Aspects of Finite Gaudin Models: Separation of Variables and Description from 3dBF Theory

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Abstract

In this thesis, we provide a separation of variables for quantum Gaudin models associated to low rank matrix Lie algebras, and a description of the finite classical Gaudin model for any semisimple Lie algebra from a gauge theory.

We separate the variables for the quantum $\mathfrak{sl}_2$-Gaudin model with irregular singularities, producing an explicit coordinate change and comparing this to the known solution provided by the Bethe Ansatz. We also produce a separation of variables of the $\mathfrak{gl}_3$-Gaudin model with irregular singularities following previous work separating the variables in the XXX-chain. We do this both by directly taking the limit of the XXX case, and by working only in the Gaudin setting.

The affine Gaudin model, associated with an untwisted affine Kac-Moody algebra, is known to arise from a certain gauge fixing of 4-dimensional mixed topological-holomorphic Chern-Simons theory in the Hamiltonian framework. We show that the finite classical Gaudin model, associated with a finite-dimensional semisimple Lie algebra, can likewise be obtained from a similar gauge fixing of 3-dimensional mixed BF theory with certain line defects in the Hamiltonian framework.

Contents

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Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references. Portions are based on the following article of mine;

B. Vicedo and J. Winstone. "3-dimensional mixed BF theory and Hitchin's integrable system". *Letters in Mathematical Physics* 112.4 (Jan. 2022). arXiv: 2201.07300v1

— 1 —

Introduction: The Gaudin Model and Integrability

1.1 Introduction

The Gaudin model, first introduced by Michel Gaudin in 1976 [32], was initially conceived as an N-site quantum integrable spin chain with long range interactions, based on the Lie algebra $\mathfrak{sl}_2$. Its dynamics are governed by the N commuting Gaudin Hamiltonians, which are given in terms of the standard basis $\{e, f, h\}$ by

$$H_i := \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\frac{1}{2}h^{(i)}h^{(j)} + e^{(i)}f^{(j)} + f^{(i)}e^{(j)}}{z_i - z_j},$$

for sites $z_i \in \mathbb{C}$. It was later realised that the construction can be adapted for any finite-dimensional and reductive Lie algebra $\mathfrak{g}$ [33], and furthermore that one can define a classical version of the model. There later came further variations, such as affine Gaudin models, dihedral [66], and cyclotomic Gaudin models [70], as well as adaptions such as adding in irregular singularities to the Lax matrix. Though originally conceived as a spin system specifically, the Gaudin model (and its generalisations) turns out to include a broad class of integrable systems with linear Lax relations

$$[L_1(z), L_2(w)] = \left[\frac{C_{12}}{z-w}, L_1(z) + L_2(w)\right].$$

That is, by choosing a specific representation of the underlying Lie algebra, one can obtain for example the Neumann model which describes the movement of particles on a sphere subject to harmonic forces [3, §2.11]. For this model we can take a classical Gaudin model with a double pole at infinity and apply the representation described in [68, §5.4].

In particular, classical *affine* Gaudin models, based on affine Kac-Moody algebras, are currently an active area of study - in part due to their ingrained description of various 2-dimensional integrable field theories [66], and their recent connections to gauge theory [65], which, as we will see, is not unrelated. This thesis, on the other hand, focuses on Gaudin models constructed from finite-dimensional Lie algebras - or "finite Gaudin models" - which are in many ways better understood, particularly with regards to their quantisation. Another reason to study this more straightforward case is therefore its potential as a toy model by which we can learn more about the affine Gaudin model.

To solve the original $\mathfrak{sl}_2$ version of his model (by which we mean finding the joint spectrum and common eigenvectors of the Gaudin Hamiltonians), Gaudin used the reliable and widely known technique of the Bethe Ansatz [31, 32], which has since been extended to Gaudin models constructed from arbitrary simple Lie algebras [31]. By this method one can construct multiple joint eigenvectors of the Gaudin Hamiltonians inductively from a starting reference vector, with dependence on some parameters subject to certain equations - known as the Bethe roots and Bethe equations respectively. While the Bethe Ansatz has been successfully applied to a wide range of models [43], it does have a few limitations; for instance, in order to guarantee the existence of the reference vector we are restricted to highest weight representations of the Gaudin model, which narrows the scope. However, a critical shortcoming of the Bethe Ansatz in the Gaudin model is that it has been shown not to provide a complete set of eigenvectors beyond rank 2 [47].

The Bethe Ansatz corresponds to a deeper story underlying the Gaudin model in terms of *opers* [29], which we might loosely describe as equivalence classes of certain connections of the underlying Lie algebra $\mathfrak{g}$. Each eigenvector provided by the Bethe Ansatz corresponds to a particular type of oper, known as a Miura oper. With a more complex set up, involving machinery such as vertex algebras and affine Kac-Moody algebras, it has recently been shown directly that there is a one-to-one correspondence between opers and the eigenspaces of the Gaudin model [55]; in other words, unlike the Bethe Ansatz, the oper story is complete.

Another approach to solving integrable models, developed by Sklyanin for a large number of systems [60] in the late 20th century including the quantum

$\mathfrak{sl}_2$-Gaudin model [58], is *Separation of Variables*, which is the subject of the first part of this thesis. Though the phrase is widely used in the context of decoupling multivariate differential equations, here we specifically mean Separation of Variables in the sense of solving integrable systems - in either case, the key idea is to reduce a multidimensional problem to a set of one-dimensional problems. Initially, this idea strictly referred to an explicit change of variables from a differential operator realisation of the model in question to some new coordinates in which the eigenvalue equations (for the quantum case) decouple. Later the notion was extended to include defining some variables $\{q_i\}_{i=1}^{\mathcal{D}}$ for which we can factorise our wavefunction Ψ into functions dependent on a single variable. That is,

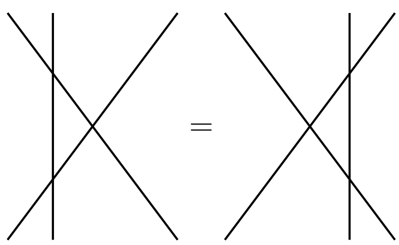
$$\Psi(q_1, \dots, q_{\mathcal{D}}) = \psi_1(q_1)\psi_2(q_2)\cdots\psi_{\mathcal{D}}(q_{\mathcal{D}}). \tag{1.1}$$

Though it has been around for longer than the oper perspective, Sklyanin's Separation of Variables for the Gaudin model miraculously produces a differential equation in each variable that exactly corresponds to the universal oper

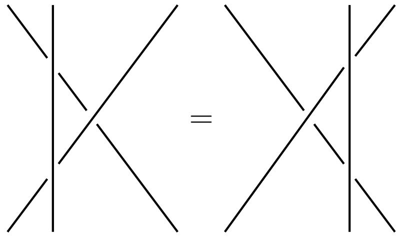
$$\operatorname{cdet}\left(\partial_z + L(z)\right), \tag{1.2}$$

where "cdet" denotes the column-ordered determinant, which we will define later. This strongly suggests that it also provides a complete solution. Hence Separation of Variables has the potential to provide something of a middle-ground between the usability of the Bethe Ansatz and the completeness of the oper story, if this were to extend to other versions of the Gaudin model such as those associated to higher rank matrix Lie algebras.

Separately from notions of solving models, another recent advance is the perspective on integrability provided by gauge theories, put forward by Costello [13]. This seemingly unlikely connection can be motivated by noticing the visual similarity between a pictorial representation of the quantum Yang Baxter equation (which underlies the integrability of the XXX spin chain)



and an allowed move in knot theory (c.f. Jones and others), with the additional structure of the lines being specified to cross "over" or "under" [71],



whose invariants are described by a 3-dimensional Chern-Simons theory. Due to the additional dependence on a complex spectral parameter, one must instead use a 4-dimensional version of the theory to describe the XXX chain - this idea was solidified alongside Witten and Yamazaki [14, 15]. This gauge theoretic perspective could not only offer a new perspective on problems in integrability, but perhaps provide us with a systematic method of defining new integrable systems.

4-dimensional Cherns-Simons Theory has also been shown [16, 40, 20] to admit a description of 2-dimensional integrable field theories; the equations of motion turn out to be the zero curvature equation, which is also the requirement of the Lax connection of an integrable field theory. Given the alternative algebraic description of integrable field theories from affine Gaudin models, this naturally led to the study of a link between this gauge theory and affine Gaudin models by Vicedo [66]. In a certain gauge, the Lax algebra of the affine Gaudin model may be found as the Dirac bracket on the theory. A noticeable missing link is therefore an analogous description of the finite classical Gaudin model from a gauge theory, which we will see later turns out to be 3-dimensional mixed BF Theory with certain defects. Since we understand the quantisation of the Gaudin model, we would hope that considering this from the 3-dimensional BF Theory perspective could lead to a framework for quantising other integrable models.

Let us discuss how we will approach these two topics within this thesis.

1.1.1 Plan of thesis

This thesis is split into three main parts. In this Chapter we introduce the quantum and classical Gaudin models and discuss integrability using these as examples. We describe the commuting quantum Hamiltonians and the Lax formulation of the model. We spend some time looking at the quantum $\mathfrak{gl}_n$ Gaudin model in particular, and introducing higher order singularities via the machinery of Takiff Lie algebras - including a double pole at infinity. We touch on related models, the Hitchin system and the XXX-chain, which appear on the peripheries of the results of this thesis.

Part I, which includes Chapters 2 and 3, concerns Separation of Variables for quantum Gaudin models associated to matrix Lie algebras. In Chapter 2 we look at the $\mathfrak{sl}_2$-Gaudin model and extend Sklyanin's Separation of Variables [58] to cover Gaudin models with irregular singularities, producing an explicit change of variables from an initial differential operator realisation. We compare this to the solution provided by the Bethe Ansatz for a version of the model with highest weight representations. We then briefly discuss how both methods relate to the underlying oper story, and what this means for completeness.

Then, in Chapter 3 we move to the rank 3 case, initially constructing the new variables and separated equation as a limit of the Separation of Variables of the XXX chain, as Ribault does in [52]. We then go on to recreate this directly in the Gaudin setting, though without the explicit change of variables we had in the $\mathfrak{sl}_2$ case. We discuss several avenues for generalising this to higher rank Gaudin models, and the pitfalls we have found in each.

Part II of this thesis, which contains Chapter 4 only, concerns the gauge theoretic perspective of integrability. In particular we present the results of the paper [67] in which we gauge fix 3-dimensional mixed topological holomorphic BF Theory and perform a Hamiltonian analysis to obtain the Lax algebra of the finite classical Gaudin model. To motivate this, we give a brief overview of the more recent development of this point of view in 4 dimensional Chern-Simons theory in particular - both by Costello, Witten, and Yamazaki for the XXX chain and integrable field theories [14, 15, 16], and by Vicedo for the affine Gaudin model [65]. We finish by suggesting future directions by which we might better understand the correspondence between gauge theories and integrable models.

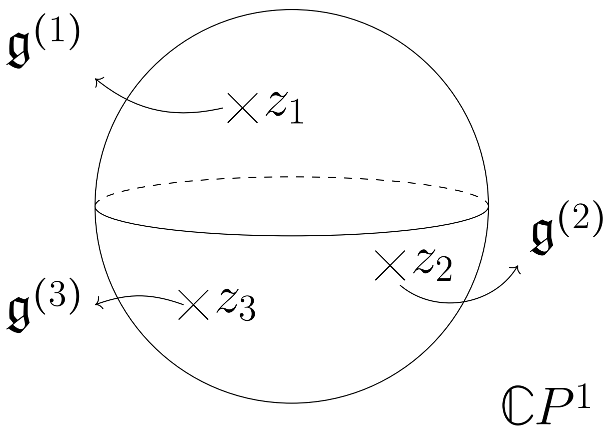
We also include an appendix detailing the method by which we constructed the realisations of Takiff Lie algebras we use in Part I. We go through the method in full for the Takiff Lie algebra $\mathfrak{sl}_2[\varepsilon]/\varepsilon^\tau$, and then state the outcome of the same method for $\mathfrak{gl}_3$ and $\mathfrak{gl}_3[\varepsilon]/\varepsilon^\tau$.

1.2 The finite Gaudin model and linear Lax relations

We define the finite Gaudin Model, and discuss the linear Lax relations. We begin with the quantum Gaudin model and later take the classical limit to define the classical version, though one could equally do this in reverse, see for example [41, §7.1].

1.2.1 The Quantum Gaudin model

We can consider the quantum finite Gaudin model as spin-particles at different points in the plane or on the complex sphere with the strength of interaction dependent on the distance between them. For a more concrete definition, let $\mathfrak{g}$ be a finite dimensional complex reductive Lie algebra, fix $N \in \mathbb{Z}_{\geq 1}$ and then fix N distinct complex numbers $z_i \in \mathbb{C}$. To each of these points, we attach a copy of $\mathfrak{g}$ such that we have N copies overall, as represented in the following image for $N = 3$.



Taking all these copies together, we have the algebra of observables for the Gaudin model $\bigotimes_{i=1}^N U(\mathfrak{g})$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. For any $X \in \mathfrak{g}$, we denote by $X^{(i)}$ the element in $\bigotimes_{i=1}^N U(\mathfrak{g})$ given by taking X on the ith factor of $\mathfrak{g}$ and the multiplicative identity of $U(\mathfrak{g})$ on all others

$$X^{(i)} = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1. \tag{1.3}$$

Let $\{I^a\}$ be a basis for our Lie algebra $\mathfrak{g}$, and we will also require a basis $\{I_a\}$ dual to it with respect to some non-degenerate symmetric bilinear form. This straightforwardly extends to a basis $\{I^{a(i)}\}$ for $\bigoplus_{i=1}^N \mathfrak{g}$ with a Lie bracket that vanishes across two different sites

$$\left[I^{a(i)}, I^{b(j)}\right] = \left[I^a, I^b\right]^{(i)} \delta^{ij}. \tag{1.4}$$

The quantum $\mathfrak{g}$-Gaudin model is then an integrable system whose commuting integrals of motion are the quadratic Gaudin Hamiltonians,

$$H_i := \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{I^{a(i)} I_a^{(j)}}{z_i - z_j}, \tag{1.5}$$

which govern the dynamics - here we are implicitly summing over the repeated index a. We can think of the factor $(z_i - z_j)$ in the denominator as encoding a greater interaction between sites that are closer together, with minimal contribution from sites that are very far apart. The Gaudin Hamiltonians are contained within a large commutative subalgebra of $\bigotimes_{i=1}^{N} U(\mathfrak{g})$ and here the notion of "solving" the model is the problem of finding the joint spectrum and simultaneous eigenstates of the Gaudin Hamiltonians H_i on the spin chain - for example by use of the Bethe-Ansatz or the Separation of Variables method we discuss in Part I. As required for integrability, the Hamiltonians all mutually commute

$$[H_i, H_j] = 0, \quad i, j = 1, \dots N. \tag{1.6}$$

Let M_i for $i = 1, \dots, N$ be a collection of $\mathfrak{g}$-modules, then $\bigotimes_{i=1}^{N} M_i$ is the "spin chain" of the $\mathfrak{g}$-Gaudin model, likening the action of $\mathfrak{g}$ on the module M_i to that of the algebra of spin ladder operators.

1.2.1.1 The Lax algebra

It is often more convenient to study integrable systems using the Lax formalism, where we package the symmetries together into one object, the Lax matrix, which in this case has an additional dependence on a complex spectral parameter $z \in \mathbb{C}$.

The Lax matrix of the Gaudin model is rational in this parameter z and valued in $\mathfrak{g} \otimes \bigoplus_{i=1}^{N} \mathfrak{g}$

$$L(z) = \sum_{i=1}^{N} \frac{I^a \otimes I_a^{(i)}}{z - z_i}, \tag{1.7}$$

the I^a with no label are in the auxiliary copy of $\mathfrak{g}$ and not attached to any particular site - or alternatively we can think of them as being attached to the site z. We can represent the auxiliary factor in some matrix representation, in which case $L(z)$ takes the form of a matrix, the entries of which are valued in $\bigoplus_{i=1}^{N} \mathfrak{g}$ and therefore not necessarily commutative.

The Lie bracket of $L(z)$ obeys the following linear Lax relations for two spectral parameters $z, w \in \mathbb{C}$

$$[L_1(z), L_2(w)] = [r(z,w), L_1(z) + L_2(w)] \tag{1.8}$$

where $L_1 = L(z) \otimes 1$ and $L_2(w) = 1 \otimes L(w)$ - the index indicating which factor the auxiliary copy of $\mathfrak{g}$ in $L(z)$ sits on. The r-matrix of the Gaudin model, on the right-hand side of (1.8) is antisymmetric and given by

$$r(z,w) = \frac{C_{12}}{z-w}, \tag{1.9}$$

where $C_{12} = I^a \otimes I_a$ is the split Casimir.

The bracket (1.8) satisfies the Jacobi identity when the r-matrix satisfies the classical Yang-Baxter equation

$$[r_{12}(z_1,z_2), r_{13}(z_1,z_3)] = [r_{23}(z_2,z_3), r_{12}(z_1,z_2)] - [r_{32}(z_3,z_2), r_{13}(z_1,z_3)]. \tag{1.10}$$

The Lax matrix still encodes the commuting Gaudin Hamiltonians (1.5), and we can recover them by considering the invariants of $L(z)$. For example if we take the bilinear form of $L(z)$ with itself, applied only in the auxiliary factor of $\mathfrak{g}$, we find the Gaudin Hamiltonians as the residues at $z = z_i$;

$$\begin{aligned} \mathrm{Res}_{z=z_i} \frac{1}{2}\langle L(z), L(z)\rangle &= \mathrm{Res}_{z=z_i} \sum_{i,j=1}^{N} \frac{1}{2} \frac{\langle I^a, I^b\rangle I_a^{(i)} I_b^{(j)}}{(z-z_i)(z-z_j)} \\ &= \mathrm{Res}_{z=z_i} \sum_{i,j=1}^{N} \frac{1}{2} \frac{I^{a(i)} I_a^{(j)}}{(z-z_i)(z-z_j)} \\ &= \mathrm{Res}_{z=z_i} \left(\sum_{i=1}^{N} \frac{C_{12}^{(i)}}{(z-z_i)^2} + \sum_{i=1}^{N} \frac{1}{2} \frac{1}{z-z_i} \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{I^{a(i)} I_a^{(j)}}{z-z_j} \right) = H_i, \end{aligned} \tag{1.11}$$

where $C_{12}^{(i)}$ is defined as $\frac{1}{2} I^{a(i)} I_{a(i)}$. If we represent $\mathfrak{g}$ in terms of matrices, $\langle L(z), L(z)\rangle$ might be $\mathrm{Tr}\, L^2(z)$ or the Killing form.

1.2.1.2 $\mathfrak{gl}_n$*-Gaudin Models*

In the first part of this thesis we will have a particular focus on quantum Gaudin models associated with matrix Lie algebras, where we will be solving these using the technique of Separation of Variables. Let us introduce the $\mathfrak{gl}_n$-Gaudin model for arbitrary rank $n \in \mathbb{Z}_{\geq 2}$.

Recall the standard basis $\{E^a_b\}^n_{a,b=1}$ of $\mathfrak{gl}_n$, whose Lie brackets are given by

$$[E^a_b, E^c_d] = \delta^c_b E^a_d - \delta^a_d E^c_b, \tag{1.12}$$

and hence the centre of the algebra is spanned by $\sum_{a=1}^n E^a_a$ and the elements E^a_b and E^b_a are dual to one another with respect to some non-degenerate symmetric bilinear form $(\cdot,\cdot) : \mathfrak{gl}_n \otimes \mathfrak{gl}_n \to \mathbb{C}$, which is given here by the trace $(A, B) = Tr(AB)$.

As discussed for the general case, we extend this basis to $\mathfrak{gl}_n^{\oplus N}$, denoting basis elements in the ith copy by $E^{a(i)}_b$ for $a, b = 1, \ldots, n$, whose Lie brackets for $i, j = 1, \ldots, N$ read

$$\left[E^{a(i)}_b, E^{c(j)}_d\right] = \delta_{ij}\left(\delta^c_b E^{a(i)}_d - \delta^a_d E^{c(i)}_b\right). \tag{1.13}$$

Again we fix N distinct $z_i \in \mathbb{C}$ and define the Lax matrix of the $\mathfrak{gl}_n$-Gaudin model associated to the direct sum $\mathfrak{gl}_n \otimes \mathfrak{gl}_n^{\oplus N}$

$$L(z) := \sum_{a,b=1}^{N} E^a_b \otimes L^a_b(z) \tag{1.14}$$

where elements without an indicated copy of $\mathfrak{gl}_n$ in the direct sum are in the auxiliary tensor factor, and the $L^a_b(z)$ are rational in z

$$L^a_b(z) = \sum_{i=1}^{N} \frac{E^{b(i)}_a}{z - z_i}. \tag{1.15}$$

When the auxiliary factor is viewed in the natural matrix representation $\pi : \mathfrak{gl}_n \to Mat_{n\times n}(\mathbb{C})$, representing the basis elements as $\pi(E^a_b) = (\delta^a_c \delta^d_b)^n_{c,d=1}$ (i.e. the matrix with 1 in the (a, b)th component and zeroes elsewhere) the Lax matrix is then represented as an $n \times n$ matrix whose (a, b)th component is $L^a_b(z)$. We remark that this matrix has non-commutative entries and therefore we cannot necessarily apply well-known results for matrices with commutative entries.

The Lax algebra in this case is defined by the Lie bracket of the components of the Lax matrix which, for two different spectral parameters $z, w \in \mathbb{C}$ is given by

$$[L^a_b(z), L^c_d(w)] = \delta^c_b \frac{L^a_d(z) - L^a_d(w)}{z - w} - \delta^a_d \frac{L^c_b(z) - L^c_b(w)}{z - w}, \tag{1.16}$$

and for the same spectral parameter by

$$[L^a_b(z), L^c_d(z)] = \delta^c_b L^{a\prime}_d(z) - \delta^a_d L^{c\prime}_b(z), \tag{1.17}$$

where $L_b^{a\prime}(z)$ denotes differentiation of the Lax matrix component $L_b^a(z)$ with respect to z.

We can compute this directly from the Lie brackets of $\mathfrak{gl}_n$

$$[L_b^a(z), L_d^c(w)] = \sum_{i,j=1}^{N} \frac{\left[E_a^{b(i)}, E_c^{d(j)}\right]}{(z-z_i)(w-z_j)} = \sum_{i=1}^{N} \left(\frac{1}{w-z_i} - \frac{1}{z-z_i}\right) \frac{\delta_d^a E_c^{b(i)}}{z-w}$$
$$= \delta_b^c \frac{L_d^a(z) - L_d^a(w)}{z-w} - \delta_d^a \frac{L_b^c(z) - L_b^c(w)}{z-w}.$$

The statement (1.17) for the case where both components have the same spectral parameter follows by taking the limit $w \to z$ of both sides.

Note that this is consistent with equation (1.8), for r-matrix

$$r(z,w) = \frac{C_{12}}{z-w},$$

in the sense that the components of the matrix $[r_{12}(z,w), L_1(z) + L_2(w)]$ correspond to the right-hand side of (1.16) for respective values of a, b, c, and d. For example, in $\mathfrak{sl}_2$ we have that

$$[L_b^a(z), L_d^c(w)] = [r_{12}(z,w), L_1(z) + L_2(w)]_{(a+2c-2, b+2d-2)}.$$

Generating Functions and Manin Matrices In the theory of classical integrable systems, the Poisson bracket used in (1.38) satisfies the Jacobi identity

$$\{L_1, \{L_2, L_3\}\} + \{L_3, \{L_1, L_2\}\} + \{L_2, \{L_3, L_1\}\} = 0,$$

if and only if the corresponding r-matrix satisfies the classical Yang-Baxter equation (1.10), both of which we will introduce later. We can see that this is indeed the case for the r-matrix of the Gaudin model, $r(z,w) = \frac{C_{12}}{z-w}$. Hence in this quantum setting, we have seen that traces of powers of $L(z)$ generate the integrals of motion, take for example $\mathfrak{sl}_2$, where we have

$$\widehat{s_1}(z) = \frac{1}{2} \operatorname{Tr} L^2(z) = \sum_{i=1}^{N} \frac{C^{(i)}}{(z-z_i)^2} + \frac{H_i}{z-z_i}. \tag{1.18}$$

In general we might find a combination of different traces of powers and derivatives, for example in $\mathfrak{gl}_2$

$$\widehat{s_1}(z) = +\frac{1}{2} (\operatorname{Tr} L(z))^2 - \frac{1}{2} \operatorname{Tr} L^2(z) + \frac{1}{2} \operatorname{Tr} L'(z). \tag{1.19}$$

These generating functions can be derived from the matrix $\left(\partial_z + L(z)\right)$ where ∂_z is multiplied by the $n \times n$ identity matrix. Note that in this matrix we have non-commuting entries, as we have seen in (1.16). It does however have the property of being a *Manin* matrix (see, for example [11, 10]), that is, a matrix A with entries in a ring R such that for $i, j, k, l = 1, \ldots, n$

$$\left[A^i_j, A^i_k\right] = 0, \tag{1.20a}$$

$$\left[A^i_j, A^k_l\right] = \left[A^k_j, A^i_l\right], \tag{1.20b}$$

i.e. if we take a rectangle of 4 elements across any two rows and two columns in the matrix, diagonal corners will have the same commutator. An advantage of Manin matrices is that we can retain some expected properties of matrices which we would otherwise lose with non-commuting elements. For example, we can define the *column-ordered determinant* of an $n \times n$ matrix A to be given such that the elements are ordered by column as the name suggests:

$$\operatorname{cdet} A = \sum_{\sigma \in S_n} (-1)^\sigma A_1^{\sigma(1)} A_2^{\sigma(2)} \cdots A_n^{\sigma(n)}. \tag{1.21}$$

Note that if A is Manin then we may swap the columns with the same effect on the determinant as we would have in a matrix with commutative entries.

To see that $\partial_z + L(z)$ is Manin, note firstly that it follows straightforwardly from (1.17) that

$$\left[\delta^i_j \partial_z + L^i_j(z), \delta^i_l \partial_z + L^i_l(z)\right] = \delta^i_l L^{i\prime}_j(z) - \delta^i_j L^{i\prime}_l(z) + \delta^i_j L^{i\prime}_l(z) - \delta^i_l L^{i\prime}_j(z) = 0 \tag{1.22}$$

and similarly that

$$\begin{aligned}\left[\delta^i_j \partial_z + L^i_j(z), \delta^k_l \partial_z + L^k_l(z)\right] &= \delta^i_j L^{k\prime}_l(z) - \delta^k_l L^{i\prime}_j(z) + \left[L^i_j(z), L^k_l(z)\right] \\ &= \left[L^k_j(z), L^i_l(z)\right] + \delta^k_j L^{i\prime}_l(z) - \delta^i_l L^{k\prime}_j(z) \\ &= \left[\delta^k_j \partial_z + L^k_j(z), \delta^i_l \partial_z + L^i_l(z)\right],\end{aligned}$$

as required.

These generating functions appear as the coefficients of ∂_z in the expansion of $\operatorname{cdet}\left(\partial_z + L(z)\right)$, that is

$$\operatorname{cdet}(\partial_z + L(z)) = \sum_{i=0}^{n} \widehat{s}_i(z) \partial_z^{n-i}. \tag{1.23}$$

For matrix Lie algebras, we will make use of the concept of minors of the Lax matrix, that is ordered determinants of sub-matrices written as

$$L^{a_1 \ldots a_k}_{b_1 \ldots b_k}(z) = \sum_{\sigma \in S_k} (-1)^\sigma L^{a_1}_{b_{\sigma(1)}}(z) L^{a_2}_{b_{\sigma(2)}}(z) \cdots L^{a_k}_{b_{\sigma(k)}}(z) \tag{1.24}$$

where $a_i, b_i \in 1, \dots, n$ for $i = 1, \dots, k$. If the minor is of length n note that (1.24) is the determinant of $L(z)$.

For $\mathfrak{gl}_3$, we therefore have three generating functions:

$$\widehat{s}_1(z) = L_1^1(z) + L_2^2(z) + L_3^3(z) = \operatorname{Tr} L(z) \tag{1.25a}$$

$$\widehat{s}_2(z) = L_{12}^{12}(z) + L_{13}^{13}(z) + L_{23}^{23}(z) + 2L_1^{1\prime}(z) + L_2^{2\prime}(z) \tag{1.25b}$$

$$\widehat{s}_3(z) = -L_{123}^{123}(z) - (L_{12}^{12}(z))' - L_1^1(z)L_3^{3\prime}(z) + L_3^1(z)L_1^{3\prime}(z) + L_1^{1\prime\prime}(z). \tag{1.25c}$$

In $\mathfrak{sl}_3$, we apply the condition $\operatorname{Tr} L(z) = 0$ and these become

$$\widehat{s}_1(z) = 0, \tag{1.26a}$$

$$\widehat{s}_2(z) = \frac{1}{2} \operatorname{Tr} L^2(z), \tag{1.26b}$$

$$\widehat{s}_3(z) = \frac{1}{3} \operatorname{Tr} L^3(z). \tag{1.26c}$$

1.2.1.3 Gaudin models with irregular singularities

So far we have introduced the most straightforward case of simple poles with no additional structure, but there exist further generalisations such as dihedral and cyclotomic Gaudin models [70, 69] or Gaudin models with irregular singularities. Let us consider the latter of these, predominantly focusing on the $\mathfrak{gl}_n$ case.

To extend to a model with poles of arbitrary strength, we replace each copy of the Lie algebra $\mathfrak{g}$ with a corresponding Takiff Lie algebra. To define this, fix a Takiff degree $\tau_i \in \mathbb{Z}_{\geq 1}$ for each site $i = 1, \dots, N$ and consider polynomials with coefficients in $\mathfrak{g}$ in some formal variables ε_i which are truncated at power $\tau_i - 1$. That is, at each site we will see the space $\mathfrak{g}[\varepsilon_i] = \mathfrak{g} \otimes \mathbb{C}[\varepsilon_i]$ quotiented by the ideal $\varepsilon_i^{\tau_i}\mathfrak{g}[\varepsilon_i] = \mathfrak{g} \otimes \varepsilon_i^{\tau_i}\mathbb{C}[\varepsilon_i]$ - this, denoted $\mathfrak{g}[\varepsilon_i]/\varepsilon_i^{\tau_i}$ is the Takiff Lie algebra at site i. For $X \in \mathfrak{g}$, we will denote the element $X\varepsilon_i^r$ for $r = 0, \dots, N$ by $X_{[r]}^{(i)}$ and say is has *mode* r. The basis $\{I^a\}$ of $\mathfrak{g}$ can then be extended to a basis $\{I_{[r]}^{a(i)} = I_a \varepsilon_i^r | r = 0 \dots, \tau_i - 1\}_{i=1}^N$ of $\mathfrak{g}[\varepsilon_i]/\varepsilon_i^{\tau_i}$.

This construction is also a Lie algebra with bracket for $\mathfrak{g}[\varepsilon]/\varepsilon^\tau$ very similar to those of $\mathfrak{g}$; if the basis elements I^a and I^b have structure constants $\alpha_c^{ab} \in \mathbb{C}$ such that

$$\left[I^a, I^b\right] = \alpha_c^{ab} I^c, \tag{1.27}$$

(where we are summing over the repeated index c) then the analogue in the Takiff algebra would be

$$\left[I_{[r]}^a, I_{[s]}^b\right] = \left[I^a, I^b\right]\varepsilon^r\varepsilon^s = \alpha_c^{ab} I^c \varepsilon^{r+s} = \alpha_c^{ab} I_{[r+s]}^c \tag{1.28}$$

with the implicit condition that $X_{[r]} = 0$ if $r \geq \tau_i$ due to the quotient. Therefore we will always have an embedded copy of $\mathfrak{g}$ found by taking all the elements of mode 0, and we can return to the simple poles case by setting the Takiff degree $\tau_i = 1$.

For example, in the $\mathfrak{gl}_n$-Gaudin model with irregular singularities, whose Lax matrix is associated to the direct sum $\bigoplus_{i=1}^N \mathfrak{gl}_n[\varepsilon]/\varepsilon^{\tau_i}$, we have the commutation relations

$$\left[E^{a(i)}_{b[r]}, E^{c(j)}_{d[s]}\right] = \delta_{ij}\left(\delta^c_b E^{a(i)}_{d[r+s]} - \delta^a_d E^{c(i)}_{b[r+s]}\right). \tag{1.29}$$

Having this additional distinction of higher modes allow us to add in higher poles in the Lax matrix at every site, with each basis element $I^{a(i)}_{[r]}$ being accompanied by a pole of strength $r+1$ at $z = z_i$ respectively. We build the Lax matrix of the $\mathfrak{g}$-Gaudin model with irregular singularities as before and using this idea to introduce the higher order poles, altogether arriving at the expression

$$L(z) = \sum_{i=1}^N \sum_{r=0}^{\tau_i - 1} \frac{I^a \otimes I^{(i)}_{a[r]}}{(z - z_i)^{r+1}}. \tag{1.30}$$

Note that in the auxiliary factor we still use the usual (non-Takiff) version of $\mathfrak{g}$, and so we can still represent the Lax operator as a matrix. For instance, in the $\mathfrak{gl}_n$-Gaudin model with irregular singularities, (1.30) tells us that (with the auxiliary factor represented in the natural representation) the (a,b)th component of the Lax matrix is given by

$$L^a_b(z) = \sum_{i=1}^N \sum_{r=0}^{\tau_i - 1} \frac{E^{b(i)}_{a[r]}}{(z - z_i)^{r+1}}. \tag{1.31}$$

The purpose of this construction is to add the higher order poles while still retaining linear Lax relations with the same r-matrix as the simple poles case. This means that any result that follows directly from the Lax relations is automatically applicable to a Gaudin model with higher order singularities as well. We will make great use of this for the $\mathfrak{gl}_3$-Gaudin model in the first part of this thesis in particular, where we use the Lax algebra to construct a Separation of Variables for certain Gaudin models.

Let us prove below these new Lax matrix elements also satisfy (1.16). From (1.31) it is immediate that

$$[L^a_b(z), L^c_d(w)] = \sum_{i,j=1}^N \sum_{r=0}^{\tau_i - 1} \sum_{s=0}^{\tau_j - 1} \frac{\left[E^{b(i)}_{a[r]}, E^{d(j)}_{c[s]}\right]}{(z - z_i)^{r+1}(w - z_j)^{s+1}}. \tag{1.32}$$

From the Lie bracket (1.29) in $\bigoplus_{i=1}^{N}\mathfrak{gl}_n[\varepsilon]/\varepsilon^{\tau_i}$ we see that we can remove the second sum over the sites (indexed by j) as anything on separate sites commute with one another. This leaves us with

$$\begin{aligned}
[L^a_b(z), L^c_d(w)] &= \sum_{i=1}^{N}\sum_{r,s=0}^{\tau_i-1}\frac{\delta^a_d E^{b(i)}_{c[r+s]} - \delta^c_b E^{d(i)}_{a[r+s]}}{(z-z_i)^{r+1}(w-z_i)^{s+1}} \\
&= \sum_{i=1}^{N}\sum_{p=0}^{\tau_i-1}\sum_{r=0}^{p}\frac{\delta^a_d E^{b(i)}_{c[p]} - \delta^c_b E^{d(i)}_{a[p]}}{(z-z_i)^{r+1}(w-z_i)^{p-r+1}},
\end{aligned}$$

where in the last equality we have relabelled to summation variable $p = r + s$. We will now use the following useful identity

$$\sum_{r=0}^{p}\frac{1}{(z-z_i)^{r+1}(w-z_i)^{p-r+1}} = \frac{-1}{(z-w)}\left(\frac{1}{(z-z_i)^{p+1}} - \frac{1}{(w-z_i)^{p+1}}\right) \tag{1.33}$$

which can be proved by induction on p, though we will not go into the details here. The base case is effectively that used in the proof of (1.16). From here it is straightforward to rearrange into the expected expression;

$$\begin{aligned}
[L^a_b(z), L^c_d(w)] &= \frac{1}{z-w}\sum_{i=1}^{N}\sum_{p=0}^{\tau_i-1}\left(\delta^c_b E^{d(i)}_{a[p]} - \delta^a_d E^{b(i)}_{c[p]}\right)\left(\frac{1}{(z-z_i)^{p+1}} - \frac{1}{(w-z_i)^{p+1}}\right) \\
&= \frac{1}{z-w}\Bigg(\delta^c_b\sum_{i=1}^{N}\sum_{p=0}^{\tau_i-1}\frac{E^{d(i)}_{a[p]}}{(z-z_i)^{p+1}} - \frac{E^{d(i)}_{a[p]}}{(w-z_i)^{p+1}} \\
&\qquad\qquad - \delta^a_d\sum_{i=1}^{N}\sum_{p=0}^{\tau_i-1}\frac{E^{b(i)}_{c[p]}}{(w-z_i)^{p+1}} - \frac{E^{b(i)}_{c[p]}}{(z-z_i)^{p+1}}\Bigg) \\
&= \delta^b_c\frac{L^a_d(z) - L^a_d(w)}{z-w} - \delta^a_d\frac{L^c_b(z) - L^c_b(w)}{z-w}.
\end{aligned}$$

As before, taking the $w \to z$ limit gives us the expression for the same spectral parameter, which is once again the same as we found in (1.17) in the previous section.

Double pole at infinity It will be convenient, when we come to the details of the Separation of Variables chapters, to have one more pole in the Lax matrix in addition to those at the sites z_i - namely a double pole placed at the point infinity as described in [68]. This will make certain processes run a little more smoothly; allowing us to have a clearer notion of "inverting" certain Lax matrix elements on the one hand and also to treat our variables more equivalently and avoiding further "twisting" steps on the other.

To include this, we add to our direct sum an abelian copy of our Lie algebra $\mathfrak{g}$ - that is an isomorphic copy of the vector space but endowed with the trivial Lie bracket - which we denote $\mathfrak{g}^{\text{comm}}$, equivalently this is also isomorphic to $\varepsilon_\infty \mathfrak{g}[\varepsilon_\infty]/\varepsilon_\infty^2$ for formal variable ε_∞, a conception that fits more smoothly into the irregular singularities setting. The Lax matrix will now instead be associated to the direct sum

$$\mathfrak{g}^{\text{comm}} \oplus \bigoplus_{i=1}^{N} \mathfrak{g}[\varepsilon_i]/\varepsilon_i^{\tau_i}, \tag{1.34}$$

with elements at this double pole denoted $X^{(\infty)}$ for $X \in \mathfrak{g}$. The Lie bracket is entirely unaltered since the additional copy of $\mathfrak{g}$ is commutative.

We can add this to the elements of our initial Lax matrix with simple poles as follows,

$$L^a_b(z) = E^{b(\infty)}_a + \sum_{i=1}^{N} \sum_{r=0}^{\tau_i - 1} \frac{E^{b(i)}_a}{(z - z_i)^{r+1}}. \tag{1.35}$$

Since a double pole at infinity turns up as a constant term in the Lax matrix, it will vanish on differentiating with respect to the spectral parameter, as is consistent with the Lax algebra relations.

