Morita Equivalence for C*-Categories

Matthew James Ferrier

Submitted for the degree of Doctor of Philosophy

School of Mathematics and Statistics

November 2022

Supervisor: Dr. Paul Mitchener

University of Sheffield

Introduction

There are a few tactics one could employ in order to motivate the need for C*-categories, to name some, we have the following;

(i) They are natural generalisations of C*-algebras in the same way that groupoids are a generalisation of groups,

(ii) They provide a framework that describes collections of C*-algebra related things that we might be interested in, such as the collection of Hilbert modules over a fixed C*-algebra, or the collection of non-degenerate representations of a C*-algebra,

(iii) They provide a technical tool for studying things like assembly maps [6, 11, 24].

Morita equivalence for rings is an equivalence relation that preserves many ring-theoretic properties, which is also substantially weaker than isomorphism. Two unital rings are Morita equivalent if their categories of right modules are equivalent. A classification theorem accredited to Eilenberg and Watts then explains that such categorical equivalences are represented by the existence of certain bimodules.

For C*-algebras, Morita equivalence was introduced by Rieffel [27, 28, 29], with a pair of C*-algebras being declared Morita equivalent if there exists a certain Hilbert bimodule between them, and then this can be used to show that Morita equivalent C*-algebras have equivalent categories of right Hilbert modules. Similarly to the theory for rings, Morita equivalence in this setting preserves many properties of interest [2], but one additional triumph is that the theory for C*-algebras works for both unital and non-unital C*-algebras.

The theory of C*-algebras and Morita equivalences can be put into the framework of bicategories [17]; there is a bicategory where the objects are C*-algebras, 1-cells are certain Hilbert bimodules called correspondences, and 2-cells are bimodule homomorphisms. In this bicategory, the invertible 1-cells are precisely the Morita equivalences. This point of view has been put into use by Albandik and Meyer [1] where colimits in this bicategory are studied, and some constructions with C*-algebras are shown to be examples of such colimits.

In the first 4 chapters of this thesis, we will develop the required theory of Hilbert modules and bimodules over C*-categories. Many of our definitions have been covered already in the literature [12, 23, 24] and we will review these, together with plenty of examples, and also set out new definitions that we will need later.

In Chapter 5 we will cover the category algebra construction. This was introduced by Joachim [12] for unital C*-categories, but we will show that many of his results still hold true for non-unital C*-categories.

In Chapter 6 we will introduce the notion of Morita equivalence for C*-categories. Just like Rieffel's original work, this will be phrased in terms of the existence of certain bimodules, which we will be calling *equivalence bimodules.* We will show that the category algebra construction relates to Morita equivalence in the following way.

Theorem. *If $\mathcal{B}$ is a small C^*-category, then there is a C^*-algebra $A(\mathcal{B})$ such that $\mathcal{B}$ and $A(\mathcal{B})$ are Morita equivalent. Moreover when $\mathcal{C}$ is another small C^*-category, $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent if and only if $A(\mathcal{B})$ and $A(\mathcal{C})$ are Morita equivalent.*

In Chapter 7 we will show that just as for C*-algebras, the theory of Morita equivalence fits nicely into the framework of a bicategory.

Theorem. *There is a bicategory $\mathfrak{Corr}_{Cat}$ where the objects are small C^*-categories, 1-cells are certain bimodules called correspondences, and 2-cells are bimodule homomorphisms. The invertible 1-cells in this bicategory are precisely the Morita equivalences. Writing $\mathfrak{Corr}_{Alg}$ for the analogous bicategory of C^*-algebras, then the category algebra construction defines a pseudofunctor*

$$A(-) : \mathfrak{Corr}_{Cat} \to \mathfrak{Corr}_{Alg}.$$

Finally, in Chapter 8 we look at certain *-functors between categories of Hilbert modules, with our end goal being the following alternative characterisation of Morita equivalence.

Theorem. *C^*-categories $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if and only if their categories* *of right Hilbert modules are equivalent, with the equivalence implemented by a pair of strongly continuous* **-functors.*

Contents

Chapter 1

Preliminaries

We will begin by giving a brief summary of certain notation and bits of theory that we will be using throughout our work.

1.1 Category Theory

Here is an overview of some the categorical notation and conventions we will be following.

(i) We will denote categories by mathscr letters $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$,

(ii) In a category $\mathscr{A}$, we will write $\mathscr{O}\mathrm{b}(\mathscr{A})$ for the collection of objects of $\mathscr{A}$, and will write $\mathscr{A}(X, Y)$ for the collection of morphisms from X to Y,

(iii) We may write $\mathscr{B} \subseteq \mathscr{A}$ to indicate that $\mathscr{B}$ is a subcategory of $\mathscr{A}$,

(iv) We denote morphisms in a category by normal arrows $\to$, and denote natural transformations of functors by double arrows $\Rightarrow$,

(v) We may write $F \simeq G$ to indicate that the functors F and G are naturally isomorphic,

(vi) When writing down the components of a natural transformation $\theta : F \Rightarrow G$, if it will increase readability and cause no confusion, then we will just write θ for a component $\theta_A : F(A) \to G(A)$,

(vii) When considering a functor F of two variables, we may notate this as $F(-,=)$, to emphasise that F takes two inputs.

1.1.1 Frequently Used Categories

Here we include a list of some categories which we will make frequent use of. Most of these will be introduced as we go along, but we provide this list as a quick reference.

(i) **Hilb**; objects are complex Hilbert spaces, morphisms are adjointable operators,

(ii) **Hilb**-A; objects are right Hilbert modules over a fixed C*-algebra A, morphisms are adjointable operators,

(iii) $\mathbf{Hilb}_{\mathcal{K}}$-$A$; objects are right Hilbert modules over a fixed C*-algebra A, morphisms are compact operators,

(iv) A-**Hilb**; objects are left Hilbert modules over a fixed C*-algebra A, morphisms are adjointable operators,

(v) A-$\mathbf{Hilb}_{\mathcal{K}}$; objects are left Hilbert modules over a fixed C*-algebra A, morphisms are compact operators,

(vi) **Vect**; objects are $\mathbb{C}$-vector spaces, morphism are linear maps.

1.2 Background Material

1.2.1 Algebroids and Their Modules

The main algebraic definition that we will need is that of an algebroid. These can be thought of as associative algebras with many objects. First we require a non-standard piece of category theory.

Definition 1.2.1. A *non-unital category* $\mathcal{A}$ is a collection of objects and morphisms which satisfies all the axioms required of a category, except we don't require each collection of endomorphisms $\mathcal{A}(A, A)$ to possess an identity morphism. Similarly, a *non-unital* functor is a mapping between (possibly non-unital) categories, which satisfies all the conditions required of a functor, except we don't require it to preserve identities.

As we introduce categories, we will tend to state whether they are unital or not.

Definition 1.2.2. An *algebroid* over $\mathbb{C}$ is a (potentially non-unital) category $\mathcal{A}$ such that each morphism set is a $\mathbb{C}$-vector space and the composition operation is bilinear. The functors of interest between algebroids are those consisting of linear maps between the vector spaces of morphisms involved. We will call such functors *linear functors.* We call a linear functor between unital algebroids *unital* if it preserves the identity morphisms.

Remarks 1.2.3. (i) A standard reference for algebroids would be [20] where their algebraic properties are studied.

(ii) Note that a unital algebroid is simply a $\mathbb{C}$-linear category. We will favour the term algebroid so that we may keep in mind that we might not have identity morphisms.

Examples 1.2.4. (i) As hinted above, an associative algebra over $\mathbb{C}$ can be viewed as an algebroid with a single object,

(ii) Let $\mathcal{G}$ be a groupoid[1]. We construct an algebroid $\mathbb{C}\,\mathcal{G}$ as follows; we set $\mathcal{O}\mathrm{b}(\mathbb{C}\,\mathcal{G}) = \mathcal{O}\mathrm{b}(\mathcal{G})$, and set

$$\mathbb{C}\,\mathcal{G}(X,Y) = \{\sum_{i=1}^{m} \lambda_i g_i;\ \lambda_i \in \mathbb{C},\ g_i \in \mathcal{G}(X,Y)\}.$$

Then we define composition law by

$$(\sum_i \lambda_i g_i) \circ (\sum_j \mu_j h_j) = \sum_{i,j} (\lambda_i \mu_j) g_i \circ h_j.$$

Note that $\mathbb{C}\,\mathcal{G}$ is a unital algebroid.

The following general definition will play a crucial role throughout most of our work.

Definition 1.2.5. If $\mathcal{A}$ is an algebroid, then a *right $\mathcal{A}$-module* is a linear functor

$$\mathcal{E} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Vect},$$

where **Vect** denotes the category of complex vector spaces. The functorial properties of $\mathcal{E}$ give us an action of $\mathcal{A}$ on the collection of vector spaces $\{\mathcal{E}(A);\ A \in \mathcal{O}\mathrm{b}(\mathcal{A})\}$. This action is given by

$$\xi \cdot a := \mathcal{E}(a)\xi,$$

[1] For this example we adopt the point of view that a groupoid is a small category where each arrow is an isomorphism.

where $a \in \mathcal{A}(A, A')$ and $\xi \in \mathcal{E}(A')$. Similarly, one can define a *left $\mathcal{A}$-module* to be a linear functor

$$\mathcal{F} : \mathcal{A} \to \mathbf{Vect},$$

this time with action given by

$$a \cdot \xi := \mathcal{F}(a)\xi.$$

If we wish to emphasise that a given module action is part of a left or right module, then we will write $a \overset{\curvearrowright}{\cdot} \xi$ for the left action of a on ξ, and $\xi \overset{\curvearrowleft}{\cdot} a$ for the right action of a on ξ. Homomorphisms of right or left $\mathcal{A}$-modules are just natural transformations of functors, and we write $\theta : \mathcal{D} \Rightarrow \mathcal{E}$ to indicate that θ is a module homomorphism from $\mathcal{D}$ to $\mathcal{E}$. We can use the terms *module homomorphism* and *natural transformation* interchangeably, but we will retain the term module homomorphism for algebraic morphisms of this form. We will write $\mathbf{Mod} - \mathcal{A}$ and $\mathcal{A} - \mathbf{Mod}$ for the categories of right and left $\mathcal{A}$-modules and module homomorphisms respectively.

Remark 1.2.6. The above definition is obviously a bit degenerate for non-unital algebroids, since we could just take any collection of vector spaces indexed by the objects of our algebroid, and define a module by sending all morphisms to zero. If one wishes to look at module theory over non-unital rings, then there are extra conditions once can impose in order to avoid this problem [25]. However for us, this will never present an issue, as we will point out later.

1.2.2 C*-categories

Here we will quickly review the basic theory of C*-categories, with a focus to fix notation and cover useful constructions that we will need later on. There are now quite a few references one can choose from for the basic theory; the original material is [10], where C*-categories are introduced as a stepping stone towards W*-categories, and Mitchener covers most of the same material but in greater detail in [23]. Dell'Ambrogio gives a very readable overview in [7], but note that there it is assumed that C*-categories are unital. Finally there is the more modern approach of Bunke in [5] which makes heavy use of category theory.

Now we can begin the process of trying to do analysis with algebroids.

Definition 1.2.7. We will say that an algebroid $\mathcal{A}$ is a *normed category* if

(i) Each morphism set is a normed $\mathbb{C}$-vector space,

(ii) For any composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{A}$, we have

$$\|gf\| \leq \|g\|\|f\|,$$

where the norms are taken in the appropriate vector spaces.

We refer to the collection of norms associated with a normed category just as a *norm*. A normed category $\mathcal{A}$ is called a *Banach category* if additionally each of its morphism sets is complete.

Remark 1.2.8. Banach categories are defined and studied in [15], however in that book Banach categories are also required to be additive. We do not assume this.

Definition 1.2.9. Let $\mathcal{A}$ be an algebroid over $\mathbb{C}$. An *involution* on $\mathcal{A}$ is a contravariant functor $(-)^* : \mathcal{A} \to \mathcal{A}$ which is the identity on objects and whose behaviour on arrows satisfies

(i) $(\lambda f + \mu g)^* = \overline{\lambda} f^* + \overline{\mu} g^*$ for all arrows $f, g \in \mathcal{A}(X,Y)$ and all scalars $\lambda, \mu \in \mathbb{C}$,

(ii) $f^{**} = f$ for every arrow of $\mathcal{A}$,

(iii) $(gf)^* = f^* g^*$ for all composable arrows f, g of $\mathcal{A}$.

We call an algebroid that is equipped with an involution a **-category*.

Now we may define a C*-category.

Definition 1.2.10. A normed *-category $\mathcal{A}$ is called a *pre-C*-category* if

(i) For each $f \in \mathcal{A}(X,Y)$, the following C*-inequality is satisfied,

$$\|f\|^2 \leq \|f^* f + g^* g\|,$$

for all morphisms $g \in \mathcal{A}(X,Y)$,

(ii) For each morphism $f \in \mathcal{A}(X,Y)$, there is an endomorphism $g \in \mathcal{A}(X,X)$ with $f^* f = g^* g$.

$\mathcal{A}$ is called a *C*-category* if it additionally satisfies

(iii) Each of its morphism sets is complete.

Before we comment on the previous definition, recall the following definition for C*-algebras.

Definition 1.2.11. If A is a complex C*-algebra, then an element $a \in A$ is called *positive* if there is $x \in A$ with $a = x^*x$. Equivalently, a is positive if it's spectrum $\sigma(a)$ is a subset of $\mathbb{R}^{\geq 0}$.

Remarks 1.2.12. Below are two frequently used observations.

(i) If $\mathcal{A}$ is a C*-category, then each endomorphism set $\mathcal{A}(A, A)$ is a complex C*-algebra,

(ii) Conditions (i) and (ii) imply that in a C*-category $\mathcal{A}$, for any morphism $a \in \mathcal{A}(A, A')$, the composite a^*a is a positive element of $\mathcal{A}(A, A)$, and moreover the C*-identity $\|a^*a\| = \|a\|^2$ holds. See [22] for a detailed discussion of this matter.

Definition 1.2.13. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-categories. A linear functor $F : \mathcal{A} \to \mathcal{B}$ is called a **-functor* if it additionally satisfies

$$F(f^*) = F(f)^*,$$

for each morphism $f \in \mathcal{A}(X, Y)$.

Approximate Units

Let A be a C*-algebra. We recall that an element $a \in A$ is called *self adjoint* if it satisfies $a = a^*$. An approximate unit for A is a net of self adjoint elements (e_λ), in the unit ball of A, satisfying

$$\|a - ae_\lambda\| \to 0, \qquad \|a - e_\lambda a\| \to 0,$$

for all $a \in A$. Since each endomorphism set of a C*-category is a C*-algebra, it possesses an approximate unit, and this will satisfy the same limits as those stated above, where this time a is taken from the relevant endomorphism C*-algebra. We will make use of the following lemma in a few places, which shows that the approximate units found in a C*-category satisfy similar limits with respect to arbitrary morphisms.

Lemma 1.2.14. *Let $\mathcal{A}$ be a C*-category. If $a \in \mathcal{A}(A, A')$ is any morphism, then if (u_λ) is a (self adjoint) approximate unit for $\mathcal{A}(A, A)$, then we have*

$$\lim_\lambda \|a - au_\lambda\| = 0.$$

Proof. Direct computations yield the following,

$$\begin{aligned}\|a - au_\lambda\|^2 &= \|(a - au_\lambda)^*(a - au_\lambda)\| \\ &= \|a^*a - a^*au_\lambda - u_\lambda a^*a + u_\lambda a^*au_\lambda\| \\ &\leq \|a^*a - a^*au_\lambda\| + \|u_\lambda\|\|a^*a - a^*au_\lambda\| \\ &\leq \|a^*a - a^*au_\lambda\| + \|a^*a - a^*au_\lambda\|.\end{aligned}$$

The morphism a^*a is an element of the C*-algebra $\mathcal{A}(A, A)$, and (u_λ) is an approximate unit for $\mathcal{A}(A, A)$, so upon taking limits the two right hand terms tend to zero and the result follows. □

Basic Constructions

One can perform various constructions with C*-categories. Here we will recap the construction of products, coproducts and additive completions.

Example 1.2.15. Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of C*-categories.

(i) The *product* of $\mathcal{A}$ and $\mathcal{B}$ is the C*-category $\mathcal{A} \times \mathcal{B}$ with

$$\mathcal{O}\mathrm{b}(\mathcal{A} \times \mathcal{B}) = \mathcal{O}\mathrm{b}(\mathcal{A}) \times \mathcal{O}\mathrm{b}(\mathcal{B}),$$

and

$$(\mathcal{A} \times \mathcal{B})((A, B), (A', B')) = \mathcal{A}(A, A') \times \mathcal{B}(B, B'),$$

where each morphism set is equipped with the max-norm

$$\|(a, b)\| = \max\{\|a\|, \|b\|\}.$$

The C*-category $\mathcal{A} \times \mathcal{B}$ together with the projection maps

$$(a, b) \mapsto a, \qquad (a, b) \mapsto b,$$

can be checked to satisfy the universal property required of a product.

(ii) The *coproduct* of $\mathcal{A}$ and $\mathcal{B}$ is the C*-category $\mathcal{A} \sqcup \mathcal{B}$ with

$$\mathcal{O}\mathrm{b}(\mathcal{A} \sqcup \mathcal{B}) = \mathcal{O}\mathrm{b}(\mathcal{A}) \sqcup \mathcal{O}\mathrm{b}(\mathcal{B}),$$

and

$$(\mathcal{A} \sqcup \mathcal{B})(X,Y) = \begin{cases} \mathcal{A}(X,Y) & \text{if } X, Y \in \mathcal{O}\mathrm{b}(\mathcal{A}) \\ \mathcal{B}(X,Y) & \text{if } X, Y \in \mathcal{O}\mathrm{b}(\mathcal{B}) \\ 0 & \text{otherwise.} \end{cases}$$

The C*-category $\mathcal{A} \sqcup \mathcal{B}$ together with the obvious inclusion maps can be checked to satisfy the universal property required of a coproduct.

Example 1.2.16. Let $\mathcal{A}$ be a unital C*-category. The *additive completion* of $\mathcal{A}$ is the category $\mathcal{A}_\oplus$ whose objects are formal words $A_1 \cdots A_m$ where $A_1, \ldots, A_m \in \mathcal{O}\mathrm{b}(\mathcal{A})$. The empty word is permitted and is denoted by 0. The morphisms in this category are matrices,

$$\mathcal{A}_\oplus(A_1 \cdots A_m, B_1 \cdots B_n) = \left\{ \begin{pmatrix} f_{11} & \cdots & f_{1m} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nm} \end{pmatrix} ;\ f_{ij} \in \mathcal{A}(A_j, B_i) \right\}.$$

The composition of morphisms is given by matrix multiplication and we further define an involution on $\mathcal{A}_\oplus$ by taking the conjugate transpose of a given matrix. To place a norm on $\mathcal{A}_\oplus$, we take a *-functor $\rho : \mathcal{A} \to \mathbf{Hilb}$ which is faithful and injective on objects[2], then define $\overline{\rho} : \mathcal{A}_\oplus \to \mathbf{Hilb}$ by

$$\overline{\rho}(A_1 \cdots A_m) = \rho(A_1) \oplus \cdots \oplus \rho(A_m), \qquad \overline{\rho} \begin{pmatrix} f_{11} & \cdots & f_{1m} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nm} \end{pmatrix} = \begin{pmatrix} \rho(f_{11}) & \cdots & \rho(f_{1m}) \\ \vdots & \ddots & \vdots \\ \rho(f_{n1}) & \cdots & \rho(f_{nm}) \end{pmatrix}.$$

The functor $\overline{\rho}$ continues to be a faithful *-functor and we may check that the mapping $f \mapsto \|\overline{\rho}(f)\|$ gives us a norm on $\mathcal{A}_\oplus$ for which $\mathcal{A}_\oplus$ is a C*-category.

Remark 1.2.17. Antoun and Voigt give constructions for additive completions of non-unital C*-categories [3], however we don't need to make use of these in our work.

Assumptions Concerning Sizes

There are a few constructions in our work where we will be forming sums indexed by the objects of a given C*-category, and also points where we wish to consider categories of C*-categories. This means that generally we will be assuming that our C*-categories

[2] Such a *-functor is guaranteed to exist by [23, Theorem 6.12].

are *small*. We say generally, because we will need to consider things like C*-categories of Hilbert modules, which won't be small. As a general rule, when we make statements like "Let $\mathcal{A}$ be a C*-category." then we are also assuming that $\mathcal{A}$ is small, and when we define categories which are large, then we will state that they are large.

Chapter 2

Hilbert Modules

2.1 The Basics

We shall begin by reviewing the definition of Hilbert modules over a C*-category. This material may be located in papers such as [12] and [24], but since Hilbert modules are going to be our main objects of interest, we will include this here for completeness. A thorough review of the theory of Hilbert modules over C*-algebras can be found in textbooks such as [16] and [26], and this is the base for many of our constructions and results.

