Aspects of endomorphism monoids of certain algebras

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PhD

University of York
Mathematics

August 2023

Abstract

This thesis is concerned with the study of endomorphism monoids of certain algebras. We first describe the semigroup structure of a family of subsemigroups of the endomorphism monoid of an independence algebra $\mathscr{A}$. Each of these subsemigroups is associated with a subalgebra $\mathscr{B}$ of $\mathscr{A}$ and is called the *subsemigroup of endomorphisms with restricted range in* $\mathscr{B}$. Denoted by $T(\mathscr{A},\mathscr{B})$, it consists of all endomorphisms of $\mathscr{A}$ whose image lies in $\mathscr{B}$. We show in particular that such semigroups are not regular in general and that they present significant differences in their structure from that of $\mathrm{End}(\mathscr{A})$.

In a similar fashion, we investigate the semigroup structure of $\mathrm{End}(\mathcal{T}_n)$, the endomorphism monoid of the full transformation monoid of a finite set with n elements. We describe the ideals of $\mathrm{End}(\mathcal{T}_n)$ and show that, in particular, $\mathcal{T}_n$ and $\mathrm{End}(\mathcal{T}_n)$ are not respectively embeddable into each other (except in the degenerate case of $n = 1$).

We then move on to the study of translational hulls of ideals of the endomorphism monoid of algebras. We start with the case of an independence algebra $\mathscr{A}$, where we discuss the translational hull $\Omega(\mathfrak{I})$ of the (0-)minimal ideal $\mathfrak{I}$ of $\mathrm{End}(\mathscr{A})$. We give conditions under which $\Omega(\mathfrak{I})$ and $\mathrm{End}(\mathscr{A})$ are isomorphic and we construct a canonical isomorphism where possible. A more general approach of translational hulls in the case where $\mathscr{A}$ is an arbitrary algebra is then presented, where we prove that any ideal $\mathfrak{I}$ of $\mathrm{End}(\mathscr{A})$ satisfying some representability and separability conditions on $\mathscr{A}$ will be such that its translational hull is isomorphic to $\mathrm{End}(\mathscr{A})$. Finally, we close this thesis by computing the translational hulls of some of the ideals of $\mathrm{End}(\mathscr{A})$, where $\mathscr{A}$ will stand either for a free algebra, an independence algebra, or $\mathcal{T}_n$.

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Acknowledgments

Cette thèse n'aurait point vu le jour sans le soutien de mes parents et grands-parents, qui m'ont sans cesse encouragé à pousser toujours plus loin les limites de mes connaissances, m'en donnant les moyens ainsi que le support nécessaire pour l'accomplir.

Another person who has undeniably been key in my development as a mathematician is Vicky. From the moment she taught me the course on Semigroup Theory here in York, and all the way throughout her supervision, Vicky has pushed me to go further in my understanding of mathematics, always keen to give me new questions (apparently some were supposed to be easy...) and to support me when things where not really moving as expected. I cannot thank her enough for all she has brought me.

To all my friends who studied alongside me (Gab, Hugo, Quentin ×2, Anto, Nico, Gauthier, Maria et autres potes de l'EPFL), the other PhDs of the department with whom I have spent much time going to pubs, and the friends whom I have met through my various musical and sporty activities (especially those from UYBC where I indubitably have spent way more time than I should have): thank you for being part of the journey.

There is however a group of people that deserves a special mention: *the extended Millfielders*. Housemates by status or by adoption, regulars of the Sunday dinners, and close friends, Andrew, Ding, Rutvij, Jenny, Ali, Georgia, Sam, Simen, Beth, and above all Cordelia, thank you very much for these years with you!

AG

Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

Barring a few results, the content of Chapters II and III has appeared as a single article in Semigroup Forum [24], while the content of Chapter IV, following a collaboration with Prof. Victoria Gould and Dr. Marianne Johnson, has been submitted and is currently pending review, with a pre-print available [23].

A York semigroupist tradition

C'est un coin de tableau où chantent des formules
Accrochant follement aux habits des débris
De craies ; où des notions, d'un cerveau somnambule,
Naissent : c'est un petit trou qui de maths est épris.

Un élève jeune, bouche ouverte, tête nue,
Et le crâne baignant dans une pluie d'idées,
Pense ; il se tient debout, en cet endroit reclu,
Considérant ces fonctions, qu'il veut étudier.

Les méninges emballées, il pense. Souriant comme
Sourirait un enfant béat, là il raisonne :
Nature, berce-le savamment : qu'il s'émeuve.

Les problèmes présents ne sont que passagers ;
Il pense à son travail, la logique aux aguets,
Fébrile. Il a deux erreurs dans cette preuve.

Le mathématicien du bureau[1]

[1] Adaptation de *Le dormeur du val* d'Arthur Rimbaud

Introduction

For any mathematical object M, the endomorphism monoid of M as well as its group of units, the automorphism group, are key to understanding the structure of M. A well-known result of G. Birkhoff [5] for groups, later extended to monoids, tells us that every group is isomorphic to the automorphism group of some unary algebra; analogously, every monoid is isomorphic to the endomorphism monoid of some unary algebra. Here the term *algebra* is understood in the sense of universal algebra, or abstract algebra as introduced in the early 20$^{\text{th}}$ century (see the introduction of the book by McKenzie, McNulty and Taylor [34] for a historical perspective of its origin). In this thesis, we study the endomorphism monoid of certain algebras through the lenses of restrictions and extensions. These are to be taken in the sense of the study of subsets of the endomorphism monoid having nice structural properties and we will see how these can generate the endomorphism monoid via fairly natural constructions that will be explained below.

A specific class of algebras that will be of particular interest in the coming work is that of *independence algebras*. These were introduced by V. Gould [22] in order to account for the similarities encountered in the structure of the endomorphism monoid of a vector space when compared to that of the full transformation monoid of a non-empty set. Independence algebras have been fully classified (see [2, 6, 54]) and include sets and vector spaces, as well as free group acts, affine algebras or quasifield algebras. The structure of the endomorphism monoid of an independence algebra has been widely studied, for example by looking at its ideals and Green's relations [22], its idempotent generated part [17, 18] or its automorphism groups and normal subgroups [1]. We take here several different directions.

First, we study the family of subsemigroups of the endomorphism monoid which arise from restricting the set of endomorphisms to those whose image lies in a specific subalgebra. More precisely, given an independence algebra $\mathscr{A}$, denote its endomorphism monoid by $\mathrm{End}(\mathscr{A})$. For each subalgebra $\mathscr{B}$ of $\mathscr{A}$, we can define the *semigroup of endomorphisms with restricted range in* $\mathscr{B}$, denoted by $T(\mathscr{A},\mathscr{B})$, as the subsemigroup which only contains endomorphisms of $\mathscr{A}$ whose image lie inside the subalgebra $\mathscr{B}$, that is,

$$T(\mathscr{A},\mathscr{B}) = \{\alpha \in \mathrm{End}(\mathscr{A}) \mid \operatorname{im}\alpha \subseteq \mathscr{B}\}\,.$$

Even though these semigroups inherit some structural properties from that of $\mathrm{End}(\mathscr{A})$, they are very different in many ways and their study will be the object of Chapters II and III. To start with, a semigroup of the form $T(\mathscr{A},\mathscr{B})$ is not a monoid unless we are in one of the extreme cases where $\mathscr{B}$ is the whole of $\mathscr{A}$ or a singleton, nor is it a regular semigroup. This leads us not only to study Green's relations and the ideal structure of these semigroups, but also the generalised versions of these notions through the so-called extended Green's relations and the $*$- and $\sim$-ideals of $T(\mathscr{A},\mathscr{B})$.

Another direction of research is to describe when some subset of $\mathrm{End}(\mathscr{A})$ contains enough information to reconstruct the endomorphism monoid exactly. In order to make this precise, we investigate the notion of translational hulls of a semigroup. Given an endomorphism $\alpha \in \mathrm{End}(\mathscr{A})$, we can define natural actions on ideals $\mathfrak{I} \subseteq \mathrm{End}(\mathscr{A})$ by multiplying elements of $\mathfrak{I}$ by α either on the left or on the right. Such actions are called the *left* and *right translations induced by* α on $\mathfrak{I}$. However, it might happen that not all left [resp. right] actions on $\mathfrak{I}$ come from the multiplication by an endomorphism in $\mathrm{End}(\mathscr{A})$. Thus we will define the translational hull of $\mathfrak{I}$, denoted by $\Omega(\mathfrak{I})$, as the set of all pairs of left and right actions which satisfy some compatibility conditions. The question is now whether the translational hull of $\mathfrak{I}$ only consists of translations induced by elements of $\mathrm{End}(\mathscr{A})$. One of the objects of this thesis is therefore to decide if this holds when $\mathfrak{I}$ is the ideal consisting of all endomorphisms whose image lie in a monogenic subalgebra of $\mathscr{A}$, which correspond in our case to the set of maps that have rank at most one. We will show in Chapter V that this holds for independence algebras with the exception of a few special cases, and will give conditions to describe when this happens.

In fact, such an approach is not particular to independence algebras, and in Chapter VI we will generalise this study to more abstract algebras. In other words, given an algebra $\mathscr{A}$ and an ideal $\mathfrak{I} \subseteq \operatorname{End}(\mathscr{A})$, we will provide sufficient conditions under which the translational hull of $\mathfrak{I}$ will be isomorphic to the endomorphism monoid $\operatorname{End}(\mathscr{A})$. In order to illustrate the discussions surrounding these conditions, we will give examples in Chapter VII where the general theory can be applied, as well as instances where the actual computation of translational hulls of ideals of $\operatorname{End}(\mathscr{A})$ can be hard to achieve. The latter happens in a prominent fashion when considering the full transformation monoid $\mathcal{T}_n$ on a finite set as our algebra $\mathscr{A}$. Since the description of the structure of $\operatorname{End}(\mathcal{T}_n)$ has not been presented before, we will spend time to do so in Chapter IV.

A FOREWORD CONCERNING THE ORDER OF THE CHAPTERS:

The content presented in this thesis does not consist of a sequence of results which needs to be read in a linear order. Far from it, the reader is advised to choose and pick the order in which they will read the coming chapters, after taking into account the following note.

Not counting the preliminary chapter (which place in this thesis will be explained in its prologue), there are three main groupings of chapters. The first one, consisting of Chapters II & III, stands separately from the two others as the only place where the semigroup $T(\mathscr{A},\mathscr{B})$ is considered. The second grouping, which only includes Chapter IV with its description of $\operatorname{End}(\mathcal{T}_n)$, sits on its own. Even though Chapters V–VII can be grouped together since they all deal with translational hulls of ideals, they can be further separated. Indeed, the ideas and techniques used in Chapter V are very different from those of Chapter VI, and the classes of algebras examined are also distinct. This makes the reading of these two chapters mostly independent from each other. The only chapter for which the reader will need to have read other parts of this thesis is Chapter VII where ideas discussed in Chapter VI are adapted to the content of Chapters IV & V.

Thus, after reading the opening of Chapter I, we encourage the reader to cherry-pick the chapter(s) they want to read based on their urge to learn about a specific topic, their mood, the time they have at hand, or even the simple roll of a die or some random thought that would cross their mind at the moment they would lay their hand on this thesis.

Happy reading!

— I —

Preliminaries

Throughout this thesis, we will assume that the reader is confident with classical semigroup theory concepts such as congruences, semigroup homomorphisms and Green's relations. If in the coming chapters, the reader needs to refresh their knowledge about a notion that is not defined in the present chapter, then they are advised to consult the excellent introductory books of Howie [28], Clifford and Preston [8, 9] or Higgins [27]. Additionally, the reader is expected to know the basic notions of universal algebras defined as pairs (A, F) where A is the universe of the algebra and F its set of fundamental operations. This includes understanding the definitions and properties of homomorphisms, terms and the subalgebra operator. A good reference in this domain is the book from McKenzie, McNulty and Taylor [34].

In this chapter we will give an introduction to the lesser-known notions used in this thesis. In particular, on the topic of general semigroup theory, Section I.2 will describe a generalisation of the notion of semigroup action through the concept of translational hulls and give the connection between this notion and that of ideal extensions. Section I.3 will present extended Green's relations, a set of equivalence relations used to study the structure of non-regular semigroups. Following this, a very important class of universal algebras called independence algebras, which generalises the concept of sets and vector spaces, will be introduced in Section I.4. Since the main object of this thesis concerns endomorphism monoids, we shall close this chapter by giving properties of the endomorphism monoid of an independence algebra in Section I.5.

Apart from a few results, we have taken the decision to include proofs of the statements in this chapter, in order to be as complete as possible and to help the reader see how one can play with the definitions and some arguments in a general context before meeting them again at a later point in the thesis. This means in particular that it makes this introductory chapter fairly long. Nevertheless, there is no need to read it all in one go since the different concepts presented here will be used in different places throughout the thesis. Indeed, Section I.2 is only needed from Chapter V onwards even though it is introduced first here. Chapters II, III and V will need the background on independence algebras of Sections I.4 and I.5. Finally, Section I.3 will only be necessary to read Chapters III and IV. For this reason, the reader is advised to first choose which grouping of chapters they want to read (II–III, IV or V–VII) before coming here to read only the relevant sections.

In order to facilitate the reading of this thesis, we start with some general notation.

I.1 NOTATION

Since this preliminary chapter does not introduce the standard semigroup and universal algebraic background, it is necessary to describe the basic notation and conventions that will be used throughout this thesis. Hopefully, we have managed to conciliate the conciseness of the writing with the clarity of the statements, meaning that a simple glance over a formula should easily convey most of the information without being cluttered by too many brackets, parentheses or multiple indices.

In order to facilitate this, a general rule is that elements of a semigroup or an algebra will use Roman letters, while maps will use Greek letters. However, since we will consider functions on endomorphism monoids, we will often have to deal with three different levels of domain and we will distinguish them (as much as the finitary aspect of our alphabets allows) as follows:

- the "bottom" level, consisting of elements from a semigroup or an algebra, will be denoted by Roman letters;
- the "middle" level, consisting of endomorphisms on our semigroup or algebra, will take the first part of the Greek alphabet; and
- the "top" level, consisting of maps acting on the endomorphisms, will be relegated to use the second part of the Greek alphabet.

Following this general guidance and mindset, we now give the appropriate notation that will be used without further mention in the following chapters, first by considering general notation, and then that which is specific to the two main areas of algebra involved in the coming work: semigroups and universal algebras.

I.1.1 General notation

Indexing sets. When it is clear from the context, the set $\{x_i \mid i \in I\}$ will be abbreviated as $\{x_i\}$ without necessarily specifying its index set. If such an abbreviation is used and we need to refer to the indexing set after the definition of this set, we will render it as the capital letter corresponding to the index used. For example, the indexing set of $\{y_j\}$ is denoted by J. Moreover, an indexing set will usually be non-empty and its size not necessarily finite. Whether each of these possibilities can happen in the given situation will be explicitly mentioned when it cannot be clearly deduced from the context. For example, one cannot have an infinite indexing set when picking distinct elements of a finite set, and if we know that the chosen set of elements is non-empty, then its indexing set is also non-empty. Without loss of generality, a non-empty set of cardinality at least $k \in \mathbb{N}$ will be assumed to contain elements $1, 2, \ldots, k$, so that if $\{y_j\}$ is non-empty and contains at least 3 elements, we can talk about y_1, y_2 and y_3 without ambiguity.

Set operations. Given two sets A and B, we usually denote their union by $A \cup B$. However, if we want to give the additional information that A and B are disjoint sets, then we will write their (disjoint) union as $A \sqcup B$. By writing $B \subseteq A$, we say that B is a subset of A, and to further indicate that this is a proper subset we will use $B \subsetneq A$.

Maps and composition. Following the standard convention in semigroup theory, maps will be *right maps*, that is, written to the right of their operand. For example, if ϕ is a map that sends x to y, we write $x\phi = y$. In the context of translational hulls, we will allow some specific maps to be written on the left in order to facilitate some later notation and computations and we will call them *left maps*. For such maps, saying that the image of x under λ is y will be written as $\lambda x = y$.

Well-defined compositions of two right [respectively left] maps will not require extra parentheses (e.g. we can write $x\alpha\beta$ in place of $(x\alpha)\beta$), whereas they might be required when both a left and a right map are involved in an expression (e.g. for a left map λ and a right map ρ, $\lambda(x\rho)$ might be different from $(\lambda x)\rho$).

Unless specified otherwise, the identity map will usually be denoted by id, or id_A when we need to explicit the underlying set A.

Two-row writing of maps. Let S be a set with disjoint subsets $\{x_i\}$ and $\{y_j\}$ such that $S = \{x_i\} \sqcup \{y_j\}$. By writing $\alpha = \begin{pmatrix} x_i & y_j \\ a_i & b_j \end{pmatrix}$ we mean that α is a map on S (with image in some set A) such that $x_i\alpha = a_i$ for all $i \in I$ and $y_j\alpha = b_j$ for all $j \in J$. Notice that this does not mean that the sets $\{a_i\}$ and $\{b_j\}$ are disjoint.

Image, kernel and endomorphisms. Let S and T be two semigroups, or two algebras and $\alpha\colon S \to T$. Then the image of α is written $\operatorname{im}\alpha = S\alpha \subseteq T$ and the kernel relation of α is $\ker\alpha = \{(s,t) \in S \times S \mid s\alpha = t\alpha\}$.

The monoid of endomorphisms of S is denoted by $\mathrm{End}(S)$, and the automorphism group by $\mathrm{Aut}(S)$. If α is an isomorphism, then we write $S \cong T$.

I.1.2 On semigroups

Thanks to the associativity property of semigroups we will drop all unnecessary parentheses.

Transformations. For a set X, we will denote by $\mathcal{T}_X$ its *full transformation monoid*, by $\mathcal{PT}_X$ its *monoid of partial transformations*, and by $\mathcal{I}_X$ its *symmetric inverse monoid.* In the case where $X = \{1, \ldots, n\}$, these will be written as $\mathcal{T}_n$, $\mathcal{PT}_n$ and $\mathcal{I}_n$ respectively.

If we want to consider left maps instead, we will add the symbol 'op' as exponent to the above notation, that is, the sets $\mathcal{T}_X^{\mathrm{op}}$, $\mathcal{PT}_X^{\mathrm{op}}$ and $\mathcal{I}_X^{\mathrm{op}}$ will correspond to the monoids of full, partial, and bijective partial transformations of X when written on the left.

Idempotents. The set of idempotents of a semigroup S is denoted by $E(S)$, which will be often shortened to E.

Green's relations. The Green's relations, of utmost importance in semigroup theory, will be denoted by the letters $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$, $\mathscr{H}$ and $\mathscr{J}$. In case there is a need to distinguish the semigroup in which these are considered, a subscript will be added to the relation, writing $\mathscr{L}_S$, $\mathscr{R}_S$, $\mathscr{D}_S$, $\mathscr{H}_S$ and $\mathscr{J}_S$ to denote that the Green's relations are to be considered using elements of S.

I.1.3 For universal algebras

Algebras and their universe. Algebras will always be denoted by calligraphic letters such as $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, while their universe will be denoted by the corresponding letter in regular font, here A, B and C. By writing that $\mathscr{B} \subseteq \mathscr{A}$, we mean that $\mathscr{B}$ is a subalgebra of $\mathscr{A}$.

Operations and terms. In order to differentiate them from maps on the algebra, all fundamental operations and terms will be operating on the left. Given an algebra $\mathscr{A}$, the set of all terms of $\mathscr{A}$ will be denoted $\mathcal{T}^{\mathscr{A}}$, and the subset of k-ary terms by $\mathcal{T}_k^{\mathscr{A}}$. We associate nullary operation with their image in A, which we call *constants*, while terms whose image is a singleton (that is, constant terms) will be called *algebraic constants*. The unary term corresponding to the projection on its variable, which is also the identity map, will be denoted by id.

Moreover, to reduce the amount of parentheses needed, we also agree that terms will have priority over functions during the evaluation of an expression in an algebra, that is, given $t \in \mathcal{T}_k^{\mathscr{A}}$, as well as elements $a_1, \ldots, a_k \in A$ and a map ϕ on A, we write $t(a_1, \ldots, a_k)\phi$ for the expression $(t(a_1, \ldots, a_k))\,\phi$. To shorten notation, when it is clear from context, given t, $a_1, \ldots, a_k$ and ϕ as before, we will write $t(\overline{a_i})$ for $t(a_1, \ldots, a_k)$ and $t\big(\overline{a_i\phi}\big)$ for $t(a_1\phi, \ldots, a_k\phi)$.

Subalgebra operator. The subalgebra operator will be denoted by $\langle \cdot \rangle$, and given an algebra $\mathscr{A}$, and a set $X \subseteq A$, we will abuse notation and will consider $\langle X \rangle$ both as a subuniverse and as a subalgebra. Notice that $\langle \emptyset \rangle$ is non-empty exactly whenever $\mathscr{A}$ has constants.

Cardinals. A cardinal which is of special interest is the size of the smallest generating set of a subalgebra, which we will denote by e. In other words, $e = 0$ if $\langle \emptyset \rangle \neq \emptyset$ and $e = 1$ otherwise. The successor of a cardinal κ will be denoted by κ^+. Moreover, following usual conventions, we denote by $\aleph_0$ the smallest infinite cardinal.

I.2 TRANSLATIONAL HULLS

One of the most natural approaches in algebra to study a mathematical object, is to consider the action of this object on itself, or on another compatible object. More precisely, given a semigroup S, it is common to look at the action of S on itself by left or right multiplication by one of its elements. One of the frameworks aiming to generalise these actions is the notion of translational hull, which is intimately related to the description of the left and right actions of elements of S.

Specifically, consider a semigroup S and define for each s in S a left map $\lambda_s \colon S \to S$ as well as a right map $\rho_s \colon S \to S$ by

$$\lambda_s t = st \qquad \text{and} \qquad t\rho_s = ts,$$

for all $t \in S$. These maps can easily be seen to satisfy the following properties for all $u, v \in S$ where $\lambda = \lambda_s$ and $\rho = \rho_s$:

1) $\lambda(uv) = (\lambda u)v$,

2) $(uv)\rho = u(v\rho)$, and

3) $u(\lambda v) = (u\rho)v$.

These observations lead to the study of pairs of functions (λ, ρ) where λ is a left map, ρ is a right map, and 1), 2) and 3) hold. The study of the translational hull of a semigroup consists exactly in finding all such pairs.

The translational hulls of many classes of semigroups have already been investigated. In some cases, they were shown to share common properties with their underlying semigroup. In particular, if a semigroup is cancellative [resp. inverse, adequate, type A, or a semilattice], then the same holds for its translational hull (see [4, 16, 21, 41, 44]). For some classes it is even possible to give a complete description of the elements of the translational hull. This is the case for example for completely 0-simple semigroups as shown by Petrich [42]. In this thesis, we will be interested in translational hulls of ideals of endomorphism monoids of certain universal algebras.

All the results present in this section are folklore and can be found in classical books on semigroup theory such as Howie [28] or Clifford and Preston [8, 9], but a more in-depth survey on translational hulls was written by Petrich [41]. Some of the proofs are located in other places of the literature but can be uncovered in [40, 42, 52, 53].

I.2.1 Definitions and properties

Throughout this section, S denotes a semigroup.

Definition I.2.1. Translations of a semigroup S are defined as follows.

- A right map $\rho\colon S \to S$ is a *right translation of S* if

$$(st)\rho = s(t\rho)$$

for all $s, t \in S$. The set of all right translations of S is denoted by $\mathrm{P}(S)$.

- Dually, a left map $\lambda\colon S \to S$ is a *left translation of S* if

$$\lambda(st) = (\lambda s)t$$

for all $s, t \in S$, and we write $\Lambda(S)$ for the set of all left translations of S.

- A left translation λ and a right translation ρ on S are said to be *linked* if for all $s, t \in S$ we have:

$$s(\lambda t) = (s\rho)t.$$

In such case, we call the pair (λ, ρ) a *linked pair* or a *bi-translation* of S.

- The set of all linked pairs (λ, ρ) of S is a subset of $\Lambda(S) \times \mathrm{P}(S)$ called the *translational hull of S* and is denoted by $\Omega(S)$.

Remark I.2.2. The main reason why we decided to write left translations as left maps is to make the linking condition easier to write and to work with. In particular, given $s, t \in S$ and $(\lambda, \rho) \in \Omega(S)$, both possible interpretations of the expressions $st\rho$ and λst coincide, while there is a unique interpretation of $s\lambda t$ and $s\rho t$ since λ acts on the left while ρ acts on the right. We can therefore remove the parentheses without making the equations ambiguous, and we will often take the choice to do so. Writing left translations as left maps instead of right maps will also facilitate the notation of the product of elements in $\Omega(S)$.

Definition I.2.3. The *natural projections* from $\Omega(S)$ into $\mathrm{P}(S)$ and $\Lambda(S)$ are the maps π_{P} and π_{Λ} defined by

$$(\lambda, \rho)\pi_{\mathrm{P}} = \rho, \qquad \text{and} \qquad (\lambda, \rho)\pi_{\Lambda} = \lambda, \qquad \text{for all } (\lambda, \rho) \in \Omega(S).$$

The images of π_{P} and π_{Λ} are respectively denoted by $\widetilde{\mathrm{P}}(S)$ and $\widetilde{\Lambda}(S)$.

The most straightforward example of translations that are linked comes from the trivial map on S.

Lemma I.2.4. *The identity mapping $\mathbb{1}_{\mathrm{P}}\colon s \mapsto s$ acting on the right of S is a right translation of S. Dually, the left identity mapping $\mathbb{1}_{\Lambda}$ is a left translation of S. Furthermore, the pair $\mathbb{1}_{\Omega} = (\mathbb{1}_{\Lambda}, \mathbb{1}_{\mathrm{P}})$ is a bi-translation of S.*

Proof. Clearly if $s,t \in S$, then $(st)\mathbb{1}_{\mathrm{P}} = st = s(t\mathbb{1}_{\mathrm{P}})$ and similarly $\mathbb{1}_{\Lambda}(st) = (\mathbb{1}_{\Lambda}s)t$ which shows that $\mathbb{1}_{\mathrm{P}} \in \mathrm{P}(S)$ and $\mathbb{1}_{\Lambda} \in \Lambda(S)$. Last, $s(\mathbb{1}_{\Lambda}t) = st = (s\mathbb{1}_{\mathrm{P}})t$ so that the pair $(\mathbb{1}_{\Lambda}, \mathbb{1}_{\mathrm{P}})$ is linked. $\square$

Is it clear from the definition of right and left translations that such maps can be seen as elements of $\mathcal{T}_S$, and the multiplication in $\mathrm{P}(S)$ and $\Lambda(S)$ is inherited from that of $\mathcal{T}_S$ as follows.

Lemma I.2.5. *The set* $\mathrm{P}(S)$ *is a submonoid of* $\mathcal{T}_S$ *while* $\Lambda(S)$ *is a submonoid of* $\mathcal{T}_S^{\mathrm{op}}$ *with respective identity* $\mathbb{1}_{\mathrm{P}}$ *and* $\mathbb{1}_{\Lambda}$ *under the multiplication given by*

$$s(\rho\rho') = (s\rho)\rho' \quad \textit{and} \quad (\lambda\lambda')s = \lambda(\lambda's),$$

where $s \in S$, $\rho, \rho' \in \mathrm{P}(S)$ *and* $\lambda, \lambda' \in \Lambda(S)$.

Additionally, the set $\Omega(S)$ *is also a monoid with identity* $\mathbb{1}_{\Omega} = (\mathbb{1}_{\Lambda}, \mathbb{1}_{\mathrm{P}})$ *when defining the multiplication for* $(\lambda, \rho), (\lambda', \rho') \in \Omega(S)$ *by*

$$(\lambda, \rho)(\lambda', \rho') = (\lambda\lambda', \rho\rho').$$

Proof. By definition of the maps $\mathbb{1}_{\mathrm{P}}$, $\mathbb{1}_{\Lambda}$ and $\mathbb{1}_{\Omega}$, it is clear that these are two-sided identities for $\mathrm{P}(S)$, $\Lambda(S)$ and $\Omega(S)$ respectively. It only remains to show that these sets are closed under products since the associativity of the product will follow directly from that of $\mathcal{T}_S$.

Let $\rho, \rho' \in \mathrm{P}(S)$ and $s,t \in S$. Then we have the following:

$$s(t(\rho\rho')) = s((t\rho)\rho') = (s(t\rho))\rho' = ((st)\rho)\rho' = (st)(\rho\rho'),$$

where the second and third equalities comes from the fact that ρ' and ρ are right translations. The dual holds for left translations and thus $\mathrm{P}(S)$ and $\Lambda(S)$ are monoids.

Now let $(\lambda, \rho), (\lambda', \rho') \in \Omega(S)$. From the first part, we have that $\lambda\lambda' \in \Lambda(S)$ and $\rho\rho' \in \mathrm{P}(S)$, so it remains to show that these form a linked pair. Indeed, using the fact that λ and ρ are linked, and similarly for λ' and ρ', for any $s,t \in S$ we have that:

$$s((\lambda\lambda')t) = s(\lambda(\lambda't)) = (s\rho)(\lambda't) = ((s\rho)\rho')t = (s(\rho\rho'))t,$$

which shows that $\lambda\lambda'$ and $\rho\rho'$ are linked and thus $\Omega(S)$ is a monoid. $\square$

As a direct consequence from the fact that $\mathrm{P}(S)$, $\Lambda(S)$ and $\Omega(S)$ are monoids, we have the following:

Corollary I.2.6. *The projections* π_{P} *and* π_{Λ} *are monoid homomorphisms and their images* $\widetilde{\mathrm{P}}(S)$ *and* $\widetilde{\Lambda}(S)$ *are submonoids of* $\mathrm{P}(S)$ *and* $\Lambda(S)$ *respectively.*

We will later be investigating when these maps are one-one, or onto.

Notation I.2.7. In light of Lemma I.2.5 we will remove any superfluous parentheses whenever no ambiguity can arise. For example, given $(\lambda, \rho) \in \Omega(S)$ and elements $s, t, u, v \in S$ we can write $st\lambda uv = st\rho uv$ since

$$st\lambda(uv) = s(t\lambda u)v = s(t\rho u)v = (st)\rho uv,$$

which means that all meaningful ways to set parentheses give the same answer.

We have seen in the introduction that some left and right translations arise from the multiplication on the left or on the right by an element of S. Such translations form a specific class and are important in their own right.

Definition I.2.8. Given $s \in S$, the map $\rho_s \in \mathrm{P}(S)$ [resp. $\lambda_s \in \Lambda(S)$] defined by

$$t\rho_s = ts \quad [\text{resp. } \lambda_s t = st]$$

for all $t \in S$ is called the *inner right [resp. left] translation of S induced by s*. We write by $\mathrm{P}_0(S)$ [resp. $\Lambda_0(S)$] the set of all right [resp. left] inner translations of S.

Additionally, since the maps λ_s and ρ_s are clearly seen to be linked for each $s \in S$, we call $\sigma_s = (\lambda_s, \rho_s)$ the *inner bi-translation induced by s*. The set of all inner bi-translations of S is denoted by $\Sigma(S)$.

It is easy to see that $\mathrm{P}_0(S)$ and $\Lambda_0(S)$ are respective subsemigroups of $\mathrm{P}(S)$ and $\Lambda(S)$ since $\rho_s\rho_t = \rho_{st}$ and $\lambda_s\lambda_t = \lambda_{st}$. Hence, $(\lambda_s, \rho_s)(\lambda_t, \rho_t) = (\lambda_{st}, \rho_{st})$, which shows that $\Sigma(S)$ is a subsemigroup of $\Omega(S)$. In fact, these sets are more than subsemigroups as given by the following lemma.

Lemma I.2.9. *Let $s \in S$ and take $\lambda' \in \Lambda(S)$ and $\rho' \in \mathrm{P}(S)$. Then we have that*

$$\lambda'\lambda_s = \lambda_{\lambda' s}, \quad \textit{and} \quad \rho_s\rho' = \rho_{s\rho'}$$

and if the pair (λ', ρ') is linked, we also get that

$$\lambda_s\lambda' = \lambda_{s\rho'}, \quad \textit{and} \quad \rho'\rho_s = \rho_{\lambda' s}.$$

Consequently, $\mathrm{P}_0(S)$ *is a right ideal of* $\mathrm{P}(S)$*,* $\Lambda_0(S)$ *a left ideal of* $\Lambda(S)$ *and* $\Sigma(S)$ *a two-sided ideal of* $\Omega(S)$*.*

Proof. The equations follows directly from the fact that if $\rho' \in \mathrm{P}(S)$ then we have $t\rho_s\rho' = ts\rho' = t\rho_{s\rho'}$ and if $\lambda' \in \Lambda(S)$ is linked to ρ' we also get

$$t\rho'\rho_s = t\rho' s = t\lambda' s = t\rho_{\lambda' s},$$

as well as their dual for λ_s. Hence $\mathrm{P}_0(S)$ and $\Lambda_0(S)$ are respectively a right ideal of $\mathrm{P}(S)$ and a left ideal of $\Lambda(S)$. Moreover, this also shows that

$$(\lambda_s, \rho_s)(\lambda', \rho') = (\lambda_{s\rho'}, \rho_{s\rho'}) \quad \text{and} \quad (\lambda', \rho')(\lambda_s, \rho_s) = (\lambda_{\lambda' s}, \rho_{\lambda' s}),$$

so that $\Sigma(S)$ is a two-sided ideal of $\Omega(S)$. $\square$

Not all translations are inner, and this relates to the existence of a right, left or two-sided identity in S.

Lemma I.2.10. *1) $\mathrm{P}(S) = \mathrm{P}_0(S)$ if and only if S has a right identity.*

2) $\Lambda(S) = \Lambda_0(S)$ if and only if S has a left identity.

3) $\Omega(S) = \Sigma(S)$ if and only if S has an identity.

Proof. The proof of 2) is dual to that of 1) so it will be omitted.

1) Suppose that $\mathrm{P}(S) = \mathrm{P}_0(S)$. Since $\mathrm{P}(S)$ is a monoid with identity $\mathbb{1}_\mathrm{P}$, we get that there exists $e \in S$ such that $\mathbb{1}_\mathrm{P} = \rho_e \in \mathrm{P}_0(S)$. Thus for all $s \in S$ we obtain that $se = s\rho_e = s\mathbb{1}_\mathrm{P} = s$, which shows that e is a right identity in S.

Conversely, let e be a right identity of S. Then for all $\rho \in \mathrm{P}(S)$ and $s \in S$, by writing $f = e\rho$ we have that:

$$s\rho = (se)\rho = s(e\rho) = sf = s\rho_f,$$

so that $\rho = \rho_f \in \mathrm{P}_0(S)$ and $\mathrm{P}(S) \subseteq \mathrm{P}_0(S)$. Since we also know that $\mathrm{P}_0(S) \subseteq \mathrm{P}(S)$ in general, we get that $\mathrm{P}(S) = \mathrm{P}_0(S)$.

3) Suppose that $\Omega(S) = \Sigma(S)$. Then, as $\Omega(S)$ is a monoid, we have that there exists $e \in S$ such that $(\lambda_e, \rho_e) = \mathbb{1}_\Omega = (\mathbb{1}_\Lambda, \mathbb{1}_\mathrm{P})$. Then, $se = s\rho_e = s$ and $es = \lambda_e s = s$ as before, from which we have that e is a left and a right identity. Thus S has a two-sided identity.

Conversely, let $e \in S$ be the identity of S and let $(\lambda, \rho) \in \Omega(S)$. By defining $x, y \in S$ as $x = \lambda e$ and $y = e\rho$, we can see that $\lambda = \lambda_x$ and $\rho = \rho_y$ using the same argument as above together with its dual. Since the pair (λ, ρ) is linked, we also have that

$$x = ex = e\lambda e = e\rho e = ye = y,$$

and thus $(\lambda, \rho) = (\lambda_x, \rho_x) \in \Sigma(S)$. Therefore $\Omega(S) = \Sigma(S)$ as required. $\square$

We mentioned at the beginning of this section that translations generalise the action of a semigroup on itself. We now make this concrete in the following lemma.

Lemma I.2.11. *The function $\sigma\colon S \to \Omega(S)$ which sends an element $s \in S$ to $\sigma_s = (\lambda_s, \rho_s)$ is a homomorphism of S onto $\Sigma(S)$.*

Similarly, $\mathrm{P}_0(S)$ and $\Lambda_0(S)$ are homomorphic images of S through the maps $\sigma_{\mathrm{P}}\colon S \to \mathrm{P}(S)$ and $\sigma_{\Lambda}\colon S \to \Lambda(S)$ defined by

$$s\sigma_{\mathrm{P}} = \rho_s \quad \text{and} \quad s\sigma_{\Lambda} = \lambda_s \qquad \text{for all } s \in S.$$

In particular, we get that $\sigma_{\mathrm{P}} = \sigma \circ \pi_{\mathrm{P}}$ and $\sigma_{\Lambda} = \sigma \circ \pi_{\Lambda}$.

Proof. It is clear from the definition of $\Sigma(S) \subseteq \Omega(S)$ in I.2.8 than σ is a well-defined map whose image is $\Sigma(S)$, and the corresponding statements for σ_{P} and σ_{Λ} also hold.

Moreover, if $s, t \in S$, then we get that $\sigma_s\sigma_t = (\lambda_s, \rho_s)(\lambda_t, \rho_t) = (\lambda_{st}, \rho_{st}) = \sigma_{st}$, that is, $s\sigma t\sigma = (st)\sigma$ and thus σ is a homomorphism. Similarly, σ_{P} and σ_{Λ} are homomorphisms. □

In order to have that every inner left, right or bi-translation comes from a unique element of our semigroup S, we need S to satisfy a weak notion of the cancellativity property.

Definition I.2.12. A semigroup S is called *weakly reductive* if for all $s, t \in S$ we have that $sx = tx$ and $xs = xt$ for all $x \in S$ implies $s = t$, or equivalently, if $\lambda_s = \lambda_t$ and $\rho_s = \rho_t$ implies $s = t$.

A semigroup S is called *left reductive* [resp. *right reductive*] if for all $s, t \in S$ we have that $xs = xt$ [resp. $sx = tx$] for all $x \in S$ implies $s = t$, that is, if $\rho_s = \rho_t$ [resp. $\lambda_s = \lambda_t$] implies $s = t$.

A semigroup S is called *reductive* if it is both a right reductive and a left reductive semigroup.

Remark I.2.13. Notice that if S has a left identity, then it is left reductive and dually for right reductivity, while if S is a monoid, then it is reductive. Moreover we have that a left or right reductive semigroup S is weakly reductive, so that reductivity implies weak reductivity, and these notions coincide on a commutative semigroup.

Lemma I.2.14. *For a semigroup S, the following occurs:*

- *S is left reductive if and only if σ_{P} is injective, and then $S \cong \mathrm{P}_0(S)$;*
- *S is right reductive if and only if σ_{Λ} is injective, and then $S \cong \Lambda_0(S)$;*

- *S is weakly reductive if and only if σ is injective.*

In all cases, we also have that $S \cong \Sigma(S)$.

Proof. The equivalences between left, right or weak reductivity with the injectivity of σ_{P}, σ_{Λ} and σ respectively follow immediately from the definition. We then obtain the isomorphisms directly from Lemma I.2.11. □

Combining the above result together with Lemma I.2.10 we get the following.

Corollary I.2.15. *A semigroup S is a monoid if and only if $S \cong \Omega(S)$.*

Proof. If S is a monoid, then we get that $\Sigma(S) = \Omega(S)$ by Lemma I.2.10. Moreover, S is reductive by Remark I.2.13. Thus S is also weakly reductive and we have that $S \cong \Sigma(S)$ by Lemma I.2.14. Therefore $S \cong \Omega(S)$.

Conversely, $\Omega(S)$ is a monoid with identity $\mathbb{1}_{\Omega}$ by Lemma I.2.5 and therefore if $S \cong \Omega(S)$, then S must also be a monoid. □

In general, not every right translation is necessarily linked to a left translation, and vice-versa, but this happens when the semigroup is commutative.

Lemma I.2.16. *If S is commutative, then $\widetilde{\mathrm{P}}(S) = \mathrm{P}(S)$ and $\widetilde{\Lambda}(S) = \Lambda(S)$. Consequently, the projections π_{P} and π_{Λ} are surjective and $\Omega(S) = \Lambda(S) \times \mathrm{P}(S)$.*

Proof. Let us assume that S is a commutative semigroup. Since we already know that $\widetilde{\mathrm{P}}(S) \subseteq \mathrm{P}(S)$ and $\widetilde{\Lambda}(S) \subseteq \Lambda(S)$ by definition, we only need to show the reverse inclusion.

Let $\rho \in \mathrm{P}(S)$ and define a left map $\lambda \colon S \to S$ by $\lambda s = s\rho$ for all $s \in S$. We aim to show that λ is a left translation linked to ρ, which will give us that $(\lambda, \rho) \in \Omega(S)$ and thus $\rho \in \widetilde{\mathrm{P}}(S)$.

Indeed, let $s, t \in S$. Then, using the commutativity in S, we have that

$$\lambda(st) = (st)\rho = (ts)\rho = t(s\rho) = s\rho t = (\lambda s)t,$$

which shows that $\lambda \in \Lambda(S)$. Furthermore, these translations are linked since

$$s\rho t = t(s\rho) = (ts)\rho = \lambda(ts) = (\lambda t)s = s\lambda t.$$

Consequently $(\lambda, \rho) \in \Omega(S)$, and we obtain that $\mathrm{P}(S) \subseteq \widetilde{\mathrm{P}}(S)$, hence the equality $\widetilde{\mathrm{P}}(S) = \mathrm{P}(S)$. We also get that $\widetilde{\Lambda}(S) = \Lambda(S)$ by duality. □

It is however not necessary for a semigroup to be commutative for every right and left translation to be part of a bi-translation and we will give an example illustrating this using rectangular bands in Section I.2.2.2.

On the other hand, one can ask when the natural projections of $\Omega(S)$ are injective. In other words, this determines when a right translation is linked to at most one left translation, or vice-versa. Once again, this is related to the reductivity properties of the semigroup S.

Lemma I.2.17. *If S is left reductive, then π_{P} is injective, and thus $\Omega(S) \cong \widetilde{\mathrm{P}}(S)$. Dually, if S is right reductive, then π_{Λ} is injective, and thus $\Omega(S) \cong \widetilde{\Lambda}(S)$.*

Consequently, if S is reductive, then the natural projections π_{P} and π_{Λ} are injective and we have that $\Omega(S) \cong \widetilde{\mathrm{P}}(S) \cong \widetilde{\Lambda}(S)$.

Proof. Suppose that S is left reductive, and let $(\lambda, \rho), (\lambda', \rho') \in \Omega(S)$ be such that $(\lambda, \rho)\pi_{\mathrm{P}} = (\lambda', \rho')\pi_{\mathrm{P}}$. Then $\rho = \rho'$ and for all $s, t \in S$ we have that

$$s\lambda t = s\rho\, t = s\rho' t = s\lambda' t.$$

By the left reductivity of S, we get that $\lambda t = \lambda' t$ for all $t \in S$, and thus $\lambda = \lambda'$. Hence $(\lambda, \rho) = (\lambda', \rho')$ and π_{P} is injective. By definition, we therefore have that $\Omega(S) \cong \operatorname{im} \pi_{\mathrm{P}} = \widetilde{\mathrm{P}}(S)$.

The proof of the statement for when S is right reductive is dual, while the second part of the lemma follows from the fact that a reductive semigroup is both left and right reductive. □

Remark I.2.18. Notice that the converse of the above statements does not hold in general. Indeed, π_{P} is injective if $(\lambda, \rho), (\lambda', \rho) \in \Omega(S)$ forces $\lambda = \lambda'$. In particular, this means that, if $s\lambda t = s\lambda' t$ for all $s, t \in S$ for some $\lambda, \lambda' \in \Lambda(S)$ linked to the same right translation, we must have $\lambda = \lambda'$. This can be rewritten as

$$\forall t \in S,\ \forall s \in S,\ \forall (\lambda, \rho), (\lambda', \rho) \in \Omega(S) : s\lambda t = s\lambda' t \implies \lambda t = \lambda' t.$$

This last condition is not equivalent to the property of S being left reductive, unless S has the additional property that for any $a, b \in S$ there exist $t \in S$ and $\lambda, \lambda' \in \Lambda(S)$ such that $a = \lambda t$, $b = \lambda' t$ and λ, λ' are both linked to some $\rho \in \mathrm{P}(S)$.

To see this, suppose that S has this additional property and that π_{P} is injective. Then, for any $a, b \in S$ and $s \in S$ such that $sa = sb$, there exists $t \in S$ with $a = \lambda t$ and $b = \lambda' t$ for some $\lambda, \lambda' \in \Lambda(S)$ linked to the same right translation $\rho \in \mathrm{P}(S)$. By injectivity of π_{P}, we have that $\lambda = \lambda'$, and thus $a = \lambda t = \lambda' t = b$, which shows that S is left reductive.

Remark I.2.19. Even though Lemma I.2.17 tells us that the projections are isomorphisms when the semigroup S is reductive, in practice, this is rarely useful on its own to actually compute the translational hull of S. Indeed, if we know that these isomorphisms hold, we still need to find the exact conditions for a right or left translation to belong to a bi-translation. This is not always easy as this process is equivalent to finding the inverse of one of the projections (π_{P} or π_{Λ}). However, if we have a good understanding, say, of how right translations behave, and that for some right translations we are able to create left translations linked to them, then, knowing that $\Omega(S)$ is isomorphic to $\widetilde{\mathrm{P}}(S)$ allows us to deduce that the left translation created is unique, which gives a description of the translational hull respectively to the right translations for which the construction process works.

Before discussing the relationship between translational hulls and ideal extensions, we give a few examples on the actual computation of translational hulls.

I.2.2 Classical examples

In this section, we will see how one can compute the translational hull of different types of semigroups, by giving results for right-zero, left-zero and null semigroups, as well as rectangular bands and Brandt semigroups.

I.2.2.1 Left-zero, right-zero and null semigroups

All of the following results can be found in the work of Tamura [52, 53], where he uses the terminology *right singular semigroup* where we use *right-zero semigroup*. In this section, unless specified otherwise, S is an arbitrary semigroup.

Lemma I.2.20. *An element $z \in S$ is a left zero if and only if $z\rho = z$ for all $\rho \in \mathrm{P}(S)$.*

Dually, $z \in S$ is a right zero if and only if $\lambda z = z$ for all $\lambda \in \Lambda(S)$.

Proof. Let $z \in S$ be a left-zero and $\rho \in \mathrm{P}(S)$. Then we have

$$z\rho = (z^2)\rho = z(z\rho) = z.$$

Conversely, suppose that $z \in S$ is such that $z\rho = z$ for all $\rho \in \mathrm{P}(S)$. In particular, this means that for all $x \in S$, we have

$$zx = z\rho_x = z,$$

which shows that z is a left-zero in S.

Dual arguments show that an element is a right zero if and only if it is fixed by every left translation of S. □

From this, we immediately obtain the following corollary.

Corollary I.2.21. *An element is a (two-sided) zero of S if and only if it is fixed by every right and left translation of S.*

As a useful corollary, we get that if a semigroup S has a zero, then in order for a map to be a right, left or bi-translation of S, it suffices to verify that the associated condition holds on all non-zero elements and that this map fixes the zero of S.

Corollary I.2.22. *Let S be a semigroup with a zero 0. Then $\rho \in \mathrm{P}(S)$ if and only if $0\rho = 0$ and $(st)\rho = s(t\rho)$ for all $s, t \in S \setminus \{0\}$. Dually $\lambda \in \Lambda(S)$ if and only if $\lambda 0 = 0$ and $\lambda(st) = (\lambda s)t$ for all $s, t \in S \setminus \{0\}$.*

Moreover $(\lambda, \rho) \in \Omega(S)$ if and only if $\lambda 0 = 0\rho = 0$ and $s\lambda t = s\rho t$ for all $s, t \in S \setminus \{0\}$.

Proof. The fact that the conditions are necessary is clear from Corollary I.2.21 above, so it suffices to prove that they are sufficient.

Let $\rho\colon S \to S$ be such that $0\rho = 0$ and $(st)\rho = s(t\rho)$ for all $s, t \in S \setminus \{0\}$. In order for ρ to be a right translation, we need to consider products st when at least one of s or t is zero. However, if $s = 0$ we get that $(st)\rho = 0\rho = 0 = 0(t\rho) = s(t\rho)$, while if $t = 0$ we obtain $(st)\rho = 0\rho = 0 = s0 = s(0\rho) = s(t\rho)$. Thus ρ is a right translation, and a dual argument holds for left translations.

Finally, let $\lambda \in \Lambda(S)$ and $\rho \in \mathrm{P}(S)$ be such that $\lambda 0 = 0\rho = 0$ and $s\lambda t = s\rho t$ for all $s, t \in S \setminus \{0\}$. As above, suppose that $s, t \in S$ are such that at least one of s or t is 0. Then either $s = 0$ and we get $s\lambda t = 0\lambda t = 0 = 0t = 0\rho t = s\rho t$, or $t = 0$ and then $s\lambda t = s\lambda 0 = s0 = 0 = s\rho 0 = s\rho t$. In both cases, we have $s\lambda t = s\rho t$ and thus $(\lambda, \rho) \in \Omega(S)$. □

We can now describe the translational hulls for right- or left-zero semigroups.

Proposition I.2.23. *The following are equivalent.*

1) *S is a right-zero semigroup,*
2) *$\mathrm{P}(S) = \mathcal{T}_S$,*
3) *$\Lambda(S) = \{\mathbb{1}_\Lambda\}$.*

Consequently, if S is a right-zero semigroup, then $\Omega(S) \cong \mathcal{T}_S$.

Dually, S is a left-zero semigroup if and only if $\Lambda(S) = \mathcal{T}_S^{\mathrm{op}}$ if and only if $\mathrm{P}(S) = \{\mathbb{1}_{\mathrm{P}}\}$, and then $\Omega(S) \cong \mathcal{T}_S^{\mathrm{op}}$.

Proof. The results for right- and left-zero semigroups are dual, so we shall only write the proofs in the case for right-zero semigroups.

1) $\Rightarrow$ 2): Since $\mathrm{P}(S) \subseteq \mathcal{T}_S$, we let $\rho \in \mathcal{T}_S$. Then for any $s, t \in S$ we have

$$(st)\rho = t\rho = s(t\rho),$$

where we used the fact that t and $t\rho$ are both right zeros in S, giving that $\rho \in \mathrm{P}(S)$.

2) $\Rightarrow$ 1): Let $s, t \in S$ and $\rho \in \mathcal{T}_S$ be such that $(st)\rho = t = t\rho$. Since $\rho \in \mathrm{P}(S)$ by assumption, it follows that $t = (st)\rho = s(t\rho) = st$. Thus S is a right-zero semigroup.

1) $\Rightarrow$ 3): Since all elements of S are right zeros, it follows from Lemma I.2.20 that every left translation $\lambda \in \Lambda(S)$ is such that $\lambda s = s$ for all $s \in S$. Thus $\lambda = \mathbb{1}_\Lambda$ and $\Lambda(S) = \{\mathbb{1}_\Lambda\}$.

3) $\Rightarrow$ 1): Let $s, t \in S$ and consider $\lambda_s \in \Lambda(S)$. Then $\lambda_s = \mathbb{1}_\Lambda$ and

$$st = \lambda_s t = \mathbb{1}_\Lambda t = t,$$

which shows that S is a right-zero semigroup.

It follows from this equivalence that if S is a right-zero semigroup, then $\mathrm{P}(S) = \mathcal{T}_S$ and $\Lambda(S) = \{\mathbb{1}_\Lambda\}$, and thus for all $\rho \in \mathrm{P}(S)$ and $s, t \in S$, we have that

$$s\rho t = t = st = s\mathbb{1}_\Lambda t.$$

Therefore $(\mathbb{1}_\Lambda, \rho) \in \Omega(S)$ and $\Omega(S) = \{(\mathbb{1}_\Lambda, \rho) \mid \rho \in \mathcal{T}_S\} \cong \mathcal{T}_S$. ☐

In a null semigroup, the conditions on being right or left translations are even more permissive, as given by the following.

Proposition I.2.24. *Let S be a null semigroup with zero 0. Then*

$$\Omega(S) = \left\{(\lambda, \rho) \in \mathcal{T}_S^{\mathrm{op}} \times \mathcal{T}_S \mid \lambda 0 = 0 = 0\rho\right\}.$$

Proof. Let $(\lambda, \rho) \in \Omega(S) \subseteq \mathcal{T}_S^{\mathrm{op}} \times \mathcal{T}_S$. Then, by Corollary I.2.21, it follows that 0 is fixed by λ and ρ, that is, $\lambda 0 = 0 = 0\rho$.

Conversely, let $\lambda \in \mathcal{T}_S^{\mathrm{op}}$ and $\rho \in \mathcal{T}_S$ be such that $\lambda 0 = 0 = 0\rho$. Together with the fact that S is a null semigroup, we get that for all $s, t \in S$:

- $\lambda(st) = \lambda 0 = 0 = (\lambda s)t$,
- $(st)\rho = 0\rho = 0 = s(t\rho)$, and
- $s\lambda t = 0 = s\rho t$.

Therefore $(\lambda, \rho) \in \Omega(S)$ as required. $\square$

I.2.2.2 Rectangular bands

Let $R = I \times J$ be a rectangular band. Then the right and left translations can be fully characterised, and respectively correspond to transformations of the underlying sets J and I.

Lemma I.2.25. *Let $\rho \in \mathrm{P}(R)$. Then there exists $\bar{\rho} \in \mathcal{T}_J$ such that $(i,j)\rho = (i, j\bar{\rho})$ for all $(i,j) \in R$. Conversely, any map $\phi\colon J \to J$ gives rise to a unique right translation ρ of R defined by $(i,j)\rho = (i, j\phi)$. Furthermore, the bijection $\mathrm{P}(R) \to \mathcal{T}_J : \rho \mapsto \bar{\rho}$ is an isomorphism.*

Dually, $\Lambda(R)$ is isomorphic to $\mathcal{T}_I^{\mathrm{op}}$ via the map $\lambda \mapsto \bar{\lambda}$ where $\lambda(i,j) = (\bar{\lambda} i, j)$ for all $(i,j) \in R$.

Proof. Let $\rho \in \mathrm{P}(R)$ and let us assume that $(i,j)\rho = (i^*, j^*)$ for some $(i,j) \in R$. Then, for all $(k,j) \in R$ we have that

$$(k,j)\rho = ((k,j)(i,j))\,\rho = (k,j)\,((i,j)\rho) = (k,j)(i^*,j^*) = (k,j^*),$$

which shows that ρ induces a map $\bar{\rho} \in \mathcal{T}_J$ defined by $j\bar{\rho} = j^*$, where j^* is the element of J such that $(i,j)\rho = (i,j^*)$ for any $i \in I$.

Conversely, let $\bar{\rho} \in \mathcal{T}_J$ and define $\rho\colon R \to R$ by $(x,y)\rho = (x, y\bar{\rho})$ for all $(x,y) \in R$. Then for $(i,j),(k,l) \in R$, we have:

$$((i,j)(k,l))\,\rho = (i,l)\rho = (i, l\bar{\rho}) = (i,j)(k, l\bar{\rho}) = (i,j)\,((k,l)\rho)\,,$$

and thus $\rho \in \mathrm{P}(R)$.

To show that $\mathrm{P}(R)$ and $\mathcal{T}_J$ are indeed isomorphic, consider $\rho, \rho' \in \mathrm{P}(R)$ and their associated maps $\bar{\rho}, \overline{\rho'} \in \mathcal{T}_J$. Then for all $(i,j) \in R$, we get that

$$(i, j\overline{\rho\rho'}) = (i,j)\rho\rho' = (i, j\bar{\rho})\rho' = (i, j\bar{\rho}\overline{\rho'}),$$

so that the bijection is also a morphism, hence an isomorphism.

Similar arguments show that $\Lambda(R) \cong \mathcal{T}_I^{\mathrm{op}}$ through the map given in the statement of the lemma. $\square$

It is clear that, unless trivial, R is not a commutative semigroup. However, we can show that every right and every left translation is part of a linked pair, which shows that the converse of Lemma I.2.16 does not hold. In fact, in a rectangular band, what can be proved is even stronger, namely, that any pair of a left and a right translation is a bi-translation.

Lemma I.2.26. *Every right translation of R is linked with every left translation of R. Consequently, $\widetilde{\mathrm{P}}(R) = \mathrm{P}(R)$, $\widetilde{\Lambda}(R) = \Lambda(R)$ and $\Omega(R) = \mathcal{T}_I^{\mathrm{op}} \times \mathcal{T}_J$.*

Proof. Let $\rho \in \mathrm{P}(R)$ and $\lambda \in \Lambda(R)$. Then, using the description of left and right translations given in Lemma I.2.25, for all $(i,j),(k,l) \in R$ we have that:

$$(i,j)\lambda(k,l) = (i,j)(\bar{\lambda}k,l) = (i,l) = (i,j\bar{\rho})(k,l) = (i,j)\rho(k,l).$$

Therefore, (λ,ρ) is a linked pair, and all the equalities in the statement follow directly from this. ☐

The previous lemma shows that the translational hull of a rectangular band is very large, and it is easy to see that the action by right multiplication by an element corresponds to a constant map in $\mathcal{T}_J$. Moreover, these actions are all distinct as can be seen by the following.

Lemma I.2.27. *The semigroup R is weakly reductive, and thus σ is injective and $R \cong \Sigma(R)$.*

Proof. Let $(i,j),(k,l) \in R$ and suppose that for all $x \in R$ we have $x(i,j) = x(k,l)$ and $(i,j)x = (k,l)x$. In particular, by letting $x = (a,b)$, we get that

$$(a,j) = (a,b)(i,j) = (a,b)(k,l) = (a,l),$$

and thus $j = l$. Similarly, we have that $i = k$ since

$$(i,b) = (i,j)(a,b) = (k,l)(a,b) = (k,b).$$

Therefore $(i,j) = (k,l)$ and R is weakly reductive. The isomorphism then follows from Lemma I.2.14. ☐

I.2.2.3 Brandt semigroups

Let $B = I \times I \cup \{0\}$ be a Brandt semigroup with trivial subgroups. Recall that the multiplication of two non-zero elements $(i,j),(k,l) \in B$ is given by

$$(i,j)(k,l) = \begin{cases} (i,l) & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly to the case of the rectangular band, it is easy to show that right and left translations are in one-to-one correspondence with the partial bijections of I as shown in the next lemma. In fact, we have here a special case of a rectangular 0-band.

Remark I.2.28. Since B has a zero, it follows from Corollary I.2.21 that any right and left translation must fix 0.

Lemma I.2.29. *For every $\rho \in \mathrm{P}(B)$, there exists $\bar{\rho} \in \mathcal{PT}_I$ with*

$$\operatorname{dom} \bar{\rho} = \{j \in I \mid (i,j)\rho \neq 0 \textit{ for some } i \in I\}$$

such that $(i,j)\rho = (i, j\bar{\rho})$ whenever $(i,j)\rho \neq 0$. Conversely, any partial transformation $\bar{\rho}$ of I gives rise to a right translation ρ of B by setting $0\rho = 0$, $(i,j)\rho = (i, j\bar{\rho})$ if $j \in \operatorname{dom} \bar{\rho}$ and $(i,j)\rho = 0$ otherwise. Moreover, this bijection is an isomorphism of $\mathrm{P}(B)$ onto $\mathcal{PT}_I$.

Proof. Let $\rho \in \mathrm{P}(B)$, and let $(i,j) \in B$ be such that $(i,j)\rho \neq 0$. Suppose that $(i,j)\rho = (i^*, j^*)$, then we have:

$$(i^*, j^*) = (i,j)\rho = ((i,i)(i,j))\,\rho = (i,i)\,((i,j)\rho) = (i,i)(i^*, j^*),$$

which shows that $(i,i)(i^*, j^*) \neq 0$ and thus $i^* = i$. Moreover, for any $(k,j) \in B$, we also have that $(k,j)\rho \neq 0$ since

$$(k,j)\rho = ((k,i)(i,j))\,\rho = (k,i)\,((i,j)\rho) = (k,i)(i,j^*) = (k,j^*).$$

Conversely, if $(i,j) \in B$ is such that $(i,j)\rho = 0$, then $(k,j)\rho = 0$ for all $(k,j) \in B$ since

$$(k,j)\rho = ((k,i)(i,j))\,\rho = (k,i)\,((i,j)\rho) = (k,i)0 = 0.$$

Hence, we have that $(i,j)\rho \neq 0$ if and only if $(k,j)\rho \neq 0$ for all $k \in I$. We now define the map $\bar{\rho} \in \mathcal{PT}_I$ by $\operatorname{dom} \bar{\rho} = \{j \in I \mid \exists i \in I : (i,j)\rho \neq 0\}$ and $j\bar{\rho} = j^*$ for all

$j \in \text{dom } \bar{\rho}$. It is clear by the previous observation that $\bar{\rho}$ is a well-defined map that satisfies the condition $(i,j)\rho = (i, j\bar{\rho})$ for all $(i,j) \in B$ such that $(i,j)\rho \neq 0$.

On the other hand, let $\bar{\rho} \in \mathcal{PT}_I$ and define $\rho\colon B \to B$ by $0\rho = 0$, and

$$(i,j)\rho = \begin{cases} (i, j\bar{\rho}) & \text{if } j \in \text{dom } \bar{\rho}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(i,j),(k,l) \in B$. Then, we have the following:

$$((i,j)(k,l))\rho = \begin{cases} (i,l)\rho & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} (i, l\bar{\rho}) & \text{if } j = k \text{ and } l \in \text{dom } \bar{\rho}, \\ 0 & \text{otherwise,} \end{cases}$$

while

$$(i,j)((k,l)\rho) = \begin{cases} (i,j)(k, l\bar{\rho}) & \text{if } l \in \text{dom } \bar{\rho}, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} (i, l\bar{\rho}) & \text{if } l \in \text{dom } \bar{\rho} \text{ and } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

which shows that $((i,j)(k,l))\rho = (i,j)((k,l)\rho)$ in all cases, and thus ρ is a right translation by Corollary I.2.22.

These arguments above show that we have a bijection between $\mathrm{P}(B)$ and $\mathcal{PT}_I$. Thus, it only remains to show that the map $\psi\colon \rho \mapsto \bar{\rho}$ is a homomorphism. Let $\rho, \rho' \in \mathrm{P}(B)$ be such that $\rho\psi = \bar{\rho}$, $\rho'\psi = \overline{\rho'}$ and $(\rho\rho')\psi = \bar{\bar{\rho}}$. It is well-known that in $\mathcal{PT}_I$ we have

$$j \in \text{dom } (\bar{\rho}\overline{\rho'}) \Leftrightarrow j \in \text{dom } \bar{\rho} \text{ and } j\bar{\rho} \in \text{dom } \overline{\rho'},$$

and since

$$\begin{aligned} \text{dom } \bar{\bar{\rho}} &= \{j \in I \mid \exists i \in I : (i,j)\rho\rho' \neq 0\} \\ &= \{j \in I \mid j \in \text{dom } \bar{\rho} \text{ and } \exists i \in I : (i, j\bar{\rho})\rho' \neq 0\} \\ &= \left\{j \in I \mid j \in \text{dom } \bar{\rho} \text{ and } j\bar{\rho} \in \text{dom } \overline{\rho'}\right\}, \end{aligned}$$

we get the equality $\text{dom } \bar{\bar{\rho}} = \text{dom } (\bar{\rho}\overline{\rho'})$. From this, for every $j \in \text{dom } \bar{\bar{\rho}}$, we have

$$(i, j\bar{\bar{\rho}}) = (i,j)\rho\rho' = (i, j\bar{\rho}\overline{\rho'}),$$

which shows that $j\bar{\bar{\rho}} = j\bar{\rho}\overline{\rho'}$ and thus ψ is an isomorphism of $\mathrm{P}(B)$ onto $\mathcal{PT}_I$. □

Using the same ideas as above, we have a dual characterisation for the left translations.

Lemma I.2.30. *For every $\lambda \in \Lambda(B)$, there exists $\bar{\lambda} \in \mathcal{PT}_I^{\mathrm{op}}$ with*

$$\mathrm{dom}\ \bar{\lambda} = \{i \in I \mid \lambda(i,j) \neq 0 \textit{ for some } j \in I\}$$

such that $\lambda(i,j) = (\bar{\lambda}i, j)$ whenever $\lambda(i,j) \neq 0$. Conversely, any partial left transformation $\bar{\lambda}$ of I gives rise to a left translation λ of B by setting $\lambda 0 = 0$, $\lambda(i,j) = (\bar{\lambda}i, j)$ if $j \in \mathrm{dom}\ \bar{\lambda}$ and $\lambda(i,j) = 0$ otherwise. Moreover, this bijection is an isomorphism of $\Lambda(B)$ onto $\mathcal{PT}_I^{\mathrm{op}}$.

Contrary to the previous examples of this section where every right and left translations were part of a linked pair, this is not the case for Brandt semigroups. In fact, only translations whose associated partial transformation is a partial bijection on I can be linked, as given by the following.

Lemma I.2.31. *Let $(\lambda, \rho) \in \Omega(B)$ with $\bar{\lambda}$ and $\bar{\rho}$ their associated partial transformation on I. Then the following equalities hold:*

1) $\mathrm{dom}\ \bar{\rho} = \mathrm{im}\, \bar{\lambda}$ *and* $\mathrm{dom}\ \bar{\lambda} = \mathrm{im}\, \bar{\rho}$,

2) $\bar{\lambda}(j\bar{\rho}) = j$ *for all* $j \in \mathrm{dom}\ \bar{\rho}$, *and*

3) $(\bar{\lambda}i)\bar{\rho} = i$ *for all* $i \in \mathrm{dom}\ \bar{\lambda}$.

Consequently $\bar{\rho} \in \mathcal{I}_I$ and $\bar{\lambda} \in \mathcal{I}_I^{\mathrm{op}}$ are inverses of each other (when viewing $\bar{\lambda}$ as a right map).

Proof. Let $(\lambda, \rho) \in \Omega(B)$. Let $j \in \mathrm{dom}\ \bar{\rho}$. Then for all $(a,b) \in B$ we have:

$$(a,b) = (a, j\bar{\rho})(j\bar{\rho}, b) = (a,j)\rho(j\bar{\rho}, b) = (a,j)\lambda(j\bar{\rho}, b).$$

Since $(a,b) \neq 0$, if follows that $\lambda(j\bar{\rho}, b) \neq 0$. Hence $j\bar{\rho} \in \mathrm{dom}\ \bar{\lambda}$ and $\mathrm{im}\, \bar{\rho} \subseteq \mathrm{dom}\ \bar{\lambda}$. Moreover, we get that $(a,b) = (a,j)(\bar{\lambda}(j\bar{\rho}), b)$, and thus $j = \bar{\lambda}(j\bar{\rho})$, which also shows that $\mathrm{dom}\ \bar{\rho} \subseteq \mathrm{im}\, \bar{\lambda}$. Together with the dual argument, we obtain 1) and then 2) and 3) follow. This shows in particular that $\bar{\lambda}$ and $\bar{\rho}$ are inverses of each other, as claimed. □

Corollary I.2.32. *The projections π_{P} and π_{Λ} are injective.*

Proof. Suppose that $(\lambda, \rho), (\lambda', \rho) \in \Omega(B)$. Then by Lemma I.2.31 we get that $\bar{\rho} \in \mathcal{I}_I$ and $\bar{\lambda}, \overline{\lambda'} \in \mathcal{I}_I^{\mathrm{op}}$ are both inverses of $\bar{\rho}$, that is, $\bar{\lambda} = \overline{\lambda'}$. Using Lemma I.2.30 we then obtain that $\lambda = \lambda'$, and therefore π_{P} is injective. Dual arguments give us that π_{Λ} is also injective. □

In fact, if the partial transformation $\bar{\rho}$ associated to a right translation ρ is a partial bijection, then this translation is always part of a linked pair as shown by:

Lemma I.2.33. *Let $\rho \in \mathrm{P}(B)$ be such that $\bar{\rho} \in \mathcal{I}_I$. Define $\lambda\colon B \to B$ by $\lambda 0 = 0$, and*

$$\lambda(i,j) = \begin{cases} (i\bar{\rho}^{-1}, j) & \textit{if } i \in \operatorname{im}\bar{\rho}, \\ 0 & \textit{otherwise.} \end{cases}$$

Then $(\lambda, \rho) \in \Omega(B)$.

Proof. Let $\lambda\colon B \to B$ be defined as above and consider $(i,j),(k,l) \in B$. Then

$$\lambda\left((i,j)(k,l)\right) = \begin{cases} \lambda(i,l) & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} (i\bar{\rho}^{-1}, l) & \text{if } j = k \text{ and } i \in \operatorname{im}\bar{\rho}, \\ 0 & \text{otherwise,} \end{cases}$$

while

$$\left(\lambda(i,j)\right)(k,l) = \begin{cases} (i\bar{\rho}^{-1}, j)(k,l) & \text{if } i \in \operatorname{im}\bar{\rho}, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} (i\bar{\rho}^{-1}, l) & \text{if } i \in \operatorname{im}\bar{\rho} \text{ and } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

which shows that λ is a left translation of B using Corollary I.2.22. In order to show that λ and ρ are linked, we can see that on one hand we have

$$(i,j)\lambda(k,l) = \begin{cases} (i,j)(k\bar{\rho}^{-1}, l) & \text{if } k \in \operatorname{im}\bar{\rho}, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} (i,l) & \text{if } k \in \operatorname{im}\bar{\rho} \text{ and } j = k\bar{\rho}^{-1}, \\ 0 & \text{otherwise} \end{cases}$$

while on the other hand we find

$$(i,j)\rho(k,l) = \begin{cases} (i,j\bar{\rho})(k,l) & \text{if } j \in \operatorname{dom}\bar{\rho}, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} (i,l) & \text{if } j \in \operatorname{dom}\bar{\rho} \text{ and } j\bar{\rho} = k, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\bar{\rho} \in \mathcal{I}_I$, it follows that $k \in \operatorname{im}\bar{\rho}$ with $j = k\bar{\rho}^{-1}$ is equivalent to $j \in \operatorname{dom}\bar{\rho}$ with $j\bar{\rho} = k$. Therefore $(i,j)\lambda(k,l) = (i,j)\rho(k,l)$ for all $(i,j),(k,l) \in B$. Using the fact that $0\rho = \lambda 0 = 0$, we get that $(\lambda, \rho) \in \Omega(B)$ by Corollary I.2.22. $\square$

Clearly, the dual of the previous lemma holds for left translations, and combining the previous results gives us the following corollary.

Corollary I.2.34. *A right translation $\rho \in \mathrm{P}(B)$ is linked if and only if $\bar{\rho} \in \mathcal{I}_I$, that is, $\widetilde{\mathrm{P}}(B) \cong \mathcal{I}_I$. Consequently, we also have $\Omega(B) \cong \widetilde{\mathrm{P}}(B) \cong \widetilde{\Lambda}(B) \cong \mathcal{I}_I$.*

Proof. If $\rho \in \widetilde{\mathrm{P}}(B)$ then $\bar{\rho} \in \mathcal{I}_I$ by Lemma I.2.31, while any map $\bar{\rho} \in \mathcal{I}_I \subseteq \mathcal{PT}_I$ gives rise to a right translation $\rho \in \mathrm{P}(B)$ by Lemma I.2.29 which is then part of a linked pair using Lemma I.2.33.

Moreover, from Corollary I.2.32 we get that π_{P} and π_{Λ} are injective, that is, $\Omega(B) \cong \widetilde{\mathrm{P}}(B) \cong \widetilde{\Lambda}(B)$. □

Remark I.2.35. We could have deduced from the start that the translational hull was isomorphic to the semigroups of left and right translations that are linked. Indeed, suppose that $x, y \in B$ are such that $xs = ys$ for all $s \in B$. Then, we either have that $x = y = 0$, or $x = (i,j)$, $y = (k,l)$ and by taking $s = (j,j)$, we get that $(i,j) = xs = ys = (k,l)(j,j)$, which forces $l = j$ and $k = i$, that is, $x = y$, so that B is right reductive. A similar argument shows that B is also left reductive, and thus reductive. The isomorphisms then follow from Lemma I.2.17.

However, the above approach would not have given us the description of the linked translations that we obtained in Corollary I.2.34.

Remark I.2.36. The description of the translational hull of Brandt semigroups is in fact a particular case of the description for completely 0-simple semigroups presented by Petrich [42, 43].

I.2.3 IDEALS AND TRANSLATIONS

Throughout this section S denotes a semigroup and $T \subseteq S$ a subsemigroup. For $x \in S$, we abuse notation and we define the maps $\rho_x \colon T \to S$ and $\lambda_x \colon T \to S$ to be the right and left action by multiplication of x on T, that is, $t\rho_x = tx$ and $\lambda_x t = xt$ for all $t \in T$.

Remark I.2.37. Notice that under this definition, the maps ρ_x and λ_x are inner translations of S if and only if $T = S$. Similarly, despite the notation, only the maps ρ_x and λ_x with x in T are inner translations of T.

Nevertheless, the maps ρ_x and λ_x above satisfy some similar properties to those held by inner translations. Indeed, by associativity in S, it is clear that for all $x, y \in S$ we have that

$$(st)\rho_x = stx = s(tx) = s(t\rho_x), \quad \lambda_x(st) = xst = (\lambda_x s)t,$$
$$t\rho_{xy} = txy = (tx)\rho_y = t\rho_x\rho_y, \quad \text{and} \quad s\lambda_x t = sxt = s\rho_x t$$

for all $s, t \in T$. Thus $\rho_{xy} = \rho_x\rho_y$, (dually, $\lambda_{xy} = \lambda_x\lambda_y$) and the pair (λ_x, ρ_x) satisfy the linking condition of bi-translations.

It is natural to ask whether the maps ρ_x and λ_x are, respectively, right and left translations of T, which corresponds to the situation where the images of ρ_x and λ_x lie in the subsemigroup T. In order to do so, we need to consider the notion of idealisers of the subsemigroup T.

Definition I.2.38. The *right idealiser [resp. left idealiser] of T in S*, denoted by $\mathbb{I}_S^r(T)$ [resp. $\mathbb{I}_S^\ell(T)$], consists of all elements $s \in S$ satisfying $ts \in T$ [resp. $st \in T$] for all $t \in T$, that is,

$$\mathbb{I}_S^r(T) = \{s \in S \mid Ts \subseteq T\}$$

and dually for $\mathbb{I}_S^\ell(T)$.

The *idealiser of T in S*, written $\mathbb{I}_S(T)$, is the set of all elements $s \in S$ such that $st, ts \in T$ for all $t \in T$, and thus $\mathbb{I}_S(T) = \mathbb{I}_S^r(T) \cap \mathbb{I}_S^\ell(T)$.

Notation I.2.39. When it is clear from the context what the over semigroup S is, we will drop the subscript and simply write $\mathbb{I}^r(T)$, $\mathbb{I}^\ell(T)$ and $\mathbb{I}(T)$ for the right, left and (two-sided) idealiser of T in S.

Remark I.2.40. The sets $\mathbb{I}^r(T)$, $\mathbb{I}^\ell(T)$ and $\mathbb{I}(T)$ are subsemigroups of S containing T. Furthermore, if T is an ideal of S, then clearly $\mathbb{I}^r(T) = \mathbb{I}^\ell(T) = \mathbb{I}(T) = S$.

We can now show that only elements from the idealisers give rise to translations when acting by multiplication.

Lemma I.2.41. *Let $x \in S$. Then we have the following:*

1) $x \in \mathbb{I}^r(T)$ if and only if $\rho_x \in \mathrm{P}(T)$;

2) $x \in \mathbb{I}^\ell(T)$ if and only if $\lambda_x \in \Lambda(T)$;

3) $x \in \mathbb{I}(T)$ if and only if $(\lambda_x, \rho_x) \in \Omega(T)$.

Proof. 1) If $x \in \mathbb{I}^r(T)$, then $tx \in T$ for all $t \in T$, that is, $t\rho_x \in T$ and thus $\rho_x\colon T \to T$. That ρ_x is a right translation of T then follows from Remark I.2.37.
Conversely, if $\rho_x \in \mathrm{P}(T)$, then $tx = t\rho_x \in T$ for all $t \in T$, that is, $x \in \mathbb{I}^r(T)$.
2) Dual of 1).
3) By the previous points, if $x \in \mathbb{I}(T) = \mathbb{I}^r(T) \cap \mathbb{I}^\ell(T)$, then $\rho_x \in \mathrm{P}(T)$ and $\lambda_x \in \Lambda(T)$. Moreover, (λ_x, ρ_x) is a linked pair by Remark I.2.37, so that $(\lambda_x, \rho_x) \in \Omega(T)$.
Conversely, if $(\lambda_x, \rho_x) \in \Omega(T)$, then $\rho_x \in \mathrm{P}(T)$ and $\lambda_x \in \Lambda(T)$, which shows that $x \in \mathbb{I}^r(T) \cap \mathbb{I}^\ell(T) = \mathbb{I}(T)$ by the previous points, as required. □

In particular, if T is an ideal, there are natural homomorphisms between S and the monoids of right, left or bi-translations of T.

Corollary I.2.42. *If T is an ideal of S, then the maps $\chi_{\mathrm{P}}\colon S \to \mathrm{P}(T)$, $\chi_{\Lambda}\colon S \to \Lambda(T)$ and $\chi\colon S \to \Omega(S)$ defined by*

$$x\chi_{\mathrm{P}} = \rho_x, \quad x\chi_{\Lambda} = \lambda_x \quad \textit{and} \quad x\chi = (\lambda_x, \rho_x)$$

are well-defined homomorphisms. Moreover, we have that

$$\mathrm{P}_0(T) \subseteq \operatorname{im}\chi_{\mathrm{P}} \subseteq \widetilde{\mathrm{P}}(T), \quad \Lambda_0(T) \subseteq \operatorname{im}\chi_{\Lambda} \subseteq \widetilde{\Lambda}(T), \quad \textit{and} \quad \Sigma(T) \subseteq \operatorname{im}\chi \subseteq \Omega(T).$$

Proof. Since T is an ideal of S, it follows from Lemma I.2.41 that for all $x \in S$, $(\lambda_x, \rho_x) \in \Omega(T)$, and thus χ_{P}, χ_{Λ} and χ are well-defined with their image lying inside $\widetilde{\mathrm{P}}(T)$, $\widetilde{\Lambda}(T)$ and $\Omega(T)$ respectively. That these maps are also homomorphisms whose image contains the inner translations of T follows directly from Remark I.2.37 together with the fact that $(\lambda_x, \rho_x)(\lambda_y, \rho_y) = (\lambda_{xy}, \rho_{xy})$ for all $x, y \in S$. □

We finish this section by giving some results from the literature showing how translational hulls are closely related to the notion of ideal extensions. We will not include their proofs, since the study of such extensions is not the objective of this thesis. Nevertheless, it is worth noting that during the '60s and '70s, there was a vast project led by Eastern-Europe mathematicians aimed at axiomatising the abstract notion of ideal extensions. This resulted in a long list of publications (as can be seen from the many references on that matter listed in Petrich's survey [41]).

Definition I.2.43. Let A and B be disjoint semigroups with B having a zero element 0. A semigroup V is called an *ideal extension of A by B* if V contains A as an ideal, and if the Rees quotient semigroup V/A is isomorphic to B.

An extension V of A is *dense* if the only congruence on V which restricts to the equality relation on A is the equality relation on V.

A semigroup A is a *densely embedded ideal* of a semigroup V if V is the largest semigroup under inclusion which is a dense extension of A.

Remark I.2.44. Another way to see dense extensions is in terms of non-injective endomorphisms. In this context, a semigroup A is a densely embedded ideal of a semigroup V if A is an ideal of V, and V is maximal under the condition that every non-injective homomorphism $\phi\colon V \to S$ for some semigroup S induces a non-injective homomorphism $\phi_{|A}\colon A \to S$.

Notice in particular that if our subsemigroup T is a densely embedded ideal of S, then it is an ideal, and the map χ is well-defined. We now have the following result from Gluskin [20] which links the translational hull of a semigroup with its dense extensions.

Lemma I.2.45 (Gluskin)**.** *A weakly reductive semigroup T is a densely embedded ideal of a semigroup S if and only if χ is an isomorphism of S onto $\Omega(T)$.*

In particular, suppose that T is a weakly reductive semigroup. Then T is isomorphic to $\Sigma(T) \subseteq \Omega(T)$ by Lemma I.2.14 so that T is densely embedded into $\Omega(T)$ and into any semigroup S isomorphic to $\Omega(T)$. Moreover, if V is a dense extension of T, then the image of V under $\chi_{|V} : V \to \Omega(T)$ is a subsemigroup of $\Omega(T)$ which contains $\Sigma(T)$ as an ideal (since $\Sigma(T)$ is an ideal of $\Omega(T)$). Thus, we can find all the dense extensions of T by looking at all the subsemigroups of $\Omega(T)$ containing $\Sigma(T)$. The condition for T to be weakly reductive cannot however be suppressed, as given by the following result of Shevrin (see [41]).

Lemma I.2.46 (Shevrin)**.** *A semigroup which is not weakly reductive cannot be a densely embedded ideal of any semigroup.*

Contrary to the approach taken by the Eastern-European schools, we are more interested in actually computing the translational hull of ideals in some semigroup of interest (such as ideals in endomorphism monoids of certain algebras) and only view the notion of extensions as a coincidental result.

I.3 EXTENDED GREEN'S RELATIONS

The study of extended Green's relations was introduced by Pastijn [39] and developed by Fountain [13], El-Qallali [45], Lawson [29] and many others (see [7, 25, 26, 46]). The idea was to complement the study of semigroups that are regular, since these extended relations coincide with the classical Green's relations in regular semigroups.

Each of Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$, $\mathscr{D}$ and $\mathscr{J}$ have a corresponding $*$- and $\sim$-relation that generalises them. The $*$-relations relate elements with mutual cancellativity properties, while the $\sim$-relations are expressed in terms of idempotent left or right identities.

In order to emulate the role that the relations $\mathscr{L}$ and $\mathscr{R}$ play in regular semigroups, Fountain [13] introduced the notion of *abundant semigroups*, which are semigroups in which each $\mathscr{L}^*$- and $\mathscr{R}^*$-classes contains an idempotent. This concept was later extended to the $\sim$ counterpart, giving rise to the now-called *Fountain semigroups*, formerly introduced as *semi-abundant semigroups* by El-Qallali [45], but also called *weakly abundant semigroups* in the literature [15].

Throughout this section S will be a semigroup with set of idempotents $E = E(S)$. We will give a precise definition of all of the extended $*$ and $\sim$ Green's relations on S, as well as some useful results, which will be needed in later chapters. The proofs included here can be found in [13, 33, 39, 45, 46].

I.3.1 DEFINITIONS

We start by defining the generalisations of the relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{D}$ as their formulations can be easily given.

Definition I.3.1. Let $a, b \in S$. The *extended Green's relations on S* are defined as follows:

$$\begin{aligned}
a\,\mathscr{L}^*\,b &\iff \left(ax = ay \Leftrightarrow bx = by \quad \forall x, y \in S^1\right),\\
a\,\mathscr{R}^*\,b &\iff \left(xa = ya \Leftrightarrow xb = yb \quad \forall x, y \in S^1\right),\\
a\,\widetilde{\mathscr{L}}\,b &\iff \left(ae = a \Leftrightarrow be = b \quad \forall e \in E\right),\\
a\,\widetilde{\mathscr{R}}\,b &\iff \left(ea = a \Leftrightarrow eb = b \quad \forall e \in E\right),\\
\mathscr{H}^* &= \mathscr{L}^* \wedge \mathscr{R}^*, \qquad \widetilde{\mathscr{H}} = \widetilde{\mathscr{L}} \wedge \widetilde{\mathscr{R}},\\
\mathscr{D}^* &= \mathscr{L}^* \vee \mathscr{R}^*, \quad \text{and} \quad \widetilde{\mathscr{D}} = \widetilde{\mathscr{L}} \vee \widetilde{\mathscr{R}}.
\end{aligned}$$

It is straightforward to see that these relations are equivalence relations. Moreover, if $a\,\mathscr{L}^*\,b$, and $s \in S$, then $asx = asy$ for some $x, y \in S^1$ if and only if $bsx = bsy$, so that $as\,\mathscr{L}^*\,bs$. Therefore $\mathscr{L}^*$ is a right congruence, and dually, $\mathscr{R}^*$ is a left congruence. However, $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}$ do not behave similarly and we will give an example in Chapter III where $\widetilde{\mathscr{R}}$ is not a left congruence on $T(\mathscr{A}, \mathscr{B})$, as well as another example in Chapter IV where $\widetilde{\mathscr{L}}$ fails to be a right congruence in $\operatorname{End}(\mathcal{T}_n)$.

On a very different note, notice that the definition of $\mathscr{L}^*$ seems closely related to the concept of left translations. In fact, if $S^1 = S$, that is, if S is a monoid, then we can see that $a\,\mathscr{L}^*\,b$ if and only if $\ker \lambda_a = \ker \lambda_b$. Otherwise, S is a semigroup without an identity and we can consider the map λ_a^1 to be the inner left translation of S^1 associated to a, that is, $\lambda_a^1\colon S^1 \to S^1$ and $\lambda_a^1 x = ax$ for all $x \in S^1$. With this notation, we can say that $a \mathscr{L}^* b$ if and only if $\ker \lambda_a^1 = \ker \lambda_b^1$. If we define ρ_a^1 dually as the inner right translation of S^1 associated with a, we also get that $a \mathscr{R}^* b$ if and only if $\ker \rho_a^1 = \ker \rho_b^1$.

Remark I.3.2. In fact, this close relation is not coincidental since the $\mathscr{L}^*$-relation was initially defined by Pastijn [39] using partial left translations. A *bijective partial left translation on* S is a map $\lambda \in \mathcal{I}_S^{\mathrm{op}}$ which satisfies $\lambda(st) = (\lambda s)t$ whenever both sides of the equality are defined. Then, Pastijn defined the relation $\mathscr{L}^*$ on a semigroup S by: $a \mathscr{L}^* b$ if and only if there exists a bijective partial left translation λ on S such that $\lambda a = b$ (and thus $\lambda^{-1} b = a$).

The relations $\mathscr{L}^*$ and $\mathscr{R}^*$ may be defined in another equivalent manner, as follows.

Lemma I.3.3. *Let $a, b \in S$. Then $a \mathscr{L}^* b$ if and only if a and b are $\mathscr{L}$-related in an oversemigroup T of S. Similarly, $a \mathscr{R}^* b$ if and only if a and b are $\mathscr{R}$-related in an oversemigroup T of S.*

Proof. Since the statement for $\mathscr{R}^*$ is dual to that of $\mathscr{L}^*$, we will only prove the case for $\mathscr{R}^*$ in order to work with right maps instead of left maps.

Let $a, b \in S$ and suppose that $a \mathscr{R}^* b$. It is well-know [1.1.2 in 28] that S embeds into $\mathcal{T}_{S^1}$ through the map $s \mapsto \rho_s^1$, where $\rho_s^1\colon S^1 \to S^1$ is such that $x\rho_s^1 = xs$ for all $x \in S^1$. As mentioned after the definition of $\mathscr{R}^*$, we have that $a \mathscr{R}^* b$ is equivalent to $\ker \rho_a^1 = \ker \rho_b^1$. But then, we get that the maps ρ_a^1 and ρ_b^1 are $\mathscr{R}$-related in $\mathcal{T}_{S^1}$ by classical arguments (see for example [2.6 in 8]). Thus a and b are $\mathscr{R}$-related in an oversemigroup $T \supseteq S$.

Conversely, suppose that $a, b \in S$ are $\mathscr{R}$-related in an oversemigroup $T \supseteq S$. Therefore, there exist $t, t' \in T^1$ such that $at = b$ and $bt' = a$. Subsequently, for any $x, y \in S^1$ we get that if $xa = ya$, then $xat = yat$, that is, $xb = yb$. Similarly, if $xb = yb$, then $xbt' = ybt'$, which means that $xa = ya$. Therefore, $a \mathscr{R}^* b$ as required. $\square$

The generalisation of Green's relation $\mathscr{J}$ is not as direct and simple, and requires the concept of ideal saturation.

Definition I.3.4. Let $\mathscr{X} \subseteq S \times S$ be an equivalence relation on S, and for all $s \in S$ denote by $\mathscr{X}_s$ the $\mathscr{X}$-class of s. We say that an ideal I of S is *saturated by* $\mathscr{X}$ if $\mathscr{X}_a \subseteq I$ for all $a \in I$.

It is easy to see that the intersection of a set of ideals saturated by an equivalence relation $\mathscr{X}$ is itself saturated by $\mathscr{X}$. Moreover, the semigroup S is clearly saturated by any equivalence relation, which means that there is a minimal ideal of S saturated by $\mathscr{X}$. Thus, given a subset I of S, we can look for the smallest ideal containing I that is saturated by $\mathscr{X}$. We can effectively construct this minimal ideal as follows:

Lemma I.3.5. *Let I be a subset of S and $\mathscr{X}$ be an equivalence relation on S. Define iteratively I_k as follows:*

1) $I_0 = I$, and

2) $I_{k+1} = \{xby \mid b \in \mathscr{X}_a$ for some $a \in I_k$ and $x, y \in S^1\}$.

Then $I^{\text{sat}} = \bigcup_{k \in \mathbb{N}^0} I_k$ is the smallest ideal of S containing I that is saturated by $\mathscr{X}$.

Proof. Let $a \in I^{\text{sat}}$. Then there exists $k \in \mathbb{N}^0$ such that $a \in I_k$. Thus $\mathscr{X}_a \subseteq I_{k+1}$ and $xay \in I_{k+1}$ for all $x, y \in S^1$, so that I^{sat} is an ideal of S saturated by $\mathscr{X}$.

Now let J be an ideal of S saturated by $\mathscr{X}$ and containing I. We show that $I^{\text{sat}} \subseteq J$. By definition, we clearly have that $I_0 \subseteq J$. Now suppose that $I_k \subseteq J$ for some $k \in \mathbb{N}$ and let $a \in I_k$. Since J is saturated by $\mathscr{X}$ it follows that $\mathscr{X}_a \subseteq J$. Moreover, for every $b \in \mathscr{X}_a$, we have that $xby \in J$ for all $x, y \in S^1$ as J is an ideal. Therefore $I_{k+1} \subseteq J$ and by induction we obtain that $I^{\text{sat}} \subseteq J$, which finishes proving that I^{sat} is the minimal ideal of S containing I and saturated by $\mathscr{X}$. $\square$

Definition I.3.6. Let $I \subseteq S$ and $\mathscr{X}$ be an equivalence relation. We call the ideal I^{sat} the *saturation of I by $\mathscr{X}$*.

Remark I.3.7. Notice that if $\mathscr{X}$ and $\mathscr{Y}$ are two equivalence relations and $I \subseteq S$, the saturation of I by $\mathscr{X}$ and $\mathscr{Y}$ correspond to the saturation of I by $\mathscr{X} \vee \mathscr{Y}$. It is easy to see that this gives the same result as alternatively saturating I by $\mathscr{X}$ first, and then saturating the resulting ideal by $\mathscr{Y}$ at each step of the saturation process.