For the Gaudin model with irregular singularities, it is perhaps more fitting to use the latter conception of $\mathfrak{g}^{\text{comm}}$ as a Takiff Lie algebra to better align with the rest, we note that objects in $\varepsilon_\infty \mathfrak{g}[\varepsilon_\infty]/\varepsilon_\infty^2$ may only have mode $r = 1$. Therefore the (a, b)th element of the Lax matrix is given by

$$L^a_b(z) = E^{b(\infty)}_{a[1]} + \sum_{i=1}^{N} \sum_{r=0}^{\tau_i - 1} \frac{E^{b(i)}_{a[r]}}{(z - z_i)^{r+1}}. \tag{1.36}$$

1.2.2 Finite Classical Gaudin models

The second part of this thesis concerns the classical field theory known as BF theory, and how it may be gauge fixed such that the equations of motion resemble the Lax equation of a classical integrable model. We will then move to the Hamiltonian formalism to confirm that the r-matrix that arises is indeed that of a classical version of the Gaudin model.

To take a classical limit of the quantum Gaudin model we have just described, we will need to introduce the parameter $\hbar$ and then take the $\hbar \to 0$ limit. Recall that the algebra of observables of the quantum Gaudin model consists of copies of the universal enveloping algebra $U(\mathfrak{g})$, which we may think of as the tensor algebra $T(\mathfrak{g})$ of all tensor products with the additional condition that the Lie bracket of any $X, Y \in \mathfrak{g}$ is the same as their commutator,

$$U(\mathfrak{g}) = T(\mathfrak{g})/(X \otimes Y - Y \otimes X - [X, Y]).$$

We introduce $\hbar$ as a factor in front of the Lie bracket, and denote this algebra $U_\hbar(\mathfrak{g})$;

$$U_\hbar(\mathfrak{g}) = T(\mathfrak{g})/(X \otimes Y - Y \otimes X - \hbar[X, Y]),$$

effectively in the previous section we had set $\hbar$ to 1. If we now take the $\hbar \to 0$ limit we are setting all the commutators to zero

$$T(\mathfrak{g})/(X \otimes Y - Y \otimes X) = S(\mathfrak{g}),$$

and we are left with N copies of the symmetric algebra $S(\mathfrak{g})^{\otimes N}$, i.e. our observables now commute and we have a classical model. The dynamical variables will be the basis elements across the copies of our Lie algebra, $I^{a(i)} \in \mathfrak{g}^{\oplus N}$.

We need to interpret the Lie bracket of $\mathfrak{g}$, for the basis elements $I^{a(i)} \in S(\mathfrak{g})^{\otimes N}$ as a Poisson bracket,

$$\{I^{a(i)}, I^{b(j)}\} = \delta_{ij}\left[I^a, I^b\right]^{(i)}, \tag{1.37}$$

and extend this bracket to the rest of the observables in $S(\mathfrak{g})^{\otimes N}$, using linearity of the bracket along with the Leibniz rule.

1.2.2.1 Classical Lax algebra

The discussion of the classical Lax algebra largely follows early chapters of [3] and the lecture notes [51].

In the classical case, our linear Lax relations take the form

$$\{L_1(z), L_2(w)\} = [(r(z, w), L_1(z) + L_2(w)] \tag{1.38}$$

where the r-matrix is $r_{12}(z, w) = \frac{C_{12}}{z-w}$ as before and the right hand side is a commutator of matrices. We introduce for convenience a new quantity $J^{(i)} \in \mathfrak{g} \otimes \mathfrak{g}^{\oplus N}$ defined as

$$J^{(i)} = I_a \otimes I^{a(i)}$$

where we implicitly sum over the index a. As before the Gaudin Lax matrix is meromorphic with the same simple poles at the points $z_i \in \mathbb{C}$

$$L(z) = \sum_{i=1}^{N} \frac{J^{(i)}}{z - z_i}. \tag{1.39}$$

The classical Lax matrix $L(z)$ satisfies (1.38) because the $J^{(i)}$ now satisfy the *Kostant-Kirillov* bracket

$$\{J_1^{(i)}, J_2^{(j)}\} = -\Big[C_{12}, J_2^{(i)}\Big]\delta_{ij} \tag{1.40}$$

where $C_{12} = I^a \otimes I_a$ is the split Casimir.

The basis of the classical Lax formalism is that $L(z)$ obeys the Lax equation

$$\frac{\mathrm{d}}{\mathrm{d}t}L(z) = [L(z), M(z)] \tag{1.41}$$

for similar matrix $M(z)$ made up of functions on the phase space, and the right-hand side is again a matrix commutator.

Solutions to (1.41) are given by conjugations of $L(z, t = 0)$ by some time-dependent invertible matrix $g(t)$

$$L(z,t) = g(t)L(z,t = 0)g(t)^{-1}, \tag{1.42}$$

and correspondingly

$$M(z) = \frac{\mathrm{d}g}{\mathrm{d}t}g(t)^{-1}. \tag{1.43}$$

These solutions preserve the spectrum of $L(z)$ in time, meaning that the Lax equation is isospectral. The integrals of motion are therefore found using adjoint-invariant polynomials $P, Q : \mathfrak{g} \to \mathbb{C}$, where the polynomial is applied to the auxiliary part of $L(w)$ for some fixed $w \in \mathbb{C}$. By (1.42),

$$P\left(L(w)\right) = P\left(g(t)L(w,t = 0)g(t)^{-1}\right) = P(L(w,t = 0)), \tag{1.44}$$

one of which will be the Hamiltonian.

For Liouville integrability, we also require that any two such integrals of motion $P(L(w)$ and $Q(L(z))$ are in involution with one another,

$$\{P(L(w)), Q(L(z))\} = 0.$$

We show this for a matrix representation of $\mathfrak{g}$. Let $P\big(L(w)\big) = \mathrm{Tr}\left(L(n)\right)^n$ and $Q\big(L\big) = \mathrm{Tr}\left(L(z)\right)^m$ for $m, n \in \mathbb{Z}_{\geq 1}$ - the generalisation to sums of such polynomials is straightforward. Let us take (1.38) and apply P to the second tensor factor of $\mathfrak{g}$ on both sides, using the Leibniz rule we see that

$$\begin{aligned}\{L_1(z), P(L_2(w))\} &= \mathrm{Tr}_2\left(\Big[r_{12}(z,w)\big(L_2(w)\big)^{n-1}, L_1(z) + L_2(w)\Big]\right)\\ &= \Big[\mathrm{Tr}_2\left(r_{12}(z,w)\big(L_2(w)\big)^{n-1}\right), \mathrm{Tr}_2\, L_1(z)\Big],\end{aligned}$$

as $\mathrm{Tr}_2\big(L_2(w)\big)$ is a multiple of the identity and hence commutative. Here Tr_2 denotes taking the trace over the second tensor factor of $\mathfrak{g}$ only. We repeat the same process when applying Q to the first tensor factor, and find similarly $\mathrm{Tr}_1\,\mathrm{Tr}_2\big(L_1(w)\big)$ is a multiple of the identity. Hence we have that

$$\{Q(L(z)), P(L(w))\} = 0. \tag{1.45}$$

This in fact goes both ways, and the invariant polynomials of $L(z)$ Poisson commuting implies the existence of an r-matrix satisfying (1.38) - see [3] or [51] for proof of this.

The integrals of motion being those quantities unchanged by conjugation of the Lax matrix is analogous to the way that in a gauge theory physically relevant quantities are those preserved by gauge transformations, an idea we will expand on in Part II of this thesis.

Our Lax matrix $L(z)$ is more fundamental than the other half of the pair $M(z)$, as the time dependence of $L(z)$ is governed by the Hamiltonian and hence once we have written the Hamiltonian in terms of Lax matrix, its partner $M(z)$ is completely determined by $L(z)$. Once we pick one of our integrals of motion $P(L(z))$ to be the Hamiltonian, then we require

$$M(z) = \frac{P'\left(L(w)\right)}{z-w} \tag{1.46}$$

to reproduce the Lax equation

$$\partial_t L(z) = \left[\frac{P'(L(w))}{z-w}, L(z)\right]. \tag{1.47}$$

For example in a matrix Lie algebra we might take $P(L(w) = \mathrm{Tr}\left(L^n(w)\right)$, and here $M(z)$ would take the form

$$M(z) = \frac{nL(w)^{n-1}}{z-w}. \tag{1.48}$$

In fact if we choose $P(L(w)) = \mathrm{Tr}\, L^2(w)$ for a matrix representation of $\mathfrak{g}$, then we recover the Gaudin Hamiltonians (1.5) exactly as in (1.11).

Let us look at one particular realisation - or general class of realisations - which we will be able to reproduce when we consider the gauge theoretic perspective. Let $u_1, \dots, u_N \in \mathfrak{g}$ and consider the orbits of these points in $\mathfrak{g}$ under conjugation by elements of the corresponding Lie group G, the adjoint orbits of the fixed points $u_i \in \mathfrak{g}$ are defined as

$$\mathcal{O}_{u_i} = \{\widehat{u} = h_i u_i h_i^{-1} | h_i \in G\}. \tag{1.49}$$

The Poisson structure on $\prod_{i=1}^N \mathcal{O}_i$ arises from the symplectic structure in the usual way (described in Chapter 14 of [3]) and in this bracket the $\widehat{u}_i$ also satisfy the Kostant-Kirillov bracket

$$\{\widehat{u}_{i1}, \widehat{u}_{j2}\} = -[C_{12}, \widehat{u}_{i2}]\delta_{ij}. \tag{1.50}$$

The realisation ϕ is therefore given by

$$\phi(J^{(i)}) = \widehat{u}_i, \tag{1.51}$$

with each basis element $I^{a(i)}$ being realised as the corresponding coordinate of $\widehat{u}_i$ in that basis. The Lax matrix hence has the Lie algebra-valued $\widehat{u}_i$ as the residues at the poles z_i

$$L(z) = \sum_{i=1}^N \frac{\widehat{u}_i}{z - z_i}. \tag{1.52}$$

1.2.2.2 Higher genus Gaudin models

One might ask whether the spectral parameter z must be constrained to $\mathbb{C}$ or $\mathbb{C}P^1$, and what changes when we alleviate that constraint. In fact this construction of a higher genus Gaudin model is exactly the Hitchin system with marked points.

Let us first sketch the Hitchin system without the marked points, as introduced in [3, §7.11]. Let C be a Riemann surface of arbitrary genus, which we will denote g, and let $\mathcal{A}$ the $(0,1)$-fields on C. If we let z be a local coordinate on C then a field $A \in \mathcal{A}$ takes the form $A_{\bar{z}} \mathrm{d}\bar{z}$. We also have gauge transformations by group-valued functions, $h : C \to G$, under which A transforms as a connection

$$A \mapsto h^{-1} A h - h^{-1} \bar{\partial} h. \tag{1.53}$$

We also introduce the Higgs field Φ as a covector to a tangent vector at $A \in \mathcal{A}$. The Hitchin system is concerned with pairs (A, Φ), and we refer the reader to [3] for the details of this. The z component of Φ, denoted $\Phi_z(z)$ in local coordinate z, will correspond to the classical Gaudin Lax matrix (1.39) when C has genus zero.

The phase space of the Hitchin system is defined by its moment map μ

$$\mu = \partial_{\bar{z}} \Phi + \Phi \wedge A + \Phi \wedge A, \tag{1.54}$$

which we can write in local coordinates as $\mu_{z\bar{z}} \mathrm{d}z \wedge \mathrm{d}\bar{z}$ where

$$\mu_{z\bar{z}} = \partial_{\bar{z}} \Phi_z + \ [A_{\bar{z}}, \Phi_z]. \tag{1.55}$$

For the phase space we set $\mu = 0$, factoring by the gauge group.

In genus zero, i.e. setting $C = \mathbb{C}P^1$, this system is trivial. To instead obtain the Gaudin model we need to introduce the data of the sites z_i, by including particular marked points at the sites of the Hitchin system. Instead of setting the moment map of the system μ to zero, we set it to

$$\mu = 2\pi i \sum_{j=1}^{N} \delta_{z z_j}. \tag{1.56}$$

In the gauge $A_{\bar{z}} = 0$, (1.56) gives us that

$$\partial_{\bar{z}} \Phi_z(z, \bar{z}) = 2\pi i \sum_{j=1}^{N} \delta_{z z_j}, \tag{1.57}$$

which implies that that Φ_z is meromorphic with simple poles at the points $z_i \in \mathbb{C}$ as required for the Lax matrix of the Gaudin model. We will go into the details of this calculation from the 3dBF perspective specifically in Part II, where some of the results apply equally to higher genus Hitchin systems beyond the Gaudin model.

1.3 Models with Quadratic Lax Relations

Though certainly not the focus of this thesis, we will also briefly introduce quadratic Lax relations (as opposed to the linear Lax relations discussed so far) using the example of the XXX spin chain, also known as the Heisenberg spin chain. In particular, we will show that in some limit we retrieve once more the linear Lax relations discussed above, which will be of great use to us in the Separation of Variables part of this thesis, and serve as a motivating example in the following part.

Classically, quadratic Lax relations would take the form

$$\{L_1(z), L_2(w)\} = [r_{12}(z, w), L_1(z) L_2(w)], \tag{1.58}$$

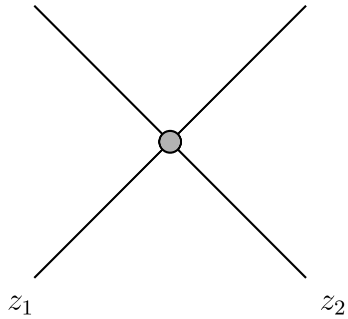
which on quantisation becomes

$$L_1(z) L_2(w) R_{12}(z, w) = R_{12}(z, w) L_2(w) L_1(z). \tag{1.59}$$

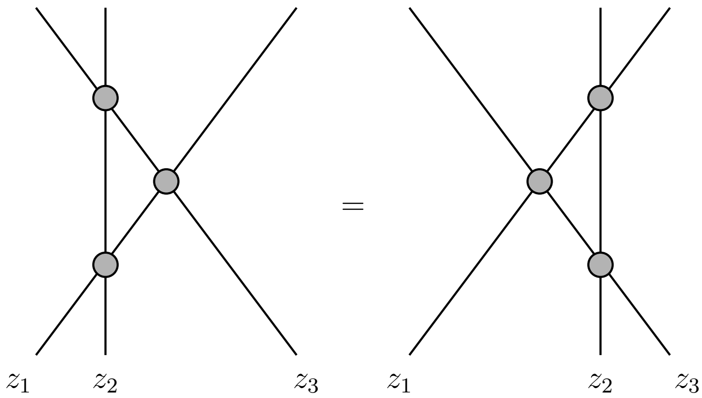
Where in the linear case, the r-matrix satisfies the classical Yang-Baxter equation (1.10), the R-matrix is a solution of the quantum Yang-Baxter equation

$$R_{12}(z_1, z_2) R_{13}(z_1, z_3) R_{23}(z_2, z_3) = R_{23}(z_2, z_3) R_{13}(z_1, z_3) R_{12}(z_1, z_2). \tag{1.60}$$

We can depict the R-matrix graphically as the crossing point of two lines, labelled by the two associated sites and the corresponding spectral parameter. We see this in the following diagram;

$R(z_1, z_2) =$ 

From this we might even interpret R as a scattering matrix, see [71]. Extending this picture, we have a geometric interpretation of (1.60), as shown in the following diagram;



In other words, the relation (1.60) allows us to pass these lines over one another. This geometric interpretation, and its visual similarity to invariants of knot theory, was a key entry point to considering integrable models from the perspective of gauge theories (see, for example, [71]) which we will discuss in Part II.

1.3.1 The XXX-Chain

We introduce the XXX-chain, following [21, 59, 51]. Solutions to (1.60) can be of three types; rational, trigonometric, and elliptic [71]. The XXX-chain - a spin chain with nearest neighbour interactions and periodic boundary conditions - is an example of the former of these; it has a rational R-matrix which produces the rational r-matrix of the Gaudin model in a certain limit. We will not consider trigonometric or elliptic R-matrices within this thesis.

Let $\{E^i_j\}^n_{i,j=1}$ be the standard basis of $\mathfrak{gl}_n$, then we have

$$R(z,w) = 1 + \hbar\frac{1}{z-w}\sum E^i_j \otimes E^j_i, \tag{1.61}$$

where $\hbar$ is an additional parameter included for the Gaudin limit later - note that we have the Gaudin r-matrix at first order in $\hbar$.

The Lax matrix at a particular site labelled i of the XXX-chain for $\mathfrak{gl}_n$ is associated to the direct sum $\mathfrak{gl}_n \otimes \mathfrak{gl}_n^{(i)}$ and takes the form

$$L^{(i)}_{XXX}(z) = 1 + \frac{\hbar}{z}\sum_{a,b=1}^{N} E^a_b \otimes E^{b(i)}_a. \tag{1.62}$$

Of course to describe the dynamics of the spin chain as a whole we require a global object, thus we combine copies of the Lax matrix across the different sites z_i into the *monodromy* matrix,

$$T(z) = L^{(1)}_{XXX}(z-z_i)L^{(2)}_{XXX}(z-z_2)\cdots L^{(N)}_{XXX}(z-z_N).$$

Here the copies of $L_{XXX}(z)$ are multiplied as matrices in the auxiliary factor of $\mathfrak{gl}_n$, hence $T(z)$ is associated to the direct sum $\mathfrak{gl}_n \otimes \bigoplus_{i=1}^N \mathfrak{gl}_n^{(i)}$. Representing this first factor of $\mathfrak{gl}_n$ we can consider $T(z)$ to be a matrix with entries $T^i_j(z)$.

By repeated application of (1.59), we see that $T(z)$ satisfies the same relation as $L_{XXX}(z)$,

$$T_1(z)T_2(w)R_{12}(z,w) = R_{12}(z,w)T_2(w)T_1(z). \tag{1.63}$$

Analogously to the minors in the Lax matrix of the $\mathfrak{gl}_n$ Gaudin model, we can take quantum minors in the monodromy matrix $T(z)$ of the $\mathfrak{gl}_n$ XXX-chain, which by convention also contain shifts in the spectral parameter by factors of $\hbar$, see [59];

$$T\left[{}^{a_1a_2\ldots a_m}_{b_1b_2\ldots b_m}\right](z) = \sum_{\sigma\in S_m}(-1)^{\sigma}T^{a_1}_{b_{\sigma(1)}}(z+(m-1)\hbar)T^{a_2}_{b_{\sigma(2)}}(z+(m-2)\hbar)\cdots T^{a_m}_{b_{\sigma(m)}}(z). \tag{1.64}$$

We distinguish (1.64) from the minors in the Gaudin model $L^{a_1\ldots a_m}_{b_1\ldots b_m}$ in (1.24) which do not contain these shifts in the spectral parameter by multiples of $\hbar$ through the use of the square brackets.

The commuting Hamiltonians of the model are generated from some of these minors [21],

$$t_k(z) = \sum_{\substack{i_1,i_2,\ldots,i_k=1 \\ i_1<i_2<\cdots<i_k}} T\left[\begin{smallmatrix} i_1\ldots i_k \\ i_1\ldots i_k \end{smallmatrix}\right]. \tag{1.65}$$

We will mainly be concerned with the $\mathfrak{gl}_3$ XXX-chain and its relation to the $\mathfrak{gl}_3$ Gaudin model, in which case the generating functions are

$$t_1(z) = \operatorname{Tr} T(z) = T_1^1(z) + T_2^2(z) + T_3^3(z), \tag{1.66a}$$

$$t_2(z) = T\left[\begin{smallmatrix} 12 \\ 12 \end{smallmatrix}\right](z) + T\left[\begin{smallmatrix} 13 \\ 13 \end{smallmatrix}\right](z) + T\left[\begin{smallmatrix} 23 \\ 23 \end{smallmatrix}\right](z) \tag{1.66b}$$

$$d(z) = T\left[\begin{smallmatrix} 12\ldots n \\ 12\ldots n \end{smallmatrix}\right](z), \tag{1.66c}$$

where (1.66c) is known as the quantum determinant.

Following from (1.63), commutation relations of elements in the monodromy $T(z)$ are given by

$$(z-w)[T_b^a(z), T_d^c(z)] = \hbar \left(T_b^c(z)T_d^a(w) - T_b^c(z)T_d^a(w)\right). \tag{1.67}$$

The $T_j^i(z)$ are therefore a realisation of the *Yangian* $Y(\mathfrak{gl}_\mathfrak{n})$, an associative algebra with generators $T_j^{i(k)}$ for $i,j = 1,\ldots,n$ and $k \in \mathbb{Z}_{\geq 1}$, see for instance [64, 59]. The elements of $T(z)$ are polynomial in these generators

$$T_j^i(z) = \delta_{ij} + \sum_k T_j^{i(k)} \hbar^k z^{-k}. \tag{1.68}$$

The centre of the Yangian is generated by the quantum determinant $d(z) = t_n(z)$.

1.3.1.1 The Gaudin limit

For this thesis, our interest in the XXX chain is mainly in how it relates to the Gaudin model. As we have already noted, the Gaudin Lax matrix appears at first order in $\hbar$ of the quantum R-matrix,

$$R(z,w) = 1 + \hbar r(z,w). \tag{1.69}$$

Taking the semi-classical limit $\hbar \to 0$ we can recover the quantum Gaudin model. Similarly, the Lax matrix (1.62) of the XXX chain resembles the Lax matrix of a single-site Gaudin model, but if we expand the monodromy matrix $T(z)$ in this small $\hbar$ limit

$$T(z) = 1 + \hbar \sum_{i=1}^N \frac{\sum_{a,b=1}^N E_b^a \otimes E_a^{b(i)}}{z - z_i} + O(\hbar^2) = 1 + \hbar L(z) + O(\hbar^2), \tag{1.70}$$

where $L(z)$ is the Gaudin Lax matrix for N sites. Furthermore, the linear Lax algebra of the quantum Gaudin model arises in the $\hbar \to 0$ limit of (1.63).

In Chapter 3, we will use this limit to move from the known result of Separation of Variables for models in the Yangian $Y(\mathfrak{sl}_3)$ [59, 21], to a corresponding result for the $\mathfrak{sl}_3$-Gaudin model as conducted by Ribault in [52].

Part I

Separation of Variables

— 2 —

Separation of variables for $\mathfrak{sl}_2$-Gaudin models

Separation of variables (SoV) can be most broadly described as reducing a multidimensional problem to a set of one-dimensional problems. This includes both in the widely used sense of transforming differential equations, and in the sense of the method of solving integrable systems developed by Evgeny Sklyanin in the 1980s [57, 59, 60, 61], and originally styled as the "functional Bethe Ansatz" - the latter of these is what we are concerned with in this thesis.

If we realise the Lie algebra underlying a given model in terms of differential operators, the equations of motion will be differential equations in all the variables of the representation. The relation to separation of variables in the former sense becomes clearer; we can transform to new coordinates in which the equations entirely decouple to solve the model. Although this is the historical origin of SoV, it is generally enough define some variables $\{q_j\}_{j=1}^{\mathcal{D}}$ such that the join eigenfunctions may be factorised with each factor depending on only one variable q_j without the explicit change of variables from some initial differential operator realisation.

In other words, for an N-dimensional quantum integrable system with integrals of motion given by $\{\widehat{H}_i\}_{i=1}^{N}$, we require a set of variables $\{q_i\}_{i=1}^{\mathcal{D}}$ and corresponding coordinates $\{p_j\}_{j=1}^{\mathcal{D}}$, which are conjugate

$$[q_i, p_j] = \delta_{ij},$$

and $\mathcal{D}$ equations f_j such that

$$f_j(q_j, p_j, \widehat{H}_1, \dots, \widehat{H}_N) = 0,$$

with a fixed ordering of the operators q_j, p_j and $\widehat{H}_i$ if we are considering a quantum model. If we then apply the f_j to a common eigenvector of the $\widehat{H}_i$, we find that

$$f_j(q_j, p_j, H_1, \ldots, H_N)\Psi = 0, \quad j = 1, \ldots, N,$$

where H_i is the eigenvalue of $\widehat{H}_i$ on Ψ for $i = 1, \ldots, N$. Thus Ψ must factorise into some functions $\psi_j(q_j)$ which each depend on only one of our variables q_j, each determined by the corresponding one-dimensional function f_j.

Advantages of the separation of variables technique in comparison to others, such as the Bethe Ansatz, include that it does not require us to assume we have some highest weight reference vector, and therefore applies to a broader class of representations. The greatest benefit to SoV for the Gaudin model in particular is perhaps that the Bethe Ansatz has been shown to not provide a complete set of eigenvectors beyond rank 2 [47], whereas solving via separation of variables is believed to be complete due to the correspondence between the separated equations and the corresponding $\mathfrak{g}$-opers - which have been shown to provide a complete solution [29] - that we will define for the $\mathfrak{sl}_2$ case later.

In the earlier days of SoV, it was suggested that that it could provide an alternative definition of quantum integrability [60, 59] , which was more practically verifiable and precisely defined. The link between classical Lax integrability and SoV has since been explored further, see [62] for Lax matrices of linear type, but this is no longer viewed as a key motivation for SoV as the understanding of quantum integrability and its relation to solvability has since improved.

In the review article New Trends [60], Sklyanin outlines a "magic recipe" used to construct a separation of variables for a range of classical integrable models. Classically, we expect the separated equation to arise from the characteristic polynomial of the Lax matrix at some value of the spectral parameter

$$\det\left(\lambda(z) - L(z)\right), \tag{2.1}$$

where $\lambda(z)$ is the corresponding eigenvalue. The separated variables $\{q_j\}_{j=1}^{\mathcal{D}}$ then arise at the poles of the the Baker-Akhiezer function $\Omega(z)$ (the eigenvalue of the Lax matrix $L(z)$) having fixed a suitable normalisation, which is not necessarily straightforward. The conjugate variables p_j are then taken to be the eigenvalue $\lambda(q_j)$ evaluated at that point. Therefore, as both lie on the spectral curve, (2.1) provides the required separated equations

$$\det\left(p_j - L(q_j)\right) = 0. \tag{2.2}$$

The difficulty then is ensuring that we have the correct number of variables and that the transformation to these variables is canonical. To achieve this, we take the separated variables to be the zeroes of some function $B(z)$, often known as the Sklyanin B-operator, which is made up of such a combination of Lax matrix elements that the order guarantees that we have the necessary number of variables q_j. For the variables to be canonical they must mutually Poisson commute, and so we require that

$$\{B(z), B(w)\} = 0, \tag{2.3}$$

for all spectral parameters $z, w \in \mathbb{C}$. We can also express p_j in terms of some other rational function of Lax matrix elements, denoted $A(z)$, and hence all properties required for SoV may be derived from the Lax algebra. For matrix models of rank n, we will need the separating function $B(z)$ to be of order $n(n-1)/2$ in the elements of the Lax matrix. In the rank 2 case this is straightforward and we simply have single Lax matrix elements for both $A(z)$ and $B(z)$, but the expressions become more cumbersome at higher rank, with inverses of Lax matrix components appearing in $A(z)$ in particular. Sklyanin includes the form of $A(z)$ and $B(z)$ for a number of models, including the classical $\mathfrak{sl}_2$ and $\mathfrak{gl}_3$ models with linear Lax relations, the $\mathfrak{gl}_3$ XXX-chain, and the classical XYZ magnet.

In the quantum case we can follow a very similar method, now with operator-valued functions $A(z)$ and $B(z)$, with the separated variables appearing as "operator zeroes" of $B(z)$. We must also be specific about how we are substituting these operator zeroes into $A(z)$, and we will follow the convention of substitution from the left in this thesis. Where classically the magic recipe provides the separated equations from the spectral curve, in the quantum case we conjecture that this becomes

$$\operatorname{cdet}\left(\partial_{q_i} + L(q_i)\right),$$

where we specify a column-ordered determinant as we have non-commuting Lax matrix elements, and $L(q_i)$ does not commute with ∂_{q_i}. It makes sense here to identify the operator p_j with the differential operator ∂_{q_i} as the coordinates are conjugate, which leaves us with a differential equation in a single separated variable q_i. Interestingly, this quantisation of the spectral curve bears a strong resemblance to the universal oper, despite the relatively straightforward method. We will briefly discuss this link further at the end of the chapter.

Otherwise, Sklyanin has constructed an SoV for a variety of integrable systems, including the quantum $\mathfrak{sl}_2$-Gaudin model [58], the quantum Toda chain [61], the Yangian $Y(\mathfrak{sl}_3)$ [59] as well as the classical $\mathfrak{sl}_3$ XXX-chain [57].

Some of the more recent interest in SoV for models with quadratic Lax relations stems from its potential to inform us about string theories - see, for example, [21, 8, 9, 34, 54, 53]. One can infer about such complicated theories by studying a related quantum integrable spin chain, hence the interest in separation of variables in spin-chains related to the Yangian $Y(\mathfrak{gl}_n)$. As well as SoV theoretically being applicable to any representation, if one takes a separation of variables basis of the spin chain (that is, a basis in which the eigenstates factorise into functions of a single variable), one finds simpler expressions for the correlation functions in terms of Baxter Q-functions, which are solutions to the Baxter equations.

For example, in [34] the authors extend the known SoV to the $Y(SU(n))$ case by twisting the usual Sklyanin operator $B_{XXX}(z)$ of the "magic recipe" by a similarity transform to a diagonalisable version, called $B^{\text{good}}_{XXX}(z)$ that cruicially still provides separated variables at the roots. They suggest an extension to the higher rank $\mathfrak{sl}_n$ case, which has been checked up to $N = 4$ sites. Having established a separated basis they go on in [8] to construct a corresponding measure, and hence take scalar products in this basis, focusing on the fundamental representation at each site. In the following chapter we will discuss an analogous method to generalise the SoV of the Gaudin model to arbitrary rank, and its limitations.

In [53] and [54], Ryan and Volin consider the XXX chain with finite dimensional irreducible representations of $\mathfrak{gl}_n$ at the sites. They form particular left eigenstates of the separated variables which diagonalise B from related eigenvectors of a Gelfand-Tsetlin subalegbra of the Yangian and show that these eigenstates factorise into product of Q-functions from Baxter equations.

In [44], Maillet and Niccoli construct the SoV basis without Sklyanin's separating function $B(z)$. Instead they take the action of combinations of the transfer matrix on some generic covector at particular values of the spectral parameter, which depend on the representation of the symmetry algebra. This eliminates the difficulty of finding expressions for $B(z)$ at higher rank, and has been applied to $Y(\mathfrak{gl}_n)$. Though these results provide a lot of insight, when we construct SoV the Gaudin model in this chapter and the next we will align more with Sklyanin's magic recipe, using the separating functions $A(z)$ and

$B(z)$.

In this part of thesis we are primarily concerned with work towards the extension of separation of variables in quantum $\mathfrak{gl}_n$ Gaudin models to both arbitrary rank and arbitrary strengths of poles. So far, an SoV has been constructed for the $\mathfrak{sl}_2$-Gaudin model [58], the XXX chain [21] and models with Yangian $Y(\mathfrak{sl}_3)$ symmetry more generally [59], and by taking the limit of this Ribault provides an SoV for the corresponding $\mathfrak{sl}_3$ Gaudin model in [52]. Furthermore we have an SoV for the *classical* $\mathfrak{gl}_n$-Gaudin model at any rank [23]. This chapter specifically recalls Sklyanin's separation of variables for the $\mathfrak{sl}_2$-Gaudin model with simple poles and extends this to cover Gaudin models with irregular singularities.

2.1 SoV for the $\mathfrak{sl}_2$-Gaudin model with irregular singularities

Recall the description of the $\mathfrak{gl}_n$-Gaudin model with irregular singularities detailed in Chapter 1, for $\bigoplus_{i=1}^N \mathfrak{sl}_2[\varepsilon_i]/\varepsilon_i^{\tau_i}$ we have basis elements $\{e^{(i)}_{[r]} | r = 0,\dots,\tau_i - 1\}_{i=1}^N$. Since we follow the simple poles case [58] very closely, we can check that the expressions correspond at each step by taking the particular case where the Takiff degree $\tau_i = 1$ on each site. There will be some small discrepancies because we also include the double pole at infinity - this both removes the need to perform the permutation of the Pauli matrices as in section 3 of the article, and better aligns with the following chapter in which we will turn to the problem of separation of variables for the $\mathfrak{gl}_3$-Gaudin model.

To explicitly compute a change of variables, we will need to realise the Takiff Lie algebra at each site $\mathfrak{sl}_2[\epsilon_i]/\epsilon_i^{\tau_i}$ as differential operators. The details of how we can construct such a realisation may be found in appendix A, with the realisation ultimately being given by (A.6a),

$$\pi(f^{(i)}_{[r]}) = x_{i,[r]}$$
$$\pi(h^{(i)}_{[r]}) = \sum_{s=0}^{\tau_i-1-r} -2x_{i,[s+r]}\partial_{i,[s]} + 2\ell_{i,[r]}.$$
$$\pi(e^{(i)}_{[r]}) = \sum_{\substack{s,t=0\\ s+t+r<\tau_i}}^{\tau_i-1} -x_{i,[r+s+t]}\partial_{i,[s]}\partial_{i,[t]} + \sum_{s=0}^{\tau_i-1-r} 2\ell_{i,[r+s]}\partial_{i,[s]},$$

where $\ell_{i,[r]}$ are the weights associated to the specific representation. Setting Takiff degrees to one, we recover a standard differential operator realisation of

$\mathfrak{sl}_2^{(i)}$

$$\pi(e^{(i)}) = -x_i\partial_{x_i}^2 + 2\ell_i\partial_{x_i}, \quad \pi(f^{(i)}) = x_i, \quad \pi(h^{(i)}) = -x_i\partial_{x_i} + \ell_i,$$

where we are identifying $x_{i,[0]}$ with x_i.

In this realisation (A.6a), we see that the problem of finding eigenvalues and eigenvectors of $\widehat{s}_1(z)$ becomes a rather complicated differential equation in all $\mathcal{D} = \sum_{i=1}^N \tau_i$ variables;

$$\begin{aligned}\widehat{s}_1(z)\Psi = \sum_{i,j=1}^{N}\sum_{r=0}^{\tau_i-1}\sum_{q=0}^{\tau_j-1}\frac{1}{(z-z_i)^{\tau_i}(z-z_j)^{\tau_j}}\Bigg(&x_{i,[r]}\sum_{\substack{s,t=0\\ s+t+q<\tau}}^{\tau_j-1} -x_{j,[q+s+t]}\partial_{j,[s]}\partial_{j,[t]}\\
&+x_{i,[r]}\sum_{s=0}^{\tau_j-1-q}2\ell_{j,[q+s]}\partial_{j,[s]}\\
&+\sum_{\substack{s,t=0\\ s+t+r<\tau_i}}^{\tau_i-1} -x_{i,[r+s+t]}\partial_{i,[s]}\partial_{i,[t]}x_{j,[s]} + \sum_{s=0}^{\tau_i-1-r}2\ell_{i,[r+s]}\partial_{i,[s]}x_{j,[s]}\\
&+\frac{1}{2}\sum_{s=0}^{\tau_i-1-r}\sum_{t=0}^{\tau_j-1-r}\left(-2x_{i,[s+r]}\partial_{i,[s]}+2\ell_{i,[r]}\right)\left(-2x_{j,[t+q]}\partial_{j,[t]}+2\ell_{j,[q]}\right)\Bigg)\Psi\\
=s_1(z)\Psi,&\end{aligned} \tag{2.4}$$

where $s_1(z)$ is the eigenvalue of $\widehat{s}_1(z)$ on Ψ. The existence of a factorised form of Ψ is quite unclear from (2.4), so we will construct a change of variables where the wavefunction Ψ naturally factorises.

2.1.1 Constructing Separated Variables

We label the four elements of the Lax matrix as operator valued functions of the complex spectral parameter z

$$L(z) = \begin{pmatrix} A(z) & B(z)\\ C(z) & D(z)\end{pmatrix}, \tag{2.5}$$

where $A(z)$ and $B(z)$ correspond to the separating functions of the magic recipe. As we discussed for the classical case of the magic recipe, the zeroes of the operator $B(z) = L^1_2(z)$ will provide the new, separated variables. $B(z)$ is a rational function of z with N poles at the points $z = z_i$, each of strength τ_i respectively. From the Lax matrix relations (1.16), we see that it trivially commutes with itself at different values of the spectral parameter,

$$[B(z), B(w)] = \left[L^1_2(z), L^1_2(w)\right] = 0.$$

It is also interesting to note that in the realisation (A.6a), $B(z)$ is written in terms of the variables $x_{i,[r]}$ only, with no $\partial_{x_{i,[r]}}$ operators. There is also a constant term arising form the realisation of $f^{(\infty)} \in \mathfrak{sl}_2^{\text{comm}}$, that is, in our initial realisation we have

$$B(z) = 1 + \sum_{j=1}^{N} \sum_{r=0}^{\tau_j - 1} \frac{x_{i,[r]}}{(z - z_i)^{r+1}}.$$

If we return to the simple poles case, identifying $x_{i,[0]}$ with x_i, this matches Sklyanin's expression;

$$B(z) = 1 + \sum_{j=1}^{N} \frac{x_i}{z - z_i}.$$

It is clear that as $z \to \infty$ we have at leading order

$$B(z) = 1 + \mathcal{O}(z^{-1}).$$

From this we surmise that there must be $\mathcal{D}$ "zeroes" of $B(z)$ to exactly cancel the factors of z in the poles - i.e. if we return to the simple poles case by setting all Takiff degrees $\tau_i = 1$ we have the N variables as described by Sklyanin. By zeroes we are here referring to operators q_j such that

$$B(q_j) = 0, \quad j = 1, \dots, \mathcal{D}. \tag{2.6}$$

We have already noted that $B(z)$ commutes with itself for different values of the spectral parameter and it follows that the new variables must also pairwise commute, leaving a factorised form of $B(z)$

$$B(z) = \frac{(z - q_1) \dots (z - q_{\mathcal{D}})}{(z - z_1)^{\tau_1} \dots (z - z_N)^{\tau_N}}. \tag{2.7}$$

Since $[q_i, q_j] = 0$, there is a realisation of $\mathfrak{sl}_2$ such that $B(z)$ is a multiplication operator of the form (2.7), and the q_i are variables in the same sense as the representation variables $x_{i,[r]}$, and act as multiplication operators on $\mathbb{C}[\{q_i\}_{i=1}^{\mathcal{D}}]$. Note that the variables are non-unique, and one can for example swap the roles of $A(z)$ and $B(z)$.