We recall from the previous chapter that when $\mathcal{A}$ is a C*-category (so is an algebroid), then a right $\mathcal{A}$-module is a linear functor

$$\mathcal{E} : \mathcal{A}^{\mathrm{op}} \to \mathbf{Vect},$$

and a left $\mathcal{A}$-module is a linear functor

$$\mathcal{F} : \mathcal{A} \to \mathbf{Vect}.$$

Definition 2.1.1. [12, p. 645] Let $\mathcal{A}$ be a C*-category and let $\mathcal{E}$ be a right $\mathcal{A}$-module. An *inner product* on $\mathcal{E}$ is a collection of maps

$$\langle -, - \rangle : \mathcal{E}(Y) \times \mathcal{E}(X) \to \mathcal{A}(X, Y),$$

such that for all objects $X, X', Y \in \mathcal{O}\mathrm{b}(\mathcal{A})$, vectors $\xi \in \mathcal{E}(Y)$, $\eta, \zeta \in \mathcal{E}(X')$, morphisms $a \in \mathcal{A}(X', X)$ and scalars $\lambda, \mu \in \mathbb{C}$, the following conditions are satisfied,

(i) $\langle \xi, \lambda\eta + \mu\zeta \rangle = \lambda\langle \xi, \eta\rangle + \mu\langle \xi, \zeta\rangle$,

(ii) $\langle \xi, \eta \cdot a\rangle = \langle \xi, \eta\rangle a$,

(iii) $\langle \xi, \eta\rangle^* = \langle \eta, \xi\rangle$,

(iv) $\langle \xi, \xi\rangle \geq 0$ and $\langle \xi, \xi\rangle = 0$ if and only if $\xi = 0$.

The inequality in point (iv) is stating that $\langle \xi, \xi\rangle$ is a positive element of the C*-algebra $\mathcal{A}(Y, Y)$. Notice how these conditions imply that an inner product is conjugate linear in its first argument. There is a similar definition for inner products on left $\mathcal{A}$-modules. In a slight abuse of terminology, we may refer to inner products on $\mathcal{A}$-modules as being *$\mathcal{A}$-valued.*

Notation 2.1.2. If we wish to emphasise that an inner product on a given $\mathcal{A}$-module is associated with the category $\mathcal{A}$ then we shall decorate the inner product with a subscript,

$$ {}_{\mathcal{A}}\langle -, -\rangle \text{ or } \langle -, -\rangle_{\mathcal{A}}, $$

if the module is a left or right module respectively. If we wish to further highlight when we have two different inner products appearing, for example if we are considering two different $\mathcal{A}$-valued inner products, then we may use different brackets such as $[-, -]$. Again, if we need to, then we will decorate these alternative brackets with subscripts.

Before we delve too deep into the theory, notice the obvious complexity involved with inner products; an inner product on a right $\mathcal{A}$-module $\mathcal{E}$ consists of a whole lot of data, and when we start trying to prove results involving them, we're going to have to keep track of which objects of $\mathcal{A}$ are involved and where various vectors are coming from. One possible way to keep track of this information is via diagrams, for example notice that condition (ii) of the previous definition simply states that the following diagram must commute for all $X, Y, Z \in \mathcal{O}\mathrm{b}(\mathcal{A})$ and all $a \in \mathcal{A}(X, Z)$,

$$\begin{tikzcd}
\mathcal{E}(Y) \times \mathcal{E}(Z) \arrow[r, "(\mathrm{id}_{\mathcal{E}(Y)}{,}\mathcal{E}(a))"] \arrow[d, "\langle -{,}-\rangle"'] & \mathcal{E}(Y) \times \mathcal{E}(X) \arrow[d, "\langle -{,}-\rangle"] \\
\mathcal{A}(Z, Y) \arrow[r, "(-)\circ a"'] & \mathcal{A}(X, Y)
\end{tikzcd}$$

Now we show how to define norms from inner products, and state a version of the Cauchy-Schwarz inequality.

Lemma 2.1.3. *[23, Corollary 8.7] If $\mathcal{E}$ is a right (or left) $\mathcal{A}$-module equipped with an inner product, then for each $A \in \mathcal{O}\mathrm{b}\,\mathcal{A}$ the assignment*

$$\mathcal{E}(A) \ni \alpha \mapsto \|\langle \alpha, \alpha \rangle\|^{\frac{1}{2}},$$

defines a norm on the vector space $\mathcal{E}(X)$.

Lemma 2.1.4. *[23, Lemma 8.6] If $\mathcal{E}$ is a right $\mathcal{A}$-module equipped with an inner product, then if $\xi \in \mathcal{E}(Y)$ and $\eta \in \mathcal{E}(X)$ we have*

$$\|\langle \xi, \eta \rangle\| \leq \|\langle \xi, \xi \rangle\|^{\frac{1}{2}} \|\langle \eta, \eta \rangle\|^{\frac{1}{2}}.$$

Remark 2.1.5. If $\alpha \in \mathcal{E}(A)$ and $a \in \mathcal{A}(A', A)$, then the computation

$$\|\alpha \cdot a\|^2 = \|\langle \alpha \cdot a, \alpha \cdot a \rangle\| = \|a^* \langle \alpha, \alpha \rangle a\| \leq \|a^*\| \|\alpha\|^2 \|a\| = \|a\|^2 \|\alpha\|^2,$$

shows that $\|\alpha \cdot a\| \leq \|\alpha\| \|a\|$ in any right $\mathcal{A}$-module equipped with an inner product.

Definition 2.1.6. If $\mathcal{E}$ is a right $\mathcal{A}$-module equipped with an inner product, then we call $\mathcal{E}$ a *right Hilbert $\mathcal{A}$-module* if for every $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, the space $\mathcal{E}(A)$ is complete with respect to the norm defined above. *Left Hilbert $\mathcal{A}$-modules* are defined analogously.

The morphisms of interest between Hilbert modules in this setting will be analogues of the adjointable operators between ordinary Hilbert modules. We will introduce such things shortly. For now we prove that analogues of simple results for Hilbert modules over a C*-algebra hold in this setting.

Lemma 2.1.7. *Let $\mathcal{E}$ be a right Hilbert $\mathcal{A}$-module, and write $\mathcal{E}(\mathcal{A}) := \cup_{A \in \mathcal{O}\mathrm{b}(\mathcal{A})} \mathcal{E}(A)$. Then*

(i) $\langle \alpha, 0 \rangle = 0$, for all $\alpha \in \mathcal{E}(\mathcal{A})$,

(ii) If for some $\beta \in \mathcal{E}(\mathcal{A})$ we have $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in \mathcal{E}(\mathcal{A})$, then $\beta = 0$,

(iii) If $\beta, \gamma \in \mathcal{E}(X)$ and $\langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle$ for all $\alpha \in \mathcal{E}(\mathcal{A})$, then $\beta = \gamma$.

Proof. The first point follows from the equality

$$\langle \alpha, 0 \rangle = \langle \alpha, 0 + 0 \rangle = \langle \alpha, 0 \rangle + \langle \alpha, 0 \rangle.$$

The second point is simple; taking $\alpha = \beta$ gives $\langle \beta, \beta \rangle = 0$, which can happen if and only if $\beta = 0$. For the final point we observe that the stated equality is the same as

$$\langle \alpha, \beta - \gamma \rangle = 0,$$

which we now know implies $\beta = \gamma$. $\square$

2.2 Examples of Hilbert Modules

Before we proceed, let's look at some examples of Hilbert modules.

Example 2.2.1. Hilbert modules over a C*-algebra A, defined in the usual way, are still Hilbert modules according to our definition.

Example 2.2.2. If $\mathcal{A}$ is a C*-category and $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, then we can consider the contravariant hom functor $\mathcal{A}(-, A)$. This functor is given by

$$\begin{aligned} \mathcal{O}\mathrm{b}(\mathcal{A}) \ni X &\mapsto \mathcal{A}(X, A) \\ \mathcal{A}(X, Y) \ni f &\mapsto (-) \circ f. \end{aligned}$$

This gives us a right $\mathcal{A}$-module and we can define an inner product by

$$\langle f, g \rangle = f^* g.$$

The fact that this gives a right Hilbert $\mathcal{A}$-module follows from the C*-identity $\|f^* f\| = \|f\|^2$.

Example 2.2.3. This example is very similar to the previous one, and gives us a nice concrete example of a Hilbert module. Let $\mathcal{H}$ be some fixed Hilbert space, let $\{\mathcal{J}_\lambda\}_{\lambda \in \Lambda}$ be a family of Hilbert spaces, and let $C^*(\mathcal{J}_\lambda)$ be the full subcategory of **Hilb** with set of objects $\{\mathcal{J}_\lambda\}_{\lambda \in \Lambda}$. We define a linear functor $\mathcal{E} : C^*(\mathcal{J}_\lambda)^{\mathrm{op}} \to$ **Vect**, by

$$\begin{aligned} \mathcal{O}\mathrm{b}(C^*(\mathcal{J}_\lambda)) \ni \mathcal{J}_\lambda &\mapsto \mathcal{L}(\mathcal{J}_\lambda, \mathcal{H}) \\ \mathcal{L}(\mathcal{J}_\lambda, \mathcal{J}_{\lambda'}) \ni S &\mapsto (-) \circ S. \end{aligned}$$

Finally we define an inner product by

$$\begin{aligned} \mathcal{L}(\mathcal{J}_{\lambda'}, \mathcal{H}) \times \mathcal{L}(\mathcal{J}_\lambda, \mathcal{H}) &\to \mathcal{L}(\mathcal{J}_\lambda, \mathcal{J}_{\lambda'}) \\ (Q, R) &\mapsto Q^* R. \end{aligned}$$

Now one may through the details to see that this is a Hilbert module.

Example 2.2.4. Suppose $\mathcal{A}$ is a C*-category and $\mathcal{E}$ is a right $\mathcal{A}$-module equipped with an inner product. In this example we will construct the *completion*, $\widehat{\mathcal{E}}$, of $\mathcal{E}$. Define $\widehat{\mathcal{E}}$ to be the linear functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Vect}$ which sends an object A to the completion of $\mathcal{E}(A)$ with respect to the norm induced by the inner product. The action of this functor on morphisms is given by

$$\alpha \cdot a = \lim(\alpha_n \cdot a),$$

where (α_n) is a sequence converging to α. Remark 2.1.5 guarantees that the above limit does converge. The inner product $[-,-]$ on $\widehat{\mathcal{E}}$ is defined similarly, by

$$[\alpha, \beta] = \lim\langle \alpha_n, \beta_n \rangle,$$

where (α_n) converges to α, and (β_n) converges to β. This limit is guaranteed to converge by the Cauchy-Schwarz inequality. It is straightforward to verify that this does actually define an inner product $\widehat{\mathcal{E}}$, and that $\widehat{\mathcal{E}}$ is a Hilbert module.

For further examples, we may consult the literature; see [24, Example 3.3] for an analogue of the standard Hilbert module over a C*-algebra, and [24, Example 3.4] for the direct sum of Hilbert modules.

2.3 Hilbert Modules are Non-degenerate

Hark back to the previous chapter containing our preliminary material, in particular Remark 1.2.6. There we stated that the existence of degenerate modules over non-unital algebroids will not pose an issue for us, this is because we are considering Hilbert modules, and here we shall justify this claim.

Lemma 2.3.1. *Given $\mathcal{A}$ is a C*-category and $\mathcal{E} \in$ **Hilb**-$\mathcal{A}$, we have that for each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ the set*

$$\mathit{Span}\{\xi \cdot \langle \eta, \zeta \rangle,\ \xi, \eta, \zeta \in \mathcal{E}(A)\},$$

is dense in $\mathcal{E}(A)$.

Proof. This is an adaptation of the one given in [16, p.5]. Let S denote the closed linear span of the set $\{\langle \eta, \zeta \rangle;\ \eta, \zeta \in \mathcal{E}(A)\} \subseteq \mathcal{A}(A, A)$, then S is an ideal in the C*-algebra $\mathcal{A}(A, A)$ and we let (e_λ) be an approximate unit for S. If $\xi \in \mathcal{E}(A)$, then for all λ we have

$$\|\xi - \xi e_\lambda\|^2 = \|\langle \xi - \xi e_\lambda, \xi - \xi e_\lambda \rangle\| = \|\langle \xi, \xi \rangle - \langle \xi, \xi \rangle e_\lambda - e_\lambda \langle \xi, \xi \rangle + e_\lambda \langle \xi, \xi \rangle e_\lambda\|.$$

This equality implies that if we take any $\epsilon > 0$, then we may find λ_0 such that whenever $\lambda_0 \leq \lambda$ we have

$$\|\xi - \xi e_\lambda\| < \frac{\epsilon}{2}.$$

Since $e_\lambda \in S$, we can find elements $\eta_1, \ldots, \eta_n, \zeta_1, \ldots, \zeta_n \in \mathcal{E}(A)$ such that

$$\left\| e_\lambda - \sum_{i=1}^{n} \langle \eta_i, \zeta_i \rangle \right\| \leq \frac{\epsilon}{2\|\xi\|}.$$

Combining inequalities we get

$$\begin{aligned}
\left\| \xi - \xi \left(\sum_{i=1}^{n} \langle \eta_i, \zeta_i \rangle \right) \right\| &= \left\| \xi - \xi e_\lambda + \xi e_\lambda - \xi \left(\sum_{i=1}^{n} \langle \eta_i, \zeta_i \rangle \right) \right\| \\
&\leq \|\xi - \xi e_\lambda\| + \left\| \xi e_\lambda - \xi \left(\sum_{i=1}^{n} \langle \eta_i, \zeta_i \rangle \right) \right\| \\
&\leq \|\xi - \xi e_\lambda\| + \|\xi\| \left\| e_\lambda - \sum_{i=1}^{n} \langle \eta_i, \zeta_i \rangle \right\| \\
&\leq \frac{\epsilon}{2} + \|\xi\| \frac{\epsilon}{2\|\xi\|} \\
&= \epsilon.
\end{aligned}$$

Hence the result follows. □

This lemma shows that the degenerate example of a module over a non-unital algebroid that we stated earlier does not work as an example of a Hilbert module over a C*-category.

Corollary 2.3.2. *If $\mathcal{A}$ is a C*-category, then the only right Hilbert $\mathcal{A}$-module $\mathcal{E}$ where the action is given by*

$$\alpha \cdot a = 0,$$

*is the zero functor $\mathcal{A}^{op} \to$ **Vect**.*

2.4 Adjointable Operators

As we stated earlier, the morphisms between Hilbert modules that we are interested in are adjointable operators.

Definition 2.4.1. Let $\mathcal{D}$ and $\mathcal{E}$ be a pair of right Hilbert $\mathcal{A}$-modules. An *adjointable operator* T from $\mathcal{D}$ to $\mathcal{E}$ consists of a collection of maps

$$\{T_A : \mathcal{D}(A) \to \mathcal{E}(A);\ A \in \mathcal{O}\mathrm{b}(\mathcal{A})\},$$

for which there exists another collection of maps

$$\{T_A^* : \mathcal{E}(A) \to \mathcal{D}(A);\ A \in \mathcal{O}\mathrm{b}(\mathcal{A})\},$$

such that for each pair $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$ we have

$$\langle T_{A'}^* \alpha, \beta\rangle = \langle \alpha, T_A \beta\rangle.$$

It's important to note that no assumptions are made about the maps involved, for example we don't assume that they are linear. We suggestively write $T : \mathcal{D} \Rrightarrow \mathcal{E}$ to indicate that T is an adjointable operator from $\mathcal{D}$ to $\mathcal{E}$. We call the corresponding collection of maps $T^* : \mathcal{E} \Rrightarrow \mathcal{D}$ the *adjoint* of T.

The adjointable operators defined above are analogues of the usual adjointable operators between Hilbert modules over C*-algebras. Unsurprisingly, one has the following result.

Lemma 2.4.2. *[21, Proposition 7.33] For an adjointable operator $T : \mathcal{D} \Rrightarrow \mathcal{E}$, each component map $T_A : \mathcal{D}(A) \to \mathcal{E}(A)$ is a bounded linear map.*

One more property of adjointable operators that is stated in [21, Proposition 7.33], but not obviously proved is the following.

Lemma 2.4.3. *If $T : \mathcal{D} \Rrightarrow \mathcal{E}$ is an adjointable operator between right Hilbert $\mathcal{A}$-modules, then T is a natural transformation.*

Proof. For each $a \in \mathcal{A}(A, A')$, we require the following square to commute,

$$\begin{tikzcd}
\mathcal{D}(A') \arrow[r, "\mathcal{D}(a)"] \arrow[d, "T_{A'}"'] & \mathcal{D}(A) \arrow[d, "T_A"] \\
\mathcal{E}(A') \arrow[r, "\mathcal{E}(a)"'] & \mathcal{E}(A).
\end{tikzcd}$$

We let $A'' \in \mathcal{O}\mathrm{b}(\mathcal{A})$ be arbitrary and take $\alpha \in \mathcal{E}(A'')$ and $\beta \in \mathcal{D}(A)$, then we have

$$\begin{aligned} \langle \alpha, T_A \mathcal{D}(a)\beta \rangle &= \langle T^*_{A''}\alpha, \mathcal{D}(a)\beta \rangle \\ &= \langle T^*_{A''}\alpha, \beta \rangle a \\ &= \langle \alpha, T_{A'}\beta \rangle a \\ &= \langle \alpha, \mathcal{E}(a) T_{A'}\beta \rangle. \end{aligned}$$

Using part (*iii*) of Lemma 2.1.7, we deduce that

$$T_A \mathcal{D}(a)\beta = \mathcal{E}(a) T_{A'}\beta,$$

as required. □

Remark 2.4.4. The fact that an adjointable operator is a natural transformation should be seen as some sort of $\mathcal{A}$-linearity statement. If we take an adjointable operator $\theta : \mathcal{D} \Rightarrow \mathcal{E}$, then naturality of θ tells us that

$$(\theta_{A'} \circ \mathcal{D}(a))\eta = (\mathcal{D}(a) \circ \theta_A)\eta.$$

This simply states that

$$\theta_{A'}(a \cdot \eta) = a \cdot \theta_A(\eta).$$

Definition 2.4.5. The *norm* of an adjointable operator $T : \mathcal{D} \Rightarrow \mathcal{E}$ is the quantity

$$\|T\| = \sup\{\|T_X\| : \ X \in \mathcal{O}\mathrm{b}(\mathcal{A})\}.$$

When this is finite, the operator T is called *bounded*. We write $\mathcal{L}(\mathcal{D}, \mathcal{E})$ for the collection of all bounded adjointable operators from $\mathcal{D}$ to $\mathcal{E}$.

Remark 2.4.6. Notice that all adjointable operators between Hilbert modules over a C*-category with finitely many objects are automatically bounded. In particular this includes Hilbert modules over a C*-algebra being viewed as a one-object category.

Lemma 2.4.7. *If $\mathcal{A}$ is a C*-category, then we have a large unital C*-category **Hilb**-$\mathcal{A}$ whose objects are the right Hilbert modules over $\mathcal{A}$ and the morphisms are bounded adjointable operators. The involution is just given by taking adjoints. Similarly we have a large unital C*-category $\mathcal{A}$-**Hilb** consisting of left Hilbert $\mathcal{A}$-modules.*

Proof. See [23, Proposition 9.4]. □

2.5 Examples of Adjointable Operators

As we did before with Hilbert modules, let's take a look at some examples of adjointable operators.

Example 2.5.1. Let $\mathcal{A}$ be a C*-category. For an object $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, we have seen that the contravariant hom-functor

$$\mathcal{A}(-, A),$$

is a right Hilbert $\mathcal{A}$-module. Given $a \in \mathcal{A}(A, A')$, we'll show that a induces a bounded adjointable operator

$$T : \mathcal{A}(-, A) \Rightarrow \mathcal{A}(-, A').$$

The components of T and T^* are defined by

$$T_{A''} := a \circ (-) : \mathcal{A}(A'', A) \to \mathcal{A}(A'', A'), \qquad T^*_{A''} := a^* \circ (-) : \mathcal{A}(A'', A') \to \mathcal{A}(A'', A).$$

These maps satisfy

$$\langle T^*_{A''} x, y\rangle = (a^* x)^* y = (x^* a)y = x^*(ay) = \langle x, T_{A''} y\rangle,$$

so that T is indeed an adjointable operator. To show that T is bounded, we fix an object $A'' \in \mathcal{O}\mathrm{b}(\mathcal{A})$, then

$$\|T_{A''}\| = \sup_{\|x\|=1} \|T_{A''} x\| = \sup_{\|x\|=1} \|ax\| \leq \sup_{\|x\|=1} \|a\|\|x\| = \|a\|.$$

Hence $\|T\| \leq \|a\|$, proving that T is bounded.

Example 2.5.2. The following is taken from [12, Example 2.2]. Let $\mathcal{A}$ be a unital C*-category, then for each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ we have a right Hilbert $\mathcal{A}$-module

$$\mathcal{A}(-, A).$$

Now let $\mathcal{E}$ be any right Hilbert $\mathcal{A}$-module and pick $\eta \in \mathcal{E}(A)$. We define an adjointable operator $T : \mathcal{A}(-, A) \Rightarrow \mathcal{E}$ as follows, for $A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$ the component $T_{A'}$ is given by

$$\mathcal{A}(A', A) \ni a \mapsto a \cdot \eta,$$

and the component $T^*_{A'}$ is given by

$$\xi \mapsto \langle \eta, \xi \rangle.$$

We check that T is adjointable, so take $\xi \in \mathcal{E}(A'')$ and $a \in \mathcal{A}(A', A)$, then

$$\langle T^*_{A''}\xi, a\rangle = \langle \langle \eta, \xi\rangle, a\rangle = \langle \eta, \xi\rangle^* a = \langle \xi, \eta\rangle a = \langle \xi, a \cdot \eta\rangle.$$

To check that T is bounded, for $A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$, we have

$$\begin{aligned} \sup_{\|a\|\leq 1} \|a \cdot \eta\|^2 &= \sup_{\|a\|\leq 1} \|\langle a \cdot \eta, a \cdot \eta\rangle\| \\ &= \sup_{\|a\|\leq 1} \|a^* \langle \eta, \eta\rangle a\| \\ &\leq \sup_{\|a\|\leq 1} \|a\|^2 \|\eta\|^2 \\ &\leq \|\eta\|^2, \end{aligned}$$

hence we conclude that T is bounded. Finally observe that by Yoneda's lemma, T is the unique adjointable operator $\mathcal{A}(-, A) \Rightarrow \mathcal{E}$ which sends $\mathrm{id}_A \mapsto \eta$.