In view of Lemma I.3.5 and Remark I.3.7 above, for every element $a \in S$, there is a minimal ideal of S which contains a and is saturated by both $\mathscr{L}^*$ and $\mathscr{R}^*$, that is, by $\mathscr{D}^*$. This ideal, denoted by $J^*(a)$ by Fountain [13], is called the *principal $*$-ideal generated by a*.

Similarly, we define the *principal $\sim$-ideal generated by $a \in S$*, denoted by $\widetilde{J}(a)$, as the smallest ideal of S containing a and saturated by both $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}$, that is, saturated by $\widetilde{\mathscr{D}}$. We can now define the relations $\mathscr{J}^*$ and $\widetilde{\mathscr{J}}$.

Definition I.3.8. Let $a, b \in S$. Then $a \mathscr{J}^* b$ if and only if $J^*(a) = J^*(b)$, and $a \widetilde{\mathscr{J}} b$ if and only if $\widetilde{J}(a) = \widetilde{J}(b)$.

Notation I.3.9. In the cases where there are multiple semigroups involved in the discussion and ambiguity can arise, we will add subscripts to the relations. For example, if S and T are semigroups, we will write $\mathscr{L}_S^*$ and $\mathscr{R}_S^*$ for the relations $\mathscr{L}^*$ and $\mathscr{R}^*$ when seen in S, while $\mathscr{L}_T^*$ and $\mathscr{R}_T^*$ will be relatively to T. This is especially important if T is a subsemigroup of S, in which case it is easy to see from the definition that we will have $\mathscr{L}_S^* \bigcap (T \times T) \subseteq \mathscr{L}_T^*$ and $\mathscr{R}_S^* \bigcap (T \times T) \subseteq \mathscr{R}_T^*$, as well as the corresponding inclusions for $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}$.

I.3.2 Characterisations and comparisons

Since the meet of two relations is their intersection, we obtain that the descriptions of $\mathscr{H}^*$ and $\widetilde{\mathscr{H}}$ will follow directly from those of $\mathscr{L}^*$, $\mathscr{R}^*$, $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}$.

On the other hand, the relations $\mathscr{D}^*$ and $\widetilde{\mathscr{D}}$ are defined in terms of join, which is more complicated to describe. Nevertheless, there is a well-known characterisation (see for example [1.5.11 in 28]) of the join of two arbitrary equivalence relations which is given as follows:

Lemma I.3.10. *Let τ, π be equivalence relations on S, and let $a, b \in S$. Then a and b are $\tau \vee \pi$-related if and only if $a\,(\tau \circ \pi)^n\, b$ for some $n \in \mathbb{N}$. Moreover, if τ and π commute, then $\tau \vee \pi = \tau \circ \pi$.*

The relations $\mathscr{L}$ and $\mathscr{R}$ commute, but in general this is not the case for $\mathscr{L}^*$ and $\mathscr{R}^*$, and neither for $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}$. Thus the characterisation of $\mathscr{D}^*$ and $\widetilde{\mathscr{D}}$ is given by:

Corollary I.3.11. *Let $a, b \in S$. Then a and b are $\mathscr{D}^*$-related in S if and only if there exist $c_0, c_1, \ldots, c_{2n} \in S$ such that*

$$a = c_0 \,\mathscr{L}^*\, c_1 \,\mathscr{R}^*\, c_2 \ldots \mathscr{L}^*\, c_{2n-1} \,\mathscr{R}^*\, c_{2n} = b.$$

Similarly, a is $\widetilde{\mathscr{D}}$-related to b if and only if there exist $c_0, \ldots, c_{2n} \in S$ such that $a = c_0 \,\widetilde{\mathscr{L}}\, c_1 \,\widetilde{\mathscr{R}}\, c_2 \ldots \widetilde{\mathscr{L}}\, c_{2n-1} \,\widetilde{\mathscr{R}}\, c_{2n} = b$.

A key result to see that the $*$ and $\sim$ equivalence relations defined in the previous section are indeed generalisations of classical Green's relations is given by:

Lemma I.3.12. *In any semigroup S, we have the following inclusions:*

1) $\mathscr{L} \subseteq \mathscr{L}^* \subseteq \widetilde{\mathscr{L}}$;

2) $\mathscr{R} \subseteq \mathscr{R}^* \subseteq \widetilde{\mathscr{R}}$;

3) $\mathscr{H} \subseteq \mathscr{H}^* \subseteq \widetilde{\mathscr{H}}$;

4) $\mathscr{D} \subseteq \mathscr{D}^* \subseteq \widetilde{\mathscr{D}}$;

5) $\mathscr{J} \subseteq \mathscr{J}^* \subseteq \widetilde{\mathscr{J}}$; *and*

6) $\mathscr{D}^* \subseteq \mathscr{J}^*$ *and* $\widetilde{\mathscr{D}} \subseteq \widetilde{\mathscr{J}}$.

Proof. Let $a, b \in S$.

1) That $\mathscr{L} \subseteq \mathscr{L}^*$ is clear from Lemma I.3.3. By taking $x = e \in E$ and $y = 1$ in the definition of $\mathscr{L}^*$, we clearly see that two elements $\mathscr{L}^*$-related will be $\widetilde{\mathscr{L}}$-related, which shows that $\mathscr{L}^* \subseteq \widetilde{\mathscr{L}}$.

2) Dual to 1).

3) Follows immediately from 1) and 2).

4) Suppose that $a \,\mathscr{D}\, b$. Then there exists $c \in S$ such that $a \,\mathscr{L}\, c \,\mathscr{R}\, b$. Since $\mathscr{L} \subseteq \mathscr{L}^*$ and $\mathscr{R} \subseteq \mathscr{R}^*$ by the points above, we get that $a \,\mathscr{L}^*\, c \,\mathscr{R}^*\, b$ and thus $a \,\mathscr{D}^*\, b$.

Now, if $a \,\mathscr{D}^*\, b$, then $a = c_0 \,\mathscr{L}^*\, c_1 \,\mathscr{R}^*\, c_2 \ldots \mathscr{L}^*\, c_{2n-1} \,\mathscr{R}^*\, c_{2n} = b$ for some elements $c_0, c_1, \ldots, c_{2n} \in S$. Since $\mathscr{L}^* \subseteq \widetilde{\mathscr{L}}$ and $\mathscr{R}^* \subseteq \widetilde{\mathscr{R}}$, it follows directly that $a \left(\widetilde{\mathscr{L}} \circ \widetilde{\mathscr{R}}\right)^n b$ and thus $a \,\widetilde{\mathscr{D}}\, b$.

5) If a and b generate the same principal ideal, say I, then it is clear that saturating I by $\mathscr{L}^*$ and $\mathscr{R}^*$ will give us that $I^{\mathrm{sat}} = J^*(a)$ and $I^{\mathrm{sat}} = J^*(b)$. Thus $a \,\mathscr{J}^*\, b$.

Suppose that $a \,\mathscr{J}^*\, b$. Then $J^*(a) = J^*(b)$. Since $a \in \widetilde{J}(a)$ and $\widetilde{J}(a)$ is saturated by $\widetilde{\mathscr{D}}$, then $\widetilde{J}(a)$ is an ideal saturated by $\mathscr{D}^* \subseteq \widetilde{\mathscr{D}}$. However, $J^*(a)$ is the smallest ideal containing a that is saturated by $\mathscr{D}^*$, which means that $J^*(a) \subseteq \widetilde{J}(a)$. Thus $b \in J^*(b) = J^*(a) \subseteq \widetilde{J}(a)$, so that $\widetilde{J}(b) \subseteq \widetilde{J}(a)$. Similarly, we get that $\widetilde{J}(a) \subseteq \widetilde{J}(b)$, so that $a \,\widetilde{\mathscr{J}}\, b$.

6) Suppose that $a \,\mathscr{D}^*\, b$ with $a = c_0 \,\mathscr{L}^*\, c_1 \,\mathscr{R}^*\, c_2 \ldots \mathscr{L}^*\, c_{2n-1} \,\mathscr{R}^*\, c_{2n} = b$ for some $c_0, c_1, \ldots, c_n \in S$. Since $J^*(a)$ is saturated by $\mathscr{L}^*$, it follows that $c_1 \in J^*(a)$, and then $c_2 \in J^*(a)$ since this ideal is also saturated by $\mathscr{R}^*$. By induction, we get that $b \in J^*(a)$. The converse gives us that $a \in J^*(b)$ and thus $a \,\mathscr{J}^*\, b$.

A similar argument shows that $\widetilde{\mathscr{D}} \subseteq \widetilde{\mathscr{J}}$. $\square$

These inclusions may be strict but some of them become equalities when we restrict our attention to regular elements, as given in the next lemma.

Lemma I.3.13. *If a and b are both regular elements of S, then*

$$a \mathscr{L} b \Leftrightarrow a \mathscr{L}^* b \Leftrightarrow a \widetilde{\mathscr{L}} b \quad \textit{and} \quad a \mathscr{R} b \Leftrightarrow a \mathscr{R}^* b \Leftrightarrow a \widetilde{\mathscr{R}} b.$$

Proof. Since $\mathscr{L} \subseteq \mathscr{L}^* \subseteq \widetilde{\mathscr{L}}$, it suffices to prove that two regular elements that are $\widetilde{\mathscr{L}}$-related must be $\mathscr{L}$-related. The result for $\mathscr{R}$ will follow by duality.

So assume that a and b are regular elements such that $a \widetilde{\mathscr{L}} b$. Then there exist $a', b' \in S$ such that $a = aa'a$ and $b = bb'b$. Thus $a'a$ and $b'b$ are idempotents and by the definition of being $\widetilde{\mathscr{L}}$-related, we get that $a = ab'b$ and $b = ba'a$, and thus $a \mathscr{L} b$. $\square$

Notice however that two regular elements which are $\mathscr{D}^*$- or $\widetilde{\mathscr{D}}$-related will not necessarily be $\mathscr{D}$-related, and the same hold for $\mathscr{J}^*$ and $\widetilde{\mathscr{J}}$ relatively to $\mathscr{J}$. However, this always happens if our semigroup is regular, in which case we have that the extended Green's relations coincide exactly with the classical Green's relations.

Corollary I.3.14. *If S is a regular semigroup, then*

$$\mathscr{L} = \mathscr{L}^* = \widetilde{\mathscr{L}}, \qquad \mathscr{R} = \mathscr{R}^* = \widetilde{\mathscr{R}}, \qquad \mathscr{H} = \mathscr{H}^* = \widetilde{\mathscr{H}},$$
$$\mathscr{D} = \mathscr{D}^* = \widetilde{\mathscr{D}}, \qquad \textit{and} \qquad \mathscr{J} = \mathscr{J}^* = \widetilde{\mathscr{J}}\,.$$

Proof. The cases for $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{H}$ follow directly from Lemma I.3.13. Then we have that $\widetilde{\mathscr{D}} = \widetilde{\mathscr{L}} \vee \widetilde{\mathscr{R}} = \mathscr{L} \vee \mathscr{R} = \mathscr{D}$, and $\mathscr{D}^* = \mathscr{D}$ as well.

Finally, since any ideal is saturated by $\mathscr{L}$ and $\mathscr{R}$, hence by $\widetilde{\mathscr{L}} = \mathscr{L}^* = \mathscr{L}$ and $\widetilde{\mathscr{R}} = \mathscr{R}^* = \mathscr{R}$, we get that the principal $*$- and $\sim$-ideal generated by an element $a \in S$ coincide with the principal ideal generated by a, so that $\mathscr{J} = \mathscr{J}^* = \widetilde{\mathscr{J}}$. $\square$

In order to facilitate our discussions when dealing with the relations $\mathscr{J}^*$ and $\widetilde{\mathscr{J}}$, we give another characterisation for the principal $*$- and $\sim$-ideals.

Lemma I.3.15. *Let $a \in S$. Then $b \in J^*(a)$ if and only if there exist $a_0, \ldots, a_n \in S$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^1$ such that $a = a_0$, $b = a_n$ and $a_i \mathscr{D}^* x_i a_{i-1} y_i$ for all $1 \leq i \leq n$.*

Similarly, $b \in \widetilde{J}(a)$ if and only if there exist elements $a_0, \ldots, a_n \in S$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^1$ such that $a = a_0$, $b = a_n$ and $a_i \widetilde{\mathscr{D}} x_i a_{i-1} y_i$ for all $1 \leq i \leq n$.

Proof. We will only prove the lemma for $J^*(a)$ since the arguments are similar for $\widetilde{J}(a)$. Let $a \in S$ and consider $B \subseteq S$ to be the set of all $b \in S$ satisfying the given conditions. We want to show that $B = J^*(a)$.

For this, let $b \in B$. Then there exist $a_0, \ldots, a_n \in S$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^1$ such that $a = a_0$, $b = a_n$ and $a_i \mathscr{D}^* x_i a_{i-1} y_i$ for all $1 \leq i \leq n$. If $a_{i-1} \in J^*(a)$ for some $1 \leq i \leq n$, then $x_i a_{i-1} y_i \in J^*(a)$ since $J^*(a)$ is an ideal. Moreover, $J^*(a)$ is saturated by $\mathscr{L}^*$ and $\mathscr{R}^*$, hence also by $\mathscr{D}^*$ and from the fact that $a_i \mathscr{D}^* x_i a_{i-1} y_i$ we obtain that $a_i \in J^*(a)$. Since $a_0 = a \in J^*(a)$, it follows by induction that $b = a_n \in J^*(a)$, so that $B \subseteq J^*(a)$.

Conversely, let $b \in B$ and $a_0, \ldots, a_n$ as before. Then for all $x, y \in S^1$, we can see that setting $a_{n+1} = xby$ we get that $a_{n+1} \mathscr{D}^* x a_n y$, so that $xby \in B$. Thus, B is an ideal. Furthermore, $\mathscr{L}^*, \mathscr{R}^* \subseteq \mathscr{D}^*$, which means that for all $c \in S$ lying in the $\mathscr{L}^*$- or $\mathscr{R}^*$-class of b, we get that $c \mathscr{D}^* b$, and thus setting $a_{n+1} = c$ and $x_{n+1} = y_{n+1} = 1$, we see that c satisfies all conditions to be in B. Therefore B is saturated by both $\mathscr{L}^*$ and $\mathscr{R}^*$, and since $a \in B$ it follows that $J^*(a) \subseteq B$. Hence $J^*(a) = B$. $\square$

I.3.3 Classes of idempotents

We finish this section on extended Green's relations by mentioning the special role of idempotents. We first can see that $\mathscr{H}^*$ and $\widetilde{\mathscr{H}}$ are idempotent separating equivalence relations, as given by:

Lemma I.3.16. *Let $e, f \in E$. Then $e \mathscr{H}^* f$ or $e \widetilde{\mathscr{H}} f$ if and only if $e = f$.*

Proof. Suppose that $e \widetilde{\mathscr{H}} f$. Then $e \widetilde{\mathscr{L}} f$ and $e^2 = e$ so that $fe = f$. Also $e \widetilde{\mathscr{R}} f$ and $f^2 = f$, which gives us $fe = e$. Hence $e = f$. $\square$

In a similar way as idempotents are right and left identities for elements in their $\mathscr{L}$- and $\mathscr{R}$-class, they have the same role for elements in their $\mathscr{L}^*$- and $\mathscr{R}^*$-class.

Lemma I.3.17. *Let $a \in S$ and $e = e^2 \in S$. Then $a \mathscr{L}^* e$ if and only if $ae = a$ and $ax = ay$ implies $ex = ey$ for all $x, y \in S^1$.*

Similarly, $a \widetilde{\mathscr{L}} e$ if and only if $ae = a$ and $af = a$ implies $ef = e$ for all $f \in E$.

Dual statements hold for $\mathscr{R}^*$ and $\widetilde{\mathscr{R}}$.

Proof. Clearly we have that $e^2 = e1$, and then $a \mathscr{L}^* e$ implies that $ae = a1 = a$. Conversely, if $ae = a$ and $ex = ey$ for some $x, y \in S^1$, then $ax = aex = aey = ay$. The result follows. $\square$

Corollary I.3.18. *Every $\mathscr{H}^*$-class of S containing an idempotent is a cancellative monoid. Consequently, every finite $\mathscr{H}^*$-class is a group.*

Proof. Let H_e^* be an $\mathscr{H}^*$-class which contains an idempotent $e \in S$. By Lemma I.3.17, we know that e is both a left and a right identity for every element in H_e^*. Let $a, b \in H_e^*$ and let $x, y \in S^1$ be such that $abx = aby$. Since $a \mathscr{L}^* e$, we get that $bx = ebx = eby = by$, and since $b \mathscr{L}^* e$ we get $ex = ey$. We also have that $abe = ab$ and thus $ab \mathscr{L}^* e$ by Lemma I.3.17. Similarly, $ab \mathscr{R}^* e$, which shows that H_e^* is a monoid.

Now, suppose that $a, b, c \in H_e^*$ are such that $ab = ac$. Then $a \mathscr{L}^* e$, and we deduce $eb = ec$, so that $b = c$. Dually $ba = ca$ implies $b = c$, and therefore H_e^* is cancellative. □

It is well-known that a semigroup is regular if and only if every $\mathscr{L}$- and $\mathscr{R}$-class contains an idempotent. We can thus make similar definitions to define wider classes of semigroups by replacing $\mathscr{L}$ and $\mathscr{R}$ by their extended versions.

Definition I.3.19. A semigroup S is called *right [resp. left] abundant* if every $\mathscr{L}^*$-class [resp. every $\mathscr{R}^*$-class] contains an idempotent and *right [resp. left] Fountain* if every $\widetilde{\mathscr{L}}$-class [resp. every $\widetilde{\mathscr{R}}$-class] contains an idempotent.

A semigroup that is both left and right abundant [resp. Fountain] is called an *abundant [resp. Fountain] semigroup.*

It is clear from the definition that left [resp. right] abundant implies left [rep. right] Fountain. The converse is not true. Further, in Chapters III and IV, we will see examples of semigroups that are right abundant but not even left Fountain, and others that are left abundant and Fountain.

Remark I.3.20. Using Lemma I.3.17 (see [14]), we can get that being right abundant for a semigroup corresponds to the notion of being right principally projective (RPP for short), where a semigroup is RPP if every principal right ideal is projective. In [7] the authors describe other classes of semigroups relative to the presence of idempotents in each class of a given extended Green's relation.

I.4 INDEPENDENCE ALGEBRAS

The class of independence algebras was introduced by Gould in [22] in order to account for the similarities between endomorphisms of sets, vector spaces and free group acts, and to allow their study under a common framework. Incidentally, she noted that these algebras also correspond to the v^*-algebras described by Narkiewicz in [36], which continued the study of different notions of independence in universal algebras initiated by Marczewski [31]. In this thesis, we will only consider the notion of independence relative to the closure operator $\langle \cdot \rangle$ (sometimes also called *C-independence*), but v^*-algebras were initially defined using another notion of independence, which is now often called *M-independence* (with regard to Marczewski). A good comparison between these two notions of independence can be found in the article of Araújo and Fountain [3].

All the results presented in this section are well-known and can be found in the literature (see [1, 2, 10, 17, 22, 30, 34]). Some proofs are included to make this thesis as self-contained as possible on this topic, particularly when those proofs are hard to track down in the literature.

I.4.1 Definition

Throughout this section $\mathscr{A} = (A, F)$ will denote an algebra. We first recall a basic fact on the subalgebra operator.

Lemma I.4.1. *The operator $\langle \cdot \rangle$ is a closure operator and is algebraic. In particular, this means that for any $X, X', Y \subseteq A$ such that $\langle X \rangle = \langle X' \rangle$, we have:*

1) $\langle X \cup Y \rangle = \Big\langle \langle X \rangle \cup \langle Y \rangle \Big\rangle = \langle X' \cup Y \rangle$*; and*
2) $\langle X \rangle = \bigcup \{ \langle Z \rangle \mid Z \subseteq X, \text{and } Z \text{ is finite} \}$.

In order to define independence algebras, we first need to talk about the notion of independence.

Definition I.4.2. A set $X \subseteq A$ is *independent* if $x \notin \langle X \setminus \{x\} \rangle$ for all $x \in X$.

If $x \in A$ and $\{x\}$ is independent, then x is an *independent element of A*.

A *maximal independent set* (or *subset*) is an independent set X such that there exist no independent set Z with $X \subsetneq Z \subseteq A$.

A subset $B \subseteq A$ is *closed* if $B = \langle X \rangle$ for some $X \subseteq A$. In such a case B is a subuniverse of A and X is said to *span* B.

It is straightforward to see that the empty set is an independent set. Similarly, any element that is not a constant is independent. A subset $Y \subseteq A$ that is not independent will be called *dependent*. We now consider the existence of maximal independent subsets.

Lemma I.4.3 ([2.2.6 in 30]). *For any subset X of A, and any independent subset Y of X, there exists a maximal independent subset of X containing Y. Consequently, every subset of A contains a maximal independent subset.*

Proof. Let Y be an independent subset of a set $X \subseteq A$ and consider the set $\mathcal{S} = \{Z \subseteq A \mid Y \subseteq Z \subseteq X,$ and Z is independent$\}$. Clearly $\mathcal{S} \neq \emptyset$ since it contains the set Y itself. The result will follow directly from Zorn's Lemma if we can prove that any ascending chain in $\mathcal{S}$ has an upper bound.

So let S be a totally ordered subset of $\mathcal{S}$ and $W = \bigcup_{S_i \in S} S_i$. Since $Y \subseteq W$, we need to show that W is independent. Suppose that $w \in W$ and $w \in \langle W \setminus \{w\}\rangle$. Using the fact that $\langle\,\cdot\,\rangle$ is an algebraic operator, it follows that

$$\langle W \setminus \{w\}\rangle = \bigcup \Big\{\langle V\rangle : V \subseteq W \setminus \{w\} \text{ and } V \text{ is finite}\Big\},$$

and thus $w \in \langle V\rangle$ for some finite subset V of $W \setminus \{w\}$. Therefore $V \cup \{w\} \subseteq W$, and since V is finite and S is an ordered chain, we must have $V \cup \{w\} \subseteq S_i$ for some $S_i \in S$. This in turn implies that $w \in \langle V\rangle \subseteq \langle S_i \setminus \{w\}\rangle$, contradicting the independence of S_i. Thus W is an independent set and is therefore an upper bound for S. Hence every ascending chain in $\mathcal{S}$ has an upper bound, which proves the first part of the statement while the second part follows directly by taking $Y = \emptyset$. □

Since the existence of maximal independent subsets is guaranteed by Lemma I.4.3 above, we will use this fact without further reference.

Definition I.4.4. The algebra $\mathscr{A}$ has the *exchange property* if any subset X of A satisfies the following condition:

(EP) for all $a, b \in A$, if $a \in \Big\langle X \cup \{b\}\Big\rangle$ and $a \notin \langle X\rangle$, then $b \in \Big\langle X \cup \{a\}\Big\rangle$.

Example I.4.5. It is clear that given a set S, the algebra $\mathcal{S} = (S, \emptyset)$ trivially satisfies the exchange property, since for any $X \subseteq S$ we have $\langle X\rangle = X$. Similarly, suppose that $\mathcal{V} = \Big(V, \Big\{+, 0, \{f_\mu\}_{\mu \in \Bbbk}\Big\}\Big)$ is a vector space over a field $\Bbbk$ with operations f_μ corresponding to the scalar multiplication by an element $\mu \in \Bbbk$. The subalgebra operator corresponds in this case to taking linear combinations. From this, it is clear that if $a, b \in V$ and $X \subseteq V$ are such that $a \in \langle X \cup \{b\}\rangle$, then there exist scalars $\mu_i, \mu_b \in \Bbbk$ such that $a = \sum_i \mu_i x_i + \mu_b b$. If $a \notin \langle X\rangle$, this means that $\mu_b \neq 0$, in which case we also have $b = \sum_i -\mu_b^{-1}\mu_i x_i + \mu_b^{-1} a$, so that $b \in \langle X \cup \{a\}\rangle$ and $\mathcal{V}$ satisfies the exchange property.

The exchange property can be formulated differently in terms of the subuniverse spanned by independent sets, as given by:

Proposition I.4.6 ([2.3.1 in 30]). *The following conditions are equivalent:*

i) $\mathscr{A}$ *has* (EP)*;*

ii) for every $X \subseteq A$ *and* $a \in A$*, if* X *is independent and* $a \notin \langle X\rangle$ *then* $X \cup \{a\}$ *is independent;*

iii) for every $X \subseteq A$*, if* Y *is a maximal independent subset of* X*, then* $\langle Y\rangle = \langle X\rangle$*;*

iv) for every $X, Y \subseteq A$ *such that* $Y \subseteq X$*, if* Y *is independent, then there is an independent set* Z *such that* $Y \subseteq Z \subseteq X$ *and* $\langle Z\rangle = \langle X\rangle$*.*

Proof. $i) \Rightarrow ii)$: Let $X \subseteq A$ be independent and $a \in A$ be such that $a \notin \langle X\rangle$. Suppose now that $X \cup \{a\}$ is dependent, that is, there exists $x \in X \cup \{a\}$ such that $x \in \langle\, (X \cup \{a\}) \setminus \{x\}\,\rangle$. Since $a \notin \langle X\rangle$, it follows that $x \neq a$ so that $x \in X$. Since X is independent, we have $x \notin \langle X \setminus \{x\}\rangle$. But $(X \cup \{a\}) \setminus \{x\} = (X \setminus \{x\}) \cup \{a\}$, and $\mathscr{A}$ has (EP) so it follows that $a \in \langle\, (X \setminus \{x\}) \cup \{x\}\,\rangle = \langle X\rangle$, a contradiction. Therefore $X \cup \{a\}$ is an independent set.
$ii) \Rightarrow iii)$: Let $X \subseteq A$, and assume that Y is a maximal independent subset of X. Since $Y \subseteq X$, it follows that $\langle Y\rangle \subseteq \langle X\rangle$. For the reverse inclusion, if an element $x \in X$ such that $x \notin \langle Y\rangle$ would exist, then $Y \cup \{x\}$ would be independent by hypothesis $ii)$, contradicting the maximality of Y in X. Therefore $X \subseteq \langle Y\rangle$ which gives that $\langle X\rangle \subseteq \langle Y\rangle$ and hence the equality.
$iii) \Rightarrow iv)$: Let $Y \subseteq X$ with Y an independent set. By Lemma I.4.3 there exists a maximal independent subset Z of X with $Y \subseteq Z$, and by $iii)$, it follows that $\langle Z\rangle = \langle X\rangle$, as desired.
$iv) \Rightarrow ii)$: Let X be an independent subset of A and $a \in A$ be such that $a \notin \langle X\rangle$. Then $\langle X\rangle \neq \langle X \cup \{a\}\rangle$ and by $iv)$ there exists $X \subseteq Z \subseteq X \cup \{a\}$ such that Z is independent with the property that $\langle Z\rangle = \langle X \cup \{a\}\rangle$. Thus $Z = X \cup \{a\}$ and so $X \cup \{a\}$ is independent.
$ii) \Rightarrow i)$: Let $X \subseteq A$, and let $a, b \in A$ such that $a \notin \langle X\rangle$ but $a \in \langle X \cup \{b\}\rangle$. Suppose that $b \notin \langle X\rangle$ since otherwise, it follows trivially that $b \in \langle X \cup \{a\}\rangle$. By Lemma I.4.3, there exists a maximal independent subset Z of X. Since we know that $ii)$ implies $iii)$, we get that $\langle Z\rangle = \langle X\rangle$. Then $a, b \notin \langle Z\rangle$ and so $Z \cup \{a\}$ and $Z \cup \{b\}$ are

independent by *ii*). By Lemma I.4.1, we have that $\langle X \cup \{b\}\rangle = \langle Z \cup \{b\}\rangle$. Thus $a \in \langle Z \cup \{b\}\rangle$ and therefore $Z \cup \{a\} \cup \{b\}$ is dependent. Hence $Z \cup \{a\}$ and $Z \cup \{b\}$ are both maximal independent subsets of $Z \cup \{a\} \cup \{b\}$. Using *iii*) again, we get

$$\Big\langle Z \cup \{a\}\Big\rangle = \Big\langle Z \cup \{a\} \cup \{b\}\Big\rangle = \Big\langle Z \cup \{b\}\Big\rangle$$

so that $b \in \langle Z \cup \{b\}\rangle = \langle Z \cup \{a\}\rangle = \langle X \cup \{a\}\rangle$ using Lemma I.4.1 again. Hence the algebra $\mathscr{A}$ has (EP). $\square$

The relationship between minimal spanning set and maximal independent set is important, and it can be shown that these coincide in any algebra that has the exchange property.

Corollary I.4.7 ([22])**.** *Suppose that $\mathscr{A}$ satisfies* (EP)*, and let $Y \subseteq X \subseteq A$. Then the following are equivalent:*

i) Y is a maximal independent subset of X;

ii) Y is independent and $\langle Y\rangle = \langle X\rangle$;

iii) Y is minimal with respect to $\langle Y\rangle = \langle X\rangle$.

Proof. $i) \Rightarrow ii)$: This is exactly part *iii*) of the above Proposition I.4.6.
$ii) \Rightarrow iii)$: Let Y be an independent set with $\langle Y\rangle = \langle X\rangle$ and let $Z \subsetneq Y$. Then taking $y \in Y \setminus Z$, we have that $y \notin \langle Z\rangle \subseteq \langle Y \setminus \{y\}\rangle$, so that $\langle Z\rangle \subsetneq \langle Y\rangle = \langle X\rangle$. Therefore Y is minimal with respect to $\langle Y\rangle = \langle X\rangle$.
$iii) \Rightarrow i)$: Let Y be minimal with the property that $\langle Y\rangle = \langle X\rangle$. Suppose that there exists $y \in Y$ such that $y \in \langle Y \setminus \{y\}\rangle$. Then $Y \subseteq \langle Y \setminus \{y\}\rangle$ which shows that $\langle X\rangle = \langle Y\rangle \subseteq \langle Y \setminus \{y\}\rangle$, so that $\langle Y \setminus \{y\}\rangle = \langle X\rangle$, contradicting the minimality of Y. Hence Y is independent. Now, for any $Y \subsetneq Z \subseteq X$ and $z \in Z \setminus Y$, we have that $z \in \langle X\rangle = \langle Y\rangle \subseteq \langle Z \setminus \{z\}\rangle$. Thus Z is dependent and so Y is a maximal independent subset of X. $\square$

With this result, we can now define the notion of basis.

Definition I.4.8. Suppose that $\mathscr{A}$ has (EP), and let $X \subseteq A$. A *basis of X* is a subset $Y \subseteq X$ satisfying the equivalent conditions of Corollary I.4.7.

Remark I.4.9. It is clear that if we take a subset X satisfying $X = \langle X\rangle$ in Corollary I.4.7, we have that Y is a maximal independent subset of X if and only if Y is an independent set that spans X. Thus, the definition of a basis given above correspond to the well-known definition of basis for a subspace X in a vector space.

It follows directly from Lemma I.4.3 and Corollary I.4.7 by taking $X = A$ that any algebra $\mathscr{A}$ in which (EP) holds has a basis Z of its universe A. In such a case, we also say that Z is a *basis of* $\mathscr{A}$. Moreover, if $\mathscr{B}$ is a subalgebra of $\mathscr{A}$, then by part *iv*) of Proposition I.4.6 and part *ii*) of Corollary I.4.7, any independent set $Y \subseteq B$ lies inside a basis X of B. Hence $Y \cup (X \setminus Y)$ is a basis of $\mathscr{B}$ and $Y \cap (X \setminus Y) = \emptyset$. This leads us to the notion of basis extension.

Definition I.4.10. Suppose that $\mathscr{A}$ has (EP). Let $\mathscr{B} \subseteq \mathscr{A}$ and $Y \subseteq B$ be an independent set. A *basis extension of* Y *(in* B*)* is an independent set $Z \subseteq B$ such that $Y \cup Z$ is a basis of B and $Y \cap Z = \emptyset$. In this case, we also say that Z *extends* Y *to a basis of* B.

Remark I.4.11. An important case is when we take $\mathscr{B} = \mathscr{A}$, as it means that any independent subset can be extended to a basis. If the mention of the subalgebra in which we are doing the basis extension is omitted, it is assumed that the extension is relative to the algebra $\mathscr{A}$ itself.

We now give properties of basis extensions, which are to be expected when thinking of this notion in the context of vector spaces. The first one relates to the cardinality of the sets involved.

Proposition I.4.12 ([2.3.6 in 30])**.** *Suppose that* $\mathscr{A}$ *satisfies (EP), and let* $\mathscr{B} \subseteq \mathscr{A}$*. If* X, X' *are independent subsets of* B *such that* $\langle X \rangle = \langle X' \rangle$ *and* Y, Z *are respective basis extensions of* X *and* X' *in* B*, then* $|Y| = |Z|$*.*

Consequently, all bases of an algebra have the same cardinality, and all basis extensions of a given set have the same cardinality.

Proof. Let X, X', Y and Z be as given in the statement. Without loss of generality, assume that $|Y| \leq |Z|$.

Suppose that $Y = \{y_1, \ldots, y_n\}$ is finite, and take $Z_n = \{z_1, \ldots, z_n\}$, a subset of Z with the same cardinality as Y. Since $z_1 \notin \langle X' \rangle = \langle X \rangle$, but $z_1 \in \langle X \cup Y \rangle$, there exists an integer $i \leq n$ that is the smallest with respect to $z_1 \in \langle X \cup \{y_1, \ldots, y_i\} \rangle$. Thus by the exchange property, $y_i \in \langle X \cup \{z_1\} \cup \{y_1, \ldots, y_{i-1}\} \rangle \subseteq \langle X \cup \{z_1\} \cup (Y \setminus \{y_i\}) \rangle$. Relabelling the elements of Y for convenience so that $i = 1$, it follows that

$$B = \langle X \cup Y \rangle = \Big\langle X \cup \{z_1\} \cup \{y_2, \ldots, y_n\} \Big\rangle.$$

Since $\langle X' \cup \{z_1\} \rangle = \langle X \cup \{z_1\} \rangle$ by Lemma I.4.1, we get that $z_2 \notin \langle X \cup \{z_1\} \rangle$ and we can now use the same argument replacing X by $X \cup \{z_1\}$ and $\{y_1, \ldots, y_n\}$ by

$\{y_2,\dots,y_n\}$, to get that

$$B = \left\langle X \cup \{z_1, z_2\} \cup \{y_3,\dots,y_n\} \right\rangle.$$

We then proceed by induction on the elements of Z_n to obtain that

$$\langle X' \cup Z\rangle = B = \langle X \cup \{z_1,\dots,z_n\}\rangle = \langle X' \cup Z_n\rangle.$$

If $z \in Z \setminus Z_n$, then $z \in B = \langle X' \cup Z_n\rangle = \left\langle X' \cup Z \setminus \{z\}\right\rangle$ which contradicts the independence of $X' \cup Z$. Thus $Z = Z_n$ and so $|Y| = |Z|$.

Now, consider the case where Y and Z are both infinite. Let $y \in Y$, then since $\langle\,\cdot\,\rangle$ is an algebraic operator there exist finite subsets $X'_y \subseteq X$ and $Z_y \subseteq Z$ such that $y \in \left\langle X'_y \cup Z_y\right\rangle$. Thus $Y \subseteq \left\langle \bigcup_{y\in Y}\left(X'_y \cup Z_y\right)\right\rangle$, and as $X \subseteq \langle X\rangle = \langle X'\rangle$, it follows that

$$X \cup Y \subseteq \left\langle X' \cup \bigcup_{y\in Y}\left(X'_y \cup Z_y\right)\right\rangle = \left\langle X' \cup \bigcup_{y\in Y} Z_y\right\rangle.$$

Consequently, $B = \left\langle X \cup Y\right\rangle \subseteq \left\langle X' \cup \bigcup_{y\in Y} Z_y\right\rangle \subseteq \left\langle X' \cup Z\right\rangle = B$, forcing the equality $\left\langle X' \cup \bigcup_{y\in Y} Z_y\right\rangle = \left\langle X' \cup Z\right\rangle$. Then, if $z \in Z \setminus \bigcup_{y\in Y} Z_y$, we get that $z \in \left\langle (X' \cup Z)\setminus\{z\}\right\rangle$, contradicting the independence of $X' \cup Z$. Therefore $Z = \bigcup_{y\in Y} Z_y$ and we also obtain

$$\begin{aligned}
|Z| = \left|\bigcup_{y\in Y} Z_y\right| &\leq \sum_{y\in Y}|Z_y| \\
&\leq |Y|\cdot\aleph_0 && \text{as } |Z_y| \text{ is finite for each } y, \\
&= |Y| && \text{since } |Y| \text{ is infinite.}
\end{aligned}$$

Together with the starting assumption that $|Y| \leq |Z|$, the equality of the two cardinals is proven.

The last part of the proposition follows directly by setting $X = \emptyset = X'$ in the first case, and $X = X'$ in the second. □

We can now define the concept of rank.

Definition I.4.13. Suppose that $\mathscr{A}$ satisfies (EP) and let $X \subseteq A$. The *rank of* X, denoted by $\operatorname{rank} X$, is the cardinality of any basis of X. If X is independent, then the *corank of* X, denoted by $\operatorname{corank} X$, is the cardinality of any basis extension Y of X. That rank and corank are well-defined comes from Proposition I.4.12.

Similarly, if $\mathscr{B} \subseteq \mathscr{A}$, then the *rank of* $\mathscr{B}$ is defined as the rank of B, and the *corank of* $\mathscr{B}$ is the corank of B in A.

If $\alpha \in \operatorname{End}(\mathscr{A})$, then $\operatorname{im}\alpha$ is a subalgebra of $\mathscr{A}$ and we define the *rank of* α, denoted by $\operatorname{rk}(\alpha)$, as the rank of the subuniverse $\operatorname{im}\alpha$.

Remark I.4.14. When talking about subalgebras or universes of subalgebras, we often refer to the rank and corank as the dimension and codimension respectively. In practice, we will use both terminologies indiscriminately, which means that we will also use $\dim \mathscr{B}$, $\dim B$, $\operatorname{codim}\mathscr{B}$ and $\operatorname{codim} B$ for $\operatorname{rank}\mathscr{B}$, $\operatorname{rank} B$, $\operatorname{corank}\mathscr{B}$ and $\operatorname{corank} B$ respectively.

Notation I.4.15. Since the minimal generating set of a subalgebra is a basis, it follows that the value e defined in Section I.1 corresponds to the minimal rank of a subalgebra of $\mathscr{A}$, as well as the smallest rank of an endomorphism of $\mathscr{A}$.

Notice that if X consists only of algebraic constants, then the empty set is a basis of X and so $\operatorname{rank} X = 0$. Moreover, for any subset X of A, it is clear that $\operatorname{rank} X \leq \operatorname{rank}\langle X\rangle$. However the reverse inequality also holds since any basis Y of X is minimal with respect to $\langle Y\rangle = \langle X\rangle = \langle\langle X\rangle\rangle$ which shows that it is also a basis of $\langle X\rangle$. Hence, we have proved the following:

Lemma I.4.16. *For any $X \subseteq A$, we have* $\operatorname{rank} X = \operatorname{rank}\langle X\rangle$.

Definition I.4.17. If $\mathscr{A}$ has (EP), then we say that $\mathscr{A}$ satisfies the *free basis property*, denoted by (F), if any function $\alpha\colon X \to A$ defined on a basis X can be extended to an endomorphism $\overline{\alpha}$ of $\mathscr{A}$.

In [22], Gould noticed that not every algebra satisfying (EP) also satisfies (F), and constructed the following example.

Example I.4.18. Let C be a chain, that is, a totally ordered set regarded as a semilattice, so $xy = x \wedge y = \min\{x, y\}$. Thus, we have that $\langle X\rangle = X$ so any set is independent. It is easy to see that C has (EP). Suppose that $|C| = 2$ and let $x, y \in C$ be such that $x < y$, that is, $x \neq y$ and $x = xy = yx$. Since $\{x, y\}$ is maximal and independent, it is a basis of C and we define $\alpha\colon \{x, y\} \to C$ by $x\alpha = y$ and $y\alpha = x$. If α could be extended to an endomorphism $\beta\colon C \to C$, then it has to agree on $\{x, y\}$. But $(xy)\beta = (xy)\alpha = x\alpha = y$ and $x\beta y\beta = x\alpha y\alpha = yx = x$, contradicting the fact that β is a homomorphism. Thus C does not satisfy the free basis property.

We can now give the definition of independence algebra.

Definition I.4.19. The algebra $\mathscr{A}$ is an *independence algebra* if it satisfies both (EP) and (F).

Example I.4.20. As we mentioned earlier, sets and vector spaces satisfy the exchange properties. Moreover, a basis of a set S is itself, so that it trivially satisfies the free basis property, while for a vector space, this property corresponds to the fact that linearly extending a map defined on a basis gives an endomorphism. Thus sets and vector spaces are particular examples of independence algebras.

It is worth noting that independence algebras are fully classified. A first classification is due to Urbanik [54] using the work of a number of authors on v^*-algebras, while a more recent one for finite independence algebras was given by Cameron and Szabó [6]. The equivalence of both classifications was a milestone achieved by Araújo, Bentz, Cameron, Kinyon and Konieczny in [2], answering a long standing question, and giving a clear presentation of the different classes. We will give more details on this classification in Section I.5.2 but for now, it is sufficient to know that free group actions (which also includes sets) and vector spaces form two of the classes present in this classification.

I.4.2 Some properties of independence algebras

Throughout this section $\mathscr{A}$ will denote an independence algebra. We give here some useful results which will be used without further mention in the rest of the thesis. We start by showing some properties of unary terms with regards to the set of algebraic constants. Recall that $\mathcal{T}_k^{\mathscr{A}}$ denotes the set of terms of arity k.

Lemma I.4.21. *Assume that $|\mathscr{A}| \geq 2$ and let $t \in \mathcal{T}_1^{\mathscr{A}}$. Then the following are equivalent:*

1) t is constant on A;

2) there exists $a \in \langle \emptyset \rangle$ such that $t(x) = a$ for all $x \in A$; and

3) $t(x) \in \langle \emptyset \rangle$ for some $x \notin \langle \emptyset \rangle$.

It follows that algebraic constants are terms whose image is a constant of A.

Proof. Clearly 2) implies 1) and 3).
1) $\Rightarrow$ 2): Let t be a constant unary term and denote by a its image. Suppose that a is an independent element and let $c \in A$ be such that $c \neq a$. Then, since $\mathscr{A}$ has (EP), we can extend $\{a\}$ to a basis $\{a\} \sqcup Y$ of A. Define a map $\alpha\colon \{a\} \sqcup Y \to A$ by $a\alpha = c$ and $y\alpha = y$ for all $y \in Y$. Since $\mathscr{A}$ has (F), we extend α to an endomorphism

$\overline{\alpha} \in \operatorname{End}(\mathscr{A})$, and we obtain that

$$a = t(c) = t(a\overline{\alpha}) = t(a)\overline{\alpha} = a\overline{\alpha} = c,$$

contradicting the choice of c. Thus a cannot be an independent element, which shows that the image of t is a constant, and thus t is an algebraic constant.

3) $\Rightarrow$ 1): Let $x \in A$ be independent such that $t(x) = a \in \langle\emptyset\rangle$. Since $\mathscr{A}$ has (EP), we extend $\{x\}$ to a basis $\{x\} \sqcup Y$ of A. Now, let $b \in A$ and define a map $\alpha_b \colon \{x\} \sqcup Y \to A$ by $x\alpha_b = y\alpha_b = b$. Using the free basis property of $\mathscr{A}$, we extend α_b to an endomorphism $\overline{\alpha}_b$, and we obtain that

$$t(b) = t(x\overline{\alpha}_b) = t(x)\overline{\alpha}_b = a\overline{\alpha}_b = a.$$

Therefore $t(b) = a$ for all $b \in A$, and t is constant on A.

Now assume that $s \in \mathcal{T}_k^{\mathscr{A}}$ for some $k \in \mathbb{N}$ is a constant term. Then we can set $t \in \mathcal{T}_1^{\mathscr{A}}$ by $t(x) = s(x, \ldots, x)$, which is a unary constant term. By the previous part, it follows that its image is an element $a \in \langle\emptyset\rangle$, so that the image of s is a constant of A. $\square$

Remark I.4.22. In view of this lemma, we will no longer distinguish between constants and algebraic constants in independence algebras.

Building upon the previous lemma, we can show that any unary term which contains an independent element in its image is somewhat invertible.

Lemma I.4.23. *Let $t \in \mathcal{T}_1^{\mathscr{A}}$ be a non-constant unary term. Then there exists a unary term $u \in \mathcal{T}_1^{\mathscr{A}}$ such that $t \circ u = u \circ t = \mathrm{id}$.*

Consequently, the set of non-constant unary terms of $\mathscr{A}$ forms a group.

Proof. Let $a \in A$ be independent and write $b = t(a)$. Then by Lemma I.4.21, we have that $b \notin \langle\emptyset\rangle$ since t is not a constant term. Since $\mathscr{A}$ has (EP), $b \in \langle\{a\}\rangle$ and b is independent, it follows that $a \in \langle\{b\}\rangle$. Thus there exists $u \in \mathcal{T}_1^{\mathscr{A}}$ such that $a = u(b)$. From this, we get that $b = t(a) = t(u(b))$ and $a = u(b) = u(t(a))$.

In order to show that $t \circ u = u \circ t = \mathrm{id}$, we extend $\{a\}$ to a basis $\{a\} \cup X$ of A. Then for $c \in A$ define $\alpha \colon \{a\} \cup X \to A$ by $a\alpha = c$ and $x\alpha = x$ for all $x \in X$. By the free basis property, we extend α to $\overline{\alpha} \in \operatorname{End}(\mathscr{A})$, and then we have

$$c = a\overline{\alpha} = u(t(a))\overline{\alpha} = u(t(a\overline{\alpha})) = u(t(c)).$$

Thus it follows that $u \circ t = \mathrm{id}$ for all $y \in A$. A similar argument gives also that $t \circ u = \mathrm{id}$. $\square$

Before talking about the endomorphism monoid of $\mathscr{A}$, we first look at properties of endomorphisms, as well as consequences of the property (F). The most common example of endomorphisms of an algebra are the identity maps on some subalgebras. In fact, they correspond exactly to the idempotents, as given by:

Lemma I.4.24. *Let $\eta \in \operatorname{End}(\mathscr{A})$. Then $\eta = \eta^2$ if and only if $\eta|_{\operatorname{im}\eta} = \operatorname{id}|_{\operatorname{im}\eta}$. Moreover, if $\mathscr{B} \subseteq \mathscr{A}$, then there exists $\eta = \eta^2 \in \operatorname{End}(\mathscr{A})$ such that $\operatorname{im}\eta = B$.*

Proof. The first part comes from the well-known characterisation of idempotents of $\mathcal{T}_S$ for any set S since $\operatorname{End}(\mathscr{A}) \subseteq \mathcal{T}_A$.

Now, let $\mathscr{B} \subseteq \mathscr{A}$ and X be a basis of B. Let $b \in B$, so that $b = u(\overline{x_i})$ for some $u \in \mathcal{T}^{\mathscr{A}}$. We extend X to a basis $X \sqcup Y$ of A, and we define the map $\varepsilon\colon X \sqcup Y \to A$ by $x\varepsilon = x$ and $y\varepsilon = b$. Using the free basis property, we let $\eta \in \operatorname{End}(\mathscr{A})$ be the extension of ε. Then we have $b\eta = u(\overline{x_i})\eta = u(\overline{x_i\eta}) = b$. Also, for all $a \in A$, there exist $t \in \mathcal{T}^{\mathscr{A}}$, $x_1, \ldots, x_k \in X$ and $y_1, \ldots, y_m \in Y$ such that $a = t\left(\overline{x_k}, \overline{y_m}\right)$, from which we get that

$$a\eta^2 = t\left(\overline{x_k\eta^2}, \overline{y_m\eta^2}\right) = t\left(\overline{x_k\eta}, \overline{b\eta}\right) = t\left(\overline{x_k\eta}, \overline{b}\right) = t\left(\overline{x_k\eta}, \overline{y_m\eta}\right) = a\eta,$$

so that η is an idempotent. $\square$

We now look more closely into the extension property of a map. We first show that any map defined on an independent set admits an extension to a homomorphism and an endomorphism.

Lemma I.4.25. *1) If $X \subseteq A$ is independent and $\alpha\colon X \to A$, then α can be uniquely extended to a homomorphism $\overline{\alpha}\colon \langle X \rangle \to A$.*

2) Moreover, if $\mathscr{B}$ is a subalgebra of $\mathscr{A}$ and $\beta\colon B \to A$ is a homomorphism, then β can be extended to an endomorphism $\gamma \in \operatorname{End}(\mathscr{A})$ with $\operatorname{im}\gamma = \operatorname{im}\beta$.

Proof. 1) If $X = \emptyset$, the result is clear from the fact that any endomorphism must fix the elements of $\langle \emptyset \rangle$ whenever this is non-empty. So we assume that $X \neq \emptyset$.

Since X is independent, let Y be a basis extension of X in A and define $\gamma\colon X \sqcup Y \to A$ by

$$x\gamma = x\alpha \quad \text{and} \quad y\gamma = x_0\alpha$$

for all $x \in X$ and $y \in Y$, where $x_0 \in X$ is arbitrary. Then, by (F) γ can be extended to an endomorphism $\overline{\gamma} \in \operatorname{End}(\mathscr{A})$. If we now set $\overline{\alpha} = \overline{\gamma}|_{\langle X \rangle}$, if follows that for all $x \in X$ we get $x\overline{\alpha} = x\gamma = x\alpha$, so that $\overline{\alpha}$ is a homomorphism extension of α.

Suppose now that $\delta\colon \langle X\rangle \to A$ is another homomorphism extending α. Then for all $y \in \langle X\rangle$, there exist $t \in \mathcal{T}^{\mathscr{A}}$ and $x_1, \dots, x_k \in X$ such that $y = t(x_1, \dots, x_k)$. From this we get that

$$\begin{aligned} y\delta &= t(x_1, \dots, x_k)\delta = t(x_1\delta, \dots, x_k\delta) \\ &= t(x_1\alpha, \dots, x_k\alpha) = t(x_1\overline{\alpha}, \dots, x_k\overline{\alpha}) \\ &= t(x_1, \dots, x_k)\overline{\alpha} = y\overline{\alpha}. \end{aligned}$$

Thus $\delta = \overline{\alpha}$ and any extension is unique.

2) Suppose now that X is a basis of the subalgebra B and extend it to a basis $X \sqcup Y$ of A. Let $\alpha = \beta|_X$ be defined as in part 1). Then $\beta = \overline{\alpha} = \gamma|_B$ and $\operatorname{im}\gamma = A\gamma = \langle X \sqcup Y\rangle\gamma = \langle (X \sqcup Y)\gamma\rangle = \langle X\beta\rangle = \langle X\rangle\beta = \operatorname{im}\beta$, as claimed. □

If the independent set in Lemma I.4.25 is in fact a basis of A, then the uniqueness of the extension gives us the immediate corollary.

Corollary I.4.26. *Two endomorphisms α and β that agree on a basis of A are equal.*

Remark I.4.27. In view of Lemma I.4.25 and Corollary I.4.26, we can identify without ambiguity a map defined on a basis of $\mathscr{A}$ with its unique endomorphism extension using the free basis property. When the map is only defined on an independent set that is not maximal, we can still use Lemma I.4.25 to extend it first to a homomorphism $\overline{\alpha}$, and then to an endomorphism $\overline{\overline{\alpha}}$. Even though in this last case we lose uniqueness of the extension, we will usually drop the notation $\overline{\alpha}$ and $\overline{\overline{\alpha}}$, and will also write α for its homomorphism or endomorphism extension.

We now look at results when the map we want to extend has the extra property that it is injective.

Lemma I.4.28. *Let $X \subseteq A$ and $\alpha \in \operatorname{End}(\mathscr{A})$. If $\alpha\colon X \to X\alpha$ is injective and $X\alpha$ is independent, then X is independent.*

Proof. Suppose that X is dependent but $X\alpha$ is independent. Then there exist distinct elements $x, x_1, \dots, x_k \in X$ and a term $t \in \mathcal{T}^{\mathscr{A}}$ such that $x = t(x_1, \cdots, x_k)$. Since α is injective, $x\alpha, x_1\alpha, \dots, x_k\alpha$ are all distinct, and since it is an endomorphism, it follows that $x\alpha = t(x_1, \dots, x_k)\alpha = t(x_1\alpha, \dots, x_k\alpha)$, contradicting the independence of the set $X\alpha$. □

Remark I.4.29. Before giving a corollary of Lemma I.4.28, we make an observation concerning terms built from independent sets. Let $X = \{x_1, \dots, x_n\} \subseteq A$ an independent set and suppose that $t, s \in \mathcal{T}^{\mathscr{A}}$ are such that $t(x_1, \dots, x_n) = s(x_1, \dots, x_n)$. Then for any $a_1, \dots, a_n \in A$, we define $\alpha \colon X \to A$ by $x_i\alpha = a_i$, and extend it to an endomorphism. Then, we get that

$$t(a_1, \dots, a_n) = t(x_1\alpha, \dots, x_n\alpha) = t(x_1, \dots, x_n)\alpha = s(x_1, \dots, x_n)\alpha = s(a_1, \dots, a_n),$$

which shows that the terms t and s agree on all n-tuples of A.

Corollary I.4.30. *Let X be an independent set and $\alpha \colon X \to A$ be injective. If $X\alpha$ is independent, then the extension of α to a homomorphism is injective.*

Proof. Suppose that there exist $a, b \in \langle X \rangle$ such that $a\overline{\alpha} = b\overline{\alpha}$ where $\overline{\alpha}$ is the extension of α. Then we have that $t(x_1, \dots, x_n)\overline{\alpha} = s(t_1, \dots, x_n)\overline{\alpha}$ for some $t, s \in \mathcal{T}^{\mathscr{A}}$ and $x_1, \dots, x_n \in X$, that is, $t(x_1\alpha, \dots, x_n\alpha) = s(x_1\alpha, \dots, x_n\alpha)$. By assumption of $X\alpha$ being independent, we can use Remark I.4.29 to get that $t(x_1, \dots, x_n) = s(x_1, \dots, x_n)$, that is, $a = b$ and $\overline{\alpha}$ is injective. □

In view of Lemma I.4.28, we have a well-defined notion of preimage basis as follows.

Definition I.4.31. Let $\alpha \in \mathrm{End}(\mathscr{A})$. A set $X \subseteq A$ is a *preimage basis of* α if α is injective on X and $X\alpha$ is a basis of $\operatorname{im}\alpha$.

Thus a set X can only be a preimage basis of a map $\alpha \in \mathrm{End}(\mathscr{A})$ if it is independent in the first place.

Remark I.4.32. For a basis Y of $\operatorname{im}\alpha$, define for each $y \in Y$ an element $x_y \in A$ such that $x_y\alpha = y$. Then α is injective on the set $X = \{x_y : y \in Y\}$, which shows that X is a preimage basis of α.

Moreover, we have that $|X| = |Y| = \operatorname{rank}(Y) = \operatorname{rank}(\operatorname{im}\alpha) = \operatorname{rk}(\alpha)$.

Notation I.4.33. Under the consideration of the remark above, given $\alpha \in \mathrm{End}(\mathscr{A})$, we will say "let $X\alpha$ be a basis of $\operatorname{im}\alpha$", to mean that we pick X to be a preimage basis of α.

We finish this section by proving that any two isomorphic subalgebras of an independence algebra must have the same rank, and that the converse also holds.

Lemma I.4.34. *Let $\mathscr{B}$ and $\mathscr{C}$ be subalgebras of $\mathscr{A}$. Then $\mathscr{B} \cong \mathscr{C}$ if and only if* $\operatorname{rank} B = \operatorname{rank} C$.

Proof. Let $\theta\colon \mathscr{B} \to \mathscr{C}$ be an isomorphism, and let $X\theta$ be a basis of $\operatorname{im}\theta = C$. Then $\operatorname{rank} C = |X\theta| = |X| \leq \operatorname{rank} B$ since $X \subseteq B$. Dually, $\operatorname{rank} B \leq \operatorname{rank} C$ using a basis of $\operatorname{im}\theta^{-1}$, and we get that $\operatorname{rank} B = \operatorname{rank} C$.

Conversely, suppose that $\operatorname{rank} B = \operatorname{rank} C$, and let X and Y be respective bases of B and C, so that $\alpha\colon X \to Y$ is a bijection. By Lemma I.4.25 extend α to the homomorphism $\theta\colon \langle X\rangle \to A$. Since $\operatorname{im}\theta = \langle X\rangle\theta = \langle X\theta\rangle = \langle X\alpha\rangle = \langle Y\rangle = C$, it follows that θ is onto. Moreover, since $Y = X\alpha$ is an independent set, we get that θ is injective by Corollary I.4.30. Thus θ is an isomorphism from B to C. □

Remark I.4.35. It follows from Lemma I.4.34 that all one-dimensional subalgebras of $\mathscr{A}$ are isomorphic. Thus, either all one-dimensional subalgebras are singletons, or they all have at least two distinct elements.

I.4.3 Additional notation

We close this introduction on independence algebras by adding to the notations presented in Section I.1 in order to facilitate the exposition of the proofs coming in subsequent chapters. All sets and elements considered below will belong to the universe of an independence algebra $\mathscr{A}$.

Closed sets and bases. By writing $C = \langle b_1, b_2\rangle$, we mean that C is the closed subset of A generated by b_1 and b_2. If we write $C = \langle\{b_1, b_2\}\rangle$ instead, it is assumed that the set $\{b_1, b_2\}$ also forms a basis of C.

Similarly, by $B = \langle\{x_i\} \sqcup \{y_j\}\rangle$ we mean that the sets $\{x_i\}$ and $\{y_j\}$ are independent sets and that together they form a basis of B. Moreover, if $D \subseteq B$ has a known basis W, we will abuse notation and write $B = D \sqcup \langle\{z_k\}\rangle$ to signify that D is a subalgebra of B and that the basis W of D can be extended to a basis of B through $\{z_k\}$.

Endomorphisms. If $A = \langle\{x_i\} \sqcup \{y_j\}\rangle$ and $b_i, d_j \in A$ for all $i \in I$ and $j \in J$, when writing $\alpha = \begin{pmatrix} x_i & y_j \\ b_i & d_j \end{pmatrix}$ we will assume that this map is an endomorphism without further considerations, since we know by Lemma I.4.25 that the map α defined above can be extended to a unique endomorphism of $\mathscr{A}$. Notice also that the sets $\{b_i\}$ and $\{d_j\}$ are not necessarily independent nor disjoint.

Terms. Given an independent set $\{x_i\}$ and $a \in \langle\{x_i\}\rangle$, we will write $a = t(\overline{x_i})$ for some term $t \in \mathcal{T}^{\mathscr{A}}$ to mean that there exists a finite independent subset $\{x_{i_1}, \dots, x_{i_k}\} \subseteq \{x_i\}$ and a k-ary term t such that $a = t(x_{i_1}, \dots, x_{i_k})$. In particular, if $\{x_i\}$ is infinite, this notation allows us to write terms concisely without requiring the subset to be made explicit.

I.5 ENDOMORPHISM MONOIDS OF INDEPENDENCE ALGEBRAS

Since this thesis is concerned with endomorphism monoids, we finish this preliminary chapter with a section giving some properties of the endomorphism monoid of an independence algebra which will be useful in the coming work. All of the results contained in this section will be given without proof and can be found in Section 4 of [22].

One of the most important structural properties of the endomorphism monoid of an independence algebra is the following.

Proposition I.5.1. *Let $\mathscr{A}$ be an independence algebra. Then* $\mathrm{End}(\mathscr{A})$ *is regular.*

I.5.1 Ideals and Green's relations

Throughout this section, $\mathscr{A}$ denotes an independence algebra of dimension κ.

Since ranks of endomorphisms correspond to ranks of subuniverses of A, it follows that for all $\alpha \in \mathrm{End}(\mathscr{A})$ we have $\mathrm{rk}(\alpha) \leq \kappa$ and $\mathrm{rk}(\alpha) \geq e$, where e is the smallest rank of a subalgebra of $\mathscr{A}$ as described in Notation I.4.15. The following lemma gives us that there is an endomorphism of each rank between these two bounds.

Lemma I.5.2. *Suppose that* $\dim \mathscr{A} = \kappa$ *and let μ be a cardinal such that $e \leq \mu \leq \kappa$. Then there exist a subalgebra $\mathscr{B} \subseteq \mathscr{A}$ of rank μ and a map $\alpha \in \mathrm{End}(\mathscr{A})$ such that* $\mathrm{im}\,\alpha = B$*, so that* $\mathrm{rk}(\alpha) = \mu$*.*

We now concentrate on particular subsets of $\mathrm{End}(\mathscr{A})$. Recall that κ^+ denotes the successor cardinal of κ. For each $e < \mu \leq \kappa^+$, we define the set $T_\mu \subseteq \mathrm{End}(\mathscr{A})$ by

$$T_\mu = \{\alpha \in \mathrm{End}(\mathscr{A}) \mid \mathrm{rk}(\alpha) < \mu\}\,.$$

Under this notation, it is clear that $T_{\kappa^+} = \mathrm{End}(\mathscr{A})$. Similarly, we have that the set $T_1 = \{\alpha \in \mathrm{End}(\mathscr{A}) \mid \mathrm{im}\,\alpha = \langle\emptyset\rangle\}$, which is non-empty if and only if $\mathscr{A}$ has constants.

Remark I.5.3. Notice that in the definition of T_μ, we take all maps of rank strictly less that μ. This is necessary if $\mathscr{A}$ is infinite dimensional since we can then have limit cardinals. In the case where $\mathscr{A}$ is finite dimensional, then $\kappa^+ < \aleph_0$ and each μ is the successor of a finite cardinal, so that we can replace each ideal T_μ by the corresponding ideal $\overline{T}_{\mu-1} := \{\alpha \in \mathscr{A} \mid \mathrm{rk}(\alpha) \leq \mu - 1\}$.

Lemma I.5.4. *For all $\alpha, \beta \in \mathrm{End}(\mathscr{A})$, we have $\mathrm{rk}(\alpha\beta) \leq \min\{\mathrm{rk}(\alpha), \mathrm{rk}(\beta)\}$.*

Consequently, T_μ is an ideal of $\mathrm{End}(\mathscr{A})$ for all $e < \mu \leq \kappa^+$.

Remark I.5.5. Notice that if $T_1 \neq \emptyset$, then it is a left-zero semigroup. Indeed, if $\alpha \in T_1$ and $\beta \in \mathrm{End}(\mathscr{A})$, then for all $a \in A$, we have $a\alpha \in \langle\emptyset\rangle$. Since β is an endomorphism, it must fix constants, and thus $(a\alpha)\beta = a\alpha$, so that $\alpha\beta = \alpha$. Thus α is a left-zero of $\mathrm{End}(\mathscr{A})$ and T_1 is a left-zero semigroup.

We now give a description of Green's relations on $\mathrm{End}(\mathscr{A})$ as follows:

Proposition I.5.6. *Let $\alpha, \beta \in \mathrm{End}(\mathscr{A})$. Then we have the following:*

1) *$\alpha \mathscr{L} \beta$ if and only if $\mathrm{im}\,\alpha = \mathrm{im}\,\beta$;*
2) *$\alpha \mathscr{R} \beta$ if and only if $\ker\alpha = \ker\beta$;*
3) *$\alpha \mathscr{D} \beta$ if and only if $\mathrm{rk}(\alpha) = \mathrm{rk}(\beta)$;*
4) *$\mathscr{J} = \mathscr{D}$.*

Since ideals are unions of $\mathscr{J}$-classes, we directly have the following corollary.

Corollary I.5.7. *Any ideal of $\mathrm{End}(\mathscr{A})$ is of the form T_μ for some cardinal $e < \mu \leq \kappa^+$. Consequently, the ideals of $\mathrm{End}(\mathscr{A})$ form a chain.*

Remark I.5.8. It follows from Corollary I.5.7 that the minimal ideal of $\mathrm{End}(\mathscr{A})$ is T_{e+1}, that is, it consists of all maps of rank 0 if the algebra $\mathscr{A}$ contains any constants and otherwise of all maps of rank 1.

I.5.2 Classification up to equivalence

We close this section by giving a classification of endomorphism monoids of independence algebras provided in [2]. Since all the work involving independence algebras in this thesis only relate to their endomorphism monoids, we will indeed want to consider two independence algebras $\mathscr{A}$ and $\mathscr{B}$ as equivalent if their endomorphism monoids are isomorphic. Formally, this is given by the following definition.

Definition I.5.9. Two algebras $\mathscr{A} = (A, F)$ and $\mathscr{B} = (B, F')$ are called *E-equivalent* if there exists a bijection $\theta\colon A \to B$ such that the mapping $\alpha \mapsto \theta^{-1} \circ \alpha \circ \theta$ is an isomorphism from $\operatorname{End}(\mathscr{A})$ to $\operatorname{End}(\mathscr{B})$.

The classification of independence algebras up to E-equivalence is then given by the following:

Proposition I.5.10 ([Theorem 2.10 of 2])**.** *Any independence algebra $\mathscr{A} = (A, F)$ is E-equivalent to a free group action algebra, a quasifield algebra, a linear algebra, or an affine algebra.*

We now describe these classes of independence algebras.

Free group action algebra. Let G be a group of permutations of a non-empty set A by an action on the left. Suppose that $C \subseteq A$ is such that all fixed points of any non-identity $g \in G$ are in C and that $gC = C$ for all $g \in G$. A *free group action algebra* is an algebra $\mathscr{A} = (A, F)$, where for each $g \in G$ and $c \in C$, the set F contains a unary operation f_g and a nullary operation f_c defined for all $a \in A$ by

$$f_g(a) = ga \quad \text{and} \quad f_c() = c.$$

With this description, it follows that $g(A \setminus C) = A \setminus C$ and that this action is free, that is, if $g, h \in G$ are such that $ga = ha$ for some $a \in A \setminus C$, then $g = h$. Thus there exists a set $X = \{x_i\} \subseteq A \setminus C$ so that we can split $A \setminus C$ into orbits of the form Gx_i.

This algebra can then be seen as the G-set

$$F_{X \sqcup C}(G) = C \sqcup \bigsqcup_{x_i \in X} Gx_i,$$

where $gx_i = hx_j$ if and only if $x_i = x_j$ and $g = h$, and the elements of C are the constants of the algebra.

Remark I.5.11. If G is trivial and C is empty, then the corresponding free group action algebra is simply a set with no operations.

Quasifield algebra. Let Q be a non-empty set with two binary operations, a multiplication $\cdot$ and a subtraction $-$. The set Q is a *quasifield* if there exists distinct elements $0, 1 \in Q$ such that $(Q, \cdot)$ is a semigroup with zero 0, the algebra $\left(Q \setminus \{0\}, \{\cdot, {}^{-1}, 1\}\right)$ is a group, and for all $x, y, z \in Q$ we have the following:

1) $x - 0 = x$,

2) $x(y-z) = xy - xz$, and

3) $x-(y-z) = \begin{cases} z & \text{if } x = y, \\ (x-y)-(x-y)(y-x)^{-1}z & \text{otherwise.} \end{cases}$

A k-ary operation f on a quasifield Q is said to be *Q-homogeneous* if for every $s, t, x_1 \dots, x_k \in Q$ we have $f(s - tx_1, s - tx_2, \dots, s - tx_k) = s - tf(x_1, \dots, x_k)$.

A *quasifield algebra* is an algebra $\mathcal{Q} = (Q, F)$ where Q is a quasifield, all operations of F are Q-homogeneous and F contains all binary Q-homogeneous operations.

Remark I.5.12. In a quasifield algebra $\mathcal{Q}$, there are no nullary operations and the only unary operation is the trivial operation. Indeed, if f is a Q-homogeneous unary operation and $x \in Q$, then $f(x) = f(x - 0 \cdot 1) = x - 0f(1) = x$. Thus, every element of Q is a closed independent set, so that the one-dimensional subalgebras of $\mathcal{Q}$ are all the singletons.

Linear algebra. Let V be a vector space over a division ring $\mathcal{k}$, and V_0 a subspace of V. By a *linear algebra*, we mean an algebra $\mathcal{V} = (V, F)$ where F consists of the following operations:

$$f(x, y) = x + y, f_\mu(x) = \mu x, \text{and } f_w() = w,$$

where $x, y \in V$, $\mu \in \mathcal{k}$ and $w \in V_0$. It is easy to see that under this definition, we have that any k-ary term $t \in \mathcal{T}^{\mathcal{V}}$ is of the form

$$t(x_1, \dots, x_k) = \sum_{i=1}^{k} \mu_i x_i + w,$$

for some $\{\mu_i\} \subseteq \mathcal{k}$ and $w \in V_0$. Moreover, if $V_0 = \{0\}$ is the trivial subspace and $\mathcal{k}$ a field, it follows that this algebra coincides with the classical notion of vector space.

Affine algebra. Let A be a non-trivial vector space over a division ring $\mathcal{k}$ and A_0 a subspace of A. An *affine algebra* is defined as an algebra $\mathcal{A} = (A, F)$ with operations

$$f(x_1, \dots, x_k) = \sum_{i=1}^{k} \lambda_i x_i + a,$$

where $k \geq 1$, the scalars $\lambda_i \in \mathcal{k}$ are such that $\sum_{i=1}^{k} \lambda_i = 1$ and $a \in A_0$.

Remark I.5.13. In an affine algebra, all unary operations are of the form $f(x) = x + a$ for some $a \in A_0$. Moreover, there are no nullary operations, since in the definition, we require $k \geq 1$. This means in particular that if A_0 is the trivial subspace of A, then all one-dimensional subalgebras are singletons.

— II —

An introduction to semigroups of endomorphisms with restricted range

In this chapter, we introduce a type of subsemigroup of the endomorphism monoid of an independence algebra and we make initial study of its structure. Most of the results presented here are already known for some types of independence algebras such as sets and vector spaces, or in the context of principal one-sided ideals. We only expand upon this work and generalise it to the context of independence algebras. For this reason, the origin of the ideas we use throughout the proofs may be traced back in the literature to existing work [35, 37, 38, 47, 50]. Some results may also be deduced from [12] by considering results on Green's relations for principal one-sided ideals of a semigroup. Some new results are also presented here for use in the next chapter.

This chapter is structured as follows: we start in Section II.1 by defining the semigroup $T(\mathscr{A},\mathscr{B})$ which will be the object of our study, after giving a brief overview of the special cases previously studied. We will then compare $T(\mathscr{A},\mathscr{B})$ to the endomorphism monoid $\mathrm{End}(\mathscr{A})$ in Section II.2, to show that these are very different objects. In Section II.3 we will focus on the regular elements of $T(\mathscr{A},\mathscr{B})$ before describing its Green's relations in Section II.4 and its ideal structure in Section II.5.

Note. Content from this chapter has already appeared in a paper [24], but we give more details and additional results here.

II.1 HISTORICAL CONSIDERATIONS

The full transformation monoid $\mathcal{T}_X$ of a set X, the monoid of linear transformations $\operatorname{End}(V)$ on a vector space V, as well as their generalisation to the endomorphism monoid $\operatorname{End}(\mathscr{A})$ of an independence algebra $\mathscr{A}$, have been the focus of a considerable amount of research during the last decades. Following Malcev's work on the automorphism group of $\mathcal{T}_X$, Symons [51] investigated the automorphism group of the subsemigroup of $\mathcal{T}_X$ consisting of all maps with range restricted to a subset $Y \subseteq X$, which he denoted by $T(X,Y)$. Later, Nenthein, Youngkhong and Kemprasit [38] started the study of the properties of $T(X,Y)$ which led to similar studies of $T(V,W)$, the semigroup of linear transformations of a vector space V with range restricted to a subspace W. As noticed in Section I.4, both sets and vector spaces are examples of independence algebras. Our aim here is to put the work on $T(X,Y)$ and $T(V,W)$ into the general context they provide.

Let $\mathscr{A}$ be an independence algebra, and $\mathscr{B}$ a subalgebra of $\mathscr{A}$. We make no global assumptions concerning the cardinalities of $\mathscr{A}$ and $\mathscr{B}$: these may be either finite or infinite dimensional algebras.

Definition II.1.1. The *semigroup of endomorphisms of* $\mathscr{A}$ *with restricted range in* $\mathscr{B}$ is the set

$$T(\mathscr{A},\mathscr{B}) = \{\alpha \in \operatorname{End}(\mathscr{A}) \mid \operatorname{im}\alpha \subseteq \mathscr{B}\},$$

with product inherited from that of $\operatorname{End}(\mathscr{A})$.

It is clear from the definition that if $\mathscr{B} = \emptyset$ then $T(\mathscr{A},\mathscr{B}) = \emptyset$ and we will therefore exclude this case. Otherwise, the set $T(\mathscr{A},\mathscr{B})$ is easily seen to be a subsemigroup of $\operatorname{End}(\mathscr{A})$, and if $\mathscr{B} = \mathscr{A}$ it is equal to the monoid $\operatorname{End}(\mathscr{A})$. In fact, we will prove in Section II.2 that unless $\mathscr{B} = \mathscr{A}$ or $\mathscr{B}$ is a singleton, then $T(\mathscr{A},\mathscr{B})$ is not isomorphic to the endomorphism monoid on any independence algebra. Thus $T(\mathscr{A},\mathscr{B})$ holds a special place in the study of subsemigroups of $\operatorname{End}(\mathscr{A})$ and is of particular interest.

In the case where our algebra $\mathscr{A}$ is simply a set, Green's relations on $T(\mathscr{A},\mathscr{B})$ have been studied by Sanwong and Sommanee [47]. In the case where $\mathscr{A}$ is a vector space, Nenthein, Youngkhong and Kemprasit [37, 38] determined the regular elements and then Sullivan [50] studied Green's relations. Subsequently, Mendes-Gonçalves and Sullivan [35, 50] gave the structure of the ideals of $T(\mathscr{A},\mathscr{B})$ for both the set and the vector space case. For each cardinal r, the set $\{\alpha \in T(\mathscr{A},\mathscr{B}) \mid \operatorname{rank}\alpha < r\}$ is an ideal of $T(\mathscr{A},\mathscr{B})$. However, unlike the case for $\mathrm{End}(\mathscr{A})$ itself, these ideals are not in general the only ones present in $T(\mathscr{A},\mathscr{B})$, nor do they form a chain. In [35, 50], the authors used the ideal structure when the dimension of the subalgebra $\mathscr{B}$ is at least 3 in order to construct two ideals that are not comparable under containment. In this way, they showed a weaker version of Corollary II.2.3, namely, that if $\dim \mathscr{B} \geq 3$ then $T(\mathscr{A},\mathscr{B})$ cannot be isomorphic to $\mathrm{End}(\mathscr{C})$ for any $\mathscr{C}$ a set or a vector space, as appropriate, since ideals of the latter always form a chain. We show via a more direct route that there cannot be such an isomorphism for any independence algebras apart from the trivial cases. Nevertheless, we will give similar examples of ideals not forming a chain to give the reader an idea on this particular behaviour.

Another way to study the semigroup $T(\mathscr{A},\mathscr{B})$ is to use the fact that this semigroup corresponds to the principal left ideal of $\mathrm{End}(\mathscr{A})$ generated by any idempotent with image the subalgebra $\mathscr{B}$. Such an approach has been taken by East [12], where he investigated some aspects of the structure of principal one-sided ideals of a semigroup. In particular, the subset of regular elements, Green's relations, as well as the idempotent-generated subsemigroup of principal one-sided ideals were exhibited. Even though some of the proofs contained in this chapter can be seen as a special case of the results of East, the context of independence algebras allows us to give explicit constructions, and we also develop other properties which will be used throughout Chapter III. From now on, $\mathscr{A}$ will denote a general independence algebra, and $\mathscr{B}$ a non-empty subalgebra of $\mathscr{A}$.

II.2 INITIAL COMPARISONS WITH THE ENDOMORPHISM MONOID

As mentioned earlier, the semigroup $T(\mathscr{A},\mathscr{B})$ is not a monoid, unless the subalgebra $\mathscr{B}$ has some specific conditions, as given by the following lemma:

Lemma II.2.1. *The semigroup $T(\mathscr{A},\mathscr{B})$ is a monoid if and only if $\mathscr{B} = \mathscr{A}$, or $\mathscr{B}$ is a singleton.*

Proof. Clearly if $\mathscr{B} = \mathscr{A}$ then $T(\mathscr{A},\mathscr{B}) = \mathrm{End}(\mathscr{A})$, and if $\mathscr{B}$ is a singleton, say $\{b\}$, then $T(\mathscr{A},\mathscr{B}) = \{c_b\}$, where c_b is the constant map with value b. Moreover, both of these are indeed monoids.

Let us now assume that $\mathscr{B} \subsetneq \mathscr{A}$ and let $b_1, b_2 \in B$. Since $\mathscr{A}$ is an independence algebra and $\mathscr{B} \neq \mathscr{A}$, there exists an independent element $a \in A \setminus B$. Let X be a basis of B and extend it to a basis $X \sqcup \{a\} \sqcup Y$ of A, where Y is possibly empty. We now define two maps $\alpha, \beta \in \operatorname{End}(\mathscr{A})$ as follows:

$$\alpha = \begin{pmatrix} x_i & a & y_j \\ x_i & b_1 & b_1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} x_i & a & y_j \\ x_i & b_2 & b_1 \end{pmatrix}.$$

Clearly both $\operatorname{im}\alpha$ and $\operatorname{im}\beta$ lie in B, which means that α and β belong to $T(\mathscr{A},\mathscr{B})$. Suppose that $T(\mathscr{A},\mathscr{B})$ is a monoid, and denote its identity by ε. Now let $c \in B$ be such that $a\varepsilon = c$. Since ε is a left identity for α and $\alpha|_B$ is the identity on B, we need $b_1 = a\alpha = a\varepsilon\alpha = c\alpha = c$. Similarly, ε is a left identity for β and thus we have that $b_2 = a\beta = a\varepsilon\beta = c$. Therefore $b_1 = b_2$, which means that $\mathscr{B}$ is a singleton as these elements were taken arbitrarily. This shows that for any proper subalgebra $\mathscr{B}$ with at least two distinct elements, the semigroup $T(\mathscr{A},\mathscr{B})$ does not contain a two-sided identity, and thus is not a monoid. □

Remark II.2.2. Notice that the proof shows a stronger result, namely that there is no general left-identity in $T(\mathscr{A},\mathscr{B})$ unless $\mathscr{A} = \mathscr{B}$ or $\mathscr{B}$ is a singleton.

This, together with the fact that the trivial monoid is the endomorphism monoid of an independence algebra with a single element, directly gives us the following result:

Corollary II.2.3. *The semigroup $T(\mathscr{A},\mathscr{B})$ is isomorphic to $\operatorname{End}(\mathscr{C})$ for $\mathscr{C}$ an independence algebra if and only if $\mathscr{B} = \mathscr{A}$, or $\mathscr{B}$ is a singleton.*

In spite of the fact that $T(\mathscr{A},\mathscr{B})$ has exhibited a very basic difference from $\operatorname{End}(\mathscr{A})$, by virtue of not being a monoid, some results are inherited directly from the structure of the independence algebra and are thus similar between these two objects. We give here two results that emphasize this fact, and that will be used later.

Notation II.2.4. Following our notation for elements of $\operatorname{End}(\mathscr{A})$ in Section I.4.3, for $\mathscr{A} = \langle \{x_i\} \sqcup \{y_j\} \rangle$ when we say that $\alpha \in T(\mathscr{A},\mathscr{B})$ is defined by $\alpha = \begin{pmatrix} x_i & y_j \\ c_i & d_j \end{pmatrix}$, it suffices to verify that $\{c_i\}, \{d_j\} \subseteq B$ as then $\operatorname{im}\alpha = \langle c_i, d_j \rangle \subseteq B$.

Lemma II.2.5. *Let $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$ be such that $\operatorname{im}\alpha \cong \operatorname{im}\beta$. Then there exist $\gamma, \mu \in T(\mathscr{A},\mathscr{B})$ such that:*

- $\operatorname{im}\gamma = \operatorname{im}\beta$ *and* $\ker\gamma = \ker\alpha$*; and*

- $\operatorname{im}\mu = \operatorname{im}\alpha$ *and* $\ker\mu = \ker\beta$.

Proof. Since $\operatorname{im}\alpha \cong \operatorname{im}\beta$, there exists an isomorphism $\phi\colon A\alpha \to A\beta$. Define $\gamma, \mu \in T(\mathscr{A}, \mathscr{B})$ by $\gamma = \alpha\phi$ and $\mu = \beta\phi^{-1}$. Then, $\operatorname{im}\gamma = A\gamma = (A\alpha)\phi = A\beta = \operatorname{im}\beta$ and similarly $\operatorname{im}\mu = \operatorname{im}\alpha$. Additionally, since ϕ is injective, we get that for all $a, b \in A$

$$(a, b) \in \ker\gamma \iff a\alpha\phi = b\alpha\phi \iff a\alpha = b\alpha \iff (a, b) \in \ker\alpha,$$

so that $\ker\gamma = \ker\alpha$ and using similar arguments we have that $\ker\mu = \ker\beta$ which concludes the proof. □

Lemma II.2.6. *Let $\mathscr{C}$ be a non-empty subalgebra of $\mathscr{B}$. Then there exists an idempotent $\eta \in T(\mathscr{A}, \mathscr{B})$ such that $\operatorname{im}\eta = \mathscr{C}$.*

Consequently, for all $\alpha \in T(\mathscr{A}, \mathscr{B})$, there exists $\eta = \eta^2 \in T(\mathscr{A}, \mathscr{B})$ such that $\operatorname{im}\eta = \operatorname{im}\alpha$ and thus $\alpha\eta = \alpha$.

Proof. Let $\mathscr{C}$ be a non-empty subalgebra of $\mathscr{B}$. Then $\mathscr{C}$ is also a subalgebra of $\mathscr{A}$ and it follows from Lemma I.4.24 that there exists an idempotent $\eta \in T(\mathscr{A}, \mathscr{B})$ such that $\operatorname{im}\eta = C \subseteq B$. Thus $\eta \in T(\mathscr{A}, \mathscr{B})$.

Let $\alpha \in T(\mathscr{A}, \mathscr{B})$ and $\mathscr{C} = \operatorname{im}\alpha$. Then by taking η as above, we have that $\operatorname{im}\eta = \operatorname{im}\alpha$. Moreover, $\eta|_{\operatorname{im}\eta} = \operatorname{id}|_{\operatorname{im}\eta}$ by Lemma I.4.24 again, so that $\alpha\eta = \alpha$ as claimed. □

II.3 REGULAR ELEMENTS

We have already mentioned in Proposition I.5.1 that the endomorphism monoid of any independence algebra is regular. Thus, in the case where $\mathscr{B} = \mathscr{A}$, it follows that $T(\mathscr{A}, \mathscr{B})$ is regular. However, in general $T(\mathscr{A}, \mathscr{B})$ is not regular and we list explicitly the cases where it is. Showing that all other cases do not give a regular semigroup will be the purpose of Corollary II.3.3, and the remainder of this section will be devoted to proving useful results on the presence of regular elements in our semigroup.

Assume first that $\langle\emptyset\rangle \neq \emptyset$ and take $\mathscr{B} = \langle\emptyset\rangle$. Then $T(\mathscr{A}, \mathscr{B})$ is a left-zero semigroup, and is thus regular. Indeed, since any endomorphism has to act as the identity on $\langle\emptyset\rangle$, it follows that for any $\alpha, \beta \in T(\mathscr{A}, \mathscr{B})$ we have that $\operatorname{im}\alpha = \langle\emptyset\rangle = \mathscr{B}$ and since $\beta|_B = \operatorname{id}|_B$, we obtain that $\alpha\beta = \alpha$.

Assume now that $\langle\emptyset\rangle = \emptyset$ and $\dim B = 1$. We show that in this case $T(\mathscr{A},\mathscr{B})$ is also regular. In order to see this, we let $\{b\}$ to be a basis of B which is then extended to a basis $\{b\} \sqcup \{x_i\}$ of A. Since $\mathscr{A}$ has no constants, by Lemma I.4.21 and I.4.23, we know that the set of unary terms is a group, which we denote G. Moreover, since the only subalgebra of $\mathscr{B}$ is $\mathscr{B}$ itself, we get that any element of B can be written as $g(b)$ for some $g \in G$. Let $\alpha \in T(\mathscr{A},\mathscr{B})$. Then, $b\alpha = g_b(b)$ for some $g_b \in G$, and for all $i \in I$ we also have $x_i\alpha = g_i(b)$ for some $g_i \in G$. If we define $\beta \in T(\mathscr{A},\mathscr{B})$ by $b\beta = g_b^{-1}(b)$ and $x_i\beta = g_i(b)$ for all $i \in I$, then we get that $b\beta\alpha = g_b^{-1}(g_b(b)) = b$ and thus

$$x_i\alpha\beta\alpha = g_i(b)\beta\alpha = g_i(b\beta\alpha) = g_i(b) = x_i\alpha \quad \text{and}$$
$$b\alpha\beta\alpha = g_b(b\beta\alpha) = g_b(b) = b\alpha,$$

so that $\alpha\beta\alpha = \alpha$ and therefore $T(\mathscr{A},\mathscr{B})$ is regular.

Remark II.3.1. In the above case where $\langle\emptyset\rangle = \emptyset$ and $\dim B = 1$, if our algebra $\mathscr{A}$ is finite dimensional, say with rank n, we can also realise the semigroup $T(\mathscr{A},\mathscr{B})$ as a wreath product as follows. Taking G, b and $X = \{x_2, \dots, x_n\}$ as before, we set $Y = \{b\} \sqcup X$ as a basis of A and we let $c_1 \in \mathcal{T}_n$ be the constant map with image the element 1, that is, $kc_1 = 1$ for all $k \in \{1, \dots, n\}$. Define $W = G \wr_n \{c_1\}$ to be the semigroup on $G^n \times \{c_1\}$, where for any $g_1, \dots, g_n, h_1, \dots, h_n \in G$ the multiplication is given by

$$\begin{aligned}((g_1, \dots, g_n), c_1)\,((h_1, \dots, h_n), c_1) &= ((g_1 h_{1c_1}, \dots, g_n h_{nc_1}), c_1)\\ &= ((g_1 h_1, \dots, g_n h_1), c_1)\,.\end{aligned}$$

Then the map $\phi\colon W \to T(\mathscr{A},\mathscr{B})$ which sends the element $((g_1, \dots, g_n), c_1)$ to the map $\alpha_g = \begin{pmatrix} b & x_i \\ g_1(b) & g_i(b) \end{pmatrix} \in T(\mathscr{A},\mathscr{B})$ is an isomorphism. To see that, write $\bar{g}$ as a shorthand for $((g_1, \dots, g_n), c_1)$, and suppose that $\bar{g}\phi = \bar{h}\phi$, that is $\alpha_g = \alpha_h$. This means in particular that for any $x_i \in X$, we have that $g_i(b) = x_i\alpha_g = x_i\alpha_h = h_i(b)$, and similarly, $g_1(b) = b\alpha_g = b\alpha_h = h_1(b)$. Since these unary terms are equal on the independent set $\{b\}$, they must be equal everywhere by Remark I.4.29. Thus $g_y = h_y$, which shows that $\bar{g} = \bar{h}$ and therefore ϕ is injective. It is obvious that ϕ is also surjective, and thus it is a bijection. Additionally, for any $x_i \in X$ we have that $x_i(\bar{g}\phi\bar{h}\phi) = x_i\alpha_g\alpha_h = g_i(b)\alpha_h = g_i h_1(b) = x_i(\bar{g}\bar{h})\phi$, and similarly $b(\bar{g}\phi\bar{h}\phi) = b(\bar{g}\bar{h})\phi$ which shows that ϕ is a morphism, and thus it is an isomorphism of W onto $T(\mathscr{A},\mathscr{B})$.

We now want to show that unless we are in the cases above, the semigroup $T(\mathscr{A},\mathscr{B})$ is not regular. Notice first that if $\alpha \in T(\mathscr{A},\mathscr{B})$ is a regular element, then there exists $\gamma \in T(\mathscr{A},\mathscr{B})$ such that $\alpha = \alpha\gamma\alpha$ and from this, we obtain that

$$A\alpha = (A\alpha\gamma)\alpha \subseteq B\alpha.$$

Following [50], we define the set Q which will contain all regular elements by

$$Q = \{\alpha \in T(\mathscr{A},\mathscr{B}) \mid A\alpha \subseteq B\alpha\}.$$

It is clear that for any element $\alpha \in T(\mathscr{A},\mathscr{B})$, the condition $A\alpha \subseteq B\alpha$ on the elements of Q can be rewritten as $A\alpha = B\alpha$ or as $(A \setminus B)\alpha \subseteq B\alpha$, and these equivalent conditions will be equally used to define an element of Q.

We can now show that the regular elements of $T(\mathscr{A},\mathscr{B})$ are exactly those of the set Q defined above.

Proposition II.3.2. *The set Q consists of all the regular elements of $T(\mathscr{A},\mathscr{B})$, and is a right ideal of $T(\mathscr{A},\mathscr{B})$.*

Proof. As per the initial remark above, any regular element of $T(\mathscr{A},\mathscr{B})$ lies in Q. Moreover, for any $\alpha \in Q$ and $\beta \in T(\mathscr{A},\mathscr{B})$ we have that

$$A(\alpha\beta) = (A\alpha)\beta \subseteq (B\alpha)\beta = B(\alpha\beta),$$

and thus $\alpha\beta \in Q$ and Q is a right ideal.

Let us now show that any element of Q is regular. Let $\alpha \in Q$ and write $B\alpha = \langle\{b_i\alpha\}\rangle$, so that $\{b_i\} \subseteq B$ is a preimage basis of α. We also take an element $c \in \langle\{b_i\}\rangle$, which exists as $\mathscr{B}$ is non-empty. Since $\operatorname{im}\alpha = A\alpha \subseteq B\alpha$, then for any $a \in A$ there exists a term t such that $a\alpha = t\left(\overline{b_i\alpha}\right) = t\left(\overline{b_i}\right)\alpha$. If we let $A = \langle\{b_i\}\sqcup\{x_j\}\rangle$ and we define $u_j = x_j\alpha = t_j\left(\overline{b_i\alpha}\right)$, then we have that the map α can be written as

$$\alpha = \begin{pmatrix} b_i & x_j \\ b_i\alpha & u_j \end{pmatrix}.$$

Additionally, let $\{a_j\} \subseteq A$ be such that $A = \langle\{b_i\alpha\} \sqcup \{a_j\}\rangle$, and define $\gamma \in T(\mathscr{A},\mathscr{B})$ by:

$$\gamma = \begin{pmatrix} b_i\alpha & a_j \\ b_i & c \end{pmatrix},$$

where $c \in B$ is arbitrary. Then $\gamma \in Q$ and we can see that $\alpha = \alpha\gamma\alpha$ since $b_i\alpha\gamma\alpha = b_i\alpha$ and $x_j\alpha\gamma\alpha = t_j\left(\overline{b_i\alpha}\right)\gamma\alpha = x_j\alpha$, which finishes the proof that any element in Q is regular. $\square$

For ease of use, the set of non-regular maps, that is, the set $T(\mathscr{A},\mathscr{B}) \setminus Q$ will be denoted by Q^c. We can now prove under which condition the semigroup $T(\mathscr{A},\mathscr{B})$ is not regular, that is, when $T(\mathscr{A},\mathscr{B}) \setminus Q \neq \emptyset$.

