σ_D denotes the isomorphism that permutes the appropriate tensor factors, and each $\mathbb{1}$ denotes the identity map on the appropriate tensor factor. Then, similarly to (2.4.14), we set:

$$\begin{aligned} \theta_\lambda^{(i,i+1)} &:= \sum_{t=1}^{\lambda_{i+1}} \theta_\lambda^{(i,i+1,t)} : \sum_{t=1}^{\lambda_{i+1}} D^{\lambda(i,i+1,t)}E \to D^\lambda E, \\ \theta_\lambda &:= \sum_{i=1}^{\ell-1} \theta_\lambda^{(i,i+1)} : \sum_{i=1}^{\ell-1}\sum_{t=1}^{\lambda_{i+1}} D^{\lambda(i,i+1,t)}E \to D^\lambda E. \end{aligned} \tag{2.4.28}$$

For $\lambda \in \Lambda^+(n)$, we have that $\Delta_{\Bbbk}(\lambda) \cong \operatorname{coker}\theta_\lambda$ [ABW, Theorem II.3.16]. Now, recall that $\Delta_{\Bbbk}(\lambda)^\circ \cong \nabla_{\Bbbk}(\lambda)$ and that $(D^\alpha E)^\circ \cong S^\alpha E$ for $\alpha \in \Lambda(n)$. By taking contravariant duals, it follows that $\nabla_{\Bbbk}(\lambda) \cong \ker{\theta_\lambda}^\circ$ and it is easy to check that we have the identifications ${\theta_\lambda}^\circ = \psi_\lambda$ and $\theta_\lambda^{(i,j,t)\circ} = \psi_\lambda^{(i,j,t)}$ for $1 \leq i < j \leq \ell$, $1 \leq t \leq \lambda_j$.

Remark 2.4.29. Note that one may now recover Examples 2.4.21(ii) without use of contravariant duality. Indeed, when $\lambda = (r)$, we have $\theta_\lambda = 0$ since $\mathrm{len}(\lambda) = 1$. Hence, according to (2.4.28), we have that $\Delta_{\Bbbk}(r) \cong \operatorname{codom}\theta_\lambda = D^rE$.

2.5. The Schur Algebra

2.5.1. **The Schur Coalgebra.** Now, recall that the polynomial algebra $A_{\Bbbk}(n)$ has a $\mathbb{N}$-grading of the form $A_{\Bbbk}(n) = \bigoplus_{r\in\mathbb{N}} A_{\Bbbk}(n,r)$, where $A_{\Bbbk}(n,r)$ is spanned by the monomials $c_{\imath\jmath}$ for $\imath,\jmath \in I(n,r)$. Note that the restriction Δ of Δ_G to $A_{\Bbbk}(n,r)$ endows $A_{\Bbbk}(n,r)$ with the structure of a $\Bbbk$-coalgebra, which we call the *Schur coalgebra (of degree r)*. Explicitly, for $\imath,\ell \in I(n,r)$, we have that $\Delta : c_{\imath k} \mapsto \sum_{\jmath\in I(n,r)} c_{\imath\jmath}\otimes c_{\jmath k}$.

Remark 2.5.1. It follows from Remark 2.4.1 that the (G,G)-bimodule structure on $\Bbbk[G]$ is determined by the comultiplication $\Delta_G : \Bbbk[G] \to \Bbbk[G]\otimes\Bbbk[G]$. Now, since $A_{\Bbbk}(n,r)$ is closed under the restriction Δ of Δ_G, it follows that $A_{\Bbbk}(n,r)$ is a (G,G)-submodule of $\Bbbk[G]$. In particular, for $\imath, k \in I(n,r)$, we have that $g \in G$ acts on $c_{\imath k}$ from the left by $g\cdot c_{\imath k} = \sum_{\jmath\in I(n,r)} c_{\jmath k}(g)c_{\imath\jmath}$, and from the right by $c_{\imath k}\cdot g = \sum_{\jmath\in I(n,r)} c_{\imath\jmath}(g)c_{\jmath k}$.

Now, note that $c_{\imath\jmath} = c_{k\ell}$ if and only if $(\imath,\jmath) \sim (k,\ell)$ where the equivalence relation on $I(n,r)\times I(n,r)$ is determined by the diagonal action of $\mathfrak{S}_r$. Accordingly, $A_{\Bbbk}(n,r)$ has a $\Bbbk$-basis parametrised by the $\mathfrak{S}_r$-orbits in $I(n,r)\times I(n,r)$.

2.5.2. **The Schur Algebra.** Now, by the *Schur algebra (of degree r)* for G, we shall mean the $\Bbbk$-linear dual $S_\Bbbk(n,r) := \mathrm{Hom}_\Bbbk(A_\Bbbk(n,r),\Bbbk)$ of the Schur coalgebra. Recall that $A_\Bbbk(n,r)$ has a $\Bbbk$-basis parametrised by $\mathfrak{S}_r$-orbits in $I(n,r)\times I(n,r)$. For $\imath,\jmath \in I(n,r)$, we write $\xi_{\imath\jmath} \in S_\Bbbk(n,r)$ for the element dual to the basis element of $A_\Bbbk(n,r)$ labelled by the $\mathfrak{S}_r$-orbit of $(\imath,\jmath)$ in $I(n,r)\times I(n,r)$. Explicitly, we have:

$$\xi_{\imath\jmath}(c_{\mathscr{s}\mathscr{t}}) = \begin{cases} 1, & \text{if } (\mathscr{s},\mathscr{t}) \sim (\imath,\jmath), \\ 0, & \text{if } (\mathscr{s},\mathscr{t}) \not\sim (\imath,\jmath), \end{cases}, \tag{2.5.2}$$

for $\mathscr{s},\mathscr{t} \in I(n,r)$. Clearly, for $\imath,\jmath,\mathscr{k},\ell \in I(n,r)$, we have that $\xi_{\imath\jmath} = \xi_{\mathscr{k}\ell}$ if and only if $(\imath,\jmath)\sim(\mathscr{k},\ell)$.

Now, recall that if A is a $\Bbbk$-coalgebra with comultiplication Δ_A, then the dual map Δ_A^* endows the dual space $A^* = \mathrm{Hom}_\Bbbk(A,\Bbbk)$ with the structure of $\Bbbk$-algebra [K, Proposition III.1.2]. Accordingly, the Schur algebra $S_\Bbbk(n,r)$ is endowed with the structure of a $\Bbbk$-algebra whose product Π is given by the $\Bbbk$-linear dual of Δ. Explicitly, the product $\Pi : S_\Bbbk(n,r)\otimes S_\Bbbk(n,r) \to S_\Bbbk(n,r)$ is given by $\Pi(\xi\otimes\xi')(c_{\mathscr{r}\mathscr{t}}) = \sum_{\mathscr{s}\in I(n,r)} \xi(c_{\mathscr{r}\mathscr{s}})\otimes\xi'(c_{\mathscr{s}\mathscr{t}})$ for $\mathscr{r},\mathscr{t} \in I(n,r)$. Henceforth, for $\xi,\xi' \in S_\Bbbk(n,r)$, we write $\xi\cdot\xi' := \Pi(\xi\otimes\xi')$.

For details on the product Π of the Schur algebra, consult [G, §2.3]. For our purposes, it will suffice to observe the following:

Lemma 2.5.3 ([G, (2.3c)]). *Let $n,r\in\mathbb{N}$. Then, the product in $S_\Bbbk(n,r)$ satisfies:*

(i) *For $\imath,\jmath,\mathscr{k},\ell \in I(n,r)$, we have that $\xi_{\imath\jmath}\cdot\xi_{\mathscr{k},\ell} = 0$ unless $\jmath\sim\mathscr{k}$.*

(ii) *For $\imath,\jmath \in I(n,r)$, we have that $\xi_{\imath\imath}\cdot\xi_{\imath\jmath} = \xi_{\imath\jmath} = \xi_{\imath\jmath}\cdot\xi_{\jmath\jmath}$.*

In particular, it follows from Lemma 2.5.3(ii) that we have that the elements of the form $\xi_{\imath\imath} \in S_\Bbbk(n,r)$ for $\imath \in I(n,r)$ are idempotents of $S_\Bbbk(n,r)$. Henceforth, for $\alpha\in\Lambda(n,r)$, we denote by $\xi_\alpha \in S_\Bbbk(n,r)$ the idempotent $\xi_{\imath\imath}$ where $\imath$ is arbitrary with $\imath\in\alpha$. Then, it follows from Lemma 2.5.3 that $\sum_{\alpha\in\Lambda(n,r)}\xi_\alpha = 1_S$ [G, (2.3d)], where 1_S denotes the multiplicative identity in $S_\Bbbk(n,r)$.

2.5.3. **Connection with the General Linear Groups.** In this section, we review the connection between the Schur algebra $S_\Bbbk(n,r)$ and the category $M_\Bbbk(n,r)$ of degree r homogenous G-modules. Firstly, for $g\in G$, we write $A_\Bbbk(n,r) \xrightarrow{\mathrm{ev}_g} \Bbbk$ for the element of $S_\Bbbk(n,r)$ with $h \xmapsto{\mathrm{ev}_g} h(g)$. Then, the $\Bbbk$-algebra homomorphism $\Bbbk G \xrightarrow{\mathrm{ev}} S_\Bbbk(n,r)$ determined by $g \xmapsto{\mathrm{ev}} \mathrm{ev}_g$ is surjective [G, Proposition (2.4b)(i)]. Now, let $V\in S_\Bbbk(n,r)$–mod. Then, we endow V with the structure of a left-G-module via $g\cdot v := \mathrm{ev}_g\cdot v$ for $g\in G$, $v\in V$. Under this identification, we have that $V\in M_\Bbbk(n,r)$. Moreover, this association establishes an equivalence of categories between $M_\Bbbk(n,r)$ and $S_\Bbbk(n,r)$–mod [G, §2.4]. In particular, recall that $E^{\otimes r}$ has a G-module structure via $g\cdot e_\jmath = \sum_{\imath\in I(n,r)} c_{\imath\jmath}(g)e_\imath$ for $g\in G$, $\jmath\in I(n,r)$. Thus, the corresponding $S_\Bbbk(n,r)$-module structure on $E^{\otimes r}$ is determined via $\xi\cdot e_\jmath = \sum_{\imath\in I(n,r)}\xi(c_{\imath\jmath})e_\imath$ for $\xi\in S_\Bbbk(n,r)$, $\jmath\in I(n,r)$.

For $V\in M_\Bbbk(n,r)$ and $\alpha\in\Lambda(n,r)$, we have the following characterisation of the weight-space V^α in terms of the $S_\Bbbk(n,r)$-module structure on V.

Lemma 2.5.4 ([G, §3.2]). *Let $V\in M_\Bbbk(n,r)$ and $\alpha\in\Lambda(n,r)$. Then, the $\Bbbk$-space $\xi_\alpha\cdot V$ is precisely the α weight-space V^α of V.*

Finally, we shall need the following alternate description of the weight-space V^α.

Lemma 2.5.5 ([D$_3$, 2.1(8)])**.** *Let* $\alpha \in \Lambda(n,r)$*. Then for* $V \in M_{\Bbbk}(n,r)$*, we have a* $\Bbbk$*-linear isomorphism* $\mathrm{Hom}_G(V, S^\alpha E) \cong V^\alpha$*.*

Remark 2.5.6. Note that in [D$_3$], Donkin is working in the more general context of the *q-Schur algebra*, where the $\Bbbk$-space V^α is given by $\xi_\alpha \cdot V$ **by definition**. Thus, in order to apply Lemma 2.5.5 to our purposes, we must make use of Lemma 2.5.4.

2.6. Representations of the Symmetric Groups

In this section, we introduce the prerequisite notation relating to the representation theory of the symmetric groups. For the most part, we follow the conventions established within [J$_1$]. Note that in [J$_1$], James works with the opposite algebra $(\Bbbk\mathfrak{S}_r)^{\mathrm{op}}$ to our symmetric group algebra. Accordingly, James constructs his modules as right-modules for $(\Bbbk\mathfrak{S}_r)^{\mathrm{op}}$, whereas we construct the equivalent left-modules for $\Bbbk\mathfrak{S}_r$.

2.6.1. **Tableaux and Tabloids.** For a partition λ, by the *Young diagram* of λ, we mean the set $[\lambda] := \{(i,j) \mid i \in [\mathrm{len}(\lambda)], j \in [\lambda_i]\}$. We refer to the elements of $[\lambda]$ as its nodes. We identify the Young diagram $[\lambda]$ with the diagram formed by placing a unit square centred at each node in $[\lambda]$. When we do so, we shall follow the *English convention* whereby each node $x = (i,j)$ is placed i units southward and j units eastward. For example, if $\lambda = (3,1)$, then the Young diagram $[\lambda]$ of λ is identified with the diagram:

$$[(3,1)] = \{(1,1),(1,2),(1,3),(2,1)\} \leftrightarrow \begin{array}{|c|c|c|} \hline (1,1) & (1,2) & (1,3) \\ \hline (2,1) & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}\,, \tag{2.6.1}$$

where here, for the sake of clarity, we have decorated each box in the diagram and each corresponding node in $[\lambda]$ with matching colours, along with superimposing the coordinate of each box.

Now, let S be a non-empty set, with $\lambda \in \mathbb{N}^n$. Then, by an *S-valued λ-tableau* (plural *λ-tableaux*), we mean a function $[\lambda] \xrightarrow{\mathsf{t}} S$. We refer to the sequence λ as the *shape* of t. We typically represent the data of t by superimposing the diagram $[\lambda]$ with the values of t. For instance, the following is an example of a $[5]$-valued $(3,2)$-tableau:

$$\mathsf{t} = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}\,. \tag{2.6.2}$$

For λ a partition of r, we denote by T_λ the set of all bijective $[r]$-valued λ-tableaux. By the *row-canonical* (resp. *column-canonical*) λ-tableau, denoted t_λ (resp. t^λ), we mean the λ-tableau obtained by filling $[\lambda]$ with the numbers from $[r]$ in sequence, left to right (resp. top to bottom) and top to bottom (resp. left to right). For instance, we have:

$$\mathsf{t}_{(2,2,1)} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \qquad \mathsf{t}^{(2,2,1)} = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,. \tag{2.6.3}$$

The left-action of $\mathfrak{S}_r$ on $[r]$ induces a left-action of $\mathfrak{S}_r$ on T_λ that is *regular*, which is to say that it is equivalent (in the sense of group-actions) to the left-regular group-action of $\mathfrak{S}_r$ on itself by left-multiplication. Then, for $\mathsf{t} \in \mathsf{T}_\lambda$, by the *row-stabiliser* (resp. *column-stabiliser*) of T, denoted $R(\mathsf{t})$ (resp. $C(\mathsf{t})$), we mean the subgroup of $\mathfrak{S}_r$ consisting of all elements that preserve, in the set-wise sense, each row (resp. column) of t. For instance, for the example tableau t given in (2.6.2), we have:

$$R(\mathsf{t}) = \mathrm{Sym}(\{1,2,4\}) \times \mathrm{Sym}(\{3,5\}), \qquad C(\mathsf{t}) = \mathrm{Sym}(\{1,3\}) \times \mathrm{Sym}(\{2,5\}).$$

Now, given $\mathsf{s}, \mathsf{t} \in \mathsf{T}_\lambda$, write $\mathsf{s} \sim_R \mathsf{t}$ (resp. $\mathsf{s} \sim_C \mathsf{t}$) if $\mathsf{s} = \sigma \cdot \mathsf{t}$ for some $\sigma \in R(\mathsf{t})$ (resp. $\sigma \in C(\mathsf{t})$). The relations $\sim_R$ and $\sim_C$ both define equivalence relations on T_λ, and for $\mathsf{t} \in \mathsf{T}_\lambda$, we denote by $\overline{\underline{\mathsf{t}}}$ (resp. $|\mathsf{t}|$) the equivalence class of t under $\sim_R$ (resp. $\sim_C$). By a *λ-row-tabloid* (resp. *λ-column-tabloid*), we mean a subset of T_λ of the form $\overline{\underline{\mathsf{t}}}$ (resp. $|\mathsf{t}|$) for some $\mathsf{t} \in \mathsf{T}_\lambda$. Just as with λ-tableaux, we refer to the partition λ as the *shape* of a given λ-(row/column)-tabloid. When it is clear from context, we will omit the λ prefix. We can represent a row-tabloid (resp. column-tabloid) by omitting the vertical (resp. horizontal) lines from the diagram of any representative of the equivalence class. For example, given t as in (2.6.2), we have:

$$\overline{\underline{\mathsf{t}}} = \begin{array}{ccc} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \cline{1-2} \end{array}\ , \qquad\qquad |\mathsf{t}| = \begin{array}{|c|c|c|} 1 & 2 & 4 \\ 3 & 5 & \end{array}\ .$$

For our purposes, we will only require row-tabloids. Thus, we refer to λ-row-tabloids simply as λ-tabloids. We denote the set of all such λ-tabloids by $\overline{\underline{\mathsf{T}}}_\lambda$.

2.6.2. **Permutation Modules.** Now, fix a partition λ of r. Then, the left-action of $\mathfrak{S}_r$ on T_λ induces a left-action of $\mathfrak{S}_r$ on $\overline{\underline{\mathsf{T}}}_\lambda$. By the *permutation module (associated to λ)*, denoted $M_{\Bbbk}(\lambda)$, we mean the left $\Bbbk\mathfrak{S}_r$-module $\Bbbk\overline{\underline{\mathsf{T}}}_\lambda$ defined by the left-action $\mathfrak{S}_r \curvearrowright \overline{\underline{\mathsf{T}}}_\lambda$. Since the action of $\mathfrak{S}_r$ on $\overline{\underline{\mathsf{T}}}_\lambda$ is transitive, $M_{\Bbbk}(\lambda)$ is a cyclic $\Bbbk\mathfrak{S}_r$-module generated by any single tabloid $\overline{\underline{\mathsf{t}}} \in \overline{\underline{\mathsf{T}}}_\lambda$. Note that there is a $\Bbbk\mathfrak{S}_r$-linear surjection $\Bbbk\mathsf{T}_\lambda \twoheadrightarrow \Bbbk\overline{\underline{\mathsf{T}}}_\lambda$ defined by $\mathsf{t} \mapsto \overline{\underline{\mathsf{t}}} \in M_{\Bbbk}(\lambda)$ for $\mathsf{t} \in \mathsf{T}_\lambda$.

Note that for $\mathsf{t} \in \mathsf{T}_\lambda$, the stabiliser $\mathfrak{S}_{\overline{\underline{\mathsf{t}}}} := \mathrm{Stab}_{\mathfrak{S}_r}(\overline{\underline{\mathsf{t}}})$ is precisely the row-stabiliser $R(\mathsf{t})$ of t, and when t is the row-canonical λ-tableau t_λ, the subgroup $\mathfrak{S}_\lambda := \mathfrak{S}_{\overline{\underline{\mathsf{t}}}_\lambda} \leq \mathfrak{S}_r$ is the (standard) *Young subgroup* (associated to λ). Observe that we have the (internal) direct product decomposition:

$$\mathfrak{S}_\lambda = \prod_{j=1}^{\mathrm{len}(\lambda)} \mathrm{Sym}\left\{ k \in [r] \;\middle|\; \sum_{i<j} \lambda_i < k \leq \sum_{i \leq j} \lambda_i \right\} \leq \mathfrak{S}_r. \tag{2.6.4}$$

Now, let α be a composition of r. Then, taking inspiration from (2.6.4), we **define** the *Young subgroup* associated to α by:

$$\mathfrak{S}_\alpha = \prod_{j=1}^{\mathrm{len}(\alpha)} \mathrm{Sym}\left\{ k \in [r] \;\middle|\; \sum_{i<j} \alpha_i < k \leq \sum_{i \leq j} \alpha_i \right\} \leq \mathfrak{S}_r. \tag{2.6.5}$$

Here, we observe that if α is a partition, then the definition (2.6.5) agrees with the observation (2.6.4). Now, note that if β is a composition of r, then β may be obtained from α by reordering terms, if and only if the Young subgroup $\mathfrak{S}_\beta$ is conjugate to $\mathfrak{S}_\alpha$ in $\mathfrak{S}_r$. In particular, every Young subgroup of $\mathfrak{S}_r$ is conjugate to a unique standard Young subgroup. Explicitly, if μ is the unique partition of r that may be obtained from α by reordering terms, then $\mathfrak{S}_\alpha$ is conjugate to the standard Young subgroup $\mathfrak{S}_\mu$.

Now, by the Orbit-Stabiliser theorem, we have a set-bijection between the set $\underline{\mathsf{T}}_\lambda$ and the set of left-cosets of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_r$. Accordingly, $M_\Bbbk(\lambda)$ is isomorphic to the left $\Bbbk\mathfrak{S}_r$-module $\Bbbk\ \mathfrak{S}_r/\mathfrak{S}_\lambda$ defined by the left-action $\mathfrak{S}_r \curvearrowright (\mathfrak{S}_r/\mathfrak{S}_\lambda)$ given by left-multiplication. Note that there is an isomorphism $M_\Bbbk(\lambda) \cong \Bbbk\ \mathfrak{S}_r/\mathfrak{S}_\lambda \cong \operatorname{ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_r} \mathsf{triv}_r$ as representations for $\mathfrak{S}_r$. Then, if α is a composition of r, we **define** the permutation module $M_\Bbbk(\alpha)$ associated to α to be the induced module $\operatorname{ind}_{\mathfrak{S}_\alpha}^{\mathfrak{S}_r} \mathsf{triv}_r$. If β is a composition of r that may be obtained from α by reordering terms, then the Young subgroups $\mathfrak{S}_\alpha$ and $\mathfrak{S}_\beta$ are conjugate, and so the permutation modules $M_\Bbbk(\alpha)$ and $M_\Bbbk(\beta)$ are isomorphic. In particular, if μ is the unique partition of r that may be obtained from α by reordering terms, then $M_\Bbbk(\alpha) \cong M_\Bbbk(\mu)$. Note that, since duality commutes with induction, permutation modules are self-dual.

Example 2.6.6. When $\lambda = (r)$, we have $\mathfrak{S}_{(r)} = \mathfrak{S}_r$, and so the permutation module $M_\Bbbk(r)$ is the trivial module. On the other hand, when $\lambda = (1^r)$, the Young subgroup $\mathfrak{S}_{(1^r)}$ is the trivial group, and so $M_\Bbbk(1^r)$ is the left-regular module for $\Bbbk\mathfrak{S}_r$.

Let α be a composition of r, and write $\ell := \operatorname{len}(\alpha)$. Here, we introduce an alternate identification of the permutation module $M_\Bbbk(\alpha)$ that shall prove useful to the constructions within this thesis. Firstly, denote by S_α the set of ordered sequences $\boldsymbol{x} = (\boldsymbol{x}_1|\dots|\boldsymbol{x}_\ell)$ whose terms $\boldsymbol{x}_i = (x_{i1},\dots,x_{i\alpha_i})$ are unordered sequences with terms in $[r]$ with the property that for each $k \in [r]$, there exists a unique pair (i,j) in the range $1 \le i \le \operatorname{len}(\alpha)$, $1 \le j \le \alpha_i$ with the property that $x_{ij} = k$. Here, $\boldsymbol{x}_i$ denotes the empty sequence whenever $\alpha_i = 0$. Note that there is a left-action of $\mathfrak{S}_r$ on S_α determined by:

$$\sigma \cdot \boldsymbol{x} := (\sigma(x_{11}),\dots,\sigma(x_{1\alpha_1})|\cdots|\sigma(x_{\ell 1}),\dots,\sigma(x_{\ell\alpha_\ell}))\,,$$

for $\boldsymbol{x} \in \mathsf{S}_\alpha$ and $\sigma \in \mathfrak{S}_r$. If λ is a partition of r, there is a well-defined set-bijection $\mathsf{S}_\lambda \xrightarrow{\nu} \underline{\mathsf{T}}_\lambda$ such that the i^{th}-row of the λ-tabloid $\nu(\boldsymbol{x})$ is given by $\boldsymbol{x}_i$ for each i in the range $1 \le i \le \ell$. Moreover, since ν commutes with the left-action of $\mathfrak{S}_r$ on each of S_λ and $\underline{\mathsf{T}}_\lambda$, it follows that ν induces a $\Bbbk\mathfrak{S}_r$-linear isomorphism of the form $\Bbbk\mathsf{S}_\lambda \cong \Bbbk\underline{\mathsf{T}}_\lambda = M_\Bbbk(\lambda)$. Accordingly, one may identify the permutation module $M_\Bbbk(\lambda)$ as the $\Bbbk$-space defined on the $\Bbbk$-basis S_λ. Further, let α be a composition of r, and denote by μ the unique partition of r that may be obtained from α by reordering its terms. Then, the corresponding conjugation automorphism $\mathfrak{S}_r \xrightarrow{\cong} \mathfrak{S}_r$ mapping the Young subgroup $\mathfrak{S}_\mu$ to $\mathfrak{S}_\alpha$ induces a $\Bbbk\mathfrak{S}_r$-module isomorphism $M_\Bbbk(\mu) \cong \Bbbk\ \mathfrak{S}_r/\mathfrak{S}_\mu \to \Bbbk\ \mathfrak{S}_r/\mathfrak{S}_\alpha \cong M_\Bbbk(\alpha)$, and it is clear that this isomorphism maps the set S_μ onto the set S_α. Hence, once again, one may identify the permutation module $M_\Bbbk(\alpha)$ as the $\Bbbk$-space defined on the $\Bbbk$-basis S_α.