To describe the new q_j more explicitly, we can find the change of variables from our original representation variables $x_{i,[r]}$. We equate the expressions for $B(z)$ in terms of the two sets of variables $\{x_{i,[r]} | r = 0, \dots, \tau_i - 1\}_{i=1}^{N}$ and $\{q_j\}_{j=1}^{\mathcal{D}}$

$$B(z) = 1 + \sum_{j=1}^{N} \sum_{r=0}^{\tau_i - 1} \frac{x_{i,[r]}}{(z - z_i)^{r+1}} = \frac{(z - q_1) \dots (z - q_{\mathcal{D}})}{(z - z_1)^{\tau_1} \dots (z - z_N)^{\tau_N}}.$$

Returning to the simple poles version for a moment, for any $i = 1, \dots, N$ we find x_i as the residue at site $z = z_i$ of $B(z)$, and by taking the residue of our two expressions for $B(z)$ we obtain the change of variables explicitly as

$$x_i = \operatorname{Res}_{z_i} B(z) = \frac{\prod_{j=1}^{N}(z_i - q_j)}{\prod_{\substack{k=1\\k\neq j}}(z_i - z_k)}. \tag{2.8}$$

To extend this to higher order poles, we simply take higher order residues for poles of strength $r = 0, \dots, \tau_j - 1$ at each site z_j, giving an expression for $x_{i,[r]}$ in terms of the new variables q_i

$$x_{i,[\tau_i-1-r]} = \frac{1}{r!}\lim_{z\to z_i}\left(\frac{d^r}{dz^r}\frac{\prod_{k=1}^{\mathcal{D}}(z-q_k)}{\prod_{\substack{j=1\\j\neq i}}^{N}(z-z_j)^{\tau_j}}\right). \tag{2.9}$$

The double pole at infinity allows us to treat the variables q_j entirely equivalently.

For a more interesting example, let us set the Takiff degree $\tau_i = 2$ at some site i. We then have two representation variables $x_{i,[0]}$ and $x_{i,[1]}$ and can write them in terms of q_i accordingly;

$$x_{i,[0]} = \frac{\sum_{m=1}^{\mathcal{D}}\prod_{k\neq m}(z_i - q_k)}{\prod_{j\neq i}(z_i - z_j)^{\tau_j}} - \sum_{l\neq i}\frac{\prod_{k=1}^{\mathcal{D}}(z_i - q_k)}{(z_i - z_l)\prod_{j\neq i}(z_i - z_j)^{\tau_j}} \tag{2.10a}$$

$$x_{i,[1]} = \frac{\prod_{k=1}^{\mathcal{D}}(z_i - q_k)}{\prod_{j\neq i}^{N}(z_i - z_j)^{\tau_j}}. \tag{2.10b}$$

The change of variables (2.9) induces an isomorphism of rings between polynomials in $\mathbb{C}[\{x_{i,[r]}\}]$ and the ring of polynomials that are symmetric in the new variables $\{q_i\}_{i=1}^{\mathcal{D}}$, which we will denote $S[\{q_i\}_{i=1}^{\mathcal{D}}]$. It is clear immediately from (2.9) that each variable $x_{i,[r]}$ in our original representation is contained within $S[\{q_i\}_{i=1}^{\mathcal{D}}]$, as the right-hand side is symmetric in the new variables $\{q_i\}_{i=1}^{\mathcal{D}}$. To show the reverse, consider the elementary symmetric polynomials in the new variables, $\sigma_0 = 1$ and $\sigma_m = \sum_{\substack{i_1,\dots,i_m=1\\ i_1<i_2<\dots<i_m}}^{\mathcal{D}} q_{i_1}\dots q_{i_m}$ for $m = 1, \dots \mathcal{D}$. We can expand the product in the numerator of our change of variables to be

$$\prod_{k=1}^{\mathcal{D}}(z - q_k) = \sum_{m=0}^{\mathcal{D}}(-1)^m z^{\mathcal{D}-m}\sigma_m,$$

hence overall we have $x_{i,[r]} = \sum_{m=0}^{\mathcal{D}} C_m\sigma_m$ where C_m are complex coefficients formed of the sites $z_i \in \mathbb{C}$, since we have taken the limit $z \to z_i$. Therefore considering all possible values of $i = 1, \dots, N$ and subsequent $r = 0, \dots, \tau_i$, we have $\mathcal{D}$ equations in the independent σ_m and may therefore rewrite these as a linear combination of the $x_{i,[r]}$ - hence it is also true that $S[\{q_i\}_{i=1}^{\mathcal{D}}] \subseteq \mathbb{C}[\{x_{i,[r]}|r = 0, \dots, \tau_i - 1\}_{i=1}^{\mathcal{D}}]$ and the two are isomorphic.

In the simple poles case, we can quotient $\mathbb{C}[x_1, \dots, x_N]$ by the ideal $I = (x_1^{2\ell_1+1}, \dots, x_N^{2\ell_N+1})$ to obtain a finite dimensional module of $\mathfrak{sl}_2^{\otimes N}$. Sklyanin demonstrates that this ideal is isomorphic to an ideal $J = (\{\prod_{j=1}^{N}(z_i - q_j)^{2\ell_i+1}\}_{i=1}^{N})$ of $S[\{q_i\}_{i=1}^{N}]$, by substituting (2.8) into I to obtain J. Therefore the quotient rings are also isomorphic,

$$\mathbb{C}[x_1, \dots, x_N]/I \cong S[\{q_i\}_{i=1}^{N}]/J. \tag{2.11}$$

Hence we can move from our original realisation of the Gaudin spin chain to this isomorphic realisation in the separated variables $\{q_i\}_{i=1}^{\mathcal{D}}$ via (2.9).

Representations of Takiff Lie algebras are far less studied, (though some simple ones are covered in [24]) and it is not clear what the equivalent of I and J would be in general. We can write down such an ideal for the most straightforward case of $\mathfrak{sl}_2[\varepsilon]/\varepsilon^2$, if we set one of the weights $\ell_{[1]} = 0$. We focus on a single site where we have six basis elements realised as

$$\begin{aligned}
\pi(f_{[0]}) &= x_{[0]}\\
\pi(f_{[1]}) &= x_{[1]}\\
\pi(h_{[0]}) &= -2x_{[0]}\partial_{[0]} + 2\ell_{[0]}\\
\pi(h_{[1]}) &= -2x_{[1]}\partial_{[0]}\\
\pi(e_{[0]}) &= -x_{[0]}\partial_{[0]}^2 - 2x_{[1]}\partial_{[0]}\partial_{[1]} + 2\ell_{[0]}\partial_{[0]}\\
\pi(e_{[1]}) &= -x_{[1]}\partial_{[0]}^2.
\end{aligned}$$

An ideal preserved by the action of $\mathfrak{sl}_2[\varepsilon]/\epsilon^2$ would then be

$$I = x_{[0]}^{2\ell_{[0]}+1}\mathbb{C}[x_{[0]}] + x_{[1]}\mathbb{C}[x_{[0]}, x_{[1]}]. \tag{2.12}$$

Let us briefly sketch the reasoning for this. In the realisation, the basis elements of mode 1, $\pi(f_{[1]})$, $\pi(h_{[1]})$ and $\pi(e_{[1]})$, will multiply by $x_{[1]}$ which moves everything in the ideal (2.12) into the $x_{[1]}\mathbb{C}[x_{[0]}, x_{[1]}]$ portion, and hence preserves it. Similarly, the action of $\pi(f_{[0]})$ and $\pi(h_{[0]})$ are trivially absorbed into $\mathbb{C}[x_{[0]}]$. This leaves us to consider

$$\begin{aligned}
&\pi(e_{[0]})\left(x_{[0]}^{2\ell_{[0]}+1}\mathbb{C}[x_{[0]}] + x_{[1]}\mathbb{C}[x_{[0]}]\right)\\
&\qquad = -2\ell_{[0]}(2\ell_{[0]}+1)x_{[0]}^{2\ell_{[0]}}\mathbb{C}[x_{[0]}] - 2x_{[1]}\mathbb{C}[x_{[0]}] + 2\ell_{[0]}(2\ell_{[0]}+1)x_{[0]}^{2\ell_{[0]}}\mathbb{C}[x_0],\\
&\qquad = x_{[1]}\mathbb{C}[x_{[0]}],
\end{aligned}$$

where we have absorbed factors back into $\mathbb{C}[x_{[0]}]$. To find the corresponding ideal of $S[\{q_l\}_{l=1}^{2N}]$ in the separated variables, we would need an ideal at each site, for simplicity let us say that $\tau_1 = 2$ with $\ell_{1,[1]} = 0$ as above and $\tau_i = 1$ for $i = 2, \ldots, N$. Therefore the ideal of $\mathbb{C}[x_{1,[0]}, x_{1,[1]}, x_{2,[0]}, \ldots, x_{N,[0]}]$ is generated by

$$\left(x_{1,[0]}^{2\ell_{1,[0]}+1}\mathbb{C}[x_{1,[0]}] + x_{1,[1]}\mathbb{C}[x_{1,[0]}], x_{2,[0]}^{2\ell_{2,[0]}+1}\mathbb{C}[x_{2,[0]}], \ldots, x_{N,[0]}^{2\ell_{N,[0]}+1}\mathbb{C}[x_{N,[0]}]\right).$$

We would then substitute in (2.8) and (2.10) to find the corresponding ideal of $S[\{q_i\}_{i=1}^{N+1}]$. However we will not include this expression here as it is quite clunky, even in this most simple Takiff Lie algebra.

Unfortunately the quotient $\mathbb{C}[x_{[0]}, x_{[1]}]/I$ is trivial in that it is isomorphic to the simple poles case anyhow, since in the quotient we remove any terms involving $x_{[1]}$. For a non-trivial ideal we could take for example

$$K = x_{[1]}^2\mathbb{C}[x_{[0]}, x_{[1]}], \tag{2.13}$$

which would give the quotient

$$\mathbb{C}[x_{[0]}, x_{[1]}]/K = \mathbb{C}[x_{[0]}] + x_{[1]}\mathbb{C}[x_{[0]}]. \tag{2.14}$$

Since $\mathbb{C}[x_{[0]}]$ is a module of $\mathfrak{sl}_2$, acting with the mode zero elements of $\mathfrak{sl}_2[\varepsilon]/\varepsilon^2$ keeps us in the same term, whereas acting with the mode one elements moves us left as we multiply by $x_{[1]}$. This could be extended by taking an arbitrary power of $x_{[1]}$ in K, where mode r Lie algebra elements would move us r terms to the left. The quotient (2.14) is infinite dimensional so the weight $\ell_{[0]}$ has no bearing in this case. It would be interesting to use this idea of combining $\mathfrak{sl}_2$-modules into a Takiff module with a finite dimensional module in each term, though the exact form of the required ideal is unclear.

2.1.2 Separating the Variables

Keeping in mind that we ultimately are looking for the joint spectrum and common eigenvectors of the Gaudin Hamiltonians, we now describe their generating function $\widehat{s}_1(z)$ in terms of these new variables. If they have been suitably chosen, then the eigenvalue equations (realised as differential operators) will decouple in the new variables, leaving a set of $\mathcal{D}$ differential equations each dependent on only one of the q_i.

Let us write $\widehat{s}_1(z)$ in terms of our separating functions,

$$\widehat{s}_1(z) = \frac{1}{2}\operatorname{Tr}(L(z))^2 = \frac{1}{2}\big(A^2(z) + D^2(z) + B(z)C(z) + C(z)B(z)\big), \tag{2.15}$$

with $A(z), B(z), C(z), D(z)$ as in equation (2.5). We can rearrange this exactly as Sklyanin does [58], since the Lax matrix with higher order singularities has the same r-matrix as the Lax matrix with simple poles. Using the tracelessness of $L(z)$, we eliminate $D(z)$

$$\widehat{s}_1(z) = \frac{1}{2}\big(2A^2(z) + B(z)C(z) + C(z)B(z)\big). \tag{2.16}$$

Note that by the Lax algebra relations (1.16) we have

$$[C(z), B(z)] = \left[L_1^2(z), L_2^1(z)\right] = L_2^{2\prime}(z) - L_1^{1\prime}(z) = -2A'(z) \tag{2.17}$$

where in the last step we have again used that $\operatorname{Tr} L(z) = 0$ in $\mathfrak{sl}_2$. Altogether, we have that

$$\widehat{s}_1(z) = A^2(z) - A'(z) + B(z)C(z). \tag{2.18}$$

Matching with general convention, we specify below and throughout this thesis that the substitution into a differential operator $Y(z)$ is *from the left*, meaning that we take the function of z to the left of the differential operators before substitution. E.g., for an operator $Y(z)$ in terms of differential operators

$$Y(z) = Y_0(z) + \sum_{i=1}^{\mathcal{D}} Y_i(z, \{q_i\}_{i=1}^{\mathcal{D}})\partial_{q_i}, \tag{2.19}$$

with functions $Y_i(z, \{q_i\}_{i=1}^N)$ of z and $\{q_i\}_{i=1}^{\mathcal{D}}$, by substitution $z \to q_j$ we mean

$$Y(z)|_{z=q_j} = Y_0(q_j) + \sum_{i=1}^{\mathcal{D}} Y_i(q_j, \{q_i\}_{i=1}^N)\partial_{q_i}. \tag{2.20}$$

This generalises to higher order differential operators in the obvious way.

To separate the eigenvalue equations of $\widehat{s}_1(z)$ we will need to carefully substitute $z \to q_i$ into it - i.e. from the left. Since $B(z)$ is the part furthest to the left and by definition of our coordinates $B(z)|_{z\to q_i} = 0$, we are left with

$$\widehat{s}_1(z)|_{z\to q_i} = \big((A(z))^2 - A'(z)\big)\Big|_{z\to q_i}.$$

Therefore to proceed we consider what happens when we substitute $z \to q_i$ into $A(z) = L_1^1(z)$ from the left. In the Lie algebra elements $A(z)$ has the form

$$A(z) = L_1^1(z) = \lambda + \sum_{l=1}^{N}\sum_{r=0}^{\tau_l-1} \frac{h_{[r]}^{(l)}}{2(z - z_l)^{r+1}}. \tag{2.21}$$

We will see that substituting any of the variables q_i defined in (2.7) into (2.21) from the left gives

$$A(z)|_{z\to q_i} = \Lambda(q_i) - \partial_{q_i} \tag{2.22}$$

for a scalar function

$$\Lambda(z) = \lambda + \sum_{j=1}^{N}\sum_{r=0}^{\tau_j-1} \frac{\ell_{j,[r]}}{(z-z_j)^{r+1}}. \tag{2.23}$$

We can show this in two separate ways. On the one hand, it follows directly from the Lax algebra relations, however this method gives no information on the scalar function $\Lambda(z)$, nor the specific form of $A(z)$ in this realisation. So we will also show this explicitly using the change of variables (2.9), as in the case for the $\mathfrak{sl}_2$-Gaudin model with simple poles.

Firstly, since $A(z) = L^1_1(z)$ and $B(z) = L^1_2(z)$ it follows from the Lax algebra that

$$[A(z), B(w)] = \frac{B(z) - B(w)}{z-w}. \tag{2.24}$$

Then, as we have noted that the variables q_j mutually commute, we may write $B(w)$ as $(w-q_j)\mathcal{B}(w)$ with $\mathcal{B}(q_j) \neq 0$ since the zeroes are distinct as operators, though their eigenvalues may coincide on a specific state. Therefore when we substitute q_j into $A(z)$ from the left, (2.24) becomes

$$(w-q_j)[A(z)|_{z\to q_i}, \mathcal{B}(w)] - [A(z)|_{z\to q_i}, q_j]\mathcal{B}(w) = \frac{(w-q_j)\mathcal{B}(w)}{w-q_i}.$$

Upon setting $w = q_j$ the left-hand side becomes

$$-[A(z)|_{z\to q_i}, q_j]\mathcal{B}(q_j),$$

and the right will depend on whether $i = j$ or not. If they are distinct, then the right hand side is immediately zero, whereas if they are the same, we have

$$-[A(z)|_{z\to q_i}, q_i]\mathcal{B}(q_i) = \mathcal{B}(q_i). \tag{2.25}$$

Hence it is clear that

$$[A(z)|_{z\to q_i}, q_j] = -\delta_{ij} \tag{2.26}$$

where δ_{ij} is the Kronecker delta. This tells us that $A(z)|_{z\to q_i}$ must be of the form $\Lambda(z) - \partial_{q_i}$ for some scalar function $\Lambda(z)$.

More directly, we look at the $A(z)$ in the realisation (A.6a);

$$A(z) = \lambda + \sum_{j=1}^{N} \sum_{r=0}^{\tau_j-1} \sum_{s=0}^{\tau_j-r-1} \frac{\ell_{j,[r]} - x_{j,[r+s]}\partial_{x_{j,[s]}}}{(z-z_j)^{r+1}}. \tag{2.27}$$

We substitute $z \to q_i$ from the left and separate the constant term and the levels into two terms

$$A(z)|_{z\to q_i} = \left(\lambda + \sum_{j=1}^{N} \sum_{r=0}^{\tau_j-1} \frac{\ell_{j,[r]}}{(z-z_j)^{r+1}}\right)\Bigg|_{z\to q_i} - \left(\sum_{j=1}^{N} \sum_{r=0}^{\tau_j-1} \sum_{s=0}^{\tau_j-r-1} \frac{x_{j,[r+s]}}{(z-z_j)^{r+1}} \frac{\partial}{\partial x_{j,[s]}}\right)\Bigg|_{z\to q_i}.$$

The first term here is a scalar function, here evaluated at $z \to q_i$ which we label $\Lambda(z)$.

To rewrite this in terms of the $\{q_j\}_{j=1}^N$, we firstly use the change of variables (2.9) we derived previously to calculate $\frac{\partial x_{l,[\tau_l-1-r]}}{\partial q_i}$;

$$\begin{aligned}\frac{\partial x_{l,[\tau_l-1-r]}}{\partial q_i} &= \frac{1}{r!} \frac{\partial}{\partial q_i} \lim_{z\to z_l} \left(\frac{\mathrm{d}^r}{\mathrm{d}z^r} \frac{\prod_{k=1}^{\mathcal{D}}(z-q_k)}{\prod_{\substack{j=1\\ j\neq l}}^{N}(z-z_j)^{\tau_j}} \right) \\ &= \frac{1}{r!} \lim_{z\to z_l} \left(\frac{\mathrm{d}^r}{\mathrm{d}z^r} \frac{1}{(q_i - z)} \frac{\prod_{k=1}^{\mathcal{D}}(z-q_k)}{\prod_{\substack{j=1\\ j\neq l}}^{N}(z-z_j)^{\tau_j}} \right).\end{aligned}$$

By repeated application of the product rule this becomes

$$\begin{aligned}\frac{\partial x_{l,[\tau_l-1-r]}}{\partial q_i} &= \frac{1}{r!} \lim_{z\to z_l} \Big(\sum_{s=0}^{r} \binom{r}{s} \frac{\mathrm{d}^s}{\mathrm{d}z^s} \frac{1}{q_i - z} \frac{\mathrm{d}^{r-s}}{\mathrm{d}z^{r-s}} \frac{\prod_{k=1}^{\mathcal{D}}(z-q_k)}{\prod_{\substack{j=1\\ j\neq l}}^{N}(z-z_j)^{\tau_j}} \Big) \\ &= \frac{1}{r!} \lim_{z\to z_l} \left(\sum_{s=0}^{r} \binom{r}{s} \frac{s!}{(q_i - z)^{s+1}} \frac{\mathrm{d}^{r-s}}{\mathrm{d}z^{r-s}} \frac{\prod_{k=1}^{\mathcal{D}}(z-q_k)}{\prod_{\substack{j=1\\ j\neq l}}^{N}(z-z_j)^{\tau_j}} \right) \\ &= \sum_{s=0}^{r} \frac{1}{(q_i - z_l)^{s+1}} \frac{1}{(r-s)!} \lim_{z\to z_l} \left(\frac{\mathrm{d}^{r-s}}{\mathrm{d}z^{r-s}} \frac{\prod_{k=1}^{\mathcal{D}}(z-q_k)}{\prod_{\substack{j=1\\ j\neq l}}^{N}(z-z_j)^{\tau_j}} \right) \\ &= \sum_{s=0}^{r} \frac{x_{i,[\tau_i-r+s-1]}}{(q_i - z_l)^{s+1}},\end{aligned}$$

where in the last line we have recognised the expression for $x_{i,[\tau_i-r+s-1]}$ again from the change of variables (2.9). We can now expand ∂_{q_i} in terms of $x_{i,[r]}$ via the chain rule

$$\frac{\partial}{\partial q_i} = \sum_{l=1}^{N} \sum_{r=0}^{\tau_l-1} \frac{\partial x_{l,[r]}}{\partial q_i} \frac{\partial}{\partial x_{l,[r]}} = \sum_{l=1}^{N} \sum_{r=0}^{\tau_l-1} \sum_{s=0}^{\tau_l-1-r} \frac{x_{l,[r+s]}}{(q_i - z_l)^{s+1}} \frac{\partial}{\partial x_{l,[r]}}.$$

This is immediately recognisable in our expression $A(z)$ in terms of the realisation variables $x_{i,[r]}$ from equation (2.27), and thus we arrive at the desired result.

Now we have seen that $A(z)|_{z\to q_i} = \Lambda(q_i) - \partial_{q_i}$, we want to consider how such a substitution works for $A^2(z)$ and $A'(z)$ as they appear in $\widehat{s}_1(z)$. Naively we might think that $A^2(z)|_{z\to q_i} = (\Lambda(q_i) - \partial_{q_i})^2$, however this does not work due to the subtleties of the convention of substitution from the left. Once we have substituted $z \to q_i$ into the first factor of $A(z)$,

$$A^2(z)\Big|_{z\to q_i} = \Big((\Lambda(q_i) - \partial_{q_i})A(z)\Big)\Big|_{z\to q_i}, \tag{2.28}$$

we need to move the other factor of $A(z)$ through to the left before substituting into it, introducing an additional term from reordering

$$[(\Lambda(q_i) - \partial_{q_i}), A(z)]\Big|_{z\to q_i} = -\Big[\partial_{q_i}, A(z)\Big|_{z\to q_i}\Big] + [\partial_z, A(z)]\Big|_{z\to q_i} = A'(z)\Big|_{z\to q_i}. \tag{2.29}$$

This additional copy of $A'(z)$ fortuitously cancels the one in (2.18), leaving

$$\widehat{s}_1(z)\Big|_{z\to q_i} = \big(A^2(z) - A'(z)\big)\Big|_{z\to q_i} = \big(\Lambda(q_i) - \partial_{q_i}\big)^2.$$

In the next chapter we will generalise this idea of combinations of powers of $A(z)$ and derivatives to "quantum powers", which behave predictably when we substitute $z \to q_i$ from the left.

Therefore if $\Psi = \Psi(q_1, \ldots, q_{\mathcal{D}})$ is an eigenfunction of $\widehat{s}_1(z)$ such that

$$\widehat{s}_1(z)\Psi = s_1(z)\Psi,$$

then by substituting $z \to q_i$ from the left we see that it is also true that

$$\partial_{q_i}^2\Psi - 2\Lambda(q_i)\partial_{q_i}\Psi + (\Lambda^2(q_i) - \Lambda'(q_i))\Psi = s_1(q_i)\Psi. \tag{2.30}$$

As this only affects one of the variables, the eigenfunctions of $\widehat{s}_1(z)$ are products of functions in one variable $\Psi = \psi_1(q_1)\cdots\psi_{\mathcal{D}}(q_{\mathcal{D}})$ (or more generally, sums of such products) where each $\psi_i(q_i)$ satisfies a differential equation of the form

$$y'' - 2\Lambda y' + (\Lambda^2 - \Lambda')y = s_1 y. \tag{2.31}$$

More explicitly, plugging in the expression for $\Lambda(z)$ (2.23) and $\widehat{s}_1(z)$ in terms of the Gaudin Hamiltonians (1.18), the equation reads

$$y'' - 2\left(\lambda + \sum_{k=1}^{N}\sum_{r=0}^{\tau_k-1}\frac{\ell_{k,[r]}}{(q_i - z_k)^{r+1}}\right)y' + \left(\sum_{k=1}^{N}\sum_{r=0}^{\tau_k-1}\frac{a_{k,[r]}}{(q_i - z_k)^{r+1}}\right)y = 0 \tag{2.32}$$

where

$$a_{k,[0]} = -H_{k,[0]} + \ell_{k,[0]} \left(\sum_{\substack{j=1 \\ j \neq i}}^{N} \sum_{s=0}^{\tau_j - 1} \frac{2\ell_{j,[s]}}{(z_k - z_j)^{s+1}} \right)$$

and, with $H_{k,[r]}$ as eigenvalues of the Gaudin Hamiltonians $\widehat{H}_{k,[r]}$,

$$a_{k,[r]} = -H_{k,[r]} + \ell_{k,[r]} \left(\sum_{\substack{j=1 \\ j \neq i}}^{N} \sum_{s=0}^{\tau_k - 1} \frac{2\ell_{j,[s]}}{(z_k - z_j)^{s+1}} \right) + r\ell_{k,[r-1]} + \sum_{s=0}^{\tau_k - 2} \ell_{k,[s]}\ell_{k,[r-s-1]}.$$

On setting the Takiff degrees to 1 and once again identifying $x_{i,[0]}$ with x_i and also $a_{k,[0]}$ with a_k, these separated equations become

$$y'' - 2\left(\lambda + \sum_{k=1}^{N} \frac{\ell_i}{(q_i - z_k)}\right) y' + \left(\sum_{k=1}^{N} \frac{a_k}{(q_i - z_k)}\right) y = 0, \tag{2.33}$$

exactly recovering [58, Eq. (1.27a)].

In (2.26), the exact form of the scalar function $\Lambda(z)$ appears to be somewhat arbitrary, since we only seem to require that it commutes with q_i. In our discussion of the Bethe Ansatz below, we will see that it is relevant as the eigenvalue of $A(z)$ on the vacuum state, however we can "hide" it in the wavefunction $\psi_i(q_i)$ to make our differential equation appear simpler, and the analogy between (2.30) and the oper clearer. If we let $\tilde{\psi}_i(q_i) = e^{-\int \Lambda(q_i)} \psi_i(q_i)$ then we can shift the differential operator ∂_{q_i} by $\Lambda(q_i)$

$$\begin{aligned} -\partial_{q_i} \tilde{\psi}_i(q_i) &= \Lambda(q_i)\tilde{\psi}_i(q_i) - e^{-\int \Lambda(q_i)} \partial_{q_i} \psi_i(q_i) \\ &= e^{-\int \Lambda(q_i)} \left(\Lambda(q_i) - \partial_{q_i}\right) \psi_i(q_i). \end{aligned} \tag{2.34}$$

The pay-off for the simpler differential equation that we ultimately achieve from this rotation is that the eigenstates are in the more complicated space $e^{-\int \Lambda(q_i)} \mathbb{C}[\{x_{j,[r]} | r = 0, \ldots \tau_j - 1\}_{j=1}^{N}$. This will be a more convenient choice in the $\mathfrak{gl}_3$-case covered in the following chapter, since we will not be working with the realisation directly it will be less clunky to simply rescale and remove $\Lambda(q_i)$. However, for the present chapter we will continue with the form of $\Lambda(z)$ given in (2.23), since it naturally arises from the realisation we have chosen and highlights the correspondence to the original simple poles case.

2.2 Comparison to Bethe Ansatz

Let us now align this SoV perspective on solving the Gaudin model with the well-known diagonalisation of the Gaudin Hamiltonians using the Bethe Ansatz, see [31] for an overview. We will see that the Bethe vectors do indeed satisfy our separated equation (2.32). Let us first briefly recall the Bethe Ansatz for the $\mathfrak{sl}_2$-Gaudin model.

2.2.1 The Bethe Ansatz

This section follows [58] and [31]. Though the Bethe Ansatz for the irregular singularities case has not been mentioned previously in the literature, the method of the Bethe Ansatz that we will outline here is derived wholly from the Lax algebra relations and thus applicable to any strength of poles.

To ensure a suitable reference vector we require the representations at each site to be highest-weight representations; for $k = 1, \ldots, N$ we take at the kth site the module V_k with highest weight vector v_k, and at infinity the module V_∞ where $\mathfrak{sl}_2$ acts by multiplication by the numbers in the matrix described in (A.10). Hence the spin-chain Ω of the Gaudin model is given by

$$\Omega = V_1 \otimes V_2 \otimes \cdots \otimes V_N \otimes V_\infty. \tag{2.35}$$

Our reference vector will then be the vacuum state found as a tensor product of the highest weight vectors at each site

$$|0\rangle = v_{\lambda_1} \otimes v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_N} \otimes v_\infty, \tag{2.36}$$

note that any state in Ω may be reached by acting with a suitable combination of $f^{(i)}_{[r]}$ on $|0\rangle$. For example, since $B(z)$ is such a combination

$$B(w)\,|0\rangle = f^{(\infty)}\,|0\rangle + \sum_{i=1}^{N} \sum_{r=0}^{\tau_i - 1} \frac{f^{(i)}_{[r]}\,|0\rangle}{(z - z_i)^{\tau_i + 1}} \in \Omega. \tag{2.37}$$

To understand the effect of the other Lax matrix elements in (2.5) on $|0\rangle$, let us consider them in the realisation (A.6a), in which the modules V_k are made up of polynomials in the variables $\{x_{i,[r]}\}_{r=0}^{\tau_k - 1}$ with corresponding highest weight vector $v_k = 1$ for $k = 1, \ldots, N$. If we apply $C(z)$ to the vacuum vector $|0\rangle$ using this realisation we see that

$$C(z)\,|0\rangle = 0, \tag{2.38}$$

as $\pi(e^{(k)}_{[r]})$ acts first by differentiation. Similarly we find that $|0\rangle$ is an eigenvector of $A(z)$,

$$A(z)\,|0\rangle = \Lambda(z)\,|0\rangle\,, \qquad \Lambda(z) = \lambda - \sum_{i=1}^{N} \sum \frac{\ell_{i,[r]}}{(z - z_i)^{r+1}}. \tag{2.39}$$

Note that $\Lambda(z)$ is the scalar function we found previously in (2.23) from the separation of variables approach. From (2.39) and (2.18) it is clear that $|0\rangle$ is an eigenvectors of $\widehat{s}_1(z)$ with eigenvalue $\Lambda^2(z) - \Lambda'(z)$.

We have noted that $B(z)|0\rangle \in \Omega$ for all $z \in \mathbb{C}$, we now determine the condition on the value w of the spectral parameter for $\Psi(w) = B(w)|0\rangle$ to be another eigenvector of $\widehat{s}_1(z)$. We can reorder the operators and use our known eigenvalue on the vacuum, but this introduces an additional term from the commutation relations;

$$\widehat{s}_1(z)\Psi(w) = \big(\Lambda^2(z) - \Lambda'(z)\big)\Psi(w) + [\widehat{s}_1(z), B(w)]\,|0\rangle\,. \tag{2.40}$$

It follows by the Lax algebra relations that

$$\begin{aligned}[\widehat{s}_1(z), B(w)]\,|0\rangle &= \frac{2}{z-w}\big(B(z)A(w) - B(w)A(z)\big)\,|0\rangle \\ &= \frac{-2\Lambda(z)}{z-w}B(w)\,|0\rangle + \frac{2\Lambda(w)}{z-w}B(z)\,|0\rangle \\ &= \frac{-2\Lambda(z)}{z-w}\Psi(w) + \frac{2\Lambda(w)}{z-w}B(z)\,|0\rangle\,.\end{aligned} \tag{2.41}$$

Regrouping this, we see that the Bethe vector $\Psi(w)$ is an eigenvector of $\widehat{s}_1(z)$ with eigenvalue

$$s_1(z,w) = \big(\chi_w(z) - \Lambda(z)\big)^2 + \frac{\mathrm{d}}{\mathrm{d}z}\big(\chi_w(z) - \Lambda(z)\big) \tag{2.42}$$

where $\chi_w(z) = \frac{1}{z-w}$, if and only if the parameter w satisfies the equation

$$\Lambda(w) = 0. \tag{2.43}$$

This is referred to as the Bethe equation and parameters w satisfying it as the Bethe-root. Note that if we have rescaled to remove $\Lambda(z)$ from $A(z)$ as in (2.34), then the vacuum vector would be of the more complex form $e^{-\int \Lambda(q_i)}$ so we would still find $\Lambda(z)$ as the eigenvalue of $A(z)$ on the vacuum, and hence the Bethe equation would not be affected.

We can extend this to further deviations from the reference vector $|0\rangle$ by applying more copies of $B(z)$ at different Bethe roots $w_1, w_2, \ldots, w_m$,

$$\Psi(w_1, w_2, \ldots, w_m) = B(w_1)B(w_2)\ldots B(w_m)\,|0\rangle\,. \tag{2.44}$$

To determine the eigenvalue and Bethe equation we again move $\widehat{s}_1(z)$ past the factors of $B(w_k)$ by repeatedly applying (2.41) along with the relation

$$\begin{aligned}&[A(z), B(w_1)B(w_2)\cdots B(w_m)]\,|0\rangle \\ &= -\chi_{w_1,\ldots,w_m}\Psi(w_1, w_2, \ldots, w_m) + \sum_{k=1}^{m}\chi_{w_k}B(z)\Psi(w_1, \ldots, \cancel{w_k}, \ldots, w_m),\end{aligned} \tag{2.45}$$

where

$$\chi_{w_1,\dots,w_m}(z) = \sum_{i=1}^{m} \frac{1}{z - w_k}. \tag{2.46}$$

Thus applying $\widehat{s}_1(z)$ to $\Psi(w_1,\dots,w_m)$ we get

$$\begin{aligned}\widehat{s}_1(z)\Psi_{w_1 w_2 \dots w_m} &= s_1(z;\{w_j\}_{j=1}^m)\Psi(w_1,\dots,w_m) \\ &+ \sum_{k=1}^{m} 2\chi_{w_k}(z)\big(\Lambda(w_k) - \chi_{w_1,\dots,\cancel{w_k},\dots,w_m}(w_k)\big)B(z)\Psi(w_1,\dots,\cancel{w_k},\dots,w_m),\end{aligned} \tag{2.47}$$

where

$$s_1(z;\{w_j\}_{j=1}^m) = (\chi_{w_1,\dots,w_m}(z) - \Lambda(z))^2 + \frac{\mathrm{d}}{\mathrm{d}z}(\chi_{w_1,\dots,w_m}(z) - \Lambda(z)). \tag{2.48}$$

Clearly from (2.47), for $\Psi(w_1,\dots,w_m)$ to be an eigenvector of $\widehat{s}_1(z)$, the Bethe roots $\{w_k\}_{k=1}^m$ must now satisfy Bethe equations

$$\Lambda(w_k) = \sum_{\substack{j=1 \\ j\neq k}} \frac{1}{w_k - w_j}. \tag{2.49}$$

Thus we have obtained the joint spectrum of the Gaudin Hamiltonians for any highest weight representation - which is complete in $\mathfrak{sl}_2$ [47].

2.2.2 Bethe Vectors and the Separated Equation

Let us check that the Bethe vectors (2.44) we know to be solutions of the Gaudin model do indeed satisfy equation (2.32), beginning with the vacuum vector $|0\rangle$ (recalling that that this is the tensor product of highest weight vectors given by $v_k = 1$ for $k = 1,\dots,N$ in our differential operator realisation of $\mathfrak{sl}_1[\varepsilon_i]/\varepsilon_i^{\tau_i}$). There is clearly no dependence on any of the separated variables, hence the separated equation is mostly trivial

$$(\Lambda(q_i) - \partial_{q_i})^2 |0\rangle - s_1(q_i)|0\rangle = \Lambda^2(q_i)|0\rangle - \Lambda'(q_i)|0\rangle - s_1(q_i)|0\rangle = 0. \tag{2.50}$$

The final equality here is simply a restatement of the fact that the eigenvalue of our generating function on the vacuum is $s_1(q_i) = \Lambda^2(q_i) - \Lambda'(q_i)$.

For cases where our eigenvector does have some dependence on the separated variables, take $\Psi(w) = B(w)|0\rangle$ - or, when factorised in terms of the variables

$$\Psi(w) = \frac{(w-q_1)\cdots(w-q_{\mathcal{D}})}{(w-z_1)^{\tau_1}\cdots(w-z_N)^{\tau_N}}|0\rangle. \tag{2.51}$$

Because we showed that the separated variables are isomorphic to our original realisation in the variables $\{x_{i,[r]} | r = 0, \ldots, \tau_i - 1\}_{i=1}^{N}$, they must also be distinct. Hence when we apply (2.30) to $\Psi(w)$ only the first derivative terms are non-zero;

$$\partial_{q_i} \Psi(w) = \frac{1}{q_i - w} \Psi(w), \tag{2.52}$$

where we have recognised that this factor in front is just $\chi_w(q_i) = \frac{1}{q_i - w}$ as defined above. Therefore the separated equation acting on $\Psi(w)$ leaves only

$$-2\Lambda \chi_w \Psi(w) + \left(\Lambda^2 - \Lambda'\right) \Psi(w) - s_1 \Psi(w) = 0 \tag{2.53}$$

which again is a rearrangement of the known eigenvalue $s_1(z)$ in (2.48).

In fact Sklyanin shows in [58] that any generic Bethe vector $\Psi(w_1, w_2, \ldots, w_m)$ satisfies equation (2.32) by rearranging to arrive at the separated equation from the Bethe Ansatz context. Using (2.7) we can immediately factorise $\Psi(w_1, \ldots, w_m)$ in terms of our separated variables

$$\Psi(w_1, \ldots, w_m) = \prod_{k=1}^{m} \prod_{j=1}^{\mathcal{D}} (w_k - q_j) |0\rangle \,. \tag{2.54}$$

Labelling $y(z) = \prod_{k=1}^{m} (z - w_k)$, $\Psi(w_1, \ldots, w_m)$, we have

$$\Psi(w_1) = y(q_1) \cdots y(q_{\mathcal{D}}) |0\rangle \,, \tag{2.55}$$

with each factor depending on one separated variable.

The first part of this argument would theoretically apply beyond $\mathfrak{sl}_2$; if one can show that the eigenfunctions of the Hamiltonians may be derived from the separating function $B(z)$ acting on the vacuum, then we have a factorised form with each depending on a single separated variable, albeit with no knowledge of the corresponding separated equation. In this lowest rank case this is immediate, as $B(z)$ is simply the lowering operator, however it becomes less trivial for higher rank cases. This idea is the crux of the method laid out by Gromov, Cavalgia, Levkovich-Maslyuk and Sizov in [8, 34] to construct the separation of variables for the the $\mathfrak{sl}_\mathfrak{n}$-XXX chain. We will discuss this with regards to the $\mathfrak{gl}_3$-Gaudin model in the next chapter when looking towards higher rank.

In this case we can go further and see that $y(q_j)$ satisfies (2.32) for all $j = 1, \ldots, \mathcal{D}$. Firstly, note that

$$\chi_{w_1, \ldots, w_m}(z) = \frac{y'(z)}{y(z)}. \tag{2.56}$$

and hence (2.48) may be written as

$$s_1 = \left(\frac{y'}{y} - \Lambda\right)^2 + \left(\frac{y'}{y} - \Lambda\right)' = \frac{(y')^2}{y^2} - 2\Lambda\frac{y'}{y} + \Lambda^2 + \frac{y''}{y} - \frac{(y')^2}{y^2} - \Lambda'. \tag{2.57}$$

Multiplying both sides by y, we reach exactly our separated equation once again

$$s_1 y = y'' - 2\Lambda y' + \left(\Lambda^2 - \Lambda'\right) y \tag{2.58}$$

so $y(q_j)$ (and therefore Ψ) satisfies the separated equation for each variable q_j. Thus when we have highest weight representations at each site, finding polynomial solutions to (2.32) is equivalent to finding the Bethe roots $w_1, \ldots, w_m$.