It should be noted that not all adjointable operators are bounded. The following example is inspired by [31, Remark 15.12].

Example 2.5.3. Consider the C*-category $\mathbb{C}_\oplus$. Note that the set $\mathcal{O}\mathrm{b}(\mathbb{C}_\oplus)$ can be viewed as $\mathbb{N}$. We fix $n \in \mathbb{N}$ and consider the right $\mathbb{C}_\oplus$-module $\mathbb{C}_\oplus(-, n)$. We'll define an adjointable operator $\mathbb{C}_\oplus(-, n) \Rightarrow \mathbb{C}_\oplus(-, n)$ to have components

$$\mathbb{C}_\oplus(m, n) \ni A \mapsto mA,$$

with adjoint given by

$$\mathbb{C}_\oplus(m, n) \ni A \mapsto \overline{m}A.$$

We have

$$\langle \overline{m}A, B\rangle = (\overline{m}A)^* B = mA^* B = A^*(mB) = \langle A, mB\rangle.$$

So this operator is adjointable, but for $m \in \mathcal{O}\mathrm{b}(\mathbb{C}_\oplus)$ we have

$$\sup_{\|A\|=1} \|mA\| = \sup_{\|A\|=1} |m|\|A\| = |m|,$$

so the operator is not bounded.

2.6 Compact Operators

An important class of operators are the compact operators.

Definition 2.6.1. Let $\mathcal{A}$ be a C*-category and $\mathcal{E}, \mathcal{F} \in \mathbf{Hilb}\text{-}\mathcal{A}$. For any object $X \in \mathcal{O}\mathrm{b}(\mathcal{A})$ and any pair $\xi \in \mathcal{F}(X)$, $\eta \in \mathcal{E}(X)$, we define a bounded adjointable operator $\theta_{\xi,\eta} : \mathcal{E} \Rightarrow \mathcal{F}$ to have components

$$\begin{aligned} \mathcal{E}(Y) &\to \mathcal{F}(Y) \\ \zeta &\mapsto \xi \cdot \langle \eta, \zeta \rangle \end{aligned}$$

Lemma 2.6.2. *With everything as above, $\theta_{\xi,\eta}$ really is a bounded adjointable operator.*

Proof. First we'll verify adjointability, and claim that $\theta_{\xi,\eta}$ has adjoint with components

$$\begin{aligned} \mathcal{F}(Y) &\to \mathcal{E}(Y) \\ \zeta &\mapsto \eta\langle \xi, \zeta \rangle. \end{aligned}$$

We observe that

$$\langle \theta_{\xi,\eta}\zeta, \zeta' \rangle = \langle \xi\langle \eta, \zeta \rangle, \zeta' \rangle = \langle \eta, \zeta \rangle^* \langle \xi, \zeta' \rangle = \langle \zeta, \eta \rangle \langle \xi, \zeta' \rangle = \langle \zeta, \eta\langle \xi, \zeta' \rangle \rangle = \langle \zeta, \theta_{\eta,\xi}\zeta' \rangle,$$

so that we have $\theta^*_{\xi,\eta} = \theta_{\eta,\xi}$. To check boundedness, we first fix $Y \in \mathcal{O}\mathrm{b}(\mathcal{A})$, then we have

$$\begin{aligned} \sup_{\|\zeta\| \leq 1} \|\theta_{\xi,\eta}\zeta\|^2 &= \sup_{\|\zeta\| \leq 1} \|\xi\langle \eta, \zeta \rangle\|^2 \\ &= \sup_{\|\zeta\| \leq 1} \|\langle \xi\langle \eta, \zeta \rangle, \xi\langle \eta, \zeta \rangle \rangle\| \\ &= \sup_{\|\zeta\| \leq 1} \|\langle \eta, \zeta \rangle^* \langle \xi, \xi \rangle \langle \eta, \zeta \rangle\| \\ &\leq \sup_{\|\zeta\| \leq 1} \|\eta\|^2 \|\xi\|^2 \|\zeta\|^2 \\ &\leq \|\xi\|^2 \|\eta\|^2. \end{aligned}$$

The first inequality follows from the Cauchy-Schwarz inequality, and since this holds for each Y, it follows that $\|\theta_{\xi,\eta}\|$ is finite, so that we have a bounded adjointable operator as claimed. ☐

Definition 2.6.3. Let $\mathcal{A}$ be a C*-category and $\mathcal{E}, \mathcal{F} \in \mathbf{Hilb}\text{-}\mathcal{A}$. The space of *finite* *rank operators* from $\mathcal{E}$ to $\mathcal{F}$ is the vector space

$$\mathrm{Span}\{\theta_{\xi,\eta};\ X \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ \xi \in \mathcal{F}(X),\ \eta \in \mathcal{E}(X)\}.$$

The closure of this space in $\mathcal{L}(\mathcal{E}, \mathcal{F})$ is the Banach space of *compact operators* from $\mathcal{E}$ to $\mathcal{F}$.

Lemma 2.6.4. *If $\mathcal{A}$ is a C^*-category, then we have a large wide*[1] *C^*-subcategory $\textbf{\textit{Hilb}}_{\mathcal{K}}$-$\mathcal{A}$ of $\textbf{\textit{Hilb}}$-$\mathcal{A}$ consisting of all the compact operators between right Hilbert $\mathcal{A}$-modules. With suitable tweaks to our definition of finite rank operators, we also have the large subcategory $\mathcal{A}$-$\textbf{\textit{Hilb}}_{\mathcal{K}}$ of left Hilbert $\mathcal{A}$-modules and compact operators.*

Proof. See [24, Proposition 3.8]. □

We record the following result here to be used later on.

Lemma 2.6.5. *Given a C^*-category $\mathcal{A}$ and right Hilbert $\mathcal{A}$-module $\mathcal{E}$, for each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ we have*

$$\mathcal{E}(A) = \{K_A(\eta);\ K \in \mathcal{K}(\mathcal{E}),\ \eta \in \mathcal{E}(A)\}.$$

Where $K_A(\eta)$ denotes the A-th component of the compact operator K evaluated at the vector η.

Proof. We let (e_λ) be an approximate unit for $\mathcal{K}(\mathcal{E})$, then observe that

$$e_\lambda(\xi \cdot \langle \eta, \zeta \rangle) = e_\lambda \theta_{\xi,\eta}(\zeta) \xrightarrow{\lambda} \theta_{\xi,\eta}(\zeta) = \xi \cdot \langle \eta, \zeta \rangle.$$

By Lemma 2.3.1, it now follows that $e_\lambda \xi \xrightarrow{\lambda} \xi$ for any $\xi \in \mathcal{E}(A)$. This shows us that the set

$$\{K_A(\xi);\ K \in \mathcal{K}(\mathcal{E}),\ \xi \in \mathcal{E}(A)\},$$

is dense in $\mathcal{E}(A)$. To finish the proof, we will appeal to the Cohen-Hewitt factorisation theorem for modules over C*-algebras which we recall below.

Theorem 2.6.6. *[26, Proposition 2.33] Let A be a C^*-algebra and let X be a left A-module. Recall the following,*

(i) X is called a Banach A-module if X is a Banach space and $\|a \cdot x\| \leq \|a\|\|x\|$ for all $a \in A$ and $x \in X$,

[1] A wide subcategory is one which contains the same objects as the parent category.

(ii) X is called non-degenerate if the set span$\{a \cdot x;\ a \in A,\ x \in X\}$ is dense in X.

If X is a non-degenerate Banach A-module, then for every $y \in X$, we can find $a \in A$ and $x \in X$ such that

$$y = a \cdot x.$$

We proceed as follows; we consider the C*-algebra $\mathcal{K}(\mathcal{E})$, and the Banach space $\mathcal{E}(A)$. We note that $\mathcal{E}(A)$ is a left $\mathcal{K}(\mathcal{E})$-module under the action

$$K \cdot \xi = K_A(\xi).$$

Moreover, our work so far in this proof shows that $\mathcal{E}(A)$ is a non-degenerate Banach $\mathcal{K}(\mathcal{E})$-module, so invoking the theorem stated above gives us what we want. □

This result tells us that "every element in a Hilbert module lies in the image of some compact operator on that Hilbert module".

2.7 Unitary Operators

One particular example of adjointable operators which will be useful later are unitary operators.

Definition 2.7.1. Let $\mathcal{D}$ and $\mathcal{E}$ be right Hilbert $\mathcal{A}$-modules, and suppose that we have a collection of maps $\{\theta_A : \mathcal{D}(A) \to \mathcal{E}(A);\ A \in \mathcal{O}\mathrm{b}(\mathcal{A})\}$. We will call such a collection of maps a *unitary operator* if

(i) For each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, the map $\theta_A : \mathcal{D}(A) \to \mathcal{E}(A)$ has dense image,

(ii) For each pair $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$, the following diagram commutes

$$\begin{tikzcd} \mathcal{D}(A') \times \mathcal{D}(A) \arrow[rr, "(\theta_{A'}, \theta_A)"] \arrow[dr, "\langle -,- \rangle"'] & & \mathcal{E}(A') \times \mathcal{E}(A) \arrow[dl, "\langle -,- \rangle"] \\ & \mathcal{A}(A, A'). & \end{tikzcd}$$

Lemma 2.7.2. *If $\theta : \mathcal{D} \Rightarrow \mathcal{E}$ is a unitary operator, then θ is an invertible, bounded adjointable operator with inverse θ^*.*

Proof. Point (ii) in the definition tells us that for each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, the map θ_A is an isometry, so is injective and has closed image. Consequently, the map θ_A must be a bijection. Furthermore, we have

$$\langle \theta_{A'}^{-1}\eta, \xi \rangle = \langle \theta_{A'}\theta_{A'}^{-1}\eta, \theta_A \xi \rangle = \langle \eta, \theta_A \xi \rangle,$$

so that θ is adjointable, where the adjoint has components $\theta_A^* = \theta_A^{-1}$. Finally, we quickly check that θ is bounded, for this we have

$$\|\theta_A\|^2 = \|\theta_A^*\theta_A\| = \| \mathrm{id} \| = 1,$$

so that $\|\theta\| = 1$, and we are done. $\square$

Chapter 3

Hilbert Bimodules

The next thing we will look at are Hilbert bimodules. If we think of a Hilbert module over a C*-category $\mathcal{A}$ as a collection of vector spaces equipped with an action of $\mathcal{A}$, then a Hilbert bimodule is a collection of vector spaces equipped with actions of a pair of C*-categories.

3.1 The Definition and Examples

As a starting point, here is the relevant definition for C*-algebras.

Definition 3.1.1. If A and B are a pair of C*-algebras, then a *right Hilbert* $A-B$ *bimodule* is simply a right Hilbert B-module X, together with a *-homomorphism

$$\varphi : A \to \mathcal{L}(X).$$

Observe that if we are given a right Hilbert $A-B$ bimodule $\varphi : A \to \mathcal{L}(X)$, then X carries the structure of a left A-module via the action

$$a \cdot x := \varphi(a)x.$$

Furthermore, the left and right module actions are compatible in the sense that

$$a \cdot (x \cdot b) = \varphi(a)x \cdot b = (\varphi(a)x) \cdot b = (a \cdot x) \cdot b.$$

Remark 3.1.2. Notice that we have imposed no conditions on the map φ in our definition, other than insisting it be a *-homomorphism. Quite often one encounters this definition but with an extra non-degeneracy assumption, but still under the name of a Hilbert bimodule. For example, see [8, Definition 1.1]. We shall call a Hilbert $A-B$ bimodule where the *-homomorphism φ is non-degenerate[1] an $A-B$ *correspondence*, and such things will feature later on.

In the above definition, we didn't have to actually specify the module X, instead we could have simply insisted that φ be a *-functor into the category of right Hilbert B-modules. We take this as our guide for what right Hilbert $\mathcal{A}-\mathcal{B}$ bimodules over C*-categories ought to look like. For our work, we will require the following general lemma.

Lemma 3.1.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebroids, and consider a functor $F_{(-)} : \mathcal{A} \to \boldsymbol{Mod}{-}\mathcal{B}$. We may construct a functor*

$$_{(-)}F : \mathcal{B}^{op} \to \mathcal{A} - \boldsymbol{Mod},$$

such that

(i) $F_A(B) = {}_BF(A)$, *for all* $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ *and* $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$,

(ii) *For* $a \in \mathcal{A}(A, A')$, $b \in \mathcal{B}(B', B)$ *and* $\xi \in F_A(B)$, *we have*

$$(a \overset{\curvearrowright}{\cdot} \xi) \overset{\curvearrowleft}{\cdot} b = a \overset{\curvearrowright}{\cdot} (\xi \overset{\curvearrowleft}{\cdot} b).$$

Proof. Take any $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$. We'll show how B gives rise to a left $\mathcal{A}$-module ${}_BF(-)$. We define this module on objects by

$$A \mapsto F_A(B).$$

Given a morphism $a \in \mathcal{A}(A, A')$, we get a homomorphism $F_{(a)}$[2] $: F_A \Rightarrow F_{A'}$. We define ${}_BF(a)$ to be the B-th component of $F_{(a)}$,

$$F_A(B) \xrightarrow{F_{(a)B}} F_{A'}(B).$$

Functoriality of ${}_BF(-)$ follows immediately from that of $F_{(-)}$. Now we need to see how a morphism $b : B' \to B$ induces a $\mathcal{A}$-module homomorphism ${}_{(b)}F : {}_BF \Rightarrow {}_{B'}F$.

[1] Flick ahead to Definition 3.5.1 for the definition of non-degeneracy.

[2] We will use the brackets when notating homomorphisms like $F_{(a)}$ since we can then refer to the components of such homomorphisms by $F_{(a)B}$.

There is only one sensible way to attempt this; we define the homomorphism to have components

$$ {}_{(b)}F_A = F_{(b)A}. $$

To visualise what is going on, and to check that this is indeed a module homomorphism, consider the following diagram, where $a \in \mathcal{A}(A, A')$ and $b \in \mathcal{B}(B', B)$,

$$\begin{tikzcd}
{}_BF(A) \arrow[rrr, "{}_{(b)}F_A"] \arrow[ddd, "{}_BF(a)"'] \arrow[dr, equal] & & & {}_{B'}F(A) \arrow[ddd, "{}_{B'}F(a)"] \arrow[dl, equal] \\
& F_A(B) \arrow[r, "F_A(b)"] \arrow[d, "F_{(a)B}"'] & F_A(B') \arrow[d, "F_{(a)B'}"] & \\
& F_{A'}(B) \arrow[r, "F_{A'}(b)"'] \arrow[dl, equal] & F_{A'}(B') \arrow[dr, equal] & \\
{}_BF(A') \arrow[rrr, "{}_{(b)}F_{A'}"'] & & & {}_{B'}F(A')
\end{tikzcd}$$

Note that for ${}_{(b)}F$ to be a module homomorphism, we require the outer square to commute. The central square commutes, simply because $F_{(a)}$ is a natural transformation, so it follows that the outer square commutes as well. Therefore ${}_{(b)}F$ is a homomorphism. Furthermore we can read off that

$$ (a \cdot \xi) \cdot b = a \cdot (\xi \cdot b), $$

for every vector $\xi \in {}_BF(A) = F_A(B)$. $\square$

Notation 3.1.4. Let $F_{(-)} : \mathcal{A} \to \mathbf{Mod} - \mathcal{B}$ be a functor as in the previous lemma. To indicate the left action of a morphism $a \in \mathcal{A}(A, A')$ on some vector $\xi \in F_A(B)$, we will use the notations $F_{(a)B}(\xi)$ and $a \cdot \xi$ interchangeably.

Similarly, one may prove the following.

Lemma 3.1.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be algebroids, and consider a functor ${}_{(-)}F : \mathcal{B}^{op} \to \mathcal{A} - \boldsymbol{Mod}$. We may construct a functor*

$$ F_{(-)} : \mathcal{A} \to \boldsymbol{Mod} - \mathcal{B}, $$

such that

(i) $F_A(B) = {}_BF(A)$, *for all* $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ *and* $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$,

(ii) For $a \in \mathcal{A}(A, A')$, $b \in \mathcal{B}(B', B)$ and $\xi \in {}_B F(A)$, we have

$$(a \overset{\curvearrowright}{\cdot} \xi) \overset{\curvearrowleft}{\cdot} b = a \overset{\curvearrowright}{\cdot} (\xi \overset{\curvearrowleft}{\cdot} b).$$

Combining [24, Definition 3.11] with the previous lemmas gives us our definition of Hilbert bimodules.

Definition 3.1.6. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-categories. A *right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule* is a *-functor

$$F_{(-)} : \mathcal{A} \to \textbf{Hilb-}\mathcal{B}.$$

Using Lemma 3.1.3 this induces a functor

$${}_{(-)}F : \mathcal{B}^{\mathrm{op}} \to \mathcal{A} - \textbf{Mod},$$

and we call $F_{(-)}$ the right action of $\mathcal{B}$ and ${}_{(-)}F$ the left action of $\mathcal{A}$, where the categories $\mathcal{A}$ and $\mathcal{B}$ act on the collection of vector spaces

$$\{F_{(A)}(B);\ A \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ B \in \mathcal{O}\mathrm{b}(\mathcal{B})\}.$$

Similarly, we will call a *-functor of the form

$${}_{(-)}F : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}\textbf{-Hilb},$$

a *left Hilbert $\mathcal{A} - \mathcal{B}$ bimodule.* Such a *-functor induces a functor

$$F_{(-)} : \mathcal{B} \to \textbf{Mod} - \mathcal{A},$$

and we call the functors ${}_{(-)}F$ and $F_{(-)}$ the left action of $\mathcal{A}$ and right action of $\mathcal{B}$ respectively.

We will notate such bimodules by either ${}_{\mathcal{A}}F_{\mathcal{B}}$, or by referring directly to one of the functors ${}_{(-)}F$ or $F_{(-)}$, but note that when writing ${}_{\mathcal{A}}F_{\mathcal{B}}$ we will have to specify whether the bimodule is a left Hilbert bimodule or a right Hilbert bimodule.

Remark 3.1.7. The inclusion of the word *left* or *right* is to indicate that the inner products involved in a given bimodule are associated with the left or right module structures.

Remark 3.1.8. In literature on modules and bimodules over a category, there are several different names and terminology for this concept; given *unital* categories $\mathcal{C}$ and

$\mathcal{D}$, a profunctor or correspondence or distributor or bimodule is a functor

$$F : \mathcal{D}^{op} \times \mathcal{C} \to \mathbf{Set}.$$

With this definition it is clear that such a thing will induce left and right modules by fixing one argument of the functor, however it is crucial to note that this process requires the identity morphisms of the categories, so we can't expect our bimodule definition to be expressible in terms of two variable functors in general.

Example 3.1.9. Let $\mathcal{A}$ be a C*-category. We have seen already that each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ gives us a right Hilbert $\mathcal{A}$-module $\mathcal{A}(-, A)$, and that each morphism $a \in \mathcal{A}(A, A')$ gives us a bounded adjointable operator $\mathcal{A}(-, A) \Rightarrow \mathcal{A}(-, A')$. If we pick back through the construction of these adjointable operators, then we see that these constructions give us a *-functor

$$\mathcal{A}(-, =) : \mathcal{A} \to \mathbf{Hilb}\text{-}\mathcal{A}.$$

Example 3.1.10. Suppose that we have a *-functor $\phi : \mathcal{A} \to \mathcal{B}$. We can use the previous example to define the composite *-functor

$$\Phi_{(-)} : \mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\mathcal{B}(-,=)} \mathbf{Hilb}\text{-}\mathcal{B}.$$

Which is a right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule.

One important example of Hilbert bimodules is the following.

Definition 3.1.11. If ${}_{\mathcal{A}}F_{\mathcal{B}}$ is a right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule, where the left action is a *-functor ${}_{(-)}F : \mathcal{B}^{\mathrm{op}} \to \mathcal{A}\text{-}\mathbf{Hilb}$, then we shall call ${}_{\mathcal{A}}F_{\mathcal{B}}$ a *bi-Hilbert* $\mathcal{A} - \mathcal{B}$ *bimodule.*

Example 3.1.12. As in our last example, let $\phi : \mathcal{A} \to \mathcal{B}$ be a *-functor, between C*-categories, but further assume that the categories are unital and that ϕ is a unitary equivalence[3] We will show that under these extra assumptions, the right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule $\Phi_{(-)}$ of the previous example is a bi-Hilbert bimodule. For each $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we need an inner product on the left $\mathcal{A}$-module $\mathcal{B}(B, \phi(-))$. We fix $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, then for $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$ we let $\phi^{-1} : \mathcal{B}(\phi(A), \phi(A')) \to \mathcal{A}(A, A')$ be the inverse to ϕ (guaranteed

[3] *-Functors between C*-categories are unitary equivalences if and only if they are equivalences of the underlying categories, so if and only if they are full, faithful and essentially surjective. See [7, Lemma 4.6].

to exist because ϕ is full and faithful), and define

$$\begin{aligned} {}_{\mathcal{A}}\langle -, - \rangle : \mathcal{B}(B, \phi(A')) \times \mathcal{B}(B, \phi(A)) &\to \mathcal{A}(A, A') \\ (f, g) &\mapsto \phi^{-1}(fg^*). \end{aligned}$$

Let's check that this is indeed an inner product.

