Series Expansion for Operators Acting on Generalized Fock Spaces

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Abstract

An expansion formula for observables, in terms of annihilation and creation operators has been proved firstly in [1] on the Fock space of a free Bose field and then in [2] for a Fock space associated to a factorizing scattering function. We are proving the expansion in a general context, where both of these results are included. We also prove that the expansion formula works for other models, too.

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Author's declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for a degree or other qualification at this University or elsewhere. All sources are acknowledged as references.

Contents

1 Introduction

Relativistic Quantum Field Theory (QFT) is our most successful and well-tested physical theory describing the fundamental constituents of matter, the fundamental particles, and their interactions. However, despite its experimental successes, from a mathematical viewpoint there are still many open questions concerning rigorous approaches of QFT. In this thesis, our focus is on certain operators in QFT known as local observables. These are associated with spacetime points (or bounded regions) and they commute with each other at spacelike separations. Rigorous mathematical constructions of local observables are notoriously difficult to achieve.

The aforementioned *local observables* play a highly important role in relativistic quantum field theory. The particular way in which these operators arise is not going to concern us, however we are interested in getting some more insight to them, i.e. a certain way of representing them so that essential information is reflected. This is achieved to some extent via the expansion formula (4.2) which is proved in a quite general context, for all quadratic forms A acting on a Fock space, which might be unbounded in high particle numbers or high energies, but with certain regularity properties which keep them bounded in a way. The expansion formula is originally traced to Araki [1], who regarded bounded operators acting on the Fock space associated to a free Bose field. Later, Bostelmann and Cadamuro [2] proved it in a quite different way for integrable models on 1+1-dimensional Minkowski space, in particular models associated to a factorizing scattering function S and they proved existence of the expansion for unbounded operators that belong to a certain class. Here, we establish the expansion formula for unbounded operators acting on Fock spaces associated to other models too, for example integrable models with several particle species [7]. We are also showing that an operator is homeomorphically associated to its expansion (statement (ii) of Theorem 4.3).

In order to be more clear, Araki proved in [1] that every bounded operator acting on a Fock space has the following expansion:

$$A = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} (a^{\dagger m} [A]_{mn} a^n) \tag{1.1}$$

which can also be stated in the form of formal integral kernels as

$$A = \sum_{m,n=0}^{\infty} \int \frac{d\theta d\eta}{m!n!} f_{m,n}(\theta, \eta) a^\dagger(\theta_1)..a^\dagger(\theta_m) a(\eta_1)..a(\eta_n)$$

where the generalized functions $f_{m,n}$ are given by

$$f_{m,n}(\theta, \eta) = \langle \Omega, [a(\theta_m), ..., [a(\theta_1), [...[A, a^\dagger(\eta_n)]..., a^\dagger(\eta_1)]...]\Omega \rangle$$

We will mainly work with the expansion (1.1), which holds in the case of a free Bose field and we will generalize it for integrable models and other cases as well, using ideas from Araki's proof. Bostelmann and Cadamuro proved in [2] a similar expansion, with the annihilation and creation operators replaced by "symmetrized" Zamolodchikov operators $z^\dagger, z$ and for A being a quadratic form satisfying some boundedness conditions. They managed to prove the following expansion for A in this class:

$$A = \sum_{m,n=0}^{\infty} \int \frac{d\theta d\eta}{m!n!} f_{m,n}(\theta, \eta) z^\dagger(\theta_1)..z^\dagger(\theta_m) z(\eta_1)..z(\eta_n) \tag{1.2}$$

In a different notation, this expansion reads:

$$A = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} (z^{\dagger m} [A]_{mn} z^n) \tag{1.3}$$

Bostelmann and Cadamuro [2] not only proved that each quadratic form in this class can be expanded as in (1.2) but also that each set of generalized functions $f_{m,n}$ with certain regularity conditions gives rise to an observable of the class mentioned before via the expansion in the right hand side of (1.2). We will also show that this one to one correspondence is homeomorphic, when the two spaces are given the right topology.

In Section 4, we prove the expansion in the general symmetrized case. In particular, we introduce our symmetrized Fock space $\mathcal{H} = \oplus_n \mathcal{H}_n$ and an associated subspace $\mathcal{E}_n$ for each particle number n. These spaces carry a different topology than the one of the Fock space, satisfying certain properties needed for existence of the expansion. Then, we distinguish between two important cases: the first one is when $\mathcal{E}_n$ has the structure of a Hilbert space, which is covered in Section 6; the second one is when $\mathcal{E}_n$ is a nuclear Fréchet space, a situation that we are dealing with in Section 8. Most of the applications we have in mind fit into the Hilbert space case, however an important example, the one where $\mathcal{E}_n$ is the Schwartz space is covered by the nuclear case.

In Section 5, we study a "smaller" class of observables, as in [2]. We call this "damping in one side", since our quadratic forms A "allow" high energy behavior in one of the two arguments, so that the damping factor $e^{-\omega(H/\mu)}$ only needs to be applied in one side. One could start from this case of course, since both proofs proceed similarly, however we chose to prove the expansion for the observables that require damping in both sides first, for simplicity reasons.

In Section 4.2, we are trying to outline how the coefficients $[A]_{mn}$ of the expansion change under symmetry transformations. We only investigate the case of space-time translations, in which the corresponding change in the associated coefficients is quite natural.

Finally, in Section 9, we describe several different models within which one can obtain the expansion. Besides the model used by Bostelmann and Cadamuro [2] we prove the expansion in the context of Lechner and Schützenhofer [7] for S being a "matrix-valued" scattering function, for the model of ordered-Fock spaces [4] and for $T-$deformed Fock spaces, as in [3].

2 Preliminaries

2.1 Hilbert Spaces

An **inner product space** $\mathcal{K}$ is a real- or complex- linear space, equipped with an inner product, i.e. a function $\langle \cdot, \cdot \rangle : \mathcal{K} \times \mathcal{K} \to \mathbb{F}$ (where $\mathbb{F}$ is the underlying field, i.e. $\mathbb{R}$ or $\mathbb{C}$) satisfying the following:

- $\langle x, x \rangle \geq 0$ for each $x \in \mathcal{K}$
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for each $x, y \in \mathcal{K}$
- $\langle \cdot, \cdot \rangle$ is anti-linear in the first argument and linear in the second, i.e.

  $$\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle$$

  and

  $$\langle \lambda x + \mu y, z \rangle = \overline{\lambda} \langle x, z \rangle + \overline{\mu} \langle y, z \rangle$$

  for each $x, y, z \in \mathcal{K}$, $\lambda, \mu \in \mathbb{F}$.

For our purposes, we will only consider complex inner product spaces. For $x \in \mathcal{K}$, we set $\|x\| := \langle x, x \rangle^{1/2}$, and this turns out to be a norm, i.e. satisfying the following:

- $\|x\| \geq 0$ for each $x \in \mathcal{K}$ and $\|x\| = 0$ if and only if $x = 0$
- $\|\lambda x\| = |\lambda| \|x\|$ for each $x \in \mathcal{K}$, $\lambda \in \mathbb{F}$
- $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in \mathcal{K}$

The last inequality is referred to as the "triangle inequality". The first two properties are trivial to verify. The last one follows from the **Cauchy-Schwarz** inequality, which holds in every inner product space, i.e.

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

which follows from $\langle x, x \rangle \geq 0$ if we set $x - (\langle y, x \rangle / \|y\|^2) y$ in place of x. Using the Cauchy-Schwarz inequality, we see

$$\begin{aligned} \|x+y\|^2 = \langle x+y, x+y \rangle = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(\langle x, y \rangle) &\leq \|x\|^2 + \|y\|^2 + 2|\langle x, y \rangle| \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2 \end{aligned}$$

and therefore $\|x + y\| \leq \|x\| + \|y\|$. Thus, $\mathcal{K}$ becomes a metric space with distance function $d(x, y) = \|x - y\|$. If $\mathcal{K}$ is in addition complete with respect to this metric, it is called a **Hilbert space**.

Two vectors x, y are called **orthogonal** if $\langle x, y \rangle = 0$. They are called **orthonormal** if in addition $\|x\| = \|y\| = 1$. By Zorn's Lemma, there exist maximal sets of pairwise orthonormal vectors and all these sets are called **orthonormal bases**. If $\{e_i \ : \ i \in I\}$ is such a set, one can show that

$$x = \sum_{i \in I} \langle e_i, x \rangle e_i$$

where all but countably many coefficients are vanished, and the sum is convergent in $\mathcal{K}$. It can be shown that each orthonormal basis has the same cardinality. If the cardinality is at most countable, $\mathcal{K}$ is called **separable**. This is equivalent to the classical topological definition of a separable space, i.e. there exists a countable subset of $\mathcal{K}$ whose closure is $\mathcal{K}$ itself.

Suppose $\mathcal{K}_1$ and $\mathcal{K}_2$ are Hilbert spaces and $T : \mathcal{K}_1 \to \mathcal{K}_2$ is linear. The following three conditions are equivalent:

- T is continuous
- T is continuous at 0.
- There exists a constant $C > 0$ such that $\|Tx\| \leq C\|x\|$ for each $x \in \mathcal{K}_1$.

If these hold, T is called a **bounded operator**. It can be shown that there exists a minimum constant $C \geq 0$ such that $\|Tx\| \leq C\|x\|$ for all x, and it is equal to

$$\sup\{\|Tx\| \ : \ x \in \mathcal{K}_1, \ \|x\| \leq 1\}$$

This constant is denoted by $\|T\|$. This is actually a norm in the space $\mathscr{L}(\mathcal{K}_1, \mathcal{K}_2)$ of all bounded operators from $\mathcal{K}_1$ into $\mathcal{K}_2$ and this actually turns $\mathscr{L}(\mathcal{K}_1, \mathcal{K}_2)$ into a **Banach space**, i.e. a complete normed space.

We state the following fundamental result in the theory of Hilbert spaces:

Theorem 2.1 (Riesz Representation Theorem)**.** *Suppose T is a bounded linear functional from a Hilbert space $\mathcal{K}$ into $\mathbb{C}$ (i.e. there exists a constant $C \geq 0$ such that $|T(x)| \leq C\|x\|$ for each $x \in \mathcal{K}$). Then, there exists a unique vector $y \in \mathcal{K}$ such that $T(x) = \langle y, x \rangle$ for all $x \in \mathcal{K}$.*

Using the previous Theorem, we are going to prove the following:

Theorem 2.2. *Suppose $\mathcal{K}_1, \mathcal{K}_2$ are Hilbert spaces and $T \in \mathscr{L}(\mathcal{K}_1, \mathcal{K}_2)$. Then, there exists a unique operator $T^* \in \mathscr{L}(\mathcal{K}_2, \mathcal{K}_1)$ satisfying*

$$\langle Tx, y \rangle_2 = \langle x, T^* y \rangle_1$$

for all $x \in \mathcal{K}_1, y \in \mathcal{K}_2$. Furthermore, we have $(T^*)^* = T$, $(\lambda T)^* = \overline{\lambda} T^*$, $(T+S)^* = T^* + S^*$, $(TS)^* = S^* T^*$ and $\|T^*\| = \|T\|$.

Proof. We only prove the first claim, and that $\|T^*\| = \|T\|$. The function $x \mapsto \overline{\langle Tx, y \rangle_2}$ is linear and bounded, thus by Riesz representation Theorem, there exists a unique vector in $\mathcal{K}_1$ (which we denote by $T^* y$) that satisfies

$\overline{\langle Tx, y\rangle_2} = \langle T^*y, x\rangle_1$ for each $x \in \mathcal{K}_1$ or $\langle Tx, y\rangle_2 = \langle x, T^*y\rangle_1$ for each $x \in \mathcal{K}_1$. For $\lambda, \mu \in \mathbb{C}$ and $x, y \in \mathcal{K}_2$, and for $z \in \mathcal{K}_1$, we have:

$$\begin{aligned}\langle z, T^*(\lambda x + \mu y)\rangle_1 = \langle Tz, \lambda x + \mu y\rangle_2 = \lambda\langle Tz, x\rangle_2 + \mu\langle Tz, \mu y\rangle_2 \\ = \lambda\langle z, T^*x\rangle_1 + \mu\langle z, T^*y\rangle_1 = \langle z, \lambda T^*x + \mu T^*y\rangle_1\end{aligned}$$

Therefore, we have

$$\langle z, T^*(\lambda x + \mu y) - \lambda T^*x - \mu T^*y\rangle_1 = 0$$

for each $z \in \mathcal{K}_1$. Setting $z = T^*(\lambda x + \mu y) - \lambda T^*x - \mu T^*y$, we get

$$\|T^*(\lambda x + \mu y) - \lambda T^*x - \mu T^*y\|^2 = 0$$

thus $T^*(\lambda x + \mu y) = \lambda T^*x + \mu T^*y$. This proves that T^* is linear. In order to show $\|T^*\| = \|T\|$, we observe that $\|x\| = \sup\{|\langle y, x\rangle| \ : \ \|y\| \le 1\}$ in every Hilbert space. Then, we have

$$\begin{aligned}\|T^*y\| = \sup\{|\langle x, T^*y\rangle| \ : \ \|x\| \le 1\} = \sup\{|\langle Tx, y\rangle| \ : \ \|x\| \le 1\} \\ \le \sup\{\|T\|\|x\|\|y\| \ : \ \|x\| \le 1\} = \|T\|\|y\|\end{aligned}$$

This proves that $T^* \in \mathscr{L}(\mathcal{K}_2, \mathcal{K}_1)$ with $\|T^*\| \le \|T\|$. Finally, since $(T^*)^* = T$, $\|T\| = \|(T^*)^*\| \le \|T^*\|$, therefore $\|T\| = \|T^*\|$. □

The operator T^* will be called the **adjoint** of T. If $T^* = T$, T is called **self-adjoint**.

For a subset $\mathcal{U}$ of $\mathcal{K}$, we define $\mathcal{U}^\perp := \{y \in \mathcal{K} \ : \ \langle y, x\rangle = 0 \text{ for each } x \in \mathcal{U}\}$. It is an easy exercise to verify that $\mathcal{U}^\perp$ is a closed subspace of $\mathcal{K}$ (even if $\mathcal{U}$ is not a subspace of $\mathcal{K}$). Furthermore, it follows that $\mathcal{U}^{\perp\perp}$ is the smallest closed subspace that contains $\mathcal{U}$. In particular, $\mathcal{U}$ is a closed subspace of $\mathcal{K}$ if and only if $\mathcal{U}^{\perp\perp} = \mathcal{U}$. For each closed subspace $\mathcal{U}$ of $\mathcal{K}$ we have the following decomposition:

$$\mathcal{K} = \mathcal{U} \oplus \mathcal{U}^\perp$$

The proofs of the above facts are contained in [9], Theorem 3.1.7 and Corollary 3.1.8.

Suppose $P \in \mathscr{L}(\mathcal{K}) := \mathscr{L}(\mathcal{K}, \mathcal{K})$ for some Hilbert space $\mathcal{K}$. $\mathcal{K}$ is called an **orthogonal projection** or just a **projection** if $P^2 = P$ and $P^* = P$. Projections play a very interesting role in the Theory of Hilbert spaces and they are a quite useful tool in the understanding of the spaces and the linear bounded operators between them. The following Proposition gives us a clue about that:

Proposition 2.3. *Let $\mathcal{K}$ be a Hilbert space and P a projection on $\mathcal{K}$. Then, the image of P is a closed subspace of $\mathcal{K}$. Conversely, for each closed subspace V of $\mathcal{K}$, there exists a unique projection on $\mathcal{K}$ with $V = P(\mathcal{K})$.*

The proof of the above Proposition can be found in [9], paragraph 3.2.13.

This allows us to identify projections acting on a Hilbert space $\mathcal{K}$ with closed subspaces of $\mathcal{K}$. For two projections P, Q, we will write $P \le Q$ meaning that $P(\mathcal{K}) \subset Q(\mathcal{K})$. We have the following Lemma:

Lemma 2.4. *For two projections P, Q acting on a Hilbert space $\mathcal{K}$, $P \leq Q$ if and only if $PQ = QP = P$*

Proof. If $PQ = QP = P$ it is clear that $P(\mathcal{K}) = QP(\mathcal{K}) \subset Q(\mathcal{K})$ and therefore $P \leq Q$. For the converse, assume $x \in \mathcal{K}$ and decompose x as $x = y + z$ with $y \in P(\mathcal{K})$ and $z \in P(\mathcal{K})^{\perp}$. We have $Pz = 0$ because

$$\|Pz\|^2 = \langle Pz, Pz\rangle = \langle P^*Pz, z\rangle = \langle Pz, z\rangle = 0$$

and since $y = Py'$ for some y', we have $Py = P^2y' = Py' = y$. Thus $Px = y$. Since $y \in P(\mathcal{K}) \subset Q(\mathcal{K})$, we also have $Qy = y$ thus $QPx = y = Px$. We have proved $QP = P$. If we take adjoints, we obtain $PQ = P$ and the proof is complete. □

Suppose that $\{\mathcal{K}_n : n \in \mathbb{N}\}$ is a given family of Hilbert spaces. We denote by $\sum_n \mathcal{K}_n$ the algebraic direct sum, i.e. the space of all sequences $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \mathcal{K}_n$ for each n and all but finitely many x_n are zero. This space has an obvious structure of a vector space. Furthermore, for $(x_n)_n, (y_n)_n \in \sum_n \mathcal{K}_n$, we define

$$\langle (x_n)_n, (y_n)_n\rangle := \sum_n \langle x_n, y_n\rangle$$

It is trivial to verify that this is in fact an inner product on the space $\sum_n \mathcal{K}_n$ and the associated norm is

$$\|(x_n)_n\| = \left(\sum_n \|x_n\|^2\right)^{1/2}$$

The Cauchy completion of this space is a Hilbert space, denoted by $\bigoplus_n \mathcal{K}_n$ (for a precise construction of the Cauchy completion, or just completion of a normed space, see [9], Proposition 2.1.12. In case of an inner product space, it is easily verified that the completion is a Hilbert space). $\bigoplus_n \mathcal{K}_n$ has another realization: it is the space of all sequences $(x_n)_{n\in\mathbb{N}}$, where $x_n \in \mathcal{K}_n$ for each n and such that

$$\sum_n \|x_n\|^2 < \infty$$

and with inner product

$$\langle (x_n)_n, (y_n)_n\rangle = \sum_n \langle x_n, y_n\rangle$$

2.2 Tensor Products of Hilbert Spaces

The concept of tensor products is very essential in the study of Hilbert spaces, and it appears that there is a natural way of defining the tensor product of any finite number of Hilbert spaces and equip this with an inner product too.

Let $\mathcal{H}$ be a Hilbert space. We denote the algebraic k-fold tensor product of $\mathcal{H}$ with itself by $\mathcal{H}^{\tilde{\otimes} k}$. For two elements

$$\eta = \sum_{j=1}^{K} \phi_1^j \otimes ... \otimes \phi_k^j \in \mathcal{H}^{\tilde{\otimes} k}$$

$$\kappa = \sum_{i=1}^{L} \psi_1^i \otimes ... \otimes \psi_k^i \in \mathcal{H}^{\tilde{\otimes} k}$$

we set

$$\langle \eta, \kappa \rangle = \sum_{j=1}^{K} \sum_{i=1}^{L} \langle \phi_1^j, \psi_1^i \rangle ... \langle \phi_k^j, \psi_k^i \rangle$$

This can be easily seen (by the universal property of tensor products) to be well-defined. Now, we shall prove that this is an inner product. For an element

$$\eta = \sum_{j=1}^{L} \phi_1^j \otimes ... \otimes \phi_k^j$$

we can pick an orthonormal basis of the finite dimensional space generated by the vectors ϕ_i^j, $\{e_\ell\}_{\ell=1}^{M}$. Then, we rewrite

$$\eta = \sum_{j_1,...,j_k} a_{j_1..j_k} e_{j_1} \otimes ... \otimes e_{j_k} \quad \text{thus}$$

$$\langle \eta, \eta \rangle = \sum_{j_1,...,j_k} \sum_{i_1,...,i_k} \overline{a_{j_1..j_k}} a_{i_1..i_k} \langle e_{j_1}, e_{i_1} \rangle ... \langle e_{j_k}, e_{i_k} \rangle = \sum_{j_1,...,j_k} |a_{j_1..j_k}|^2$$

Now, it's obvious that $\langle \eta, \eta \rangle \geq 0$ with equality if and only if $\eta = 0$. The other properties are immediate.

We define $\mathcal{H}^{\otimes k}$ as the Cauchy completion of $\mathcal{H}^{\tilde{\otimes} k}$ with respect to this inner product.

For bounded operators $A_1, ..., A_k \in \mathscr{L}(\mathcal{H})$, we set

$$A_1 \otimes ... \otimes A_k(\phi_1 \otimes ... \otimes \phi_k) = (A_1\phi_1) \otimes ... \otimes (A_k\phi_k)$$

and extend it by linearity to $\mathcal{H}^{\tilde{\otimes} k}$. We would like to extend it by continuity to the whole space $\mathcal{H}^{\otimes k}$.