Finally, for α a composition of r, we denote by $M_{\Bbbk,\mathsf{sgn}}(\alpha)$ the *signed permutation module* (associated to α), that is $M_{\Bbbk,\mathsf{sgn}}(\alpha) := M_\Bbbk(\alpha) \otimes \mathsf{sgn}_r$. Note that in characteristic 2, the signed permutation module coincides with the standard permutation module.

2.6.3. **Polytabloids and Specht Modules.** Now, let $r \in \mathbb{N}$ with λ a partition of r. Then, for $\mathsf{t} \in \mathsf{T}_\lambda$, by the *polytabloid* associated to t, we mean the element of $M_\Bbbk(\lambda)$ given by $\{\mathsf{t}\} := \{C(\mathsf{t})\} \cdot \underline{\mathsf{t}} = \sum_{\sigma \in C(\mathsf{t})} \mathsf{sgn}(\sigma)(\sigma \cdot \underline{\mathsf{t}}) \in M_\Bbbk(\lambda)$.

Remark 2.6.7. Note that a polytabloid $\{\mathsf{t}\}$ depends not only on the equivalence class $\overline{\underline{\mathsf{t}}}$, but actually on the tableau t itself, as the following example demonstrates:

$$\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \right\} = \begin{array}{c}\hline 1\ \ 2 \\ \hline 3 \\ \hline\end{array} - \begin{array}{c}\hline 2\ \ 3 \\ \hline 1 \\ \hline\end{array} \neq \begin{array}{c}\hline 1\ \ 3 \\ \hline 2 \\ \hline\end{array} - \begin{array}{c}\hline 2\ \ 3 \\ \hline 1 \\ \hline\end{array} = \left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \right\}. \tag{2.6.8}$$

Remark 2.6.9. Let $r \in \mathbb{Z}_{>0}$, with λ a partition of r. Then, for $\mathsf{t} \in \mathsf{T}_\lambda$, it is clear to see that we have that $\sigma \cdot \{\mathsf{t}\} = \{\sigma \cdot \mathsf{t}\}$ for $\sigma \in \mathfrak{S}_r$. In particular, if $\sigma \in C(\mathsf{t})$, then $\{\sigma \cdot \mathsf{t}\} = \pm \{\mathsf{t}\}$.

It follows from Remark 2.6.9 that the $\Bbbk$-space $\mathrm{Sp}_\Bbbk(\lambda)$ spanned by the polytabloids $\{\mathsf{t}\}$ for $\mathsf{t} \in \mathsf{T}_\lambda$ is actually a submodule of $M_\Bbbk(\lambda)$, which we refer to as the *Specht module* (associated to λ). Once again, since the action of $\mathfrak{S}_r$ on T_λ is transitive, it follows that the Specht module $\mathrm{Sp}_\Bbbk(\lambda)$ is cyclic, generated by any single λ-polytabloid.

Example 2.6.10. When $\lambda = (r)$, since $M_\Bbbk(r)$ is already the trivial module, the same is true for $\mathrm{Sp}_\Bbbk(\lambda)$. On the other hand, when $\lambda = (1^r)$, we see that there is a single $\sim_C$-equivalence class in T_λ, and so it follows that $\mathrm{Sp}(1^r) \cong \mathsf{sgn}_r$.

2.6.4. **Standard Results.** In this section, we list some standard results from the literature relating to the representation theory of the symmetric groups. Here, we fix $r \in \mathbb{Z}_{>0}$ with λ a partition of r.

Firstly, we have the *standard basis theorem*, which describes a $\Bbbk$-basis for a Specht module $\mathrm{Sp}_\Bbbk(\lambda)$ parametrised by the *standard* λ-tableaux. Recall that we say that a λ-tableau $\mathsf{t} \in \mathsf{T}_\lambda$ is *standard* if the entries of t increase along rows and columns.

Theorem 2.6.11 ([J$_1$, Theorem 8.4])**.** *The set of λ-polytabloids $\{\mathsf{t}\} \in \mathrm{Sp}_\Bbbk(\lambda)$ corresponding to standard λ-tableaux $\mathsf{t} \in \mathsf{T}_\lambda$ forms a $\Bbbk$-basis for the Specht module $\mathrm{Sp}_\Bbbk(\lambda)$.*

Theorem 2.6.12 ([J$_1$, Theorem 4.12])**.** *Suppose that the field $\Bbbk$ is of characteristic zero. Then, the Specht modules labelled by partitions of r form a complete set of representatives of the isomorphism classes of irreducible $\Bbbk\mathfrak{S}_r$-modules.*

Remark 2.6.13. The Example proceeding this remark shows that the condition that $\Bbbk$ has characteristic zero in Theorem 2.6.12 is certainly necessary.

Example 2.6.14. Suppose that the field $\Bbbk$ has characteristic $p = 3$, and write $\lambda := (2, 1)$. Then, by a result of James [J$_1$, §24.4], we have that the Specht module $\mathrm{Sp}_\Bbbk(2, 1)$ contains a submodule isomorphic to the trivial module. Since the Specht module $\mathrm{Sp}_\Bbbk(2, 1)$ is clearly not itself the trivial module, we may immediately conclude that this Specht module is in fact reducible. Nevertheless, let us verify this fact directly. Note that according to Theorem 2.6.11, the Specht module $\mathrm{Sp}_\Bbbk(2, 1)$ has a $\Bbbk$-basis given by the two polytabloids stated in (2.6.8):

$$\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \right\} = \begin{array}{c}\hline 1\ \ 2 \\ \hline 3 \\ \hline\end{array} - \begin{array}{c}\hline 2\ \ 3 \\ \hline 1 \\ \hline\end{array}, \qquad \left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \right\} = \begin{array}{c}\hline 1\ \ 3 \\ \hline 2 \\ \hline\end{array} - \begin{array}{c}\hline 2\ \ 3 \\ \hline 1 \\ \hline\end{array}.$$

Then, since $-2 = 1 \in \Bbbk$, it follows that we have that:

$$v := \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \right\} + \left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \right\} = \begin{array}{c}\hline 1\ \ 2 \\ \hline 3 \\ \hline\end{array} + \begin{array}{c}\hline 1\ \ 3 \\ \hline 2 \\ \hline\end{array} + \begin{array}{c}\hline 2\ \ 3 \\ \hline 1 \\ \hline\end{array} = \left[\overline{\underline{\mathsf{T}}}_{(2,1)}\right] \in \mathrm{Sp}_\Bbbk(2, 1),$$

and so v spans a 1-dimensional submodule of $\mathrm{Sp}_{\Bbbk}(2,1)$ isomorphic to the trivial module. Meanwhile, notice that:

$$w := \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \right\} - \left\{ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \right\} = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \in \mathrm{Sp}_{\Bbbk}(2,1),$$

and so $(1\ 2)\cdot w = -w - v$, whilst $(2\ 3)\cdot w = -w$. It follows that the quotient of $\mathrm{Sp}_{\Bbbk}(2,1)$ by the submodule $\Bbbk \cdot v$ is isomorphic to the sign module $\mathtt{sgn}_3$ for $\Bbbk\mathfrak{S}_3$.

Now, we shall make use of the following result that characterises the dual of a Specht module:

Theorem 2.6.15 ([J$_1$, Theorem 8.15])**.** *Let λ be a partition of r. Then, we have a $\Bbbk\mathfrak{S}_r$-linear isomorphism of the form $\mathrm{Sp}_{\Bbbk}(\lambda)^* \cong \mathrm{Sp}_{\Bbbk}(\lambda') \otimes \mathtt{sgn}_r$.*

The following result of James is fundamental for addressing (in)decomposability of Specht modules. Recall that, for $e \in \mathbb{N}$ with $e > 1$, we say that a partition λ is *e-regular* if no term of λ is repeated e or more times. In particular, a partition λ is 2-regular if and only if it has distinct terms.

Theorem 2.6.16 ([J$_1$, Corollary 13.17])**.** *Suppose that either $\Bbbk$ is not of characteristic 2, or that $\Bbbk$ has characteristic 2 and λ is 2-regular. Then, the $\Bbbk$-space $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda), M(\lambda))$ is one-dimensional.*

In particular, it follows from Theorem 2.6.16 that $\mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}_{\Bbbk}(\lambda)) \cong \Bbbk$ for all such λ as in the statement of the Theorem. Thus, we arrive at:

Theorem 2.6.17 ([J$_1$, Corollary 13.18])**.** *Suppose that either $\Bbbk$ is not of characteristic 2, or that $\Bbbk$ has characteristic 2 and λ is 2-regular. Then, the Specht module $\mathrm{Sp}_{\Bbbk}(\lambda)$ is indecomposable.*

Remark 2.6.18. Now, as remarked in the introduction to this thesis, in [J$_1$, 23.10(iii)], James showed that $\mathrm{Sp}_{\Bbbk}(5,1^2)$ decomposes over fields of characteristic 2, thereby providing the first known example of a decomposable Specht module.

2.7. The Schur Functor

In this section we review the construction of the Schur functor f for the general linear group, along with some of its properties à la [G, §6]. We note that the functor f is a special case of a more general construction, that is to say, of a functor associated to an idempotent. With this in mind, we will proceed at this level of generality before specialising to our purposes.

2.7.1. **Schur Functor Associated to an Idempotent.** In the following, we fix a finite dimensional associative unital $\Bbbk$-algebra S with $\xi \in S$ a non-zero idempotent. Note that the $\Bbbk$-subspace $\xi S \xi \subseteq S$ of S has the structure of a $\Bbbk$-algebra whose product is inherited from that of S. However, it is important to note that $\xi S \xi$ is not a subalgebra of S (unless $\xi = 1_S$) since they have different multiplicative identities.

Let $V, W \in S$–mod with some S-linear map $V \xrightarrow{\phi} W$. Then, the $\Bbbk$-subspaces $\xi V \subseteq V$, $\xi W \subseteq W$ each have the structure of a left $\xi S \xi$-module, and the restriction $\phi|_{\xi V}$ gives a $\xi S\xi$-linear map $\phi|_{\xi V} : \xi V \to \xi W$. Accordingly, we may associate to ξ a functor S–mod $\xrightarrow{f_\xi} \xi S\xi$–mod, which we call *the Schur functor (associated to ξ).*

Meanwhile, note that the left S-module $S\xi$ has the structure of a right $\xi S\xi$-module. Moreover, for $W \in S\xi$–mod, the $\Bbbk$-space $S\xi \otimes_{\xi S\xi} W$ inherits the structure of a left S-module from that of S. Accordingly, we also associate to ξ a functor of the form $\xi S\xi$–mod $\xrightarrow{g_\xi}$ S–mod that is given on objects by $g_\xi W = S\xi \otimes_{\xi S\xi} W$, which we call the *$g$-functor (associated to ξ).* Note that this g-functor g_ξ is a right-inverse of f_ξ [G, Theorem (6.2d)].

Lemma 2.7.1. *The g-functor g_ξ is left-adjoint to the Schur functor f_ξ.*

Proof. Firstly, for $V, V' \in S$–mod, $v \in V$, write $\mathrm{ev}_v^{V,V'} : \mathrm{Hom}_S(V, W) \to V'$ for the $\Bbbk$-linear map with $(V \xrightarrow{\phi} V') \mapsto \phi(v) \in V'$. Note that, for $V \in S$–mod, the $\Bbbk$-linear map $\mathrm{ev}_\xi^V := \mathrm{ev}_\xi^{S\xi,V}$ is an embedding with image $\xi V \subseteq V$. Moreover, for such V, it is clear that the aforementioned $\Bbbk$-linear isomorphism $\xi V \cong \mathrm{Hom}_S(S\xi, V)$ is an isomorphism of $\xi S\xi$-modules. Thus, for $W \in \xi S\xi$–mod, $V \in S$–mod, we have a $\Bbbk$-linear isomorphism of the form $\mathrm{Hom}_{\xi S\xi}(W, f_\xi V) \cong \mathrm{Hom}_{\xi S\xi}(W, \mathrm{Hom}_S(S\xi, V))$. On the other hand, for such V, W, we have a $\Bbbk$-linear isomorphism $\mathrm{Hom}_{\xi S\xi}(W, \mathrm{Hom}_S(S\xi, V)) \cong \mathrm{Hom}_S(S\xi \otimes_{\xi S\xi} W, V)$ given by the *Hom-tensor adjunction*, and so we are done since $g_\xi W := S\xi \otimes_{\xi S\xi} W$. $\square$

Lemma 2.7.2. *Let S be a finite dimensional $\Bbbk$-algebra with $\xi \in S$ a non-zero idempotent. Then:*

(i) *The Schur functor $f_\xi : S$–mod $\to \xi S\xi$–mod is exact* [G, (6.2a)]

(ii) *The g-functor $g_\xi : \xi S\xi$–mod $\to S$–mod is right-exact.*

Finally, we have the main result describing how the Schur functor relates the categories S–mod and $\xi S\xi$–mod.

Theorem 2.7.3 ([G, Theorem (6.2g)])**.** *Suppose that $\{V_\lambda \mid \lambda \in \Lambda\}$ is a complete set of representatives of the isomorphism classes of irreducible modules in S–mod, where Λ denotes some parametrising set. Write $\Lambda^\xi := \{\lambda \in \Lambda \mid \xi V_\lambda \neq \{0\}\}$. Then, the set $\{f_\xi V_\lambda \mid \lambda \in \Lambda^\xi\}$ is a complete set of representatives of the isomorphism classes of irreducible modules in $\xi S\xi$–mod.*

2.7.2. **Connections to the Symmetric Groups.** Fix $n, r \in \mathbb{N}$ with $n \geq r$ and write $\mathscr{r} := (1, \ldots, r) \in I(n, r)$ with $\omega := \mathsf{c}(\mathscr{r}) = (1^r) \in \Lambda(n, r)$. Here, we specialise the results of this section to the case that $S = S_\Bbbk(n, r)$ is the Schur algebra, with the idempotent ξ given by $\xi_\omega = \xi_{\mathscr{r}\mathscr{r}}$.

Firstly, recall that $S_\Bbbk(\omega) := \xi_\omega S_\Bbbk(n, r)\xi_\omega$ is $\Bbbk$-spanned by elements of the form $\xi_{\mathscr{i}\mathscr{j}}$ with $\mathscr{i}, \mathscr{j} \in I(n, r)$. Moreover, since $\mathrm{Stab}_{\mathfrak{S}_r}(\mathscr{r}) = \{1\}$, it follows that the $\mathfrak{S}_r$-orbits in $\omega \times \omega \subseteq I(n, r) \times I(n, r)$ are parametrised by the set $\{(\mathscr{r}\sigma, \mathscr{r}) \mid \sigma \in \mathfrak{S}_r\}$. Thus, $S_\Bbbk(\omega)$ has a $\Bbbk$-basis given by $\xi_{(\mathscr{r}\sigma)\mathscr{r}}$ for $\sigma \in \mathfrak{S}_r$. Moreover, for $\sigma, \tau \in \mathfrak{S}_r$, we have that $(\xi_{(\mathscr{r}\sigma)\mathscr{r}} \cdot \xi_{(\mathscr{r}\tau)\mathscr{r}})(c_{\mathscr{i}\mathscr{k}}) = \delta_{\mathsf{c}(\mathscr{k}),\omega}\delta_{(\mathscr{r}\sigma),(\mathscr{i}\tau^{-1})}$ for $\mathscr{i}, \mathscr{k} \in I(n, r)$, and so it follows that we have $\xi_{(\mathscr{r}\sigma)\mathscr{r}} \cdot \xi_{(\mathscr{r}\tau)\mathscr{r}} = \sum_{\mathscr{i} \in I(n,r)} \delta_{(\mathscr{r}\sigma),(\mathscr{i}\tau^{-1})}\xi_{\mathscr{i}\mathscr{r}} = \xi_{(\mathscr{r}\sigma\tau)\mathscr{r}}$. Thus, the $\Bbbk$-linear isomorphism $\Bbbk\mathfrak{S}_r \to S_\Bbbk(\omega)$ determined by $\sigma \mapsto \xi_{(\mathscr{r}\sigma)\mathscr{r}}$ is an isomorphism of $\Bbbk$-algebras [G, (6.1d)].

Remark 2.7.4. It is worth explicitly stating the significance behind the requirement that $n \geq r$. Note that when $n \geq r$, the action of $\mathfrak{S}_r$ on $I(n,r)$ is freely transitive, that is to say, no permutation in $\mathfrak{S}_r$ fixes $I(n,r)$ point-wise. In particular, no element of $\mathfrak{S}_r$ fixes $\not{r}$. This fails spectacularly when $n < r$ since then, any $\sigma \in \mathrm{Sym}(\{n+1,\dots,r\})$ fixes $I(n,r)$ point-wise. Thus, when $n < r$, it is not possible to recover an element of $\mathfrak{S}_r$ from its action on $I(n,r)$.

Now, after applying the identification $\Bbbk\mathfrak{S}_r \cong S_{\Bbbk}(\omega)$, we have that the Schur functor f is given by $f := f_{\xi_\omega} : S_{\Bbbk}(n,r)$–mod $\to \Bbbk\mathfrak{S}_r$–mod, whilst the g-functor g is given by $g := g_{\xi_\omega} : \Bbbk\mathfrak{S}_r$–mod $\to S_{\Bbbk}(n,r)$–mod.

Remark 2.7.5. Firstly, note that since $\mathrm{Stab}_{\mathfrak{S}_r}(\not{r}) = \{1\}$, we have that the $\Bbbk$-linear map $E^{\otimes r} \to S_{\Bbbk}(n,r)$ with $e_{\not{j}} \mapsto \xi_{\not{j}\not{r}}$ is an embedding with image $S_{\Bbbk}(n,r)\xi_\omega$. Moreover, by comparing the left $S_{\Bbbk}(n,r)$-module structures of $E^{\otimes r}$ and $S_{\Bbbk}(n,r)\xi_\omega$, we see that this embedding provides a $S_{\Bbbk}(n,r)$-linear isomorphism $E^{\otimes r} \cong S_{\Bbbk}(n,r)\xi_\omega$. Now, let $V \in S_{\Bbbk}(n,r)$–mod. Then, recall that in the proof of Lemma 2.7.1, we saw that we have a $\Bbbk\mathfrak{S}_r$-module isomorphism $fV = \xi_\omega V \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(E^{\otimes r}, V)$, where here, we have applied the identifications $S_{\Bbbk}(\omega) \cong \Bbbk\mathfrak{S}_r$ ($\Bbbk$–alg) and $S_{\Bbbk}(n,r)\xi_\omega \cong E^{\otimes r}$ ($S_{\Bbbk}(n,r)$–mod). Moreover, this isomorphism is functorial, and so in this case, one may identify the Schur functor f with the Hom-functor $\mathrm{Hom}_{S_{\Bbbk}(n,r)}(E^{\otimes r}, -)$.

Remark 2.7.6. Note that in characteristic zero, the algebras $S_{\Bbbk}(n,r)$ and $\Bbbk\mathfrak{S}_r$ are both semisimple, and so the same applies to the categories $S_{\Bbbk}(n,r)$–mod, $\Bbbk\mathfrak{S}_r$–mod.

Note that due to Remark 2.7.6, the functors f and g have a stronger relationship in characteristic zero:

Lemma 2.7.7. *Suppose that $\Bbbk$ has characteristic zero. Then:*

(i) *The g-functor is exact.*

(ii) *The functors f and g are inverse equivalence of categories* [D$_3$, §2.1 Remarks(iii)].

We conclude by collecting results from the literature describing how these functors acts on the particular modules pertinent to this thesis.

Lemma 2.7.8. *Let $n, r \in \mathbb{N}$ with $n \geq r$. Then, for $\lambda \in \Lambda^+(n,r)$, $\alpha \in \Lambda(n,r)$, we have:*

(i) $fS^\alpha E \cong M_{\Bbbk}(\alpha)$ [D$_2$, Lemma (3.5)(i)].

(ii) $f\Lambda^\alpha E \cong M_{\Bbbk}(\alpha) \otimes \mathtt{sgn}_r = M_{\Bbbk,\mathtt{sgn}}(\alpha)$ [D$_2$, Lemma (3.5)(ii)].

(iii) $gM_{\Bbbk}(\alpha) \cong S^\alpha E$ [DG$_1$, Appendix A].

(iv) $f\nabla_{\Bbbk}(\lambda) \cong \mathrm{Sp}_{\Bbbk}(\lambda)$ [G, (6.3c)].

3 Homomorphisms

For the remainder of this thesis, we drop the dependence on the field $\Bbbk$ in the notation for induced modules, Weyl modules, (signed) permutation modules, and Specht modules.

Now, we move on to the crux of this thesis. In this chapter, for $\lambda, \mu \in \Lambda^+(n,r)$, we construct, in general characteristic, an identification of the $\Bbbk$-space of $\Bbbk\mathfrak{S}_r$-linear homomorphisms $\mathrm{Sp}(\lambda) \to \mathrm{Sp}(\mu)$ in terms of a certain subspace of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M_{\mathtt{sgn}}(\lambda'), M(\mu))$. Then, we conclude by examining, in characteristic 2, this identification in the case that $\lambda = \mu = (a, m-1, \ldots, 2, 1^b)$ for parameters a, b, m satisfying $a - m \equiv b \pmod 2$.

3.1. General Constructions

3.1.1. **Constructions of the Specht Module.** We set $\ell \coloneqq \mathrm{len}(\lambda)$. By applying the Schur functor f to the maps ϕ_λ and ψ_λ from (2.4.14) and (2.4.16) respectively, we obtain the $\Bbbk\mathfrak{S}_r$-homomorphisms:

$$\bar{\phi}_\lambda \coloneqq f(\phi_\lambda) : \bigoplus_{i=1}^{\ell-1} \bigoplus_{s=1}^{\lambda_{i+1}} M_{\mathtt{sgn}}(\lambda^{(i,i+1,s)}) \to M_{\mathtt{sgn}}(\lambda), \tag{3.1.1}$$

$$\bar{\psi}_\lambda \coloneqq f(\psi_\lambda) : M(\lambda) \to \bigoplus_{i=1}^{\ell-1} \bigoplus_{t=1}^{\lambda_{i+1}} M(\lambda^{(i,i+1,t)}). \tag{3.1.2}$$

As a consequence of the exactness of the Schur functor f, it follows that $\mathrm{Sp}(\lambda) \cong \mathrm{coker}\, \bar{\phi}_{\lambda'}$ and $\mathrm{Sp}(\lambda) \cong \ker \bar{\psi}_\lambda$. This second isomorphism is an alternative realisation of James' Kernel Intersection Theorem [J_1, Corollary 17.18]. These two descriptions of the Specht module $\mathrm{Sp}(\lambda)$ will be crucial for the considerations in this thesis.