Corollary II.3.3. *For $\mathscr{B} \neq \mathscr{A}$, the semigroup $T(\mathscr{A},\mathscr{B})$ is not regular if and only if one of the following happen:*

- $\dim B \geq 2$*; or*
- $\dim B = 1$ *and* $\langle \emptyset \rangle \neq \emptyset$.

Proof. If either $\dim B = 1$ and $\langle \emptyset \rangle = \emptyset$, or $B = \langle \emptyset \rangle \neq \emptyset$, then we have shown that $T(\mathscr{A},\mathscr{B})$ is regular at the beginning of Section II.3.

Now suppose that $\dim B \geq 2$ or that $\dim B = 1$ and $\langle \emptyset \rangle \neq \emptyset$. If $\dim B \geq 2$, then there exists an independent set $\{b_1, b_2\} \subseteq B$. On the other hand, if $\dim B = 1$ and $\langle \emptyset \rangle \neq \emptyset$, then there exists an independent element $b_1 \in B$, and a constant $b_2 \in B$. In both cases, we have two elements $b_1, b_2 \in B$ such that $b_1 \notin \langle b_2 \rangle$. Now let $B = \langle \{x_j\} \rangle$, $A = \langle \{x_j\} \sqcup \{y_i\} \rangle$ and define $\alpha \in T(\mathscr{A},\mathscr{B})$ by

$$\alpha = \begin{pmatrix} x_j & y_i \\ b_2 & b_1 \end{pmatrix}.$$

Then we have that $B\alpha = \langle b_2 \rangle \subsetneq \langle b_1, b_2 \rangle = A\alpha$ and thus α is not in Q. □

If the semigroup $T(\mathscr{A},\mathscr{B})$ is regular, then we automatically obtain most of its structure such as Green's relations and the ideals from the study of $\mathrm{End}(\mathscr{A})$. For the remainder of this chapter, we will concentrate on the study of $T(\mathscr{A},\mathscr{B})$ in the cases when it is not regular, that is, we will assume throughout this chapter that $\mathscr{B} \neq \mathscr{A}$ and that either $\dim B \geq 2$ or we have $\dim B = 1$ with $\langle \emptyset \rangle \neq \emptyset$.

We would like to avoid having to distinguish these two cases in the proofs, so we give a useful tool in the following lemma. Recall from Section I.4 that we denote by e the minimal rank of a subalgebra of $\mathscr{A}$, which also corresponds to the minimal rank of maps in $T(\mathscr{A},\mathscr{B})$.

Lemma II.3.4. *There exist two distinct elements b_1 and b_2 in B such that $b_1 \notin \langle b_2 \rangle$.*

Let $\alpha \in T(\mathscr{A},\mathscr{B})$. If $\mathrm{rk}(\alpha) > e$, then there exist $\{a_1, a_2\} \subseteq A$ such that $a_1 \notin \langle a_2 \rangle$ and $a_1\alpha \notin \langle a_2\alpha \rangle$.

Proof. All the algebras we are considering have either $\dim B \geq 2$, or $\dim B = 1$ and $\langle \emptyset \rangle \neq \emptyset$. Thus the same argument as in the proof of the Corollary II.3.3 works, and we find two elements $b_1, b_2 \in B$ such that $b_1 \notin \langle b_2 \rangle$.

Now let $\alpha \in T(\mathscr{A},\mathscr{B})$ be such that $\mathrm{rk}(\alpha) > e$. If $\mathrm{rk}(\alpha) \geq 2$, then α has a pre-image basis of size at least two, that is, there exist $a_1, a_2 \in A$ such that $\{a_1, a_2\}$ is part of a pre-image basis of α. But then, by definition of pre-image basis, we have that $a_1 \notin \langle a_2 \rangle$ and $a_1\alpha \notin \langle a_2\alpha \rangle$. If $\mathrm{rk}(\alpha) = 1$, this means that $e = 0$, and thus $\langle \emptyset \rangle \neq \emptyset$. Then, by taking $\{a_1\}$ to be a pre-image basis of α and $a_2 \in \langle \emptyset \rangle$, we get that $a_1 \notin \langle \emptyset \rangle = \langle b_2 \rangle$ and $a_1\alpha \notin \langle \emptyset \rangle = \langle b_2\alpha \rangle$. □

In view of Lemma II.3.4, we can treat in a similar way algebras of a given rank that contain constants with algebras of a rank one higher that do not have them.

Remark II.3.5. A few notes on the structure of Q worth mentioning are the following:

- *Q is always non-empty:* let $B = \langle \{b_i\} \rangle$ and $A = \langle \{b_i\} \sqcup \{a_j\} \rangle$, then the map $\alpha = \begin{pmatrix} b_i & a_j \\ b_i & b_1 \end{pmatrix}$ is an idempotent and is clearly in Q.
- *Q is not a left ideal:* let $b_1, b_2 \in B$ such that $b_1 \notin \langle b_2 \rangle$ and define the algebras $C = \langle b_1, b_2 \rangle$, $B = C \sqcup \langle \{y_j\} \rangle$, and $A = B \sqcup \langle \{x_i\} \rangle$ together with the following maps:
$$\alpha = \begin{pmatrix} b_1 & b_2 & y_j & x_i \\ b_2 & b_2 & b_2 & b_1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} b_1 & b_2 & y_j & x_i \\ b_1 & b_2 & b_1 & b_1 \end{pmatrix}.$$
Then it follows that $\alpha \notin Q$ and $\beta \in Q$ are such that $\alpha\beta = \alpha \notin Q$.
- *Any map in Q has a preimage basis in B:* for $\alpha \in Q$ we have $\operatorname{im}\alpha = A\alpha = B\alpha$, so there exist $\{b_i\} \subseteq B$ such that $\operatorname{im}\alpha = \langle \{b_i\alpha\} \rangle$, and then $\{b_i\}$ is a preimage basis for α. Consequently, the elements a_1 and a_2 in Lemma II.3.4 can be assumed to be in B directly. From now on, these two facts will be used without explicit mention.

Since regular maps will be of the utmost importance in the description of Green's relations in $T(\mathscr{A},\mathscr{B})$, some lemmas are given here for later use. The first one gives us a necessary and sufficient condition on when the product of two maps lies in Q.

Lemma II.3.6. *For $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$ we have that $\alpha\beta \in Q$ if and only if for all $x \in A$, there exists an element $y \in B$ such that $(x\alpha, y\alpha) \in \ker\beta \cap (B \times B)$.*

Proof. Suppose first that $\alpha\beta \in Q$. From the remarks above, there exists a preimage basis of $\alpha\beta$ in B, say $\{b_i\}$. Therefore, for any $x \in A$, we have that $x\alpha\beta \in \langle \{b_i\alpha\beta\} \rangle$ and thus there exists a term t such that $x\alpha\beta = t(\overline{b_i\alpha\beta}) = t(\overline{b_i})\alpha\beta$. Since $t(\overline{b_i}) \in B$ and $\big(x\alpha, t(\overline{b_i})\alpha\big) \in \ker\beta \cap (B \times B)$, the first direction is proved.

For the converse, we have that $A\alpha\beta = \{x\alpha \mid x \in A\}\,\beta$. However, each element of the set $\{x\alpha \mid x \in A\}$ lies in B and is $\ker\beta$-related to some element $y\alpha$ in $B\alpha$ by assumption. Thus $A\alpha\beta = \{y\alpha \mid y \in B\}\,\beta = B\alpha\beta$ which gives us that $\alpha\beta \in Q$. □

The next lemma shows that we can always create a map α' in Q from a non-regular map α with the same image, and that the converse is also true given that the rank of α is not minimal.

Lemma II.3.7. *For any map $\alpha \in Q^c$, there exists a map $\alpha' \in Q$ such that $\operatorname{im}\alpha' = \operatorname{im}\alpha$. Similarly, for any map $\beta \in Q$ such that $\operatorname{rk}(\beta) > e$, there exists $\beta' \in Q^c$ such that $\operatorname{im}\beta' = \operatorname{im}\beta$.*

Proof. Let us assume first that $\alpha \in Q^c$. Then $B\alpha = \langle\{b_i\alpha\}\rangle$ (with I possibly empty) and we extend this to a basis $\{b_i\alpha\} \sqcup \{a_j\alpha\}$ of $\operatorname{im}\alpha \neq B\alpha$, so that $J \neq \emptyset$. Since $\dim B \geq |I| + |J|$, we can write $B = \langle\{c_i\} \sqcup \{x_j\} \sqcup \{y_k\}\rangle$ where the set K is possibly empty. Finally letting $A = B \sqcup \langle\{z_s\}\rangle$ and defining α' as the following:

$$\alpha' = \begin{pmatrix} c_i & x_j & y_k & z_s \\ b_i\alpha & a_j\alpha & y_k\alpha & z_s\alpha \end{pmatrix},$$

we have that

$$A\alpha' \subseteq \operatorname{im}\alpha = \langle\{b_i\alpha\} \sqcup \{a_j\alpha\}\rangle = \langle\{c_i\} \sqcup \{x_j\}\rangle\alpha' \subseteq B\alpha',$$

and thus $\alpha' \in Q$ with $\operatorname{im}\alpha' = \operatorname{im}\alpha$.

For the second part of the lemma, assume that $\beta \in Q$ is such that $\operatorname{rk}(\beta) > e$. Then, $\operatorname{im}\beta = B\beta$ and there exist two elements $b_1, b_2 \in B$ such that $b_1\beta \notin \langle b_2\beta\rangle$ together with a set $\{x_j\} \subseteq B$ such that $A\beta = B\beta = \langle b_1\beta, b_2\beta\rangle \sqcup \langle\{x_j\beta\}\rangle$. Write $B = \langle b_1, b_2\rangle \sqcup \langle\{x_j\}\rangle \sqcup \langle\{y_k\}\rangle$, $A = B \sqcup \langle\{a_i\}\rangle$ and define

$$\beta' = \begin{pmatrix} \{b_1, b_2\} & x_j & y_k & a_i \\ b_2\beta & x_j\beta & b_2\beta & b_1\beta \end{pmatrix}.$$

Then $A\beta' = \langle b_1\beta, b_2\beta\rangle \sqcup \langle\{x_j\beta\}\rangle$ and $B\beta' = \langle b_2\beta\rangle \sqcup \langle\{x_j\beta\}\rangle = (B \setminus \{b_1\})\beta$. Hence $b_1\beta \notin B\beta'$ which gives us that $A\beta' \neq B\beta'$. Therefore $\beta' \notin Q$, and we also have that $\operatorname{im}\beta' = \operatorname{im}\beta$ as required. □

When we consider a non-regular map α, even though the equation $A\alpha = B\alpha$ is not satisfied, we can nonetheless have that $A\alpha \cong B\alpha$, which happens precisely when $A\alpha$ and $B\alpha$ have the same rank as given in Lemma I.4.34. As an example, consider $A = \langle \{x_i\} \rangle$, $B = \langle \{x_i\} \setminus \{x_1\} \rangle$ where $I = \mathbb{N}$, and define $\alpha \in T(\mathscr{A}, \mathscr{B})$ by $x_k\alpha = x_{k+1}$. Then clearly $x_2 \in A\alpha \setminus B\alpha$, so that $\alpha \notin Q$, but we nevertheless have that $\operatorname{rk}(\alpha) = \dim B = \aleph_0 = \dim(B\alpha)$. This peculiar behaviour of non-regular maps can however only happen if α has infinite rank. This is made clear in the following lemma.

Lemma II.3.8. *Let $\alpha \in T(\mathscr{A}, \mathscr{B})$ be such that $A\alpha \cong B\alpha$. Then either $\alpha \in Q$, or $\operatorname{rk}(\alpha) \geq \aleph_0$. Consequently, $\operatorname{rk}(\alpha) = \dim(B\alpha)$ implies $\alpha \in Q$ only if $\operatorname{rk}(\alpha)$ is finite.*

Proof. If $A\alpha \cong B\alpha$, then $\operatorname{rank}(A\alpha) = \operatorname{rank}(B\alpha) = \kappa$. Since $B\alpha \subseteq A\alpha$, if κ is finite then we must have $B\alpha = A\alpha$ and thus $\alpha \in Q$. □

The contrapositive statement immediately gives us the following:

Corollary II.3.9. *If $\alpha \notin Q$ has finite rank, then*

$$\operatorname{rk}(\alpha|_B) = \dim(B\alpha) < \dim(A\alpha) = \operatorname{rk}(\alpha).$$

Another important property of regular maps is that they are the only ones that have left identities, which will be a useful consideration in the coming study of the semigroup $T(\mathscr{A}, \mathscr{B})$. This is formally given by the following lemma.

Lemma II.3.10. *Let $\alpha \in T(\mathscr{A}, \mathscr{B})$. Then the following are equivalent:*

1) α is regular;

2) $\alpha = \eta\alpha$ for some idempotent $\eta \in T(\mathscr{A}, \mathscr{B})$ with $\operatorname{rk}(\eta) = \operatorname{rk}(\alpha)$; and

3) $\alpha = \gamma\alpha$ for some $\gamma \in T(\mathscr{A}, \mathscr{B})$.

Proof. $1 \Rightarrow 2$*:* Since α is regular, then $\alpha = \alpha\beta\alpha$ for some $\beta \in T(\mathscr{A}, \mathscr{B})$ and taking $\eta = \alpha\beta$ gives the desired result.
$2 \Rightarrow 3$*:* Putting $\gamma = \eta$ gives the result immediately.
$3 \Rightarrow 1$*:* If $\alpha = \gamma\alpha$, then $A\alpha = (A\gamma)\alpha \subseteq B\alpha$ and thus $\alpha \in Q$. □

Remark II.3.11. Given a regular map $\alpha \in T(\mathscr{A}, \mathscr{B})$, one can easily construct an idempotent left identity for α. Indeed, let $\{b_i\alpha\}$, $\{b_i\} \sqcup \{x_j\}$ and $\{b_i\} \sqcup \{x_j\} \sqcup \{a_k\}$ be bases for $\operatorname{im}\alpha$, B and A respectively. With this notation, we can write α as

$$\alpha = \begin{pmatrix} b_i & x_j & a_k \\ b_i\alpha & u_j\big(\overline{b_i\alpha}\big) & v_k\big(\overline{b_i\alpha}\big) \end{pmatrix},$$

for some terms u_j and v_k. Define $\eta \in T(\mathscr{A},\mathscr{B})$ by

$$\eta = \begin{pmatrix} b_i & x_j & a_k \\ b_i & u_j\big(\overline{b_i}\big) & v_k\big(\overline{b_i}\big) \end{pmatrix}.$$

Then, we have that η is an idempotent such that $\eta\alpha = \alpha$ and $\operatorname{im}\eta = \langle\{b_i\}\rangle$ so that $\operatorname{rk}(\eta) = |I| = \operatorname{rk}(\alpha)$. Notice that we have constructed η in such a way that we also have that $\ker\alpha = \ker\eta$.

II.4 GREEN'S RELATIONS

In order to better understand the structure of $T(\mathscr{A},\mathscr{B})$, as for any semigroup, we start by looking at Green's relations. There are three relevant semigroups in question here: $\operatorname{End}(\mathscr{A})$, $T(\mathscr{A},\mathscr{B})$ and Q. To avoid confusion, where the relation is on $\operatorname{End}(\mathscr{A})$ or Q, we use a subscript (we consider $T(\mathscr{A},\mathscr{B})$ as our base case and so do not use a subscript here). For example, $\mathscr{R}_A$, $\mathscr{R}$ and $\mathscr{R}_Q$ denote Green's relation $\mathscr{R}$ on $\operatorname{End}(\mathscr{A})$, $T(\mathscr{A},\mathscr{B})$ and Q respectively. Recall from Proposition I.5.6 that Green's relations are given on $\operatorname{End}(\mathscr{A})$ by: $\alpha\,\mathscr{R}_A\,\beta$ if and only if $\ker\alpha = \ker\beta$, $\alpha\,\mathscr{L}_A\,\beta$ if and only if $\operatorname{im}\alpha = \operatorname{im}\beta$, and $\alpha\,\mathscr{D}_A\,\beta$ if and only if $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$. Since Q is a regular subsemigroup of $\operatorname{End}(\mathscr{A})$, we directly have the following:

Lemma II.4.1. *In Q, we have that $\mathscr{R}_Q = \mathscr{R}_A \cap (Q\times Q)$, $\mathscr{L}_Q = \mathscr{L}_A \cap (Q\times Q)$ and thus also $\mathscr{D}_Q = \mathscr{D}_A \cap (Q\times Q)$.*

The description of Green's relations in the semigroup $T(\mathscr{A},\mathscr{B})$ is however slightly different. This will be the purpose of the remainder of this section. We thus extend the results of Sullivan, Sanwong and Sommanee [47, 50], using similar techniques to theirs.

We start by showing that $\mathscr{R} = \mathscr{R}_A \cap (T(\mathscr{A},\mathscr{B}) \times T(\mathscr{A},\mathscr{B}))$.

Proposition II.4.2. *Let $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$. Then $\alpha = \beta\mu$ for some $\mu \in T(\mathscr{A},\mathscr{B})$ if and only if $\ker\beta \subseteq \ker\alpha$.*

Consequently, $\alpha\,\mathscr{R}\,\beta$ in $T(\mathscr{A},\mathscr{B})$ if and only if $\ker\alpha = \ker\beta$.

Proof. Clearly if $\alpha = \beta\mu$ then $\ker\beta \subseteq \ker\alpha$. Now suppose that $\ker\beta \subseteq \ker\alpha$, and let $A\beta = \langle\{a_i\beta\}\rangle$. We also let $A = \langle\{a_i\beta\} \sqcup \{x_j\}\rangle$ and define $\gamma \in T(\mathscr{A},\mathscr{B})$ by:

$$\gamma = \begin{pmatrix} a_i\beta & x_j \\ a_i\alpha & x_j\beta \end{pmatrix}.$$

By definition of $\{a_i\beta\}$, for any $z \in A$, there exists a term t such that $z\beta = t(\overline{a_i\beta}) = t(\overline{a_i})\beta$ which shows that $(z, t(\overline{a_i})) \in \ker\beta \subseteq \ker\alpha$, and thus

$$z\beta\gamma = t\left(\overline{a_i\beta}\right)\gamma = t\left(\overline{(a_i\beta)\gamma}\right) = t\left(\overline{a_i\alpha}\right) = t(\overline{a_i})\alpha = z\alpha.$$

Therefore we have that $\alpha = \beta\gamma$.

It follows that $\ker\alpha = \ker\beta$ if and only if $\alpha = \beta\mu$ and $\beta = \alpha\mu'$ for some $\mu, \mu' \in T(\mathscr{A}, \mathscr{B})^1$ if and only if $\alpha \mathscr{R} \beta$. □

Remark II.4.3. Notice that regular maps can only be $\mathscr{R}$-related to other regular maps since $\mathscr{R} \subseteq \mathscr{D}$ and in any semigroup either all elements in a $\mathscr{D}$-class are regular, or none of them are. Thus, if $\alpha, \beta \in T(\mathscr{A}, \mathscr{B})$ are such that $\ker\alpha = \ker\beta$, then either $\alpha, \beta \in Q$, or $\alpha, \beta \in Q^c$.

The relation $\mathscr{L}$, however, does not behave exactly as in $\operatorname{End}(\mathscr{A})$ and is more restrictive on the non-regular part of $T(\mathscr{A}, \mathscr{B})$.

Proposition II.4.4. *If $\alpha \in T(\mathscr{A}, \mathscr{B})$ and $\beta \in Q$, then $\alpha = \lambda\beta$ for some $\lambda \in T(\mathscr{A}, \mathscr{B})$ if and only if $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$.*

Consequently, $\alpha \mathscr{L} \beta$ in $T(\mathscr{A}, \mathscr{B})$ if and only if one of the following happen:

- *$\alpha = \beta$; or*
- *$\alpha, \beta \in Q$ and $\operatorname{im}\alpha = \operatorname{im}\beta$.*

Proof. Let $\alpha \in T(\mathscr{A}, \mathscr{B})$ and $\beta \in Q$. Clearly if $\alpha = \lambda\beta$, then $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$. Conversely, let us assume that $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and write $A\alpha = \langle\{a_i\alpha\}\rangle$. Since $\beta \in Q$, we have that $\operatorname{im}\alpha \subseteq B\beta$ and thus $\{a_i\alpha\}$ is an independent set in $B\beta$. This means that we can rewrite each element of this set as some $f_i\beta$ where $f_i \in B$, that is, we have a set $\{f_i\} \subseteq B$ such that $a_i\alpha = f_i\beta$ for all $i \in I$ and the set $\{f_i\beta\}$ is independent. We now take $\{f_j\} \subseteq B$ such that $A\beta = B\beta = \langle\{f_i\beta\} \sqcup \{f_j\beta\}\rangle$ so that $\{f_i\} \sqcup \{f_j\} \subseteq B$ is independent, and we let $\{x_k\}$ and $\{y_\ell\}$ be subsets of A such that $A = \langle\{a_i\} \sqcup \{x_k\}\rangle$ and $A = \langle\{f_i\} \sqcup \{f_j\} \sqcup \{y_\ell\}\rangle$. With this notation, we can write the maps α and β as follows:

$$\alpha = \begin{pmatrix} a_i & x_k \\ a_i\alpha & t_k(\overline{a_i\alpha}) \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} f_i & f_j & y_\ell \\ f_i\beta & f_j\beta & y_\ell\beta \end{pmatrix}.$$

where each t_k is a term of our algebra. If we define a map $\lambda \in T(\mathscr{A}, \mathscr{B})$ by

$$\lambda = \begin{pmatrix} a_i & x_k \\ f_i & t_k(\overline{f_i}) \end{pmatrix},$$

we get that $a_i\lambda\beta = f_i\beta = a_i\alpha$ and $x_k\lambda\beta = t_k(\overline{f_i})\beta = t_k(\overline{f_i\beta}) = t_k(\overline{a_i\alpha}) = x_k\alpha$. Thus $\alpha = \lambda\beta$, which concludes the first statement of the proposition.

Now, let us assume that $\alpha \mathscr{L} \beta$ in $T(\mathscr{A},\mathscr{B})$. Then, there exist $\lambda, \lambda' \in T(\mathscr{A},\mathscr{B})^1$ such that $\alpha = \lambda\beta$ and $\beta = \lambda'\alpha$. If $\lambda = 1$ or $\lambda' = 1$ then we have that $\alpha = \beta$. Otherwise, we have that

$$\alpha = \lambda\lambda'\alpha \quad \text{and} \quad \beta = \lambda'\lambda\beta,$$

from which we get that α and β are regular elements using Lemma II.3.10. From the first part of the proposition we also have that $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $\operatorname{im}\beta \subseteq \operatorname{im}\alpha$, hence the equality. The converse follows directly from the description of $\mathscr{L}_Q$ given above whenever $\alpha \neq \beta$. □

From $\mathscr{R}$ and $\mathscr{L}$ we immediately get the characterisation of the $\mathscr{H}$ relation as:

Corollary II.4.5. *Let $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$. Then $\alpha \mathscr{H} \beta$ if and only if one of the following happen:*

- *$\alpha = \beta$; or*
- *$\alpha, \beta \in Q$ with $\ker\alpha = \ker\beta$ and $\operatorname{im}\alpha = \operatorname{im}\beta$.* □

We know that in $\operatorname{End}(\mathscr{A})$ the relations $\mathscr{D}_A$ and $\mathscr{J}_A$ coincide. However, this is not the case in $T(\mathscr{A},\mathscr{B})$ and these relations are described in the following two propositions.

Proposition II.4.6. *If $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$, then $\alpha \mathscr{D} \beta$ in $T(\mathscr{A},\mathscr{B})$ if and only if one of the following happen:*

- *$\ker\alpha = \ker\beta$; or*
- *$\alpha, \beta \in Q$ with $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$.*

Proof. Assume that $\alpha \mathscr{D} \beta$. Then there exists $\gamma \in T(\mathscr{A},\mathscr{B})$ such that $\alpha \mathscr{R} \gamma \mathscr{L} \beta$. By Propositions II.4.2 and II.4.4 we have that $\ker\alpha = \ker\gamma$, and either $\gamma = \beta$ or γ and β are both regular with $\operatorname{im}\gamma = \operatorname{im}\beta$. In the case where $\gamma = \beta$ we obtain that $\ker\alpha = \ker\gamma = \ker\beta$, as required. Assume now that we are in the case where $\gamma, \beta \in Q$ and $\operatorname{im}\gamma = \operatorname{im}\beta$. Then, by Remark II.4.3, α is also in Q since $\alpha \mathscr{R} \gamma$, and we also get that

$$\operatorname{rk}(\beta) = \operatorname{rk}(\gamma) = \dim(A/\ker\gamma) = \dim(A/\ker\alpha) = \operatorname{rk}(\alpha).$$

Conversely, if $\ker\alpha = \ker\beta$, then $\alpha \mathscr{R} \beta$ and thus $\alpha \mathscr{D} \beta$. Otherwise, assume that $\mathrm{rk}(\alpha) = \mathrm{rk}(\beta)$ and $\alpha, \beta \in Q$. Therefore, by Lemmas I.4.34 and II.2.5 we have that $\mathrm{im}\,\alpha \cong \mathrm{im}\,\beta$ and then there exists $\gamma \in T(\mathscr{A}, \mathscr{B})$ such that $\mathrm{im}\,\gamma = \mathrm{im}\,\beta$, and $\ker\gamma = \ker\alpha$. Furthermore, since $\alpha \in Q$, we get that $\gamma \in Q$ by Remark II.4.3, which finishes showing that $\alpha \mathscr{R} \gamma \mathscr{L} \beta$ for some $\gamma \in T(\mathscr{A}, \mathscr{B})$, and therefore $\alpha \mathscr{D} \beta$. □

Proposition II.4.7. *Let $\alpha, \beta \in T(\mathscr{A}, \mathscr{B})$. Then $\alpha = \lambda\beta\mu$ for some $\lambda \in T(\mathscr{A}, \mathscr{B})$ and $\mu \in T(\mathscr{A}, \mathscr{B})^1$ if and only if $\mathrm{rk}(\alpha) \leq \dim(B\beta)$. Consequently, $\alpha \mathscr{J} \beta$ in $T(\mathscr{A}, \mathscr{B})$ if and only if one of the following happens:*

- $\ker\alpha = \ker\beta$, *or*
- $\mathrm{rk}(\alpha) = \dim(B\alpha) = \dim(B\beta) = \mathrm{rk}(\beta)$.

Proof. Let us assume first that $\alpha = \lambda\beta\mu$ for some $\lambda \in T(\mathscr{A}, \mathscr{B})$ and $\mu \in T(\mathscr{A}, \mathscr{B})^1$. Then $A\alpha = (A\lambda)\beta\mu \subseteq (B\beta)\mu$, from which we get that

$$\mathrm{rk}(\alpha) = \dim(A\alpha) \leq \dim\left((B\beta)\mu\right) \leq \dim(B\beta).$$

Conversely, suppose that $\mathrm{rk}(\alpha) \leq \dim(B\beta)$. If $\mathrm{rk}(\alpha) = 0$, then α is a left zero of $T(\mathscr{A}, \mathscr{B})$ and thus $\alpha = \alpha\beta$, giving us the result with $\lambda = \alpha$ and $\mu = 1$. Otherwise, suppose that $\mathrm{rk}(\alpha) \geq 1$ and write $A\alpha = \langle\{x_i\alpha\}\rangle$. Since $\mathrm{rk}(\alpha) \leq \dim(B\beta)$, there exists an independent set $\{b_i\} \subseteq B$ such that $A\beta = \langle\{b_i\beta\} \sqcup \{y_k\beta\}\rangle$. Thus we can write $A = \langle\{x_i\} \sqcup \{x'_j\}\rangle = \langle\{b_i\} \sqcup \{y_k\} \sqcup \{w_\ell\}\rangle$ and the maps α and β may be defined by

$$\alpha = \begin{pmatrix} x_i & x'_j \\ x_i\alpha & u_j(\overline{x_i\alpha}) \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} b_i & y_k & w_\ell \\ b_i\beta & y_k\beta & v_\ell(\overline{b_i\beta}, \overline{y_k\beta}) \end{pmatrix},$$

for some terms u_j and v_ℓ. Extend $\{b_i\beta\}$ into a basis of A via $\{z_m\}$ and define λ and μ in $T(\mathscr{A}, \mathscr{B})$ by the following:

$$\lambda = \begin{pmatrix} x_i & x'_j \\ b_i & u_j(\overline{b_i}) \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} b_i\beta & z_m \\ x_i\alpha & x_1\alpha \end{pmatrix}.$$

Then it can be easily seen that $x_i\lambda\beta\mu = x_i\alpha$ and $x'_j\lambda\beta\mu = u_j(\overline{x_i\alpha}) = x'_j\alpha$, which shows that $\lambda\beta\mu = \alpha$.

In order to prove the second part of the proposition, let us first assume that $\alpha \mathscr{J} \beta$, that is, there exist $\lambda, \mu, \lambda', \mu' \in T(\mathscr{A}, \mathscr{B})^1$ such that $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$. If $\lambda = \lambda' = 1$, then we have that $\alpha \mathscr{R} \beta$ and thus Proposition II.4.2 gives us that $\ker\alpha = \ker\beta$. If we have that only one of λ and λ' is 1, we show that we can find

$\gamma, \gamma' \in T(\mathscr{A},\mathscr{B})$ and $\delta, \delta' \in T(\mathscr{A},\mathscr{B})^1$ such that $\alpha = \gamma\beta\delta$ and $\beta = \gamma'\alpha\delta'$. Indeed, assuming without loss of generality that $\lambda = 1 \neq \lambda'$, we get that

$$\alpha = \beta\mu = \lambda'\alpha\mu'\mu = \lambda'\beta(\mu\mu'\mu)$$

and the equations above are satisfied by taking $\gamma = \gamma' = \lambda'$, $\delta = \mu\mu'\mu$ and $\delta' = \mu'$. By the previous part of the proposition we therefore have that $\mathrm{rk}(\alpha) \leq \dim(B\beta)$ and $\mathrm{rk}(\beta) \leq \dim(B\alpha)$. From this, we get that:

$$\dim(A\alpha) = \mathrm{rk}(\alpha) \leq \dim(B\beta) \leq \dim(A\beta) = \mathrm{rk}(\beta) \leq \dim(B\alpha) \leq \dim(A\alpha),$$

thus forcing the required equalities.

Conversely, assume that $\ker\alpha = \ker\beta$. Then $\alpha \mathscr{R} \beta$ by Proposition II.4.2 and thus $\alpha \mathscr{J} \beta$ since $\mathscr{R} \subseteq \mathscr{J}$. On the other hand, if $\mathrm{rk}(\alpha) = \dim(B\beta) = \dim(B\alpha) = \mathrm{rk}(\beta)$, then we use the first part of the proof in order to obtain the desired result. □

Remark II.4.8. In the proof of Proposition II.4.7 given above, notice that the maps λ and μ are constructed in such a way that $\mu \in Q$, and whenever $\{x_i\} \subseteq B$ we also have $\lambda \in Q$.

From Lemma II.3.8, we know that the condition $\mathrm{rk}(\alpha) = \dim(B\alpha)$ is equivalent to $\alpha \in Q$ only if α has finite rank, which means that $\mathscr{D} = \mathscr{J}$ whenever $\mathscr{B}$ is finite dimensional. On the other hand, if $\mathscr{B}$ is infinite dimensional, then there exist non-regular maps of infinite rank that are $\mathscr{J}$-related to regular maps of the same rank. Indeed, assume that $A = \langle\{x_i\}\rangle$ with $I = \mathbb{N}$, $B = \langle\{x_{i\geq 2}\}\rangle$ and define $\alpha = \begin{pmatrix} \{x_1, x_2\} & x_{i\geq 3} \\ x_2 & x_i \end{pmatrix}$ and $\beta = \begin{pmatrix} x_i \\ x_{i+1} \end{pmatrix}$. Then we have that $\alpha \in Q$, $\beta \notin Q$ but $\mathrm{rk}(\alpha) = \dim(B\alpha) = \mathrm{rk}(\beta) = \dim(B\beta) = \aleph_0$, which means that $\alpha \mathscr{J} \beta$. However, α is not $\mathscr{D}$-related to β by Proposition II.4.6, which shows that $\mathscr{D} \subsetneq \mathscr{J}$ in that case.

Nevertheless, on the regular subsemigroup Q the relations $\mathscr{D}$ and $\mathscr{J}$ coincide, as is given by the following:

Lemma II.4.9. *Let $\alpha, \beta \in Q$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in Q$ if and only if $\mathrm{rk}(\alpha) \leq \mathrm{rk}(\beta)$. Consequently, $\alpha \mathscr{J}_Q \beta$ if and only if $\mathrm{rk}(\alpha) = \mathrm{rk}(\beta)$, and thus $\mathscr{J}_Q = \mathscr{D}_Q$.*

Proof. Clearly if $\alpha = \lambda\beta\mu$ then $\mathrm{rk}(\alpha) \leq \mathrm{rk}(\beta\mu) \leq \mathrm{rk}(\beta)$ by Lemma I.5.4. Conversely, assume that $\mathrm{rk}(\alpha) \leq \mathrm{rk}(\beta)$. Since both α and β lie in Q, we have that $A\alpha = B\alpha$ and $\mathrm{rk}(\beta) = \dim(A\beta) = \dim(B\beta)$. Thus we let $A\alpha = \langle\{x_i\alpha\}\rangle$ and $A\beta = \langle\{b_i\beta\} \sqcup \{y_k\beta\}\rangle$

where $\{x_i\}$ and $\{b_i\} \sqcup \{y_k\}$ are preimage bases of α and β respectively, which are both taken to be in B. We also write $A = \langle \{x_i\} \sqcup \left\{x'_j\right\} \rangle = \langle \{b_i\beta\} \sqcup \{z_m\} \rangle$ and define $\lambda, \mu \in T(\mathscr{A}, \mathscr{B})$ by:

$$\lambda = \begin{pmatrix} x_i & x'_j \\ b_i & u_j(\overline{b_i}) \end{pmatrix} \quad \text{and} \quad \mu = \begin{pmatrix} b_i\beta & z_m \\ x_i\alpha & x_1\alpha \end{pmatrix}$$

where the terms u_j are such that $x'_j\alpha = u_j(\overline{x_i\alpha})$. Clearly λ and μ are in Q since they have a preimage basis in B, and we also have $x_i\lambda\beta\mu = x_i\alpha$ and $x'_j\lambda\beta\mu = u_j(\overline{x_i\alpha}) = x'_j\alpha$, so that $\alpha = \lambda\beta\mu$.

We know that $\mathscr{D}_Q \subseteq \mathscr{J}_Q$, so it suffices to prove the converse. Suppose that $\alpha \mathrel{\mathscr{J}_Q} \beta$. Then $\mathrm{rk}(\alpha) = \mathrm{rk}(\beta)$ by the first part, and by Proposition II.4.6 we know that α and β are $\mathscr{D}$-related in $T(\mathscr{A}, \mathscr{B})$, that is, there exists $\gamma \in T(\mathscr{A}, \mathscr{B})$ with $\operatorname{im}\alpha = \operatorname{im}\gamma$ and $\ker\gamma = \ker\beta$. By Remark II.4.3 we then have $\gamma \in Q$ since β is regular, and thus $\alpha \mathrel{\mathscr{D}_Q} \beta$, which finishes showing $\mathscr{D}_Q = \mathscr{J}_Q$. $\square$

II.5 IDEALS

In the same way that Section II.4 generalised the approach to Green's relations exhibited in the cases of vector spaces and sets, this section is generalising the description of the ideals of $T(\mathscr{A}, \mathscr{B})$ using the same ideas developed by Sullivan and Mendes-Gonçalves [35, 50]. Recall from Corollary I.5.7 that the ideals of the endomorphism monoid of an independence algebra are precisely the sets of the form $\{\alpha \in \mathrm{End}(\mathscr{A}) \mid \mathrm{rk}(\alpha) < k\}$ for each $k \leq (\dim A)^+$. However, in the context of $T(\mathscr{A}, \mathscr{B})$, the ideals are not solely determined by ranks. Nevertheless, the ideals of the subsemigroup Q are in one-to-one correspondence with the cardinals not greater than $(\dim B)^+$.

Recall that e denotes the smallest rank of a non-empty subalgebra of $\mathscr{A}$, or equivalently, the smallest rank of a map in $T(\mathscr{A}, \mathscr{B})$.

Proposition II.5.1. *The ideals of Q are precisely the sets*

$$Q_r = \{\alpha \in Q \mid \mathrm{rk}(\alpha) < r\}$$

where $e < r \leq (\dim B)^+$.

Proof. Let us assume that I is an ideal of Q and denote by r the cardinal defined by

$$r = \min\left\{e < \kappa \leq (\dim B)^+ \mid \mathrm{rk}(\beta) < \kappa \text{ for all } \beta \in I\right\}.$$

We claim that $I = Q_r$. By definition of r, we clearly have that $I \subseteq Q_r$.

For the reverse inclusion, consider a map $\alpha \in Q_r$. If $\mathrm{rk}(\beta) < \mathrm{rk}(\alpha)$ for all $\beta \in I$, this forces $\mathrm{rk}(\alpha) \geq r$ by minimality of r, contradicting the fact that $\alpha \in Q_r$. Thus, there exists some $\beta \in I$ with $\mathrm{rk}(\alpha) \leq \mathrm{rk}(\beta)$. By Lemma II.4.9, it follows that there exist $\lambda, \mu \in Q$ such that $\alpha = \lambda\beta\mu$. This gives us that $\alpha \in I$ since I is an ideal and thus $Q_r \subseteq I$. Therefore $I = Q_r$, completing the proof. □

Following the usual definition in $\mathrm{End}(\mathscr{A})$, we define the sets T_k for any $k > e$ by

$$T_k = \{\alpha \in T(\mathscr{A},\mathscr{B}) \mid \mathrm{rk}(\alpha) < k\}.$$

These are easily seen to be ideals of $T(\mathscr{A},\mathscr{B})$ using Lemma I.5.4. It is obvious that for all $k \geq (\dim B)^+$ we have that $T_k = T_{(\dim B)^+} = T(\mathscr{A},\mathscr{B})$. Moreover, we have a minimal ideal, as given by:

Lemma II.5.2. *The ideal T_{e^+} is the minimal ideal of $T(\mathscr{A},\mathscr{B})$.*

Proof. Clearly any element α of T_{e^+} is regular using the converse of Corollary II.3.9 together with the fact that $e \leq \dim(B\alpha) \leq \dim(A\alpha) = e$.

Now let I be an ideal of $T(\mathscr{A},\mathscr{B})$, $\beta \in I$ and $\alpha \in T_{e^+}$. Then $\beta\alpha \in I \cap T_{e^+}$, so that we can assume that $\mathrm{rk}(\beta) = e$ in the first place. Then for all $\delta \in T_{e^+}$, we have that $\mathrm{rk}(\beta) = \mathrm{rk}(\delta)$. Thus $\beta \mathscr{J}_Q \delta$ by Lemma II.4.9, so that $\delta \in I$ since I is an ideal. Hence $T_{e^+} \subseteq I$, which shows that T_{e^+} is the minimal ideal of $T(\mathscr{A},\mathscr{B})$. □

Following the footsteps of [35, 50], we define for any non-empty subset S of $T(\mathscr{A},\mathscr{B})$ the cardinal $r(S)$ and the subset $K(S) \subseteq T(\mathscr{A},\mathscr{B})$ as follows:

$$r(S) = \min\{\kappa \leq (\dim B)^+ \mid \dim(B\alpha) < \kappa,\ \forall \alpha \in S\},$$
$$\text{and } K(S) = \{\beta \in T(\mathscr{A},\mathscr{B}) \mid \ker\alpha \subseteq \ker\beta,\ \text{for some } \alpha \in S\}.$$

Notice that $S \subseteq K(S)$ and by Proposition II.4.2, we can also express $K(S)$ as

$$K(S) = \{\beta \in T(\mathscr{A},\mathscr{B}) \mid \beta = \alpha\mu,\ \text{for some } \alpha \in S \text{ and } \mu \in T(\mathscr{A},\mathscr{B})\}.$$

From this, it is clear that if $\beta \in K(S)$ and $\lambda \in T(\mathscr{A},\mathscr{B})$, then we have that $\beta\lambda \in K(S)$ and thus $K(S)$ is a right ideal, namely, $K(S) = S\,T(\mathscr{A},\mathscr{B})$.

We now want to show that any ideal of $T(\mathscr{A},\mathscr{B})$ is of the form $T_{r(S)} \cup K(S)$ or $T_{r(S)^+} \cup K(S)$ for some non-empty subset $S \subseteq T(\mathscr{A},\mathscr{B})$, where we can see that

$$T_{r(S)} \cup K(S) = \{\beta \in T(\mathscr{A},\mathscr{B}) \mid \operatorname{rk}(\beta) < r(S), \text{ or } \ker\alpha \subseteq \ker\beta \text{ for some } \alpha \in S\}.$$

Since the union of two right ideals is again a right ideal, it follows that $T_{r(S)} \cup K(S)$ and $T_{r(S)^+} \cup K(S)$ are right ideals of $T(\mathscr{A},\mathscr{B})$. We first show that all the ideals of the form T_k can be written as of one of these two forms.

Lemma II.5.3. *For any cardinal $e < k \leq (\dim B)^+$, we have that $T_k = T_{r(S)} \cup K(S)$ with $S = T_k$.*

Proof. Let $S = T_k$. Then for each $m < k$, there exists a subalgebra $\mathscr{C} \subseteq \mathscr{B}$ of rank m and by Lemma II.2.6, we have an idempotent $\eta_m \in T(\mathscr{A},\mathscr{B})$ with $\operatorname{im}\eta_m = C$, so that $\operatorname{rk}(\eta) = m$. Thus $\eta_m \in T_k$, which shows that $r(S) > m$. Since this is true for each $m < k$, we get that $r(S) = k$. Additionally, if $\beta \in K(S)$, then $\beta = \alpha\mu$ for some $\alpha \in T_k$ and $\mu \in T(\mathscr{A},\mathscr{B})$. Since T_k is an ideal, we get that $\beta \in T_k$ and thus $K(S) \subseteq T_k$. Therefore $T_k = T_{r(S)} \cup K(S)$ as claimed. $\square$

Before we can show that all ideals are of the form $T_{r(S)} \cup K(S)$ or $T_{r(S)^+} \cup K(S)$ for non-empty $S \subseteq T(\mathscr{A},\mathscr{B})$, we first need to show that such sets are indeed ideals.

Lemma II.5.4. *For each non-empty subset S of $T(\mathscr{A},\mathscr{B})$, the sets $T_{r(S)} \cup K(S)$ and $T_{r(S)^+} \cup K(S)$ are ideals of $T(\mathscr{A},\mathscr{B})$.*

Proof. Let $\emptyset \neq S \subseteq T(\mathscr{A},\mathscr{B})$. As mentioned earlier, we have that $T_{r(S)}$ is an ideal and $T_{r(S)} \cup K(S)$ is a right ideal. Now let $\beta \in K(S)$; then there exists $\alpha \in S$ and $\mu \in T(\mathscr{A},\mathscr{B})$ such that $\beta = \alpha\mu$. For $\lambda \in T(\mathscr{A},\mathscr{B})$, we then get

$$\operatorname{rk}(\lambda\beta) = \dim(A\lambda\beta) \leq \dim(B\beta) = \dim(B\alpha\mu) \leq \dim(B\alpha) < r(S),$$

and therefore $\lambda\beta \in T_{r(S)}$. Hence $T_{r(S)} \cup K(S)$ and $T_{r(S)^+} \cup K(S)$ are ideals. $\square$

In order to show the reverse statement, we need two small lemmas beforehand which will become handy when choosing an adequate set S for each ideal of $T(\mathscr{A},\mathscr{B})$.

Lemma II.5.5. *If $\alpha \in Q$ and $e \leq s < \operatorname{rk}(\alpha)$, then there exists a map $\lambda \in T(\mathscr{A},\mathscr{B})$ such that $\lambda\alpha \notin Q$ and $\dim(B\lambda\alpha) = s$.*

Proof. Since $\alpha \in Q$, then $A\alpha = B\alpha = \langle\{b_i\alpha\}\rangle$ for some $\{b_i\} \subseteq B$, and by letting $A = \langle\{b_i\} \sqcup \{x_j\}\rangle$ the map α can be written as

$$\alpha = \begin{pmatrix} b_i & x_j \\ b_i\alpha & u_j(\overline{b_i\alpha}) \end{pmatrix},$$

for some terms u_j. Now take $\{b'_k\} \sqcup \{b_1\} \subseteq \{b_i\}$ such that $|K| = s$ (which is possible by the assumption on the value of s), and take some $z \in A \setminus B$ such that $A = \langle\{b'_k\} \sqcup \{z\} \sqcup \{y_\ell\}\rangle$. Define $\lambda \in T(\mathscr{A},\mathscr{B})$ by

$$\lambda = \begin{pmatrix} b'_k & z & y_\ell \\ b'_k & b_1 & c \end{pmatrix},$$

where c can be taken in $\langle\emptyset\rangle$ whenever this is non-empty, and otherwise we can take c to be any element in $\{b'_k\} \neq \emptyset$ since $s \geq e = 1$ in that case. Then we have

$$B\lambda\alpha = \langle\{b'_k\alpha\}\rangle \neq \langle\{b'_k\alpha\} \sqcup \{b_1\alpha\}\rangle = A\lambda\alpha,$$

which gives that $\lambda\alpha \notin Q$. Moreover, $\dim(B\lambda\alpha) = |K| = s$ as required. □

Lemma II.5.6. *Let $I \neq T_{e^+}$ be an ideal of $T(\mathscr{A},\mathscr{B})$. Then there exists a map $\gamma \in I$ such that $\gamma \notin Q$.*

Proof. Let $I \neq T_{e^+}$ be an ideal of $T(\mathscr{A},\mathscr{B})$. Since $T_{e^+} \subseteq I$ by Lemma II.5.2, there exists $\alpha \in I \backslash T_{e^+}$. Then $\mathrm{rk}(\alpha) > e$ and by Lemma II.3.4, there exist $z_1, z_2 \in \operatorname{im}\alpha \subseteq B$ such that $z_1 \notin \langle z_2\rangle$. If we can find two elements satisfying this property such that one of z_1 and z_2 does not lie in $B\alpha$, then we have that $A\alpha \neq B\alpha$ and thus $\alpha \notin Q$ and we can take $\gamma = \alpha$.

On the other hand, assume that any two elements z_1 and z_2 satisfying the property above lie in $B\alpha$. Let $\operatorname{im}\alpha = \langle\{b_i\alpha\}\rangle$. If there exists $a \in \{b_i\}$ such that $a \in A \setminus B$, then we have that $a\alpha \notin \langle\{b_i\alpha \mid b_i \neq a\}\rangle \neq \emptyset$, which contradicts the assumption on z_1 and z_2. Thus $\{b_i\} \subseteq B$, which means that $A\alpha \subseteq B\alpha$ and $\alpha \in Q$. Since $\mathrm{rk}(\alpha) > e$, invoking Lemma II.5.5 with $s = e$ gives us the existence of a map $\lambda \in T(\mathscr{A},\mathscr{B})$ such that $\lambda\alpha \notin Q$. Moreover, we also have that $\lambda\alpha \in I$ because I is an ideal. Therefore the map $\gamma := \lambda\alpha$ satisfies the requirements of the lemma. □

We can now finally give the characterisation of the ideals in $T(\mathscr{A},\mathscr{B})$.

Theorem II.5.7. *The ideals of $T(\mathscr{A},\mathscr{B})$ are precisely the sets $T_{r(S)} \cup K(S)$ and $T_{r(S)^+} \cup K(S)$ where S is a non-empty subset of $T(\mathscr{A},\mathscr{B})$.*

In particular, if I is an ideal of $T(\mathscr{A},\mathscr{B})$ and $S = I \setminus Q$, then

- $I = T_{r(S)} \cup K(S)$ *if* $\mathrm{rk}(\gamma) < r(S)$ *for all* $\gamma \in I \cap Q$*; or*
- $I = T_{r(S)^+} \cup K(S)$ *otherwise.*

Proof. From Lemma II.5.4, any set of the form $T_{r(S)} \cup K(S)$ or $T_{r(S)^+} \cup K(S)$ with $\emptyset \neq S \subseteq T(\mathscr{A},\mathscr{B})$ is an ideal. Suppose now that I is an ideal of $T(\mathscr{A},\mathscr{B})$. We show that there is a non-empty set S such that $I = T_{r(S)} \cup K(S)$ or $I = T_{r(S)^+} \cup K(S)$.

If I is the minimal ideal of $T(\mathscr{A},\mathscr{B})$, that is, if $I = T_{e^+}$, then we set $S = T_{e^+}$. By Lemma II.5.3, we obtain that $I = T_{r(S)} \cup K(S)$ as required.

From now on, let us assume that $I \neq T_{e^+}$. Then $S = I \setminus Q$ is non-empty by Lemma II.5.6, and by setting $r = r(S)$, we show that I is equal to either $T_r \cup K(S)$ or $T_{r^+} \cup K(S)$.

First, we have that $T_r \cup K(S) \subseteq I$. Indeed, if $\beta \in K(S)$, then $\beta = \alpha\mu$ for some $\alpha \in S \subseteq I$ and $\mu \in T(\mathscr{A},\mathscr{B})$, and thus $\beta \in I$. On the other hand, let $\beta \in T_r$. If $\mathrm{rk}(\beta) > \dim(B\alpha)$ for all $\alpha \in I$, then, in particular, $\mathrm{rk}(\beta) > \dim(B\alpha)$ for all $\alpha \in S$, which contradicts the minimality of r since $\mathrm{rk}(\beta) < r$. Therefore, there exists $\alpha \in I$ such that $\mathrm{rk}(\beta) \leq \dim(B\alpha)$. By Proposition II.4.7, this means that $\beta = \lambda\alpha\mu$ for some $\lambda \in T(\mathscr{A},\mathscr{B})$, $\mu \in T(\mathscr{A},\mathscr{B})^1$ which shows that $\beta \in I$. Therefore $T_r \cup K(S) \subseteq I$.

It is also clear that $I \setminus Q = S \subseteq K(S)$ and that for any $\gamma \in I \cap Q$ such that $\dim(B\gamma) < r$, we have that $r > \dim(B\gamma) = \mathrm{rk}(\gamma)$, so that $\gamma \in T_r$. We now need to distinguish between the case where $I = T_r \cup K(S)$ and $I = T_{r^+} \cup K(S)$ by looking at the possible values of $\dim(B\gamma)$ for $\gamma \in I \cap Q$.

On one hand, if $\dim(B\gamma) < r$ for all $\gamma \in I \cap Q$ then we have shown that $I \cap Q \subseteq T_r$ and thus $I = T_r \cup K(S)$ by combining all previous inclusions.

On the other hand, assume that there exists at least one $\beta \in I \cap Q$ such that $\dim(B\beta) \geq r$ and set $\kappa = \mathrm{rk}(\beta) = \dim(B\beta)$. If $\kappa > r$, then using Lemma II.5.5 with $s = r$, we get that there exists $\lambda \in T(\mathscr{A},\mathscr{B})$ such that $\dim(B\lambda\beta) = r$ and $\lambda\beta \notin Q$. But then $\lambda\beta \in I \setminus Q = S$ and this contradicts the definition of $r = r(S)$. Therefore we must have that $\kappa = r$ from which we have that $\beta \in T_{r^+}$. This gives us that $I \cap Q \subseteq T_{r^+}$ and therefore $I \subseteq T_{r^+} \cup K(S)$. In order to get an equality in this last equation, we only need to show that $T_{r^+} \subseteq I$ since we already know that $T_r \cup K(S) \subseteq I$. For this, consider $\gamma \in T_{r^+}$. Since in the current case there exists a map $\beta \in I$ with $\dim(B\beta) = r$, we have that $\mathrm{rk}(\gamma) \leq r = \dim(B\beta)$. Then by Proposition II.4.7, $\gamma = \lambda\beta\mu$ for some $\lambda \in T(\mathscr{A},\mathscr{B}), \mu \in T(\mathscr{A},\mathscr{B})^1$ which shows that

$\gamma \in I$ and gives us that $T_{r^+} \subseteq I$. Therefore, when there exists a map $\gamma \in I \cap Q$ with $\mathrm{rk}(\gamma) \geq r$, we have that $I = T_{r^+} \cup K(S)$. □

Remark II.5.8. Notice that for $k > e^+$ and two distinct sets S and S', one can have $T_{r(S)} \cup K(S) = T_{r(S')} \cup K(S')$. Similarly, for an ideal I, there might exist sets S and S' such that $I = T_{r(S)} \cup K(S)$ and $I = T_{r(S')^+} \cup K(S')$. Indeed, if we take $I = T_k$, then we know that $I = T_{r(S)} \cup K(S)$ for $S = T_k$ by Lemma II.5.3, but we can also obtain I using Theorem II.5.7 with the set $S' = I \setminus Q$. More precisely, if k is finite, then by Corollary II.3.9, we have that $\mathrm{rk}(\alpha|_B) < \mathrm{rk}(\alpha) < r$ for all $\alpha \in S'$, so that $r(S') \leq k-1$. Now, T_k contains idempotents of rank $k-1$ corresponding to each subalgebra of $\mathscr{B}$ of rank $k-1$. Thus we fall into the second case of Theorem II.5.7 and we get that $T_k = I = T_{r(S')^+} \cup K(S')$. On the other hand, if k is infinite, then we directly have that $r(S') = k$ and that all elements of I are such that $\dim(B\gamma) \leq \mathrm{rk}(\gamma) < k$. Thus the first case of Theorem II.5.7 gives us that $T_k = I = T_{r(S')} \cup K(S')$ in this case.

We can now give examples of the construction of two ideals that are not comparable under inclusion, as long as we are not in the case where the algebra $\mathscr{A}$ is a set with 3 elements $\{x_1, x_2, x_3\}$ and its subalgebra $\mathscr{B}$ has dimension exactly 2, that is, $B = \{x_1, x_2\}$. In that specific case, $T(\mathscr{A},\mathscr{B})$ is composed of 8 maps given by

$$\alpha_1 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_1 & x_1 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_2 & x_2 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_1 & x_2 \end{pmatrix},$$
$$\gamma_2 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_2 & x_1 \end{pmatrix}, \quad \delta_{1,k} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_k \end{pmatrix}, \quad \text{and} \quad \delta_{2,k} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & x_k \end{pmatrix},$$

for $k \in \{1,2\}$. It is easy to see that there are only three ideals, namely the sets $T_2 = \{\alpha_1, \alpha_2\}$, $T_2 \cup \{\alpha \in Q^c \mid \mathrm{rk}(\alpha) = 2\} = \{\alpha_1, \alpha_2, \gamma_1, \gamma_2\}$ and $T(\mathscr{A},\mathscr{B})$ itself, and that these ideals form a chain.

Example II.5.9. Let us assume that there exist independent elements $b_1, b_2 \in B$, that $\mathscr{A}$ does not have any constants and that B has codimension at least 2 in A. Let $B = \langle \{b_1, b_2\} \sqcup \{d_j\} \rangle$ and $A = \langle \{b_1, b_2\} \sqcup \{d_j\} \sqcup \{x_1, x_2\} \sqcup \{y_k\} \rangle$. Define the following maps in $T(\mathscr{A},\mathscr{B})$:

$$\alpha = \begin{pmatrix} x_1 & \{b_i, d_j, x_2, y_k\} \\ b_1 & b_2 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} x_2 & \{b_i, d_j, x_1, y_k\} \\ b_1 & b_2 \end{pmatrix}.$$

From this we have that $\mathrm{rk}(\alpha) = \mathrm{rk}(\beta) = 2$ with $\dim(B\alpha) = \dim(B\beta) = 1$ and $(x_1, b_1) \in \ker\beta \setminus \ker\alpha$ while $(x_2, b_1) \in \ker\alpha \setminus \ker\beta$, which shows that $\ker\alpha$ and

$\ker\beta$ are incomparable. Since $r(\{\alpha\}) = r(\{\beta\}) = 2$, we let $I_\alpha = T_2 \cup K(\{\alpha\})$ and $I_\beta = T_2 \cup K(\{\beta\})$, which are both ideals from the previous theorem. Now it is easy to see that $\alpha \in I_\alpha \setminus I_\beta$ and $\beta \in I_\beta \setminus I_\alpha$, which shows that these two ideals are incomparable. Notice that if b_2 is a constant in the above, the maps α and β are still well-defined elements of $T(\mathscr{A},\mathscr{B})$ and the conclusion still hold.

Similarly, if $B = \langle\{b_1, b_2, b_3\} \sqcup \{d_j\}\rangle$ and $A = \langle\{b_1, b_2, b_3\} \sqcup \{d_j\} \sqcup \{x_1\} \sqcup \{y_k\}\rangle$, then we modify the maps above as follows:

$$\alpha = \begin{pmatrix} x_1 & b_1 & \{b_2, b_3, d_j, y_k\} \\ b_1 & b_2 & b_3 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} x_1 & b_2 & \{b_1, b_3, d_j, y_k\} \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Then we have that $\mathrm{rk}(\alpha) = \mathrm{rk}(\beta) = 3$ but $\dim(B\alpha) = \dim(B\beta) = 2$, while we also get that $(b_2, b_3) \in \ker\alpha \setminus \ker\beta$ and $(b_1, b_3) \in \ker\beta \setminus \ker\alpha$. Hence $\ker\alpha$ and $\ker\beta$ are incomparable and we have that $r(\{\alpha\}) = r(\{\beta\}) = 3$. Then the ideals $I_\alpha = T_3 \cup K(\{\alpha\})$ and $I_\beta = T_3 \cup K(\{\beta\})$ are incomparable since $\alpha \in I_\alpha \setminus I_\beta$ and $\beta \in I_\beta \setminus I_\alpha$, as required. Once again, setting b_3 to be a constant with the same maps still give the same result.

— III —

Extended Green's relations on semigroups of endomorphisms with restricted range

It was shown earlier in Corollary II.3.3 that the semigroup of endomorphisms with restricted range is not regular in general, so it makes sense to look into its extended Green's relations. We thus restrict ourselves to the cases where this semigroup is not regular. Hence, throughout this chapter, $\mathscr{A}$ will denote an independence algebra, and $\mathscr{B} \subsetneq \mathscr{A}$ has either dimension at least two, or has dimension one and a non-empty set of constants. By Lemma II.3.4, this means in particular that there exist $b_1, b_2 \in B$ such that $b_1 \notin \langle b_2 \rangle$, and we will use this fact without further mention. As before, we will also denote by Q the set of regular elements, and we write $E = E(T(\mathscr{A}, \mathscr{B}))$ for the set of idempotents of $T(\mathscr{A}, \mathscr{B})$.

In this chapter we will describe all extended Green's relations defined in Section I.3 on the semigroup $T(\mathscr{A}, \mathscr{B})$, starting with the relations $\mathscr{L}^*$, $\mathscr{R}^*$, $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}$ in Section III.1. From this, we get that $T(\mathscr{A}, \mathscr{B})$ is a right abundant semigroup, but is not left Fountain, and that the relation $\widetilde{\mathscr{R}}$ is not a left congruence in general. Moreover, the relations $\mathscr{L}^*$ and $\mathscr{R}^*$ do not commute in $T(\mathscr{A}, \mathscr{B})$, making the description of their join much harder to compute. However, by looking at their composition $\mathscr{L}^* \circ \mathscr{R}^*$ in Section III.2, we fully characterise both $\mathscr{D}^* = \mathscr{L}^* \vee \mathscr{R}^*$ and $\widetilde{\mathscr{D}} = \widetilde{\mathscr{L}} \vee \widetilde{\mathscr{R}}$ in Section III.3 and show that $\mathscr{J}^* = \mathscr{D}^*$ and $\widetilde{\mathscr{J}} = \widetilde{\mathscr{D}}$.

Note. The results present in this chapter constitute the second part of the article published in Semigroup Forum [24]. However, the long proofs have been reworked to provide greater clarity and make them easier to follow.

III.1 THE RELATIONS $\mathscr{L}^*$, $\widetilde{\mathscr{L}}$, $\mathscr{R}^*$ AND $\widetilde{\mathscr{R}}$

As a direct consequence of Lemma I.3.13, we have that the relations $\mathscr{L}_Q$, $\mathscr{L}_Q^*$, and $\widetilde{\mathscr{L}}_Q$ are equal (and similarly $\mathscr{R}_Q = \mathscr{R}_Q^* = \widetilde{\mathscr{R}}_Q$), but this is not true for $T(\mathscr{A}, \mathscr{B})$. Indeed, the following propositions will show that on $T(\mathscr{A}, \mathscr{B})$ the extended Green's relations $\mathscr{L}^*$, $\mathscr{R}^*$, $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}$ differ from the relations $\mathscr{L}$ and $\mathscr{R}$ given in Propositions II.4.4 and II.4.2.

We have seen in Lemma II.3.10 that in $T(\mathscr{A}, \mathscr{B})$, a map admits an idempotent as a left identity if and only if it is regular. We give here an alternative equivalence for a map to have an idempotent as a right identity.

Lemma III.1.1. *Let $\alpha \in T(\mathscr{A}, \mathscr{B})$ and $\eta \in E$. Then $\alpha\eta = \alpha$ if and only if $\operatorname{im}\alpha \subseteq \operatorname{im}\eta$.*

Proof. Let $\alpha \in T(\mathscr{A}, \mathscr{B})$ and $\eta \in E$. If $\alpha\eta = \alpha$, then it is clear that $\operatorname{im}\alpha \subseteq \operatorname{im}\eta$. Conversely, assume that $\operatorname{im}\alpha \subseteq \operatorname{im}\eta$. Since η is an idempotent, it follows that $\eta|_{\operatorname{im}\eta} = \mathrm{id}_{\operatorname{im}\eta}$ by Lemma I.4.24. Hence, $\eta|_{\operatorname{im}\alpha} = \mathrm{id}_{\operatorname{im}\alpha}$ which shows that $\alpha\eta = \alpha$. □

We can now give the description of $\mathscr{L}^*$ and $\widetilde{\mathscr{L}}$, which happen to be the same for $T(\mathscr{A}, \mathscr{B})$, as follows:

Proposition III.1.2. *Let $\alpha, \beta \in T(\mathscr{A}, \mathscr{B})$. Then $\alpha \,\widetilde{\mathscr{L}}\, \beta$ if and only if $\operatorname{im}\alpha = \operatorname{im}\beta$. Consequently, $\mathscr{L}^* = \widetilde{\mathscr{L}}$ in $T(\mathscr{A}, \mathscr{B})$.*

Proof. Assume first that $\alpha \,\widetilde{\mathscr{L}}\, \beta$. By Lemma II.2.6, there exist two idempotents $\eta, \theta \in T(\mathscr{A}, \mathscr{B})$ such that $\operatorname{im}\eta = \operatorname{im}\alpha$, $\operatorname{im}\theta = \operatorname{im}\beta$, so that $\alpha\eta = \alpha$ and $\beta\theta = \beta$. Since $\alpha \,\widetilde{\mathscr{L}}\, \beta$, it follows that $\alpha\theta = \alpha$ and $\beta\eta = \beta$. Hence, by Lemma III.1.1 we have that $\operatorname{im}\alpha \subseteq \operatorname{im}\theta = \operatorname{im}\beta$ and $\operatorname{im}\beta \subseteq \operatorname{im}\eta = \operatorname{im}\alpha$. Therefore $\operatorname{im}\alpha = \operatorname{im}\beta$.

Conversely, suppose that $\operatorname{im}\alpha = \operatorname{im}\beta$ and let η be an idempotent such that $\alpha\eta = \alpha$. Then, using Lemma III.1.1, we have that $\operatorname{im}\alpha \subseteq \operatorname{im}\eta$ so that $\operatorname{im}\beta \subseteq \operatorname{im}\eta$. Therefore $\beta\eta = \beta$, and exchanging the role of α and β in this argument gives us that for any idempotent θ, having $\beta\theta = \beta$ implies $\alpha\theta = \alpha$. Thus α and β are $\widetilde{\mathscr{L}}$-related.

To show that $\mathscr{L}^* = \widetilde{\mathscr{L}}$, notice that any two maps $\mathscr{L}$-related in $\operatorname{End}(\mathscr{A})$ will be $\mathscr{L}^*$-related in $T(\mathscr{A}, \mathscr{B})$ by Lemma I.3.3. Using Proposition I.5.6, we get the following inclusions:

$$\{(\alpha, \beta) \mid \operatorname{im}\alpha = \operatorname{im}\beta\} \subseteq \mathscr{L}^* \subseteq \widetilde{\mathscr{L}} = \{(\alpha, \beta) \mid \operatorname{im}\alpha = \operatorname{im}\beta\},$$

which forces the equalities, and so $\mathscr{L}^* = \widetilde{\mathscr{L}}$. □

Whereas the description of $\mathscr{L}^*$ and $\widetilde{\mathscr{L}}$ follows the description of $\mathscr{L}_Q$, the same cannot be said for $\mathscr{R}^*$ and $\widetilde{\mathscr{R}}$. Their explicit descriptions are given by Propositions III.1.3 and III.1.4.

Proposition III.1.3. *Let $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$. Then $\alpha \mathscr{R}^* \beta$ if and only if one of the following occurs:*

1) $\ker\alpha = \ker\beta$ *and $\alpha, \beta \in Q$; or*

2) $\ker\alpha \cap (B \times B) = \ker\beta \cap (B \times B)$ *and $\alpha, \beta \notin Q$.*

Proof. Let $\alpha \in Q$ and $\beta \in T(\mathscr{A},\mathscr{B})$ and assume that $\alpha \mathscr{R}^* \beta$. From Lemma II.3.10, we know that there exists $\gamma \in T(\mathscr{A},\mathscr{B})$ such that $\alpha = \gamma\alpha$. Since $\alpha \mathscr{R}^* \beta$ we also get that $\beta = \gamma\beta$, so that γ is a left-identity for β. Using Lemma II.3.10 again, this forces $\beta \in Q$. This means in particular that regular maps can only be $\mathscr{R}^*$-related to regular maps. Then, using Proposition II.4.2 together with the fact that $\mathscr{R}_Q = \mathscr{R}^*_Q$, we get that two regular elements α and β are $\mathscr{R}^*$-related if and only if $\ker\alpha = \ker\beta$.

We now consider the case where both α and β are non-regular elements. Notice first that for any map $\alpha \in T(\mathscr{A},\mathscr{B})$, if $\gamma\alpha = \delta\alpha$ with $\gamma \neq \delta = 1$, we obtain that $A\alpha = A\gamma\alpha \subseteq B\alpha$ and then $\alpha \in Q$. For this reason, in what follows, either $\gamma = \delta = 1$ or neither γ nor δ is equal to 1.

Let $\alpha, \beta \in Q^c$ and assume that $\alpha \mathscr{R}^* \beta$, that is, $\gamma\alpha = \delta\alpha$ if and only if $\gamma\beta = \delta\beta$ for $\gamma, \delta \in T(\mathscr{A},\mathscr{B})^1$. In order to show that $\ker\alpha \cap (B \times B) \subseteq \ker\beta \cap (B \times B)$, we construct for each pair of elements in the the kernel of α two specific maps $\gamma, \delta \in T(\mathscr{A},\mathscr{B})$ satisfying the relation $\gamma\alpha = \delta\alpha$. To this end, let $B = \langle \{y_k\} \rangle$ and $A = \langle \{y_k\} \sqcup \{u\} \sqcup \{x_j\} \rangle$. Then for any pair $(b_1, b_2) \in \ker\alpha \cap (B \times B)$ with $b_1 \neq b_2$, we define $\gamma, \delta \in T(\mathscr{A},\mathscr{B})$ by:

$$\gamma = \begin{pmatrix} y_k & u & x_j \\ y_k & b_1 & y_1 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} y_k & u & x_j \\ y_k & b_2 & y_1 \end{pmatrix}.$$

It is clear that $\gamma\alpha = \delta\alpha$, and since $\alpha \mathscr{R}^* \beta$, it follows that $\gamma\beta = \delta\beta$. Therefore $b_1\beta = u\gamma\beta = u\delta\beta = b_2\beta$ and thus $(b_1, b_2) \in \ker\beta \cap (B \times B)$. Using the same argument interchanging the roles of α and β, we deduce that $\ker\alpha \cap (B \times B) = \ker\beta \cap (B \times B)$.

Conversely, assume that $\ker\alpha \cap (B \times B) = \ker\beta \cap (B \times B)$ and that $\gamma\alpha = \delta\alpha$ for some $\gamma, \delta \in T(\mathscr{A},\mathscr{B})$. Then for any $a \in A$ we have $a\gamma\alpha = a\delta\alpha$ from which $(a\gamma, a\delta) \in \ker\alpha$. But since $a\gamma, a\delta \in B$, it follows that $(a\gamma, a\delta) \in \ker\beta$. Therefore $a\gamma\beta = a\delta\beta$, and since this is true for any $a \in A$ we get that $\gamma\beta = \delta\beta$. By symmetry of the argument, we see that $\gamma\alpha = \delta\alpha$ if and only if $\gamma\beta = \delta\beta$ for any $\gamma, \delta \in T(\mathscr{A}, \mathscr{B})$. Since this equivalence also holds when $\gamma = \delta = 1$ and using the note made earlier in the proof, we conclude that $\alpha \, \mathscr{R}^* \, \beta$ as required. □

Proposition III.1.4. *Let $\alpha, \beta \in T(\mathscr{A}, \mathscr{B})$. Then $\alpha \, \widetilde{\mathscr{R}} \, \beta$ if and only if one of the following happens:*

1) $\ker \alpha = \ker \beta$ *and $\alpha, \beta \in Q$; or*

2) $\alpha, \beta \in Q^c$.

Proof. From Lemma II.3.10 we know that $\alpha = \eta\alpha$ for some $\eta \in E$ if and only if α is regular. It follows that regular maps can only be $\widetilde{\mathscr{R}}$-related to regular maps. Using this, the fact that $\mathscr{R}_Q = \widetilde{\mathscr{R}}_Q$, and Proposition II.4.2, the proposition follows easily. □

From the description of our extended Green's relations, we can see that $T(\mathscr{A}, \mathscr{B})$ is not abundant nor Fountain since no idempotent lies in the $\mathscr{R}^*$- or $\widetilde{\mathscr{R}}$-class of an element in Q^c. However, using Lemma II.2.6, we can see that each $\mathscr{L}^*$-class of $T(\mathscr{A}, \mathscr{B})$ contains an idempotent and thus $T(\mathscr{A}, \mathscr{B})$ is a right-abundant semigroup.

As noticed in Section I.3.1 the relations $\mathscr{L}^*$ and $\mathscr{R}^*$ are respectively right and left congruences, and thus $\widetilde{\mathscr{L}}$ is also a right congruence. However, $\widetilde{\mathscr{R}}$ fails to be a left congruence in $T(\mathscr{A}, \mathscr{B})$ as soon as $\dim B \geq 3$, as given by the following lemma.

Lemma III.1.5. *If $\dim B \geq 3$, then there exist $\alpha, \beta, \gamma \in T(\mathscr{A}, \mathscr{B})$ such $\alpha \, \widetilde{\mathscr{R}} \, \beta$, but $\gamma\alpha \, \cancel{\widetilde{\mathscr{R}}} \, \gamma\beta$.*

Consequently, $\widetilde{\mathscr{R}}$ is not a congruence relation on $T(\mathscr{A}, \mathscr{B})$ if $\dim B \geq 3$.

Proof. Suppose that $\dim B \geq 3$ and write $B = \langle \{y_m\} \sqcup \{b_i\} \rangle$ and $A = B \sqcup \langle \{x\} \sqcup \{a_j\} \rangle$ where $|M| = 3$ and the sets $\{a_j\}$ and $\{b_i\}$ are possibly empty. Consider the following maps:

$$\alpha = \begin{pmatrix} y_1 & \{y_2, y_3, b_i, a_j\} & x \\ y_3 & y_3 & y_1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} y_1 & \{y_2, y_3, b_i, a_j\} & x \\ y_2 & y_3 & y_1 \end{pmatrix}.$$

Since neither α nor β is regular, it follows that $\alpha \, \widetilde{\mathscr{R}} \, \beta$. But we also have that

$$\alpha^2 = \begin{pmatrix} \{x, y_m, a_j, b_i\} \\ y_3 \end{pmatrix} \quad \text{and} \quad \alpha\beta = \begin{pmatrix} x & \{y_m, a_j, b_i\} \\ y_2 & y_3 \end{pmatrix},$$

which shows that $\alpha^2 \in Q$ while $\alpha\beta \notin Q$ and thus they cannot be $\widetilde{\mathscr{R}}$-related. □

It is possible to determine exactly the conditions on the subalgebra $\mathscr{B}$ under which $\widetilde{\mathscr{R}}$ becomes a left congruence, which we do in Lemma III.1.7. However, before that, we give the following lemma hinting that one needs to look outside of $\mathscr{R}^*$-classes to find examples where the congruence property fails.

Lemma III.1.6. *Let $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$ be such that $\ker\alpha \cap (B \times B) = \ker\beta \cap (B \times B)$. Then $\ker\gamma\alpha = \ker\gamma\beta$ and $\gamma\alpha \,\widetilde{\mathscr{R}}\, \gamma\beta$ for all $\gamma \in T(\mathscr{A},\mathscr{B})$.*

Proof. Consider $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$ such that $\ker\alpha \cap (B \times B) = \ker\beta \cap (B \times B)$. Let $\gamma \in T(\mathscr{A},\mathscr{B})$ and $x, y \in A$. Then:

$$\begin{aligned} (x,y) \in \ker\gamma\alpha &\iff (x\gamma, y\gamma) \in \ker\alpha \\ &\iff (x\gamma, y\gamma) \in \ker\beta \qquad \text{(since } x\gamma \text{ and } y\gamma \text{ are in } B\text{)} \\ &\iff (x,y) \in \ker\gamma\beta, \end{aligned}$$

so that $\ker\gamma\alpha = \ker\gamma\beta$. Additionally, Lemma II.3.6 tells us that $\gamma\alpha \in Q$ if and only if $\gamma\beta \in Q$. Thus either both of $\gamma\alpha$ and $\gamma\beta$ lie in Q and have the same kernel, or neither of them is in Q. In both cases Proposition III.1.4 gives that $\gamma\alpha$ and $\gamma\beta$ are $\widetilde{\mathscr{R}}$-related, as required. □

We can now give the exact characterisation of the subalgebras B that will permit $\widetilde{\mathscr{R}}$ to be a left congruence.

Lemma III.1.7. *The equivalence $\widetilde{\mathscr{R}}$ is a left congruence if and only if one of the following occurs:*

1) $\dim B = 2$ *and one-dimensional subalgebras are singletons; or*

2) $\dim B = 1$ *and the constant subalgebra is a singleton.*

Proof. First notice that since $\widetilde{\mathscr{R}}_Q = \mathscr{R}_Q$, it follows that $\widetilde{\mathscr{R}}$ is a left congruence on $Q \times Q$. Thus $\widetilde{\mathscr{R}}$ can only fail to be a left congruence if we use non-regular elements. This will happen if we can find maps $\alpha, \beta \in Q^c$ (which are then $\widetilde{\mathscr{R}}$-related by Proposition III.1.4) and a map $\gamma \in T(\mathscr{A},\mathscr{B})$ such that $\gamma\alpha$ and $\gamma\beta$ are not $\widetilde{\mathscr{R}}$-related. This, in turn, will be the case either if only one of $\gamma\alpha$ and $\gamma\beta$ lie in Q, or if both products are regular but have a different kernel.

With this in mind, let us assume that $\dim B = 2$ and that one-dimensional subalgebras are singletons and let $\alpha, \beta \notin Q$. Then $B\alpha \subsetneq A\alpha \subseteq B$ and similarly for β so that $\operatorname{rk}(\alpha|_B) = \operatorname{rk}(\beta|_B) = 1$. Also, since one-dimensional subalgebras are singletons, this forces $B\alpha = \{c\}$ and $B\beta = \{c'\}$ for some $c, c' \in B$. This means in particular that $\ker\alpha \cap (B \times B) = \ker\beta \cap (B \times B) = B \times B$. By Lemma III.1.6 it follows that for any map $\gamma \in T(\mathscr{A}, \mathscr{B})$, we have that $\gamma\alpha \,\widetilde{\mathscr{R}}\, \gamma\beta$, and thus $\widetilde{\mathscr{R}}$ is a left congruence for such an algebra.

Similarly, if $\dim B = 1$ and the constant subalgebra only consists of a single element, say 0, then it is clear that for any map $\alpha \notin Q$, we have that $B\alpha = \{0\}$. Therefore, the argument used above can be applied in a similar manner to give that $\widetilde{\mathscr{R}}$ is a left congruence in this case.

In order to show that outside of these cases $\widetilde{\mathscr{R}}$ fails to be a left congruence, we exhibit counterexamples. The first one was given in Lemma III.1.5 whenever $\dim B \geq 3$, so we can focus on the cases where $\dim B \leq 2$.

So assume that $\dim B \leq 2$ and write $B = \langle b_1, b_2 \rangle$, $A = B \sqcup \langle \{a_j\} \rangle$ where $b_1 \notin \langle b_2 \rangle$ and there exists a term $g \in \mathcal{T}_1^{\mathscr{A}}$ with $g(b_2) \neq b_2$. This means that either $\{b_1, b_2\}$ is an independent set, and thus we are in the case where B has dimension 2 and subalgebras are not singletons (since $\langle \{b_2\} \rangle$ contains at least two elements); or b_2 is a constant, and we are in the case where B has dimension 1 but contains at least two constants (namely, b_2 and $g(b_2)$). We now define the following maps:

$$\alpha = \begin{pmatrix} a_j & b_1 & b_2 \\ b_1 & b_2 & b_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} a_j & b_1 & b_2 \\ b_1 & g(b_2) & b_2 \end{pmatrix}, \quad \text{and} \quad \gamma = \begin{pmatrix} a_j & b_1 & b_2 \\ b_1 & b_1 & b_2 \end{pmatrix}.$$

Notice that since $b_2\alpha = b_2\beta = b_2\gamma = b_2$, this maps are well-defined endomorphisms even if b_2 is a constant. Then, we have the following:

- $\alpha, \beta \notin Q$ and thus $\alpha \,\widetilde{\mathscr{R}}\, \beta$;
- $\gamma\alpha = \begin{pmatrix} a_j & b_1 & b_2 \\ b_2 & b_2 & b_2 \end{pmatrix}$ and thus $\gamma\alpha \in Q$;
- $\gamma\beta = \begin{pmatrix} a_j & b_1 & b_2 \\ g(b_2) & g(b_2) & b_2 \end{pmatrix}$ and thus $\gamma\beta \in Q$;
- $(b_1, b_2) \in \ker\gamma\alpha$, but $(b_1, b_2) \notin \ker\gamma\beta$.

Therefore we have that $\gamma\alpha$ and $\gamma\beta$ are not $\widetilde{\mathscr{R}}$-related, whereas α and β are which shows that $\widetilde{\mathscr{R}}$ is not a left congruence in these two cases. This concludes the proof since all cases have now been covered. □

Before moving on to working with compositions of the relations $\mathscr{L}^*$ and $\mathscr{R}^*$, we give a technical lemma that will allow us to compare kernels of maps by looking at their definition on a basis.

Lemma III.1.8. *Let $B = \langle\{b_k\} \sqcup \{b_i\}\rangle$ and $A = B \sqcup \langle\{a_j\}\rangle$. Suppose that the maps $\gamma, \delta \in T(\mathscr{A},\mathscr{B})$ can be expressed as follows:*

$$\gamma = \begin{pmatrix} b_k & b_i & a_j \\ f_k & t_i\left(\overline{f_k}\right) & g_j \end{pmatrix} \qquad \text{and} \qquad \delta = \begin{pmatrix} b_k & b_i & a_j \\ x_k & t_i(\overline{x_k}) & y_j \end{pmatrix},$$

for some terms t_i, where the sets $\{f_k\}$ and $\{x_k\}$ are respective bases of $\operatorname{im}(B\gamma)$ *and* $\operatorname{im}(B\delta)$*. Then we have that* $\ker\gamma \cap (B \times B) = \ker\delta \cap (B \times B)$*.*
Consequently, if $\gamma, \delta \in Q^c$ we also obtain that $\gamma \mathscr{R}^* \delta$.

Proof. Let $\gamma, \delta \in T(\mathscr{A},\mathscr{B})$ be given as above and let $(a, b) \in \ker\gamma \cap (B \times B)$. Then there exist terms u and v such that $a = u\left(\overline{b_k}, \overline{b_i}\right)$ and $b = v\left(\overline{b_k}, \overline{b_i}\right)$, which gives us that

$$u\left(\overline{f_k}, \overline{t_i\left(\overline{f_k}\right)}\right) = a\gamma = b\gamma = v\left(\overline{f_k}, \overline{t_i\left(\overline{f_k}\right)}\right).$$

Now we extend $\{f_k\}$ to a basis of A through $\{d_m\}$, and we define $\eta \in T(\mathscr{A},\mathscr{B})$ by

$$\eta = \begin{pmatrix} f_k & d_m \\ x_k & x_1 \end{pmatrix}.$$

From this we get the following:

$$\begin{aligned} a\delta &= u\left(\overline{b_k}, \overline{b_i}\right)\delta = u\left(\overline{x_k}, \overline{t_i(\overline{x_k})}\right) \\ &= u\left(\overline{f_k\eta}, \overline{t_i\left(\overline{f_k\eta}\right)}\right) = u\left(\overline{f_k}, \overline{t_i\left(\overline{f_k}\right)}\right)\eta \\ &= a\gamma\eta = b\gamma\eta \\ &= v\left(\overline{f_k}, \overline{t_i\left(\overline{f_k}\right)}\right)\eta = v\left(\overline{x_k}, \overline{t_i(\overline{x_k})}\right) = b\delta, \end{aligned}$$

which means that $(a, b) \in \ker\delta \cap (B \times B)$. Hence $\ker\gamma \cap (B \times B) \subseteq \ker\delta \cap (B \times B)$. The reverse inclusion works similarly by using the map $\theta = \begin{pmatrix} x_k & e_m \\ f_k & f_1 \end{pmatrix} \in T(\mathscr{A},\mathscr{B})$. Therefore we have that $\ker\gamma \cap (B \times B) = \ker\delta \cap (B \times B)$. Moreover, if $\gamma, \delta \in Q^c$ then Proposition III.1.3 allows us to conclude that $\gamma \mathscr{R}^* \delta$. □

III.2 WORKING ON THE COMPOSITION $\mathscr{L}^* \circ \mathscr{R}^*$

As mentioned in Section I.3, the relations $\mathscr{L}^*$ and $\mathscr{R}^*$ do not commute in general, as can be seen in the following example.

Example III.2.1. Let $A = \langle\{x_1, x_2, x_3, x_4, x_5\}\rangle$ and $B = \langle\{x_3, x_4, x_5\}\rangle$. Define $\alpha, \beta, \gamma \in T(\mathscr{A}, \mathscr{B})$ by the following:

$$\alpha = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_3 & x_3 & x_3 & x_4 & x_5 \end{pmatrix}, \qquad \beta = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_3 & x_3 & x_5 & x_5 & x_5 \end{pmatrix},$$
$$\text{and} \quad \gamma = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_3 & x_4 & x_5 & x_5 & x_5 \end{pmatrix}.$$

Then we clearly have that $\alpha \in Q$, $\beta \notin Q$, $\gamma \notin Q$ and these maps satisfy the relations $\operatorname{im}\alpha = \operatorname{im}\gamma$ and $\ker\beta \cap (B \times B) = \ker\gamma \cap (B \times B)$. From Propositions III.1.2 and III.1.3 we therefore have that $\alpha\, \mathscr{L}^*\circ\mathscr{R}^*\, \beta$ through γ.

Now, in order to have $\alpha\, \mathscr{R}^*\circ\mathscr{L}^*\, \beta$, we need to find $\delta \in T(\mathscr{A}, \mathscr{B})$ such that $\operatorname{im}\delta = \operatorname{im}\beta = \langle\{x_3, x_5\}\rangle$ and $\ker\alpha = \ker\delta$ (since $\alpha \in Q$). However, for any such δ we would need

$$\operatorname{im}\delta \cong A/\ker\delta = A/\ker\alpha \cong \operatorname{im}\alpha,$$

giving that $\dim(\operatorname{im}\delta) = 3$ which is impossible. Therefore, no such $\delta \in T(\mathscr{A}, \mathscr{B})$ can exist and $\mathscr{L}^*$ and $\mathscr{R}^*$ do not commute in $T(\mathscr{A}, \mathscr{B})$.

When replacing in the example above the relation $\mathscr{R}^*$ by $\widetilde{\mathscr{R}}$, the same arguments hold since $\gamma, \beta \notin Q$ implies that $\gamma\, \widetilde{\mathscr{R}}\, \beta$. Therefore, similarly to $\mathscr{L}^*$ and $\mathscr{R}^*$, we can see that the relations $\widetilde{\mathscr{L}} = \mathscr{L}^*$ and $\widetilde{\mathscr{R}}$ do not commute in $T(\mathscr{A}, \mathscr{B})$, exhibiting a different behaviour from the usual Green's relations.

Nevertheless, in the case of $T(\mathscr{A}, \mathscr{B})$ it is possible to give a precise characterisation for $\mathscr{D}^*$. We do this in Theorem III.3.1. Somewhat surprisingly, it depends on the corank of the subalgebra $\mathscr{B}$ inside $\mathscr{A}$. In order to achieve this, we will look closely into a single composition of the relations $\mathscr{L}^*$ and $\mathscr{R}^*$ to give an exact description in Proposition III.2.6. We will then look at consequences of this result when considering only maps of finite rank in Section III.2.2, or the opposite, when focusing on maps having infinite rank in Section III.2.3.

III.2.1 A step towards $\mathscr{D}^*$

Since we know that two maps are $\mathscr{D}$-related in $\operatorname{End}(\mathscr{A})$ if and only if they have the same rank, one could ask if this is a sufficient condition to be $\mathscr{D}^*$ related in $T(\mathscr{A}, \mathscr{B})$. That is indeed the case as given by the following lemma.

Lemma III.2.2. *Let $\alpha, \beta \in T(\mathscr{A}, \mathscr{B})$ be such that $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$. Then $\alpha\, \mathscr{L}^*\circ\mathscr{R}^*\, \beta$ and thus $\alpha\, \mathscr{D}^*\, \beta$.*

Proof. For any $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$ with $\mathrm{rk}(\alpha) = \mathrm{rk}(\beta)$, Lemmas I.4.34 and II.2.5 give us that there exists $\mu \in T(\mathscr{A},\mathscr{B})$ with $\mathrm{im}\,\alpha = \mathrm{im}\,\mu$ and $\ker\beta = \ker\mu$ (so that also $\ker\beta \cap (B\times B) = \ker\mu \cap (B\times B)$). Thus $\alpha\,\mathscr{L}^*\,\mu$ by Proposition III.1.2. Additionally, since two maps with the same kernel are either both regular or both non-regular by Remark II.4.3, using the appropriate case in Proposition III.1.3 gives that $\mu\,\mathscr{R}^*\,\beta$. Therefore we have $\alpha\,\mathscr{L}^*{\circ}\mathscr{R}^*\,\beta$ and $\alpha\,\mathscr{D}^*\,\beta$. ☐

The full characterisation of $\mathscr{D}^*$ requires us to also concentrate on the composition $\mathscr{L}^*{\circ}\mathscr{R}^*$. The next two lemmas give us sufficient conditions for two maps to be $\mathscr{L}^*{\circ}\mathscr{R}^*$-related.

Lemma III.2.3. *Let $\alpha \in T(\mathscr{A},\mathscr{B})$. If a non-regular map $\beta \in T(\mathscr{A},\mathscr{B})$ is such that $\mathrm{rk}(\beta|_B) \geq \aleph_0$ and $B\beta \cong \mathrm{im}\,\alpha$, then $\alpha\,\mathscr{L}^*{\circ}\mathscr{R}^*\,\beta$.*

Proof. Assume that $\beta \notin Q$ is such that $\mathrm{rk}(\beta|_B) \geq \aleph_0$ and $B\beta \cong \mathrm{im}\,\alpha$. We then write $B\beta = \langle\{b_k\beta\}\rangle$, $B = \langle\{b_k\} \sqcup \{b_i\}\rangle$ and $A = B \sqcup \langle\{a_j\}\rangle$. By the assumption on the image of α, we also can write $A\alpha = \langle\{x_k\alpha\}\rangle$ for some $\{x_k\} \subseteq A$. Since $|K| = \mathrm{rk}(\beta|_B) \geq \aleph_0$, there exists a bijection ϕ between K and $K' = K \setminus \{1\}$ and for each $k \in K$, we set $z_k = x_{k\phi}\alpha$. We now define a map γ in $T(\mathscr{A},\mathscr{B})$ by:

$$\gamma = \begin{pmatrix} b_k & b_i & a_j \\ z_k & t_i(\overline{z_k}) & x_1\alpha \end{pmatrix},$$

where the terms t_i are such that $b_i\beta = t_i(\overline{b_k\beta})$. Then, we have that $\mathrm{im}\,\alpha = \langle\{x_k\alpha\}\rangle = \mathrm{im}\,\gamma \neq \langle\{z_k\}\rangle = \mathrm{im}\,(\gamma|_B)$ and thus $\alpha\,\mathscr{L}^*\,\gamma$ with $\gamma \notin Q$. Also, looking at the expression for γ and noting that $\beta = \begin{pmatrix} b_k & b_i & a_j \\ b_k\beta & t_i\left(\overline{b_k\beta}\right) & a_j\beta \end{pmatrix}$, we have that $\gamma\,\mathscr{R}^*\,\beta$ by Lemma III.1.8. Therefore $\alpha\,\mathscr{L}^*{\circ}\mathscr{R}^*\,\beta$ as expected. ☐

Remark III.2.4. For any map $\gamma \in T(\mathscr{A},\mathscr{B})$, we clearly have that $\mathrm{rk}(\gamma|_B) \leq \mathrm{rk}(\gamma)$. Moreover, by writing $B\gamma = \langle\{b_k\gamma\}\rangle$ and $\mathrm{im}\,\gamma = \langle\{b_k\gamma\} \sqcup \{a_j\gamma\}\rangle$, we have that $|J| \leq \mathrm{codim}_{\mathrm{A}}\,B$, and therefore $\mathrm{rk}(\gamma) \leq \mathrm{rk}(\gamma|_B) + \mathrm{codim}_{\mathrm{A}}\,B$.

Lemma III.2.5. *Let $\alpha \in T(\mathscr{A},\mathscr{B})$. If a non-regular map $\beta \in T(\mathscr{A},\mathscr{B})$ satisfies the inequality*

$$\mathrm{rk}(\beta|_B) < \mathrm{rk}(\alpha) \leq \mathrm{rk}(\beta|_B) + \mathrm{codim}_{\mathrm{A}}\,B,$$

then $\alpha\,\mathscr{L}^*{\circ}\mathscr{R}^*\,\beta$.