In order to do that, it suffices to show that the operator $A_1 \otimes ... \otimes A_k$ is bounded in $\mathcal{H}^{\tilde{\otimes} k}$. Actually, we will show that $\|A_1 \otimes ... \otimes A_k\| = \|A_1\| ... \|A_k\|$. It is clear that $(A_1 \otimes .. \otimes A_k)(B_1 \otimes .. \otimes B_k) = (A_1B_1) \otimes .. \otimes (A_kB_k)$ for operators $A_1, .., A_k$ and $B_1, .., B_k$ in $\mathscr{L}(\mathcal{H})$. Now, for the " $\leq$ " part, we notice that

$$A_1 \otimes ... \otimes A_k = (A_1 \otimes I \otimes ... \otimes I)...(I \otimes I \otimes ... \otimes I \otimes A_k)$$

Therefore, it is sufficient to show that $\|A_1 \otimes I \otimes ... \otimes I\| \leq \|A_1\|$. The general case $\|I \otimes ... \otimes A_i \otimes ... \otimes I\| \leq \|A_i\|$ is proved the same way. Suppose

$$\eta = \sum_{j=1}^{L} \phi_1^j \otimes ... \otimes \phi_k^j$$

After finding an orthonormal basis $\{e_j\}$ for the finite dimensional space spanned by the vectors ϕ_j^i, we can write

$$\eta = \sum_{j_1,...,j_k} a_{j_1..j_k} e_{j_1} \otimes ... \otimes e_{j_k} = \sum_{j_2,...,j_k} \left(\sum_{j_1} a_{j_1..j_k} e_{j_1} \right) \otimes e_{j_2} \otimes ... \otimes e_{j_k} =$$

$$= \sum_{j_2,...,j_k} x_{j_2..j_k} \otimes e_{j_2} \otimes ... \otimes e_{j_k}$$

By orthogonality, it follows that

$$\|\eta\|^2 = \sum_{j_2,...,j_k} \|x_{j_2..j_k}\|^2$$

Also,

$$\|(A_1 \otimes I \otimes ... \otimes I)\eta\|^2 = \sum_{j_2,...,j_k} \|A_1 x_{j_2..j_k}\|^2 \leq \|A_1\|^2 \|x_{j_2..j_k}\|^2 = \|A_1\|^2 \|\eta\|^2$$

Which proves indeed that $\|A_1 \otimes I \otimes ... \otimes I\| \leq \|A_1\|$ and by our earlier observation, we have that $\|A_1 \otimes ... \otimes A_k\| \leq \|A_1\| ... \|A_k\|$.

To show equality, for each $\varepsilon > 0$ we can peak unit vectors $e_1, ..., e_k$ so that $\|A_i e_i\| > \|A_i\| - \varepsilon$. Then,

$$\|(A_1 \otimes ... \otimes A_k)(e_1 \otimes ... \otimes e_k)\| > (\|A_1\| - \varepsilon) ... (\|A_k\| - \varepsilon)$$

Thus, $A_1 \otimes ... \otimes A_k$ is an operator in $\mathscr{L}(\mathcal{H}^{\otimes k})$.

Finally, it can be easily verified that for $A_1, .., A_k \in \mathscr{L}(\mathcal{H})$, we have

$$(A_1 \otimes .. \otimes A_k)^* = A_1^* \otimes .. \otimes A_k^*$$

(it is first shown for the algebraic tensor product $\mathcal{H}^{\tilde{\otimes} k}$ and then it must hold in the whole space by density).

2.3 L^2-space

A particular example of a Hilbert space that we will be dealing a lot with, is the case of $L^2(\mathbb{R}^n)$. We equip $\mathbb{R}^n$ with the Lebesgue measure (for a detailed construction of the Lebesgue measure, the reader may see [12], pages 49-54). For each $1 \leq p < \infty$, we define $L^p(\mathbb{R}^n)$ as the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ for which

$$\int_{\mathbb{R}^n} |f(x)|^p dx < \infty$$

and equip it with norm

$$\|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p}$$

This is actually a seminorm, since $\|f\|_p = 0$ if and only if $f = 0$ almost everywhere, i.e. there exists a set $S \subset \mathbb{R}^n$ of zero Lebesgue-measure such that $f(x) = 0$ for each $x \in \mathbb{R}^n \setminus S$. In order to fix this subtlety, we say that two measurable functions f, g are equivalent if $f = g$ almost everywhere. This is an equivalence relation in the set of all measurable functions, so that the elements of $L^p(\mathbb{R}^n)$ are in fact equivalence classes of functions (we identify functions that agree almost everywhere). Then, $\|f\|_p$ actually becomes a norm. Triangle inequality follows from Minkowski's inequality ([12] Theorem 3.5). Since this space is complete, this is actually a Banach space ([12], Theorem 3.11). In the particular case when $p = 2$, this is actually a Hilbert space, with inner product

$$\langle f, g \rangle_2 := \int_{\mathbb{R}^n} \overline{f(x)} g(x) dx$$

The fact that this is actually well-defined when $f, g \in L^2(\mathbb{R}^n)$ can be seen from Hölder's inequality ([12] Theorem 3.5). But $L^2(\mathbb{R}^n)$ has another very interesting property, which we are proving here:

Proposition 2.5. *For each m, n, the spaces $L^2(\mathbb{R}^m) \otimes_H L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^{m+n})$ are isometrically isomorphic, in the sense that there exists a linear bijective map between them that is also an isometry (i.e. preserves norms). Here, $\otimes_H$ denotes the Hilbert-space tensor product as we defined it in the previous subsection.*

Proof. First, we prove that $L^2(\mathbb{R}^n)$ is separable. The space $C_c(\mathbb{R}^n)$ of all continuous functions with compact support is dense in $L^2(\mathbb{R}^n)$ ([12], Theorem 3.14), and it is not hard to show that $C_c(\mathbb{R}^n)$ is separable. We define a map F from the algebraic tensor product $L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{m+n})$ as $F(f \otimes g)(\theta, \eta) = f(\theta) g(\eta))$. It is clear that this is a well-defined map. We show now that it is an isometry: For an element $\Psi = \sum_i \psi_i \otimes \phi_i \in L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^n)$, we can find a finite orthonormal set $\{e_j\}$ in $L^2(\mathbb{R}^n)$ that has the same span as the set $\{\psi_i\}$. Then, we can write $\Psi = \sum_j a_j \otimes e_j$ for certain functions $a_j \in L^2(\mathbb{R}^m)$, and we have

$$\|\Psi\| = \left(\sum_j \|a_j\|_2^2 \right)^{1/2}$$

We have:

$$\|F(\Psi)\|_2^2 = \int_{\mathbb{R}^{m+n}} \left| \sum_j a_j(\theta)e_j(\eta) \right|^2 d\theta d\eta$$
$$= \int_{\mathbb{R}^{m+n}} \overline{\sum_j a_j(\theta)e_j(\eta)} \sum_k a_k(\theta)e_k(\eta)d\theta d\eta$$
$$= \sum_{j,k} \left(\int_{\mathbb{R}^m} \overline{a_j(\theta)} a_k(\theta)d\theta \right) \left(\int_{\mathbb{R}^n} \overline{e_j(\eta)} e_k(\eta)d\eta \right) = \sum_j \int_{\mathbb{R}^m} |a_j(\theta)|^2 d\theta$$
$$= \sum_j \|a_j\|_2^2 = \|\Psi\|^2$$

This proves that the map F is an isometry. It suffices then to show that its image is dense in $L^2(\mathbb{R}^{m+n})$. To this end, since the spaces $L^2(\mathbb{R}^m)$ and $L^2(\mathbb{R}^n)$ are separable, we can find orthonormal bases $\{\phi_j\}_{j\in\mathbb{N}}$, $\{\psi_j\}_{j\in\mathbb{N}}$ respectively. We are going to prove that the set $\{F(\phi_j \otimes \psi_k)\}_{j,k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^{m+n})$. It is clear that it is an orthonormal set. Now, in order to prove it is a basis, it is sufficient to prove that if $f \in L^2(\mathbb{R}^{m+n})$ satisfies $\langle F(\phi_j \otimes \psi_k), f\rangle = 0$ for each $j, k \in \mathbb{N}$, then $f = 0$. We have for each j, k that

$$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(\theta, \eta)\psi_k(\eta)d\eta \right) \phi_j(\theta)d\theta = 0$$

therefore, since $\{\phi_j\}$ is an ON basis, we have

$$\int_{\mathbb{R}^n} f(\theta, \eta)\psi_k(\eta)d\eta = 0$$

almost everywhere in $\mathbb{R}^m$, i.e. for each k there exists a set $E_k \subset \mathbb{R}^m$ of measure 0 such that $\int_{\mathbb{R}^n} f(\theta, \eta)\psi_k(\eta)d\eta = 0$ for each $\theta \in \mathbb{R}^m \backslash E_k$. We set $E = \cup_k E_k$. Since this is a countable union, E also has measure 0 and we have $\int_{\mathbb{R}^n} f(\theta, \eta)\psi_k(\eta)d\eta = 0$ for each $k \in \mathbb{N}$, and each $\theta \in \mathbb{R}^m \backslash E$. Therefore, for almost every θ, the function $\eta \mapsto f(\theta, \eta)$ is almost everywhere zero. This actually means that f is zero almost everywhere, and this proves that $\{F(\phi_j \otimes \psi_k)\}_{j,k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^{m+n})$. Since the subspace spanned by an ON basis is dense in a Hilbert space, the proof is complete.

□

2.4 Bochner spaces

There is a generalization of the theory of measurable and integrable functions to functions taking values in Banach spaces (instead of $\mathbb{C}$). Of course the generalization follows in a rather obvious way when we have functions taking values in a finite-dimensional space, but the situation is quite different when the space is infinite-dimensional. We are going to state a few fundamental results of this theory, that will be needed later.

Suppose that X is a complex Banach space. Its (continuous) dual space X^* is defined as the space of all linear continuous functionals from X into $\mathbb{C}$, i.e. $\mathscr{L}(X, \mathbb{C})$. We also suppose that $(S, \mu, \mathfrak{M})$ is a positive measure space. There are three different concepts of measurability for functions $f : S \to X$. For the first one, we will need the definition of a simple function:

Definition 2.1. A function $f : S \to X$ is called **simple** if it takes finitely many values and if $f^{-1}(\{x\})$ is a measurable set for each $x \in X$, i.e. belongs to $\mathfrak{M}$. The set of all simple functions will be denoted by $\Sigma(S; X)$.

We observe that except for finitely many $x \in X$, the set $f^{-1}(\{x\})$ is empty, so we only have to bother for the $x \in X$ that belong to the image of f.

We now state the three notions for measurability:

Definition 2.2. Suppose that $f : S \to X$.

- f is called **strongly measurable** or **Bochner measurable** if it is the pointwise limit of a sequence of simple functions, i.e. there exists a sequence $(s_n) \subset \Sigma(S; X)$ such that $\lim s_n(t) = f(t)$ for each $t \in S$.
- f is called **measurable** if for each Borel subset $A \subset X$, its preimage $f^{-1}(A)$ is measurable.
- f is called **weakly measurable** if for each $x^* \in X^*$, the function $x^* \circ f : S \to \mathbb{C}$ is measurable (in the usual sense).

In the case $X = \mathbb{C}$, the three definitions are equivalent. In general this is not true, however this holds in many cases. In general, it can be verified that

$$f \text{ strongly measurable} \Rightarrow f \text{ measurable} \Rightarrow f \text{ weakly measurable}$$

However, we have the following very important result, whose proof can be found in [10], Theorem 1.1.20:

Theorem 2.6 (Pettis measurability Theorem)**.** *If $f : S \to$ is weakly measurable and separably valued, i.e. $f(S) \subset X'$ for a subspace X' of X that is separable, then f is strongly measurable.*

In particular, if X is separable, the three definitions are equivalent. We shall only consider cases where X is separable, therefore we are not going to distinguish between the three definitions. We are now going to introduce **Bochner spaces**, which are the generalization of L^p-spaces. From now on, for this section, we are assuming that X is separable.

If $f : S \to X$ is measurable, then the function $\|f\| : S \to \mathbb{C}$, where $\|f\|(s) := \|f(s)\|$ is measurable too, because $\|\cdot\|$ is continuous, thus measurable. For $1 \leq p < \infty$, we define $L^p(S; X)$ as the space of all measurable functions $f : S \to X$, for which

$$\int_S \|f(s)\|^p d\mu(s) < \infty$$

i.e. all measurable functions $f : S \to X$ for which $\|f\| \in L^p(S)$ (as before, we identify functions that agree in almost every point). For $p = \infty$ we set $L^\infty(S; X)$ be the set of all measurable functions $f : S \to X$ for which there exists a set $E \subset S$ of measure 0 such that $\sup\{\|f(s)\| \ : \ s \in S \setminus E\} < \infty$. It is trivial, using Minkowski's inequality, to verify that $L^p(S; X)$ is a Banach space for each $1 \leq p \leq \infty$ with norm

$$\|f\|_p = \left(\int_S \|f(s)\|^p d\mu(s) \right)^{1/p}$$

in case $p < \infty$ and

$$\|f\|_\infty = \text{ess sup}\{\|f(s)\| \ : \ s \in S\}$$

The proof that these spaces are Banach uses the same arguments as for the usual case $X = \mathbb{C}$.

An example of particular interest for us is the space $L^2(S; \mathcal{K})$ when $\mathcal{K}$ is a Hilbert space. This is itself a Hilbert space, since the norm comes from the inner product

$$\langle f, g \rangle = \int_S \langle f(s), g(s) \rangle d\mu(s)$$

It is easy to see that if f, g are measurable, then $s \mapsto \langle f(s), g(s) \rangle$ is also measurable. The fact that the integral converges absolutely follows from Cauchy-Schwarz inequality and Hölder's inequality. It is then trivial to check that this is in fact an inner product.

We shall now prove a result that we are going to need later, to describe the model studied by Lechner and Schützenhofer in [7].

Proposition 2.7. *Suppose that $\mathcal{K}$ is a separable Hilbert space and $(S, \mathfrak{M}, \mu)$ a positive measure space. Then, the spaces $L^2(S; \mathcal{K})$ and $L^2(S) \otimes_H \mathcal{K}$ are naturally isometrically isomorphic, via the map*

$$f \otimes h \mapsto (\theta \mapsto f(\theta)h)$$

Here, $\otimes_H$ denotes the Hilbert space tensor product of the two spaces.

The proof follows easily if we use the following lemma:

Lemma 2.8. *Suppose $(S, \mathfrak{M}, \mu)$ is a positive measure space and X is a separable Banach space. Then, for each $p \in [1, \infty)$, the space $\Sigma(S; X) \cap L^p(S; X)$ of $p-$integrable simple functions is $\| \cdot \|_p$-dense in $L^p(S; X)$.*

The proof can be found in [5], Lemma 1.2.19.

Proof of Proposition 2.7. We define a map $F : L^2(S) \otimes \mathcal{K} \to L^2(S; \mathcal{K})$ where $L^2(S) \otimes \mathcal{K}$ denotes the algebraic tensor product of the two spaces as follows:

$$F(f \otimes h)(x) = f(x)h$$

This is clearly well-defined and extends by linearity to the algebraic tensor product. To see that this is actually an isometry, for any element $\Psi \in L^2(S) \otimes \mathcal{K}$, we can write it in the form

$$\Psi = \sum_j f_j \otimes e_j$$

where $\{e_j\}$ is an ON set in $\mathcal{K}$. Then, it follows that

$$\|\Psi\| = \left(\sum_j \|f_j\|_2^2 \right)^{1/2}$$

Now, we compute:

$$\begin{aligned} \|F(\Psi)\|_2^2 &= \int_S \left\| \sum_j f_j(s) e_j \right\|^2 d\mu(s) \\ &= \int_S \sum_{j,i} \overline{f_j(s)} f_i(s) \langle e_j, e_i \rangle d\mu(s) = \int_S \sum_j |f_j(s)|^2 d\mu(s) = \sum_j \|f_j\|^2 \\ &= \|\Psi\|^2 \end{aligned}$$

Hence, we can extend this map to an isometry from $L^2(S) \otimes_H \mathcal{K}$ by density and continuity. From Lemma 2.8, it follows that (the image of) the algebraic tensor product $L^2(S) \otimes \mathcal{K}$ is dense in $L^2(S; \mathcal{K})$, since it contains all simple functions that are $p-$integrable. Therefore, the image of the extended map is the whole space and the proof is complete. □

Later, we are going to be dealing with the space $\mathcal{L} := L^2(\mathbb{R}; \mathcal{K})$ for some separable Hilbert space $\mathcal{K}$, and the spaces $\mathcal{L}^{\otimes n}$. By Proposition 2.7, $\mathcal{L} \simeq L^2(\mathbb{R}) \otimes_H \mathcal{K}$ and therefore

$$\begin{aligned} \mathcal{L}^{\otimes n} &\simeq (L^2(\mathbb{R}) \otimes_H \mathcal{K}) \otimes_H .. \otimes_H (L^2(\mathbb{R}) \otimes_H \mathcal{K}) \\ &\simeq (L^2(\mathbb{R}) \otimes_H .. \otimes_H L^2(\mathbb{R})) \otimes_H (\mathcal{K} \otimes_H .. \otimes_H \mathcal{K}) \\ &\simeq L^2(\mathbb{R}^n) \otimes_H \mathcal{K}^{\otimes n} \simeq L^2(\mathbb{R}^n; \mathcal{K}^{\otimes n}) \end{aligned}$$

where $\simeq$ means that the spaces are isometrically isomorphic. Of course all these isomorphisms are natural. Another useful Corollary of Proposition 2.7 is the following:

Corollary 2.9. *The space $C_c(\mathbb{R}^n; \mathcal{K})$ of continuous compactly supported functions on $\mathbb{R}^n$ with values in $\mathcal{K}$ is dense in $L^2(\mathbb{R}^n; \mathcal{K})$.*

Proof. Since $C_c(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, it is a simple exercise to show that the algebraic tensor product $C_c(\mathbb{R}^n) \otimes \mathcal{K}$ is dense in $L^2(\mathbb{R}^n) \otimes \mathcal{K}$. Since the latter is dense in $L^2(\mathbb{R}^n) \otimes_H \mathcal{K} = L^2(\mathbb{R}^n; \mathcal{K})$, $C_c(\mathbb{R}^n) \otimes \mathcal{K}$ is dense in $L^2(\mathbb{R}^n; \mathcal{K})$. Since we clearly have $C_c(\mathbb{R}^n) \otimes \mathcal{K} \subset C_c(\mathbb{R}^n; \mathcal{K})$, the proof is complete. □

2.5 Symmetric group

In this section, we shall establish a finite presentation of the symmetric group $\mathscr{G}_n$ that will be proven very useful later. By $\mathscr{G}_n$, we mean the set of all bijections from the set $\{1, .., n\}$ onto itself. This has the structure of a finite group, with multiplication given by composition and the identity element being the identity function. For each k, the element $\tau_k = (k \;\; k+1)$ will denote the permutation that keeps all numbers besides k and $k+1$ fixed, and sends k to $k+1$ and vice versa (this is an element of $\mathscr{G}_n$ for each $n \geq k+1$). It is trivial to verify the following relations:

- $\tau_k^2 = 1$ for all k
- $\tau_i \tau_j = \tau_j \tau_i$ for all i, j such that $|i - j| \geq 2$
- $\tau_k \tau_{k+1} \tau_k = \tau_{k+1} \tau_k \tau_{k+1}$ for all k

We are going to prove that these relations are enough to describe the group $\mathscr{G}_n$.

For each n, we define

$$G_n = \langle g_1, .., g_{n-1} \mid g_i^2 = 1,\; g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},\; g_i g_j = g_j g_i \text{ for } |i-j| \geq 2 \rangle$$

We are going to prove that the groups G_n and $\mathscr{G}_n$ are isomorphic for each n. Since $\mathscr{G}_n$ satisfies the above relations, there exists a (unique) well-defined homorphism from G_n into $\mathscr{G}_n$ such that $g_i \mapsto \tau_i$ for each i. For any transposition $(m \;\; n)$ (with $n > m$), it is easy to verify that

$$(m \;\; n) = \tau_m \tau_{m+1} .. \tau_{n-2} \tau_{n-1} \tau_{n-2} .. \tau_{m+1} \tau_m$$

and it is well known that $\mathscr{G}_n$ is generated by its transpositions. Therefore, $\mathscr{G}_n$ is generated by the elements $\tau_1, .., \tau_{n-1}$. Hence, the above homomorphism from G_n into $\mathscr{G}_n$ is also surjective. All that remains is to show that it is also injective. We are going to prove this after proving two Lemmas first and after we observe that there is a natural homomorphism f_n from G_n into G_{n+1} that maps g_i to itseld for each $i \leq n-1$.