2.2.3 Relation to Opers

Underlying both the Separation of Variables and the Bethe Ansatz for the Gaudin model is a description of the spectrum of the Gaudin model using opers, put forth by Feigin, Frenkel, and Reshetikhin in [26, 27, 29]. Crucially, this construction produces a complete set of joint eigenvectors for the Gaudin Hamiltonians [55]. These same opers can be used to describe the Bethe Ansatz in the Gaudin model, and also appear (after some rearranging) in the SoV of both this Chapter and the next. We briefly introduce $\mathfrak{sl}_2$-opers, largely following [41].

Consider the space of meromorphic $\mathfrak{sl}_2$-connections on $\mathbb{C}P^1$, which take the following form when we view $\mathfrak{sl}_2$ in its usual matrix representation;

$$\nabla = \partial_z + \begin{pmatrix} a(z) & b(z) \\ 1 & -a(z) \end{pmatrix} \tag{2.59}$$

where z is a coordinate on $\mathbb{C}P^1$ and $a(z)$ and $b(z)$ are meromorphic functions in z. We can gauge transform a connection by a matrix $g(z)$ valued in the associated Lie group SL_2;

$$g(z)\nabla g(z)^{-1} = \partial_z + g(z)\begin{pmatrix} a(z) & b(z) \\ 1 & -a(z) \end{pmatrix} g(z)^{-1} - g'(z)g(z)^{-1} \tag{2.60}$$

and the space of connections like (2.59) are invariant under gauge transforms of the form

$$g(z) = \begin{pmatrix} 1 & f(z) \\ 0 & 1 \end{pmatrix}, \tag{2.61}$$

where $f(z)$ is meromorphic. Under such a transformation, they become

$$g(z)\nabla g(z)^{-1} = \partial_z + \begin{pmatrix} a(z) + f(z) & b(z) - 2a(z)f(z) - f^2(z) - f'(z) \\ 1 & -a(z) - f(z) \end{pmatrix}. \tag{2.62}$$

An $\mathfrak{sl}_2$-*oper* is the equivalence class $[\nabla]$ of $\mathfrak{sl}_2$-connections (2.59) under gauge transformations of the form (2.61). We can choose a specific canonical representative of such an equivalence class by taking a gauge transform with $f(z) = -a(z)$, leaving us with zeroes on the diagonal

$$\partial_z + \begin{pmatrix} 0 & c(z) \\ 1 & 0 \end{pmatrix}, \tag{2.63}$$

for some meromorphic function $c(z)$.

A *Miura* oper is a connection of the special form

$$\nabla = \partial_z + \begin{pmatrix} m(z) & 0 \\ 1 & -m(z) \end{pmatrix}. \tag{2.64}$$

The canonical representative in the equiavalence class with a Miura oper is

$$\nabla_c = \partial_z + \begin{pmatrix} 0 & m^2(z) - m'(z) \\ 1 & 0 \end{pmatrix}, \tag{2.65}$$

where we call $m^2(z) - m'(z)$ the Miura transformation of $m(z)$.

2.2.3.1 Opers and the Bethe Ansatz

In [26] we see an alternative formulation of the Bethe Ansatz for the Gaudin model in terms of opers.

Let $w_1, \ldots, w_m$ be the Bethe roots, defined as solutions to (2.49), and let $\chi_{w_1,\ldots,w_m}$ and $\Lambda(z)$ be as in (2.46) and (2.23) respectively. We may then construct the Miura oper

$$\nabla_{w_1,\ldots,w_m} = \partial_z + \begin{pmatrix} \Lambda(z) - \chi_{w_1,\ldots,w_m}(z) & 0 \\ 1 & \chi_{w_1,\ldots,w_m}(z) - \Lambda(z) \end{pmatrix}. \tag{2.66}$$

If we take instead the canonical representative of the oper containing $\nabla_{w_1,\ldots,w_m}$, we see that it is

$$\nabla_{c,w_1,\ldots,w_m} = \partial_z + \begin{pmatrix} 0 & (\chi_{w_1,\ldots,w_m}(z) - \Lambda(z))^2 - (\chi_{w_1,\ldots,w_m}(z) - \Lambda(z))' \\ 1 & 0 \end{pmatrix}. \tag{2.67}$$

In other words, the eigenvalue $s_1(z)$ on $\Psi(w_1,\ldots,w_m)$ is the Miura transform of $(\chi_{w_1,\ldots,w_m}(z) - \Lambda(z))$.

To recover the Bethe equations (2.49) in this formalism, consider the behaviour of the eigenvalue $s_1(z)$ at the poles $z = w_i$. The double pole arising from the $\chi^2_{w_1,\ldots,w_m}(z)$ term cancels with that from the $\chi'_{w_1,\ldots,w_m}(z)$, hence we have a simple pole with residue

$$\mathrm{Res}_{z=w_i}\, s_1(z) = \sum_{i=1}^{N} \frac{-2}{w_i - w_j} - 2\Lambda(w_i). \tag{2.68}$$

Therefore in this oper formulation, the Bethe equations satisfy the requirement that $s_1(z)$ is regular at the Bethe roots $z = w_j$.

2.2.3.2 Opers in Separation of Variables

Interestingly opers also appear in the separated equation (2.30) itself. This may not be initially obvious, since our second order differential equation does not bear much resemblance to (2.59), but we can rearrange to see the similarity. We firstly let the oper act upon some generic vector and set this to zero

$$\left(\partial_z + \begin{pmatrix} a(z) & b(z) \\ 1 & -a(z) \end{pmatrix}\right)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0. \tag{2.69}$$

We can rewrite (2.69) as two simultaneous differential equations

$$\partial_z\phi_1 + a(z)\phi_1 + b(z)\phi_2 = 0, \tag{2.70a}$$
$$\partial_z\phi_2 + \phi_1 - a(z)\phi_2 = 0, \tag{2.70b}$$

and combine these as a single second order differential equation

$$\partial_z^2\phi_2 + \left(b(z) - a^2(z) - a'(z)\right)\phi_2 = 0 \tag{2.71}$$

- note that the coefficient of $\partial_z\phi_2$ is 0, as we are in $\mathfrak{sl}_2$. If we label the coefficient of ϕ_2 as $\tau(z)$ then (2.71) is

$$\left(\partial_z^2 + \tau(z)\right)\phi_2 = 0. \tag{2.72}$$

Let us now compare this to the quantity $\mathrm{cdet}(\partial_z\mathbb{1} + L(z))$ acting on an eigenvector Ψ of $\widehat{s_1}$,

$$\mathrm{cdet}\left(\partial_z + L(z)\right)\Psi = \left(\partial_z^2 + \widehat{s_1}(z)\right)\Psi = \left(\partial_z^2 + s_1(z)\right)\Psi. \tag{2.73}$$

This bears a clear resemblance not only to the second order differential equation (2.72), but also to the separated equations if we ignore the terms containing the scalar function $\Lambda(z)$ and replace z by q_i. We will find in the following chapter this trend continues at the next rank, and that the separated equation may again be found from

$$\operatorname{cdet}\big(\partial_z + L(z)\big)\Big|_{z\to q_i \text{from left}} \Psi = 0 \tag{2.74}$$

in the separated variables q_i. The quantity $\operatorname{cdet}\big(\partial_z + L(z)\big)$ is known as the universal oper, with $\partial_z^2 + \tau_\Psi$ being the oper associated to the state Ψ.

As we remarked above, the scalar function $\Lambda(z)$ does not appear naturally from this perspective, however we can still include it using the transformation (2.34). If we multiply on the left by $e^{\int \Lambda(q_i)}$ then we have

$$e^{\int \Lambda(q_i)} \partial_{q_i} \phi = \big(\partial_{q_i} - \Lambda(q_i)\big) e^{\int \Lambda(q_i)} \phi. \tag{2.75}$$

Therefore, performing this rotation on (2.74) gives

$$e^{\Lambda q_i}\big(\partial_{q_i}^2 + s_1\big) e^{-\Lambda q_i} = (\partial_{q_i} - \Lambda)^2 + s_1, \tag{2.76}$$

which leads naturally to our separated equation by expanding the brackets and acting on y

$$y'' - 2\Lambda y' + (\Lambda^2 - \Lambda')y = s_1 y. \tag{2.77}$$

As discussed above, an important advantage of the Separation of variables technique over the Bethe Ansatz is that the latter does not provide the complete set of eigenvectors, whereas separation of variables is generally believed to be complete. It has been shown [55] that there is a bijective correspondence between the space of opers and the eigenspaces of the Gaudin Hamiltonians, and we have seen that opers correspond to these second order differential equations which separate the variables. In contrast, the Bethe Ansatz has been shown to be incomplete for higher rank [47]. Though we have seen that the Bethe vectors also correspond to opers (specifically, to Miura opers of the form (2.66)) there are joint eigenvectors of the Gaudin Hamiltonians which cannot be described in this way - i.e they correspond to opers that are not Miura opers.

It is interesting that, via SoV, we rediscover this complete description of the Gaudin eigenstates using opers, without needing to delve into the complicated machinery. In addition, the appearance of the universal oper $\operatorname{cdet}(\partial_z + L(z))$ in the separated equation might prove useful for systematically constructing an SoV for higher rank Gaudin models.

— 3 —

Separation of variables for $\mathfrak{gl}_3$-Gaudin models

We now consider separation of variables for the quantum $\mathfrak{sl}_3$ and $\mathfrak{gl}_3$ Gaudin models. As a guide we have the known SoV for the $\mathfrak{sl}_3$ XXX-chain [21, 59], from which we can take the Gaudin limit as described in Section 1.3.1.1. Firstly we construct the SoV for the $\mathfrak{sl}_3$ Gaudin model entirely by taking the Gaudin limit of the separating functions and the separated equations, which was conducted independently by Ribault in [52]. Following this we can recreate the SoV entirely within the Gaudin model itself, where it is straightforward to extend it to the more general $\mathfrak{gl}_3$ Gaudin model with irregular singularities as it is a consequence of the Lax algebra. As we discussed in the previous chapter, the separating functions $A(z)$ and $B(z)$ become increasingly involved expressions at higher rank, but taking the limit of the XXX chain provides a starting point which we can check against the classical expressions from [60] and [23]. We will later discuss a variety of methods to construct the separating functions and separated equations more systematically at arbitrary rank without the need to refer to other models.

Our initial realisation for the Takiff Lie algebra $\mathfrak{gl}_3[\varepsilon_i]/\varepsilon_i^{\tau_i}$ is detailed in appendix A, (A.8). We now have $3\tau_i$ variables at each site labelled $x_{i,[r]}$,$y_{i,[r]}$ and $z_{i,[r]}$ for modes $r = 0, \ldots, \tau_i - 1$ and sites $i = 1, \ldots N$, hence we now require $3\mathcal{D}$ separated variables q_j. Note that there is a notational conflict between the sitez z_i and the realisation variables $z_{i,[r]}$, but as we rarely use the realisation directly it will not prove to be overly troublesome. In this chapter we do not construct an explicit change of variables analogous to (2.9), as both the realisation (A.8) and $B(z)$ are more complex and it is not clear how to isolate the variables as we did in the rank 2 case. While the change of variables provided the basis of the idea of SoV, particularly in earlier examples, we do not need to restrict the definition of SoV to include it, as was discussed in [60]. One place it is useful to have the perspective of the initial realisation is when defining the Gaudin separating function $A(z)$, as the inverse terms that arise may then be viewed as fractions.

3.1 As a limit of the XXX-chain

We take the Gaudin limit of the separating functions and the separated equation for the XXX chain, as Ribault does in [52]. Let us firstly briefly present the SoV for the quadratic case.

3.1.1 Summary of SoV for the sl3 XXX-chain

We will follow [59] and [21]. In the former Sklyanin provides a separation of variables applicable to any quantum integrable model associated to $Y(\mathfrak{sl}_3)$, whereas in the latter Derkachov and Valinevich focus specifically on the XXX-chain and provide a starting differential operator realisation for this model, before going on to find eigenvectors of $B_{XXX}(z)$. Here the separated equation will be a finite difference equation, which will become a third order differential equation in the limit, the rank 3 analogue to (2.32). We will include the parameter $\hbar$ in order to take the Gaudin limit later.

As the Lax matrix $L_{XXX}(z)$ (1.62) of the XXX chain covers only single site, the separating functions will instead be made up of elements of the global monodromy matrix $T(z)$. Representing the auxiliary factor in the fundamental representation we have a 3×3 matrix,

$$T(z) = \begin{pmatrix} T^1_1(z) & T^1_2(z) & T^1_3(z) \\ T^2_1(z) & T^2_2(z) & T^2_3(z) \\ T^3_1(z) & T^3_2(z) & T^3_3(z) \end{pmatrix}.$$

The separating function $B_{XXX}(z)$ is then given by

$$B_{XXX}(z) = T^1_3(z) T\left[\begin{smallmatrix}12\\23\end{smallmatrix}\right](z-\hbar) - T^1_3(z) T\left[\begin{smallmatrix}12\\13\end{smallmatrix}\right](z-\hbar), \tag{3.1}$$

recalling the definition of the quantum minor $T\left[\begin{smallmatrix}ac\\bd\end{smallmatrix}\right](z)$ given in (1.64). The separated variables, which we label $\{q_i\}_{i=1}^{3N}$, are as usual defined as the operator zeroes of $B_{XXX}(z)$, that is

$$B_{XXX}(z) = B_0(z)(z-q_1)(z-q_2)\cdots(z-q_{3N}), \tag{3.2}$$

Where $B_0(z)$ is just some rational function of z. Note that at rank 3 we have $3N$ separated variables. As we would hope, $B(z)$ commutes with itself [59, Eq. (21)]

$$[B_{XXX}(z), B_{XXX}(w)] = 0, \tag{3.3}$$

and hence the $\{q_i\}_{i=1}^{3N}$ also mutually commute.

Similarly, the corresponding separating function $A_{XXX}(z)$ which provides the conjugate coordinates is given by

$$A_{XXX}(z) = -T[^{12}_{13}](z-\hbar)(T_3^2(z-\hbar))^{-1}, \tag{3.4}$$

assuming that the operator $T_3^2(z)$ is invertible.

For reference, let us also briefly note the commutation relations of our separating functions $A_{XXX}(z)$ and $B_{XXX}(z)$ to guide us when we find the corresponding relation in the Gaudin model later;

$$\begin{aligned}(z-w)A_{XXX}(z)B_{XXX}(z) - (z-w-\hbar)B_{XXX}(w)A_{XXX}(z) \\ = \hbar B_{XXX}(z)A_{XXX}(w)(T_3^2(z-\hbar))^{-1}(T_3^2(z))^{-1}T_3^2(w-\hbar)T_3^2(w).\end{aligned} \tag{3.5}$$

In the XXX-chain, the separated equation will be a finite difference equation and so $A_{XXX}|_{z\to q_i}$ must be a finite difference operator; if $F(q_1,\ldots,q_{3N})$ is polynomial in the separated variables q_i then as in [59, Eq.(36)], [21, Eq.(4.37)] we can act with $A_{XXX}(z)$ to get

$$A(z)\Big|_{z=q_i} F(q_1,\ldots,q_{3N})) = F(q_1,\ldots,q_i-\hbar,\ldots,q_{3N}). \tag{3.6}$$

Recall the definition of the generating functions of the integrals of motion $t_1(z), t_2(z), t_3(z)$ given in (1.66). The separated equation of an eigenvector Φ of the XXX-chain is therefore a difference equation given in [59, Eq.(48)]

$$\begin{aligned}\Phi(q_1,q_2,\ldots,q_i-3\hbar,\ldots) - t_1(q_i+2\hbar)\Phi(q_1,q_2,\ldots,q_i-2\hbar,\ldots) \\ +t_2(q_i-2\hbar)\Phi(q_1,q_2,\ldots,q_i-\hbar,\ldots) - d(q_i-2\hbar)\Phi(q_1,q_2,\ldots) = 0\end{aligned} \tag{3.7}$$

for which $\Phi(q_1,q_2,\ldots.q_{3N}) = \phi_1(q_1)\phi_2(q_2),\cdots\phi_{3N}(q_{3N})$ is clearly a solution.

3.1.2 Gaudin limit

We can now take the Gaudin limit, as described in section 1.3.1.1 to obtain an SoV for the corresponding $\mathfrak{gl}_3$ Gaudin model with simple poles, as in [52]. Recall that we recover the Lax matrix of the Gaudin model from the monodromy $T(z)$ at first order in the $\hbar \to 0$ limit,

$$T(z) = 1 + \hbar L(z) + \hbar^2 T^{(2)}(z) + \hbar^3 T^{(3)}(z) + O(\hbar^4); \tag{3.8}$$

we have expanded up to order $\hbar^3$ as we will need to check that these higher order parts cancel off, leaving us with a result truly in the Gaudin setting. We firstly define our separated variables by constructing suitable separating functions for the Gaudin model by taking the limit of (3.4) and (3.1), as they no longer arise naturally as single elements of the Lax matrix;

$$\begin{aligned} A_{\text{XXX}}(z) &= 1 + \hbar A(z) + \mathcal{O}(\hbar^2) \\ B_{\text{XXX}}(z) &= \hbar^3 B(z) + \mathcal{O}(\hbar^4). \end{aligned}$$

This leaves us with rational separating functions $A(z)$ and $B(z)$ in the components of the Gaudin Lax matrix only.

$$A(z) = (L^2_3(z))^{-1} L^{12}_{13}(z) = L^{12}_{23}(z)(L^2_3(z))^{-1} \tag{3.9a}$$

$$B(z) = L^{21}_{31}(z) L^1_3(z) - L^2_3(z) L^{12}_{32}(z) \tag{3.9b}$$

where now the minors do not have shifts in the spectral parameter as we have Taylor expanded in $\hbar$. Fortunately we do not need to worry about the ordering in the $A(z)$ definition too much as $L^2_3(z)$ commutes with the minor $L^{12}_{13}(z)$ by repeated application of (1.17). We also note that on taking the classical limit (i.e. letting the Lax matrix elements commute with one another) the expressions for the separating functions given in (3.9) become exactly their analogues for the classical Gaudin model as described in [62].

Entirely as before we may take our separated variables q_i to be the $3N$ operator zeroes of $B(z)$;

$$B(z) = \frac{(z - q_1)\cdots(z - q_{3N})}{(z - z_1)^3 \cdots (z - z_N)^3}, \tag{3.10}$$

as expected we have $3N$ variables which aligns with our $3N$ variables x_i, y_i, z_i for $i = 1, \ldots, N$ in our original realisation of ${\mathfrak{sl}_3}^{\oplus N}$ (A.7). It is straightforward to see that $B(z)$ and hence the separated variables commute by taking the limit of (3.3)

$$\hbar^6[B(z), B(w)] = 0. \tag{3.11}$$

We could consider the effect of $A(z)$ by taking limits of (3.5), and indeed Ribault does [52, Eq.(4.13)], however we do not strictly need this information to complete the SoV. We will consider this later directly from the Gaudin model, for now let us instead take the $\hbar \to 0$ limit of the separated equation (3.7) directly to produce the Gaudin separated equation. We begin by Taylor expanding the eigenfunction Φ in powers of $\hbar$,

$$\begin{aligned}
0 =& (1 - t_1(q_i + 2\hbar) + t_2(q_i + 2\hbar) - d(q_i + 2\hbar))\Phi \\
&+ \hbar(3 - 2t_1(q_i + 2\hbar) + t_2(q_i + 2\hbar))\partial_{q_i}\Phi \\
&+ \hbar^2(\frac{9}{2} - 2t_1(q_i + 2\hbar) + \frac{1}{2}t_2(q_i + 2\hbar))\partial^2_{q_i}\Phi \\
&+ \hbar^3\left(\frac{9}{2} - \frac{4}{3}t_1(q_i + 2\hbar) + \frac{1}{6}t_2(q_i + 2\hbar)\right)\partial^3_{q_i}\Phi,
\end{aligned}$$

where we are ignoring anything of order $\hbar^4$ or higher, and we will then deal with the coefficients of Φ and each of its derivatives separately leaving us with the separated equation of the Gaudin model at order $\hbar^3$.

Expand each of the generating functions in $\hbar$ using (3.8), ignoring $\hbar^3$ and higher for the moment, gives

$$t_1(z) = 3 + \hbar \operatorname{Tr} L(z) + \hbar^2 \operatorname{Tr} T^{(2)}(z) \tag{3.12a}$$

$$\begin{aligned}
t_2(z) =& 3 + 2\hbar \operatorname{Tr} L(z) \\
&+ \hbar^2\big(L^{12}_{12}(z) + L^{13}_{13}(z) + L^{23}_{23}(z) + 2L^{1\prime}_1(z) + L^{2\prime}_2(z) + \operatorname{Tr} T^{(2)}(z)\big).
\end{aligned} \tag{3.12b}$$

Applying (3.12) to the coefficient of $\partial^3_{q_i}\Phi$ we have

$$\hbar^3\left(\frac{9}{2} - \frac{4}{3}t_1(q_i) + \frac{1}{6}t_2(q_i)\right) = \hbar^3\left(\frac{9}{2} - 4 + \frac{1}{2}\right) + O(\hbar^4) = 1 + O(\hbar^4).$$

Similarly for the coefficient of $\partial^2_{q_i}$, it follows simply from equations (3.12) that there is no contribution at order $\hbar^2$ and we are left with $-\hbar^3 \operatorname{Tr} L(q_i) = 0$ due to the tracelessness of the Lax matrix for the $\mathfrak{sl}_3$-Gaudin model. We will treat with the $\mathfrak{gl}_3$ case later in the chapter when we work only in the Gaudin model.

Moving on to the $\partial_{q_i}\Phi$ coefficient, we once again see that terms of order $\hbar$ and $\hbar^2$ drop out rather easily, leaving us with the following expression

$$\hbar^3\big(L^{12}_{12}(q_i) + L^{13}_{13}(q_i) + L^{23}_{23}(q_i) + L^{1\prime}_1(q_i) + L^{2\prime}_2(q_i)\big), \tag{3.13}$$

which we recognise as the generating function $\widehat{s}_2(z)$ of the Gaudin model from equation (1.25b). As we have seen in (1.26b) we can rewrite this for the special linear case as $\frac{1}{2}\operatorname{Tr} L^2(z)$.

Finally for the coefficient of Φ in the separating equation,

$$1 - t_1(q_i) + t_2(q_i) - d(q_i), \tag{3.14}$$

we will need to consider the limit of the quantum determinant $d(z)$ in addition to the behaviour of $t_1(z)$ and $t_2(z)$ up to order $\hbar^3$. It turns out that the cross terms that arise in the limit involving $T^{(2)}(z)$ and $T^{(3)}(z)$ cancel off neatly, but the calculation is quite tedious so we will not include them explicitly in our expansion;

$$\begin{aligned} d(z) =& 1 + \hbar \operatorname{Tr} L(z) + \hbar^2\big(L^{12}_{12}(z) + L^{13}_{13}(z) + L^{23}_{23}(z) + 2L^{1\prime}_{1}(z) + L^{2\prime}_{2}(z)\big) \\ &+ \hbar^3\big(L^{123}_{123}(z) + (L^{12}_{12})' + L^{1\prime}_{1}(z)L^{3}_{3}(z) - L^{1\prime}_{3}(z)L^{3}_{1}(z) + L^{1\prime\prime}_{1}(z) + \frac{1}{2}L^{2\prime\prime}_{2}(z)\big) \\ &+ \text{ higher order cross terms.} \end{aligned}$$

Once again it is straightforward to show that the terms of order $\hbar^2$ and below cancel off, leaving the relevant remaining part of $t_2(q_i)$,

$$\hbar^3\big(\frac{1}{2}L^{2\prime\prime}_{2}(q_i) + L^{3\prime\prime}_{3}(q_i)\big). \tag{3.15}$$

Therefore the coefficient of Φ in terms of the Lax matrix components is

$$\hbar^3\big(- L^{123}_{123}(q_i) - (L^{12}_{12}(q_i))' - L^{1}_{1}(q_i)L^{3\prime}_{3}(q_i) + L^{1}_{3}(q_i)L^{3\prime}_{1}(q_i) + L^{1\prime\prime}_{1}(q_i)\big). \tag{3.16}$$

Again we recognise the expression within the brackets as the generating function of the Gaudin model $\widehat{s}_3(q_i)$ from (1.25c), which we have seen becomes

$$\frac{\hbar^3}{3}\operatorname{Tr} L^3(z), \tag{3.17}$$

once we apply the $\mathfrak{sl}_3$ condition that $\operatorname{Tr} L(z) = 0$.

Overall we have $3N$ third order differential equations at order $\hbar^3$ with coefficients made up of the eigenvalues of the Gaudin Hamiltonians

$$\partial^3_{q_i}\Phi + s_2(q_i)\partial_{q_i}\Phi - s_3(q_i)\Phi = 0, \tag{3.18}$$

which are the separated equations for the $\mathfrak{sl}_3$-Gaudin model and we have $\Phi = \phi_1(q_1)\cdots\phi_{3N-1}(q_{3N})$. Note that this is again the universal $\mathfrak{sl}_3$-oper cdet $\big(\partial_{q_i} + L(q_i)\big)$ applied to Φ with generating functions replaced by their eigenvalues.

3.2 SoV Directly

Our aim in considering the $\mathfrak{sl}_3$ case was in part to then generalise to arbitrary rank, making worthwhile to reproduce this result directly in the Gaudin setting - where it can apply to $\mathfrak{gl}_3$ with minimal adjustment. Furthermore, since we will derive the SoV from the Lax algebra alone it applies to the Gaudin model with irregular singularities. We can still use the XXX SoV as a guide, in particular we use the suitable separating functions $A(z)$ and $B(z)$ we found in (3.9) as a limit of $A_{XXX}(z)$ and $B_{XXX}(z)$, but we now take the $\hbar \to 0$ limit at the start of the process instead of at the end.

3.2.1 Constructing the variables

Let us consider the behaviour of $B(z)$ in the limit $z \to \infty$; from the realisation of the commutative copy of $\mathfrak{gl}_3$ at infinity (A.11) we get an additional constant from the term $L_3^2(z)L_2^1(z)L_3^2(z)$, leaving

$$B(z) = 1 + O\left(\frac{1}{z}\right), \tag{3.19}$$

hence we expect as many operator zeroes as we have poles, which means $3\mathcal{D}$ separated variables. In general we note that for the $\mathfrak{gl}_n$ Gaudin model we would need $B(z)$ to be made up of multiples of $n(n-1)/2$ Lax matrix elements to provide enough variables for a representation of $\mathfrak{gl}_n$, with $A(z)$ of order 1 in Lax matrix elements overall. We define the separated variables q_i in the usual way, as the $3\mathcal{D}$ operator zeroes of $B(z)$,

$$B(z) = \frac{(z-q_1)\dots(z-q_{3\mathcal{D}})}{(z-z_1)^{3\tau_1}\dots(z-z_N)^{3\tau_N}}. \tag{3.20}$$

It is no longer straightforward to construct the associated change of variables from our original representation variables $\{x_{i,[r]}, y_{i,[r]}, z_{i,[r]}\}$ to the new coordinates $\{q_i\}_{i=1}^{\mathcal{D}}$. Let us consider for the moment the $\mathfrak{gl}_3$-Gaudin model with simple poles in its Lax matrix - as usual equating $x_{i,[0]}, y_{i,[0]}, z_{i,[0]}$ at the ith site with x_i, y_i, z_i respectively. In terms of these representation variables $B(w)$ has the rather involved form

$$\begin{aligned} B(w) = 1 + \sum_{i,j,k=1}^{N} \frac{1}{(w-z_i)(w-z_j)(w-z_k)} \Big(& x_i(u_3 + y_j\partial_{y_j} + z_j\partial_{z_j})y_k \\ & - (x_i\partial_{y_i} - \partial_{z_i}(u_3 - u_2 + z_i\partial_{z_i}))y_jy_k + x_i(z_j + y_j\partial_{x_j})x_k \\ & - x_iy_j(u_1 + 1 + x_k\partial_{x_k} - z_k\partial_{z_k})\Big). \end{aligned}$$

If we again try and equate this to our expression (3.20) for $B(z)$ factorised in terms of separated variables to determine a relationship it is not clear how to isolate one of our original coordinates, as we did in the rank 2 case by taking a residue. For this reason we will continue the process of separating the variables without this explicit part of the construction.

To ensure that the variables $\{q_i\}_{i=1}^{\mathcal{D}}$ commute, we must recreate (3.11) using only the Lax algebra (1.16), that is we require

$$[B(z), B(w)] = 0 \quad z, w \in \mathbb{C}. \tag{3.21}$$

We prove this by direct calculation;

$$\begin{aligned}
[B(z), B(w)] = {} & L_3^2(z)\big[L_{32}^{12}(z), L_3^2(w)\big]L_{32}^{12}(w) + L_3^2(w)\big[L_3^2(z), L_{32}^{12}(w)\big]L_{32}^{12}(z) \\
& - L_3^2(z)\big[L_{32}^{12}(z), L_{31}^{21}(w)\big]L_3^1(w) - L_3^2(w)\big[L_{31}^{21}(z), L_{32}^{12}(w)\big]L_3^1(z) \\
& + L_{31}^{21}(z)\big[L_3^1(z), L_{31}^{21}(w)\big]L_3^1(w) + L_{31}^{21}(z)\big[L_3^1(z), L_{31}^{21}(w)\big]L_3^1(w).
\end{aligned}$$

The relevant commutation relations are those between components of the Lax matrix and minors

$$\big[L_3^1(z), L_{31}^{21}(w)\big] = \big[L_3^2(z), L_{32}^{12}(w)\big] = \frac{1}{z-w}(L_3^2(z)L_3^1(w) - L_3^1(z)L_3^2(w)), \tag{3.22}$$

and between the two minors themselves

$$\begin{aligned}
\big[L_{31}^{21}(z), L_{32}^{12}(w)\big] = {} & \frac{1}{z-w}\Big(L_{32}^{12}(z)L_3^2(w) \\
& - L_3^2(z)L_{32}^{12}(w) + L_{31}^{21}(w)L_3^1(z) - L_3^1(w)L_{31}^{21}(z)\Big).
\end{aligned} \tag{3.23}$$

On substituting these into $[B(z), B(w)]$ all terms cancel leaving

$$\begin{aligned}
(z-w)[B(z), B(w)] = {} & \\
& L_3^2(z)\Big(L_3^1(w)L_3^2(z) - L_3^1(z)L_3^2(w)\Big)L_{32}^{12}(w) \\
& + L_3^2(w)\Big(L_3^1(z)L_3^2(w) - L_3^1(w)L_3^2(z)\Big)L_{32}^{12}(z) \\
& - L_3^2(z)\Big(L_{32}^{12}(w)L_3^2(z) - L_3^2(w)L_{32}^{12}(z) \\
& \qquad + L_{31}^{21}(z)L_3^1(w) - L_3^1(z)L_{31}^{21}(w)\Big)L_3^1(w)
\end{aligned}$$

$$
\begin{aligned}
&- L_3^2(w)\Big(L_{32}^{12}(z)L_3^2(w) - L_3^2(z)L_{32}^{12}(w) \\
&\qquad\qquad\qquad + L_{31}^{21}(w)L_3^1(z) - L_3^1(w)L_{31}^{21}(z)\Big)L_3^1(z) \\
&+ L_{31}^{21}(z)\Big(L_3^1(w)L_3^2(z) - L_3^1(z)L_3^2(w)\Big)L_3^1(w) \\
&+ L_{31}^{21}(w)\Big(L_3^1(z)L_3^2(w) - L_3^1(w)L_3^2(z)\Big)L_3^1(z) \\
=& L_3^2(z)L_3^1(w)\big[L_3^2(z), L_{32}^{12}(w)\big] - L_3^2(w)L_3^1(z)\big[L_{32}^{12}(z), L_3^2(w)\big] \\
&\quad + \big[L_3^1(z), L_{31}^{21}(z)\big]L_3^2(z)L_3^1(w) - \big[L_{31}^{21}(z), L_3^1(w)\big]L_3^2(w)L_3^1(z) \\
=& \ 0.
\end{aligned}
$$

It therefore follows that for the separated variables $\{q_i\}_{i=1}^{3\mathcal{D}}$ we have

$$
\big[q_i, q_j\big] = 0, \quad i, j = 1, \ldots, 3\mathcal{D}, \tag{3.24}
$$

as required.

Our conjugate coordinates will be defined by $A(z)|_{z\to q_i}$ for $i = 1, \ldots, 3\mathcal{D}$, and must also commute with one another - note that as before the substitution is from the left. Firstly note the trivial commutation relations

$$
\begin{aligned}
\big[L_3^2(z), L_3^2(w)\big] &= 0 \\
\big[L_3^2(z), L_{13}^{12}(w)\big] &= 0,
\end{aligned}
$$

and also that the minor $L_{13}^{12}(z)$ commutes with itself at different spectral parameters $z, w, \in \mathbb{C}$

$$
\begin{aligned}
\big[L_{13}^{12}(z), L_{13}^{12}(w)\big] =& \big[L_1^1(z)L_3^2(z), L_1^1(w)L_3^2(w)\big] - \big[L_1^1(z)L_3^2(z), L_3^1(w)L_1^2(w)\big] \\
&- \big[L_3^1(z)L_1^2(z), L_1^1(w)L_3^2(w)\big] + \big[L_3^1(z)L_1^2(z), L_3^1(w)L_1^2(w)\big] \\
= \frac{1}{z-w}\Big(& -\Big(L_3^1(z) - L_3^1(w)\Big)L_3^2(z)L_1^2(w) + L_3^1(w)\Big(L_1^2(z) - L_1^2(w)\Big)L_3^2(z) \\
&+ \Big(L_3^1(z) - L_3^1(w)\Big)L_1^2(z)L_3^2(w) - L_3^1(z)\Big(L_1^2(z) - L_1^2(w)\Big)L_3^2(w) \\
&- L_3^1(w)\Big(L_3^2(z) - L_3^2(w)\Big)L_1^2(z) + L_3^1(z)\Big(L_3^2(z) - L_3^2(w)\Big)L_1^2(w)\Big) \\
=& \ 0.
\end{aligned}
$$

Hence is clear that

$$
[A(z), A(w)] = 0, \quad z, w \in \mathbb{C}. \tag{3.25}
$$

It is immediate from (3.25) that the conjugate variables $A(z)|_{z\to q_i}$ also commute.

Finally, we also require the variables q_i and $A(z)|_{z\to q_i}$ to satisfy the third canonical commutation relation

$$\left[q_i, A(z)|_{z\to q_j}\right] = \delta_{ij}. \tag{3.26}$$

It is this condition that will allow us to write

$$A(z)|_{z\to q_i} = -\partial_{q_i} \tag{3.27}$$

in the interpretation of the separated variables as a differential operator realisation of the $\mathfrak{gl}_3$-Gaudin model. Equation (3.26) will follow from the Lax algebra - in particular from the commutation relations of our separating functions $[A(z), B(w)]$.

To calculate $[A(z), B(w)]$, we first establish a related identity without inverse terms

$$\begin{aligned}(z-w)\Big(L_{13}^{12}(z)B(w)L_3^2(z) - L_3^2(z)B(w)L_{13}^{12}(z)\Big)\\ = \tilde{B}(z)(L_3^2(w))^2 - L_3^2(z)B(w)L_3^2(z)\end{aligned} \tag{3.28}$$

where we have introduced $\tilde{B}(z)$

$$\tilde{B}(z) = L_{13}^{12}(z)L_3^1(z) - L_3^2(z)L_{23}^{21}(z). \tag{3.29}$$

Note that (3.28) is the Gaudin limit of the identity [59, Eq.(29)] from the corresponding XXX chain proof.

Using the definition of $B(w)$, we see that the left-hand side of (3.28) can be written in terms of the following Lie brackets

$$\begin{aligned}(z-w)\Big(&L_{31}^{21}(w)\big[L_{13}^{12}(z), L_3^1(w)\big]L_3^2(z)\\ &+ L_3^2(w)\big[L_3^2(z), L_{32}^{12}(w)\big]L_{13}^{12}(z) - L_3^2(w)\big[L_{13}^{12}(z), L_{32}^{12}(w)\big]L_3^2(z)\Big),\end{aligned}$$

which by (3.22) and (3.23) becomes

$$\begin{aligned}&L_{13}^{12}(w)\Big(L_3^1(z)L_3^2(w) - L_3^1(w)L_3^2(z)\Big)L_3^2(z)\\ &+ L_3^2(w)\Big(L_3^1(z)L_3^2(w) - L_3^1(w)L_3^2(z)\Big)L_{13}^{12}(z)\\ &- L_3^2(w)\Bigg(L_{23}^{21}(z)L_3^2(w) - L_3^2(z)L_{32}^{12}(w) + L_{31}^{21}(w)L_3^1(z)\\ &\qquad - L_3^1(w)L_{13}^{12}(z) + \frac{1}{z-w}\Big(L_3^1(z)L_3^2(w) - L_3^2(z)L_3^1(w)\Big)\Bigg)L_3^2(z)\\ =\ &L_3^1(z)L_{13}^{12}(z)\big(L_3^2(w)\big)^2 - L_3^2(z)L_3^1(w)L_{31}^{21}(w)L_3^2(z)\\ &+ L_3^2(z)L_{23}^{21}(z)\big(L_3^2(w)\big)^2 - L_3^2(z)L_3^2(w)L_{32}^{12}(w)L_3^2(z).\end{aligned}$$

Using the definitions of $B(z)$ and $\tilde{B}(z)$, we can regroup this to give exactly the right-hand side of identity (3.28).

This identity (3.28) is relevant because if we multiply by $(L_3^2(z))^{-1}$ from both the left and the right, the left-hand side becomes

$$(z-w)[A(z), B(w)],$$

where we recall that $L_3^2(z)$ and $L_{13}^{12}(z)$ commute so the ordering in the definition of $A(z)$ is unimportant.