Proof. Let α and β be as in the statement. Using the same notation as in the previous lemma, we write $B\beta = \langle\{b_k\beta\}\rangle$, $B = \langle\{b_k\}\sqcup\{b_i\}\rangle$ and $A = B\sqcup\langle\{a_j\}\rangle$. Since $\operatorname{rk}(\alpha) > \operatorname{rk}(\beta|_B)$, there exists a set $L \neq \emptyset$ such that we also have $A\alpha = \langle\{x_k\alpha\}\sqcup\{y_\ell\alpha\}\rangle$ for some $\{x_k\}, \{y_\ell\} \subseteq A$. By noticing that $\operatorname{codim}_{\mathrm{A}} B = |J|$, we can now rewrite the condition $\operatorname{rk}(\beta|_B) < \operatorname{rk}(\alpha) \leq \operatorname{rk}(\beta|_B) + \operatorname{codim}_{\mathrm{A}} B$ in terms of the underlying sets as

$$|K| < |K \sqcup L| \leq |K \sqcup J|.$$

Clearly, if $\dim(B\beta) = |K|$ is finite, then we obtain from this equation that $|L| \leq |J|$. Otherwise, $|K|$ is infinite and we have that $|K| < |K \sqcup L| = \max\{|K|, |L|\}$, so that $|L| > |K|$. So $|L| = |K \sqcup L| \leq |K \sqcup J| = |J|$. Thus in both the finite and the infinite dimensional cases, we have that $|L| \leq |J|$ and we can extract a subset $J' \subseteq J$ such that there exists a bijection $\phi\colon J' \to L$. From this, we define elements $\{z_j\} \subseteq B$ by $z_j = y_{j\phi}\alpha$ if $j \in J'$ and $z_j = y_1\alpha$ (which necessarily exists) otherwise, which implies that $\langle z_j\rangle = \langle\{y_\ell\alpha\}\rangle$. Now we define a map $\gamma \in T(\mathscr{A}, \mathscr{B})$ as follows:

$$\gamma = \begin{pmatrix} b_k & b_i & a_j \\ x_k\alpha & t_i(\overline{x_k\alpha}) & z_j \end{pmatrix},$$

where the terms t_i are such that $b_i\beta = t_i\Big(\overline{b_k\beta}\Big)$. Then we have that

$$\operatorname{im}\gamma = \langle\{x_k\alpha\} \sqcup \{y_\ell\alpha\}\rangle = \operatorname{im}\alpha,$$

and thus $\alpha\,\mathscr{L}^*\,\gamma$ by Proposition III.1.2. We can also see that $y_1\alpha \in \operatorname{im}\gamma \setminus B\gamma$ so that $\gamma \notin Q$, while $\gamma\,\mathscr{R}^*\,\beta$ by Lemma III.1.8. Therefore $\alpha\,\mathscr{L}^*\circ\mathscr{R}^*\,\beta$, which concludes the proof. $\square$

We can now give the exact description of the relation $\mathscr{L}^*\circ\mathscr{R}^*$.

Proposition III.2.6. *Let $\alpha \in T(\mathscr{A}, \mathscr{B})$. Then $\alpha\,\mathscr{L}^*\circ\mathscr{R}^*\,\beta$ for some $\beta \in T(\mathscr{A}, \mathscr{B})$ if and only if one of the following happens:*

1) $\beta \in Q$ and $\operatorname{im}\alpha \cong \operatorname{im}\beta$;

2) $\beta \notin Q$ and $B\beta \cong \operatorname{im}\alpha$ with $\operatorname{rk}(\beta|_B) \geq \aleph_0$;

3) $\beta \notin Q$ and $\operatorname{rk}(\beta|_B) < \operatorname{rk}(\alpha) \leq \operatorname{rk}(\beta|_B) + \operatorname{codim}_{\mathrm{A}} B$.

Exchanging the roles of α and β, we have the dual characterisation for when $\alpha\,\mathscr{R}^*\circ\mathscr{L}^*\,\beta$.

Proof. Let $\alpha \in T(\mathscr{A},\mathscr{B})$ and consider the three given conditions. The cases when $\beta \notin Q$ are directly given by Lemmas III.2.3 and III.2.5, whereas when $\beta \in Q$ and $\operatorname{im}\alpha \cong \operatorname{im}\beta$ we use Lemma III.2.2 to also obtain that $\alpha \mathscr{L}^*\circ\mathscr{R}^* \beta$.

To show the converse, assume that $\beta \in T(\mathscr{A},\mathscr{B})$ is such that $\alpha \mathscr{L}^*\circ\mathscr{R}^* \beta$. Then there exists $\gamma \in T(\mathscr{A},\mathscr{B})$ such that $\alpha \mathscr{L}^* \gamma$ and $\gamma \mathscr{R}^* \beta$. Therefore $\operatorname{im}\alpha = \operatorname{im}\gamma$ by Proposition III.1.2, and Proposition III.1.3 tells us that either $\beta, \gamma \in Q$ are such that $\ker\gamma = \ker\beta$, or $\beta, \gamma \notin Q$ and $\ker\gamma \cap (B \times B) = \ker\beta \cap (B \times B)$. If $\beta \in Q$ (and hence also $\gamma \in Q$), then we have that

$$\operatorname{im}\alpha = \operatorname{im}\gamma \cong A/\ker\gamma = A/\ker\beta \cong \operatorname{im}\beta,$$

which gives us the first case.

We now assume that $\beta, \gamma \notin Q$. Thus

$$\operatorname{im}\beta|_B \cong B/(\ker\beta \cap (B \times B)) = B/(\ker\gamma \cap (B \times B)) \cong \operatorname{im}\gamma|_B,$$

so that $\operatorname{rk}(\beta|_B) = \operatorname{rk}(\gamma|_B)$. If $\operatorname{rk}(\gamma) = \operatorname{rk}(\gamma|_B)$, then $\operatorname{rk}(\gamma) \geq \aleph_0$ by Lemma II.3.8 and from $\operatorname{rk}(\beta|_B) = \operatorname{rk}(\gamma) = \operatorname{rk}(\alpha)$ we get that $B\beta \cong \operatorname{im}\alpha$ which corresponds to the second case. Otherwise, $\operatorname{rk}(\gamma|_B) < \operatorname{rk}(\gamma)$ and then, setting Z to be a basis extension of B in A, we have that

$$\begin{aligned}\operatorname{rk}(\beta|_B) = \operatorname{rk}(\gamma|_B) < \operatorname{rk}(\alpha) = \operatorname{rk}(\gamma) = \dim(A\gamma) &\leq \dim(B\gamma) + \dim(Z\gamma)\\ &\leq \operatorname{rk}(\gamma|_B) + \dim Z = \operatorname{rk}(\beta|_B) + \operatorname{codim}_{\mathrm{A}} B,\end{aligned}$$

giving us the remaining case. □

Remark III.2.7. Notice that if $\beta \in Q^c$ is such that $\operatorname{rk}(\beta|_B) \geq \max\{\aleph_0, \operatorname{codim}_{\mathrm{A}} B\}$, then the third case in Proposition III.2.6 cannot happen. For, in this situation, we obtain that $\operatorname{rk}(\beta|_B) + \operatorname{codim}_{\mathrm{A}} B = \operatorname{rk}(\beta|_B)$. Using the inequalities of Remark III.2.4, this also means that $\operatorname{rk}(\beta) = \operatorname{rk}(\beta|_B)$.

As a direct corollary from this proposition we get that two maps cannot be $\mathscr{L}^*\circ\mathscr{R}^*$-related if their ranks are more than $\operatorname{codim}_{\mathrm{A}} B$ apart. This also shows how important the codimension of $\mathscr{B}$ is in order to characterise $\mathscr{D}^*$.

Corollary III.2.8. *Let $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$. If $\alpha \mathscr{L}^*\circ\mathscr{R}^* \beta$, then*

$$\operatorname{rk}(\alpha) \leq \operatorname{rk}(\beta) + \operatorname{codim}_{\mathrm{A}} B \quad \text{and} \quad \operatorname{rk}(\beta) \leq \operatorname{rk}(\alpha) + \operatorname{codim}_{\mathrm{A}} B.$$

Consequently, if $\alpha \mathscr{D}^* \beta$, then for some natural number n, we have that

$$\operatorname{rk}(\alpha) \leq \operatorname{rk}(\beta) + n \cdot \operatorname{codim}_{\mathrm{A}} B \quad \text{and} \quad \operatorname{rk}(\beta) \leq \operatorname{rk}(\alpha) + n \cdot \operatorname{codim}_{\mathrm{A}} B.$$

Proof. Let $\alpha, \beta \in T(\mathscr{A}, \mathscr{B})$ be such that $\alpha \,\mathscr{L}^*\circ\mathscr{R}^*\, \beta$, and consider the different cases of Proposition III.2.6:

1) if $\beta \in Q$ and $\operatorname{im}\alpha \cong \operatorname{im}\beta$, then $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$;
2) if $\beta \notin Q$ and $B\beta \cong \operatorname{im}\alpha$, then we get $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta|_B) \leq \operatorname{rk}(\beta)$ and also $\operatorname{rk}(\beta) \leq \operatorname{rk}(\beta|_B) + \operatorname{codim}_{\mathrm{A}} B = \operatorname{rk}(\alpha) + \operatorname{codim}_{\mathrm{A}} B$;
3) otherwise, $\beta \notin Q$ and we have that $\operatorname{rk}(\beta|_B) < \operatorname{rk}(\alpha) \leq \operatorname{rk}(\beta|_B) + \operatorname{codim}_{\mathrm{A}} B$. Then we get that $\operatorname{rk}(\beta) \leq \operatorname{rk}(\beta|_B) + \operatorname{codim}_{\mathrm{A}} B \leq \operatorname{rk}(\alpha) + \operatorname{codim}_{\mathrm{A}} B$ and also $\operatorname{rk}(\alpha) \leq \operatorname{rk}(\beta) + \operatorname{codim}_{\mathrm{A}} B$.

In all cases we can see that the two inequalities $\operatorname{rk}(\alpha) \leq \operatorname{rk}(\beta) + \operatorname{codim}_{\mathrm{A}} B$ and $\operatorname{rk}(\beta) \leq \operatorname{rk}(\alpha) + \operatorname{codim}_{\mathrm{A}} B$ always hold as expected.

For the second part of the proposition, we now assume that $\alpha \,\mathscr{D}^*\, \beta$. Then there exists a finite sequence $\gamma_0, \gamma_1, \ldots, \gamma_{2n}$ in $T(\mathscr{A}, \mathscr{B})$ such that

$$\alpha = \gamma_0 \,\mathscr{L}^*\, \gamma_1 \,\mathscr{R}^*\, \gamma_2 \,\mathscr{L}^* \ldots \mathscr{L}^*\, \gamma_{2n-1} \,\mathscr{R}^*\, \gamma_{2n} = \beta.$$

Thus $\alpha \,\mathscr{L}^*\circ\mathscr{R}^*\, \gamma_2$ and $\gamma_{2i} \,\mathscr{L}^*\circ\mathscr{R}^*\, \gamma_{2i+2}$ for all $1 \leq i < n$. Therefore we obtain the left inequalities of the previous part for each of these γ_{2i}, that is, we have $\operatorname{rk}(\alpha) \leq \operatorname{rk}(\gamma_2) + \operatorname{codim}_{\mathrm{A}} B$ and $\operatorname{rk}(\gamma_{2i}) \leq \operatorname{rk}(\gamma_{2i+2}) + \operatorname{codim}_{\mathrm{A}} B$ for all $1 \leq i < n$. Combining them all together, we get that

$$\operatorname{rk}(\alpha) \leq \operatorname{rk}(\gamma_{2n}) + n \cdot \operatorname{codim}_{\mathrm{A}} B = \operatorname{rk}(\beta) + n \cdot \operatorname{codim}_{\mathrm{A}} B.$$

Similarly, $\operatorname{rk}(\beta) \leq \operatorname{rk}(\alpha) + n \cdot \operatorname{codim}_{\mathrm{A}} B$ by using the same argument on the right inequalities, which concludes the proof. □

III.2.2 Equivalence classes in finite ranks

In this section, we restrict ourselves only to maps of finite rank. First of all, we can show that maps with minimal rank e can only be related to another map with minimal rank.

Lemma III.2.9. *Let* $\alpha \in T(\mathscr{A}, \mathscr{B})$ *be such that* $\operatorname{rk}(\alpha) = e$. *Then* $\alpha \,\mathscr{L}^*\circ\mathscr{R}^*\, \beta$ *for* $\beta \in T(\mathscr{A}, \mathscr{B})$ *if and only if* $\operatorname{rk}(\beta) = e$. *Consequently,* $\alpha \,\mathscr{D}^*\, \beta$ *if and only if* $\beta \in T_{e^+}$.

Proof. For both statements, one direction is given by Lemma III.2.2, so we are only left to show that having minimal rank is a necessary condition to be related to a map of rank e.

Let $\beta \in T(\mathscr{A},\mathscr{B})$ be such that $\alpha\,\mathscr{L}^*\circ\mathscr{R}^*\,\beta$. Then there exists $\gamma \in T(\mathscr{A},\mathscr{B})$ such that $\alpha\,\mathscr{L}^*\,\gamma$ and $\gamma\,\mathscr{R}^*\,\beta$. By Proposition III.1.2, we have that $\operatorname{im}\alpha = \operatorname{im}\gamma$ and thus $\operatorname{rk}(\gamma) = e$ and $\gamma \in Q$. From this, Proposition III.1.3 gives us that $\beta \in Q$ and $\ker\gamma = \ker\beta$ which implies that $\operatorname{im}\gamma \cong \operatorname{im}\beta$ so that $\operatorname{rk}(\beta) = e$ as required.

Similarly, if $\alpha\,\mathscr{D}^*\,\beta$, then there exist $\gamma_1, \dots, \gamma_n$ such that

$$\alpha\,\mathscr{L}^*\circ\mathscr{R}^*\,\gamma_1\,\mathscr{L}^*\circ\mathscr{R}^*\,\gamma_2\cdots\gamma_{n-1}\,\mathscr{L}^*\circ\mathscr{R}^*\,\gamma_n = \beta,$$

and by induction on the argument above, we obtain that $\operatorname{rk}(\beta) = e$. Therefore, a map in T_{e^+} can only be $\mathscr{D}^*$-related to another one with same rank e. □

If $\mathscr{B}$ is a maximal proper subalgebra of $\mathscr{A}$, then the set of maps that can be reached through a series of composition of $\mathscr{L}^*$ and $\mathscr{R}^*$ starting from a map α with finite rank is restricted to those having the same rank as α. This is given formally by the following:

Lemma III.2.10. *Assume that* $\operatorname{codim}_{\mathrm{A}} B = 1$ *and let* $\alpha \in T(\mathscr{A},\mathscr{B})$ *be a map of finite rank. Then* $\alpha\,\mathscr{L}^*\circ\mathscr{R}^*\,\beta$ *for some* $\beta \in T(\mathscr{A},\mathscr{B})$ *if and only if* $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$. *Consequently,* $\alpha\,\mathscr{D}^*\,\beta$ *for* $\beta \in T(\mathscr{A},\mathscr{B})$ *if and only if* $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$.

Proof. Since the sufficient condition was given by Lemma III.2.2, we only need to show one direction. So we let $\beta \in T(\mathscr{A},\mathscr{B})$ be such that $\alpha\,\mathscr{L}^*\circ\mathscr{R}^*\,\beta$.

We first suppose that $\beta \in Q$. Then by Proposition III.2.6, $\alpha\,\mathscr{L}^*\circ\mathscr{R}^*\,\beta$ if and only if $\operatorname{im}\alpha \cong \operatorname{im}\beta$ which is equivalent to $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$.

Now suppose that $\beta \notin Q$. Since α has finite rank, only case 3) of Proposition III.2.6 can occur. Hence we have that

$$\operatorname{rk}(\beta|_B) < \operatorname{rk}(\alpha) \le \operatorname{rk}(\beta|_B) + \operatorname{codim}_{\mathrm{A}} B = \operatorname{rk}(\beta|_B) + 1.$$

This forces $\operatorname{rk}(\beta|_B)$ to be finite and using the contrapositive of Lemma II.3.8 we get that

$$\begin{aligned}\operatorname{rk}(\beta|_B) = \dim(B\beta) < \operatorname{rk}(\beta) &= \dim(A\beta)\\ &\le \dim(B\beta) + \operatorname{codim}_{\mathrm{A}} B\\ &= \operatorname{rk}(\beta|_B) + 1,\end{aligned}$$

which in turn forces $\operatorname{rk}(\beta) = \operatorname{rk}(\beta|_B) + 1$. This also forces $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta|_B) + 1$, and thus $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$, as required.

As a direct consequence, if $\alpha\,\mathscr{D}^*\,\beta$ for some $\beta \in T(\mathscr{A},\mathscr{B})$, then $\alpha\,(\mathscr{L}^*\circ\mathscr{R}^*)^n\,\beta$ for some $n \in \mathbb{N}$, and by induction on n, we get that $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$, which concludes the proof. □

If we now look at a subalgebra that is not a maximal proper subalgebra then, by consecutive compositions of $\mathscr{L}^*$ and $\mathscr{R}^*$, we are able to go up and down the finite ranks as long as we keep clear of T_{e^+}. This process is given formally by the following lemma.

Lemma III.2.11. *Assume that* $\operatorname{codim}_{\mathrm{A}} B \geq 2$ *and let* $\alpha \in Q$ *be a map of finite rank strictly greater than* e.

If $\operatorname{rk}(\alpha) \geq e+2$ *then there exist* $\delta_1 \notin Q$ *and* $\gamma_1 \in Q$ *such that* $\operatorname{rk}(\delta_1) = \operatorname{rk}(\gamma_1) = \operatorname{rk}(\alpha) - 1$ *and* $\alpha \,\mathscr{L}^*\circ\mathscr{R}^*\, \delta_1 \,\mathscr{L}^*\, \gamma_1$.

If $\operatorname{rk}(\alpha) < \dim B$, *then there exist* $\delta_2 \notin Q$ *and* $\gamma_2 \in Q$ *such that* $\operatorname{rk}(\delta_2) = \operatorname{rk}(\alpha)$, $\operatorname{rk}(\gamma_2) = \operatorname{rk}(\alpha) + 1$ *and* $\alpha \,\mathscr{L}^*\, \delta_2 \,\mathscr{R}^*\circ\mathscr{L}^*\, \gamma_2$.

Consequently, for all $\beta \in T(\mathscr{A}, \mathscr{B})$ *such that* $e < \operatorname{rk}(\beta) < \aleph_0$, *there exists* $n \in \mathbb{N}$ *such that* $\alpha \,(\mathscr{L}^*\circ\mathscr{R}^*)^n\, \beta$.

Proof. Suppose that $\operatorname{codim}_{\mathrm{A}} B \geq 2$ and consider $\alpha \in Q$ such that $e < \operatorname{rk}(\alpha) < \aleph_0$. Since α is regular, we let $A\alpha = B\alpha = \langle\{b_i\alpha\}\rangle$, $B = \langle\{b_i\} \sqcup \{c_k\}\rangle$ and $A = B \sqcup \langle\{x_j\}\rangle$ with $|J| \geq 2$. Then we have that

$$\alpha = \begin{pmatrix} b_i & c_k & x_j \\ b_i\alpha & u_k(\overline{b_i\alpha}) & v_j(\overline{b_i\alpha}) \end{pmatrix}$$

for some terms u_k and v_j.

For the first part, since $|I| = \operatorname{rk}(\alpha) \geq e+2$, define δ_1 and γ_1 as follows:

$$\delta_1 = \begin{pmatrix} b_{i\geq 3} & \{b_1, b_2\} & c_k & x_j \\ b_i\alpha & d & d & b_2\alpha \end{pmatrix} \quad \text{and} \quad \gamma_1 = \begin{pmatrix} b_{i\geq 2} & b_1 & c_k & x_j \\ b_i\alpha & b_2\alpha & b_2\alpha & b_2\alpha \end{pmatrix},$$

where the set $\{b_{i\geq 3}\}$ is possibly empty and the element $d \in B$ is taken as a constant if $e = 0$, and $d = b_3\alpha$ otherwise (which necessarily exists since $|I| = \operatorname{rk}(\alpha) \geq 3$ in that case). Thus $\delta_1 \notin Q$, $\gamma_1 \in Q$ and we have that $\operatorname{im}\delta_1 = \langle\{b_i\alpha\} \setminus \{b_1\alpha\}\rangle = \operatorname{im}\gamma_1$, so that $\delta_1 \,\mathscr{L}^*\, \gamma_1$ and $\operatorname{rk}(\delta_1) = \operatorname{rk}(\gamma_1) = |I| - 1$. Also $\operatorname{rk}(\delta_1|_B) = |I| - 2 < |I| = \operatorname{rk}(\alpha)$ and $\operatorname{rk}(\delta_1|_B) + \operatorname{codim}_{\mathrm{A}} B \geq |I| - 2 + 2 = \operatorname{rk}(\alpha)$, which shows that $\alpha \,\mathscr{L}^*\circ\mathscr{R}^*\, \delta_1$ by Proposition III.2.6, finishing the first part of the proof.

Now assume that $e < \operatorname{rk}(\alpha) < \dim B$, then there exists an element $z \in B$ such that $z \notin \operatorname{im}\alpha$. Using the same notation as above with this time $|I| = \operatorname{rk}(\alpha) \geq 1$, define δ_2 and γ_2 by

$$\delta_2 = \begin{pmatrix} b_{i\geq 2} & b_1 & c_k & x_j \\ b_i\alpha & d & d & b_1\alpha \end{pmatrix} \quad \text{and} \quad \gamma_2 = \begin{pmatrix} b_i & c_k & x_j \\ b_i\alpha & z & x_j\alpha \end{pmatrix},$$

where the set $\{b_{i\geq 2}\}$ is again possibly empty and the element $d \in B$ is defined in a similar way as before, that is, d is chosen as any constant if $e = 0$ and is set to $b_2\alpha$ (which then exists) otherwise. Then clearly $\delta_2 \notin Q$ and $\operatorname{im}\alpha = \operatorname{im}\delta_2$ so $\alpha \,\mathscr{L}^*\, \delta_2$. Also $\{x_j\alpha\} \subseteq \langle\{b_i\alpha\}\rangle$ so $\gamma_2 \in Q$. Finally, since $\operatorname{codim}_{\mathrm{A}} B \geq 2$, we have that

$$\operatorname{rk}(\delta_2|_B) = |I| - 1 < \operatorname{rk}(\gamma_2) = |I| + 1 \leq \operatorname{rk}(\delta_2|_B) + \operatorname{codim}_{\mathrm{A}} B.$$

Hence $\gamma_2 \,\mathscr{L}^*{\circ}\mathscr{R}^*\, \delta_2$ by Proposition III.2.6, finishing the proof of the second part of the lemma.

Now consider $\beta \in T(\mathscr{A},\mathscr{B})$ such that $e < \operatorname{rk}(\beta) < \aleph_0$. If $\beta \notin Q$, then by Lemma II.3.7, there exists $\beta' \in Q$ such that $\operatorname{im}\beta' = \operatorname{im}\beta$ and thus, $\beta' \,\mathscr{L}^*\, \beta \,\mathscr{R}^*\, \beta$, so we can assume that $\beta \in Q$ in the first place. Similarly, we can assume that $\operatorname{rk}(\alpha) \geq \operatorname{rk}(\beta)$ since otherwise we can exchange the role of α and β. If $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$, then $\alpha \,\mathscr{L}^*{\circ}\mathscr{R}^*\, \beta$ by Lemma III.2.2, so we can assume that $\operatorname{rk}(\alpha) > \operatorname{rk}(\beta)$. Set $m = \operatorname{rk}(\alpha) - \operatorname{rk}(\beta)$ and construct $\gamma_1, \ldots, \gamma_m \in Q$ by the process described in the first part of the lemma with the following properties:

- $\operatorname{rk}(\gamma_1) = \operatorname{rk}(\alpha) - 1$ and $\alpha \,(\mathscr{L}^*{\circ}\mathscr{R}^*)^2\, \gamma_1$;
- $\operatorname{rk}(\gamma_m) = \operatorname{rk}(\beta)$; and
- $\operatorname{rk}(\gamma_{r+1}) = \operatorname{rk}(\gamma_r) - 1$ and $\gamma_r \,(\mathscr{L}^*{\circ}\mathscr{R}^*)^2\, \gamma_{r+1}$ for all $1 \leq r \leq m-1$.

Then we have that $\alpha \,(\mathscr{L}^*{\circ}\mathscr{R}^*)^{2m}\, \gamma_m$ and $\gamma_m \,\mathscr{L}^*{\circ}\mathscr{R}^*\, \beta$. Therefore, in all cases, we have that $\alpha \,(\mathscr{L}^*{\circ}\mathscr{R}^*)^n\, \beta$ for some integer n. □

III.2.3 Dealing with infinite ranks

While the previous section was concerned by maps of finite rank, on the other hand, this one is focused on maps of infinite rank, whenever this is possible. For it to make sense, we are assuming that the dimension of $\mathscr{A}$ and $\mathscr{B}$ are both infinite. We first show that if the subalgebra $\mathscr{B}$ has a codimension smaller than its dimension, then a map with rank larger that the codimension of B cannot be $\mathscr{L}^*{\circ}\mathscr{R}^*$-related with maps of a different rank.

Lemma III.2.12. *Assume that* $\operatorname{codim}_{\mathrm{A}} B < \kappa \leq \dim B$ *for some infinite cardinal* κ *and let* $\alpha \in T(\mathscr{A},\mathscr{B})$ *be such that* $\operatorname{rk}(\alpha) = \kappa$*. Let* $\beta \in T(\mathscr{A},\mathscr{B})$*. Then* $\alpha \,\mathscr{L}^*{\circ}\mathscr{R}^*\, \beta$ *if and only if* $\operatorname{rk}(\beta) = \operatorname{rk}(\alpha)$*.*

Proof. One direction is already given by Lemma III.2.2. We therefore assume that $\alpha \,\mathscr{L}^*{\circ}\mathscr{R}^*\, \beta$ and we go through the cases of Proposition III.2.6. If $\operatorname{im}\alpha \cong \operatorname{im}\beta$, then we directly have that $\mathrm{rk}(\alpha) = \mathrm{rk}(\beta)$. On the other hand, if $\operatorname{im}\alpha \cong B\beta$ then $\mathrm{rk}(\beta|_B) = \kappa$. But then $\kappa = \mathrm{rk}(\beta|_B) \leq \mathrm{rk}(\beta) \leq \mathrm{rk}(\beta|_B) + \operatorname{codim}_{\mathrm{A}} B = \kappa$, and therefore $\mathrm{rk}(\beta) = \kappa = \mathrm{rk}(\alpha)$. Notice that the third case of the proposition cannot occur since we would have that $\kappa = \mathrm{rk}(\alpha) \leq \mathrm{rk}(\beta|_B) + \operatorname{codim}_{\mathrm{A}} B$ which forces $\mathrm{rk}(\beta|_B) = \kappa$ and thus the condition $\mathrm{rk}(\beta|_B) < \mathrm{rk}(\alpha)$ is not satisfied. □

In another way, if the dimension and codimension of $\mathscr{B}$ are both infinite cardinals, then from a map with rank at least $\aleph_0$ we can reach maps with larger infinite rank through the relation $\mathscr{L}^*\circ\mathscr{R}^*$, as long as we do not go further than the codimension of B. This idea is given more formally in the following lemma.

Lemma III.2.13. *Assume that* $\dim B$ *and* $\operatorname{codim}_{\mathrm{A}} B$ *are both infinite cardinals, and set* $M = \min\{\dim B, \operatorname{codim}_{\mathrm{A}} B\}$. *Let* $\alpha \in Q$ *be such that* $\aleph_0 \leq \mathrm{rk}(\alpha) < M$. *Then for all* ν *with* $\mathrm{rk}(\alpha) < \nu \leq M$ *there exists* $\beta \in T(\mathscr{A}, \mathscr{B})$ *such that* $\mathrm{rk}(\beta) = \nu$ *and* $\alpha\, \mathscr{L}^*\circ\mathscr{R}^*\, \beta$.

Proof. Let $\alpha \in Q$. Then we can write $A\alpha = B\alpha = \langle\{b_k\alpha\}\rangle$, $B = \langle\{b_k\} \sqcup \{c_i\}\rangle$ and $A = B \sqcup \langle\{a_j\}\rangle$. By assumption on the rank of α, we have that $|K \sqcup I| = \dim B > \mathrm{rk}(\alpha) = |K| \geq \aleph_0$, and thus it follows that $|I| = \dim B > \aleph_0$. Now, let ν be such that $\mathrm{rk}(\alpha) < \nu \leq M = \min\{|I|, |J|\}$. Then there exist sets $S \subseteq J$ and $S' \subseteq I$ such that $|S| = |S'| = \nu$, and we let $\phi\colon S \to S'$ be a bijection between them. For all $j \in J$ we now set elements $z_j \in B$ by $z_j = c_{j\phi}$ if $j \in S$ and $z_j = c_1$ otherwise, and we define the map $\beta \in T(\mathscr{A}, \mathscr{B})$ as:

$$\beta = \begin{pmatrix} b_k & c_i & a_j \\ b_k\alpha & b_1\alpha & z_j \end{pmatrix}.$$

Clearly we have that $\beta \notin Q$ and $\mathrm{rk}(\beta) = |K \sqcup S| = |K| + \nu = \mathrm{rk}(\alpha) + \nu = \nu$. Also $B\beta = \langle\{b_k\alpha\}\rangle = \operatorname{im}\alpha$ and from the second case of Proposition III.2.6 we have that $\alpha\, \mathscr{L}^*\circ\mathscr{R}^*\, \beta$, which concludes the proof. □

Finally, we can show that if the codimension of $\mathscr{B}$ is infinite, then it is possible to bridge the gap between any map of finite rank with a map of infinite rank smaller than the codimension of $\mathscr{B}$ with only a few iterations of the composition $\mathscr{L}^*\circ\mathscr{R}^*$.

Lemma III.2.14. *Assume that* $\operatorname{codim}_{\mathrm{A}} B = \kappa$ *for some infinite cardinal* κ *and let* $\alpha, \beta \in T(\mathscr{A}, \mathscr{B})$ *be such that the following inequalities hold:*

$$e < \mathrm{rk}(\alpha) < \aleph_0 \leq \mathrm{rk}(\beta) \leq \kappa.$$

Then $\alpha\, (\mathscr{L}^*\circ\mathscr{R}^*)^4\, \beta$ *and thus* $\alpha\, \mathscr{D}^*\, \beta$.

Proof. Given $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$ as in the statement, we create maps γ_1 and γ_2 in $T(\mathscr{A},\mathscr{B})$ such that $\alpha\, \mathscr{L}^*\circ\mathscr{R}^*\, \gamma_1\, \mathscr{L}^*\circ\mathscr{R}^*\, \gamma_2\, \mathscr{L}^*\circ\mathscr{R}^*\, \beta$.

For this, we write $A\alpha = \langle\{x_k\alpha\}\rangle$, $B = \langle\{y_k\} \sqcup \{b_i\}\rangle$ and $A = B \sqcup \langle\{a_j\}\rangle$. Since $e < \operatorname{rk}(\alpha) = |K| < \aleph_0$ we set $L = K \setminus \{1\}$, so that $|L| = |K| - 1$ and $\{y_k\} = \{y_1\} \sqcup \{y_\ell\}$. By assumption, we have that

$$|K \sqcup I| = \dim B \geq \operatorname{rk}(\beta) \geq \aleph_0,$$

so that $|I| \geq \aleph_0$, and $|J| = \operatorname{codim}_{\mathrm{A}} B = \kappa$. Therefore, there exist two subsets $S \subseteq J$ and $S' \subseteq I$ such that $|S| = |S'| = \aleph_0$ and we have a bijection $\phi\colon S \to S'$ between these sets. For all $j \in J$, we now set elements $z_j \in B$ by $z_j = b_{j\phi}$ if $j \in S$ and $z_j = b_1$ otherwise. Under this setup, we define $\gamma_1 \in T(\mathscr{A},\mathscr{B})$ as:

$$\gamma_1 = \begin{pmatrix} y_1 & y_\ell & b_i & a_j \\ c & y_\ell & c & z_j \end{pmatrix},$$

where $c \in \langle\{y_\ell\}\rangle$ (which necessarily exists). Then we have that $\operatorname{rk}(\gamma_1) = \aleph_0$ and $B\gamma_1 = \langle\{y_\ell\}\rangle \subsetneq \langle\{y_\ell\} \sqcup \{z_j\}\rangle = A\gamma_1$, so that $\gamma_1 \notin Q$. Moreover,

$$\operatorname{rk}(\gamma_1|_B) = |L| < |K| = \operatorname{rk}(\alpha) < \aleph_0 \leq \kappa = \operatorname{rk}(\gamma_1|_B) + \operatorname{codim}_{\mathrm{A}} B,$$

and thus $\alpha\, \mathscr{L}^*\circ\mathscr{R}^*\, \gamma_1$ by the third case of Proposition III.2.6. Now, either we have that $\aleph_0 = \operatorname{rk}(\gamma_1) = \operatorname{rk}(\beta)$ and we directly get that $\gamma_1\, \mathscr{L}^*\circ\mathscr{R}^*\, \beta$ by Lemma III.2.2, or we have that $\aleph_0 = \operatorname{rk}(\gamma_1) < \operatorname{rk}(\beta)$. If the latter occurs, then we also have that $\operatorname{rk}(\beta) \leq \min\{\dim B, \operatorname{codim}_{\mathrm{A}} B\} \leq \kappa$ by the initial assumptions. Since $\gamma_1 \notin Q$, by Lemma II.3.7, there exists $\gamma_2 \in Q$ with $\operatorname{im}\gamma_1 = \gamma_2$. Thus $\gamma_1\, \mathscr{L}^*\circ\mathscr{R}^*\, \gamma_2$ by Proposition III.1.2, and we also have that $\aleph_0 = \operatorname{rk}(\gamma_2) = \operatorname{rk}(\gamma_1) < \kappa$. We can now invoke Lemma III.2.13 using γ_2 and $\nu = \operatorname{rk}(\beta)$ to get a map $\gamma_3 \in T(\mathscr{A},\mathscr{B})$ such that $\operatorname{rk}(\gamma_3) = \operatorname{rk}(\beta)$ and $\gamma_2\, \mathscr{L}^*\circ\mathscr{R}^*\, \gamma_3$, which means that $\gamma_2(\mathscr{L}^*\circ\mathscr{R}^*)^2\beta$ by Lemma III.2.2. Therefore, in both situations, we have that $\alpha(\mathscr{L}^*\circ\mathscr{R}^*)^4\beta$ and thus $\alpha\, \mathscr{D}^*\, \beta$, as required. □

III.3 THE HIGHER EXTENDED GREEN'S RELATIONS

We now have all the tools needed to prove the characterisation of all of the remaining extended Green's relation. We start by giving the theorem we have been working towards over the last section: the description of the relation $\mathscr{D}^*$.

Theorem III.3.1. *Let $\alpha, \beta \in T(\mathscr{A}, \mathscr{B})$. Then $\alpha \, \mathscr{D}^* \, \beta$ if and only if one of the following happens:*

(i) $\operatorname{codim}_{\mathrm{A}} B = 1$ *and* $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$*;*

(ii) $2 \leq \operatorname{codim}_{\mathrm{A}} B < \aleph_0$ *and either* $e < \operatorname{rk}(\alpha), \operatorname{rk}(\beta) < \aleph_0$ *or* $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$*;*

(iii) $\operatorname{codim}_{\mathrm{A}} B = \kappa$ *for some infinite cardinal* κ*, and either* $e < \operatorname{rk}(\alpha), \operatorname{rk}(\beta) \leq \kappa$ *or* $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$*.*

Proof. Let $\alpha, \beta \in T(\mathscr{A}, \mathscr{B})$ be such that $\alpha \, \mathscr{D}^* \, \beta$. In order to show that the conditions of the theorem are necessary, we focus on the different situations. Since we know from Lemma III.2.9 that T_{e^+} is a $\mathscr{D}^*$-class, this gives us the appropriate part of each condition, and we can assume from now on that all maps have rank larger than e.

Similarly, assume that $\operatorname{codim}_{\mathrm{A}} B = 1$. Then, if $\operatorname{rk}(\alpha)$ is finite, Lemma III.2.10 directly gives us that $\operatorname{rk}(\beta) = \operatorname{rk}(\alpha)$. On the other hand, if $\operatorname{rk}(\alpha) \geq \aleph_0$, then by Lemma III.2.12, any map $\delta \in T(\mathscr{A}, \mathscr{B})$ satisfying $\alpha \, \mathscr{L}^* \circ \mathscr{R}^* \, \delta$, will be such that $\operatorname{rk}(\alpha) = \operatorname{rk}(\delta)$. Since we have that $\alpha \, (\mathscr{L}^* \circ \mathscr{R}^*)^n \, \beta$ for some $n \in \mathbb{N}$, we obtain by induction that $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$, and case (i) is therefore proved.

We now assume that $\operatorname{codim}_{\mathrm{A}} B \geq 2$ and we let κ be an infinite cardinal. From Corollary III.2.8, we know that for some $n \in \mathbb{N}$

$$\operatorname{rk}(\alpha) \leq \operatorname{rk}(\beta) + n \cdot \operatorname{codim}_{\mathrm{A}} B \quad \text{and} \quad \operatorname{rk}(\beta) \leq \operatorname{rk}(\alpha) + n \cdot \operatorname{codim}_{\mathrm{A}} B. \tag{$\star$}$$

If both $\operatorname{rk}(\alpha)$ and $\operatorname{codim}_{\mathrm{A}} B$ are finite, then we have that $\operatorname{rk}(\beta) \leq \operatorname{rk}(\alpha) + n \cdot \operatorname{codim}_{\mathrm{A}} B < \aleph_0$, which shows that in this case we have $e < \operatorname{rk}(\alpha), \operatorname{rk}(\beta) < \aleph_0$ as presented in case (ii).

In a similar manner, suppose that $\operatorname{codim}_{\mathrm{A}} B = \kappa$ for some infinite cardinal κ and $\operatorname{rk}(\alpha) \leq \kappa$. Then we obtain that $\operatorname{rk}(\beta) \leq \operatorname{rk}(\alpha) + n \cdot \operatorname{codim}_{\mathrm{A}} B = \operatorname{codim}_{\mathrm{A}} B$. This gives us that $e < \operatorname{rk}(\alpha), \operatorname{rk}(\beta) \leq \kappa$, which corresponds to the first part of case (iii).

Lastly, if $\operatorname{rk}(\alpha) \geq \aleph_0$ and $\operatorname{rk}(\alpha) > \operatorname{codim}_{\mathrm{A}} B$, then using the left inequality of ($\star$), we get that $\operatorname{rk}(\alpha) \leq \operatorname{rk}(\beta) + n \cdot \operatorname{codim}_{\mathrm{A}} B = \max \{\operatorname{rk}(\beta), n \cdot \operatorname{codim}_{\mathrm{A}} B\}$ which forces $\operatorname{rk}(\beta) \geq \aleph_0$ and then $\operatorname{rk}(\alpha) \geq \operatorname{rk}(\beta)$. Using the right inequality of ($\star$), we also obtain $\operatorname{rk}(\beta) \leq \operatorname{rk}(\alpha) + n \cdot \operatorname{codim}_{\mathrm{A}} B = \operatorname{rk}(\alpha)$, and thus $\operatorname{rk}(\beta) = \operatorname{rk}(\alpha)$. This argument shows in particular that if $2 \leq \operatorname{codim}_{\mathrm{A}} B < \aleph_0$ and $\operatorname{rk}(\alpha) \geq \aleph_0$, or that $\operatorname{codim}_{\mathrm{A}} B = \kappa$ and $\operatorname{rk}(\alpha) > \kappa$, then $\operatorname{rk}(\beta) = \operatorname{rk}(\alpha)$ which corresponds respectively to the second part of cases (ii) and (iii).

Since all the possible values for $\operatorname{codim}_A B$ and $\operatorname{rk}(\alpha)$ are covered in the different arguments above, we have therefore proved that the conditions (i)-(iii) given in the theorem are necessary conditions to obtain $\alpha\,\mathscr{D}^*\,\beta$.

Conversely, we now assume that one of conditions (i), (ii) or (iii) hold and we verify that this is sufficient to get $\alpha\,\mathscr{D}^*\,\beta$. In other words, for any $\alpha,\beta \in T(\mathscr{A},\mathscr{B})$, we want to show that $\alpha(\mathscr{L}^*\circ\mathscr{R}^*)^n\beta$ for some natural number n whenever one of these conditions is satisfied.

Notice that if $\beta \notin Q$, then there exists $\beta' \in Q$ such that $\operatorname{im}\beta' = \operatorname{im}\beta$ by Lemma II.3.7. This means that $\beta'\,\mathscr{L}^*\,\beta$ by Proposition III.1.2 and thus $\beta'\,\mathscr{L}^*\circ\mathscr{R}^*\,\beta$. If we get that $\alpha(\mathscr{L}^*\circ\mathscr{R}^*)^m\beta'$ for some $m \in \mathbb{N}$, we then get $\alpha(\mathscr{L}^*\circ\mathscr{R}^*)^{m+1}\beta$. Therefore we can assume from now on that $\beta \in Q$.

We already know from Lemma III.2.2 that if $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$, then $\alpha\,\mathscr{D}^*\,\beta$, which shows that the appropriate part of cases (i), (ii) and (iii) are sufficient conditions to get that the two maps are $\mathscr{D}^*$-related. The only possibilities left to verify are those where the ranks of α and β are different and lie in the intervals given in conditions (ii) and (iii).

Assume that $2 \le \operatorname{codim}_A B$ and that $e < \operatorname{rk}(\alpha), \operatorname{rk}(\beta) < \aleph_0$, that is, either condition (ii) holds, or we have condition (iii) with two maps of finite rank. Then, by invoking the third part of Lemma III.2.11, there exists a $n \in \mathbb{N}$ such that $\alpha(\mathscr{L}^*\circ\mathscr{R}^*)^n\beta$ and therefore $\alpha\,\mathscr{D}^*\,\beta$.

From now on, we assume that condition (iii) holds with $\operatorname{codim}_A B = \kappa$ and $e < \operatorname{rk}(\alpha), \operatorname{rk}(\beta) \le \kappa$ for some infinite cardinal κ. Without loss of generality, we also assume that $\operatorname{rk}(\alpha) < \operatorname{rk}(\beta)$. If both ranks are infinite, then we can use Lemma III.2.13 with $\nu = \operatorname{rk}(\beta)$ (since $\dim B \ge \operatorname{rk}(\beta) > \operatorname{rk}(\alpha) \ge \aleph_0$ in that case) to get a map $\gamma \in T(\mathscr{A},\mathscr{B})$ such that $\alpha\,\mathscr{L}^*\circ\mathscr{R}^*\,\gamma$ and $\operatorname{rk}(\gamma) = \operatorname{rk}(\beta)$. But then $\alpha(\mathscr{L}^*\circ\mathscr{R}^*)^2\beta$ using Lemma III.2.2 and thus $\alpha\,\mathscr{D}^*\,\beta$.

Otherwise, we are in the situation where $\operatorname{rk}(\alpha) < \aleph_0 \le \operatorname{rk}(\beta)$ and we can conclude that $\alpha\,\mathscr{D}^*\,\beta$ using Lemma III.2.14, which finishes the proof of the characterisation of $\mathscr{D}^*$. □

From the characterisation of $\mathscr{D}^*$ in Theorem III.3.1 it is easy to see that if B is finite dimensional and $\operatorname{codim}_A B \ge 2$, then $\mathscr{D}^*$ is made of only 2 classes, namely T_{e^+} and $T_{e^+}^c$. In fact, the same goes for the $\mathscr{J}^*$ classes since these are equal, as given by the following proposition.

Proposition III.3.2. *In $T(\mathscr{A},\mathscr{B})$, we have that $\mathscr{D}^* = \mathscr{J}^*$.*

Proof. Since we know that $\mathscr{D}^* \subseteq \mathscr{J}^*$, it remains to show the converse. To this end, we are going to determine the $\mathscr{J}^*$-classes of certain cases by describing the principal $*$-ideals generated by specific elements. Combining these cases with the description of the $\mathscr{D}^*$-classes given in Theorem III.3.1 will finish the proof.

As a general setup, we consider $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$ with $\beta \in J^*(\alpha)$. Then, by Lemma I.3.15, there exist $\gamma_0, \dots, \gamma_n \in T(\mathscr{A},\mathscr{B})$ and $\{\lambda_i\}, \{\mu_i\} \subseteq T(\mathscr{A},\mathscr{B})^1$ such that:

$$\gamma_0 = \alpha, \quad \gamma_n = \beta \quad \text{and} \quad (\gamma_i, \lambda_i\gamma_{i-1}\mu_i) \in \mathscr{D}^* \quad \text{for all } 1 \leq i \leq n.$$

Assume first that $\operatorname{codim}_{\mathrm{A}} B = 1$. Then, by Theorem III.3.1, we get that $\operatorname{rk}(\gamma_1) = \operatorname{rk}(\lambda_1\alpha\mu_1) \leq \operatorname{rk}(\alpha)$ and by induction, we obtain $\operatorname{rk}(\gamma_i) \leq \operatorname{rk}(\alpha)$ for all $1 \leq i \leq n$, from which conclude that $\operatorname{rk}(\beta) \leq \operatorname{rk}(\alpha)$. Reversing the roles of α and β we have that if $\alpha \in J^*(\beta)$, then $\operatorname{rk}(\alpha) \leq \operatorname{rk}(\beta)$. Therefore if $\alpha \mathrel{\mathscr{J}^*} \beta$, then $\beta \in J^*(\alpha)$ and $\alpha \in J^*(\beta)$, which forces $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$.

From now on, we assume that $\operatorname{codim}_{\mathrm{A}} B \geq 2$. In the case where $\operatorname{rk}(\alpha) = e$, then we also have that $\operatorname{rk}(\lambda_1\alpha\mu_1) = e$. Since $\gamma_1 \mathrel{\mathscr{D}^*} \lambda_1\alpha\mu_1$, this forces $\operatorname{rk}(\gamma_1) = \operatorname{rk}(\lambda_1\alpha\mu_1)$ by condition (ii) of Theorem III.3.1, and thus $\operatorname{rk}(\gamma_1) = e$. By induction on the γ_i's, we get that $\operatorname{rk}(\beta) = e = \operatorname{rk}(\alpha)$. Therefore $J^*(\alpha) = T_{e^+}$, from which we have that if $\alpha \mathrel{\mathscr{J}^*} \beta$ with $\operatorname{rk}(\alpha) = e$, then $\operatorname{rk}(\beta) = \operatorname{rk}(\alpha)$.

From now on, we assume $\operatorname{rk}(\alpha) > e$. Consider the case when $\operatorname{codim}_{\mathrm{A}} B = \kappa$ for some infinite cardinal κ, and suppose that $\operatorname{rk}(\beta) > \operatorname{codim}_{\mathrm{A}} B$. Then, from the fact that $\beta = \gamma_n \mathrel{\mathscr{D}^*} \lambda_n\gamma_{n-1}\mu_n$, we necessarily have that $\operatorname{rk}(\beta) = \operatorname{rk}(\lambda_n\gamma_{n-1}\mu_n)$ from (iii) of Theorem III.3.1 and therefore $\operatorname{rk}(\gamma_{n-1}) \geq \operatorname{rk}(\beta) > \operatorname{codim}_{\mathrm{A}} B$. By reverse induction on i from $n-1$ to 1, we get that $\operatorname{rk}(\gamma_{i-1}) \geq \operatorname{rk}(\gamma_i) > \operatorname{codim}_{\mathrm{A}} B$ for all i and thus $\operatorname{rk}(\alpha) \geq \operatorname{rk}(\beta) > \operatorname{codim}_{\mathrm{A}} B$. This shows that $\beta \in J^*(\alpha)$ with $\operatorname{rk}(\beta) > \operatorname{codim}_{\mathrm{A}} B$ only if $\operatorname{rk}(\alpha) > \operatorname{codim}_{\mathrm{A}} B$ and $\operatorname{rk}(\alpha) \geq \operatorname{rk}(\beta)$. Together with the reverse statement, we conclude that if $\operatorname{rk}(\alpha) > \operatorname{codim}_{\mathrm{A}} B \geq \aleph_0$, then $\alpha \mathrel{\mathscr{J}^*} \beta$ implies $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$.

Using a similar argument when $\operatorname{codim}_{\mathrm{A}} B < \aleph_0$, we also have that if $\beta \in J^*(\alpha)$ with $\operatorname{rk}(\beta) \geq \aleph_0$, then we get $\operatorname{rk}(\alpha) \geq \operatorname{rk}(\beta)$. Therefore, this case gives us that if $\operatorname{rk}(\alpha) \geq \aleph_0 > \operatorname{codim}_{\mathrm{A}} B$ and $\alpha \mathrel{\mathscr{J}^*} \beta$, then $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta)$.

In all of the above cases, we can see that if two maps α and β are $\mathscr{J}^*$-related, then they have the same rank and thus they are $\mathscr{D}^*$-related by Lemma III.2.2. The remaining cases to consider are when we either have that $2 \leq \operatorname{codim}_{\mathrm{A}} B < \aleph_0$ and $e < \operatorname{rk}(\alpha), \operatorname{rk}(\beta) < \aleph_0$, or we have that $\operatorname{codim}_{\mathrm{A}} B = \kappa$ and $e < \operatorname{rk}(\alpha), \operatorname{rk}(\beta) \leq \kappa$.

However, any two maps α and β which are $\mathscr{J}^*$-related and satisfy these conditions will be $\mathscr{D}^*$-related by the first part of conditions (ii) and (iii) in Theorem III.3.1. Thus the $\mathscr{J}^*$-classes of $T(\mathscr{A},\mathscr{B})$ coincide with the $\mathscr{D}^*$-classes, which completes the proof that $\mathscr{J}^* = \mathscr{D}^*$. □

Remark III.3.3. Another way to determine $J^*(\alpha)$ for an element $\alpha \in T(\mathscr{A},\mathscr{B})$ with finite rank would have been to use the description of the ideals of $T(\mathscr{A},\mathscr{B})$ given in Section II.5 before saturating them by $\mathscr{L}^*$ and $\mathscr{R}^*$. If we let $I = T_{r(S)} \cup K(S)$ be an ideal generated by $S = \{\alpha\}$, we can see that in order for it to be saturated by $\mathscr{L}^*$, we always need to have $K(S) \subseteq T_{r(S)}$. On the other hand, as long as $r(S) > e^+$ and $\operatorname{codim}_{\mathrm{A}} B > 1$, saturating by $\mathscr{R}^*$ allows us to reach maps with a rank up to the limit cardinal that is the maximum between the codimension of B and $\aleph_0$. Two maps would then be $\mathscr{J}^*$-related if the associated limit cardinal is the same. Similar arguments can be used to give all the possible descriptions of $*$-ideals by using the general description of ideals in $T(\mathscr{A},\mathscr{B})$.

To finish off the study of these extended Green's relations in $T(\mathscr{A},\mathscr{B})$, we show that the relations $\widetilde{\mathscr{D}}$ and $\widetilde{\mathscr{J}}$ only have two equivalence classes, since the corank of $\mathscr{B}$ has no impact in that situation.

Proposition III.3.4. *In $T(\mathscr{A},\mathscr{B})$, the only $\widetilde{\mathscr{D}}$ and $\widetilde{\mathscr{J}}$ classes are T_{e^+} and $T_{e^+}^c$.*

Proof. Let $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$ be such that $\operatorname{rk}(\alpha) = \operatorname{rk}(\beta) = e$. Then $\alpha \mathscr{D}^* \beta$ by Lemma III.2.2, and thus $\alpha \widetilde{\mathscr{D}} \beta$ since $\mathscr{D}^* \subseteq \widetilde{\mathscr{D}}$. On the other hand, assume that the rank of both α and β is strictly greater than e. If $\alpha \in Q$ then, by Lemma II.3.7, there exists $\alpha' \in Q^c$ such that $\operatorname{im} \alpha = \operatorname{im} \alpha'$, while if $\alpha \in Q^c$ in the first place, we simply set $\alpha' = \alpha$. In both cases, we have that $\alpha \widetilde{\mathscr{L}} \alpha'$ and similarly, $\beta \widetilde{\mathscr{L}} \beta'$ for some $\beta' \in Q^c$. Then, by Proposition III.1.4, we get that $\alpha' \widetilde{\mathscr{R}} \beta'$, and so $\alpha \widetilde{\mathscr{L}} \circ \widetilde{\mathscr{R}} \circ \widetilde{\mathscr{L}} \beta$. Therefore $\alpha \widetilde{\mathscr{D}} \beta$ for any $\alpha, \beta \in T_{e^+}^c$.

For the converse, notice first that if $\gamma \in T(\mathscr{A},\mathscr{B})$ is such that $\operatorname{rk}(\gamma) = e$, then for any $\delta \in T(\mathscr{A},\mathscr{B})$ we have that $\operatorname{rk}(\delta) = e$ whenever $\gamma \widetilde{\mathscr{R}} \delta$ or $\gamma \widetilde{\mathscr{L}} \delta$. Indeed, if $\gamma \widetilde{\mathscr{R}} \delta$ then, by Proposition III.1.4 together with the fact that $\gamma \in Q$, we get that $\delta \in Q$ and $\ker \delta = \ker \gamma$. Consequently, $\operatorname{im} \delta \cong \operatorname{im} \gamma$ and so $\operatorname{rk}(\delta) = \operatorname{rk}(\gamma) = e$. Similarly, if $\gamma \widetilde{\mathscr{L}} \delta$, then Proposition III.1.2 gives us that $\operatorname{im} \delta = \operatorname{im} \gamma$ and thus $\operatorname{rk}(\delta) = \operatorname{rk}(\gamma) = e$, proving the claim. Now consider $\alpha, \beta \in T(\mathscr{A},\mathscr{B})$ such that $\alpha \widetilde{\mathscr{D}} \beta$. Then there exists a finite sequence of compositions of $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}$ relating α to β. If $\operatorname{rk}(\alpha) = e$, then finite induction gives us that all maps in that sequence have rank e, and thus $\operatorname{rk}(\beta) = e$.

Similarly, if $\mathrm{rk}(\alpha) \neq e$, then by symmetry of the arguments, we necessarily get that $\mathrm{rk}(\beta) \neq e$, which concludes the proof of the characterisation of $\widetilde{\mathscr{D}}$.

Since $\widetilde{\mathscr{D}} \subseteq \widetilde{\mathscr{J}}$, in order to determine the $\widetilde{\mathscr{J}}$-classes in $T(\mathscr{A}, \mathscr{B})$, it suffices to show whether an element of T_{e^+} can be $\widetilde{\mathscr{J}}$-related to an element of $T_{e^+}^c$. So let $\alpha, \beta \in T(\mathscr{A}, \mathscr{B})$ with $\alpha \mathrel{\widetilde{\mathscr{J}}} \beta$ and $\alpha \in T_{e^+}$. Then by Lemma I.3.15, there exist $\gamma_0, \dots, \gamma_n \in T(\mathscr{A}, \mathscr{B})$ and $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \in T(\mathscr{A}, \mathscr{B})^1$ such that $\gamma_0 = \alpha$, $\gamma_n = \beta$ and $(\gamma_i, \lambda_i \gamma_{i-1} \mu_i) \in \widetilde{\mathscr{D}}$ for all $1 \leq i \leq n$. But then, $\mathrm{rk}(\lambda_1 \gamma_0 \mu_1) = \mathrm{rk}(\alpha) = e$ since e is the minimal rank. Moreover, since $\gamma_1 \mathrel{\widetilde{\mathscr{D}}} \lambda_1 \gamma_0 \mu_1$, we get by the first part of this proof that $\mathrm{rk}(\gamma_1) = \mathrm{rk}(\lambda_1 \gamma_0 \mu_1) = e$. By induction, $\mathrm{rk}(\gamma_i) = e$ for all $1 \leq i \leq n$, so that $\mathrm{rk}(\beta) = e$. Therefore, α can only be related to maps in T_{e^+}, which shows that $\widetilde{\mathscr{D}}$ has two equivalence classes and that $\widetilde{\mathscr{J}} = \widetilde{\mathscr{D}}$. $\square$

— IV —

The structure of End($\mathcal{T}_n$)

The full transformation semigroup $\mathcal{T}_n$ on a finite set $\{1, \ldots, n\}$ is an important object in algebra. It is therefore natural to study its endomorphism monoid End($\mathcal{T}_n$). Even though the elements of End($\mathcal{T}_n$) were described by Schein and Teclezghi [48], surprisingly, the algebraic structure of End($\mathcal{T}_n$) has not been further explored. The main focus of our work will be on the general behaviour of End($\mathcal{T}_n$) which emerges for $n \geq 5$. The cases $n \leq 4$ exhibit several degenerate or exceptional behaviours: for instance, it is immediate that End($\mathcal{T}_1$) $=$ Aut($\mathcal{T}_1$) is the trivial group, and we will see that End($\mathcal{T}_4$) is unique in that it contains endomorphisms which only exist in End($\mathcal{T}_n$) if $n = 4$.

In this chapter, we will follow a similar approach to that used to study the semigroup $T(\mathscr{A}, \mathscr{B})$ in Chapters II and III. We will start in Section IV.1 by describing the elements of the semigroup End($\mathcal{T}_n$) we will study, using the results from [48]. In Section IV.2 we will study more closely the singular endomorphisms of $\mathcal{T}_n$, that is, the endomorphisms which are not automorphisms, leading us to the exhibition in Section IV.3 of a partition of End($\mathcal{T}_n$) into sets containing maps which present a similar behaviour. In order to allow general arguments to be used, in Sections IV.4 to IV.7 we restrict ourselves to the study of End($\mathcal{T}_n$) for $n \geq 5$. We first describe some properties of the idempotents and determine the regular elements of End($\mathcal{T}_n$) in Section IV.4. We then give a description of Green's relations in Section IV.5 and use this to determine the ideal structure of End($\mathcal{T}_n$) in Section IV.6. Since the monoid End($\mathcal{T}_n$) is not regular for $n \geq 5$, we consider its extended Green's relations in Section IV.7. To complete the picture, in Section IV.8, we analyse the structure of End($\mathcal{T}_n$) for $n \leq 4$.

Note. The work in this chapter comes from a collaboration with Prof. Victoria Gould and Dr. Marianne Johnson, which has been submitted for publication [23]. The present chapter includes some additional results and expand proofs for the case $n = 4$ that were only sketched in the paper.

IV.1 INTRODUCTION

IV.1.1 Notation and conventions

Throughout this chapter, we write $\mathcal{S}_n \subseteq \mathcal{T}_n$ to denote the symmetric group and $\mathcal{A}_n \subseteq \mathcal{S}_n$ the alternating group, that is, the subgroup of $\mathcal{S}_n$ of all even permutations. By a slight abuse of notation, we suppress the dependence on n and write simply id to denote the identity element of $\mathcal{T}_n$. For an element $g \in \mathcal{S}_n$, and $s \in \mathcal{T}_n$, we denote by s^g the product $g^{-1}sg$. For $1 \leq i \neq j \leq n$ we also write $(i\,j)$ for the transposition swapping i and j in $\mathcal{S}_n$, and c_i for the constant map with image i in $\mathcal{T}_n$.

For $n \neq 4$, the alternating group is the only non-trivial proper normal subgroup of $\mathcal{S}_n$, whilst for $n = 4$ there is one additional normal subgroup, namely, the Klein subgroup $\mathcal{K} = \{\mathrm{id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$. By direct calculation, we have the following result concerning the cosets of $\mathcal{K}$.

Lemma IV.1.1. *For any $s \in \mathcal{S}_4$ there is a unique element of $\mathcal{K}s$ which fixes 4. More explicitly, if we denote this element by p_s, we have that*

$$p_s = \begin{cases} \mathrm{id} & \text{if } s \in \mathcal{K}, \\ (1\,2) & \text{if } s \in \{(1\,2), (3\,4), (1\,3\,2\,4), (1\,4\,2\,3)\}, \\ (1\,3) & \text{if } s \in \{(1\,3), (2\,4), (1\,3\,2\,4), (1\,4\,3\,2)\}, \\ (2\,3) & \text{if } s \in \{(2\,3), (1\,4), (1\,2\,4\,3), (1\,3\,4\,2)\}, \\ (1\,2\,3) & \text{if } s \in \{(1\,2\,3), (1\,4\,2), (1\,3\,4), (2\,4\,3)\}, \text{ and} \\ (1\,3\,2) & \text{if } s \in \{(1\,3\,2), (1\,2\,4), (1\,4\,3), (2\,3\,4)\}. \end{cases}$$

In particular, $\{p_s : s \in \mathcal{S}_4\} = \{h \in \mathcal{S}_4 : 4h = 4\}$ and $p_s = s$ if and only if $4s = 4$.

To facilitate readability, we will consider that the semigroup $\mathcal{T}_n$ will be our "bottom level" so that, following our convention, its elements will be written using Roman letters (elements of $\{1, \ldots, n\}$ will be constrained to letters i to l). We will also use the short-hand notation $\mathcal{E}_n = \mathrm{End}(\mathcal{T}_n)$ and $\mathcal{G}_n = \mathrm{Aut}(\mathcal{T}_n)$ for the endomorphism monoid and automorphism group of $\mathcal{T}_n$ respectively. The identity element of $\mathcal{E}_n$ (and hence also of $\mathcal{G}_n$) is the trivial automorphism, which will be denoted by ε (again, suppressing the dependence on n where convenient).

IV.1.2 Description of the endomorphisms

We are interested in the semigroup endomorphisms of $\mathcal{T}_n$, that is, the maps on $\mathcal{T}_n$ which preserve the multiplication, but not necessarily the identity. Their characterisation is due to Schein and Teclezghi [48] which we recall below.

Theorem IV.1.2 ([48]). *Let $g \in \mathcal{S}_n$ and $t, e \in \mathcal{T}_n$ and define maps $\psi_g, \phi_{t,e} \colon \mathcal{T}_n \to \mathcal{T}_n$ for any $s \in \mathcal{T}_n$ by:*

$$s\psi_g = s^g \quad \text{and} \quad s\phi_{t,e} = \begin{cases} t & \text{if } s \in \mathcal{S}_n \setminus \mathcal{A}_n, \\ t^2 & \text{if } s \in \mathcal{A}_n, \\ e & \text{if } s \in \mathcal{T}_n \setminus \mathcal{S}_n. \end{cases}$$

Then the automorphisms and endomorphisms of $\mathcal{T}_n$ are described as follows.

1) $\mathcal{G}_n = \{\psi_g : g \in \mathcal{S}_n\}$.
2) *For $n \neq 4$,*

$$\mathcal{E}_n = \mathcal{G}_n \cup \{\phi_{t,e} : t, e \in \mathcal{T}_n,\ t^3 = t,\ te = et = e^2 = e\}.$$

3) *For $n = 4$*

$$\mathcal{E}_4 = \mathcal{G}_4 \cup \{\phi_{t,e} : t, e \in \mathcal{T}_4,\ t^3 = t,\ te = et = e^2 = e\} \cup \{\sigma^g : g \in \mathcal{S}_4\},$$

where $s\sigma = p_s$ if $s \in \mathcal{S}_4$, $s\sigma = c_4$ if $s \in \mathcal{T}_4 \setminus \mathcal{S}_4$ and for all $s \in \mathcal{T}_4$ and all $g \in \mathcal{S}_4$ $s\sigma^g = (s\sigma)^g$.

Notice that with the above notation, we have that $\sigma = \sigma^{\mathrm{id}}$ and $\psi_{\mathrm{id}} = \varepsilon$. For $\alpha \in \mathcal{E}_n$ we write $\operatorname{im}\alpha$ to denote the *image of* α, and define the *rank of* α, denoted by $\mathrm{rk}(\alpha)$, to be the cardinality of $\operatorname{im}\alpha$. Then, it is easy to see that the image of each automorphism ψ_g is the whole of $\mathcal{T}_n$, and hence has rank $|\mathcal{T}_n| = n^n$. On the other end, the image of each endomorphism of the form $\phi_{t,e}$ is the set $\{t, t^2, e\}$, which can have up to three distinct elements, so that its rank is 1, 2 or 3. If $n = 4$, we can also see that $\operatorname{im}\sigma^g = \{p_s^g \colon s \in \mathcal{S}_4\} \cup \{c_{4g}\}$, which is isomorphic to $\mathcal{S}_3 \cup \{c_4\}$ and thus each endomorphism of the form σ^g has rank exactly 7.

Definition IV.1.3. An endomorphism $\alpha \in \mathcal{E}_n$ is *singular* if $\operatorname{rk}(\alpha) < n^n$, that is, if α is not an automorphism.

The set of all singular endomorphisms is denoted by $\mathcal{E}_n \setminus \mathcal{G}_n$.

In fact the group of automorphisms of $\mathcal{T}_n$ is isomorphic to $\mathcal{S}_n$ as given by:

Lemma IV.1.4. *Let $g, h \in \mathcal{S}_n$. Then $\psi_g = \psi_h$ if and only if $g = h$. Consequently,* $\operatorname{Aut}(\mathcal{T}_n) = \mathcal{G}_n = \{\psi_g \colon g \in \mathcal{S}_n\}$ *is isomorphic to* $\mathcal{S}_n = \operatorname{Aut}(\{1, \dots, n\})$.

Proof. Let $\psi_g, \psi_h \in \mathcal{G}_n$ and suppose that $\psi_g = \psi_h$. Then for any $1 \leq i \leq n$ we have $c_i\psi_g = g^{-1}c_i g = c_i g = c_{ig}$, and similarly $c_i\psi_h = c_{ih}$. This gives us $ig = ih$ for all $1 \leq i \leq n$ and thus $g = h$.

Additionally, it is clear from their definition that for all $\psi_g, \psi_h \in \mathcal{E}_n$ we have that $\psi_g\psi_h = \psi_{gh} \in \mathcal{E}_n$. Hence, the map $\mathcal{S}_n \to \mathcal{G}_n$ sending g to ψ_g is an isomorphism. □

Remark IV.1.5. When working with the symmetric group $\mathcal{S}_n$, it is often expected to encounter exceptions when $n = 4$ or $n = 6$. We have already seen in Theorem IV.1.2 that there are some additional endomorphisms which only exist in $\mathcal{E}_n$ if $n = 4$. However, for $n = 6$, Lemma IV.1.4 shows that we need not worry about the outer automorphisms of S_6 when considering $\mathcal{T}_6$.

IV.1.3 Properties of the endomorphism σ

We finish this introductory section by giving some properties of the map σ in the case when $n = 4$.

Lemma IV.1.6. *For $s, t, g \in \mathcal{S}_4$, we have $p_s\sigma^g = p_s^g$, $p_{st} = p_s p_t$ and $p_{s^{-1}} = p_s^{-1}$.*

Proof. That $p_s\sigma = p_s$ follows from Lemma IV.1.1. Thus $p_s\sigma^g = (p_s\sigma)^g = p_s^g$ for all $g \in \mathcal{S}_4$. Also, since σ is an endomorphism, we have that $p_{st} = (st)\sigma = s\sigma t\sigma = p_s p_t$, and then, $p_s p_{s^{-1}} = p_{\mathrm{id}} = \mathrm{id}$, which shows that $p_{s^{-1}} = p_s^{-1}$. □

These facts will now be used without further comment. We can also see that the maps σ^g keeps the special subgroups of $\mathcal{T}_n$ separated, as given by:

Lemma IV.1.7. *Let $t \in \mathcal{T}_4$ and $g \in \mathcal{S}_4$. Then $t\sigma^g \in \mathcal{S}_4 \setminus \mathcal{A}_4$ [resp. $t\sigma \in \mathcal{A}_4$, $t\sigma \in \mathcal{T}_4 \setminus \mathcal{S}_4$] if and only if $t \in \mathcal{S}_4 \setminus \mathcal{A}_4$ [resp. $t \in \mathcal{A}_4$, $t \in \mathcal{T}_4 \setminus \mathcal{S}_4$].*

Proof. If $t \in \mathcal{T}_4 \setminus \mathcal{S}_4$, then $t\sigma^g = c_{4g}$, hence $t\sigma^g \in \mathcal{T}_4 \setminus \mathcal{S}_4$. Conversely, for all $s \in \mathcal{S}_4$, we have $s\sigma = p_s \in \mathcal{S}_4$ and thus $s\sigma^g \in \mathcal{S}_4$, so that $t\sigma^g \in \mathcal{T}_4 \setminus \mathcal{S}_4$ only if $t \in \mathcal{T}_4 \setminus \mathcal{S}_4$.

Now, let $t \in \mathcal{S}_4$. It follows from the description of p_t in Lemma IV.1.1 that $p_t \in \mathcal{A}_4$ if and only if $t \in \mathcal{A}_4$. Since conjugating by an element g of $\mathcal{S}_4$ does not change the cycle structure, we get that $t\sigma \in \mathcal{A}_4$ if and only if $t\sigma^g \in \mathcal{A}_4$, which concludes the proof. $\square$

IV.2 SINGULAR ENDOMORPHISMS

In this section, we focus on the set of singular endomorphisms $\mathcal{E}_n \setminus \mathcal{G}_n$, that is, on endomorphisms of rank 1, 2, 3 or 7.

IV.2.1 Endomorphisms of rank at most 3

In order to encapsulate the conditions of the elements t and e in $\mathcal{T}_n$ so that $\phi_{t,e}$ is an endomorphism, we let

$$U_n = \left\{ t \in \mathcal{T}_n \mid t^3 = t \text{ with } te = et = e^2 = e \text{ for some } e \in \mathcal{T}_n \right\}.$$

We say that (t, e) form a *permissible pair*, if $\phi_{t,e} \in \mathcal{E}_n$, and we denote by P_n the set of all permissible pairs, that is,

$$P_n = \left\{ (t, e) \mid t \in U_n,\ te = et = e^2 = e \right\}.$$

Before considering the endomorphisms in $\mathcal{E}_n$ further, we give some important properties of the sets U_n and P_n.

Lemma IV.2.1. *The set U_n consists of all elements $t = t^3$ satisfying $kt = k$ for at least one $1 \leq k \leq n$. Moreover, for $t \in U_n$ the number of permissible pairs with first component equal to t is*

$$\sum_{r=1}^{|J|} \binom{|J|}{r} r^{|I|+|J|-r}$$

where $J = \{k : kt = k\}$ and I is maximal such that t restricts to a fixed point free permutation on $I \cup It$.

Proof. Let $t \in U_n$. By definition we have $t^3 = t$. Note that if there exists $e \in \mathcal{T}_n$ such that $et = e$, then then for all $k \in \{1, \ldots, n\}$ we must have $ket = ke$, giving that all elements in the image of e are fixed by t. Conversely, suppose that $t^3 = t$ and $kt = k$. Then it is easy to see that $c_k t = tc_k = c_k = c_k^2$, and hence $t \in U_n$.

Note that if $t \in \mathcal{T}_n$ satisfies $t^3 = t$ then for all $k \in \{1, \dots, n\}$ there exist i_k, j_k such that $kt = j_k$, $kt^2 = j_k t = i_k$ and $kt^3 = i_k t = j_k t^2 = kt$. If $j_k = k$ then also $i_k = k$. It follows that $J = \{k : i_k = j_k = k\}$, $K = \{k : k \neq j_k = i_k\}$, $L = \{k : k = i_k \neq j_k\}$ and $M = \{k : k \neq j_k, k \neq i_k \text{ and } j_k \neq i_k\}$, partition the domain $\{1, \dots, n\}$ of t. It is clear from these definitions that t restricts to the identity on J, $Kt \subseteq J$ and $Mt \subseteq L$. Moreover, L is maximal such that t restricts to a fixed-point free permutation of order 2 on L.

The general picture to have in mind is as follows. For each transformation $t \in \mathcal{T}_n$ satisfying $t^3 = t$, we may partition the domain of t as:

$$\begin{array}{rcll}
\{1, \dots, n\} &=& J \cup K \cup L \cup M, \text{ where} & \\
J &=& \{j : jt = jt^2 = j\} & j \circlearrowleft \\
K &=& \{k : kt = kt^2 \neq k\} & k \longrightarrow kt \circlearrowleft \\
L &=& \{l : lt \neq lt^2 = l\} & l = lt^2 \rightleftarrows lt \\
M &=& \{m : m \neq mt \neq mt^2 \neq m\} & m \longrightarrow mt \rightleftarrows mt^2
\end{array}$$

Noting that $l \in L$ if and only if $lt \in L$, it is clear that L may be further partitioned into two sets of equal size, $I = \{i_s\}$ and $It = \{i_s t\}$. Thus we may write each t such that $t^3 = t$ in the form:

$$t = \begin{pmatrix} j & k & i_s & i_s t & m \\ j & kt & i_s t & i_s & mt \end{pmatrix}, \text{ with } kt \in J \text{ and } mt \in I \cup It,$$

where here j, k and m denote arbitrary elements of the sets J, K and M respectively. We claim that the elements e satisfying $e = e^2 = te = et$ are precisely those of the form:

$$e = \begin{pmatrix} j & k & i_s & i_s t & m \\ jf & ktf & i_s f & i_s f & mtf' \end{pmatrix},$$

where f is any function $f : J \cup I \to R$ fixing a non-empty subset $R \subseteq J$ pointwise, and $f' : I \cup It \to R$ is defined by $i_r t f' = i_r f' = i_r f$.

For example, given the transformation $t = t^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 1 & 2 \end{pmatrix}$, this description yields that $e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ is the unique idempotent satisfying $te = et = e$.

Indeed, if t fixes exactly one element then we have no choice but to take $R = J$, and since this is a singleton set this leaves no choice for the function f.

Now to prove the claim. If $t \in U_n$, then by the first paragraph of the proof we may assume that $|J| > 0$. It is then straightforward to describe the elements e which satisfy $et = te = e = e^2$. The condition that e is idempotent implies that e fixes each element of its image. Further, if R is a subset of $\{1, \ldots, n\}$, then there is a bijection between functions $f : \{1, \ldots, n\} \setminus R \to R$ and idempotents with image equal to R. Note that (as observed above) the condition $et = e$ implies that the image of e is contained in J. Let R be a non-empty subset of J. We aim to show that every function $f : (J \setminus R) \cup I \to R$ extends uniquely to the whole of $\{1, \ldots, n\}$ to give an idempotent $e \in \mathcal{T}_n$ with image R satisfying the constraints $et = te = e$.

We have observed that every idempotent must fix its image. Thus, if e is to be an idempotent with image R extending f, we have no choice on how e must act on $J \cup I$. Now the condition $te = e$ forces $ke = kte$ for all $k \in \{1, \ldots, n\} = J \cup K \cup I \cup It \cup M$. If $k \in K$ then $kt \in J$ and $ke = kte = (kt)f$. If $k \in It$, then $kt \in I$ and the condition $e = te$ forces $ke = kte = (kt)f$. If $k \in M$ then $kt \in I \cup It$ and hence either $kt \in I$ and $ke = kte = (kt)f$, or $kt \in It$. In the latter case, certainly $kt^2 \in I$, so that $kt = kt^3 = i_k t$ for $i_k = kt^2 \in I$, and then $ke = kte = i_k f$. Since the sets $J \cup K \cup I \cup It \cup M$ partition the domain of t, this shows that there is exactly one way to extend f to an idempotent $e \in \mathcal{T}_n$ with image R satisfying $et = te = e$.

For a fixed t, the number of idempotents such that $(t, e) \in P_n$ is therefore found by summing the total number of functions $f : (J \setminus R) \cup I \to R$ as R ranges over non-empty subsets of J. That is, for $t \in U_n$ with partition as given above we have that the number of idempotents e satisfying $(t, e) \in P_n$ is $\sum_{r=1}^{|J|} \binom{|J|}{r} r^{|I|+|J|-r}$. $\square$

We now gather together some routine but useful facts concerning the set P_n.

Lemma IV.2.2. *Let $k \in \mathbb{N}$, $t, e \in \mathcal{T}_n$ and $g \in \mathcal{S}_n$.*

1) If t is idempotent, then $t \in U_n$ and $(t, t) \in P_n$.

2) If $(t, e) \in P_n$ with $t^2 = e$, then $t = e$.

3) We have $t \in \mathcal{S}_n \cap U_n$ if and only if $t^2 = \mathrm{id}$. Consequently, $t \in \mathcal{S}_n \cap U_n$ is a product of an odd number [resp. even number] no more than $(n-1)/2$ of disjoint transpositions if $t \in \mathcal{S}_n \setminus \mathcal{A}_n$ [resp. if $t \in \mathcal{A}_n$].

4) We have $(t^g)^k = t^g$ if and only if $t^k = t$. Additionally, $t^g = \mathrm{id}$ if and only if $t = \mathrm{id}$.

5) If $(t, e) \in P_n$, then $(t^g, e^g) \in P_n$.

6) If $(t, e) \in P_n$ and $e = \mathrm{id}$, then $t = \mathrm{id}$.

7) If $(t, e) \in P_n$, then t^2 is idempotent and $(t^2, e) \in P_n$.

8) If $(t, e) \in P_4$ and $g \in \mathcal{S}_4$, then $(t\sigma^g, e\sigma^g) \in P_4$.

Proof. 1) Let $t \in \mathcal{T}_n$ be an idempotent. Then clearly $t^3 = t = t^2$, and hence $(t, t) \in P_n$.

2) Let $(t, e) \in P_n$ be such that $t^2 = e$. Then we immediately obtain that $t = t^3 = tt^2 = te = e$.

3) If $t \in \mathcal{S}_n$, from $t^3 = t$ we get that $t^2 = tt^{-1} = \mathrm{id}$, while $t \notin \mathcal{S}_n$ directly implies that $t^2 \neq \mathrm{id}$. The second part follows from the fact that a permutation of order 2 is a product of disjoint transpositions, and that $t \in U_n$ must fix at least one element of $\{1, \ldots, n\}$ by Lemma IV.2.1.

4) If $t^k = t$, then we have that $(t^g)^k = (g^{-1}tg)^k = g^{-1}t^kg = g^{-1}tg = t^g$. Conversely, if $(t^g)^k = t^g$, then we have $g^{-1}t^kg = g^{-1}tg$ which immediately gives us that $t^k = t$. Finally, $g^{-1}tg = t^g = \mathrm{id}$ if and only if $t = g\,\mathrm{id}\,g^{-1} = \mathrm{id}$ as required.

5) Let $(t, e) \in P_n$ and $g \in \mathcal{S}_n$. Since $t^3 = t$, by the previous argument, we get that $(t^g)^3 = t^g$. Also, $t^ge^g = g^{-1}teg = g^{-1}eg = e^g$ and similarly $e^gt^g = e^g = (e^g)^2$, which shows that $(t^g, e^g) \in P_n$.

6) Let $(t, e) \in P_n$ so that $te = e$. Then, if $e = \mathrm{id}$ we get that $t = t\,\mathrm{id} = \mathrm{id}$, as required.

7) Let $(t, e) \in P_n$. Then from $t^3 = t$ we directly obtain that $(t^2)^2 = t^3t = t^2$ and thus t^2 is an idempotent and lies in U_n by point 1). Since $te = et = e$, we have that $t^2e = t(te) = te = e$, and similarly $et^2 = e$. Hence $t^2e = et^2 = e = e^2$, which shows that $(t^2, e) \in P_n$.

8) Let $(t, e) \in P_4$ and $g \in \mathcal{S}_4$. We first show that $(t\sigma, e\sigma) \in P_4$ and then apply 5) to obtain the full result. Notice that if $t, e \in \mathcal{T}_n \setminus \mathcal{S}_n$, then $t\sigma = e\sigma = c_4$, while if $t = e = \mathrm{id}$, then $t\sigma = e\sigma = \mathrm{id}$, which by part 1) shows in both cases that $(t\sigma, e\sigma) \in P_4$. Otherwise, $t \in \mathcal{S}_4$ and $e \neq \mathrm{id}$, and then $t\sigma = p_t$ and $e\sigma = c_4$. Since σ is an endomorphism, we get that $(t\sigma)^3 = t^3\sigma = t\sigma$. Moreover, as $p_t \in \mathcal{S}_4$ fixes 4 by definition, it follows that $p_tc_4 = c_4p_t = c_4 = c_4^2$ and we thus have $(t\sigma, e\sigma) = (p_t, c_4) \in P_4$. □

We now give some characteristics of the endomorphisms of $\mathcal{T}_n$ with rank at most three that will play an important part in the discussions to come. Elements of the proofs of the following two results given explicitly can be found scattered throughout the proof of Theorem IV.1.2 in [48]. They are only included here for convenience, and to prepare the reader for similar arguments to come.

Corollary IV.2.3. *1) An endomorphism $\alpha \in \mathcal{E}_n$ has rank 1 if and only if $\alpha = \phi_{e,e}$ for some $e^2 = e \in \mathcal{T}_n$. There is a one-to-one correspondence between the endomorphisms of rank 1 and the one-element subsemigroups of $\mathcal{T}_n$.*

2) An endomorphism $\alpha \in \mathcal{E}_n$ has rank 2 if and only if $\alpha = \phi_{t,e}$ for some $(t, e) \in P_n$ with $t^2 = t \neq e$. There is a one-to-one correspondence between the endomorphisms of rank 2 and the two-element semilattices $\{t, e\} \subseteq \mathcal{T}_n$ with $e < t$.

3) An endomorphism $\alpha \in \mathcal{E}_n$ has rank 3 if and only if $\alpha = \phi_{t,e}$ for some $(t, e) \in P_n$ with $t \neq t^2 \neq e$. There is a one-to one correspondence between the endomorphisms of rank 3 and the three-element subsemigroups of $\mathcal{T}_n$ consisting of a two-element subgroup $\{t, t^2\}$ having identity element t^2, together with an adjoined zero e.

4) The map $\phi_{t,\mathrm{id}} \in \mathcal{E}_n$ if and only if $t = \mathrm{id}$.

5) If $\phi_{u,f} \in \mathcal{E}_n$ for some $u, f \in \mathcal{T}_n$, then $\phi_{u^2,f}$, $\phi_{f,f}$ and ϕ_{u^2,u^2} are also in $\mathcal{E}_n$.

Proof. Let $\alpha \in \mathcal{E}_n$ be an element of rank at most three. Thus $\alpha = \phi_{t,e}$ for some $(t, e) \in P_n$ and it follows from the definition of $\phi_{t,e}$ that $\operatorname{im} \phi_{t,e} = \{t, t^2, e\}$.

For part 1) it is clear that α has rank 1 if and only if $t = t^2 = e$. In this case, since e is idempotent it is clear that the image of $\phi_{e,e}$ is the trivial semigroup $\{e\}$. Conversely, for each idempotent e there is a unique endomorphism $\phi_{e,e}$ of rank 1 with image $\{e\}$.

For part 2), we note that it follows from Lemma IV.2.2 part 2) that $\phi_{t,e}$ having rank 2 is equivalent to the condition that $t = t^2 \neq e$ (indeed, this tells us that it is not possible to simultaneously have $t \neq t^2$ and $t^2 = e$, and since e is idempotent it is also not possible to simultaneously have $t = e$ and $t^2 \neq t$). In this case, the image of $\phi_{t,e}$ is $\{e, t\}$ and the relations $e = e^2 = te = et$ and $t = t^2$ yield that this is a two element semilattice with $e < t$. Conversely, for each pair of distinct comparable idempotents t, e with $t > e$ there is a unique endomorphism $\phi_{t,e}$ of rank 2 with image $\{t, e\}$.

For part 3), it is clear that $\phi_{t,e}$ has rank 3 if and only if $t \neq t^2 \neq e$. In this case $\phi_{t,e}$ has image $\{e, t, t^2\}$ and using the fact that t^2 is idempotent and $(t^2, e) \in P_n$ by Lemma IV.2.2 part 7), we obtain that $\{t, t^2, e\}$ is a subsemigroup of $\mathcal{T}_n$, where the idempotent t^2 acts identically on the left and right of t, and the idempotent e acts as a left and right zero on all three elements. Conversely, for each two-element subgroup $\{t, t^2\}$ where t^2 is idempotent, and each idempotent e such that $\{t, t^2, e\}$ is a group with a zero e adjoined, we have a unique endomorphism $\phi_{t,e}$ of rank 3 with image $\{t, t^2, e\}$. This proves parts 1)–3).

To see that 4) holds note that for the map $\phi_{t,\mathrm{id}}$ to be in $\mathcal{E}_n$, we require $(t, \mathrm{id}) \in P_n$, which forces $t = \mathrm{id}$ by 6) of Lemma IV.2.2, and then $\phi_{t,\mathrm{id}} = \phi_{\mathrm{id},\mathrm{id}}$.

Finally, for part 5), we know that $\phi_{u,f} \in \mathcal{E}_n$ if and only if $(u, f) \in P_n$. But then, using parts 7) and 1) of Lemma IV.2.2 we have that u^2 and f are idempotents, and we therefore have that $u^2, f \in U_n$ as well as $(u^2, f) \in P_n$. Hence $\phi_{u^2,f}$, $\phi_{f,f}$ and ϕ_{u^2,u^2} satisfy all conditions to be endomorphisms. □

The next result is surprising in that singular elements of $\mathcal{E}_n$ of rank no greater than 3 (so, all singular elements in the case $n \geq 5$) are entirely determined by their images. This has significant consequences later when we consider Green's relations.

Lemma IV.2.4. *Let $\phi_{t,e}, \phi_{u,f} \in \mathcal{E}_n$. Then $\phi_{t,e} = \phi_{u,f}$ if and only if $\operatorname{im}\phi_{t,e} = \operatorname{im}\phi_{u,f}$ if and only if $t = u$ and $e = f$.*

Proof. Suppose that $\operatorname{im}\phi_{t,e} = \operatorname{im}\phi_{u,f}$ and consider the description of the images corresponding to the possible ranks of these maps as given in Corollary IV.2.3. Clearly if $\phi_{t,e}$ and $\phi_{u,f}$ have rank 1, then $t = e = u = f$. Suppose that they have rank 2 so that $\operatorname{im}\phi_{t,e} = \{t, e\}$ and $\operatorname{im}\phi_{u,f} = \{u, f\}$ where t, e, u and f are all idempotents. Then, since their images are two element semilattices with $e < t$ and $f < u$, we get that $e = f$ and $t = u$. Finally, if $\phi_{t,e}$ and $\phi_{u,f}$ have rank 3, then $\{t, t^2, e\} = \{u, u^2, f\}$ where t and u are the only non idempotent elements, and e and f are the zeros, respectively, which together forces $t = u$ and $e = f$. In all cases, we have shown that if $\operatorname{im}\phi_{t,e} = \operatorname{im}\phi_{u,f}$ then $t = u$ and $e = f$, and therefore $\phi_{t,e} = \phi_{u,f}$. All the other equivalences follows directly from this. □

We also describe below the explicit multiplication of elements in $\mathcal{E}_n$ for $n \neq 4$ as this will be a cornerstone of many later proofs.

Corollary IV.2.5. *Let $g, h \in \mathcal{S}_n$ and $(t, e), (u, f) \in P_n$. Then we have the following compositions in $\mathcal{E}_n$:*

1) $\psi_g \psi_h = \psi_{gh}$*;*

2) $\psi_g \phi_{t,e} = \phi_{t,e}$*;*

3) $\phi_{t,e} \psi_g = \phi_{t^g, e^g}$*; and*

4)
$$\phi_{t,e}\phi_{u,f} = \begin{cases} \phi_{u,f} & \text{if } t \in \mathcal{S}_n \setminus \mathcal{A}_n \text{ and } e \neq \mathrm{id}, \\ \phi_{u^2,f} & \text{if } t \in \mathcal{A}_n \text{ and } e \neq \mathrm{id}, \\ \phi_{f,f} & \text{if } t \in \mathcal{T}_n \setminus \mathcal{S}_n \text{ and } e \neq \mathrm{id}, \\ \phi_{u^2,u^2} & \text{if } t = e = \mathrm{id}. \end{cases}$$

Proof. All of the products are straightforward computations using the definition in Theorem IV.1.2; we only detail that for 4). It is nonetheless worth noting that the map on the right-hand side of product 3) is well-defined and indeed belongs to $\mathcal{E}_n$ by point 5) of Lemma IV.2.2.

So consider $(t, e), (u, f) \in P_n$ so that $\phi_{t,e}, \phi_{u,f} \in \mathcal{E}_n$, and let $s \in \mathcal{T}_n$. Then:

$$s\phi_{t,e}\phi_{u,f} = \begin{cases} t\phi_{u,f} & \text{if } s \in \mathcal{S}_n \setminus \mathcal{A}_n, \\ t^2\phi_{u,f} & \text{if } s \in \mathcal{A}_n, \\ e\phi_{u,f} & \text{if } s \in \mathcal{T}_n \setminus \mathcal{S}_n. \end{cases}$$

Recall from Lemma IV.2.2 part 6) that if $e = \mathrm{id}$ then we must also have $t = \mathrm{id}$. Clearly, if $t = e = \mathrm{id}$, then we have that $t\phi_{u,f} = t^2\phi_{u,f} = e\phi_{u,f} = \mathrm{id}\phi_{u,f} = u^2$ so that $\phi_{t,e}\phi_{u,f} = \phi_{u^2,u^2}$ in this case. Thus in all remaining cases we may assume that $e \neq \mathrm{id}$ and hence $e \in \mathcal{T}_n \setminus \mathcal{S}_n$. If $t \in \mathcal{T}_n \setminus \mathcal{S}_n$, then $t^2, e \in \mathcal{T}_n \setminus \mathcal{S}_n$ so that $t\phi_{u,f} = t^2\phi_{u,f} = e\phi_{u,f} = f$. Therefore we get that $\phi_{t,e}\phi_{u,f} = \phi_{f,f}$ whenever $t \in \mathcal{T}_n \setminus \mathcal{S}_n$. If $t \in \mathcal{S}_n$, then $t^2 = \mathrm{id}$ by Lemma IV.2.2 part 3), so that $t^2\phi_{u,f} = \mathrm{id}\phi_{u,f} = u^2$. In the case where $t \in \mathcal{A}_n$, we get that

$$s\phi_{t,e}\phi_{u,f} = \begin{cases} t\phi_{u,f} = u^2 & \text{if } s \in \mathcal{S}_n \setminus \mathcal{A}_n, \\ \mathrm{id}\phi_{u,f} = u^2 & \text{if } s \in \mathcal{A}_n, \\ e\phi_{u,f} = f & \text{if } s \in \mathcal{T}_n \setminus \mathcal{S}_n, \end{cases}$$

which shows that $\phi_{t,e}\phi_{u,f} = \phi_{u^2,f}$. Otherwise, $t \in \mathcal{S}_n \setminus \mathcal{A}_n$ and we obtain

$$s\phi_{t,e}\phi_{u,f} = \begin{cases} t\phi_{u,f} = u & \text{if } s \in \mathcal{S}_n \setminus \mathcal{A}_n, \\ \mathrm{id}\phi_{u,f} = u^2 & \text{if } s \in \mathcal{A}_n, \\ e\phi_{u,f} = f & \text{if } s \in \mathcal{T}_n \setminus \mathcal{S}_n, \end{cases}$$

so that $\phi_{t,e}\phi_{u,f} = \phi_{u,f}$ in that case. □

Notice that for $n \neq 4$ the previous result encapsulates the multiplication table of $\mathcal{E}_n$. In the next section we record the remaining products in the case $n = 4$.

IV.2.2 Endomorphisms of rank 7 in End($\mathcal{T}_4$)

Let $D(4) = \{\sigma^g : g \in \mathcal{S}_4\}$ denote the set of all endomorphisms of rank 7 in $\mathcal{E}_4$. We complete the multiplication table presented in Corollary IV.2.5 to account for these additional elements in $\mathcal{E}_4$.