Lemma 2.10. *Each element of G_n can be written as a word in the letters $g_1, .., g_{n-1}$ with g_{n-1} appearing at most once.*

Proof. We are going to prove it by induction on n. For G_1 and G_2 it is obvious (G_1 is the trivial group and G_2 contains only two elements, 1 and g_1). We let $n \geq 3$ and we assume that the claim holds for G_{n-1}. Since $g_i^2 = 1$, i.e. $g_i^{-1} = g_i$, it is clear that every element can be written as a word in the letters $g_1, .., g_{n-1}$. Suppose g is any word in the letters $g_1, .., g_{n-1}$ and assume g_{n-1} appears more than once, i.e. $g = w_1 g_{n-1} w_2 g_{n-1} w_3$ for words w_1, w_2, w_3. We can assume that w_2 does not contain the letter g_{n-1}. If w_2 does not contain the letter g_{n-2} either, then it commutes with g_{n-1}, therefore $g = w_1 w_2 w_3$ and we have eliminated the letter g_{n-1} twice. In case it contains g_{n-2}, since it is in the image of G_{n-1}, by the assumption hypothesis we can assume that it contains the letter g_{n-2} only once, i.e. $w_2 = wg_{n-2}w'$ with w, w' being words in the letters $g_1, .., g_{n-3}$. Since w, w' commute with g_{n-1}, we get

$$g = w_1 g_{n-1} w g_{n-2} w' g_{n-1} w_3 = w_1 w g_{n-1} g_{n-2} g_{n-1} w' w_3$$
$$= w_1 w g_{n-2} g_{n-1} g_{n-2} w' w_3$$

Thus we have eliminated the letter g_{n-1} once. Continuing this way, we can reduce the number of times it appears to at most 1. □

We introduce the following subsets of G_n:

$$\Sigma_1 = \{1, g_1\}$$
$$\Sigma_2 = \{1, g_2, g_2 g_1\}$$
$$...$$
$$\Sigma_{n-1} = \{1, g_{n-1}, g_{n-1}g_{n-2}, .., g_{n-1}g_{n-2}..g_1\}$$

It is clear that the cardinality of Σ_i, $|\Sigma_i|$ is $\leq i + 1$ for each i.

Lemma 2.11. *Every element of G_n can be written as a word $u_1..u_{n-1}$, where $u_i \in \Sigma_i$ for each i.*

Proof. We prove it again by induction on n. For $n = 1$ there is nothing to prove and $G_2 = \Sigma_1$. Now, assume $n \geq 3$ and that the claim holds for the group G_{n-1}. Let $g \in G_n$. By Lemma 2.10, g can be written in a way that it contains the letter g_{n-1} at most once.

If it does not contain the letter g_{n-1}, it is in the image of G_{n-1}, thus it can be written as $g = u_1..u_{n-2}$ with $u_i \in \Sigma_i$ for each i. Setting $u_{n-1} = 1 \in \Sigma_{n-1}$, we have written g in the desired way.

Now, assume that g contains the letter g_{n-1} exactly once, i.e. $g = w_1 g_{n-1} w_2$ for words w_1, w_2 which do not include g_{n-1}. Since w_2 comes from G_{n-1}, by the induction hypothesis it can be written as $w_2 = u_1..u_{n-2}$ with $u_i \in \Sigma_i$ for each i. It is clear that g_{n-1} commutes with $u_1, .., u_{n-3}$, thus

$$g = w_1 u_1 u_2..u_{n-3} g_{n-1} u_{n-2}$$

The word $w_1 u_1 u_2..u_{n-3}$ also comes from G_{n-1}, thus it can be written as $v_1..v_{n-2}$ with $v_i \in \Sigma_i$ for each i. Finally, since $u_{n-2} \in \Sigma_{n-2}$, it follows that $g_{n-1}u_{n-2} \in \Sigma_{n-1}$. This completes the proof. □

Now, we can finally prove the main result:

Theorem 2.12. *The groups G_n and $\mathscr{G}_n$ are isomorphic for each n.*

Proof. From the discussion above, we have seen that there exists a surjective homomorphism from G_n onto $\mathscr{G}_n$. Since $\mathscr{G}_n$ is finite, in order to show that it is also injective, it suffices to show $|G_n| \leq |\mathscr{G}_n| = n!$ From Lemma 2.11, the map $(u_1, .., u_{n-1}) \mapsto u_1..u_{n-2}$ from $\Sigma_1 \times .. \times \Sigma_{n-1}$ into G_n is surjective, therefore

$$|G_n| \leq |\Sigma_1 \times .. \times \Sigma_{n-1}| = \prod_{i=1}^{n-1} |\Sigma_i| \leq \prod_{i=1}^{n-1} (i + 1) = n!$$

□

2.6 Topological Vector Spaces

In this section we shall build towards the definition of nuclear spaces and some properties that we are going to use.

A **topological vector space** X is a vector space over $\mathbb{F}$ $(= \mathbb{R}$ or $\mathbb{C})$, equipped with a Hausdorff topology, such that the maps $(x, y) \mapsto x + y$ from $X \times X$ onto X and $(\lambda, x) \mapsto \lambda x$ from $\mathbb{F} \times X$ onto X are both continuous. We will restrict our attention only in the case $\mathbb{F} = \mathbb{C}$. By definition, U is a neighbourhood of 0 if and only if $x + U$ is a neighbourhood of x for each $x \in X$. Therefore, the topology is described completely by the set of neighbourhoods of 0. We introduce a few definitions that play an important role in the Theory of topological vector spaces.

A set $A \subset X$ is called **convex** if for every $x, y \in A$ and $0 \leq t \leq 1$,
$tx + (1 - t)y \in A$.

A set $A \subset X$ is called **balanced** if $\lambda A \subset A$ for every $|\lambda| \leq 1$.

A set $A \subset X$ is called **absorbing** if for each $x \in X$ there exists some $t > 0$ such that $x \in tA$.

A set $A \subset X$ is called **bounded** if for every neighbourhood U of 0, there exists some $\rho > 0$ such that $A \subset \rho U$.

2.6.1 Locally convex spaces

In order to define locally convex spaces, we need to fix some terms first. A **neighbourhood** of a point $x \in X$ is an open set that contains x. A set $\mathscr{B}$ of neighbourhoods of x is called a **neighbourhood basis** of x if every neighbourhood of x contains some element of $\mathscr{B}$.

Definition 2.3. Let X be a topological vector space over $\mathbb{C}$. X is called **locally convex** if there exists a neighbourhood basis of 0 consisting only of convex balanced open sets.

There exists an equivalent characterization of locally convex spaces that is quite more convenient to work with.

If X is a vector space over $\mathbb{F}$, a **seminorm** on X is a function $p : X \to [0, \infty)$ such that:

- $p(\lambda x) = |\lambda| p(x)$ for each $\lambda \in \mathbb{F}$ and $x \in X$
- $p(x + y) \leq p(x) + p(y)$ for each $x, y \in X$

In other words, p is like a norm, with the only difference being that there might be nonzero elements x with $p(x) = 0$. Now suppose that there exists a separating family of seminorms $\{p_i \ : \ i \in I\}$ on X, i.e. for each $x \neq 0$ there exists an $i \in I$ with $p_i(x) \neq 0$. For each $x \in X$, any finite subset $\{i_1, .., i_n\}$ of I and $\varepsilon > 0$, we set

$$U_{x,i_1,..,i_n,\varepsilon} = \bigcap_{k=1}^{n} \{y \in X \ : \ p_{i_k}(x - y) < \varepsilon\}$$

It can be seen that these sets form a basis for a certain topology in X. The equivalent characterization of a locally convex space is the following:

Theorem 2.13. *A topological vector space X over $\mathbb{F}$ is locally convex if and only if its topology is induced by a separating family of seminorms.*

Sketch of the proof. In case the topology of X is given by a separating family of seminorms, it is easy to verify that it is locally convex. For the other direction, suppose $\mathscr{B}$ is a neighbourhood basis of 0 consisting of convex balanced sets. For every $U \in \mathscr{B}$, the *Minkowski functional* of U is defined as

$$p_U(x) := \inf\{t > 0 \ : \ x \in tU\}$$

To see that this is finite, one has to observe that every neighbourhood of 0 is absorbing. For any $x \in X$, the map $\lambda \mapsto \lambda x$ from $\mathbb{F}$ to X is continuous, and since 0 gets mapped to 0, there exists a $t > 0$ such that the image of $(-t, t)$ is contained in U, i.e. $rx \in U$ for each $0 < r < t$. This proves the claim. The fact that U is balanced and convex can be used to prove that p_U is actually a seminorm. It can also be verified that for each $U \in \mathscr{B}$, $U = \{x \in X \ : \ p_U(x) < 1\}$. If $p_U(x) = 0$ for each $U \in \mathscr{B}$, then $x \in U$ for each $U \in \mathscr{B}$ which means $x = 0$, since X is Hausdorff. Thus the family is also separating. Let $\mathscr{B}'$ be the family of all sets of the form

$$\bigcap_{k=1}^{n} \{x \in X \ : \ p_{U_k}(x) < \varepsilon\}$$

for $U_1, .., U_n \in \mathscr{B}$. To show that the two topologies coincide, it suffices to show that for each $V \in \mathscr{B}$ there exists $V' \in \mathscr{B}'$ with $V' \subset V$ and vice versa. One assertion is trivial, since $U = [p_U < 1]$ for each $U \in \mathscr{B}$. Now, suppose

$$V' = \bigcap_{k=1}^{n} [p_{U_k} < \varepsilon] \in \mathscr{B}'$$

Clearly, $\varepsilon U_k = [p_{U_k} < \varepsilon]$, thus $V' = \bigcap \varepsilon U_k$, which is open in the original locally convex topology, as a finite intersection of open sets, and a neighbourhood of 0. Therefore, it contains some subset of $\mathscr{B}$. ☐

A locally convex topological vector space that is also metrizable (i.e. its topology is induced by some metric) is called a **Fréchet** space. The following Proposition gives us a good insight into Fréchet spaces:

Proposition 2.14. *Suppose X is a locally convex topological vector space. X is metrizable if and only if its topology is induced by a countable separating family of seminorms.*

Proof. We only prove one direction. The other is Remark 1.38 (c) in [11].

Suppose that X is metrizable and $\mathcal{P}$ is the family of seminorms that induces its topology. Then, for each $n \in \mathbb{N}$, since the ball of radius $1/n$ around 0, $B(0, 1/n)$ is a neighbourhood of 0, there exist seminorms $p_1^n, .., p_{k_n}^n \in \mathcal{P}$ and a $\varepsilon_n > 0$ such that

$$\bigcap_{i=1}^{k_n} [p_i^n < \varepsilon_n] \subset B(0, 1/n)$$

We then consider the family $\mathcal{Q} = \bigcup_{n=1}^{\infty} \{p_1^n, .., p_{k_n}^n\}$. Clearly, this is a countable family, and since $\{B(0, 1/n) \ : \ n \in \mathbb{N}\}$ is a neighbourhood basis of 0, the topology is induced by the family $\mathcal{Q}$. It is obvious that this family is also separating. □

2.6.2 Tensor products and bilinear mappings

For a locally convex topological vector space X over $\mathbb{F}$, we denote by X' the topological dual of X, i.e. the set of all continuous linear mappings $f : X \to \mathbb{F}$. There are many different ways to endow X' with a topology that makes it a locally convex space. We are going to focus on the **weak dual topology** and the **strong dual topology**. For each $x \in X$, the function $p_x : X' \to [0, \infty)$, $p_x(x') = |x'(x)|$ is clearly a seminorm on X' and the family $\{p_x \ : \ x \in X\}$ is clearly separating. The induced locally convex topology on X' is called **weak dual topology** or just **weak topology**. It is clear that a net (x'_λ) in X' converges to $x' \in X'$ if and only if $x'_\lambda(x) \to x'(x)$ for every $x \in X$. When X' is endowed with this topology, it will be denoted by X'_σ.

There exists another topology that we shall introduce. For every bounded set $B \subset X$, we define the function $p_B : X' \to [0, \infty)$, by

$$p_B(x') = \sup_{x \in B} |x'(x)|$$

Since continuous images of bounded sets are bounded, the former quantity is finite and it follows trivially that p_B is actually a seminorm. It is also clear that the family $\{p_B \ : \ B \text{ bounded}\}$ is separating. The induced locally convex topology is called **strong dual topology** or just **strong topology**. When X' is endowed with it, it is denoted by X'_b. Observe that in case X is a normed space, this topology is actually the usual norm topology on X^*.

If E, F, G are topological spaces, a map $f : E \times F \to G$ is said to be **separately continuous** if the maps $f_x = f(x, \cdot) : F \to G$ and $f_y = f(\cdot, y) : E \to G$ are continuous for each x, y. f is said to be **jointly continuous** (or just continuous) if it is continuous when $E \times F$ is endowed with the product topology. It is straightforward to verify that if f is jointly continuous, then it is separately continuous. The converse is not true in general. If the spaces E, F, G are topological vector spaces, we denote by $B(E, F; G)$ the space of all jointly continuous bilinear forms from $E \times F$ to G and by $\mathscr{B}(E, F; G)$ the space of all separately continuous bilinear forms. The spaces have an obvious linear structure and $B(E, F; G)$ is a subspace of $\mathscr{B}(E, F; G)$. In case $G = \mathbb{F}$, we will denote the spaces by $B(E, F)$ and $\mathscr{B}(E, F)$. We would like to endow them with a certain topology.

For every A bounded subset of E, B a bounded subset of F and W a neighbourhood of 0, we set

$$U(A, B, W) := \{\Phi \in B(E, F; G) \ : \ \Phi(A, B) \subset W\}$$

The family of all such subsets can be proved to be a neighbourhood basis of 0 in $B(E, F; G)$ for a locally convex topology, compatible with the linear structure of

$B(E,F;G)$. This is due to the fact that $U(A,B,W)$ is absorbing, which follows from the fact that $\Phi(A,B)$ is bounded. The induced topology is often referred to as **topology of bi-bounded convergence**. This is no longer true in case $\Phi \in \mathscr{B}(E,F;G)$ and one has to be more careful.

We are going to equip the space $\mathscr{B}(E'_b, F'_b)$ with some locally convex topology. First, we need the following definition:

Definition 2.4. Suppose X is a Hausdorff topological space and Y is a topological vector space. Suppose that S is a set of continuous functions from X into Y and $x_0 \in X$. We say that the set S is *equicontinuous* at x_0 if for every neighbourhood V of 0 in Y, there exists a neighbourhood U of x_0 such that $f(x) - f(x_0) \in V$ for every $f \in S$ and every $x \in U$.

We say that S is *equicontinuous* if it is equicontinuous at each $x_0 \in X$.

Then, we have the following result ([15] Proposition 42.1):

Proposition 2.15. *Suppose that E,F,G are locally convex spaces and that E'_b, F'_b are the strong duals of E and F respectively. Suppose also that $\Phi \in \mathscr{B}(E'_b, F'_b; G)$ and that A and B are equicontinuous subsets of E' and F' respectively. Then, $\Phi(A,B)$ is bounded in G.*

Now, the family of the sets

$$U(A,B,W) := \{\Phi \in \mathscr{B}(E'_b, F'_b; G) \ : \ \Phi(A,B) \subset W\}$$

for A,B equicontinuous subsets of E' and F' respectively and W a neighbourhood of 0 in G is actually a neighbourhood basis of 0 for a locally convex topology on $\mathscr{B}(E'_b, F'_b; G)$ that is compatible with its linear structure.

Since the weak dual topology is clearly weaker than the strong topology, we have the following inclusions:

$$B(E'_\sigma, F'_\sigma) \subset \mathscr{B}(E'_\sigma, F'_\sigma) \subset \mathscr{B}(E'_b, F'_b) \tag{2.1}$$

Now, for E,F locally convex spaces we are finally able to endow their algebraic tensor product $E \otimes F$ with a locally convex topology. For $x \in E$, $y \in F$, the map $(x', y') \mapsto x'(x)y'(y)$ from $E' \times F'$ into $\mathbb{F}$, is clearly bilinear and continuous when E' and F' are equipped with the weak dual topology. This gives us a well defined map from the algebraic tensor product $E \otimes F$ into $B(E'_\sigma, F'_\sigma)$. In [15], Proposition 42.4 it is actually shown that this map is an isomorphism of $E \otimes F$ onto $B(E'_\sigma, F'_\sigma)$. We call $E \otimes F$ equipped with the topology that it induces from $\mathscr{B}(E'_b, F'_b)$ (see 2.1) **injective tensor product** of the spaces E and F and we denote it by $E \otimes_\varepsilon F$.

Remark 2.1. We observe that the function $(x,y) \mapsto x \otimes y$ from $E \times F$ into $E \otimes_\varepsilon F$ is continuous. To see this, suppose that the net (x_λ, y_λ) converges to 0. Then $x_\lambda \to 0$ and $y_\lambda \to 0$. We need to show that $x_\lambda \otimes y_\lambda$, considered as an element of $B(E'_\sigma, F'_\sigma)$ converges to 0 in the topology of uniform convergence on equicontinuous subsets. Equivalently, we need to show that for each $\epsilon > 0$ and every equicontinuous subsets $A \subset E'$, $B \subset F'$, there exists λ_0 such that

$|x_\lambda \otimes y_\lambda(A, B)| < \epsilon$ for every $\lambda \geq \lambda_0$. Since A (resp. B) is equicontinuous, there exists a neighbourhood U (resp. V) of 0 in E (resp. F) such that $|x'(x)| < \sqrt{\epsilon}$ (resp. $|y'(y)| < \sqrt{\epsilon}$) for every $x \in U$ and $x' \in A$ (resp. every $y \in V$ and $y' \in B$). Since we have $x_\lambda \to 0$ and $y_\lambda \to 0$, there exists a λ_0 such that $x_\lambda \in U$ and $y_\lambda \in V$ for every $\lambda \geq \lambda_0$. Then, for every $x' \in A$, $y' \in B$ and $\lambda \geq \lambda_0$, we have

$$|x_\lambda \otimes y_\lambda(x', y')| = |x'(x_\lambda)y'(y_\lambda)| < \epsilon$$

which proves our claim.

The injective topology is one of the two main topologies defined on tensor products of locally convex spaces. The second one is the **projective topology**.

Definition 2.5. The strongest locally convex topology on $E \otimes F$ that makes the map $(x, y) \mapsto x \otimes y$ from $E \times F$ into $E \otimes F$ continuous is called *projective topology*. The space $E \otimes F$ equipped with this topology is denoted by $E \otimes_\pi F$.

By Remark 2.1, this topology exists and it is stronger than the $\varepsilon-$topology.

There is an explicit description of the family of seminorms that induces the topology of $E \otimes_\pi F$, in terms of the seminorms that induce the topologies of E and F. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are the families of seminorms that induce the topologies of E and F respectively. Then, for each $p \in \mathcal{P}$, $q \in \mathcal{Q}$ and $a \in E \otimes F$, we define

$$(p \otimes q)(a) := \inf \left\{ \sum_i p(x_i)q(y_i) \ : \ a = \sum_i x_i \otimes y_i \right\}$$

It can be shown that this is a seminorm and that the family

$$\{p \otimes q \ : \ p \in \mathcal{P}, \ q \in \mathcal{Q}\}$$

induces the topology of $E \otimes_\pi F$. For a detailed proof, see [15], Proposition 43.1.

The projective tensor product is also described by its universal property:

Theorem 2.16. *Suppose E, F and G are locally convex topological vector spaces and $f : E \times F \to G$ is a continuous bilinear mapping. Then, the induced linear map $\tilde{f} : E \otimes_\pi F \to G$ is continuous with respect to the projective topology.*

The correspondence $f \leftrightarrow \tilde{f}$ provides an (algebraic) isomorphism between the spaces $B(E, F; G)$ and $L(E \otimes_\pi F; G)$, the latter being the space of all continuous linear maps from $E \otimes_\pi F$ into G. Furthermore, the projective topology is the only one with this property.

The proof follows easily from the definition of the projective topology. For a detailed proof, see [15] Proposition 43.4.

We also need to introduce the **completion** of a topological vector space. It is a generalization of the completion of a metric space, for topological vector spaces that might not be metrizable. Suppose E is a topological vector space. A net $\{x_\lambda\}_{\lambda \in \Lambda}$ of E is called a **Cauchy net** if for every neighbourhood U of 0, there exists some $\lambda_0 \in \Lambda$ such that $x_{\lambda_1} - x_{\lambda_2} \in U$ for every $\lambda_1, \lambda_2 \geq \lambda_0$.

The space E is called **complete** if every Cauchy net in E converges to some point $x \in E$. Now, suppose E is metrizable. By [12], Theorem 1.24, the metric that induces its topology can be assumed to be translation-invariant, i.e. $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in E$. It is not very difficult to show that if E is metrizable, then E is complete if and only if it is sequentially complete, i.e. if every Cauchy sequence converges.

We have the following remarkable result, whose proof can be found in [15], Theorem 5.2:

Theorem 2.17. *Let E be a topological vector space. There exists a complete topological vector space $\widehat{E}$ and a linear continuous and a linear embedding (i.e. a continuous injection that is a homeomorphism onto its image) $\iota : E \to \widehat{E}$ such that $\iota(E)$ is dense in $\widehat{E}$ and so that the following holds:*

For every complete topological vector space F and any continuous linear map $f : E \to F$, there exists a unique linear and continuous map $\widehat{f} : \widehat{E} \to F$ extending f, i.e. satisfying $\widehat{f} \circ \iota = f$. Furthermore, for every pair $(\widehat{E}_1, \iota_1)$ consisting of a complete topological vector space $\widehat{E}_1$ and a linear dense embedding ι_1 of E into $\widehat{E}_1$, there exists a linear homeomorphism $j : \widehat{E} \to \widehat{E}_1$ such that $j \circ \iota = \iota_1$

We call the space $\widehat{E}$ of the previous Theorem the **Cauchy completion** of E. In case E is metrizable, the Cauchy completion coincides with the usual completion of a metric space.

Now that we have introduced the Cauchy completion of a topological vector space, for E, F locally convex spaces, we denote by $E\hat{\otimes}_\varepsilon F$ and $E\hat{\otimes}_\pi F$ respectively, the Cauchy completions of the spaces $E \otimes_\varepsilon F$ and $E \otimes_\pi F$.