For the right-hand side we first note that it follows straightforwardly from the Lax algebra that $B(z)$ and $\tilde{B}(z)$ satisfy

$$L_3^2(z)\tilde{B}(z) = B(z)L_3^2(z),$$

and therefore

$$(L_3^2(z))^{-1}\tilde{B}(z)(L_3^2(w))^2(L_3^2(z))^{-1} = B(z)(L_3^2(z))^{-2}(L_3^2(w))^2.$$

Overall, this leaves us with the relation

$$[A(z), B(w)] = \frac{B(z)(L_3^2(z))^{-2}(L_3^2(w))^2 - B(w)}{z-w}. \tag{3.30}$$

Using (3.30) we will prove that the separated variables satisfy (3.26). Firstly, since we have seen that the variables commute we can pull the factor $(w-q_j)$ out to the front of $B(w)$ to rewrite the latter as

$$B(w) = (w-q_j)\mathcal{B}(w). \tag{3.31}$$

Note that since the variables are all distinct zeroes of $B(z)$, there is no repeated factor and we must have that $\mathcal{B}(q_j) \neq 0$. Therefore we can expand the left-hand side of equation (3.30) while substituting $z \to q_i$ as

$$[A(z)|_{z\to q_i}, B(w)] = (w-q_j)[A(z)|_{z\to q_i}, \mathcal{B}(w)] - [A(z)|_{z\to q_i}, q_j]\mathcal{B}(w). \tag{3.32}$$

If we set $w \to q_j$, the first term on the left-hand side disappears, leaving

$$-[A(z)|_{z\to q_i}, q_j]\mathcal{B}(q_j). \tag{3.33}$$

For the right hand side of (3.30), the definition of the separated variables tells us that substituting $z \to q_i$ into $B(z)$ is simply zero, which leaves

$$\frac{(w-q_j)\mathcal{B}(w)}{w-q_i}. \tag{3.34}$$

If $q_i = q_j$, then we can simplify this to just $\mathcal{B}(q_j)$. Therefore, equating left and right hand sides tells us that

$$[q_j, A(z)|_{z\to q_i}] = 1$$

On the other hand, the right-hand side vanishes on substituting $w \to q_j$ if $q_i \neq q_j$. Combining these two results we reach equation (3.26).

Going forward we will work in the separated variable realisation of the Gaudin model and write the conjugate coordinates as $A(z)|_{z\to q_i} = -\partial_{q_i}$, as is consistent with the canonical commutation relations (3.24) - (3.26) that we have shown here. We do not include a scalar function analogous to $\Lambda(z)$ as in the $\mathfrak{sl}_2$ case (although this would also be consistent with the canonical commutation relations) as it does not arise naturally here, and obscures the relation to the oper.

3.2.2 The separated equation

We now turn to reconstructing equation (3.18) directly from the Lax algebra of the Gaudin model. Analogously to the $\mathfrak{sl}_2$-case in the previous chapter we do this by relating the generating functions for the integrals of motion to our separating functions $A(z)$ and $B(z)$, and substituting in each variable. Since we are looking to find a third order differential equation we find an expression cubic in $A(z)$, as (2.18) was quadratic in $A(z)$ in the previous chapter.

In the rank 2 case we also found an additional derivative term of $A(z)$ arising from the rearranging which handily cancelled terms arising in the subtleties of the substitution from the left. It is useful to introduce particular groupings of such combinations of $A(z)$ and its derivatives called "quantum powers" of $A(z)$, based on the very similar quantum powers of the Lax matrix in [11]. Denoted by square brackets, the quantum powers are defined as

$$A^{[0]}(z) = 1, \quad A^{[k+1]}(z) = A(z)A^{[k]}(z) - \left(A^{[k]}(z)\right)', \tag{3.35}$$

where the dashes denote differentiation with respect to z. In particular we recognise $A^{[2]}(z) = A^2(z) - A'(z)$ from (2.28), and recall that we showed that

$$\left(A^{[2]}(z)\right)\Big|_{z\to q_i} = \partial_{q_i}^2.$$

The purpose of defining the quantum powers is to generalise this idea to higher powers of ∂_{q_i}, that is

$$A^{[k]}(z)|_{z\to q_i \textbf{from left}} = (-1)^k \partial_{q_i}^{\,k}. \tag{3.36}$$

We can show this inductively, the base case $k = 1$ being given by (3.27). Assume that $A^{[k]}(z)|_{z\to q_i} = (-1)^k \partial_{q_i}$, then by definition $A^{[k+1]}(z)|_{z\to q_i}$ is given by

$$\left(A^{[k+1]}(z)\right)|_{z\to q_i} = \left(A(z)A^{[k]}(z) - (A^{[k]}(z))'\right)|_{z\to q_i}.$$

As $A(z)|_{z\to q_i}$ are precisely the conjugate variables $-\partial_{q_i}$ we can substitute into the leftmost part of the expression and rearrange

$$\begin{aligned}\left(A^{[k+1]}(z)\right)|_{z\to q_i} &= \left(-\partial_{q_i} A^{[k]}(z) - (A^{[k]}(z))'\right)|_{z\to q_i}\\ &= \left(-A^{[k]}(z)\partial_{q_i} - \left[\partial_{q_i}, A^{[k]}(z)\right] - (A^{[k]}(z))'\right)|_{z\to q_i}.\end{aligned}$$

We can now both use our assumption and rewrite the bracket $[\partial_{q_i}, A^{[k]}(z)]|_{z\to q_i}$ as the difference between q_i dependence after substitution and the z dependence before substitution;

$$\begin{aligned}&\left(A^{[k+1]}(z)\right)|_{z\to q_i}\\ &\quad= (-1)^{k+1}\partial_{q_i}^{k+1} - \left[\partial_{q_i}, A^{[k]}(z)|_{z\to q_i}\right] + \left[\partial_z, A^{[k]}(z)\right]|_{z\to q_i} - (A^{[k]}(z))'|_{z\to q_i}\\ &\quad= (-1)^{k+1}\partial_{q_i}^{k+1} - \left[\partial_{q_i}, \partial_{q_i}^{k}\right] + (A^{[k]}(z))'|_{z\to q_i} - (A^{[k]}(z))'|_{z\to q_i}\\ &\quad= (-1)^{k+1}\partial_{q_i}^{k+1},\end{aligned}$$

all additional terms cancel nicely leaving the desired expression.

In particular for rank 3 SoV we will require

$$A^{[2]}(z)|_{z\to q_i} = \left(A^2(z) - A'(z)\right)|_{z\to q_i} = \partial_{q_i}^2, \tag{3.37a}$$

$$A^{[3]}(z)|_{z\to q_i} = \left(A^3(z) - 2A(z)A'(z) - A(z)A'(z) + A''(z)\right)|_{z\to q_i} = -\partial_{q_i}^3. \tag{3.37b}$$

Using these quantum powers, we can write down the relation between our separating functions that will ultimately lead to our separated equation

$$-A^{[3]}(z) + A^{[2]}(z)\widehat{s}_1(z) - A(z)\widehat{s}_2(z) + \widehat{s}_3(z) = -B(z)\left(L_3^2(z)\right)^{-3} R(z) \tag{3.38}$$

where

$$R(z) = L_{13}^{12}(z)L_1^2(z) - L_3^2(z)L_{31}^{32}(z). \tag{3.39}$$

We want to show (3.38) using only the Lax algebra in order to ensure the SoV we found as a limit of the XXX SoV is equally applicable to both $\mathfrak{gl}_3$ and the irregular singularities case.

As we did for $[A(z), B(z)]$, we will first show a related identity of Lax matrix elements that does not involve the more complicated inverse terms;

$$\begin{aligned}
&-\big(L^{12}_{13}(z)\big)^3 - 3\big(L^{12}_{13}(z)\big)^2 L^{2\prime}_3(z) + 3\big(L^{12}_{13}(z)\big)' L^2_3(z) L^{12}_{13}(z) - 2L^{12}_{13}(z)\big(L^{2\prime}_3(z)\big)^2 \\
&+ L^{12}_{13}(z) L^{2\prime\prime}_3(z) L^2_3(z) - \big(L^{12}_{13}(z)\big)''\big(L^2_3(z)\big)^2 + 2L^{2\prime}_3(z)\big(L^{12}_{13}(z)\big)' L^2_3(z) \\
&+ \Big((L^{12}_{13}(z))^2 L^2_3 - (L^{12}_{13}(z))'(L^2_3(z))^2 + L^{2\prime}_3(z) L^2_3(z) L^{12}_{13}(z)\Big)\widehat{s}_1(z) \\
&- L^{12}_{13}(z)\big(L^2_3(z)\big)^2 \widehat{s}_2(z) + \big(L^2_3(z)\big)^3 \widehat{s}_3(z) = \widehat{B}(z) R(z),
\end{aligned} \tag{3.40}$$

where

$$\widehat{B}(z) = L^{21}_{31}(z) L^1_3(z) - L^2_3(z) L^{12}_{32}(z) - 3L^2_3(z) L^{1\prime}_3(z) + 3L^{2\prime}_3(z) L^1_3(z).$$

This identity (3.40) has been checked analytically using Mathematica, but not included here since it is a long-winded application of the Lax algebra relations and not particularly insightful.

Having confirmed this, we can multiply on the left by $\big(-L^2_3(z)\big)^{-3}$, and regroup the left-hand side in terms of $A(z)$

$$-A^3(z) + 3A(z)A'(z) - A''(z) + \Big(A^2(z) - A'(z)\Big)\widehat{s}_1(z) - A(z)\widehat{s}_2(z) + \widehat{s}_3(z).$$

We recognise the quantum powers $A^{[3]}(z)$ and $A^{[2]}(z)$ from equations (3.37) above to recover the left-hand side of (3.38).

For the right-hand side, note that

$$\widehat{B}(z)\big(L^2_3(z)\big)^3 = \big(L^2_3(z)\big)^3 B(z),$$

which follows straightforwardly from the commutation relations

$$\Big[\big(L^2_3(z)\big)^3, L^{12}_{32}(z)\Big] = 3\big(L^2_3(z)\big)^3 L^{1\prime}_3(z) - 3L^1_3(z) L^{2\prime}_3(z)\big(L^2_3(z)\big)^2.$$

Therefore, when we multiply the right-hand side by $(L^2_3(z))^{-3}$, we have

$$\big(L^2_3(z)\big)^{-3}\widehat{B}(z)R(z) = B(z)\big(L^2_3(z)\big)^{-3}R(z)$$

as required.

Thus, simply by substituting each of our separated variables q_i into our cubic identity (3.38) from the left, we find that by definition of the variable q_i as a zero of $B(z)$, the right-hand side immediately vanishes

$$\Big(B(z)\big(L^2_3(z)\big)^{-3}R(z)\Big)\Big|_{z\to q_i} = \Big(B(z)\Big|_{z\to q_i}\big(L^2_3(z)\big)^{-3}R(z)\Big)\Big|_{z\to q_i} = 0. \tag{3.41}$$

Whereas on the left-hand side of (3.38), we can use the property (3.36) of the quantum powers of $A(z)$

$$\left(\partial_{q_i}^3 + \partial_{q_i}^2 \widehat{s}_1(z) + \partial_{q_i} \widehat{s}_2(z) + \widehat{s}_3(z)\right)\Big|_{z\to q_i}. \tag{3.42}$$

This gives us overall $3\mathcal{D}$ polynomials each in only one conjugate pair of the separated variables and integrals of motion, so already provides a separation of variables for the system. We now go on to shape it into equation (3.18). If we apply the above to a common eigenfunction Ψ of the quantum Hamiltonians (and therefore an eigenfunction of $\widehat{s}_1(z)$, $\widehat{s}_2(z)$,and $\widehat{s}_3(z)$) we can exchange the operator generating functions $\widehat{s}_k(z)$ for their corresponding eigenvalues $s_k(z)$

$$\begin{aligned}&\left(\partial_{q_i}^3 + \partial_{q_i}^2 \widehat{s}_1(z) + \partial_{q_i} \widehat{s}_2(z) + \widehat{s}_3(z)\right)\Big|_{z\to q_i}\Psi \\ &= \left(\partial_{q_i}^3 + \partial_{q_i}^2 s_1(z) + \partial_{q_i} s_2(z) + s_3(z)\right)\Big|_{z\to q_i}\Psi = 0.\end{aligned}$$

Without the operators, we may move $s_1(z)$, $s_2(z)$, and $s_3(z)$ past the partial derivatives without problem, and we need not be so careful when making the $z \to q_i$ substitution. This fully recovers (3.18), with the additional ∂_{q_i} term as the Lax matrix is no longer necessarily traceless,

$$\left(\partial_{q_i}^3 + s_1(q_i)\partial_{q_i}^2 + s_2(q_i)\partial_{q_i} + s_3(q_i)\right)\Psi = 0. \tag{3.43}$$

Thus we have a separation of variables for the $\mathfrak{gl}_3$ Gaudin model with irregular singularities in the variables $\{q_i\}_{i=1}^{3\mathcal{D}}$. The problem of finding the common eigenfunction $\Psi(q_1, \ldots, q_{3\mathcal{D}})$ may be reduced to one-dimensional problems using the Ansatz

$$\Psi(q_1, \ldots, q_{3\mathcal{D}}) = \prod_{i=1}^{3\mathcal{D}} \psi_i(q_i), \tag{3.44}$$

where each $\psi_i(q_i)$ satisfies an equation of the form

$$\psi_i''' + s_1\psi'' + s_2\psi + s_3 = 0. \tag{3.45}$$

Note again the similarity of 3.43 to the universal oper

$$\operatorname{cdet}\left(\partial_z + L(z)\right) = \partial_z^3 + \widehat{s}_1(z)\partial_z^2 + \widehat{s}_2(z)\partial_z + \widehat{s}_3(z), \tag{3.46}$$

which we also saw in the rank 2 case.

3.3 Approaches to Higher rank

Now we have constructed the separation of variables in these two lowest rank case, we can think about how to generalise this to cover Gaudin models of matrix Lie algebras of some arbitrary rank n. In this section we will discuss several approaches we have tried to systematically construct the separation of variables such that it would generalise. While none of these have led to a generalised separation of variables for the $\mathfrak{gl}_n$-Gaudin model, none are entirely unfruitful and together they form a picture of what we might expect the operator $B(z)$ and the separated equation might look like in this more general setting.

Let us discuss some general trends across the two cases we have looked at so far that might continue. In both cases the separation of variables hinges on the two operator separating functions $A(z)$ and $B(z)$ - simply elements of the Lax matrix for the $\mathfrak{sl}_2$-Gaudin model, and already much more complicated rational functions for the next rank up. We see that $A(z)$ is of order 1 overall in Lax matrix elements, whilst in each of the cases $B(z)$ is of order $\frac{n(n-1)}{2}$ in these - that is for $n=2$, $B(z)$ consists of just one Lax matrix element, and for $n=3$ it is cubic in them. This is necessary to ensure that $B(z)$ has $\frac{1}{2}n(n-1)\mathcal{D}$ operator zeroes, which as we have seen provide the separated variables.

In both cases, what makes $A(z)$ and $B(z)$ good candidates for separating functions ultimately arises from the commutation relations. We would require that

$$[B(z), B(w)] = 0 \tag{3.47a}$$

$$[A(z), A(w)] = 0 \tag{3.47b}$$

$$[A(z), B(z)] = \frac{B(z)Q(z,w) - B(w)}{z-w} \tag{3.47c}$$

where $Q(z,w)$ is some rational function of matrix elements of $L(z)$ and $L(w)$. For the cases considered thus far $Q(z,w)=1$ and $Q(z,w)=\left(L_3^2(z)\right)^{-2}\left(L_3^2(w)\right)^2$ - in both we can note that $Q(z)$ is made up of the denominator of $A(z)$. If these commutation relations are satisfied then the argument we used previously applies here as well; separated variables defined as the operator zeroes q_j of $B(z)$ and the conjugates as $A(z)|_{z\to q_j}$ (where substitution is from the left) satisfy canonical commutation relations

$$[q_i, q_j] = 0, \quad \left[A(z)|_{z\to q_i}, A(z)|_{z\to q_j}\right] = 0, \quad \left[q_i, A(z)|_{z\to q_j}\right] = \delta_{ij}, \tag{3.48}$$

for all $i, j = 1, \ldots, \frac{n(n-1)}{2}\mathcal{D}$.

Another trend across the past two chapters has been the clear similarity between the equation in the separated variables and the universal oper cdet $\big(\partial_z + L(z)\big)$ in each case. The prevalence of the oper in determining a separated equation for the Gaudin model (at least for the $\mathfrak{sl}_2$ and $\mathfrak{gl}_3$ cases) suggests that there may be some deeper connection worthy of further investigation, and as discussed in Chapter 2 gives some credence to the general belief that SoV is complete. Were this pattern to continue, we would expect at higher rank a simultaneous eigenfunction Ψ of the Gaudin Hamiltonians to satisfy an equation of the form

$$\sum_{k=0}^{n} s_{n-k}(q_i)\partial_{q_i}^{k}\Psi = 0 \tag{3.49}$$

in each of the separated variables q_i, where $s_0(z) = 1$. (This notation is slightly inconsistent with that used in the $\mathfrak{sl}_2$ case discussed in the previous chapter, as the quantity that would be $\widehat{s}_1(z)$ by the conventions above vanishes due to tracelessness, though the similarity to the oper is still present.)

We know that using the quantum powers of $A(z)$, this would follow simply by substitution $z \to q_i$ from the left if suitable separating functions $A(z)$ and $B(z)$ satisfy some identity

$$\sum_{k=0}^{n}(-1)^k A^{[k]}(z)\widehat{s}_{n-k}(z) = B(z)R(z). \tag{3.50}$$

Here $R(z)$ is some rational function of Lax operator elements - and must certainly contain the inverse of the Lax matrix for any rank higher than $n = 2$ to balance the order on both sides. We note that the property (3.36) of the quantum powers would still apply as long as $A(z)|_{z\to q_i} = -\partial_{q_i}$.

The difficulty then, is finding a method applicable at any rank to construct separating functions of the right order that also satisfy relations akin to (3.47) and (3.50). Below we detail some methods, drawing again from the attempts put forward for quadratic Lax relations, and of the separation of variables for the classical $\mathfrak{gl}_n$-Gaudin model (which has been proven for arbitrary n by Skrypnyk and Dubrovin in their article [23]). It seems that $A(z)$ is the more complicated of the two separating functions to consistently construct, possibly due to the requirement that it is order 1 in Lax matrix components and must involve inverses in some sense.

3.3.1 Manin matrix approach

Chervov and Falqui [10, 11] lay out a procedure to construct $A(z)$ and $B(z)$ in the case of the XXX chain by making use of Manin matrices. Following the analogue of this method for the Gaudin model, we define a new matrix $M(z)$ by taking the coefficients of powers of ∂_z in the last column of the classical adjoint of $\partial_z + L(z)$, and wish to use this same matrix to construct both $A(z)$ and $B(z)$. For the rank 2 and rank 3 cases we've already covered, $M(z)$ would respectively be

$$M(z) = \begin{pmatrix} -L^1_2(z) & L^1_1(z) \\ 0 & 1 \end{pmatrix}, \quad M(z) = \begin{pmatrix} -L^{21}_{23}(z) & -L^{12}_{13}(z) & L^{12}_{12}(z) + L^{1\prime}_1(z) \\ L^1_3(z) & -L^2_3(z) & L^1_1(z) + L^2_2(z) \\ 0 & 0 & 1 \end{pmatrix}.$$

In both of these cases, $M(z)$ is a Manin matrix and the separating functions $B(z)$ is found (up to a negative sign) as the column ordered determinant

$$B(z) = \operatorname{cdet} M(z), \tag{3.51}$$

which is in accordance with the expressions for $B(z)$ we have found so far in (2.5) and (3.9b). That $B(z)$ is constructed in the same way for each rank is hopeful, but we still need to define $A(z)$. In this method we only define $A(z)$ at the separated variables themselves, we will label these $p_i = A(z)_{z\to q_i}$. We make a vector of powers of p_i, and define it by the equation

$$M(z)^t|_{z\mapsto q_i} \begin{pmatrix} 1 \\ p_i \\ \vdots \\ p_i^{n-1}(z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.52}$$

For $\mathfrak{sl}_2$, this does give us the expected result (again up to a sign) of $A(z) = -L^1_1$, but we run into problems for $\mathfrak{gl}_3$ where from substituting the correct $M(z)$ into (3.52) we have that both of the following hold

$$-L^{21}_{23}(z)|_{z\to q_i} + L^1_3(z)|_{z\to q_i} p_i = 0 \tag{3.53a}$$

$$-L^{12}_{13}(z)|_{z\to q_i} + L^2_3(z)|_{z\to q_i} p_i = 0. \tag{3.53b}$$

Clearly the latter is the definition of $A(z)$ we have found to work in this chapter, but these two statements are not equivalent as we would hope.

To see this let us start with the second expression and multiply by $L_3^1(z)|_{z\to q_i}$ on the left, leaving

$$-L_3^1(z)|_{z\to q_i}L_{12}^{12}(z)|_{z\to q_i} + L_3^1(z)|_{z\to q_i}L_3^2(z)|_{z\to q_i}p_i = 0.$$

Note that we can reorder $B(z)$ to be

$$B(z) = L_3^1(z)L_{13}^{12}(z) - L_{23}^{21}(z)L_3^2(z), \tag{3.54}$$

with additional commutation terms cancelling off. Using this, and that $B(z)|_{z\to q_i} = 0$ by definition, we have that

$$-L_{23}^{21}(z)|_{z\to q_i}L_3^2(z)|_{z\to q_i} + L_3^2(z)|_{z\to q_i}L_3^1(z)_{z\to q_i}p_i = 0.$$

Which we rewrite as

$$L_3^2(z)|_{z\to q_i}\Big(-L_{23}^{21}(z)|_{z\to q_i} + L_3^1(z)|_{z\to q_i}p_i\Big) + \big[L_3^2(z)|_{z\to q_i}, L_{23}^{21}(z)|_{z\to q_i}\big].$$

Since

$$\big[L_3^2(z)|_{z\to q_i}, L_{23}^{21}(z)|_{z\to q_i}\big] = L_3^2(z)|_{z\to q_i}L_3^{1\prime}(z)|_{z\to q_i} - L_3^1(z)|_{z\to q_i}L_3^{2\prime}(z)|_{z\to q_i} \neq 0,$$

we see that the two expressions in (3.53) are in fact not equivalent. Therefore it is unclear which equation to choose when defining $A(z)$ more generally and this approach reaches a stumbling block. Interestingly, if the condition of the separated variables were instead that $\widetilde{B}(z)|_{z\to q_i} = 0$, as defined in (3.29) then the two equations for $A(z)$ *would* be equivalent.

3.3.2 SKRYPNYK-DUBROVIN APPROACH

Skrypnyk and Dubrovin constructed a separation of variables for the *classical* $\mathfrak{gl}_n$-Gaudin model at any rank [23], and so the next approach is to use this as our guide to systematically separate the variables for the *quantum* $\mathfrak{sl}_2$ and $\mathfrak{gl}_3$ Gaudin models and beyond.

In their work, both of the separating functions are once again constructed from determinants of matrices built this time out of powers of the Lax matrix. We can carefully quantise this, and get back once again the low rank separating functions in the form familiar to us. A naive quantisation - by which we mean straightforwardly replacing the Lax matrix of the classical Gaudin model with its quantum counterpart - fails to give us all of the expected expression for our separating functions. To get around this, we will return to the Chervov and Falqui definition of *quantum powers* of $L(z)$ [11];

$$L^{[0]} = 1, \quad L^{[i+1]}(z) = L^{[i]}(z)L(z) - (L^{[i]}(z))'. \tag{3.55}$$

We can now simply take the Skrypnyk and Dubrovin expression for $B(z)$ in [63, Eq. 3.5] and replace powers of $L(z)$ with quantum powers.

Take a vector $\vec{k} = \begin{pmatrix} 0, & 0, & \dots, & 1 \end{pmatrix}^t$, the expression for $B(z)$ is given by

$$B(z) = \operatorname{rdet}\begin{pmatrix} \vec{k} & L(z)\vec{k} & L^{[2]}(z)\vec{k} & \cdots & L^{[n-1]}(z)\vec{k} \end{pmatrix}, \tag{3.56}$$

where the listed vectors are the columns of the matrix. We are using "rdet" (the *row-* ordered determinant) instead of cdet here to avoid adding transposes that may over-complicate the expression. In fact this matrix (or rather its transpose) is not Manin at rank as low as the $\mathfrak{gl}_3$ case, though at that particular rank the zeroes in the vector $\vec{k}$ sort out any problematic parts within the determinant. It is also worth remarking that while this non-Maninness prevents us from easily applying a wealth of techniques for matrices with commutative entries, we can still take a row-ordered determinant so long as we carefully specify an ordering of the columns and rows at the start of the process as we have here, with no guarantee that switching such an ordering produces the same result.

The other separating function $A(z)$ is written as a fraction of two determinants

$$A(z) = \frac{C(z)}{D(z)}, \tag{3.57}$$

each of which we can quantise with the same procedure as we just did with $B(z)$. Let $\vec{\xi} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^t$

$$C(z) = \operatorname{rdet}\left(\vec{k}, L(z)\vec{\xi}, L(z)\vec{k}, \cdots, L^{[n-2]}(z)\vec{k}\right), \tag{3.58a}$$

$$D(z) = \operatorname{rdet}\left(\vec{k}, \vec{\xi}, L(z)\vec{k}, \cdots, L^{[n-2]}(z)\vec{k}\right), \tag{3.58b}$$

(This is a slightly different but equivalent form to [23, Eq.(3.8)].) These are both non-unique and different choices on the vectors $\vec{k}$ and $\vec{\xi}$ will produce new separating functions.

For $\mathfrak{sl}_2$ (and $\mathfrak{gl}_2$), we obtain the expected expressions from (3.58) up to a sign

$$B(z) = \operatorname{rdet}\begin{pmatrix} 0 & L^1_2(z) \\ 1 & L^2_2(z) \end{pmatrix} = -L^1_2(z) \tag{3.59}$$

and (3.56)

$$C(z) = \operatorname{rdet}\begin{pmatrix} 0 & L_1^1(z) \\ 1 & L_1^2(z) \end{pmatrix} = L_1^1(z)$$
$$D(z) = \operatorname{rdet}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1.$$

Hence $A(z) = -L_1^1(z)$ as we saw in the previous chapter. They also provide the same expressions for $A(z)$ and $B(z)$ in the $\mathfrak{gl}_3$-Gaudin model that we had previously;

$$B(z) = \operatorname{rdet}\begin{pmatrix} 0 & L_3^1(z) & \sum_{i=1}^3 L_i^1(z)L_3^i(z) - L_3^{1\prime}(z) \\ 0 & L_3^2(z) & \sum_{i=1}^3 L_i^2(z)L_3^i(z) - L_3^{2\prime}(z) \\ 1 & L_3^3(z) & \sum_{i=1}^3 L_i^3(z)L_3^i(z) - L_3^{3\prime}(z) \end{pmatrix} = L_{32}^{12}(z)L_3^2 - L_3^1(z)L_{31}^{21}(z)$$

and

$$C(z) = \operatorname{rdet}\begin{pmatrix} 0 & L_1^1(z) & L_3^1(z) \\ 0 & L_1^2(z) & L_3^2(z) \\ 1 & L_1^3(z) & L_3^3(z) \end{pmatrix} = L_{13}^{12}(z) \tag{3.61a}$$
$$D(z) = \det\begin{pmatrix} 0 & 1 & L_3^1(z) \\ 0 & 0 & L_3^2(z) \\ 1 & 0 & L_3^3(z) \end{pmatrix} = L_3^2(z), \tag{3.61b}$$

hence again $A(z) = L_{13}^{12}(z)(L_3^2(z))^{-1}$.

Problems start arising when we attempt the $\mathfrak{gl}_4$ case - not only do we find that more relevant parts of the matrix are no longer Manin, but also the results we get for $C(z)$ and $D(z)$ no longer commute so there is some ambiguity in how to interpret the fraction (3.57) in terms of ordering. Furthermore, while the systematic construction of the operators that look like separating functions is certainly a start, the rest of the process does not straightforwardly translate to the quantum Gaudin model; we do not know if they are effective as separating functions beyond the rank 2 and 3 cases and what link, if any, there is between the opers $\operatorname{cdet}(\partial_z + L(z))$ and the separated equations.

In future, it would be interesting to check whether the two approaches, following Chervov-Falqui or Skrypnyk-Dubrovin, produce the same expression for $B(z)$ in the $n = 4$ case.

3.3.3 Cavaglia, Gromov, Levkovich-Maslyuk, Sizov Approach

In the approaches we have discussed so far it has been $A(z)$ that has proven more difficult to generalise to higher rank, so we now look to the work of Gromov, Cavaglia, Levkovich-Maslyuk, and Sizov on SoV in [34, 8] on $\mathfrak{sl}_n$ quantum spin chains, which only relies on the B-operator. Instead of trying to find a separated equation as we have thus far, we aim to describe eigenvectors of the Gaudin Hamiltonians as copies of $B(z)$ acting on some vacuum vector $|0\rangle$, and construct a separated eigenbasis. This of course limits us to highest weight representations to ensure such a reference vector.

In [34], the authors show that one may write the Bethe Vectors of Ψ of the XXX chain in terms of $B_{XXX}(z)$, and they therefore factorise

$$\Psi(w_1, \dots, w_m) = B(w_1)\cdots B(w_m)\,|0\rangle = B_0 \prod_{j=1}^{m}\prod_{k=1}^{N}(w_j - q_k), \tag{3.62}$$

where $w_1, \dots, w_m$ are Bethe roots in the XXX chain and B_0 is some constant. They go on to produce left eigenvectors $\langle w_1, \dots, w_m|$, however since $B_{XXX}(z)$ is not diagonalisable, this requires performing a generic twist

$$B^{\text{good}}_{XXX}(z) = K^{-1} B_{XXX}(z) K$$

for some matrix K. $B^{\text{good}}(z)$ still has separated variables as its zeroes but now has the advantage of producing a full basis of eigenvectors in which the Bethe vectors Ψ fully separate.

To bring these ideas across to the Gaudin model, we take a highest weight representation, which will be the fundamental representation in this case and try and express Bethe vectors as in (3.62). We have already seen in Section 2.2 that in the $\mathfrak{sl}_2$-Gaudin model the Bethe vectors can be written in exactly this form, because $B(z)$ is made up of lowering operators. We have the same problem of $B(z)$ not being diagonalisable, and so we can go on to perform a similar twist if we wish to find the separated eigenbasis.

For rank 3, $B(z)$ is more complicated but we can check this computationally for low N in the fundamental representation of $\mathfrak{gl}_3$ by comparing the action of $B(w)$ on the reference vector to the Bethe vector $\Psi_1(w)$.

In the representation, E^a_b are 3×3 matrices with a 1 in the (a, b)th slot and zeroes elsewhere, the reference vector is given by

$$|0\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\0\\0\end{pmatrix},$$

where the right-hand side has N tensor copies.

The first Bethe vector $\Psi_1(w)$ is given by

$$\Psi_1(w) = \Big(L_2^1(w) + L_3^2(w)\Big)\,|0\rangle\,, \tag{3.63}$$

which we have checked computationally for low N is the same as $B(w)\,|0\rangle$ if and only if w satisfies the Bethe equation

$$\sum_{k=1}^{N} \frac{1}{w - z_k} = 0. \tag{3.64}$$

Therefore, $\Psi_1(w)$ factorises, with each factor depending on only one of the separated variables;

$$\Psi_1(w) = B(w)\,|0\rangle = B_0 \prod_i (w - q_i)\,|0\rangle\,,$$

where B_0 is some constant factor. This has only been checked for the simple poles Gaudin model with $N = 2$ and $N = 3$.

We would hope, following the example of the XXX chain and the rank 2 SoV, that other Bethe vectors could also be written as

$$\Psi_m(w_1, \ldots, w_m) = B(w_1) \ldots B(w_m)\,|0\rangle\,, \tag{3.65}$$

however we found when we add in more Bethe roots that the Bethe vectors are given by a sum of copies of $B(w_k)$ acting on the reference vector in different combinations of the Bethe roots, and so does not clearly factorise.

For rank 4 and higher, [34] provides expressions for $B_{XXX}(z)$, of which we could potentially take the Gaudin limit to find a general expression for $B(z)$. As we are working from Bethe vectors this method does not produce any new vectors of the Gaudin model, but it can produce a separated basis in which to view them.

3.3.4 An Observation on Opers and SoV

Finally we have an observation from the oper perspective in the $\mathfrak{sl}_2$ case. We note that the quantum power $A^{[2]}(z)$ resembles a Miura transform of

$-A(z)$, though of course we have only truly defined Muira transformations for meromorphic functions, not operator valued functions.

Therefore if one could take an oper-like object with non-commutative matrix entries given by

$$\partial_z + \begin{pmatrix} A(z) & B(z)C(z) \\ 1 & -A(z) \end{pmatrix}$$

(which interestingly would be Miura-like if we substitute $z \to q_i$ from the left as the top right-hand entry would be zero by definition), then the canonical representative would be

$$\partial_z + \begin{pmatrix} 0 & -\widehat{s}_1(z) \\ 1 & 0 \end{pmatrix},$$

where we have used that

$$\widehat{s}_1(z) = A^{[2]}(z) - B(z)C(z). \tag{3.66}$$

Therefore, if we transform this to a second order differential equation as we did for a generic $\mathfrak{sl}_2$-oper in (2.72), we recover the separated equation for the $\mathfrak{sl}_2$ Gaudin model;

$$\partial_{q_i}^2 \phi - \widehat{s}_1(z)|_{z \to q_i} \phi = 0.$$

To adapt this to the $\mathfrak{gl}_2$-Gaudin model we can instead begin with

$$\partial_z + \begin{pmatrix} A(z) & B(z)C(z) \\ 1 & \operatorname{Tr} L(z) - A(z) \end{pmatrix}. \tag{3.67}$$

While this remark does seem to tie together several elements of the $\mathfrak{sl}_2$ SoV, it is not clear how we would justify the starting differential operator. It also does not offer us any construction of the separating functions $A(z)$ and $B(z)$ at higher rank.

Part II

Gaudin Models from 3dBF Theory

— 4 —

3dBF Theory and Finite Gaudin models

In the second part of this thesis we turn to look at the classical Gaudin model for arbitrary Lie algebra $\mathfrak{g}$, and explore integrability from the more recent perspective of gauge theories. We present the work we have previously published in the article [67]. This alternative gauge theoretic perspective was put forth by Costello and later developed alongside Witten and Yamazaki with regards to 4-dimensional Chern-Simons Theory and its encoded description of the Yangian [14, 15] - or see [71] for an introduction. Furthermore, this was extended to a study of 2 dimensional integrable field theories, also from 4-dimensional Chern-Simons Theory [16]. Another area of study provides a more algebraic description of integrable field theories from the perspective of affine Gaudin models [66, 25, 17]. Naturally, this led to consideration of a link between 4 dimensional Chern-Simons and affine Gaudin models - and indeed Vicedo has shown [65] that one can obtain the Lax algebra of the affine Gaudin model by a certain gauge fixing of 4 dimensional Chern-Simons Theory. This is our chief motivation for analagously describing the finite Gaudin model from a 3-dimensional gauge theory.

In particular, we will show that the finite Gaudin model can be described using a collection of line defects on a 3-dimensional mixed BF Theory on $\mathbb{R} \times \mathbb{C}P^1$. Moreover, we can describe tamely ramified Hitchin systems [3, 37] on higher genus surfaces in the same way by taking an arbitrary Riemann surface C instead of $\mathbb{C}P^1$.

The main interest in both approaches lies in their potential to offer new perspectives on *quantum* integrable field theory. By contrast with the affine case, the quantisation of the finite Gaudin model, and more generally of the Hitchin system, is extremely well understood; see *e.g.* [5, 26, 28, 29, 46, 48, 49, 47, 45, 24, 27, 55]. The connection between 3d mixed BF theory and finite Gaudin models should therefore provide a useful toy model for further developing our understanding of the gauge theoretic approach to integrable models and more generally integrable field theories in the sense of [16]. In particular, it would be very desirable to understand the Bethe ansatz construction in quantum Gaudin models, and more generally the Gaudin/oper correspondence from the point of view of quantum 3d mixed BF theory. It is expected that the quantum Gaudin model, and more generally the quantisation of the Hitchin system, should arise from critical level quantum 3d Chern-Simons theory [30, 72], which we will see may be viewed as a certain deformation of quantum 3d mixed BF theory.

4.1 4-dimensional Chern-Simons and integrability

Let us motivate the gauge theoretic perspective by discussing integrability through the lens of 4-dimensional semi-holomorphic 4-dimensional Chern-Simons theory (4dCS) in particular. The results presented in this chapter are largely inspired by the connection between 4dCS and affine Gaudin models described by Vicedo in [65].

To define the theory, we let G be a semisimple Lie group over $\mathbb{C}$, with Lie algebra $\mathfrak{g}$ and fix a non-degenerate invariant symmetric bilinear form $\langle\cdot,\cdot\rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ on $\mathfrak{g}$ - for example in a matrix representation this could be $\langle X, Y\rangle = \mathrm{Tr}(XY)$. We let Σ be a 2-manifold (it will be either $\mathbb{R}^2$ or $\mathbb{C}^* = \mathbb{R}\times S^1$ here) with coordinates t and σ and let z be a holomorphic coordinate on $\mathbb{C}P^1$, the theory will be over $\Sigma\times\mathbb{C}P^1$. The Chern-Simons 3-form is then given by

$$CS(\mathcal{A}) = \langle\mathcal{A}, \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}\rangle\,. \tag{4.1}$$

To make the 4-dimensional version of the theory we also need to introduce a one-form ω

$$\omega = \varphi(z)\mathrm{d}z, \tag{4.2}$$

in the examples that follow we will choose the function $\varphi(z)$ to be holomorphic or meromorphic depending on the situation. Putting the above together we reach the action of 4dCS

$$S_{4dCS}[\mathcal{A}] = \frac{i}{4\pi}\int_{\Sigma\times\mathbb{C}P^1}\omega\wedge CS(\mathcal{A}). \tag{4.3}$$

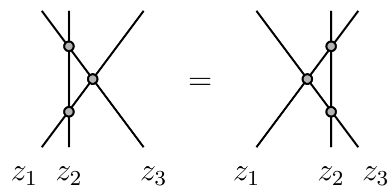
The equations of motion are the condition that the gauge field $\mathcal{A}$ has zero curvature away from any zeroes ω might have;

$$\omega \wedge F(\mathcal{A}) = 0, \tag{4.4}$$

with the curvature 2-form $F(\mathcal{A}) = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$.