Lemma IV.2.6. *The sets $D(4)$ and $\mathcal{E}_4 \setminus D(4)$ are subsemigroups of $\mathcal{E}_4$. Moreover, we have the following compositions in $\mathcal{E}_4$:*

1) $\sigma^g \sigma^h = \sigma^{p_g h}$;
2) $\sigma^g \psi_h = \sigma^{gh}$;
3) $\psi_h \sigma^g = \sigma^{p_h g}$;
4) $\sigma^g \phi_{t,e} = \phi_{t,e}$; *and*
5) $\phi_{t,e} \sigma^g = \phi_{t\sigma^g, e\sigma^g}$.

Proof. That $\mathcal{E}_4 \setminus D(4)$ is a subsemigroup follows from Corollary IV.2.5. To see that $D(4)$ is a subsemigroup, consider $\sigma^g, \sigma^h \in D(4)$. Using the fact that σ is an endomorphism, for all $s \in \mathcal{T}_4$ we have:

$$s\sigma^g\sigma^h = \begin{cases} (g^{-1}p_s g)\sigma^h & \text{if } s \in \mathcal{S}_4, \\ c_{4g}\sigma^h & \text{if } s \in \mathcal{T}_4 \setminus \mathcal{S}_4, \end{cases} = \begin{cases} h^{-1}(p_{g^{-1}} p_s p_g)h & \text{if } s \in \mathcal{S}_4, \\ c_{4h} & \text{if } s \in \mathcal{T}_4 \setminus \mathcal{S}_4, \end{cases}$$
$$= s\sigma^{p_g h},$$

where we use the fact that both p_s and p_g fix 4 and hence $p_s\sigma = p_s$ and $c_{4h} = c_{4p_g h}$. This shows that 1) holds, and hence that $D(4)$ is a subsemigroup.

For all $s \in \mathcal{T}_4$, it is clear that $s\sigma^g\psi_h = h^{-1}s\sigma^g h = s\sigma^{gh}$, and so 2) holds. Similarly, using the fact that σ is an endomorphism, we get

$$s\psi_h\sigma^g = (h^{-1}sh)\sigma^g = g^{-1}p_h^{-1}s\sigma p_h g = s\sigma^{p_h g},$$

as given in 3).

Using Lemma IV.1.7 we have that for all $s \in \mathcal{T}_4$ and $\sigma^g, \phi_{t,e} \in \mathcal{E}_4$

$$s\sigma^g\phi_{t,e} = \begin{cases} t & \text{if } s\sigma^g \in \mathcal{S}_4 \setminus \mathcal{A}_4, \\ t^2 & \text{if } s\sigma^g \in \mathcal{A}_4, \\ e & \text{if } s\sigma^g \in \mathcal{T}_4 \setminus \mathcal{S}_4, \end{cases} = \begin{cases} t & \text{if } s \in \mathcal{S}_4 \setminus \mathcal{A}_4, \\ t^2 & \text{if } s \in \mathcal{A}_4, \\ e & \text{if } s \in \mathcal{T}_4 \setminus \mathcal{S}_4, \end{cases}$$

so that $\sigma^g\phi_{t,e} = \phi_{t,e}$ and 4) holds.

Finally, for all $s \in \mathcal{T}_4$ we have

$$s\phi_{t,e}\sigma^g = \begin{cases} t\sigma^g & \text{if } s \in \mathcal{S}_4 \setminus \mathcal{A}_4, \\ t^2\sigma^g & \text{if } s \in \mathcal{A}_4, \\ e\sigma^g & \text{if } s \in \mathcal{T}_4 \setminus \mathcal{S}_4. \end{cases}$$

Notice that since σ^g is an endomorphism, we have that $t^2\sigma^g = (t\sigma^g)^2$. The result of 5) then follows from the fact that $(t\sigma^g, e\sigma^g) \in P_4$ by 8) of Lemma IV.2.2. □

IV.3 A DECOMPOSITION VIA RANK AND TYPE

The monoid $\mathcal{E}_n$ can be partitioned in a convenient way by considering the following subsets of $\mathcal{E}_n \setminus \mathcal{G}_n$:

$$\begin{aligned} E_3(n) &= \{\phi_{t,e} \in \mathcal{E}_n \colon t \in \mathcal{S}_n \setminus \mathcal{A}_n, e \neq \mathrm{id}\}, \\ A(n) &= \{\phi_{t,e} \in \mathcal{E}_n \colon t \in \mathcal{A}_n, t \neq \mathrm{id} \neq e\}, \\ B(n) &= \left\{\phi_{t,e} \in \mathcal{E}_n \colon t \in \mathcal{T}_n \setminus \mathcal{S}_n, t \neq t^2 \neq e \neq \mathrm{id}\right\}, \\ E_2(n) &= \{\phi_{\mathrm{id},e} \in \mathcal{E}_n \colon e \neq \mathrm{id}\}, \\ C(n) &= \left\{\phi_{t,e} \in \mathcal{E}_n \colon t \in \mathcal{T}_n \setminus \mathcal{S}_n, t = t^2 \neq e \neq \mathrm{id}\right\} \text{ and} \\ E_1(n) &= \{\phi_{e,e} \in \mathcal{E}_n\}. \end{aligned}$$

For $n = 4$ we also define $E_7(4) = \{\sigma^g : g \in \mathcal{K}\} \subsetneq D(4) = \{\sigma^g : g \in \mathcal{S}_4\}$. To reduce notation, when it is clear from context we will suppress the dependence on n and write simply $E_3, A, B, E_2, C, E_1, E_7$ and D. These subsets group together maps that share similar properties, such as the idempotents of a given rank:

Lemma IV.3.1. *For $k = 1, 2, 3, 7$ the set E_k consists of all the idempotents of rank k in $\mathcal{E}_n$. The set of idempotents of $\mathcal{E}_n$ is therefore*

$$E(\mathcal{E}_n) = \begin{cases} \{\varepsilon\} \cup E_7 \cup E_3 \cup E_2 \cup E_1 & \text{if } n = 4, \\ \{\varepsilon\} \cup E_3 \cup E_2 \cup E_1 & \text{otherwise.} \end{cases}$$

Proof. Clearly, the only idempotent element of $\mathcal{G}_n$ is ε. It follows from the multiplication in Corollary IV.2.5 that $\alpha = \phi_{t,e} \in \mathcal{E}_n$ is idempotent if and only if either (i) t is odd and $e \neq \mathrm{id}$ (in which case $\alpha \in E_3$), or (ii) $t = t^2 \neq e$ and $t \in \mathcal{S}_n$ (in which case

$t = \mathrm{id}$ and $\alpha \in E_2$) or (iii) $t = e$ from the last two cases of 4) in Corollary IV.2.5 (in which case $\alpha \in E_1$). For $n = 4$ it follows from Lemma IV.2.6 that $\sigma^g \in D$ is idempotent if and only if $p_g = \mathrm{id}$ or, in other words, if and only if $g \in \mathcal{K}$. $\square$

The remaining sets A, B, C (together with $D \setminus E_7$ in the case $n = 4$) account for all the non-idempotent singular maps; the reasoning behind this grouping shall be made apparent shortly.

Lemma IV.3.2. *The endomorphism monoid $\mathcal{E}_n$ can be written as:*

$$\mathcal{E}_n = \begin{cases} \mathcal{G}_n \cup D \cup (E_3 \cup B) \cup (E_2 \cup C) \cup (E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\} \cup \{\phi_{\mathrm{id},\mathrm{id}}\}) & \text{if } n = 4, \\ \mathcal{G}_n \cup (E_3 \cup A \cup B) \cup (E_2 \cup C) \cup (E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\} \cup \{\phi_{\mathrm{id},\mathrm{id}}\}) & \text{otherwise,} \end{cases}$$

where subsets containing endomorphisms of the same rank are bracketed together. For $n \geq 2$ this union is disjoint and the set $\mathcal{E}_n \setminus \mathcal{G}_n$ is an ideal of $\mathcal{E}_n$.

Proof. That $\mathcal{E}_n$ is the union of the given sets follows from Theorem IV.1.2 and Corollary IV.2.3, noting that the constraints of the given sets cover all eventualities and that the set A is empty in the case of $n = 4$ (since if $t \in \mathcal{A}_4 \setminus \{\mathrm{id}\}$, then t has no fixed points, and thus $t \notin U_n$ by Lemma IV.2.1). The fact that the bracketed expressions are the sets of endomorphisms of the same rank also follows from Corollary IV.2.3. Each automorphism of $\mathcal{E}_n$ has rank n^n and hence the union is disjoint for $n \geq 2$. $\square$

As we have seen in Corollary IV.2.5, there is an important distinction in the multiplicative behaviour of elements $\phi_{t,e}$ depending on where the element $t \in \mathcal{T}_n$ lies and whether $e = \mathrm{id}$. Since this will be of great importance to determine Green's relations and the ideal structure of $\mathcal{E}_n$, we define the *type* of an endomorphism θ relative to the type of the underlying transformations associated with θ.

Definition IV.3.3. We say that $\theta \in \mathcal{E}_n$ is of:

- *group type* if $\theta \in \mathcal{G}_n$;
- *exceptional type* if $n = 4$ and $\theta \in D$;
- *odd type* if $\theta \in E_3$;
- *even type* if $\theta \in A \cup E_2$;
- *non-permutation type* if $\theta \in B \cup C \cup (E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\})$;
- *trivial type* if $\theta = \phi_{\mathrm{id},\mathrm{id}}$.

Notice that the partition of $\mathcal{E}_n$ given by Lemma IV.3.2 is therefore a partition into subsets of elements having the same rank *and* type.

The notion of type is a good one since this characteristic of a map is stable under multiplication by automorphisms.

Lemma IV.3.4. *1) For any $\phi_{t,e} \in \mathcal{E}_n$ and $\psi_g \in \mathcal{G}_n$ we have that*

$$\phi_{t,e}, \quad \psi_g\phi_{t,e} = \phi_{t,e} \quad \textit{and} \quad \phi_{t,e}\psi_g = \phi_{t^g,e^g}$$

have the same type.

2) For any $\sigma^h \in D$ and any $\psi_g \in \mathcal{G}_4$ we have that

$$\sigma^h, \quad \psi_g\sigma^h = \sigma^{p_g h} \quad \textit{and} \quad \sigma^h\psi_g = \sigma^{hg}$$

have the same (exceptional) type.

3) Let $n \neq 4$ and $\gamma \in \mathcal{E}_n$. For any $\phi_{t,e} \in X$ where X is one of A, B or C, we have $\phi_{t,e}\gamma \in X$ if and only if $\gamma \in \mathcal{G}_n$.

Proof. The proof of parts 1) and 2) follow immediately from Corollary IV.2.5 and Lemma IV.2.6, together with the observation that conjugation in $\mathcal{E}_n$ preserves the parity of elements in $\mathcal{S}_n$ and the rank of all elements.

To show that 3) holds, it only remains to show the converse. Let $\phi_{t,e}$ be in A, so that it is of even type and rank 3. By Corollary IV.2.5 we have that for any $\phi_{u,f} \in \mathcal{E}_n$, the product $\phi_{t,e}\phi_{u,f} = \phi_{u^2,f}$ has rank at most 2, so cannot lie in A. Similarly, if $\phi_{t,e}$ is in B or C, so that it has non-permutation type and is of rank 3 or 2, then $\phi_{t,e}\phi_{u,f} = \phi_{f,f}$ has rank 1, so cannot lie in B or C. $\square$

Definition IV.3.5. For $\alpha \in \mathcal{E}_n$ we define the *orbit* of α to be $\alpha\mathcal{G}_n$. It is easy to see that all elements of a given orbit have the same rank and (by Lemma IV.3.4) the same type.

In view of the decomposition given in Lemma IV.3.2, we note that each of the sets $\mathcal{G}_n, E_3, A, B, E_2, C, E_1$ (and D in the case $n = 4$) is a union of orbits. For $\phi_{t,e} \in X$ where X is one of A, B or C, it will sometimes be convenient to write $X_{t,e}$ to denote the orbit of $\phi_{t,e}$, in order to easily recall the rank and type of elements in this orbit without specific mention of the corresponding properties of t and e.

For all $n \geq 5$, being of the same type is equivalent to acting in the same way by multiplication on the left on the singular part of $\mathcal{E}_n$.

Lemma IV.3.6. *Let $n \geq 5$ and let $\alpha, \beta \in \mathcal{E}_n \setminus \mathcal{G}_n$. Then α and β are of the same type if and only if $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathcal{E}_n \setminus \mathcal{G}_n$.*

Proof. Since $\alpha, \beta, \gamma \in \mathcal{E}_n \setminus \mathcal{G}_n$, only 4) in Corollary IV.2.5 is relevant. It follows immediately from the description of this multiplication that if α and β are of the same type, then $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathcal{E}_n \setminus \mathcal{G}_n$.

Conversely, suppose $\alpha, \beta \in \mathcal{E}_n \setminus \mathcal{G}_n$ are such that $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathcal{E}_n \setminus \mathcal{G}_n$. If $\gamma = \phi_{u,f} \in \mathcal{E}_n$ where $u = (2\,3) \neq \mathrm{id} = u^2 \neq f = c_1 \neq u$, then the maps $\phi_{u,f}$, $\phi_{u^2,f}$, $\phi_{f,f}$ and ϕ_{u^2,u^2} are all distinct. From this, it is clear that if α and β have different types, then we fall into into a different case for the multiplication and therefore $\alpha\gamma \neq \beta\gamma$, which gives us the equivalence. □

Remark IV.3.7. The authors of [48] gave explicit formulae to count the number of endomorphisms of each rank. Since $\mathcal{G}_n$ and $\mathcal{S}_n$ are isomorphic we have $|\mathcal{G}_n| = n!$. Meanwhile, it is clear that for any $e = e^2 \in \mathcal{E}_n$ we have that $(e, e) \in P_n$ and if $e \neq \mathrm{id}$ then also $(\mathrm{id}, e) \in P_n$. Consequently, $|E_1(n)| = |E_2(n)| + 1 = |\{e \in \mathcal{T}_n : e^2 = e\}|$. We shall not attempt to give formulae for the cardinality of the remaining sets in our partition of $\mathcal{E}_n$, but it is useful to note that for $n \geq 5$ each set in this partition is non-empty. We conclude this section by recording examples to demonstrate this below; some of these examples will be utilised in later proofs.

Example IV.3.8. Let $n \geq 5$ and let $t, u, p, q, e, f \in \mathcal{T}_n$ be defined as follows:

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 & i_{\geq 5} \\ 1 & 3 & 2 & 1 & i \end{pmatrix}, \quad p = \begin{pmatrix} 1 & 2 & 3 & 4 & i_{\geq 5} \\ 2 & 1 & 4 & 3 & i \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 2 & 3 & 4 & i_{\geq 5} \\ 5 & 5 & 5 & 5 & i \end{pmatrix},$$

$$u = \begin{pmatrix} 1 & 2 & 3 & 4 & i_{\geq 5} \\ 1 & 1 & 1 & 4 & 4 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 2 & 3 & 4 & i_{\geq 5} \\ 1 & 3 & 2 & 4 & i \end{pmatrix}, \quad f = \begin{pmatrix} 1 & 2 & 3 & 4 & i_{\geq 5} \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It is easy to verify that $p, q \in \mathcal{S}_n$, $p \in \mathcal{A}_n$, $q \in \mathcal{S}_n \setminus \mathcal{A}_n$ and $t, u, e, f \in \mathcal{T}_n \setminus \mathcal{S}_n$ are such that:

- $e^2 = e \neq \mathrm{id} \neq f = f^2$;
- $p^2 = \mathrm{id} = q^2$, $pe = e = ep$ and $qf = f = fq$ so that $(p, e), (q, f) \in P_n$;
- $t^3 = t$ and $te = e = et$ so that $(t, e) \in P_n$.
- $u^2 = u$ and $uf = f = fu$ so that $(u, f) \in P_n$.

It then follows that the maps $\phi_{p,e}, \phi_{q,f}, \phi_{t,e}, \phi_{\mathrm{id},f}$ and $\phi_{u,f}$ are endomorphisms of $\mathcal{T}_n$. Moreover, it is clear from definition that $\phi_{q,f} \in E_3(n)$, $\phi_{p,e} \in A(n)$, $\phi_{t,e} \in B(n)$, $\phi_{\mathrm{id},f} \in E_2(n)$, $\phi_{u,f} \in C(n)$, and $\phi_{f,f} \in E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\}$.

Remark IV.3.9. We will make use of the preceding results repeatedly in the following sections where we describe the regular elements, Green's relations, ideal structure and extended Green's relations for $\mathcal{E}_n$. In order to state our results in their most general form, it will be convenient to assume that $n \geq 5$; this, of course, bypasses the case when $n = 4$, where the structure of $\mathcal{E}_4$ is more complicated due to the additional maps of rank 7, as well as some degenerate behaviour for $n \leq 3$ (where there are in some sense too few maps for the general behaviour to emerge). We will return to these special cases in Section IV.8.

IV.4 IDEMPOTENTS AND REGULARITY

Throughout this section we assume that $n \geq 5$. In particular, this means that the set $\mathcal{E}_n \setminus \mathcal{G}_n$ of singular endomorphisms is equal to the set of endomorphisms of rank at most three:

$$\mathcal{E}_n \setminus \mathcal{G}_n = \{\phi_{t,e} : (t,e) \in P_n\} = \left\{\phi_{t,e} \mid t, e \in \mathcal{T}_n \text{ with } t^3 = t \text{ and } te = et = e^2 = e\right\}.$$

IV.4.1 THE LEFT ACTION OF THE ENDOMORPHISM MONOID ON THE SINGULAR PART

For $(t,e) \in P_n$ the maps $\phi_{t^2,e}$, $\phi_{e,e}$ and ϕ_{t^2,t^2} are all closely related to the map $\phi_{t,e}$. Indeed their images are all contained in that of $\phi_{t,e}$ and by Corollary IV.2.5 they are all in $\mathcal{E}_n\phi_{t,e}$. Lemma IV.3.6 allows us to define a notation facilitating this identification.

Definition IV.4.1. Let $\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$. Then we define α^+, α^- and α^0 as:

- $\alpha^+ = \gamma\alpha$ for any $\gamma \in \mathcal{E}_n \setminus \mathcal{G}_n$ of even type;
- $\alpha^- = \gamma\alpha$ for any $\gamma \in \mathcal{E}_n \setminus \mathcal{G}_n$ of non-permutation type; and
- $\alpha^0 = \phi_{\mathrm{id},\mathrm{id}}\alpha$ (that is, $\gamma\alpha$ for $\gamma \in \mathcal{E}_n \setminus \mathcal{G}_n$ of trivial type).

Additionally, for $X \subseteq \mathcal{E}_n \setminus \mathcal{G}_n$ and $\dagger \in \{+, -, 0\}$, we define $X^\dagger$ to be the set $\left\{\alpha^\dagger : \alpha \in X\right\}$.

Remark IV.4.2. Under this definition, we can now see that if $\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$, then $\gamma\alpha \in \{\alpha, \alpha^+, \alpha^-, \alpha^0\}$ for any $\gamma \in \mathcal{E}_n$ and thus $\mathcal{E}_n\alpha = \{\alpha, \alpha^+, \alpha^-, \alpha^0\}$. Additionally, if we write α as $\phi_{t,e}$, then we have that

$$\gamma\alpha = \begin{cases} \alpha = \phi_{t,e} & \text{if } \gamma \text{ has group type or odd type,} \\ \alpha^+ = \phi_{t^2,e} & \text{if } \gamma \text{ has even type,} \\ \alpha^- = \phi_{e,e} & \text{if } \gamma \text{ has non-permutation type,} \\ \alpha^0 = \phi_{t^2,t^2} & \text{if } \gamma \text{ has trivial type.} \end{cases}$$

We now show how each of the sets involved in the decomposition of $\mathcal{E}_n$ behave under the operations mapping α to α^+, α^- or α^0.

Lemma IV.4.3. *Let $\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$. Then*

1) $1 = \mathrm{rk}(\alpha^0) = \mathrm{rk}(\alpha^-) \leq \mathrm{rk}(\alpha^+) \leq \mathrm{rk}(\alpha) \leq 3$*;*

2) $\alpha^+ = \alpha$ *if and only if* $\mathrm{rk}(\alpha) \leq 2$ *if and only if* $\alpha \in E_2 \cup C \cup E_1$*;*

3) $\alpha^- = \alpha$ *if and only if* $\alpha^0 = \alpha$ *if and only if* $\mathrm{rk}(\alpha) = 1$ *if and only if* $\alpha \in E_1$*;*

4) $E_3^+ \cup A^+ \subseteq E_2 = E_2^+$, $\quad B^+ \subseteq C = C^+$, *and* $E_1^+ = E_1$*;*

5) $E_3^- \cup A^- \cup B^- \cup C^- \subseteq E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\} = E_2^-$, *and* $E_1^- = E_1$*;*

6) $B^0 \cup C^0 \subseteq E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\} = (E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\})^0$, *and*
$$A^0 = E_3^0 = E_2^0 = \{\phi_{\mathrm{id},\mathrm{id}}\} = \{\phi_{\mathrm{id},\mathrm{id}}\}^0.$$

Consequently, we have $(\mathcal{E}_n \setminus \mathcal{G}_n)^+ = E_2 \cup C \cup E_1$ *and* $(\mathcal{E}_n \setminus \mathcal{G}_n)^- = (\mathcal{E}_n \setminus \mathcal{G}_n)^0 = E_1$.

Proof. Let $\alpha = \phi_{t,e} \in \mathcal{E}_n$ for some $(t,e) \in P_n$. Throughout this proof, we use the description of α^+, α^- and α^0 given in Remark IV.4.2.

Part 1) follows immediately from Remark IV.4.2. For part 2), suppose first that $\alpha = \alpha^+$, which means that $\phi_{t,e} = \phi_{t^2,e}$, so that $t = t^2$ and $\mathrm{rk}(\alpha) \leq 2$. If $t \in \mathcal{S}_n$, this forces $t = \mathrm{id}$, and thus $\alpha = \phi_{\mathrm{id},e} \in E_2$. Otherwise, $t \in \mathcal{T}_n \setminus \mathcal{S}_n$ and either $t \neq e$, which means that $\alpha \in C$, or $t = e$ and then $\alpha = \phi_{e,e} \in E_1$. Conversely, if $\alpha \in E_2 \cup C \cup E_1$, then $t = t^2$ from which we have that α has rank at most 2 and $\alpha^+ = \phi_{t^2,e} = \phi_{t,e} = \alpha$.

For part 3), if $\alpha = \alpha^-$, then we have that $\phi_{t,e} = \phi_{e,e}$ which forces $t = e$, while if $\alpha = \alpha^0$, we require $\phi_{t,e} = \phi_{t^2,t^2}$ which implies that $t = t^2 = e$. In both cases, this gives us that $\alpha \in E_1$, that is, has rank 1. Conversely, if $\alpha \in E_1$, then $t = t^2 = e$ and $\alpha^- = \phi_{e,e} = \alpha$ while $\alpha^0 = \phi_{t^2,t^2} = \phi_{e,e} = \alpha$.

For part 4), notice that we have already seen that $E_2^+ = E_2$, $C^+ = C$ and $E_1^+ = E_1$. If $\alpha = \phi_{t,e} \in E_3 \cup A$ so that $t \in \mathcal{S}_n$, then $\alpha^+ = \phi_{t^2,e} = \phi_{\mathrm{id},e} \in E_2$. On the other hand, if $\alpha = \phi_{t,e} \in B$, we have that $t^2 \neq e$ and then $\alpha^+ = \phi_{t^2,e} \in C$.

Similarly, for part 5) if $\alpha = \phi_{t,e} \in E_3 \cup A \cup B \cup E_2 \cup C$ then $e \neq \mathrm{id}$ and so $\alpha^- = \phi_{e,e} \in E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\}$, whilst for each idempotent $e \neq \mathrm{id}$ we have that $\phi_{\mathrm{id},e}^- = \phi_{e,e}$ and so $E_2^- = E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\}$. The remaining equality is given by part 3). Likewise, for part 6) if $\alpha = \phi_{t,e} \in B \cup C \cup E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\}$ then $t^2 \neq \mathrm{id}$ and so $\alpha^0 = \phi_{t^2,t^2} \in E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\}$, whilst for $\alpha = \phi_{t,e} \in E_3 \cup A \cup E_2 \cup \{\phi_{\mathrm{id},\mathrm{id}}\}$ we have $t^2 = \mathrm{id}$ and so $\alpha^0 = \phi_{\mathrm{id},\mathrm{id}}$. □

The inclusions given in the previous lemma, as well as the representation of the action of the maps $\alpha \mapsto \alpha^x$ for $x \in \{+,-,0\}$ can be seen in Figure IV.1 alongside the depiction of the $\mathcal{J}$-order of the monoid $\mathcal{E}_n$. The latter will be given explicitly in Proposition IV.6.1.

As a direct consequence of the previous lemma, we can describe the composition of the operations mapping $\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$ to α^+, α^- or α^0.

Lemma IV.4.4. *Let $\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$. Then for any $x, y \in \{+,-,0\}$ we have $(\alpha^x)^y = \alpha^y$ if $x = +$ and $(\alpha^x)^y = \alpha^x$ otherwise.*

Proof. For any $\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$, we have that $\alpha^-, \alpha^0 \in E_1$. It follows from Lemma IV.4.3 that if $x \in \{-,0\}$, then we get that $(\alpha^x)^y = \alpha^x$ for all $y \in \{+,-,0\}$. On the other hand, since $\alpha^+ \in E_2 \cup C \cup E_1$ we get that $(\alpha^+)^+ = \alpha^+$. If we now assume that $\alpha = \phi_{t,e}$, then using Remark IV.4.2 we have that $(\alpha^+)^- = \phi_{t^2,e}^- = \phi_{e,e} = \alpha^-$ and $(\alpha^+)^0 = \phi_{t^2,e}^0 = \phi_{t^2,t^2} = \alpha^0$. Hence, we have proved that $(\alpha^x)^y = \alpha^y$ for all $y \in \{+,-,0\}$ whenever $x = +$. □

IV.4.2 Idempotents

The set of idempotents of $\mathcal{E}_n$, which we denote by $E = E(\mathcal{E}_n)$, has very nice properties as given by:

Lemma IV.4.5. *In $\mathcal{E}_n$ we have the following:*

1) If $\alpha \in \mathcal{E}_n$ and $\beta \in \mathcal{E}_n \setminus \mathcal{G}_n$ is idempotent, then $\alpha\beta \in E$.

2) If $\alpha, \beta \in E$ are idempotents, then $\alpha\beta \in E$.

3) Let $\alpha = \phi_{t,e} \in \mathcal{E}_n \setminus \mathcal{G}_n$ and $g \in \mathcal{S}_n$. Then, $\alpha \in E_i$ for some $1 \leq i \leq 3$ if and only if $\phi_{t,e}\psi_g = \phi_{t^g,e^g} \in E_i$.

4) Let $\alpha, \beta \in E$. Then β is a left identity for α, that is, $\beta\alpha = \alpha$, if and only if one of the following hold:

- $\alpha = \varepsilon = \beta$;
- $\alpha \in E_3$ *and* $\beta \in \{\varepsilon\} \cup E_3$;
- $\alpha \in E_2$ *and* $\beta \in \{\varepsilon\} \cup E_3 \cup E_2$;
- $\alpha \in E_1$ *and* $\beta \in E$.

On the other hand, β is a right identity for α, that is, $\alpha\beta = \alpha$, if and only if one of the following hold:

- $\alpha = \varepsilon = \beta$;
- $\alpha \in E_3$ *and* $\beta \in \{\varepsilon, \alpha\}$;
- $\alpha = \phi_{\mathrm{id},e} \in E_2$ *and* $\beta \in \{\varepsilon, \phi_{\mathrm{id},e}\} \cup \{\phi_{u,e} \colon u \in \mathcal{S}_n$ *is odd and* $(u, e) \in P_n\}$;
- $\alpha = \phi_{e,e} \in E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\}$ *and*
  $\beta \in \{\varepsilon, \phi_{\mathrm{id},e}, \phi_{e,e}\} \cup \{\phi_{u,e} \colon u \in \mathcal{S}_n$ *is odd and* $(u, e) \in P_n\}$; *or*
- $\alpha = \phi_{\mathrm{id},\mathrm{id}} \in E_1$ *and* $\beta \in \{\varepsilon, \phi_{\mathrm{id},\mathrm{id}}\} \cup E_3 \cup E_2$.

Proof. 1) Let $\alpha \in \mathcal{E}_n$ and $\beta \in \mathcal{E}_n \setminus \mathcal{G}_n$. If $\alpha \in \mathcal{G}_n$, then $\alpha\beta = \beta$ which shows that $\alpha\beta \in E$ whenever $\beta \in E$. On the other hand, if $\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$, then $\alpha\beta \in \{\beta, \beta^+, \beta^-, \beta^0\}$. Thus, it suffices to show that if β is idempotent, then so are each of β^+, β^- and β^0. By Lemma IV.4.3, we have that $\beta^-, \beta^0 \in E_1$, while $\beta^+ \in E_2$ if $\beta \in E_3$ and $\beta^+ = \beta$ if $\beta \in E_2 \cup E_1$.

2) If either $\alpha = \varepsilon$ or $\beta = \varepsilon$, then the result is trivial. Otherwise, $\alpha, \beta \in \mathcal{E}_n \setminus \mathcal{G}_n$ and the result follows from part 1).

3) Let $\alpha = \phi_{t,e} \in E$ and $\psi_g \in \mathcal{G}_n$. Since $\phi_{t,e}$ and ϕ_{t^g,e^g} are of the same type by Lemma IV.3.4, it follows that if $\alpha \in E_3$ then $\alpha\psi_g \in E_3$. If $\alpha = \phi_{\mathrm{id},e} \in E_2$, then $\mathrm{id}^g = \mathrm{id}$ so that $\phi_{t^g,e^g} = \phi_{\mathrm{id},e^g} \in E_2$, while $\alpha = \phi_{e,e} \in E_1$ forces $\phi_{t^g,e^g} = \phi_{e^g,e^g} \in E_1$. The converse is clear from the fact that $\phi_{t,e} = \phi_{t^g,e^g}\psi_{g^{-1}}$.

4) Suppose that $\alpha, \beta \in E$, so that $\alpha, \beta \in \{\varepsilon\} \cup E_3 \cup E_2 \cup E_1$. Notice that in all cases, we can have $\beta = \varepsilon$ and it is the only possibility whenever $\alpha = \varepsilon$. Thus we can assume that $\beta = \phi_{u,f} \in E_3 \cup E_2 \cup E_1$ and we consider the products $\alpha\beta$ and $\beta\alpha$ using the expressions given in Remark IV.4.2 and the uniqueness of writing exposed in Lemma IV.2.4.

- Suppose that $\alpha = \phi_{t,e} \in E_3$ (and hence is of odd type). If $\beta\alpha = \alpha$, then this means that we need $\phi_{u,f}\phi_{t,e} = \phi_{t,e}$. Since $t \neq t^2 \neq e$, it follows that $\phi_{u,f}$ must be of odd type, which shows that $\beta \in E_3$. On the other hand, for any $\beta \in E_3 \cup E_2 \cup E_1$, we have that $\alpha\beta = \beta$ since α has odd type, and thus the equation $\alpha\beta = \alpha$ forces $\beta = \alpha$.
- Suppose that $\alpha = \phi_{\mathrm{id},e} \in E_2$ (and hence is of even type). Since $\alpha = \alpha^+$ but $e \neq \mathrm{id}$, it follows that $\beta\alpha = \alpha$ only if β is either of odd or even type, that is, $\beta \in E_3 \cup E_2$. Additionally, $\alpha\beta = \beta^+$ and if we assume that $\alpha\beta = \alpha$, then we require $\phi_{\mathrm{id},e} = \alpha = \beta^+ = \phi_{u^2,f}$. This means that $u^2 = \mathrm{id}$ and $f = e$, so that either $\beta = \alpha$, or $u \in \mathcal{S}_n$ is such that $(u, e) \in P_n$.
- If $\alpha = \phi_{e,e} \in E_1$, then $\alpha = \alpha^+ = \alpha^- = \alpha^0$ by Lemma IV.4.3, and thus $\beta\alpha = \alpha$ for all $\beta \in E$.
- If $\alpha = \phi_{e,e} \in E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\}$ (and hence is of non-permutation type) then we get that $\alpha\beta = \beta^- = \phi_{f,f}$. Thus if $\alpha\beta = \alpha$, we get that $f = e$ and $(u, e) \in P_n$. This occurs whenever $u = \mathrm{id}$, $u = e$ or $u \in \mathcal{S}_n$ is odd with $(u, e) \in P_n$.
- If $\alpha = \phi_{\mathrm{id},\mathrm{id}}$ (and hence is of trivial type) then we have $\alpha\beta = \beta^0 = \phi_{u^2,u^2}$. Thus $\alpha\beta = \alpha$ forces $u^2 = \mathrm{id}$ and thus $\beta \in E_3 \cup E_2 \cup \{\phi_{\mathrm{id},\mathrm{id}}\}$.

Finally, for each point above the converse directly follows from the description of the multiplication in Remark IV.4.2 according to the type of the maps considered. □

As a direct corollary, we give some properties of the set of idempotents of a given rank.

Corollary IV.4.6. *1) Each element of E_3 is a left identity of $\mathcal{E}_n \setminus \mathcal{G}_n$.*

2) Each element of E_1 is a right zero of $\mathcal{E}_n \setminus \mathcal{G}_n$.

3) The minimal ideal of $\mathcal{E}_n$ is E_1.

4) The set of all singular idempotents $E_3 \cup E_2 \cup E_1$ forms a left ideal of $\mathcal{E}_n$.

5) The set of all idempotents $E = \{\varepsilon\} \cup E_3 \cup E_2 \cup E_1$ is a band, and forms a chain of right-zero semigroups.

Proof. The fact that elements of E_3 are left identities for $\mathcal{E}_n \setminus \mathcal{G}_n$ is immediate from Corollary IV.2.5, as is the fact that elements of E_1 are right zeroes for $\mathcal{E}_n \setminus \mathcal{G}_n$. Since $\mathrm{rk}(\alpha\beta) \leq \min\{\mathrm{rk}(\alpha), \mathrm{rk}(\beta)\}$ for all $\alpha, \beta \in \mathcal{E}_n$, and that elements of E_1 have rank 1, it follows that E_1 is an ideal of $\mathcal{E}_n$. Since E_1 is a right-zero semigroup, it has no proper ideals and is therefore the minimal ideal of $\mathcal{E}_n$. That $E_3 \cup E_2 \cup E_1$ forms a left ideal of $\mathcal{E}_n$ follows directly from part 1) of Lemma IV.4.5, while the fact that E is a band is immediate from the fact that it is a subsemigroup by part 2) of the same lemma and only consists of idempotents. Finally, it is easy to see from part 4) of Lemma IV.4.5 that E_3 and E_2 are right-zero semigroups. Since ε is the identity of the monoid $\mathcal{E}_n$, it only remains to show that $E_2E_3 \subseteq E_2$. However, this is immediate from the fact that each $\phi_{\mathrm{id},e} \in E_2$ is of even type, and then, for all $\phi_{u,f} \in E_3$, we have that $\phi_{\mathrm{id},e}\phi_{u,f} = \phi_{u^2,f} = \phi_{\mathrm{id},f} \in E_2$. □

Remark IV.4.7. Combining the results of Lemma IV.4.5 and Corollary IV.4.6, we have shown that the idempotents of $\mathcal{E}_n$ form a *left regular band*, that is, a band that satisfies the identity $xyx = yx$.

IV.4.3 Regular elements

We now have all the tools necessary to describe the regular elements of $\mathcal{E}_n$. Recall that we have a standing assumption that $n \geq 5$.

Proposition IV.4.8. *The set of all regular elements of $\mathcal{E}_n$ is $\mathcal{G}_n \cup E(\mathcal{E}_n)$. Furthermore, this is a proper subsemigroup of $\mathcal{E}_n$. In particular, $\mathcal{E}_n$ is not regular.*

Proof. Clearly if $\alpha \in \mathcal{E}_n$ is an automorphism or an idempotent, then it is a regular element. The fact that $\mathcal{G}_n \cup E(\mathcal{E}_n)$ forms a (proper) subsemigroup is immediate from Lemma IV.3.4 and Corollary IV.4.6.

Conversely, let $\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$ be a regular element of $\mathcal{E}_n$ so that $\alpha\gamma\alpha = \alpha$ where $\gamma \in \mathcal{E}_n$. Suppose for contradiction that α is not idempotent (that is, $\alpha \in A \cup B \cup C$). Then it follows from Corollary IV.2.5 part 2) that $\gamma \in \mathcal{E}_n \setminus \mathcal{G}_n$, as it would otherwise contradict the fact that $\alpha^2 \neq \alpha$. Note that $\alpha\gamma$ must be an idempotent left identity for α and hence of the same rank as α. If $\alpha \in A \cup B$, then α has rank 3; but since $\gamma \in \mathcal{E}_n \setminus \mathcal{G}_n$ and α has even or non-permutation type, the rank of $\alpha\gamma$ is no greater than 2, contradicting that $\alpha\gamma$ has the same rank as α. Similarly, if $\alpha \in C$, then α has rank 2; but since α has non-permutation type, the rank of $\alpha\gamma$ is 1. This shows that the only regular elements of $\mathcal{E}_n \setminus \mathcal{G}_n$ are the idempotents.

The fact that $\mathcal{G}_n \cup E(\mathcal{E}_n)$ is a subsemigroup of $\mathcal{E}_n$ follows from part 1) and 3) of Lemma IV.4.5, and since $A \cup B \cup C \neq \emptyset$, this finishes to show that regular elements form a proper subsemigroup of $\mathcal{E}_n$, so that $\mathcal{E}_n$ is not regular. □

Remark IV.4.9. We will see in Section IV.8 that the *only* values of $n \in \mathbb{N}$ such that $\mathcal{E}_n$ is regular are $n = 1$ and $n = 2$.

A consequence that the regular elements of $\mathcal{E}_n$ lie inside $\mathcal{G}_n \cup E$ is that $\mathcal{T}_n$ cannot embed into $\mathcal{E}_n$. In fact the converse is also true as given by the following.

Corollary IV.4.10. *The semigroup $\mathcal{T}_n$ does not embed into* $\operatorname{End}(\mathcal{T}_n)$. *Conversely,* $\operatorname{End}(\mathcal{T}_n)$ *does not embed into $\mathcal{T}_n$ either.*

Proof. Suppose first that there exists an embedding θ from $\mathcal{T}_n$ into $\operatorname{End}(\mathcal{T}_n)$. Since all elements of $\mathcal{T}_n$ are regular, it follows that the image of θ lies in $\mathcal{G}_n \cup E$. Now consider

$$t = \begin{pmatrix} 1 & 2 & k_{\geq 3} \\ 2 & 3 & 3 \end{pmatrix} \in \mathcal{T}_n,$$

so that $t^2 = c_3 \neq t$. Since $\mathcal{G}_n \cong \mathcal{S}_n$, we must have that $t\theta \in E_3 \cup E_2 \cup E_1$. But then $t\theta = (t\theta)^2 = (t^2)\theta$, a contradiction with the fact that θ is supposed to be injective.

For the converse, notice that for any morphism $\theta\colon \operatorname{End}(\mathcal{T}_n) \to \mathcal{T}_n$, idempotents of $\operatorname{End}(\mathcal{T}_n)$ must map to idempotents of $\mathcal{T}_n$. However, by Remark IV.3.7, we know that $|E_1| = |E_2| + 1 = |\{e = e^2 \in \mathcal{T}_n\}| \geq 3$ so that θ cannot be injective, and thus cannot be an embedding. $\square$

IV.5 GREEN'S RELATIONS

Throughout this section we again assume that $n \geq 5$ so that the set $\mathcal{E}_n \setminus \mathcal{G}_n$ of singular endomorphisms is equal to the set of all endomorphisms of the form $\phi_{t,e}$ each having rank at most three. Here we turn our attention to Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{D}$, $\mathscr{H}$ and $\mathscr{J}$. It is well known that in a finite monoid, the $\mathscr{R}$-class and the $\mathscr{L}$-class of the identity coincide, and are hence equal to the $\mathscr{H}$-class of the identity, which is the group of invertible elements (see for example [49, Corollary 1.5]). In the case of $\mathcal{E}_n$ is is easy to see this directly, using considerations of rank.

We first show that the $\mathscr{L}$-classes of $\mathcal{E}_n$ are singletons except for elements of $\mathcal{G}_n$, which from the above form a single $\mathscr{L}$-class.

Proposition IV.5.1. *Let $\alpha, \beta \in \mathcal{E}_n$.*

1) *If $\alpha \in \mathcal{E}_n \backslash \mathcal{G}_n$ then the principal left ideal generated by α is $\mathcal{E}_n\alpha = \{\alpha, \alpha^+, \alpha^-, \alpha^0\}$.*

2) *$\alpha \mathscr{L} \beta$ if and only if $\operatorname{im}\alpha = \operatorname{im}\beta$ if and only if $\alpha = \beta$ or $\alpha, \beta \in \mathcal{G}_n$.*

Proof. 1) By Example IV.3.8 we know that $\mathcal{E}_n$ contains an element of each type, and so Remark IV.4.2 applies here to give that the principal left ideal is indeed $\mathcal{E}_n\alpha = \{\alpha, \alpha^+, \alpha^-, \alpha^0\}$.

2) Clearly if $\alpha = \beta$, then $\alpha \mathscr{L} \beta$. On the other hand, if $\alpha, \beta \in \mathcal{G}_n$, then as remarked above $\alpha \mathscr{L} \beta$.

Conversely, suppose that $\alpha \mathscr{L} \beta$, so that $\alpha = \gamma\beta$ and $\beta = \delta\alpha$ for some $\gamma, \delta \in \mathcal{E}_n$. It follows that $\operatorname{im}\alpha = \operatorname{im}\beta$. Clearly if this image is the whole of $\mathcal{T}_n$, then $\alpha, \beta \in \mathcal{G}_n$. Otherwise $\alpha, \beta \in \mathcal{E}_n \setminus \mathcal{G}_n$, so that $\alpha = \phi_{t,e}$ and $\beta = \phi_{u,f}$ for some $(t, e), (u, f) \in P_n$ which gives us that $\alpha = \beta$ by Lemma IV.2.4. □

Remark IV.5.2. Recall that if X is one of A,B or C, then X is the union of orbits $X_{t,e}$ where $(t, e) \in P_n$ is such that $\phi_{t,e} \in X$. Moreover, it is easy to see that

$$\begin{aligned}(X_{t,e})^+ &= \left\{(\phi_{t^g,e^g})^+ : g \in \mathcal{S}_n\right\} = \left\{\phi_{(t^g)^2,e^g} : g \in \mathcal{S}_n\right\} \\ &= \{\phi_{t^2,e}\psi_g : g \in \mathcal{S}_n\} = \phi_{t^2,e}\mathcal{G}_n = \phi^+_{t,e}\mathcal{G}_n,\end{aligned}$$

and we simply write this set as $X^+_{t,e}$. Notice that Lemma IV.4.3 gives $C^+_{t,e} \subseteq C$ whilst $A^+_{t,e} \nsubseteq A$ and $B^+_{t,e} \nsubseteq B$.

Proposition IV.5.1 shows that $\mathscr{L}$ that is surprisingly restrictive. However, the $\mathscr{R}$-classes of $\mathcal{E}_n$ are much larger, as given by the following.

Proposition IV.5.3. *Let $\alpha \in \mathcal{E}_n$. The principal right ideal of $\mathcal{E}_n$ generated by α is equal to:*

$$\alpha\mathcal{E}_n = \begin{cases} \mathcal{E}_n & \text{if } \alpha \in \mathcal{G}_n, \\ \mathcal{E}_n \setminus \mathcal{G}_n & \text{if } \alpha \in E_3, \\ A_{t,e} \cup E_2 \cup C \cup E_1 & \text{if } \alpha = \phi_{t,e} \in A, \\ E_2 \cup C \cup E_1 & \text{if } \alpha \in E_2, \\ B_{t,e} \cup E_1 & \text{if } \alpha = \phi_{t,e} \in B, \\ C_{t,e} \cup E_1 & \text{if } \alpha = \phi_{t,e} \in C, \text{or} \\ E_1 & \text{if } \alpha \in E_1. \end{cases}$$

Consequently, $\mathcal{G}_n$, E_3, E_2 and E_1 each consist of a single $\mathscr{R}$-class, whilst each remaining $\mathscr{R}$-class is the orbit of an element in A, B or C.

Proof. First note that if $\alpha \in \mathcal{G}_n$, then $\alpha\mathcal{E}_n = \mathcal{E}_n$ while if $\alpha = \phi_{t,e} \in \mathcal{E}_n \setminus \mathcal{G}_n$, then by Corollary IV.2.5 and Remark IV.4.2, we have that

$$\alpha\mathcal{E}_n = \{\phi_{t^g,e^g} : g \in \mathcal{G}_n\} \cup \{\gamma^x : \gamma \in \mathcal{E}_n \setminus \mathcal{G}_n\},$$

where γ^x is one of γ, γ^+, γ^-, or γ^0, depending on the type of α. For example, if $\alpha = \phi_{t,e} \in B$, then

$$\alpha\mathcal{E}_n = \phi_{t,e}\mathcal{G}_n \cup \left\{\gamma^- : \gamma \in \mathcal{E}_n \setminus \mathcal{G}_n\right\} = B_{t,e} \cup \left\{\phi_{f,f} : f^2 = f \in \mathcal{T}_n\right\} = B_{t,e} \cup E_1.$$

Similar reasoning demonstrates that the principal right ideal generated by α is as stated in each of the remaining cases.

Since $\alpha \mathscr{R} \beta$ if and only if $\alpha\mathcal{E}_n = \beta\mathcal{E}_n$, it follows directly that $\mathcal{G}_n$, E_3, E_2 and E_1 each consist of a single $\mathscr{R}$-class, and that $\alpha, \beta \in A \cup B \cup C$ are $\mathscr{R}$-related if and only if they lie in the same orbit of the form $\phi_{t,e}\mathcal{G}_n$. □

Corollary IV.5.4. *In $\mathcal{E}_n$, we have that $\mathscr{H} = \mathscr{L} \subseteq \mathscr{R} = \mathscr{D} = \mathscr{J}$.*

Proof. From Propositions IV.5.1 and IV.5.3, one can directly see that $\mathscr{L} \subseteq \mathscr{R}$, and therefore $\mathscr{H} = \mathscr{L} \cap \mathscr{R} = \mathscr{L}$ while $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{L} \vee \mathscr{R} = \mathscr{R}$. Additionally, since $\mathcal{E}_n$ is finite, we have that $\mathscr{D} = \mathscr{J}$ (see [28, Proposition II 1.4]). □

IV.6 IDEAL STRUCTURE AND $\mathscr{J}$-ORDER

We have already described the principal left and right ideals of $\mathcal{E}_n$ in Propositions IV.5.1 and IV.5.3. Corollary IV.5.4 determines the $\mathscr{J}$-relation, and hence when two principal ideals coincide. Nevertheless, it is worthwhile recording their form explicitly.

Proposition IV.6.1. *Let $\alpha \in \mathcal{E}_n$. Then the principal two-sided ideal generated by α is*

$$\mathcal{E}_n\alpha\mathcal{E}_n = \begin{cases} \mathcal{E}_n & \text{if } \alpha \in \mathcal{G}_n, \\ \mathcal{E}_n \setminus \mathcal{G}_n & \text{if } \alpha \in E_3, \\ A_{t,e} \cup E_2 \cup C \cup E_1 & \text{if } \alpha = \phi_{t,e} \in A, \\ E_2 \cup C \cup E_1 & \text{if } \alpha \in E_2, \\ B_{t,e} \cup C_{t^2,e} \cup E_1 & \text{if } \alpha = \phi_{t,e} \in B, \\ C_{t,e} \cup E_1 & \text{if } \alpha = \phi_{t,e} \in C, \\ E_1 & \text{if } \alpha \in E_1. \end{cases}$$

Proof. We have already given a description of the principal right ideals $\alpha\mathcal{E}_n$ in Proposition IV.5.3. It is clear that $\mathcal{E}_n\alpha\mathcal{E}_n = \alpha\mathcal{E}_n$ for all $\alpha \in \mathcal{G}_n$. Suppose then that

$\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$. By Corollary IV.2.5 and Remark IV.4.2 the description of the principal two-sided ideals $\mathcal{E}_n\alpha\mathcal{E}_n$ can therefore be found by taking the closure of the principal right ideal $\alpha\mathcal{E}_n$ under the operations $\gamma \mapsto \gamma^+, \gamma \mapsto \gamma^-$ and $\gamma \mapsto \gamma^0$. By Lemma IV.4.3 we note that each ideal $\alpha\mathcal{E}_n$ is closed under the latter two operations, and if $\alpha \notin B$ then it is closed under all three operations, giving $\mathcal{E}_n\alpha\mathcal{E}_n = \alpha\mathcal{E}_n$. For $\alpha = \phi_{t^g,e^g} \in B_{t,e}$ we note that $\alpha^+ = \phi_{(t^g)^2,e^g} = \phi_{(t^2)^g,e^g} \in C_{t^2,e}$, giving that $\mathcal{E}_n\alpha\mathcal{E}_n = \alpha\mathcal{E}_n \cup C_{t^2,e}$ as required. □

As an immediate consequence of the description of all the principal ideals of $\mathcal{E}_n$, we can describe the $\mathscr{J}$-order of elements in $\mathcal{E}_n$ as follows:

Corollary IV.6.2. *Let $\alpha, \beta \in \mathcal{E}_n$. Then $\beta \leq_{\mathscr{J}} \alpha$ if and only if one of the following holds:*

1) $\alpha \in \mathcal{G}_n$;
2) $\alpha \in E_3$ *and* $\beta \in \mathcal{E}_n \setminus \mathcal{G}_n$;
3) $\alpha = \phi_{t,e} \in A$, *and* $\beta \in A_{t,e} \cup E_2 \cup C \cup E_1$;
4) $\alpha \in E_2$ *and* $\beta \in E_2 \cup C \cup E_1$;
5) $\alpha = \phi_{t,e} \in B$ *and* $\beta \in B_{t,e} \cup C_{t^2,e} \cup E_1$;
6) $\alpha = \phi_{t,e} \in C$ *and* $\beta \in C_{t,e} \cup E_1$; *or*
7) $\alpha, \beta \in E_1$.

Using this, we can now see the structure of the $\mathscr{J}$-order of $\mathcal{E}_n$ as laid out in Figure IV.1 below, where we have added how the maps $\alpha \mapsto \alpha^x$ for $x \in \{+,-,0\}$ act on the different components of $\mathcal{E}_n$.

From the description of the principal ideals of $\mathcal{E}_n$, we can also give an explicit formulation for each ideal of $\mathcal{E}_n$. In order to do so, we use the notation introduced in Definition IV.3.5. Notice in particular that if Y is a union of orbits of elements of B, that is, $Y = \bigcup_{\phi_{t,e}\in B'} \phi_{t,e}\mathcal{G}_n = \bigcup_{\phi_{t,e}\in B'} B_{t,e}$ for some $B' \subseteq B$, then

$$\begin{aligned} Y^+ &= \bigcup_{\phi_{t,e}\in B'} \left\{(\phi_{t^g,e^g})^+ \colon g \in \mathcal{S}_n\right\} = \bigcup_{\phi_{t,e}\in B'} \{\phi_{t^2,e}\psi_g \colon g \in \mathcal{S}_n\} \\ &= \bigcup_{\phi_{t,e}\in B'} \phi_{t^2,e}\mathcal{G}_n = \bigcup_{\phi_{t,e}\in B'} C_{t^2,e}. \end{aligned}$$

Thus, $Y^+ \subseteq C$ is again a union of orbits. Since each ideal is a union of principal ideals, we can call upon Proposition IV.6.1 to describe the ideals of $\mathcal{E}_n$, as follows.

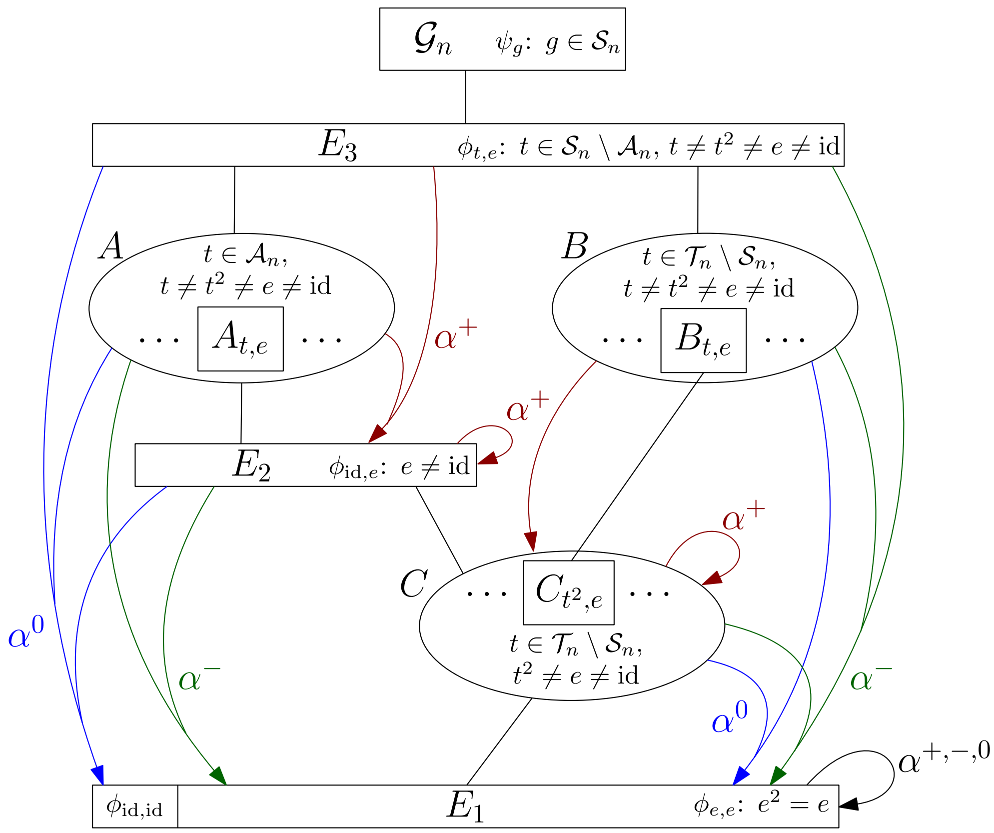

Figure IV.1: The $\mathscr{J}$-order of $\mathcal{E}_n$ for $n \geq 5$. Each rectangle represents a $\mathscr{J}$-class, whilst the ovals represent the groupings of elements according to the sets A, B, C. Black lines indicate the $\mathscr{J}$-order, whilst directed lines indicate the action of the maps $\alpha \mapsto \alpha^x$ for $x \in \{+, -, 0\}$.

Corollary IV.6.3. *Any ideal of $\mathcal{E}_n$ takes one of the following forms:*

1) $\mathcal{E}_n$ (the only ideal containing automorphisms);

2) $\mathcal{E}_n \setminus \mathcal{G}_n$ (the only proper ideal containing elements of odd type);

3) $X \cup Y \cup E_2 \cup C \cup E_1$ (ideals containing elements of even type, but no elements of odd type); or

4) $Y \cup Y^+ \cup Z \cup E_1$ (ideals containing only elements of trivial or non-permutation type),

where the sets X, Y and Z are (possibly empty) unions of orbits taken from sets A, B and C respectively.

Proof. Part 1) and 2) are clear from the description of the principal ideals of $\mathcal{E}_n$ in Proposition IV.6.1.

For 3), suppose that J is an ideal of $\mathcal{E}_n$ containing an element of even type $\phi_{t,e} \in A \cup E_2$ but no elements of odd type. Then clearly, $J \subseteq \mathcal{E}_n \setminus (\mathcal{G}_n \cup E_3)$ and by Proposition IV.6.1, we have that $A_{t,e} \cup E_2 \cup C \cup E_1 \subseteq J$. Moreover, if we let

$$X = \bigcup_{\beta \in J \cap A} \beta\mathcal{G}_n = \bigcup_{\phi_{u,f} \in J \cap A} A_{u,f} \quad \text{and} \quad Y = \bigcup_{\beta \in J \cap B} \beta\mathcal{G}_n = \bigcup_{\phi_{u,f} \in J \cap B} B_{u,f},$$

we can see that $X \cup Y \subseteq J$. Since $X^+, X^-, X^0, Y^+, Y^-, Y^0 \subseteq E_2 \cup C \cup E_1$ by Lemma IV.4.3 and X and Y are stable under multiplication by automorphisms of $\mathcal{G}_n$, it follows that $X \cup Y \cup E_2 \cup C \cup E_1$ is an ideal of $\mathcal{E}_n$, and therefore $J = X \cup Y \cup E_2 \cup C \cup E_1$.

Similarly, for 4), suppose that J is an ideal that contains only elements of trivial or non-permutation type, that is, $J \subseteq B \cup C \cup E_1$. Clearly, since E_1 is the minimal ideal of $\mathcal{E}_n$, we have that $E_1 \subseteq J$. Additionally, if J contains an element $\phi_{t,e} \in B$, then we have that $B_{t,e} \cup C_{t^2,e} \cup E_1 \subseteq J$. In particular, if we define $Y = \bigcup_{\beta \in J \cap B} \beta\mathcal{G}_n$ as before, we see that we need $Y \cup Y^+ \subseteq J$. Since it is possible for J to contain elements of C that are not in $\mathcal{E}_n\beta\mathcal{E}_n$ for any $\beta \in Y$, we define Z as the union of all of their orbits, that is,

$$Z = \bigcup_{\substack{\phi_{t^2,e} \in J \cap C, \\ \phi_{t,e} \notin J}} \phi_{t^2,e}\mathcal{G}_n.$$

Then $Z \subseteq J$. Since $Y^-, Y^0, Z^-, Z^0 \subseteq E_1 \subseteq J$ and $Z^+ = Z$ by Lemma IV.4.3, it follows that $Y \cup Y^+ \cup Z \cup E_1$ is closed under multiplication and therefore we obtain that $J = Y \cup Y^+ \cup Z \cup E_1$. □

IV.7 EXTENDED GREEN'S RELATIONS AND GENERALISED REGULARITY PROPERTIES

We assume once more that $n \geq 5$ so that, as shown in Proposition IV.4.8, the monoid $\mathcal{E}_n$ is not regular. In order to better understand the structure of these endomorphism monoids, we turn to the extended Green's relations. Recall their definition from

Section I.3:

$$\alpha \mathscr{R}^* \beta \iff \Big(\gamma\alpha = \delta\alpha \Leftrightarrow \gamma\beta = \delta\beta \quad \forall \gamma, \delta \in \mathcal{E}_n\Big),$$
$$\alpha \mathscr{L}^* \beta \iff \Big(\alpha\gamma = \alpha\delta \Leftrightarrow \beta\gamma = \beta\delta \quad \forall \gamma, \delta \in \mathcal{E}_n\Big),$$
$$\alpha \widetilde{\mathscr{R}} \beta \iff \Big(\eta\alpha = \alpha \Leftrightarrow \eta\beta = \beta \quad \forall \eta = \eta^2 \in \mathcal{E}_n\Big),$$
$$\alpha \widetilde{\mathscr{L}} \beta \iff \Big(\alpha\eta = \alpha \Leftrightarrow \beta\eta = \beta \quad \forall \eta = \eta^2 \in \mathcal{E}_n\Big),$$
$$\mathscr{H}^* = \mathscr{L}^* \wedge \mathscr{R}^* = \mathscr{L}^* \cap \mathscr{R}^*, \qquad \widetilde{\mathscr{H}} = \widetilde{\mathscr{L}} \wedge \widetilde{\mathscr{R}} = \widetilde{\mathscr{L}} \cap \widetilde{\mathscr{R}},$$
$$\mathscr{D}^* = \mathscr{L}^* \vee \mathscr{R}^*, \qquad \widetilde{\mathscr{D}} = \widetilde{\mathscr{L}} \vee \widetilde{\mathscr{R}},$$
$$\alpha \mathscr{J}^* \beta \iff J^*(\alpha) = J^*(\beta), \quad \text{and}$$
$$\alpha \widetilde{\mathscr{J}} \beta \iff \widetilde{J}(\alpha) = \widetilde{J}(\beta),$$

where $J^*(\alpha)$ [resp. $\widetilde{J}(\alpha)$] is the smallest ideal containing α that is saturated by $\mathscr{L}^*$ and $\mathscr{R}^*$ $\Big[$resp. by $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}\Big]$.

Before describing these relations on $\mathcal{E}_n$, notice that $\mathcal{G}_n$ is an $\mathscr{H}$-class and therefore $\mathcal{G}_n$ is contained in an $\mathscr{H}^*$-class and hence in an $\widetilde{\mathscr{H}}$-class.

Lemma IV.7.1. *The group $\mathcal{G}_n$ is an $\mathscr{R}^*$-class and an $\widetilde{\mathscr{R}}$-class of $\mathcal{E}_n$.*

Proof. Since $\mathcal{G}_n$ lies in an $\mathscr{R}^*$-class, and that $\mathscr{R}^* \subseteq \widetilde{\mathscr{R}}$, it only remains to show that if $\beta \notin \mathcal{G}_n$ then β is not $\widetilde{\mathscr{R}}$-related to ε. But this is clear since $\beta = \phi_{t,e}\beta$ for any odd $\phi_{t,e}$, but certainly $\varepsilon \neq \phi_{t,e}\varepsilon$. □

For the description of the other $\mathscr{R}^*$-classes we can show, using the type of maps in $\mathcal{E}_n \setminus \mathcal{G}_n$, that it is sufficient to consider idempotents acting on the left in order to characterise the relation $\mathscr{R}^*$ for elements in $\mathcal{E}_n \setminus \mathcal{G}_n$.

Lemma IV.7.2. *For any $\alpha, \beta \in \mathcal{E}_n$, $\alpha \mathscr{R}^* \beta$ is equivalent to*

$$\eta\alpha = \zeta\alpha \iff \eta\beta = \zeta\beta \quad \forall \eta, \zeta \in E(\mathcal{E}_n).$$

Proof. Suppose first that $\alpha \mathscr{R}^* \beta$. By definition, for all $\gamma, \delta \in \mathcal{E}_n$ we have that $\gamma\alpha = \delta\alpha$ if and only if $\gamma\beta = \delta\beta$. In particular, this statement holds for all idempotents $\gamma, \delta \in \mathcal{E}_n$.

Conversely, suppose that for all $\eta, \zeta \in E(\mathcal{E}_n)$ we have $\eta\alpha = \zeta\alpha \iff \eta\beta = \zeta\beta$. Notice that for all $\xi \in \mathcal{E}_n \setminus \mathcal{G}_n$, we have that $\varepsilon\xi = \phi_{t,e}\xi$ for all $\phi_{t,e} \in E_3$. We immediately get from this that is $\alpha \in \mathcal{G}_n$ then the above condition is satisfied only if $\beta \in \mathcal{G}_n$, which by the previous result gives that $\alpha \mathscr{R}^* \beta$ in this case. Suppose then that $\alpha, \beta \in \mathcal{E}_n \setminus \mathcal{G}_n$, and let $\gamma, \delta \in \mathcal{E}_n$ be such that $\gamma\alpha = \delta\alpha$. Since there is an idempotent of each type, it follows from Lemma IV.3.6 that $\eta\alpha = \zeta\alpha$ where η is an idempotent of the same type as γ and ζ is an idempotent of the same type as δ, and hence $\gamma\beta = \eta\beta = \zeta\beta = \delta\beta$. A dual argument shows that $\gamma\beta = \delta\beta$ implies $\gamma\alpha = \delta\alpha$, and hence $\alpha \mathscr{R}^* \beta$. □

Since $\mathscr{R}^*$ and $\widetilde{\mathscr{R}}$ only depend on idempotents, we start by showing when a map admits a left identity.

Lemma IV.7.3. *Let $\alpha \in \mathcal{E}_n$. Then $\eta\alpha = \alpha$ for $\eta \in \mathcal{E}_n$ if and only if one of the following holds:*

1) *α has rank n^n and $\eta = \varepsilon$ (equivalently, $\alpha \in \mathcal{G}_n$, and $\eta = \varepsilon$);*
2) *α has rank 3 and η has group or odd type (equivalently, $\alpha \in E_3 \cup A \cup B$ and $\eta \in \mathcal{G}_n \cup E_3$);*
3) *α has rank 2 and η has group, odd, or even type (equivalently, $\alpha \in E_2 \cup C$ and $\eta \in \mathcal{G}_n \cup E_3 \cup A \cup E_2$); or*
4) *α has rank 1 and $\eta \in \mathcal{E}_n$ (equivalently, $\alpha \in E_1$ and $\eta \in \mathcal{E}_n$) .*

In particular, if η is an idempotent then, $\eta\alpha = \alpha$ if and only if $\alpha \leq_{\mathscr{J}} \eta$.

Proof. It is clear that ε is the only left identity for $\alpha \in \mathcal{G}_n$. Assume now that $\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$. By consideration of rank, we can immediately determine the idempotent left identities. The result follows from Corollary IV.6.2 and Lemma IV.7.2. □

Furthermore, two idempotents η and ζ satisfying $\eta\alpha = \zeta\alpha$ for a given map $\alpha \in \mathcal{E}_n$ must lie above α in the $\mathscr{J}$ order, or have the same type. This is formally given by the following.

Lemma IV.7.4. *Let $\alpha = \phi_{t,e} \in \mathcal{E}_n \setminus \mathcal{G}_n$. Then $\eta\alpha = \zeta\alpha$ for some $\eta, \zeta \in E(\mathcal{E}_n)$ if and only if one of the following happens:*

- *$\alpha \leq_{\mathscr{J}} \eta$ and $\alpha \leq_{\mathscr{J}} \zeta$; or*
- *η and ζ have the same type.*

Proof. If $\eta, \zeta \in E(\mathcal{E}_n)$ are such that $\alpha \leq_{\mathscr{J}} \eta$ and $\alpha \leq_{\mathscr{J}} \zeta$, then $\eta\alpha = \alpha$ and $\zeta\alpha = \alpha$ by Lemma IV.7.3, which shows that $\eta\alpha = \zeta\alpha$. Similarly, if η and ζ have the same type, then $\eta\alpha = \zeta\alpha$ by Lemma IV.3.6 and the fact that ε is the only idempotent of group type.

For the converse, suppose that that $\eta\alpha = \zeta\alpha$. If $\alpha \leq_{\mathscr{J}} \eta$, then $\eta\alpha = \alpha$ by Lemma IV.7.3. Therefore $\zeta\alpha = \eta\alpha = \alpha$ so that $\alpha \leq_{\mathscr{J}} \zeta$, which corresponds to the first case. Thus, we can now assume that $\alpha \not\leq_{\mathscr{J}} \eta$ and $\alpha \not\leq_{\mathscr{J}} \zeta$. In particular, this means that $\alpha \notin E_1$ and so $t \neq e$ and that $\eta, \zeta \in E_2 \cup E_1$. Assume that η is of even type and ζ is of trivial or non-permutation type, that is, $\eta \in E_2$ and $\zeta \in E_1$. Then $\eta\alpha = \zeta\alpha$ gives

$$\phi_{t^2,e} = \begin{cases} \phi_{t^2,t^2} & \text{if } \zeta = \phi_{\mathrm{id},\mathrm{id}}, \\ \phi_{e,e} & \text{otherwise}, \end{cases}$$

which shows in either case that $t^2 = e$, so that $t = e$ by Lemma IV.2.2 part 2), a contradiction. A similar contradiction arises when considering η of trivial type and ζ of non-permutation type. Therefore η and ζ must have the same type. □

We can now easily determine the $\mathscr{R}^*$ and $\widetilde{\mathscr{R}}$ relations.

Proposition IV.7.5. *Let $\alpha, \beta \in \mathcal{E}_n$. Then the following conditions are equivalent:*

1) $\alpha \,\mathscr{R}^*\, \beta$;

2) $\alpha \,\widetilde{\mathscr{R}}\, \beta$;

3) α and β are $\mathscr{R}$-below the same idempotents;

4) α and β are $\mathscr{J}$-below the same idempotents;

5) α and β have the same rank.

Consequently, $\widetilde{\mathscr{R}} = \mathscr{R}^*$ is a left congruence and the $\mathscr{R}^*$-classes of $\mathcal{E}_n$ are $\mathcal{G}_n$, $E_3 \cup A \cup B$, $E_2 \cup C$ and E_1.

Proof. If 1) holds, then so also does 2), since $\mathscr{R}^* \subseteq \widetilde{\mathscr{R}}$. In any semigroup S, if $ea = a$ for some $e, a \in S$, then clearly $a \leq_{\mathscr{R}} e$. On the other hand, if $a \leq_{\mathscr{R}} f$ for some idempotent $f \in S$, then from $a = fb$ for $b \in S^1$ we obtain $fa = ffb = fb = a$. Applying this to $\mathcal{E}_n$ gives that 2) and 3) are equivalent. Examination of Lemma IV.7.3 now yields that 3), 4) and 5) are equivalent.

It remains to show that if $\alpha \,\widetilde{\mathscr{R}}\, \beta$ then $\alpha \,\mathscr{R}^*\, \beta$. Suppose therefore that $\alpha \,\widetilde{\mathscr{R}}\, \beta$ and $\eta\alpha = \zeta\alpha$ where η, ζ are idempotent. Using Lemma IV.7.4, either $\alpha \leq_{\mathscr{J}} \eta$ and $\alpha \leq_{\mathscr{J}} \zeta$ so that the same is true for β and $\eta\beta = \beta = \zeta\beta$; or η and ζ have the same type, in which case certainly $\eta\beta = \zeta\beta$ by Lemma IV.3.6. Lemma IV.7.2 finishes the proof that $\alpha \,\mathscr{R}^*\, \beta$, that is, 1) holds.

That $\widetilde{\mathscr{R}}$ is a left congruence then follows from the fact that in any semigroup S the relation $\mathscr{R}^*$ is a left congruence. By point 5) the congruence classes are as given. $\square$

We now turn our attention toward the relations $\mathscr{L}^*$ and $\widetilde{\mathscr{L}}$. In what follows, for each $(t, e) \in P_n$ we write $\mathrm{Fix}(t, e) = \{g \in \mathcal{S}_n : t^g = t \text{ and } e^g = e\}$. Notice in particular that $\mathrm{Fix}(\mathrm{id}, e) = \mathrm{Fix}(e, e)$ for all $e^2 = e$ and $\mathrm{Fix}(\mathrm{id}, \mathrm{id}) = \mathcal{G}_n$. The next lemma is immediate.

Lemma IV.7.6. *For any $\phi_{t,e} \in \mathcal{E}_n \setminus \mathcal{G}_n$ and $\psi_g \in \mathcal{G}_n$ we have that $\phi_{t,e}\psi_g = \phi_{t,e}$ if and only if $g \in \mathrm{Fix}(t, e)$.*

We can also describe when a map admits a right identity.

Lemma IV.7.7. *Let $\alpha \in \mathcal{E}_n$. Then $\alpha\eta = \alpha$ for $\eta \in \mathcal{E}_n$ if and only if one of the following holds:*

1) $\alpha \in \mathcal{G}_n$ and $\eta = \varepsilon$;

2) $\alpha = \phi_{t,e} \in E_3$ and $\eta \in \{\psi_g : g \in \mathrm{Fix}(t, e)\} \cup \{\alpha\} \subseteq \mathcal{G}_n \cup \{\alpha\}$;

3) $\alpha = \phi_{\mathrm{id},e} \in E_2$ and
$\eta \in \{\psi_g : g \in \mathrm{Fix}(e, e)\} \cup \{\phi_{u,e} : u^2 = \mathrm{id} \text{ and } (u, e) \in P_n\} \subseteq \mathcal{G}_n \cup E_3 \cup A \cup \{\alpha\}$;

4) $\alpha = \phi_{e,e} \in E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\}$ and
$\eta \in \{\psi_g : g \in \mathrm{Fix}(e, e)\} \cup \{\phi_{u,e} : (u, e) \in P_n\} \subseteq (\mathcal{E}_n \setminus E_1) \cup \{\alpha\}$;

5) $\alpha = \phi_{\mathrm{id},\mathrm{id}} \in E_1$ and $\eta \in \mathcal{G}_n \cup E_3 \cup A \cup E_2 \cup \{\phi_{\mathrm{id},\mathrm{id}}\}$;

6) $\alpha = \phi_{t,e} \notin E(\mathcal{E}_n)$ and $\eta \in \{\psi_g : g \in \mathrm{Fix}(t, e)\} \subseteq \mathcal{G}_n$.

Proof. Part 1) is immediate from the multiplication in Corollary IV.2.5.

Let us now assume that $\alpha = \phi_{t,e} \in \mathcal{E}_n \setminus \mathcal{G}_n$. The case where $\eta \in \mathcal{G}_n$ is given by Lemma IV.7.6, so we therefore assume that $\eta = \phi_{u,f} \in \mathcal{E}_n \setminus \mathcal{G}_n$ and we look at the product $\alpha\eta$ as in Remark IV.4.2.

- Since elements of E_3 are left identities for $\mathcal{E}_n \setminus \mathcal{G}_n$ by Corollary IV.4.6, it follows that if $\alpha \in E_3$, then $\alpha\eta = \alpha$ if and only if $\eta = \alpha$.
- If $\alpha = \phi_{\mathrm{id},e}$, then $\alpha\eta = \phi_{u^2,f}$ and thus $\alpha\eta = \alpha$ if and only if $\eta = \phi_{u,e}$ for some $u^2 = \mathrm{id}$.
- If $\alpha = \phi_{e,e} \neq \phi_{\mathrm{id},\mathrm{id}}$, then $\alpha\eta = \phi_{f,f}$ which gives us that $\alpha\eta = \alpha$ if and only if $f = e$, that is, $\eta = \phi_{u,e}$ for any $u \in U_n$ such that $(u, e) \in P_n$.

- If $\alpha = \phi_{\mathrm{id},\mathrm{id}}$, then $\alpha\eta = \phi_{u^2,u^2}$ and therefore $\alpha\eta = \alpha$ if and only if $u^2 = \mathrm{id}$, that is, if and only if $\eta \in E_3 \cup A \cup E_2 \cup \{\phi_{\mathrm{id},\mathrm{id}}\}$.

By consideration of rank, it is clear that if $\alpha \in A \cup B \cup C$, then $\mathrm{rk}(\alpha\eta) < \mathrm{rk}(\alpha)$, and thus $\alpha\eta \neq \alpha$ for all $\eta \in \mathcal{E}_n \setminus \mathcal{G}_n$, as required. □

As a direct consequence, we obtain the description of the relation $\widetilde{\mathscr{L}}$.

Proposition IV.7.8. *The $\widetilde{\mathscr{L}}$-classes of $\mathcal{E}_n$ are $\mathcal{E}_n \setminus (E_3 \cup E_2 \cup E_1)$ and all the singletons $\{\eta\}$ where $\eta = \eta^2 \neq \varepsilon$.*

Proof. In order to show that $\alpha \mathrel{\widetilde{\mathscr{L}}} \beta$ for some $\alpha, \beta \in \mathcal{E}_n$, we need to show that they have the same idempotents as right identities. It is clear by specialising Lemma IV.7.3 to the case where η is idempotent that $\mathcal{G}_n \cup A \cup B \cup C$ form a single $\widetilde{\mathscr{L}}$-class. Moreover, from Lemma I.3.13, we know that idempotents are $\widetilde{\mathscr{L}}$-related if and only if they are $\mathscr{L}$-related, and hence by Proposition IV.5.1, if and only if they are equal. Since any idempotent is a right identity for itself, the result follows. □

Remark IV.7.9. Unlike the situation for $\mathscr{R}^*$ and $\widetilde{\mathscr{R}}$, we find that $\mathscr{L}^*$ is a strictly smaller relation than $\widetilde{\mathscr{L}}$. Indeed, in general, the relation $\mathscr{L}^*$ on a semigroup S is well-known to be a right congruence. However, it is easy to see that $\widetilde{\mathscr{L}}$ is *not* a right congruence. Indeed, taking $\alpha \in A$, $\beta \in B$ and $\phi_{t,e} \in E_3$ we find that $\alpha \mathrel{\widetilde{\mathscr{L}}} \beta$ whilst $\alpha\phi_{t,e} = \phi_{\mathrm{id},\mathrm{id}}$ and $\beta\phi_{t,e} = \phi_{e,e}$ are distinct idempotents, and hence not $\widetilde{\mathscr{L}}$-related.

We can now give the description of $\mathscr{L}^*$.

Proposition IV.7.10. *Let $\alpha, \beta \in \mathcal{E}_n$. Then $\alpha \mathrel{\mathscr{L}^*} \beta$ if and only if one of the following occurs:*

- *$\alpha, \beta \in \mathcal{G}_n$;*
- *$\alpha, \beta \in E_3 \cup E_2 \cup E_1$ and $\alpha = \beta$; or*
- *$\alpha, \beta \in A \cup B \cup C$ are such that $\alpha = \phi_{t,e}$, $\beta = \phi_{u,f}$ have the same type and $\mathrm{Fix}(t, e) = \mathrm{Fix}(u, f)$.*

Proof. Since $\mathscr{L} \subseteq \mathscr{L}^* \subseteq \widetilde{\mathscr{L}}$, we have that all elements of $\mathcal{G}_n$ are $\mathscr{L}^*$-related, and that idempotents of $\mathcal{E}_n$ distinct from ε form their own $\widetilde{\mathscr{L}}$-class by Proposition IV.7.8 and thus they also form their own $\mathscr{L}^*$-class. Therefore, it only remains to show that elements of $\mathcal{G}_n$ cannot be $\mathscr{L}^*$-related to non-regular elements (that is, elements of $A \cup B \cup C$) and that two non-regular elements are $\mathscr{L}^*$-related if and only if they have the same type and are fixed by the same automorphisms of $\mathcal{G}_n$.

To see first that no elements of $\mathcal{G}_n$ can be $\mathscr{L}^*$-related to elements of $A \cup B \cup C$, consider $\alpha \in \mathcal{G}_n$ and $\beta \in A \cup B \cup C$, and let $t = (1\,2)$, $e = c_3 \in \mathcal{T}_n$. Since $t^2 = \mathrm{id}$ and $(t, e) \in P_n$ we have $\phi_{t,e}, \phi_{\mathrm{id},e}, \phi_{e,e} \in \mathcal{E}_n$. Clearly $\alpha\phi_{t,e}, \alpha\phi_{\mathrm{id},e}$ and $\alpha\phi_{e,e}$ are all distinct. If $\beta \in A$, then we have that $\beta\phi_{t,e} = \phi_{\mathrm{id},e} = \beta\phi_{\mathrm{id},e}$, while if $\beta \in B \cup C$ we have $\beta\phi_{t,e} = \phi_{e,e} = \beta\phi_{\mathrm{id},e}$. Therefore elements of $\mathcal{G}_n$ cannot be $\mathscr{L}^*$-related to elements of $A \cup B \cup C$.

From now on, we assume that α, β are non-regular (i.e. contained in $A \cup B \cup C$), and that $\alpha = \phi_{v,k}$ and $\beta = \phi_{u,f}$ for some $(v, k), (u, f) \in P_n$.

Suppose that $\alpha \mathscr{L}^* \beta$ and consider the maps $\phi_{t,e}$, $\phi_{\mathrm{id},e}$ and $\phi_{e,e}$ in $\mathcal{E}_n$ as above. To see that α and β must have the same type, notice that if $\alpha \in A$ and $\beta \in B \cup C$, then $\alpha\phi_{t,e} = \phi_{\mathrm{id},e} \neq \phi_{e,e} = \alpha\phi_{e,e}$ while $\beta\phi_{t,e} = \phi_{e,e} = \beta\phi_{e,e}$ which contradicts the fact that $\alpha \mathscr{L}^* \beta$. Thus either $\alpha, \beta \in A$, or $\alpha, \beta \in B \cup C$ which shows that $\mathscr{L}^*$-related maps must be of the same type. Lemma IV.7.6 gives that $\mathrm{Fix}(v, k) = \mathrm{Fix}(u, f)$.

Conversely, assume that α and β have the same type and that $\mathrm{Fix}(v, k) = \mathrm{Fix}(u, f)$. By Lemma IV.3.6 we have that $\alpha\eta = \beta\eta$ for all $\eta \in \mathcal{E}_n \setminus \mathcal{G}_n$. Suppose now that ψ_g, ψ_h in $\mathcal{G}_n$. If $\alpha\psi_g = \alpha\psi_h$, then $\alpha\psi_{gh^{-1}} = \alpha$. It follows that $gh^{-1} \in \mathrm{Fix}(v, k) = \mathrm{Fix}(u, f)$ and so $\beta\psi_g = \beta\psi_h$. Finally, it is easy to see that if $\alpha \in A \cup B \cup C$, $\gamma \in \mathcal{G}_n$ and $\delta \in \mathcal{E}_n \setminus \mathcal{G}_n$, then as the rank of $\alpha\gamma$ is the same as the rank of α, but the rank of $\alpha\delta$ is strictly less than the rank of α, we cannot have that $\alpha\gamma = \alpha\delta$. Using the symmetry in the arguments used above concludes the proof. $\square$

Using Propositions IV.7.5, IV.7.8 and IV.7.10, we immediately get the following.

Proposition IV.7.11. *For $n \geq 5$ the semigroup $\mathcal{E}_n$ is left abundant (and hence left Fountain), right Fountain but not right abundant.*

We can now describe the relations $\mathscr{D}^*$ and $\mathscr{J}^*$.

Proposition IV.7.12. *The $\mathscr{D}^*$-classes of $\mathcal{E}_n$ are $\mathcal{G}_n$, E_1 and $\mathcal{E}_n \setminus (\mathcal{G}_n \cup E_1)$ and further, $\mathscr{D}^* = \mathscr{J}^*$.*

Proof. By Propositions IV.7.5 and IV.7.10 we know that the elements of $\mathcal{G}_n$ form a single $\mathscr{R}^*$-class and a single $\mathscr{L}^*$-class. Thus it follows that $\mathcal{G}_n$ is also a $\mathscr{D}^*$-class.

Let $\alpha \in E_1$ and suppose that $\alpha \mathscr{L}^* \circ \mathscr{R}^* \delta$ for some $\delta \in \mathcal{E}_n \setminus \mathcal{G}_n$. Then there exists $\eta \in \mathcal{E}_n \setminus \mathcal{G}_n$ such that $\alpha \mathscr{L}^* \eta \mathscr{R}^* \delta$. By Proposition IV.7.10 we find that $\eta = \alpha \in E_1$, and by Proposition IV.7.5 we see that $\delta \in E_1$. It follows from this argument together with the fact that E_1 is an $\mathscr{R}^*$-class that E_1 is also a $\mathscr{D}^*$-class.

Suppose then that $\alpha \in \mathcal{E}_n \setminus (\mathcal{G}_n \cup E_1)$ and $\alpha \mathscr{D}^* \delta$. It follows from the above that $\delta \in \mathcal{E}_n \setminus (\mathcal{G}_n \cup E_1)$. We aim to show that if $\alpha, \delta \in \mathcal{E}_n \setminus (\mathcal{G}_n \cup E_1)$ then $\alpha \mathscr{D}^* \delta$.

Notice that by our partition of $\mathcal{E}_n$ we have that

$$\mathcal{E}_n \setminus (\mathcal{G}_n \cup E_1) = (E_3 \cup A \cup B) \cup (E_2 \cup C),$$

is the union of all elements of rank 3 and all elements of rank 2. Moreover, it follows from Proposition IV.7.5 that if α, δ either both have rank 3 or both have rank 2, then $\alpha \mathscr{R}^* \delta$, and hence $\alpha \mathscr{D}^* \delta$. We show that there is an element $\beta \in B$ of rank 3 and $\gamma \in C$ of rank 2 such that $\beta \mathscr{L}^* \gamma$, which will complete the description of $\mathscr{D}^*$.

We recall from Example IV.3.8 that for

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 & i_{\geq 5} \\ 1 & 3 & 2 & 1 & i \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 2 & 3 & 4 & i_{\geq 5} \\ 1 & 1 & 1 & 4 & 4 \end{pmatrix}, \quad f = \begin{pmatrix} 1 & 2 & 3 & 4 & i_{\geq 5} \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

we have $\beta := \phi_{t,f} \in B$ and $\gamma := \phi_{u,f} \in C$, so that β and γ have the same type. We show that $\mathrm{Fix}(t, f) = \mathrm{Fix}(u, f)$, so that $\beta \mathscr{L}^* \gamma$ by Proposition IV.7.10.

Since $f = c_1$ we see that that $\mathrm{Fix}(t, f) = \{g \in \mathcal{S}_n : gt = tg \text{ and } 1g = 1\}$. If $g \in \mathrm{Fix}(t, f)$ we therefore have $1 = 1g = 4tg = 4gt$ and hence (since g is a permutation and $1g = 1$) $4g = 4$. For all $i \geq 5$ we have $ig = itg = igt$, that is, ig is fixed by t from which it follows that $ig \geq 5$ for all $i \geq 5$. Finally we have $2g = 3gt$ and $3g = 2gt$. Thus

$$\mathrm{Fix}(t, f) = \{g, (2\,3)g : g \in \mathcal{S}_n, ig = i \text{ for } 1 \leq i \leq 4\}.$$

Similarly $\mathrm{Fix}(u, f) = \{g \in \mathcal{S}_n : gu = ug \text{ and } 1g = 1\}$. If $g \in \mathrm{Fix}(u, f)$ we therefore have $1 = 1g = 1gu = 2gu = 3gu$ and hence (since g is a permutation and $1g = 1$) we must have $\{2g, 3g\} = \{2, 3\}$. Thus for all $i \geq 4$ we have that $ig \geq 4$. Moreover, since u fixes $4g$ we must have $4g = 4$ since 4 is the only value distinct from 1 that is fixed by u. It is then easy to see that

$$\mathrm{Fix}(u, f) = \{g, (2\,3)g : g \in \mathcal{S}_n, ig = i \text{ for } 1 \leq i \leq 4\} = \mathrm{Fix}(t, f).$$

We now look at the $\mathscr{J}^*$ relation. Since $\mathscr{D}^* \subseteq \mathscr{J}^*$ and that $\mathcal{E}_n$, $\mathcal{E}_n \setminus \mathcal{G}_n$ and E_1 are distinct ideals of $\mathcal{E}_n$ saturated by $\mathscr{D}^*$, it follows that these ideals are associated with distinct $\mathscr{J}^*$-classes, and thus $\mathscr{J}^* = \mathscr{D}^*$. $\square$

Finally, the $\widetilde{\mathscr{D}}$ and $\widetilde{\mathscr{J}}$-relations are only composed of two classes: the minimal ideal E_1 and its complement $\mathcal{E}_n \setminus E_1$.

Proposition IV.7.13. *The $\widetilde{\mathscr{D}}$-classes of $\mathcal{E}_n$ are $\mathcal{E}_n \setminus E_1$ and E_1 and further, $\widetilde{\mathscr{J}} = \widetilde{\mathscr{D}}$.*

Proof. Propositions IV.7.5 and IV.7.8 immediately give us that there are two $\widetilde{\mathscr{D}}$-classes, namely $\mathcal{E}_n \setminus E_1$ and E_1. Since E_1 is an ideal saturated by $\widetilde{\mathscr{D}}$, it is a $\widetilde{\mathscr{J}}$-class and the result follows. □

IV.8 THE STRUCTURE OF END($\mathcal{T}_n$) FOR $n \leq 4$

In order to have clean statements with uniform proofs, in the previous sections we focussed on the case where $n \geq 5$. To complete the picture, in this section we describe the structure of $\mathcal{E}_n$ in the cases where $n \leq 4$. We note that the decomposition in terms of rank and type given in Lemma IV.3.2 can also be used to describe the structure of $\mathcal{E}_n$ in these small cases, however, some of the sets turn out to be empty.

Indeed, $\mathcal{E}_1 = \mathcal{G}_1 = \{\varepsilon\}$ is a trivial group, and $\mathcal{E}_2$ decomposes as a disjoint union $\mathcal{E}_2 = \mathcal{G}_2 \cup E_2(2) \cup E_1(2)$ where $\mathcal{G}_2 = \{\varepsilon, \psi_{(1\,2)}\}$ is the automorphism group, whilst $E_2(2) = \{\phi_{\mathrm{id},c_1}, \phi_{\mathrm{id},c_2}\}$ and $E_1(2) = \{\phi_{\mathrm{id},\mathrm{id}}, \phi_{c_1,c_1}, \phi_{c_2,c_2}\}$ are the idempotents of rank 2 and 1 respectively. We note in particular that these two semigroups (consisting of group elements and idempotents only) are regular.

For $n = 3$, the endomorphism monoid $\mathcal{E}_3$ decomposes as the disjoint union $\mathcal{E}_3 = \mathcal{G}_3 \cup E_3(3) \cup E_2(3) \cup C(3) \cup E_1(3)$. The elements of $C(3)$ are not regular (note that the reasoning given in the proof of Proposition IV.4.8 also applies here), so that $\mathcal{E}_3$ is not a regular semigroup.

For the case $n = 4$, recall that the endomorphism monoid $\mathcal{E}_4$ was described in Lemma IV.3.2 as $\mathcal{E}_4 = \mathcal{G}_4 \cup D(4) \cup E_3(4) \cup B(4) \cup E_2(4) \cup C(4) \cup E_1(4)$ where each set is non-empty and $D(4)$ contains idempotents of rank 7, namely the elements in the set $E_7(4) = \{\sigma^g : g \in \mathcal{K}\}$.

In spite of these differences, certain properties turn out to be common to all endomorphism monoids $\mathcal{E}_n$. Their proofs are often akin to those presented in Sections IV.4–IV.7 and these results could be obtained by direct enumeration using a computer program such as GAP, but we will give here a detailed account for the case $n = 4$, which happens to be more complicated to handle.

IV.8.1 General considerations

In order to use the same techniques as in the case for $n \geq 5$, we start by giving a weakened version of Lemma IV.3.6.

Lemma IV.8.1. *Let $\alpha, \beta \in \mathcal{E}_n \setminus \mathcal{G}_n$. If α and β are of the same type, then $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathcal{E}_n \setminus \mathcal{G}_n$ if $n \neq 4$ and $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathcal{E}_4 \setminus (\mathcal{G}_4 \cup D(4))$ if $n = 4$.*

Proof. If $n \neq 4$, this follows directly from 4) of Corollary IV.2.5, while if $n = 4$, it suffices to add the results in Lemma IV.2.6. □

Using this lemma, Definition IV.4.1 is now valid for all $n \geq 2$, where for $n = 4$, we restrict ourselves to define α^+, α^- and α^0 to elements of $\mathcal{E}_4 \setminus (\mathcal{G}_4 \cup D(4))$.

We now give a result containing all the properties that are common in $\mathcal{E}_n$ for all values of n.