Remark 2.2. Since the projective topology on $E \otimes F$ is finer than the injective topology, the identity map $i : E \otimes_\pi F \to E \otimes_\varepsilon F$ is continuous. If we denote by ι_ε and ι_π the inclusions of $E \otimes_\varepsilon F$ and $E \otimes_\pi F$ respectively into their completions, the map $\iota_\varepsilon \circ i$ is linear and continuous from $E \otimes_\pi F$ into $E\hat{\otimes}_\varepsilon F$, so by Theorem 2.17, it extends uniquely to a linear and continuous map $\tilde{i} : E\hat{\otimes}_\pi F \to E\hat{\otimes}_\varepsilon F$.

2.6.3 Tensor product of Banach spaces

In order to introduce the definition of nuclear spaces, one needs to talk about projective tensor products of Banach spaces. This is a special case of the projective tensor product of locally convex spaces, as we will see. If X, Y are Banach spaces, for each $a \in X \otimes Y$ (the algebraic tensor product), we define

$$\|a\|_\pi := \inf \left\{ \sum_i \|x_i\| \|y_i\| \ : \ a = \sum_i x_i \otimes y_i \right\}$$

In [13] Proposition 2.1 it is shown that this is indeed a norm, that satisfies $\|x \otimes y\|_\pi = \|x\| \|y\|$. The completion of $X \otimes Y$ with respect to this norm is denoted by $X\hat{\otimes}_\pi Y$ and it is called the **projective tensor product** of X and Y. Like the projective tensor product in locally convex spaces, it possesses a universal property itself.

For Banach spaces X, Y and Z, it is easy to verify that a bilinear form $F : X \times Y \to Z$ is jointly continuous if and only if there exists a constant $C > 0$ such that $\|F(x, y)\| \leq C\|x\|\|y\|$ for all $x \in X$, $y \in Y$. Therefore, the space $B(X, Y; Z)$ of all jointly continuous bilinear forms can be equipped with the following norm:

$$\|F\| := \sup\{\|F(x, y)\| \ : \ \|x\| \leq 1, \ \|y\| \leq 1\}$$

With this in mind, we can state the universal property for the projective tensor product of Banach spaces:

Theorem 2.18. *Suppose X, Y, Z are Banach spaces. For every continuous bilinear map $F : X \times Y \to Z$, there exists a unique bounded linear map $\tilde{F} : X\hat{\otimes}_\pi Y \to Z$ satisfying $\tilde{F}(x \otimes y) = F(x, y)$ for every $x \in X, y \in Y$. Furthermore, $\|\tilde{F}\| = \|F\|$. Therefore, the spaces $B(X, Y; Z)$ and $\mathscr{L}(X\hat{\otimes}_\pi Y, Z)$ are isometrically isomorphic.*

The proof of the above result can be found in [13], Theorem 2.9. There are lots of other important results about the projective tensor product of Banach spaces, but we shall only use its universal property, in order to define nuclear spaces.

2.6.4 Nuclear mappings and nuclear spaces

Suppose X and Y are Banach spaces. We define a map $F : X^* \times Y \to \mathscr{L}(X; Y)$ by

$$F(x^*, y)(x) = x^*(x)y$$

It is clear that this is a well-defined, bilinear map. Furthermore, we have

$$\|F(x^*, y)(x)\| = \|x^*(x)y\| = |x^*(x)|\|y\| \leq \|x^*\|\|x\|\|y\|$$

This proves that $\|F(x^*, y)\| \leq \|x^*\|\|y\|$, hence $F \in B(X^*, Y; \mathscr{L}(X, Y))$ with $\|F\| \leq 1$. By Theorem 2.18, there is an induced bounded linear map $\tilde{F} : X^*\hat{\otimes}_\pi Y \to \mathscr{L}(X, Y)$ with $\tilde{F}(x^* \otimes y) = F(x^*, y)$ for each $x^* \in X^*$ and $y \in Y$, and furthermore, $\|\tilde{F}\| = \|F\| \leq 1$.

Definition 2.6. The image of $X^*\hat{\otimes}_\pi Y$ under $\tilde{F}$ into $\mathscr{L}(X, Y)$ is denoted by $\mathscr{L}^1(X, Y)$. The elements of $\mathscr{L}^1(X, Y)$ are called the **nuclear mappings** of X into Y.

Now, suppose that E is a locally convex topological vector space. Pick any continuous seminorm p on E (the topology of E does not change if we add p to the family of seminorms that induces its topology). The set

$$\ker p := \{x \in E \ : \ p(x) = 0\}$$

is a closed linear subspace of E. Therefore, $E/\ker p$ has a natural linear structure, and p is a norm on that space. We denote by $\widehat{E}_p$ the completion of that normed space. Observe that for any pair of continuous seminorms p, q such that $p \leq q$, we have a well-defined map $f_{qp} : \widehat{E}_q \to \widehat{E}_p$ (since $x \in \ker q$ implies $x \in \ker p$), which is also a contraction (i.e. $\|f_{qp}\| \leq 1$).

Definition 2.7. Suppose E is a locally convex space. E is called **nuclear** if for every pair of continuous seminorms p, q on E, such that $p \leq q$, the induced map $f_{qp} : \widehat{E}_q \to \widehat{E}_p$ is a nuclear mapping.

Although the definition is not very intuitive of what nuclear spaces actually are, they possess many interesting properties, a lot of which are associated to tensor products. A few of them, that we shall use later, are stated in the following Proposition:

Proposition 2.19. *(i) The space E is nuclear if and only if for every locally convex space F the canonical map $\tilde{i} : E\hat{\otimes}_\pi F \to E\hat{\otimes}_\varepsilon F$ of Remark 2.2 is an (onto) homeomorphism.*

(ii) If E is a Fréchet space, E is nuclear if and only if its strong dual E'_b is nuclear

(iii) If E and F are nuclear spaces, then $E\hat{\otimes}F$ is also nuclear.

For detailed proofs, the reader can see [15], Theorem 50.1 and Propositions 50.1, 50.6. Observe that the fact that $\tilde{i}$ is a homeomorphism implies that the identity map id: $E \otimes_\pi F \to E \otimes_\varepsilon F$ is also a homeomorphism, therefore the spaces $E \otimes_\pi F$ and $E \otimes_\varepsilon F$ carry the same topology.

2.6.5 Schwartz space

A famous example of a nuclear Fréchet space, which we are going to deal with later, is the space $\mathcal{S}(\mathbb{R}^n)$ of **Schwartz functions** on $\mathbb{R}^n$. This space consists of all smooth functions $f : \mathbb{R}^n \to \mathbb{C}$, such that

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty$$

for all multi-indices $\alpha = (\alpha_1, .., \alpha_n)$, $\beta = (\beta_1, .., \beta_n)$, where $x^\alpha = x_1^{\alpha_1}..x_n^{\alpha_n}$, $D^\beta f = \partial^{\beta_1}..\partial^{\beta_n} f$. We equip $\mathcal{S}(\mathbb{R}^n)$ with the locally convex topology induced by this family of seminorms. Since this is a countable family of seminorms, this is a Fréchet space. The proof that $\mathcal{S}(\mathbb{R}^n)$ is nuclear for each n is quite involved and we are not going to show this here. For a proof, the reader is referred to [15], Corollary of Theorem 51.5.

It is easy to see that $\mathcal{S}(\mathbb{R}^n)$ is complete. Suppose $(f_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}(\mathbb{R}^n)$. First, it is clear that (f_m) is uniformly Cauchy, as a function from $\mathbb{R}^n \to \mathbb{C}$, since for every $\varepsilon > 0$, $U = \{f \in \mathcal{S}(\mathbb{R}^n) \ : \ \|f\|_\infty < \varepsilon\}$ is a neighbourhood of 0, hence there exists M such that $f_k - f_m \in U$ for every $k, m > M$ or equivalently $\|f_k - f_m\|_\infty < \varepsilon$ for all $k, m > M$. We know by Mathematical Analysis that in that case, the sequence (f_m) converges uniformly to some function f and since (f_m) is clearly uniformly bounded, $\|f\|_\infty < \infty$. Repeating the same argument for all derivatives $D^\beta f_m$, for every multi-index β, the sequence $(D^\beta f_m)_{m\in\mathbb{N}}$ is uniformly Cauchy, therefore it converges uniformly to some function f_β, with $\|f_\beta\|_\infty < \infty$. Again, by undergraduate Analysis, we know this means that the derivative $D^\beta f$ exists and $D^\beta f = f_\beta$. Thus, f

is smooth. Again, if we repeat the argument for any multi-index α, we get that $(x^\alpha D^\beta f_m)$ converges uniformly to some function $f_{\alpha\beta}$ with $\|f_{\alpha\beta}\|_\infty < \infty$. Since clearly $x^\alpha D^\beta f_m \to x^\alpha D^\beta f$ pointwise, we must have $f_{\alpha\beta} = x^\alpha D^\beta f$. Thus, $f \in \mathcal{S}(\mathbb{R}^n)$ and since $\|f_m - f\|_{\alpha,\beta} \to 0$ for every multi-indices α, β, f_m converges to f in $\mathcal{S}(\mathbb{R}^n)$.

It is easy to see that if $f \in \mathcal{S}(\mathbb{R}^n)$, $g \in \mathcal{S}(\mathbb{R}^m)$, then the function $f \otimes g$ is in $\mathcal{S}(\mathbb{R}^{m+n})$ (we identify $f \otimes g$ with the function $(\theta, \eta) \mapsto f(\theta)g(\eta)$). Therefore, we get a bilinear map from $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^m)$ into $\mathcal{S}(\mathbb{R}^{m+n})$. The map is also continuous, therefore it induces a continuous map from $\mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^m)$ into $\mathcal{S}(\mathbb{R}^{n+m})$ (continuous with respect to both tensor products, thanks to nuclearity), which is clearly injective. In [15], Theorem 51.6 it is shown that the map is actually an embedding, with a dense image inside $\mathcal{S}(\mathbb{R}^{n+m})$, therefore, by the universal property in Theorem 2.17, the spaces $\mathcal{S}(\mathbb{R}^n)\hat{\otimes}\mathcal{S}(\mathbb{R}^m)$ and $\mathcal{S}(\mathbb{R}^{n+m})$ are actually homeomorphic.

3 Fock Space Structure and Unsymmetrized Expansion

In this section, we are going to introduce the unsymmetrized Fock space model that we are going to work with, and introduce the class of observables that we are going to expand, in terms of annihilation and creation operators.

We assume that $\mathcal{L}$ is a given complex Hilbert space, with inner product denoted by $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. The unsymmetrized Fock space is the Hilbert space direct sum

$$\mathcal{L}^{\oplus} := \bigoplus_{n=0}^{\infty} \mathcal{L}^{\otimes n}$$

where $\mathcal{L}^{\otimes n}$ is the usual Hilbert space tensor product of n copies of $\mathcal{L}$ (we set $\mathcal{L}^{\otimes 0} := \mathbb{C}$).

We define the annihilation and creation operators $(a, a^{\dagger})$ as (unbounded) operators acting on the Fock space in the following way:
For $f \in \mathcal{L}, \Phi = (\Phi_n)_n \in \mathcal{L}^{\oplus}$,

$$(a^{\dagger}(f)\Phi)_n = \sqrt{n} f \otimes \Phi_{n-1} \tag{3.1}$$

$$a(f) = (a^{\dagger}(f))^* \tag{3.2}$$

In particular, if $\phi = \phi_1 \otimes ... \otimes \phi_n$, we have

$$a(f)\phi = \sqrt{n} \langle f, \phi_1 \rangle \phi_2 \otimes ... \otimes \phi_n$$

In order to define the class of quadratic forms we are interested in, we assume that each space $\mathcal{L}^{\otimes n}$ contains a subspace $\mathcal{E}_n$, equipped with a topology τ_n and such that

(1) $\mathcal{E}_n \otimes \mathcal{E}_m$ is a dense subspace of $\mathcal{E}_{m+n}$ (in the topology of the latter).

(2) τ_n is finer than the inner product topology induced by $\mathcal{L}^{\otimes n}$

For each m, n let Q_{mn} denote the space of all (jointly) continuous quadratic (i.e. sesquilinear) forms $A : \mathcal{E}_m \times \mathcal{E}_n \to \mathbb{C}$, where $\mathcal{E}_m \times \mathcal{E}_n$ is equipped with the product topology of the topologies τ_n and τ_m. As usual, we are assuming that A is antilinear in the first argument and linear in the second. Note that A may not be continuous with respect to the Hilbert space topology that $\mathcal{E}_n$ and $\mathcal{E}_m$ inherit from $\mathcal{L}^{\otimes n}$ and $\mathcal{L}^{\otimes m}$ respectively.

Furthermore, we assume that each Q_{mn} is equipped with some locally convex topology and that for each m, n, t there exists a linear and continuous map

$$\pi_{mnt} : Q_{mn} \to Q_{m+t,n+t}$$

$$A \mapsto A \otimes 1_t$$

such that:

(i) $\langle a\otimes b,(A\otimes 1_t)(c\otimes d)\rangle = \langle a, Ac\rangle\langle b,d\rangle$ for each $a\in\mathcal{E}_m, c\in\mathcal{E}_n,\ b,d\in\mathcal{E}_t$, $A\in Q_{mn}$, where $\langle b,d\rangle$ denotes the inner product in $\mathcal{L}^{\otimes t}$

(ii) $(A\otimes 1_t)\otimes 1_s = A\otimes 1_{t+s}$ for each $A\in Q_{mn}$ and each $t,s\in\mathbb{N}$.

Remark 3.1. If we further assume that for any fixed $b\in\mathcal{E}_{k'}$ the map $a\mapsto a\otimes b$ from $\mathcal{E}_k$ into $\mathcal{E}_{k+k'}$ is continuous (with respect to the $\tau_k,\tau_{k+k'}-$topologies) for each k,k', (ii) actually follows from (i). It is easy to see that by the first requirement,

$$\langle\Psi,((A\otimes 1_t)\otimes 1_s)\Phi\rangle = \langle\Psi,(A\otimes 1_{t+s})\Phi\rangle \tag{3.3}$$

whenever $\Psi\in\mathcal{E}_m\otimes\mathcal{E}_t\otimes\mathcal{E}_s$ and $\Phi\in\mathcal{E}_n\otimes\mathcal{E}_t\otimes\mathcal{E}_s$. For $\Psi=\Psi_1\otimes\Psi_2$, where $\Psi_1\in\mathcal{E}_m$ and $\Psi_2\in\mathcal{E}_{t+s}$, we can find a net $(\Psi_\lambda)\subset\mathcal{E}_t\otimes\mathcal{E}_s$ converging to Ψ_2, so $\Psi_1\otimes\Psi_\lambda$ converges to $\Psi_1\otimes\Psi_2$. Since $\Psi_\lambda\otimes\Psi_2\in\mathcal{E}_m\otimes\mathcal{E}_t\otimes\mathcal{E}_s$, for $\Phi\in\mathcal{E}_n\otimes\mathcal{E}_t\otimes\mathcal{E}_s$, we have

$$\langle\Psi_1\otimes\Psi_\lambda,((A\otimes 1_t)\otimes 1_s)\Phi\rangle = \langle\Psi_1\otimes\Psi_\lambda,(A\otimes 1_{t+s})\Phi\rangle$$

and by continuity, if we take limits on both sides we have

$$\langle\Psi_1\otimes\Psi_2,((A\otimes 1_t)\otimes 1_s)\Phi\rangle = \langle\Psi_1\otimes\Psi_2,(A\otimes 1_{t+s})\Phi\rangle$$

Repeating the same argument for Φ, we can conclude that 3.3 holds for each $\Psi\in\mathcal{E}_m\otimes\mathcal{E}_{t+s}$ and $\Phi\in\mathcal{E}_n\otimes\mathcal{E}_{t+s}$. Thus, by the same density argument, we get 3.3 for all $\Psi\in\mathcal{E}_{m+t+s}$, $\Phi\in\mathcal{E}_{n+t+s}$, and therefore condition (ii) holds automatically.

We define $\mathcal{E}_f$ as the algebraic direct sum of all spaces $\mathcal{E}_n$, $\sum_{n=0}^{\infty}\mathcal{E}_n$ (for $n=0$, we set $\mathcal{E}_0:=\mathbb{C}$) and define Q_f as

$$Q_f := \prod_{m,n} Q_{mn}$$

equipped with the product topology. We note that each form in Q_f gives rise to a quadratic form from $\mathcal{E}_f\times\mathcal{E}_f$ into $\mathbb{C}$, which however is not necessarily continuous if we equip $\mathcal{E}_f$ with the product topology it inherits from $\prod_n\mathcal{E}_n$. However, if we restrict to $\prod_{k=1}^{n}\mathcal{E}_k\times\prod_{k=1}^{m}\mathcal{E}_k$, for any fixed n,m, the induced form is continuous.

Now, for $D\in Q_{mn}$, $\Psi\in\mathcal{E}_\ell,\Phi\in\mathcal{E}_k$ with $\ell=k-n+m$ and $k\geq n$.

$$\langle\Psi,(a^{\dagger m}Da^n)\Phi\rangle := \frac{\sqrt{k!\ell!}}{(k-n)!}\langle\Psi,(D\otimes 1_{k-n})\Phi\rangle$$

and in any other case for Ψ,Φ, it is equal to 0. Observe that this is a generalized form of the annihilation and creation operators as we defined them in 3.1 (resp. 3.2), in case $m=1$, $n=0$ (resp. $m=0$, $n=1$) and for $A:\mathcal{E}_1\times\mathbb{C}\to\mathbb{C}$ (resp. $A:\mathbb{C}\times\mathcal{E}_1\to\mathbb{C}$) being of the form $\langle\psi,Az\rangle = z\langle\psi,f\rangle$ (resp. $\langle z,A\psi\rangle=\overline{z}\langle f,\psi\rangle$) for some $f\in\mathcal{L}$. It is clear that $(a^{\dagger m}Da^n)\in Q_f$.

Finally, for $A \in Q_f$ and m, n any pair of naturals, we define

$$[A]_{mn} := \sum_{t=0}^{\min(m,n)} a(m,n;t)(A_{m-t,n-t} \otimes 1_t)$$

where:

$$a(m,n;t) := C_t m! n! ((m-t)!(n-t)!)^{-1/2} \quad \text{for}$$

$$C_0 := 1 \quad \text{and}$$

$$C_t := \sum_{n=1}^{t} (-1)^n \sum_{\substack{(t_1,..,t_n) \\ t_1+..+t_n=t \\ t_i \geq 1}} (t_1!..t_n!)^{-1} \quad \text{for } t \geq 1$$

Lemma 3.1.

$$\sum_{t=0}^{L} \frac{1}{(L-t)!} C_t = \begin{cases} 1 & L=0 \\ 0 & L \geq 1 \end{cases}$$

Proof. For $L = 0$ or $L = 1$, it can be directly verified. Suppose that $L \geq 2$. We compute:

$$\begin{aligned} \sum_{t=0}^{L} \frac{1}{(L-t)!} C_t &= \frac{1}{L!} + C_L + \sum_{t=1}^{L-1} \sum_{n=1}^{t} \frac{(-1)^n}{(L-t)!} \sum_{\substack{(t_1,..,t_n) \\ t_1+..+t_n=t}} (t_1!..t_n!)^{-1} \\ &= \frac{1}{L!} + C_L + \sum_{n=1}^{L-1} (-1)^n \sum_{t=n}^{L-1} \frac{1}{(L-t)!} \sum_{\substack{(t_1,..,t_n) \\ t_1+..+t_n=t}} (t_1!..t_n!)^{-1} \\ &= \frac{1}{L!} + C_L + \sum_{n=1}^{L-1} (-1)^n \sum_{t=1}^{L-n} \frac{1}{t!} \sum_{\substack{(t_1,..,t_n) \\ t_1+..+t_n=L-t}} (t_1!..t_n!)^{-1} \qquad (3.4) \end{aligned}$$

Now, we observe that the sum

$$\sum_{t=1}^{L-n} \frac{1}{t!} \sum_{\substack{(t_1,..,t_n) \\ t_1+..+t_n=L-t}} (t_1!..t_n!)^{-1}$$

can also be written as

$$\sum_{\substack{(t_1,..,t_n,t_{n+1}) \\ t_1+..+t_n+t_{n+1}=L}} (t_1!..t_n!t_{n+1}!)^{-1}$$

Therefore, we continue our calculations in 3.4 as follows:

$$\frac{1}{L!} + C_L + \sum_{n=1}^{L-1}(-1)^n \sum_{t=1}^{L-n} \frac{1}{t!} \sum_{\substack{(t_1,..,t_n) \\ t_1+..+t_n=L-t}} (t_1!..t_n!)^{-1}$$

$$= \frac{1}{L!} + C_L + \sum_{n=1}^{L-1}(-1)^n \sum_{\substack{(t_1,..,t_n,t_{n+1}) \\ t_1+..+t_n+t_{n+1}=L}} (t_1!..t_n!t_{n+1}!)^{-1}$$

$$= \sum_{n=2}^{L}(-1)^n \sum_{\substack{(t_1,..,t_n) \\ t_1+..+t_n=L}} (t_1!..t_n!)^{-1} + \sum_{n=2}^{L}(-1)^{n-1} \sum_{\substack{(t_1,..,t_n) \\ t_1+..+t_n=L}} (t_1!..t_n!)^{-1}$$

$$= 0$$

□

Theorem 3.2. *Suppose $D_{mn} \in Q_{m,n}$ for each $m, n \geq 0$. Then, the formula*

$$A = \sum_{m,n=0}^{\infty} \frac{1}{m!n!}(a^{\dagger m} D_{mn} a^n)$$

defines a form in Q_f such that $[A]_{mn} = D_{mn}$.