3.1.2. **The g-Functor.** First, in Proposition 3.1.3(i), we point out a new property of the g-functor, which we immediately apply in Proposition 3.1.3(ii) to obtain a new short proof of the fact that $g\,\mathrm{Sp}(\lambda) \cong \nabla(\lambda)$ when $p \neq 2$ [D_4, Proposition 10.6(i)], [McD, Theorem 1.1].

Proposition 3.1.3. *Assume that $p \neq 2$. Then:*

(i) *For $\alpha \in \Lambda(n,r)$, we have $gM_{\mathtt{sgn}}(\alpha) \cong \Lambda^\alpha E$.*

(ii) *For $\lambda \in \Lambda^+(n,r)$, we have $g\,\mathrm{Sp}(\lambda) \cong \nabla(\lambda)$.*

Proof. (i) Recall that for $\beta \in \Lambda(n,r)$ and $V \in M_\Bbbk(n,r)$, we have a $\Bbbk$-isomorphism $\mathrm{Hom}_G(V, S^\beta E) \cong V^\beta$, and so in particular $\dim V^\beta = \dim \mathrm{Hom}_G(V, S^\beta E)$. Moreover, $fS^\alpha E \cong M(\alpha)$ and so it follows that:

$$\mathrm{Hom}_G(gM_{\mathtt{sgn}}(\alpha), S^\beta E) \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M_{\mathtt{sgn}}(\alpha), fS^\beta E) \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M_{\mathtt{sgn}}(\alpha), M(\beta)).$$

Now, since $p \neq 2$, the dimension of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M_{\mathtt{sgn}}(\alpha), M(\beta))$ does not depend on the value of p [DJ, Theorem 3.3(ii)], and so in order to calculate the dimension of $gM_{\mathtt{sgn}}(\alpha)^\beta$, we may assume that $p = 0$. However, in characteristic 0, the functors f and g are inverse equivalences of categories and so $gM_{\mathtt{sgn}}(\alpha) \cong \Lambda^\alpha E$. Therefore, for $p \neq 2$, we deduce that $\dim gM_{\mathtt{sgn}}(\alpha)^\beta = \dim \Lambda^\alpha E^\beta$ for all $\beta \in \Lambda(n,r)$. Now, recall that for $V \in M_\Bbbk(n,r)$, we have the weight space decomposition $V = \bigoplus_{\beta\in\Lambda(n,r)} V^\beta$ as in §2.4.4, and so it follows that, for $p \neq 2$, we have $\dim gM_{\mathtt{sgn}}(\alpha) = \dim \Lambda^\alpha E$.

Now, we have that $M(1^r) \cong eSe$ and so $gM(1^r) \cong Se \otimes_{eSe} eSe \cong Se \cong E^{\otimes r}$ [G, (6.4f)]. For $\alpha \in \Lambda(n,r)$ we have a surjective G-homomorphism $E^{\otimes r} \to \Lambda^\alpha E$ and so via the Schur functor, we get a surjective $\Bbbk\mathfrak{S}_r$-homomorphism $M(1^r) \to M_{\mathtt{sgn}}(\alpha)$. The functor g, being right-exact, preserves surjections, and so the G-homomorphism $gM(1^r) \to gM_{\mathtt{sgn}}(\alpha)$ is surjective. We consider the commutative diagram:

$$\begin{tikzcd} gM(1^r) \arrow[r, "\cong"] \arrow[d] & E^{\otimes r} \arrow[d] \\ gM_{\mathtt{sgn}}(\alpha) \arrow[r] & \Lambda^\alpha E \end{tikzcd}, \tag{3.1.4}$$

where the horizontal maps in (3.1.4) are induced from the $\Bbbk\mathfrak{S}_r$-linear embeddings: $M(1^r) \cong fE^{\otimes r} \hookrightarrow E^{\otimes r}$ and $M_{\mathtt{sgn}}(\alpha) \cong f\Lambda^\alpha E \hookrightarrow \Lambda^\alpha E$. The top horizontal map is an isomorphism and the right-hand vertical map is surjective, and so the bottom horizontal map is hence surjective. Since $\dim gM_{\mathtt{sgn}}(\alpha) = \dim \Lambda^\alpha E$ away from characteristic 2, we obtain the isomorphism $gM_{\mathtt{sgn}}(\alpha) \cong \Lambda^\alpha E$ for $p \neq 2$.

(ii) Recall that $\nabla(\lambda) \cong \operatorname{coker}\phi_{\lambda'}$, where $\phi_{\lambda'} : K(\lambda') \to \Lambda^{\lambda'}E$ and $K(\lambda')$ is the direct sum of tensor products of exterior powers given in (2.4.14), where here we replace the partition λ with λ'. By applying the Schur functor f to $\phi_{\lambda'}$, we obtain the $\Bbbk\mathfrak{S}_r$-homomorphism $\bar{\phi}_{\lambda'} : \bar{K}(\lambda') \to M_{\mathtt{sgn}}(\lambda')$, where $\bar{K}(\lambda')$ is the direct sum of signed permutation modules given in (2.4.16), again substituting λ with λ'. Also, recall that $\mathrm{Sp}(\lambda) \cong \operatorname{coker}\bar{\phi}_{\lambda'}$. By part (i), we have that $gM_{\mathtt{sgn}}(\lambda') \cong \Lambda^{\lambda'}E$ and so $g\bar{K}(\lambda') \cong K(\lambda')$. Hence, we obtain the commutative diagram:

$$\begin{tikzcd} g\bar{K}(\lambda') \arrow[r, "g(\bar{\phi}_{\lambda'})"] \arrow[d, "\cong"'] & gM_{\mathtt{sgn}}(\lambda') \arrow[d, "\cong"] \\ K(\lambda') \arrow[r, "\phi_{\lambda'}"] & \Lambda^{\lambda'}E \end{tikzcd}. \tag{3.1.5}$$

The image of $g(\bar{\phi}_{\lambda'})$ is mapped isomorphically onto the image of $\phi_{\lambda'}$, and so in particular $\operatorname{coker}\phi_{\lambda'} \cong \operatorname{coker} g(\bar{\phi}_{\lambda'})$. Finally, g preserves cokernels since it is right-exact, and so we deduce that $\nabla(\lambda) \cong \operatorname{coker}\phi_{\lambda'} \cong \operatorname{coker} g(\bar{\phi}_{\lambda'}) \cong g \operatorname{coker}\bar{\phi}_{\lambda'} \cong g\,\mathrm{Sp}(\lambda)$. □

3.2. HOMOMORPHISMS

Here, we fix $n, r \in \mathbb{N}$ with $n \geq r \geq 1$.

Lemma 3.2.1. *Let $\alpha, \beta \in \Lambda(n,r)$. Then:*

(i) $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \cong \mathrm{Hom}_G(S^\alpha E, S^\beta E) \cong (S^\alpha E)^\beta$.

(ii) *For $p \neq 2$, we have* $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M_{\mathtt{sgn}}(\alpha), M(\beta)) \cong \mathrm{Hom}_G(\Lambda^\alpha E, S^\beta E) \cong (\Lambda^\alpha E)^\beta$.

Proof. Recall from Lemma 2.7.1, that for $V \in M_\Bbbk(n,r)$ and $W \in \Bbbk\mathfrak{S}_r$–mod, we have a $\Bbbk$-isomorphism of the form $\mathrm{Hom}_G(gW, V) \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(W, fV)$. Parts (i)-(ii) then both follow from our comments in §3.1.1, and Proposition 3.1.3(i). □

3.2.1. **Homomorphisms Between Specht Modules.** Now, we move on to our general description of the $\Bbbk$-space of $\Bbbk\mathfrak{S}_r$-linear homomorphisms $\mathrm{Sp}(\lambda) \to \mathrm{Sp}(\mu)$ for partitions $\lambda, \mu \in \Lambda^+(n,r)$. Unless otherwise stated, $\Bbbk$ denotes an algebraically closed field of characteristic $p \geq 0$. Note that the condition that the field $\Bbbk$ is algebraically closed is imposed to ensure certain nicety properties of the connection between the categories $M_\Bbbk(n,r)$ and $\Bbbk\mathfrak{S}_r$–mod, and may not be necessary for some of the results contained within this thesis.

Lemma 3.2.2. *Let $\lambda, \mu \in \Lambda^+(n,r)$. Then:*

(i) *There is a $\Bbbk$-linear isomorphism:*

$$\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda), \mathrm{Sp}(\mu)) \cong \left\{ h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M_{\mathtt{sgn}}(\lambda'), M(\mu)) \;\middle|\; \begin{array}{c} h \circ \phi_{\lambda'} = 0, \\ \psi_\mu \circ h = 0. \end{array} \right\}.$$

(ii) *In particular, when $p = 2$, there is a $\Bbbk$-linear isomorphism:*

$$\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda), \mathrm{Sp}(\mu)) \cong \left\{ h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu)) \;\middle|\; \begin{array}{c} h \circ \phi_{\lambda'} = 0, \\ \psi_\mu \circ h = 0. \end{array} \right\}.$$

Proof. Part (i) follows immediately from the two descriptions of the Specht module: $\mathrm{Sp}(\lambda) \cong \operatorname{coker} \bar{\phi}_{\lambda'}$ and $\mathrm{Sp}(\mu) \cong \ker \bar{\psi}_\mu$ from §3.1.1. Part (ii) then follows from part (i) and the fact that the permutation module and the signed permutation module coincide in characteristic 2. $\square$

Now, for the remainder of this thesis, we shall assume that the underlying field $\Bbbk$ has characteristic 2. Also, we fix $n, r \in \mathbb{N}$ with $n \geq r \geq 1$. Recall the G-homomorphisms $\phi_\lambda^{(i,j,s)}$ and $\psi_\lambda^{(i,j,t)}$ from (2.4.13) and (2.4.15) respectively.

Lemma 3.2.3. *Let $\lambda \in \Lambda^+(n)$ with $\ell := \mathrm{len}(\lambda)$. Then:*

(i) $\operatorname{im} \phi_\lambda^{(i,j,s)} \subseteq \operatorname{im} \phi_\lambda$ *for $1 \leq i < j \leq \ell$, $1 \leq s \leq \lambda_j$.*

(ii) $\ker \psi_\lambda \subseteq \ker \psi_\lambda^{(i,j,t)}$ *for $1 \leq i < j \leq \ell$, $1 \leq t \leq \lambda_j$.*

Proof. For part (i), from [ABW, Theorem II.2.16], we have that $\operatorname{im} \phi_\lambda = \ker d_\lambda$, where the map $\Lambda^\lambda E \xrightarrow{d_\lambda} S^{\lambda'} E$ is a G-homomorphism that arises as a composition of (tensor products of) comultiplications between exterior powers and (tensor products of) multiplications between symmetric powers [ABW, Definition II.1.3]. Now, from [ABW, Lemma II.2.3], we have that for each $1 \leq i < \ell$, the map d_λ may be factored through the G-homomorphism:

$$\Lambda^\lambda E \xrightarrow{\mathbb{1} \otimes \cdots \otimes d_{(\lambda_i, \lambda_{i+1})} \otimes \cdots \otimes \mathbb{1}} \Lambda^{\lambda_1} E \otimes \cdots \otimes \Lambda^{\lambda_{i-1}} E \otimes (S^2 E)^{\otimes \lambda_{i+1}} \otimes E^{\otimes(\lambda_i - \lambda_{i+1})} \otimes \Lambda^{\lambda_{i+2}} E \otimes \cdots \otimes \Lambda^{\lambda_\ell} E,$$

where $d_{(\lambda_i, \lambda_{i+1})}$ is the corresponding map associated to the partition $(\lambda_i, \lambda_{i+1})$, and each $\mathbb{1}$ refers to the identity map on the corresponding tensor factor. Now, it is clear that one may replace $i+1$ with any $j > i$ in the statement of [ABW, Lemma II.2.3] without any harm. Then, part (i) follows by applying [ABW, Theorem II.2.16] for the partition (λ_i, λ_j).

For part (ii), we use the ABW-construction of the Weyl module $\Delta(\lambda)$ (2.4.28). Similarly to part (i), from [ABW, Theorem II.3.16] and the comment before [ABW, Definition II.3.4], we deduce that $\operatorname{im}\theta_\lambda^{(i,j,t)} \subseteq \operatorname{im}\theta_\lambda$ for $1 \le i < j \le \ell$ and $1 \le t \le \lambda_j$. Taking contravariant duals, we have that $\ker\theta_\lambda{}^\circ \subseteq \ker\theta_\lambda^{(i,j,t)\circ}$ for all such i, j, t. The result follows by recalling the identifications $\theta_\lambda{}^\circ = \psi_\lambda$ and $\theta_\lambda^{(i,j,t)\circ} = \psi_\lambda^{(i,j,t)}$ from the end of §2.4.6. $\square$

Let $\lambda \in \Lambda^+(n,r)$. By applying the Schur functor f to the maps $\phi_\lambda^{(i,j,s)}$ and $\psi_\lambda^{(i,j,t)}$ of (2.4.13) and (2.4.15) respectively, we obtain the $\Bbbk\mathfrak{S}_r$-homomorphisms:

$$\bar{\phi}_\lambda^{(i,j,s)} : M_{\mathsf{sgn}}(\lambda^{(i,j,s)}) \to M_{\mathsf{sgn}}(\lambda), \qquad \bar{\psi}_\lambda^{(i,j,t)} : M(\lambda) \to M(\lambda^{(i,j,t)}).$$

Remark 3.2.4. We may view any partition $\lambda \in \Lambda^+(n,r)$ as an n-tuple by appending an appropriate number of zeros to λ. Accordingly, we may relax the dependence on $\operatorname{len}(\lambda)$ of the maps $\bar{\phi}_\lambda$ and $\bar{\psi}_\lambda$. We do so by setting $\bar{\phi}_\lambda^{(i,j,s)} := 0$ and $\bar{\psi}_\lambda^{(i,j,t)} := 0$ if $\operatorname{len}(\lambda) < j \le n$.

By Lemma 3.2.2(ii) and Lemma 3.2.3, we obtain the following Corollary:

Corollary 3.2.5. *Assume that the characteristic of the underlying field* $\Bbbk$ *is* 2*. Then, for* $\lambda, \mu \in \Lambda^+(n,r)$*, the* $\Bbbk$*-space of* $\Bbbk\mathfrak{S}_r$*-linear homomorphisms* $\operatorname{Hom}_{\Bbbk\mathfrak{S}_r}(\operatorname{Sp}(\lambda), \operatorname{Sp}(\mu))$ *may be identified with the* $\Bbbk$*-subspace of* $\operatorname{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu))$ *consisting of those elements* h *that satisfy:*

(i) $h \circ \bar{\phi}_{\lambda'}^{(i,j,s)} = 0$ *for* $1 \le i < j \le n$ *and* $1 \le s \le \lambda'_j$*,*

(ii) $\bar{\psi}_\mu^{(i,j,t)} \circ h = 0$ *for* $1 \le i < j \le n$ *and* $1 \le t \le \mu_j$*.*

In light of Corollary 3.2.5, we introduce the following definition.

Definition 3.2.6. Let $\lambda, \mu \in \Lambda^+(n,r)$. Then, we say that a $\Bbbk\mathfrak{S}_r$-linear homomorphism $M(\lambda') \xrightarrow{h} M(\mu)$ is *essential* if h satisfies the composition relations of Corollary 3.2.5. Accordingly, we denote by $\operatorname{ERel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu))$ the $\Bbbk$-subspace of $\operatorname{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu))$ consisting of the essential homomorphisms $M(\lambda') \xrightarrow{h} M(\mu)$.

3.2.2. **Homomorphisms Between Permutation Modules.** In this section, we assume that the underlying field $\Bbbk$ has characteristic 2. Here, we provide a matrix description of a $\Bbbk$-basis of $\operatorname{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ for $\alpha, \beta \in \Lambda(n,r)$.

Now, let $\alpha, \beta \in \Lambda(n,r)$. Then, according to Lemma 3.2.1, we have a $\Bbbk$-linear isomorphism of the form $\operatorname{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \cong (S^\alpha E)^\beta$. Now, recall that the tensor product $S^\alpha E$ has a $\Bbbk$-basis of the form $\{\bigotimes_{i=1}^n \prod_{j=1}^n e_i^{a_{ij}} \mid \sum_j a_{ij} = \alpha_i\}$, where the i^{th}-tensor factor is defined to be 1 if $\alpha_i = 0$ for some $1 \le i \le n$. We may parametrise this $\Bbbk$-basis by the set of all elements of $M_{n\times n}(\mathbb{N})$ whose sequence of row-sums is equal to α. Accordingly, for $\beta \in \Lambda(n,r)$, the β-weight space $(S^\alpha E)^\beta$ has a $\Bbbk$-basis parametrised by the set of all matrices in $M_{n\times n}(\mathbb{N})$ whose sequence of row-sums is equal to α, and whose sequence of column-sums is equal to β. But this description is precisely the set $\operatorname{Tab}(\alpha, \beta)$. In sum, we deduce that the dimension of the $\Bbbk$-space $\operatorname{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ is equal to $\dim(S^\alpha E)^\beta = |\operatorname{Tab}(\alpha, \beta)|$. For further reading regarding the dimension of $\operatorname{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$, the reader may wish to refer to [J_1, §13].

Now, we associate to each $A \in \operatorname{Tab}(\alpha, \beta)$, the $\Bbbk\mathfrak{S}_r$-homomorphism $M(\alpha) \xrightarrow{\rho[A]} M(\beta)$ defined as follows. Given a basis element $\boldsymbol{x} := (\boldsymbol{x}_1 | \dots | \boldsymbol{x}_n) \in \mathsf{S}_\alpha \subseteq M_\Bbbk(\alpha)$ as in §2.6.2, we set $\rho[A](\boldsymbol{x})$ to be the sum of all basis elements of $M(\beta)$ that may be obtained from $\boldsymbol{x}$ by moving, in concert, a_{ij} entries from its i^{th}-position $\boldsymbol{x}_i$ to its j^{th}-position $\boldsymbol{x}_j$ for every $1 \leq i,j \leq n$. The set $\{\rho[A] \mid A \in \mathrm{Tab}(\alpha,\beta)\}$ is linearly independent. Indeed, take any linear combination of the $\rho[A]$s, say $h = \sum_A h[A]\rho[A]$ ($h[A] \in \Bbbk$), along with any basis element $\boldsymbol{x}$ of $M(\alpha)$, and then consider the coefficients of the basis elements of $M(\beta)$ in $h(\boldsymbol{x})$. Thus, the subspace of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ spanned by the homomorphisms $\{\rho[A] \mid A \in \mathrm{Tab}(\alpha,\beta)\}$ has dimension $|\mathrm{Tab}(\alpha,\beta)|$, and so we indeed see that the set $\{\rho[A] \mid A \in \mathrm{Tab}(\alpha,\beta)\}$ forms a $\Bbbk$-basis of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$. Accordingly, given $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ and $A \in \mathrm{Tab}(\alpha,\beta)$, we shall denote by $h[A] \in \Bbbk$ the coefficient of $\rho[A]$ in h so that $h = \sum_{A\in\mathrm{Tab}(\alpha,\beta)} h[A]\rho[A]$.

Remark 3.2.7. Here, we highlight a link with the maps Θ_{t} defined within [J$_1$, §13] (note that here, the author uses T in place of t). Firstly, given $\lambda \in \Lambda^+(n,r)$ and a (not necessarily injective) $[r]$-valued λ-tableau t, James defines the *type* of t, which we denote by $\mathrm{type}(\mathsf{t}) = (\mathrm{type}_k(\mathsf{t}))_k$, to be the sequence whose entries are defined by $\mathrm{type}_k(\mathsf{t}) := |\{(i,j) \in [\lambda] \mid \mathsf{t}(i,j) = k\}|$ for each k, and for $\lambda, \mu \in \Lambda^+(n,r)$, we denote by $\mathscr{T}(\lambda,\mu)$ the set of all such λ-tableaux of type μ. In [J$_1$, §13], James associates to each such $\mathsf{t} \in \mathscr{T}(\lambda,\mu)$ a homomorphism $\Theta_{\mathsf{t}} \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda), M(\mu))$, and in [J$_1$, Theorem 13.19], James showed that the set of Θ_{t} parametrised by those t within the set:

$$\{\mathsf{t} \in \mathscr{T}(\lambda,\mu) \mid \mathsf{t} \text{ has weakly increasing rows}\} \tag{3.2.8}$$

forms an alternate $\Bbbk$-basis for $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda), M(\mu))$. The link between James' basis and the basis $\{\rho[A] \mid A \in \mathrm{Tab}(\alpha,\beta)\}$ is as follows. Given $\lambda \in \Lambda^+(n,r)$ with a $[r]$-valued λ-tableau t, we define the *content* $\mathsf{c}(\mathsf{t}) = (\mathsf{c}_{ij}(\mathsf{t}))_{i,j}$ to be the $(n \times n)$-matrix with entries:

$$\mathsf{c}_{ij}(\mathsf{t}) := |\{1 \leq u \leq \lambda_i \mid \mathsf{t}(i,u) = j\}| = \left|\mathsf{t}^{-1}(j)\right|.$$

Then, clearly, we have that:

$$\begin{aligned}
\sum_j \mathsf{c}_{ij}(\mathsf{t}) &= |\{1 \leq u \leq \lambda_i\}| = \lambda_i, \\
\sum_i \mathsf{c}_{ij}(\mathsf{t}) &= |\{(u,v) \in [\lambda] \mid \mathsf{t}(u,v) = j\}| = \mathrm{type}_j(\mathsf{t}),
\end{aligned}$$

and so c defines a mapping $\mathscr{T}(\lambda,\mu) \xrightarrow{\mathsf{c}} \mathrm{Tab}(\lambda,\mu)$ for $\lambda, \mu \in \Lambda^+(n,r)$. Moreover, it is clear that the restriction of c to those $\mathsf{t} \in \mathscr{T}(\lambda,\mu)$ as in (3.2.8) defines a bijection onto $\mathrm{Tab}(\lambda,\mu)$. Finally, it is clear to see that for such t, James' Θ_{t} is precisely $\rho[\mathsf{c}(\mathsf{t})]$.

Examples 3.2.9. Let $\lambda \in \Lambda^+(n,r)$. For $1 \leq i,j \leq n$, denote by $E_{ij} \in M_{n\times n}(\mathbb{N})$ the matrix with a 1 in its $(i,j)^{\text{th}}$-position and 0s elsewhere. Notice that:

(i) $\bar{\phi}_{\lambda}^{(i,j,s)} = \rho[A]$, where $A := \mathrm{diag}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j - s, \ldots, \lambda_n) + sE_{ij}$.

(ii) $\bar{\psi}_{\lambda}^{(i,j,t)} = \rho[B]$, where $B := \mathrm{diag}(\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j - t, \ldots, \lambda_n) + tE_{ji}$.

Remark 3.2.10. Consider the $\Bbbk$-basis $\{\rho[A] \mid A \in \mathrm{Tab}(\alpha,\beta)\}$ of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$. If $A \in \mathrm{Tab}(\alpha,\beta)$, then it is clear that $A' \in \mathrm{Tab}(\beta,\alpha)$. Moreover, it is also clear that the set $\{\rho[A'] \mid A \in \mathrm{Tab}(\alpha,\beta)\}$ forms a $\Bbbk$-basis of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$.

Now, for $\alpha \in \Lambda(n,r)$, recall that the permutation module $M(\alpha)$ is self-dual. We write $M(\alpha) \xrightarrow{\delta_\alpha} M(\alpha)^*$ for the $\Bbbk\mathfrak{S}_r$-isomorphism that sends each basis element $\boldsymbol{x} \in \mathsf{S}_\alpha$ of $M(\alpha)$ to the corresponding basis element of $M(\alpha)^*$ dual to $\boldsymbol{x}$. We shall denote by $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \xrightarrow{\zeta_{\alpha,\beta}} \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta)^*, M(\alpha)^*)$ the natural $\Bbbk$-isomorphism, and by $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \xrightarrow{\eta_{\alpha,\beta}} \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$ the $\Bbbk$-linear isomorphism given by $\eta_{\alpha,\beta}(h) = \delta_\alpha^{-1} \circ \zeta_{\alpha,\beta}(h) \circ \delta_\beta$ for $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$.