Proposition IV.8.2. *Let $n \in \mathbb{N}$. In the endomorphism monoid $\mathcal{E}_n$, the following statements hold:*

1) *the set of all idempotents is a band, and forms a rank-ordered chain of right zero semigroups;*
2) *the set of idempotents of rank 1 is the minimal ideal of $\mathcal{E}_n$;*
3) *$\mathscr{H} = \mathscr{L} \subseteq \mathscr{R} = \mathscr{D} = \mathscr{J}$;*
4) *$\mathscr{R}^* = \widetilde{\mathscr{R}}$ and the classes are the sets of elements with the same rank;*
5) *$\mathcal{E}_n$ is a left abundant semigroup.*

Proof. These facts have already been explicitly proven for $n \geq 5$ in the previous sections, and it is straightforward to check that the details go through in just the same way for all $n \neq 4$. Indeed, for all $n \neq 4$, statements 1) and 2) follow in exactly the same way as in the proof of Corollary IV.4.6, whilst it is readily verified that the characterisation of the $\mathscr{L}$ and $\mathscr{R}$ relations given in Section IV.5 ($\alpha \mathrel{\mathscr{L}} \beta$ if and only if $\alpha = \beta$ or $\alpha, \beta \in \mathcal{G}_n$; and $\alpha \mathrel{\mathscr{R}} \beta$ if and only if $\alpha, \beta \in E_k$ for some k or $\alpha, \beta \in \mathcal{G}_n$ or $\alpha\mathcal{G}_n = \beta\mathcal{G}_n$) also hold in these cases, from which statement 3) follows (since $\mathscr{L} \subseteq \mathscr{R}$). Part 4) and 5) are trivial in the case where $n = 1$ or $n = 2$ since these semigroups are regular. The astute reader will also notice that the proofs of the results present between Lemma IV.7.1 and Proposition IV.7.5 hold without modifications in the case $n = 3$ since there is an idempotent of every type.

The case $n = 4$ is a little different, since there are extra endomorphisms to consider. We use extensively the multiplication of elements described in Corollary IV.2.5 and Lemma IV.2.6. It is clear from these results that $E_1(4)$ is the minimal ideal of $\mathcal{E}_4$ by consideration of ranks and that $E_1(4) < E_2(4) < E_3(4) < \{\varepsilon\}$ is a chain of right-zero semigroups. Since $p_g = \mathrm{id}$ for all $g \in \mathcal{K}$, it follows that $\sigma^g\sigma^h = \sigma^h$ for all

$\sigma^g, \sigma^h \in E_7(4)$, and thus $E_7(4)$ is also a right-zero semigroup. To complete the proof of 1), it suffices to verify that $E_7(4)E_k(4) \subseteq E_k(4)$ and $E_k(4)E_7(4) \subseteq E_k(4)$ for all $k \leq 3$, which follows from Lemma IV.2.6 together with Lemma IV.1.7.

The fact that parts 3) and 4) hold when $n = 4$ will be shown in Corollary IV.8.11 and Lemma IV.8.12 below, since their proofs are more involved and require specific attention. □

Since $\mathcal{E}_2$ is a regular semigroup, we know that Green's and extended Green's relations coincide on this semigroup, and thus Proposition IV.8.2 contains all the information of the algebraic structure of $\mathcal{E}_2$ that have been studied in this chapter. On the other hand, we mentioned that $\mathcal{E}_3$ is not regular, and so it makes sense to consider the extended Green's relations.

Proposition IV.8.3. *In $\mathcal{E}_3$ the following statements hold:*

1) $\mathscr{L}^* \subseteq \mathscr{R}^* = \mathscr{D}^* = \mathscr{J}^* = \widetilde{\mathscr{R}}$ *and the $\mathscr{R}^*$-classes are $\mathcal{G}_3$, $E_3(3)$, $E_2(3) \cup C(3)$ and $E_1(3)$ (i.e. elements of the same rank);*
2) $\alpha \widetilde{\mathscr{L}} \beta$ *if and only if $\alpha = \beta$ or $\alpha, \beta \in \mathcal{G}_3 \cup C(3)$;*
3) $\alpha \mathscr{L}^* \beta$ *if and only if $\alpha, \beta \in \mathcal{G}_3$ or $\alpha^2 = \alpha = \beta = \beta^2$ or $\alpha = \phi_{t,e}, \beta = \phi_{u,f}$ both lie in $C(3)$ with* $\mathrm{Fix}(t, e) = \mathrm{Fix}(u, f)$*;*
4) *the $\widetilde{\mathscr{D}}$-classes are $\mathcal{G}_3 \cup E_2(3) \cup C(3)$, $E_3(3)$ and $E_1(3)$;*
5) *the $\widetilde{\mathscr{J}}$-classes are $\mathcal{E}_3 \setminus E_1(3)$ and $E_1(3)$.*

Proof. Since $\mathcal{E}_3$ contains an idempotent of each type as defined in Section IV.3, almost all of the arguments given in Section IV.7 go through verbatim. The only two notable differences concern the relations $\mathscr{D}^*$ and $\widetilde{\mathscr{D}}$. Indeed for $\mathscr{D}^*$ the proof of Proposition IV.7.12 utilises an element of $B(n)$ to deduce that certain elements are $\mathscr{D}^*$-related, but since $B(3) = \emptyset$, this argument is not valid for $n = 3$. The result however follows directly from the observation that $\mathscr{L}^* \subseteq \mathscr{R}^*$ so that $\mathscr{D}^* = \mathscr{R}^*$, while the proof that $\mathscr{D}^* = \mathscr{J}^*$ is just as before. In a similar manner in the case of $\widetilde{\mathscr{D}}$, the proof of Proposition IV.7.13 relies on the fact that elements of $A(n)$, $B(n)$ and $C(n)$ are $\widetilde{\mathscr{L}}$-related in order to show that elements of $E_2(n)$, $E_3(n)$ and $\mathcal{G}_n$ are $\widetilde{\mathscr{D}}$-related. Since $A(3) = B(3) = \emptyset$ this argument does not hold for $n = 3$. However, it is easy to see that elements of $E_3(3)$ form a single $\widetilde{\mathscr{D}}$-class, while elements of $\mathcal{G}_3$ and $E_2(3)$ are $\widetilde{\mathscr{L}} \circ \widetilde{\mathscr{R}}$-related. Thus the classes of $\widetilde{\mathscr{D}}$ are as given in the statement. Finally $\widetilde{\mathscr{D}} \subseteq \widetilde{\mathscr{J}}$, and since classes of $\widetilde{\mathscr{J}}$ are ideals saturated by $\widetilde{\mathscr{L}}$ and $\widetilde{\mathscr{R}}$, it follows that the only two classes are $E_1(3)$ and $\mathcal{E}_3 \setminus E_1(3)$. □

We can generalise Corollary IV.4.10 to the smaller values of n as follows.

Proposition IV.8.4. *Let $n \in \mathbb{N}$. Then the following conditions are mutually exclusive:*

1) $\mathcal{T}_n$ is isomorphic to $\mathcal{E}_n$, and then $n = 1$;

2) $\mathcal{T}_n$ embeds into $\mathcal{E}_n$, but $\mathcal{E}_n$ does not embed into $\mathcal{T}_n$, and then $n = 2$; and

3) there is no embedding between $\mathcal{T}_n$ and $\mathcal{E}_n$, and then $n \geq 3$.

Proof. Notice that the arguments used in Corollary IV.4.10 are in fact valid for all values of $n \geq 3$, since the element considered lie in $\mathcal{T}_3$ together with the fact that $\mathcal{T}_n$ has at least 3 idempotents in this case.

Suppose now that $\mathcal{T}_n$ and $\mathcal{E}_n$ are isomorphic. Since $\mathcal{T}_n$ is regular and $\mathcal{E}_n$ is not regular for $n \geq 3$, we must have $n \leq 2$. But $|\mathcal{T}_2| = 4 < 7 = |\mathcal{E}_2|$, so that the only possible value is $n = 1$, where we know that $\mathcal{T}_1 = \mathcal{E}_1$ is the trivial group.

Finally, suppose that $\phi\colon \mathcal{T}_n \to \mathcal{E}_n$ is an embedding but that there is no embedding of $\mathcal{E}_n$ into $\mathcal{T}_n$. Then the only possibility is that $n = 2$. In this case, such an embedding exists, for example:

$$\mathrm{id} \mapsto \varepsilon, \quad (1\,2) \mapsto \varepsilon_{(1\,2)}, \quad c_1 \mapsto \phi_{c_1,c_1} \quad \text{and} \quad c_2 \mapsto \phi_{c_2,c_2}. \qquad \square$$

IV.8.2 THE CASE OF $\mathcal{E}_4$

Throughout this section, we set $n = 4$.

Because of the presence of additional endomorphisms of rank 7, namely, the elements of $D(4)$, we cannot deduce the results for $\mathcal{E}_4$ directly from the results of the previous sections in the same way we have done for $\mathcal{E}_3$. Nonetheless, the structure of $\mathcal{E}_4$ is not too far away from the structure of $\mathcal{E}_n$ when $n \geq 5$, and it is possible to use the earlier proofs by following the strategy that we now give.

Recall first from Lemma IV.2.6 that $D(4)$ and $\mathcal{E}_4 \setminus D(4)$ are subsemigroups of $\mathcal{E}_4$, and write $T(4)$ for $\mathcal{E}_4 \setminus D(4)$. In order to describe the algebraic structure of $\mathcal{E}_4$, we want to show that the presence of the elements of $D(4)$ does not influence the algebraic structure of $T(4)$, so that we can work with $T(4)$ and $D(4)$ separately in a perfectly sound manner.

Remark IV.8.5. Recall that for any subsemigroup Q of a semigroup S, we have

$$\mathscr{L}_S \cap (Q \times Q) \subseteq \mathscr{L}_Q \quad \text{and} \quad \mathscr{R}_S \cap (Q \times Q) \subseteq \mathscr{R}_Q,$$

and similarly,

$$\mathscr{L}^*_S \cap (Q \times Q) \subseteq \mathscr{L}^*_Q, \qquad \mathscr{R}^*_S \cap (Q \times Q) \subseteq \mathscr{R}^*_Q,$$
$$\widetilde{\mathscr{L}}_S \cap (Q \times Q) \subseteq \widetilde{\mathscr{L}}_Q \quad \text{and} \quad \widetilde{\mathscr{R}}_S \cap (Q \times Q) \subseteq \widetilde{\mathscr{R}}_Q\,.$$

To work with Green's and extended Green's relations, we will consider that any relation written without a subscript will be relatively to the semigroup $\mathcal{E}_4$, but we will add a subscript D or T when they are to be considered in the subsemigroups $D(4)$ and $T(4) = \mathcal{E}_4 \setminus D(4)$ respectively, that is, we will write $\mathscr{L}$, $\mathscr{L}_D$ and $\mathscr{L}_T$ for the $\mathscr{L}$-relation on $\mathcal{E}_4$, $D(4)$ and $T(4)$ respectively. We will show that, if $\mathscr{X}$ is one of $\mathscr{L}$, $\mathscr{R}$, $\mathscr{L}^*$, $\mathscr{R}^*$, $\widetilde{\mathscr{L}}$ or $\widetilde{\mathscr{R}}$, we have:

$$\mathscr{X} \cap (T(4) \times T(4)) = \mathscr{X}_T \quad \text{and} \quad \mathscr{X} \cap (D(4) \times D(4)) = \mathscr{X}_D,$$

so that $\mathscr{X} = \mathscr{X}_T \cup \mathscr{X}_D$. This way, it will suffice to describe $\mathscr{X}_D$ to obtain a description of the $\mathscr{X}$-relation on $\mathcal{E}_4$, since the description of $\mathscr{X}_T$ will be obtained from the work done in Sections IV.5 and IV.7.

In order to facilitate readability, we will remove the dependence on 4 in the proofs of this section for the sets $T(4)$, $D(4)$, $E_3(4)$, $E_2(4)$, $B(4)$, $C(4)$ and $E_1(4)$. We will make heavy use of the results contained in Sections IV.1 and IV.2.2 which concern properties of elements of $D(4)$, in particular with respect to the elements p_s and the multiplication in $\mathcal{E}_4$. A direct consequence of Lemma IV.1.7 is that $\alpha\sigma^g$ has the same type as α for all $\alpha = \phi_{t,e} \in \mathcal{E}_4 \setminus (\mathcal{G}_4 \cup D(4))$, and moreover, if $\alpha = \phi_{t,e}$ is of non-permutation type, then $\alpha\sigma^g = \phi_{c_{4g},c_{4g}} \in E_1(4)$. The aim of this section is to prove the results contained in the following proposition.

Proposition IV.8.6. *In the endomorphism monoid $\mathcal{E}_4 = \mathrm{End}(\mathcal{T}_4)$, the following statements hold:*

- *the regular elements of $\mathcal{E}_4$ are $\mathcal{G}_4 \cup D(4) \cup E(\mathcal{E}_4)$, so that $\mathcal{E}_4$ is not regular;*
- *$\alpha \mathscr{L} \beta$ if and only if $\alpha = \beta$ or $\alpha, \beta \in \mathcal{G}_4$ or $\alpha = \sigma^g$, $\beta = \sigma^h$ both lie in $D(4)$ with $4g = 4h$;*
- *$\alpha \mathscr{R} \beta$ if and only if $\alpha, \beta \in E_k(4)$ for some $1 \leq k \leq 3$ or $\alpha, \beta \in \mathcal{G}_4$ or $\alpha, \beta \in D(4)$ or $\alpha\mathcal{G}_4 = \beta\mathcal{G}_4$;*
- *$\mathscr{H} = \mathscr{L} \subsetneq \mathscr{R} = \mathscr{D} = \mathscr{J}$;*
- *$\mathscr{R}^* = \widetilde{\mathscr{R}}$ and the $\mathscr{R}^*$-classes are $\mathcal{G}_4$, $D(4)$, $E_3(4) \cup B(4)$, $E_2 \cup C(4)$ and $E_1(4)$;*

- $\alpha \widetilde{\mathscr{L}} \beta$ *if and only if* $\alpha = \beta$ *or* $\alpha, \beta \in \mathcal{G}_4 \cup B(4) \cup C(4)$ *or* $\alpha = \sigma^g$, $\beta = \sigma^h$ *both lie in* $D(4)$ *with* $4g = 4h$*;*
- $\alpha \mathscr{L}^* \beta$ *if and only if* $\alpha, \beta \in \mathcal{G}_4$ *or* $\alpha = \beta$ *or* $\alpha = \sigma^g$, $\beta = \sigma^h$ *both lie in* $D(4)$ *with* $4g = 4h$ *or* $\alpha = \phi_{t,e}, \beta = \phi_{u,f}$ *both lie in* $B(4) \cup C(4)$ *with* $\mathrm{Fix}(t, e) = \mathrm{Fix}(u, f)$*;*
- $\mathscr{D}^* = \mathscr{J}^*$ *and the* $\mathscr{D}^*$*-classes are* $\mathcal{G}_4, D(4), E_3(4) \cup B(4) \cup E_2(4) \cup C(4)$ *and* $E_1(4)$*;*
- *the* $\widetilde{\mathscr{D}}$*-classes are* $D(4)$*,* $E_1(4)$ *and* $\mathcal{E}_4 \setminus (D(4) \cup E_1(4))$*;*
- *the* $\widetilde{\mathscr{J}}$*-classes are* $\mathcal{E}_4 \setminus E_1(4)$ *and* $E_1(4)$*.*

We start by showing that $\mathcal{E}_4$ is not regular.

Lemma IV.8.7. *The regular elements of* $\mathcal{E}_4$ *are* $\mathcal{G}_4 \cup D(4) \cup E(\mathcal{E}_4)$*. Consequently,* $D(4)$ *is a regular subsemigroup and* $\mathcal{E}_4$ *is not regular.*

Proof. Clearly, elements of $\mathcal{G}_4 \cup E(\mathcal{E}_4)$ are regular. Now let $\sigma^g \in D$, and let $h = g^{-1}$ so that $p_h = p_g^{-1}$. Then we get

$$\sigma^g \sigma^h \sigma^g = \sigma^{p_g h} \sigma^g = \sigma^{p_g p_h g} = \sigma^g,$$

which shows that all elements of D are regular.

To see that an element $\alpha \in B \cup C$ is not regular, notice that $\alpha \sigma^g \alpha = \alpha^2 \neq \alpha$, and using the arguments of the proof of Proposition IV.4.8, we also get that α does not have an inverse in T. Hence $B \cup C$ consists of non-regular elements and $\mathcal{E}_4$ is not a regular semigroup. $\square$

Remark IV.8.8. Since $D(4)$ is a regular subsemigroup, it is well-known (see [2.4.2 in 28]) that $\mathscr{L}_D = \mathscr{L} \cap (D(4) \times D(4))$ and $\mathscr{R}_D = \mathscr{R} \cap (D(4) \times D(4))$.

IV.8.2.1 Green's relations

We now prove the description of Green's relations in $\mathcal{E}_4$.

Lemma IV.8.9. *Let* $\alpha, \beta \in \mathcal{E}_4$*. Then:*

1) *the principal left ideal generated by* $\alpha \in \mathcal{E}_n \setminus \mathcal{G}_4$ *is:*
   a) $\mathcal{E}_4 \alpha = \{\alpha, \alpha^+, \alpha^-, \alpha^0\}$ *if* $\alpha \in \mathcal{E}_4 \setminus (\mathcal{G}_4 \cup D(4))$*, and*
   b) $\mathcal{E}_4 \alpha = \left\{\sigma^{p_t g}, \phi_{p_t^g, c_{4g}}, \phi_{c_{4g}, c_{4g}}, \phi_{\mathrm{id},\mathrm{id}} : t \in \mathcal{S}_4\right\}$ *if* $\alpha = \sigma^g \in D(4)$*;*
2) $\alpha \mathscr{L} \beta$ *if and only if* $\alpha = \beta$ *or* $\alpha, \beta \in \mathcal{G}_4$ *or* $\alpha = \sigma^g$*,* $\beta = \sigma^h$ *both lie in* $D(4)$ *with* $4g = 4h$*;*

3) $\mathscr{L}_D = \mathscr{L} \cap (D(4) \times D(4))$ *and* $\mathscr{L}_T = \mathscr{L} \cap (T(4) \times T(4))$*, so that* $\mathscr{L} = \mathscr{L}_T \cup \mathscr{L}_D$.

Proof. We already know that elements of D cannot be $\mathscr{L}$-related to elements of T since $\mathscr{L}_D = \mathscr{L} \cap (D \times D)$. Since $\sigma^g \phi_{t,e} = \phi_{t,e}$ for all $\sigma^g \in D$ and $\phi_{t,e} \in \mathcal{E}_4 \setminus (\mathcal{G}_4 \cup D)$, it follows from Proposition IV.5.1 that if $\alpha \in \mathcal{E}_4 \setminus (\mathcal{G}_4 \cup D)$, then $\mathcal{E}_4 \alpha = \{\alpha, \alpha^+, \alpha^-, \alpha^0\}$, while $\mathcal{E}_4 \alpha = \mathcal{E}_4$ is $\alpha \in \mathcal{G}_4$, so that $\mathscr{L}_T = \mathscr{L} \cap (T \times T)$.

On the other hand, the principal left ideal generated by $\sigma^g \in D$ is

$$\begin{aligned}\mathcal{E}_4 \sigma^g = &\left\{\psi_t \sigma^g, \sigma^t \sigma^g : t \in \mathcal{S}_4\right\} \cup \{\phi_{t,e}\sigma^g : t \in \mathcal{S}_4 \text{ and } e \neq \mathrm{id}\} \cup \\ &\{\phi_{t,e}\sigma^g : t, e \in \mathcal{T}_4 \setminus \mathcal{S}_4\} \cup \{\phi_{\mathrm{id},\mathrm{id}}\sigma^g\} \\ = &\{\sigma^{p_t g} : t \in \mathcal{S}_4\} \cup \left\{\phi_{p_t^g, c_{4g}} : t \in \mathcal{S}_4\right\} \cup \left\{\phi_{c_{4g}, c_{4g}}\right\} \cup \{\phi_{\mathrm{id},\mathrm{id}}\}.\end{aligned}$$

Moreover, since $\operatorname{im} \sigma^g = \{t^g : t \in \mathcal{S}_4, 4t = 4\} \cup \{c_{4g}\}$, it follows that if $\sigma^g \mathscr{L} \sigma^h$, then we have $\operatorname{im} \sigma^g = \operatorname{im} \sigma^h$, which forces $4g = 4h$. Conversely, let $\sigma^g, \sigma^h \in D$ be such that $4g = 4h$. Then hg^{-1} fixes 4, which shows that $(hg^{-1})\sigma = hg^{-1}$ and $\sigma^{hg^{-1}}\sigma^g = \sigma^{hg^{-1}g} = \sigma^h$, and similarly for gh^{-1} giving $\sigma^{gh^{-1}}\sigma^h = \sigma^g$. Thus the $\mathscr{L}$-classes are as given in the statement and $\mathscr{L} = \mathscr{L}_T \cup \mathscr{L}_D$. □

Lemma IV.8.10. *Let $\alpha \in \mathcal{E}_4$. The principal right ideal of $\mathcal{E}_4$ generated by α is equal to:*

$$\alpha\mathcal{E}_4 = \begin{cases} \mathcal{E}_4 & \text{if } \alpha \in \mathcal{G}_4, \\ \mathcal{E}_4 \setminus \mathcal{G}_4 & \text{if } \alpha \in D(4), \\ \mathcal{E}_4 \setminus (\mathcal{G}_4 \cup D(4)) & \text{if } \alpha \in E_3(4), \\ E_2(4) \cup C(4) \cup E_1(4) & \text{if } \alpha \in E_2(4), \\ B_{t,e}(4) \cup E_1(4) & \text{if } \alpha = \phi_{t,e} \in B(4), \\ C_{t,e}(4) \cup E_1(4) & \text{if } \alpha = \phi_{t,e} \in C(4), \text{or} \\ E_1(4) & \text{if } \alpha \in E_1(4). \end{cases}$$

Consequently, for $\alpha, \beta \in \mathcal{E}_4$, we have $\alpha \mathscr{R} \beta$ if and only if $\alpha, \beta \in E_k(4)$ for some $1 \leq k \leq 3$ or $\alpha, \beta \in \mathcal{G}_4$ or $\alpha, \beta \in D(4)$ or $\alpha\mathcal{G}_4 = \beta\mathcal{G}_4$.

Furthermore, $\mathscr{R}_D = \mathscr{R} \cap (D(4) \times D(4))$ and $\mathscr{R}_T = \mathscr{R} \cap (T(4) \times T(4))$.

Proof. We already know that D is a union of $\mathscr{R}$-classes and that its elements are left identities for $\mathcal{E}_4 \setminus (\mathcal{G}_4 \cup D)$, so that $\sigma^g \mathcal{E}_4 = \mathcal{E}_4 \setminus \mathcal{G}_4$. Furthermore, for all $g, h \in \mathcal{S}_4$ we have that $p_g^{-1}h, p_h^{-1}g \in \mathcal{S}_4$, which gives that $\sigma^g \sigma^{p_g^{-1}h} = \sigma^{p_g p_g^{-1}h} = \sigma^h$ and $\sigma^h \sigma^{p_h^{-1}g} = \sigma^{p_h p_h^{-1}g} = \sigma^g$. Hence D is an $\mathscr{R}$-class.

By the remarks made at the beginning of this section, for all $\alpha \in E_k$ with $1 \leq k \leq 3$, we have that $\alpha D \subseteq E_k$. Also for $\alpha \in B \cup C$, we have $\alpha D \subseteq E_1$. Thus $\alpha \mathcal{E}_4 = \alpha T$, and the result then follows from Proposition IV.5.3. □

The proof of the following corollary is then immediate.

Corollary IV.8.11. *In $\mathcal{E}_4$, we have $\mathscr{H} = \mathscr{L} \subsetneq \mathscr{R} = \mathscr{D} = \mathscr{J}$.*

IV.8.2.2 Extended Green's relations

We start by looking at the relations $\mathscr{R}^*$ and $\widetilde{\mathscr{R}}$.

Lemma IV.8.12. *In $\mathcal{E}_4$, $\mathscr{R}^* = \widetilde{\mathscr{R}}$ and the $\mathscr{R}^*$-classes are $\mathcal{G}_4$, $D(4)$, $E_3(4) \cup B(4)$, $E_2 \cup C(4)$ and $E_1(4)$.*

Consequently, $\alpha \mathscr{R}^* \beta$ if and only if α and β lie $\mathscr{R}$-below the same idempotents if and only if α and β lie $\mathscr{J}$-below the same idempotents if and only if $\mathrm{rk}(\alpha) = \mathrm{rk}(\beta)$.

Furthermore, we have $\mathscr{R}^*_D = \mathscr{R}^* \cap (D(4) \times D(4))$ and $\mathscr{R}^*_T = \mathscr{R}^* \cap (T(4) \times T(4))$.

Proof. Notice first that elements of E_7 are left identities for all elements of $\mathcal{E}_n \setminus \mathcal{G}_4$ but they are not left identities for elements of $\mathcal{G}_4$. Similarly, elements of E_3 are left identities for $\mathcal{E}_4 \setminus (\mathcal{G}_4 \cup D)$ but are not left identities for $\mathcal{G}_4 \cup D$. Thus, elements of D cannot be $\widetilde{\mathscr{R}}$-related to any element of T.

Moreover, D is regular, so that by Corollary I.3.14, we have that $\mathscr{R}_D = \mathscr{R}^*_D = \widetilde{\mathscr{R}}_D$ which shows D is an $\widetilde{\mathscr{R}}$- and an $\mathscr{R}^*$-class.

It follows from this that T is a union of $\mathscr{R}^*$- and $\widetilde{\mathscr{R}}$-classes. Using the same arguments as in the proofs of Lemma IV.7.1 and Proposition IV.7.5, we get that the $\mathscr{R}^*_T$- and $\widetilde{\mathscr{R}}_T$-classes are $\mathcal{G}_4$, $E_3 \cup B$, $E_2 \cup C$ and E_1. By the remark at the beginning of this proof, it is clear that these are $\widetilde{\mathscr{R}}$-classes in $\mathcal{E}_4$. We aim to show that the presence of elements from D does not split these classes further for $\mathscr{R}^*$. In order to do so, it suffices to show that if two elements of $E_3 \cup B$ or $E_2 \cup C$ are $\mathscr{R}^*_T$-related, then they must be $\mathscr{R}^*$-related.

So let $\phi_{t,e}, \phi_{u,f} \in \mathcal{E}_4$ be such that $\phi_{t,e} \mathscr{R}^*_T \phi_{u,f}$. Since elements of D act as left identities on $\mathcal{E}_4 \setminus (\mathcal{G}_4 \cup D)$, it follows that if $\gamma\phi_{t,e} = \delta\phi_{t,e}$ for some $\gamma, \delta \in \mathcal{E}_4$, either $\gamma, \delta \in D$ and then trivially $\gamma\phi_{u,f} = \delta\phi_{u,f}$, or $\gamma, \delta \in T$ and thus $\gamma\phi_{u,f} = \delta\phi_{u,f}$ by assumption, or $\gamma \in D$ and $\delta \in T$ in which case we have $\gamma\phi_{t,e} = \eta\phi_{t,e} = \delta\phi_{t,e}$. Now, using the assumption that $\phi_{t,e} \mathscr{R}^*_T \phi_{u,f}$, we get $\eta\phi_{u,f} = \delta\phi_{u,f}$, so that $\gamma\phi_{u,f} = \eta\phi_{u,f} = \delta\phi_{u,f}$. Exchanging the role of $\phi_{t,e}$ and $\phi_{u,f}$, we get that if $\phi_{t,e} \mathscr{R}^*_T \phi_{u,f}$, then $\phi_{t,e} \mathscr{R}^* \phi_{u,f}$.

Therefore, the $\mathscr{R}^*$-classes are as given, and the other equivalences follow directly from Lemma IV.8.10 and Corollary IV.8.11.

The last part follows from the intermediary results present in the proof. $\square$

The relations $\widetilde{\mathscr{L}}$ and $\mathscr{L}^*$ are also closely following the descriptions of $\widetilde{\mathscr{L}}_T$ and $\mathscr{L}^*_T$.

Lemma IV.8.13. *In* $\mathcal{E}_4$ *we have the following:*

1) $\alpha \, \widetilde{\mathscr{L}} \, \beta$ *if and only if* $\alpha = \beta$ *or* $\alpha, \beta \in \mathcal{G}_4 \cup B(4) \cup C(4)$ *or* $\alpha = \sigma^g$, $\beta = \sigma^h$ *both lie in* $D(4)$ *with* $4g = 4h$;
2) $\widetilde{\mathscr{L}}_D = \widetilde{\mathscr{L}} \cap (D(4) \times D(4))$ *and* $\widetilde{\mathscr{L}}_T = \widetilde{\mathscr{L}} \cap (T(4) \times T(4))$;
3) $\alpha \, \mathscr{L}^* \, \beta$ *if and only if* $\alpha, \beta \in \mathcal{G}_4$ *or* $\alpha = \beta$ *or* $\alpha = \sigma^g$, $\beta = \sigma^h$ *both lie in* $D(4)$ *with* $4g = 4h$ *or* $\alpha = \phi_{t,e}$, $\beta = \phi_{u,f}$ *both lie in* $B(4) \cup C(4)$ *with* $\mathrm{Fix}(t,e) = \mathrm{Fix}(u,f)$;
4) $\mathscr{L}^*_D = \mathscr{L}^* \cap (D(4) \times D(4))$ *and* $\mathscr{L}^*_T = \mathscr{L}^* \cap (T(4) \times T(4))$.

Proof. Notice that since D is a regular semigroup, each element of D has an idempotent of E_7 as a right identity. Moreover, elements of E_7 cannot be right identities for elements of $\mathcal{G}_4$, nor of $B \cup C$ (since $\phi_{t,e}\sigma^g \in E_1$ for all $\phi_{t,e} \in B \cup C$), while every element of $E_3 \cup E_2 \cup E_1$ is a right identity for itself but not for elements of D. Therefore we obtain that elements of D cannot be $\widetilde{\mathscr{L}}$-related (and hence cannot be $\mathscr{L}^*$-related) to elements of T so that $\sigma^g \, \widetilde{\mathscr{L}} \, \sigma^h$ if and only if $\sigma^g \, \widetilde{\mathscr{L}}_D \, \sigma^h$. But $\widetilde{\mathscr{L}}_D = \mathscr{L}^*_D = \mathscr{L}_D$ by Corollary I.3.14 which gives the $\widetilde{\mathscr{L}}$- and $\mathscr{L}^*$-classes of D by Lemma IV.8.9.

It follows that T is a union of $\widetilde{\mathscr{L}}$- and $\mathscr{L}^*$-classes. Using the remarks above and Proposition IV.7.8, we get that $\mathcal{G}_4 \cup B \cup C$ is an $\widetilde{\mathscr{L}}$-class since the only idempotent acting as a right identity for elements of this set is ε, while idempotents of $E_3 \cup E_2 \cup E_1$ form singleton $\widetilde{\mathscr{L}}$-classes in T and thus also in $\mathcal{E}_4$.

Since idempotents form their own $\widetilde{\mathscr{L}}$-classes, then they also form singletons $\mathscr{L}^*$-classes. From Proposition IV.7.10, we also know that $\mathcal{G}_4$ is an $\mathscr{L}^*_T$-class, and since it is already an $\mathscr{L}$-class and $\mathscr{L} \subseteq \mathscr{L}^*$, we get that $\mathcal{G}_4$ is an $\mathscr{L}^*$-class. Finally, $B \cup C$ splits into smaller $\mathscr{L}^*_T$-classes where two elements are $\mathscr{L}^*_T$-related if they are fixed by the same automorphisms. Hence, in order to finish the description of the $\mathscr{L}^*$-classes of $\mathcal{E}_4$, it suffices to show that two elements of $B \cup C$ which are $\mathscr{L}^*_T$-related will be $\mathscr{L}^*$-related.

Let $\alpha, \beta \in B \cup C$ be such that $\alpha \, \mathscr{L}^*_T \, \beta$. Consider $\gamma, \delta \in \mathcal{E}_4$ satisfying $\alpha\gamma = \alpha\delta$. If $\gamma \in D$, then $\alpha\gamma = \phi_{c_{4h}, c_{4h}} = \beta\gamma$, and similarly if $\gamma = \phi_{u,f} \in \mathcal{E}_4$, then $\alpha\gamma = \phi_{f,f} = \beta\gamma$.

Thus if $\gamma, \delta \in \mathcal{E}_n \setminus \mathcal{G}_4$, then we get that $\beta\gamma = \alpha\gamma = \alpha\delta = \beta\delta$ directly. Note that we cannot have $\gamma \in \mathcal{G}_4$ and $\delta \in D$ since $\alpha\psi_g \in B \cup C$ while $\alpha\sigma^g = \in E_1$. For the remaining cases, we have $\gamma, \delta \in T$, and by assumption of α and β being $\mathscr{L}^*_T$-related, we get $\beta\gamma = \beta\delta$. Interchanging the role of α and β, we obtain that $\alpha \mathscr{L}^* \beta$, finishing to show that $\mathscr{L}^*_T = \mathscr{L}^* \cap (T(4) \times T(4))$. Therefore the $\mathscr{L}^*$-classes of $\mathcal{E}_4$ are as given in the statement while the other parts can be deduced from the steps of this proof. □

An immediate consequence of Lemma IV.8.13 is the following.

Corollary IV.8.14. *The semigroup $\mathcal{E}_4$ is right Fountain but not right abundant.*

We can now easily deduce the description of $\mathscr{D}^*$ and $\mathscr{J}^*$.

Corollary IV.8.15. *Let $\alpha, \beta \in \mathcal{E}_4$. Then $\alpha \mathscr{D}^* \beta$ if and only if $\alpha \mathscr{D}^*_T \beta$ or $\alpha \mathscr{D}^*_D \beta$.*

Moreover, the $\mathscr{D}^*$-classes are $\mathcal{G}_4$, $D(4)$, $E_3(4) \cup B(4) \cup E_2(4) \cup C(4)$ and $E_1(4)$, and $\mathscr{J}^* = \mathscr{D}^*$.

Proof. Let $\alpha, \beta \in \mathcal{E}_4$ and assume that $\alpha \mathscr{D}^* \beta$. Then there exist $\gamma_0, \gamma_1, \ldots, \gamma_{2k} \in \mathcal{E}_4$ such that

$$\alpha = \gamma_0 \mathscr{R}^* \gamma_1 \mathscr{L}^* \gamma_2 \cdots \mathscr{R}^* \gamma_{2k-1} \mathscr{L}^* \gamma_{2k} = \beta.$$

By Lemma IV.8.12 we have that if $\alpha \in D$ then $\gamma_1 \in D$, and then using Lemma IV.8.13, this also forces $\gamma_2 \in D$. By induction, we get that $\gamma_i \in D$ for all $1 \le i \le n$, so that $\beta \in D$ and $\alpha \mathscr{D}^*_D \beta$. Similarly, if $\alpha \in T$, then we get that each γ_i is in T and thus $\beta \in T$ and $\alpha \mathscr{D}^*_T \beta$.

Since $\mathscr{L}^*_D \subseteq \mathscr{R}^*_D$ and that D is an $\mathscr{R}^*$-class, it follows that it is a $\mathscr{D}^*$-class. For the other classes, we use the fact that the $\mathscr{R}^*$- and $\mathscr{L}^*$-classes of elements of T are the $\mathscr{R}^*_T$- and $\mathscr{L}^*_T$-classes to deduce that the proof of Proposition IV.7.12 can be followed entirely, with the only caveat that we need to slightly change the elements used to show that there is an element of B that is $\mathscr{L}^*$-related to an element of C. For this, it suffices to consider the restriction of the elements t, u and f to the set $\{1, 2, 3, 4\}$, that is, to use the elements

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 4 \end{pmatrix}, \quad \text{and} \quad f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

since we then have that $\phi_{t,f} \in B$ and $\phi_{u,f} \in C$ are fixed by the same automorphisms and are thus $\mathscr{L}^*$-related.

Since we know that $\mathscr{D}^* \subseteq \mathscr{J}^*$, it only remains to show that elements of distinct $\mathscr{D}^*$-classes cannot be $\mathscr{J}^*$-related. But this follows from the fact that E_1, $\mathcal{E}_4 \setminus (\mathcal{G}_4 \cup D)$, $\mathcal{E}_4 \setminus \mathcal{G}_4$ and $\mathcal{E}_4$ are all distinct ideals of $\mathcal{E}_4$ saturated by $\mathscr{D}^*$, and thus correspond to the principal $*$-ideals of $\mathcal{E}_4$. □

We finally give the description of $\widetilde{\mathscr{D}}$ and $\widetilde{\mathscr{J}}$ which differ in $\mathcal{E}_4$.

Lemma IV.8.16. *In $\mathcal{E}_4$ we have that:*

- *the $\widetilde{\mathscr{D}}$-classes are $D(4)$, $E_1(4)$ and $\mathcal{E}_4 \setminus (D(4) \cup E_1(4))$; and*
- *the $\widetilde{\mathscr{J}}$-classes are $\mathcal{E}_4 \setminus E_1(4)$ and $E_1(4)$.*

Proof. By Lemmas IV.8.12 and IV.8.13, it is clear that E_1 and D are both $\widetilde{\mathscr{D}}$-classes of $\mathcal{E}_4$. Moreover, any element of $\mathcal{E}_4 \setminus (D \cup E_1)$ is $\mathscr{R}^* \circ \mathscr{L}^*$-related to ε, which shows that this also constitutes a $\widetilde{\mathscr{D}}$-class.

Since $\widetilde{\mathscr{D}} \subseteq \widetilde{\mathscr{J}}$ and E_1 is an ideal saturated by $\widetilde{\mathscr{D}}$, it suffices to prove that elements of D are $\widetilde{\mathscr{J}}$-related to elements of $\mathcal{G}_4$. If $\alpha \in D$, then the principal ideal generated by α is $\mathcal{E}_4 \alpha \mathcal{E}_4 = \mathcal{E}_4 \setminus \mathcal{G}_4$. Since elements of $\mathcal{G}_4$ are $\widetilde{\mathscr{L}}$-related to elements of B, it follows that this ideal is not $\sim$-saturated, so that $\widetilde{J}(\alpha) = \mathcal{E}_4 = \widetilde{J}(\varepsilon)$. Hence, α is $\widetilde{\mathscr{J}}$-related to ε, which shows that $\mathcal{E}_4 \setminus E_1$ is a single $\widetilde{\mathscr{J}}$-class. □

IV.9 FURTHER CONSIDERATIONS

It is clear that this chapter only constitutes an opening in the discussion of endomorphism monoids of special algebras, and as such there are many possible extensions for this work. We list here a few potential directions.

Presentations and orbits. Since $\mathcal{E}_n$ is a finite semigroup with a fairly nice structure, we would like to give a finite presentation of this monoid. In [23], we give some results on how to find a set of generators and a presentation. However, we would like to be able to exactly describe the elements a minimal generating set must contain. This boils down to understanding the orbits of elements of $\mathcal{E}_n$, and how the sets $A(n)$, $B(n)$ and $C(n)$ are split relative to these. In particular, we know that all elements inside a given orbit have the same rank, so that we can talk about orbits of a given rank. It is clear that orbits of rank 3 cannot be generated from orbits of higher rank, or other orbits of rank 3. However, in $C(n)$, we can see that some orbits of rank 2 will be generated by orbits of rank 3, while others will not. Being able to describe all orbits of rank 2 that cannot be generated by orbits of higher rank would allow us to give a presentation of $\mathcal{E}_n$ with a possibly minimal number of generators.

Endomorphism monoids of other semigroups. In this chapter, we have looked at the monoid $\mathcal{E}_n = \mathrm{End}(\mathcal{T}_n)$ as $\mathcal{T}_n$ is arguably the most natural semigroup of *finite* transformations. Since the automorphisms and endomorphisms of $\mathcal{PT}_n$ and $\mathcal{I}_n$ have already been described by different authors (see [19, Chap. 7]), a similar study could be made to describe the structure of the monoids $\mathrm{End}(\mathcal{PT}_n)$, $\mathrm{End}(\mathcal{I}_n)$ and $\mathrm{End}(\mathcal{S}_n)$. What similarities and differences in the structure of the monoid would these have when compared to $\mathrm{End}(\mathcal{T}_n)$?

Moving away from transformation semigroups, one could consider related semigroups, such as Brauer monoids and partition monoids (see, for example, [11]), where here the endomorphisms have been determined [32].

We have assumed throughout that we were looking at the transformation semigroup of a finite set. What would be the endomorphism monoid of the full transformation monoid (and related semigroups) of an *infinite* set, such as $\mathbb{N}$?

Iteration. We have mentioned that $\mathcal{T}_n$ and $\mathcal{E}_n$ are not mutually embeddable in each other in general. One could pursue our iteration procedure by now considering the monoid $\mathrm{End}(\mathcal{E}_n) = \mathrm{End}(\mathrm{End}(\mathcal{T}_n))$. How would that monoid compare relatively to $\mathcal{T}_n$ or $\mathcal{E}_n$? Even more generally, what can be said about the sequence of monoids

$$\mathcal{T}_n, \quad \mathrm{End}(\mathcal{T}_n), \quad \mathrm{End}(\mathrm{End}(\mathcal{T}_n)), \quad \ldots?$$

— V —

The translational hull of the 0-minimal ideal of the endomorphism monoid of an independence algebra

This chapter is concerned with translational hulls and ideal extensions described in Section I.2 of the preliminaries when studied in the context of the endomorphism monoid of an independence algebra. Let $\mathscr{A}$ be an independence algebra and $\mathfrak{M}$ be the smallest ideal of $\operatorname{End}(\mathscr{A})$ containing maps of rank at least 1, and call this ideal the (0-)minimal ideal of $\operatorname{End}(\mathscr{A})$. The following question was asked by Prof. Stuart Margolis:

> *Is the translational hull of the (0-)minimal ideal* $\mathfrak{M}$ *isomorphic to the endomorphism monoid* $\operatorname{End}(\mathscr{A})$*?*

which can be reformulated as:

> *Is the monoid* $\operatorname{End}(\mathscr{A})$ *the largest semigroup that contains its (0-)minimal ideal* $\mathfrak{M}$ *as a dense ideal?*

Whenever $\operatorname{End}(\mathscr{A})$ has no other ideals than itself, we trivially get a positive answer from Corollary I.2.15 since $\operatorname{End}(\mathscr{A})$ is a monoid, so we will only consider independence algebras for which $\operatorname{End}(\mathscr{A})$ has a proper ideal. Notice also that if the set of constants of $\mathscr{A}$ is non-empty, then it follows that the set

$$\mathfrak{M} = \{\delta \in \operatorname{End}(\mathscr{A}) \mid \operatorname{im}\delta = \langle\emptyset\rangle\} = T_1,$$

is the minimal ideal of $\operatorname{End}(\mathscr{A})$ by Remark I.5.8. However, this ideal is also a left-zero semigroup by Remark I.5.5, which means that its translational hull is isomorphic to $\mathcal{T}_{T_1}$ by Proposition I.2.23. In particular, this shows that $\Omega(\mathfrak{M})$ in this case is not isomorphic to $\mathrm{End}(\mathscr{A})$ and is fully determined. For this reason we will not look at the minimal ideal of $\mathrm{End}(\mathscr{A})$ if our algebra $\mathscr{A}$ has constants. Thus, we restrict ourselves to algebras that have a proper (0-)minimal ideal, that is, algebras of rank at least 2.

From the results expressed in Section I.2, by Corollaries I.2.6 and I.2.42 we know that for any ideal $\mathfrak{I} \subseteq \mathrm{End}(\mathscr{A})$, the maps $\pi_{\mathrm{P}} \colon \Omega(\mathfrak{I}) \to \mathrm{P}(\mathfrak{I})$, $\chi_{\mathrm{P}} \colon \mathrm{End}(\mathscr{A}) \to \mathrm{P}(\mathfrak{I})$ and $\chi \colon \mathrm{End}(\mathscr{A}) \to \Omega(\mathfrak{I})$ defined by

$$(\lambda, \rho)\pi_{\mathrm{P}} = \rho, \qquad \phi\chi_{\mathrm{P}} = \rho_\phi \qquad \text{and} \quad \phi\chi = (\lambda_\phi, \rho_\phi)$$

are all morphisms. This chapter aims to give necessary and sufficient conditions for these morphisms to be isomorphisms in the case where $\mathfrak{I}$ is the (0-)minimal ideal of $\mathrm{End}(\mathscr{A})$, that is, finding conditions so that the following commuting diagram only consists of isomorphisms:

$$\begin{tikzcd} & \mathrm{P}(\mathfrak{I}) & \\ \Omega(\mathfrak{I}) \arrow[ur, "\pi_{\mathrm{P}}"] & & \mathrm{End}(\mathscr{A}) \arrow[ul, "\chi_{\mathrm{P}}"'] \arrow[ll, dashed, "\chi"] \end{tikzcd}$$

We study the maps π_{P} and χ_{P} in Section V.1 and exhibit a necessary and sufficient condition in Theorem V.1.18 under which $\Omega(\mathfrak{I})$ and $\mathrm{End}(\mathscr{A})$ are isomorphic through the composition of π_{P} and χ_{P}^{-1}. By looking closely at examples of independence algebras in Section V.2, we see that the conditions exhibited are not always easy to verify, and can be too restrictive if we only care to obtain an isomorphism between $\Omega(\mathfrak{I})$ and $\mathrm{End}(\mathscr{A})$.

Even though this approach is generalised in Section V.3 to a larger class of universal algebras which are free on their basis, we search for conditions that are less reliant on the special structure of our algebra. This will be used in Chapter VI in order to find when the map χ is an isomorphism in the context of a general universal algebra.

Remark. Our approach in this chapter is to focus on right translations, and a curious reader could wonder why this choice was preferred compared to the one that would involve left translations instead. A quick answer would be to say that it is more convenient to only deal with right maps (being either translations or morphisms), rather than having a combination of left and right maps in our expressions. However, there is also a more subtle reason why looking at right translations when dealing with independence algebras is the most practical and efficient way to describe translational hull, relying on the fact that such algebras are free on their basis, but this reason will only become apparent through the developments carried in Chapters VI and VII.

V.1 TRANSLATIONAL HULLS IN END($\mathscr{A}$)

Let $\mathscr{A}$ be an arbitrary universal algebra. We start by adapting the common notion of 0-minimal ideals defined for semigroups with a zero to the context at hand.

V.1.1 The (0-)minimal ideal

Definition V.1.1. If $\mathscr{A}$ has constants, we let $\mathfrak{O}$ denote the ideal of $\mathrm{End}(\mathscr{A})$ consisting of maps whose image is the constant subalgebra $\langle\emptyset\rangle$. We note that $\mathfrak{O}$ is a left-zero semigroup.

Let $\mathfrak{T}$ be a (two-sided) ideal of the endomorphism monoid $\mathrm{End}(\mathscr{A})$. Then $\mathfrak{T}$ is called *minimal* if it does not properly contain any ideal of $\mathrm{End}(\mathscr{A})$.

Furthermore, $\mathfrak{T}$ is called *0-minimal* if:

i) $\mathfrak{T} \neq \mathfrak{O}$, and

ii) $\mathfrak{O}$ is the only ideal properly contained in $\mathfrak{T}$.

As a shorthand, we will talk about the *(0-)minimal ideal* of $\mathrm{End}(\mathscr{A})$ to denote its minimal ideal whenever the constant subalgebra $\langle\emptyset\rangle$ of $\mathscr{A}$ is empty, and to be its 0-minimal ideal otherwise.

We now define a set $\mathfrak{I}$ of endomorphisms as follows:

$$\begin{aligned}\mathfrak{I} &:= \{\alpha \in \mathrm{End}(\mathscr{A}) \mid \mathrm{im}\,\alpha \text{ lies inside a monogenic subalgebra}\}\\ &= \{\alpha \in \mathrm{End}(\mathscr{A}) \mid \exists r_\alpha \in A \text{ with } \mathrm{im}\,\alpha \subseteq \langle r_\alpha\rangle\} \subseteq \mathrm{End}(\mathscr{A}),\end{aligned}$$

which is an ideal as given by the following.

Lemma V.1.2. *The set* $\mathfrak{I}$ *is a two-sided ideal of* $\mathrm{End}(\mathscr{A})$*.*

Proof. Let $\alpha \in \mathfrak{I}$ and $\gamma \in \mathrm{End}(\mathscr{A})$. Then we have that $\mathrm{im}\,(\gamma\alpha) \subseteq \mathrm{im}\,\alpha$, which lies in a monogenic subalgebra, so that $\gamma\alpha \in \mathfrak{I}$.

Similarly, since $\operatorname{im}\alpha$ lies inside a monogenic subalgebra, there exists an element $r_\alpha \in A$ such that $\operatorname{im}\alpha \subseteq \langle r_\alpha \rangle$. Thus

$$\operatorname{im}(\alpha\gamma) = A\alpha\gamma \subseteq \langle r_\alpha \rangle\gamma = \langle r_\alpha\gamma \rangle,$$

which gives us that $\alpha\gamma \in \mathfrak{I}$. Therefore $\mathfrak{I}$ is a two-sided ideal of $\operatorname{End}(\mathscr{A})$. $\square$

In general, the fact that the image of a map α lies inside a monogenic subalgebra $\mathscr{B}$ does not imply that $\operatorname{im}\alpha = \mathscr{B}$ whenever $\operatorname{im}\alpha \neq \langle\emptyset\rangle$. However, this is the case in independence algebras as shown by:

Lemma V.1.3. *If $\mathscr{A}$ is an independence algebra, then for all $\alpha \in \operatorname{End}(\mathscr{A})$ we have that $\operatorname{im}\alpha \subseteq \langle r_\alpha \rangle$ for some $r_\alpha \in A$ if and only if $\operatorname{im}\alpha = \langle\emptyset\rangle$ or $\operatorname{im}\alpha = \langle r_\alpha \rangle$.*

Proof. Clearly, if $\operatorname{im}\alpha = \langle\emptyset\rangle$ or $\operatorname{im}\alpha = \langle r_\alpha \rangle$ where $r_\alpha \in A$, then $\operatorname{im}\alpha \subseteq \langle r_\alpha \rangle$.

Conversely, suppose that $\operatorname{im}\alpha \subseteq \langle r_\alpha \rangle$ for some $r_\alpha \in A$. If $r_\alpha \in \langle\emptyset\rangle$, we get that $\langle\emptyset\rangle \subseteq \operatorname{im}\alpha \subseteq \langle r_\alpha \rangle = \langle\emptyset\rangle$, hence $\operatorname{im}\alpha = \langle\emptyset\rangle$. Otherwise, r_α is an independent element. If for all $x \in \operatorname{im}\alpha$, we have that $x \in \langle\emptyset\rangle$, then $\operatorname{im}\alpha = \langle\emptyset\rangle$. Otherwise there exists an independent element $x \in \operatorname{im}\alpha \subseteq \langle r_\alpha \rangle$, so that $x = t(r_\alpha)$ for some $t \in \mathcal{T}^{\mathscr{A}}$ which shows that t is not constant on A by Lemma I.4.21, and thus there exists $u \in \mathcal{T}^{\mathscr{A}}$ such that $u \circ t = \mathrm{id}$ by Lemma I.4.23. Then we get that

$$r_\alpha = u(t(r_\alpha)) = u(x) \in \operatorname{im}\alpha,$$

which shows that $\langle r_\alpha \rangle \subseteq \operatorname{im}\alpha$, and therefore we have equality. $\square$

We now restrict ourselves to the framework of independence algebras. For the remainder of Section V.1, we therefore assume that $\mathscr{A}$ is an independence algebra of rank at least 2. Moreover we let $X = \{x_i\}$ be a basis of $\mathscr{A}$, where I is a (possibly infinite) indexing set.

Corollary V.1.4. *The set $\mathfrak{I}$ is the (0-)minimal ideal of* $\operatorname{End}(\mathscr{A})$.

Proof. Since $\mathscr{A}$ is an independence algebra, by Lemma V.1.3 we get that

$$\begin{aligned}\mathfrak{I} &= \{\alpha \in \operatorname{End}(\mathscr{A}) \mid \operatorname{im}\alpha = \langle\emptyset\rangle \text{ or } \operatorname{im}\alpha = \langle r_\alpha \rangle \text{ for some } r_\alpha \in A \setminus \langle\emptyset\rangle\} \\ &= \{\alpha \in \operatorname{End}(\mathscr{A}) \mid \operatorname{rk}(\alpha) \leq 1\}.\end{aligned}$$

Moreover, since the ideals of $\operatorname{End}(\mathscr{A})$ are of the form T_κ for some cardinal κ by Corollary I.5.7, and that they form a chain, it follows that $\mathfrak{I} = T_2$ is (0-)minimal. $\square$

Remark V.1.5. Even though we now have that $\mathfrak{I} = T_2$, we prefer to look at this ideal with respect to monogenic subalgebras since this will allow us to generalise the results obtained in the rest of this section to algebras outside of the context of independence algebra, an approach that we will take in Section V.3.

Since any endomorphism in $\mathrm{End}(\mathscr{A})$ is uniquely determined by its description on a basis of our algebra $\mathscr{A}$, those that sends all basis elements to a single element of A are of particular interest, and will be granted a specific notation.

Definition V.1.6. For each $i \in I$ and $r \in A$, we define the maps $\alpha_i, \alpha_r \in \mathrm{End}(\mathscr{A})$ for all $x_k \in X$ by:

$$x_k\alpha_i = x_i \qquad \text{and} \qquad x_k\alpha_r = r.$$

Under this definition, we immediately obtain the following facts:

Lemma V.1.7. *For any $i \in I$ and $r \in A$, we have that:*

1) $\alpha_i = \alpha_k$ *if and only if* $i = k$*, and* $\alpha_i = \alpha_r$ *if and only if* $r = x_i$*;*
2) $\mathrm{im}\,\alpha_i = \langle x_i \rangle$, $\mathrm{rk}(\alpha_i) = 1$ *and* $\alpha_i^2 = \alpha_i$ *since* α_i *is the identity map on* $\langle x_i \rangle$*;*
3) $\mathrm{im}\,\alpha_r = \langle r \rangle$ *so that* $\mathrm{rk}(\alpha_r) = 0$ *if* $r \in \langle \emptyset \rangle$ *and* $\mathrm{rk}(\alpha_r) = 1$ *otherwise;*
4) $\alpha_i, \alpha_r \in \mathfrak{I}$*;*
5) *for all* $\gamma \in \mathrm{End}(\mathscr{A})$*, we have* $x_i\gamma = x_i\alpha_i\gamma$*.*

We will now investigate the translational hull $\Omega(\mathfrak{I})$ of this (0-)minimal ideal as well as its semigroup of right translations $\mathrm{P}(\mathfrak{I})$, and their relations to the endomorphism monoid $\mathrm{End}(\mathscr{A})$ through the maps π_{P} and χ_{P}, in order to get an isomorphism χ by composition.

V.1.2 Linked pairs

Consider a linked pair $(\lambda, \rho) \in \Omega(\mathfrak{I})$. Then for any $x_i \in X$ and $\beta \in \mathfrak{I}$ we have that:

$$x_i\lambda\beta = x_i\alpha_i\lambda\beta = x_i\alpha_i\rho\beta. \tag{V.1.1}$$

Furthermore, if we take $\alpha \in \mathfrak{I}$ and denote for each $i \in I$ the term $t_i^\alpha \in \mathcal{T}^{\mathscr{A}}$ defined by $x_i\alpha = t_i^\alpha(\overline{x_j})$, then we have:

$$x_i\alpha\lambda\beta = t_i^\alpha(\overline{x_j})\lambda\beta = t_i^\alpha\left(\overline{x_j\lambda\beta}\right) = t_i^\alpha\left(\overline{x_j\alpha_j\rho\beta}\right) = t_i^\alpha(\overline{x_j\alpha_j\rho})\beta.$$

We have therefore shown the following:

Lemma V.1.8. *Let $(\lambda, \rho) \in \Omega(\mathfrak{I})$. Then the equation*

$$(x_i\alpha\rho)\beta = t_i^\alpha(\overline{x_j\alpha_j\rho})\beta \tag{V.1.2}$$

holds for all $\alpha, \beta \in \mathfrak{I}$ and all $i \in I$. □

Notation V.1.9. For a map $\alpha \in \mathrm{End}(\mathscr{A})$, we denote by $\mathcal{T}^\alpha$ the set of terms which correspond to the basis elements, that is,

$$\mathcal{T}^\alpha = \{t_i^\alpha \in \mathcal{T}^{\mathscr{A}} \mid x_i\alpha = t_i^\alpha(\overline{x_j}) \text{ for } x_i \in X\}.$$

Conversely, if equation (V.1.2) hold for a right translation ρ, then it is possible to create a left translation λ such that (λ, ρ) forms a linked pair.

Lemma V.1.10. *Let $\rho \in \mathrm{P}(\mathfrak{I})$. Then the map $\lambda\colon \mathfrak{I} \to \mathrm{End}(\mathscr{A})$ defined by*

$$x_i\lambda\gamma = x_i(\alpha_i\rho)\gamma$$

for all $x_i \in X$ and all $\gamma \in \mathfrak{I}$ is a left translation of $\mathfrak{I}$.

Furthermore, suppose that for all $\alpha, \beta \in \mathfrak{I}$ and $x_i \in X$ the equation

$$(x_i\alpha\rho)\beta = t_i^\alpha(\overline{x_j\alpha_j\rho})\beta \tag{V.1.2}$$

holds, where each $t_i^\alpha \in \mathcal{T}^\alpha$ is defined as above. Then $(\lambda, \rho) \in \Omega(\mathfrak{I})$, so that π_{P} is surjective.

Proof. First of all, let $\gamma \in \mathfrak{I}$ and $a \in A$. Then there exists a term u such that $a = u(\overline{x_i})$. From this, we have that

$$a\lambda\gamma = u(\overline{x_i})\lambda\gamma = u\Big(\overline{x_i\lambda\gamma}\Big) = u\Big(\overline{x_i(\alpha_i\rho)\gamma}\Big) = u(\overline{x_i\alpha_i\rho})\gamma \in A\gamma,$$

and thus $\operatorname{im}\lambda\gamma \subseteq \operatorname{im}\gamma$ which lies in a monogenic subalgebra of A. Therefore $\lambda\gamma \in \mathfrak{I}$.

Additionally, using the definition of λ and the associativity in $\mathrm{End}(\mathscr{A})$, we have that for any $\beta, \delta \in \mathfrak{I}$

$$x_i\lambda(\beta\delta) = x_i(\alpha_i\rho)(\beta\delta) = x_i(\alpha_i\rho\beta)\delta = (x_i\lambda\beta)\delta,$$

which shows that $\lambda(\beta\delta) = (\lambda\beta)\delta$ so that λ is a left translation of $\mathfrak{I}$.

Finally, if equation (V.1.2) holds, then for all $\alpha, \beta \in \mathfrak{I}$ we obtain

$$(x_i\alpha\rho)\beta = t_i^\alpha(\overline{x_j\alpha_j\rho})\beta = t_i^\alpha\Big(\overline{x_j\alpha_j\rho\beta}\Big) = t_i^\alpha\Big(\overline{x_j\lambda\beta}\Big) = t_i^\alpha(\overline{x_j})\lambda\beta = x_i\alpha\lambda\beta,$$

which gives us that $\alpha\rho\beta = \alpha\lambda\beta$, so that the pair (λ, ρ) is linked as required. □

V.1.3 Isomorphism of the projection map

We now look closely at the map π_{P}, which is easily shown to be injective in the context of independence algebras.

Lemma V.1.11. *The projection* $\pi_{\mathrm{P}}\colon \Omega(\mathfrak{I}) \to \mathrm{P}(\mathfrak{I})$ *is injective.*

Proof. Let $(\lambda, \rho), (\lambda', \rho) \in \Omega(\mathfrak{I})$. Since λ and λ' are linked to ρ, then for all $\beta \in \mathfrak{I}$, equation (V.1.1) gives us that

$$x_i\lambda\beta = x_i\alpha_i\lambda\beta = x_i\alpha_i\rho\beta = x_i\alpha_i\lambda'\beta = x_i\lambda'\beta,$$

which shows that $\lambda\beta = \lambda'\beta$, and thus $\lambda = \lambda'$. □

Remark V.1.12. In fact, we could have proved that π_{P} is injective using Lemma I.2.17 by arguing that $\mathfrak{I}$ is left reductive. Indeed, let $\beta, \delta \in \mathfrak{I}$ and suppose that $\alpha\beta = \alpha\delta$ for all $\alpha \in \mathfrak{I}$. Then for all $x_i \in X$, we get $x_i\beta = x_i\alpha_i\beta = x_i\alpha_i\delta = x_i\delta$, which shows that $\beta = \delta$, and thus $\mathfrak{I}$ is left reductive.

The equation exhibited in Lemma V.1.10 is in fact exactly the condition corresponding to π_{P} being an isomorphism, as we show below.

Proposition V.1.13. *The projection* $\pi_{\mathrm{P}}\colon \Omega(\mathfrak{I}) \to \mathrm{P}(\mathfrak{I})$ *is an isomorphism if and only if the independence algebra* $\mathscr{A}$ *satisfies the equation*

$$(x_i\alpha\rho)\beta = t_i^\alpha(\overline{x_j\alpha_j\rho})\beta \tag{$\star$}$$

for all $\alpha, \beta \in \mathfrak{I}$ *and* $\rho \in \mathrm{P}(\mathfrak{I})$*, with* $t_i^\alpha \in \mathscr{T}^\alpha$.

Proof. The fact that the map π_{P} is injective is given by Lemma V.1.11. Moreover, if $\mathscr{A}$ satisfies equation ($\star$), then π_{P} is surjective by Lemma V.1.10, so that $\Omega(\mathfrak{I}) \cong \mathrm{P}(\mathfrak{I})$.

Conversely, if π_{P} is an isomorphism, then it is surjective, and for all $\rho \in \mathrm{P}(\mathfrak{I})$, there exists $\lambda \in \Lambda(\mathfrak{I})$ such that $(\lambda, \rho) \in \Omega(\mathfrak{I})$. Using equation (V.1.1) together with Lemma V.1.8 we obtain that condition ($\star$) holds in $\mathscr{A}$. □

V.1.4 Isomorphism of induced right translation

We now focus our attention on the second isomorphism we are looking for, by considering the map $\chi_{\mathrm{P}}\colon \mathrm{End}(\mathscr{A}) \to \mathrm{P}(\mathfrak{I})$ which sends an endomorphism ϕ to the right translation ρ_ϕ defined by $\alpha\rho_\phi = \alpha\phi$ for all $\alpha \in \mathfrak{I}$. Since $\mathfrak{I}$ is an ideal of $\mathrm{End}(\mathscr{A})$, it follows from Corollary I.2.42 that χ_{P} is a well-defined homomorphism. Thus it only remains to show that it is a bijection.

Lemma V.1.14. *The map χ_{P} is injective. In other words $\rho_\phi = \rho_{\phi'}$ implies $\phi = \phi'$.*

Proof. Let $\phi, \phi' \in \mathrm{End}(\mathscr{A})$ and suppose that $\phi\chi_{\mathrm{P}} = \phi'\chi_{\mathrm{P}} = \rho$. Then for all $x_i \in X$, we get that $x_i\phi = x_i\alpha_i\phi = x_i\alpha_i\rho = x_i\alpha_i\phi' = x_i\phi'$, so that $\phi = \phi'$. Hence χ_{P} is injective. □

In order for the map χ_{P} to be surjective, we need each $\rho \in \mathrm{P}(\mathfrak{I})$ to behave like a homomorphism. In other words, we want the definition of ρ on any map $\alpha \in \mathfrak{I}$ to be uniquely determined by the definition of ρ on the maps α_i. This condition is given in the following lemma:

Lemma V.1.15. *The map χ_{P} is surjective if and only if the algebra $\mathscr{A}$ satisfies the equation*

$$x_i\alpha\rho = t_i^\alpha(\overline{x_j\alpha_j\rho}) \tag{V.1.3}$$

for all map $\alpha \in \mathfrak{I}$ and $\rho \in \mathrm{P}(\mathfrak{I})$, where $t_i^\alpha \in \mathcal{T}^\alpha$.

Proof. If χ_{P} is surjective, then we have that for any $\rho \in \mathrm{P}(\mathfrak{I})$ there exists $\phi \in \mathrm{End}(\mathscr{A})$ such that $\phi\chi_{\mathrm{P}} = \rho$. Thus, for any $\alpha \in \mathfrak{I}$ and its associated terms $t_i^\alpha \in \mathcal{T}^\alpha$ we get that

$$x_i\alpha\rho = x_i\alpha\phi = t_i^\alpha(\overline{x_j})\phi = t_i^\alpha\left(\overline{x_j\phi}\right) = t_i^\alpha\left(\overline{x_j\alpha_j\phi}\right) = t_i^\alpha(\overline{x_j\alpha_j\rho}),$$

that is, equation (V.1.3) holds for all $\alpha \in \mathfrak{I}$ and $\rho \in \mathrm{P}(\mathfrak{I})$.

Conversely, assume that $\mathscr{A}$ satisfies (V.1.3). For each $\rho \in \mathrm{P}(\mathfrak{I})$, we define $\phi_\rho \in \mathrm{End}(\mathscr{A})$ by $x_i\phi_\rho = x_i\alpha_i\rho$. Then for $\alpha \in \mathfrak{I}$, we have

$$x_i\alpha\phi_\rho = t_i^\alpha(\overline{x_j})\phi_\rho = t_i^\alpha\left(\overline{x_j\phi_\rho}\right) = t_i^\alpha\left(\overline{x_j\alpha_j\rho}\right) \overset{(\mathrm{V.1.3})}{=\!=} x_i\alpha\rho,$$

which shows that $\alpha\rho = \alpha\phi_\rho$ for all $\alpha \in \mathfrak{I}$. Therefore $\rho = \phi_\rho\chi_{\mathrm{P}}$ and χ_{P} is surjective. □

Using the previous lemmas, we can now show when our map χ_{P} is an isomorphism.

Proposition V.1.16. *The map $\chi_{\mathrm{P}} \colon \mathrm{End}(\mathscr{A}) \to \mathrm{P}(\mathfrak{I})$ is an isomorphism if and only if the independence algebra $\mathscr{A}$ satisfies equation (V.1.3) for all $\alpha \in \mathfrak{I}$ and $\rho \in \mathrm{P}(\mathfrak{I})$.*

Proof. Clearly, if χ_{P} is an isomorphism, then it is surjective, and we get by Lemma V.1.15 that equation (V.1.3) is satisfied.

Conversely, we know that χ_{P} is an injective homomorphism by Lemma V.1.14. Furthermore, if equation (V.1.3) holds, then it follows χ_{P} is also surjective by Lemma V.1.15, which means that it is an isomorphism of $\mathrm{End}(\mathscr{A})$ onto $\mathrm{P}(\mathfrak{I})$. □

Remark V.1.17. It is easy to see that equation (V.1.3) implies equation ($\star$), so that by Propositions V.1.13 and V.1.16, we get that if χ_{P} is an isomorphism, then π_{P} is an isomorphism as well.

V.1.5 Isomorphism between the endomorphism monoid and the translational hull of its (0-)minimal ideal

In view of the previous sections, we can now answer the original question, by giving a condition under which the endomorphism monoid of an independence algebra is isomorphic to the translational hull of its (0-)minimal ideal.

Theorem V.1.18. *Let $\mathscr{A}$ be an independence algebra with basis X, and consider the (0-)minimal ideal $\mathfrak{I}$ of its endomorphism monoid* $\mathrm{End}(\mathscr{A})$*, that is, the ideal $\mathfrak{I} := \{\alpha \in \mathrm{End}(\mathscr{A}) \mid \mathrm{rk}(\alpha) \leq 1\}$. Assume that $\mathscr{A}$ satisfies the condition*

$$x_i\alpha\rho = t_i^{\alpha}(\overline{x_j\alpha_j\rho}) \tag{$\star\star$}$$

for all $\alpha \in \mathfrak{I}$ and $\rho \in \mathrm{P}(\mathfrak{I})$, where $t_i^{\alpha} \in \mathcal{T}^{\alpha}$. Then the translational hull of $\mathfrak{I}$ is isomorphic to the endomorphism monoid of $\mathscr{A}$, that is, $\Omega(\mathfrak{I}) \cong \mathrm{End}(\mathscr{A})$.

Conversely, if $\mathrm{End}(\mathscr{A})$ is isomorphic to $\Omega(\mathfrak{I})$ through the composition of isomorphisms χ_{P} and π_{P}^{-1}, then the condition ($\star\star$) holds for all $\alpha \in \mathfrak{I}$ and $\rho \in \mathrm{P}(\mathfrak{I})$.

Proof. Notice that condition ($\star\star$) was denoted by (V.1.3) in the previous results. The theorem then follows directly from Remark V.1.17 together with Propositions V.1.13 and V.1.16. □

Remark V.1.19. From the fact that any $\alpha \in \mathfrak{I}$ with $\mathrm{im}\,\alpha \subseteq \langle\emptyset\rangle$ is a left-zero endomorphism by Remark I.5.5, and that any right translation must fix left zero elements by Lemma I.2.20, it follows that condition ($\star\star$) simply corresponds to the definition of the terms t_i^{α}, and thus equations ($\star\star$) and ($\star$) trivially hold for such endomorphisms. Furthermore, we will see in Lemma V.3.8 that the two conditions are equivalent if our ideal $\mathfrak{I}$ satisfies some sort of separation property on the algebra $\mathscr{A}$, namely that if $a \neq b \in A$, then there exists $\gamma \in \mathfrak{I}$ such that $a\gamma \neq b\gamma$. The fact that condition ($\star\star$) is strictly stronger is not obvious, but this will be shown in Corollary V.2.10.

V.2 APPLICATION TO SPECIFIC INDEPENDENCE ALGEBRAS

The equations ($\star$) and ($\star\star$) hide the structure of the algebras considered by bypassing the core of the computations, which makes it hard to see which algebras satisfy them. In order to illustrate the utility of Theorem V.1.18 we show in this section how it can be applied to different independence algebras, by showing which of the above conditions hold.

Because of the classification given in Proposition I.5.10 and the fact that we are only interested in independence algebras of rank at least 2, we will only give the result for some independence algebras since the result for the others can either be deduced from the corresponding E-equivalent algebras, or follows directly from the remarks given in the introduction of this chapter. The most common examples of independence algebras which are sets and vector spaces will be treated first, where we will show that the isomorphism between $\Omega(\mathfrak{I})$ and $\operatorname{End}(\mathscr{A})$ hold, but in different ways. In affine and quasi-field algebras however, the singular structure of one-dimensional subalgebras that they share makes condition ($\star\star$) fail to hold even though both types of algebras satisfy equation ($\star$). Nonetheless, it is possible to give an exact description of the translational hull of their minimal ideal.

V.2.1 FREE GROUP ALGEBRAS

Let $\mathscr{A} = F_{X,C}(G)$ be a free group algebra of a group G with basis X and set of constants C. From this, we get that

$$\begin{aligned}\mathfrak{I} &= \{\alpha \in \operatorname{End}(\mathscr{A}) \mid \operatorname{im}\alpha = C \text{ or } \operatorname{im}\alpha = \langle x_k\rangle \text{ for some } x_k \in X\}\\ &= \{\alpha \in \operatorname{End}(\mathscr{A}) \mid \operatorname{im}\alpha = C \text{ or } \operatorname{im}\alpha = G\cdot x_k \text{ for some } x_k \in X\}\,.\end{aligned}$$

Notice that if $\alpha \in \mathfrak{I}$ is such that $\operatorname{im}\alpha = C$, then $\alpha\beta = \alpha$ for all $\beta \in \mathfrak{I}$ and otherwise, there exists $k \in I$ such that $\alpha = \alpha\alpha_k$ where $\operatorname{im}\alpha = \langle x_k\rangle$.

With this observation, we obtain the following proposition:

Proposition V.2.1. *For any free group algebra $\mathscr{A} = F_{X,C}(G)$, we have an isomorphism $\Omega(\mathfrak{I}) \cong \operatorname{End}(\mathscr{A})$ via the maps π_{P} and χ_{P}.*

Proof. In order to prove the proposition, it suffices to show that condition ($\star\star$) of Theorem V.1.18 holds. From Remark V.1.19, it is enough to show this when $\operatorname{im}\alpha = \langle x_k\rangle$ for some $x_k \in X$. Hence, let $\alpha \in \mathfrak{I}$ be such that $\operatorname{im}\alpha = \langle x_k\rangle$ with

$x_k \in X$. Then, for any $x_i \in X$ we have that $x_i\alpha = g_i(x_k)$ for some $g_i \in G$ or $x_i\alpha = c_i \in \langle\emptyset\rangle$. Thus $x_i\alpha = t_i^\alpha(x_k)$ for some unary terms $t_i^\alpha \in \mathcal{T}^\alpha$, and we have for any $\rho \in \mathrm{P}(\mathfrak{I})$ that:

$$x_i(\alpha\rho) = x_i\left((\alpha\alpha_k)\rho\right) = x_i\alpha(\alpha_k\rho) = t_i^\alpha(x_k)\alpha_k\rho = t_i^\alpha(x_k\alpha_k\rho),$$

which shows that the condition ($\star\star$) holds as required. $\square$

V.2.2 Linear algebras

Let $\mathscr{A}$ be a linear algebra over a field $\mathscr{k}$ with basis $X = \{x_i\}$ and fixed subspace A_0. For each one-dimensional subalgebra $\mathscr{B}$ of $\mathscr{A}$, we pick an independent element r of B such that its projection on A_0 is 0, to be considered as the representative of $\mathscr{B}$ and we denote by $\mathcal{R} \supseteq X$ the set of all representatives of one-dimensional subalgebras of $\mathscr{A}$. In particular, this means that each one-dimensional subalgebra has a canonical basis, which corresponds to the singleton set containing its representative in $\mathcal{R}$. Using this notation, we have that:

$$\begin{aligned}\mathfrak{I} &= \{\alpha \in \mathrm{End}(\mathscr{A}) \mid \operatorname{im}\alpha = A_0 \text{ or } \operatorname{im}\alpha = \langle r\rangle \text{ for some } r \in \mathcal{R}\}\\ &= \left\{\alpha \in \mathrm{End}(\mathscr{A}) \mid \operatorname{im}\alpha = A_0 \text{ or } \operatorname{im}\alpha = \left\{\mu r + a \colon \mu \in \mathscr{k}, r \in \mathcal{R}, a \in A_0\right\}\right\}.\end{aligned}$$

In order to facilitate the computations, for each $i \in I$ we define $\beta_i \in \mathfrak{I}$ by

$$x_i\beta_i = x_i \qquad \text{and} \qquad x_k\beta_i = 0 \quad \text{for all } x_k \neq x_i,$$

where 0 is the zero of our vector space $\mathscr{A}$. Additionally, for any $r \in \mathcal{R}$ there exists a finite subset $J_r \subseteq I$ such that $r = \sum_{J_r} \nu_j x_j$ where $\{\nu_j\}_{J_r} \subseteq \mathscr{k}^*$, and we define $\gamma_r \in \mathfrak{I}$ by

$$x_i\gamma_r = \frac{1}{|J_r|}\nu_i^{-1}x_1 \quad \text{if } i \in J_r, \qquad \text{and} \qquad x_i\gamma_r = x_1 \quad \text{otherwise.}$$

In particular, $\operatorname{im}\gamma_r = \langle x_1\rangle$ as well as $r\gamma_r = x_1$ since

$$r\gamma_r = \sum_{j\in J_r} \nu_j x_j\gamma_r = \frac{1}{|J_r|}\sum_{j\in J_r}\nu_j\nu_j^{-1}x_1 = \frac{|J_r|}{|J_r|}x_1 = x_1,$$

and if $r \in X$, we have that $\gamma_r = \alpha_1$.

Lemma V.2.2. *For any $\alpha \in \mathfrak{I}$ with $\operatorname{im}\alpha \neq A_0$, we have that $\alpha = \alpha\gamma_r\alpha_r$, where $r \in \mathcal{R}$ is the representative of* $\operatorname{im}\alpha$*.*

Proof. Let $\alpha \in \mathfrak{I}$ with $\operatorname{im}\alpha \neq A_0$. Then there exists $r \in \mathcal{R}$ such that $\operatorname{im}\alpha = \langle r\rangle$. Moreover, for all $i \in I$ we have $x_i\alpha = a_i$ for some $a_i \in A_0$, or $x_i\alpha = \zeta_i r$ for some $\zeta_i \in \mathcal{k}$. If $x_i\alpha = a_i \in A_0$, then we directly have that $x_i\alpha\gamma_r\alpha_r = a_i\gamma_r\alpha_r = a_i = x_i\alpha$. Otherwise, we get that

$$x_i\alpha\gamma_r\alpha_r = \zeta_i r\gamma_r\alpha_r = \zeta_i x_1\alpha_r = \zeta_i r = x_i\alpha,$$

which shows that $\alpha = \alpha\gamma_r\alpha_r$ as required. □

Remark V.2.3. Let $\rho \in \mathrm{P}(\mathfrak{I})$ and $\alpha \in \mathfrak{I}$, then by Lemma V.2.2 we have that

$$\alpha\rho = (\alpha\gamma_r\alpha_r)\rho = \alpha\gamma_r(\alpha_r\rho),$$

for some $r \in \mathcal{R}$, which shows that the action of ρ on $\mathfrak{I}$ is uniquely determined by its action on the set $\{\alpha_r \in \mathfrak{I} \mid r \in \mathcal{R}\}$. Thus, it is enough to study the action of the right translations on this subset of $\mathfrak{I}$.

In order to obtain that $\Omega(\mathfrak{I}) \cong \operatorname{End}(\mathscr{A})$ by using Theorem V.1.18, we would need to show that condition $(\star\star)$ holds for all $\alpha \in \mathfrak{I}$. Notice first that if $r = \sum_{J_r} \nu_j x_j \in \mathcal{R}$, then for all $x_i \in X$ we have that

$$x_i\alpha_r = r = \sum_{J_r} \nu_j x_j = t_i^{\alpha_r}(\overline{x_j}).$$

Thus, for $(\star\star)$ to hold for α_r, we need

$$x_i\alpha_r\rho = t_i^{\alpha_r}(\overline{x_j\alpha_j\rho}) = \sum_{J_r} \nu_j(x_j\alpha_j\rho) \tag{V.2.1}$$

for all $\rho \in \mathrm{P}(\mathfrak{I})$. In fact, it suffices to verify this holds for all α_r with $r \in \mathcal{R}$, in order to get that $(\star\star)$ will hold for all $\alpha \in \mathfrak{I}$, as given by the following:

Lemma V.2.4. *Let $r \in \mathcal{R}$, $\alpha \in \mathfrak{I}$ with $\operatorname{im}\alpha = \langle r\rangle$ and $\rho \in \mathrm{P}(\mathfrak{I})$. If we have that $x_i\alpha_r\rho = t_i^{\alpha_r}(\overline{x_j\alpha_j\rho})$ for all $x_i \in X$ where $t_i^{\alpha_r} \in \mathcal{T}^{\alpha_r}$, then $x_i\alpha\rho = t_i^{\alpha}(\overline{x_j\alpha_j\rho})$ for all $x_i \in X$ where $t_i^{\alpha} \in \mathcal{T}^{\alpha}$.*

Consequently, condition $(\star\star)$ holds for all $\alpha \in \mathfrak{I}$ if and only if (V.2.1) holds for all $\alpha_r \in \mathfrak{I}$ and all $\rho \in \mathrm{P}(\mathfrak{I})$, where $r \in \mathcal{R}$.

Proof. Let $\alpha \in \mathfrak{I}$ with $\operatorname{im}\alpha = \langle r\rangle$ for some $r \in \mathcal{R}$. Then, for each $x_i \in X$, there exist $\mu_i \in \mathcal{k}$ and $a_i \in A_0$ such that

$$x_i\alpha = \mu_i r + a_i = \sum_{J_r} \mu_i\nu_j x_j + a_i =: t_i^{\alpha}(\overline{x_j}).$$

Moreover, using Lemma V.2.2, we get that $\alpha = \alpha\gamma_r\alpha_r$ and thus $\alpha\rho = \alpha\gamma_r(\alpha_r\rho)$. Let $\rho \in \mathrm{P}(\mathfrak{I})$ and suppose that $x_i\alpha_r\rho = t_i^{\alpha_r}(\overline{x_j\alpha_j\rho})$ for all $x_i \in X$, where $t_i^{\alpha_r} \in \mathcal{T}^{\alpha_r}$. In particular, by equation (V.2.1) above, we get that $x_1\alpha_r\rho = \sum_{J_r} \nu_j(x_j\alpha_j\rho)$. Therefore

$$\begin{aligned} x_i\alpha\rho = x_i\alpha\gamma_r(\alpha_r\rho) &= (\mu_i r + a_i)\gamma_r(\alpha_r\rho) \\ &= \mu_i(r\gamma_r(\alpha_r\rho)) + a_i = \mu_i(x_1\alpha_r\rho) + a_i \\ &= \mu_i\left(\sum_{J_r} \nu_j(x_j\alpha_j\rho)\right) + a_i = \sum_{J_r} \mu_i\nu_j(x_j\alpha_j\rho) + a_i \\ &= t_i^{\alpha}(\overline{x_j\alpha_j\rho}), \end{aligned}$$

as required, and the second part follows immediately from requiring this to hold for all $\rho \in \mathrm{P}(\mathfrak{I})$. $\square$

The approach to obtain the isomorphism $\Omega(\mathfrak{I}) \cong \mathrm{P}(\mathfrak{I}) \cong \mathrm{End}(\mathscr{A})$ by showing that equation (V.2.1) holds for all $\alpha_r \in \mathfrak{I}$ and all $\rho \in \mathrm{P}(\mathfrak{I})$ requires to know in the first place which maps from $\mathfrak{I}$ to $\mathfrak{I}$ are right translations. At this stage, there is no easy way of describing them directly without going through an extensive search of all the possible maps $\rho\colon \mathfrak{I} \to \mathfrak{I}$. This can possibly be achieved for small finite fields and low-dimensional vector spaces on a computer by using some program like GAP, but it quickly exceeds the capacities of any computer.

Nonetheless, it is possible to show that there is an isomorphism between the translational hull $\Omega(\mathfrak{I})$ and the endomorphism monoid $\mathrm{End}(\mathscr{A})$ directly, without requiring them to be isomorphic to the set of right translations $\mathrm{P}(\mathfrak{I})$. In order to do so, we first prove that equation V.2.1 is satisfied by all right translations which are part of a linked pair.

Lemma V.2.5. *Let $(\lambda, \rho) \in \Omega(\mathfrak{I})$. Let $i \in I$, $r \in \mathcal{R}$, $\gamma \in \mathfrak{I}$ and $x_j \in X$. Then we have:*

- $x_i\lambda\gamma = x_j\alpha_i\rho\gamma$;
- $r\lambda\gamma = x_j\alpha_r\rho\gamma$; *and*
- $x_i\alpha_r\rho = \sum_{J_r} \nu_j x_j\alpha_j\rho$ *where* $r = \sum_{J_r} \nu_j x_j$, *that is, condition (V.2.1) holds.*

Proof. Let $(\lambda, \rho) \in \Omega(\mathfrak{I})$. Then we have that

$$x_i\lambda\gamma = x_j\alpha_i\lambda\gamma = x_j\alpha_i\rho\gamma,$$

and if $r \in \mathcal{R}$ we also get:

$$r\lambda\gamma = x_j\alpha_r\lambda\gamma = x_j\alpha_r\rho\gamma.$$

Let $p \in \mathcal{R} \cup \{0\}$ be such that $\operatorname{im}(\alpha_r\rho) = \langle p\rangle$, and denote by $K_p \subseteq I$ the finite set such that $p = \sum_{K_p} \mu_k x_k$. Similarly, for each $j \in J_r$, there exists a finite set $K_j \subseteq I$ and an element $q_j \in \mathcal{R} \cup \{0\}$ such that $\operatorname{im}(\alpha_j\rho) = \langle q_j\rangle$ and $q_j = \sum_{K_j} \mu_k x_k$.

Set $K := \bigcup_{J_r} K_j \cup K_p \subseteq I$. Then K is finite and $\xi = \sum_K \beta_k \in \operatorname{End}(\mathscr{A})$ is a right identity for all maps in the set $\{\alpha_j\rho, \alpha_r\rho \mid j \in J_r\}$. Indeed, if we take $j \in J_r$, then for all $x_i \in X$ we have that $x_i\alpha_j\rho = \sum_{\ell \in K_j} \mu_\ell x_\ell + a_i$ for some $\mu_\ell \in \Bbbk$ and $a_i \in A_0$, and from this we get:

$$x_i\alpha_j\rho\xi = \Big(\sum_{\ell\in K_j} \mu_\ell x_\ell + a_i\Big)\xi = \sum_{\ell\in K_j} \mu_\ell \left(\sum_{k\in K} x_\ell\beta_k\right) + a_i = \sum_{\ell\in K_j} \mu_\ell x_\ell + a_i = x_i\alpha_j\rho,$$

so that $\alpha_j\rho\xi = \alpha_j\rho$, and similarly $\alpha_r\rho\xi = \alpha_r\rho$. Notice in particular that since $\beta_k \in \mathfrak{I}$ and $\alpha_j\rho$ is an endomorphism for all $j \in J_r$, it follows that

$$\alpha_j\rho = \alpha_j\rho\xi = \alpha_j\rho\left(\sum_{k\in K} \beta_k\right) = \sum_{k\in K} \alpha_j\rho\beta_k = \sum_{k\in K} \alpha_j\lambda\beta_k,$$

and similarly $\alpha_r\rho = \sum_{k\in K} \alpha_r\lambda\beta_k$.

We can now show that (V.2.1) holds for all right translations ρ belonging to a linked pair (λ, ρ). Indeed for any $x_i \in X$ we have:

$$\begin{aligned} x_i\alpha_r\rho &= \sum_{k\in K} x_i\alpha_r\lambda\beta_k = \sum_{k\in K} r\lambda\beta_k \\ &= \sum_{k\in K}\left(\sum_{j\in J_r} \nu_j x_j\right)\lambda\beta_k = \sum_{j\in J_r}\sum_{k\in K} \nu_j x_j\lambda\beta_k \\ &= \sum_{j\in J_r}\sum_{k\in K} \nu_j x_j\alpha_j\lambda\beta_k = \sum_{j\in J_r} \nu_j x_j\left(\sum_{k\in K} \alpha_j\lambda\beta_k\right) \\ &= \sum_{j\in J_r} \nu_j x_j\alpha_j\rho, \end{aligned}$$

as required. □

Using the previous lemma we can create the isomorphism we wanted.

Proposition V.2.6. *In any linear algebra $\mathscr{A}$, $\operatorname{End}(\mathscr{A})$ is isomorphic to $\Omega(\mathfrak{I})$ through the map $\chi : \phi \mapsto (\lambda_\phi, \rho_\phi)$, where $\lambda_\phi\alpha = \phi\alpha$ and $\alpha\rho_\phi = \alpha\phi$ for all $\alpha \in \mathfrak{I}$.*

Proof. Let $\chi\colon \operatorname{End}(\mathscr{A}) \to \Omega(\mathfrak{I})$ be defined as in the statement. Since $\mathfrak{I}$ is an ideal of $\operatorname{End}(\mathscr{A})$, it follows from Corollary I.2.42 that χ is well-defined homomorphism.

Let $\phi, \phi' \in \operatorname{End}(\mathscr{A})$ and suppose that $\phi\chi = \phi'\chi = (\lambda, \rho)$. Then for all $i \in I$, we get that $x_i\phi = x_i\alpha_i\phi = x_i\alpha_i\rho = x_i\alpha_i\phi' = x_i\phi'$, so that $\phi = \phi'$ and χ is injective.

Now consider $(\lambda, \rho) \in \Omega(\mathfrak{I})$ and define $\phi \in \operatorname{End}(\mathscr{A})$ by $x_i\phi = x_i\alpha_i\rho$ for all $i \in I$. Then $\phi\chi = (\lambda, \rho)$ if $(\lambda, \rho) = (\lambda_\phi, \rho_\phi)$, that is, if for all $\alpha \in \mathfrak{I}$ we have $\lambda\alpha = \phi\alpha$ and $\alpha\rho = \alpha\phi$. Notice that if we have $\alpha\rho = \alpha\phi$ for all $\alpha \in \mathfrak{I}$, then for all $\beta \in \mathfrak{I}$ and $i \in I$, we get

$$x_i\lambda\beta = x_i\alpha_i\rho\beta = x_i\alpha_i\phi\beta = x_i\phi\beta,$$

so that $\lambda\beta = \phi\beta$. Combining this with Remark V.2.3, it follows that if $\alpha_r\rho = \alpha_r\phi$ for all $r \in \mathcal{R}$ then we get that $\alpha\rho = \alpha\phi$ for all $\alpha \in \mathfrak{I}$ and we obtain $\phi\chi = (\lambda, \rho)$. So, let $r = \sum_{J_r} \nu_j x_j \in \mathcal{R}$. Then, using Lemma V.2.5 and the definition of ϕ, we have that for all $x_i \in X$

$$x_i\alpha_r\rho = \sum_{j\in J_r} \nu_j x_j \alpha_j \rho = \sum_{j\in J_r} \nu_j x_j \phi = \left(\sum_{j\in J_r} \nu_j x_j\right)\phi = r\phi = x_i\alpha_r\phi.$$

Hence $\alpha_r\rho = \alpha_r\phi$ for all $r \in \mathcal{R}$ and thus $\phi\chi = (\lambda, \rho)$. Therefore χ is surjective, which completes the proof that χ is an isomorphism from $\operatorname{End}(\mathscr{A})$ onto $\Omega(\mathfrak{I})$. □

Open Problem V.2.7. Proposition V.2.6 gives the explicit isomorphism between $\Omega(\mathfrak{I})$ and $\operatorname{End}(\mathscr{A})$ but does not provides an answer on whether these are isomorphic to $\mathrm{P}(\mathfrak{I})$. It is therefore natural to ask the following question:

> *Is there an equivalent condition to equation ($\star$) that only depends on the field $\mathfrak{k}$ and the fixed subspace A_0 in order to get that π_{P} is surjective?*

V.2.3 Algebras with singleton subalgebras

In this section we will treat both the quasifield algebras and the affine algebras since their one-dimensional subalgebras are singletons. Thus we consider an independence algebra $\mathscr{A}$ whose monogenic subalgebras are all singletons, and with a basis $X \neq A$ so that we are not in the situation of an algebra that is merely a set. This means in particular that A has at least three distinct elements and that there exist at least one element that does not lie in X. Then, the minimal ideal of $\operatorname{End}(\mathscr{A})$ is

$$\mathfrak{I} = \{\alpha \in \operatorname{End}(\mathscr{A}) \mid \operatorname{im}\alpha = \{a\} \text{ for some } a \in A\},$$

which consists of the constant maps γ_a defined as $b\gamma_a = a$ for all $b \in A$. Thus $\mathfrak{I}$ is a right-zero semigroup and if $x_i \in X$, then the map γ_{x_i} corresponds to the map α_i of Section V.1.

Remark V.2.8. From Proposition I.2.23, we have that any map from $\rho\colon \mathfrak{I} \to \mathfrak{I}$ is a right translation, linked to the only left translation, which is the identity map. Thus $\Omega(\mathfrak{I})$ is isomorphic to $\mathrm{P}(\mathfrak{I})$. This can also be seen from the fact that ($\star$) is straightforward to verify on right-zero semigroups and is proven here in generality.