The XXX chain The XXX spin chain can be obtained from 4-dimensional mixed topological-holomorphic Chern-Simons theory on $\mathbb{R}^2 \times \mathbb{C}$ by introducing certain line defects along the topological plane $\mathbb{R}^2$ for each site of the spin chain [13, 12, 14, 15]. This elegant description of the Heisenberg spin chain is ultimately possible because the integrable structure of the latter is underpinned by the *quantum* Yang-Baxter equation

$$\begin{aligned} &R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3) \\ &\qquad = R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2) \end{aligned}$$

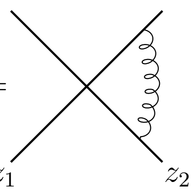


We can recreate this visual description of the integrable structure in the plane Σ of 4dCS, with the spectral parameters z_1, z_2, z_3 being fixed points in $\mathbb{C}P^1$. The symmetry of Σ under diffeomorphisms allows us to place gauge-invariant Wilson lines along Σ in the plane. That is, we take one-dimensional lines $\ell \subset \Sigma$ and a suitable representation ρ and define the Wilson line as

$$W(\ell) = \mathrm{Tr}_\rho P \exp \oint_\ell \mathcal{A}_i(t, \sigma, z, \bar{z}) \mathrm{d}\xi^i, \tag{4.5}$$

where $\xi^i = t, \sigma$ respectively, and $P\exp$ denotes the path ordered exponential. Each loop is labelled by a point $z = z_i$ in $\mathbb{C}$, and the picture that emerges when putting multiple Wilson lines onto Σ is already reminiscent of our pictorial understanding of the Yang-Baxter equation. The leading order quantum correction to the product of two Wilson loops (4.5) gives the classical r-matrix $r(z_1, z_2) = \frac{C_{12}}{z_1 - z_2}$, as shown in [14, Eq.(4.13)], or diagrammatically;

$$r(z_1, z_2) =$$



Therefore we can view the crossing points of the lines in Σ as the quantum R-matrix, $R(z_1, z_2) = 1 + \hbar r(z_1, z_2) + O(\hbar^2)$. Adding a third Wilson line to the picture, we note that we can slide them over one another and change the order of the crossing points due to the diffeomorphism invariance, with the distinct spectral parameters preventing any problems caused by a triple intersection of the lines, as they do not cross in the $\mathbb{C}P^1$ direction. We have therefore created a picture visually very reminiscent of the pictorial Yang-Baxter equation above. In [14], Costello, Witten, and Yamazaki go on to define and fully solidify this notion into a description of the XXX-chain from 4dCS.

Integrable Field Theories As well as lattice models, one can construct integrable field theories from 4dCS such as the principal chiral model - see [40] for a detailed introductory course. This is not entirely surprising when we recall that the condition on the gauge field $\mathcal{A}$ imposed by the equations of motion (namely that it has zero curvature away from the zeroes of ω) is exactly the same as the condition on the Lax connection of an integrable field theory in 2 dimensions (see e.g. [51]);

$$\partial_t \mathcal{L}_t - \partial_x \mathcal{L}_x + [\mathcal{L}_t, \mathcal{L}_x] = 0, \tag{4.6}$$

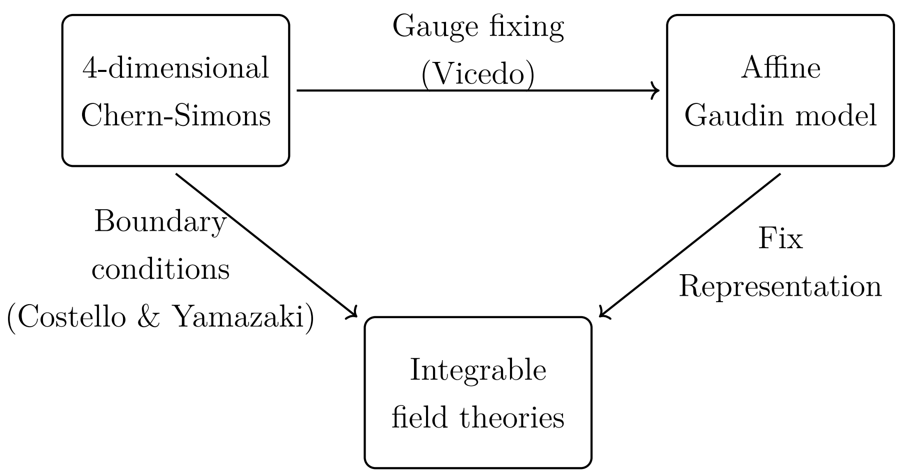
the analogue to the Lax equation of a finite dimensional integrable system. We can go on to construct the infinite integrable hierarchies from the Lax connection $\mathcal{L}$. Therefore if we identify the components of $\mathcal{A}$ in a gauge where $\mathcal{A}_{\bar{z}} = 0$ with $\mathcal{L}_t$ and $\mathcal{L}_x$ we have a flat, meromorphic Lax connection and hence an integrable field theory. To determine the specifics of the theory and find the fields involved, one can look into the boundary conditions given by the zeroes and poles of ω and the choice of gauge.

Since it is not always straightforward to specify a Lax connection [51], this gauge theoretic perspective may provide a more systematic way to find Lax connections for new integrable field theories.

Affine Gaudin models In contrast to the XXX-chain, the integrability of the (classical and quantum) Gaudin model, or more generally the Hitchin system, is underpinned by the *classical* Yang-Baxter equation

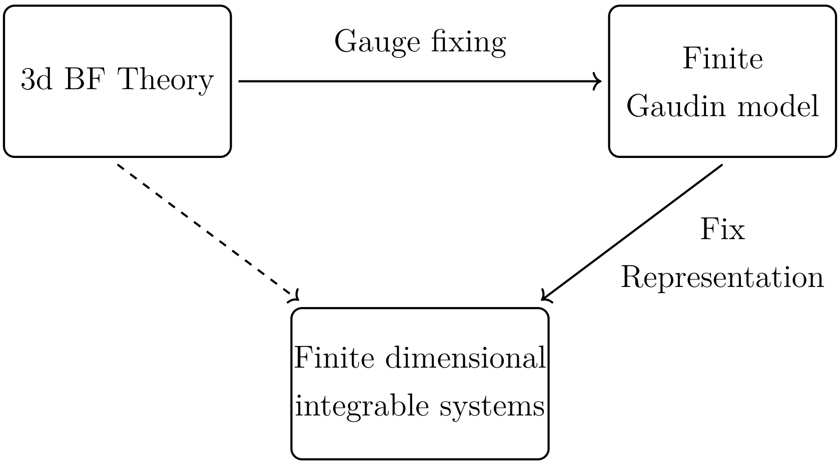
$$[r_{12}(z_1, z_2), r_{13}(z_1, z_3)] = [r_{23}(z_2, z_3), r_{12}(z_1, z_2)] - [r_{32}(z_3, z_2), r_{13}(z_1, z_3)], \tag{4.7}$$

whose topological origin is less clear. On the other hand, affine Gaudin models, whose integrability is also underpinned by the classical Yang-Baxter equation (4.7) (see [66]) can be obtained [65] from the same 4-dimensional mixed topological-holomorphic Chern-Simons theory on $\mathbb{R}^2 \times \mathbb{C}P^1$, this time by introducing surface defects along $\mathbb{R}^2$ placed at the marked points $z_i \in \mathbb{C}$ of the affine Gaudin model. Effectively, the $z, \bar{z}$ coordinates on $\mathbb{C}P^1$ play the role of the complex spectral parameter, while coordinates (t, σ) on $\mathbb{R}^2$ act as a time coordinate in the Lax equation and the additional loop parameter of the affine algebra $\widetilde{\mathfrak{g}}$ respectively. As affine Gaudin models can describe integrable field theories when we fix a representation of $\widetilde{\mathfrak{g}}$, we can summarise this part of the 4dCS story in the following diagram;



4.2 Finite Gaudin models and gauge theory

A natural question is therefore whether the ordinary Gaudin model, associated with a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ rather than an affine Kac-Moody algebra, can be described in a similar way. As we no longer require a coordinate to parameterise the loop algebra we can expect this to be a 3-dimensional theory, and the correct candidate turns out to be 3dBF theory - which we will introduce in the next section. We can summarise this too in a diagram analogous to the above;



The gauge fixing process is what forms the majority of this chapter, whereas the move from finite Gaudin models to finite dimensional integrable systems was described for example in [68, §5.4]. The final dotted arrow has not been separately constructed as it was in the four-dimensional case [40] but could effectively be considered as the composition of the other two processes - we include it here only to complete the analogy with the previous picture.

More precisely, we will perform a Hamiltonian analysis of the 3d mixed BF theory, whose fields are a partial connection 1-form A and a $(1,0)$-form B, with suitably chosen line defects. Using the condition $A_{\bar{z}} = 0$ to fix the gauge invariance, we find that the dynamics on the reduced phase space coincides with that of the finite Gaudin model, or for arbitrary genus tamely ramified Hitchin system [3, 37]. In particular, the $(1,0)$-form B becomes meromorphic and gets identified with the Higgs field - or the meromorphic Lax matrix in the genus 0 Gaudin case. This is completely analogous to the relationship found in [65] between 4d mixed topological-holomorphic Chern-Simons theory on $\Sigma \times \mathbb{C}P^1$, with the cylinder $\Sigma = \mathbb{R} \times S^1$, and the affine Gaudin model. In other words, our analysis will show that 3d mixed BF theory is to the Gaudin model what 4d Chern-Simons theory is to the affine Gaudin model.

4.3 3d mixed BF theory

Let us begin by defining the theory in question. Let G be a semisimple Lie group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$ and a fixed non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ on $\mathfrak{g}$. We also let C be a Riemann surface, to correspond to the Gaudin model we would take $C = \mathbb{C}P^1$, with higher genus surfaces corresponding to higher genus Hitchin systems.

We shall consider the 3-dimensional classical mixed topological-holomorphic BF theory on $\mathbb{R} \times C$, or 3*d mixed BF theory* for short – see *e.g.* [36, 35, 50] where the theory is discussed using the BV formalism. There are two $\mathfrak{g}$-valued fields; a $(1,0)$-form B on $\mathbb{R} \times C$, together with a $\mathfrak{g}$-valued connection 1-form A on $\mathbb{R} \times C$. This bigrading (p,q) is the one induced by the complex structure of C, such that in a local holomorphic coordinate z on some open subset of C they might be written as

$$B = B_z \mathrm{d}z, \quad A = A_t \mathrm{d}t + A_{\bar{z}} \mathrm{d}\bar{z} + A_z \mathrm{d}z.$$

We denote the curvature of the field A as $F(A) = \mathrm{d}A + \frac{1}{2}[A, A]$. The action of 3d mixed BF theory is then given simply by

$$S_{3d}[A, B] = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R} \times C} \langle B, F(A) \rangle. \tag{4.8}$$

4.3.1 Gauge invariance

The 3d mixed BF action (4.8) is trivially invariant under gauge transformations of the form $A \to A + \chi$ for any $\mathfrak{g}$-valued $(1,0)$-form χ on $\mathbb{R} \times C$. Indeed, χ drops out from the action since B is a $(1,0)$-form. We can fix this invariance by requiring that A has no $(1,0)$-component so that it locally takes the form $A = A_{\bar{z}} \mathrm{d}\bar{z} + A_t \mathrm{d}t$ for some local coordinate t on $\mathbb{R}$ and a local holomorphic coordinate z on C. From now on we will always take A to be a partial connection of this form.

More interestingly, the action (4.8) is invariant under the action of any G-valued function g on $\mathbb{R} \times C$ acting by gauge transformations on the connection 1-form A and by conjugation on the field B, namely

$$A \longmapsto {}^g A \coloneqq -\bar{\partial} g g^{-1} - \mathrm{d}_{\mathbb{R}} g g^{-1} + g A g^{-1}, \tag{4.9a}$$

$$B \longmapsto g B g^{-1}, \tag{4.9b}$$

where $\bar{\partial}$ is the $(0,1)$ component of the differential on C, and $d_{\mathbb{R}}$ is the de Rham differential on $\mathbb{R}$. Under these gauge transformations the curvature 2-form $F(A)$ transforms by conjugation $F({}^g A) = g F(A) g^{-1}$ and so the invariance of the action follows from the adjoint G-invariance of the bilinear form $\langle \cdot, \cdot \rangle$.

4.3.2 Equations of Motion

To derive the equations of motion we consider variations in the fields $B \to B + \epsilon$ and $A \to A + \eta$ by an arbitrary $(1,0)$-form ϵ and 1-form η on $\mathbb{R} \times C$, and then consider the resulting variation of the 3dBF action. Here $\mathrm{d} = \mathrm{d}_{\mathbb{R}} + \bar{\partial}$. Varying the action we find

$$\begin{aligned}\delta S_{3d}[A,B] &\coloneqq S_{3d}[A+\eta, B+\epsilon] - S_{3d}[A,B] \\ &= \frac{1}{2\pi \mathsf{i}} \int_{\mathbb{R}\times C} \Big(\langle \epsilon, F(A) \rangle + \langle B, \mathrm{d}\eta + \tfrac{1}{2}[A,\eta] + \tfrac{1}{2}[\eta, A] \rangle + O(\eta^2) \Big) \\ &= \frac{1}{2\pi \mathsf{i}} \int_{\mathbb{R}\times C} \Big(\langle \epsilon, F(A) \rangle + \langle B, \mathrm{d}\eta + [A,\eta] \rangle + O(\eta^2) \Big) \\ &= \frac{1}{2\pi \mathsf{i}} \int_{\mathbb{R}\times C} \Big(\langle \epsilon, F(A) \rangle + \langle \mathrm{d}B + [B,A], \eta \rangle + O(\eta^2) \Big),\end{aligned}$$

where in the third equality we have used the fact that A and η are both $\mathfrak{g}$-valued 1-forms, so that $[\eta, A] = [A, \eta]$. In the last equality we used Stokes's theorem, noting that $\langle B, \mathrm{d}\eta \rangle = \langle \mathrm{d}B, \eta \rangle - \mathrm{d}\langle B, \eta \rangle$, and integrating out the total derivative term, which we can ignore if we choose a variation which tends to zero sufficiently fast in the $\mathbb{R}$ direction. We also use the adjoint invariance of the bilinear form, from which we see that $\langle B, [A,\eta] \rangle = \langle [B,A] \eta \rangle$ in the last equality.

The equation of motion from varying B is therefore $F(A) = 0$, or explicitly

$$\bar{\partial} A + \mathrm{d}_{\mathbb{R}} A + \tfrac{1}{2}[A,A] = 0, \tag{4.10a}$$

while the equation of motion from varying A reads

$$\bar{\partial} B + \mathrm{d}_{\mathbb{R}} B + [B,A] = 0. \tag{4.10b}$$

The connection to integrability becomes clearer when working in local coordinates; Let z be a local holomorphic coordinate on C and t a global coordinate on $\mathbb{R}$, and write the two fields in components as $B = B_z \mathrm{d}z$ and $A = A_{\bar{z}} \mathrm{d}\bar{z} + A_t \mathrm{d}t$. We can then write the equations of motion (4.10) more explicitly in the components of the fields as

$$\partial_{\bar{z}} A_t - \partial_t A_{\bar{z}} = [A_t, A_{\bar{z}}], \tag{4.11a}$$

$$\partial_{\bar{z}} B_z = [B_z, A_{\bar{z}}], \tag{4.11b}$$

$$\partial_t B_z = [-A_t, B_z]. \tag{4.11c}$$

The first key observation to make here is that the equation of motion (4.11c) is very reminiscent of the Lax equation, particularly with the telling naming of the coordinate on $\mathbb{R}$ to be t

$$\partial_t L = [M, L]. \tag{4.12}$$

However, to make this superficial resemblance more precise we would need B_z and $-A_t$ to both be holomorphic (or more generally meromorphic) in order to identify them with the Lax pair L and M of an integrable system respectively - in the Gaudin model in particular the Lax matrix would need to be meromorphic as we have seen in Chapter 1.

The second observation, based on the other two equations of motion (4.11a) and (4.11b), is that this can be achieved by working in the gauge where $A_{\bar{z}} = 0$. Indeed, in this gauge the two equations (4.11a) and (4.11b) reduce to $\partial_{\bar{z}} A_t = 0$ and $\partial_{\bar{z}} B_z = 0$, respectively, which express the fact that A_t and B_z are both holomorphic on C.

4.3.3 Adding defects

In the Lax equation (4.12) of an integrable system, however, L and M are more generally $\mathfrak{g}$-valued *meromorphic* functions with poles at certain marked points. This is, in fact, necessary if C has genus zero, *i.e.* when $C = \mathbb{C}P^1$. Moreover, as it stands there is no relation between B_z and $-A_t$ in (4.11c), while in (4.12) the matrix M is typically built out of the Lax matrix L. We can fix both of these issues by introducing two different types of line defects in the action (4.8). We will refer to these as type A and type B line defects, since these will depend on the fields A and B, respectively.

4.3.3.1 Type A defects

As we have seen, the Lax pair of the Gaudin model is formed of two $\mathfrak{g}$-valued meromorphic functions L and M on $\mathbb{C}P^1$ with L having poles at certain marked points $z_i \in \mathbb{C}$ for $i = 1, \ldots, N$ so we need to find or add this data into 3d Mixed BF Theory somewhere. In order to view B_z and $-A_t$ as such a Lax pair, but working on a more general Riemann surface C, we would like them to be meromorphic instead of holomorphic, with B_z having poles at certain marked points $z_i \in C$. To this end, we pick and fix elements $u_i \in \mathfrak{g}$ and introduce G-valued fields h_i on $\mathbb{R}$ for $i = 1, \ldots, N$. Following [16], see also the surface defects of [7], we add to the action (4.8) the following sum of line defects

$$S_{A\text{-}def}\big[A,\{h_i\}_{i=1}^N\big] = -\sum_{i=1}^N \int_{\mathbb{R}\times\{z_i\}} \big\langle u_i, h_i^{-1}(\mathrm{d}_{\mathbb{R}} + \iota_{z_i}^* A)h_i\big\rangle \tag{4.13}$$

where $\iota_{z_i} : \mathbb{R}\times\{z_i\} \hookrightarrow \mathbb{R}\times C$ is the embedding of the line defect at z_i into the total space. In particular, the pullback $\iota_{z_i}^* A$ is just the evaluation of the component $A_t \mathrm{d}t$ at the point $z_i \in C$ so that we can rewrite the defect action (4.13) more explicitly as

$$S_{A\text{-}def}\big[A,\{h_i\}_{i=1}^N\big] = -\sum_{i=1}^N \int_{\mathbb{R}} \big\langle u_i, h_i^{-1}\big(\partial_t + A_t(z_i)\big)h_i\big\rangle \mathrm{d}t.$$

In order to maintain the gauge invariance of the action (4.8) after adding (4.13) to it, we should require that the latter is itself gauge invariant. This can easily be achieved by supplementing the gauge transformations (4.9) of the fields A and B by the transformation

$$h_i \longmapsto g h_i \tag{4.14}$$

for the G-valued fields h_i, $i = 1,\dots,N$. The gauge invariance of the type-A defect action is straightforward from here.

We can now consider the extended action

$$\widetilde{S}\big[A,B,\{h_i\}_{i=1}^N\big] \coloneqq S_{3d}\big[A,B\big] + S_{A\text{-}def}\big[A,\{h_i\}_{i=1}^N\big]. \tag{4.15}$$

Since the defect action (4.13) does not depend on B, the equations of motion (4.10a) from varying B are unchanged. On the other hand, the equation of motion (4.11b) from A in a local holomorphic coordinate z on an open neighbourhood U of the point z_i is now replaced by

$$\partial_{\bar z} B_z = \big[B_z, A_{\bar z}\big] - 2\pi \mathrm{i}\, \widehat{u}_i \delta_{z z_i}, \tag{4.16}$$

where we introduced $\widehat{u}_i \coloneqq h_i u_i h_i^{-1}$ for each $i = 1,\dots,N$ and δ_{zz_i} denotes the Dirac δ-distribution at the marked point z_i with the property that

$$\int_U f(z)\delta_{zz_i} \mathrm{d}z \wedge \mathrm{d}\bar z = f(z_i) \tag{4.17}$$

for any function $f : U \to \mathbb{C}$ on the neighbourhood $U \subset C$ of z_i equipped with the local holomorphic coordinate z.

In the gauge $A_{\bar z} = 0$, the modified equation of motion (4.16) reads

$$\partial_{\bar z} B_z = -2\pi \mathrm{i}\, \widehat{u}_i \delta_{z z_i}. \tag{4.18}$$

Using the fact that $\partial_{\bar{z}}(z-z_i)^{-1} = -2\pi\mathrm{i}\delta_{zz_i}$ we may rewrite this equation as

$$\partial_{\bar{z}}\left(B_z - \frac{\widehat{u}_i}{z-z_i}\right) = 0$$

which tells us that B_z has a simple pole at z_i with residue $\widehat{u}_i$ there, *i.e.*

$$B = \frac{\widehat{u}_i}{z-z_i}\mathrm{d}z + O(1) \tag{4.19}$$

where $O(1)$ denotes terms which are holomorphic at the point z_i.

When $C = \mathbb{C}P^1$, corresponding to the Gaudin model, if we fix a global coordinate z on $\mathbb{C} \subset \mathbb{C}P^1$ and require B to have a simple pole also at infinity then we can explicitly write B as the $\mathfrak{g}$-valued meromorphic $(1,0)$-form

$$B = \sum_{i=1}^{N} \frac{\widehat{u}_i}{z-z_i}\mathrm{d}z. \tag{4.20}$$

By varying the action (4.15) with respect to $h_i \to e^{\epsilon_i}h_i$ for some $\mathfrak{g}$-valued function ϵ_i on $\mathbb{R}$ we find N further equations of motion

$$\partial_t\widehat{u}_i = [-A_t(z_i), \widehat{u}_i] \tag{4.21}$$

for $i = 1, \ldots, N$. But given the meromorphic behaviour (4.19) of the $(1,0)$-form B at each of the marked points z_i, these are merely consequences of the equation of motion (4.11c) given by taking the residue at each z_i, assuming that A_t is regular at z_i, as will be the case in the following section.

4.3.3.2 Type B defects

The type A line defects we have just introduced ensure that B_z is no longer holomorphic in the gauge $A_{\bar{z}} = 0$ but rather meromorphic with poles at certain marked points $z_i \in C$. The type B line defects will have a similar effect on the field A_t. However, since $-A_t$ is meant to play the role of M in the Lax pair (4.12), we want it to be built out of B_z, which plays the role of the Lax matrix L.

Let $P : \mathfrak{g} \to \mathbb{C}$ be a G-invariant polynomial on $\mathfrak{g}$ and fix a point $w \in C$ distinct from the marked points $z_i \in C$ for $i = 1, \ldots, N$ at which the type A line defects were inserted in section 4.3.3.1. We consider the following line defect

$$S_{B\text{-}def}[B] = -\int_{\mathbb{R}\times\{w\}} P(B_z)\mathrm{d}t = -\int_{\mathbb{R}} P\big(B_z(w)\big)\mathrm{d}t \tag{4.22}$$

where z is a local holomorphic coordinate around the point $w \in C$ and, writing $B = B_z \mathrm{d}z$ in this coordinate, $B_z(w)$ denotes the evaluation of B_z at the point w.

The G-invariance of the polynomial P ensures that the action (4.22) is gauge invariant. Therefore, adding it to the gauge invariant action (4.15) obtained so far, we obtain the full gauge invariant action

$$S\big[A, B, \{h_i\}_{i=1}^N\big] \coloneqq S_{3d}[A, B] + S_{A-def}\big[A, \{h_i\}_{i=1}^N\big] + S_{B-def}[B]. \tag{4.23}$$

Since the defect term (4.22) only depends on B, it will not modify the equations of motion for A. Only the equation of motion for B, namely (4.11a) which has so far remained unchanged, will be altered. To derive it we note that the variation of the defect action (4.22), under the variation $B \to B + \epsilon$ considered in section 4.3.2 with $\epsilon = \epsilon_z \mathrm{d}z$ in the local holomorphic coordinate z, reads

$$\begin{aligned} \delta S_{B-def}[B] &\coloneqq S_{B-def}[B+\epsilon] - S_{B-def}[B] \\ &= -\int_{\mathbb{R}} \Big(P\big(B_z(w) + \epsilon_z(w)\big) - P\big(B_z(w)\big)\Big) \mathrm{d}t \\ &= -\int_{\mathbb{R}} \Big(\big\langle P'(B_z(w)), \epsilon_z(w)\big\rangle + O\big(\epsilon_z(w)^2\big)\Big)\mathrm{d}t \end{aligned}$$

where in the third line we introduced the element $P'(B_z(w)) \in \mathfrak{g}$ such that the linear map $\langle P'(B_z(w)), \cdot\rangle : \mathfrak{g} \to \mathbb{C}$ is the derivative of $P : \mathfrak{g} \to \mathbb{C}$ at $B_z(w)$ and kept only the terms linear in $\epsilon_z(w)$. It follows that (4.11a) is now replaced by

$$\partial_{\bar z} A_t - \partial_t A_{\bar z} = [A_t, A_{\bar z}] + 2\pi \mathrm{i}\, P'\big(B_z(w)\big)\delta_{zw}. \tag{4.24}$$

In the gauge $A_{\bar z} = 0$ this simplifies to

$$\partial_{\bar z} A_t = 2\pi \mathrm{i}\, P'\big(B_z(w)\big)\delta_{zw} \tag{4.25}$$

or in other words,

$$\partial_{\bar z}\left(A_t + \frac{P'\big(B_z(w)\big)}{z - w}\right) = 0.$$

In the case $C = \mathbb{C}P^1$ this tells us that the expression in brackets is a constant. Taking this constant to be zero we therefore obtain

$$-A_t(z) = \frac{P'\big(B_z(w)\big)}{z - w}, \tag{4.26}$$

which coincides with the usual expression for $M = -A_t$ in terms of $L = B_z$, see for instance [3, Eq. (3.33)] in the case when $\mathfrak{g} = \mathfrak{gl}_r$ and the polynomial $P : \mathfrak{gl}_r \to \mathbb{C}$ is given by $X \mapsto \operatorname{tr}(X^n)$ for some $n \in \mathbb{Z}_{\geq 1}$. Indeed, in this case we have $P'(X) = nX^{n-1}$ for any $X \in \mathfrak{gl}_r$ so that (4.26) becomes

$$-A_t(z) = n\frac{B_z(w)^{n-1}}{z-w}. \tag{4.27}$$

In connection with the Hamiltonian analysis to be performed later in the chapter, where the classical r-matrix $r_{12}(z,w) = \frac{C_{12}}{w-z}$ will be introduced in (4.52), note that we can rewrite (4.27) in the more recognisable form

$$-A_t(z) = -n\,\operatorname{tr}_2\big(r_{12}(z,w)B_z(w)_2^{n-1}\big).$$

We can substitute the expression (4.26) for A_t into the equation of motion (4.11c) to obtain the desired Lax equation

$$\partial_t B_z(z) = \left[\frac{P'\big(B_z(w)\big)}{z-w}, B_z(z)\right], \tag{4.28}$$

where we have explicitly written the dependence of B_z on the spectral parameters.We thus expect from the general theory of integrable systems, see for instance the Proposition [3, p.47], that the time coordinate t along the topological direction of the 3-dimensional space $\mathbb{R} \times C$ corresponds, through the introduction of the type B defect (4.22), with the time induced by the Hamiltonian

$$H_w^P := P(B_z(w)). \tag{4.29}$$

To confirm this we will move to the Hamiltonian formalism through the Hamiltonian analysis in Section 4.4.

4.3.4 UNIFYING 1D ACTION

We have now shown that the gauge fixed equations of motion for 3d mixed BF theory in the presence of type A and B defects correspond exactly to the Lax equation (4.28) of the Gaudin model with Lax matrix $L(z) = B_z(z)$ given by (4.20), where the residues $\widehat{u}_i = h_i u_i h_i^{-1}$ are coadjoint orbits through the fixed elements $u_i \in \mathfrak{g}$ and parameterised by the dynamical G-valued variables $h_i \in G$.

At this stage we can take a brief detour to proceed along the lines of [20], where a unifying 2d action for integrable field theories of affine Gaudin type was derived from the 4d Chern-Simons action of [16]. In a similar spirit, in the present context we would like to obtain a 1d action for the Gaudin model with Lax matrix (4.20) starting from the 3d mixed BF theory with both type A and type B defects. In fact, the procedure followed in [7] is closer in spirit to the present case since we do not have to deal with the presence of a meromorphic 1-form ω having zeroes, as in the 4d Chern-Simons action considered in [20].

Following [7, §2.6], we will therefore substitute the solutions to the equations of motion (4.16) and (4.24) (but crucially not (4.11c)) in the gauge $A_{\bar{z}} = 0$, namely (4.20) and (4.26) respectively, into the full action (4.23). We will do this for the three pieces in the action separately. For the bulk action (4.8) we find

$$
\begin{aligned}
S_{3d}[A,B] &= \frac{1}{2\pi\mathsf{i}} \int_{\mathbb{R}\times\mathbb{C}} \langle B_z, \partial_{\bar{z}} A_t - \partial_t A_{\bar{z}} - [A_t, A_{\bar{z}}] \rangle \mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}t \\
&= \frac{1}{2\pi\mathsf{i}} \int_{\mathbb{R}\times\mathbb{C}} \langle B_z, \partial_{\bar{z}} A_t \rangle \mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \mathrm{d}t \\
&= \int_{\mathbb{R}} \big\langle L(w), P'\big(L(w)\big)\big\rangle \mathrm{d}t
\end{aligned}
$$

where in the second equality we used the gauge $A_{\bar{z}} = 0$. In the last equality we used the fact that B_z is identified with the Lax matrix L together with the identity (4.25), and then performed the integral over $\mathbb{C}$ using the presence of the δ-function.

For the type A defect action (4.13) we have

$$
\begin{aligned}
S_{A\text{-}def}\big[A, \{h_i\}_{i=1}^N\big] &= -\sum_{i=1}^N \int_{\mathbb{R}} \langle u_i, h_i^{-1}\partial_t h_i \rangle \mathrm{d}t - \sum_{i=1}^N \int_{\mathbb{R}} \langle \widehat{u}_i, A_t(z_i) \rangle \mathrm{d}t \\
&= -\sum_{i=1}^N \int_{\mathbb{R}} \langle u_i, h_i^{-1}\partial_t h_i \rangle \mathrm{d}t - \sum_{i=1}^N \int_{\mathbb{R}} \left\langle \widehat{u}_i, \frac{P'(L(w))}{w - z_i} \right\rangle \mathrm{d}t \\
&= -\sum_{i=1}^N \int_{\mathbb{R}} \langle u_i, h_i^{-1}\partial_t h_i \rangle \mathrm{d}t - \int_{\mathbb{R}} \big\langle L(w), P'\big(L(w)\big)\big\rangle \mathrm{d}t,
\end{aligned}
$$

where in the second equality we used (4.26) evaluated at $z = z_i$ and in the last line we recognised the sum over i in the second term as the expression for the Lax matrix $L(w) = B_z(w)$ in (4.20). Note that the second term on the right hand side exactly cancels the expression found above for the bulk action $S_{3d}[A,B]$.

Finally, the type B defect action (4.22) is simply $S_{B\text{-}def}[B] = -\int_{\mathbb{R}} H_w^P \mathrm{d}t$ using the expression (4.29) for the Hamiltonian alluded to in the previous section and to be confirmed in the analysis of the following section. Putting all the above together, we deduce that the full action (4.23) reduces to the simple form

$$S_{1d}\big[\{h_i\}_{i=1}^N\big] = -\sum_{i=1}^N \int_{\mathbb{R}} \langle u_i, h_i^{-1}\partial_t h_i\rangle \mathrm{d}t - \int_{\mathbb{R}} H_w^P \mathrm{d}t, \tag{4.30}$$

where we have suppressed the dependence on the fields A and B since these have now been expressed in terms of the dynamical variables $h_i \in G$ and the fixed elements $u_i \in \mathfrak{g}$ for $i = 1, \dots, N$. We recognise (4.30) as the first order action

$$S\big[\{h_i\}_{i=1}^N\big] = \sum_{i=1}^N \int_{\mathbb{R}} \langle X_i, h_i^{-1}\partial_t h_i\rangle \mathrm{d}t - \int_{\mathbb{R}} H_w^P \mathrm{d}t,$$

associated with the Hamiltonian H_w^P in (4.29) but where the conjugate momentum $X_i \in \mathfrak{g}$ of $h_i \in G$ has been fixed to the constant element $X_i = -u_i$. This is consistent with the Hamiltonian analysis to be performed in the next section. Namely, we will find in section 4.4.1.2 that there is a primary constraint $X_i + u_i \approx 0$ on the conjugate momentum $X_i \in \mathfrak{g}$ of the dynamical variable $h_i \in G$.

We can check directly that the equations of motion of the 1d action (4.30) are given by (4.21), with A_t as in (4.26), by varying it with respect to $h_i \to e^{\epsilon_i} h_i$ for some arbitrary $\mathfrak{g}$-valued variable ϵ_i. Under this variation, the Lax matrix $L(w)$ transforms to

$$\sum_{i=1}^N \frac{e^{\epsilon_i}\widehat{u}_i e^{-\epsilon_i}}{w - z_i} = L(w) + \sum_{i=1}^N \frac{[\epsilon_i, \widehat{u}_i]}{w - z_i} + O(\epsilon_i^2).$$

Hence, using the explicit expression $H = P(L(w))$ for the Hamiltonian, the variation of the action is given by

$$\begin{aligned}
\delta S_{1d} &\coloneqq S_{1d}[\{e^{\epsilon_i} h_i\}_{i=1}^N] - S_{1d}[\{h_i\}_{i=1}^N] \\
&= -\sum_{i=1}^N \int_{\mathbb{R}} \big\langle u_i, h_i^{-1} e^{-\epsilon_i}\partial_t (e^{\epsilon_i} h_i) - h_i^{-1}\partial_t h_i \big\rangle \mathrm{d}t \\
&\qquad\qquad - \int_{\mathbb{R}} \Bigg(P\Bigg(L(w) + \sum_{i=1}^N \frac{[\epsilon_i, \widehat{u}_i]}{w - z_i}\Bigg) - P\big(L(w)\big)\Bigg) \mathrm{d}t \\
&= -\sum_{i=1}^N \int_{\mathbb{R}} \Bigg(\langle \widehat{u}_i, \partial_t \epsilon_i\rangle + \Bigg\langle P'\big(L(w)\big), \frac{[\epsilon_i, \widehat{u}_i]}{w - z_i}\Bigg\rangle + O(\epsilon_i^2)\Bigg) \mathrm{d}t \\
&= \sum_{i=1}^N \int_{\mathbb{R}} \Bigg(\Bigg\langle \partial_t \widehat{u}_i - \Bigg[\frac{P'(L(w))}{z_i - w}, \widehat{u}_i \Bigg], \epsilon_i \Bigg\rangle + O(\epsilon_i^2)\Bigg) \mathrm{d}t,
\end{aligned}$$

where in the last equality we have used Stokes theorem and the adjoint invariance of the bilinear form. The N equations of motion for the h_i are therefore

$$\partial_t \widehat{u}_i = \Bigg[\frac{P'(L(w))}{z_i - w}, \widehat{u}_i \Bigg],$$

and using (4.26), we do indeed recover the equations of motion (4.21) found previously from adding in type A defects in section 4.3.3.1.

4.4 Hamiltonian analysis

Throughout this section we shall work in some local coordinate z on some open subset of C. Our starting point is the Lagrangian density of the action (4.23) written in terms of the components of the $\mathfrak{g}$-valued bulk fields $A = A_{\bar z}\mathrm{d}\bar z + A_t \mathrm{d}t$ and $B = B_z \mathrm{d}z$ and in terms of the G-valued defect variables h_i for $i = 1, \dots, N$ which we introduced at the type A defects above, namely

$$\mathcal{L}\big(A, B, \{h_i\}_{i=1}^N\big) = \frac{1}{2\pi\mathsf{i}}\big\langle B_z, \partial_{\bar z} A_t - \partial_t A_{\bar z} + [A_{\bar z}, A_t]\big\rangle \\ - \sum_{i=1}^N \big\langle u_i, h_i^{-1}(\partial_t + A_t) h_i\big\rangle \delta_{z z_i} - P(B_z(w))\delta_{zw}. \quad (4.31)$$

4.4.1 Conjugate momenta and primary constraints

To move to the Hamiltonian formalism we first determine the conjugate momenta of the bulk fields $A_{\bar z}$, A_t and B_z and the defect variables h_i. We shall find various primary constraints, some of which will be second class. We shall impose the latter strongly at this stage by introducing a corresponding Dirac bracket. To alleviate the notation, all Dirac brackets computed in this section will ultimately be renamed simply as $\{\cdot,\cdot\}$ before moving on to the following section where we work out the secondary constraints.

We will begin by considering the conjugate momenta of the bulk fields $A_{\bar z}$, A_t and B_z, as the conjugate momenta to the G-valued defect variables h_i will need to be handled with more care, as discussed below.

4.4.1.1 Bulk canonical fields

The conjugate momenta to the $\mathfrak{g}$-valued bulk fields $A_{\bar z}$, A_t and B_z are the $\mathfrak{g}$-valued fields given respectively by

$$\Pi_t = \frac{\partial \mathcal{L}}{\partial(\partial_t A_t)} = 0, \quad \Pi_{\bar z} = \frac{\partial \mathcal{L}}{\partial(\partial_t A_{\bar z})} = -\frac{1}{2\pi\mathsf{i}} B_z, \quad P_z = \frac{\partial \mathcal{L}}{\partial(\partial_t B_z)} = 0,$$

which satisfy the canonical Poisson bracket relations

$$\{\Pi_{t\mathbf{1}}(z), A_{t\mathbf{2}}(z')\} = C_{\mathbf{12}}\delta_{zz'},$$
$$\{\Pi_{\bar z\mathbf{1}}(z), A_{\bar z\mathbf{2}}(z')\} = C_{\mathbf{12}}\delta_{zz'},$$
$$\{P_{z\mathbf{1}}(z), B_{z\mathbf{2}}(z')\} = C_{\mathbf{12}}\delta_{zz'}.$$

Here and in what follows we use the standard tensor notation. In particular, if we fix dual bases $\{I^a\}$ and $\{I_a\}$ of $\mathfrak{g}$ with respect to the non-degenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ introduced at the start of the Chapter, then $C_{12} = I^a \otimes I_a$ is the split Casimir. Also for $\mathfrak{g}$-valued fields F and G, which can be written in components as $F = F_a I^a$ and $G = G_a I^a$, we have $\{F_1(z), G_2(z')\} \coloneqq \{F_a(z), G_b(z')\} I^a \otimes I^b$.

We have three primary constraints associated with the bulk fields, namely

$$\Pi_t \approx 0, \qquad \mathcal{C}_z \coloneqq B_z + 2\pi \mathrm{i} \Pi_{\bar{z}} \approx 0, \qquad P_z \approx 0. \tag{4.32}$$

Throughout this Chapter we use the conventional notation '≈' to denote equality on the constraint surface [38], or "weak equality". The first constraint in (4.32) is clearly first class and the latter two are second class with Poisson bracket

$$\{P_{z1}(z), \mathcal{C}_{z2}(z')\} = C_{12} \delta_{zz'}.$$

We set these both strongly to zero immediately, by which we mean restricting to the submanifold of phase space specified by $P_z = 0$ and $\mathcal{C}_z = 0$ and replacing the Poisson bracket with the corresponding Dirac bracket [38]. With respect to the latter we still have the same relations between the remaining fields, *i.e.*

$$\{\Pi_{t1}(z), A_{t2}(z')\} = C_{12} \delta_{zz'}, \tag{4.33a}$$

$$\{\Pi_{\bar{z}1}(z), A_{\bar{z}2}(z')\} = C_{12} \delta_{zz'}, \tag{4.33b}$$

and hence, by an abuse of notation, we will continue to denote this Dirac bracket as $\{\cdot, \cdot\}$.