Lemma 3.2.11. *Let $\alpha, \beta \in \Lambda(n,r)$. Then $\eta_{\alpha,\beta}(\rho[A]) = \rho[A']$ for all $A \in \mathrm{Tab}(\alpha,\beta)$.*

Proof. This is a simple calculation which we leave to the reader. □

Definition 3.2.12. For $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$, by the *transpose homomorphism h' of h*, we mean $h' := \eta_{\alpha,\beta}(h) \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$.

Notice that if $h = \sum_{A \in \mathrm{Tab}(\alpha,\beta)} h[A]\rho[A]$, then $h' = \sum_{A \in \mathrm{Tab}(\alpha,\beta)} h[A]\rho[A']$ by Lemma 3.2.11.

Lemma 3.2.13. *Let $\alpha, \beta, \gamma \in \Lambda(n,r)$. Then we have the identity $(h_2 \circ h_1)' = h_1' \circ h_2'$ for all $h_1 \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ and $h_2 \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\gamma))$.*

Proof. Since $\zeta_{\alpha,\gamma}(h_2 \circ h_1) = \zeta_{\alpha,\beta}(h_1) \circ \zeta_{\beta,\gamma}(h_2)$, we have:

$$\begin{aligned}(h_2 \circ h_1)' &= \delta_\alpha^{-1} \circ \zeta_{\alpha,\beta}(h_1) \circ \zeta_{\beta,\gamma}(h_2) \circ \delta_\gamma \\ &= (\delta_\alpha^{-1} \circ \zeta_{\alpha,\beta}(h_1) \circ \delta_\beta) \circ (\delta_\beta^{-1} \circ \zeta_{\beta,\gamma}(h_2) \circ \delta_\gamma) = h_1' \circ h_2'.\end{aligned}$$ □

3.2.3. **Relevant Homomorphisms.** Here, we introduce the notion of a *relevant homomorphism* $M(\lambda') \to M(\mu)$ for partitions $\lambda, \mu \in \Lambda^+(n,r)$. Then, we investigate the connection between relevant homomorphisms $M(\lambda') \to M(\mu)$ and homomorphisms $\mathrm{Sp}(\lambda) \to \mathrm{Sp}(\mu)$.

Lemma 3.2.14. *Let $\lambda, \mu \in \Lambda^+(n,r)$ and $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu))$. Then:*

(i) $(h \circ \bar{\phi}_{\lambda'}^{(i,j,s)})' = \bar{\psi}_{\lambda'}^{(i,j,s)} \circ h'$ *for $1 \leq i < j \leq n$, $1 \leq s \leq \lambda_j'$.*

(ii) $(\bar{\psi}_{\mu}^{(i,j,t)} \circ h)' = h' \circ \bar{\phi}_{\mu}^{(i,j,t)}$ *for $1 \leq i < j \leq n$, $1 \leq t \leq \mu_j$.*

(iii) *The map $\eta_{\lambda',\mu}$ induces a $\Bbbk$-isomorphism:*

$$\bar{\eta}_{\lambda,\mu} : \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda), \mathrm{Sp}(\mu)) \to \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\mu'), \mathrm{Sp}(\lambda')).$$

Proof. By Lemma 3.2.11 and the examples in Examples 3.2.9, it follows that we have the equality $(\bar{\phi}_{\lambda}^{(i,j,t)})' = \bar{\psi}_{\lambda}^{(i,j,t)}$. Now, parts (i)-(ii) follow directly from Lemma 3.2.13. Now, for part (iii), according to Corollary 3.2.19, we identify each $\bar{h} \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda), \mathrm{Sp}(\mu))$, with an essential homomorphism $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu))$. Then, for each such h, recall that we have the transpose homomorphism $h' = \eta_{\lambda',\mu}(h)$. Then, by applying parts (i)-(ii), along with the symmetric results obtained by swapping the roles of λ and μ in the statement of the Lemma, we see that h' is essential. Hence, once again according to Corollary 3.2.19, each such h' induces a homomorphism $\bar{h}' \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\mu'), \mathrm{Sp}(\lambda'))$. Similarly to Lemma 3.2.14(iii), it follows that $\eta_{\lambda',\mu}$ induces a $\Bbbk$-linear homomorphism $\bar{\eta}_{\lambda,\mu} : \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda), \mathrm{Sp}(\mu)) \to \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\mu'), \mathrm{Sp}(\lambda'))$ sending $\bar{h} \mapsto \bar{h}'$. By applying the same procedure to the map $\eta_{\mu',\lambda}$, we see that $\bar{\eta}_{\lambda,\mu}$ is a $\Bbbk$-isomorphism with inverse $\bar{\eta}_{\mu,\lambda}$ as required. □

For $A = (a_{ij})_{i,j} \in M_{n\times n}(\mathbb{Z})$ and $1 \le k,l \le n$, we denote by $A^{(k,l)}$ (resp. $A_{(k,l)}$) the matrix whose entries $a_{ij}^{(k,l)}$ (resp. $a_{(k,l)ij}$) are given by $a_{ij}^{(k,l)} := a_{ij} + \delta_{(i,j),(k,l)}$, (resp. $a_{(k,l)ij} := a_{ij} - \delta_{(i,j),(k,l)}$). Let $\alpha, \beta \in \Lambda(n,r)$ with $A \in \mathrm{Tab}(\alpha,\beta)$, and let $1 \le i < j \le n$, $1 \le k,l \le n$. Note that $A_{(j,l)}^{(i,l)} \in \mathrm{Tab}(\alpha^{(i,j,1)}, \beta)$ if $a_{jl} \ne 0$, whilst $A_{(k,j)}^{(k,i)} \in \mathrm{Tab}(\alpha, \beta^{(i,j,1)})$ if $a_{kj} \ne 0$.

Henceforth, for $\lambda, \mu \in \Lambda^+(n,r)$, we write $\mathcal{T}_{\lambda,\mu} := \mathrm{Tab}(\lambda', \mu)$.

Lemma 3.2.15. *Let $\lambda, \mu \in \Lambda^+(n,r)$ and $1 \le i < j \le n$. For $A \in \mathcal{T}_{\lambda,\mu}$ we have:*

(i) $\rho[A] \circ \bar{\phi}_{\lambda'}^{(i,j,1)} = \sum_l (a_{il}+1)\rho\left[A_{(j,l)}^{(i,l)}\right]$*, where the sum is over all l such that $a_{jl} \ne 0$.*

(ii) $\bar{\psi}_{\mu}^{(i,j,1)} \circ \rho[A] = \sum_k (a_{ki}+1)\rho\left[A_{(k,j)}^{(k,i)}\right]$*, where the sum is over all k such that $a_{kj} \ne 0$.*

Proof. We prove part (i), and then part (ii) is similar. We may assume that $j \le \mathrm{len}(\lambda')$. Fix $1 \le i < j \le \mathrm{len}(\lambda')$, and we denote by $\boldsymbol{x} := (\boldsymbol{x}_1|\dots|\boldsymbol{x}_i|\dots|\boldsymbol{x}_j|\dots|\boldsymbol{x}_n)$ a basis element of $M(\lambda'^{(i,j,1)})$, where $\boldsymbol{x}_i = (x_{i1},\dots,x_{i(\lambda_i'+1)})$ say. Then $\bar{\phi}_{\lambda'}^{(i,j,1)}(\boldsymbol{x}) = \sum_{k=1}^{\lambda_i'+1} \boldsymbol{x}^k$, where $\boldsymbol{x}^k$ denotes the basis element of $M(\lambda')$ that is obtained from $\boldsymbol{x}$ by omitting the entry x_{ik} from the sequence $\boldsymbol{x}_i$ and placing it in the (unordered) sequence $\boldsymbol{x}_j$. For $1 \le k \le \lambda_i'+1$, we have $\rho[A](\boldsymbol{x}^k) = \sum_t c_{kt}\boldsymbol{z}[t]$, where the $\boldsymbol{z}[t]$ are the basis elements of $M(\mu)$ and the c_{kt} are constants with $c_{kt} \in \{0,1\}$. Then $\rho[A] \circ \bar{\phi}_{\lambda'}^{(i,j,1)}(\boldsymbol{x}) = \sum_t c_t \boldsymbol{z}[t]$ where $c_t := \sum_{k=1}^{\lambda_i'+1} c_{kt}$. Now, fix $1 \le k \le \lambda_i'+1$ and some s with $c_{ks} = 1$. Then, suppose that the entry x_{ik} appears in the l^{th}-position $\boldsymbol{z}[s]_l$ of $\boldsymbol{z}[s]$ and hence $a_{jl} \ne 0$. Note that the sequence $\boldsymbol{z}[s]_l$ contains a_{il} entries from $\{x_{i1},\dots,x_{i(k-1)},x_{i(k+1)},\dots,x_{i(\lambda_i'+1)}\}$. If x_{iv} is such an entry with $v \ne k$, then $c_{vs} = 1$. On the other hand, given $1 \le q \le \lambda_i'+1$, if x_{iq} does not appear as an entry in $\boldsymbol{z}[s]_l$, then $c_{qs} = 0$. It follows that $c_s = a_{il}+1$. Meanwhile, given $1 \le l' \le n$, $\boldsymbol{z}[s]$ appears in $\rho\left[A_{(j,l')}^{(i,l')}\right](\boldsymbol{x})$ if and only if $l' = l$, in which case it appears with a coefficient of 1. The result follows. □

Lemma 3.2.16. *Let $\lambda, \mu \in \Lambda^+(n,r)$ and denote by $M(\lambda') \xrightarrow{h} M(\mu)$ a $\Bbbk\mathfrak{S}_r$-linear homomorphism with $h = \sum_{A \in \mathcal{T}_{\lambda,\mu}} h[A]\rho[A]$ say. Then for $1 \le i < j \le n$, we have:*

(i) $h \circ \bar{\phi}_{\lambda'}^{(i,j,1)} = 0$ *if and only if* $\sum_l b_{il} h\left[B_{(i,l)}^{(j,l)}\right] = 0$ *for all* $B \in \mathrm{Tab}(\lambda'^{(i,j,1)}, \mu)$.

(ii) $\bar{\psi}_{\mu}^{(i,j,1)} \circ h = 0$ *if and only if* $\sum_k d_{ki} h\left[D_{(k,i)}^{(k,j)}\right] = 0$ *for all* $D \in \mathrm{Tab}(\lambda', \mu^{(i,j,1)})$.

Proof. We shall only prove part (i) since part (ii) is similar. By Lemma 3.2.15 we have:

$$\begin{aligned} h \circ \bar{\phi}_{\lambda'}^{(i,j,1)} &= \sum_{A \in \mathcal{T}_{\lambda,\mu}} h[A](\rho[A] \circ \bar{\phi}_{\lambda'}^{(i,j,1)}) = \sum_{A \in \mathcal{T}_{\lambda,\mu}} h[A]\left(\sum_l (a_{il}+1)\rho\left[A_{(j,l)}^{(i,l)}\right]\right) \\ &= \sum_{A \in \mathcal{T}_{\lambda,\mu}} \sum_l (a_{il}+1)h[A]\rho\left[A_{(j,l)}^{(i,l)}\right] = \sum_{B \in \mathrm{Tab}(\lambda'^{(i,j,1)},\lambda)} \left(\sum_l b_{il} h\left[B_{(i,l)}^{(j,l)}\right]\right)\rho[B]. \end{aligned}$$

The result now follows from the linear independence of $\{\rho[B] \mid B \in \mathrm{Tab}(\lambda'^{(i,j,1)},\lambda)\}$. □

Now, we introduce the following weakening of Definition 3.2.6.

Definition 3.2.17. Let $\lambda, \mu \in \Lambda^+(n,r)$. Then, we say that a $\Bbbk\mathfrak{S}_r$-linear homomorphism $M(\lambda') \xrightarrow{h} M(\mu)$ is *relevant* if $h \circ \bar{\phi}_{\lambda'}^{(i,j,1)} = 0$ and $\bar{\psi}_{\mu}^{(i,j,1)} \circ h = 0$ for all $1 \le i < j \le n$. Accordingly, we denote by $\mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu))$ the $\Bbbk$-subspace of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu))$ consisting of the relevant homomorphisms $M(\lambda') \to M(\mu)$.

With Corollary 3.2.5 in mind, the following Remark is clear:

Remark 3.2.18. Let $\lambda, \mu \in \Lambda^+(n,r)$. Then:

(i) There is a $\Bbbk$-isomorphism $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda), \mathrm{Sp}(\mu)) \cong \mathrm{ERel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu))$.

(ii) There is a $\Bbbk$-embedding $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda), \mathrm{Sp}(\mu)) \hookrightarrow \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu))$.

Now, by Lemma 3.2.16, we deduce the following Corollary:

Corollary 3.2.19. *Let $\lambda, \mu \in \Lambda^+(n,r)$ and $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu))$. Then we have that $h \in \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\mu))$ if and only if the coefficients $h[A]$ of the $\rho[A]$ in h satisfy:*

(i) *For all $1 \le i < j \le n$, $1 \le k \le n$, and all $A \in \mathcal{T}_{\lambda,\mu}$ with $a_{jk} \neq 0$, we have:*

$$(a_{ik}+1)h[A] = \sum_{l \neq k} a_{il} h\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big], \qquad (R^k_{i,j}(A))$$

(ii) *For all $1 \le i < j \le n$, $1 \le k \le n$, and all $A \in \mathcal{T}_{\lambda,\mu}$ with $a_{kj} \neq 0$, we have:*

$$(a_{ki}+1)h[A] = \sum_{l \neq k} a_{li} h\Big[A^{(k,i)(l,j)}_{(k,j)(l,i)}\Big]. \qquad (C^k_{i,j}(A))$$

3.3. A Reduction Trick

3.3.1. **Flattening the Partition.** Now, we fix integers a, b, m with $a \ge m \ge 2$, and we write $\tilde{a} := b+m-1$, $\tilde{b} := a-m+1$. We denote by λ the partition $(a, m-1, \dots, 2, 1^b)$, and we fix $r := \deg(\lambda)$. Note that $\lambda' = (\tilde{a}, m-1, \dots, 2, 1^{\tilde{b}})$. We write $\nu_m := (m-1, \dots, 2)$ for $m > 2$, and with ν_2 the empty sequence.

Recall that through the ABW-construction of the induced module, we see that $\nabla(\lambda)$ is isomorphic to a G-quotient of $\Lambda^{\lambda'}E = \Lambda^{\tilde{a}}E \otimes \Lambda^{\nu_m}E \otimes E^{\otimes \tilde{b}}$, namely by the submodule $\mathrm{im}\,\phi_{\lambda'}$ (2.4.14). We claim that we can replace the factor $E^{\otimes \tilde{b}}$ with the symmetric power $S^{\tilde{b}}E$. This process is in fact independent of the characteristic of the field $\Bbbk$. To this end, we construct from $E^{\otimes \tilde{b}} \xrightarrow{\Pi} S^{\tilde{b}}E$, the G-surjection $\Lambda^{\lambda'}E \xrightarrow{\mathbb{1}\otimes\Pi} \Lambda^{\tilde{a}}E \otimes \Lambda^{\nu_m}E \otimes S^{\tilde{b}}E$.

Lemma 3.3.1. *For $m \ge 2$ and $\lambda = (a, m-1, m-2, \dots, 2, 1^b)$, we have:*

(i) $\ker(\mathbb{1} \otimes \Pi) = \sum_{k=1}^{\tilde{b}-1} \mathrm{im}\,\phi_{\lambda'}^{(m+k-1,m+k,1)} \subseteq \mathrm{im}\,\phi_{\lambda'}$.

(ii) $\nabla(\lambda) \cong \mathrm{coker}((\mathbb{1} \otimes \Pi) \circ \phi_{\lambda'})$ *as G-modules.*

Proof. (i) Firstly, that $\mathrm{im}\,\phi_{\lambda'}^{(m+k-1,m+k,1)} \subseteq \mathrm{im}\,\phi_{\lambda'}$ for $1 \le k < \tilde{b}$ follows from the definition of $\phi_{\lambda'}$. Then, note that by the definition of the symmetric power $S^{\tilde{b}}E$, the $\Bbbk$-space $\ker \Pi$ is generated by elements of the form $e_{\imath}^{[k]} = e_{\imath} \cdot (1 - \sigma_k)$ for $1 \le k < \tilde{b}$ and $\imath \in I(n, \tilde{b})$. Then, it follows that the $\Bbbk$-space $\ker(\mathbb{1} \otimes \Pi)$ is generated by elements of the form $x \otimes e_{\imath}^{[k]}$ for $x \in \Lambda^{\tilde{a}}E \otimes \Lambda^{\nu_m}E$, and such k and $\imath$. But given such x, k and $\imath$, the image of the element $x \otimes e_{\imath_1} \otimes \cdots \otimes (e_{\imath_k} \wedge e_{\imath_{k+1}}) \otimes \cdots \otimes e_{\imath_{\tilde{b}}}$ under $\phi_{\lambda'}^{(m+k-1,m+k,1)}$ is precisely $x \otimes e_{\imath}^{[k]}$, and so $x \otimes e_{\imath}^{[k]} \in \mathrm{im}\,\phi_{\lambda'}^{(m+k-1,m+k,1)}$. On the other hand, it is clear that the elements of the form $x \otimes e_{\imath}^{[k]}$ generate the $\Bbbk$-space $\mathrm{im}\,\phi_{\lambda'}^{(m+k-1,m+k,1)}$, from which part (i) follows.

(ii) Now, the map $\mathbb{1} \otimes \Pi : \Lambda^{\lambda'}E \twoheadrightarrow \Lambda^{\tilde{a}}E \otimes \Lambda^{\nu_m}E \otimes S^{\tilde{b}}E$ induces a G-surjection:

$$\pi : \frac{\Lambda^{\lambda'}E}{\ker(\mathbb{1}\otimes\Pi)} \twoheadrightarrow \frac{\Lambda^{\tilde{a}}E \otimes \Lambda^{\nu_m}E \otimes S^{\tilde{b}}E}{\mathrm{im}((\mathbb{1}\otimes\Pi)\circ\phi_{\lambda'})} \,.$$

Moreover, it follows from part (i) that $\ker \pi = \operatorname{im} \phi_{\lambda'}/\ker(\mathbb{1} \otimes \Pi)$, and so we deduce that $\nabla(\lambda) \cong \operatorname{coker}((\mathbb{1} \otimes \Pi) \circ \phi_{\lambda'})$. □

On the other hand, recall that through the James-construction of the induced module, we see that that $\nabla(\lambda)$ is isomorphic to a submodule of $S^\lambda E$, namely as the kernel of the G-homomorphism ψ_λ (2.4.16). We claim that we may replace the factor $E^{\otimes b}$ with the exterior power $\Lambda^b E$. Once again, this process is independent of the characteristic of $\Bbbk$. For this, we construct from the comultiplication $\Delta : \Lambda^b E \hookrightarrow E^{\otimes b}$, the injective G-homomorphism $\mathbb{1} \otimes \Delta : S^a E \otimes S^{\nu_m} E \otimes \Lambda^b E \hookrightarrow S^\lambda E$.

Lemma 3.3.2. *For $m \geq 2$ and $\lambda = (a, m-1, m-2, \ldots, 2, 1^b)$, we have:*

(i) $\ker \psi_\lambda \subseteq \bigcap_{k=1}^{b-1} \ker \psi_\lambda^{(m+k-1,m+k,1)} = \operatorname{im}(\mathbb{1} \otimes \Delta)$.

(ii) $\nabla(\lambda) \cong \ker(\psi_\lambda \circ (\mathbb{1} \otimes \Delta))$ *as G-modules.*

Proof. (i) Firstly, it follows from the definition of ψ_λ that $\ker \psi_\lambda \subseteq \ker \psi_\lambda^{(m+k-1,m+k,1)}$ for $1 \leq k < b$. Then, the $\Bbbk$-space $\ker \psi_\lambda^{(m+k-1,m+k,1)}$ is generated by elements of the form $x \otimes e_{\imath}^{[k]}$ for $x \in S^a E \otimes S^{\nu_m} E$, $1 \leq k < b$, and $\imath \in I(n, b)$, where $e_{\imath}^{[k]} \coloneqq e_{\imath} \cdot (1 - \sigma_k)$. It follows that the $\Bbbk$-space $\bigcap_{k=1}^{b-1} \ker \psi_\lambda^{(m+k-1,m+k,1)}$ is generated by elements of the form:

$$\sum_{\sigma \in \mathfrak{S}_b} \mathtt{sgn}(\sigma) \left(x \otimes e_{\imath_{\sigma(1)}} \otimes \cdots \otimes e_{\imath_{\sigma(b)}} \right) = x \otimes \Delta(e_{\imath_1} \wedge \cdots \wedge e_{\imath_b}) \in \operatorname{im}(\mathbb{1} \otimes \Delta).$$

Moreover, it is clear that elements of the form $x \otimes \Delta(e_{\imath_1} \wedge \cdots \wedge e_{\imath_b})$ generate the $\Bbbk$-space $\operatorname{im}(1 \otimes \Delta)$, from which part (i) follows.

(ii) Now, the map $\mathbb{1} \otimes \Delta : S^a E \otimes S^{\nu_m} E \otimes \Lambda^b E \hookrightarrow S^\lambda E$ induces a G-embedding $\nu : \ker(\psi_\lambda \circ (\mathbb{1} \otimes \Delta)) \hookrightarrow \ker \psi_\lambda$. Moreover, it follows from part (i) that ν is surjective, and so we have a G-isomorphism $\ker(\psi_\lambda \circ (\mathbb{1} \otimes \Delta)) \cong \ker \psi_\lambda \cong \nabla(\lambda)$. □

Now, we shall return to the situation where the underlying field $\Bbbk$ has characteristic 2. We fix the sequences $\alpha \coloneqq (\tilde{a}, m-1, \ldots, 2, \tilde{b})$ and $\beta \coloneqq (a, m-1, \ldots, 2, b)$.

Remark 3.3.3. We shall consider the constructions of this section from the perspective of the Specht module $\operatorname{Sp}(\lambda)$.

(i) By Lemma 3.3.1(ii) we have that $\nabla(\lambda) \cong \operatorname{coker}((\mathbb{1} \otimes \Pi) \circ \phi_{\lambda'})$. By applying the Schur functor f, we obtain that $\operatorname{Sp}(\lambda) \cong \operatorname{coker}(f(\mathbb{1} \otimes \Pi) \circ \bar{\phi}_{\lambda'})$. Now, since we are in characteristic 2, we identify $f(\Lambda^{\tilde{a}} E \otimes \Lambda^{\nu_m} E \otimes S^{\tilde{b}} E)$ with $f(S^{\tilde{a}} E \otimes S^{\nu_m} E \otimes S^{\tilde{b}} E)$ which in turn is isomorphic to $M(\alpha)$. Then, we denote by $M(\lambda') \xrightarrow{\pi_\alpha} M(\alpha)$ the $\Bbbk\mathfrak{S}_r$-surjection that is obtained from $f(\mathbb{1} \otimes \Pi)$ under these identifications. Finally, we deduce that $\operatorname{Sp}(\lambda) \cong \operatorname{coker} \bar{\phi}_\alpha$, where $\bar{\phi}_\alpha \coloneqq \pi_\alpha \circ \bar{\phi}_{\lambda'}$.

(ii) On the other hand, by Lemma 3.3.2(ii) we have that $\nabla(\lambda) \cong \ker(\psi_\lambda \circ (\mathbb{1} \otimes \Delta))$. By applying the Schur functor f, we deduce that $\operatorname{Sp}(\lambda) \cong \ker(\bar{\psi}_\lambda \circ f(\mathbb{1} \otimes \Delta))$. But once again, since we are in characteristic 2, $f(S^a E \otimes S^{\nu_m} E \otimes \Lambda^b E)$ is identified with $f(S^a E \otimes S^{\nu_m} E \otimes S^b E)$ which in turn is isomorphic to $M(\beta)$. We denote by $M(\beta) \xhookrightarrow{\iota_\beta} M(\lambda)$ the $\Bbbk\mathfrak{S}_r$-embedding that is obtained from $f(\mathbb{1} \otimes \Delta)$ under these identifications. Once again, we deduce that $\operatorname{Sp}(\lambda) \cong \ker \bar{\psi}_\beta$ where $\bar{\psi}_\beta \coloneqq \bar{\psi}_\lambda \circ \iota_\beta$.