(i) ${}_{\mathcal{A}}\langle \alpha f + \beta f', g \rangle = \alpha \phi^{-1}(fg^*) + \beta \phi^{-1}(f' g^*)$,

(ii) ${}_{\mathcal{A}}\langle a \cdot f, g \rangle = {}_{\mathcal{A}}\langle \phi(a) \circ f, g \rangle = \phi^{-1}(\phi(a) f g^*) = a \phi^{-1}(fg^*)$,

(iii) ${}_{\mathcal{A}}\langle f, g \rangle = \phi^{-1}(fg^*) = \phi^{-1}((gf^*)^*) = \phi^{-1}(gf^*)^* = {}_{\mathcal{A}}\langle g, f \rangle^*$,

(iv) Note ${}_{\mathcal{A}}\langle f, f \rangle = \phi^{-1}(ff^*)$. On endomorphism sets, ϕ is a *-homomorphism, so $\phi^{-1}(ff^*) = \phi^{-1}(f)\phi^{-1}(f)^*$ which is positive.

Completeness of the $\mathcal{A}$-modules in the induced norm is due to the identity

$$\|f\| = \|\phi^{-1}(ff^*)\|_{\mathcal{A}}^{\frac{1}{2}} = \|ff^*\|_{\mathcal{B}}^{\frac{1}{2}} = \|f\|_{\mathcal{B}}.$$

Finally, we must verify that each ${}_{(b)}\Phi$ is a bounded adjointable operator. For this, let $b \in \mathcal{B}(B, B')$ and unpack the definition of ${}_{(b)}\Phi$; we have components

$${}_{(b)}\Phi_A = \Phi_A(b) : \mathcal{B}(B', \phi(A)) \xrightarrow{(-)\circ b} \mathcal{B}(B, \phi(A)),$$

and we propose that the adjoint has components

$${}_{(b^*)}\Phi_A = \Phi_A(b^*) : \mathcal{B}(B, \phi(A)) \xrightarrow{(-)\circ b^*} \mathcal{B}(B', \phi(A)).$$

If we take $f \in \mathcal{B}(B, \phi(A'))$ and $g \in \mathcal{B}(B', \phi(A))$, then we have

$$\begin{aligned} {}_{\mathcal{A}}\langle {}_{(b^*)}\Phi_{A'}(f), g \rangle &= {}_{\mathcal{A}}\langle fb^*, g \rangle \\ &= fb^* g^* \\ &= f(gb)^* \\ &= {}_{\mathcal{A}}\langle f, gb \rangle \\ &= {}_{\mathcal{A}}\langle f, {}_{(b)}\Phi_A(g) \rangle, \end{aligned}$$

so ${}_{(b)}\Phi$ is adjointable as claimed. It is straightforward to verify that each ${}_{(b)}\Phi$ is also bounded.

3.2 Bimodule Homomorphisms and Isomorphisms

When working with C*-algebras, defining morphisms and isomorphisms of Hilbert bimodules is straightforward - put simply, they are certain maps between the underlying vector spaces of a pair of bimodules which are linear with respect to the left and right module actions. In this section we will introduce the definition of such morphisms between bimodules over C*-categories, and provide some shortcuts for checking if we have a bimodule isomorphism.

Definition 3.2.1. For a pair of right Hilbert $\mathcal{A}-\mathcal{B}$ bimodules ${}_{\mathcal{A}}F_{\mathcal{B}}$ and ${}_{\mathcal{A}}G_{\mathcal{B}}$, a *homomorphism* from ${}_{\mathcal{A}}F_{\mathcal{B}}$ to ${}_{\mathcal{A}}G_{\mathcal{B}}$ consists of a collection of maps

$$\{{}_B\Phi_A : F_A(B) \to G_A(B);\ A \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ B \in \mathcal{O}\mathrm{b}(\mathcal{B})\},$$

such that

(i) For each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, the collection of maps

$$\{{}_B\Phi_A : F_A(B) \to G_A(B);\ B \in \mathcal{O}\mathrm{b}(\mathcal{B})\},$$

assemble to give a bounded adjointable operator $\Phi_A : F_A \Rightarrow G_A$,

(ii) For each $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, then collection of maps

$$\{{}_B\Phi_A : F_A(B) \to G_A(B);\ A \in \mathcal{O}\mathrm{b}(\mathcal{A}))\},$$

assemble to give a module homomorphism ${}_B\Phi : {}_BF \Rightarrow {}_BG$.

We will notate such a homomorphism by $\Phi : {}_{\mathcal{A}}F_{\mathcal{B}} \Rightarrow {}_{\mathcal{A}}G_{\mathcal{B}}$. An *isomorphism* of right Hilbert $\mathcal{A}-\mathcal{B}$ bimodules is a homomorphism where the components assemble to give unitary operators between the right Hilbert $\mathcal{B}$-modules, and isomorphisms between the left $\mathcal{A}$-modules.

A homomorphism/isomorphism of bi-Hilbert bimodules is a bimodule homomorphism/isomorphism where the maps in point (ii) assemble to give bounded adjointable/unitary operators.

Lemma 3.2.2. *Given a homomorphism of right Hilbert $\mathcal{A}-\mathcal{B}$ bimodules $\Phi : {}_{\mathcal{A}}F_{\mathcal{B}} \to {}_{\mathcal{A}}G_{\mathcal{B}}$, we have natural transformations*

$$_{(-)}F \Rightarrow {}_{(-)}G, \qquad F_{(-)} \Rightarrow G_{(-)}.$$

If Φ is an isomorphism, then these transformations are natural isomorphisms.

Proof. By assumption, we have for each $A \in \mathcal{O}\mathrm{b}(\mathcal{B})$ a bounded adjointable operator ${}_{(-)}\Phi_A : F_{(A)} \to G_{(A)}$. When we unwrap this definition, we find that we have commuting squares of the following form, for each $b \in \mathcal{B}(B', B)$,

$$\begin{tikzcd} F_A(B) \arrow[r, "{}_B\Phi_A"] \arrow[d, "F_A(b)"'] & G_A)(B) \arrow[d, "G_A(b)"] \\ F_A(B') \arrow[r, "{}_{B'}\Phi_A"'] & G_A(B'). \end{tikzcd}$$

Using the definition of ${}_{(-)}F$ and ${}_{(-)}G$, this tells us that for each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, the following diagram commutes,

$$\begin{tikzcd} {}_BF(A) \arrow[r, "{}_B\Phi_A"] \arrow[d, "{}_{(b)}F_A"'] & {}_BG(A) \arrow[d, "{}_{(b)}G_A"] \\ {}_{B'}F(A) \arrow[r, "{}_{B'}\Phi_A"'] & {}_{B'}G(A), \end{tikzcd}$$

which in turn tells us that the following diagram commutes

$$\begin{tikzcd} {}_BF \arrow[r, Rightarrow, "{}_B\Phi_{(-)}"] \arrow[d, Rightarrow, "{}_{(b)}F"'] & {}_BG \arrow[d, Rightarrow, "{}_{(b)}G"] \\ {}_{B'}F \arrow[r, Rightarrow, "{}_{B'}\Phi_{(-)}"'] & {}_{B'}G. \end{tikzcd}$$

This final diagram tells us that we have a natural transformation ${}_{(-)}F \Rightarrow {}_{(-)}G$. Similarly, one may show that we have a natural transformation $F_{(-)} \Rightarrow G_{(-)}$. The final claim should be clear. □

Remark 3.2.3. One might be tempted to take the existence of natural transformations/isomorphisms ${}_{(-)}F \Rightarrow {}_{(-)}G$ and $F_{(-)} \Rightarrow G_{(-)}$ to be the definition of a bimodule homomorphism/isomorphism, however without any extra assumptions, this would be too weak of a notion. As the following example will show, this wrong definition would fail to capture isomorphisms of bimodules over C*-algebras.

Example 3.2.4. Consider the C*-algebra $\mathbb{M}_2$ and let u be the unitary matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We let ϕ be the *-automorphism of $\mathbb{M}_2$ given by $\phi(A) = uAu^*$. Note that $\phi^{-1} = \phi$, because $u^2 = (u^*)^2 = -\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We can of course consider $\mathbb{M}_2$ as a bi-Hilbert bimodule in the usual way, and we let X be the following Hilbert $\mathbb{M}_2 - \mathbb{M}_2$ bimodule,

$$\mathbb{M}_2 \xrightarrow{\phi} \mathbb{M}_2 \to \mathcal{L}(\mathbb{M}_2),$$

where the unlabelled arrow is the *-homomorphism sending a matrix to the relevant left multiplication operator. The right action on X is the same as that on $\mathbb{M}_2$, but now the left action is given by

$$A \cdot B = \phi(A)B.$$

Note that because ϕ is a *-isomorphism, this new bimodule is a bi-Hilbert bimodule, with left inner product given by

$$_{\mathbb{M}_2}\langle A, B\rangle = \phi^{-1}(AB^*).$$

The identity $X \to \mathbb{M}_2$ provides an isomorphism of this pair when viewed as right Hilbert $\mathbb{M}_2$-modules, and the map $\phi^{-1} : X \to \mathbb{M}_2$ provides an isomorphism when viewed as left Hilbert $\mathbb{M}_2$-modules. If we assume that $f : X \to \mathbb{M}_2$ is a bimodule isomorphism, then of course this means that we will have identities

$$f(A \overset{\curvearrowright}{\cdot} B) = A \overset{\curvearrowright}{\cdot} f(B), \qquad F(A \overset{\curvearrowleft}{\cdot} B) = f(A) \overset{\curvearrowleft}{\cdot} B.$$

Examining the first of these, we see that

$$f(A \overset{\curvearrowright}{\cdot} B) = f(\phi(A)B) = f(\phi(A) \overset{\curvearrowleft}{\cdot} B),$$

so that f must satisfy

$$A \overset{\curvearrowright}{\cdot} f(B) = f(\phi(A)) \overset{\curvearrowleft}{\cdot} B.$$

If we put $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we see that $A = f(\phi(A))$ for all A, and in particular we get $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = f(\phi(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))) = f(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$. Setting $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ shows that we must have $f(B) = B$ for all B, implying that $f = \mathrm{id}$. However the identity map $X \to \mathbb{M}_2$ cannot be an isomorphism of left Hilbert modules, since it fails to be $\mathbb{M}_2$-linear with respect to the left action; consider the following,

$$\mathrm{id}(A \overset{\curvearrowright}{\cdot} B) = \mathrm{id}(\phi(A)B) = \phi(A)B,$$

in general, the above will fail to be equal to $A \overset{\curvearrowright}{\cdot} \mathrm{id}(B)$ because ϕ is non-trivial, so that id cannot be an adjointable operator between left Hilbert $\mathbb{M}_2$-modules so cannot be a bimodule isomorphism.

For later situations where we have to verify that we have a bimodule isomorphism, it would be useful to have a shorter set of things to check.

Lemma 3.2.5. *Suppose that we have bi-Hilbert bimodules ${}_{\mathcal{A}}F_{\mathcal{B}}$ and ${}_{\mathcal{A}}G_{\mathcal{B}}$ and a collection of maps*

$$\{{}_A\Phi_B : F_A(B) \to G_A(B);\ A \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ B \in \mathcal{O}\mathrm{b}(\mathcal{B})\},$$

such that

(i) Each map ${}_A\Phi_B$ has dense image,

(ii) For all $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$ and $B, B' \in \mathcal{O}\mathrm{b}(\mathcal{B})$ we have commuting triangles like so,

$$\begin{tikzcd} {}_BF(A') \times {}_BF(A) \arrow[rr, "({}_{A'}\Phi_B, {}_A\Phi_B)"] \arrow[dr, "{}_{\mathcal{A}}\langle -,- \rangle"'] & & {}_BG(A') \times {}_BG(A) \arrow[dl, "{}_{\mathcal{A}}\langle -,- \rangle"] \\ & \mathcal{A}(A, A'), & \end{tikzcd}$$

$$\begin{tikzcd} F_A(B') \times F_A(B) \arrow[rr, "({}_A\Phi_{B'}, {}_A\Phi_B)"] \arrow[dr, "\langle -,- \rangle_{\mathcal{B}}"'] & & G_A(B') \times G_A(B) \arrow[dl, "\langle -,- \rangle_{\mathcal{B}}"] \\ & \mathcal{B}(B, B'). & \end{tikzcd}$$

Then the stated collection of maps is a bimodule isomorphism.

Proof. Fix $A \in \mathcal{O}\mathrm{b}\,\mathcal{A}$. The assumptions made above, and our characterisation of unitary operators tell us that the maps

$$\{{}_A\Phi_B;\ B \in \mathcal{O}\mathrm{b}\,\mathcal{B}\},$$

compile to give a unitary operator

$${}_A\Phi_{(-)} : F_A \Rightarrow G_A.$$

Similarly, we get for each $B \in \mathcal{O}\mathrm{b}\,\mathcal{B}$ a unitary operator

$${}_{(-)}\Phi_B : {}_BF \Rightarrow {}_BG.$$

So we have a bimodule isomorphism as claimed. $\square$

3.3 Conjugate Modules

Here we will introduce the procedure of forming conjugate modules. This gives us a way of converting right Hilbert modules into left Hilbert modules, and vice versa. It will be of most use to us when we apply this to bimodules, allowing us to turn $\mathcal{A} - \mathcal{B}$ bimodules into $\mathcal{B} - \mathcal{A}$ bimodules.

Let V be a vector space. The conjugate vector space of V is denoted by $\widetilde{V}$ and has the same abelian group structure as V. We denote the identity map (isomorphism) by $\flat : V \to \widetilde{V}$, and will write the elements of $\widetilde{V}$ in the form $\flat(v)$. The scalar multiplication on $\widetilde{V}$ is then given by

$$\lambda \flat(v) := \flat(\overline{\lambda} v).$$

A linear map $f : V \to W$ between vector spaces induces a linear map $\widetilde{f}$ between the conjugate spaces, given by

$$\widetilde{f}(\flat(v)) = \flat(f(v)),$$

and one may further check that this gives a functor $\widetilde{(-)} : \mathbf{Vect} \to \mathbf{Vect}$.

Definition 3.3.1. If $\mathcal{A}$ is a C*-category and $\mathcal{E}$ is a right module, then the *conjugate module* of $\mathcal{E}$ is a left $\mathcal{A}$-module $\widetilde{\mathcal{E}}$, defined to be the following composite

$$\mathcal{A} \xrightarrow{(-)^*} \mathcal{A}^{\mathbf{op}} \xrightarrow{\mathcal{E}} \mathbf{Vect} \xrightarrow{\widetilde{(-)}} \mathbf{Vect}.$$

Picking our way through the functors involved, we see that for $a \in \mathcal{A}(A, A')$, the linear map $\widetilde{\mathcal{E}}(a)$ is given by

$$\widetilde{\mathcal{E}(A)} \ni \flat(\xi) \mapsto \flat(\mathcal{E}(a^*)\xi).$$

In other words, the left action of $\mathcal{A}$ is given by

$$a \cdot \xi = \xi \cdot a^*.$$

If $\mathcal{E}$ was in fact a right Hilbert $\mathcal{A}$-module, then we turn $\widetilde{\mathcal{E}}$ into a left Hilbert $\mathcal{A}$-module with inner product ${}_{\mathcal{A}}\langle -, - \rangle$ defined to be the composite

$$\widetilde{\mathcal{E}(A')} \times \widetilde{\mathcal{E}(A)} \xrightarrow{(\flat,\flat)} \mathcal{E}(A') \times \mathcal{E}(A) \xrightarrow{\langle -,- \rangle_{\mathcal{A}}} \mathcal{A}(A, A').$$

Lemma 3.3.2. *Any adjointable operator $T : \mathcal{E} \Rightarrow \mathcal{E}'$ between right Hilbert $\mathcal{A}$-modules induces an adjointable operator*

$$\widetilde{T} : \widetilde{\mathcal{E}} \Rightarrow \widetilde{\mathcal{E}'},$$

between the corresponding conjugate modules.

Proof. We define the components of $\widetilde{T}$ by

$$\widetilde{T}_A := \widetilde{T_A}, \qquad \widetilde{T}^*_A := \widetilde{T^*_A}.$$

We must check that the following diagram commutes for each pair $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$,

$$\begin{tikzcd}
 & \widetilde{\mathcal{E}'(A')} \times \widetilde{\mathcal{E}(A)} \arrow[dl, "(\widetilde{T}^*_{A'}, \mathrm{id})"'] \arrow[dr, "(\mathrm{id}, \widetilde{T}_A)"] & \\
\widetilde{\mathcal{E}(A')} \times \widetilde{\mathcal{E}(A)} \arrow[d, "(\flat, \flat)"'] & & \widetilde{\mathcal{E}'(A')} \times \widetilde{\mathcal{E}'(A)} \arrow[d, "(\flat, \flat)"] \\
\mathcal{E}(A') \times \mathcal{E}(A) \arrow[dr, "\langle -,- \rangle"'] & & \mathcal{E}'(A') \times \mathcal{E}'(A) \arrow[dl, "\langle -,- \rangle"] \\
 & \mathcal{A}(A, A') &
\end{tikzcd}$$

First we have

$$\begin{aligned}
\langle \widetilde{T}^*_{A'} \flat(\alpha), \flat(\beta) \rangle &= \langle \widetilde{T^*_{A'}} \flat(\alpha), \flat(\beta) \rangle \\
&= \langle \flat(T^*_{A'} \alpha), \flat(\beta) \rangle \\
&= \langle T^*_{A'} \alpha, \beta \rangle \\
&= \langle \alpha, T_A \beta \rangle.
\end{aligned}$$

Second we have

$$\begin{aligned}
\langle \flat(\alpha), \widetilde{T}_A \flat(\beta) \rangle &= \langle \flat(\alpha), \widetilde{T_A} \flat(\beta) \rangle \\
&= \langle \flat(\alpha), \flat(T_A \beta) \rangle \\
&= \langle \alpha, T_A \beta \rangle,
\end{aligned}$$

so that both paths through the diagram are equal, and the result follows. $\square$

Remark 3.3.3. Since we define the components of the induced operator $\widetilde{T}$ using the functor $\widetilde{(-)}$, it follows that we have a functor **Hilb**-$\mathcal{A} \to \mathcal{A}$-**Hilb**. Furthermore we can see that this is a *-functor.

Definition 3.3.4. Given a right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule ${}_{\mathcal{A}}F_{\mathcal{B}}$, we define the conjugate bimodule, ${}_{\mathcal{B}}\widetilde{F}_{\mathcal{A}}$, to be the left Hilbert $\mathcal{B} - \mathcal{A}$ bimodule with left action given by the following composite,

$$_{(-)}\widetilde{F} : \mathcal{A}^{\mathrm{op}} \xrightarrow{(-)^*} \mathcal{A} \xrightarrow{F_{(-)}} \textbf{Hilb}\text{-}\mathcal{B} \xrightarrow{\widetilde{(-)}} \mathcal{B}\text{-}\textbf{Hilb}.$$

Example 3.3.5. Consider a right Hilbert $A - B$ bimodule $\varphi : A \to \textbf{Hilb}\text{-}B$ where A and B are a pair of C*-algebras. Let's examine what the conjugate bimodule of φ is according to our definition. We denote by X the vector space $\varphi(*)$, so that $\widetilde{\varphi}$ sends $* \mapsto \widetilde{X}$. The right A-action on $\widetilde{X}$ will be given by

$$\flat(x) \cdot a := \widetilde{\varphi}(a)\flat(x).$$

The adjointable operator $\widetilde{\varphi}(a)$ is defined by

$$\widetilde{\varphi}(a)\flat(x) = \widetilde{\varphi(a^*)}\flat(x) = \flat(\varphi(a^*)x),$$

so that

$$\flat(x) \cdot a = \flat(\varphi(a^*)x) = \flat(a^* \cdot x).$$

As noted previously, the left action of B on $\widetilde{X}$ is given by

$$b \cdot \flat(x) = \flat(x \cdot b^*),$$

so that the conjugate bimodule here coincides with what one might have encountered previously, for example in [26, p.49].

Suppose now that $_{\mathcal{A}}F_{\mathcal{B}}$ is a bi-Hilbert $\mathcal{A} - \mathcal{B}$ bimodule. In this case we could just as easily defined the conjugate bimodule to be the right Hilbert $\mathcal{B} - \mathcal{A}$ bimodule with right action

$$\mathcal{B} \xrightarrow{(-)^*} \mathcal{B}^{\mathrm{op}} \xrightarrow{_{(-)}F} \mathcal{A}\text{-}\textbf{Hilb} \xrightarrow{\widetilde{(-)}} \textbf{Hilb}\text{-}\mathcal{A}.$$

We need to check that this is consistent with the definition we gave above, so far as the induced right $\mathcal{A}$-action from the above definition coincides with this alternative.