4.4.1.2 Defect canonical variables

We have yet to find the conjugate momenta to the G-valued variables h_i, $i = 1, \dots, N$ introduced at the type A defects. This can be done by working in local coordinates ϕ^α on the group G where α ranges from 1 to $\dim G$, the dimension of G. We refer the reader, for instance, to [41, §3.1.2] for details. Each variable $h_i \in G$ can then be described locally in terms of the $\dim G$ variables $\phi_i^\alpha \coloneqq \phi^\alpha(h_i)$.

The relevant part of the Lagrangian in finding the conjugate momenta is

$$-\langle u_i, h_i^{-1} \partial_t h_i \rangle = -\langle u_i, \partial_t \phi_i^\alpha h_i^{-1} \partial_\alpha h_i \rangle,$$

which on the right-hand side we have rewritten in terms of the local coordinates ϕ_i^α, where ∂_α denotes the partial derivative with respect to the coordinate ϕ^α.

The corresponding conjugate momenta are therefore given by

$$\pi_{i,\alpha} = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi_i^\alpha)} = -\langle u_i, h_i^{-1}\partial_\alpha h_i \rangle, \tag{4.34}$$

and these have the usual canonical Poisson bracket relations

$$\{\phi_i^\alpha, \phi_j^\beta\} = 0, \qquad \{\pi_{i,\alpha}, \pi_{j,\beta}\} = 0, \qquad \{\pi_{i,\alpha}, \phi_j^\beta\} = \delta_\alpha^\beta \delta_{ij}.$$

To return to a coordinate free description of the phase space, we define a matrix ${L^a}_\alpha$ for some fixed basis $\{I_a\}$ of $\mathfrak{g}$ such that

$$h_i^{-1}\partial_\alpha h_i = {L^a}_\alpha I_a. \tag{4.35}$$

This ${L^a}_\alpha$ is invertable and we denote the inverse as ${L^\alpha}_a$ following the conventions of [41, §3.1.2]. We can then introduce a coordinate-free $\mathfrak{g}$-valued variable X_i which encodes the conjugate momentum $\pi_{i,\alpha}$ as

$$X_i \coloneqq {L^\alpha}_a \pi_{i,\alpha} I^a, \tag{4.36}$$

where $\{I^a\}$ is the basis of $\mathfrak{g}$ dual to $\{I_a\}$ with respect to the bilinear form $\langle \cdot, \cdot \rangle$. We therefore have a coordinate free description of the phase space, parameterised by a pair of fields (X_i, h_i) at each defect valued in $TG \simeq \mathfrak{g} \times G$, with the canonical Poisson brackets in local coordinates being equivalent to, see for instance [41],

$$\{h_{i\mathbf{1}}, h_{j\mathbf{2}}\} = 0, \tag{4.37a}$$
$$\{X_{i\mathbf{1}}, h_{j\mathbf{2}}\} = h_{i\mathbf{2}} C_{\mathbf{12}} \delta_{ij}, \tag{4.37b}$$
$$\{X_{i\mathbf{1}}, X_{j\mathbf{2}}\} = -[C_{\mathbf{12}}, X_{i\mathbf{2}}]\delta_{ij}. \tag{4.37c}$$

for each $i, j = 1, \dots, N$.

Using the definition of the matrix ${L^a}_\alpha$ in (4.35) we have

$${L^\alpha}_a \langle u_i, h_i^{-1}\partial_\alpha h_i \rangle I^a = \langle u_i, {L^\alpha}_a {L^b}_\alpha I_b \rangle I^a = \langle u_i, I_a \rangle I^a = u_i.$$

It then follows from the expression (4.34) for $\pi_{i,\alpha}$ above, derived from the Lagrangian, and the definition (4.36) of X_i that we have a primary constraint of the form

$$\mathcal{C}_i \coloneqq X_i + u_i \approx 0 \tag{4.38}$$

for each defect $i = 1, \dots, N$. These N primary constraints are not entirely first or second class. Indeed, their Poisson brackets

$$\{\mathcal{C}_{i\mathbf{1}}, \mathcal{C}_{j\mathbf{2}}\} = \{X_{i\mathbf{1}}, X_{j\mathbf{2}}\} = -[C_{\mathbf{12}}, \mathcal{C}_{i\mathbf{2}} - u_{i\mathbf{2}}]\delta_{ij} \approx [C_{\mathbf{12}}, u_{i\mathbf{2}}]\delta_{ij}, \tag{4.39}$$

are non-vanishing on the constraint surface (4.38) and are not generally invertible.

Let $\{v^i_p\}_{p=1}^{d_i}$ be a basis of the centraliser $\mathfrak{g}^{u_i} \coloneqq \ker(\mathrm{ad}_{u_i})$ of the element $u_i \in \mathfrak{g}$, with $d_i \coloneqq \dim \mathfrak{g}^{u_i}$ for each $i = 1, \dots, N$. The first class part of each $\mathcal{C}_i$ is given by the set of constraints $\mathcal{C}^p_i \coloneqq \langle v^i_p, \mathcal{C}_i \rangle$ for $p = 1, \dots, d_i$. These satisfy the relations

$$\{\mathcal{C}^p_i, \mathcal{C}_j\} \approx \left\langle v^i_{p1}, [C_{12}, u_{i2}] \right\rangle_1 \delta_{ij} = [v^i_p, u_i]\delta_{ij} = 0, \tag{4.40}$$

for every $i, j = 1, \dots, N$ and $p = 1, \dots, d_i$, where the last equality uses the fact that $v^i_p \in \mathfrak{g}^{u_i}$. In particular, we have $\{\mathcal{C}^p_i, \mathcal{C}^q_j\} \approx 0$ for any $q = 1, \dots, d_j$ so that the set of constraints $\mathcal{C}^p_i$ for $p = 1, \dots, d_i$, $i = 1, \dots, N$ are indeed first class. It also follows from (4.37b) that the first class constraints $\mathcal{C}^p_i$ generate right multiplication of the h_i by elements $e^{\epsilon v^i_p}$ of the centraliser G^{u_i} of u_i in G – note that under such transformations the $\mathfrak{g}$-valued variables $\widehat{u}_i$ are invariant

$$h_i e^{\epsilon v^j_p} u_i e^{-\epsilon v^j_p} h_i^{-1} = h_i u_i h_i^{-1} = \widehat{u}_i. \tag{4.41}$$

Let us extend the basis $\{v^i_p\}_{p=1}^{d_i}$ of the centraliser $\mathfrak{g}^{u_i}$ to a basis $\{v^i_p\}_{p=1}^{d_i} \cup \{\widetilde{v}^i_r\}_{r=1}^{c_i}$ of $\mathfrak{g}$ where $c_i \coloneqq \dim \mathfrak{g} - d_i$. We claim that the remaining constraints $\widetilde{\mathcal{C}}^r_i \coloneqq \langle \widetilde{v}^i_r, \mathcal{C}_i \rangle$ for $r = 1, \dots, c_i$ contained in $\mathcal{C}_i$ are second class. We need to show that the matrix $\{\widetilde{\mathcal{C}}^r_i, \widetilde{\mathcal{C}}^s_i\}$ for $r, s = 1, \dots, c_i$ is invertible on the constraint surface $\mathcal{C}_i \approx 0$. If this were not the case then we would have a linearly dependent column $\sum_{s=1}^{c_i} \{\widetilde{\mathcal{C}}^r_i, \widetilde{\mathcal{C}}^s_i\} a_s \approx 0$ for some $a_s \in \mathbb{C}$ with $s = 1, \dots, c_i$. On the other hand, we also know from (4.40) that $\sum_{s=1}^{c_i} \{\mathcal{C}^p_i, \widetilde{\mathcal{C}}^s_i\} a_s \approx 0$ for all $p = 1, \dots, d_i$. Combining these statements we have

$$0 \approx \sum_{s=1}^{c_i} \{\mathcal{C}_i, \widetilde{\mathcal{C}}^s_i\} a_s = \sum_{s=1}^{c_i} \left\{\mathcal{C}_i, \langle \widetilde{v}^i_s, \mathcal{C}_i \rangle\right\} a_s \approx \sum_{s=1}^{c_i} \left\langle \widetilde{v}^i_{s2}, [C_{12}, u_{i2}] \right\rangle_2 a_s = \left[u_i, \sum_{s=1}^{c_i} a_s \widetilde{v}^i_s \right],$$

where in the third step we used (4.39). It follows that $\sum_{s=1}^{c_i} a_s \widetilde{v}^i_s \in \mathfrak{g}^{u_i}$ which contradicts the assumption that $\{\widetilde{v}^i_r\}_{r=1}^{c_i}$ is the basis of some complement of $\mathfrak{g}^{u_i}$ in $\mathfrak{g}$.

We would like to impose suitable gauge fixing conditions $\mathcal{D}^p_i \approx 0$, for $p = 1, \dots, d_i$, to fix the first class constraints $\mathcal{C}^p_i$ and move to a Dirac bracket $\{\cdot, \cdot\}^*$ which fixes the constraints $\mathcal{C}_i \approx 0$ strongly. In particular, we would like to compute the Dirac bracket $\{\widehat{u}_{i1}, \widehat{u}_{j2}\}^*$ of the $\mathfrak{g}$-valued variables $\widehat{u}_i = h_i u_i h_i^{-1}$ for $i = 1, \dots, N$. It turns out that the result is independent of the choice of gauge fixing condition $\mathcal{D}^p_i \approx 0$. Indeed, consider the variables $\widehat{X}_i \coloneqq h_i X_i h_i^{-1}$. One deduces from (4.37) that they have the Poisson brackets

$$\{\widehat{X}_{i1}, \widehat{X}_{j2}\} = [C_{12}, \widehat{X}_{i2}]\delta_{ij}, \tag{4.42a}$$

$$\{X_{i1}, \widehat{X}_{j2}\} = 0 \tag{4.42b}$$

for each $i, j = 1, \dots, N$. In particular, it follows from (4.42b) that $\{\mathcal{C}_{i1}, \widehat{X}_{j2}\} = 0$ for any $i, j = 1, \dots, N$. Now the matrix of Poisson brackets between the set of all second class constraints $\mathcal{C}_i^p$, $\mathcal{D}_i^p$ for $p = 1, \dots, d_i$ and $\widetilde{\mathcal{C}}_i^r$ for $r = 1, \dots, c_i$ is of the block form

$$\begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix} \tag{4.43}$$

where the first, second and third block rows and columns correspond to the set of constraints $\mathcal{C}_i^p$, $\mathcal{D}_i^p$ and $\widetilde{\mathcal{C}}_i^r$, respectively. The zeroes of the matrix are all a direct consequence of (4.40) and each '$*$' denotes a possibly non-zero block matrix. The matrix (4.43) is invertible since the blocks in position $(1,2)$, $(2,1)$ and $(3,3)$ are all invertible by design. Its inverse is then schematically of the block form

$$\begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}^{-1} = \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & * \end{pmatrix}. \tag{4.44}$$

Since $\{\mathcal{C}_i^p, \widehat{X}_j\} = \{\widetilde{\mathcal{C}}_i^r, \widehat{X}_j\} = 0$ for all $p = 1, \dots, d_i$ and $r = 1, \dots, c_i$, the only non-zero part of the sum within the Dirac bracket will be paired with the middle element of (4.44) - we see on the righthand side that this is just zero . Together this implies that the Poisson brackets (4.42a) will remain unchanged when passing to the Dirac bracket, *i.e.* we have

$$\{\widehat{X}_{i1}, \widehat{X}_{j2}\}^* = [C_{12}, \widehat{X}_{i2}]\delta_{ij}.$$

Finally, using the fact that $\widehat{X}_i = -\widehat{u}_i$ after imposing the constraint $\mathcal{C}_i = 0$ strongly, we deduce that the $\mathfrak{g}$-valued variables $\widehat{u}_i$ for $i = 1, \dots, N$ satisfy N commuting copies of the Kostant-Kirillov bracket

$$\{\widehat{u}_{i1}, \widehat{u}_{j2}\}^* = -[C_{12}, \widehat{u}_{i2}]\delta_{ij}. \tag{4.45}$$

To avoid overburdening the notation, and since we shall need to introduce a further Dirac bracket in section 4.4.3, we will denote the Dirac bracket $\{\cdot,\cdot\}^*$ introduced above simply as $\{\cdot,\cdot\}$ from now on.

4.4.2 Hamiltonian and secondary constraints

The Hamiltonian density is defined as the Legendre transform of the Lagrangian density (4.31). However, since the field A_t is non-dynamical, *i.e.* there are no time derivatives of A_t in the action, we shall perform the Legendre transform only with respect to the dynamical fields $A_{\bar{z}}$, B_z and the dynamical variables h_i. So we define

$$\begin{aligned}\mathcal{H} &\coloneqq \langle \Pi_{\bar{z}}, \partial_t A_{\bar{z}} \rangle + \langle P_z, \partial_t B_z \rangle + \langle X_i, h_i^{-1}\partial_t h_i \rangle - \mathcal{L}\big(A, B, \{h_i\}_{i=1}^N\big)\\ &= \frac{1}{2\pi \mathsf{i}}\langle \mathcal{C}_z, \partial_t A_{\bar{z}} \rangle + \langle P_z, \partial_t B_z \rangle + \sum_{i=1}^N \langle \mathcal{C}_i, h_i^{-1}\partial_t h_i \rangle\\ &\qquad\qquad - \frac{1}{2\pi \mathsf{i}}\langle B_z, \partial_{\bar{z}} A_t + [A_{\bar{z}}, A_t] \rangle + \sum_{i=1}^N \langle \widehat{u}_i, A_t \rangle \delta_{zz_i} + H_w^P \delta_{zw}\end{aligned}$$

where in the second line we have used the definition of the bulk constraint $\mathcal{C}_z$ in (4.32) and of the defect constraints $\mathcal{C}_i$ for $i = 1, \dots, N$ in (4.38). Since we have already set these along with P_z strongly to zero, we can drop the corresponding terms in the Hamiltonian density.

The Hamiltonian is the integral of the Hamiltonian density over C, namely

$$\begin{aligned}H &\coloneqq \int_C \mathcal{H}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}\\ &= -\frac{1}{2\pi \mathsf{i}}\langle\!\langle B_z, \partial_{\bar{z}} A_t + [A_{\bar{z}}, A_t] \rangle\!\rangle + \int_C \left(\sum_{i=1}^N \langle \widehat{u}_i, A_t \rangle \delta_{zz_i} \right) \mathrm{d}z \wedge \mathrm{d}\bar{z} + H_w^P,\end{aligned}$$

where in the first term of the right hand side we introduced the notation

$$\langle\!\langle X, Y \rangle\!\rangle \coloneqq \int_C \langle X, Y \rangle\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

for any $\mathfrak{g}$-valued fields X and Y on C.

It is convenient to introduce the bulk $\mathfrak{g}$-valued field

$$\mu \coloneqq \frac{1}{2\pi \mathsf{i}}\big(\partial_{\bar{z}} B_z + [A_{\bar{z}}, B_z]\big) = -\partial_{\bar{z}} \Pi_{\bar{z}} - [A_{\bar{z}}, \Pi_{\bar{z}}], \tag{4.46}$$

where in the second equality we have used the constraint $\mathcal{C}_z = 0$ in (4.32) which is now imposed strongly. If we also introduce the $\mathfrak{g}$-valued field

$$\widehat{\mu} \coloneqq \mu + \sum_{i=1}^N \widehat{u}_i \delta_{zz_i}, \tag{4.47}$$

the Hamiltonian can be rewritten succinctly as

$$H = \langle\!\langle \widehat{\mu}, A_t \rangle\!\rangle + H_w^P. \tag{4.48}$$

At this point it is interesting to note the similarity between the $\mathfrak{g}$-valued fields μ and $\widehat{\mu}$ just introduced and the moment map of the Hitchin system [39] (we refer the reader to [3, §7.11] for a concise review of Hitchin systems). This is the moment map on the cotangent bundle $T^*\mathcal{A}$ of the space $\mathcal{A}$ of $(0,1)$-forms on the Riemann surface C, parameterised by the $(0,1)$-form A and the Higgs field Φ, defined by $\mu_{Hit} \coloneqq \bar{\partial}_A \Phi = \bar{\partial}\Phi + [A, \Phi]$. The phase space of the Hitchin system without marked points is defined as the symplectic reduction to the level surface $\mu_{Hit} = 0$. This would coincide exactly, upon identifying B_z with the Higgs field Φ and the $(0,1)$-form A with $A_{\bar{z}}$, with the condition $\mu = 0$. In the presence of marked points z_i the level of the moment map of the Hitchin system is chosen instead to be $\sum_{i=1}^N \widehat{u}_i \delta_{zz_i}$. As we will see, this level surface corresponds exactly to the constraint $\widehat{\mu} \approx 0$ coming from the gauge invariance in 3d mixed BF theory with the defects introduced in section 4.3.3. Without the defects the constraint would reduce to $\mu \approx 0$ in pure BF theory.

4.4.2.1 Gauge invariance

We need to ensure that the remaining primary constraint, $\Pi_t \approx 0$, is preserved under time evolution. That is,

$$\{H, \Pi_t\} = \widehat{\mu} \approx 0,$$

giving rise to the secondary constraint $\widehat{\mu} \approx 0$. We see from the canonical brackets (4.33) that $-\widehat{\mu}$ is the generator of gauge transformations (4.9) on the fields $A_{\bar{z}}$ and B_z since

$$\{\widehat{\mu}_1(z), A_{\bar{z}2}(z')\} = -[C_{12}, A_{\bar{z}2}(z)]\delta_{zz'} - \partial_{\bar{z}}(C_{12}\delta_{zz'}) \tag{4.49a}$$

$$\{\widehat{\mu}_1(z), B_{z2}(z')\} = -[C_{12}, B_{z2}(z)]\delta_{zz'}, \tag{4.49b}$$

where we have used the identity $[C_{12}, A_{\bar{z}2}(z) + A_{\bar{z}1}(z)] = 0$ for (4.49a). Note that the moment map μ satisfies the following Poisson bracket

$$\begin{aligned}
\{\mu_1(z), \mu_2(z')\} &= \frac{1}{2\pi \mathsf{i}}\{\mu_1(z), \partial_{\bar{z}'}B_{z2}(z')\} + \frac{1}{2\pi \mathsf{i}}\big\{\mu_1(z), [A_{\bar{z}2}(z'), B_{z2}(z')]\big\} \\
&= \frac{1}{2\pi \mathsf{i}}\Big(- \partial_{\bar{z}'}[C_{12}\delta_{zz'}, B_{z2}(z')] - \big[A_{\bar{z}2}(z), [C_{12}, B_{z2}(z')]\big]\delta_{zz'} \\
&\qquad - \big[[C_{12}, A_{\bar{z}2}(z)]\delta_{zz'} + \partial_{\bar{z}}(C_{12}\delta_{zz'}), B_{z2}(z')\big]\Big) \\
&= \frac{1}{2\pi \mathsf{i}}\Big(- [C_{12}, \partial_{\bar{z}}B_{z2}(z)]\delta_{zz'} - \big[C_{12}, [A_{\bar{z}2}(z), B_{z2}(z')]\big]\delta_{zz'}\Big) \\
&= -[C_{12}, \mu_2(z)]\delta_{zz'},
\end{aligned}$$

where in the second equality we used the relations (4.49), which also trivially hold with $\widehat{\mu}$ replaced by μ. In the third equality we have used the Jacobi identity and the fact that $\partial_{\bar{z}}\delta_{zz'} + \partial_{\bar{z}'}\delta_{zz'} = 0$, which follows using the identity $\partial_{\bar{z}}(z - z')^{-1} = -2\pi\mathsf{i}\delta_{zz'}$.

The Poisson bracket of $\widehat{\mu}$ with itself is therefore

$$\begin{aligned}\{\widehat{\mu}_1(z), \widehat{\mu}_2(z')\} &= \{\mu_1(z), \mu_2(z')\} + \sum_{i,j=1}^{N} \{\widehat{u}_{i1}, \widehat{u}_{j2}\}\delta_{zz_i}\delta_{z'z_j} \\ &= -[C_{12}, \mu_2(z)]\delta_{zz'} - \sum_{i=1}^{N}[C_{12}, \widehat{u}_{i2}\delta_{zz_i}]\delta_{z'z_i} = -[C_{12}, \widehat{\mu}_2(z)]\delta_{zz'}\end{aligned}$$

where in the second equality we have used (4.45) for the second term. This vanishes on the constraint surface so $\widehat{\mu}$ is first class – we will set it strongly to zero with an appropriate gauge fixing condition in the following section.

The time evolution of $\widehat{\mu}$ is given by

$$\begin{aligned}\{H, \widehat{\mu}(z)\} &\approx \frac{1}{2\pi\mathsf{i}}\big\{H^P_w, [A_{\bar{z}}(z), B_z(z)]\big\} \\ &= \frac{1}{2\pi\mathsf{i}}\big[\{H^P_w, A_{\bar{z}}(z)\}, B_z(z)\big] = -[P'(B_z(w)), B_z(z)]\delta_{zw} = 0,\end{aligned}$$

and therefore we have no tertiary constraints.

4.4.3 Gauge fixing and Lax formalism

Recall that so far we have fixed the pair of second class constraints $P_z \approx 0$ and $\mathcal{C}_z \approx 0$ by introducing the corresponding Dirac bracket in section 4.4.1.1. We kept the notation $\{\cdot,\cdot\}$ for this Dirac bracket. In section 4.4.1.2 we introduced a further Dirac bracket $\{\cdot,\cdot\}^*$ to fix the constraints $\mathcal{C}_i \approx 0$. As mentioned at the end of that section, by abuse of notation we continued to call this Dirac bracket $\{\cdot,\cdot\}$ since the Dirac bracket of the bulk fields is unaffected. In this section we start with the latter Dirac bracket and wish to fix the gauge invariance arising from the constraint $\widehat{\mu} \approx 0$.

We will use the gauge fixing condition $A_{\bar{z}} \approx 0$ and simultaneously impose this condition and the constraint $\widehat{\mu} \approx 0$ strongly by defining a new Dirac bracket. To this end, recall that

$$\{\widehat{\mu}_1(z), A_{\bar{z}2}(z')\} = -[C_{12}, A_{\bar{z}2}(z)]\delta_{zz'} - \partial_{\bar{z}}(C_{12}\delta_{zz'}) \approx -\partial_{\bar{z}}(C_{12}\delta_{zz'})$$

where the first equality is (4.49a) and in the last step we have used the new constraint $A_{\bar{z}} \approx 0$. This can certainly be inverted, since

$$\left\langle\!\!\left\langle -\partial_{\bar{z}}(C_{12}\delta_{zz'}), \frac{1}{2\pi\mathrm{i}}\frac{C_{23}}{z'-z''} \right\rangle\!\!\right\rangle_{(z',2)} = \frac{\mathrm{i}}{2\pi}C_{13}\partial_{\bar{z}}\left(\frac{1}{z-z''}\right) = \frac{\mathrm{i}}{2\pi}C_{13}(-2\pi\mathrm{i}\delta_{zz''}) = C_{13}\delta_{zz''}. \tag{4.50}$$

Here the subscript $(z',2)$ means that the pairing $\langle\cdot,\cdot\rangle$ is taken in the second tensor space and the integration is with respect to z'. We therefore define the new Dirac bracket, denoted $\{\cdot,\cdot\}^\star$ for $\mathfrak{g}$-valued functions U and V on C, by the usual formula [38, §1.3.3], see also [65, §2.6] for the analogous derivation in the 4d Chern-Simons theory context, namely

$$\begin{aligned}\{U_1(z),V_2(z')\}^\star &= \{U_1(z),V_2(z')\}\\ &- \left\langle\!\!\left\langle \{U_1(z),\widehat{\mu}_3(z'')\}, \left\langle\!\!\left\langle \frac{1}{2\pi\mathrm{i}}\frac{C_{34}}{z''-z'''}, \{A_{\bar{z}4}(z'''),V_2(z')\} \right\rangle\!\!\right\rangle_{(z''',4)} \right\rangle\!\!\right\rangle_{(z'',3)}\\ &- \left\langle\!\!\left\langle \{U_1(z),A_{\bar{z}3}(z''')\}, \left\langle\!\!\left\langle \frac{1}{2\pi\mathrm{i}}\frac{C_{34}}{z''-z'''}, \{\widehat{\mu}_4(z'''),V_2(z')\} \right\rangle\!\!\right\rangle_{(z''',4)} \right\rangle\!\!\right\rangle_{(z'',3)}.\end{aligned}$$

By construction, working with this Dirac bracket allows us to set the pair of constraints $\widehat{\mu} \approx 0$ and $A_{\bar{z}} \approx 0$ strongly to zero.

4.4.3.1 Lax algebra

We will show that the Dirac bracket of B_z with itself satisfies the Lax algebra

$$\{B_{z1}(z),B_{z2}(z')\}^\star = \big[r_{12}(z,z'),B_{z1}(z)+B_{z2}(z')\big], \tag{4.51}$$

where $r_{12}(z,z')$ is the standard classical r-matrix

$$r_{12}(z,z') = \frac{C_{12}}{z'-z}. \tag{4.52}$$

To compute this Dirac bracket, we begin by noting that (4.49b) implies

$$\{B_{z1}(z),\widehat{\mu}_2(z')\} = [C_{12},B_{z1}(z)]\delta_{zz'}.$$

Using this and the bracket $\{B_{z1}(z),A_{\bar{z}2}(z')\} = -2\pi\mathrm{i}C_{12}\delta_{zz'}$ which follows from

(4.33b) along with the constraint $\mathcal{C}_z = 0$ in (4.32), we find

$$\begin{aligned}
&\{B_{z1}(z), B_{z2}(z')\}^\star \\
&= -\left\langle\!\!\left\langle [C_{13}, B_{z1}(z)]\delta_{zz''}, \left\langle\!\!\left\langle \frac{1}{2\pi\mathrm{i}} \frac{C_{34}}{z''-z'''}, 2\pi\mathrm{i} C_{24}\delta_{z'z'''} \right\rangle\!\!\right\rangle_{(z''',4)} \right\rangle\!\!\right\rangle_{(z'',3)} \\
&\quad - \left\langle\!\!\left\langle -2\pi\mathrm{i} C_{13}\delta_{zz''}, \left\langle\!\!\left\langle \frac{1}{2\pi\mathrm{i}} \frac{C_{34}}{z''-z'''}, -[C_{24}, B_{z2}(z')]\delta_{z'z'''} \right\rangle\!\!\right\rangle_{(z''',4)} \right\rangle\!\!\right\rangle_{(z'',3)} \\
&= -\left\langle\!\!\left\langle [C_{13}, B_{z1}(z)]\delta_{zz''}, \frac{C_{23}}{z''-z'} \right\rangle\!\!\right\rangle_{(z'',3)} - \left\langle\!\!\left\langle C_{13}\delta_{zz''}, \left[\frac{C_{23}}{z''-z'}, B_{z2}(z')\right] \right\rangle\!\!\right\rangle_{(z'',3)} \\
&= -\left[\frac{C_{12}}{z-z'}, B_{z1}(z)\right] - \left[\frac{C_{12}}{z-z'}, B_{z2}(z')\right] = \left[\frac{C_{12}}{z'-z}, B_{z1}(z) + B_{z2}(z')\right].
\end{aligned}$$

In other words, we recover the Lax algebra (4.51).

4.4.3.2 Lax matrix

By definition of $\widehat{\mu}$ in (4.47), it follows that setting this constraint and its gauge fixing condition to zero strongly, *i.e.* $\widehat{\mu} = 0$ and $A_{\bar{z}} = 0$, leads to the equation

$$\partial_{\bar{z}} B_z = -2\pi\mathrm{i} \sum_{i=1}^{N} \widehat{u}_i \delta_{zz_i}, \tag{4.53}$$

or $\partial_{\bar{z}} B_z = -2\pi\mathrm{i}\, \widehat{u}_i \delta_{zz_i}$ in a small neighbourhood of the point z_i, which is equivalent to (4.18). This then leads to the local meromorphic behaviour (4.19) of the $(1,0)$-form B, namely

$$B = \frac{\widehat{u}_i}{z - z_i} \mathrm{d}z + O(1).$$

The Kostant-Kirillov bracket (4.45) for the residues $\widehat{u}_i$ obtained in section 4.4.1.2 (recall that we are now denoting the Dirac bracket $\{\cdot,\cdot\}^*$ of section 4.4.1.2 simply as $\{\cdot,\cdot\}$) is equivalent to the Lax algebra (4.51) derived in section 4.4.3.1.

4.4.3.3 Lax equation

At this point we have now fixed all the constraints strongly except for the primary constraint $\Pi_t \approx 0$. However, now that $\widehat{\mu} = 0$ is imposed strongly, the Hamiltonian (4.48) no longer involves the field A_t and simply reduces to

$$H = H^P_w.$$

In particular, together with the Dirac bracket (4.51) this now implies the Lax equation (4.28) in the Hamiltonian formalism

$$\left\{H^P_w, B_z(z)\right\}^\star = \left[\frac{P'\big(B_z(w)\big)}{z-w}, B_z(z)\right]. \tag{4.54}$$

We deduce, as claimed at the end of Section 4.3, that the time flow ∂_t along the topological direction of the 3-dimensional space $\mathbb{R} \times C$ is the one induced by the Hamiltonian $H_w^P = P(B_z(w))$ with respect to the Dirac bracket, *i.e.* $\partial_t f = \{H_w^P, f\}^\star$ for any function f of the Lax matrix B_z. Focusing on such observables, we are also free to set $\Pi_t = 0$ strongly since these all Poisson commute with Π_t under the Dirac bracket $\{\cdot,\cdot\}^\star$ and so their bracket will remain unchanged after introducing a further Dirac bracket to fix the constraint $\Pi_t \approx 0$.

4.4.3.4 Involution

It is well known [3] that the Lax algebra (4.51) implies the involution property

$$\left\{H_w^P, H_z^Q\right\}^\star = 0, \tag{4.55}$$

for any pair of G-invariant polynomials $P, Q : \mathfrak{g} \to \mathbb{C}$ and distinct points $w, z \in C$.

This can also be seen more directly from the above Hamiltonian analysis of 3d mixed BF theory as follows. Since $H_w^P = P(B_z(w))$ only depends on the field B_z we have the involution property

$$\left\{H_w^P, H_z^Q\right\} = 0 \tag{4.56}$$

with respect to the Poisson bracket (more precisely, recall that $\{\cdot,\cdot\}$ denotes the Dirac bracket introduced in section 4.4.1), for *any* polynomials $P, Q : \mathfrak{g} \to \mathbb{C}$ and distinct points $w, z \in C$. But since H_w^P is gauge invariant for G-*invariant* polynomials P and $-\widehat{\mu}$ is the generator of gauge transformations, see (4.49), we have $\{\widehat{\mu}(z), H_w^P\} = 0$. The involution property (4.56), for any polynomials $P, Q : \mathfrak{g} \to \mathbb{C}$, therefore immediately implies the involution property (4.55), for any G-invariant polynomials $P, Q : \mathfrak{g} \to \mathbb{C}$.

4.5 Future directions

We have now shown that classical Gaudin models associated with a finite-dimensional semisimple Lie algebra, and more generally tamely ramified Hitchin systems, can be obtained from 3d mixed BF theory in the presence of certain line defects by moving to the Hamiltonian framework and fixing the gauge symmetry using the gauge fixing condition $A_{\bar{z}} \approx 0$ - exactly analogously to the 4-dimensional gauge theoretic origins of the affine Gaudin model from 4dCS [65].

4.5.1 Alternative realisations

The Lax matrix of the Gaudin model, or the Higgs field of the Hitchin system, arises from the $(1,0)$-form B of the 3d mixed BF theory. In particular, after going to the gauge $A_{\bar{z}} = 0$ the latter becomes meromorphic with simple poles (4.19) at each z_i, the location of the type A line defects. The specific choice of line defect (4.13) led to the residues of B at these simple poles being coadjoint orbits $\widehat{u}_i = h_i u_i h_i^{-1}$ of some fixed Lie algebra elements $u_i \in \mathfrak{g}$. As is well known, and as we have rederived in the present setting in section 4.4.1.2, such coadjoint orbits provide a realisation of the Kostant-Kirillov Poisson bracket (4.45).

It would be interesting to see if other realisations of the Kostant-Kirillov Poisson bracket can be obtained by making other choices of type A defects than (4.13). Indeed, since the field B_z satisfies the Lax algebra (4.51) regardless of the choice of type A line defects we make, the residues $\widehat{u}_i$ at each simple pole z_i of B_z will necessarily satisfy the Kostant-Kirillov bracket. As mentioned in the affine case in [65, §4.1], it would be desirable to find the precise dictionary between the possible choices of type A line defects one can introduce in 3d mixed BF theory and the different types of possible representations of the Kostant-Kirillov bracket.

4.5.2 Generalised Gaudin models

We have focused in this paper on the case when the Lax matrix of the Gaudin model, or the Higgs field of the Hitchin system, has simple poles at the marked points z_i.

It would be interesting to consider also type A line defects which would give rise to higher order poles in the Lax matrix in order to construct Gaudin models with irregular singularities as we saw in previous chapters. In the affine setting, generalised surface defects in 4-dimensional Chern-Simons theory leading to affine Gaudin models with irregular singularities were considered in [6, 42].

Other generalisations of the Gaudin model which one could try to relate to 3d mixed BF theory, or some generalisation thereof, include cyclotomic Gaudin models [62, 70, 69] or dihedral Gaudin models (see [66] in the affine case), whose Lax matrices are equivariant under the action of cyclic or dihedral groups, respectively. In the affine case, such a generalisation was considered recently in [56] where the symmetric space λ-model, which can be described as a $\mathbb{Z}_4$-cyclotomic affine Gaudin model, was obtained along the lines of [65] starting from 4d Chern-Simons theory with a $\mathbb{Z}_4$-equivariance condition imposed on the gauge field.

Both the gauge theoretic and algebraic approaches to 2-dimensional integrable field theories, which are of course intimately related [65], have been used to construct many new examples of 2-dimensional classical integrable field theories in recent years; see for instance [18, 17, 4, 2] in the affine Gaudin model setting and the references above in the 4d Chern-Simons theory setting. Finite Gaudin models, or equivalently 3d mixed BF theory, could similarly be used to extend the list of known finite-dimensional integrable systems.

4.5.3 4dBF Theory

As we have seen, we can describe the finite Gaudin model via 3dBF theory and the affine Gaudin model through 4dCS, we might question why we have this mix of theories, instead of using 3-dimensional Chern-Simons theory or a 4-dimensional version of BF theory. One perspective suggests that the description of affine Gaudin models though 4dBF and though 4dCS might be somehow equivalent, as both can be viewed as a dimensional reduction (in a loose sense) of 5dCS along a copy of S^1[*]- either the first factor, or the copy hiding in $\mathbb{C}^* = S^1 \times \mathbb{R}$. In summary it would suggest that theories on the same row of the following (speculative) diagram may be gauge-fixed to describe the same model;

$$\begin{tikzcd}
& \boxed{\begin{array}{c}\mathfrak{g}\text{-5dCS on}\\ S^1\times\mathbb{C}^*\times C\end{array}} \arrow[dl] \arrow[dr] & \\
\boxed{\begin{array}{c}\mathfrak{g}\text{-4dBF on}\\ \mathbb{C}^*\times C\end{array}} \arrow[rr, dashed, leftrightarrow] \arrow[d] & & \boxed{\begin{array}{c}\mathfrak{g}\text{-4dCS on}\\ S^1\times\mathbb{R}\times C\end{array}} \arrow[d] \\
\boxed{\begin{array}{c}\mathfrak{g}\text{-3dBF on}\\ \mathbb{R}\times C\end{array}} \arrow[rr, dashed, leftrightarrow] & & \boxed{\begin{array}{c}\mathfrak{g}\text{-3dCS on}\\ \mathbb{R}\times C\end{array}}
\end{tikzcd}$$

where the solid arrows represent reducing the dimensions of the theory along a copy of S^1 (recalling that $\mathbb{C}^* = \mathbb{R} \times S^1$) and the dashed lines represent adding

[*]We are grateful to K. Costello for private communication about this.

term to the action of the BF theory in order to deform it into a Chern-Simons theory.

Let us describe this deformation in the 3 dimensional case, following the argument of [1, §2.3]. For Riemann surface C and field $\mathcal{A} \in \Omega^1(\mathbb{R} \times C, \mathfrak{g})$, the 3d Chern-Simons action is given by

$$S_{3dCS}[\mathcal{A}] = \frac{k}{4\pi i} \int_{\mathbb{R}\times C} CS(\mathcal{A}), \tag{4.57}$$

where $CS(\mathcal{A})$ is the Chern Simons three-form defined in (4.1). The level k is a complex multiple of the Killing form , which we will be able to use to "switch off" the deformation. Let us rescale the $(1,0)$-component of $\mathcal{A}$ to be nontrivial at level zero, that is we write $\mathcal{A}_z = \frac{1}{k} B_z$, and the remainder we label as the $(0,1)$-form $A = \mathcal{A}_{\bar{z}} \mathrm{d}\bar{z} + \mathcal{A}_t \mathrm{d}t$. In terms of A and B the Chern-Simons three-form (4.1) becomes

$$CS(\mathcal{A}) = \langle \frac{1}{k} B, \mathrm{d}A + \frac{1}{3}[A,A] \rangle + \frac{1}{k} \langle A, \mathrm{d}B + \frac{2}{3}[A,B] \rangle + \langle A, \partial A \rangle, \tag{4.58}$$

as $\langle A, \bar{\partial} A \rangle = 0$. We then reintroduce the curvature 2-form $F(A) = \mathrm{d}A + \frac{1}{2}[A,A]$ and reorder the second term,

$$CS(\mathcal{A}) = \frac{2}{k} \langle B, F(A) \rangle + \langle A, \partial A \rangle - \frac{1}{k} \mathrm{d} \langle A, B \rangle. \tag{4.59}$$

Hence the 3dCS action can be rewritten as

$$S_{3dCS}[\mathcal{A}] = \frac{1}{2\pi i} \int_{\mathbb{R}\times C} \langle B, F(A) \rangle + \frac{k}{4\pi i} \langle A, \partial A \rangle, \tag{4.60}$$

which clearly produces the 3d mixed BF action at level zero. We can therefore view 3dCS as a deformation of 3d mixed BF by $\frac{k}{2} \langle A, \partial A \rangle$, as in the diagram above.