Proof. Using the fact that $\otimes$ is associative and without loss of generality $m \leq n$ we compute:

$$[A]_{mn} = \sum_{t=0}^{m} a(m,n;t)(A_{m-t,n-t} \otimes 1_t)$$

$$= \sum_{t=0}^{m} a(m,n;t) \sum_{k=0}^{m-t} \frac{\sqrt{(n-t)!(m-t)!}}{(m-t-k)!k!(k-m+n)!}((D_{k,k-m+n} \otimes 1_{m-k-t}) \otimes 1_t)$$

$$= \sum_{t=0}^{m} a(m,n;t) \sum_{k=0}^{m-t} \frac{\sqrt{(n-t)!(m-t)!}}{(m-t-k)!k!(k-m+n)!}(D_{k,k-m+n} \otimes 1_{m-k})$$

$$= \sum_{k=0}^{m} \frac{1}{k!(k-m+n)!} \left(\sum_{t=0}^{m-k} a(m,n;t) \frac{\sqrt{(n-t)!(m-t)!}}{(m-t-k)!} \right) (D_{k,k-m+n} \otimes 1_{m-k})$$

$$= \sum_{k=0}^{m} \frac{m!n!}{k!(k-m+n)!} \left(\sum_{t=0}^{m-k} \frac{1}{(m-k-t)!} C_t \right) (D_{k,k-m+n} \otimes 1_{m-k})$$

By Lemma 3.1, the latter is equal to D_{mn} and the proof is complete.

□

Theorem 3.3. *Suppose $A \in Q_f$. We have the following:*

(i)

$$A = \sum_{m,n=0}^{\infty} \frac{1}{m!n!}(a^{\dagger m}[A]_{mn}a^n) \tag{3.5}$$

(ii) The map $A \mapsto ([A]_{mn})_{m,n}$ *is a homeomorphism from* Q_f *onto itself, with its inverse given by* $(D_{mn}) \mapsto \sum_{m,n=0}^{\infty} \frac{1}{m!n!}(a^{\dagger m}D_{mn}a^n)$

Proof. Without loss of generality, let $m \leq n$, let D denote the RHS of 3.5 and $\ell = k - m + n$. We compute:

$$\begin{aligned}
D_{mn} &= \sum_{k=0}^{m} \frac{1}{k!\ell!}(a^{\dagger k}[A]_{k\ell}a^{\ell}) = \sum_{k=0}^{m} \frac{\sqrt{m!n!}}{k!\ell!(m-k)!}([A]_{k\ell} \otimes 1_{m-k}) \\
&= \sum_{k=0}^{m} \frac{\sqrt{m!n!}}{k!\ell!(m-k)!} \sum_{t=0}^{k} a(k,\ell;t)((A_{k-t,\ell-t} \otimes 1_t) \otimes 1_{m-k}) \\
&= \sum_{k=0}^{m} \frac{\sqrt{m!n!}}{k!\ell!(m-k)!} \sum_{t=0}^{k} a(k,\ell;t)(A_{k-t,\ell-t} \otimes 1_{m-k+t}) \\
&= \sum_{k=0}^{m} \frac{\sqrt{m!n!}}{(m-k)!(n-k)!k!} \sum_{t=0}^{m-k} a(m-k,n-k;t)(A_{m-k-t,n-k-t} \otimes 1_{k+t}) \\
&= \sum_{L=0}^{m} \sum_{k=0}^{L} \frac{\sqrt{m!n!}}{(m-k)!(n-k)!k!} a(m-k,n-k;L-k)(A_{m-L,n-L} \otimes 1_L) \\
&= \sum_{L=0}^{m} \frac{\sqrt{m!n!}}{\sqrt{(m-L)!(n-L)!}} \left(\sum_{k=0}^{L} \frac{1}{k!} C_{L-k} \right) (A_{m-L,n-L} \otimes 1_L) = A_{mn}
\end{aligned}$$

which proves that

$$A = \sum_{m,n=0}^{\infty} \frac{1}{m!n!}(a^{\dagger m}[A]_{mn}a^n)$$

Now, we are going to prove (ii). It is clear by the previous Theorems that this map is bijective and its inverse is given by

$$(D_{mn})_{m,n} \mapsto \sum_{m,n} \frac{1}{m!n!}(a^{\dagger m}D_{mn}a^n)$$

Continuity of the map is equivalent to continuity of the map $A \mapsto [A]_{mn}$ for each m, n. Since

$$[A]_{mn} = \sum_{t=0}^{\min(m,n)} a(m,n;t)(A_{m-t,n-t} \otimes 1_t)$$

it is sufficient to show that the map $A \mapsto A_{mn} \otimes 1_t$ is continuous for each m, n, t. This is the composition of the projection map $A \mapsto A_{mn}$, which is continuous since Q_f is given the product topology, and the map $S \mapsto S \otimes 1_t$, which is continuous by our assumptions.

Now, we will show that the inverse map $(D_{mn})_{m,n} \mapsto \sum_{m,n} \frac{1}{m!n!}(a^{\dagger m} D_{mn} a^n)$ is also continuous. If we identify Q_f with $\prod_{m,n} Q_{mn}$, after a direct computation, one sees that the (k,ℓ)-coordinate of $\sum_{m,n} \frac{1}{m!n!}(a^{\dagger m} D_{mn} a^n)$ is

$$\sum \frac{1}{n!m!}(a^{\dagger m} D_{mn} a^n) \quad \text{for} \quad m = n - k + \ell$$

where the sum runs through $n = 0, .., k$ in case $\ell \geq k$ and through $m = 0, .., \ell$ in case $k \geq \ell$. Hence, it is sufficient to show that for all m, n, the map $(D_{k\ell})_{k,\ell} \mapsto (a^{\dagger m} D_{mn} a^n)$ is continuous. Since this map is the composition of the continuous projection map $(D_{k\ell})_{k,\ell} \mapsto D_{mn}$ with $D \mapsto (a^{\dagger m} D a^n)$, it suffices to show that the latter is continuous. This follows from the fact that for each k, ℓ such that $\ell = k - n + m$, $k \geq n$, the map

$$D \mapsto \frac{\sqrt{k!\ell!}}{(k-n)!} D \otimes 1_{k-n}$$

is continuous. $\square$

4 Symmetrized Fock space

4.1 Expansion formula

In order to define the symmetrized Fock space, we assume the existence of a sequence of orthogonal projections $P_n : \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}$, such that:

(1) $P_{n+m} \leq P_n \otimes 1_m$ for each n, m.

(2) $P_n(\mathcal{E}_n) \subset \mathcal{E}_n$ and $P_n|_{\mathcal{E}_n} : \mathcal{E}_n \to \mathcal{E}_n$ is τ_n-continuous for each n.

(3) $P_n \otimes 1_m : \mathcal{E}_{n+m} \to \mathcal{E}_{n+m}$ is τ_{n+m}- continuous for each n, m.

(4) The map $A \mapsto P_m A P_n$ from Q_{mn} into Q_{mn} is continuous.

By Lemma 2.4, (1) is equivalent to

$$(P_n \otimes 1_m)P_{n+m} = P_{n+m}(P_n \otimes 1_m) = P_{n+m} \quad \text{for each } n, m \geq 0 \tag{4.1}$$

The associated symmetrized Fock space is

$$\mathcal{L}_P^{\oplus} = \bigoplus_{n=0}^{\infty} P_n(\mathcal{L}^{\otimes n})$$

In this section, we are going to prove a symmetrized analogue of the expansion formula in Theorem 3.3. For $A \in Q_{mn}$, we define the sesquilinear form $P_m A P_n$ as

$$\langle \Psi, (P_m A P_n)\Phi \rangle := \langle P_m \Psi, A P_n \Phi \rangle$$

It is clear that $P_m A P_n \in Q_{mn}$. For $A = \prod_{m,n} A_{mn} \in Q_f$, we set $PAP := \prod_{m,n} P_m A_{mn} P_n$, which is also in Q_f. Then, we have the following:

Proposition 4.1. *For $A \in Q_{mn}$,*

$$P(a^{\dagger m}(P_m A P_n)a^n)P = P(a^{\dagger m} A a^n)P$$

Proof. First, we will show that $(P_m A P_n) \otimes 1_t = (P_m \otimes 1_t)(A_{mn} \otimes 1_t)(P_n \otimes 1_t)$, where the RHS form is defined the obvious way, and it is a form in $Q_{m+t,n+t}$, thanks to continuity of $P_n \otimes 1_t$ (condition (3)). Let $a \in \mathcal{E}_m, b \in \mathcal{E}_n, x, y \in \mathcal{E}_t$. We compute:

$$\begin{aligned} &\langle a \otimes x, ((P_m A P_n) \otimes 1_t)(b \otimes y) \rangle = \langle P_m a, A P_n b \rangle \langle x, y \rangle \\ &= \langle (P_m a \otimes x), (A \otimes 1_t)(P_n b \otimes y) \rangle = \langle (P_m \otimes 1_t)(a \otimes x), (A \otimes 1_t)(P_n \otimes 1_t)(b \otimes y) \rangle \end{aligned}$$

Thus, by linearity we have $(P_m A P_n) \otimes 1_t = (P_m \otimes 1_t)(A_{mn} \otimes 1_t)(P_n \otimes 1_t)$ on $(\mathcal{E}_m \otimes \mathcal{E}_t) \times (\mathcal{E}_n \otimes \mathcal{E}_t)$. By continuity of the two forms and since $\mathcal{E}_m \otimes \mathcal{E}_t$ and

$\mathcal{E}_n \otimes \mathcal{E}_t$ are dense in $\mathcal{E}_{m+t}$ and $\mathcal{E}_{n+t}$ respectively, the claim follows. Now, let $\Psi \in \mathcal{E}_\ell, \Phi \in \mathcal{E}_k$, for $\ell = k - n + m$, $k \geq n$. We compute:

$$\langle \Psi, P(a^{\dagger m}(P_m A P_n)a^n)P\Phi\rangle = \langle P_\ell \Psi, (a^{\dagger m}(P_m A P_n)a^n)P_k \Phi\rangle$$

$$= \frac{\sqrt{k!\ell!}}{(k-n)!}\langle P_\ell \Psi, ((P_m A P_n) \otimes 1_{k-n})P_k\Phi\rangle$$

$$= \frac{\sqrt{k!\ell!}}{(k-n)!}\langle P_\ell \Psi, (P_m \otimes 1_{k-n})(A \otimes 1_{k-n})(P_n \otimes 1_{k-n})P_k\Phi\rangle$$

$$= \frac{\sqrt{k!\ell!}}{(k-n)!}\langle P_\ell \Psi, (A \otimes 1_{k-n})P_k\Phi\rangle = \langle P_\ell\Psi, (a^{\dagger m} A a^n)P_k\Phi\rangle = \langle \Psi, P(a^{\dagger m} A a^n)P\Phi\rangle$$

where we have used 4.1. □

Motivated by Proposition 4.1, we are going to define the symmetrized annihilation and creation operators as

$$(z^{\dagger m} A z^n) := P(a^{\dagger m} A a^n)P$$

which are forms in Q_f when $A \in Q_{mn}$.

We also set for $A \in Q_f$

$$[A]^P_{mn} := P_m [A]_{mn} P_n \in Q_{mn}$$

We can prove now the main Theorems of this section:

Theorem 4.2. *Suppose $D_{mn} \in Q_{mn}$ for each $m, n \geq 0$. Then, the formula*

$$A = \sum_{m,n=0}^{\infty} \frac{1}{m!n!}(z^{\dagger m} D_{mn} z^n)$$

defines a form in Q_f such that $[A]^P_{mn} = P_m D_{mn} P_n$

Proof. We will prove it for $m \leq n$ and the other case follows similarly. Throughout the proof, we set $\ell = k - m + n$. The fact that $A \in Q_f$ is clear. Now, one can compute that

$$A_{mn} = \sum_{k=0}^{m} \frac{\sqrt{m!n!}}{k!\ell!(m-k)!} P_m (D_{k,k-m+n} \otimes 1_{m-k})P_n$$

Therefore, we have that

$$[A]^P_{mn} = \sum_{t=0}^{m} a(m,n;t) P_m (A_{m-t,n-t} \otimes 1_t) P_n$$

$$= \sum_{t=0}^{m} a(m,n;t) \sum_{k=0}^{m-t} \frac{\sqrt{(m-t)!(n-t)!}}{k!\ell!(m-k-t)!} P_m((P_{m-t}(D_{k\ell} \otimes 1_{m-k-t})P_{n-t}) \otimes 1_t)P_n$$

Now, using the same arguments as in the proof of Proposition 4.1, we have that

$$\begin{aligned} &P_m((P_{m-t}(D_{k\ell} \otimes 1_{m-k-t})P_{n-t}) \otimes 1_t)P_n \\ &= P_m(P_{m-t} \otimes 1_t)((D_{k\ell} \otimes 1_{m-k-t}) \otimes 1_t)(P_{n-t} \otimes 1_t)P_n \\ &= P_m(D_{k\ell} \otimes 1_{m-k})P_n \end{aligned}$$

where we have used 4.1 and associativity of $\otimes$. The rest of the proof is identical to the proof of Theorem 3.2 □

Theorem 4.3. *Suppose $A \in PQ_fP = \prod_{m,n} P_m Q_{mn} P_n$. We have the following:*

(i)

$$A = \sum_{m,n=0}^{\infty} \frac{1}{m!n!}(z^{\dagger m}[A]^P_{mn} z^n) \tag{4.2}$$

(ii) The map $A \mapsto ([A]^P_{mn})_{m,n}$ is a homeomorphism from PQ_fP onto itself, with its inverse given by $(D_{mn}) \mapsto \sum_{m,n=0}^{\infty} \frac{1}{m!n!}(z^{\dagger m} D_{mn} z^n)$. Here, the space PQ_fP is given the subspace topology, considered as a subspace of Q_f.

Proof. First, we prove (i). Let $D = \sum_{m,n=0}^{\infty} \frac{1}{m!n!}(z^{\dagger m}[A]^P_{mn} z^n)$. By Proposition 4.1, we have that $D = \sum_{m,n=0}^{\infty} \frac{1}{m!n!}(z^{\dagger m}[A]_{mn} z^n)$. For $m \leq n$ without loss of generality, and $\ell = k - m + n$ we compute:

$$\begin{aligned} D_{mn} &= \sum_{k=0}^{m} \frac{\sqrt{m!n!}}{k!\ell!(m-k)!} P_m([A]_{k\ell} \otimes 1_{m-k})P_n \\ &= \sum_{k=0}^{m} \frac{\sqrt{m!n!}}{k!\ell!(m-k)!} \sum_{t=0}^{k} a(k,\ell;t) P_m((A_{k-t,\ell-t} \otimes 1_t) \otimes 1_{m-k})P_n \\ &= \sum_{k=0}^{m} \frac{\sqrt{m!n!}}{k!\ell!(m-k)!} \sum_{t=0}^{k} a(k,\ell;t) P_m(A_{k-t,\ell-t} \otimes 1_{m-k+t})P_n \end{aligned}$$

and then after identical computations as in the proof of Theorem 3.3 (i), the latter equals $P_m A_{mn} P_n$ which is equal to A_{mn}.

Now, we will prove (ii). Continuity of the map $A \mapsto ([A]^P_{mn})_{m,n}$ is equivalent to continuity of the map $A \mapsto [A]^P_{mn} = P_m[A]_{mn}P_n$ for each m, n. In Theorem 3.3 (ii), we proved that the map $A \mapsto [A]_{mn}$ is continuous (as a map from Q_f), therefore its restriction to PQ_fP is also continuous. This fact combined with property (4) shows that $A \mapsto [A]^P_{mn}$ is continuous.

To show that the inverse is continuous, it suffices to show that the inverse composed with the (m,n)-projection (with $m \leq n$ without loss of generality and $\ell = k - m + n$), i.e. the map

$$(D_{mn})_{m,n} \mapsto \sum_{k=0}^{m} \frac{\sqrt{m!n!}}{k!\ell!(m-k)!} P_m(D_{k\ell} \otimes 1_{m-k})P_n$$

is continuous for each m, n. To this end, it is sufficient to show that $(D_{mn})_{m,n} \mapsto P_m(D_{k\ell} \otimes 1_{m-k})P_n$ is continuous for every m, n and $k \leq m$, $\ell = k-m+n$. Again by property (4), it is sufficient to show that the map $(D_{mn})_{m,n} \mapsto D_{k\ell} \otimes 1_{m-k}$ is continuous and we argued for this in the proof of Theorem 3.3 (ii), when $(D_{mn}) \in Q_f$. Therefore the restriction to PQ_fP is also continuous and this fact completes the proof. □

4.2 Symmetry transformation

We are going to generalize Proposition 3.9 in [2], regarding the way the coefficients $[A]^P_{mn}$ change in case a space-time translation acts on the Fock space. This will reveal how the expansion changes too.

Suppose that $U \in \mathscr{L}(\mathcal{L})$ is a unitary operator acting on $\mathcal{L}$. We set $U_n := U \otimes .. \otimes U \in \mathscr{L}(\mathcal{L}^{\otimes n})$. It is clear that U_n is also unitary for each n. We also assume that $U_n(\mathcal{E}_n) \subset \mathcal{E}_n$ with $U_n|_{\mathcal{E}_n}$ being continuous and that $U_nP_n = P_nU_n$ for each n. Then, for any $A \in Q_f$, we define the form $UAU^* = \prod_{m,n} U_m A_{mn} U_n^*$, where

$$\langle \Psi, (U_m A_{mn} U_n^*)\Phi \rangle := \langle U_m^* \Psi, A_{mn} U_n^* \Phi \rangle$$

for every $\Psi \in \mathcal{E}_m, \Phi \in \mathcal{E}_n$. It is clear that $UAU^* \in Q_f$. We compute:

$$[UAU^*]^P_{mn} = P_m \left(\sum_{t=0}^{\min(m,n)} a(m,n;t)((UAU^*)_{m-t,n-t} \otimes 1_t) \right) P_n$$

$$= P_m \left(\sum_{t=0}^{\min(m,n)} a(m,n;t)((U_{m-t}A_{m-t,n-t}U^*_{n-t}) \otimes 1_t) \right) P_n$$

Let $\Psi = \Psi_1 \otimes \Psi_2$ and $\Phi = \Phi_1 \otimes \Phi_2$, where $\Psi_1 \in \mathcal{E}_{m-t}, \Phi_1 \in \mathcal{E}_{n-t}$ and $\Psi_2, \Phi_2 \in \mathcal{E}_t$. Using that $U_k \otimes U_\ell = U_{k+\ell}$ for each k, ℓ and the fact that U_k (as well as U_k^*) is unitary for each k, we compute:

$$\begin{aligned}
&\langle \Psi_1 \otimes \Psi_2, ((U_{m-t}A_{m-t,n-t}U^*_{n-t}) \otimes 1_t)(\Phi_1 \otimes \Phi_2) \rangle \\
&\quad = \langle U^*_{m-t}\Psi_1, A_{m-t,n-t}U^*_{n-t}\Phi_1 \rangle \langle \Psi_2, \Phi_2 \rangle \\
&\quad = \langle U^*_{m-t}\Psi_1, A_{m-t,n-t}U^*_{n-t}\Phi_1 \rangle \langle U_t^*\Psi_2, U_t^*\Phi_2 \rangle \\
&\quad = \langle (U^*_{m-t} \otimes U^*_t)(\Psi_1 \otimes \Psi_2), (A_{m-t,n-t} \otimes 1_t)(U^*_{n-t} \otimes U^*_t)(\Phi_1 \otimes \Phi_2) \rangle \\
&\quad = \langle U^*_m(\Psi_1 \otimes \Psi_2), (A_{m-t,n-t} \otimes 1_t)U^*_n(\Phi_1 \otimes \Phi_2) \rangle
\end{aligned}$$

This proves that $(U_{m-t}A_{m-t,n-t}U^*_{n-t}) \otimes 1_t$ and $U_m(A_{m-t,n-t} \otimes 1_t)U_n^*$ agree on $(\mathcal{E}_{m-t} \otimes \mathcal{E}_t) \times (\mathcal{E}_{n-t} \otimes \mathcal{E}_t)$. By a density argument, we conclude that the two forms are equal. Using that U_k commutes with P_k for each k, we derive that

$$[UAU^*]^P_{mn} = P_m \left(\sum_{t=0}^{\min(m,n)} a(m,n;t)(U_m(A_{m-t,n-t} \otimes 1_t)U_n^*) \right) P_n$$

$$= U_m [A]^P_{mn} U_n^*$$

5 Damping in one side

In this section, we are going to work on a similar expansion, where the "damping factor" is only used in one side. To be more precise, whereas in the previous sections the space Q_{mn} included quadratic forms from $\mathcal{E}_m \times \mathcal{E}_n$, here we are considering forms where one of the two sides can also take values from $\mathcal{L}^{\otimes k}$. To be more precise, we define for each m, n:

$$C_{mn} := (\mathcal{E}_m \times \mathcal{L}^{\otimes n}) \cup (\mathcal{L}^{\otimes m} \times \mathcal{E}_n)$$