Lemma 3.3.6. *Suppose that $_{\mathcal{A}}F_{\mathcal{B}}$ is bi-Hilbert $\mathcal{A}-\mathcal{B}$ bimodule. Then the right $\mathcal{A}$-action of the conjugate bimodule $_{\mathcal{B}}\widetilde{F}_{\mathcal{A}}$, constructed in the usual way in Lemma 3.1.3, is equal to the composite*

$$\mathcal{B} \xrightarrow{(-)^*} \mathcal{B}^{op} \xrightarrow{_{(-)}F} \mathcal{A}\text{-}\textbf{\textit{Hilb}} \xrightarrow{\widetilde{(-)}} \textbf{\textit{Hilb}}\text{-}\mathcal{A}.$$

Proof. Take $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$. The right $\mathcal{A}$-module $\widetilde{F}_B$ constructed in Lemma 3.1.3 is given on objects by

$$A \mapsto {}_A\widetilde{F}(B).$$

We unpack this expression to get

$${}_A\widetilde{F}(B) = \widetilde{F_A(B)} = \widetilde{{}_BF(A)}.$$

The $\mathcal{A}$-module $\widetilde{F}_B$ is given on morphisms by

$$\mathcal{A}(A, A') \ni a \mapsto {}_{(a)}\widetilde{F}_B.$$

Again, we unpack this to see that

$${}_{(a)}\widetilde{F}_B = \widetilde{F_{(a^*)B}} = \widetilde{{}_BF(a^*)}.$$

Therefore, we have

$$\widetilde{F}_B = \widetilde{{}_BF}.$$

Finally, we take a morphism $b \in \mathcal{B}(B, B')$. Lemma 3.1.3 now tells us that we have a homomorphism $\widetilde{F}_{(b)} : \widetilde{F}_B \Rightarrow \widetilde{F}_{B'}$ with components

$$\widetilde{F}_{(b)A} = {}_A\widetilde{F}(b) : \widetilde{F}_B(A) \to \widetilde{F}_{B'}(A).$$

Unpacking the definition of ${}_{(-)}\widetilde{F}$ shows us that

$${}_A\widetilde{F}(b) = \widetilde{{}_AF(b)},$$

and the right hand side of this equality is the map

$$\flat(\eta) \mapsto \flat({}_AF(b^*)\eta) = \flat(F_{(b^*)A}\eta.$$

Hence it follows that

$$\widetilde{F}_{(b)A} = \widetilde{F_{(b)A}}.$$

□

3.4 Hilb-$\mathcal{A}$ is the Multiplier Category of Hilb$_{\mathcal{K}}$-$\mathcal{A}$

The multiplier category of a C*-category is introduced in a few places, for example [3], [14], [30]. It is a categorified version of the the multiplier algebra of a C*-algebra. One canonical example of a multiplier algebra is that when A is a C*-algebra and $X \in$ **Hilb**-A, then $\mathcal{L}(X)$ is the multiplier algebra of $\mathcal{K}(X)$. In this section we will demonstrate that when $\mathcal{A}$ is a C*-category, the multiplier category of **Hilb**$_{\mathcal{K}}$-$\mathcal{A}$ is **Hilb**-$\mathcal{A}$. We will need the following definition, which we will revisit in the following section on correspondences.

Definition 3.4.1. If $\mathcal{A}$ and $\mathcal{B}$ are C*-categories, then we call a *-functor $F_{(-)} : \mathcal{A} \to$ **Hilb**-$\mathcal{B}$ *non-degenerate* if for each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, the set

$$\mathrm{Span}\{F_{(a)}T,\ a \in \mathcal{A}(A, A),\ T \in \mathcal{K}(F_A)\},$$

is dense in $\mathcal{K}(F_A)$.

Definition 3.4.2. [30, p.5] If $\mathcal{A}$ is a C*-category and $\mathcal{I}$ is an ideal in $\mathcal{A}$, then we say that $\mathcal{I}$ is an *essential ideal* if whenever $\mathcal{J}$ is an ideal in $\mathcal{A}$ whose morphism sets are not all zero, we have

$$\mathcal{I}(A, A') \cap \mathcal{J}(A, A') \neq \{0\},$$

for all $A, A' \in \mathcal{A}$.

Definition 3.4.3. [30, Proposition 2.2] If $\mathcal{A}$ is a C*-category, then there exists a unital C*-category $\mathbb{M}\mathcal{A}$, such that $\mathcal{A}$ is an essential ideal in $\mathbb{M}\mathcal{A}$, and with the universal property that whenever $\mathcal{A}$ is an essential ideal in some unital C*-category $\mathcal{B}$, then there exists a faithful *-functor $\mathcal{B} \to \mathbb{M}\mathcal{A}$ making the following diagram commute,

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[dr] & \mathcal{B} \arrow[d, dashed] \\ & \mathbb{M}\mathcal{A}. \end{tikzcd}$$

The C*-category $\mathbb{M}\mathcal{A}$ is called the *multiplier category* of $\mathcal{A}$.

Proposition 3.4.4. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be C*-categories, with $\mathcal{A}$ an ideal in $\mathcal{C}$, and let ${}_{\mathcal{A}}F_{\mathcal{B}}$ be a correspondence. Then there exists a unique *-functor $\overline{F}$ which makes the following diagram commute,*

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[dr, "F_{(-)}"'] & \mathcal{C} \arrow[d, dashed, "\overline{F}_{(-)}"] \\ & \textbf{\textit{Hilb}}\text{-}\mathcal{B}. \end{tikzcd}$$

Moreover, if $\mathcal{A}$ is essential in $\mathcal{C}$, and F is faithful, then $\overline{F}$ is also faithful.

Proof. By definition, we have that $\mathcal{O}\mathrm{b}(\mathcal{A}) = \mathcal{O}\mathrm{b}(\mathcal{C})$ and we define $\overline{F}$ on objects by $\overline{F}_A = F_A$. For each $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, Lemma 2.6.5 tells us that

$$F_A(B) = \{K_B(\eta);\ K \in \mathcal{K}(F_A),\ \eta \in F_A(B)\},$$

and coupling this with non-degeneracy of F shows that

$$\mathrm{Span}\{\sum_i (F_{(a_i)}K_i)_B(\eta);\ a_i \in \mathcal{A}(A,A),\ K_i \in \mathcal{K}(F_A),\ \eta \in F_A(B)\},$$

is dense in $F_A(B)$. We define the component $\overline{F}_{(c)B}$ by

$$\overline{F}_{(c)B}(\sum_i (F_{(a_i)}K_i)_B(\eta)) = \sum_i (F_{(ca_i)}K_i)_B(\eta).$$

This satisfies the following,

$$\begin{aligned} \|\sum_i (F_{(ca_i)}K_i)_B(\eta)\| &= \|\sum_i (F_{(c)}F_{(a_i)}K_i)_B(\eta)\| \\ &\leq \|c\|\|\sum_i (F_{(a_i)}K_i)_B(\eta)\|, \end{aligned}$$

so that $\overline{F}_{(c)B}$ is continuous on a dense subset of $F_A(B)$, hence extends to a map

$$\overline{F}_{(c)B} : F_A(B) \to F_{A'}(B).$$

Moreover, we may check that the collections of maps $\{\overline{F}_{(c)B}\}$ and $\{\overline{F}_{(c^*)B}\}$ are adjoints of each other, hence we have $\overline{F}_{(c)} \in \mathcal{L}(F_A, F_{A'})$. We can easily check that $\overline{F}$ is indeed a *-functor, and that the uniqueness claim is satisfied. For the final claim, we consider the kernels[4] of the *-functors F and $\overline{F}$, which are ideals in $\mathcal{A}$ and $\mathcal{C}$ respectively. For $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$, we have

$$\ker(\overline{F})(A,A') \cap \mathcal{A}(A,A') = \ker(F)(A,A') \cap \mathcal{A}(A,A') = \{0\}.$$

[4]The kernel of a *-functor $F : \mathcal{A} \to \mathcal{B}$ is the ideal of $\mathcal{A}$ with morphism sets $\ker(F)(A,A') = \{a \in \mathcal{A}(A,A');\ F(a) = 0\}$. A *-functor is faithful if and only if it's kernel is the trivial ideal.

The first of these equalities follows from the construction of $\overline{F}$, and the second follows from the assumption that F is faithful. We are assuming that $\mathcal{A}$ is an essential ideal in $\mathcal{C}$, so the fact that the above holds for all A, A' implies $\overline{F}$ is faithful as required. □

Corollary 3.4.5. *If $\mathcal{A}$ is a C^*-category, then **Hilb**-$\mathcal{A}$ is the multiplier category of **Hilb**$_{\mathcal{K}}$-$\mathcal{A}$.*

Proof. The inclusion $\mathbf{Hilb}_{\mathcal{K}}$-$\mathcal{A} \hookrightarrow \mathbf{Hilb}$-$\mathcal{A}$ is non-degenerate, so the desired result follows from the previous proposition. □

3.5 Correspondences

One example of Hilbert bimodules which we will keep returning to is that of correspondences. These are bimodules which satisfy an additional non-degeneracy condition, and secretly we introduced these in the previous section.

Definition 3.5.1. A right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule is called an $\mathcal{A} - \mathcal{B}$ *correspondence* if the *-functor $F_{(-)} : \mathcal{A} \to \mathbf{Hilb}$-$\mathcal{B}$ is non-degenerate.

We will now look to get some alternative pictures of non-degeneracy. For this we need to make use of multiplier categories and the following notion.

Definition 3.5.2. [3, p.4] If $\mathcal{A}$ is a C*-category, then a net (a_λ) of morphisms in $\mathbb{M}\mathcal{A}(A, A')$ converges *strictly* to $a \in \mathbb{M}\mathcal{A}(A, A')$ if

$$\|a_\lambda a' - aa'\| \to 0, \qquad \|a'' a_\lambda - a'' a\| \to 0,$$

for all $a' \in \mathcal{A}(X, A)$ and $a'' \in \mathcal{A}(A', Y)$. We call a *-functor $F : \mathbb{M}\mathcal{A} \to \mathbb{M}\mathcal{B}$ *strictly continuous* if the each of the maps $F : \mathbb{M}\mathcal{A}(A, A') \to \mathbb{M}\mathcal{B}(F(A), F(A'))$ are strictly continuous on bounded subsets.

Using our previous work, we have the following special case of this definition.

Example 3.5.3. For a C*-category $\mathcal{B}$ and $\mathcal{D}, \mathcal{E} \in \mathbf{Hilb}$-$\mathcal{B}$. A net of operators $(T_\lambda) \subseteq \mathcal{L}(\mathcal{D}, \mathcal{E})$ converges strictly to $T \in \mathcal{L}(\mathcal{D}, \mathcal{E})$ if

$$\|T_\lambda P - TP\| \to 0, \qquad \|QT_\lambda - QT\| \to 0,$$

for all $P \in \mathcal{K}(\mathcal{C}, \mathcal{D})$ and $Q \in \mathcal{K}(\mathcal{E}, \mathcal{F})$.

Lemma 3.5.4. *For a *-functor $F_{(-)} : \mathcal{A} \to \boldsymbol{Hilb}\text{-}\mathcal{B}$, the following are equivalent:*

(i) *F is non-degenerate as above,*

(ii) *For each $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, the set Span$\{F_{(a)B}(\xi);\ a \in \mathcal{A}(A, A),\ \xi \in F_A(B)\}$ is dense in $F_A(B)$,*

(iii) *F extends to a strictly continuous, unital *-functor $\overline{F} : \mathbb{M}\mathcal{A} \to \boldsymbol{Hilb}\text{-}\mathcal{B}$,*

(iv) *For each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ and each approximate unit (e_λ) for $\mathcal{A}(A, A)$, the net $(F_{(e_\lambda)})$ converges strictly to $\mathrm{id}_{F(A)}$*[5].

Proof.

(i) $\Rightarrow$ *(iii)* Suppose that $a_\lambda \to a$ strictly in some bounded subset of $M\mathcal{A}(A, A)$. Note that this implies that for all λ we have $\|a_\lambda - a\| \leq M$, for a fixed constant M. Take a sum of compact operators $\sum_i F_{(x_i)}K_i$, where $a_i \in \mathcal{A}(A, A)$ and $K_i \in \mathcal{K}(F_A)$. We let $\overline{F}$ be the extension of F constructed in the proof of Proposition 3.4.4, then given $\epsilon > 0$ we use the strict convergence $a_\lambda \to a$ to find λ_0 such that whenever $\lambda \geq \lambda_0$ we have

$$\|(\overline{F}_{(a_\lambda)} - \overline{F}_{(a)}) \sum_i F_{(x_i)}K_i\| = \| \sum_i F_{((a_\lambda - a)x_i)}K_i\| \leq \sum_i \|(a_\lambda - a)x_i\|\|K_i\| < \frac{\epsilon}{2}.$$

If $K \in \mathcal{K}(F_A)$, then by non-degeneracy of F we can approximate K,

$$\|K - \sum_i F_{(x_i)}K_i\| < \frac{\epsilon}{2M},$$

so we have

$$\|(\overline{F}_{(a_\lambda)} - \overline{F}_{(a)})(K - \sum_i F_{(x_i)}K_i)\| \leq \|a_\lambda - a\|\|(K - \sum_i F_{(x_i)}K_i)\| < \frac{\epsilon}{2}.$$

To finish, we have

$$\begin{aligned}\|(\overline{F}_{(a_\lambda)} - \overline{F}_{(a)})K\| &\leq \|(\overline{F}_{(a_\lambda)} - \overline{F}_{(a)}) \sum_i F_{(x_i)}K_i\| + \|(\overline{F}_{(a_\lambda)} - \overline{F}_{(a)})(K - \sum_i F_{(x_i)}K_i)\| \\ &< \epsilon.\end{aligned}$$

[5] An alternative way of phrasing this is to say that the net $(F_{(e_\lambda)})$ is an approximate unit for $\mathcal{K}(F_A, F_A)$.

so that $\|\overline{F}_{(a_\lambda)} - \overline{F}_{(a)})K\| \to 0$, showing that the extension $\overline{F}$ is strictly continuous.

$(iii) \Rightarrow (iv)$ If (e_λ) is an approximate unit for $\mathcal{A}(A,A)$, then (e_λ) converges strictly to the identity in $M\mathcal{A}(A,A)$, so by assumption $(F_{(e_\lambda)})$ converges strictly to $\mathrm{id}_{F(A)}$. The result follows once we note that $\overline{F}(e_\lambda) = F(e_\lambda)$.

$(iv) \Rightarrow (i)$ Given $K \in \mathcal{K}(F(A))$, we have by assumption that $F_{(e_\lambda)}K \to K$, so the *-functor F must be non-degenerate.

$(i) \Rightarrow (ii)$ Given $\eta \in F_A(B)$, we use Lemma 2.6.5 find a compact operator $K \in \mathcal{K}(F_A)$ and an element $\xi \in F_A(B)$ such that $\eta = K_B(\xi)$. By assumption we can estimate K like so;

$$\|K - \sum_{i=1}^{m} F(a_i)T_i\| < \frac{\epsilon}{\|\xi\|}.$$

Then we have

$$\|\eta - \sum_{i=1}^{m} F(a_i)(T_i\xi)\| = \|(K_B - \sum_{i=1}^{m} F(a_i)T_i)\xi\| < \|K_B - \sum_{i=1}^{m} F(a_i)T_i)\|\|\xi\| < \epsilon,$$

so it follows that condition (ii) is satisfied.

$(ii) \Rightarrow (i)$ For $a \in \mathcal{A}(A,A)$ and $\xi, \eta \in F_A(B)$, we first note that because $F_{(a)}$ is a natural transformation, we have

$$F_{(a)} \circ \theta_{\xi,\eta} = F_{(a)B}(\xi) \cdot \langle \eta, - \rangle_{\mathcal{B}} = \theta_{a\cdot\xi,\eta}.$$

Given a compact operator $K \in \mathcal{K}(F_A)$ and $\epsilon > 0$, we can write

$$\frac{\epsilon}{2} > \|K - \sum_{i=1}^{m} \theta_{\xi_i,\eta_i}\|,$$

where $\xi_i, \eta_i \in F_A(B)$, then for each i, we use our assumption on F to write

$$\frac{\epsilon}{2m\|\eta_i\|} > \|\xi_i - \sum_{j_i} F_{(a_{j_i})B}(\zeta_{j_i})\|,$$

where $a_{j_i} \in \mathcal{A}(A,A)$ and $\zeta_{j_i} \in F_A(B)$. Now we observe that

$$\|\theta_{\xi_i,\eta_i} - \theta_{\sum_{j_i} a_{j_i}\cdot\zeta_{j_i},\eta_i}\| = \|\theta_{(\xi_i - \sum_{j_i} a_{j_i}\cdot\zeta_{j_i}),\eta_i}\| \leq \|\xi_i - \sum_{j_i} a_{j_i}\cdot\zeta_{j_i}\|\|\eta_i\| < \frac{\epsilon}{2m}.$$

To finish the proof, we now make a final computation,

$$\begin{aligned}\|K - \sum_{i,j_i} a_{j_i} F_{(a_{j_i})} \circ \theta_{\zeta_{j_i},\eta_i}\| &\leq \|K - \sum_i \theta_{\xi_i,\eta_i}\| + \sum_i \|\theta_{\xi_i,\eta_i} - \theta_{\sum_{j_i} a_{j_i}\cdot\zeta_{j_i},\eta_i}\| \\ &< \frac{\epsilon}{2} + \frac{m\epsilon}{2m} \\ &= \epsilon.\end{aligned}$$

□

Antoun and Voigt [3] define a notion of non-degeneracy for *-functors of the form $\mathcal{A} \to \mathbb{M}\mathcal{B}$. We have seen that $\mathbb{M}\mathbf{Hilb}_{\mathcal{K}}$-$\mathcal{B} = \mathbf{Hilb}$-$\mathcal{B}$, so we had better check that our notion of non-degeneracy coincides with the existing definition.

Definition 3.5.5. [3] A *-functor $F : \mathcal{A} \to \mathbb{M}\mathcal{B}$ is called *non-degenerate* if for all $A, A' \in \mathcal{A}$, the sets

$$\begin{aligned}&\mathrm{Span}\{bF(a);\ a \in \mathcal{A}(A, A),\ b \in \mathcal{B}(F(A), F(A'))\}, \\ &\mathrm{Span}\{F(a')b;\ a' \in \mathcal{A}(A', A'),\ b \in \mathcal{B}(F(A), F(A'))\},\end{aligned}$$

are dense in $\mathcal{B}(F(A), F(A'))$.

Note that any *-functor $\mathcal{A} \to \mathbf{Hilb}$-$\mathcal{B}$ which is satisfies the above definition of non-degeneracy is a correspondence. The converse is true, and we prove this below.

Lemma 3.5.6. *Any $\mathcal{A} - \mathcal{B}$ correspondence is also non-degenerate in the sense of the definition stated above.*

Proof. For $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ we let (e_λ) be an approximate unit for $\mathcal{A}(A, A)$, then by Lemma 3.5.4 we know that $(F_{(e_\lambda)})$ is an approximate unit for $\mathcal{B}(F_A, F_A)$. By Lemma 1.2.14 it follows that for all $B, B' \in \mathcal{O}\mathrm{b}(\mathcal{B})$, $b \in \mathcal{B}(B, F_A)$ and $b' \in \mathcal{B}(F_A, B')$ we have

$$\|F_{(e_\lambda)} b - b\| \to 0, \qquad \|b' F_{(e_\lambda)} - b'\| \to 0,$$

so the result follows. □

Chapter 4

Tensor Products

4.1 Algebraic Constructions

Tensor products of Hilbert modules and bimodules will be very useful for us later on. Ultimately we will construct a bicategory where 1-morphisms are certain bimodules, with the composition given by certain tensor products. Tensor products will also feature in our characterisation of certain functors between categories of Hilbert modules.

4.1.1 The Algebraic Tensor Product of Modules

The first steps we take into tensor products are to look at things purely algebraically. We will define algebraic tensor products of modules as certain coends, then show how these coends can be assembled into functors. As a gentle introduction to this section, let's briefly review how one defines the algebraic tensor product of modules over algebras.

Consider an algebra A, a right A-module $\mathcal{E}$ and a left A-module $\mathcal{F}$. If X is a vector space, then a bilinear map $\varphi : \mathcal{E} \times \mathcal{F} \to X$ is called A-balanced if it satisfies

$$\varphi(e \cdot a, f) = \varphi(e, a \cdot f),$$

for any $e \in \mathcal{E}$ and $f \in \mathcal{F}$. A vector space X is called the algebraic tensor product of $\mathcal{E}$ and $\mathcal{F}$, if there is a bilinear, A-balanced map

$$\varphi : \mathcal{E} \times \mathcal{F} \to X,$$

with the property that if Y is any other vector space with bilinear, A-balanced map

$$\pi : \mathcal{E} \times \mathcal{F} \to Y,$$

then there is a unique linear map θ which makes the following diagram commute,

$$\begin{tikzcd}
\mathcal{E} \times \mathcal{F} \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & X \arrow[d, dashed, "\theta"] \\
& Y
\end{tikzcd}$$

Remark 4.1.1. The property of a bilinear map $\varphi : \mathcal{E} \times \mathcal{F} \to X$ being A-balanced may be expressed by requiring the following diagram to commute for every $a \in A$,

$$\begin{tikzcd}
\mathcal{E} \times \mathcal{F} \arrow[r, "(\mathrm{id}, a\cdot(-))"] \arrow[d, "((-)\cdot a, \mathrm{id})"'] & \mathcal{E} \times \mathcal{F} \arrow[d, "\varphi"] \\
\mathcal{E} \times \mathcal{F} \arrow[r, "\varphi"'] & X
\end{tikzcd}$$

This leads us nicely to our next definition. Note that for now we will favour the terminology *coend* rather than *algebraic tensor product*. This will be explained later.