Lemma V.2.9. *If an ideal $\mathfrak{I}$ of an independence algebra $\mathscr{A}$ is a right-zero semigroup, then condition ($\star$) is satisfied.*

Proof. Let $\mathfrak{I}$ be an ideal that is a right-zero semigroup, and X be a basis of $\mathscr{A}$ as usual. Take $\alpha \in \mathfrak{I}$ with $\mathcal{T}^\alpha$ defined as before. Then for all $\beta \in \mathfrak{I}$ and $x_i \in X$, we have the following:

$$\begin{aligned}
(x_i\alpha\rho)\beta = (x_i\alpha^2\rho)\beta = x_i\alpha(\alpha\rho\beta) & \qquad \text{since } \alpha^2 = \alpha \\
= x_i\alpha\beta & \qquad \text{since } \mathfrak{I} \text{ is a right-zero semigroup} \\
= t_i^\alpha(\overline{x_j})\beta = t_i^\alpha\left(\overline{x_j\beta}\right) & \\
= t_i^\alpha\left(\overline{x_j\alpha_j\rho\beta}\right) & \qquad \text{since } \mathfrak{I} \text{ is a right-zero semigroup} \\
= t_i^\alpha\left(\overline{x_j\alpha_j\rho}\right)\beta, &
\end{aligned}$$

which shows that equation ($\star$) holds. $\square$

On the other end, condition ($\star\star$) does not hold for all right translations ρ in these algebras, as given by the following counterexample.

Let $x \neq y \in X$, and let $a \in A \setminus \{x, y\}$ be such that $a = t(x, y)$ for some term t. Denote by γ_x, γ_y and γ_a the maps with image $\{x\}$, $\{y\}$ and $\{a\}$ respectively and let $\rho\colon \mathfrak{I} \to \mathfrak{I}$ be a right translation such that

$$\gamma_x\rho = \gamma_y\rho = \gamma_a \qquad \text{and} \qquad \gamma_a\rho = \gamma_x.$$

Now take any basis element $z \in X$ and consider $z\gamma_a = a = t(x, y)$. Then we obtain that $z\gamma_a\rho = z\gamma_x = x$ and

$$t(x\gamma_x\rho, y\gamma_y\rho) = t(x\gamma_a, y\gamma_a) = t(a, a) = a,$$

where the last equality comes from the fact that all monogenic subalgebras are singletons. This shows that $z\gamma_a\rho \neq t(x\gamma_x\rho, y\gamma_y\rho)$, so that the equation ($\star\star$) does not hold and Theorem V.1.18 cannot be applied here. This also shows the following corollary:

Corollary V.2.10. *For an independence algebra, conditions* ($\star$) *and* ($\star\star$) *are not equivalent in general.* □

In the current case, it is however possible to explicitly compute the translational hull of $\mathfrak{I}$, as given by the following lemma.

Lemma V.2.11. *For any independence algebra $\mathscr{A}$ that is not a set where each monogenic subalgebra is a singleton, the translational hull $\Omega(\mathfrak{I})$ is isomorphic to the full transformation monoid $\mathcal{T}_A$.*

Proof. Notice first that since $\mathfrak{I} = \{\gamma_a \mid a \in A\}$ and $\gamma_a \neq \gamma_b$ if $a \neq b$, it immediately follows that $\mathfrak{I}$ is in bijection with A. Since $\mathfrak{I}$ is a right-zero semigroup, we get that $\Omega(\mathfrak{I}) \cong \mathcal{T}_{\mathfrak{I}}$ by Proposition I.2.23, and thus $\Omega(\mathfrak{I}) \cong \mathcal{T}_A$. □

V.3 EXTENDING THE RESULTS

Since the initial question was asked in the setting of independence algebras, it was natural to look for an answer inside this framework. Nonetheless, it directly prompts the question of whether the above results are only valid on such algebras, or if they are true for a more general class of algebras. As it happens, many of the arguments used in Section V.1 only rely on the fact that our endomorphisms can be uniquely defined from their action on a basis of our algebra, which gives an easy way to extend this to algebras that are freely generated on a basis. This will be the purpose of Section V.3.1. Another direction to extend the earlier results in the context of an independence algebra is to look at other ideals rather than focusing on the (0-)minimal one. In Section V.3.2.1 we show that we can get some information about the translational hull of other ideals from the translational hull of the ideals of lower ranks, and that most of these translational hulls are in fact isomorphic to the endomorphism monoid. However, in order to state these results for other classes of universal algebras where we cannot define properly the endomorphisms α_i, we need to look at our conditions under a different light, which will be the goal of Section V.3.2.3.

V.3.1 Algebras free on a basis

Most of the results contained in Section V.1 hold in a setting broader than independence algebras. Indeed, let $\mathscr{A}$ be a free algebra over a set $X = \{x_i\} \subseteq A$, that is, an algebra generated by X in such a way that any map from X to A extends to an endomorphism of $\mathscr{A}$, and define the set $\mathfrak{I}$ as before by

$$\mathfrak{I} = \{\alpha \in \mathrm{End}(\mathscr{A}) \mid \mathrm{im}\,\alpha \text{ lies inside a monogenic subalgebra}\}\,.$$

By Lemma V.1.2 we have that $\mathfrak{I}$ is an ideal of $\mathrm{End}(\mathscr{A})$. However, this ideal is no longer necessarily (0-)minimal since there could exist ideals different from $\mathfrak{O}$ strictly contained in $\mathfrak{I}$ (see Example V.3.3 below). This happens in particular whenever having $\mathrm{im}\,\alpha \subseteq \langle r\rangle$ for some $\alpha \in \mathrm{End}(\mathscr{A})$ and $r \in A$ does not imply $\mathrm{im}\,\alpha = \langle\emptyset\rangle$ or $\mathrm{im}\,\alpha = \langle r\rangle$, or equivalently, when there exists $t \in \mathcal{T}_1^{\mathscr{A}}$ such that $r \notin \langle t(r)\rangle$ for some $r \in A$, that is, when the set of non-constant unary terms in $\mathcal{T}^{\mathscr{A}}$ is not a group. Notice also that the notion of rank of a subalgebra is not well-defined, since not all minimal generating sets have the same cardinality (for example $\{1\}$ and $\{2,3\}$ are both minimal generating sets for $\mathbb{Z}$), and thus the notion of rank for an endomorphism is not well-defined either.

Nonetheless, the maps α_i and α_r introduced in Definition V.1.6 as well as their properties given in Lemma V.1.7 still hold (except for the mention of the rank of α_r), and $\mathfrak{I}$ has some sort of minimality property, as given by:

Lemma V.3.1. *The set $\mathfrak{I}$ is the smallest ideal of* $\mathrm{End}(\mathscr{A})$ *containing at least one of the α_i.*

Proof. Since $\mathfrak{I}$ is an ideal of $\mathrm{End}(\mathscr{A})$ by Lemma V.1.2 and clearly contains all of the α_i, it suffices to show this is the smallest ideal that contains at least one of them.

Let $\mathfrak{T}$ be an ideal of $\mathrm{End}(\mathscr{A})$ containing one of the α_i, which we denote by α_k for some $k \in I$. Then we have that $\alpha_i = \alpha_k\alpha_i \in \mathfrak{T}$ for all $i \in I$ since $\mathfrak{T}$ is an ideal. In particular, the ideal $\mathfrak{T}$ is independent of the initial choice of α_k. Similarly, we also get that $\alpha_r = \alpha_k\alpha_r \in \mathfrak{T}$ for all $r \in A$.

Thus, we now consider $\mathfrak{M}$ as the smallest ideal of $\mathrm{End}(\mathscr{A})$ containing all of the α_i and we let $\xi \in \mathfrak{I}$. Then $\mathrm{im}\,\xi \subseteq \langle r\rangle$ for some element $r \in A$ and for all $x_i \in X$ we get that $x_i\xi = t_i(r)$ for some $t_i \in \mathcal{T}^{\mathscr{A}}$. Pick an element $x_k \in X$ and define $\beta \in \mathfrak{I}$ by $x_i\beta = t_i(x_k)$ for all $x_i \in X$. Then we have

$$x_i\beta\alpha_r = t_i(x_k)\alpha_r = t_i(x_k\alpha_r) = t_i(r) = x_i\xi$$

for all $x_i \in X$, which shows that $\xi = \beta\alpha_r$. Using the fact that $\alpha_r \in \mathfrak{M}$ by the previous point and that $\mathfrak{M}$ is an ideal of $\mathrm{End}(\mathscr{A})$, it follows that $\xi \in \mathfrak{M}$. This shows that $\mathfrak{I} \subseteq \mathfrak{M}$ and then, by minimality of $\mathfrak{M}$, it follows that $\mathfrak{I} = \mathfrak{M}$, so that $\mathfrak{I}$ is the smallest ideal of $\operatorname{End}(\mathscr{A})$ containing at least one, and hence all, of the α_i. □

An astute reader can now verify that all the subsequent results contained in Sections V.1.2 to V.1.5 hold without any modifications of the proofs if they are stated within the setting of an algebra freely generated on a basis. This leads us to the more general result:

Theorem V.3.2. *Let $\mathscr{A}$ be a universal algebra freely generated on a basis X, and consider the ideal $\mathfrak{I} = \{\alpha \in \operatorname{End}(\mathscr{A}) \mid \operatorname{im}\alpha \subseteq \langle r\rangle$ for some $r \in A\}$. Assume that $\mathscr{A}$ satisfies the condition*

$$x_i\alpha\rho = t_i^{\alpha}(\overline{x_j\alpha_j\rho}) \tag{V.3.1}$$

for all $\alpha \in \mathfrak{I}$ and $\rho \in \mathrm{P}(\mathfrak{I})$, where $t_i^{\alpha} \in \mathcal{T}^{\alpha}$. Then the translational hull of $\mathfrak{I}$ is isomorphic to the endomorphism monoid of $\mathscr{A}$, that is, $\Omega(\mathfrak{I}) \cong \operatorname{End}(\mathscr{A})$.

Conversely, if $\Omega(\mathfrak{I})$, $\mathrm{P}(\mathfrak{I})$ and $\operatorname{End}(\mathscr{A})$ are all isomorphic to each other, then the condition (V.3.1) holds for all $\alpha \in \mathfrak{I}$ and $\rho \in \mathrm{P}(\mathfrak{I})$.

Example V.3.3. A good example of it would be to consider the free action of a monoid M on a set X, denoted by $F_X(M)$, which provides a generalisation of the example of free group algebras exposed earlier in Section V.2.1. More precisely, $F_X(M)$ is the algebra composed of the set of all formal elements $\{mx_i \mid m \in M \text{ and } x_i \in X\}$ together with the set of unary operations $\{f_a \mid a \in M\}$ defined as $f_a(mx_i) = (am)x_i$ for all $m \in M$ and $x_i \in X$. It is clear from the description that all monogenic subalgebras of $F_X(M)$ are of the form $\langle ax_k\rangle$ for some $a \in M$ and $x_k \in X$ and that these are not all generated by elements of the basis unless M is a group.

Now if we let $\mathfrak{I}$ be defined as above and $\alpha \in \mathfrak{I}$, then there exist $m_\alpha \in M$ and $x_k \in X$ such that $\operatorname{im}\alpha = \langle m_\alpha x_k\rangle$. Moreover, for any $x_i \in X$, we have that $x_i\alpha = f_{a_i}(m_\alpha x_k)$ for some $a_i \in M$ and thus $x_i\alpha = f_{a_i m_\alpha}(x_k)$. This shows that if m_α is not left invertible in M, then the ideal generated by α will not contain the map α_k, and hence $\mathfrak{I}$ is not a minimal ideal. However, we still have that the translational hull of $\mathfrak{I}$ is isomorphic to the whole endomorphism monoid. Indeed, for any $\rho \in \mathrm{P}(\mathfrak{I})$, we have that

$$\begin{aligned} x_i(\alpha\rho) &= x_i\left((\alpha\alpha_k)\rho\right) = (x_i\alpha)\alpha_k\rho \\ &= f_{a_i}(m_\alpha x_k)\alpha_k\rho = f_{a_i m_\alpha}(x_k)\alpha_k\rho = f_{a_i m_\alpha}(x_k\alpha_k\rho). \end{aligned}$$

Therefore, condition ($\star\star$) of Theorem V.1.18 holds, and thus $\Omega(\mathfrak{I}) \cong \operatorname{End}(F_X(M))$.

V.3.2 Further considerations

In this section, we go back to the setting where $\mathscr{A}$ is an arbitrary independence algebra with a basis $X = \{x_i\}$.

V.3.2.1 Additional results

Recall from Section I.5 that in $\operatorname{End}(\mathscr{A})$, all maps of a given rank k form a single $\mathscr{D}$-class denoted D_k and that the ideals of $\operatorname{End}(\mathscr{A})$ have the form

$$T_\kappa = \{\alpha \in \operatorname{End}(\mathscr{A}) \mid \operatorname{rk}(\alpha) < \kappa\} = \bigcup_{\mu<\kappa} D_\mu,$$

for any given cardinal $\kappa > 0$.

In Section V.1 we only dealt with the (0-)minimal ideal $\mathfrak{I}$ of $\operatorname{End}(\mathscr{A})$. It is natural to wonder if the translational hull of another ideal T_κ can be compared to that of $\mathfrak{I}$. We know that if we consider $\operatorname{End}(\mathscr{A})$ as an ideal in itself, then it is isomorphic to its translational hull by Lemma I.2.15 since it is a monoid. Moreover, we have shown in Section V.2 that in the context of independence algebras, either $\Omega(\mathfrak{I}) \cong \operatorname{End}(\mathscr{A})$ or $\Omega(\mathfrak{I}) \cong \mathcal{T}_A$. We aim to show that in fact, if the former happens, we have $\Omega(T_\kappa) \cong \operatorname{End}(\mathscr{A})$ for all $\kappa \geq 2$.

We start by giving a useful lemma which shows that left and right translations cannot map an endomorphism to one of higher rank.

Lemma V.3.4. *Let $\mathfrak{T} = T_\kappa$ be an ideal of* $\operatorname{End}(\mathscr{A})$*. Then for any $\alpha \in \mathfrak{T}$ we have that* $\operatorname{rk}(\alpha\rho) \leq \operatorname{rk}(\alpha)$ *[resp.* $\operatorname{rk}(\lambda\alpha) \leq \operatorname{rk}(\alpha)$*] for all $\rho \in \mathrm{P}(\mathfrak{T})$ [resp. for all $\lambda \in \Lambda(\mathfrak{T})$].*

Consequently, for all $\rho \in \mathrm{P}(\mathfrak{T})$ and $\lambda \in \Lambda(\mathfrak{T})$ we have that $T_k\rho, \lambda T_k \subseteq T_k$ for all cardinals $k \leq \kappa$.

Proof. Let $\mathfrak{T} = T_\kappa$ be an ideal of $\operatorname{End}(\mathscr{A})$ and consider a map $\alpha \in T_k$ with $k \leq \kappa$. Since $\operatorname{End}(\mathscr{A})$ is regular by Proposition I.5.1, it follows that $\alpha = \alpha\gamma\alpha$ for some $\gamma \in \operatorname{End}(\mathscr{A})$. Since $\operatorname{rk}(\gamma\alpha) \leq \operatorname{rk}(\alpha) < k$, we get that $\gamma\alpha \in \mathfrak{T}$.

Now, let $\rho \in \mathrm{P}(\mathfrak{T})$ and $\lambda \in \Lambda(\mathfrak{T})$. Then we have the following:

$$\begin{aligned} \operatorname{rk}(\alpha\rho) &= \operatorname{rk}((\alpha\gamma\alpha)\rho) = \operatorname{rk}(\alpha\gamma(\alpha\rho)) \leq \operatorname{rk}(\alpha) < k, \quad \text{and} \\ \operatorname{rk}(\lambda\alpha) &= \operatorname{rk}(\lambda(\alpha\gamma\alpha)) = \operatorname{rk}((\lambda\alpha)\gamma\alpha) \leq \operatorname{rk}(\alpha) < k, \end{aligned}$$

which shows that $\alpha\rho, \lambda\alpha \in T_k$ as required. □

Now we can prove that the monoid of right translations of a larger ideal embeds in the monoid of right translations of a smaller one.

Lemma V.3.5. *For each* $1 < k < \dim \mathscr{A}$ *and* $k < \nu \leq (\dim \mathscr{A})^+$*, the monoid* $\mathrm{P}(T_\nu)$ *embeds into* $\mathrm{P}(T_k)$ *through the morphism* $\theta\colon \rho \mapsto \rho|_{T_k}$*.*

Proof. Let $1 < k < \dim \mathscr{A}$, $k < \nu \leq (\dim \mathscr{A})^+$ and let $\rho \in \mathrm{P}(T_\nu)$. Consider the map $\widetilde{\rho} = \rho|_{T_k}$. By Lemma V.3.4 we know that for any $\alpha \in T_k$ we have $\alpha\widetilde{\rho} \in T_k$, so that $\operatorname{im}\widetilde{\rho} \subseteq T_k$. Moreover, for any $\alpha, \beta \in T_k$, we have $\alpha\beta \in T_k$ and thus

$$(\alpha\beta)\widetilde{\rho} = (\alpha\beta)\rho = \alpha(\beta\rho) = \alpha(\beta\widetilde{\rho}),$$

which shows that $\widetilde{\rho} \in \mathrm{P}(T_k)$. Hence the map $\theta\colon \mathrm{P}(T_\nu) \to \mathrm{P}(T_k)$ is well-defined. The fact that this is also a morphism follows directly from the composition in $\mathrm{P}(T_\nu)$ and Lemma V.3.4.

Now let $\rho' \in \mathrm{P}(T_\nu)$ be such that $\rho'\theta = \widetilde{\rho} = \rho\theta$. For $\alpha \in T_\nu$, we let $Y = \{y_j\}$ be a basis of $\mathscr{B} = \operatorname{im}\alpha$, which we extend via $Z = \{z_s\}$ to a basis of A. Since $|Y| \geq 1$, we define a map $\gamma \in T_\nu$ by:

$$\gamma = \begin{pmatrix} y_j & z_s \\ y_j & y_1 \end{pmatrix}.$$

Then clearly, we have that γ is idempotent and $\alpha\gamma = \alpha$ since γ is the identity on $\operatorname{im}\alpha$. Moreover, for all $y_j \in Y$, the map γ is the identity on the one-dimensional subalgebra $\langle y_j \rangle = \operatorname{im}\alpha_{y_j}$, so that $\alpha_{y_j}\gamma = \alpha_{y_j} \in T_2 \subseteq T_k$. Thus we obtain that:

$$\begin{aligned} y_j\gamma\rho &= x_1\alpha_{y_j}(\gamma\rho) = x_1(\alpha_{y_j}\gamma)\rho \\ &= x_1\alpha_{y_j}\rho = x_1\alpha_{y_j}\widetilde{\rho} \\ &= x_1\alpha_{y_j}\rho' = x_1(\alpha_{y_j}\gamma)\rho' \\ &= y_j\gamma\rho'. \end{aligned}$$

Using the fact that $\gamma^2 = \gamma$ together with the previous equation, we see that

$$\begin{aligned} z_s\gamma\rho &= z_s\gamma(\gamma\rho) = y_1\gamma\rho \\ &= y_1\gamma\rho' = z_s\gamma(\gamma\rho') \\ &= z_s\gamma\rho' \end{aligned}$$

for all $z_s \in Z$. This means in particular that $\gamma\rho$ and $\gamma\rho'$ agree on the basis $Y \sqcup Z$ of A and thus $\gamma\rho = \gamma\rho'$. Finally,

$$\alpha\rho = (\alpha\gamma)\rho = \alpha(\gamma\rho) = \alpha(\gamma\rho') = (\alpha\gamma)\rho' = \alpha\rho',$$

which shows that $\rho = \rho'$ since $\alpha \in T_\nu$ was chosen arbitrarily. Therefore θ is injective and is an embedding. $\square$

Corollary V.3.6. *If* $\mathrm{End}(\mathscr{A}) \cong \mathrm{P}(T_k)$ *via* $\phi \mapsto \rho_\phi$ *for some* $1 < k \leq \dim \mathscr{A}$, *then* $\mathrm{P}(T_\nu) \cong \mathrm{End}(\mathscr{A})$ *for all* $k < \nu \leq (\dim \mathscr{A})^+$.

Proof. Notice first that for any ideal T of $\mathrm{End}(\mathscr{A})$, by Corollary I.2.42, we know that there is a well-defined morphism $\chi_{\mathrm{P}} \colon \mathrm{End}(\mathscr{A}) \to \mathrm{P}(T) \colon \phi \mapsto \rho_\phi$, where ρ_ϕ is defined by $\alpha\rho_\phi = \alpha\phi$ for all $\alpha \in T$. Thus, in order to distinguish the morphism for T_ν from that for T_k, we will write the cardinal associated to each ideal in exponent of the morphism and the right translation, that is, χ_{P}^ν will denote the map sending $\phi \in \mathrm{End}(\mathscr{A})$ to $\rho_\phi^\nu \in \mathrm{P}(T_\nu)$, while χ_{P}^k will be for the map sending $\phi \in \mathrm{End}(\mathscr{A})$ to $\rho_\phi^k \in \mathrm{P}(T_k)$.

Now assume that $\mathrm{End}(\mathscr{A}) \cong \mathrm{P}(T_k)$ via χ_{P}^k and let $\theta \colon \mathrm{P}(T_\nu) \to \mathrm{P}(T_k)$ be defined by $\rho^\nu\theta = \rho^\nu|_{T_k}$. Since for all $\alpha \in T_k$, we have

$$\alpha\rho_\phi^\nu|_{T_k} = \alpha\phi = \alpha\rho_\phi^k,$$

and that χ_{P}^k is an isomorphism, it follows that $(\chi_{\mathrm{P}}^k)^{-1}$ is injective and thus $\rho_\phi^\nu|_{T_k} = \rho_\phi^k$ as $\rho_\phi^k(\chi_{\mathrm{P}}^k)^{-1} = \phi$. Hence $\rho_\phi^\nu\theta = \rho_\phi^k$.

Define $\tau \colon \mathrm{P}(T_\nu) \to \mathrm{End}(\mathscr{A})$ by $\tau = \theta(\chi_{\mathrm{P}}^k)^{-1}$. Since both θ and $(\chi_{\mathrm{P}}^k)^{-1}$ are injective morphisms by Lemma V.3.5 and assumption on the fact that χ_{P}^k is an isomorphism, it follows that τ is an injective morphism. Now, let $\phi \in \mathrm{End}(\mathscr{A})$ and consider $\rho_\phi^\nu \in \mathrm{P}(T_\nu)$ defined by $\alpha\rho_\phi^\nu = \alpha\phi$ for all $\alpha \in T_\nu$. Then we have

$$\rho_\phi^\nu\tau = \rho_\phi^\nu\theta(\chi_{\mathrm{P}}^k)^{-1} = \rho_\phi^k(\chi_{\mathrm{P}}^k)^{-1} = \phi,$$

which shows that τ is surjective, and therefore we have that τ is an isomorphism of $\mathrm{P}(T_\nu)$ onto $\mathrm{End}(\mathscr{A})$ whose inverse is χ_{P}^ν. □

V.3.2.2 Equivalence of conditions ($\star$) *and* ($\star\star$)

We have seen earlier in Corollary V.2.10 that condition ($\star$) of Proposition V.1.13 is not equivalent in general to condition ($\star\star$) of Theorem V.1.18. It is therefore natural to ask when this is the case. Since the latter condition implies the former, we only need to find requirements on our algebra so that the converse also holds. In other words, given an independence algebra $\mathscr{A}$ on a basis X with $\mathfrak{I}$ the (0-)minimal ideal of $\mathrm{End}(\mathscr{A})$, and given any $\alpha \in \mathfrak{I}$ and $\rho \in \mathrm{P}(\mathfrak{I})$, we want to get a condition under which we have that

$$(x_i\alpha\rho)\beta = t_i^\alpha(\overline{x_j\alpha_j\rho})\beta \quad \forall\beta \in \mathfrak{I}, \tag{V.3.2}$$

implies that

$$x_i\alpha\rho = t_i^\alpha(\overline{x_j\alpha_j\rho}) \tag{V.3.3}$$

where $x_i \in X$ and $t_i^\alpha \in \mathcal{T}^\alpha$.

In order to do so, we introduce the concept of separative ideals.

Definition V.3.7. For an algebra $\mathscr{A}$, an ideal $\mathfrak{T}$ of $\mathrm{End}(\mathscr{A})$ is called *separative for* A if any two distinct elements of A can be separated by an element of $\mathfrak{T}$, that is, if for all $a \neq b \in A$, there exists $\gamma \in \mathfrak{T}$ such that $a\gamma \neq b\gamma$.

We can now give an answer to the question above, as follows.

Lemma V.3.8. *Let $\mathscr{A}$ be an independence algebra and $\mathfrak{I}$ the (0-)minimal ideal of its endomorphism monoid* $\mathrm{End}(\mathscr{A})$*. If $\mathfrak{I}$ is separative for A, then conditions* (V.3.3) *and* (V.3.2) *are equivalent.*

Proof. Suppose that $\mathfrak{I}$ is separative for A. One direction for the equivalence of the conditions is clear, so we want to show that if equation (V.3.2) holds, then so does (V.3.3). By contradiction, let us assume that (V.3.2) holds and that there exist $x_i \in X$, $\alpha \in \mathfrak{I}$ and $\rho \in \mathrm{P}(\mathfrak{I})$ such that $x_i\alpha\rho \neq t_i^\alpha(\overline{x_j\alpha_j\rho})$. Now set $a = x_i\alpha\rho \in A$ and $b = t_i^\alpha(\overline{x_j\alpha_j\rho}) \in A$. Thus $a \neq b$ and by assumption on $\mathfrak{I}$ being separative for A, there exists $\gamma \in \mathfrak{I}$ such that $a\gamma \neq b\gamma$. But this contradicts equation (V.3.2), which shows that the two conditions are equivalent in this case. □

V.3.2.3 Towards a more general approach

The arguments developed in Section V.1 required both $\Omega(\mathfrak{I})$ and $\mathrm{End}(\mathscr{A})$ to be isomorphic to $\mathrm{P}(\mathfrak{I})$. However, this requirement appears to be unnecessarily strong, since we showed that in the linear algebra case $\Omega(\mathfrak{I}) \cong \mathrm{End}(\mathscr{A})$, without showing that they are isomorphic to $\mathrm{P}(\mathfrak{I})$. Thus, we want to find other conditions that will allow us to appropriately decide when is the translational hull of $\mathfrak{I}$ isomorphic to the endomorphism monoid $\mathrm{End}(\mathscr{A})$.

The first thing to notice is that for any ideal $\mathfrak{T} \subseteq \mathrm{End}(\mathscr{A})$, we have that the maps $\pi_{\mathrm{P}} \colon \Omega(\mathfrak{T}) \to \mathrm{P}(\mathfrak{T})$ and $\chi_{\mathrm{P}} \colon \mathrm{End}(\mathscr{A}) \to \mathrm{P}(\mathfrak{T})$ are both injective morphisms by Lemmas V.1.11 and V.1.14. In fact, the proofs for these results used the existence of the special maps α_i which exist as long as our algebra is free on a basis. Nevertheless, this can be generalised by introducing a new notion as follows.

Definition V.3.9. Let $\mathscr{A}$ be an algebra, and $\mathfrak{T}$ an ideal of $\mathrm{End}(\mathscr{A})$. We say that $\mathscr{A}$ is *representable by* $\mathfrak{T}$ if $A = \bigcup_{\gamma \in \mathfrak{T}} \operatorname{im} \gamma$.

In other words, if $\mathscr{A}$ is representable by $\mathfrak{T}$, we have that any element of A lies in the image of some $\gamma \in \mathfrak{T}$, that is, any element $a \in A$ can be written as $a = b\gamma$ for some $b \in A$ and $\gamma \in \mathfrak{T}$.

Lemma V.3.10. *If $\mathscr{A}$ is representable by $\mathfrak{T}$, then the maps $\pi_{\mathrm{P}} \colon \Omega(\mathfrak{T}) \to \mathrm{P}(\mathfrak{T})$ and $\chi_{\mathrm{P}} \colon \mathrm{End}(\mathscr{A}) \to \mathrm{P}(\mathfrak{T})$ are injective.*

Proof. Consider an element $a \in A$. Since $\mathscr{A}$ is representable by $\mathfrak{T}$, we get that $a = b\gamma$ for some $b \in A$ and $\gamma \in \mathfrak{T}$.

To see that π_{P} is injective, let $(\lambda, \rho), (\lambda', \rho) \in \Omega(\mathfrak{T})$. Since λ and λ' are both linked to ρ, then for all $\beta \in \mathfrak{T}$ we obtain that

$$a\lambda\beta = b\gamma\lambda\beta = b\gamma\rho\beta = b\gamma\lambda'\beta = a\lambda'\beta.$$

Since $a \in A$ was chosen arbitrarily, this shows that $\lambda\beta = \lambda'\beta$ for all $\beta \in \mathfrak{T}$, and thus $\lambda = \lambda'$.

Similarly, suppose that $\phi\chi_{\mathrm{P}} = \phi'\chi_{\mathrm{P}} = \rho$ for some $\phi, \phi' \in \mathrm{End}(\mathscr{A})$. Then we get that $a\phi = b\gamma\phi = b\gamma\rho = b\gamma\phi' = a\phi'$, and therefore $\phi = \phi'$, finishing the proof that χ_{P} is injective. $\square$

In light of Lemma V.3.10, we can have injectivity of the maps π_{P} and χ_{P} using a condition on an algebra relative to an ideal of its endomorphism monoid. Since this can be expressed in the general context of universal algebras, it makes more sense to consider conditions of that form rather than those presented in ($\star$) and ($\star\star$).

Moreover, since the images of π_{P} and χ_{P} both lie in $\mathrm{P}(\mathfrak{T})$, it seems a better approach to exhibit conditions under which they will coincide in $\mathrm{P}(\mathfrak{T})$, without requiring them to be equal to the full monoid $\mathrm{P}(\mathfrak{T})$. This is the approach we will take in Chapter VI in order to generalise the results developed throughout Section V.1.

— VI —

Translational hulls of ideals of the endomorphism monoid of a universal algebra

Following the ideas developed in Chapter V, we wish to study the translational hulls of ideals of the endomorphism monoid of a universal algebra $\mathscr{A}$ in order to find out when this translational hull is isomorphic to the endomorphism monoid $\operatorname{End}(\mathscr{A})$. More precisely, given an ideal $\mathfrak{I}$ of $\operatorname{End}(\mathscr{A})$, we will consider two properties of $\mathfrak{I}$ relative to $\mathscr{A}$, called REP and SEP, which are very similar to those introduced in Section V.3.2.3, in the sense that REP is concerned with the property that elements of A can be written using the image of endomorphisms in $\mathfrak{I}$ while SEP is saying that $\mathfrak{I}$ can separate any two elements of A.

In Section VI.1 we will define these two properties and discuss their connection with the notion of reductivity of $\mathfrak{I}$ introduced in Section I.2. We will then show in Section VI.2 that if $\mathscr{A}$ and $\mathfrak{I}$ are such that the conditions REP and SEP hold, then the translational hull $\Omega(\mathfrak{I})$ is isomorphic to the endomorphism monoid $\operatorname{End}(\mathscr{A})$. After that, Sections VI.3 and VI.4 will consider the cases where we only have one of these two conditions, to see how much we can understand from having one but not the other. Since there are still many gaps in the development of this theory, we close each section with some open questions that we consider good directions of research.

Note. The content of this chapter is the result of an ongoing collaboration with Prof. Victoria Gould, Dr. Marianne Johnson and Prof. Mark Kambites.

VI.1 DEFINITIONS

Throughout this chapter, $\mathscr{A}$ will denote a universal algebra and $\mathfrak{I}$ will be an ideal of $\operatorname{End}(\mathscr{A})$.

Definition VI.1.1. We say that the pair $(\mathscr{A}, \mathfrak{I})$ satisfies:

- REP if A is generated by the images of the endomorphisms of $\mathfrak{I}$, that is, if $A = \left\langle \bigcup_{\alpha \in \mathfrak{I}} \operatorname{im} \alpha \right\rangle$,
- SEP if every pair of distinct elements of A can be distinguished by an endomorphism of $\mathfrak{I}$, that is, if for all $a \neq b \in A$, there exists $\gamma \in \mathfrak{I}$ such that $a\gamma \neq b\gamma$.

Remark VI.1.2. Notice that SEP corresponds to the definition of $\mathfrak{I}$ being separative for A as defined in Section V.3.2.3. On the other hand, condition REP is weaker than requiring $\mathscr{A}$ to be representable by $\mathfrak{I}$, since we do not require every element of $\mathscr{A}$ to lie in the image of an endomorphism of $\mathfrak{I}$. In the context of independence algebras, it can however be shown that having REP or having representability is equivalent. We will delay further comments and comparisons concerning the specific case of independence algebras to the corresponding section in Chapter VII.

Remark VI.1.3. Notice that if $\mathfrak{I} = \operatorname{End}(\mathscr{A})$, then $\mathrm{id} \in \mathfrak{I}$, so that $(\mathscr{A}, \mathfrak{I})$ has both REP and SEP. Also $\mathfrak{I}$ is then a monoid and by Corollary I.2.15, we have that $\Omega(\mathfrak{I}) \cong \mathfrak{I} = \operatorname{End}(\mathscr{A})$. Thus we are truly interested in only proper ideals of $\operatorname{End}(\mathscr{A})$.

We now show that the properties REP and SEP are closely related to the notions of left and right reductivity.

Lemma VI.1.4. *1) If the pair $(\mathscr{A}, \mathfrak{I})$ satisfies REP, then $\mathfrak{I}$ is left reductive.*

2) If the pair $(\mathscr{A}, \mathfrak{I})$ satisfies SEP, then $\mathfrak{I}$ is right reductive.

Proof. 1) Let $\alpha, \beta \in \mathfrak{I}$ and suppose that $\gamma\alpha = \gamma\beta$ for all $\gamma \in \mathfrak{I}$. To show that $\mathfrak{I}$ is left reductive, we need to argue that $\alpha = \beta$. Let $a \in A$. Since the pair $(\mathscr{A}, \mathfrak{I})$ satisfies REP, there exist some $t \in \mathcal{T}^{\mathscr{A}}$, $\{x_i\} \subseteq A$ and $\{\delta_i\} \subseteq \mathfrak{I}$ such that $a = t\left(\overline{x_i\delta_i}\right)$. Then we have

$$\begin{aligned} a\alpha &= t\left(\overline{x_i\delta_i}\right)\alpha = t\left(\overline{x_i\delta_i\alpha}\right) \\ &= t\left(\overline{x_i\delta_i\beta}\right) = t\left(\overline{x_i\delta_i}\right)\beta \\ &= a\beta. \end{aligned}$$

Since $a \in A$ was arbitrary, it follows that $\alpha = \beta$ as required.

2) Let $\alpha, \beta \in \mathfrak{I}$ and suppose that $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathfrak{I}$. Assume that $\alpha \neq \beta$. Then there exists $a \in A$ such that $a\alpha \neq a\beta$. Since the pair $(\mathscr{A}, \mathfrak{I})$ has SEP, it follows that there exists $\gamma \in \mathfrak{I}$ such that $a\alpha\gamma \neq a\beta\gamma$, contradicting the fact that $\alpha\gamma = \beta\gamma$. Thus $\alpha = \beta$ and $\mathfrak{I}$ is right reductive. $\square$

Throughout this chapter, we will use the following maps, introduced in Section I.2:

- $\pi_{\mathrm{P}} \colon \Omega(\mathfrak{I}) \to \mathrm{P}(\mathfrak{I})$ is defined by $(\lambda, \rho)\pi_{\mathrm{P}} = \rho$ and has image $\operatorname{im} \pi_{\mathrm{P}} = \widetilde{\mathrm{P}}(\mathfrak{I})$;
- $\pi_{\Lambda} \colon \Omega(\mathfrak{I}) \to \Lambda(\mathfrak{I})$ is defined by $(\lambda, \rho)\pi_{\Lambda} = \lambda$ and has image $\operatorname{im} \pi_{\Lambda} = \widetilde{\Lambda}(\mathfrak{I})$;
- $\chi_{\mathrm{P}} \colon \operatorname{End}(\mathscr{A}) \to \widetilde{\mathrm{P}}(\mathfrak{I})$ defined by $\phi\chi_{\mathrm{P}} = \rho_\phi$, where $\alpha\rho_\phi = \alpha\phi$ for all $\alpha \in \mathfrak{I}$;
- $\chi_{\Lambda} \colon \operatorname{End}(\mathscr{A}) \to \widetilde{\Lambda}(\mathfrak{I})$ defined by $\phi\chi_{\Lambda} = \lambda_\phi$ where $\lambda_\phi\alpha = \phi\alpha$ for all $\alpha \in \mathfrak{I}$; and
- $\chi \colon \operatorname{End}(\mathscr{A}) \to \Omega(\mathfrak{I})$ defined by $\phi\chi = (\lambda_\phi, \rho_\phi)$.

The fact that these are all well-defined morphisms comes from Corollaries I.2.6 and I.2.42, since $\mathfrak{I}$ is an ideal of $\operatorname{End}(\mathscr{A})$. Moreover, it is clear that the image of the identity endomorphism of $\operatorname{End}(\mathscr{A})$ under χ_{P} is the trivial right translation $\mathbb{1}_{\mathrm{P}}$. Similarly, $\mathrm{id}\chi_{\Lambda} = \mathbb{1}_{\Lambda}$ so that $\mathrm{id}\chi = \mathbb{1}_{\Omega}$ and we have that χ_{P}, χ_{Λ} and χ are monoid homomorphisms.

Definition VI.1.5. We say that $\Omega(\mathfrak{I})$ is *naturally isomorphic* to $\widetilde{\mathrm{P}}(\mathfrak{I})$ [resp. to $\widetilde{\Lambda}(\mathfrak{I})$] if π_{P} [resp. π_{Λ}] is an isomorphism.

We now get an immediate corollary.

Corollary VI.1.6. *If $(\mathscr{A}, \mathfrak{I})$ satisfies REP, then $\Omega(\mathfrak{I})$ is naturally isomorphic to $\widetilde{\mathrm{P}}(\mathfrak{I})$. Dually, if $(\mathscr{A}, \mathfrak{I})$ satisfies SEP, then $\Omega(\mathfrak{I})$ is naturally isomorphic to $\widetilde{\Lambda}(\mathfrak{I})$. Consequently, if the pair $(\mathscr{A}, \mathfrak{I})$ satisfies REP and SEP, then $\Omega(\mathfrak{I}) \cong \widetilde{\mathrm{P}}(\mathfrak{I})$ and $\Omega(\mathfrak{I}) \cong \widetilde{\Lambda}(\mathfrak{I})$ via natural isomorphisms π_{P} and π_{Λ}.*

Proof. If the pair $(\mathscr{A}, \mathfrak{I})$ has REP, then $\mathfrak{I}$ is left reductive by Lemma VI.1.4, and by Lemma I.2.17, we obtain that π_{P} is injective and $\Omega(\mathfrak{I}) \cong \widetilde{\mathrm{P}}(\mathfrak{I})$. Dually, if $(\mathscr{A}, \mathfrak{I})$ has SEP, we get that $\mathfrak{I}$ is right reductive and thus $\Omega(\mathfrak{I})$ is naturally isomorphic to $\widetilde{\Lambda}(\mathfrak{I})$. $\square$

We now show that the properties REP and SEP respectively imply the injectivity of χ_{P} and χ_{Λ}.

Lemma VI.1.7. *If the pair $(\mathscr{A}, \mathfrak{I})$ satisfies REP, then χ_{P} is injective.*

Proof. Let $\phi, \psi \in \operatorname{End}(\mathscr{A})$ and suppose that $\rho_\phi = \rho_\psi$. We want to show that $a\phi = a\psi$ for all $a \in A$. Since $(\mathscr{A}, \mathfrak{I})$ has REP, for all $a \in A$ we have that $a = t\big(\overline{x_i\alpha_i}\big)$ for some $t \in \mathcal{T}^{\mathscr{A}}$, $x_i \in A$ and $\alpha_i \in \mathfrak{I}$. Thus,

$$\begin{aligned} a\phi &= t\big(\overline{x_i\alpha_i}\big)\phi = t\big(\overline{x_i\alpha_i\phi}\big) \\ &= t\big(\overline{x_i\alpha_i\rho_\phi}\big) = t\big(\overline{x_i\alpha_i\rho_\psi}\big) \\ &= t\big(\overline{x_i\alpha_i\psi}\big) = t\big(\overline{x_i\alpha_i}\big)\psi \\ &= a\psi, \end{aligned}$$

so that $\phi = \psi$ and χ_{P} is injective. □

Lemma VI.1.8. *If the pair* $(\mathscr{A}, \mathfrak{I})$ *satisfies SEP, then* χ_Λ *is injective.*

Proof. Let $\phi, \psi \in \operatorname{End}(\mathscr{A})$ and suppose that $\lambda_\phi = \lambda_\psi$. If there exists $a \in A$ such that $a\phi \neq a\psi$, then by the fact that $(\mathscr{A}, \mathfrak{I})$ has SEP, we get that $a\phi\gamma \neq a\psi\gamma$ for some $\gamma \in \mathfrak{I}$, that is, $a\lambda_\phi\gamma \neq a\lambda_\psi\gamma$, contradicting the fact that $\lambda_\phi = \lambda_\psi$. Thus $\phi = \psi$ and χ_Λ is injective. □

Remark VI.1.9. In fact, part 1) of Lemma VI.1.4 can be seen as a consequence of Lemma VI.1.7. Indeed, suppose that $(\mathscr{A}, \mathfrak{I})$ has REP, so that χ_{P} is injective. Let $\phi, \psi \in \mathfrak{I}$ be such that $\alpha\phi = \alpha\psi$ for all $\alpha \in \mathfrak{I}$. Then $\alpha\rho_\phi = \alpha\rho_\psi$ for all $\alpha \in \mathfrak{I}$, which shows that $\rho_\phi = \rho_\psi$ and thus $\phi = \psi$ by injectivity of χ_{P}. Hence $\mathfrak{I}$ is left reductive.

Dually, we can see that Lemma VI.1.8 effectively proves part 2) of Lemma VI.1.4.

We finish this section by defining some terminology that we shall use later.

Definition VI.1.10. We say that $\operatorname{End}(\mathscr{A})$ is *naturally isomorphic to* $\widetilde{\mathrm{P}}(\mathfrak{I})$ [resp. *to* $\widetilde{\Lambda}(\mathfrak{I})$] if χ_{P} [resp. χ_Λ] is an isomorphism.

VI.2 TRANSLATIONAL HULLS UNDER REP AND SEP

In Section V.1.5, we gave a necessary and sufficient condition in the context of independence algebras to obtain an isomorphism of the endomorphism monoid with the right translations and the bitranslations of the (0-)minimal ideal. Since we are now considering a larger class of universal algebras, and we are not restricting ourselves to a specific ideal of the endomorphism monoid, obtaining necessary and sufficient conditions in all generality will be be harder to achieve. However, we can show that REP and SEP are at least sufficient conditions to obtain the desired isomorphisms.

Theorem VI.2.1. *If $(\mathscr{A}, \mathfrak{I})$ satisfies REP and SEP, then*

$$\Omega(\mathfrak{I}) \cong \widetilde{\mathrm{P}}(\mathfrak{I}) \cong \widetilde{\Lambda}(\mathfrak{I}) \cong \mathrm{End}(\mathscr{A})$$

via natural isomorphisms. In particular, every bitranslation of $\mathfrak{I}$ is realised by an endomorphism of $\mathscr{A}$.

Proof. Since $(\mathscr{A}, \mathfrak{I})$ satisfies REP and SEP, we get that $\Omega(\mathfrak{I}) \cong \widetilde{\mathrm{P}}(\mathfrak{I}) \cong \widetilde{\Lambda}(\mathfrak{I})$ by Corollary VI.1.6. Moreover, $\chi_{\mathrm{P}} \colon \mathrm{End}(\mathscr{A}) \to \widetilde{\mathrm{P}}(\mathfrak{I})$ is injective by Lemma VI.1.7. We show that χ_{P} is surjective, so that we will have an isomorphism from $\mathrm{End}(\mathscr{A})$ to $\Omega(\mathfrak{I})$ via the composition $\chi_{\mathrm{P}}\pi_{\mathrm{P}}^{-1}$.

Let $\rho \in \widetilde{\mathrm{P}}(\mathfrak{I})$, so that there exists $\lambda \in \Lambda(\mathfrak{I})$ such that $(\lambda, \rho) \in \Omega(\mathfrak{I})$. We aim to show that there exists an endomorphism $\phi \in \mathrm{End}(\mathscr{A})$ such that $\rho = \rho_\phi$, and hence $\phi\chi_{\mathrm{P}} = \rho$ and χ_{P} will be surjective.

Since $(\mathscr{A}, \mathfrak{I})$ has REP, for all $a \in A$, we have $a = t\left(\overline{x_i\alpha_i}\right)$ for some $t \in \mathcal{T}^{\mathscr{A}}$, $\{x_i\} \subseteq A$ and $\{\alpha_i\} \subseteq \mathfrak{I}$. We now define $\phi \colon A \to A$ by

$$a\phi = t\left(\overline{x_i\alpha_i}\right)\phi := t\left(\overline{x_i\alpha_i\rho}\right).$$

In order to show that ϕ is well-defined, we need to show that if $t\left(\overline{x_i\alpha_i}\right) = s\left(\overline{y_j\beta_j}\right)$ for some $s \in \mathcal{T}^{\mathscr{A}}$, $\{y_j\} \subseteq A$ and $\{\beta_j\} \subseteq \mathfrak{I}$, then we get $t\left(\overline{x_i\alpha_i\rho}\right) = s\left(\overline{y_j\beta_j\rho}\right)$. So, let $\gamma \in \mathfrak{I}$. Then we have that

$$\begin{aligned} t\left(\overline{x_i\alpha_i\rho}\right)\gamma &= t\left(\overline{x_i\alpha_i\rho\gamma}\right) = t\left(\overline{x_i\alpha_i\lambda\gamma}\right) \\ &= t\left(\overline{x_i\alpha_i}\right)\lambda\gamma = s\left(\overline{y_j\beta_j}\right)\lambda\gamma \\ &= s\left(\overline{y_j\beta_j\lambda\gamma}\right) = s\left(\overline{y_j\beta_j\rho}\right)\gamma. \end{aligned}$$

Since this is valid for all $\gamma \in \mathfrak{I}$ and that $(\mathscr{A}, \mathfrak{I})$ has SEP, we get $t\left(\overline{x_i\alpha_i\rho}\right) = s\left(\overline{y_j\beta_j\rho}\right)$, that is, $t\left(\overline{x_i\alpha_i}\right)\phi = s\left(\overline{y_j\beta_j}\right)\phi$ and therefore ϕ is well-defined.

We now need to show that ϕ is an endomorphism, that is, we need to show that $u(a_1, \ldots, a_n)\phi = u(a_1\phi, \ldots, a_n\phi)$ for all $u \in \mathcal{T}_n^{\mathscr{A}}$ and $a_1, \ldots, a_n \in A$. However, since $(\mathscr{A}, \mathfrak{I})$ has REP, for each $1 \leq i \leq n$, there exists $k_i \in \mathbb{N}$ such that $a_i = t^i\left(\overline{b_{k_i}^i\beta_{k_i}^i}\right)$, where $t^i \in \mathcal{T}_{k_i}^{\mathscr{A}}$, $b_1^i, \ldots, b_{k_i}^i \in A$ and $\beta_1^i, \ldots, \beta_{k_i}^i \in \mathfrak{I}$. Then, by writing $s \in \mathcal{T}^{\mathscr{A}}$ for the term composition of u with all the t^i (so that s has arity $\sum_{i=1}^n k_i$), we get that:

$$\begin{aligned}
u(a_1,\dots,a_n)\phi &= u\left(t^1\left(\overline{b^1_{k_1}\beta^1_{k_1}}\right),\dots,t^n\left(\overline{b^n_{k_n}\beta^n_{k_n}}\right)\right)\phi \\
&= s\left(\overline{b^1_{k_1}\beta^1_{k_1}},\dots,\overline{b^n_{k_n}\beta^n_{k_n}}\right)\phi && \text{writing as a single term,} \\
&= s\left(\overline{b^1_{k_1}\beta^1_{k_1}\rho},\dots,\overline{b^n_{k_n}\beta^n_{k_n}\rho}\right) && \text{by definition of } \phi, \\
&= u\left(t^1\left(\overline{b^1_{k_1}\beta^1_{k_1}\rho}\right),\dots,t^n\left(\overline{b^n_{k_n}\beta^n_{k_n}\rho}\right)\right) && \text{by definition of } s, \\
&= u\left(t^1\left(\overline{b^1_{k_1}\beta^1_{k_1}}\right)\phi,\dots,t^n\left(\overline{b^n_{k_n}\beta^n_{k_n}}\right)\phi\right) && \text{by definition of } \phi, \\
&= u(a_1\phi,\dots,a_n\phi) && \text{by definition of the } a_i\text{'s.}
\end{aligned}$$

Thus $\phi \in \mathrm{End}(\mathscr{A})$. It only remains to show that $\gamma\rho_\phi = \gamma\rho$ for all $\gamma \in \mathfrak{I}$. But we can see that

$$u\left(\overline{a_i}\right)\gamma\rho = u\left(\overline{a_i\gamma\rho}\right) = u\left(\overline{a_i\gamma}\right)\phi = u\left(\overline{a_i}\right)\gamma\phi = u\left(\overline{a_i}\right)\gamma\rho_\phi.$$

Hence, $\rho_\phi = \rho$, so that χ_{P} is surjective, and we get that $\mathrm{End}(\mathscr{A}) \cong \widetilde{\mathrm{P}}(\mathfrak{I})$. $\square$

Even though we cannot prove whether or not REP and SEP are necessary conditions to obtain the isomorphisms, we can nonetheless show that the ideal has to satisfy the weaker condition of being reductive for the isomorphisms to happen.

Lemma VI.2.2. *If* $\mathrm{End}(\mathscr{A})$ *and* $\widetilde{\mathrm{P}}(\mathfrak{I})$ *are naturally isomorphic via* χ_{P}*, then* $\mathfrak{I}$ *is left reductive and then* $\mathrm{End}(\mathscr{A}) \cong \Omega(\mathfrak{I})$*. Dually, if* $\mathrm{End}(\mathscr{A})$ *and* $\widetilde{\Lambda}(\mathfrak{I})$ *are naturally isomorphic via* χ_Λ*, then* $\mathfrak{I}$ *is right reductive and then* $\mathrm{End}(\mathscr{A}) \cong \Omega(\mathfrak{I})$*.*

Consequently, if $\Omega(\mathfrak{I}) \cong \widetilde{\Lambda}(\mathfrak{I}) \cong \widetilde{\mathrm{P}}(\mathfrak{I}) \cong \mathrm{End}(\mathscr{A})$ *via natural isomorphisms, then* $\mathfrak{I}$ *is reductive.*

Proof. Suppose that $\chi_{\mathrm{P}} \colon \mathrm{End}(\mathscr{A}) \to \widetilde{\mathrm{P}}(\mathfrak{I})$ is an isomorphism and let $\alpha, \beta \in \mathfrak{I}$ be such that $\gamma\alpha = \gamma\beta$ for all $\gamma \in \mathfrak{I}$. Since $\rho_\alpha, \rho_\beta \in \mathrm{P}_0(\mathfrak{I})$, we get that $\gamma\rho_\alpha = \gamma\rho_\beta$ for all $\gamma \in \mathfrak{I}$, and thus $\rho_\alpha = \rho_\beta$. However, χ_{P} is injective, which means that $\alpha = \beta$ and therefore $\mathfrak{I}$ is left reductive. Using Lemma I.2.17, we then get that π_{P} is injective and $\Omega(\mathfrak{I}) \cong \widetilde{\mathrm{P}}(\mathfrak{I})$, as required.

The proof that $\mathfrak{I}$ is right reductive when χ_Λ is an isomorphism is dual, and the last result follows immediately. $\square$

Notice that in the proof of Theorem VI.2.1, we used multiple times the argument that $(\mathscr{A}, \mathfrak{I})$ had REP but only used once the fact that it had SEP in order to show that the map ϕ created from the right translation ρ is well-defined.

Question VI.2.3. Is it possible to weaken the assumption that $(\mathscr{A}, \mathfrak{I})$ has SEP in Theorem VI.2.1?

In view of Lemma VI.2.2, if we want to have the four-fold isomorphisms between $\widetilde{\mathrm{P}}(\mathfrak{I})$, $\widetilde{\Lambda}(\mathfrak{I})$, $\Omega(\mathfrak{I})$ and $\mathrm{End}(\mathscr{A})$ we need to start with an ideal that is reductive.

Question VI.2.4. Considering a reductive ideal $\mathfrak{I}$, is there an assumption weaker than REP and SEP that would allow us to get the same result as in Theorem VI.2.1?

VI.3 REP WITHOUT SEP

In this section, we will assume that our pair $(\mathscr{A}, \mathfrak{I})$ satisfies REP but not SEP. This means that by Corollary VI.1.6 and Lemma VI.1.7 we have that π_{P} is an isomorphism and χ_{P} is injective, which means that we get

$$\mathrm{End}(\mathscr{A}) \hookrightarrow \widetilde{\mathrm{P}}(\mathfrak{I}) \cong \Omega(\mathfrak{I}).$$

This means in particular that every right translation of $\widetilde{\mathrm{P}}(\mathfrak{I})$ is linked to exactly one left translation of $\mathfrak{I}$ and comes from at most one endomorphism of $\mathscr{A}$.

The main work of Theorem VI.2.1 was to construct this endomorphism of $\mathscr{A}$ from a right translation that is linked. Since we no longer have SEP, we want to show that we can at least construct a well-defined endomorphism of a quotient algebra of $\mathscr{A}$ by defining a congruence relation which will play the role of SEP in the proof to show that the map constructed is well-defined.

Definition VI.3.1. Let $\sim$ be the relation on $\mathscr{A}$ defined by $a \sim b$ for $a, b \in A$ if and only if $a\alpha = b\alpha$ for all $\alpha \in \mathfrak{I}$. The $\sim$-class of an element $a \in A$ will be denoted $[a]$.

Remark VI.3.2. It follows immediately from the definition that $\sim$ is an equivalence relation. Moreover, this relation is equality exactly when $(\mathscr{A}, \mathfrak{I})$ has SEP.

Lemma VI.3.3. *The relation $\sim$ is a congruence on $\mathscr{A}$.*

Proof. Let $\{a_i\}, \{b_i\} \subseteq A$ and suppose that $a_i \sim b_i$ for all i. Then for all $\alpha \in \mathfrak{I}$ and all terms $t \in \mathcal{T}^{\mathscr{A}}$ we have that $t\big(\overline{a_i}\big)\alpha = t\big(\overline{a_i\alpha}\big) = t\big(\overline{b_i\alpha}\big) = t\big(\overline{b_i}\big)\alpha$, as required. $\square$

We now turn our attention towards the algebra $\mathscr{A}/_{\sim}$.

Lemma VI.3.4. *The ideal $\mathfrak{I}$ acts on the right of $\mathscr{A}/_{\sim}$ by endomorphisms of $\mathscr{A}/_{\sim}$ via $[a]\alpha = [a\alpha]$ for all $a \in A$ and all $\alpha \in \mathfrak{I}$, that is, the equation $t\Big(\overline{[a_i]}\Big)\alpha = t\Big(\overline{[a_i]\alpha}\Big)$ holds for all $\{a_i\} \subseteq A$, $t \in \mathcal{T}^{\mathscr{A}}$ and $\alpha \in \mathfrak{I}$.*

Proof. Let $a, b \in A$ be such that $[a] = [b]$. Then by definition of $\sim$ we have that $a\alpha = b\alpha$ for all $\alpha \in \mathfrak{I}$, so that $[a\alpha] = [b\alpha]$. Hence $[a]\alpha$ is well-defined.

Moreover, for all $\alpha, \beta \in \mathfrak{I}$ and $a \in A$, we have that

$$[a](\alpha\beta) = [a\alpha\beta] = [a\alpha]\beta = ([a]\alpha)\beta,$$

which shows that $\mathfrak{I}$ acts by semigroup transformations on $\mathscr{A}/_{\sim}$.

Now, let $\{a_i\} \in A$, $t \in \mathcal{T}^{\mathscr{A}}$ and $\alpha \in \mathfrak{I}$. Since $\sim$ is a congruence on $\mathscr{A}$, we have that $t\Big(\overline{[a_i]}\Big) = \Big[t(\overline{a_i})\Big]$, from which we get that

$$t\Big(\overline{[a_i]}\Big)\alpha = \Big[t(\overline{a_i})\Big]\alpha = \Big[t(\overline{a_i})\alpha\Big] = \Big[t(\overline{a_i\alpha})\Big] = t\Big(\overline{[a_i\alpha]}\Big) = t\Big(\overline{[a_i]\alpha}\Big),$$

where we used the definition of the action and the fact that $\alpha \in \mathrm{End}(\mathscr{A})$. $\square$

We now construct endomorphisms of $\mathscr{A}/_{\sim}$ from right translations which are linked.

Lemma VI.3.5. *Suppose that $(\mathscr{A}, \mathfrak{I})$ satisfies REP. Then every $\rho \in \widetilde{\mathrm{P}}(\mathfrak{I})$ induces a map $\phi\colon \mathscr{A}/_{\sim} \to \mathscr{A}/_{\sim}$ defined by $[a]\phi = \Big[t(\overline{x_i\alpha_i\rho})\Big]$ where $a = t(\overline{x_i\alpha_i})$, such that for all $\alpha \in \mathfrak{I}$ and $m \in \mathscr{A}/_{\sim}$ we have:*

- *ϕ is a well-defined morphism;*
- *$m\alpha\rho = m\alpha\phi$; and*
- *$m\lambda\alpha = m\phi\alpha$;*

where $\lambda \in \widetilde{\Lambda}(\mathfrak{I})$ is the unique left translation linked to ρ.

Proof. Let $\rho \in \widetilde{\mathrm{P}}(\mathfrak{I})$. Then there exists a unique $\lambda \in \Lambda(\mathfrak{I})$ such that $(\lambda, \rho) \in \Omega(\mathfrak{I})$. Also, from the fact that $(\mathscr{A}, \mathfrak{I})$ satisfies REP, every $[a] \in \mathscr{A}/_{\sim}$ can be expressed as $[a] = \Big[t(\overline{x_i\alpha_i})\Big]$ for some $t \in \mathcal{T}^{\mathscr{A}}$, $\{x_i\} \subseteq A$ and $\{\alpha_i\} \subseteq \mathfrak{I}$. We now define $\phi\colon \mathscr{A}/_{\sim} \to \mathscr{A}/_{\sim}$ by

$$[a]\phi = \Big[t(\overline{x_i\alpha_i})\Big]\phi := \Big[t(\overline{x_i\alpha_i\rho})\Big].$$

Notice in particular that under this definition, for all $\alpha \in \mathfrak{I}$ we have that

$$[a\alpha\rho] = \Big[t(\overline{x_i\alpha_i})\alpha\rho\Big] = \Big[t\Big(\overline{x_i\alpha_i(\alpha\rho)}\Big)\Big] = \Big[t\Big(\overline{x_i(\alpha_i\alpha)\rho}\Big)\Big] = \Big[t(\overline{x_i\alpha_i\alpha})\Big]\phi = [a\alpha]\phi.$$

To show that ϕ is well-defined, suppose that $t\left(\overline{x_i\alpha_i}\right) \sim s\left(\overline{y_j\beta_j}\right)$ for some $s \in \mathcal{T}^{\mathscr{A}}$, $\{y_j\} \subseteq A$ and $\{\beta_j\} \subseteq \mathfrak{I}$. Then for all $\gamma \in \mathfrak{I}$ we get that

$$\begin{aligned} t\left(\overline{x_i\alpha_i\rho}\right)\gamma &= t\left(\overline{x_i\alpha_i\rho\gamma}\right) = t\left(\overline{x_i\alpha_i\lambda\gamma}\right) \\ &= t\left(\overline{x_i\alpha_i}\right)\lambda\gamma = s\left(\overline{y_j\beta_j}\right)\lambda\gamma \\ &= s\left(\overline{y_j\beta_j\lambda\gamma}\right) = s\left(\overline{y_j\beta_j\rho}\right)\gamma, \end{aligned}$$

which shows that $t\left(\overline{x_i\alpha_i\rho}\right) \sim s\left(\overline{y_j\beta_j\rho}\right)$, that is, $\left[t\left(\overline{x_i\alpha_i}\right)\right]\phi = \left[s\left(\overline{y_j\beta_j}\right)\right]\phi$, so that ϕ is well-defined.

Moreover, for all $[a] \in \mathscr{A}/_\sim$ and $\alpha \in \mathfrak{I}$, we have $[a]\alpha\rho = [a\alpha\rho] = [a\alpha]\phi = [a]\alpha\phi$, while

$$\begin{aligned} [a]\lambda\alpha &= \left[t(\overline{x_i\alpha_i})\right]\lambda\alpha = \left[t\left(\overline{x_i\alpha_i\lambda\alpha}\right)\right] \\ &= \left[t(\overline{x_i\alpha_i\rho\alpha})\right] = \left[t(\overline{x_i\alpha_i\rho})\right]\alpha \\ &= \left[t(\overline{x_i\alpha_i})\right]\phi\alpha = [a]\phi\alpha. \end{aligned}$$

Therefore, for all $m \in \mathscr{A}/_\sim$, we have $m\alpha\rho = m\alpha\phi$ and $m\lambda\alpha = m\phi\alpha$.

It only remains to show that ϕ is a morphism. So consider the expression $u([a_1], \ldots, [a_n])$ where $u \in \mathcal{T}^{\mathscr{A}}$ and $[a_1], \ldots, [a_n] \in \mathscr{A}/_\sim$. Using REP, for each $1 \leq i \leq n$ we write $a_i = t^i\left(\overline{b^i_{k_i}\beta^i_{k_i}}\right)$ for some $k_i \in \mathbb{N}$, $t^i \in \mathcal{T}^{\mathscr{A}}_{k_i}$, $\left\{b^i_{k_i}\right\} \subseteq A$ and $\left\{\beta^i_{k_i}\right\} \subseteq \mathfrak{I}$. Then, we get

$$\begin{aligned} u\left([a_1], \ldots, [a_n]\right)\phi &= \left[u(a_1, \ldots, a_n)\right]\phi && \text{since } \sim \text{ is a congruence,} \\ &= \left[u\left(t^1\left(\overline{b^1_{k_1}\beta^1_{k_1}}\right), \ldots, t^n\left(\overline{b^n_{k_n}\beta^n_{k_n}}\right)\right)\right]\phi && \text{by definition of } a_i, \\ &= \left[s\left(\overline{b^1_{k_1}\beta^1_{k_1}}, \ldots, \overline{b^n_{k_n}\beta^n_{k_n}}\right)\right]\phi && \text{writing as a single term,} \\ &= \left[s\left(\overline{b^1_{k_1}\beta^1_{k_1}\rho}, \ldots, \overline{b^n_{k_n}\beta^n_{k_n}\rho}\right)\right] && \text{by definition of } \phi, \\ &= \left[u\left(t^1\left(\overline{b^1_{k_1}\beta^1_{k_1}\rho}\right), \ldots, t^n\left(\overline{b^n_{k_n}\beta^n_{k_n}\rho}\right)\right)\right] && \text{by definition of } s, \\ &= u\left(\left[t^1\left(\overline{b^1_{k_1}\beta^1_{k_1}\rho}\right)\right], \ldots, \left[t^n\left(\overline{b^n_{k_n}\beta^n_{k_n}\rho}\right)\right]\right) && \text{since } \sim \text{ is a congruence,} \\ &= u\left(\left[t^1\left(\overline{b^1_{k_1}\beta^1_{k_1}}\right)\right]\phi, \ldots, \left[t^n\left(\overline{b^n_{k_n}\beta^n_{k_n}}\right)\right]\phi\right) && \text{by definition of } \phi, \\ &= u\left([a_1]\phi, \ldots, [a_n]\phi\right) && \text{by definition of } a_i. \end{aligned}$$

Thus ϕ is indeed an endomorphism of $\mathscr{A}/_\sim$ and all the results are proved. □

Remark VI.3.6. Notice that if $\sim$ is equality, that is, if $(\mathscr{A}, \mathfrak{I})$ has SEP, then this proof reproduces exactly the arguments of the proof of Theorem VI.2.1.

Moreover, we can show that the map ϕ of Lemma VI.3.5 is uniquely tied to the left translation of the linked pair considered.

Lemma VI.3.7. *Suppose that $(\mathscr{A}, \mathfrak{I})$ satisfies REP. Define $\theta\colon \widetilde{\mathrm{P}}(\mathfrak{I}) \to \mathrm{End}\big(\mathscr{A}/_{\sim}\big)$ by $\rho\theta = \phi$, where ϕ is as given in Lemma VI.3.5. Then θ is a morphism such that $\rho\theta = \rho'\theta$ if and only if ρ and ρ' are linked to the same left translation $\lambda \in \widetilde{\Lambda}(\mathfrak{I})$.*

Proof. Since $(\mathscr{A}, \mathfrak{I})$ has REP, then each $[a] \in \mathscr{A}/_{\sim}$ can be written as $[a] = [t(\overline{x_i\alpha_i})]$ for some $t \in \mathcal{T}^{\mathscr{A}}$, $\{x_i\} \subseteq A$ and $\{\alpha_i\} \subseteq \mathfrak{I}$. Then we let $\theta\colon \widetilde{\mathrm{P}}(\mathfrak{I}) \to \mathrm{End}(\mathscr{A}/_{\sim})$ be such that $\rho\theta = \phi$, where ϕ is defined by $[a]\phi = [t(\overline{x_i\alpha_i\rho})]$ for $[a] \in \mathscr{A}/_{\sim}$. The fact that θ is well-defined follows from Lemma VI.3.5.

Moreover, if $\rho\theta = \phi$ and $\rho'\theta = \phi'$, then by definition of ϕ and ϕ', we have that

$$\Big[t(\overline{x_i\alpha_i})\Big]\phi\phi' = \Big[t(\overline{x_i\alpha_i\rho})\Big]\phi' = \Big[t\Big(\overline{x_i(\alpha_i\rho)\rho'}\Big)\Big] = \Big[t\Big(\overline{x_i\alpha_i(\rho\rho')}\Big)\Big],$$

which shows that $\rho\theta\rho'\theta = (\rho\rho')\theta$ and thus θ is a homomorphism.

Finally, let $\lambda, \lambda' \in \widetilde{\Lambda}(\mathfrak{I})$ be the unique left translations linked to ρ and ρ' respectively, and suppose that $\phi = \phi'$. Then for all $s \in \mathcal{T}^{\mathscr{A}}$, $\{y_j\} \subseteq A$ and $\{\beta_j\} \subseteq \mathfrak{I}$, we have that

$$\Big[s\Big(\overline{y_j\beta_j\rho}\Big)\Big] = \Big[s\Big(\overline{y_j\beta_j}\Big)\Big]\phi = \Big[s\Big(\overline{y_j\beta_j}\Big)\Big]\phi' = \Big[s\Big(\overline{y_j\beta_j\rho'}\Big)\Big],$$

which means that $s(\overline{y_j\beta_j\rho})\gamma = s(\overline{y_j\beta_j\rho'})\gamma$ for all $\gamma \in \mathfrak{I}$. Thus, for $a = t(\overline{x_i\alpha_i}) \in A$ and $\beta \in \mathfrak{I}$, we get

$$a\lambda\beta = t\Big(\overline{x_i\alpha_i\lambda\beta}\Big) = t\Big(\overline{x_i\alpha_i\rho}\Big)\beta = t\Big(\overline{x_i\alpha_i\rho'}\Big)\beta = t\Big(\overline{x_i\alpha_i\lambda'\beta}\Big) = a\lambda'\beta.$$

Since this is valid for all $a \in A$ and $\beta \in \mathfrak{I}$, it follows that $\lambda = \lambda'$.

For the converse, suppose that $(\lambda, \rho), (\lambda, \rho') \in \Omega(\mathfrak{I})$ and let $\phi = \rho\theta$ and $\phi' = \rho'\theta$. Let $[a] \in \mathscr{A}/_{\sim}$ with $a = t(\overline{x_i\alpha_i})$ for some $t \in \mathcal{T}^{\mathscr{A}}$, $\{x_i\} \subseteq A$ and $\{\alpha_i\} \subseteq \mathfrak{I}$ by using the assumption that $(\mathscr{A}, \mathfrak{I})$ has REP. Then for all $\beta \in \mathfrak{I}$, we get:

$$t\Big(\overline{x_i\alpha_i\rho}\Big)\beta = t\Big(\overline{x_i\alpha_i\rho\beta}\Big) = t\Big(\overline{x_i\alpha_i\lambda\beta}\Big) = t\Big(\overline{x_i\alpha_i\rho'\beta}\Big) = t\Big(\overline{x_i\alpha_i\rho'}\Big)\beta,$$

from which we get that $t(\overline{x_i\alpha_i\rho}) \sim t\Big(\overline{x_i\alpha_i\rho'}\Big)$, and thus

$$[a]\phi = \Big[t(\overline{x_i\alpha_i})\Big]\phi = \Big[t(\overline{x_i\alpha_i\rho})\Big] = \Big[t\Big(\overline{x_i\alpha_i\rho'}\Big)\Big] = \Big[t(\overline{x_i\alpha_i})\Big]\phi' = [a]\phi'.$$

Since this is valid for all $[a] \in \mathscr{A}/_{\sim}$, it follows that $\phi = \phi'$. $\square$

We have seen that $\mathfrak{I}$ acts on $\mathscr{A}/_{\sim}$, but there is no reason to suppose that it acts faithfully. So we consider a relation $\approx$ on $\mathfrak{I}$ defined as follows.

Definition VI.3.8. Let $\approx$ be the congruence on $\mathfrak{I}$ induced by the action of $\mathfrak{I}$ on $\mathscr{A}/_{\sim}$, that is, for $\alpha, \beta \in \mathfrak{I}$ we have $\alpha \approx \beta$ if and only if $a\alpha \sim a\beta$ for all $a \in A$. The $\approx$-class of an element $\alpha \in \mathfrak{I}$ will be denoted $[[\alpha]]$.

We easily get another characterisation of $\approx$ in terms of multiplication by elements of $\mathfrak{I}$, as given by:

Lemma VI.3.9. *Let $\alpha, \beta \in \mathfrak{I}$. Then $\alpha \approx \beta$ if and only if $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathfrak{I}$.*

Proof. This is immediate since

$$\begin{aligned} \alpha \approx \beta &\iff a\alpha \sim a\beta \quad \text{for all } a \in A, \\ &\iff a\alpha\gamma = a\beta\gamma \quad \text{for all } a \in A \text{ and } \gamma \in \mathfrak{I}, \\ &\iff \alpha\gamma = \beta\gamma \quad \text{for all } \gamma \in \mathfrak{I}. \end{aligned}$$ □

Under this new relation, we can now show that we have a faithful action on $\mathscr{A}/_{\sim}$.

Lemma VI.3.10. *The semigroup $\mathfrak{I}/_{\approx}$ acts faithfully on the right of $\mathscr{A}/_{\sim}$ by endomorphisms of $\mathscr{A}/_{\sim}$ via $[a][[\alpha]] = [a\alpha]$ for all $a \in A$ and $\alpha \in \mathfrak{I}$.*

Proof. Let $a, b \in A$ and $\alpha, \beta \in \mathfrak{I}$ be such that $[a] = [b]$ and $[[\alpha]] = [[\beta]]$. Then by definition of $\sim$ we get that $a\alpha = b\alpha$ and $a\beta = b\beta$, so that

$$[a\alpha] = [b\alpha] = [b\beta] = [a\beta],$$

where the middle equality follows from $\alpha \approx \beta$. This gives us that the action is well-defined. Moreover, if $[a][[\alpha]] = [a][[\beta]]$ for all $a \in A$, then we get that $[a\alpha] = [a\beta]$, that is, $a\alpha \sim a\beta$, which shows that $\alpha \approx \beta$ and thus $[[\alpha]] = [[\beta]]$, so that the action is faithful.

Finally, let $\{[a_i]\} \in \mathscr{A}/_{\sim}$, $t \in \mathcal{T}^{\mathscr{A}}$ and $[[\alpha]] \in \mathfrak{I}/_{\approx}$. Since $\sim$ is a congruence on $\mathscr{A}$, we have that $t\left(\overline{[a_i]}\right) = \left[t(\overline{a_i})\right]$, from which we get that

$$t\left(\overline{[a_i]}\right)[[\alpha]] = \left[t(\overline{a_i})\right][[\alpha]] = \left[t(\overline{a_i})\alpha\right] = \left[t(\overline{a_i\alpha})\right] = t\left(\overline{[a_i\alpha]}\right) = t\left(\overline{[a_i][[\alpha]]}\right),$$

and thus $\mathfrak{I}/_{\approx}$ acts on $\mathscr{A}/_{\sim}$ by morphisms. □

In view of Lemma VI.3.10, since $\mathfrak{I}/_{\approx}$ acts faithfully on $\mathscr{A}/_{\sim}$ by morphisms, we can view $\mathfrak{I}/_{\approx}$ as embedded in $\mathrm{End}\left(\mathscr{A}/_{\sim}\right)$. However, $\mathfrak{I}/_{\approx}$ is not necessarily a right or a left ideal of $\mathrm{End}\left(\mathscr{A}/_{\sim}\right)$. Thus, by Lemma I.2.41, we have that only the endomorphisms of $\mathscr{A}/_{\sim}$ lying in the idealiser of $\mathfrak{I}/_{\approx}$ will induce a bi-translation of $\mathfrak{I}/_{\approx}$.

Nevertheless, if we consider the definition of the property REP extended to subsemigroups, we can see that we have a new pair satisfying this condition, as given by the following lemma.

Lemma VI.3.11. *The pair* $\left(\mathscr{A}/_{\sim}, \mathfrak{I}/_{\approx}\right)$ *satisfies REP.*

Proof. Let $[a] \in \mathscr{A}/_{\sim}$. Since $(\mathscr{A}, \mathfrak{I})$ has REP, then $a = t(\overline{x_i\alpha_i})$ for some $t \in \mathcal{T}^{\mathscr{A}}$, $\{x_i\} \subseteq A$ and $\{\alpha_i\} \subseteq \mathfrak{I}$. Thus, we have that

$$[a] = [t(\overline{x_i\alpha_i})] = t\left(\overline{[x_i\alpha_i]}\right) = t\left(\overline{[x_i][[\alpha_i]]}\right).$$

Since $[x_i] \in \mathscr{A}/_{\sim}$ and $[[\alpha_i]] \in \mathfrak{I}/_{\approx}$, it follows that $\mathscr{A}/_{\sim}$ is generated by $\bigcup_{[[\alpha]] \in \mathfrak{I}/_{\approx}} \operatorname{im}[[\alpha]]$, and thus $\left(\mathscr{A}/_{\sim}, \mathfrak{I}/_{\approx}\right)$ has the property REP. □

On the other hand, we may not have that $\left(\mathscr{A}/_{\sim}, \mathfrak{I}/_{\approx}\right)$ has SEP in general, but this might happen under some conditions on $\mathfrak{I}$.

For example, suppose that $\mathfrak{I} = \mathfrak{I}^2$. Then for all $[a] \neq [b] \in \mathscr{A}/_{\sim}$, there exists $\alpha \in \mathfrak{I}$ such that $a\alpha \neq b\alpha$. Since $\alpha \in \mathfrak{I} = \mathfrak{I}^2$, there exist $\beta, \delta \in \mathfrak{I}$ such that $\alpha = \beta\delta$. Then $a\beta\delta = a\alpha \neq b\alpha = b\beta\delta$, and thus $[a\beta] \neq [b\beta]$, that is, $[a][[\beta]] \neq [b][[\beta]]$. Thus every pair of distinct elements in $\mathscr{A}/_{\sim}$ can be separated by an element of $\mathfrak{I}/_{\approx}$, which means that $\left(\mathscr{A}/_{\sim}, \mathfrak{I}/_{\approx}\right)$ has SEP.

Question VI.3.12. Is it possible to adapt $\sim$ and $\approx$ such that $\mathfrak{I}/_{\approx}$ is an ideal of $\operatorname{End}\left(\mathscr{A}/_{\sim}\right)$ and the pair $\left(\mathscr{A}/_{\sim}, \mathfrak{I}/_{\approx}\right)$ has both REP and SEP?

Recall that the initial aim was to compute the translational hull of $\mathfrak{I}$.

Question VI.3.13. What is the relationship between $\Omega(\mathfrak{I})$ and $\Omega\left(\mathfrak{I}/_{\approx}\right)$? Since we have lost information by taking the quotient using the relation $\approx$, is there a canonical way to reconstruct $\Omega(\mathfrak{I})$ from $\Omega\left(\mathfrak{I}/_{\approx}\right)$ and $\mathscr{A}/_{\sim}$?

VI.4 SEP WITHOUT REP

In this section, we will assume that our pair $(\mathscr{A}, \mathfrak{I})$ satisfies SEP but not REP. Then $\left\langle \bigcup_{\alpha\in\mathfrak{I}} \operatorname{im}\alpha \right\rangle$ is a proper subalgebra of $\mathscr{A}$, which we denote by $\mathscr{B}$. Since $(\mathscr{A}, \mathfrak{I})$ satisfies SEP, we have by Corollary VI.1.6 and Lemma VI.1.8 that π_Λ is an isomorphism and χ_Λ is injective, so that

$$\operatorname{End}(\mathscr{A}) \hookrightarrow \widetilde{\Lambda}(\mathfrak{I}) \cong \Omega(\mathfrak{I}).$$

This means in particular that every left translation of $\widetilde{\Lambda}(\mathfrak{I})$ is linked to exactly one right translation of $\mathfrak{I}$ and comes from at most one endomorphism of $\mathscr{A}$. We would like to show that every $\lambda \in \widetilde{\Lambda}(\mathfrak{I})$ comes from an endomorphism of $\mathscr{A}$. However, we get the following weaker statement.

Lemma VI.4.1. *Suppose that $(\mathscr{A}, \mathfrak{I})$ satisfies SEP, and let $\mathscr{B}$ be the subalgebra of $\mathscr{A}$ generated by $\bigcup_{\alpha\in\mathfrak{I}} \operatorname{im}\alpha$. Then every $\lambda \in \widetilde{\Lambda}(\mathfrak{I})$ induces a map $\psi\colon B \to B$ such that for all $\alpha \in \mathfrak{I}$ we have:*

- *ψ is a well-defined homomorphism;*
- *$\psi(\alpha|_B) = (\lambda\alpha)|_B$;*
- *$\alpha\psi = \alpha\rho$;*

where $\rho \in \mathrm{P}(\mathfrak{I})$ is the unique right translation linked to λ.

Proof. Let $\lambda \in \widetilde{\Lambda}(\mathfrak{I})$. Then there exists a unique $\rho \in \mathrm{P}(\mathfrak{I})$ such that $(\lambda, \rho) \in \Omega(\mathfrak{I})$. By definition of $\mathscr{B}$, for all $b \in B$, there exist $t \in \mathcal{T}^{\mathscr{B}}$, $\{y_j\} \subseteq A$ and $\{\beta_j\} \subseteq \mathfrak{I}$ such that $b = t\left(\overline{y_j\beta_j}\right)$. We now define $\psi\colon B \to B$ by:

$$b\psi = t\left(\overline{y_j\beta_j}\right)\psi := t\left(\overline{y_j\beta_j\rho}\right).$$

Notice that since $\beta_j\rho \in \mathfrak{I}$, we get that $y_j\beta_j\rho \in B$, and using the fact that $\mathscr{B}$ is a subalgebra, we obtain $t\left(\overline{y_j\beta_j\rho}\right) \in B$ so that $\operatorname{im}\psi \subseteq B$.

To show that ψ is well-defined, suppose that $t\left(\overline{y_j\beta_j}\right) = s\left(\overline{z_k\delta_k}\right)$ for some $s \in \mathcal{T}^{\mathscr{B}}$, $\{z_k\} \subseteq A$ and $\{\delta_k\} \subseteq \mathfrak{I}$. Then for all $\gamma \in \mathfrak{I}$ we get that

$$\begin{aligned} t\left(\overline{y_j\beta_j\rho}\right)\gamma &= t\left(\overline{y_j\beta_j\rho\gamma}\right) = t\left(\overline{y_j\beta_j\lambda\gamma}\right) \\ &= t\left(\overline{y_j\beta_j}\right)\lambda\gamma = s\left(\overline{z_k\delta_k}\right)\lambda\gamma \\ &= s\left(\overline{z_k\delta_k\lambda\gamma}\right) = s\left(\overline{z_k\delta_k\rho}\right)\gamma \end{aligned}$$

Since $(\mathscr{A}, \mathfrak{I})$ has SEP, it follows that $t\left(\overline{y_j\beta_j\rho}\right) = s\left(\overline{z_k\delta_k\rho}\right)$, which means that $t\left(\overline{y_j\beta_j}\right)\psi = s\left(\overline{z_k\delta_k}\right)\psi$ and thus ψ is well-defined on B.

Moreover, for $\alpha \in \mathfrak{I}$, we have

$$b\psi\alpha = t\left(\overline{y_j\beta_j}\right)\psi\alpha = t\left(\overline{y_j\beta_j\rho}\right)\alpha = t\left(\overline{y_j\beta_j\rho\alpha}\right) = t\left(\overline{y_j\beta_j\lambda\alpha}\right) = t\left(\overline{y_j\beta_j}\right)\lambda\alpha = b\lambda\alpha$$

so that $\psi(\alpha|_B) = (\lambda\alpha)|_B$. Also, for $\alpha, \gamma \in \mathfrak{I}$ and $a \in A$, by using the fact that $a\alpha \in B$, we get that

$$a\alpha\psi\gamma = a\alpha\lambda\gamma = a\alpha\rho\gamma.$$

Thus $\alpha\psi\gamma = \alpha\rho\gamma$, and since $(\mathscr{A}, \mathfrak{I})$ has SEP, we obtain that $\alpha\psi = \alpha\rho$.

It only remains to show that ψ is a homomorphism of B. Let $u \in \mathcal{T}^{\mathscr{B}}$ and $\{b_i\} \subseteq B$. Then $u\left(\overline{b_i}\right) \in B$ and for all $\gamma \in \mathfrak{I}$ we have:

$$u\left(\overline{b_i}\right)\psi\gamma = u\left(\overline{b_i}\right)\lambda\gamma = u\left(\overline{b_i\lambda\gamma}\right) = u\left(\overline{b_i\psi\gamma}\right) = u\left(\overline{b_i\psi}\right)\gamma,$$

which shows that $u\left(\overline{b_i}\right)\psi = u\left(\overline{b_i\psi}\right)$ by SEP. Therefore $\psi \in \mathrm{End}(\mathscr{B})$ as required. $\square$

Remark VI.4.2. In fact, the above result tells us that every $\rho \in \widetilde{\mathrm{P}}(\mathfrak{I})$ arises from a morphism $\psi\colon B \to B$. And then, if ρ is linked to the left translations λ and λ', we obtain that the restriction of $\lambda\alpha$ and $\lambda'\alpha$ to the subalgebra $\mathscr{B}$ must agree for all $\alpha \in \mathfrak{I}$. However, this does not necessarily mean that if ψ could be extended to an endomorphism $\overline{\psi}$ of $\mathrm{End}(\mathscr{A})$, we would have that $\overline{\psi}\alpha = \lambda\alpha$ for all $\alpha \in \mathfrak{I}$.

We now look into the converse of Lemma VI.4.1, by considering under which conditions do we have that an endomorphism ψ of $\mathscr{B}$ gives rise to a linked pair (λ, ρ) of $\Omega(\mathfrak{I})$ such that $\psi\alpha = (\lambda\alpha)|_B$ and $\alpha\psi = \alpha\rho$ for all $\alpha \in \mathfrak{I}$. Notice first that if we denote by $T(\mathscr{A}, \mathscr{B})$ the subsemigroup of $\mathrm{End}(\mathscr{A})$ consisting of all endomorphisms whose image lie in $\mathscr{B}$, as we did in Chapter II, that is, the semigroup

$$T(\mathscr{A}, \mathscr{B}) = \{\alpha \in \mathrm{End}(\mathscr{A}) \mid \mathrm{im}\,\alpha \subseteq \mathscr{B}\},$$

then it is easy to see that $\mathfrak{I} \subseteq T(\mathscr{A}, \mathscr{B})$ and that $\mathfrak{I}$ is an ideal of $T(\mathscr{A}, \mathscr{B})$. Since for any $\alpha \in \mathfrak{I}$ and $\psi\colon B \to B$, the composition of maps $\alpha\psi$ is well-defined, we then get that $\alpha\psi \in T(\mathscr{A}, \mathscr{B})$. However, because $\psi \notin \mathrm{End}(\mathscr{A})$, there is no reason to assume that $\alpha\psi$ actually lies in $\mathfrak{I}$.

Definition VI.4.3. We say that a morphism $\psi\colon B \to B$ is $\mathfrak{I}$-*closed* if $\alpha\psi \in \mathfrak{I}$ for all $\alpha \in \mathfrak{I}$.

Remark VI.4.4. Given a left translation $\lambda \in \widetilde{\Lambda}(\mathfrak{I})$, we say that the map ψ described in Lemma VI.4.1 is *the morphism on B induced by* λ. Moreover, notice that ψ is then $\mathfrak{I}$-closed, since for all $\alpha \in \mathfrak{I}$, we have that $\alpha\psi = \alpha\rho \in \mathfrak{I}$, where ρ is the unique right translation linked with λ.

Lemma VI.4.5. *Let $\psi\colon B \to B$ be an $\mathfrak{I}$-closed morphism. Then the map $\rho_\psi\colon \mathfrak{I} \to \mathfrak{I}$ defined by $\alpha\rho_\psi = \alpha\psi$ is a right translation of $\mathfrak{I}$. Moreover, distinct $\mathfrak{I}$-closed morphisms determine distinct right translations.*

Proof. Since ψ is $\mathfrak{I}$-closed, we have that $\alpha\psi \in \mathfrak{I}$ for all $\alpha \in \mathfrak{I}$, so that $\operatorname{im}\rho_\psi \subseteq \mathfrak{I}$. Moreover, for any $\gamma, \delta \in \mathfrak{I}$, using the associativity of function composition, we have that

$$(\gamma\delta)\rho_\psi = (\gamma\delta)\psi = \gamma(\delta\psi) = \gamma(\delta\rho_\psi)$$

and thus $\rho_\psi \in \mathrm{P}(\mathfrak{I})$.

Now let $\psi, \phi \in \operatorname{End}(\mathscr{B})$ be $\mathfrak{I}$-closed with $\rho_\psi = \rho_\phi$. By definition of $\mathscr{B}$, let $b \in B$ be such that $b = t\left(\overline{x_i\alpha_i}\right)$ for some $t \in \mathcal{T}^{\mathscr{B}}$, $\{x_i\} \subseteq A$ and $\{\alpha_i\} \subseteq \mathfrak{I}$. Using the fact that $x_i\alpha_i \in B$, we have that

$$b\psi = t\left(\overline{x_i\alpha_i}\right)\psi = t\left(\overline{x_i\alpha_i\psi}\right) = t\left(\overline{x_i\alpha_i\rho_\psi}\right) = t\left(\overline{x_i\alpha_i\rho_\phi}\right) = t\left(\overline{x_i\alpha_i\phi}\right) = t\left(\overline{x_i\alpha_i}\right)\phi = b\phi,$$

and since this holds for all $b \in B$, we get that $\psi = \phi$ as required. □

Combining the two previous results, we obtain the following corollary.

Corollary VI.4.6. *There exists a subsemigroup* $\mathfrak{C} \subseteq \operatorname{End}(\mathscr{B})$ *consisting of* $\mathfrak{I}$*-closed morphisms, such that* $\mathfrak{C}$ *is isomorphic to* $\widetilde{\mathrm{P}}(\mathfrak{I})$*.*

Proof. Using Lemma VI.4.1 and Remark VI.4.4, if we define the set $\mathfrak{C}$ by

$$\mathfrak{C} = \left\{\psi\colon B \to B \mid \psi \text{ is induced by } \lambda, \text{ for some } \lambda \in \widetilde{\Lambda}(\mathfrak{I})\right\},$$

we get that all elements of $\mathfrak{C}$ are $\mathfrak{I}$-closed morphisms.

Moreover, suppose that $\lambda \in \widetilde{\Lambda}(\mathfrak{I})$, and let $\psi \in \mathfrak{C}$ be its induced endomorphism of $\mathscr{B}$. By Lemma VI.4.5, we have that ρ_ψ is a right translation. Moreover, if we let $\rho \in \widetilde{\mathrm{P}}(\mathfrak{I})$ be the unique right translation linked to λ, then we get that $\alpha\rho_\psi = \alpha\psi = \alpha\rho$ for all $\alpha \in \mathfrak{I}$ by definition of ρ_ψ, so that $\rho_\psi = \rho \in \widetilde{\mathrm{P}}(\mathfrak{I})$. In particular, this shows that $\mathfrak{C}$ is a subsemigroup of $\operatorname{End}(\mathscr{B})$.

Thus, let $\xi\colon \mathfrak{C} \to \widetilde{\mathrm{P}}(\mathfrak{I})$ be the map which sends ψ to ρ_ψ. The fact that this is surjective and well-defined follows from the arguments above, while injectivity comes from Lemma VI.4.5.

Finally, if $\psi, \phi \in \mathfrak{C}$, then for all $\alpha \in \mathfrak{I}$, we get that

$$\alpha\rho_{\psi\phi} = \alpha\psi\phi = (\alpha\psi)\rho_\phi = \alpha\rho_\psi\rho_\phi,$$

where the second equality comes from the fact that $\alpha\psi \in \mathfrak{I}$ since ψ is $\mathfrak{I}$-closed. Thus ξ is an isomorphism from $\mathfrak{C}$ onto $\widetilde{\mathrm{P}}(\mathfrak{I})$. □

The construction above immediately raises the following question.

Question VI.4.7. Is it possible to give a characterisation for the elements of the semigroup $\mathfrak{C}$ of Corollary VI.4.6?

We know that every left translation is linked to at most one right translation. On the contrary, a right translation could be linked to multiple left translations. Since Lemma VI.4.5 and Corollary VI.4.6 are concerned with right translations that are linked, we can ask the following.

Question VI.4.8. Given a right translation $\rho \in \widetilde{\mathrm{P}}(\mathfrak{I})$, can we describe all left translations $\lambda \in \widetilde{\Lambda}(\mathfrak{I})$ such that $(\lambda, \rho) \in \Omega(\mathfrak{I})$?

— VII —

Computing translational hulls

We have seen in Chapter VI that given an algebra $\mathscr{A}$ and $\mathfrak{I}$ an ideal of its endomorphism monoid $\mathrm{End}(\mathscr{A})$, if the pair $(\mathscr{A}, \mathfrak{I})$ satisfies REP and SEP, then we obtain an isomorphism between the translational hull $\Omega(\mathfrak{I})$ and $\mathrm{End}(\mathscr{A})$. In this chapter, we will give applications of this result for certain algebras, and present some limits to this approach when either or both conditions are not met, and where the translational hulls are hard to compute.