We summarise the content of Remark 3.3.3 in the following Lemma:

Lemma 3.3.4. *Let $m \geq 2$ and $\lambda = (a, m-1, m-2, \dots, 2, 1^b)$. Then, we have:*

(i) $\mathrm{Sp}(\lambda) \cong \operatorname{coker} \bar{\phi}_\alpha$ *as $\Bbbk\mathfrak{S}_r$-modules.*

(ii) $\mathrm{Sp}(\lambda) \cong \ker \bar{\psi}_\beta$ *as $\Bbbk\mathfrak{S}_r$-modules.*

We define the following $\Bbbk\mathfrak{S}_r$-homomorphisms:

$$\bar{\phi}_\alpha^{(i,j,s)} := \pi_\alpha \circ \bar{\phi}_{\lambda'}^{(i,j,s)} : M(\lambda'^{(i,j,s)}) \to M(\alpha), \quad \bar{\psi}_\beta^{(i,j,t)} := \bar{\psi}_\lambda^{(i,j,t)} \circ \iota_\beta : M(\beta) \to M(\lambda^{(i,j,t)}),$$

where π_α and ι_β are as defined in Remark 3.3.3.

Lemma 3.3.5. *For $m \geq 2$ and $\lambda = (a, m-1, m-2, \dots, 2, 1^b)$, we have:*

(i) $\bar{\phi}_\alpha^{(i,j,1)} = 0$ *for $m \leq i < j \leq n$.*

(ii) $\bar{\psi}_\beta^{(i,j,1)} = 0$ *for $m \leq i < j \leq n$.*

(iii) $\bar{\phi}_\alpha = \sum_{i=1}^{m-1} \sum_{s=1}^{\lambda'_{i+1}} \bar{\phi}_\alpha^{(i,i+1,s)}$.

(iv) $\bar{\psi}_\beta = \sum_{i=1}^{m-1} \sum_{t=1}^{\lambda_{i+1}} \bar{\psi}_\beta^{(i,i+1,t)}$.

Proof. Parts (i)-(ii) follow from Lemma 3.3.1(i) and Lemma 3.3.2(i) respectively. Then, parts (iii)-(iv) follow immediately from parts (i)-(ii). □

Now, the following Lemma provides an analogue of Lemma 3.2.3:

Lemma 3.3.6. *For $m \geq 2$ and $\lambda = (a, m-1, m-2, \dots, 2, 1^b)$, we have:*

(i) $\operatorname{im} \bar{\phi}_\alpha^{(i,j,s)} \subseteq \operatorname{im} \bar{\phi}_\alpha$ *for $1 \leq i < j \leq m$, $1 \leq s \leq \lambda'_j$.*

(ii) $\ker \bar{\psi}_\beta \subseteq \ker \bar{\psi}_\beta^{(i,j,t)}$ *for $1 \leq i < j \leq m$, $1 \leq t \leq \lambda_j$.*

Proof. Firstly, recall the $\Bbbk\mathfrak{S}_r$-homomorphisms π_α and ι_β defined within Remark 3.3.3. Then, part (i) follows from Lemma 3.2.3(i) by applying the Schur functor and post-composing by π_α. Similarly, we see that part (ii) follows from Lemma 3.2.3(ii) by applying the Schur functor and pre-composing by ι_β. □

Then, by combining the results of Lemma 3.3.4, Lemma 3.3.5, and Lemma 3.3.6, we obtain the following description of the endomorphism algebra of $\mathrm{Sp}(\lambda)$:

Corollary 3.3.7. *The endomorphism algebra of $\mathrm{Sp}(\lambda)$ may be identified with the $\Bbbk$-subspace of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ consisting of those elements h that satisfy:*

(i) $h \circ \bar{\phi}_\alpha^{(i,j,s)} = 0$ *for $1 \leq i < j \leq m$ and $1 \leq s \leq \lambda'_j$,*

(ii) $\bar{\psi}_\beta^{(i,j,t)} \circ h = 0$ *for $1 \leq i < j \leq m$ and $1 \leq t \leq \lambda_j$.*

Definition 3.3.8. Let $m \geq 2$, $\lambda = (a, m-1, m-2, \dots, 2, 1^b)$, $\alpha = (\tilde{a}, m-1, \dots, 2, \tilde{b})$, and $\beta = (a, m-1, \dots, 2, b)$. Then:

(i) We say that an element $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ is *semirelevant* if $h \circ \bar{\phi}_{\lambda'}^{(i,j,1)} = 0$ and $\bar{\psi}_\lambda^{(i,j,1)} \circ h = 0$ for all $m \leq i < j \leq n$.

(ii) We say that an element $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ is *relevant* if $h \circ \bar{\phi}_\alpha^{(i,j,1)} = 0$ and $\bar{\psi}_\beta^{(i,j,1)} \circ h = 0$ for all $1 \leq i < j \leq m$.

(iii) We say that an element $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ is *essential* if h satisfies the composition relations of Corollary 3.3.7.

Accordingly, we introduce the following notation:

Definition 3.3.9.

- Denote by $\mathrm{SRel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ the $\Bbbk$-subspace of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ consisting of the semirelevant homomorphisms $M(\lambda') \to M(\lambda)$.
- Denote by $\mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ the $\Bbbk$-subspace of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ consisting of the relevant homomorphisms $M(\alpha) \to M(\beta)$.
- Denote by $\mathrm{ERel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ the $\Bbbk$-subspace of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ consisting of the essential homomorphisms $M(\alpha) \to M(\beta)$.

Lemma 3.3.10. *Denote by* $\omega : \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \to \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$ *the* $\Bbbk$*-linear homomorphism with* $\omega(h) := \iota_\beta \circ h \circ \pi_\alpha$*. Then* ω *induces the following* $\Bbbk$*-linear isomorphisms:*

(i) $\hat{\omega} : \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \to \mathrm{SRel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$.

(ii) $\bar{\omega} : \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \to \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$.

(iii) $\breve{\omega} : \mathrm{ERel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta)) \to \mathrm{ERel}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\lambda))$.

Proof. Firstly, notice that the stated domains of the maps $\hat{\omega}$, $\bar{\omega}$, and $\breve{\omega}$ follow from Lemma 3.3.5. Meanwhile, it is clear that the maps $\hat{\omega}$, $\bar{\omega}$, $\breve{\omega}$ are injective, whilst the surjectivity of each of the three maps follows from Lemma 3.3.1(i) and Lemma 3.3.2(i), along with the particular forms of λ and λ'. $\square$

Remark 3.3.11. Let $\gamma \in \Lambda(n, r)$ with $\ell := \mathrm{len}(\gamma)$. Then:

(i) Fix $B \in \mathrm{Tab}(\alpha, \gamma)$. Then $\rho[B] \circ \pi_\alpha \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\lambda'), M(\gamma))$ and one can easily check that $\rho[B] \circ \pi_\alpha = \sum_A \rho[A]$, where the sum is over those $A \in \mathrm{Tab}(\lambda', \gamma)$ whose first $(m-1)$ rows agree with those of B, and also $\sum_{i=m}^{a} a_{ij} = b_{mj}$ for $1 \leq j \leq \ell$. Informally, these A are obtained from B by distributing, along columns, each non-zero entry within the m^{th}-row of B into rows m through a of A such that these rows of A contain exactly one non-zero, and hence equal to 1, entry.

(ii) Now, let $B \in \mathrm{Tab}(\gamma, \beta)$. Then $\iota_\beta \circ \rho[B] \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\gamma), M(\lambda))$ and one can easily check that $\iota_\beta \circ \rho[B] = \sum_A \rho[A]$, where the sum is over those $A \in \mathrm{Tab}(\gamma, \lambda)$ whose first $(m-1)$ columns agree with those of B, and also $\sum_{j=m}^{\tilde{a}} a_{ij} = b_{im}$ for $1 \leq i \leq \ell$. Informally, these A are obtained from B by distributing, along rows, each non-zero entry within the m^{th}-column of B into columns m through $\tilde{a}$ of A such that these columns of A contain exactly one non-zero, and hence equal to 1, entry.

The following Example details the forms of the compositions of maps discussed in Remark 3.3.11.

Example 3.3.12. Let $\lambda = (3, 1^3)$. Then, we have:

$$\begin{aligned}
\rho\begin{bmatrix} 2\,2 \\ 1\,1 \end{bmatrix} \circ \pi_{(4,2)} &= \rho\begin{bmatrix} 2\,2 \\ 1\,0 \\ 0\,1 \end{bmatrix} + \rho\begin{bmatrix} 2\,2 \\ 0\,1 \\ 1\,0 \end{bmatrix}, \\
\iota_{(3,3)} \circ \rho\begin{bmatrix} 2\,2 \\ 1\,1 \end{bmatrix} &= \rho\begin{bmatrix} 2\,1\,1\,0 \\ 1\,0\,0\,1 \end{bmatrix} + \rho\begin{bmatrix} 2\,1\,0\,1 \\ 1\,0\,1\,0 \end{bmatrix} + \rho\begin{bmatrix} 2\,0\,1\,1 \\ 1\,1\,0\,0 \end{bmatrix}, \\
\iota_{(3,3)} \circ \rho\begin{bmatrix} 2\,2 \\ 1\,1 \end{bmatrix} \circ \pi_{(4,2)} &= \rho\begin{bmatrix} 2\,1\,1\,0 \\ 1\,0\,0\,0 \\ 0\,0\,0\,1 \end{bmatrix} + \rho\begin{bmatrix} 2\,1\,0\,1 \\ 1\,0\,0\,0 \\ 0\,0\,1\,0 \end{bmatrix} + \rho\begin{bmatrix} 2\,0\,1\,1 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{bmatrix} \\
&\quad + \rho\begin{bmatrix} 2\,1\,1\,0 \\ 0\,0\,0\,1 \\ 1\,0\,0\,0 \end{bmatrix} + \rho\begin{bmatrix} 2\,1\,0\,1 \\ 0\,0\,1\,0 \\ 1\,0\,0\,0 \end{bmatrix} + \rho\begin{bmatrix} 2\,0\,1\,1 \\ 0\,1\,0\,0 \\ 1\,0\,0\,0 \end{bmatrix}.
\end{aligned}$$

The following Lemma provides an analogue of Corollary 3.2.19:

Lemma 3.3.13. *Let $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$. Then $h \in \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ if and only if the coefficients $h[B]$ of the $\rho[B]$ in h satisfy:*

(i) *For all $1 \le i < j \le m$, $1 \le k \le m$, and all $B \in \mathrm{Tab}(\alpha, \beta)$ with $b_{jk} \neq 0$, we have:*

$$(b_{ik} + 1)h[B] = \sum_{l \neq k} b_{il} h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big], \qquad (R^k_{i,j}(B))$$

(ii) *For all $1 \le i < j \le m$, $1 \le k \le m$, and all $B \in \mathrm{Tab}(\alpha, \beta)$ with $b_{kj} \neq 0$, we have:*

$$(b_{ki} + 1)h[B] = \sum_{l \neq k} b_{li} h\Big[B^{(k,i)(l,j)}_{(k,j)(l,i)}\Big]. \qquad (C^k_{i,j}(B))$$

Proof. For $B \in \mathrm{Tab}(\alpha, \beta)$, we denote by $\Omega(B)$ the subset of matrices in $\mathrm{Tab}(\lambda', \lambda)$ with:

$$\omega(\rho[B]) = \iota_\beta \circ \rho[B] \circ \pi_\alpha = \sum_{A \in \Omega(B)} \rho[A]. \qquad (3.3.14)$$

Clearly, given $B \neq B' \in \mathrm{Tab}(\alpha, \beta)$, we have that $\Omega(B) \cap \Omega(B') = \varnothing$. Now, we fix $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ with $h = \sum_{B \in \mathrm{Tab}(\alpha,\beta)} h[B]\rho[B]$, and we shall also fix the notation $\tilde{h} \coloneqq \omega(h) = \iota_\beta \circ h \circ \pi_\alpha$. Then, it follows from Remark 3.3.11 that the coefficients $\tilde{h}[A]$ of the $\rho[A]$ in $\tilde{h}$ satisfy:

$$\tilde{h}[A] = \begin{cases} h[B], & \text{if } A \in \Omega(B) \text{ for some } B \in \mathrm{Tab}(\alpha, \beta), \\ 0, & \text{otherwise.} \end{cases} \qquad (3.3.15)$$

Now, suppose that h is relevant and we shall show that the coefficients $h[B]$ of the $\rho[B]$ in h satisfy the relations stated in (i), and it may be shown in a similar manner that they also satisfy the relations stated in (ii). Firstly, note that $\tilde{h}$ is relevant by Lemma 3.3.10(ii). We fix $1 \le i < j \le m$, $1 \le k \le m$, and $B \in \mathrm{Tab}(\alpha, \beta)$ with $b_{jk} \neq 0$. Then, there exists $A \in \Omega(B)$ with $a_{jk} \neq 0$. For such an A, since $\tilde{h}$ is relevant, the relation $R^k_{i,j}(A)$ of Corollary 3.2.19(ii) gives that:

$$(a_{ik} + 1)\tilde{h}[A] = \sum_{l \neq k} a_{il} \tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big]. \qquad (3.3.16)$$

Now, take any $1 \le l \le n$ with $l \neq k$ such that $a_{il} \neq 0$. If $l < m$, then $a_{il} = b_{il}$ and

$A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,k)(j,l)}_{(j,k)(i,l)})$, so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big]$. On the other hand, if $l \geq m$, then $a_{il} = 1$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,k)(j,m)}_{(j,k)(i,m)})$ so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h\Big[B^{(i,k)(j,m)}_{(j,k)(i,m)}\Big]$. Therefore, we may rewrite (3.3.16) as:

$$(a_{ik}+1)h[B] = \sum_{\substack{l<m\\ l\neq k}} b_{il} h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] + \left(\sum_{\substack{l\geq m\\ l\neq k}} a_{il}\right) h\Big[B^{(i,k)(j,m)}_{(j,k)(i,m)}\Big]. \tag{3.3.17}$$

Now, if $k < m$, then $a_{ik} = b_{ik}$ and $\sum_{l\geq m} a_{il} = b_{im}$. Thus, (3.3.17) becomes:

$$(b_{ik}+1)h[B] = \sum_{\substack{l<m\\ l\neq k}} b_{il} h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] + b_{im}\Big[B^{(i,k)(j,m)}_{(j,k)(i,m)}\Big] = \sum_{l\neq k} b_{il} h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big],$$

which is precisely the relation $R^k_{i,j}(B)$.

On the other hand, if $k = m$, then $a_{im} = 0$, since $a_{jm} \neq 0$, and so $\sum_{l>m} a_{il} = b_{im}$. Moreover, $B^{(i,k)(j,m)}_{(j,k)(i,m)} = B$, and so (3.3.17) becomes:

$$h[B] = \sum_{l<m} b_{il} h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] + b_{im} h[B],$$

which in turn gives the relation $R^m_{i,j}(B)$:

$$(b_{im}+1)h[B] = \sum_{l\neq m} b_{il} h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big].$$

Conversely, suppose that the coefficients $h[B]$ of the $\rho[B]$ in h satisfy the relations stated in the Lemma. Note that by Lemma 3.3.10(ii), in order to show that h is relevant, it suffices to show that $\tilde{h}$ is relevant. To this end, we shall show that $\tilde{h} \circ \bar{\phi}^{(i,j,1)}_{\lambda'} = 0$ for $1 \leq i < j \leq n$, and it shall follow similarly that $\bar{\psi}^{(i,j,1)}_{\lambda} \circ \tilde{h} = 0$ for such i, j. Note that $\tilde{h}$ is semirelevant by Lemma 3.3.10(i) and so $\tilde{h} \circ \bar{\phi}^{(i,j,1)}_{\lambda'} = 0$ for $i \geq m$. Therefore, we may assume that $i < m$. Accordingly, fix some $1 \leq i < j \leq n$ with $i < m$. Then, as in the proof of Lemma 3.2.16, we have:

$$\tilde{h} \circ \bar{\phi}^{(i,j,1)}_{\lambda'} = \sum_{C\in \mathrm{Tab}(\lambda'^{(i,j,1)},\lambda)} \left(\sum_{1\leq l\leq n} c_{il} \tilde{h}\Big[C^{(j,l)}_{(i,l)}\Big]\right)\rho[C]. \tag{3.3.18}$$

Let $C \in \mathrm{Tab}(\lambda'^{(i,j,1)}, \lambda)$, and we wish to show that the coefficient of $\rho[C]$ in $\tilde{h} \circ \bar{\phi}^{(i,j,1)}_{\lambda'}$ is equal to 0. According to (3.3.15) and (3.3.18), we may assume that there exists some $1 \leq k \leq n$ with $c_{ik} \neq 0$ such that $A := C^{(j,k)}_{(i,k)} \in \Omega(B)$ for some $B \in \mathrm{Tab}(\alpha,\beta)$, where $\Omega(B)$ is as in (3.3.14), since otherwise, each summand $c_{il}\tilde{h}\Big[C^{(j,l)}_{(i,l)}\Big]$ appearing in the coefficient of $\rho[C]$ in (3.3.18) is equal to zero. Then, it follows from (3.3.18) that the coefficient of $\rho[C]$ in $\tilde{h} \circ \bar{\phi}^{(i,j,1)}_{\lambda'}$ is:

$$c_{ik} h[B] + \sum_{\substack{1\leq l\leq n\\ l\neq k}} c_{il} \tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big]. \tag{3.3.19}$$

We split our consideration into the following cases:

(i) ($j < m$; $k < m$): We have $c_{ik} = a_{ik}+1 = b_{ik}+1$. Now, if $1 \leq l < m$ with $l \neq k$, then

$c_{il} = a_{il} = b_{il}$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,k)(j,l)}_{(j,k)(i,l)})$ so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big]$. On the other hand, if $l \geq m$ with $c_{il} \neq 0$, then $c_{il} = a_{il} = 1$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,k)(j,m)}_{(j,k)(i,m)})$ so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h\Big[B^{(i,k)(j,m)}_{(j,k)(i,m)}\Big]$. Note that there are precisely b_{im} such values of l. Hence, we may rewrite (3.3.19) as:

$$(b_{ik}+1)h[B] + \sum_{\substack{1\leq l<m\\ l\neq k}} b_{il}h\Big[B^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] + b_{im}h\Big[B^{(i,k)(j,m)}_{(j,k)(i,m)}\Big] = 0,$$

since the coefficient $h[B]$ satisfies the relation $R^k_{i,j}(B)$.

(ii) ($j < m$; $k \geq m$): Here, we have $c_{ik} = 1$ and also $b_{jm} \neq 0$ since $A \in \Omega(B)$. Now, if $1 \leq l < m$, then $c_{il} = a_{il} = b_{il}$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,m)(j,l)}_{(j,m)(i,l)})$ so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h\Big[B^{(i,m)(j,l)}_{(j,m)(i,l)}\Big]$. On the other hand, if $l \geq m$ with $l \neq k$ and $c_{il} \neq 0$, then $c_{il} = a_{il} = 1$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B)$ so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h[B]$. Note that there are precisely b_{im} such values of l. Hence, we may rewrite (3.3.19) as:

$$h[B] + \sum_{1\leq l<m} b_{il}h\Big[B^{(i,m)(j,l)}_{(j,m)(i,l)}\Big] + b_{im}h[B] = 0,$$

since the coefficient $h[B]$ satisfies the relation $R^m_{i,j}(B)$.

(iii) ($j \geq m$; $k < m$): Now, we have $c_{ik} = a_{ik} + 1 = b_{ik} + 1$ and also $b_{mk} \neq 0$ since $A \in \Omega(B)$. Now, if $1 \leq l < m$ with $l \neq k$, then $c_{il} = a_{il} = b_{il}$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,k)(m,l)}_{(m,k)(i,l)})$ so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h\Big[B^{(i,k)(m,l)}_{(m,k)(i,l)}\Big]$. On the other hand, if $l \geq m$ with $c_{il} \neq 0$, then $c_{il} = a_{il} = 1$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,k)(m,m)}_{(m,k)(i,m)})$ so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h\Big[B^{(i,k)(m,m)}_{(m,k)(i,m)}\Big]$. Note that there are precisely b_{im} such values of l. Hence, we may rewrite (3.3.19) as:

$$(b_{ik}+1)h[B] + \sum_{\substack{1\leq l<m\\ l\neq k}} b_{il}h\Big[B^{(i,k)(m,l)}_{(m,k)(i,l)}\Big] + b_{im}h\Big[B^{(i,k)(m,m)}_{(m,k)(i,m)}\Big] = 0,$$

since the coefficient $h[B]$ satisfies the relation $R^k_{i,m}(B)$.

(iv) ($j \geq m$; $k \geq m$): Finally, in this case, we have $c_{ik} = 1$ and also $b_{mm} \neq 0$ since $A \in \Omega(B)$. Now, if $1 \leq l < m$, then $c_{il} = a_{il} = b_{il}$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B^{(i,m)(m,l)}_{(m,m)(i,l)})$ so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h\Big[B^{(i,m)(m,l)}_{(m,m)(i,l)}\Big]$. On the other hand, if $l \geq m$ with $l \neq k$ and $c_{il} \neq 0$, then $c_{il} = a_{il} = 1$ with $A^{(i,k)(j,l)}_{(j,k)(i,l)} \in \Omega(B)$ so that $\tilde{h}\Big[A^{(i,k)(j,l)}_{(j,k)(i,l)}\Big] = h[B]$. Note that there are precisely b_{im} such values of l. Hence, we may rewrite (3.3.19) as:

$$h[B] + \sum_{1\leq l<m} b_{il}h\Big[B^{(i,m)(m,l)}_{(m,m)(i,l)}\Big] + b_{im}h[B] = 0,$$

since the coefficient $h[B]$ satisfies the relation $R^m_{i,m}(B)$.

Thus, we have shown that the coefficient of $\rho[C]$ in $\tilde{h} \circ \bar{\phi}^{(i,j,1)}_{\lambda'}$ is zero in all possible cases, and so we are done. $\square$

Now, since α and β both have length m, we may ignore the final $(n-m)$ rows and columns of each matrix in $\mathrm{Tab}(\alpha,\beta)$ and $\mathrm{Tab}(\beta,\alpha)$. Accordingly, we identify each of

$\mathrm{Tab}(\alpha,\beta)$ and $\mathrm{Tab}(\beta,\alpha)$ with the sets $\mathcal{T}$ and $\mathcal{T}'$ respectively, where:

$$\mathcal{T} := \left\{ A \in M_{m\times m}(\mathbb{N}) \;\middle|\; \begin{array}{l} \sum_j a_{ij} = \alpha_i, \\ \sum_i a_{ij} = \beta_j. \end{array} \right\}, \quad \mathcal{T}' := \left\{ A \in M_{m\times m}(\mathbb{N}) \;\middle|\; \begin{array}{l} \sum_j a_{ij} = \beta_i, \\ \sum_i a_{ij} = \alpha_j. \end{array} \right\}.$$

Remark 3.3.20. Note that λ and its transpose λ' are of the same form. That is to say, the swap $\lambda \leftrightarrow \lambda'$ is equivalent to the swap $(a,b) \leftrightarrow (\tilde{a},\tilde{b})$, where $\tilde{a} = b+m-1$, $\tilde{b} = a-m+1$ respectively, which in turn is equivalent to the swap $\alpha \leftrightarrow \beta$. Therefore, after defining the notion of *relevance* for elements $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$, similarly to Definition 3.3.8(ii), and also swapping $\mathcal{T}$ with $\mathcal{T}'$, we obtain the following analogue of Lemma 3.3.13:

Corollary 3.3.21. *Let $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$. Then $h \in \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$ if and only if the coefficients $h[B]$ of the $\rho[B]$ in h satisfy:*

(i) *$R^k_{i,j}(B)$ for all $1 \le i < j \le m$, $1 \le k \le m$, and $B \in \mathcal{T}'$ with $b_{jk} \neq 0$,*

(ii) *$C^k_{i,j}(B)$ for all $1 \le i < j \le m$, $1 \le k \le m$, and $B \in \mathcal{T}'$ with $b_{kj} \neq 0$.*

The following Remark is clear:

Remark 3.3.22. Let $m \ge 2$ and $\lambda = (a, m-1, m-2, \dots, 2, 1^b)$. Then:

(i) We have a $\Bbbk$-linear embedding of the endomorphism algebra of $\mathrm{Sp}(\lambda)$ into the $\Bbbk$-space $\mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$.