Definition 4.1.2. Let $\mathcal{A}$ be a C*-category, $\mathcal{D}$ be a right $\mathcal{A}$-module and let $\mathcal{E}$ be a left $\mathcal{A}$-module. A vector space X along with a family of bilinear maps

$$\{h_A : \mathcal{D}(A) \times \mathcal{E}(A) \to X;\ A \in \mathcal{O}\mathrm{b}(\mathcal{A})\},$$

will be called the *coend* of $\mathcal{D}$ and $\mathcal{E}$ if:

(i) For each morphism $a \in \mathcal{A}(A, A')$, the following diagram commutes,

$$\begin{tikzcd}
\mathcal{D}(A') \times \mathcal{E}(A) \arrow[r, "(\mathrm{id}, \mathcal{E}(a))"] \arrow[d, "(\mathcal{D}(a), \mathrm{id})"'] & \mathcal{D}(A') \times \mathcal{E}(A') \arrow[d, "h_{A'}"] \\
\mathcal{D}(A) \times \mathcal{E}(A) \arrow[r, "h_A"'] & X.
\end{tikzcd}$$

We may refer to this property by saying that the family of maps $\{h_A;\ A \in \mathcal{O}\mathrm{b}(\mathcal{A})\}$ is $\mathcal{A}$-*balanced*,

(ii) If Y is any other vector space with bilinear maps

$$\{i_A : \mathcal{D}(A) \times \mathcal{E}(A) \to Y;\ A \in \mathcal{O}\mathrm{b}(\mathcal{A})\},$$

making the analogous diagrams commute, there is a unique linear map $\theta : X \to Y$ which makes the following diagram commute

$$\begin{tikzcd}
\mathcal{D}(A') \times \mathcal{E}(A) \arrow[r, "(\mathrm{id},\mathcal{E}(a))"] \arrow[d, "(\mathcal{D}(a),\mathrm{id})"'] & \mathcal{D}(A') \times \mathcal{E}(A') \arrow[d, "h_{A'}"] \arrow[ddr, bend left, "i_{A'}"] & \\
\mathcal{D}(A) \times \mathcal{E}(A) \arrow[r, "h_A"'] \arrow[drr, bend right, "i_A"'] & X \arrow[dr, dashed, "\theta"] & \\
 & & Y.
\end{tikzcd}$$

It should be no surprise that coends always exist.

Lemma 4.1.3. *If $\mathcal{A}$ is a C^*-category, then the coend of a right $\mathcal{A}$-module $\mathcal{D}$ and a left $\mathcal{A}$-module $\mathcal{E}$ always exists, and we denote it by $\mathcal{D} \boxdot_{\mathcal{A}} \mathcal{E}$.*

Proof. We consider the following vector space

$$X := \bigoplus_{A \in \mathcal{O}\mathrm{b}(\mathcal{A})} \mathcal{D}(A) \otimes_{\mathbf{Vect}} \mathcal{E}(A),$$

where $\otimes_{\mathbf{Vect}}$ denotes the ordinary tensor product of vector spaces. We let Y be the subspace generated by the elements

$$(\eta, f \cdot \xi) - (\eta \cdot f, \xi),$$

for all $\eta \in \mathcal{D}(A')$, $\xi \in \mathcal{E}(A)$ and $f \in \mathcal{A}(A, A')$. Then we define $\mathcal{D} \boxdot_{\mathcal{A}} \mathcal{E}$ to be the quotient X/Y. For each object $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, we obtain a bilinear map

$$\begin{aligned} h_A ; \mathcal{D}(A) \times \mathcal{E}(A) &\to \mathcal{D} \boxdot_{\mathcal{A}} \mathcal{E} \\ (\eta, \xi) &\mapsto \eta \otimes \xi. \end{aligned}$$

Note that strictly speaking $\eta \otimes \xi$ is an equivalence class. We will brush this technicality aside, and it should cause us no problems. Henceforth we will call vectors of the form $\eta \otimes \xi$ *elementary tensors*. For every $a \in \mathcal{A}(A, A')$, the following diagram commutes,

$$\begin{tikzcd}
\mathcal{D}(A') \times \mathcal{E}(A) \arrow[r, "(\mathrm{id},\mathcal{E}(a))"] \arrow[d, "(\mathcal{D}(a),\mathrm{id})"'] & \mathcal{D}(A') \times \mathcal{E}(A') \arrow[d, "h_{A'}"] \\
\mathcal{D}(A) \times \mathcal{E}(A) \arrow[r, "h_A"'] & \mathcal{D} \boxdot_{\mathcal{A}} \mathcal{E},
\end{tikzcd}$$

So it only remains for us to check the universal property, so assume that we have another vector space Z with a suitable family of bilinear maps making the following diagram commute,

$$\begin{tikzcd}
\mathcal{D}(A') \times \mathcal{E}(A) \arrow[r, "(\mathrm{id},\mathcal{E}(a))"] \arrow[d, "(\mathcal{D}(a),\mathrm{id})"'] & \mathcal{D}(A') \times \mathcal{E}(A') \arrow[d, "h_{A'}"] \arrow[ddr, bend left, "i_{A'}"] & \\
\mathcal{D}(A) \times \mathcal{E}(A) \arrow[r, "h_A"'] \arrow[drr, bend right, "i_A"'] & \mathcal{D} \boxdot_{\mathcal{A}} \mathcal{E} & \\
& & Z.
\end{tikzcd}$$

Now we define a map $\theta : \mathcal{D} \boxdot_{\mathcal{A}} \mathcal{E} \to Z$ on elementary tensors by

$$\theta(\eta \otimes \xi) = i_A(\eta, \xi),$$

where $(\eta, \xi) \in \mathcal{D}(A) \times \mathcal{E}(A)$. This ought to be a well defined linear map, and uniqueness should be obvious. $\square$

Notation 4.1.4. Let us comment briefly on our notation used in the previous section, and how it will feature in what follows.

(i) For purely algebraic coends and tensor products, we will use the dotted symbols $\boxdot$ and $\odot$,

(ii) We will add subscripts where necessary to indicate which category we are "tensoring over", e.g. $\boxdot_{\mathcal{A}}$ means we are tensoring over $\mathcal{A}$,

(iii) For elementary tensors in a coend, we will always use the $\otimes$ notation.

4.1.2 Functoriality

Note that the coend of two modules is just a vector space. Later on when we construct tensor products involving bimodules, we will be producing modules, and one vital ingredient for these constructions are the following functorial properties of coends.

Lemma 4.1.5. *Let $\mathcal{A}$ be a C^*-category, let $\mathcal{D}, \mathcal{D}'$ be right $\mathcal{A}$-modules, let $\mathcal{E}$ be a left $\mathcal{A}$-module and suppose that we have a module homomorphism $T : \mathcal{D} \Rightarrow \mathcal{D}'$. The homo-*

morphism T induces a linear map

$$\mathcal{D} \boxdot_{\mathcal{A}} \mathcal{E} \to \mathcal{D}' \boxdot_{\mathcal{A}} \mathcal{E},$$

which is given on elementary tensors by

$$\eta \otimes \xi \mapsto (T_A \eta) \otimes \xi.$$

This construction extends to a functor

$$(-) \boxdot_{\mathcal{A}} \mathcal{E} : \boldsymbol{Mod}\text{–}\mathcal{A} \to \boldsymbol{Vect}.$$

Proof. Given a morphism $a \in \mathcal{A}(A, A')$, we draw the following diagram, where we'll explain the dotted and dashed arrows later.

$$\begin{tikzcd}
 & \mathcal{D}'(A') \times \mathcal{E}(A) \arrow[rr, "(\mathrm{id},\mathcal{E}(a))"] \arrow[dd, "(\mathcal{D}'(a),\mathrm{id})"] & & \mathcal{D}'(A') \times \mathcal{E}(A') \arrow[dd, dotted, "h'_{A'}"] \\
\mathcal{D}(A') \times \mathcal{E}(A) \arrow[ur, "(T_{A'},\mathrm{id})"] \arrow[rr, dotted, "(\mathrm{id},\mathcal{E}(a))"'] \arrow[dd, dotted, "(\mathcal{D}(a),\mathrm{id})"'] & & \mathcal{D}(A') \times \mathcal{E}(A') \arrow[ur, dotted, "(T_{A'},\mathrm{id})"] \arrow[dd, "h_{A'}"] & \\
 & \mathcal{D}'(A) \times \mathcal{E}(A) \arrow[rr, dotted, "h'_A"'] & & \mathcal{D}' \boxdot_{\mathcal{A}} \mathcal{E} \\
\mathcal{D}(A) \times \mathcal{E}(A) \arrow[ur, dotted, "(T_A,\mathrm{id})"'] \arrow[rr, "h_A"'] & & \mathcal{D} \boxdot_{\mathcal{A}} \mathcal{E} \arrow[ur, dashed] &
\end{tikzcd}$$

We make some observations;

(i) The left face commutes by naturality of $T : \mathcal{D} \Rightarrow \mathcal{D}'$,

(ii) The upper face commutes by direct computation,

(iii) The front and rear faces commute by construction of coends.

The two dotted paths through the cube and all of this commutativity allows us to invoke the universal property of coends, and we fill in the dashed arrow uniquely.

To extend this to get a functor, we will need to tack on an extra cube and appeal to the uniqueness of the dashed arrows we generate. $\square$

Similarly, one may prove the following.

Lemma 4.1.6. *Let $\mathcal{A}$ be a C^*-category, let $\mathcal{D}$ be a right $\mathcal{A}$-module, let $\mathcal{E}, \mathcal{E}'$ be left $\mathcal{A}$-modules and suppose that we have a module homomorphism $T : \mathcal{E} \Rightarrow \mathcal{E}'$. The homomorphism T induces a linear map*

$$\mathcal{D} \boxdot_{\mathcal{A}} \mathcal{E} \to \mathcal{D} \boxdot_{\mathcal{A}} \mathcal{E}',$$

which is given on elementary tensors by

$$\eta \otimes \xi \mapsto \eta \otimes (T_A \xi).$$

This construction extends to a functor

$$\mathcal{D} \boxdot_{\mathcal{A}} (-) : \mathcal{A} - \boldsymbol{Mod} \to \boldsymbol{Vect}.$$

4.1.3 More Functoriality: Building Modules From Coends

The functoriality results presented in the previous section tell us how we can impose module structures on certain tensor products.

Lemma 4.1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-categories, let $\mathcal{D}$ be a left $\mathcal{B}$-module and let $F_{(-)} : \mathcal{A} \to \boldsymbol{Mod} - \mathcal{B}$ be a functor. The collection of coends*

$$\{F_A \boxdot_{\mathcal{B}} \mathcal{D};\ A \in \mathcal{O}\mathrm{b}(\mathcal{A})\},$$

may be assembled to give a left $\mathcal{A}$-module.

Proof. By considering the composite

$$\mathcal{A} \xrightarrow{F_{(-)}} \mathbf{Mod} - \mathcal{B} \xrightarrow{(-)\boxdot_{\mathcal{B}}\mathcal{D}} \mathbf{Vect}.$$

We see that we have a left $\mathcal{A}$-module as claimed. By unpacking the definition of the rightmost arrow above, we see that this module has action given on elementary tensors by

$$a \cdot (\xi \otimes \eta) = (a \cdot \xi) \otimes \eta.$$

□

Lemma 4.1.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-categories, let $\mathcal{E}$ be a right $\mathcal{A}$-module, and let $F_{(-)} : \mathcal{A} \to$ **Mod** $- \mathcal{B}$ be a functor. The collection of coends*

$$\{\mathcal{E} \boxdot_{\mathcal{A}} {}_B F;\ B \in \mathcal{O}\mathrm{b}(\mathcal{B})\},$$

may be assembled to give a right $\mathcal{B}$-module.

Proof. As above, for existence we look at the composite

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{(-)F} \mathcal{A} - \mathbf{Mod} \xrightarrow{\mathcal{E}\boxdot_{\mathcal{A}}(-)} \mathbf{Vect}.$$

And the action is given on elementary tensors by

$$(\eta \otimes \xi) \cdot b = \eta \otimes (\xi \cdot b).$$

□

Definition 4.1.9. The *algebraic tensor product* of a left $\mathcal{B}$-module $\mathcal{D}$ and a functor $F_{(-)} : \mathcal{A} \to \mathbf{Mod} - \mathcal{B}$ is the left $\mathcal{A}$-module constructed above. Similarly, the *algebraic tensor product* of a right $\mathcal{A}$-module $\mathcal{E}$ and a functor $F_{(-)} : \mathcal{A} \to \mathbf{Mod} - \mathcal{B}$ is the right $\mathcal{B}$-module constructed above.

The final result of this section is the following.

Proposition 4.1.10. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be C^*-categories and consider a pair of functors*

$$D_{(-)} : \mathcal{A} \to \boldsymbol{Mod} - \mathcal{B}, \qquad E_{(-)} : \mathcal{B} \to \boldsymbol{Mod} - \mathcal{C}.$$

The assignment

$$\mathcal{O}\mathrm{b}(\mathcal{A}) \ni A \mapsto D_A \odot_{\mathcal{B}} E,$$

extends to a functor $D \odot_{\mathcal{B}} E : \mathcal{A} \to \boldsymbol{Mod} - \mathcal{C}$.

Proof. The assignment

$$\mathcal{O}\mathrm{b}(\mathcal{A}) \ni A \mapsto D_A \odot_{\mathcal{B}} E,$$

gives us a right $\mathcal{C}$-module for each object of $\mathcal{A}$, so what we need to do is check that this assignment is functorial with respect to morphisms. Given $a \in \mathcal{A}(A, A')$, we are assuming that a gives us a homomorphism

$$D_{(a)} : D_A \Rightarrow D_{A'},$$

so by previous functoriality results we have, for each $C \in \mathcal{O}\mathrm{b}(\mathcal{C})$, a linear map between coends

$$\begin{gathered} D_A \boxdot_{\mathcal{B}} {}_C E \to D_{A'} \boxdot_{\mathcal{B}} {}_C E \\ \eta \otimes \xi \mapsto a \cdot \eta \otimes \xi, \end{gathered}$$

and we define our module homomorphism using these linear maps; the component $D \odot_{\mathcal{B}} E(a)_C$ will be the map of coends mentioned above. Using the previous result, we know that for each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, $D_A \odot_{\mathcal{B}} E$ is a right $\mathcal{C}$-module, so if we take $c \in \mathcal{C}(C', C)$ then we can draw the following diagram,

$$\begin{tikzcd}
D_A \odot_{\mathcal{B}} E(C) \arrow[dr, equal] & & & D_A \odot_{\mathcal{B}} E(C') \arrow[dl, equal] \\
& D_A \boxdot_{\mathcal{B}} {}_C E \arrow[r] \arrow[d] & D_A \boxdot_{\mathcal{B}} {}_{C'} E \arrow[d] & \\
& D_{A'} \boxdot_{\mathcal{B}} {}_C E \arrow[r] \arrow[dl, equal] & D_{A'} \boxdot_{\mathcal{B}} {}_{C'} E \arrow[dr, equal] & \\
D_{A'} \odot_{\mathcal{B}} E(C) & & & D_{A'} \odot_{\mathcal{B}} E(C')
\end{tikzcd}$$

The horizontal arrows come from the right $\mathcal{C}$ action, and the vertical arrows are the components of our proposed module homomorphism. It is straightforward to verify that this diagram commutes, so that we have the required homomorphism.

To finish the proof, one needs to check that compositions and any identities are preserved, but this is straightforward because of how everything is defined. □

Definition 4.1.11. For functors D and E as above, we call the functor $D \odot_{\mathcal{B}} E$ the *algebraic tensor product* of D and E.

4.2 The Interior Tensor Product

Since Hilbert modules are just special examples of modules, we can of course perform the constructions of the previous section with them. When we do this, we require the resulting modules to be Hilbert modules, and this requires a bit of work.

Lemma 4.2.1. *If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories, $\mathcal{E}$ is a right Hilbert $\mathcal{A}$-module, and ${}_{\mathcal{A}}F_{\mathcal{B}}$ is a right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule, then we have an inner product $[-,-]_{\mathcal{B}}$ on the right $\mathcal{B}$-module $\mathcal{E} \odot_{\mathcal{A}} F$ defined on simple tensors*[1] *by*

$$[\xi \otimes \eta, \xi' \otimes \eta]_{\mathcal{B}} = \langle \eta, \langle \xi, \xi' \rangle_{\mathcal{A}} \cdot \eta' \rangle_{\mathcal{B}}.$$

Where $\xi \otimes \eta \in (\mathcal{E} \odot_{\mathcal{A}} F)(B')$ and $\xi' \otimes \eta' \in (\mathcal{E} \odot_{\mathcal{A}} F)(B)$. Upon completing, we end up with a right Hilbert $\mathcal{B}$-module $\mathcal{E} \otimes_{\mathcal{A}} F$.

Proof. Use [24, Lemma 3.17] to see that this formula gives a semi-inner product, then use [24, Lemma 3.19] together with our construction of coends in Lemma 4.1.3 to see that it is in fact an inner product. $\square$

Similarly, one could prove the following.

Lemma 4.2.2. *If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories, $\mathcal{D}$ is a left Hilbert $\mathcal{A}$-module, and ${}_{\mathcal{A}}F_{\mathcal{B}}$ is a right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule, then we have an inner product ${}_{\mathcal{A}}[-,-]$ on the left $\mathcal{A}$-module $F \odot_{\mathcal{B}} \mathcal{D}$ defined on simple tensors by*

$${}_{\mathcal{A}}[\xi \otimes \eta, \xi' \otimes \eta'] = {}_{\mathcal{A}}\langle \xi \cdot {}_{\mathcal{B}}\langle \eta, \eta' \rangle, \xi' \rangle.$$

Where $\xi \otimes \eta \in (F \odot_{\mathcal{B}} \mathcal{D})(A')$ and $\xi' \otimes \eta' \in (F \odot_{\mathcal{B}} \mathcal{D})(A)$. Upon completing, we end up with a left Hilbert $\mathcal{A}$-module $F \otimes_{\mathcal{B}} \mathcal{E}$.

Definition 4.2.3. With $\mathcal{E}$, $\mathcal{D}$ and $F_{(-)}$ as above, we call the right Hilbert $\mathcal{B}$-module $\mathcal{E} \otimes_{\mathcal{A}} F$ the *interior tensor product* of $\mathcal{E}$ and F and call the left Hilbert $\mathcal{A}$-module $F \otimes_{\mathcal{B}} \mathcal{D}$ the *interior tensor product* of F and $\mathcal{D}$.

Remark 4.2.4. Above we used $[-,-]$ to denote the inner product on a tensor product. Henceforth we will tend to just notate this tensor product by $\langle -,- \rangle$, and it should be clear from the context when this refers to an inner product on a tensor product. However when we need additional clarity, we will use both notations.

We have just seen that the tensor product of a right Hilbert module with a right Hilbert bimodule gives back a right Hilbert module. For our later work, we will need to know that the tensor product of two right Hilbert bimodules is again a right Hilbert bimodule.

[1] We extend this rule in the obvious manner to sums of simple tensors.

Lemma 4.2.5. *If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are C^*-categories, ${}_{\mathcal{A}}F_{\mathcal{B}}$ and ${}_{\mathcal{B}}G_{\mathcal{C}}$ are right Hilbert $\mathcal{A}-\mathcal{B}$ and $\mathcal{B}-\mathcal{C}$ bimodules respectively, then the assignment*

$$\mathcal{O}\mathrm{b}(\mathcal{A}) \ni A \mapsto F_A \otimes_{\mathcal{B}} G,$$

*extends to a *-functor $\mathcal{A} \to$ **Hilb**-$\mathcal{C}$, i.e. a right Hilbert $\mathcal{A}-\mathcal{C}$ bimodule, which we denote by $F \otimes_{\mathcal{B}} G$.*

Proof. From the algebraic constructions covered in the previous section, a morphism $a \in \mathcal{A}(A, A')$ gives, for each $C \in \mathcal{O}\mathrm{b}(\mathcal{C})$ a well defined linear map of coends

$$\begin{aligned} F_A \boxdot_{\mathcal{B}} {}_C G &\to F_{A'} \boxdot_{\mathcal{B}} {}_C G \\ \xi \otimes \eta &\mapsto (a \cdot \xi) \otimes \eta. \end{aligned}$$

And moreover these linear maps may be assembled to give a functor $\mathcal{A} \to \mathbf{Mod} - \mathcal{C}$. As we did before, we write the action of this functor on morphisms as $a \cdot (\xi \otimes \eta) = (a \cdot \xi) \otimes \eta$. What remains for us to do here, is checking that these linear maps extend to ones between the completions of the relevant spaces, that they induce adjointable operators and that this all assembles to give a *-functor. Towards this end, first let $T \in \mathcal{L}(F_A, F_{A'})$, take sums $\sum_i \xi_i \otimes \eta_i \in (F_A \odot G)(C')$ and $\sum_j \xi'_j \otimes \eta'_j \in (F_{A'} \odot G)(C)$, then we make the following computation,

$$\begin{aligned} \langle \sum_i T(\xi_i) \otimes \eta_i, \sum_j \xi'_j \otimes \eta'_j \rangle &= \sum_{i,j} \langle T(\xi_i) \otimes \eta_i, \xi'_j \otimes \eta'_j \rangle \\ &= \sum_{i,j} \langle \eta_i, \langle T(\xi_i), \xi'_j \rangle_{\mathcal{A}} \cdot \eta'_j \rangle \\ &= \sum_{i,j} \langle \eta_i, \langle \xi_i, T^*(\xi'_j) \rangle_{\mathcal{A}} \cdot \eta'_j \rangle \\ &= \langle \sum_i \xi_i \otimes \eta_i, \sum_j T^*(\xi'_j) \otimes \eta'_j \rangle. \end{aligned}$$

Now let $x = \sum_i \xi_i \otimes \eta_i \in (F_A \odot G)(C)$, and we make another computation, where we will explain the steps afterwards,

$$\begin{aligned}
\|a\|^2\langle x,x\rangle - \langle a\cdot x, a\cdot x\rangle &\overset{!}{=} \langle x, \|a\|^2 x\rangle - \langle x, a^*a\cdot x\rangle \\
&= \langle x, \sum_i \|a\|^2 \xi_i \otimes \eta_i - \sum_i (a^*a\cdot \xi_i)\otimes \eta_i\rangle \\
&= \langle x, \sum_i (\|a\|^2 \operatorname{id}_{F_A} - F_{a^*a})_{B_i}\xi_i \otimes \eta_i\rangle \\
&\overset{!!}{=} \langle x, \sum_i (S^*S)_{B_i}(\xi_i)\otimes \eta_i\rangle \\
&= \langle \sum_i S_{B_i}(\xi_i)\otimes \eta_i, \sum_i S_{B_i}(\xi_i)\otimes\eta_i\rangle \\
&\geq 0.
\end{aligned}$$

The equality marked ! follows from the previous calculation, together with functoriality of F. The equality marked !! follows from the following claim.