For each function $A : C_{mn} \to \mathbb{C}$, we set A^ℓ and A^r be the restrictions of A to $\mathcal{E}_m \times \mathcal{L}^{\otimes n}$ and $\mathcal{L}^{\otimes m} \times \mathcal{E}_n$ respectively. We set

$$Q_{mn} := \{A : C_{mn} \to \mathbb{C} \ : \ A^\ell, A^r \text{ are sesquilinear and continuous}\}$$

where $\mathcal{E}_m \times \mathcal{L}^{\otimes n}$ is equipped with the product topology of τ_m and the Hilbert space topology of $\mathcal{L}^{\otimes n}$ and similarly for $\mathcal{L}^{\otimes m} \times \mathcal{E}_n$. It is clear that Q_{mn} is a linear space, and we assume that it is equipped with some locally convex topology. We also assume for any integers m, n, t the existence of a linear and continuous map

$$\pi_{mnt} : Q_{mn} \to Q_{m+t,n+t}$$
$$A \mapsto A \otimes 1_t$$

such that

- $\langle a \otimes b, (A \otimes 1_t)(c \otimes d)\rangle = \langle a, Ac\rangle\langle b, d\rangle$ whenever $a \in \mathcal{L}^{\otimes m}, b \in \mathcal{L}^{\otimes t}, c \in \mathcal{E}_n, d \in \mathcal{E}_t$ or $a \in \mathcal{E}_m, b \in \mathcal{E}_t, c \in \mathcal{L}^{\otimes m}, d \in \mathcal{L}^{\otimes t}$
- $(A \otimes 1_t) \otimes 1_s = A \otimes 1_{t+s}$

Remark 5.1. As in Remark 3.1, the second requirement follows from the first, if we also assume that for fixed $b \in \mathcal{E}_n, c \in \mathcal{E}_m$, the maps $a \mapsto a \otimes b$ and $d \mapsto c \otimes d$ are continuous as maps from $\mathcal{E}_m$ (resp. $\mathcal{E}_n$) into $\mathcal{E}_{m+n}$ for each m, n. It is easy to see using the first property that

$$\langle \Psi, ((A \otimes 1_t) \otimes 1_s)\Phi\rangle = \langle \Psi, (A \otimes 1_{t+s})\Phi\rangle \tag{5.1}$$

whenever $\Psi \in \mathcal{E}_m \otimes \mathcal{E}_t \otimes \mathcal{E}_s$ and $\Phi \in \mathcal{L}^{\otimes n} \otimes \mathcal{L}^{\otimes t} \otimes \mathcal{L}^{\otimes s}$. We know that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}^{\otimes t} \otimes \mathcal{L}^{\otimes s}$ is dense in $\mathcal{L}^{\otimes (n+t+s)}$ with respect to the original topology, therefore by continuity with respect to this topology, we get 5.1 for all $\Phi \in \mathcal{L}^{\otimes (n+t+s)}$. Now, for $\Psi_1 \in \mathcal{E}_m$, $\Psi_2 \in \mathcal{E}_{t+s}$, there is a net $(\Psi_\lambda) \subset \mathcal{E}_t \otimes \mathcal{E}_s$ converging to Ψ_2 in the τ_{t+s}-topology. By continuity, $\Psi_1 \otimes \Psi_\lambda$ converges to $\Psi_1 \otimes \Psi_2$ in the τ_{m+t+s}-topology. Since 5.1 holds for $\Psi = \Psi_1 \otimes \Psi_\lambda$, we get 5.1 for all $\Psi \in \mathcal{E}_m \otimes \mathcal{E}_{t+s}$. Again, using density of $\mathcal{E}_m \otimes \mathcal{E}_{t+s}$ in $\mathcal{E}_{m+t+s}$, 5.1 holds for all $\Psi \in \mathcal{E}_{m+t+s}, \phi \in \mathcal{L}^{\otimes (n+t+s)}$. We can do the exact same procedure for $\Psi \in \mathcal{L}^{\otimes (m+t+s)}, \Phi \in \mathcal{E}_{n+t+s}$, and then the claim is proved.

The rest of the definitions (Q_f, $(a^{\dagger m} A a^n)$ and $[A]_{mn}$) are defined as in Section 3, with the new notion of the tensor product of sesquilinear forms. The reader can verify that the proofs proceed exactly like before, so Theorems 3.2 and 3.3 hold for the case of one-side damping too. One can also check that the results of Section 4 apply to this case too, with small modifications to the proofs, using the same arguments.

6 Expansion in Hilbertizable Spaces

Throughout this Section, we assume that the spaces $\mathcal{E}_n$ are hilbertizable spaces, in the sense that their topology τ_n comes from an inner product, denoted by $\langle\cdot,\cdot\rangle_{\omega,n}$ and that they are complete with respect to the corresponding norm $\|\cdot\|_{\omega,n}$. We note that continuity of the inclusion map $\iota : (\mathcal{E}_n, \tau_n) \hookrightarrow \mathcal{L}^{\otimes n}$ means that there exists a constant C_n such that $\|x\| \le C_n\|x\|_{\omega,n}$ for each $x \in \mathcal{E}_n$. Furthermore, we assume that the natural inclusions $\mathcal{E}_m \otimes \mathcal{E}_n \hookrightarrow \mathcal{E}_m \otimes_H \mathcal{L}^{\otimes n}$ and $\mathcal{E}_m \otimes \mathcal{E}_n \hookrightarrow \mathcal{L}^{\otimes m} \otimes_H \mathcal{E}_n$ are continuous, where $\otimes_H$ denotes the Hilbert space tensor product and $\mathcal{E}_{m+n}$ is equipped with the $\langle\cdot,\cdot\rangle_{\omega,m+n}$-topology. Note, that since $\mathcal{E}_m \otimes \mathcal{E}_n$ is dense in $\mathcal{E}_{m+n}$, the maps $\iota_1 : \mathcal{E}_m \otimes \mathcal{E}_n \hookrightarrow \mathcal{E}_m \otimes_H \mathcal{L}^{\otimes n}$ and $\iota_2 : \mathcal{E}_m \otimes \mathcal{E}_n \hookrightarrow \mathcal{L}^{\otimes m} \otimes_H \mathcal{E}_n$ can be extended to continuous maps $\tilde{\iota}_1 : \mathcal{E}_{m+n} \hookrightarrow \mathcal{E}_m \otimes_H \mathcal{L}^{\otimes n}$ and $\tilde{\iota}_2 : \mathcal{E}_{m+n} \hookrightarrow \mathcal{L}^{\otimes m} \otimes_H \mathcal{E}_n$.

We also assume that the natural inclusion $\mathcal{E}_m \otimes \mathcal{E}_n \hookrightarrow \mathcal{E}_{m+n}$, where $\mathcal{E}_m \otimes \mathcal{E}_n$ is equipped with the Hilbert space-tensor product topology, is continuous, and thus can be extended to a continuous function from $\mathcal{E}_m \otimes_H \mathcal{E}_n$ into $\mathcal{E}_{m+n}$.

Since we are dealing with Hilbert spaces, we can equip Q_{mn} with a natural topology, namely the norm topology coming from

$$\|A\|_{mn} := \sup\{|\langle\Psi, A\Phi\rangle| \ : \ \|\Psi\|_{\omega,m} \le 1, \|\Phi\|_{\omega,n} \le 1\}$$

Now, we are going to define $A \otimes 1_t$ for $A \in Q_{mn}$. For $A \in Q_{mn}$, by Riesz representation Theorem (2.1), there exists a linear and bounded operator $T_A : \mathcal{E}_n \to \mathcal{E}_m$, with $\|T_A\| \le \|A\|_{mn}$ such that

$$\langle\Psi, T_A(\Phi)\rangle_{\omega,m} = \langle\Psi, A\Phi\rangle \ \text{ for each } \Psi \in \mathcal{E}_m, \Phi \in \mathcal{E}_n$$

We obtain the operator $T_A \otimes 1_t : \mathcal{E}_n \otimes_H \mathcal{L}^{\otimes t} \to \mathcal{E}_m \otimes_H \mathcal{L}^{\otimes t}$, where 1_t is the identity map of $\mathcal{L}^{\otimes t}$. Since we have the continuous inclusions $\tilde{\iota}_{n+t} : \mathcal{E}_{n+t} \hookrightarrow \mathcal{E}_n \otimes_H \mathcal{L}^{\otimes t}$ and $\tilde{\iota}_{m+t} : \mathcal{E}_{m+t} \hookrightarrow \mathcal{E}_m \otimes_H \mathcal{L}^{\otimes t}$, we define

$$A \otimes 1_t := \langle \tilde{\iota}_{m+t}(\cdot), (T_A \otimes 1_t)\tilde{\iota}_{n+t}(\cdot)\rangle_{\otimes H}$$

We have

$$\begin{aligned}
|\langle\Psi, (A \otimes 1_t)\Phi\rangle| &\le \|T_A \otimes 1_t\| \|\tilde{\iota}_{m+t}(\Psi)\|_{\otimes H} \|\tilde{\iota}_{n+t}(\Phi)\|_{\otimes H} \\
&\le \|T_A\| \|\tilde{\iota}_{m+t}\| \|\tilde{\iota}_{n+t}\| \|\Psi\|_{\omega,m} \|\Phi\|_{\omega,n} \le C_{mnt}\|A\|_{mn}\|\Psi\|_{\omega,m}\|\Phi\|_{\omega,n}
\end{aligned}$$

for some constant C_{mnt} depending only on m, n, t, therefore $A \otimes 1_t \in Q_{m+t,n+t}$ and $\|A \otimes 1_t\|_{m+t,n+t} \le C_{mnt}\|A\|_{mn}$ which proves that the map $A \mapsto A \otimes 1_t$ is also continuous. Now, for $a \in \mathcal{E}_m, c \in \mathcal{E}_n, b, d \in \mathcal{E}_t$, we compute:

$$\begin{aligned}
\langle a \otimes b, (A \otimes 1_t)(c \otimes d)\rangle &= \langle \tilde{\iota}_{m+t}(a \otimes b), (T_A \otimes 1_t)\tilde{\iota}_{n+t}(c \otimes d)\rangle_{\otimes H} \\
&= \langle a \otimes b, T_A(c) \otimes d\rangle_{\otimes H} = \langle a, T_A(c)\rangle_{\omega,m}\langle b, d\rangle = \langle a, Ac\rangle\langle b, d\rangle
\end{aligned}$$

For fixed $b \in \mathcal{E}_n$, the map $a \mapsto a \otimes b$ from $\mathcal{E}_m$ into $\mathcal{E}_{m+n}$ is continuous, which follows from continuity of the inclusion $\mathcal{E}_m \otimes_H \mathcal{E}_n \hookrightarrow \mathcal{E}_{m+n}$. By Remark 3.1, we deduce that $(A \otimes 1_t) \otimes 1_s = A \otimes 1_{t+s}$. Therefore, since all required properties hold, the Expansion Theorem 3.3 applies.

7 One-side damping in Hilbertizable Spaces

In this section, we make the same assumptions as in the previous one. For $C_{mn} = (\mathcal{E}_m \times \mathcal{L}^{\otimes n}) \cup (\mathcal{L}^{\otimes m} \times \mathcal{E}_n)$ and

$$Q_{mn} = \{A : C_{mn} \to \mathbb{C} \ : \ A^r, A^\ell \text{ sesquilinear, continuous}\}$$

we equip Q_{mn} with the following norm:

$$\begin{aligned}\|A\|_{mn} := \frac{1}{2}\sup\{|\langle\Psi, A^\ell\Phi\rangle| \ : \ \Psi \in \mathcal{E}_m, \Phi \in \mathcal{L}^{\otimes n}, \|\Psi\|_{\omega,m} \leq 1, \|\Phi\| \leq 1\} \\ + \frac{1}{2}\sup\{|\langle\Psi, A^r\Phi\rangle| \ : \ \Psi \in \mathcal{L}^{\otimes m}, \Phi \in \mathcal{E}_n, \|\Psi\| \leq 1, \|\Phi\|_{\omega,n} \leq 1\}\end{aligned}$$

Suppose that $A \in Q_{mn}$ and let A^r, A^ℓ be the restrictions of A to $\mathcal{L}^{\otimes m} \times \mathcal{E}_n$ and $\mathcal{E}_m \times \mathcal{L}^{\otimes n}$ respectively. By Riesz representation Theorem (2.1), by continuity of the forms A^r, A^ℓ, there exist maps $T_A^r \in \mathscr{L}(\mathcal{E}_n, \mathcal{L}^{\otimes m})$ and $T_A^\ell \in \mathscr{L}(\mathcal{E}_m, \mathcal{L}^{\otimes n})$ such that

$$\langle\Psi, T_A^r(\Phi)\rangle = \langle\Psi, A^r\Phi\rangle \quad \text{for every } \Psi \in \mathcal{L}^{\otimes m},\ \Phi \in \mathcal{E}_n$$

$$\langle\Psi, A^\ell\Phi\rangle = \langle T_A^\ell(\Psi), \Phi\rangle \quad \text{for every} \quad \Psi \in \mathcal{E}_m, \Phi \in \mathcal{L}^{\otimes n}$$

with $\|T_A^r\|, \|T_A^\ell\| \leq 2\|A\|_{mn}$. Then, after tensoring with the identity map 1_t of $\mathcal{L}^{\otimes t}$, we get the operators $T_A^r \otimes 1_t : \mathcal{E}_n \otimes_H \mathcal{L}^{\otimes t} \to \mathcal{L}^{\otimes(m+t)}$ and $T_A^\ell \otimes 1_t : \mathcal{E}_m \otimes_H \mathcal{L}^{\otimes t} \to \mathcal{L}^{\otimes(n+t)}$, with $\|T_A^r \otimes 1_t\| = \|T_A^r\|$ and $\|T_A^\ell \otimes 1_t\| = \|T_A^\ell\|$. Now, let $\tilde{\imath}_{m+t}$ and $\tilde{\imath}_{n+t}$ denote the continuous inclusions $\mathcal{E}_{m+t} \hookrightarrow \mathcal{E}_m \otimes_H \mathcal{L}^{\otimes t}$ and $\mathcal{E}_{n+t} \hookrightarrow \mathcal{E}_n \otimes_H \mathcal{L}^{\otimes t}$ respectively. We define the sesquilinear forms $(A \otimes 1_t)^r : \mathcal{L}^{\otimes(m+t)} \times \mathcal{E}_{n+t} \to \mathbb{C}$ and $(A \otimes 1_t)^\ell : \mathcal{E}_{m+t} \times \mathcal{L}^{\otimes(n+t)} \to \mathbb{C}$ as

$$(A \otimes 1_t)^r = \langle\cdot, (T_A^r \otimes 1_t)\tilde{\imath}_{n+t}(\cdot)\rangle$$

$$(A \otimes 1_t)^\ell = \langle(T_A^\ell \otimes 1_t)\tilde{\imath}_{m+t}(\cdot), \cdot\rangle$$

with the notation $\langle\cdot,\cdot\rangle$ denoting the tensor product in $\mathcal{L}^{\otimes(m+t)}$ in the first case and in $\mathcal{L}^{\otimes(n+t)}$ in the second case. For $\Psi \in \mathcal{L}^{\otimes(m+t)}$ and $\Phi \in \mathcal{E}_{n+t}$,

$$\begin{aligned}|\langle\Psi, (A \otimes 1_t)^r\Phi\rangle| = |\langle\Psi, (T_A^r \otimes 1_t)\tilde{\imath}_{n+t}(\Phi)\rangle| \leq \|\Psi\|\|(T_A^r \otimes 1_t)\tilde{\imath}_{n+t}(\Phi)\| \\ \leq \|T_A^r\|\|\tilde{\imath}_{n+t}(\Phi)\|\|\Psi\| \leq C_{nt}\|A\|_{mn}\|\Phi\|_{\omega,n+t}\|\Psi\|\end{aligned}$$

thus $(A \otimes 1_t)^r$ is a continuous sesquilinear form and similarly $(A \otimes 1_t)^\ell$ is too.

A straightforward computation shows that

$$\langle\Psi, (A \otimes 1_t)^r\Phi\rangle = \langle\Psi, (A \otimes 1_t)^\ell\Phi\rangle \tag{7.1}$$

for all $\Psi \in \mathcal{E}_m \otimes \mathcal{E}_t$, $\Phi \in \mathcal{E}_n \otimes \mathcal{E}_t$. If $\Psi \in \mathcal{E}_{m+t}$, we can find a net $(\Psi_\lambda) \subset \mathcal{E}_m \otimes \mathcal{E}_t$ converging to Ψ in the τ_{m+t}-topology. By τ_{m+t}-left continuity of $(A \otimes 1_t)^\ell$, we have

$$\lim_\lambda \langle\Psi_\lambda, (A \otimes 1_t)^\ell\Phi\rangle = \langle\Psi, (A \otimes 1_t)^\ell\Phi\rangle$$

Since τ_{m+t} is finer than the original topology in $\mathcal{L}^{\otimes(m+t)}$, we also have that Ψ_λ converges to Ψ in this topology. By $\mathcal{L}^{\otimes(m+t)}$-left continuity of $(A \otimes 1_t)^r$, we have

$$\lim_\lambda \langle \Psi_\lambda, (A \otimes 1_t)^r \Phi \rangle = \langle \Psi, (A \otimes 1_t)^r \Phi \rangle$$

We can then use the same argument for Ψ, and we get that 7.1 holds for all $\Psi \in \mathcal{E}_{m+t}, \Phi \in \mathcal{E}_{n+t}$. Since $(A \otimes 1_t)^r$ and $(A \otimes 1_t)^\ell$ agree on $\mathcal{E}_{m+t} \times \mathcal{E}_{n+t}$, we can define $A \otimes 1_t$ as a form in $Q_{m+t,n+t}$, whose restrictions in $\mathcal{E}_{m+t} \times \mathcal{L}^{\otimes(n+t)}$ and $\mathcal{L}^{\otimes(m+t)} \times \mathcal{E}_{n+t}$ are $(A \otimes 1_t)^\ell$ and $(A \otimes 1_t)^r$ respectively.

It is also straightforward to verify that

$$\langle a \otimes b, (A \otimes 1_t)(c \otimes d) \rangle = \langle a, Ac \rangle \langle b, d \rangle$$

whenever $(a, b, c, d) \in \mathcal{E}_m \times \mathcal{E}_t \times \mathcal{L}^{\otimes n} \times \mathcal{L}^{\otimes t}$ or $(a, b, c, d) \in \mathcal{L}^{\otimes m} \times \mathcal{L}^{\otimes t} \times \mathcal{E}_n \times \mathcal{E}_t$.

Finally, as we verified in the previous section, the map $a \mapsto a \otimes b$ is continuous as a map from $\mathcal{E}_k$ into $\mathcal{E}_{k+k'}$ for each $b \in \mathcal{E}_{k'}$, therefore, by Remark 5.1

$$(A \otimes 1_t) \otimes 1_s = A \otimes 1_{t+s}$$

for every t, s and $A \in Q_{mn}$.

8 Expansion in Nuclear Spaces

The second case is somewhat more complicated. We suppose that the spaces $\mathcal{E}_n$ are complete nuclear Fréchet spaces, with the topology τ_n that they carry. We also assume that for each n, m, the projective (and also injective) tensor topology of $\mathcal{E}_m \otimes \mathcal{E}_n$ is the same as the topology it inherits from $\mathcal{E}_{m+n}$, i.e. τ_{m+n}. Therefore the spaces $\mathcal{E}_m \hat{\otimes} \mathcal{E}_n$ (where $\hat{\otimes}$ denotes the Cauchy completion of the tensor product) and $\mathcal{E}_{m+n}$ are homeomorphic. In this case, for each m, n, m', n', we are going to construct a continuous bilinear map

$$\pi_{mnm'n'} : Q_{mn} \times Q_{m'n'} \to Q_{m+m',n+n'}$$

$$(S, T) \mapsto S \otimes T$$

satisfying $\langle a \otimes b, (S \otimes T)(c \otimes d)\rangle = \langle a, Sc\rangle\langle b, Td\rangle$.

We are going to denote the conjugate linear space of $\mathcal{E}_n$ by $\overline{\mathcal{E}_n}$, that is the set $\{\overline{v} \ : \ v \in \mathcal{E}_n\}$ (where the notation $\overline{v}$ is formal), equipped with addition $\overline{v_1} + \overline{v_2} := \overline{v_1 + v_2}$ and multiplication $\lambda\overline{v} := \overline{\overline{\lambda}v}$. Then the space Q_{mn} is the space $B(\overline{\mathcal{E}_m}, \mathcal{E}_n)$ of all continuous bilinear forms from $\overline{\mathcal{E}_m} \times \mathcal{E}_n \to$ into $\mathbb{C}$. The space $B(\overline{\mathcal{E}_m}, \mathcal{E}_n)$ can be equipped with the topology of bi-bounded convergence (see Section 2.6.2) and by the identification with Q_{mn}, we have the desired topology.