(ii) We have a $\Bbbk$-linear embedding of the endomorphism algebra of $\mathrm{Sp}(\lambda')$ into the $\Bbbk$-space $\mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$.

Remark 3.3.23. Let $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ and consider the transpose homomorphism $h' \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$. We have:

(i) For $1 \le i < j \le m$, $1 \le k \le m$, and $A \in \mathcal{T}$ with $a_{jk} \neq 0$, the relation $R^k_{i,j}(A)$ concerning the coefficient of $\rho[A]$ in h coincides with the relation $C^k_{i,j}(A')$ concerning the coefficient of $\rho[A']$ in h'.

(ii) For $1 \le i < j \le m$, $1 \le k \le m$, and $A \in \mathcal{T}$ with $a_{kj} \neq 0$, the relation $C^k_{i,j}(A)$ concerning the coefficient of $\rho[A]$ in h coincides with the relation $R^k_{i,j}(A')$ concerning the coefficient of $\rho[A']$ in h'.

(iii) The transpose homomorphism h' is relevant if and only if h is relevant.

3.3.2. **A Critical Relation.** Here, we shall highlight a new relation that occurs as a combination of the relations $R^k_{i,j}(A)$ and $C^k_{i,j}(A)$ of Lemma 3.3.13 that will play an important role in our considerations below.

Lemma 3.3.24. *Suppose that $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ is a relevant homomorphism. Then the coefficients $h[A]$ of the $\rho[A]$ in h satisfy the relations:*

$$z_{j,k}(A)h[A] = \sum_{\substack{i<j \\ l>k}} a_{il} h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right] + \sum_{\substack{i>j \\ l<k}} a_{il} h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right], \tag{$Z_{j,k}(A)$}$$

for all $1 \le j,k \le m$ and $A \in \mathcal{T}$ with $a_{jk} \neq 0$, where $z_{j,k}(A) := \sum\limits_{i<j} a_{ik} + \sum\limits_{l<k} a_{jl} + j + k \in \Bbbk$.

Proof. Since h is relevant, the coefficients $h[A]$ of the $\rho[A]$ in h satisfy the relations of Lemma 3.3.13, and so in particular, given $1 \leq j, k \leq m$, the coefficients satisfy the relation $\sum_{i<j} R^k_{i,j}(A) + \sum_{l<k} C^j_{l,k}(A)$ for all $A \in \mathcal{T}$ with $a_{jk} \neq 0$. But, the left-hand side of this relation is given by:

$$\sum_{i<j}(a_{ik}+1)h[A] + \sum_{l<k}(a_{jl}+1)h[A] = z_{j,k}(A)h[A], \tag{3.3.25}$$

by definition of $z_{j,k}(A)$. On the other hand, the right-hand side of this relation is:

$$\sum_{\substack{i<j\\ l\neq k}} a_{il} h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right] + \sum_{\substack{l<k\\ i\neq j}} a_{il} h\left[A^{(j,l)(i,k)}_{(j,k)(i,l)}\right]. \tag{3.3.26}$$

Now, notice that for $i < j$, $l < k$ we have $A^{(j,l)(i,k)}_{(j,k)(i,l)} = A^{(i,k)(j,l)}_{(j,k)(i,l)}$ and so after cancelling those terms that appear twice, we may rewrite (3.3.26) as:

$$\sum_{\substack{i<j\\ l\neq k}} a_{il} h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right] + \sum_{\substack{l<k\\ i\neq j}} a_{il} h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right] = \sum_{\substack{i<j\\ l>k}} a_{il} h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right] + \sum_{\substack{i>j\\ l<k}} a_{il} h\left[A^{(i,k)(j,l)}_{(j,k)(i,l)}\right],$$

which, along with (3.3.25), gives the required expression. □

3.4. One-Dimensional Endomorphism Algebra

From here, we shall assume that the parameters a, b, and m satisfy the parity condition: $a - m \equiv b \pmod 2$. Note that this condition is preserved by the swap $(a, b) \leftrightarrow (\tilde{a}, \tilde{b})$, where $\tilde{a} := b + m - 1$, $\tilde{b} := a - m + 1$.

Firstly, we highlight some basic properties of the coefficients $z_{j,k}(A)$ of Lemma 3.3.24.

Lemma 3.4.1. *Let $A \in \mathcal{T}$. Then:*

(i) $z_{j,k}(A) = \sum_{i>j} a_{ik} + \sum_{l>k} a_{jl} + \alpha_j + \beta_k + j + k$ *for* $1 \leq j, k \leq m$.

(ii) $z_{j,k}(A) = \sum_{i>j} a_{ik} + \sum_{l>k} a_{jl}$ *for* $1 < j, k < m$.

(iii) $z_{j,m}(A) = b + 1 + \sum_{i>j} a_{im}$ *and* $z_{m,k}(A) = a + m + \sum_{i>k} a_{mi}$ *for* $1 < j, k < m$.

(iv) $z_{m,m}(A) = 1$.

(v) $z_{1,m}(A) = \sum_{i>1} a_{im}$ *and* $z_{m,1}(A) = \sum_{i>1} a_{mi}$.

Proof. Part (i) follows from substituting the two expressions: $\sum_{i<j} a_{ik} = \beta_k - \sum_{i\geq j} a_{ik}$ and $\sum_{l<k} a_{jl} = \alpha_j - \sum_{l\geq k} a_{ji}$ into the definition of $z_{i,j}(A)$. Parts (ii)-(v) then follow immediately from part (i) along with the forms of α and β. □

Definition 3.4.2. Let $A, B \in \mathcal{T}$. Then:

(i) We write $A <_R B$ to mean that B follows A under the induced lexicographical order on rows, reading left to right and bottom to top. This is a total order and we call it the *row-order*.

(ii) We write $A <_C B$ to mean that B follows A under the induced lexicographical order on columns, reading top to bottom and right to left. This is a total order and we call it the *column-order*.

Remark 3.4.3. Let $1 \leq j, k \leq m$ and let $A \in \mathcal{T}$ with $a_{jk} \neq 0$. Then any $B = A^{(i,k)(j,l)}_{(j,k)(i,l)}$ that appears in the relation $Z_{j,k}(A)$ of Lemma 3.3.24 satisfies both $B <_R A$ and $B <_C A$.

From now on, we fix a relevant homomorphism $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$.

Lemma 3.4.4. *Let $A \in \mathcal{T}$ and suppose that $a_{mm} \neq 0$. Then $h[A] = 0$.*

Proof. Firstly, $z_{m,m}(A) = 1$ by Lemma 3.4.1(iv), and the result follows by $Z_{m,m}(A)$. □

Remark 3.4.5. Assume that $m = 2$, where then $\alpha = (b+1, a-1)$ and $\beta = (a, b)$. Suppose that $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ is a non-zero relevant homomorphism, and suppose that $A \in \mathcal{T}$ is such that $h[A] \neq 0$. We may assume that $a_{22} = 0$ by Lemma 3.4.4. Now, since $a_{12} + a_{22} = b$ and $a_{21} + a_{22} = a - 1$, we deduce that $a_{12} = b$ and $a_{21} = a - 1$. Moreover, since $a_{11} + a_{12} = b + 1$, we have that $a_{11} = 1$. Hence, there is a unique matrix A for which $h[A] \neq 0$, namely:

$$A = \begin{array}{|c|c|} \hline 1 & b \\ \hline a-1 & 0 \\ \hline \end{array}\,.$$

Hence for $\lambda = (a, 1^b)$ with $a \equiv b \pmod 2$, we deduce that $\mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda)) \cong \Bbbk$, and in this way we recover Murphy's result [Mur, Theorem 4.1].

Lemma 3.4.6. *Let $A \in \mathcal{T}$ and suppose that there exist some $1 < j, k < m$ such that $a_{jm} \neq 0$ and $a_{mk} \neq 0$. Then $h[A] = 0$.*

Proof. Suppose for contradiction that the claim is false and let $A \in \mathcal{T}$ be a counterexample that is minimal with respect to the column-order $<_C$. We choose $1 < j, k < m$ to be maximal such that $a_{jm}, a_{mk} \neq 0$. We may assume that $a_{mm} = 0$ by Lemma 3.4.4. Now, by Lemma 3.4.1(iii) we have $z_{j,m}(A) + z_{m,k}(A) = 1$ and so the relation $Z_{j,m}(A) + Z_{m,k}(A)$ gives:

$$h[A] = \sum_{\substack{i>j \\ l<m}} a_{il} h[B^{[i,l]}] + \sum_{\substack{i<m \\ l>k}} a_{il} h[D^{[i,l]}],$$

where $B^{[i,l]} := A^{(i,m)(j,l)}_{(j,m)(i,l)}$ for $i > j$, $l < m$ with $a_{il} \not\equiv 0$, and $D^{[i,l]} := A^{(i,k)(m,l)}_{(m,k)(i,l)}$ for $i < m$, $l > k$ with $a_{il} \not\equiv 0$.

Suppose that $i > j$, $l < m$ are such that $a_{il} \not\equiv 0$, and consider the matrix $B^{[i,l]}$. If $i = m$, then $b^{[m,l]}_{mm} \neq 0$ and so $h[B^{[m,l]}] = 0$ by Lemma 3.4.4. On the other hand, if $i < m$ then $b^{[i,l]}_{im}, b^{[i,l]}_{mk} \neq 0$, and notice also that $B^{[i,l]} <_C A$ by Remark 3.4.3. Therefore, by minimality of A, we have that $h[B^{[i,l]}] = 0$. Similarly, one may show that $h[D^{[i,l]}] = 0$ for $i < m$, $l > k$ with $a_{il} \not\equiv 0$, and so we deduce that $h[A] = 0$. □

Definition 3.4.7. We define the sets:

(i) $\mathcal{TR} := \{A \in \mathcal{T} \mid a_{i1} = 1 \text{ for } 1 \leq i < m, \text{ and } a_{mk} = 0 \text{ for } 1 < k \leq m\}$.

(ii) $\mathcal{TC} := \{A \in \mathcal{T} \mid a_{1k} = 1 \text{ for } 1 \leq k < m, \text{ and } a_{im} = 0 \text{ for } 1 < i \leq m\}$.

Lemma 3.4.8. *Let $A \in \mathcal{T}$ and suppose that $A \notin \mathcal{TR} \cup \mathcal{TC}$. Then $h[A] = 0$.*

Proof. By Lemma 3.4.4 we may assume that $a_{mm} = 0$. Suppose that $a_{mk} \neq 0$ for some k with $1 < k < m$. Then, by Lemma 3.4.6, we may assume that $a_{jm} = 0$ for $1 < j < m$. But then $a_{1m} = b$ and so $\sum_{l<m} a_{1l} = m - 1$. Since $A \notin \mathcal{TC}$ we deduce

that there exists some $1 \leq l < m$ with $a_{1l} = 0$. Now, the relation $C^1_{l,m}(A)$ gives that $h[A] = \sum_{j>1} a_{jl} h[B^{[j]}]$ where $B^{[j]} := A^{(1,l)(j,m)}_{(1,m)(j,l)}$ for $j > 1$ with $a_{jl} \not\equiv 0$. Suppose that $j > 1$ is such that $a_{jl} \not\equiv 0$. If $j = m$ then $b^{[m]}_{mm} \neq 0$ and so $h[B^{[m]}] = 0$ by Lemma 3.4.4. Moreover, for $1 < j < m$ we have that $b^{[j]}_{mk}, b^{[j]}_{jm} \neq 0$ and so $h[B^{[j]}] = 0$ by Lemma 3.4.6. Therefore, we deduce that $h[A] = 0$.

Hence, we may assume that $a_{mk} = 0$ for all $1 < k \leq m$ and so it follows that $a_{m1} = a - m + 1$ and that $\sum_{j<m} a_{j1} = m - 1$. However, since $A \notin \mathcal{TR}$ we must have that $a_{j1} = 0$ for some j with $1 \leq j < m$. Now, the relation $R^1_{j,m}(A)$ gives $h[A] = \sum_{l>1} a_{jl} h[D^{[l]}]$ where $D^{[l]} := A^{(j,1)(m,l)}_{(m,1)(j,l)}$ for $l > 1$ with $a_{jl} \not\equiv 0$. Suppose that $l > 1$ is such that $a_{jl} \not\equiv 0$. If $l = m$, then $d^{[m]}_{mm} \neq 0$ and so $h[D^{[m]}] = 0$ by Lemma 3.4.4. On the other hand, if $1 < l < m$ then $d^{[l]}_{ml} \neq 0$. Now, if $d^{[l]}_{um} \neq 0$ for some $1 < u < m$, then $h[D^{[l]}] = 0$ by Lemma 3.4.6. Hence, we may assume that $d^{[l]}_{um} = 0$ for all $1 < u < m$ and so we deduce that $d^{[l]}_{1m} = a_{1m} = b$. Since $A \notin \mathcal{TC}$ we have that there exists some $1 \leq k < m$ with $a_{1k} = 0$ and hence $d^{[l]}_{1k} = 0$. Then, the relation $C^1_{k,m}(D^{[l]})$ expresses $h[D^{[l]}]$ as a linear combination of $h[F]$s where either $f_{mm} \neq 0$, or $f_{ml} \neq 0$ and $f_{vm} \neq 0$ for some v with $1 < v < m$. Once again, Lemma 3.4.4 and Lemma 3.4.6 give that $h[F] = 0$ for all such F and so $h[D^{[l]}] = 0$. Hence $h[A] = 0$. $\square$

Definition 3.4.9. We shall require some additional notation that we shall introduce here:

(i) In order to assist with counting in reverse, set $\tau(i) := m - (i - 1)$ for $1 \leq i \leq m$.

(ii) For $1 < i < m$, we define:

$$\mathcal{TR}_i := \{A \in \mathcal{TR} \mid \text{the } \tau(j)^{\text{th}}\text{-row of } A \text{ contains } j \text{ odd entries for } 1 < j \leq i\}.$$

(iii) For $1 < i < m$, we define $\overline{\mathcal{TR}_i} := \mathcal{TR}_i \setminus \mathcal{TR}_{i+1}$, where we set $\mathcal{TR}_m := \varnothing$.

Remark 3.4.10. Let $A \in \mathcal{T}$. Recall that $\sum_l a_{\tau(i)l} = i$ for $1 < i < m$. Therefore, if $A \in \mathcal{TR}_i$ for some $1 < i < m$, then the $\tau(j)^{\text{th}}$-row of A consists entirely of ones and zeros for all $1 < j \leq i$.

Definition 3.4.11. Let $1 < i < m$ and $A \in \overline{\mathcal{TR}_i}$. Then:

(i) We set $\mathcal{K}_A := \{2 \leq k \leq i \mid a_{uk} = 1 \text{ for } \tau(i) \leq u \leq \tau(k)\}$.

(ii) We set $k_A := \min\{2 \leq k \leq i + 1 \mid k \notin \mathcal{K}_A\}$.

(iii) If $k_A \leq i$, we set $j_A := \min\{k_A \leq j \leq i \mid a_{\tau(j)k_A} = 0\}$.

(iv) If $k_A \leq i$ and $k_A \leq j \leq i$, we denote by $w^j(A) := (w^j_1(A), w^j_2(A), \ldots)$ the decreasing sequence of column-indices within the final $\tau(k_A)$ columns of A that satisfy $a_{\tau(j)w^j_s(A)} = 1$ for $s \geq 1$.

Notice that the sequence $w^j(A)$ has $j - k_A + 1$ terms.

Example 3.4.12. We have $k_A = 4$, $j_A = 4$, and $w^5(A) = (7, 5)$, where:

$$A := \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 1 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a-m+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \in \overline{\mathcal{TR}}_5.$$

Lemma 3.4.13. *Let $2 < i < m$ and let $A \in \overline{\mathcal{TR}}_i$ with $k_A \leq i$. Suppose that there exists some index k with $k_A < k \leq i$ such that $w^j_t(A) = w^{j-1}_{t-1}(A)$ for all $k_A < j \leq k$ and all even t. Then for $l \geq k_A$, $k_A \leq j \leq k$, we have $\sum_{u \geq \tau(j)} a_{ul} \equiv 1$ if and only if $l = w^j_s(A)$ for some odd s.*

Proof. We proceed by induction on j. The case $j = k_A$ is clear and so we may assume that $j > k_A$ and that the claim holds for all smaller values of j in the given range. Let $l \geq k_A$ and suppose that $\sum_{u \geq \tau(j)} a_{ul} \equiv 1$. Suppose, for the moment, that $a_{\tau(j)l} = 0$. Then $\sum_{u \geq \tau(j)} a_{ul} = \sum_{u \geq \tau(j-1)} a_{ul}$, and so $l = w^{j-1}_s(A)$ for some odd s by the inductive hypothesis. However, $w^j_{s+1}(A) = w^{j-1}_s(A) = l$ and so $a_{\tau(j)l} = 1$, contradicting that $a_{\tau(j)l} = 0$. Hence, $a_{\tau(j)l} = 1$ and so $l = w^j_s(A)$ for some s. Moreover, $\sum_{u \geq \tau(j)} a_{ul} \equiv 1$ if and only if $\sum_{u \geq \tau(j-1)} a_{ul} \equiv 0$ and so by Lemma 3.4.13 we deduce that $l \neq w^{j-1}_{s'}(A)$ for any odd s'. Now, if s is even then $w^j_s(A) = w^{j-1}_{s-1}(A)$, leading to a contradiction. Hence, s must be odd. Conversely, suppose that $l = w^j_s(A)$ for some odd s, and suppose, for the sake of contradiction, that $\sum_{u \geq \tau(j)} a_{ul} \equiv 0$. Then, there exists some $k_A \leq j' < j$ such that $a_{\tau(j')l} = 1$, and we choose j' to be maximal with this property. Therefore, $a_{ul} = 0$ for $\tau(j) < u < \tau(j')$ and $\sum_{u \geq \tau(j')} a_{ul} \equiv 1$. Then, by the inductive hypothesis, $l = w^{j'}_{s'}(A)$ for some odd s'. But then $w^{j'+1}_{s'+1}(A) = w^{j'}_{s'}(A) = l$, by our assumption, and so $a_{\tau(j'+1)l} = 1$. Now, by the maximality of j', we must have $j' + 1 = j$. Thus, $l = w^{j'+1}_{s'+1}(A) = w^j_{s'+1}(A) = w^j_s(A)$ and so $s' + 1 = s$, which is impossible since s' and s are both odd. Hence $\sum_{u \geq \tau(j)} a_{ul} \equiv 1$, and so we are done. $\square$

Lemma 3.4.14. *Let $2 < i < m$ and let $A \in \overline{\mathcal{TR}}_i$ with $k_A \leq i$. Suppose that $z_{\tau(j),l}(A) = 0$ for all $k_A \leq j \leq i$, $k_A \leq l < m$ with $a_{\tau(j)l} = 1$. Then $w^j_s(A) = w^{j-1}_{s-1}(A)$ for $k_A < j \leq i$ and even s with $s \leq j - k_A + 1$.*

Proof. We fix i and we proceed by induction on j, increasing from $j = k_A + 1$. Here $w^j(A) = (w^j_1(A), w^j_2(A))$ and for $w := w^j_2(A)$ we have $z_{\tau(j),w}(A) = 0$. Now, by applying Lemma 3.4.1(ii), we have that $z_{\tau(j),w}(A) = \sum_{u > \tau(j)} a_{uw} + \sum_{v > w} a_{\tau(j)v} = a_{\tau(j-1)w} + 1$. Therefore, the entry $a_{\tau(j-1)w}$ is odd and so $w = w^{k_A}_1(A)$ as required. Suppose now that $k_A + 1 < j \leq i$ and that the claim holds for smaller values of j in the given range. Note that this implies that the hypotheses of Lemma 3.4.13 are met for $k = j - 1$.

Suppose that s is even and set $l := w^j_s(A)$. Then $\sum_{u > \tau(j)} a_{ul} + s - 1 \equiv 0$ by Lemma 3.4.1(ii) since $z_{\tau(j),l}(A) = 0$. Therefore, $\sum_{u \geq \tau(j-1)} a_{ul} \equiv 1$ and so by the inductive hypothesis, Lemma 3.4.13 applies and gives that $l = w^{j-1}_{s'}(A)$ for some odd s' with $s' \leq j - k_A$. Now, the sequence $w^j(A)$ has exactly one extra term compared to

$w^{j-1}(A)$ and so the number of even indices in $w^j(A)$ equals the number of odd indices in $w^{j-1}(A)$. It follows that $s' = s - 1$ and so we are done. □

Lemma 3.4.15. *Let $1 < i < m$ and let $A \in \overline{\mathcal{TR}_i}$ with $k_A \leq i$. Suppose that A satisfies the inequality $w_1^j(A) > w_1^{j-1}(A)$ for all $j_A < j \leq i$. Then we may express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ where either:*

(i) *$B \in \overline{\mathcal{TR}_{i'}}$ for some $i' < i$,*

(ii) *$B \in \overline{\mathcal{TR}_i}$ with $k_B > k_A$,*

(iii) *$B \in \overline{\mathcal{TR}_i}$ with $k_B = k_A$ and $B <_C A$, which is witnessed within the final $\tau(w_1^{j_A}(A))$ columns of A and B.*

Moreover, if $A \notin \mathcal{TC}$ then $B \notin \mathcal{TC}$ for all such B listed above.

Proof. To ease notation we set $u := \tau(j_A) > 1$, $k := k_A$, and $w := w_1^{j_A}(A)$. Notice that $w > k$, and that $a_{uk} = 0$ and $a_{uw} = 1$. The relation $C_{k,w}^u(A)$ gives $h[A] = \sum_{l \neq u} a_{lk} h[B^{[l]}]$ where $B^{[l]} := A_{(u,w)(l,k)}^{(u,k)(l,w)}$ for $l \neq u$ with $a_{lk} \not\equiv 0$. Let $l \neq u$ be such that $a_{lk} \not\equiv 0$, and let $k^{[l]} := k_{B^{[l]}}$, $j^{[l]} := j_{B^{[l]}}$, and $w^{[l]} := w_1^{j^{[l]}}(B^{[l]})$. We shall proceed by induction on j_A, decreasing from $j_A = i$.

Firstly, suppose that $j_A = i$. If $l > u$ and $a_{lw} \neq 0$, then $B^{[l]} \in \overline{\mathcal{TR}_{i'}}$ for some $i' < i$, and so $B^{[l]}$ is as described in case (i). Now, if $l > u$ with $a_{lw} = 0$, then $k^{[l]} = k$, $B^{[l]} <_C A$, and the final column in which $B^{[l]}$ and A differ is the w^{th}-column. Hence, here $B^{[l]}$ is as described in case (iii). On the other hand, if $l < u$, then $k^{[l]} > k$ and $B^{[l]}$ is as described in case (ii).

Now, suppose that $j_A < i$ and that the claim holds for all $D \in \overline{\mathcal{TR}_i}$ with $j_A < j_D \leq i$. We split our consideration into steps:

Step 1: If $l > u$ and $a_{lw} \neq 0$, then $B^{[l]} \in \overline{\mathcal{TR}_{i'}}$ for some $i' < i$, and so $B^{[l]}$ is as described in case (i). On the other hand, if $l > u$ and $a_{lw} = 0$, then $B^{[l]} \in \overline{\mathcal{TR}_i}$ with $k^{[l]} = k$ and $B^{[l]} <_C A$. Moreover, the final column in which $B^{[l]}$ and A differ in this case is the w^{th}-column and so $B^{[l]}$ is as described in case (iii).