We will start by looking at free algebras in Section VII.1 before considering the special case of independence algebras in Section VII.2, where we will compare the results obtained through this approach with those given in Chapter V. Finally, in Section VII.3, we will focus on the endomorphism monoid $\mathrm{End}(\mathcal{T}_n)$ described in Chapter IV. Even though we are able to describe all left and right translations on each ideal of $\mathrm{End}(\mathcal{T}_n)$, we will show that bi-translations are very hard to understand and to compute, since they are not necessarily coming from transformations of the underlying algebra $\mathcal{T}_n$.

Note. Work present on this chapter follows from collaborations with Prof. Victoria Gould, Dr. Marianne Johnson and Prof. Mark Kambites.

VII.1 FREE ALGEBRAS

Throughout this section, we assume that $\mathscr{A}$ is a free algebra over $X = \{x_i\} \subseteq A$. Hence, we can write any $a \in A$ as $a = t(\overline{x_i})$ for some $t \in \mathcal{T}^{\mathscr{A}}$, and we will make use of this fact without further mention. Recall that e denote the smallest cardinality of a generating set for a subalgebra of $\mathscr{A}$, and that κ^+ denotes the successor cardinal of κ.

Lemma VII.1.1. *For each $e < \mu \leq |X|^+$, the set*

$$\mathfrak{I}_\mu := \{\alpha \in \operatorname{End}(\mathscr{A}) \mid \operatorname{im}\alpha \subseteq \langle Y \rangle \textit{ for some } Y \subseteq A \textit{ with } |Y| < \mu\}$$

is an ideal of $\operatorname{End}(\mathscr{A})$*.*

Proof. If $\alpha \in \mathfrak{I}_\mu$ with $\operatorname{im}\alpha \subseteq \langle Y \rangle$, and $\beta \in \operatorname{End}(\mathscr{A})$, then we have that $\operatorname{im}(\beta\alpha) \subseteq \operatorname{im}\alpha \subseteq \langle Y \rangle$ and $\operatorname{im}(\alpha\beta) \subseteq \langle Y \rangle\beta = \langle Y\beta \rangle$. Since $|Y| < \mu$ we also have that $|Y\beta| < \mu$ and thus $\alpha\beta, \beta\alpha \in \mathfrak{I}_\mu$, so that $\mathfrak{I}_\mu$ is an ideal of $\operatorname{End}(\mathscr{A})$. □

We now consider the properties REP and SEP. Clearly, if $\mu = |X|^+$, we get that $\mathfrak{I}_\mu = \operatorname{End}(\mathscr{A})$, so that both conditions hold for the pair $(\mathscr{A}, \mathfrak{I}_\mu)$. In fact, it is easy to show that we will always have REP if $\mu > 1$.

Lemma VII.1.2. *The pair $(\mathscr{A}, \mathfrak{I}_\mu)$ has REP unless $\mu = 1$ and $\emptyset \neq \langle \emptyset \rangle \neq A$.*

Proof. Let us assume first that $\mu \geq 2$, and define for each $a \in A$ a map $\alpha_a \in \operatorname{End}(\mathscr{A})$ by $x_i\alpha_a = a$ for all $x_i \in X$. Then $\operatorname{im}\alpha_a \subseteq \langle a \rangle$, which shows that $\alpha_a \in \mathfrak{I}_\mu$, and we immediately get that $A = \bigcup_{a \in A} \operatorname{im}\alpha_a \subseteq \langle \bigcup_{\alpha \in \mathfrak{I}_\mu} \operatorname{im}\alpha \rangle$, so that $(\mathscr{A}, \mathfrak{I}_\mu)$ has REP.

The only possibility left is when $\mu = 1$ which can only happens when $\langle \emptyset \rangle \neq \emptyset$. Then either $\langle \emptyset \rangle = A$, in which case the argument above works and $(\mathscr{A}, \mathfrak{I}_1)$ has REP, or $\langle \emptyset \rangle \neq A$, which means that there exists some $a \in A$ such that $a \notin \langle \emptyset \rangle$, so that $a \notin \left\langle \bigcup_{\alpha \in \mathfrak{I}_1} \operatorname{im}\alpha \right\rangle$, and we do not have REP in this case. □

On the other hand, when considering SEP, we have that if $\mu = 2$ and $|\operatorname{im}\alpha| = 1$ for all $\alpha \in \mathfrak{I}_\mu$, then no two elements of A can be separated by an element of $\mathfrak{I}_\mu$, so that $(\mathscr{A}, \mathfrak{I}_\mu)$ does not satisfy SEP in that case. In fact, the property SEP is almost always equivalent to the fact that the ideal considered is right reductive, as given by the following.

Lemma VII.1.3. *Suppose that $\mu > 1$. Then $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP if and only if $\mathfrak{I}_\mu$ is right reductive.*

Proof. One direction is directly given by Lemma VI.1.4. So assume that $\mathfrak{I}_\mu$ is right reductive. Let $a \neq b \in A$, and consider the maps $\alpha, \beta \in \operatorname{End}(\mathscr{A})$ defined by $x_i\alpha = a$ and $x_i\beta = b$ for all $x_i \in X$. Then $\operatorname{im}\alpha \subseteq \langle a \rangle$ and $\operatorname{im}\beta \subseteq \langle b \rangle$, which shows that $\alpha, \beta \in \mathfrak{I}_\mu$ since $\mu \geq 2$. Let us assume that $a\gamma = b\gamma$ for all $\gamma \in \mathfrak{I}_\mu$. Then we get that

$x_i\alpha\gamma = x_i\beta\gamma$ for all $x_i \in X$, and thus $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathfrak{I}_\mu$ as $A = \langle X \rangle$. Since $\mathfrak{I}_\mu$ is right reductive, it follows that $\alpha = \beta$, that is, $a = x_i\alpha = x_i\beta = b$, a contradiction. Therefore for all $a \neq b \in A$, there exists $\gamma \in \mathfrak{I}_\mu$ such that $a\gamma \neq b\gamma$, that is, $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP. □

When we do have that $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP, we can of course use Theorem VI.2.1 to get that $\Omega(\mathfrak{I}_\mu) \cong \mathscr{A}$. Finding out when SEP is satisfied relies a lot on the actual structure of the algebra considered, as is shown by the following example.

Example VII.1.4. We give two different free algebras where we consider $\mathfrak{I}_2$, and show that SEP will be satisfied in one case but not in the other.

1) Let $\mathscr{A}$ be a free group action algebra of a non-trivial group G over a set X (as defined in Section I.5.2). Then $\alpha \in \mathfrak{I}_2$ if and only if $\operatorname{im}\alpha = \langle x \rangle = Gx$ for some $x \in X$. Now let $a \neq b \in A$ be such that $a = g(x_j)$ and $b = h(x_k)$ for some $g, h \in G$ and $x_j, x_k \in X$. If $x_j = x_k$, it follows that $h \neq g$, and we can define $\alpha \in \mathfrak{I}_2$ by $x_i\alpha = x_j$ for all $x_i \in X$, from which we get that $a\alpha = g(x_j) \neq h(x_j) = b\alpha$. Otherwise, $x_j \neq x_k$, and since G is not trivial, there exists $f \in G$ such that $hf \neq g$. Then define $\alpha \in \mathfrak{I}_2$ by $x_k\alpha = f(x_j)$ and $x_i\alpha = x_j$ for all $x_k \neq x_i \in X$. It follows that $a\alpha = g(x_j) \neq hf(x_j) = b\alpha$. In both cases, we get that there exists $\alpha \in \mathfrak{I}_2$ such that $a\alpha \neq b\alpha$, so that $(\mathscr{A}, \mathfrak{I}_2)$ has SEP.

2) Let $\mathscr{A}$ be the free monoid on two elements, that is, $A = \{a, b\}^*$. Then for each $\alpha \in \mathfrak{I}_2$, there exists a word $u \in A$ such that $\operatorname{im}\alpha \subseteq \left\{u^k : k \geq 0\right\}$. Then, if $a\alpha = u^m$ and $b\alpha = u^n$ for some $m, n \in \mathbb{N}_0$, it follows that α is completely determined by u, m and n. Thus for each word $u \in A$, and each pair of non-negative integers m and n, we write $u^{(m,n)}$ to denote the unique element of $\mathfrak{I}_2$ mapping a to u^m and b to u^n. Now for $x \in \{a, b\}$, denote by $|u|_x$ the number of occurrences of letter x in u, and let $v \neq w \in A$ be such that $|v|_a = |w|_a$ and $|v|_b = |w|_b$. Then it is easy to see that for all $u^{(m,n)} \in \mathfrak{I}_2$, we have

$$vu^{(m,n)} = u^{(m|v|_a + n|v|_b)} = u^{(m|w|_a + n|w|_b)} = wu^{(m,n)},$$

so that $(\mathscr{A}, \mathfrak{I}_2)$ does not satisfy SEP.

Since we have REP but not SEP, we could follow the process in Section VI.3 to quotient $\mathscr{A}$ and $\mathfrak{I}_\mu$. However, it is possible to describe the translational hull of $\mathfrak{I}_\mu$ precisely in free algebras as coming from maps $\phi\colon A \to A$ which behave well with respect to the endomorphisms in $\mathfrak{I}_\mu$. We start by some definitions, where we write $\sim_\mu$ for the congruence relation on $\mathscr{A}$ induced by elements of $\mathfrak{I}_\mu$ as given in Definition VI.3.1, that is, $a \sim_\mu b$ if and only if $a\gamma = b\gamma$ for all $\gamma \in \mathfrak{I}_\mu$.

Definition VII.1.5. Let $\phi\colon A \to A$. Then the map ϕ is a

- *right μ-morphism of A* if for all $Y \subseteq A$ with $|Y| < \mu$ we have that $\phi|_{\langle Y\rangle}\colon \langle Y\rangle \to A$ is a morphism;
- *left μ-morphism of A* if $t\left(\overline{x_i\phi}\right) \sim_\mu t\left(\overline{x_i}\right)\phi$ for all $t \in \mathcal{T}^{\mathscr{A}}$; or
- *μ-morphism of A* if ϕ is a right μ-morphism and a left μ-morphism.

Remark VII.1.6. Notice that if $\sim_\mu$ is equality, that is, if $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP, then a left μ-morphism is precisely a morphism.

Using these definitions, we can now describe the right and left translations of $\mathfrak{I}_\mu$ in terms of maps on A.

Lemma VII.1.7. *A map $\phi\colon A \to A$ is a right μ-morphism if and only if the map $\rho_\phi\colon \mathfrak{I}_\mu \to \mathfrak{I}_\mu$ given by $\alpha\rho_\phi = \alpha\phi$ is a right translation of $\mathfrak{I}_\mu$.*

Proof. Let $\phi\colon A \to A$ be a right μ-morphism and $\alpha \in \mathfrak{I}_\mu$. Then $\operatorname{im}\alpha \subseteq \langle Y\rangle = B$ for some $Y \subseteq A$ with $|Y| < \mu$. By definition, we also have that $\phi|_B$ is a morphism. Moreover, $\operatorname{im}(\alpha\phi) \subseteq \operatorname{im}(\phi|_B) = \langle Y\phi\rangle$ with $|Y\phi| \leq |Y| < \mu$. Therefore $\alpha\phi \in \mathfrak{I}_\mu$ for all $\alpha \in \mathfrak{I}_\mu$. It follows from this observation that the map $\rho_\phi\colon \mathfrak{I}_\mu \to \mathfrak{I}_\mu$ given by $\alpha\rho_\phi = \alpha\phi$ is well-defined. Using the associativity of composition of maps, for all $\gamma, \delta \in \mathfrak{I}_\mu$ we also have that $(\gamma\delta)\rho_\phi = \gamma\delta\phi = \gamma(\delta\rho_\phi)$, which shows that ρ_ϕ is indeed a right translation of $\mathfrak{I}_\mu$.

Conversely, suppose that $\phi\colon A \to A$ is such that $\rho_\phi \in \mathrm{P}(\mathfrak{I}_\mu)$. Let $\{y_j\} = Y \subseteq A$ be such that $|Y| < \mu$. Then $|Y| \leq |X|$ and for each $y_j \in Y$, we pick some $x_j \in X$ to define $\alpha \in \operatorname{End}(\mathscr{A})$ by $x_j\alpha = y_j$ and $x_k\alpha = c$ for some $c \in \langle Y\rangle$ if $x_k \notin \{x_j\}$. Therefore $\operatorname{im}\alpha \subseteq \langle Y\rangle$, which shows that $\alpha \in \mathfrak{I}_\mu$. Now, for all $t \in \mathcal{T}^{\mathscr{A}}$ we have that

$$t(\overline{y_j})\phi = t(\overline{x_j\alpha})\phi = t(\overline{x_j})\alpha\phi = t(\overline{x_j})\alpha\rho_\phi = t\left(\overline{x_j\alpha\rho_\phi}\right) = t\left(\overline{x_j\alpha\phi}\right) = t\left(\overline{y_j\phi}\right),$$

where we used the fact that $\alpha\rho_\phi \in \mathfrak{I}_\mu$. Hence, the map ϕ restricts to a morphism on $\langle Y\rangle$. Since $Y \subseteq A$ was arbitrary under the property that $|Y| < \mu$, it follows that ϕ is a right μ-morphism. $\square$

Lemma VII.1.8. *A map $\phi\colon A \to A$ is a left μ-morphism if and only if the map $\lambda_\phi\colon \mathfrak{I}_\mu \to \mathfrak{I}_\mu$ given by $\lambda\alpha = \phi\alpha$ is a left translation of $\mathfrak{I}_\mu$.*

Proof. Let $\phi\colon A \to A$ be a left μ-morphism and $\alpha \in \mathfrak{I}_\mu$. Then for all $t \in \mathcal{T}^{\mathscr{A}}$, we get

$$t\Big(\overline{x_i\phi\alpha}\Big) = t\Big(\overline{x_i\phi}\Big)\alpha = t\Big(\overline{x_i}\Big)\phi\alpha,$$

where the last equality comes from the fact that $t(\overline{x_i\phi}) \sim_\mu t(\overline{x_i})\phi$ by definition. Hence, $\phi\alpha$ is a morphism of A. Moreover, $\operatorname{im}(\phi\alpha) \subseteq \operatorname{im}\alpha \subseteq \langle Y\rangle$ with $|Y| < \mu$, so that $\phi\alpha \in \mathfrak{I}_\mu$. This shows that the map $\lambda_\phi\colon \mathfrak{I}_\mu \to \mathfrak{I}_\mu$ is well-defined. Now, by associativity of composition of maps, we have that $\lambda_\phi(\gamma\delta) = \phi\gamma\delta = (\lambda_\phi\gamma)\delta$ for all $\gamma, \delta \in \mathfrak{I}_\mu$. Therefore λ_ϕ is a left translation of $\mathfrak{I}_\mu$.

Conversely, let $\phi\colon A \to A$ be such that $\lambda_\phi \in \Lambda(\mathfrak{I}_\mu)$. Then for all $\alpha \in \mathfrak{I}_\mu$ and $t \in \mathcal{T}^{\mathscr{A}}$, we have that $\phi\alpha = \lambda_\phi\alpha \in \mathfrak{I}_\mu$ and then

$$t\Big(\overline{x_i\phi}\Big)\alpha = t\Big(\overline{x_i\phi\alpha}\Big) = t\Big(\overline{x_i\lambda_\phi\alpha}\Big) = t\Big(\overline{x_i}\Big)\lambda_\phi\alpha = t\Big(\overline{x_i}\Big)\phi\alpha,$$

which shows that $t\Big(\overline{x_i\phi}\Big) \sim_\mu t\Big(\overline{x_i}\Big)\phi$. Therefore ϕ is a left μ-morphism. □

We can now give an exact description of the translational hull of $\mathfrak{I}_\mu$.

Theorem VII.1.9. *For $\mu \geq 2$, the translational hull of $\mathfrak{I}_\mu$ is given by*

$$\Omega(\mathfrak{I}_\mu) = \{(\lambda_\phi, \rho_\phi) : \phi \textit{ is a } \mu\textit{-morphism of } A\}.$$

Proof. If $\phi\colon A \to A$ is a μ-morphism, then it is a right and a left μ-morphism, so that by Lemmas VII.1.7 and VII.1.8, we obtain the translations $\rho_\phi \in \mathrm{P}(\mathfrak{I}_\mu)$ and $\lambda_\phi \in \Lambda(\mathfrak{I}_\mu)$. Moreover, for all $\gamma, \delta \in \mathfrak{I}_\mu$, we have that

$$\gamma\lambda_\phi\delta = \gamma\phi\delta = \gamma\rho_\phi\delta,$$

and thus $(\lambda_\phi, \rho_\phi) \in \Omega(\mathfrak{I}_\mu)$.

Conversely, we want to show that if $(\lambda, \rho) \in \Omega(\mathfrak{I}_\mu)$, then there exists a μ-morphism ϕ of A such that $\rho = \rho_\phi$ and $\lambda = \lambda_\phi$. We start with some notation. Since $\mathscr{A}$ is a free algebra over $X = \{x_i\}$, for $\alpha \in \operatorname{End}(\mathscr{A})$ such that $x_i\alpha = a_i$ with $\{a_i\} \subseteq A$, we write $\alpha = (a_i)_X$ as a shorthand. In particular, $(a)_X$ denotes the morphism that maps all generators in X to the same element $a \in A$.

Let $\rho \in \widetilde{\mathrm{P}}(\mathfrak{I}_\mu)$. We first show that for all $a \in A$, there exists $b \in A$ such that $(a)_X\rho = (b)_X$. By definition, since $(a)_X\rho \in \mathfrak{I}_\mu$, there exists $\{b_i\} \subseteq A$ such that $(a)_X\rho = (b_i)_X$. Then for all $y \in X$, we have that $(y)_X(a)_X = (a)_X$, and thus

$$(y)_X(b_i)_X = (y)_X((a)_X\rho) = ((y)_X(a)_X)\rho = (a)_X\rho = (b_i)_X.$$

In particular, for all $b_k \in \{b_i\}$, we get

$$b_k = x_k(b_i)_X = x_1(x_k)_X(b_i)_X = x_1(b_i)_X = b_1,$$

which shows that $(a)_X\rho = (b)_X$ for some $b \in A$.

We can now define $\phi\colon A \to A$ by $a\phi = b$ where $b \in A$ is such that $(a)_X\rho = (b)_X$ or, equivalently, $a\phi = y((a)_X\rho)$ for any $y \in X$. In particular, given $\{a_i\}, \{b_i\} \subseteq A$ such that $(a_i)_X, (b_i)_X \in \mathfrak{I}_\mu$ and $(a_i)_X\rho = (b_i)_X$, we obtain that for all $y \in X$ and $a_k \in \{a_i\}$:

$$\begin{aligned} a_k\phi &= y(a_k\phi)_X = y\Big((a_k)_X\rho\Big) = y\Big((x_k)_X(a_i)_X\Big)\rho \\ &= y(x_k)_X\Big((a_i)_X\rho\Big) = y(x_k)_X(b_i)_X = y(b_k)_X \\ &= b_k, \end{aligned}$$

and thus $(a_i)_X\rho = (a_i\phi)_X$. We shall now see that ϕ is the desired μ-morphism. Since $\rho \in \widetilde{\mathrm{P}}(\mathfrak{I}_\mu)$, there exists $\lambda \in \widetilde{\Lambda}(\mathfrak{I}_\mu)$ such that (λ, ρ) is a linked pair. We now show that $\rho = \rho_\phi$ and $\lambda = \lambda_\phi$.

• *$\rho = \rho_\phi$:* Let $(a_i)_X \in \mathfrak{I}_\mu$. Then for all $x_j \in X$, we have:

$$x_j(a_i)_X\rho = x_j(a_i\phi)_X = a_j\phi = x_j(a_i)_X\phi,$$

which shows that $(a_i)_X\rho = (a_i)_X\phi$, and therefore $\rho = \rho_\phi$.

• *$\lambda = \lambda_\phi$:* For any $a \in A$, we have that $(a)_X \in \mathfrak{I}_\mu$. Then, for any $\beta \in \mathfrak{I}_\mu$ and $y \in X$, we get:

$$a\phi\beta = y(a)_X\phi\beta = y(a)_X\rho\beta = y(a)_X\lambda\beta = a\lambda\beta,$$

which shows that $\phi\beta = \lambda\beta$, and thus $\lambda = \lambda_\phi$.

Since ρ_ϕ and λ_ϕ are respectively a right and a left translation, it follows by Lemmas VII.1.7 and VII.1.8 that ϕ is a right μ-morphism as well as a left μ-morphism, hence a μ-morphism. Therefore $\Omega(\mathfrak{I}) \subseteq \{(\lambda_\phi, \rho_\phi) : \phi \text{ is a } \mu\text{-morphism of } A\}$, which finishes the proof. □

As an immediate consequence, we get the following corollary.

Corollary VII.1.10. *For $\mu \geq 2$, $\Omega(\mathfrak{I}) \cong \mathrm{End}(\mathscr{A})$ if and only if all μ-morphisms of A are endomorphisms.*

Proof. Since $(\mathscr{A}, \mathfrak{I}_\mu)$ has REP by Lemma VII.1.2, it follows that $\chi_{\mathrm{P}}\colon \mathrm{End}(\mathscr{A}) \to \Omega(\mathfrak{I}_\mu)$ sending ϕ to $(\lambda_\phi, \rho_\phi)$ is an injective morphism by Lemma VI.1.7. Hence, using Theorem VII.1.9, we get that χ_{P} is an isomorphism if and only if it is surjective, if and only if all μ-morphisms of A are endomorphisms. □

Remark VII.1.11. Notice that if $\mu = 1$, then $\mathfrak{I}_\mu$ is a left-zero semigroup, and by Proposition I.2.23, we get that $\Omega(\mathfrak{I}_\mu) \cong \mathcal{T}_{\mathfrak{I}_\mu}^{\mathrm{op}}$.

Example VII.1.12. Coming back to 2) in Example VII.1.4, it can be shown that $\phi\colon A \to A$ is a right 2-morphism if and only if for all words $u \in A$ and $k \in \mathbb{N}_0$, we have $(u^k)\phi = (u\phi)^k$, and the right translation associated to it is defined by $u^{(m,n)}\rho_\phi = (u\phi)^{(m,n)}$. On the other hand $\phi\colon A \to A$ is a left 2-morphism if and only if for all $u \in A$ we have

$$\begin{cases} |u\phi|_a = |u|_a|a\phi|_a + |u|_b|b\phi|_a & \text{and} \\ |u\phi|_b = |u|_a|a\phi|_b + |u|_b|b\phi|_b, \end{cases}$$

that is, ϕ is linearly increasing the number of letters a and b in the word, but does not recognise the order in which they appear. Because elements of $\mathfrak{I}_2$ acts on the same way on words with the same letters, it follows that the left translation associated to a left 2-morphism ϕ is defined by $\lambda_\phi u^{(m,n)} = u^{(m|a\phi|_a+n|a\phi|_b,\, m|b\phi|_a+n|b\phi|_b)}$. Thus we have bi-translations that do not come from endomorphisms since the 2-morphisms of A do not "see" the word they act upon, but only the number of letters it contains. For example, if $\phi\colon A \to A$ is the map reversing a word, then ϕ is a 2-morphism, but it is not an endomorphism.

VII.2 INDEPENDENCE ALGEBRAS

Throughout this section, let $\mathscr{A}$ be an independence algebra. We know by Corollary I.5.7 that any ideal of $\mathrm{End}(\mathscr{A})$ is of the form $T_\mu = \{\alpha \in \mathrm{End}(\mathscr{A}) \mid \mathrm{rk}(\alpha) < \mu\}$, which in this case corresponds exactly to the ideals $\mathfrak{I}_\mu$ defined above. We have already treated the case of the (0-)minimal ideal of $\mathrm{End}(\mathscr{A})$ in Chapter V, but we can now consider any ideals of $\mathrm{End}(\mathscr{A})$ by using the properties REP and SEP. Since an independence algebra is free on its basis, it follows that all the results of Section VII.1 above hold, and thus by Lemma VII.1.2, we have that $(\mathscr{A}, \mathfrak{I}_\mu)$ has REP, for all $\mu \geq 2$.

Remark VII.2.1. Note that for independence algebras, the definition of REP is equivalent to that of representability given in Section V.3.2. Indeed, suppose that $X = \{x_i\} \subseteq A$ is a basis of A, and that $(\mathscr{A}, \mathfrak{I}_\mu)$ has REP. Let $a \in A$ and define $\alpha_a \in \mathrm{End}(\mathscr{A})$ by $x_i\alpha_a = a$ for all $x_i \in X$. Clearly, if $\mu > 1$, then $\alpha_a \in \mathfrak{I}_\mu$, so that

$$A = \bigcup_{a\in A} \operatorname{im}\alpha_a = \bigcup_{\gamma\in\mathfrak{I}_\mu} \operatorname{im}\gamma,$$

and $\mathscr{A}$ is representable by $\mathfrak{I}_\mu$.

Now, suppose that $\mu = 1$. If $a \notin \langle\emptyset\rangle$, then $a \notin \left\langle \bigcup_{\gamma\in\mathfrak{I}_\mu} \operatorname{im}\gamma \right\rangle$ since $\operatorname{im}\gamma \subseteq \langle\emptyset\rangle$ for all $\gamma \in \mathfrak{I}_1$, which contradicts the fact that $(\mathscr{A}, \mathfrak{I}_\mu)$ has REP. Thus we must have that $a \in \langle\emptyset\rangle$ for all $a \in A$ in this case, and therefore $\alpha_a \in \mathfrak{I}_\mu$ so that $\mathscr{A}$ is representable by $\mathfrak{I}_\mu$.

Moreover, in an independence algebra, it is easy to see that $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP for all $\mu > 1$, unless 1-dimensional subalgebras are singletons, as given by the following:

Lemma VII.2.2. *Let $\mu > 1$. Then $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP if and only if one of the following happens:*

1) $\mu > 2$; or

2) every 1-dimensional subalgebra of $\mathscr{A}$ contains at least two elements.

Proof. Suppose that $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP and $\mu = 2$. Let $a \neq b \in A$. Then there exists $\gamma \in \mathfrak{I}_\mu$, such that $a\gamma \neq b\gamma$. Since $\mu = 2$, it follows that $\operatorname{im}\gamma$ is a one-dimensional subalgebra which contains at least two elements. Since all one-dimensional subalgebras are isomorphic, it follows from Remark I.4.35 that they must all contain at least two elements, as required.

We now want to show that if either of conditions 1) or 2) hold, then we must have that $(\mathscr{A}, \mathfrak{I}_\mu)$ satisfies SEP. Let $a \neq b \in \mathscr{A}$ and write $B = \langle a, b\rangle$ and $A = B \sqcup \langle\{x_i\}\rangle$. If $a, b \in \langle\emptyset\rangle$, then $a\gamma = a \neq b = b\gamma$ for all $\gamma \in \mathfrak{I}_\mu$, so assume from now on that $a \notin \langle\emptyset\rangle$, so that either $b = t(a)$ for some $t \in \mathcal{T}_1^{\mathscr{A}}$, or $\{a, b\}$ is independent.

If $b = t(a)$, we define $\gamma = \begin{pmatrix} a & x_i \\ a & a \end{pmatrix}$, so that $\operatorname{rk}(\gamma) = 1 < \mu$, and thus $\gamma \in \mathfrak{I}_\mu$. Moreover, $b\gamma = t(a)\gamma = t(a\gamma) = t(a) = b \neq a = a\gamma$.

Suppose now that $\{a, b\}$ is independent. If $\mu > 2$, we let $\gamma = \begin{pmatrix} a & b & x_i \\ a & b & a \end{pmatrix}$, so that $\operatorname{rk}(\gamma) = 2$ and then $\gamma \in \mathfrak{I}_\mu$ with $a\gamma = a \neq b = b\gamma$. If $\mu = 2$ and 1-dimensional subalgebras are not singleton, it follows that there exists $s \in \mathcal{T}_1^{\mathscr{A}}$ such that $a \neq s(a)$. Then, if we define $\gamma = \begin{pmatrix} a & b & x_i \\ a & s(a) & a \end{pmatrix}$, we get that $\operatorname{rk}(\gamma) = 1$ so that $\gamma \in \mathfrak{I}_\mu$ and we also have $a\gamma = a \neq s(a) = b\gamma$.

Hence, in all cases, there exists a map $\gamma \in \mathfrak{I}_\mu$ such that $a\gamma \neq b\gamma$, which shows that $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP under either condition. □

The case when $\langle\emptyset\rangle \neq \emptyset$ and $\mu = 1$ is very different and is given by:

Lemma VII.2.3. *Let $\mu = 1$ and $C = \langle\emptyset\rangle$. Then $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP if and only if $|C| \geq 2$ and for all non-constant unary terms $t \neq \mathrm{id}$ there exists $c \in C$ with $c \neq t(c)$.*

Proof. Notice first that since $\mu = 1$, it follows that $\operatorname{im}\gamma = C$ for all $\gamma \in \mathfrak{I}_\mu$.

If $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP, then for $a \neq b \in A$, there exists $\gamma \in \mathfrak{I}_\mu$ such that $a\gamma \neq b\gamma$. Thus $|C| \geq 2$. Moreover, if $t \in \mathcal{T}^{\mathscr{A}}$ is a non-constant unary term such that $t \neq \mathrm{id}$, then for all independent elements $a \in A$, we have that $t(a) \neq a$. Then, by SEP, there exists $\gamma \in \mathfrak{I}_\mu$ such that $a\gamma \neq t(a)\gamma = t(a\gamma)$, that is, there exists $c \in C$ such that $c \neq t(c)$.

Conversely, suppose that $|C| \geq 2$ and that for each non-constant unary term $t \in \mathcal{T}_1^{\mathscr{A}} \setminus \{\mathrm{id}\}$, we have an element $c_t \in C$ such that $c_t \neq t(c_t)$. Let $a \neq b \in A$. If $a, b \in C$, then clearly $a\gamma \neq b\gamma$ for all $\gamma \in \mathfrak{I}_\mu$, so suppose that $a \notin C$, so that a is independent and we can write $B = \langle a, b\rangle$ and $A = B \sqcup \langle\{x_i\}\rangle$. We then have one of the following situations:

- $b \in C$, so that $b \neq d$ for some $d \in C$ and we define $\gamma = \begin{pmatrix} a & x_i \\ d & d \end{pmatrix}$;
- $b = t(a)$ for some non-constant $t \in \mathcal{T}_1^{\mathscr{A}}$, and we let $\gamma = \begin{pmatrix} a & x_i \\ c_t & c_t \end{pmatrix}$; or
- $\{a, b\}$ is independent, and we define $\gamma = \begin{pmatrix} a & b & x_i \\ c & d & d \end{pmatrix}$ for some $c \neq d \in C$.

In all cases, we get that $\gamma \in \mathfrak{I}_\mu$ with $a\gamma \neq b\gamma$ and thus $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP. □

We can now give equivalent conditions for when the translational hull of $\mathfrak{I}_\mu$ is naturally isomorphic to $\widetilde{\mathrm{P}}(\mathfrak{I}_\mu)$, $\widetilde{\Lambda}(\mathfrak{I}_\mu)$ and $\mathrm{End}(\mathscr{A})$.

Corollary VII.2.4. *Let $\mu > 1$. Then the following are equivalent:*

1) *$(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP;*
2) *$\Omega(\mathfrak{I}_\mu) \cong \widetilde{\mathrm{P}}(\mathfrak{I}_\mu) \cong \widetilde{\Lambda}(\mathfrak{I}_\mu) \cong \mathrm{End}(\mathscr{A})$;*
3) *$\mathfrak{I}_\mu$ is a reductive ideal;*
4) *$\mu > 2$, or $\mu = 2$ and every 1-dimensional subalgebra contains at least two elements.*

Proof. Recall that $(\mathscr{A}, \mathfrak{I}_\mu)$ always has REP when $\mu > 1$ by Lemma VII.1.2.

Under this observation, we have that 1) ⇒ 2) by Theorem VI.2.1 and 2) ⇒ 3) by Lemma VI.2.2. The fact that 3) ⇒ 1) comes directly from Lemma VII.1.3 since a reductive ideal is right reductive. Finally, the equivalence of 1) and 4) is exactly Lemma VII.2.2. □

Remark VII.2.5. When $\mu = 1$ the situation is slightly more complicated and we do not have equivalence between statements 1) and 3) of Corollary VII.2.4. Since $\mathfrak{I}_\mu$ is a left-zero semigroup, it is right reductive, but $(\mathscr{A}, \mathfrak{I}_\mu)$ has SEP only under the conditions given in Lemma VII.2.3.

Similarly, if $C = \langle \emptyset \rangle \neq A$, then $(\mathscr{A}, \mathfrak{I}_\mu)$ does not have REP by Lemma VII.1.2 but $\mathfrak{I}_\mu$ is left reductive if and only if $\langle \emptyset \rangle$ is a singleton. Indeed, $\mathfrak{I}_\mu$ is a left-zero semigroup and thus $\gamma\alpha = \gamma\beta = \gamma$ for all $\alpha, \beta, \gamma \in \mathfrak{I}_\mu$. Hence, we either have that $\mathfrak{I}_\mu$ is not left reductive, or $\alpha = \beta$ for all $\alpha, \beta \in \mathfrak{I}_\mu$, that is, $\mathfrak{I}_\mu$ is a singleton. In the latter case, since $C \neq A$, there exists an independent element $a \in A$, and thus for all $c \in C$, there exists a map $\gamma \in \mathfrak{I}_\mu$ which sends a to c. The fact that $\mathfrak{I}_\mu$ is a singleton forces C to be a singleton, as required.

To finish this section, we recall that in Chapter V we were interested in the situation when $\Omega(\mathfrak{I}) \cong \mathrm{P}(\mathfrak{I}) \cong \mathrm{End}(\mathscr{A})$. The result of Corollary VII.2.4 is slightly different. It is clear that if $\Omega(\mathfrak{I}) \cong \mathrm{P}(\mathfrak{I})$, then $\mathrm{P}(\mathfrak{I}) = \widetilde{\mathrm{P}}(\mathfrak{I})$. However, even though from each right translation $\rho \in \mathrm{P}(\mathfrak{I})$, we can define a left translation $\lambda \colon \mathfrak{I} \to \mathfrak{I}$ by $x_i\lambda\beta = x_i\alpha_i\rho\beta$ for all $x_i \in X$, Lemma V.1.10 tells us that these will form a linked pair only if equation V.1.2 holds.

Therefore, working with REP and SEP presents the advantage that we can treat all ideals directly in a more general abstraction whereas the approach with equations ($\star$) and ($\star\star$) in Section V.1 is only made for the ideal of rank 1 but gives us the additional information that all right translations are part of a linked pair.

VII.3 THE FULL TRANSFORMATION MONOID $\mathcal{T}_n$

The algebras we have considered so far are types of free algebras, which means that REP has always been satisfied. In order to better understand the intricacies and limits of the approach with REP and SEP, we consider here an example of an algebra that is far from being a free algebra. For this reason, in this section we will look at the translational hull of ideals of $\mathrm{End}(\mathcal{T}_n)$ for some $n \in \mathbb{N}$, using the descriptions provided in Chapter IV. In order to avoid pathological cases, let us assume throughout this section that $n \geq 5$. We quickly recall here the structure and the ideals of $\mathrm{End}(\mathcal{T}_n)$.

From Lemma IV.3.2, the monoid $\mathcal{E}_n = \mathrm{End}(\mathcal{T}_n)$ can be decomposed as

$$\mathcal{E}_n = \mathcal{G}_n \cup E_3 \cup \Big(A \cup E_2\Big) \cup \Big(B \cup C \cup E_1 \backslash \{\phi_{\mathrm{id,id}}\}\Big) \cup \{\phi_{\mathrm{id,id}}\},$$

where subsets containing endomorphisms of the same type are bracketed together. Moreover, by Corollary IV.6.3, the ideals of $\mathcal{E}_n$ are of the following form:

1) $\mathcal{E}_n$ (the only ideal containing automorphisms);
2) $\mathcal{E}_n \setminus \mathcal{G}_n$ (the only proper ideal containing elements of odd type);
3) $X \cup Y \cup E_2 \cup C \cup E_1$ (ideals containing elements of even type, but no elements of odd type); and
4) $Y \cup Y^+ \cup Z \cup E_1$ (ideals containing only elements of trivial or non-permutation type);

where the sets X, Y, and Z are (possibly empty) union of orbits taken from the sets A, B and C respectively. Last, we remind the reader of the notation given in Remark IV.4.2: if $\alpha = \phi_{t,e} \in \mathcal{E}_n \setminus \mathcal{G}_n$ and $\gamma \in \mathcal{E}_n$, then we have that

$$\gamma\alpha = \begin{cases} \alpha = \phi_{t,e} & \text{if } \gamma \text{ has group type or odd type (i.e. } \gamma \in \mathcal{G}_n \cup E_3),\\ \alpha^+ = \phi_{t^2,e} & \text{if } \gamma \text{ has even type (i.e. } \gamma \in A \cup E_2),\\ \alpha^- = \phi_{e,e} & \text{if } \gamma \text{ has non-permutation type (i.e. } \gamma \in B \cup C \cup E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}\}),\\ \alpha^0 = \phi_{t^2,t^2} & \text{if } \gamma \text{ has trivial type (i.e. } \gamma = \phi_{\mathrm{id},\mathrm{id}}).\end{cases}$$

From now on, let $\mathfrak{I}$ be an ideal of $\mathcal{E}_n$. It is easy to show that the only ideal such that $(\mathcal{E}_n, \mathfrak{I})$ has REP and SEP is $\mathcal{E}_n$ itself, as given by the following.

Lemma VII.3.1. *The pair $(\mathcal{E}_n, \mathfrak{I})$ satisfies REP if and only if $\mathfrak{I} = \mathcal{E}_n$ or $\mathfrak{I} = \mathcal{E}_n \setminus \mathcal{G}_n$, while $(\mathcal{E}_n, \mathfrak{I})$ satisfies SEP if and only if $\mathfrak{I} = \mathcal{E}_n$.*

Proof. Let $e = e^2 \in \mathcal{T}_n$. Then we have that $\phi_{\mathrm{id},e} \in \mathcal{E}_n \setminus \mathcal{G}_n$ is such that $e\phi_{\mathrm{id},e} = e$. Moreover, for all $u = (i\,j) \in \mathcal{S}_n$ there exists $1 \leq k \leq n$ with $i \neq k \neq j$, and then if we let $f = c_k \in \mathcal{T}_n$, the map with constant image k, we get that $fu = uf = f = f^2$, so that $\phi_{u,f} \in \mathcal{E}_n \setminus \mathcal{G}_n$ and $u\phi_{u,f} = u$. Since every element of $\mathcal{T}_n$ can be generated using idempotents and transpositions, it follows that if $\mathfrak{I} = \mathcal{E}_n$ or $\mathfrak{I} = \mathcal{E}_n \setminus \mathcal{G}_n$, then $(\mathcal{E}_n, \mathfrak{I})$ has REP.

Conversely, notice that the image of elements in $\mathcal{E}_n \setminus (\mathcal{G}_n \cup E_3)$ only contains elements of $\mathcal{T}_n \setminus \mathcal{S}_n$ or of $\mathcal{A}_n$. Since transpositions cannot be generated using even permutations and singular transformations, it follows that for any ideal $\mathfrak{I} \subsetneq \mathcal{E}_n \setminus \mathcal{G}_n$, we get that $(1\,2) \notin \left\langle \bigcup_{\phi_{t,e} \in \mathfrak{I}} \operatorname{im} \phi_{t,e} \right\rangle$. Therefore $(\mathcal{E}_n, \mathfrak{I})$ does not have REP if $\mathfrak{I} \neq \mathcal{E}_n$ and $\mathfrak{I} \neq \mathcal{E}_n \setminus \mathcal{G}_n$.

We already know that $(\mathcal{E}_n, \mathfrak{I})$ has SEP if $\mathfrak{I} = \mathcal{E}_n$ since $\mathcal{E}_n$ is a monoid. Conversely, if $\mathfrak{I} \neq \mathcal{E}_n$ and $s_1, s_2 \in \mathcal{S}_n \setminus \mathcal{A}_n$ then $s_1\phi_{t,e} = s_2\phi_{t,e}$ for all $\phi_{t,e} \in \mathcal{E}_n \setminus \mathcal{G}_n$, which shows that $(\mathcal{E}_n, \mathfrak{I})$ does not have SEP in this case. □

It follows from Lemma VII.3.1 that Theorem VI.2.1 cannot be applied if $\mathfrak{I} \neq \mathcal{E}_n$. We now look at the conditions under which $\mathfrak{I}$ is left or right reductive.

Lemma VII.3.2. *Suppose that $\mathfrak{I} \neq \mathcal{E}_n$. Then $\mathfrak{I}$ is not right reductive, and $\mathfrak{I}$ is left reductive if and only if $\mathfrak{I} = \mathcal{E}_n \setminus \mathcal{G}_n$ or $\mathfrak{I} \subseteq E_2 \cup C \cup E_1$.*

Proof. Recall that since $\mathfrak{I} \neq \mathcal{E}_n$, all endomorphisms of $\mathfrak{I}$ are of the form $\phi_{t,e}$ for some $(t,e) \in P_n$.

In particular, by Lemma IV.3.6, we get that if $\phi_{t,e}, \phi_{u,f} \in \mathfrak{I}$ have the same type, then $\phi_{t,e}\alpha = \phi_{u,f}\alpha$ for all $\alpha \in \mathfrak{I}$, and thus $\mathfrak{I}$ cannot be right reductive.

Now suppose that $\mathfrak{I} = \mathcal{E}_n \setminus \mathcal{G}_n$. Let $\phi_{t,e}, \phi_{u,f} \in \mathfrak{I}$ be such that $\alpha\phi_{t,e} = \alpha\phi_{u,f}$ for all $\alpha \in \mathfrak{I}$. In particular, this applies to all $\alpha \in E_3$, and since elements of E_3 are left identities for elements of $\mathcal{E}_n \setminus \mathcal{G}_n$ by Corollary IV.4.6, it follows that $\phi_{t,e} = \alpha\phi_{t,e} = \alpha\phi_{u,f} = \phi_{u,f}$, and therefore $\mathfrak{I}$ is left reductive.

Suppose now that $\mathfrak{I} \neq \mathcal{E}_n \setminus \mathcal{G}_n$, that is, $\mathfrak{I}$ does not contain elements of group or odd type, and let $\phi_{t,e}, \phi_{u,f} \in \mathfrak{I}$. Notice first that if $\phi^-_{t,e} = \phi^-_{u,f}$ and $\phi^0_{t,e} = \phi^0_{u,f}$, then we get that $\phi_{e,e} = \phi_{f,f}$ and $\phi_{t^2,t^2} = \phi_{u^2,u^2}$, so that $e = f$ and $t^2 = u^2$ by Lemma IV.2.4 and thus $\phi^+_{t,e} = \phi_{t^2,e} = \phi_{u^2,f} = \phi^+_{u,f}$. Since $E_1 \subseteq \mathfrak{I}$, it follows that $\alpha\phi_{t,e} = \alpha\phi_{u,f}$ for all $\alpha \in \mathfrak{I}$ if and only if $\phi^-_{t,e} = \phi^-_{u,f}$ and $\phi^0_{t,e} = \phi^0_{u,f}$. Therefore $\mathfrak{I}$ is left reductive if and only if $\phi_{t^2,e} = \phi_{u^2,f}$ implies $\phi_{t,e} = \phi_{u,f}$, that is, if and only if there exist no $\phi_{t,e} \neq \phi_{u,e} \in \mathfrak{I}$ such that $t^2 = u^2$.

As a consequence, if there exists $\phi_{t,e} \in \mathfrak{I} \cap (A \cup B)$, then $\phi_{t^2,e} \in \mathfrak{I}$ by the description of the ideals. Since t^2 is idempotent, it follows that $\phi^-_{t,e} = \phi_{e,e} = \phi^-_{t^2,e}$ and $\phi^0_{t,e} = \phi_{t^2,t^2} = \phi^0_{t^2,e}$, so that $\mathfrak{I}$ is not left reductive. On the other hand, if $\mathfrak{I} \subseteq E_2 \cup C \cup E_1$, then any $\phi_{t,e} \in \mathfrak{I}$ is such that $t^2 = t$, which means that $\phi^-_{t,e} = \phi^-_{u,f}$ and $\phi^0_{t,e} = \phi^0_{u,f}$ if and only if $e = f$ and $t = t^2 = u^2 = u$, which shows that $\phi_{t,e} = \phi_{u,f}$, and therefore $\mathfrak{I}$ is left reductive. □

Corollary VII.3.3. *In $\mathcal{E}_n$, have that $\Omega(\mathfrak{I}) \cong \widetilde{\mathrm{P}}(\mathfrak{I}) \cong \widetilde{\Lambda}(\mathfrak{I}) \cong \mathrm{End}(\mathscr{A})$ if and only if $\mathfrak{I} = \mathcal{E}_n$.*

Proof. If $\mathfrak{I} = \mathcal{E}_n$, then $(\mathcal{E}_n, \mathfrak{I})$ has REP and SEP, and thus $\Omega(\mathfrak{I}) \cong \widetilde{\mathrm{P}}(\mathfrak{I}) \cong \widetilde{\Lambda}(\mathfrak{I}) \cong \mathrm{End}(\mathscr{A})$ by Theorem VI.2.1.

Conversely, if $\mathfrak{I}$ is an ideal such that $\Omega(\mathfrak{I}) \cong \widetilde{\mathrm{P}}(\mathfrak{I}) \cong \widetilde{\Lambda}(\mathfrak{I}) \cong \mathrm{End}(\mathscr{A})$, then by Lemma VI.2.2, we get that $\mathfrak{I}$ is right and left reductive. Since no proper ideal is right reductive by Lemma VII.3.2 and that a monoid is always reductive by Remark I.2.13, we get that $\mathfrak{I} = \mathcal{E}_n$. □

Since the case of $\mathcal{E}_n$ is special and has now been treated, for the remainder of this chapter we will assume that $\mathfrak{I}$ is a proper ideal of $\mathcal{E}_n$.

Another ideal for which we can directly describe its translational hull is the minimal ideal E_1. Indeed, E_1 is a right-zero semigroup by Corollary IV.4.6, which means that $\Omega(E_1) \cong \mathcal{T}_{E_1}$ by Proposition I.2.23. Since elements of E_1 are in one-to-one correspondence with the idempotents of $\mathcal{T}_n$, it follows that $\Omega(E_1) \cong \{\alpha\colon E(\mathcal{T}_n) \to E(\mathcal{T}_n)\}$.

In order to look at the translational hulls of other ideals of $\mathcal{E}_n$, we need first to focus on the left and right translations.

VII.3.1 Left translations

We start by giving some properties of left translations on all proper ideals of $\mathcal{E}_n$.

Lemma VII.3.4. *Let $\lambda \in \Lambda(\mathfrak{I})$ and $\alpha, \beta \in \mathfrak{I} \subseteq \mathcal{E}_n \setminus \mathcal{G}_n$.*

1) If α and β are of the same type, then $(\lambda\alpha)\gamma = (\lambda\beta)\gamma$ for all $\gamma \in \mathfrak{I}$.

2) If α is idempotent, then $\lambda\alpha$ is idempotent.

3) If α has a left identity in $\mathfrak{I}$, then $\lambda\alpha \in \{\alpha, \alpha^+, \alpha^-, \alpha^0\}$.

4) If α is idempotent, then the restriction of λ to $\mathcal{E}_n\alpha\mathcal{E}_n \subseteq \mathfrak{I}$ is one of the following maps:

$$\gamma \mapsto \gamma, \quad \gamma \mapsto \gamma^+, \quad \gamma \mapsto \gamma^-, \quad \text{or} \quad \gamma \mapsto \gamma^0.$$

Proof. 1) If α and β have the same type, it follows by Lemma IV.3.6 that $\alpha\gamma = \beta\gamma$ for all $\gamma \in \mathfrak{I} \subseteq \mathcal{E}_n \setminus \mathcal{G}_n$, which shows that $(\lambda\alpha)\gamma = \lambda(\alpha\gamma) = \lambda(\beta\gamma) = (\lambda\beta)\gamma$.

2) If $\alpha^2 = \alpha$, we get that $\lambda\alpha = (\lambda\alpha)\alpha$. Since $\lambda\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$ and α is idempotent, it follows from part 1) of Lemma IV.4.5 that $(\lambda\alpha)\alpha$ is idempotent, that is, $\lambda\alpha$ is idempotent.

3) This follows directly from the fact that if $\alpha = \beta\alpha$ for some $\beta \in \mathfrak{I}$, then $\lambda\alpha = (\lambda\beta)\alpha \in \{\alpha, \alpha^+, \alpha^-, \alpha^0\}$ since $\alpha \in \mathcal{E}_n \setminus \mathcal{G}_n$.

4) Suppose that α is idempotent and $\beta \in \mathcal{E}_n\alpha\mathcal{E}_n \subseteq \mathfrak{I}$. Since idempotents act as left identities on all elements $\mathscr{J}$-below themselves by Lemma IV.7.3, it follows that $\alpha\beta = \beta$. Hence, by part 3), we get that $\lambda\beta$ is one of β, β^+, β^- or β^0 which

only depends on the type of $\lambda\alpha$. Since this holds for all $\beta \in \mathcal{E}_n\alpha\mathcal{E}_n$, it follows that λ restricts to one of the maps given on $\mathcal{E}_n\alpha\mathcal{E}_n$. □

We can now describe the left translations of all proper ideals, starting with the ideal $\mathcal{E}_n \setminus \mathcal{G}_n$.

Corollary VII.3.5. *If $\mathfrak{I}$ contains an element of odd type, then $\mathfrak{I} = \mathcal{E}_n \setminus \mathcal{G}_n$ and $\Lambda(\mathfrak{I}) = \left\{\lambda^{\mathrm{id}}, \lambda^+, \lambda^-, \lambda^0\right\}$ which are defined for all $\alpha \in \mathfrak{I}$ by:*

$$\lambda^{\mathrm{id}}\alpha = \alpha, \quad \lambda^+\alpha = \alpha^+, \quad \lambda^-\alpha = \alpha^- \quad \textit{and} \quad \lambda^0\alpha = \alpha^0.$$

Proof. Let $\alpha = \phi_{t,e} \in \mathfrak{I}$ be an element of odd type, so that $\mathfrak{I} = \mathcal{E}_n \setminus \mathcal{G}_n$ by the description of the ideals given at the beginning of this section. Moreover, by Proposition IV.6.1, we have that $\mathcal{E}_n\alpha\mathcal{E}_n = \mathcal{E}_n \setminus \mathcal{G}_n$ and since elements of E_3 are left identities for all elements of $\mathcal{E}_n \setminus \mathcal{G}_n$ by Corollary IV.4.6, it follows by part 4) of Lemma VII.3.4 that the maps λ^{id}, λ^+, λ^- and λ^0 defined above are the only possible left translations of $\mathfrak{I}$. Finally these four maps are all distinct because α, $\alpha^+ = \phi_{\mathrm{id},e}$, $\alpha^- = \phi_{e,e}$ and $\alpha^0 = \phi_{\mathrm{id},\mathrm{id}}$ are all distinct elements of $\mathfrak{I}$. □

The next ideals we consider are those that do not contain elements of odd type, but have elements of even type.

Lemma VII.3.6. *Suppose that $\mathfrak{I}$ contains an element of even type, but does not contain any element of odd type, and let $\mathfrak{T} := E_2 \cup C \cup E_1 \subseteq \mathfrak{I}$. Then, the left translations of $\mathfrak{I}$ are precisely the maps $\lambda\colon \mathfrak{I} \to \mathfrak{I}$ such that the following conditions are satisfied for some $y \in \{+, -, 0\}$:*

i) for all $\alpha \in \mathfrak{T}$, we have that $\lambda\alpha = \alpha^y$;

ii) for all $\alpha \in \mathfrak{I} \cap A$, we have that $(\lambda\alpha)\beta = \beta^y$ for all $\beta \in \mathfrak{I}$; and

iii) for all $\alpha \in \mathfrak{I} \cap B$, we have that $\lambda\alpha$ has non-permutation type.

Proof. Notice first that by using the description of the ideals given at the beginning of this section, we have that if $\mathfrak{I}$ contains an element of even type, but no element of odd type, then $\mathfrak{I} = X \cup Y \cup E_2 \cup C \cup E_1$ for some $X \subseteq A$ and $Y \subseteq B$. Hence, conditions *i)*–*iii)* cover all possible elements of $\mathfrak{I}$.

Let $\eta \in E_2$ and suppose that $\lambda \in \Lambda(\mathfrak{I})$. Then $\mathcal{E}_n\eta\mathcal{E}_n = \mathfrak{T}$ and by part 4) of Lemma VII.3.4, it follows that the restriction of λ to $\mathfrak{T}$ is a map sending γ to one of γ, γ^+, γ^- or γ^0. Since $\gamma = \gamma^+$ for all $\gamma \in \mathfrak{T}$ by Lemma IV.4.3, it follows that λ

restricts to $\gamma \mapsto \gamma^y$ on $\mathfrak{T}$ for some $y \in \{+,-,0\}$, and we get condition *i*). Moreover, for all $\alpha, \beta \in \mathfrak{I}$, we have that $\alpha\beta \in \mathfrak{T}$, and thus

$$(\lambda\alpha)\beta = \lambda(\alpha\beta) = (\alpha\beta)^y = (\beta^x)^y,$$

for some $x \in \{+,-,0\}$ depending on the type of α. In particular, if $\alpha \in \mathfrak{I} \cap A$, we have that $\alpha\beta = \beta^+$, and then by Lemma IV.4.4, we get that $(\beta^+)^y = \beta^y$. Hence, $(\lambda\alpha)\beta = \beta^y$, which is condition *ii*). On the other hand, if we consider $\alpha \in \mathfrak{I} \cap B$, we have that $\alpha\beta = \beta^-$ and by Lemma IV.4.4 we get that $(\lambda\alpha)\beta = (\beta^-)^y = \beta^-$, which shows that $\lambda\alpha$ has non-permutation type and condition *iii*) holds.

Conversely, suppose that $\lambda\colon \mathfrak{I} \to \mathfrak{I}$ satisfies conditions *i*)–*iii*) with $y \in \{+,-,0\}$. Then for all $\alpha, \beta \in \mathfrak{I}$ we have that $\alpha\beta \in \mathfrak{T}$ so that $\lambda(\alpha\beta) = (\alpha\beta)^y$. We now consider all cases for elements in $\mathfrak{I}$, and use Lemma IV.4.4 appropriately:

- if $\alpha \in E_2 \subseteq \mathfrak{T}$, then $(\alpha\beta)^y = (\beta^+)^y = \beta^y$ and since α^+, α^- and α^0 are respectively of even, non-permutation and trivial type, it follows by *i*) that for all $\beta \in \mathfrak{I}$, we have $(\lambda\alpha)\beta = \alpha^y\beta = \beta^y$;
- if $\alpha \in C \cup E_1 \setminus \{\phi_{\mathrm{id,id}}\} \subseteq \mathfrak{T}$, then $(\alpha\beta)^y = (\beta^-)^y = \beta^-$. Moreover, α^y has non-permutation type for all $y \in \{+,-,0\}$ and we get by *i*) that $(\lambda\alpha)\beta = \alpha^y\beta = \beta^-$;
- if $\alpha = \phi_{\mathrm{id,id}} \subseteq \mathfrak{T}$, then $(\alpha\beta)^y = (\beta^0)^y = \beta^0$. Furthermore, $\alpha^y = \alpha = \phi_{\mathrm{id,id}}$ and by condition *i*) we obtain $(\lambda\alpha)\beta = \alpha^y\beta = \alpha\beta = \beta^0$;
- if $\alpha \in \mathfrak{I} \cap A$, then $(\alpha\beta)^y = (\beta^+)^y = \beta^y$, and by *ii*) we also have that $(\lambda\alpha)\beta = \beta^y$;
- if $\alpha \in \mathfrak{I} \cap B$, then $(\alpha\beta)^y = (\beta^-)^y = \beta^-$, and by *iii*), we have that $\lambda\alpha$ has non-permutation type, so that $(\lambda\alpha)\beta = \beta^-$.

In all cases, we see that $\lambda(\alpha\beta) = (\alpha\beta)^y = (\lambda\alpha)\beta$, and therefore $\lambda \in \Lambda(\mathfrak{I})$. □

Finally, for ideals which do not contain any element of even type, we get the following:

Lemma VII.3.7. *Suppose that $\mathfrak{I}$ only contains elements of trivial or non-permutation type. Then*

1) *either $\mathfrak{I} = E_1$, and then the only left translation of $\mathfrak{I}$ is $\mathbb{1}_\Lambda$;*
2) *otherwise $\mathfrak{I} \cap C \neq \emptyset$, and then the left translations of $\mathfrak{I}$ are precisely the maps $\lambda\colon \mathfrak{I} \to \mathfrak{I}$ such that $\lambda\alpha = \alpha$ for all $\alpha \in E_1$ and $\lambda\alpha \neq \phi_{\mathrm{id,id}}$ if $\alpha \neq \phi_{\mathrm{id,id}}$.*

Proof. Notice first that the only element of trivial type is $\phi_{\mathrm{id,id}} \in E_1$, and thus all other elements of $\mathfrak{I}$ have non-permutation type.

If $\mathfrak{I} = E_1$, then $\mathfrak{I}$ is a right-zero semigroup by Corollary IV.4.6 and thus we have that $\Lambda(\mathfrak{I}) = \{\mathbb{1}_\Lambda\}$ by Proposition I.2.23.

Suppose then that $\mathfrak{I} \cap C \neq \emptyset$. If $\lambda \in \Lambda(\mathfrak{I})$, then by Lemma I.2.20 we have that $\lambda\alpha = \alpha$ for all $\alpha \in E_1$ since all elements of E_1 are right zeros. By assumption, there exists $\phi_{t,e} \in \mathfrak{I} \cap C$. Now, if $\gamma \in \mathfrak{I} \setminus E_1$ is such that $\lambda\gamma = \phi_{\mathrm{id},\mathrm{id}}$, we get that

$$\phi_{e,e} = \lambda\phi_{e,e} = \lambda(\gamma\phi_{t,e}) = (\lambda\gamma)\phi_{t,e} = \phi_{\mathrm{id},\mathrm{id}}\phi_{t,e} = \phi_{t^2,t^2},$$

which forces $e = t^2 = t$, a contradiction. Hence, we have that $\lambda\gamma \neq \phi_{\mathrm{id},\mathrm{id}}$ for all $\gamma \notin E_1$, and for $\gamma \in E_1$, $\lambda\gamma = \phi_{\mathrm{id},\mathrm{id}}$ is equivalent to $\gamma = \phi_{\mathrm{id},\mathrm{id}}$, that is $\lambda\alpha \neq \phi_{\mathrm{id},\mathrm{id}}$ if $\alpha \neq \phi_{\mathrm{id},\mathrm{id}}$.

Conversely, suppose that $\lambda \colon \mathfrak{I} \to \mathfrak{I}$ is such that $\lambda\alpha = \alpha$ for all $\alpha \in E_1$ and $\lambda\alpha \neq \phi_{\mathrm{id},\mathrm{id}}$ for all $\alpha \neq \phi_{\mathrm{id},\mathrm{id}}$. Then, for all $\alpha, \beta \in \mathfrak{I}$, we have that $\alpha\beta \in E_1$ and thus

$$\lambda(\alpha\beta) = \alpha\beta = \begin{cases} \beta^- & \text{if } \alpha \neq \phi_{\mathrm{id},\mathrm{id}} \\ \beta^0 & \text{if } \alpha = \phi_{\mathrm{id},\mathrm{id}}, \end{cases} \quad \text{whilst} \quad (\lambda\alpha)\beta = \begin{cases} \beta^- & \text{if } \lambda\alpha \neq \phi_{\mathrm{id},\mathrm{id}} \\ \beta^0 & \text{if } \lambda\alpha = \phi_{\mathrm{id},\mathrm{id}}. \end{cases}$$

However, since $\lambda\alpha \neq \phi_{\mathrm{id},\mathrm{id}}$ for all $\alpha \neq \phi_{\mathrm{id},\mathrm{id}}$, and $\lambda\phi_{\mathrm{id},\mathrm{id}} = \phi_{\mathrm{id},\mathrm{id}}$, it follows that $\lambda(\alpha\beta) = (\lambda\alpha)\beta$ for all $\alpha, \beta \in \mathfrak{I}$, that is, $\lambda \in \Lambda(\mathfrak{I})$. □

Remark VII.3.8. We can see from Corollary VII.3.5 that all left translations of the ideal $\mathcal{E}_n \setminus \mathcal{G}_n$ come from a multiplication by a map on the left. However, this is not the case for the other proper ideals of $\mathcal{E}_n$. For example, suppose that $\phi_{t,e} \in \mathfrak{I}$ with $t = (1\,2)(3\,4)$ and $e = c_5$ and let $u = t^{(1\,3)} = (1\,4)(2\,3)$. Then $\phi_{u,e} \in \phi_{t,e}\mathcal{G}_n \subseteq \mathfrak{I}$ and if we define $\lambda \colon \mathfrak{I} \to \mathfrak{I}$ by $\lambda\phi_{t,e} = \phi_{u,e}$ and $\lambda\alpha = \alpha$ for all $\alpha \neq \phi_{t,e}$, we get that $\lambda \in \Lambda(\mathfrak{I})$ by Lemma VII.3.6 (with $y = +$). However, this cannot come from a multiplication by a map on the left since $\phi_{u,e} \notin \mathcal{E}_n\phi_{t,e} = \{\phi_{t,e}, \phi_{\mathrm{id},e}, \phi_{e,e}, \phi_{\mathrm{id},\mathrm{id}}\}$. A similar example when there are no even elements can be created by sending an element of $B \cup C$ to one of its conjugate and fixing all the others. Such a map will be a left translation by Lemma VII.3.7 but is not coming from a multiplication on the left.

VII.3.2 Right translations

The description of the right translations is slightly more complicated. Throughout this section we will write F for the set of idempotents of $\mathcal{T}_n$ to distinguish it from the sets E_i which contain idempotents of $\mathcal{E}_n$. It is well-known that there is a partial order on the idempotents of $\mathcal{T}_n$ which is defined as $e \leq f$ if and only if $e = ef = fe$

for $e, f \in F$. In particular, it follows that for any $e, f \in F$, we have that $e \leq f$ if and only if $\phi_{f,e} \in \mathcal{E}_n$.

Recall first that $E_1 \subseteq \mathfrak{I}$ for all ideals $\mathfrak{I} \subseteq \mathcal{E}_n$, and thus $\phi_{e,e} \in \mathfrak{I}$ for all $e \in F$. Moreover, if $\rho \in \mathrm{P}(\mathfrak{I})$ and $e \in F$, then we have that $\phi_{e,e} \in E_1 \subseteq \mathfrak{I}$ and

$$\phi_{e,e}\rho = (\phi_{e,e}\phi_{e,e})\rho = \phi_{e,e}(\phi_{e,e}\rho) \in E_1.$$

Thus there exists a unique $f \in F$ such that $\phi_{e,e}\rho = \phi_{f,f}$. This shows in particular that $\rho|_{E_1} : E_1 \to E_1$ induces a map $d\colon F \to F$ sending e to the unique element f such that $\phi_{e,e}\rho = \phi_{f,f}$. To simplify notation, we will write this as $d_e = f$, so that $\phi_{e,e}\rho = \phi_{d_e,d_e}$.

We now gives some properties that any right translation on a proper ideal of $\mathcal{E}_n$ must satisfy.

Lemma VII.3.9. *Let ρ be a right translation of $\mathfrak{I} \subseteq \mathcal{E}_n \setminus \mathcal{G}_n$. Then we have that:*

1) *$d_e = t\phi_{e,e}\rho$ for all $\phi_{e,e} \in \mathfrak{I}$ and $t \in \mathcal{T}_n$;*
2) *for all $\phi_{t,e} \in \mathfrak{I}$, there exists $r_{t,e} \in \mathcal{T}_n$ with $r_{t,e}^2 = d_{t^2}$ such that $\phi_{t,e}\rho = \phi_{r_{t,e},d_e}$;*
3) *for all $\phi_{f,e} \in \mathfrak{I}$ with $f^2 = f$, we have $r_{f,e}^2 = d_f$ and $d_e \leq d_f$;*
4) *if $\mathfrak{I}$ contains an element of even type, then $r_{f,e} = d_f$ for all idempotents $e \leq f$.*

Proof. 1) For $e \in F$ and $t \in \mathcal{T}_n$, we have $t\phi_{e,e}\rho = t\phi_{d_e,d_e} = d_e$.

2) Let $\phi_{t,e} \in \mathfrak{I}$ and $h = h^2 \neq \mathrm{id}$ and suppose that $\phi_{t,e}\rho = \phi_{u,f}$. Then we have

$$\phi_{d_e,d_e} = \phi_{e,e}\rho = (\phi_{h,h}\phi_{t,e})\rho = \phi_{h,h}(\phi_{t,e}\rho) = \phi_{h,h}\phi_{u,f} = \phi_{f,f}$$

so that $f = d_e$, whilst

$$\phi_{d_{t^2},d_{t^2}} = \phi_{t^2,t^2}\rho = (\phi_{\mathrm{id},\mathrm{id}}\phi_{t,e})\rho = \phi_{\mathrm{id},\mathrm{id}}(\phi_{t,e}\rho) = \phi_{\mathrm{id},\mathrm{id}}\phi_{u,f} = \phi_{u^2,u^2},$$

which gives that $u^2 = d_{t^2}$. Hence, we can pick $r_{t,e} = u \in \mathcal{T}_n$ to obtain that $\phi_{t,e}\rho = \phi_{r_{t,e},d_e}$ is such that $r_{t,e}^2 = d_{t^2}$.

3) If we set $t = f = f^2$ in the previous part, we get that $\phi_{f,e}\rho = \phi_{u,d_e}$ with $u = r_{f,e}$ and $u^2 = d_{f^2} = d_f$. Moreover, since $\mathfrak{I}$ is an ideal and $\phi_{u,d_e} \in \mathfrak{I}$, it follows that $\phi_{\mathrm{id},e}\phi_{u,d_e} = \phi_{u^2,d_e} \in \mathfrak{I}$. Then we get that $u^2 d_e = d_e u^2 = d_e$, that is, $d_f d_e = d_e d_f = d_e$, which shows that $d_e \leq d_f$.

4) If $\mathfrak{I}$ contains an element of even type, then we know that $E_2 \cup C \cup E_1 \subseteq \mathfrak{I}$. Hence, two idempotents $e, f \in F$ are such that $e \leq f$ if and only if $\phi_{f,e} \in E_2 \cup C \cup E_1$. Then we have that

$$\phi_{r_{f,e},d_e} = \phi_{f,e}\rho = (\phi_{\mathrm{id},e}\phi_{f,e})\rho = \phi_{\mathrm{id},e}(\phi_{f,e}\rho) = \phi_{\mathrm{id},e}\phi_{r_{f,e},d_e} = \phi_{r_{f,e}^2,d_e},$$

which shows that $r_{f,e} = r_{f,e}^2 = d_f$ as required. □

This allows us to give the description of all the right translations on ideals that do not contain elements of even type.

Lemma VII.3.10. *Suppose that $\mathfrak{I}$ only contains elements of trivial and non-permutation type. Then the right translations of $\mathfrak{I}$ are precisely the maps ρ defined by $\phi_{t,e}\rho = \phi_{r_{t,e},d_e}$, where:*

i) $d\colon F \to F$ *with* $d_e \leq d_f$ *whenever* $\phi_{f,e} \in \mathfrak{I}$;

ii) *if* $\phi_{t,e} \in \mathfrak{I}$, *then* $\phi_{r_{t,e},d_e} \in \mathfrak{I}$;

iii) $r_{e,e} = d_e$ *for all idempotents* e;

iv) $r_{t,e}^2 = d_{t^2}$.

Proof. Suppose first that $\rho \in \mathrm{P}(\mathfrak{I})$. Then using Lemma VII.3.9 we get *i*) and *iv*) directly, while *ii*) comes from the fact that if $\phi_{t,e} \in \mathfrak{I}$, then $\phi_{t,e}\rho \in \mathfrak{I}$ and condition *iii*) is obtained by noticing that $d_e = \mathrm{id}\phi_{e,e}\rho = \mathrm{id}\phi_{r_{e,e},d_e} = r_{e,e}$ using the definition of d and part 2).

Conversely, let us assume that $\rho\colon \mathfrak{I} \to \mathfrak{I}$ is defined by $\phi_{t,e}\rho = \phi_{r_{t,e},d_e}$ for all $\phi_{t,e} \in \mathfrak{I}$ and that it satisfies conditions *i*)–*iv*). Condition *ii*) ensures us that ρ is well-defined. In order for ρ to be a right translation, we need to show that $(\alpha\phi_{t,e})\rho = \alpha(\phi_{t,e}\rho)$ for all $\alpha, \phi_{t,e} \in \mathfrak{I}$. Since both sides of the equality only depend on the type of α, and that $\mathfrak{I}$ only contains elements of trivial type and non-permutation type, we look into the two possible cases.

- If $\alpha = \phi_{\mathrm{id},\mathrm{id}}$, that is, α has trivial type, then we have that

  $$(\alpha\phi_{t,e})\rho = \phi_{t^2,t^2}\rho = \phi_{r_{t^2,t^2},d_{t^2}} = \phi_{d_{t^2},d_{t^2}},$$

  where the last equality comes from *iii*) using the fact that t^2 is idempotent. Also,

  $$\alpha(\phi_{t,e}\rho) = \alpha\phi_{r_{t,e},d_e} = \phi_{r_{t,e}^2,r_{t,e}^2},$$

  and using *iv*), we get that $\alpha(\phi_{t,e}\rho) = \phi_{d_{t^2},d_{t^2}} = (\alpha\phi_{t,e})\rho$.

- If α has non-permutation type, then $\alpha = \phi_{u,f}$ with $f \neq \mathrm{id}$, and then we have

  $$(\alpha\phi_{t,e})\rho = \phi_{e,e}\rho = \phi_{r_{e,e},d_e} = \phi_{d_e,d_e} = \alpha\phi_{r_{t,e},d_e} = \alpha(\phi_{t,e}\rho),$$

  where the middle equality uses *iii*) since e is idempotent.

Hence in all cases, we have that $(\alpha\phi_{t,e})\rho = \alpha(\phi_{t,e}\rho)$ for all $\alpha, \phi_{t,e} \in \mathfrak{I}$, so that ρ is a right translation of $\mathfrak{I}$. □

The description of the right translations for ideals that contain endomorphisms of even type is almost identical, with an added restriction on idempotents of F.

Lemma VII.3.11. *Suppose that $\mathfrak{I}$ contains an element of even type. Then the right translations of $\mathfrak{I}$ are precisely the maps $\rho\colon \phi_{t,e} \mapsto \phi_{r_{t,e},d_e}$ where:*

i) $d\colon F \to F$ is an order preserving map;

ii) if $\phi_{t,e} \in \mathfrak{I}$, then $\phi_{r_{t,e},d_e} \in \mathfrak{I}$;

iii) if $f^2 = f$, then $r_{f,e} = d_f$ for all idempotents $e \leq f$;

iv) $r_{t,e}^2 = d_{t^2}$.

Proof. Notice first that since $\mathfrak{I}$ contains an element of even type, then we have that $E_2 \cup C \cup E_1 \subseteq \mathfrak{I}$, which means that for all idempotents $e, f \in F$ such that $e \leq f$, we get $\phi_{f,e} \in \mathfrak{I}$.

Then it is clear that if $\rho \in \mathrm{P}(\mathfrak{I})$, we get by Lemma VII.3.9 that ρ must satisfy the stated conditions.

Conversely, suppose that $\rho\colon \mathfrak{I} \to \mathfrak{I}$ is defined by $\phi_{t,e}\rho = \phi_{r_{t,e},d_e}$ for all $\phi_{t,e} \in \mathfrak{I}$ and that r and d satisfy conditions $i)$–$iv)$. In particular, ρ is well-defined by condition $ii)$. Let $\phi_{t,e} \in \mathfrak{I}$ and consider the type of $\alpha \in \mathfrak{I}$.

- If $\alpha \in \mathfrak{I}$ has odd type, then α is a left identity for all elements of $\mathfrak{I}$ so that $(\alpha\phi_{t,e})\rho = \phi_{t,e}\rho = \alpha(\phi_{t,e}\rho)$.
- If $\alpha \in \mathfrak{I}$ has even type then we have that

  $$(\alpha\phi_{t,e})\rho = \phi_{t^2,e}\rho = \phi_{r_{t^2,e},d_e} = \phi_{d_{t^2},d_e},$$

  where the last equality comes from $iii)$ together with the fact that t^2 is idempotent. On the other hand,

  $$\alpha(\phi_{t,e}\rho) = \alpha\phi_{r_{t,e},d_e} = \phi_{r_{t,e}^2,d_e} = \phi_{d_{t^2},d_e}$$

  by using $iv)$, so that $(\alpha\phi_{t,e})\rho = \alpha(\phi_{t,e}\rho)$.
- If $\alpha \in \mathfrak{I}$ has trivial or non-permutation type, then $(\alpha\phi_{t,e})\rho = \alpha(\phi_{t,e}\rho)$ by using the same arguments made in the proof of Lemma VII.3.10 above, since $iii)$ implies that $r_{f,f} = d_f$ for all idempotents $f \in F$ and all other conditions are similar.

Thus, in all cases, we get that $(\alpha\phi_{t,e})\rho = \alpha(\phi_{t,e}\rho)$, which shows that ρ is a right translation of $\mathfrak{I}$. $\square$

Remark VII.3.12. From Lemmas VII.3.10 and VII.3.11, we get a description of the right translations for all proper ideals of $\mathcal{E}_n$, and we can see that right translations do not necessarily come from multiplication by an element of $\mathcal{E}_n$ on the right since the conditions on d and r are too loose to enforce this. In particular, we can send elements of B of the form $\phi_{t,e}$ to the element $\phi_{t^2,e} \in C$, but since $B(\mathcal{E}_n \setminus \mathcal{G}_n) \cap C = \emptyset$ such a right translation cannot come from a multiplication by an element on the right. For a more involved example, see Remark VII.3.15 below.

VII.3.3 Translational hulls

In view of the results above, the right and left translations of arbitrary ideals of $\mathcal{E}_n$ can be pretty wild, and we will therefore not attempt to characterise all the linked pairs. In fact, the condition for a left and a right translation to be linked does not even ensure that they come from functions on the algebra $\mathcal{T}_n$, as given by the following.

Lemma VII.3.13. *If there exist elements $\phi_{u,f} \neq \phi_{u,h} \in \mathfrak{I} \cap (A \cup B)$, then $\Omega(\mathfrak{I})$ is not realised by transformations of $\mathcal{T}_n$.*

Proof. Notice first that since $\phi_{u,f} \in A \cup B$, it follows that $\phi_{u^2,f} \neq \phi_{u,f}$ but $\phi_{u^2,f}$ has the same type as $\phi_{u,f}$.

We now define $\rho\colon \mathfrak{I} \to \mathfrak{I}$ by $\phi_{u,f}\rho = \phi_{u^2,f}$ and $\phi_{t,e}\rho = \phi_{t,e}$ for all $\phi_{t,e} \in \mathfrak{I} \setminus \{\phi_{u,f}\}$. By direct computation, or by taking d to be the identity map (which is then order preserving), and letting $r_{u,f} = u^2$ and $r_{t,e} = t$ for all $\phi_{t,e} \neq \phi_{u,f}$, we get that ρ satisfies all conditions of Lemmas VII.3.10 and VII.3.11, so that $\rho \in \mathrm{P}(\mathfrak{I})$.

Moreover, for all $\alpha, \beta \in \mathfrak{I}$ we have that $\alpha\rho\beta = \alpha\beta$ if $\alpha \neq \phi_{u,f}$ and otherwise $\phi_{u,f}\rho\beta = \phi_{u^2,f}\beta = \phi_{u,f}\beta$, where the last equality comes from Lemma IV.3.6 since $\phi_{u^2,f}$ and $\phi_{u,f}$ have the same type. This shows in particular that ρ is linked to the left translation $\mathbb{1}_\Lambda$.

Now, let $s \in \mathcal{T}_n$ be an odd permutation. Then we get that $s\phi_{u,f}\rho = s\phi_{u^2,f} = u^2$. On the other hand, we have that $\phi_{u,h} \neq \phi_{u,f}$ so that $s\phi_{u,h}\rho = s\phi_{u,h} = u$. Suppose now that there exists $\theta\colon \mathcal{T}_n \to \mathcal{T}_n$ such that $\phi_{t,e}\rho = \phi_{t,e} \circ \theta$ for all $\phi_{t,e} \in \mathfrak{I}$. Then we get that $u^2 = s\phi_{u,f}\rho = s\phi_{u,f} \circ \theta = u\theta$ and simultaneously $u = s\phi_{u,h}\rho = s\phi_{u,h} \circ \theta = u\theta$, which is not possible since $u \neq u^2$. Hence $(\mathbb{1}_\Lambda, \rho) \in \Omega(\mathfrak{I})$ is not realised by a transformation of $\mathcal{T}_n$. □

As shown by Lemma VII.3.13, we have that even right translations linked to the identity left translation are hard to tame and to understand since there could be so many of them.

We close this chapter by showing that not every right or left translation can be part of a linked pair, and that some restrictions will occur.

Lemma VII.3.14. *Suppose that $\mathfrak{I} = \mathcal{E}_n \setminus \mathcal{G}_n$ and let $(\lambda, \rho) \in \Omega(\mathfrak{I})$. Then for each $\phi_{t,e} \in \mathfrak{I}$, the type of $\phi_{t,e}\rho$ is fully determined by the left translation λ.*

Proof. By Corollary VII.3.5, we know that λ is one of λ^{id}, λ^+, λ^- or λ^0, which is equivalent to having that $\lambda\gamma = \alpha\gamma$ for some specific α of a given type.

Since $(\lambda, \rho) \in \Omega(\mathfrak{I})$, we have that for all $\phi_{t,e}, \phi_{u,f} \in \mathfrak{I}$

$$\phi_{r_{t,e},d_e}\phi_{u,f} = \phi_{t,e}\rho\phi_{u,f} = \phi_{t,e}\lambda\phi_{u,f} = \phi_{t,e}\alpha\phi_{u,f}.$$

We now look at the possible types for α.

Clearly, if α has odd type, then $\phi_{t,e}\alpha\phi_{u,f} = \phi_{t,e}\phi_{u,f}$, and thus by Lemma IV.3.6, we get that $\phi_{r_{t,e},d_e}$ must have the same type as $\phi_{t,e}$ for all $\phi_{t,e} \in \mathfrak{I}$.

If α has even type, then we get $\phi_{t,e}\alpha\phi_{u,f} = \phi_{t,e}\phi_{u^2,f}$, and the type of $\phi_{r_{t,e},d_e}$ is determined as follows:

- if $\phi_{t,e} \in E_3 \cup A \cup E_2$, then $\phi_{t,e}\phi_{u^2,f} = \phi_{u^2,f}$, and thus we must have $r_{t,e} \in \mathcal{A}_n$;
- if $\phi_{t,e} \in B \cup C \cup E_1 \setminus \{\phi_{\text{id},\text{id}}\}$, then $\phi_{t,e}\phi_{u^2,f} = \phi_{f,f}$, so that $r_{t,e} \in \mathcal{T}_n \setminus \mathcal{S}_n$; and
- if $\phi_{t,e} = \phi_{\text{id},\text{id}}$, then $\phi_{t,e}\phi_{u^2,f} = \phi_{u^2,u^2}$ which forces $r_{t,e} = d_e = \text{id}$.

Now if α has non-permutation type, then we get that $\phi_{t,e}\alpha\phi_{u,f} = \phi_{t,e}\phi_{f,f} = \phi_{f,f}$ since elements of E_1 are right zeros, and therefore $r_{t,e} \in \mathcal{T}_n \setminus \mathcal{S}_n$ for all $\phi_{t,e} \in \mathfrak{I}$. Similarly, if α is of trivial type, then $\phi_{t,e}\alpha\phi_{u,f} = \phi_{t,e}\phi_{u^2,u^2} = \phi_{u^2,u^2}$, and thus $r_{t,e} = d_e = \text{id}$, that is, $\phi_{t,e}\rho = \phi_{\text{id},\text{id}}$ for all $\phi_{t,e} \in \mathfrak{I}$.

This shows that for each possible left translation λ on $\mathcal{E}_n \setminus \mathcal{G}_n$, the type of the map $\phi_{t,e}\rho$ for $\phi_{t,e} \in \mathfrak{I}$ and ρ a right translation linked with λ is fully determined by λ. □

Remark VII.3.15. By looking at the proof of Lemma VII.3.14, we can notice that a right translation of $\mathcal{E}_n \setminus \mathcal{G}_n$ that is linked must send the endomorphisms of non-permutation type to a subset of $\mathcal{E}_n \setminus \mathcal{G}_n$ which only contains one of the types. To see that some right translations are not linked, we create a right translation ρ that does not satisfy this condition.

Consider $h = \begin{pmatrix} 1 & i_{\geq 2} \\ 2 & i \end{pmatrix} \in F$. Then we have that $vh = hv = h$ if and only if $v = h$ or $v = \mathrm{id}$ for $v \in \mathcal{T}_n$, that is, $\phi_{v,h} \in \mathcal{E}_n$ if and only if $v = h$ or $v = \mathrm{id}$. Now define d and r as follows:

$$d_e = \begin{cases} \mathrm{id} & \text{if } e = h, \\ e & \text{otherwise,} \end{cases} \qquad \text{and} \qquad r_{t,e} = \begin{cases} \mathrm{id} & \text{if } t^2 = h, \\ t & \text{otherwise.} \end{cases} \tag{VII.3.1}$$

Then one can verify that all the conditions from Lemma VII.3.11 are satisfied, so that the map $\rho\colon \phi_{t,e} \mapsto \phi_{r_{t,e},d_e}$ is a right translation. Explicitly, we have

$$\phi_{t,e}\rho = \begin{cases} \phi_{\mathrm{id},e} & \text{if } t^2 = h \neq e, \\ \phi_{\mathrm{id},\mathrm{id}} & \text{if } h = e \text{ (i.e if } \phi_{t,e} = \phi_{\mathrm{id},h} \text{ or } \phi_{t,e} = \phi_{h,h}), \\ \phi_{t,e} & \text{otherwise,} \end{cases}$$

from which it is easy to see that all elements of $E_1 \setminus \{\phi_{\mathrm{id},\mathrm{id}}, \phi_{h,h}\}$ are sent to elements of non-permutation type, while $\phi_{h,h}$ is sent to the element of trivial type, and elements of the form $\phi_{t,e} \in B$ with $t^2 = h$ are sent to elements of even type. Hence, ρ cannot be linked to any of the four left translations of $\mathcal{E}_n \setminus \mathcal{G}_n$.

Lemma VII.3.16. *Suppose that $\mathfrak{I} = C \cup E_1$ and let $(\lambda, \rho) \in \Omega(\mathfrak{I})$. Then one of the following occurs:*

1) $\lambda\phi_{u,f} = \phi_{u,u}$ *and* $r_{t,e} = r_{\mathrm{id},\mathrm{id}} = \mathrm{id}$*;*
2) $\lambda\phi_{u,f} = \phi_{f,f}$ *and* $r_{t,e} \neq \mathrm{id} \neq r_{\mathrm{id},\mathrm{id}}$*; or*
3) $\lambda\phi_{u,f} = \phi_{u,f}$ *and* $r_{\mathrm{id},\mathrm{id}} = \mathrm{id} \neq r_{t,e}$*,*

for all $\phi_{u,f} \in C$ and $\phi_{t,e} \in \mathfrak{I}$.

Consequently, the only left translations that are linked come from the multiplication by an endomorphism on the left, while right translations which are linked have additional restrictions on the maps d and r they induce.

Proof. Let $\phi_{u,f} \in C$, so that $u^2 = u \neq f = f^2$, and let $\lambda\phi_{u,f} = \phi_{p,q}$. Then since (λ, ρ) is a linked pair, we have that for all $\phi_{t,e} \in \mathfrak{I}$

$$\phi_{r_{t,e},d_e}\phi_{u,f} = \phi_{t,e}\rho\phi_{u,f} = \phi_{t,e}\lambda\phi_{u,f} = \phi_{t,e}\phi_{p,q}.$$

It follows in particular that $p, q \in \{u, f\}$ and we cannot have $p = f$ and $q = u$ since $uf = fu = f$. The different cases will now follow from a study of the two possible types for $\phi_{r_{t,e},d_e}$:
1) Suppose that $r_{t,e} = \mathrm{id}$ for some $\phi_{t,e} \in \mathfrak{I} \setminus \phi_{\mathrm{id},\mathrm{id}}$. Then we get that $d_e = \mathrm{id}$ and $\phi_{u^2,u^2} = \phi_{q,q}$. Thus $q = u^2 = u$, which forces $p = u$ and therefore $\lambda\phi_{u,f} = \phi_{u,u}$ for all $\phi_{u,f} \in C$. Moreover, for all $\phi_{v,h} \in \mathfrak{I}$, we then get $\phi_{r_{v,h},d_h}\phi_{u,f} = \phi_{v,h}\phi_{u,u} = \phi_{u,u}$, which shows that we also get $r_{v,h} = d_h = \mathrm{id}$.
2) Suppose that $r_{\mathrm{id},\mathrm{id}} \neq \mathrm{id}$. Then $\phi_{f,f} = \phi_{r_{\mathrm{id},\mathrm{id}},d_{\mathrm{id}}}\phi_{u,f} = \phi_{\mathrm{id},\mathrm{id}}\phi_{p,q} = \phi_{p^2,p^2} = \phi_{p,p}$, which shows that $p = f$ and thus $q = f$. Hence $\lambda\phi_{u,f} = \phi_{f,f}$ for all $\phi_{u,f} \in C$, and we also get that $\phi_{r_{t,e},d_e}\phi_{u,f} = \phi_{f,f}$, which means that $r_{t,e} \neq \mathrm{id}$ for all $\phi_{t,e} \in \mathfrak{I}$.
3) Lastly, suppose that $r_{\mathrm{id},\mathrm{id}} = \mathrm{id}$ but $r_{t,e} \neq \mathrm{id}$ for some $\phi_{t,e} \in \mathfrak{I} \setminus \{\phi_{\mathrm{id},\mathrm{id}}\}$. On one hand, we get that $\phi_{u,u} = \phi_{r_{\mathrm{id},\mathrm{id}},d_{\mathrm{id}}}\phi_{u,f} = \phi_{p,p}$, while on the other hand we have $\phi_{f,f} = \phi_{r_{t,e},d_e}\phi_{u,f} = \phi_{q,q}$, giving that $\phi_{p,q} = \phi_{u,f}$. Thus $\lambda = \mathbb{1}_\Lambda$, and for all $\phi_{v,h} \in \mathfrak{I} \setminus \{\phi_{\mathrm{id},\mathrm{id}}\}$ we have that $\phi_{r_{v,h},d_h}\phi_{u,f} = \phi_{v,h}\phi_{u,f} = \phi_{f,f}$, which means that $r_{v,h} \neq \mathrm{id}$.

From the description of the cases above, it is then clear that λ comes from the multiplication by an element of $\mathcal{E}_n$ of trivial type for case 1), of non-permutation type for case 2) and of group, odd or even type for case 3). Similarly, a right translation ρ will not be linked if the conditions on the value of $r_{\mathrm{id},\mathrm{id}}$ coming from the cases above are not satisfied, which shows that many right translations will not be linked (since there are barely any restrictions on this element given in the conditions of Lemma VII.3.10). $\square$

List of symbols and index

In what follows S is a semigroup, $T \subseteq S$ and $\mathscr{X}$ is a Green's relation.

LIST OF COMMON SYMBOLS

$\langle X \rangle$: subalgebra generated by X, 8
$\langle \emptyset \rangle$: constant subalgebra, 8
$\mathrm{rk}(\alpha)$: rank of α, 44
$\mathbb{1}_{\Lambda}$: identity left translation, 10
$\mathbb{1}_{\mathrm{P}}$: identity right translation, 10
$\mathbb{1}_{\Omega}$: identity bi-translation, 10
$\aleph_0$: smallest infinite cardinal, 8
e: smallest cardinality of a generating set for a subalgebra, 8
κ^+: successor cardinal of κ, 8
λ_x: left translation induced by an element x, 26
ρ_x: right translation induced by an element x, 26
π_{Λ}: map $(\lambda, \rho) \mapsto \lambda$, 10
π_{P}: map $(\lambda, \rho) \mapsto \rho$, 10
χ_{Λ}: map $x \mapsto \lambda_x$, 28
χ_{P}: map $x \mapsto \rho_x$, 28
χ: map $x \mapsto (\lambda_x, \rho_x)$, 28

$\mathscr{A}$, $\mathscr{B}$, $\ldots$: universal algebras, 8

$\mathcal{E}_n$: endomorphism monoid of $\mathcal{T}_n$, 102
$E(S)$: set of idempotents, 7

$\mathcal{A}_n$: alternating group, 101
$\mathcal{K}$: Klein 4 subgroup, 101
$\mathcal{S}_n$: symmetric group, 101
$\mathcal{G}_n$: automorphism group of $\mathcal{T}_n$, 102

$\daleth^{\ell}_S(T)$: left idealiser of T in S, 27
$\daleth^{r}_S(T)$: right idealiser of T in S, 27
$\daleth_S(T)$: idealiser of T in S, 27

$J^*(a)$: principal $*$-ideal generated by a, 32
$\widetilde{J}(a)$: principal $\sim$-ideal generated by a, 32

$\Lambda(S)$: monoid of left translations, 10
$\widetilde{\Lambda}(S)$: set of linked left translations, 10
$\Lambda_0(S)$: set of inner left translations, 12
$\mathrm{P}(S)$: monoid of right translations, 10
$\widetilde{\mathrm{P}}(S)$: set of linked right translations, 10

$\mathrm{P}_0(S)$: set of inner right translations, 12
$\Omega(S)$: translational hull, 10
$\Sigma(S)$: set of inner bi-translations, 12

Q: regular elements of $T(\mathscr{A},\mathscr{B})$, 61

$T(\mathscr{A},\mathscr{B})$: semigroup of endomorphisms with restricted range, 56
T_μ: ideal of $\mathrm{End}(\mathscr{A})$, 51

$\mathcal{T}^{\mathscr{A}}$: set of all terms of $\mathscr{A}$, 8
$\mathcal{T}^{\alpha}$: set of terms defining α on a basis, 152

$\mathcal{T}_k^{\mathscr{A}}$: set of all k-ary terms of $\mathscr{A}$, 8
$\mathcal{T}_X$, $\mathcal{PT}_X$, $\mathcal{I}_X$: monoids of full, partial and bijective partial transformations, 7
$\mathcal{T}_X^{\mathrm{op}}$, $\mathcal{PT}_X^{\mathrm{op}}$, $\mathcal{I}_X^{\mathrm{op}}$: ——— when composed of left maps, 7

$\mathscr{X}_s$: $\mathscr{X}$-class of an element s, 31
$\mathscr{X}_S$: relation $\mathscr{X}$ on the semigroup S, 7
$\mathscr{X}^*$: extended $*$-relation of $\mathscr{X}$, 30, 33
$\widetilde{\mathscr{X}}$: extended $\sim$-relation of $\mathscr{X}$, 30, 33

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