For E, F being nuclear Fréchet spaces, we have the following:

- The canonical map from $(E\hat{\otimes}F)'$ into $B(E, F)$,

$$x' \mapsto ((a, b) \mapsto x'(a \otimes b))$$

  is a homeomorphism onto $B(E, F)$ in case $B(E, F)$ is equipped with the topology of bi-bounded convergence and $(E\hat{\otimes}F)'$ is equipped with the strong dual topology.

- The canonical map from $E'_b \otimes F'_b$ into $(E\hat{\otimes}F)'_b$,

$$x' \otimes y' \mapsto (x \otimes y \mapsto x'(x)y'(y))$$

  is continuous and it extends to a homeomorphism of $E'_b\hat{\otimes}F'_b$ onto $(E\hat{\otimes}F)'_b$.

For detailed proofs of the above facts, the reader is referred to [14] Sections 9.8, 9.9. We note that by Proposition 2.19, the spaces E'_b and F'_b are nuclear, therefore the notation $E'_b\hat{\otimes}F'_b$ makes perfect sense. We also see that since $B(E, F)$ is identified with $(E\hat{\otimes}F)'$ and the latter is nuclear, again by Proposition 2.19, $B(E, F)$ is also nuclear.

Using these facts, we have the following identifications:

$$\begin{aligned} B(\overline{\mathcal{E}_m}, \mathcal{E}_n)\hat{\otimes}B(\overline{\mathcal{E}_{m'}}, \mathcal{E}_{n'}) &\cong (\overline{\mathcal{E}_m}\hat{\otimes}\mathcal{E}_n)'\hat{\otimes}(\overline{\mathcal{E}_{m'}}\hat{\otimes}\mathcal{E}_{n'})' \cong (\overline{\mathcal{E}_m}'\hat{\otimes}\mathcal{E}'_n)\hat{\otimes}(\overline{\mathcal{E}_{m'}}'\hat{\otimes}\mathcal{E}'_{n'}) \\ &\cong (\overline{\mathcal{E}_m}'\hat{\otimes}\overline{\mathcal{E}_{m'}}')\hat{\otimes}(\mathcal{E}'_n\hat{\otimes}\mathcal{E}'_{n'}) \cong (\overline{\mathcal{E}_m}\hat{\otimes}\overline{\mathcal{E}_{m'}})'\hat{\otimes}(\mathcal{E}_n\hat{\otimes}\mathcal{E}_{n'})' \cong \overline{\mathcal{E}_{m+m'}}'\hat{\otimes}\mathcal{E}'_{n+n'} \\ &\cong (\overline{\mathcal{E}_{m+m'}}\hat{\otimes}\mathcal{E}_{n+n'})' \cong B(\overline{\mathcal{E}_{m+m'}}, \mathcal{E}_{n+n'}) \quad (8.1) \end{aligned}$$

The canonical map $\chi : B(\overline{\mathcal{E}_m}, \mathcal{E}_n) \times B(\overline{\mathcal{E}_{m'}}, \mathcal{E}_{n'}) \to B(\overline{\mathcal{E}_m}, \mathcal{E}_n)\hat{\otimes}B(\overline{\mathcal{E}_{m'}}, \mathcal{E}_{n'})$, $\chi(C, D) = C \otimes D$ is continuous, by definition of the tensor product topology, thus using the identification in 8.1, we obtain the desired map $\pi_{mnm'n'}$. We need to see that $\langle a \otimes b, (S \otimes T)(c \otimes d)\rangle = \langle a, Sc\rangle\langle b, Td\rangle$ (recall that we use the notation $S \otimes T$ for $\pi_{mnm'n'}(S, T)$).

Since we can identify the spaces $B(\overline{\mathcal{E}_m}, \mathcal{E}_n)$ and $B(\overline{\mathcal{E}_{m'}}, \mathcal{E}_{n'})$ with $\overline{\mathcal{E}_m}'\hat{\otimes}\mathcal{E}_n'$ and $\overline{\mathcal{E}_{m'}}'\hat{\otimes}\mathcal{E}_{n'}'$, we first assume that $S = S_1 \otimes S_2$, $T = T_1 \otimes T_2$, where S_1, S_2, T_1, T_2 are in $\overline{\mathcal{E}_m}', \mathcal{E}_n', \overline{\mathcal{E}_{m'}}', \mathcal{E}_{n'}'$ respectively. Then, through the chain of maps in 8.1, $S \otimes T$ is first mapped to $(S_1 \otimes T_1) \otimes (S_2 \otimes T_2) \in (\overline{\mathcal{E}_m}'\hat{\otimes}\overline{\mathcal{E}_{m'}}')\hat{\otimes}(\mathcal{E}_n'\hat{\otimes}\mathcal{E}_{n'}')$. Then, if $a \in \mathcal{E}_m$, $b \in \mathcal{E}_{m'}$, $c \in \mathcal{E}_n$ and $d \in \mathcal{E}_{n'}$,

$$\langle a \otimes b, (S \otimes T)(c \otimes d)\rangle =$$

$$\begin{aligned} ((S_1 \otimes T_1) \otimes (S_2 \otimes T_2))(\overline{a \otimes b}, c \otimes d) &= (S_1 \otimes T_1)(\overline{a \otimes b})(S_2 \otimes T_2)(c \otimes d) \\ &= S_1(\overline{a})T_1(\overline{b})S_2(c)T_2(d) = S(\overline{a}, c)T(\overline{b}, d) = \langle a, Sc\rangle\langle b, Td\rangle \end{aligned}$$

Therefore, we have $\langle a \otimes b, \pi_{mnm'n'}(S, T)(c \otimes d)\rangle = \langle a, Sc\rangle\langle b, Td\rangle$ as desired. The identity follows by linearity for $S \in \overline{\mathcal{E}_m}' \otimes \mathcal{E}_n'$, $T \in \overline{\mathcal{E}_{m'}}' \otimes \mathcal{E}_{n'}'$. Now, if $S \in \overline{\mathcal{E}_m}'\hat{\otimes}\mathcal{E}_n'$, $T \in \overline{\mathcal{E}_{m'}}' \otimes \mathcal{E}_{n'}'$, we can find a net $(S_\lambda) \subset \overline{\mathcal{E}_m}' \otimes \mathcal{E}_n'$ such that $S_\lambda \to S$ in the topology of bi-bounded convergence of $B(\overline{\mathcal{E}_m}, \mathcal{E}_n)$. By definition of the topology, for any bounded subsets $X \subset \mathcal{E}_m$ and $Y \subset \mathcal{E}_n$, we have

$$\lim_\lambda \sup_{a \in X, b \in Y} |\langle a, (S_\lambda - S)b\rangle| = 0$$

In particular, for any $a \in \mathcal{E}_m, b \in \mathcal{E}_n$, $\langle a, S_\lambda b\rangle \to \langle a, Sb\rangle$. By continuity of the map $\pi_{mnm'n'}$, we have that

$$\pi_{mnm'n'}(S_\lambda, T) \to \pi_{mnm'n'}(S, T)$$

in the topology of bi-bounded convergence of $B(\overline{\mathcal{E}_{m+m'}}, \mathcal{E}_{n+n'})$, and in particular

$$\langle \Psi, \pi_{mnm'n'}(S_\lambda, T)\Phi\rangle \to \langle \Psi, \pi_{mnm'n'}(S, T)\Phi\rangle$$

for each $\Psi \in \mathcal{E}_{m+m'}, \Phi \in \mathcal{E}_{n+n'}$. For a, b, c, d in $\mathcal{E}_m, \mathcal{E}_n, \mathcal{E}_{m'}, \mathcal{E}_{n'}$ respectively, combining everything so far, we have:

$$\begin{aligned} \langle a \otimes b, \pi_{mnm'n'}(S, T)(c \otimes d)\rangle &= \lim_\lambda \langle a \otimes b, \pi_{mnm'n'}(S_\lambda, T)(c \otimes d)\rangle \\ &= \lim_\lambda \langle a, S_\lambda c\rangle\langle b, Td\rangle = \langle a, Sc\rangle\langle b, Td\rangle \end{aligned}$$

Since we have established the property for $S \in \overline{\mathcal{E}_m}'\hat{\otimes}\mathcal{E}_n'$, $T \in \overline{\mathcal{E}_{m'}}' \otimes \mathcal{E}_{n'}'$, we can use the same argument, this time with a net $(T_\mu) \subset \overline{\mathcal{E}_{m'}}' \otimes \mathcal{E}_{n'}'$ converging to a $T \in \overline{\mathcal{E}_{m'}}'\hat{\otimes}\mathcal{E}_{n'}'$ and we have proved our claim.

If we denote by 1_t the bilinear form $(a, b) \mapsto \langle a, b\rangle$, which is clearly in Q_{tt} since τ_t is stronger than the $\mathcal{L}^{\otimes t}$-topology, we get that the map

$$\pi_{mnt} := \pi_{mntt}(\cdot, 1_t) : Q_{mn} \to Q_{m+t,n+t}$$

is continuous and satisfies $\langle a \otimes b, (A \otimes 1_t)(c \otimes d)\rangle = \langle a, Ac\rangle\langle b, d\rangle$ for all $a \in \mathcal{E}_m$, $c \in \mathcal{E}_m$, $b, d \in \mathcal{E}_t$. Since the tensor topology of $\mathcal{E}_k \otimes \mathcal{E}_{k'}$ is the same as the one it inherits from $\mathcal{E}_{k+k'}$, it is immediate that the map $a \mapsto (a \otimes b)$ is continuous, therefore by Remark 3.1, the map π_{mnt} is also associative, i.e. $(A \otimes 1_t) \otimes 1_s = A \otimes 1_{t+s}$.

9 Applications

9.1 Scalar Integrable Models of QFT

9.1.1 Symmetrized expansion

In the model described by Bostelmann and Cadamuro [2] the S-symmetrized Fock space is formed with respect to a factorizing scattering function S. The assumptions made on S are that it is a smooth function $S : \mathbb{R} \to \mathbb{C}$, with values in the complex unit circle, satisfying

$$\overline{S(\theta)} = S(\theta)^{-1} = S(-\theta) \quad \text{for each } \theta \in \mathbb{R} \tag{9.1}$$

Suppose $\mathcal{L} = L^2(\mathbb{R})$, so that $\mathcal{L}^{\otimes n} = L^2(\mathbb{R}^n)$ for each n. For a smooth, increasing, sublinear function $\omega : [0, +\infty) \to [0, +\infty)$ called the *indicatrix* and the energy function $E(\theta) = \sum_{j=1}^{n} \cosh \theta_j$, we introduce the space

$$\mathcal{E}_n := \{f \in \mathcal{L}^{\otimes n} \mid \|e^{\omega(E(\cdot))} f\|_2 < \infty\}$$

The fact that this is a dense subspace of $\mathcal{L}^{\otimes n}$ is obvious, because it contains the space $C_c(\mathbb{R}^n)$ of all compactly supported continuous functions and we know that the latter is dense in $L^2(\mathbb{R}^n)$. It is also clear that this space can be equipped with an inner product, namely

$$\langle f, g \rangle_{\omega,n} := \int_{\mathbb{R}^n} e^{2\omega(E(\theta))} \overline{f(\theta)} g(\theta) d\theta$$

This is well-defined when f, g lie in $\mathcal{E}_n$, because

$$\int_{\mathbb{R}^n} \left| e^{2\omega(E(\theta))} \overline{f(\theta)} g(\theta) \right| d\theta \le \|e^{\omega(E(\cdot))} f\|_2 \|e^{\omega(E(\cdot))} g\|_2$$

due to Hölder's inequality. Another useful characterization of $\mathcal{E}_n$ is that it is isometrically isomorphic to $\mathcal{L}^{\otimes n}$ via the unitary map

$$e^{\omega(H/\mu)} : \mathcal{E}_n \to L^2(\mathbb{R}^n)$$
$$f \mapsto e^{\omega(E(\cdot))} f$$

It follows automatically that $\mathcal{E}_n$ is complete, thus a Hilbert space.

The $\|\cdot\|_\omega$-topology is finer than the $\|\cdot\|_2$-topology. This is immediate, due to the (obvious) fact that $\|\cdot\|_2 \le \|\cdot\|_\omega$.

The space $\mathcal{E}_m \otimes \mathcal{E}_n$ is $\|\cdot\|_\omega$-dense in $\mathcal{E}_{m+n}$. It is clear that $\mathcal{E}_m \otimes \mathcal{E}_n$ consists of all finite linear combinations of functions of the form $e^{-\omega(E(\theta))-\omega(E(\eta))} f(\theta) g(\eta)$, where $\theta = (\theta_1, .., \theta_m), \eta = (\eta_1, .., \eta_n)$ and $f \in L^2(\mathbb{R}^m)$, $g \in L^2(\mathbb{R}^n)$. Thus, its image in $L^2(\mathbb{R}^{m+n})$ via $e^{\omega(H/\mu)}$ consists of finite linear combinations of functions of the form

$$e^{\omega(E(\theta,\eta))-\omega(E(\theta))-\omega(E(\eta))} f(\theta) g(\eta)$$

Let $h \in L^2(\mathbb{R}^{m+n})$. We know there exists a sequence $(h_n) \subset L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^n)$ $\|\cdot\|_2$-converging to h. By Hölder, using the fact that

$$|e^{\omega(E(\theta,\eta))-\omega(E(\theta))-\omega(E(\eta))}| \leq 1$$

we get that

$$e^{\omega(E(\theta,\eta))-\omega(E(\theta))-\omega(E(\eta))} h_n \xrightarrow{\|\cdot\|_2} e^{\omega(E(\theta,\eta))-\omega(E(\theta))-\omega(E(\eta))} h$$

Therefore, $e^{\omega(H/\mu)}(\mathcal{E}_m \otimes \mathcal{E}_n)$ is $\|\cdot\|_2$-dense in $e^{\omega(E(\theta,\eta))-\omega(E(\theta))-\omega(E(\eta))} L^2(\mathbb{R}^{m+n})$. However, the latter contains $C_c(\mathbb{R}^{m+n})$, which is dense in $L^2(\mathbb{R}^{m+n})$, thus $e^{\omega(H/\mu)}(\mathcal{E}_m \otimes \mathcal{E}_n)$ is dense in $L^2(\mathbb{R}^{m+n})$. Using that $e^{\omega(H/\mu)}$ is an isometric isomorphism, we conclude that $\mathcal{E}_m \otimes \mathcal{E}_n$ is $\|\cdot\|_{\omega,m+n}$-dense in $(e^{\omega(H/\mu)})^{-1}(L^2(\mathbb{R}^{m+n}))$ i.e. in $\mathcal{E}_{m+n}$.

The natural inclusion $\mathcal{E}_m \otimes \mathcal{E}_n \hookrightarrow \mathcal{E}_m \otimes_H \mathcal{L}^{\otimes n}$ is continuous, with respect to the $\|\cdot\|_\omega$-norm on the domain and the Hilbert space tensor norm on $\mathcal{E}_m \otimes \mathcal{L}^{\otimes n}$. To see this, suppose $\Psi = \sum_j \psi_j \otimes \phi_j \in \mathcal{E}_m \otimes \mathcal{E}_n$. We compute:

$$\begin{aligned}
\|\Psi\|^2_{\otimes H} = \langle \Psi, \Psi \rangle_{\otimes H} &= \sum_{i,j} \langle \psi_i, \psi_j \rangle_{\omega,m} \langle \phi_i, \phi_j \rangle \\
&= \sum_{i,j} \int_{\mathbb{R}^m} e^{2\omega(E(\theta))} \overline{\psi_i(\theta)} \psi_j(\theta) d\theta \int_{\mathbb{R}^n} \overline{\phi_i(\eta)} \phi_j(\eta) d\eta \\
= \int_{\mathbb{R}^{m+n}} e^{2\omega(E(\theta))} \overline{\Psi(\theta,\eta)} \Psi(\theta,\eta) d\theta d\eta &\leq \int_{\mathbb{R}^{m+n}} e^{2\omega(E(\theta,\eta))} |\Psi(\theta,\eta)|^2 d\theta d\eta \\
&= \|\Psi\|^2_{\omega,m+n}
\end{aligned}$$

where, in the inequality, we have used that ω is increasing.

The natural inclusion $\mathcal{E}_m \otimes \mathcal{E}_n \hookrightarrow \mathcal{E}_{m+n}$ where the domain is equipped with the Hilbert space tensor product topology, is continuous. To see this, suppose $\Psi = \sum_j \psi_j \otimes \phi_j \in \mathcal{E}_m \otimes \mathcal{E}_n$ like before. If we denote the Hilbert space tensor product of the spaces $\mathcal{E}_m$ and $\mathcal{E}_n$ by $\langle \cdot, \cdot \rangle_{\otimes \omega}$ We compute:

$$\begin{aligned}
\|\Psi\|^2_{\otimes \omega} = \langle \Psi, \Psi \rangle_{\otimes \omega} &= \sum_{i,j} \langle \psi_i, \psi_j \rangle_{\omega,m} \langle \phi_i, \phi_j \rangle_{\omega,n} \\
&= \sum_{i,j} \int_{\mathbb{R}^m} e^{2\omega(E(\theta))} \overline{\psi_i(\theta)} \psi_j(\theta) d\theta \int_{\mathbb{R}^n} e^{2\omega(E(\eta))} \overline{\phi_i(\eta)} \phi_j(\eta) d\eta \\
&= \int_{\mathbb{R}^{m+n}} e^{2\omega(E(\theta))+2\omega(E(\eta))} \overline{\Psi(\theta,\eta)} \Psi(\theta,\eta) d\theta d\eta \\
&\geq \int_{\mathbb{R}^{m+n}} e^{2\omega(E(\theta,\eta))} |\Psi(\theta,\eta)|^2 d\theta d\eta = \|\Psi\|^2_{\omega,m+n}
\end{aligned}$$

where, in the inequality, we have used that ω is sublinear.

Since all conditions are satisfied, one obtains the unsymmetrized expansion for Hilbert spaces. Now, we are going to define the projections P_n associated to the scattering function S, in order to obtain the symmetrized expansion for sesquilinear forms.

We define an action D_n of the group $\mathscr{G}_n$ of permutations of n elements on $\mathcal{L}^{\otimes n}$ as follows: for $\tau_k = (k \;\; k+1)$ being an adjacent transposition, we set

$$D_n(\tau_k)(\Phi)(\theta_1, ..., \theta_n) = S(\theta_{k+1} - \theta_k)\Phi(\theta_1, ..., \theta_{k+1}, \theta_k, ..., \theta_n)$$

Since the permutation group $\mathscr{G}_n$ can be described as

$$\mathscr{G}_n = \langle \tau_1, .., \tau_{n-1} \mid \tau_j^2 = 1, \tau_i\tau_j = \tau_j\tau_i \text{ for } |i-j| > 1, \tau_j\tau_{j+1}\tau_j = \tau_{j+1}\tau_j\tau_{j+1} \rangle$$

(see Section 2.5 for details), in order to extend the action to all permutations, it suffices to verify that:

(i) $D_n(\tau_j)^2 = I$ for each $1 \le j \le n-1$

(ii) $D_n(\tau_i)D_n(\tau_j) = D_n(\tau_j)D_n(\tau_i)$ for all $i, j \in \{1, .., n-1\}$ such that $|i-j| > 1$

(iii) $D_n(\tau_j)D_n(\tau_{j+1})D_n(\tau_j) = D_n(\tau_{j+1})D_n(\tau_j)D_n(\tau_{j+1})$ for each $1 \le j \le n-2$

These can be verified by direct computations, using (9.1) for (i), and just the definition of $D_n(\tau_k)$ for (ii) and (iii). Since we also have $D_n(\tau_j)^* = D_n(\tau_j^{-1}) = D_n(\tau_j)$ for all j, it follows that the action is unitary.

The associated projections are defined as:

$$P_n := \frac{1}{n!}\sum_{\sigma\in\mathscr{G}_n} D_n(\sigma)$$

It can be easily seen that it is an orthogonal projection. We have to verify that it satisfies conditions (1)-(4) in Section 4.

First of all, one can easily verify that for each n, k with $1 \le k \le n-1$, $D_n(\tau_k) \otimes 1_m = D_{n+m}(\tau_k)$ (τ_k can also be considered as an element in $\mathscr{G}_{n+m}$). Then, for each $\tau \in \mathscr{G}_n$, we can express τ as a product of adjacent transpositions, say $\tau = \tau_{i_1}..\tau_{i_N}$, and then we have

$$\begin{aligned} D_n(\tau) \otimes 1_m = (D_n(\tau_{i_1})..D_n(\tau_{i_N})) \otimes 1_m = (D_n(\tau_{i_1}) \otimes 1_m)..(D_n(\tau_{i_N}) \otimes 1_m) \\ = D_{n+m}(\tau_{i_1})..D_{n+m}(\tau_{i_N}) = D_{n+m}(\tau) \end{aligned}$$

Now, using this, as well as that $\mathscr{G}_n \subset \mathscr{G}_{n+m}$, we compute:

$$\begin{aligned} &P_{n+m}(P_n \otimes 1_m) = \frac{1}{(n+m)!n!}\sum_{\sigma\in\mathscr{G}_{n+m}} D_{n+m}(\sigma)\sum_{\tau\in\mathscr{G}_n} D_{n+m}(\tau) \\ &= \frac{1}{(n+m)!n!}\sum_{\tau\in\mathscr{G}_n}\sum_{\sigma\in\mathscr{G}_{n+m}} D_{n+m}(\sigma\circ\tau) = \frac{1}{(n+m)!n!}\sum_{\tau\in\mathscr{G}_n}\sum_{\sigma\in\mathscr{G}_{n+m}} D_{n+m}(\sigma) \\ &= \frac{1}{(n+m)!}\sum_{\sigma\in\mathscr{G}_{n+m}} D_{n+m}(\sigma) = P_{n+m} \end{aligned}$$

and this proves that $P_{n+m} \leq P_n \otimes 1_m$ for each n, m.