Step 2: Now, if $\tau(i) \leq l < u$ with $a_{lw} \neq 0$. Then $B^{[l]} \in \overline{\mathcal{TR}_{m-l}}$ with $m - l < i$ since $l \geq \tau(i) = m - i + 1$, and so $B^{[l]}$ is as described as in case (i).

Step 3: On the other hand, if $\tau(i) \leq l < u$ and $a_{lw} = 0$, then $B^{[l]} \in \overline{\mathcal{TR}_i}$ with $k^{[l]} = k$ and $j^{[l]} > j_A$. Moreover, the final column in which A and B differ is the w^{th}-column, and so $w_1^j(B^{[l]}) = w_1^j(A)$ for all $j_A < j \leq i$ since $w_1^j(A) > w_1^{j-1}(A)$ for all $j_A < j \leq i$, and so in particular $w_1^j(B^{[l]}) > w_1^{j-1}(B^{[l]})$ for each $j^{[l]} < j \leq i$. Hence, by the inductive hypothesis, $B^{[l]}$ must satisfy the claim, and so $h[B^{[l]}]$ may be written as a linear combination of $h[D]$s for some $D \in \mathcal{T}$ where either:

(iv) $D \in \overline{\mathcal{TR}_{i'}}$ for some $i' < i$,

(v) $D \in \overline{\mathcal{TR}_i}$ with $k_D > k^{[l]}$,

(vi) $D \in \overline{\mathcal{TR}_i}$ with $k_D = k^{[l]}$ and $D <_C B^{[l]}$, which is witnessed within the final $\tau(w^{[l]})$ columns of $B^{[l]}$ and D.

Any such D as in (iv) is as described in case (i), whereas any such D as in (v) is as described in case (ii) since $k^{[l]} = k_A$. Now, notice that the final $\tau(w^{[l]})$ columns of A and $B^{[l]}$ match since $w^{[l]} > w$, and so any such D as in (vi) also satisfies $D <_C A$ (witnessed within the final $\tau(w)$ columns of A and D), and so is as described in case (iii).

Step 4: Finally, if $l < \tau(i)$, then $B^{[l]} \in \overline{\mathcal{TR}}_i$. Moreover, if $a_{tk} = 1$ for all t in the range $\tau(i) \leq t < \tau(j_A)$, then $k^{[l]} > k$ and so $B^{[l]}$ is as described in case (ii). On the other hand, if $a_{tk} = 0$ for some t in this range, then $k^{[l]} = k$ with $j^{[l]} > j_A$ and then one may proceed as in Step 3 above.

Now, suppose that $A \notin \mathcal{TC}$ but $B^{[l]} \in \mathcal{TC}$ for some $l \neq u$ with $a_{lk} \not\equiv 0$. Notice that this forces $l = 1$ and $a_{lk} = 2$, which contradicts that $a_{lk} \not\equiv 0$. Hence if $A \notin \mathcal{TC}$, then $B^{[l]} \notin \mathcal{TC}$ for all $l \neq u$ with $a_{lk} \not\equiv 0$. By applying this argument recursively, it follows that if $A \notin \mathcal{TC}$, then all such B produced by this procedure satisfy $B \notin \mathcal{TC}$ as well. □

Lemma 3.4.16. *Let $1 < i < m-1$ and let $A \in \overline{\mathcal{TR}}_i$ with $k_A = i+1$. Then we may express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ where either:*

(i) *$B \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$,*

(ii) *$B \notin \mathcal{TR}$.*

Moreover, if $A \notin \mathcal{TC}$ then $B \notin \mathcal{TC}$ for all such B listed above.

Proof. Firstly, recall that the sum of the entries in the $\tau(i+1)^{\text{th}}$-row of A is $i+1$. Now, since $A \notin \mathcal{TR}_{i+1}$, we deduce that the $\tau(i+1)^{\text{th}}$-row of A contains at most $i-1$ odd entries. Hence, there exists some $1 < s \leq i$ such that $a_{\tau(i+1)s}$ is even and we choose s be minimal with this property. To ease notation, we set $q := \tau(i+1)$ and $u := \tau(s)$. Note that $a_{us} = 1$. The relation $R^s_{q,u}(A)$ gives that $h[A] = \sum_{l \neq s} a_{ql} h[B^{[l]}]$ where $B^{[l]} := A^{(q,s)(u,l)}_{(u,s)(q,l)}$ for $l \neq s$ with $a_{ql} \not\equiv 0$.

If $l = 1$, then $B^{[1]} \notin \mathcal{TR}$, and so $B^{[1]}$ is as described in case (ii). Now, if $1 < l < s$, then $B^{[l]} \in \overline{\mathcal{TR}}_{s-1}$ with $s - 1 < i$, and so $B^{[l]}$ is as described in case (i). Meanwhile, if $l > s$, then $B^{[l]} \in \overline{\mathcal{TR}}_i$ and, as in the previous paragraph, we may find some $s < t \leq i$ (depending on l) such that $b^{[l]}_{qt}$ is even, and we take t to be minimal with this property. The relation $R^t_{q,\tau(t)}(B^{[l]})$ expresses $h[B^{[l]}]$ as a linear combination of $h[D]$s for some $D \in \mathcal{T}$ that must either fit into one of the cases described in the statement of the claim, or otherwise once again $D \in \overline{\mathcal{TR}}_i$ and there exists some $t < v \leq i$ such that d_{qv} is even, and we take v to be minimal with this property. Noting that $v > t > s$, it is clear that this process must terminate, hence providing the desired expression for $h[A]$.

Now, suppose that $A \notin \mathcal{TC}$ but $B^{[l]} \in \mathcal{TC}$ for some $l \neq s$ with $a_{ql} \not\equiv 0$. Then, notice that $B^{[l]}$ agrees with A outside the $\tau(i+1)^{\text{th}}$-row and $\tau(s)^{\text{th}}$-row, and so in particular they agree in the first row since $i < m-1$. Hence $a_{1v} = b^{[l]}_{1v} = 1$ for $1 \leq v < m$ since $B^{[l]} \in \mathcal{TC}$. Now, by considering the first row-sum and the last column-sum of A, we deduce that $a_{1m} = b$ and $a_{vm} = 0$ for $1 < v \leq m$. However, this implies that $A \in \mathcal{TC}$, which is a contradiction. Once again, by applying this argument recursively, it follows that if $A \notin \mathcal{TC}$, then all such B produced by this procedure satisfy $B \notin \mathcal{TC}$ as well. □

Lemma 3.4.17. *Let $1 < i < m-1$ and let $A \in \overline{\mathcal{TR}}_i$. Then we may express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T} \setminus \mathcal{TR}$. Moreover, if $A \notin \mathcal{TC}$ then all such B satisfy $B \notin \mathcal{TR} \cup \mathcal{TC}$.*

Proof. We proceed by induction on $i \geq 2$. Firstly, suppose that $i = 2$. Since $A \notin \mathcal{TR}_3$ with $\sum_l a_{(m-2)l} = 3$, the $(m-2)^{\text{th}}$-row of A must contain a single odd entry, which must then be equal to 1, and be located in the first column of A. On the other hand, since $A \in \mathcal{TR}_2$, there exists a unique $l > 1$ with $a_{(m-1)l} = 1$. The relation $R^l_{m-2,m-1}(A)$ gives $h[A] = h[B]$ for $B := A^{(m-2,l)(m-1,1)}_{(m-1,l)(m-2,1)}$. Evidently, $B \notin \mathcal{TR}$, and so the claim holds for $i = 2$.

Now, we suppose that $i > 2$ and that the claim holds for all $B \in \mathcal{T}$ such that $B \in \overline{\mathcal{TR}}_{i'}$ for some $2 \leq i' < i$. Suppose, for the sake of contradiction, that the claim fails for this particular value of i and consider the set of counterexamples $A \in \overline{\mathcal{TR}}_i$ whose value of k_A is maximal amongst all counterexamples. Now, we choose A to be the element of this set that is minimal with respect to the column-ordering. In other words, if $D \in \overline{\mathcal{TR}}_i$ is a counterexample to the claim, then either $k_D < k_A$, or $k_D = k_A$ and $D \geq_C A$.

Now if $k_A = i + 1$, then Lemma 3.4.16 states that we may express $h[A]$ as a linear combination of some $h[B]$s for some $B \in \mathcal{T}$ where either $B \in \overline{\mathcal{TR}}_{i'}$ with $i' < i$, or $B \notin \mathcal{TR}$. In the first case the inductive hypothesis states that $h[B]$ can be expressed as a linear combination of some $h[D]$s with $D \notin \mathcal{TR}$, whilst in the second case we have $B \in \mathcal{T} \setminus \mathcal{TR}$. Thus, $h[A]$ satisfies the statement of the claim which contradicts that A was chosen to be a counterexample.

Hence, we may assume that $k_A \leq i$. Suppose, for the sake of contradiction, that there exists $k_A \leq j \leq i$, $k_A \leq k < m$ such that $a_{\tau(j)k} = 1$ and $z_{\tau(j),k}(A) = 1$. The relation $Z_{\tau(j),k}(A)$ gives the expression:

$$h[A] = \sum_{\substack{u<\tau(j) \\ l>k}} a_{ul} h[B^{[u,l]}] + \sum_{\substack{u>\tau(j) \\ l<k}} a_{ul} h[B^{[u,l]}], \tag{3.4.18}$$

where $B^{[u,l]} := A^{(u,k)(\tau(j),l)}_{(\tau(j),k)(u,l)}$ for all such (u, l) satisfying $a_{ul} \not\equiv 0$.

Now, set $B := B^{[u,l]}$ where (u, l) is as in (3.4.18) with $a_{ul} \not\equiv 0$. We claim that B fits into one of the following cases: $B \notin \mathcal{TR}$, $B \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$, or $B \in \overline{\mathcal{TR}}_i$ with $k_B = k_A$ and $B <_C A$. We provide full details for the case where $u > \tau(j)$, $l < k$ with the other case, that is $u < \tau(j)$, $l > k$, being similar.

If $l = 1$ then $B \notin \mathcal{TR}$ and so B is of the desired form. Now, if $1 < l < k_A$, then either $u \geq \tau(k_A)$ or $\tau(j) < u < \tau(k_A)$. In the first case, we have $B \in \overline{\mathcal{TR}}_{j-1}$, whilst in the second case we have $B \in \overline{\mathcal{TR}}_{\tau(u)-1}$ if $a_{uk} = 1$ and $B \in \overline{\mathcal{TR}}_{j-1}$ if $a_{uk} = 0$. Hence, in either case, we deduce that $B \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$. Suppose now that $k_A \leq l < k$, then we must have $\tau(j) < u \leq \tau(k_A)$ since $a_{ul} \not\equiv 0$. Now, if $a_{uk} = 1$ then $B \in \overline{\mathcal{TR}}_{\tau(u)-1}$, whilst if $a_{uk} = 0$ and $a_{\tau(j)l} = 1$, then $B \in \overline{\mathcal{TR}}_{j-1}$. Finally, if $a_{uk} = 0$ and $a_{\tau(j)l} = 0$, then $B \in \overline{\mathcal{TR}}_i$ with $k_B = k_A$ and $B <_C A$. But then, either by the inductive hypothesis on i, or by the minimality of A, all such B produced in this procedure must satisfy the statement of the claim, and hence so must A, which contradicts that A was chosen to be a counterexample.

Therefore, we may assume that that $z_{\tau(j),k}(A) = 0$ for all $k_A \leq j \leq i$, $k_A \leq k < m$ such that $a_{\tau(j)k} = 1$. Then, by Lemma 3.4.14 and Lemma 3.4.15, we may express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ where either: $B \in \overline{\mathcal{TR}}_{i'}$ for some $i' < i$,

$B \in \overline{\mathcal{TR}}_i$ with $k_B > k_A$, or $B \in \overline{\mathcal{TR}}_i$ with $k_B = k_A$ and $B <_C A$. But then, either by the inductive hypothesis on i, maximality of k_A, or minimality of A, each such B must satisfy the statement of the claim, and hence so must A, which contradicts that A was chosen to be a counterexample. Thus, no such counterexample may exist. Finally, once again, it is clear to see from the steps taken above that if $A \notin \mathcal{TC}$, then all such B produced by this procedure satisfy $B \notin \mathcal{TC}$ as well. □

Corollary 3.4.19. *Let $1 < i < m-1$ and let $A \in \overline{\mathcal{TR}}_i$ with $A \notin \mathcal{TC}$. Then $h[A] = 0$.*

Proof. By Lemma 3.4.17, we may express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ with $B \notin \mathcal{TR} \cup \mathcal{TC}$. But $h[B] = 0$ for all such B by Lemma 3.4.8, and so the result follows. □

Lemma 3.4.20. *Let $A \in \mathcal{TR} \setminus \mathcal{TC}$. Then $h[A] = 0$.*

Proof. Suppose, for the sake of contradiction, that the claim is false, and let $A \in \mathcal{T}$ be a counterexample that is minimal with respect to the column-ordering $<_C$ defined in Definition 3.4.2(ii). By Corollary 3.4.19, we may assume that $A \notin \overline{\mathcal{TR}}_i$ for any $i < m-1$, and so we must have that $A \in \mathcal{TR}_{m-1} \setminus \mathcal{TC}$ since $A \in \mathcal{TR}$. Hence, for each $1 < u < m$, either $a_{um} = 0$ or $a_{um} = 1$, and we claim that there exists at least one u in this range with $a_{um} = 1$. Indeed, suppose otherwise, then there exists some $1 < v < m$ with a_{1v} even since $A \notin \mathcal{TC}$. But then the relation $C^1_{vm}(A)$ expresses $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ with $B <_C A$ and $B \in \mathcal{TR} \setminus \mathcal{TC}$. But $h[B] = 0$ for all such B by minimality of A, which contradicts that A was chosen to be a counterexample. We hence write $(u_1, \ldots, u_s)$ for the increasing sequence whose terms are given by all u in the range $1 < u < m$ with $a_{um} = 1$. Firstly, suppose that $s > 1$ and set $u := u_{s-1}$ and $u' := u_s$. By Lemma 3.4.1(iii), we have that $z_{u,m}(A) + z_{u',m}(A) = 1$ and so the relation $Z_{u,m}(A) + Z_{u',m}(A)$ is given by:

$$h[A] = \sum_{\substack{v>u \\ l<m}} a_{vl} h[B^{[v,l]}] + \sum_{\substack{v>u' \\ l<m}} a_{vl} h[D^{[v,l]}], \tag{3.4.21}$$

where $B^{[v,l]} := A^{(v,m)(u,l)}_{(u,m)(v,l)}$ and $D^{[v,l]} := A^{(v,m)(u',l)}_{(u',m)(v,l)}$ for all such (v,l) with $a_{vl} \not\equiv 0$. Now, let (v,l) be as in (3.4.21) with $a_{vl} \not\equiv 0$.

If $l = 1$, then $B^{[v,1]}, D^{[v,1]} \notin \mathcal{TR} \cup \mathcal{TC}$ and so $h[B^{[v,1]}] = h[D^{[v,1]}] = 0$ by Lemma 3.4.8. On the other hand, if $l > 1$, then $B^{[v,l]}, D^{[v,l]} \in \mathcal{TR} \setminus \mathcal{TC}$ and $A <_C B^{[v,l]}, D^{[v,l]}$. Hence, by the minimality of A, once again we deduce that $h[B^{[v,l]}] = h[D^{[v,l]}] = 0$. Thus $h[A] = 0$, which contradicts that A was chosen to be a counterexample.

Hence we may assume that $s = 1$, or in other words that there exists a unique u in the range $1 < u < m$ such that $a_{um} = 1$, and so then $z_{1,m}(A) = 1$ by Lemma 3.4.1(v). By applying similar considerations to the above to the relation $Z_{1,m}(A)$, we once again reach a contradiction, and so no such counterexample may exist. □

Definition 3.4.22. For $1 < i < m$, similarly to $\mathcal{TR}_i$ of Definition 3.4.9(ii), we define:

$$\mathcal{TC}_i := \{A \in \mathcal{TC} \mid \text{the } \tau(j)^{\text{th}}\text{-column of } A \text{ contains } j \text{ odd entries for } 1 < j \leq i\}.$$

Remark 3.4.23. Firstly, note that by Remark 3.3.23, we see that the transpose homomorphism $h' \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\beta), M(\alpha))$ of h is relevant. Now, the results proven above are independent of the values of a and b, provided that they satisfy the parity condition $a - m \equiv b$. In particular, note that this condition is preserved under the swap $(a, b) \leftrightarrow (\tilde{a}, \tilde{b})$, where $\tilde{a} := b + m - 1$, $\tilde{b} := a - m + 1$. But, as in Remark 3.3.20, this swap is equivalent to the swap $\lambda \leftrightarrow \lambda'$ and accordingly $\alpha \leftrightarrow \beta$ and $\mathcal{T} \leftrightarrow \mathcal{T}'$. Therefore, by defining the subsets $\mathcal{TR}', \mathcal{TC}' \subseteq \mathcal{T}'$ analogously to $\mathcal{TR}, \mathcal{TC} \subseteq \mathcal{T}$, we obtain results analogous to those shown in this section for the coefficients $h'[A']$ of the $\rho[A']$ in h'.

Proposition 3.4.24. *Let $A \in \mathcal{T}$ and suppose that $A \notin \mathcal{TR}_{m-1} \cap \mathcal{TC}_{m-1}$. Then $h[A] = 0$.*

Proof. Suppose that $D \in \mathcal{T}$ is such that $h[D] \neq 0$. Then, we may assume that we have $D \in \mathcal{TR} \cup \mathcal{TC}$ since otherwise $h[D] = 0$ by Lemma 3.4.8. Moreover, we may assume that $D \notin \mathcal{TR} \setminus \mathcal{TC}$ since otherwise $h[D] = 0$ by Lemma 3.4.20. On the other hand, if $D \in \mathcal{TC} \setminus \mathcal{TR}$, then $D' \in \mathcal{TR}' \setminus \mathcal{TC}'$, where $\mathcal{TR}', \mathcal{TC}' \subseteq \mathcal{T}'$ are as defined in Remark 3.4.23. But then we have $h[D] = h'[D'] = 0$ á la Lemma 3.4.20, which contradicts our choice of D, and so we may assume that $D \notin \mathcal{TC} \setminus \mathcal{TR}$. In sum, we have shown that $h[D] = 0$ for all $D \in \mathcal{T}$ with $D \notin \mathcal{TR} \cap \mathcal{TC}$. In particular, to prove the Proposition, we may assume that $A \in \mathcal{TR} \cap \mathcal{TC}$. Now, if $A \notin \mathcal{TR}_{m-1}$, then there exists some i with $1 < i < m - 1$ such that $A \in \overline{\mathcal{TR}_i}$. But then Lemma 3.4.17 allows one to express $h[A]$ as a linear combination of $h[B]$s for some $B \in \mathcal{T}$ with $B \notin \mathcal{TR}$. But then every such B satisfies $B \notin \mathcal{TR} \cap \mathcal{TC}$ and hence that $h[B] = 0$ as shown above, and so $h[A] = 0$. On the other hand, if $A \notin \mathcal{TC}_{m-1}$, then $A' \notin \mathcal{TR}'_{m-1}$ where $\mathcal{TR}'_{m-1} \subseteq \mathcal{T}'$ is defined analogously to $\mathcal{TR}_{m-1} \subseteq \mathcal{T}$. But then $h[A] = h'[A'] = 0$ by the $'$-decorated analogue to the argument outlined above, and so we are done. □

Theorem 3.4.25. *Let $\lambda = (a, m-1, \ldots, 2, 1^b)$ and $a \geq m \geq 2$, $b \geq 1$, such that $a - m \equiv b \pmod 2$, and write $r := \deg(\lambda)$. Then $\mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda)) \cong \Bbbk$.*

Proof. Let $\bar{h}$ be a non-zero endomorphism of $\mathrm{Sp}(\lambda)$, which we identify with a relevant homomorphism $h \in \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ as in Remark 3.3.22. If $A \in \mathcal{T}$ with $h[A] \neq 0$, then $A \in \mathcal{TR}_{m-1} \cap \mathcal{TC}_{m-1}$ by Proposition 3.4.24. But since $\sum_v a_{\tau(i)v} = i$, $\sum_u a_{u\tau(j)} = j$ for $1 < i, j < m$, this set consists solely of the matrix:

$$A_0 := \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \cdots & 1 & 1 & b \\ \hline 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \hline \tilde{b} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline \end{array},$$

where $\tilde{b} := \tau(a) = a - m + 1$. Therefore, we have $h = h[A_0]\rho[A_0]$, and so we are done. □

4 Hook Specht Modules

In the following, we assume that the characteristic of the field $\Bbbk$ is 2.

4.1. Reduction for Homomorphisms

Here, we shall consider the $\Bbbk$-space of $\Bbbk\mathfrak{S}_r$-linear homomorphisms $\mathrm{Sp}(\lambda) \to \mathrm{Sp}(\mu)$ where $\lambda, \mu \in \Lambda(n,r)$ are both hook partitions of r, say $\lambda = (a, 1^b)$ and $\mu = (a', 1^{b'})$ for some integers $a, b, a', b' \geq 1$ with $a + b = r = a' + b'$.

Remark 4.1.1. At first glance, it may appear to the reader that we have neglected to consider the cases that $b = 0$ or $b' = 0$. However, if $b = 0$ say, then since we are in characteristic 2, we have that $\mathrm{Sp}(a, 1^b) = \mathrm{Sp}(r) \cong \mathrm{Sp}(1^r) = \mathrm{Sp}(1, 1^{r-1})$, and so we may swap $(a, b) = (r, 0)$ with $(a, b) = (1, r-1)$ and proceed accordingly. Similar adjustments may be made in the case that $b' = 0$.

Remark 4.1.2. It is clear to see that we may generalise the reduction process described in §3.3.1 to the case that $\lambda = (a, 1^b)$, $\mu = (a', 1^{b'})$ with $a + b = a' + b'$. Firstly, note that $\lambda' = (b+1, 1^{a-1})$, and so we fix $\alpha := (b+1, a-1)$ with $\beta := (a', b')$. Then, according to our reduction process, we identify $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda), \mathrm{Sp}(\mu))$ with the $\Bbbk$-subspace $\mathrm{ERel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ of $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ consisting of the essential relevant homomorphisms. That is to say, those $\Bbbk\mathfrak{S}_r$-linear homomorphisms $M(\alpha) \xrightarrow{h} M(\beta)$ that satisfy $h \circ \bar{\phi}_\alpha = 0 = \bar{\psi}_\beta \circ h$. Now, notice $\bar{\phi}_\alpha = \bar{\phi}_\alpha^{(1,2,1)}$ and $\bar{\psi}_\beta = \bar{\psi}_\beta^{(1,2,1)}$. Thus, by appropriate analogues of Lemma 3.3.10 and Remark 3.2.18, we have the $\Bbbk$-linear isomorphism $\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(a, 1^b), \mathrm{Sp}(a', 1^{b'})) \cong \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(\alpha), M(\beta))$ as $\Bbbk$-spaces, which is precisely:

$$\left\{ h \in \mathrm{Hom}_\Bbbk(M(b+1, a-1), M(a', b')) \;\middle|\; \begin{array}{ll} h \circ \bar{\phi}^{(1,2,1)}_{(b+1,a-1)} & = 0, \\ \bar{\psi}^{(1,2,1)}_{(a',b')} \circ h & = 0. \end{array} \right\}. \tag{4.1.3}$$

4.2. Endomorphisms

Recall that in Remark 3.4.5, we (partially) recovered Murphy's result; that is to say that $\mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(1, a^b))$ is one-dimensional when $a \equiv b \pmod 2$ [Mur, Theorem 4.1]. However, in fact, Murphy goes further than this by providing the dimension of this endomorphism algebra in the alternate parity cases, thereby providing an explicit dimension formula of the endomorphism algebra of any Specht module labelled by hooks. In the following result, we shall do the same by explicitly finding a basis for the $\Bbbk$-space of relevant homomorphisms $M(b+1, a-1) \to M(a, b)$ given in (4.1.3).