Claim 4.2.6. *The operator $\|a\|^2 \operatorname{id}_{F_A} - F_{a^*a}$ is positive.*

Proof of claim: Since F is a *-functor, we have $F_{a^*a} = (F_a)^*F_a$, so we can find $T \in \mathcal{L}(F_A)$ such that $T^*T = (F_a)^*F_a$. Note that we have $\|T\|^2 = \|F_a\|^2 \leq \|a\|^2$. The operator T^*T is positive, so $\sigma(T^*T) \subseteq [0, \|T\|^2] \subseteq [0, \|a\|^2]$ and by spectral mapping for polynomials we have

$$\sigma(\|a\|^2 \operatorname{id}_{F_A} - T^*T) = \{\|a\|^2 - \lambda;\ \lambda \in \sigma(T^*T)\}.$$

When $\lambda \in \sigma(T^*T)$, we have the inequality $0 \leq \|T\|^2 - \lambda \leq \|a\|^2 - \lambda$, so that $\|a\|^2 \operatorname{id}_{F_A} - T^*T = \|a\|^2 \operatorname{id}_{F_A} - F_{a^*a}$ is a positive operator. ■

Consequently, this shows that we have the inequality

$$\|\langle a\cdot x, a\cdot x\rangle\| \leq \|a\|^2 \|\langle x,x\rangle\|,$$

so that each map $F_A \boxdot_C G \to F_{A'} \boxdot_C G$ extends to a well defined bounded linear map $F_A \boxtimes_C G \to F_{A'} \boxtimes_C G$. Now we need to verify that we get, for each $a \in \mathcal{A}(A, A')$ an adjointable operator $F_A \otimes G \Rightarrow F_{A'} \otimes G$. If $x \in (F_A \odot G)(C')$ and $y \in (F_{A'} \odot G)(C)$, then using the Cauchy-Schwarz inequality we get

$$\|\langle a\cdot x, y\rangle\| \leq \|a\cdot x\|\|y\| \leq \|a\|\|x\|\|y\|,$$

and this shows that the equality $\langle a\cdot x, y\rangle = \langle x, a^*\cdot y\rangle$ remains true when we take $x \in (F_A \otimes G)(C')$ and $y \in (F_{A'} \otimes G)(C)$, hence we do get an adjointable operator $F_A \otimes G \Rightarrow F_{A'} \otimes G$, and our starting functor $\mathcal{A} \to \mathbf{Mod} - \mathcal{C}$ extends to a *-functor $\mathcal{A} \to \mathbf{Hilb}\text{-}\mathcal{C}$. $\square$

Let's collect a useful fact about the norm on the interior tensor product.

Lemma 4.2.7. *Suppose we have a pair of C^*-categories $\mathcal{A}$ and $\mathcal{B}$, together with $\mathcal{D} \in$ **Hilb**-$\mathcal{A}$ and a right Hilbert bimodule ${}_{\mathcal{A}}F_{\mathcal{B}}$. Then the norm on the interior tensor product $\mathcal{D} \otimes_{\mathcal{A}} F$ satisfies*

$$\|\xi \otimes \eta\| \leq \|\xi\|\|\eta\|,$$

whenever $\xi \otimes \eta$ is an elementary tensor in $(\mathcal{D} \otimes_{\mathcal{A}} F)(B)$ for some $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$.

Proof. Take $\xi \otimes \eta \in (\mathcal{D} \otimes_{\mathcal{A}} F)(B)$, then using the definition of the norm on the tensor product $\mathcal{D} \otimes_{\mathcal{A}} F$, we have

$$\begin{aligned}
\|\xi \otimes \eta\|^2 &= \|\langle \eta, \langle \xi, \xi\rangle_{\mathcal{A}} \cdot \eta\rangle_{\mathcal{B}}\| \\
&= \|\langle \eta, (F_{\langle \xi,\xi\rangle_{\mathcal{A}}})_B \eta\rangle_{\mathcal{B}}\| \\
&\leq \|\eta\|\|(F_{\langle \xi,\xi\rangle_{\mathcal{A}}})_B \eta\| \\
&\leq \|\eta\|\|(F_{\langle \xi,\xi\rangle_{\mathcal{A}}})_B\|\|\eta\| \\
&\overset{!}{\leq} \|\eta\|\|F_{\langle \xi,\xi\rangle_{\mathcal{A}}}\|\|\eta\| \\
&\overset{!!}{\leq} \|\eta\|\|\langle \xi,\xi\rangle_{\mathcal{A}}\|\|\eta\| \\
&= \|\eta\|^2\|\xi\|^2.
\end{aligned}$$

The inequalities stated above follow from standard facts, for example the one marked ! uses the definition of the norm of the bounded adjointable operator $F_{\langle \xi,\xi\rangle_{\mathcal{A}}}$ and the one marked !! uses the fact that *-functors are automatically continuous. $\square$

One more result that will be crucial for us later on is the following, telling us that the interior tensor product of two correspondences is a correspondence.

Lemma 4.2.8. *If ${}_{\mathcal{A}}F_{\mathcal{B}}$ and ${}_{\mathcal{B}}G_{\mathcal{C}}$ are correspondences, then the interior tensor product $F \otimes_{\mathcal{B}} G$ is an $\mathcal{A} - \mathcal{C}$ correspondence.*

Proof. We already know that $F \otimes_{\mathcal{B}} G$ is a right Hilbert $\mathcal{A} - \mathcal{C}$ bimodule, so all that remains to check is that the relevant *-functor $\mathcal{A} \to \mathbf{Hilb}\text{-}\mathcal{C}$ is non-degenerate. For this we will show that this *-functor satisfies condition (ii) in Lemma 3.5.4. If we take $\zeta \in (F \otimes_{\mathcal{B}})(C)$, then for $\epsilon > 0$ we can make an estimate as follows,

$$\|\zeta - \sum_{i=1}^{m} \xi_i \otimes \eta_i\| < \frac{\epsilon}{2},$$

where each $\xi \in F_A(B)$ and each $\eta_i \in {}_C G(B)$. Non-degeneracy of $F : \mathcal{A} \to \textbf{Hilb}\text{-}\mathcal{B}$ (in particular, point (ii) of Lemma 3.5.4) means that for each i we can make another estimate like so,

$$\|\xi_i - \sum_{j_i=1}^{n_i} (F_{a_{j_i}})_B \xi'_{j_i}\| < \frac{\epsilon}{2m\|\eta_i\|}.$$

Then we have

$$\begin{aligned}
\|\sum_i \xi_i \otimes \eta_i - (\sum_{j_i} (F_{a_{j_i}})_B \xi'_{j_i}) \otimes \eta_i\| &= \|\sum_i (\xi_i - (\sum_{j_i} (F_{a_{j_i}})_B \xi'_{j_i})) \otimes \eta_i\| \\
&\leq \sum_i \|(\xi_i - (\sum_{j_i} (F_{a_{j_i}})_B \xi'_{j_i})) \otimes \eta_i\| \\
&\leq \sum_i \|(\xi_i - (\sum_{j_i} (F_{a_{j_i}})_B \xi'_{j_i}))\| \|\eta_i\| \\
&< \sum_i \frac{\epsilon}{2m} \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

So it follows that

$$\|\zeta - \sum_i (\sum_{j_i} (F_{a_{j_i}})_B \xi'_{j_i} \otimes \eta_i)\| \leq \|\zeta - \sum_i \xi_i \otimes \eta_i\| + \|\sum_i \xi_i \otimes \eta_i - (\sum_{j_i} (F_{a_{j_i}})_B \xi'_{j_i}) \otimes \eta_i\| < \epsilon,$$

which confirms that $F \otimes_{\mathcal{B}} G$ is a correspondence. □

We will revisit tensor products later, when we are looking at equivalence bimodules.

Chapter 5

The Category Algebra of a C*-Category

Here we shall review the constructions made in [12]. It's important to note that in the cited paper there is a blanket assumption that C*-categories be unital. We're not making this assumption so we will need to tweak things accordingly.

Notation 5.0.1. In this chapter we are mostly going to be using the same notation as Joachim in [12]. In an attempt to avoid repeating letters in different fonts in our statements, for this chapter we will be naming our C*-categories starting at the letter B, i.e. $\mathcal{B}, \mathcal{C}, \mathcal{D}, \dots$

5.1 The $A(\mathcal{B})$ Construction

This construction associates to a given C*-category, a C*-algebra, in a manner which is only functorial if we restrict our attention to *-functors whose object map is injective.

Definition 5.1.1. Let $\mathcal{B}$ be a C*-category. We construct a *-algebra $A^0(\mathcal{B})$ as follows; we consider the vector space

$$A^0(\mathcal{B}) := \bigoplus_{B, B' \in \mathcal{O}\mathrm{b}(\mathcal{B})} \mathcal{B}(B, B').$$

The elements in this vector space are of course finite sums of morphisms from $\mathcal{B}$, e.g.

$\sum_{k=1}^{m} b_k$, and we define a multiplication on $A^0(\mathcal{B})$ by the rule

$$\mathcal{B}(B'', B''') \times \mathcal{B}(B, B') \ni (b_1, b_2) \mapsto \begin{cases} b_1 \circ b_2 & \text{if } B' = B'', \\ 0 & \text{otherwise.} \end{cases}$$

we then just extend this linearly to the whole of $A^0(\mathcal{B})$. We define an involution on this algebra by

$$\left(\sum_{k=1}^{m} b_k \right)^* = \sum_{k=1}^{m} b_k^*.$$

To try and get a feeling for what the algebra $A^0(\mathcal{B})$ looks like, note that the definition of the vector space can be thought of as the collection of $\mathcal{O}\mathrm{b}(\mathcal{B}) \times \mathcal{O}\mathrm{b}(\mathcal{B})$ matrices, where the B-th row is populated by morphisms with target B, and the B-th colum is populated by morphisms with source B. Consequently, an entry of one of these matrices may be given subscripts, e.g. $b_{B,B'}$, to indicate that it lives in the B-th row and B'-th column, so is a morphism $B' \to B$. With point of view, the multiplication is just matrix multiplication and the involution is a sort of conjugate transpose. Of course this is modulo the technicality that $\mathcal{O}\mathrm{b}(\mathcal{B})$ may not be finite, or even countable, so thinking about matrices indexed by it is potentially painful. We may also find it useful at times to view an element $f \in A^0(\mathcal{B})$ as a sum

$$f = \sum_{B' \in \mathcal{O}\mathrm{b}(\mathcal{B})} \sum_{B \in \mathcal{O}\mathrm{b}(\mathcal{B})} f_{B'B},$$

where $f_{B'B} \in \mathcal{B}(B, B')$ and only finitely many of the $f_{B'B}$ are non-zero.

Example 5.1.2. Consider a pair of C*-algebras B and C, and form their coproduct $B \sqcup C$ where we view B and C as one-object C*-categories. The *-algebra $A^0(B \sqcup C)$ can be seen to be the space of matrices

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

under matrix multiplication and involution

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^* = \begin{pmatrix} a^* & 0 \\ 0 & b^* \end{pmatrix}.$$

Clearly we have an isomorphism $A^0(B \sqcup C) \cong A \oplus B$.

Extending this construction to one which produces C*-algebras will require us to place a suitable norm on $A^0(\mathcal{B})$. In order to do this we first construct a useful Hilbert module over the category $\mathcal{B}$. This module is constructed in [12] for unital C*-categories.

Definition 5.1.3. Let $\mathcal{B}$ be a C*-category, we will construct a right Hilbert $\mathcal{B}$-module $\mathcal{I}$. For $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we let $\mathcal{I}^0(B)$ be the vector space

$$\bigoplus_{B' \in \mathcal{O}\mathrm{b}(\mathcal{B})} \mathcal{B}(B, B').$$

Given a morphism $b \in \mathcal{B}(B, B')$, we define $\mathcal{I}^0(b) : \mathcal{I}^0(B') \to \mathcal{I}^0(B)$ by composition;

$$\mathcal{I}^0(b)\left(\sum_{k=1}^{m} b_k\right) = \sum_{k=1}^{m} b_k \circ b,$$

where $\sum_{k=1}^{m} b_k \in \mathcal{I}^0(B')$. Note that all the composites $b_k \circ b$ are indeed defined. So far this gives us a right $\mathcal{B}$-module

$$\mathcal{I}^0 : \mathcal{B}^{\mathrm{op}} \to \mathbf{Vect}.$$

We define an inner product on this module by the rule

$$(b, b') \mapsto b^* b',$$

where the product b^*b' is taken in the *-algebra $A^0(\mathcal{B})$, and we extend this rule linearly. This will give us a right $\mathcal{B}$-module, equipped with an inner product, that may be completed to give a right Hilbert $\mathcal{B}$-module.

The following lemma has two parts, one easy and one fiddly. This lemma will be useful for us later on.

Lemma 5.1.4. *If $\mathcal{B}$ is a C*-category, then*

*(i) We have a functor $\mathcal{I}^0 : A^0(\mathcal{B}) \to$ **Mod**$-\mathcal{B}$ where the right $\mathcal{B}$-module $\mathcal{I}^0(*)$ is the one constructed above.*

(ii) If $x, y \in \mathcal{I}^0(B')$ and $z \in \mathcal{I}(B)$, then we have the identity

$$x \cdot \langle y, z \rangle = (xy^*) \cdot z,$$

where the product xy^* is taken in $A^0(\mathcal{B})$.

Proof. (i) In a slight abuse of notation, henceforth we will write $\mathcal{I}^0$ for $\mathcal{I}^0(*)$. All that we need to do is check that each $f \in A^0(\mathcal{B})$ induces a module homomorphism $\mathcal{I}^0 \Rightarrow \mathcal{I}^0$, and we define these homomorphisms via left multiplication, so for $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$ we put

$$L^f_B(u) = fu,$$

where the product on the right is being taking in $A^0(\mathcal{B})$. By observing that the $\mathcal{B}$-module action can be viewed as multiplication taking place in $A^0(\mathcal{B})$, it follows immediately that the components $\{L^f_B;\ B \in \mathcal{O}\mathrm{b}(\mathcal{B})\}$ assemble into a module homomorphism.

(ii) Suppose first that $z \in \mathcal{I}^0(B)$. By definition we have $x\cdot\langle y, z\rangle = x\cdot(y^*z)$. Supposing that $x = \sum_i x_i$, $y = \sum_j y_j$ and $z = \sum_k z_k$, this gives us

$$x \cdot (y^*z) = \sum_i x_i \cdot (\sum_j \sum_k y_j^* z_k) = \sum_i (x_i \circ \sum_j \sum_k y_j^* z_k) = \sum_i \sum_j \sum_k x_i \circ (y_j^* z_k).$$

We note that each of the composites $x_i \circ (y_j^* z_k)$ and $x_i \circ y_j^*$ are always defined, and that $y_j^* z_k = 0 \iff (x_i \circ y_j^*)z_k = 0$. Whenever $y_j^* z_k \neq 0$, we clearly have $x_i \circ (y_j^* z_k) = (x_i \circ y_j^*)z_k$, hence we can re-bracket like so,

$$\sum_i \sum_j \sum_k x_i \circ (y_j^* z_k) = \sum_i \sum_j \sum_k (x_i \circ y_j^*) z_k.$$

Where now the product of $x_i \circ y_j^*$ with z_k is being computed in $A^0(\mathcal{B})$. Carrying on, we have

$$\sum_i \sum_j \sum_k (x_i \circ y_j^*) z_k = (\sum_i \sum_j x_i \circ y_j^*) \cdot \sum_k z_k = (xy^*) \cdot z.$$

Finally we take $z \in \mathcal{I}(B)$ and let (z_n) be a sequence in $\mathcal{I}^0(B)$ converging to z, then

$$x \cdot \langle y, z\rangle = \lim_{n\to\infty} x \cdot \langle y, z_n\rangle = \lim_{n\to\infty} (xy^*) \cdot z_n = (xy^*) \cdot z,$$

and we are done.

□

We will use the Hilbert $\mathcal{B}$-module $\mathcal{I}$ to define a C*-norm on $A^0(\mathcal{B})$. Alternatively we could show that the maximal C*-seminorm on this *-algebra is a norm, hence we can complete it to a C*-algebra. This procedure would leave us with the same C*-algebra, for example see the proof of [12, Lemma 3.6] for details of this argument. The following result is a generalisation of [12, Lemma 3.1], for categories which need not be unital.

Proposition 5.1.5. *If $\mathcal{B}$ is a C*-category and $\mathcal{I}$ is the right Hilbert $\mathcal{B}$-module constructed earlier, then $A^0(\mathcal{B})$ is isomorphic to a dense *-subalgebra of $\mathcal{K}(\mathcal{I})$.*

Proof. By the previous lemma we have an algebra homomorphism $A^0(\mathcal{B}) \to \mathrm{Hom}(\mathcal{I}^0, \mathcal{I}^0)$, where $\mathrm{Hom}(\mathcal{I}^0, \mathcal{I}^0)$ denotes the space of module homomorphisms from $\mathcal{I}^0$ to $\mathcal{I}^0$. The first thing that we will do is show that this extends to a *-homomorphism $A^0(\mathcal{B}) \to \mathcal{L}(\mathcal{I})$, then we will show that the image of this *-homomorphism is contained in $\mathcal{K}(\mathcal{I})$, then finally argue that this image is dense.

Claim 5.1.6. *The homomorphism $A^0(\mathcal{B}) \to Hom(\mathcal{I}^0, \mathcal{I}^0)$ extends to a *-homomorphism $A^0(\mathcal{B}) \to \mathcal{L}(\mathcal{I})$.*

Proof of claim: We take $f \in A^0(\mathcal{B})$, then observe that for $u \in \mathcal{I}^0(B)$ we have

$$\|fu\|^2 = \|(fu)^* fu\| = \|u^* f^* fu\| \leq \|u\| \|f^* f\| \|u\| = \|u\|^2 \|f^* f\|,$$

so that the module homomorphism $L^f : \mathcal{I}^0 \Rightarrow \mathcal{I}^0$ extends to a homomorphism $L^f : \mathcal{I} \Rightarrow \mathcal{I}$ and moreover one may easily check that L^f is an adjointable operator with adjoint L^{f^*}, hence we get a *-homomorphism $A^0(\mathcal{B}) \to \mathcal{L}(\mathcal{I})$. ∎

Claim 5.1.7. *For $f \in A^0(\mathcal{B})$, the operator $L^f : \mathcal{I} \Rightarrow \mathcal{I}$ is a finite linear combination of compact operators.*

Proof of claim: Writing $f = \sum_{i=1}^m \sum_{j=1}^n f_{ij}$ with $f_{ij} \in \mathcal{B}(X_j, X_i)$, we fix $j \in \{1, \ldots, n\}$, let $(e^j_\lambda)_\lambda$ be an approximate unit for the C*-algebra $\mathcal{B}(X_j, X_j)$, then consider the net of operators defined by

$$T^j_\lambda = \sum_{i=1}^m f_{ij} \cdot \langle e^j_\lambda, - \rangle.$$

Note that for each i, $f_{ij} \cdot \langle e^j_\lambda, - \rangle$ is simply the rank one operator $\theta_{f_{ij}, e^j_\lambda}$, in particular we can rest assured that T^j_λ is a finite rank operator on $\mathcal{I}$ (so is compact). Note further that we can view the sum $\sum_i f_{ij}$ as an element of $\mathcal{I}(X_j)$, since each morphism f_{ij}

belongs to $\mathcal{B}(X_j, X_i)$. This point of view will prove useful later on. Furthermore, we have

$$\begin{aligned}\|T_\lambda^j - L^{\sum_i f_{ij}}\| &= \sup_{\|b\|=1} \|\sum_i (f_{ij} \cdot \langle e_\lambda^j, b\rangle) - (\sum_i f_{ij})b\| \\ &= \sup_{\|b\|=1} \|\sum_i (f_{ij} e_\lambda^j)b - \sum_i f_{ij} b\|.\end{aligned}$$

We have used part (ii) of the earlier lemma in the second equality here. It follows that

$$\|T_\lambda^j - (\sum_i f_{ij})(-)\| \leq \sup_{\|b\|=1} \|\sum_i f_{ij} e_\lambda^j - \sum_i f_{ij}\|\|b\|.$$

Since $(e_\lambda^j)_\lambda$ is an approximate unit, this shows that $\|T_\lambda^j - (\sum_i f_{ij})(-)\| \to 0$, so the net (T_λ^j) converges to the left multiplication operator $L^{\sum_i f_{ij}}$, and because each T_λ^j is compact it then follows that $L^{\sum_i f_{ij}}$ is compact. We note that

$$L^f = \sum_{j=1} L^{\sum_i f_{ij}},$$

so that L^f is a finite sum of compact operators, thus is itself compact. ■

So far we have shown that we in fact have a *-homomorphism $A^0(\mathcal{B}) \to \mathcal{K}(\mathcal{I})$.