Next, we need to show $P_n(\mathcal{E}_n) \subset \mathcal{E}_n$ and $P_n|_{\mathcal{E}_n} : \mathcal{E}_n \to \mathcal{E}_n$ is continuous. To this end, let $f \in L^2(\mathbb{R}^n)$ and $1 \leq k \leq n-1$. Then,

$$
\begin{aligned}
&D_n(\tau_k)(e^{-\omega(H/\mu)}f)(\theta) \\
&\quad = S(\theta_{k+1} - \theta_k)e^{-\omega(E(\theta_1,..,\theta_{k+1},\theta_k,..,\theta_n))}f(\theta_1,..,\theta_{k+1},\theta_k,..,\theta_n) \\
&\quad = S(\theta_{k+1} - \theta_k)e^{-\omega(E(\theta))}f(\theta_1,..,\theta_{k+1},\theta_k,..,\theta_n) = e^{-\omega(E(\theta))}D_n(\tau_k)f(\theta)
\end{aligned}
$$

Therefore,

$$D_n(\tau_k)e^{-\omega(H/\mu)} = e^{-\omega(H/\mu)}D_n(\tau_k)$$

which proves that

$$D_n(\tau)e^{-\omega(H/\mu)} = e^{-\omega(H/\mu)}D_n(\tau)$$

for each $\tau \in \mathscr{G}_n$ and therefore

$$P_n e^{-\omega(H/\mu)} = e^{-\omega(H/\mu)}P_n$$

Since $\mathcal{E}_n = e^{-\omega(H/\mu)}(L^2(\mathbb{R}^n))$, the first part is proved. The second part follows too, since

$$\|P_n(e^{-\omega(H/\mu)}f)\|_{\omega,n} = \|e^{-\omega(H/\mu)}(P_n f)\|_{\omega,n} = \|P_n f\| \leq \|f\| = \|e^{-\omega(H/\mu)}f\|_{\omega,n}$$

for each $f \in L^2(\mathbb{R}^n)$. Although we do not need it, one can also prove that P_n is a projection of $\mathcal{E}_n$, too.

We shall also show that $P_n \otimes 1_m$ is $\|\cdot\|_{\omega,n+m}$-continuous. We have already proved that

$$P_n \otimes 1_m = \frac{1}{n!}\sum_{\sigma \in \mathscr{G}_n} D_{n+m}(\sigma)$$

hence it is sufficient to show that $D_{n+m}(\sigma)$ is $\|\cdot\|_{\omega,n+m}$-continuous for each $\sigma \in \mathscr{G}_n$. For each $f \in L^2(\mathbb{R}^{n+m})$, we have

$$
\begin{aligned}
\|D_{n+m}(\sigma)(e^{-\omega(H/\mu)}f)\|_{\omega,n+m} &= \|e^{-\omega(H/\mu)}D_{n+m}(\sigma)f\|_{\omega,n+m} \\
&= \|D_{n+m}(\sigma)f\| = \|f\| = \|e^{-\omega(H/\mu)}f\|_{\omega,n+m}
\end{aligned}
$$

which proves our claim.

The final assertion that the map $A \mapsto P_m A P_n$ from Q_{mn} into itself is continuous, is trivial, and we actually have $\|P_m A P_n\|_{mn} \leq \|A\|_{mn}$. These facts imply the existence of the symmetrized expansion of any form in Q_{mn}.

9.1.2 Symmetry transformation under space-time translations

Space-time translations act on a vector $\psi \in \mathcal{L}^{\otimes n} = L^2(\mathbb{R}^n)$ as

$$(U_n(x)\psi)(\theta) = e^{ip(\theta)\cdot x}\psi(\theta)$$

where $\theta = (\theta_1, .., \theta_n) \in \mathbb{R}^n$, $x \in \mathbb{R}^2$ and

$$p(\theta) = \sum_{k=1}^{n} \mu \begin{pmatrix} \cosh \theta_k \\ \sinh \theta_k \end{pmatrix}$$

and $\mu > 0$ is the mass of a scalar particle. It is straightforward to verify that $U_n(x)$ is unitary, with $U_n(x)^* = U_n(x)^{-1} = U_n(-x)$. If we set $U := U_1$, it is also easy to verify that $U_n = U^{\otimes n}$. The verifications that $U_n(\mathcal{E}_n) \subset \mathcal{E}_n$, that $U_n|_{\mathcal{E}_n}$ is continuous and $U_n D_n(\tau_k) = D_n(\tau_k) U_n$ for each n, k are trivial. From the latter, it follows that $P_n U_n = U_n P_n$. By what we have shown in 4.2, for each $A \in Q_f$, $[UAU^*]^P_{mn} = U_m [A]^P_{mn} U_n^*$ and the expansion changes analogously.

9.2 Integrable Models of QFT with Several Particle Species

In this section, we are going to prove the expansion for the model studied by Lechner, Schützenhofer [7], which is a generalization of the previous case. For a separable Hilbert space $\mathcal{K}$, we set $\mathcal{L} := L^2(\mathbb{R}; \mathcal{K})$ which can be identified with $L^2(\mathbb{R}) \otimes_H \mathcal{K}$. As we saw in Section 2.4, the $n-$particle space $\mathcal{L}^{\otimes n}$ is naturally isomorphic to $L^2(\mathbb{R}^n; \mathcal{K}^{\otimes n})$ In particular, the inner product in $\mathcal{L}^{\otimes n}$ is

$$\langle f, g \rangle = \int_{\mathbb{R}^n} \langle f(\theta), g(\theta) \rangle d\theta$$

where the inner product inside the integral is the one of $\mathcal{K}^{\otimes n}$. We will denote the corresponding norm as $\| \cdot \|_2$

Let $\omega : [0, +\infty) \to [0, +\infty)$ be as in 9.1, as well as the energy function E. We set

$$\mathcal{E}_n := \left\{ f \in L^2(\mathbb{R}^n; \mathcal{K}^{\otimes n}) \ : \ \int_{\mathbb{R}^n} \| e^{\omega(E(\theta))} f(\theta) \|^2 d\theta < \infty \right\}$$

where the norm inside the integral is the one of $\mathcal{K}^{\otimes n}$ and we equip it with the following inner product:

$$\langle f, g \rangle_{\omega, n} = \langle e^{\omega(E(\cdot))} f, e^{\omega(E(\cdot))} g \rangle$$

Again, using the fact that the space $C_c(\mathbb{R}^n; \mathcal{K}^{\otimes n})$ of all compactly supported continuous functions from $\mathbb{R}^n$ into $\mathcal{K}^{\otimes n}$ is dense in $L^2(\mathbb{R}^n; \mathcal{K}^{\otimes n})$, one can show that the space $\mathcal{E}_n$ satisfies all necessary conditions for the expansion studied in Section 3, using the exact same arguments as in 9.1.

Now we shall work towards establishing the symmetrized expansion for projections P_n associated to an S-matrix. An S-matrix [7] is defined as a continuous function $S : \mathbb{R} \to \mathscr{L}(\mathcal{K} \otimes \mathcal{K})$, which satisfies for each θ :

$$S(\theta)^* = S(\theta)^{-1} = S(-\theta) \tag{9.2}$$

and the Yang-Baxter equation:

$$(S(\theta)\otimes 1_1)(1_1\otimes S(\theta+\theta'))(S(\theta')\otimes 1_1) = (1_1\otimes S(\theta'))(S(\theta+\theta')\otimes 1_1)(1_1\otimes S(\theta)) \tag{9.3}$$

for all $\theta, \theta' \in \mathbb{R}$. Although there are more assumptions made in [7], we do not need more assumptions for our purposes.

For an operator $T \in \mathscr{L}(\mathcal{K}\otimes\mathcal{K})$ and $1 \leq k \leq n-1$, we set

$$T_{n,k} := 1_{k-1}\otimes T\otimes 1_{n-k-1} \in \mathscr{L}(\mathcal{K}^{\otimes n})$$

Now, we define an action D_n of $\mathscr{G}_n$ on $\mathcal{L}^{\otimes n}$ by setting

$$D_n(\tau_k)(\Psi)(\theta_1,..,\theta_n) := S(\theta_{k+1}-\theta_k)_{n,k}(\Psi(\theta_1,..,\theta_{k+1},\theta_k,..,\theta_n))$$

for each adjacent transposition $\tau_k = (k \quad k+1) \in \mathscr{G}_n$. As in 9.1, in order to extend this to an action of $\mathscr{G}_n$, the following identities must be true:

- $D_n(\tau_j)^2 = I$
- $D_n(\tau_i)D_n(\tau_j) = D_n(\tau_j)D_n(\tau_j)$ for $|i-j|>1$
- $D_n(\tau_j)D_n(\tau_{j+1})D_n(\tau_j) = D_n(\tau_{j+1})D_n(\tau_j)D_n(\tau_{j+1})$

After direct but long computations, one can verify that the three identities hold, the first one thanks to 9.2, the second one by definition, and the third one thanks to the Yang-Baxter equation (9.3).

This extends to a well defined action $D_n : \mathscr{G}_n \to \mathscr{L}(\mathcal{L}^{\otimes n})$, thanks to (9.2) and (9.3) which is actually a unitary representation, since $D_n(\tau_k)^* = D_n(\tau_k^{-1}) = D_n(\tau_k)$ for each $1 \leq k \leq n-1$, after a straightforward verification.

Now, we define the projections $P_n : \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}$ as

$$P_n := \frac{1}{n!}\sum_{\sigma\in\mathscr{G}_n} D_n(\sigma)$$

First, we are going to show that $P_{n+m} \leq P_n \otimes 1_m$. Let $\Psi = \Psi_1 \otimes \Psi_2$ with $\Psi_1 \in \mathcal{L}^{\otimes n}$, $\Psi_2 \in \mathcal{L}^{\otimes m}$ and $1 \leq k \leq n-1$. We compute:

$$\begin{aligned}(D_n(\tau_k)\otimes 1_m)(\Psi)(\theta,\eta) &= D_n(\tau_k)(\Psi_1)(\theta)\otimes\Psi_2(\eta)\\ &= S(\theta_{k+1}-\theta_k)_{n,k}(\Psi_1(\theta_1,..,\theta_{k+1},\theta_k,..,\theta_n))\otimes\Psi_2(\eta)\\ &= (1_{k-1}\otimes S(\theta_{k+1}-\theta_k)\otimes 1_{n-k-1})(\Psi_1(\theta_1,..,\theta_{k+1},\theta_k,..,\theta_n))\otimes\Psi_2(\eta)\\ &= (1_{k-1}\otimes S(\theta_{k+1}-\theta_k)\otimes 1_{n+m-k-1})(\Psi_1(\theta_1,..,\theta_{k+1},\theta_k,..,\theta_n)\otimes\Psi_2(\eta))\\ &= S(\theta_{k+1}-\theta_k)_{n+m,k}(\Psi(\theta_1,..,\theta_{k+1},\theta_k,..,\theta_n,\eta)) = D_{n+m}(\tau_k)(\Psi)(\theta,\eta)\end{aligned}$$

Where in the last expression, we consider τ_k as an element of $\mathscr{G}_{n+m}$ in the obvious way. It follows that

$$D_n(\tau_k)\otimes 1_m = D_{n+m}(\tau_k)$$

and therefore, for each $\tau \in \mathscr{G}_n$ we have

$$D_n(\tau) \otimes 1_m = D_{n+m}(\tau)$$

Then, the same computation as in the previous subsection shows that

$$P_{n+m} \leq P_n \otimes 1_m$$

The proofs of the rest of the required properties of the projections P_n proceed exactly as in Section 9.1.

9.3 Rapidity-Ordered Spaces

In this case, we are following the model used by Duell [4]. Suppose we are given a Hilbert space $\mathcal{L}$ and a partial order $\prec$ on it. We let the spaces $\mathcal{E}_n$ be the same as $\mathcal{L}^{\otimes n}$, and one can easily be convinced that all conditions hold. For each n, we define $W_n \subset \mathcal{L}^{\otimes n}$ to be the closed linear span of the following set:

$$\{\Phi_1 \otimes ... \otimes \Phi_n \mid \Phi_1 \prec ... \prec \Phi_n\}$$

and we define P_n as the projection onto the subspace W_n. In order to see that $P_{n+m} \leq P_n \otimes 1_m$, we first observe that if $\Phi_1 \prec ... \prec \Phi_n \prec \Phi_{n+1} \prec ... \prec \Phi_{n+m}$ then

$$\begin{aligned}(P_n \otimes 1_m)(\Phi_1 \otimes .. \otimes \Phi_{n+m}) &= P_n(\Phi_1 \otimes ... \otimes \Phi_n) \otimes \Phi_{n+1} \otimes ... \otimes \Phi_{n+m} \\ &= \Phi_1 \otimes ... \otimes \Phi_{n+m}\end{aligned}$$

Hence, we have $\Phi_1 \otimes ... \otimes \Phi_{n+m} \in \mathrm{Ran}(P_n \otimes 1_m)$ for $\Phi_1 \prec ... \prec \Phi_{n+m}$, and therefore $W_{n+m} = \mathrm{Ran}(P_{n+m}) \subset \mathrm{Ran}(P_n \otimes 1_m)$, hence

$$P_{n+m} \leq P_n \otimes 1_m$$

9.4 T-deformed Spaces

Finally, we have the case of [3], where the authors considered a general Hilbert space $\mathcal{L}$ and an operator $T \in \mathscr{L}(\mathcal{L} \otimes \mathcal{L})$ which is unitary, self-adjoint and which satisfies the Yang-Baxter equation:

$$(T \otimes I_1)(I_1 \otimes T)(T \otimes I_1) = (I_1 \otimes T)(T \otimes I_1)(I_1 \otimes T)$$

Then, they define $T_i \in \mathscr{L}(\mathcal{L}^{\otimes n})$ as

$$T_i := I_{i-1} \otimes T \otimes I_{n-i-1}$$

and for $\tau_k = (k \;\; k+1)$, $D_n(\tau_k) := T_k$ It follows directly that this action can be extended to the whole space $\mathscr{G}_n$ thanks to the properties of T. Then, they set

$$P_n := \frac{1}{n!} \sum_{\sigma \in \mathscr{G}_n} D_n(\sigma)$$

The reader might easily check that $D_n(\tau) \otimes 1_m = D_{n+m}(\tau)$ and it follows that $P_{n+m} \leq P_n \otimes 1_m$. For any sequence of subspaces $\mathcal{E}_n$ satisfying the necessary properties, we get a symmetrized expansion with respect to the family (P_n). In particular, it holds for $\mathcal{E}_n := \mathcal{L}^{\otimes n}$ equipped with the original topology.

9.5 Schwartz functions

For this example, we set $\mathcal{L} := L^2(\mathbb{R})$, i.e. $\mathcal{L}^{\otimes n} = L^2(\mathbb{R}^n)$. For each n, let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions (see section 2.6.5). We will prove that $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. Since the space of all compactly supported smooth functions $C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ (for a detailed proof, see [8] Theorem 2.16 and Lemma 2.19) and it is clearly contained in $\mathcal{S}(\mathbb{R}^n)$, it follows that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. For $f \in \mathcal{S}(\mathbb{R}^n)$, we have:

$$\begin{aligned}\|f\|_2^2 &= \int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} \frac{1}{1+|x|^2}(1+|x|^2)|f(x)|^2 dx \\ &\leq C \sup_{x\in\mathbb{R}^n} \{(1+|x|^2)|f(x)|^2\} \leq C\|f\|_\infty(\|f\|_\infty + \|f\|_{\alpha_1,0} + .. + \|f\|_{\alpha_n,0})\end{aligned}$$

where α_k is the multi-index consisting of 2 in the k-place and 0's everywhere else, and

$$C = \int_{\mathbb{R}^n} \frac{1}{1+|x|^2} dx < \infty$$

This proves that $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ and that the Schwartz topology is finer than the L^2 topology. By the discussion in Section 2.6.5, it follows that $\mathcal{S}(\mathbb{R}^n)$ satisfies all necessary conditions of Section 8 regarding the expansion in nuclear spaces.

For the symmetrized expansion, we are going to use the family (P_n) of the projections that we defined in 9.1, however we need to pose stronger conditions on the scattering function S. In particular, we need that all derivatives of the scattering function S are bounded (not necessarily uniformly bounded).

Lechner [6] defines scattering functions as functions $S : \overline{S(0,\pi)} \to \mathbb{C}$, where $S(0,\pi) := \{z \in C \ : \ 0 < \text{Im}z < \pi\}$, which are analytic in the interior of the strip and bounded and continuous in the closure $\overline{S(0,\pi)}$, and with the assumptions that

$$\overline{S(\theta)} = S(\theta)^{-1} = S(\theta + i\pi) = S(-\theta)$$

for all $\theta \in \mathbb{R}$. So far we have not needed more than the properties of the restriction of S to the real line. In [6] Definition 3.3, Lechner defines **regular** scattering functions as the ones who have a bounded and analytic continuation to a strip $S(-\kappa, \pi+\kappa)$ for some $\kappa > 0$. With this assumption, for every $\theta \in \mathbb{R}$, the closed ball of radius $\kappa/2$ around θ is contained in $S(-\kappa, \pi+\kappa)$, therefore by Cauchy's integral formula, we have for each n:

$$S^{(n)}(\theta) = \frac{n!}{2\pi i} \oint_\gamma \frac{S(z)}{(z-\theta)^{n+1}} dz$$

for γ being the boundary of the above ball. We conclude that

$$|S^{(n)}(\theta)| \leq \frac{n!\|S\|_\infty}{(\kappa/2)^n}$$

which proves that the derivatives of S are all bounded in $\mathbb{R}$. So from now on, we assume that S is a regular scattering function.

We have to show that $P_n(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$ for each n and it is continuous with respect to the Schwartz topology, and that $P_n \otimes 1_m$ is also continuous as a map from $\mathcal{S}(\mathbb{R}^{n+m})$ into itself. Since we have

$$P_n = \frac{1}{n!} \sum_{\sigma \in \mathscr{G}_n} D_n(\sigma) \quad \text{and}$$

$$P_n \otimes 1_m = \frac{1}{n!} \sum_{\sigma \in \mathscr{G}_n} D_{n+m}(\sigma)$$

and furthermore $\mathscr{G}_n$ is generated by adjacent transpositions, it is sufficient to show for each n and $1 \leq k \leq n-1$ that $D_n(\tau_k)(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^n)$ and that $D_n(\tau_k)$ is continuous as a function from $\mathcal{S}(\mathbb{R}^n)$ into itself. We recall that

$$D_n(\tau_k)(f)(\theta) = S(\theta_{k+1} - \theta_k) f(\theta_1, .., \theta_{k+1}, \theta_k, .., \theta_n)$$

Since all derivatives of S are bounded, one can easily see that for any multi-indices α, β,

$$\|D_n(\tau_k)(f)\|_{\alpha,\beta} \leq C_{|\alpha|,|\beta|} \sum_{|\beta'| \leq |\beta|} \|f\|_{\alpha',\beta'}$$

Where $|\beta'| \leq |\beta|$ means that every index of β' is less or equal than the corresponding index of β and α' denotes α with the indices k and $k+1$ interchanged. Then our claim is proved, and the symmetrized expansion holds for all quadratic forms.

10 Summary and Discussion

In this thesis, we established the existence of a certain way of expanding any quadratic form A (which we call an *observable*) acting on a Fock space, satisfying certain boundedness conditions, with main applications the cases of integrable models with an associated factorizing scattering function (or scattering matrix). The expansion reads

$$A = \sum_{m,n=0}^{\infty} \frac{1}{m!n!}(z^{\dagger m}[A]_{mn}z^n)$$

where $z^\dagger$ and z are *Zamolodchikov* operators. The terms of the expansion and the forms $[A]_{mn}$ can be studied in order to collect useful information on the observable A.

In Sections 3 and 4, we proved the existence of the expansion formula and the homeomorphic nature of the correspondence it has with the observable. We tried to pose as few conditions as possible in order for the proof to work in a general context.

In Sections 6 and 8, we saw two general cases, quite different from each other, in which the conditions are met and all specific examples we have in mind (all included in Section 9) fit into one of the two cases.

The large class of examples we provided indicates how important it is to pose as general conditions as possible. Also, we believe that the proof was much simpler than the proof of the expansion in [2], however we have not generalized all results of [2] to our context. One can imagine how terribly complicated it would be to prove an expansion formula for S-matrices, for the model [7], using a method similar to [2].

We did not discuss about what happens in the level of distributions and integral kernels. We did not manage to look into this problem yet, although it would be of great importance to have insight into the associated kernels, since they can probably provide us with more information on the observables. We hope to deal with this problem later, or that someone else does.

Finally, we saw how the expansion changes in case a unitary operator of a specific kind acts on the observable, motivated by [2], where the authors considered space-time translations and boosts. Although we tried to consider doing the same for antiunitary operators, generalizing space-time reflections, we were not able to come up with a satisfying answer. However, we did not spend much time on this problem, so there might be something interesting in case it is further investigated.

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