Proposition 4.2.1. *Let $a, b \geq 1$ with $a \not\equiv b \pmod 2$, and $r := a + b$. Then:*

$$\dim \mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(a, 1^b)) = \begin{cases} \frac{1}{2}\min(a+1, b+2), & \textit{if} \quad b \equiv 0 \pmod 2, \\ \frac{1}{2}\min(a, b+1), & \textit{if} \quad b \equiv 1 \pmod 2. \end{cases} \tag{4.2.2}$$

Proof. Firstly, if $a = 1$, then $\mathrm{Sp}(a, 1^b) = \mathtt{sgn}_r$, and so clearly we have, in accordance with (4.2.2), that $\dim \mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(a, 1^b)) = 1$. Henceforth, we assume that $a \geq 2$, and then fix $\alpha := (b+1, a-1)$ and $\beta := (a, b)$, with $c := \min(a-1, b)$, and note that $c \equiv b \equiv a-1$ since $a \not\equiv b$. Then, we have $\mathcal{T} := \mathrm{Tab}(\alpha, \beta) = \{\rho[A^{[t]}] \mid 0 \leq t \leq c\}$, where for $0 \leq t \leq c$, we write:

$$A^{[t]} := \begin{array}{|c|c|}\hline t+1 & b-t \\ \hline a-1-t & t \\ \hline\end{array} \in \mathcal{T}.$$

Now, fix some $\Bbbk\mathfrak{S}_r$-linear homomorphism $M(\alpha) \xrightarrow{h} M(\beta)$ given by $h = \sum_{t=0}^{c} h[A^{[t]}]\rho[A^{[t]}]$ say. Then h is relevant if and only if the coefficients $h[A^{[t]}]$ satisfy the relations:

$$(\bigstar)\left\{\begin{array}{rrl}
(t \neq a-1) & (t+2)h[A^{[t]}] = (b-t)h[A^{[t+1]}], & (R^1_{1,2}(A^{[t]})) \\
(t \neq 0) & (b+1-t)h[A^{[t]}] = (t+1)h[A^{[t-1]}], & (R^2_{1,2}(A^{[t]})) \\
(t \neq b) & (t+2)h[A^{[t]}] = (a-1-t)h[A^{[t+1]}], & (C^1_{1,2}(A^{[t]})) \\
(t \neq 0) & (a-t)h[A^{[t]}] = (t+1)h[A^{[t-1]}]. & (C^2_{1,2}(A^{[t]}))
\end{array}\right.$$

Now, since $c \equiv b \equiv a-1$, after reducing coefficients modulo 2, we see that the system of equations $(\star)$ reduces to:

$$(\bar{\bigstar})\left\{\begin{array}{rrrl}
 & (0 \leq t < c) & th[A^{[t]}] = (t+c)h[A^{[t+1]}], & (X_t) \\
a \neq b-1 & (t = c) & ch[A^{[c]}] \equiv 0. & (X_c)
\end{array}\right.$$

Then, we split into the following cases:

- $(c \equiv 0)$: Here, if t is even with $0 \leq t \leq c$, then the relation (X_t) is superfluous, and so the relations $(\bar{\star})$ are satisfied if and only if $h[A^{[t]}] = h[A^{[t+1]}]$ for odd t with $0 < t < c$. It follows that the element $\rho[A^{[0]}]$, along with the elements of the form $\rho[A^{[2s+1]}] + \rho[A^{[2s+2]}]$ for $0 \leq s \leq \frac{1}{2}(c-2)$, form a $\Bbbk$-basis for the $\Bbbk$-space of relevant homomorphisms $M(\alpha) \to M(\beta)$. Hence:

$$\dim \mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(a, 1^b)) = 2 + \frac{1}{2}(c-2) = \frac{1}{2}\min(a+1, b+2).$$

- $(c \equiv 1)$: Now, in this case, note that the relation (X_{c-1}) gives $h[A^{[c]}] = 0$, which is precisely the relation (X_c). Hence, whether or not $a = b-1$, we see that the relations $(\bar{\star})$ are satisfied if and only if the relations (X_t) are satisfied for all $0 \leq t < c$. But this is equivalent to $h[A^{[t]}] = 0$ for odd $0 \leq t \leq c$. It follows that the elements $\rho[A^{[2s]}]$ for $0 \leq s \leq \frac{1}{2}(c-1)$ form a $\Bbbk$-basis for the $\Bbbk$-space of relevant homomorphisms $M(\alpha) \to M(\beta)$. Hence:

$$\dim \mathrm{End}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(a, 1^b)) = 1 + \frac{1}{2}(c-1) = \frac{1}{2}\min(a, b+1). \qquad \square$$

4.3. Homomorphisms

Now, we shall generalise the calculation of §4.2 in order to calculate the dimension of the $\Bbbk$-space of the $\Bbbk\mathfrak{S}_r$-linear homomorphisms $\mathrm{Sp}(a, 1^b) \to \mathrm{Sp}(a', 1^{b'})$. Once again, as discussed in Remark 4.1.2, we do so by calculating the dimension of the $\Bbbk$-space of relevant homomorphisms $M(\alpha) \to M(\beta)$, where $\alpha = (b+1, a-1)$ and $\beta = (a', b')$ given in (4.1.3).

Proposition 4.3.1. *Let $a, a', b, b' \geq 1$ with the property that $r := a+b = a'+b'$, and write $d := a'-a = b-b'$. Then, the dimension, denoted $\delta(a, b', d)$, of the $\Bbbk$-space of*

$\Bbbk\mathfrak{S}_r$-linear homomorphisms $\mathrm{Sp}(a,1^b) \to \mathrm{Sp}(a',1^{b'})$ *is given by:*

$$\delta(a,b',d) = \begin{cases} 0, & \text{if } a \equiv b',\ d < -1, \\ 0, & \text{if } a \equiv b',\ d \geq -1,\ d \equiv 1, \\ 1, & \text{if } a \equiv b',\ d \geq -1,\ d \equiv 0, \\ \frac{1}{2}\min(a, b'+1), & \text{if } a \not\equiv b',\ d \geq -1,\ (a,b',d) \equiv (0,1,1),\ (0,1,0), \\ \frac{1}{2}\min(a+1, b'+2), & \text{if } a \not\equiv b',\ d \geq -1,\ (a,b',d) \equiv (1,0,0), \\ \frac{1}{2}\min(a-1, b'), & \text{if } a \not\equiv b',\ d \geq -1,\ (a,b',d) \equiv (1,0,1), \\ \frac{1}{2}\min(a'+1, b+2), & \text{if } a \not\equiv b',\ d < -1,\ (a,b',d) \equiv (0,1,1),\ (1,0,0), \\ \frac{1}{2}\min(a', b+1), & \text{if } a \not\equiv b',\ d < -1,\ (a,b',d) \equiv (0,1,0),\ (1,0,1). \end{cases} \tag{4.3.2}$$

Proof. Firstly, suppose that $a = 1$. Then, by [J_1, Theorem 24.4] (see also [DG_1, Proposition 3.5]), we have that:

$$\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(1^r), \mathrm{Sp}(a',1^{b'})) \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(r), \mathrm{Sp}(a',1^{b'})) \cong \begin{cases} \Bbbk, & \text{if } a' \equiv 1, \\ 0, & \text{if } a' \equiv 0, \end{cases},$$

and so, in accordance with (4.3.2), we $\delta(1,b',d) = 1$ if $d \equiv 0$, and $\delta(1,b',d) = 0$ if $d \equiv 1$. Now, suppose that $a \geq 2$ and $a' = 1$. Then, once again by [J_1, Theorem 24.4], we have that:

$$\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(a,1^b), \mathrm{Sp}(1^r)) \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(r), \mathrm{Sp}(b+1,1^{a-1})) \cong \begin{cases} \Bbbk, & \text{if } b \equiv 0, \\ 0, & \text{if } b \equiv 1, \end{cases},$$

and so, once again in accordance with (4.3.2), we indeed see that $\delta(a,b',d) = 1$ if $d \equiv b'$ and $\delta(a,b',d) = 0$ if $d \not\equiv b'$.

Now, we suppose that $a, a' \geq 2$ and we fix $\alpha := (b+1, a-1)$ and $\beta := (a', b')$. Then, note that we have that $\mathcal{T} := \mathrm{Tab}(\alpha,\beta) = \{\rho[A^{[t]}] \mid c' \leq t \leq c\}$, where $c := \min(a-1, b')$, $c' := \max(0, -d-1)$, and:

$$A^{[t]} := \begin{array}{|c|c|} \hline t+1+d & b'-t \\ \hline a-1-t & t \\ \hline \end{array} \in \mathcal{T}.$$

Now, fix some $\Bbbk\mathfrak{S}_r$-linear homomorphism $M(\alpha) \xrightarrow{h} M(\beta)$ given by $h = \sum_{t=c'}^{c} h[A^{[t]}]\rho[A^{[t]}]$ say. Note that h is relevant if and only if the coefficients $h[A^{[t]}]$ satisfy the relations:

$$(\star)\begin{cases} (t \neq a-1) & (t+2+d)h[A^{[t]}] = (b'-t)h[A^{[t+1]}], & (R^1_{1,2}(A^{[t]})) \\ (t \neq 0) & (b'+1-t)h[A^{[t]}] = (t+1+d)h[A^{[t-1]}], & (R^2_{1,2}(A^{[t]})) \\ (t \neq b') & (t+2+d)h[A^{[t]}] = (a-1-t)h[A^{[t+1]}], & (C^1_{1,2}(A^{[t]})) \\ (t \neq 0) & (a-t)h[A^{[t]}] = (t+1+d)h[A^{[t-1]}]. & (C^2_{1,2}(A^{[t]})) \end{cases}$$

We split into cases:

- ($a \not\equiv b'$): In this case, since $a \equiv b'-1$, we see that after reducing coefficients modulo 2, the system of equations $(\star)$ reduces to:

$$(\overline{\star})\begin{cases} & (c' \leq t < c) & (t+d)h[A^{[t]}] = (t+c)h[A^{[t+1]}], & (X_t) \\ a \neq b'-1 & (t = c) & (c+d)h[A^{[c]}] \equiv 0. & (X_c) \end{cases}$$

Then, we split this case into the following subcases:

- ($a \not\equiv b'$; $c \equiv d$): Here, if $t \equiv d$ with $c' \leq t \leq c$, then the relation (X_t) is superfluous, and so the relations $(\bar{\star})$ are satisfied if and only if we have that $h[A^{[t]}] = h[A^{[t+1]}]$ for $c' \leq t < c$ with $t \not\equiv d$. It follows that the elements $\rho[A^{[s]}] + \rho[A^{[s+1]}]$ for $c' \leq s < c$ with $s \not\equiv d$, along with the element $\rho[A^{[0]}]$ when $d \geq -1$ with $d \equiv 0$, form a $\Bbbk$-basis for the $\Bbbk$-space of relevant homomorphisms $M(\alpha) \to M(\beta)$.
- ($a \not\equiv b'$; $c \not\equiv d$): Now, here, the relation (X_{c-1}) gives $h[A^{[c]}] = 0$, which is precisely the relation (X_c). Hence, whether or not $a = b' - 1$, we see that the relations $(\bar{\star})$ are satisfied if and only if the relations (X_t) are satisfied for all $c' \leq t < c$. But this is equivalent to $h[A^{[t]}] = 0$ for $c' \leq t \leq c$ with $t \not\equiv d$. It follows that the elements $\rho[A^{[s]}]$ for $c' \leq s \leq c$ with $s \equiv d$ form a $\Bbbk$-basis for the $\Bbbk$-space of relevant homomorphisms $M(\alpha) \to M(\beta)$.

The case $a \not\equiv b'$ follows by counting, in each parity case, the size of the corresponding basis provided by the above analysis. See Remark 4.3.7 for details.

- ($a \equiv b'$): Now, if h is relevant, then the relation: $Z_{2,2}(A^{[t]}) = R^2_{1,2}(A^{[t]}) + C^2_{1,2}(A^{[t]})$ gives that $h[A^{[t]}] = 0$ unless $t = 0$. It follows that $\delta(a, b', d) \leq 1$, and $\delta(a, b', d) = 1$ if and only if $A^{[0]} \in \mathcal{T}$ and $\rho[A^{[0]}]$ is relevant. Finally, we note that $A^{[0]} \in \mathcal{T}$ if and only if $d \geq -1$, whilst for $d \geq -1$, we see that $h := \rho[A^{[0]}]$ satisfies the relations $R^1_{1,2}(A^{[0]})$, $C^1_{1,2}(A^{[0]})$ of $(\star)$ only when $d \equiv 0$, and conversely, when $d \equiv 0$, we see that h satisfy all of the relations of $(\star)$, and so is hence relevant. □

We finish this section by collecting a series of remarks relating to Proposition 4.3.1.

Remark 4.3.3. We begin by observing that one may tackle each of the cases $a = 1$ or $a' = 1$ in the proof of Proposition 4.3.1 using methods in the spirit of the thesis, and in particular without relying on James' result [J_1, Theorem 24.4]. To do so, we first note that throughout this thesis, during the investigation of the case $\lambda = (a, m-1, \ldots, 2, 1^b)$, for the sake of notation, we set $a \geq m \geq 2$. In particular, when $m = 2$, this excludes the case where $a = 1$, that it to say, the sign module $\mathrm{Sp}(1^r) \cong \mathtt{sgn}_r$ for $\Bbbk\mathfrak{S}_r$. That said, one may tackle such cases by applying a minor modification to the reduction process detailed in §3.3.1. Indeed, consider the following:

- Firstly, suppose that $a = 1$ with $a' > 1$, and $b, b' \geq 1$, and write $r := 1 + b = a' + b'$, and also $d := a' - 1 = b - b' \geq 1$. Then, we have a $\Bbbk$-linear isomorphism:

$$\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(1^r), \mathrm{Sp}(a', 1^{b'})) \cong \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(r), M(a', b')),$$

with $\mathrm{Tab}((r), (a', b')) = \{A := \begin{array}{|c|c|}\hline a' & b' \\ \hline\end{array}\}$. But, since $b' \neq 0$, we see that $\rho[A]$ is relevant if and only if a' is odd, and so indeed, $\delta(1, b', d) = 1$ if $d \equiv 0$, whilst $\delta(1, b', d) = 0$ if $d \equiv 1$.

- On the other hand, suppose that $a' = 1$ with $a > 1$, and $b, b' \geq 1$, and write $r := a + b = 1 + b'$, and also $d := -(a-1) = b - b' \leq -1$. Then, once again, we have a $\Bbbk$-linear isomorphism:

$$\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(a,1^b),\mathrm{Sp}(1^r)) \cong \mathrm{Rel}_{\Bbbk\mathfrak{S}_r}(M(b+1,a-1),M(r)),$$

with $\mathrm{Tab}((b+1,a-1),(r)) = \left\{A := \begin{array}{|c|}\hline b+1 \\ \hline a-1 \\ \hline\end{array}\right\}$. But, since $a-1 \neq 0$, $\rho[A]$ is relevant if and only if b is even, and so indeed, $\delta(1-d,b',d) = 1$ if $d \equiv b'$, whilst on the other hand, we have that $\delta(1-d,b',d) = 0$ if $d \not\equiv b'$.

- Finally, if $a = a' = 1$ with $b, b' \geq 1$, then $\mathrm{Sp}(a,1^b) = \mathrm{Sp}(a',1^{b'}) = \mathrm{Sp}(1^r) \cong \mathtt{sgn}_r$, and here, the result is clearly in accordance with (4.3.2).

Remark 4.3.4. Recall that for $\lambda,\mu \in \Lambda^+(n,r)$, we have a $\Bbbk$-linear isomorphism induced by duality of the form:

$$\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\lambda),\mathrm{Sp}(\mu)) \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(\mu'),\mathrm{Sp}(\lambda)). \tag{4.3.5}$$

Now, let $a,a',b,b' \geq 1$ such that $r := a+b = a'+b'$. Then, applying (4.3.5) to $\lambda := (a,1^b)$, $\mu := (a',1^{b'})$, we see that we have a $\Bbbk$-linear isomorphism:

$$\mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(a,1^b),\mathrm{Sp}(a',1^{b'})) \cong \mathrm{Hom}_{\Bbbk\mathfrak{S}_r}(\mathrm{Sp}(b'+1,1^{a'-1}),\mathrm{Sp}(b+1,1^{a-1})). \tag{4.3.6}$$

Accordingly, we deduce that $\delta(a,b',d) = \delta(b'+1,a-1,d)$, where δ is as in (4.3.2). It is worth observing that in each case described in the statement of Proposition 4.3.1, we see that the corresponding conditions are indeed preserved by the swap $(a,b',d) \leftrightarrow (b'+1,a-1,d)$, provided that $a,a' \neq 1$.

Remark 4.3.7. Now, by applying the analysis of Proposition 4.3.1 for each case where $\delta(a,b',d) \neq 0$, along with Remark 4.3.3 in the appropriate cases, we arrive at the following explicit $\Bbbk$-basis for the $\Bbbk$-space of relevant homomorphisms $M(\alpha) \to M(\beta)$ for the appropriate choice of α,β.

- $\underline{a = 1,\ d \equiv 0}$:
  - $d = 0$: $\{\rho[\,\begin{array}{|c|}\hline r \\ \hline\end{array}\,]\}$,
  - $d > 0$: $\{\rho[\,\begin{array}{|c|c|}\hline a' & b' \\ \hline\end{array}\,]\}$.
- $\underline{a \neq 1,\ a' = 1}$:
  - $d \equiv b'$: $\left\{\rho\left[\begin{array}{|c|}\hline b+1 \\ \hline a-1 \\ \hline\end{array}\right]\right\}$.
- $\underline{a,a' \neq 1,\ a \equiv b',\ d \geq -1}$:
  - $d \equiv 0$: $\{\rho[A^{[0]}]\}$.
- $\underline{a,a' \neq 1,\ a \not\equiv b',\ d \geq -1}$:
  - $(a,b',d) \equiv (0,1,1)$: $\{\rho[A^{[2s]}] + \rho[A^{[2s+1]}] \mid 0 \leq s \leq \frac{1}{2}\min(a-2,b'-1)\}$,
  - $(a,b',d) \equiv (0,1,0)$: $\{\rho[A^{[0]}], \rho[A^{[2s]}] \mid 0 < s \leq \frac{1}{2}\min(a-2,b'-1)\}$,
  - $(a,b',d) \equiv (1,0,0)$: $\{\rho[A^{[2s+1]}] + \rho[A^{[2s+2]}], \rho[A^{[0]}] \mid 0 \leq s \leq \frac{1}{2}\min(a-3,b'-2)\}$,
  - $(a,b',d) \equiv (1,0,1)$: $\{\rho[A^{[2s+1]}] \mid 0 \leq s \leq \frac{1}{2}\min(a-3,b'-2)\}$.
- $\underline{a,a' \neq 1,\ a \not\equiv b',\ d < -1}$:
  - $(a,b',d) \equiv (0,1,1)$: $\{\rho[A^{[2s]}] + \rho[A^{[2s+1]}] \mid -\frac{1}{2}(d+1) \leq s \leq \frac{1}{2}\min(a-2,b'-1)\}$,
  - $(a,b',d) \equiv (0,1,0)$: $\{\rho[A^{[2s]}] \mid -\frac{1}{2}d \leq s \leq \frac{1}{2}\min(a-2,b'-1)\}$,
  - $(a,b',d) \equiv (1,0,0)$: $\{\rho[A^{[2s+1]}] + \rho[A^{[2s+2]}] \mid -\frac{1}{2}(d+2) \leq s \leq \frac{1}{2}\min(a-3,b'-2)\}$,
  - $(a,b',d) \equiv (1,0,1)$: $\{\rho[A^{[2s+1]}] \mid -\frac{1}{2}(d+1) \leq s \leq \frac{1}{2}\min(a-3,b'-2)\}$.

References

[ABW] K. Akin, D. A. Buchsbaum, and J. Weyman, *Schur Functors and Schur Complexes*, Advances in Mathematics, **44(3)**, (1982), 207–278.

[BBS] C. Bessenrodt, C. Bowman, and L. Sutton, *Kronecker positivity and* 2*-modular representation theory*, Transactions of the American Mathematical Society, Series B, **8(33)**, (2021), 1024–1055.

[D_1] S. Donkin, *Finite Resolutions of Modules for Reductive Algebraic Groups*, Journal of Algebra, **101(2)**, (1986), 473–488.

[D_2] S. Donkin, *On Tilting Modules for Algebraic Groups*, Mathematische Zeitschrift, **212**, (1993), 39–60.

[D_3] S. Donkin, *The q-Schur Algebra*, London Mathematical Society Lecture Note Series, Cambridge University Press (1998).

[D_4] S. Donkin, *Tilting Modules for Algebraic Groups and Finite Dimensional Algebras*, Handbook of tilting theory, **13**, (2007), 215–257.

[DF] C. J. Dodge and M. Fayers, *Some new decomposable Specht modules*, Journal of Algebra, **357**, (2012), 235–262.

[DG_1] S. Donkin and H. Geranios, *First Degree Cohomology of Specht Modules and Extensions of Symmetric Powers*, Advances in Mathematics, **345**, (2019), 618–701.

[DG_2] S. Donkin and H. Geranios, *Decompositions of Some Specht Modules I*, Journal of Algebra, **550**, (2020), 1–22.

[DJ] R. Dipper and G. James, *Representations of Hecke Algebras of General Linear Groups*, Proceedings of the London Mathematical Society, **3(1)**, (1986), 20–52.

[F_1] M. Fayers, *Reducible Specht modules*, Journal of Algebra, **280(2)**, (2004), 500–504.

[F_2] M. Fayers, *Irreducible Specht modules for Hecke algebras of type A*, Advances in Mathematics, **193(2)**, (2005), 438–452.

[G] J. A. Green, *Polynomial Representations of* GL_n*: With an Appendix on Schensted Correspondence and Littelmann Paths*, volume 830, Springer (2006).

[GH] H. Geranios and A. Higgins, *On the Endomorphism Algebra of Specht Modules in Even Characteristic*, arXiv preprint arXiv:2301.12582v2.

[J_1] G. James, *The Representation Theory of the Symmetric Groups*, volume 682, Springer-Verlag (1978).

[J_2] G. D. James, *On a conjecture of Carter concerning irreducible Specht modules*, in *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 83, Cambridge University Press, pp. 11–17.

[J_3] G. D. James, *Some combinatorial results involving Young diagrams*, in *Mathematical proceedings of the cambridge philosophical society*, volume 83, Cambridge University Press, pp. 1–10.

[Jan] J. C. Jantzen, *Representations of Algebraic Groups*, volume 107, American Mathematical Soc. (2003).

[JMur] G. James and G. Murphy, *The determinant of the Gram matrix for a Specht module*, Journal of Algebra, **59(1)**, (1979), 222–235.

[K] C. Kassel, *Quantum groups*, volume 155, Springer Science & Business Media (2012).

[L] S. Lyle, *Some reducible Specht modules*, Journal of Algebra, **269(2)**, (2003), 536–543.

[Mar] S. Martin, *Schur Algebras and Representation Theory*, 112, Cambridge University Press (1993).

[Mat] A. Mathas, *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, volume 15, American Mathematical Soc. (1999).

[McD] E. McDowell, *The Image of the Specht Module Under the Inverse Schur Functor in Arbitrary Characteristic*, Journal of Algebra, **586**, (2021), 865–898.

[Mur] G. Murphy, *On Decomposability of Some Specht Modules for Symmetric Groups*, J. Algebra, **66(1)**, (1980), 156–168.

[Spe] L. Speyer, *Decomposable Specht modules for the Iwahori–Hecke algebra* $\mathscr{H}_{\mathbb{F},-1}(\mathfrak{S}_r)$, Journal of Algebra, **418**, (2014), 227–264.

[Spr] T. A. Springer, *Linear algebraic groups.*, volume 9 of *Prog. Math.*, Boston, MA: Birkhäuser, 2nd ed. edition (1998).