To write the action of 4dBF theory, we let C be a Reimann surface and take the $\mathfrak{g}$-valued fields $\alpha \in \Omega^{(0,1)}(\mathbb{C}^* \times C)$ and $\beta \in \Omega(2,0)(\mathbb{C}^* \times C)$. The action is entirely analogous to the 3dBF action

$$S_{4\mathrm{dBF}}[\alpha, \beta] = \frac{1}{4\pi \mathrm{i}} \int_{\mathbb{C}^* \times C} \langle \beta, F^{(0,2)}(\alpha) \rangle$$

where $F^{(0,2)}(\alpha)$ is the holomorphic curvature,

$$F^{(0,2)}(\alpha) = \bar{\partial} \alpha + \frac{1}{2}[\alpha, \alpha].$$

We can write the fields α, β in terms of some holomorphic coordinates $\lambda = e^{t+\mathrm{i}\theta}, \bar{\lambda}$ on $\mathbb{C}^*$ and $z, \bar{z}$ on C,

$$\alpha = \alpha_{\bar{\lambda}}\bar{\lambda}^{-1}\mathrm{d}\bar{\lambda} + \alpha_z \mathrm{d}z + \alpha_{\bar{z}}\mathrm{d}\bar{z}, \tag{4.61}$$

$$\beta = \beta_z \mathrm{d}z \wedge \lambda^{-1}\mathrm{d}\lambda. \tag{4.62}$$

Performing the same Hamiltonian analysis as we did for 3dBF theory, we arrive at the Lax equations of the finite $\mathfrak{g}$-Gaudin model, this time with an additional degree of freedom in the variable θ;

$$\{\beta_{z1}(\theta, z), \beta_{z1}(\theta', z')\}^* = \delta_{zz'}\delta_{\theta\theta'}\left[\frac{C_{12}}{z - z'}, \beta_{z1}(\theta, z) + \beta_{z2}(\theta, z)\right]. \tag{4.63}$$

In other words this is the the Gaudin model of the Loop algebra $\mathcal{L}_{\mathfrak{g}}$, which makes sense given that we can rewrite 4dBF associated with $\mathfrak{g}$ to be written as 3dBF associated with $\mathcal{L}_{\mathfrak{g}}$. This theory is *ultralocal* in the sense that its Poisson bracket (4.63) does not contain any terms proportionl to a derivative of the Dirac delta [19].

On the other hand, the classical affine Gaudin model is non-ultralocal, with Poisson bracket given by [17, Eq.(2.18)]

$$\begin{aligned}\{L_1(z,\theta), L_2(z',\theta')\} = &\left[\frac{C_{12}}{z' - z}^{-1}, L_1(z,\theta) - L_2(z',\theta)\right]\delta_{\theta\theta'} \\ &- (\varphi(z) - \varphi(z'))\frac{C_{12}}{z' - z}\delta'_{\theta\theta'}.\end{aligned} \tag{4.64}$$

As it has been shown that the 4dCS theory produces the affine Gaudin model on suitable gauge fixing, we would need to deform the 4dBF theory in order to introduce this non-ultralocal term. Guided by (4.60) in the 3d case, for some meromorphic 1-form $\omega = \varphi(z)\mathrm{d}z$ let us write

$$S_{\text{4dBF-def}}[\alpha, \beta] = \frac{1}{4\pi\mathrm{i}}\int_{\mathbb{C}^*\times C}\langle\beta, F^{(0,2)}(\alpha)\rangle + \frac{1}{4\pi i}\int_{\mathbb{C}^*\times C}\omega \wedge \langle\alpha, \partial\alpha\rangle. \tag{4.65}$$

where the level k has been replaced with the meromorphic function $\varphi(z)$. For the theory to be non-ultralocal we expect ω to have zeroes [65, §4.3]. In analogy with the 3d case, we would hope that this deformed 4dBF action is equivalent to the 4dCS action. This is suggested by the equations of motion; if we let

$$\beta = \omega \wedge \alpha_\lambda \lambda^{-1}\mathrm{d}\lambda$$

and recall that $\alpha_\lambda = \frac{1}{2}(\alpha_t - i\alpha_\theta)$, then one of he equations of of motion of the action (4.65) is the zero curvature equation

$$\partial_t \alpha_\theta - \partial_\theta \alpha_t + \alpha_t \alpha_\theta = 0. \tag{4.66}$$

This is consistent with the equations of motion of 4dCS as we have seen earlier in the chapter, however further work would be required to show that this deformation is fully equivalent to the 4dCS action. A next step might then be to perform the Hamiltonian Analysis on this deformed action in order to obtain the affine Gaudin model from the 4dBF perspective.

4.5.4 Potential Relation to Separation of Variables

We finish with a brief note on how this 3dBF perspective might relate to the work on Separation of Variables we discussed in the previous part. Since we focused on SoV in *quantum* Gaudin models, the 3dBF story would in reality relate to the classical SoV conducted by Skrypnyk and Dubrovin in [23], and the quantum version would instead most likely be from the perspective of 3-dimensional Chern-Simons Theory [72].

Recall that we found that the separated equation in the quantum $\mathfrak{sl}_2$ and $\mathfrak{gl}_3$ Gaudin models to be given by the universal oper

$$\operatorname{cdet}\left(\partial_{q_i} + L(q_i)\right) = 0, \tag{4.67}$$

and for the classical Gaudin models by the spectral curve

$$\det\left(\lambda(q_i) - L(q_i)\right) = 0, \tag{4.68}$$

where $\lambda(z)$ is the eigenvalue of $L(z)$. To bring this into the 3dBF perspective we can choose our type-B defect (4.22) to be this classical spectral curve, using our identification of the Lax matrix $L(z)$ with the component B_z of the gauge-field B;

$$S_{\text{B-def}} = -\int_{\mathbb{R}\times w} \det(\lambda(w) - B_z(w))\mathrm{d}t. \tag{4.69}$$

Since the spectral curve generates as its coefficients the integrals of motion of the classical Gaudin model, which are given by the gauge invariant polynomials P, this would be the most general form of the type-B defect. Pursuing this connection could lead to a new approach for SoV, and more widely for opers in the quantum case, from the gauge-theoretic perspective.

— A —

Constructing Differential Operator Realisations of Takiff Lie Algebras

For the separation of variables portion of this thesis, we will need to consider the Lie algebra $\mathfrak{g}$ underlying the Gaudin model as an algebra of differential operators, in order for the eigenvalue equations of the Gaudin Hamiltonians to read as differential equations so we can apply the separation of variables technique.

For example, it is well-known that the basis $\{e, f, h\}$ for $\mathfrak{sl}_2$ may be realised as differential operators on the space $\mathbb{C}[[x]]$ of formal power series in x

$$\phi(e) = -x\partial_x^2 + 2\ell\partial_x, \quad \phi(f) = x, \quad \phi(h) = -x\partial_x + \ell \tag{A.1}$$

where $\ell \in$ is the weight. The space $\mathbb{C}[[x]]$ is then a Verma module of $\mathfrak{sl}_2$, with the corresponding highest weight vector being $1 \in \mathbb{C}[[x]]$. We can construct finite-dimensional highest weight modules of weight $2\ell+1$ by taking the quotient by the ideal generated by $x^{2\ell+1}$. Extending this, we can realise $U(\mathfrak{sl}_2)^{\otimes N}$in $\mathbb{C}[x_1, \ldots, x_N]$ with the ith copy of $\mathfrak{sl}_2$ as in equation (A.1) in the variable x_i, and therefore all of the observables of the $\mathfrak{sl}_2$-Gaudin model.

On the other hand, when considering the Gaudin model with irregular singularities we will need a differential operator realisation analogous to (A.1) for the Takiff Lie algebra $\mathfrak{sl}_2[\varepsilon_i]/\varepsilon_i^{\tau_i}$ as defined in Chapter 1, as well as for $\mathfrak{gl}_3[\varepsilon_i]/\varepsilon_i^{\tau_i}$ when we turn to consider the higher rank scenario. Fortunately we can use the formula, outlined by Draisma in [22], to realise any Lie algebra in terms of first order differential operators. The stipulation that they are first order will not actually be necessary for our purposes, and in fact we will perform the transformation $x \leftrightarrow \frac{\partial}{\partial x}$ to ensure other conveniences in the final expression, however it is still the basis of the construction. Let us outline in detail here the method for constructing the $\mathfrak{sl}_2[\varepsilon_i]/\varepsilon_i^{\tau_i}$ realisation in particular, as the algorithm is very similar for $\mathfrak{gl}_3[\varepsilon_i]/\varepsilon_i^{\tau_i}$ so we need only point out some minor differences. A useful final check will be to recover (A.1) on setting the Takiff degrees $\tau_i = 1$.

A.1 Realisation formula for $\mathfrak{sl}_2[\varepsilon_i]/\varepsilon_i^{\tau_i}$

For the realisation formula [22, Thm 2.1], we let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra such that the codimension $\dim \mathfrak{g} - \dim \mathfrak{h} = n$ is finite, this will be the number of variables in which we represent $\mathfrak{g}$.

For our example, we will take $\mathfrak{h} = \mathfrak{b}^+[\varepsilon_i]/\varepsilon_i^{\tau_i}$ generated by $\{e_{[r]}\}_{r=0}^{\tau_i-1}$ and $\{h_{[r]}\}_{r=0}^{\tau_i-1}$, the Takiff equivalent to the (non-strictly) upper triangular matrices for $\mathfrak{sl}_2$ in the fundamental representation. The codimension of $\mathfrak{b}^+[\varepsilon_i]/\varepsilon_i^{\tau_i}$ is τ_i so as expected our realisation of $\mathfrak{s}l_2$ will be in differential operators in τ_i variables which we label $x_{[0]}, \dots, x_{[\tau_i-1]}$. We must choose a realisation with a suitable number of variables, for example one could take $\pi(E^a_b) = x_a \partial_{x_b}$ as a realisation for $\mathfrak{gl}_\mathfrak{n}$ at any rank, but we would then only have n variables.

The formula then requires that we choose an ordered basis $\mathcal{B}$ the first part of which is a basis $\mathcal{C}$ of $\mathfrak{h}$, (which in our example case will be given by $\{h_{[0]}, \dots, h_{[\tau_i-1]}, e_{[0]}, \dots, e_{[\tau_i-1]}\}$) and the second part is an ordered set of elements $\{Y_1, \dots, Y_n\}$ spanning the rest of $\mathfrak{g}$ (which here will be $\{f_{[0]}, \dots, f_{[\tau_i-1]}\}$ ordering by mode).

For the generic formula, we extend $\mathcal{B}$ to the corresponding PBW-basis of the universal enveloping algebra $U(\mathfrak{g})$ - that is, the basis given by all correctly ordered monomials in $\mathcal{B}$. We then define functions $\chi_i : U(\mathfrak{g}) \to \mathbb{C}$ which take an element in the enveloping algebra to the coefficient of Y_i when it is reordered to be written in the PBW basis.The realisation of $X \in U(\mathfrak{g})$ in terms of differential operators in some variables $y_1, \dots, y_n$ is then given by

$$\psi(X) = \sum_{i=1}^{N} \Big(\sum_{m_1,\dots,m_n \in \mathbb{Z}_{\geq 0} N} \chi_i(Y_1^{m_1} \cdots Y_n^{m_n} X) \frac{y_1^{m_1} \cdots y_n^{m_n}}{m_1! \cdots m_n!} \Big) \partial_i, \tag{A.2}$$

where ∂_i denotes $\frac{\partial}{\partial y_i}$.

Let us consider what the value of $\chi_i(f_{[0]}^{m_0} \dots f_{[\tau_i-1]}^{m_{\tau_i-1}} X)$ will be for each of our basis elements $e_{[r]}, f_{[r]}, h_{[r]}$. To correctly order these we will need to move the $f_{[s]}$ for all modes $s = 0, \dots, \tau_i - 1$ to the end of the expression, and at this point we can count the single f terms arising from the commutation relations (1.29) - if indeed there are any. The most straightforward case is to consider basis elements $f_{[r]}$, as these all commute at different modes we obtain no additional terms from the reordering at all, and so χ_i will only return a non-zero result if $m_0 = m_1 = \cdots = m_{\tau_i - 1} = 0$ and if $i = r$. Hence it is straightforward to plug this in to the formula (A.2) and see that $\psi(f_{[r]}) = \partial_{x_{[r]}}$ which we denote $\partial_{[r]}$.

When considering $\psi(h_{[r]})$, we note from the Lie bracket

$$\left[h_{[r]}, f_{[s]}\right] = -2f_{[r+s]}$$

that the χ_i functions will only have a non-zero output if exactly one of the $m_j = 1$ and the rest are zero, at which point we see that $\chi_i(f_{[s]}h_{[r]}) = 2\delta_{i,r+s}$. A similar line of reasoning follows for $e_{[r]}$ leaving us with the overall realisation of $\mathfrak{sl}_2[\varepsilon_i]/\varepsilon_i^{\tau_i}$

$$\psi(f_{[r]}) = \partial_{[r]}, \tag{A.3a}$$

$$\psi(h_{[r]}) = 2\sum_{s=0}^{\tau_i - 1} x_{[s]}\partial_{[r+s]}, \tag{A.3b}$$

$$\psi(e_{[r]}) = -\sum_{\substack{s,t,=0 \\ s+t+r<\tau_i}}^{\tau_i - 1} x_{[s]}x_{[t]}\partial_{[r+s+t]}. \tag{A.3c}$$

If we set the degree $\tau_i = 1$ and return to the non-Takiff case, we can see that we have clearly not yet reconstructed (A.1) in the variable $x_{[0]}$, as the weight ℓ is not present anywhere in (A.3). Though this remains a valid differential operator realisation (one can check that the expected commutation relations apply), the weights will play a role in Chapter 2. To add these in, we turn again to Draisma who provides a further algorithm [22, Thm 3.2] for this purpose;

$$\widetilde{\phi}(X) = \psi(X) + \sum_{m_1,\dots,m_n} \eta(\pi_{\mathfrak{h}}(Y_1^{m_1}\cdots Y_n^{m_n}X))\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\cdots m_n!} \tag{A.4}$$

where $\pi_{\mathfrak{h}}$ is the projector onto $\mathfrak{h}$ and $\eta : \mathfrak{h} \to \mathbb{C}$ are the functions that provides the weights with the requirement that $\eta([\mathfrak{h}, \mathfrak{h}]) = 0$. For the $\mathfrak{sl}_2[\varepsilon_i]/\varepsilon_i^{\tau_i}$ realisation, this condition forces $\eta(e_{[r]}) = 0$ for $r = 0, \dots, \tau_i$, leaving us with the effect of η on the diagonals when viewed as matrices at all modes, which we label as the weights such that $\eta(h_{[r]}) = 2\ell_{[r]}$. The projector $\pi_{\mathfrak{h}}$ will kill any additional term not in $\mathfrak{h}$, so we do not expect the weights to appear in $\tilde{\phi}(f_{[r]})$ as is consistent with (A.1). By the same process as above, now considering reorderings that leave a single copy of $h_{[s]}$, we find the realisation

$$\widetilde{\phi}(f_{[r]}) = \partial_{[r]}, \tag{A.5a}$$

$$\widetilde{\phi}(h_{[r]}) = 2\sum_{s=0}^{\tau_i-1-r} x_{[s]}\partial_{[r+s]} + 2\ell_{[r]}, \tag{A.5b}$$

$$\widetilde{\phi}(e_{[r]}) = -\sum_{\substack{s,t,=0\\ s+t+r<\tau_i}}^{\tau_i-1} x_{[s]}x_{[t]}\partial_{[r+s+t]} + 2\sum_{s=0}^{\tau_i-1-r} \ell_{[r+s]}x_{[s]}. \tag{A.5c}$$

Finally, it will be convenient for our purposes in the separation of variables part of this thesis for our realisation of $f_{[r]}$ to be in one of the variables only and involving no differential operators $\partial_{x_{[r]}}$. To this end we make the transformation $x \leftrightarrow \frac{\partial}{\partial x}$ and rearrange, absorbing any constants that arise from this into the weights $\ell_{[r]}$ - we also add a subscript i to the variables and weights because we will be using this realisation across the N sites of the Gaudin model;

$$\pi(f_{[r]}^{(i)}) = x_{i,[r]} \tag{A.6a}$$

$$\pi(h_{[r]}^{(i)}) = \sum_{s=0}^{\tau_i-1-r} -2x_{i,[s+r]}\partial_{i,[s]} + 2\ell_{i,[r]}. \tag{A.6b}$$

$$\pi(e_{[r]}^{(i)}) = \sum_{\substack{s,t=0\\ s+t+r<\tau_i}}^{\tau_i-1} -x_{i,[r+s+t]}\partial_{i,[s]}\partial_{i,[t]} + \sum_{s=0}^{\tau_i-1-r} 2\ell_{i,[r+s]}\partial_{i,[s]}, \tag{A.6c}$$

which is entirely consistent with the commutation relations of $\mathfrak{sl}_2[\varepsilon_i]/\varepsilon_i^{\tau_i}$ and with (A.1) on setting $\tau_i = 1$.

A.2 Realisations of $\mathfrak{gl}_3$ and $\mathfrak{gl}_3(\varepsilon_\mathfrak{i})/\varepsilon_\mathfrak{i}^\tau$

We will also require realisations of $\mathfrak{gl}_3$ and the Takiff Lie algebra $\mathfrak{gl}_3(\varepsilon)/\varepsilon^\tau$ in Chapter 2. Derkachov and Valeinivich [21] provide such a realisation for $\mathfrak{sl}_3$ and we can construct a similar one here using Draisma's method that we described above.

Recall the standard basis $\{E^i_j\}_{i,j=1}^3$ of $\mathfrak{gl}_3$. In this case we take $\mathfrak{h}$ to again be the equivalent of the upper triangular matrices, hence $\{Y_1,\ldots,Y_n\} = \{E^2_1, E^3_1, E^3_2\}$ and so we expect three variables (x_i, y_i, z_i) on each site. Following the same steps as we did for the $\mathfrak{sl}_2$ Takiff algebra we arrive at the following realisation:

$$\begin{aligned}
\pi(E_1^{1(i)}) &= \ell_i^3 + y_i\partial_{y_i} + z_i\partial_{z_i} \\
\pi(E_1^{2(i)}) &= z_i + y_i\partial_{x_i} \\
\pi(E_1^{3(i)}) &= y_i \\
\pi(E_2^{1(i)}) &= x_i\partial_{y_i} + \partial_{z_i}(\ell_i^2 - \ell_i^3 - z_i\partial_{z_i}) \\
\pi(E_2^{2(i)}) &= \ell_i^1 + 1 + x\partial_{x_i} - z_i\partial_{z_i} \\
\pi(E_2^{3(i)}) &= x_i \\
\pi(E_3^{1(i)}) &= \partial_y(-x_i\partial_{x_i} - y_i\partial_{y_i} - z_i\partial_{z_i} - 1 + \ell_i^1 - \ell_i^3) \\
&\quad - \partial_{x_i}\partial_{z_i}(z_i\partial_{z_i} + u_2 - u_3) \\
\pi(E_3^{2(i)}) &= -z_i\partial_{y_i} + \partial_{x_i}(-x_i\partial_{x_i} - y_i\partial_{y_i} + z_i\partial_{z_i} + \ell_i^1 - \ell_i^2) \\
&\quad + z_i\partial_{z_i} + \ell_i^1 - \ell_i^2 \\
\pi(E_3^{3(i)}) &= \ell_i^2 + x_i\partial_{x_i} - z_i\partial_{z_i}.
\end{aligned} \tag{A.7}$$

where $\ell_i^1, \ell_i^2, \ell_i^3$ are the three weights per site.

Similarly to realise $\mathfrak{gl}_3[\varepsilon)]/\varepsilon^\tau$, we have the set of Y_i given by $\{E_{1,[r]}^{2(i)}, E_{1[r]}^{3(i)}, E_{2[r]}^{3(i)}\}_{r=0}^{\tau_i-1}$, so we expect $3\tau_i$ variables to realise this algebra, or $3\sum_{i=1}^N \tau_i = 3\mathcal{D}$ variables across all the sites.

$$\begin{aligned}
\pi(E_{1[r]}^{1(i)}) &= \sum_{s=0}^{\tau_i-r-1} -\partial_{x_{i,[s]}}x_{i,[s+r]} - \partial_{z_{i,[s]}}z_{i,[s+r]} + \sigma_{i,r}^1 \\
\pi(E_{1[r]}^{2(i)}) &= -x_{i,[r]} - \sum_{s=0}^{\tau_i-r-1} \partial_{y_{i,[s]}}z_{i,[r+s]} \\
\pi(E_{1[r]}^{3(i)}) &= -z_{i,[r]} \\
\pi(E_{2[r]}^{1(i)}) &= \sum_{s=0}^{\tau_i-r-1} \partial z_{i,[s]}y_{i,[s+r]} + (\sigma_{i,r+s}^2 - \sigma_{i,r+s}^1)\partial_{z_{i,[s]}} \\
&\quad + \sum_{\substack{s,t=0 \\ r+s+t<\tau_i}}^{\tau_i-1} \partial_{x_{i,[s]}}\partial_{x_{i,[t]}}x_{i,[r+s+t]} \\
\pi(E_{2[r]}^{2(i)}) &= \sigma_{i,[r]}^2 + \sum_{s=0}^{\tau_i-r-1} \partial_{x_{i,[s]}}x_{i,[s+r]} - \partial_{y_{i,[s]}}y_{i,[s+r]} \\
\pi(E_{2[r]}^{3(i)}) &= -y_{i,[r]}
\end{aligned} \tag{A.8}$$

$$
\begin{aligned}
\pi(E^{1(i)}_{3[r]}) &= \sum_{s=0}(\sigma^3_{i,r+s} - \sigma^1_{i,r+s})\partial_{z_{i,[s]}} \\
&\quad + \sum_{\substack{s,t=0\\ r+s+t<\tau_i}} \partial_{z_{i,[s]}}\partial_{z_{i,[t]}} z_{i,[r+s+t]} + \partial_{x_{i,[s]}}\partial_{z_{i,[t]}} x_{i,[r+s+t]} + \partial_{y_{i,[s]}}\partial_{z_{i,[t]}} y_{i,[r+s+t]} \\
&\quad + (\sigma^2_{i,r+s} - \sigma^1_{i,r+s})\partial_{x_{i,[s]}}\partial_{y_{i,[t]}} + \sum_{\substack{s,t,q=0\\ q+r+s+t<\tau_i}} \partial_{x_{i,[s]}}\partial_{x_{i,[t]}}\partial_{y_{i,[q]}} x_{i,[q+r+s+t]} \\
\pi(E^{2(i)}_{3[r]}) &= \sum_{s=0}\partial_{z_{i,[s]}} y_{i,[r+s]} + (\sigma^3_{i,[r+s]} - \sigma^2_{i,[r+s]})\partial_{y_{i,[s]}} \\
&\quad + \sum_{s,t=0} \partial_{y_{i,[s]}}\partial_{y_{i,[ts]}} y_{i,[r+s+t]} + \partial_{x_{i,[s]}}\partial_{y_{i,[y]}} x_{i,[r+s+t]} + \partial_{y_{i,[s]}}\partial_{z_{i,[t]}} z_{i,[r+s+t]} \\
\pi(E^{3(i)}_{3[r]}) &= \sigma^3_{i,[r]} + \sum_{s=0}^{\tau_i-r-1} \partial_{y_{i,[s]}} y_{i,[r+s]} + \partial_{z_{i,[s]}} z_{i,[r+s]}.
\end{aligned}
$$

Here the weights are denoted by $\sigma^k_{i,[r]}$, where $k = 1, \ldots, 3$, r denotes the mode and i the site. Note that if we set the Takiff degree to zero, we do again recover (A.7) exactly after reordering.

A.3 Realisation of the Double Pole at Infinity

When including a double pole at infinity in the model, we have an additional commutative copy of $\mathfrak{g}$ to realise. Each basis element is realised by a complex number and no new variables, as they all commute. For $\mathfrak{sl}_2$ and $\mathfrak{sl}_2[\varepsilon_i]/\varepsilon_i^{\tau_i}$ we have

$$\phi(e^\infty) = 0, \quad \phi(f^\infty) = 1, \quad \phi(h^\infty) = 2\lambda \tag{A.9}$$

for $\lambda \in \mathbb{C}$. It is important that $\phi(f^{(\infty)})$ is non-zero in particular to ensure that we can treat our separated variables equivalently. We can write this compactly in the following matrix

$$\phi(E^{a(\infty)}_b)^2_{a,b=1} = \begin{pmatrix} \lambda & 0 \\ 1 & -\lambda \end{pmatrix}. \tag{A.10}$$

And similarly for $\mathfrak{gl}_3$ and $\mathfrak{gl}_3[\varepsilon]/\varepsilon^\tau$ we summarise this as in [68] by

$$\phi(E^{a(\infty)}_b)^3_{a,b=1} = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{pmatrix}, \tag{A.11}$$

where $\lambda \in \mathbb{C}$. The non-zero constant term for $E^{3(\infty)}_2$ is particularly important here when it comes to sensibly inverting the Lax matrix element $L^2_3(z)$ in the definition of our separating function $A(z)$ in Chapter 3.

References

[1] M. Aganagic, K. Costello, J. McNamara, and C. Vafa. “Topological Chern-Simons/Matter Theories” (June 2017). arXiv: 1706.09977.

[2] G. Arutyunov, C. Bassi, and S. Lacroix. “New integrable coset sigma models”. *JHEP* 03.3 (Mar. 2021), p. 062. ISSN: 10298479.

[3] O. Babelon, D. Bernard, and M. Talon. “Introduction to Classical Integrable Systems”. *Introduction to Classical Integrable Systems* (Apr. 2003).

[4] C. Bassi and S. Lacroix. “Integrable deformations of coupled -models”. *JHEP* 05.5 (May 2020), p. 059. ISSN: 10298479.

[5] A. Beilinson and V. Drinfeld. “Quantization of Hitchin’s Integrable System and Hecke Eigensheaves” (1991). available at www.math.uchicago.edu/~mitya/langlands/hitchin/BD-hitchin.pdf.

[6] M. Benini, A. Schenkel, and B. Vicedo. “Homotopical Analysis of 4d Chern-Simons Theory and Integrable Field Theories”. *Communications in Mathematical Physics* 389.3 (Feb. 2022), pp. 1417–1443. ISSN: 14320916. arXiv: 2008.01829.

[7] V. Caudrelier, M. Stoppato, and B. Vicedo. “On the Zakharov–Mikhailov action:” *Lett. Math. Phys.* 111.3 (June 2021), p. 82. ISSN: 15730530. arXiv: 2012.04431.

[8] A. Cavaglià, N. Gromov, and F. Levkovich-Maslyuk. “Separation of variables and scalar products at any rank”. *Journal of High Energy Physics* 2019.9 (July 2019). arXiv: 1907.03788.

[9] A. Cavaglià, N. Gromov, and F. Levkovich-Maslyuk. "Separation of Variables in AdS/CFT: Functional Approach for the Fishnet CFT". *Journal of High Energy Physics* 2021.6 (Mar. 2021). arXiv: 2103.15800v2.

[10] A. Chervov, G. Falqui, and V. Rubtsov. "Algebraic properties of Manin matrices 1". *Advances in Applied Mathematics* 43.3 (Sept. 2009), pp. 239–315. ISSN: 0196-8858.

[11] A. Chervov and G. Falqui. "Manin matrices and Talalaev's formula". *Journal of Physics A: Mathematical and Theoretical* 41.19 (Nov. 2007). arXiv: 0711.2236.

[12] K. Costello. "Integrable lattice models from four-dimensional field theories". *Proc. Symp. Pure Math.* 88 (Dec. 2014), p. 3. arXiv: 1308.0370.

[13] K. Costello. "Supersymmetric gauge theory and the Yangian" (Mar. 2013). arXiv: 1303.2632.

[14] K. Costello, E. Witten, and M. Yamazaki. "Gauge Theory And Integrability, I". *Notices of the International Congress of Chinese Mathematicians* 6.1 (2018), pp. 46–119. ISSN: 23264810. arXiv: 1709.09993.

[15] K. Costello, E. Witten, and M. Yamazaki. "Gauge Theory And Integrability, II". *Notices of the International Congress of Chinese Mathematicians* 6.1 (Feb. 2018), pp. 120–146. ISSN: 23264810. arXiv: 1908.02289.

[16] K. Costello and M. Yamazaki. "Gauge Theory And Integrability, III" (2019). arXiv: 1908.02289v1.

[17] F. Delduc, S. Lacroix, M. Magro, and B. Vicedo. "Assembling integrable -models as affine Gaudin models". *JHEP* 06.6 (June 2019), p. 017. ISSN: 10298479.

[18] F. Delduc, S. Lacroix, M. Magro, and B. Vicedo. "Integrable Coupled σ-Models". *Phys. Rev. Lett.* 122.4 (2019). ISSN: 10797114.

[19] F. Delduc, T. Kameyama, S. Lacroix, M. Magro, and B. Vicedo. "Ultralocal Lax connection for para-complex $\mathbb{Z}_T$-cosets". *Nuclear Physics B* 949 (Sept. 2019). arXiv: 1909.00742v1.

[20] F. Delduc, S. Lacroix, M. Magro, and B. Vicedo. "A unifying 2d action for integrable -models from 4d Chern–Simons theory". *Lett. Math. Phys.* 110.7 (July 2020), pp. 1645–1687. ISSN: 15730530. arXiv: 1909.13824.

[21] S. E. Derkachov and P. A. Valinevich. "Separation of variables for the quantum SL(3,C) spin magnet: eigenfunctions of Sklyanin B-operator" (July 2018). arXiv: 1807.00302.

[22] J. Draisma. "Constructing Lie algebras of first order differential operators". *Journal of Symbolic Computation* 36.5 (2003), pp. 685–698.

[23] B. Dubrovin and T. Skrypnyk. "Separation of variables for linear Lax algebras and classical r -matrices". *Journal of Mathematical Physics* 59.9 (Sept. 2018). ISSN: 00222488.

[24] B. Feigin, E. Frenkel, and V. Toledano Laredo. "Gaudin models with irregular singularities". *Adv. Math.* 223.3 (Feb. 2010), p. 873. ISSN: 00018708. arXiv: math/0612798.

[25] B. Feigin and E. Frenkel. "Quantization of soliton systems and Langlands duality". *Exploring New Structures and Natural Constructions in Mathematical Physics* (May 2007), pp. 185–274. arXiv: 0705.2486.

[26] B. Feigin, E. Frenkel, and N. Reshetikhin. "Gaudin model, Bethe ansatz and correlation functions at the critical level". *Commun. Math. Phys.* 166.1 (Dec. 1994), pp. 27–62. ISSN: 00103616.

[27] B. Feigin, E. Frenkel, and L. Rybnikov. "Opers with irregular singularity and spectra of the shift of argument subalgebra". *Duke Math. J.* 155.2 (Nov. 2010), pp. 337–363. ISSN: 00127094.

[28] E. Frenkel. "Opers on the projective line, flag manifolds and Bethe Ansatz". *Mosc. Math. J.* 4.3 (2004), pp. 655–705. ISSN: 16093321.

[29] E. Frenkel. "Gaudin model and opers". *Progress in Mathematics* 237 (July 2004), pp. 1–58. ISSN: 2296505X. arXiv: math/0407524.

[30] D. Gaiotto and E. Witten. "Gauge Theory and the Analytic Form of the Geometric Langlands Program". *Annales Henri Poincare* (2022). ISSN: 14240637. arXiv: 2107.01732.

[31] D. Garajeu. "Bethe ansatz for the Gaudin model and its relation with Knizhnik-Zamolodchikov equations". *Journal of Mathematical Physics* 43.11 (Nov. 2002), pp. 5732–5756. ISSN: 00222488.

[32] M. Gaudin. "Diagonalisation d'une classe d'Hamiltoniens de spin". *J. Phys.* 37.10 (1976), pp. 1087–1098. ISSN: 0302-0738.

[33] M. Gaudin. “La fonction d’onde de Bethe”. *Collection du Commissariat à l’énergie atomique. Série scientifique* (1983).

[34] N. Gromov, F. Levkovich-Maslyuk, and G. Sizov. “New Construction of Eigenstates and Separation of Variables for SU(N) Quantum Spin Chains”. *Journal of High Energy Physics* 2017.9 (Oct. 2016). arXiv: 1610.08032.

[35] O. Gwilliam, E. Rabinovich, and B. R. Williams. “Quantization of topological-holomorphic field theories: local aspects” (July 2021). arXiv: 2107.06734.

[36] O. Gwilliam and B. R. Williams. “A one-loop exact quantization of Chern-Simons theory” (Oct. 2019). arXiv: 1910.05230.

[37] R. Hartshorne. “Algebraic Geometry”. Graduate Texts in Mathematics 52 (1977).

[38] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems*. 1992.

[39] N. Hitchin. “Stable bundles and integrable systems”. *Duke Mathematical Journal* 54.1 (1987). ISSN: 00127094.

[40] S. Lacroix. “Four-dimensional Chern-Simons theory and integrable field theories”. *Journal of Physics A: Mathematical and Theoretical* 55.8 (Feb. 2022). ISSN: 17518121. arXiv: 2109.14278.

[41] S. Lacroix. “Integrable models with twist function and affine Gaudin models” (2018). arXiv: 1809.06811.

[42] S. Lacroix and B. Vicedo. “Integrable $\mathcal{E}$-Models, 4d Chern-Simons Theory and Affine Gaudin Models. I. Lagrangian Aspects”. *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)* 17 (Nov. 2020), p. 45. arXiv: 2011.13809v2.

[43] F. Levkovich-Maslyuk. “Lectures on the Bethe Ansatz” (2016). arXiv: 1606.02950v2.

[44] J. M. Maillet and G. Niccoli. “On quantum separation of variables”. *Journal of Mathematical Physics* 59.9 (July 2018). arXiv: 1807.11572v2.

[45] E. Mukhin, V. Tarasov, and A. Varchenko. “Schubert calculus and representations of the general linear group”. *J. Am. Math. Soc.* 22.4 (Apr. 2009), pp. 909–940. ISSN: 0894-0347. arXiv: 0711.4079.

[46] E. Mukhin and A. Varchenko. "Critical points of master functions and flag varieties". *Commun. Contemp. Math.* 6.1 (Feb. 2004), pp. 111–163. ISSN: 02191997.

[47] E. Mukhin and A. Varchenko. "Multiple orthogonal polynomials and a counterexample to Gaudin Bethe Ansatz Conjecture". *Trans. Am. Math. Soc.* 359.11 (June 2007), pp. 5383–5418. ISSN: 0002-9947. arXiv: math/0501144.

[48] E. Mukhin and A. Varchenko. "Miura opers and critical points of master functions". *Cent. Eur. J. Math.* 3.2 (June 2005), pp. 155–182. ISSN: 1895-1074.

[49] E. Mukhin and A. Varchenko. "Norm of a Bethe vector and the Hessian of the master function". *Compos. Math.* 141.4 (July 2005), pp. 1012–1028. ISSN: 0010437X.

[50] E. Rabinovich. "Factorization Algebras for Bulk-Boundary Systems" (Nov. 2021). arXiv: 2111.01757.

[51] A. L. Retore. "Introduction to classical and quantum integrability" (2022). arXiv: 2109.14280v2.

[52] S. Ribault. "On sl3 Knizhnik-Zamolodchikov equations and W3 null-vector equations". *Journal of High Energy Physics* 2009.10 (Nov. 2008). arXiv: 0811.4587.

[53] P. Ryan and D. Volin. "Separated variables and wave functions for rational gl(N) spin chains in the companion twist frame". *Journal of Mathematical Physics* 60.3 (Mar. 2018). ISSN: 00222488.

[54] P. Ryan and D. Volin. "Separation of variables for rational gl(n) spin chains in any compact representation, via fusion, embedding morphism and Bäcklund flow" (2020). arXiv: 2002.12341v2.

[55] L. Rybnikov. "A proof of the Gaudin Bethe Ansatz conjecture". *Int. Math. Res. Not.* 22 (2020), pp. 8766–8785.

[56] D. M. Schmidtt. "Symmetric space λ-model exchange algebra from 4d holomorphic Chern-Simons theory". *JHEP* 12.12 (Dec. 2021), p. 004. ISSN: 10298479.

[57] E. K. Sklyanin. “Separation of Variables in the Classical Integrable SL(3) Magnetic Chain”. *Communications in Mathematical Physics* 150.1 (Nov. 1992), pp. 181–191. arXiv: hep-th/9211126v1.

[58] E. K. Sklyanin. “Separation of variables in the Gaudin model”. *Journal of Soviet Mathematics* 47.2 (Oct. 1989), pp. 2473–2488. ISSN: 00904104.

[59] E. K. Sklyanin. “Separation of variables in the quantum integrable models related to the Yangian Y[sl(3)]”. *Journal of Mathematical Sciences* 80.3 (Dec. 1992), pp. 1861–1871. arXiv: hep-th/9212076.

[60] E. K. Sklyanin. “Separation of Variables. New Trends.” *Progress of Theoretical Physics Supplement* 118 (Apr. 1995), pp. 35–60. arXiv: solv-int/9504001v1.

[61] E. K. Sklyanin. “The quantum Toda chain”. *Non-Linear Equations in Classical and Quantum Field Theory* (Apr. 6334), pp. 196–233.

[62] T. Skrypnyk. “"Z2-graded" Gaudin models and analytical Bethe ansatz”. *Nucl. Phys. B* 870.3 (May 2013), pp. 495–529. ISSN: 05503213.

[63] T. Skrypnyk. “On a class of $gl(n) \otimes gl(n)$ -valued classical r -matrices and separation of variables”. *Journal of Mathematical Physics* 62.6 (June 2021), p. 63508. ISSN: 00222488.

[64] D. V. Talalaev. “The quantum Gaudin system”. *Functional Analysis and its Applications* 40.1 (Jan. 2006), pp. 73–77. ISSN: 00162663.

[65] B. Vicedo. “4d Chern–Simons theory and affine Gaudin models”. *Lett. Math. Phys.* 111.1 (Feb. 2021), p. 24. ISSN: 15730530.

[66] B. Vicedo. “On integrable field theories as dihedral affine Gaudin models”. *Int. Math. Res. Not.* 2020.15 (2020), pp. 4513–4601. ISSN: 16870247. arXiv: 1701.04856.

[67] B. Vicedo and J. Winstone. “3-dimensional mixed BF theory and Hitchin’s integrable system”. *Letters in Mathematical Physics* 112.4 (Jan. 2022). arXiv: 2201.07300v1.

[68] B. Vicedo and C. Young. “(glm, gln)-Dualities in Gaudin Models with Irregular Singularities”. *Symmetry Integrability and Geometry-methods and Applications* 14 (May 2017). ISSN: 18150659.

[69] B. Vicedo and C. Young. “Cyclotomic Gaudin models with irregular singularities”. *J. Geom. Phys.* 121 (Nov. 2017), pp. 247–278. ISSN: 03930440.

[70] B. Vicedo and C. Young. “Cyclotomic Gaudin models: construction and Bethe ansatz”. *Commun. Math. Phys.* 343.3 (May 2016), pp. 971–1024. ISSN: 14320916. arXiv: 1409.6937v1.

[71] E. Witten. “Integrable Lattice Models From Gauge Theory”. *Advances in Theoretical and Mathematical Physics* 21.7 (Nov. 2016), pp. 1819–1843. ISSN: 10950753. arXiv: 1611.00592.

[72] K. Zeng. “Monopole Operators and Bulk-Boundary Relation in Holomorphic Topological Theories” (2021). arXiv: 2111.00955.