Claim 5.1.8. *The *-homomorphism $A^0(\mathcal{B}) \to \mathcal{K}(\mathcal{I})$ which sends $f \mapsto L^f$ is injective.*

Proof of claim: First we note that if $b \in \mathcal{B}(B, X_j)$, then

$$L_B^f(b) = L_B^{\sum_i f_{ij}}(b).$$

We now suppose that $f = \sum_{i=1}^m \sum_{j=1}^n f_{ij}$ is such that $L^f = 0$, then we fix $j \in \{1, \ldots, n\}$ and let (e_λ^j) be an approximate unit for $\mathcal{B}(X_j, X_j)$. Given $\epsilon > 0$, we can find for each $i \in \{1, \ldots, m\}$ some λ_i for which

$$\lambda \geq \lambda_i \Rightarrow \|f_{ij} - f_{ij} e_\lambda^j\| < \frac{\epsilon}{m}.$$

Majorizing all of the λ_i's then gives us λ_0 such that

$$\lambda \geq \lambda_0 \Rightarrow \|f_{ij} - f_{ij} e_\lambda^j\| < \frac{\epsilon}{m},$$

for all $i \in \{1, \dots, m\}$. This implies that

$$\|\sum_i f_{ij} - \sum_i f_{ij} e^j_\lambda\| \overset{?}{\leq} \sum_i \|f_{ij} - f_{ij} e^j_\lambda\| < m \times \frac{\epsilon}{m} = \epsilon.$$

The inquality marked ? perhaps needs some justification; if we take $\sum_k x_k \in \mathcal{I}(B)$, then by definition of the norm on $\mathcal{I}$, we have $\|f\| = \|\sum_k x_k^* \sum_k x_k\|^{\frac{1}{2}} = \|\sum_k \sum_l x_k^* x_l\|^{\frac{1}{2}}$, and whenever $k \neq l$ the morphisms x_k^* and x_l are non-composable, hence $\|f\| = \|\sum_k x_k^* x_k\|^{\frac{1}{2}} \leq \sum_k \|x_k\|$. From an earlier note, we see that

$$L^{\sum_i f_{ij}}_{X_j}(e^j_\lambda) = L^f_{X_j}(e^j_\lambda) = 0,$$

and once we incorporate this with the inequality we just derived, we see that for any $\epsilon > 0$, $\|\sum_i f_{ij}\| < \epsilon$ so that $\sum_i f_{ij} = 0$. Repeating the argument for each j then shows us that $f = \sum_i \sum_j f_{ij} = \sum_j 0 = 0$. ■

To finish the proof, we need to check one final claim.

Claim 5.1.9. *The image of the *-homomorphism $A^0(\mathcal{B}) \to \mathcal{K}(\mathcal{I})$ is dense (so that the image of this *-homomorphism is the dense*-subalgebra of $\mathcal{K}(\mathcal{I})$ which $A^0(\mathcal{B})$ is isomorphic to).*

Proof of claim: We take $T \in \mathcal{K}(\mathcal{I})$, then by construction of the compact operators, if we are given $\epsilon > 0$ then we may find a sum $\sum_{k=1}^n \theta_{x_k, y_k}$ such that

$$\|T - \sum_{k=1}^n \theta_{x_k, y_k}\| \leq \epsilon.$$

Note here that the x_k, y_k belong to $\mathcal{I}(B')$. If we can show that the finite rank operator $\sum_{k=1}^n \theta_{x_k, y_k}$ can be approximated by a finite rank operator $\sum_{k=1}^m \theta_{x'_k, y'_k}$ where $x'_k, y'_k \in \mathcal{I}^0(B')$, then it will follow that

$$L^{\sum_k x'_k y'^*_k} = \sum_k (x'_k y'^*_k) \cdot - = \sum_k \theta_{x'_k, y'_k}.$$

This step seems to be straightforward. We let $(x_n^{(k)})$ and $(y_n^{(k)})$ be sequences in $\mathcal{I}^0(B)$ converging to x_k and y_k respectively, then we have

$$\begin{aligned}\|\theta_{x_k,y_k} - \theta_{x_n^{(k)},y_n^{(k)}}\| &\leq \sup_{\|z\|=1\|} \|x_k \cdot \langle y_k, z\rangle - \langle x_n^{(k)} \cdot \langle y_k, z\rangle\| + \sup_{\|z\|=1} \|x_n^{(k)} \cdot \langle y_k, z\rangle - x_n^{(k)} \langle y_n^{(k)}, z\rangle\| \\ &\leq \sup_{\|z\|=1} \|x_k - x_n^{(k)}\| \|\langle y, z\rangle\| + \sup_{\|z\|=1} \|x_n^{(k)}\| \|\langle y_k - y_n^{(k)}, z\rangle\| \\ &\leq \|x_k - x_n^{(k)}\| \|y_k\| + \|x_n^{(k)}\| \|y_k - y_n^{(k)}\|,\end{aligned}$$

and the expression on the right hand side converges to 0. Hence the sequence $(\sum_k \theta_{x_n^{(k)},y_n^{(k)}})_{n\in\mathbb{N}}$ converges to $\sum_k \theta_{x_k,y_k}$. Now we can choose our $\sum_{k=1}^m \theta_{x'_k,y'_k}$ such that

$$\|\sum_{k=1}^m \theta_{x_k,y_k} - \sum_{k=1}^m \theta_{x'_k,y'_k}\| < \epsilon,$$

and we get

$$\|T - \sum_{k=1}^n \theta_{x_k,y_k} - \sum_{k=1}^m \theta_{x'_k,y'_k}\| \leq \|T - \sum_{k=1}^n \theta_{x_k,y_k}\| + \|\|\sum_{k=1}^m \theta_{x_k,y_k} - \sum_{k=1}^m \theta_{x'_k,y'_k}\| < 2\epsilon.$$

■

□

Definition 5.1.10. When $\mathcal{B}$ is a C*-category, we may now use the previous proposition to identify $A^0(\mathcal{B})$ as a dense subalgebra of $\mathcal{K}(\mathcal{I})$. We take the closure of $A^0(\mathcal{B})$ and are left with a C*-algebra $A(\mathcal{B})$. We will call this algebra the *category algebra* of $\mathcal{B}$. Clearly we have the identification $A(\mathcal{B}) \cong \mathcal{K}(\mathcal{I})$.

The following is an immediate corollary.

Corollary 5.1.11. *The functor $\mathcal{I} : A^0(\mathcal{B}) \to$ **Mod**$-\mathcal{B}$ extends to a *-functor $A(\mathcal{B}) \to$ **Hilb**-$\mathcal{B}$, hence is a right Hilbert $A(\mathcal{B}) - \mathcal{B}$ bimodule..*

We will revisit this construction after we have introduced equivalence bimodules.

Chapter 6

Equivalence Bimodules

We now arrive at the crux of our work: equivalence bimodules. These are going to be analogues of Rieffel's imprimitivity bimodules, introduced in [27, Definition 6.10] and we will use these bimodules to define Morita equivalences between C*-categories.

6.1 Full Hilbert Bimodules

We need a supplementary definition.

Lemma 6.1.1. *Let $F_{(-)} : \mathcal{A} \to$ **Hilb**-$\mathcal{B}$ be a right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule. For $B, B' \in \mathcal{O}\mathrm{b}(\mathcal{B})$ we define*

$$\mathcal{J}^0_{\mathcal{B}}(B, B') := \mathit{Span}\{\langle y, x\rangle_{\mathcal{B}};\ A \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ y \in F_A(B'),\ x \in F_A(B)\} \subseteq \mathcal{B}(B, B'),$$

then the wide subcategory $\mathcal{J}_{\mathcal{B}} \subseteq \mathcal{B}$ with morphism sets

$$\mathcal{J}_{\mathcal{B}}(B, B') = \overline{\mathcal{J}^0_{\mathcal{B}}(B, B')},$$

is an ideal in $\mathcal{B}$.

Proof. By design, we have that each $\mathcal{J}_{\mathcal{B}}(B, B')$ is a closed subspace of $\mathcal{B}(B, B')$. If we take $\sum_k \langle \xi_k, \eta_k\rangle \in \mathcal{J}^0_{\mathcal{B}}(B, B')$, $b \in \mathcal{B}(B', B'')$ and $b' \in \mathcal{B}(B'', B)$ then we have

$$b \sum_k \langle \xi_k, \eta_k\rangle = \sum_k \langle \xi_k \cdot b^*, \eta_k\rangle \in \mathcal{J}^0_{\mathcal{B}}(B, B''),$$

and

$$\left(\sum_k \langle \xi_k, \eta_k \rangle\right) b' = \sum_k \langle \xi_k, \eta_k \cdot b' \rangle \in \mathcal{J}^0_{\mathcal{B}}(B'', B').$$

So if $f \in \mathcal{J}_{\mathcal{B}}(B, B')$ with a sequence (f_n) in $\mathcal{J}^0_{\mathcal{B}}(B, B')$ converging to f, then by the above, the sequence (bf_n) lies in $\mathcal{J}^0_{\mathcal{B}}(B, B'')$, so it follows that $bf \in \mathcal{J}_{\mathcal{B}}(B, B'')$. Similarly, $fb' \in \mathcal{J}_{\mathcal{B}}(B'', B')$. Hence we can conclude that $\mathcal{J}_{\mathcal{B}}$ is an ideal in $\mathcal{B}$. □

Definition 6.1.2. We'll say that a right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule $F_{(-)} : \mathcal{A} \to \mathbf{Hilb} - \mathcal{B}$ is *full* if the ideal $\mathcal{J}_{\mathcal{B}}$ is isomorphic to $\mathcal{B}$. With an appropriate tweak to the statement of Lemma 6.1.1 we also get a notion of fullness for left Hilbert bimodules.

6.2 The Definition and Examples

Definition 6.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-categories. An $\mathcal{A} - \mathcal{B}$ *equivalence bimodule* is a bi-Hilbert $\mathcal{A} - \mathcal{B}$ bimodule ${}_{\mathcal{A}}F_{\mathcal{B}}$, such that

(i) The right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule $F_{\mathcal{B}}$ is full in the sense of Definition 6.1.2,

(ii) The left Hilbert $\mathcal{B} - \mathcal{A}$ bimodule ${}_{\mathcal{A}}F$ is full in the sense of Definition 6.1.2,

(iii) For all $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$ and $B, B' \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we have the identity

$${}_{\mathcal{A}}\langle \xi, \eta \rangle \cdot \zeta = \xi \cdot \langle \eta, \zeta \rangle_{\mathcal{B}},$$

where

$$\xi \in F_{A'}(B'), \qquad \eta \in F_A(B'), \qquad \zeta \in F_A(B).$$

We will say that a pair of C*-categories $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* if there exists an equivalence bimodule ${}_{\mathcal{A}}F_{\mathcal{B}}$. In this case we may also say that there is an equivalence bimodule between $\mathcal{A}$ and $\mathcal{B}$.

We'd best show that there are plenty of examples of this phenomenon.

Example 6.2.2. If $\mathcal{A}$ is a C*-category, then recall from Example 3.1.9 that we have a *-functor

$$\mathcal{A}_{\mathcal{A}} : \mathcal{A} \to \mathbf{Hilb}\text{-}\mathcal{A}.$$

The left action induced by this functor sends $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ to the covariant hom functor $\mathcal{A}(A,-)$, which is a left $\mathcal{A}$-module. We define an inner product by

$$_{\mathcal{A}}\langle a', a\rangle = a'a^*.$$

With $a \in \mathcal{A}(A, A')$, the string of equalities

$$\|_{\mathcal{A}}\langle a, a\rangle\|^{\frac{1}{2}} = \|aa^*\|^{\frac{1}{2}} = \|(a^*)^*a^*\|^{\frac{1}{2}} = \|a^*\| = \|a\|,$$

shows that the norm on $\mathcal{A}(A, A')$ induced by the inner product coincides with the existing norm, so that $\mathcal{A}(A,-)$ is a left Hilbert $\mathcal{A}$-module. Similarly to Example 2.5.1, one may check that a morphism $a \in \mathcal{A}(A', A)$ defines a bounded adjointable operator $\mathcal{A}(A,-) \Rightarrow \mathcal{A}(A',-)$ with components

$$\mathcal{A}(A, A'') \xrightarrow{(-)\circ a} \mathcal{A}(A', A''), \qquad \mathcal{A}(A', A'') \xrightarrow{(-)\circ a^*} \mathcal{A}(A, A'').$$

Therefore, the left action associated with this bimodule is a *-functor

$$_{\mathcal{A}}\mathcal{A} : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}\text{-}\mathbf{Hilb}.$$

To check that conditions (i) and (ii) of Definition 6.2.1 hold true, we let (u_λ) be a (self adjoint) approximate unit for $\mathcal{A}(A, A)$, then for any $a \in \mathcal{A}(A, A')$ and $a' \in \mathcal{A}(A', A)$, we have

$$_{\mathcal{A}}\langle a, u_\lambda\rangle = au_\lambda, \qquad \langle u_\lambda, a'\rangle_{\mathcal{A}} = u_\lambda a'.$$

So if we consider the sets

$$\mathrm{span}\{_{\mathcal{A}}\langle x, y\rangle;\ x \in \mathcal{A}(A, A'),\ y \in \mathcal{A}(A, A)\}, \qquad \mathrm{span}\{\langle x, y\rangle_{\mathcal{A}};\ x \in \mathcal{A}(A, A),\ y \in \mathcal{A}(A', A)\},$$

then we see that we have what we need; for example if we look at the left hand set, then because au_λ belongs to the stated span for all $a \in \mathcal{A}(A, A')$, then the closure of this set is $\mathcal{A}(A, A')$. It follows that the ideal $\mathcal{J}$ in $\mathcal{A}$ with morphism sets

$$\mathcal{J}(A, A') = \overline{\mathrm{span}\{_{\mathcal{A}}\langle x, y\rangle;\ A'' \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ x \in \mathcal{A}(A'', A'),\ y \in \mathcal{A}(A'', A)\}},$$

is isomorphic to $\mathcal{A}$, so that the bimodule $_{\mathcal{A}}\mathcal{A}$ is full. Finally, to check condition (iii) we have

$$_{\mathcal{A}}\langle a, a'\rangle \cdot a'' = (a(a')^*)a'' = a((a')^*a'') = a \cdot \langle a', a''\rangle_{\mathcal{A}},$$

for all $a \in \mathcal{A}(A''', A')$, $a' \in \mathcal{A}(A''', A)$ and $a'' \in \mathcal{A}(A'', A)$. So it follows that any C*-category is Morita equivalent to itself.

Example 6.2.3. Suppose that $\phi : \mathcal{A} \to \mathcal{B}$ is a unitary equivalence between unital C*-categories. We have seen in Example 3.1.12 how this gives a bi-Hilbert $\mathcal{A} - \mathcal{B}$ bimodule $\Phi_{(-)} : \mathcal{A} \to \textbf{Hilb}\text{-}\mathcal{B}$. Now we will verify the other properties required of an equivalence bimodule. Given $A, A' \in \mathcal{O}\text{b}(\mathcal{A})$, we claim that

$$\operatorname{span}\{{}_{\mathcal{A}}\langle x, y\rangle;\ x \in \mathcal{B}(\phi(A), \phi(A')),\ y \in \mathcal{B}(\phi(A), \phi(A))\},$$

is dense in $\mathcal{A}(A, A')$. Indeed for $a \in \mathcal{A}(A, A')$ one has

$${}_{\mathcal{A}}\langle \phi(a), \mathrm{id}_{\phi(A)}\rangle = \phi^{-1}(\phi(a)\phi(\mathrm{id}_A)) = a.$$

We further claim that if $B, B' \in \mathcal{O}\text{b}(\mathcal{B})$, and $A \in \mathcal{O}\text{b}(\mathcal{A})$ is chosen such that $\phi(A) \cong B'$ the set

$$\operatorname{span}\{\langle x, y\rangle_{\mathcal{B}};\ x \in \mathcal{B}(B', \phi(A)),\ y \in \mathcal{B}(B, \phi(A))\},$$

is dense in $\mathcal{B}(B, B')$. For this, we take $b \in \mathcal{B}(B, B')$, then we let $f : \phi(A) \to B'$ denote the relevant unitary isomorphism, we have

$$\langle f^*, f^*b\rangle_{\mathcal{B}} = ff^*b = b,$$

and as before, the density claim now follows. To check the final property, we take

$$b' \in \mathcal{B}(B', \phi(A)), \qquad b \in \mathcal{B}(B', \phi(A)), \qquad b'' \in \mathcal{B}(B, \phi(A)),$$

then we have

$$\begin{aligned}
{}_{\mathcal{A}}\langle b', b\rangle \cdot b'' &= \phi^{-1}(b'b^*) \cdot b'' \\
&= \phi\phi^{-1}(b'b^*)b'' \\
&= b'b^*b'' \\
&= b'(b^*b'') \\
&= b' \cdot \langle b, b''\rangle_{\mathcal{B}}.
\end{aligned}$$

Remark 6.2.4. Since an isomorphism of categories is an equivalence of categories, this shows that isomorphic C*-categories are also Morita equivalent.

Example 6.2.5. Suppose that we have an $\mathcal{A} - \mathcal{B}$ equivalence bimodule ${}_{\mathcal{A}}F_{\mathcal{B}}$ and consider the conjugate bimodule ${}_{\mathcal{B}}\widetilde{F}_{\mathcal{A}}$. Recall that the conjugate bimodule has left and right actions given by the following composites

$$
\begin{aligned}
{}_{(-)}\widetilde{F} &: \mathcal{A}^{\mathrm{op}} \xrightarrow{(-)^*} \mathcal{A} \xrightarrow{F_{(-)}} \textbf{Hilb-}\mathcal{B} \xrightarrow{\widetilde{(-)}} \mathcal{B}\textbf{-Hilb} \\
\widetilde{F}_{(-)} &: \mathcal{B} \xrightarrow{(-)^*} \mathcal{B}^{\mathrm{op}} \xrightarrow{{}_{(-)}F} \mathcal{A}\textbf{-Hilb} \xrightarrow{\widetilde{(-)}} \textbf{Hilb-}\mathcal{A}.
\end{aligned}
$$

We will show that this bimodule is a $\mathcal{B}-\mathcal{A}$ equivalence bimodule. First we note that the conjugate module is clearly a bi-Hilbert $\mathcal{B}-\mathcal{A}$ bimodule. We will show that the right action $\widetilde{F}_{(-)}$ is full, so we need to check that for all $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$, we have

$$\mathcal{A}(A, A') = \overline{\mathrm{Span}}\{\langle \flat(y), \flat(x)\rangle_{\mathcal{A}};\ B \in \mathcal{O}\mathrm{b}(\mathcal{B}),\ \flat(y) \in \widetilde{F}_B(A'),\ \flat(x) \in \widetilde{F}_B(A)\}.$$

Recall that the inner product $\langle \flat(y), \flat(x)\rangle_{\mathcal{A}}$ is defined by

$$\langle \flat(y), \flat(x)\rangle_{\mathcal{A}} = {}_{\mathcal{A}}\langle y, x\rangle.$$

So because the $\flat$ maps are isomorphisms, and the left action ${}_{(-)}F$ is full, the claim follows. Similarly one gets fullness of the left action ${}_{(-)}\widetilde{F}$.

This leaves us only one property left to check; that we have the identity

$${}_{\mathcal{B}}\langle \flat(x), \flat(y)\rangle \cdot \flat(z) = \flat(x) \cdot \langle \flat(y), \flat(z)\rangle_{\mathcal{A}},$$

where

$$\flat(x) \in \widetilde{F}_{B'}(A'), \qquad \flat(y) \in \widetilde{F}_B(A'), \qquad \flat(z) \in \widetilde{F}_B(A).$$

To this end, we have

$$
\begin{aligned}
{}_{\mathcal{B}}\langle \flat(z), \flat(y)\rangle \cdot \flat(x) &= x \cdot \langle z, y\rangle_{\mathcal{B}}^* \\
&= x \cdot \langle y, z\rangle_{\mathcal{B}} \\
&= {}_{\mathcal{A}}\langle x, y\rangle \cdot z \\
&= \flat(z) \cdot \langle \flat(y), \flat(x)\rangle_{\mathcal{A}}.
\end{aligned}
$$

Example 6.2.6. This will be an interesting example, and one which shows that Morita equivalence is weaker than equivalence of categories. We will show that the algebra $\mathcal{K}$ of compact operators is Morita equivalent to $\mathbb{C}_{\oplus}$, the additive completion of $\mathbb{C}$. Ultimately we are going to construct a functor $F : \mathcal{K} \to \textbf{Hilb-}\,\mathbb{C}_{\oplus}$, such a functor is going to pick out a single right Hilbert $\mathbb{C}_\oplus$-module, so we begin by coming up with a functor

$$\mathcal{D} : \mathbb{C}_\oplus^{\mathrm{op}} \to \mathrm{Vect}.$$

We let $\mathcal{D}$ have object map

$$n \mapsto \bigoplus_{k=1}^{n} \ell^2(\mathbb{N}),$$

for each $n \in \mathcal{O}\mathrm{b}(\mathbb{C}_\oplus) = \mathbb{N}$, and if (a_{ij}) is a $n \times m$ matrix (so a morphism $m \to n$), we put

$$\mathcal{D}(a_{ij})(x_1, \ldots, x_n) := (x_1, \ldots, x_n)(a_{ij}) = \left(\sum_{i=1}^{n} x_i a_{i1}, \ldots, \sum_{i=1}^{n} x_i a_{im} \right).$$

Note that each x_i is a *sequence* in $\ell^2(\mathbb{N})$ and that each a_{ij} is merely a complex number, so each term in the expression on the right hand side is a finite sum of sequences, each of which has been scalar multiplied by a complex number. This defines a linear functor $\mathcal{D} : \mathbb{C}_\oplus^{\mathrm{op}} \to \mathrm{Vect}$.

Now we need an inner product. Denote by $\langle -, - \rangle_{\ell^2}$ the standard inner product on $\ell^2(\mathbb{N})$, then for each pair $(m, n) \in \mathbb{N} \times \mathbb{N}$, we define $\langle -, - \rangle : F(n) \times F(m) \to \mathbb{C}_\oplus(m, n)$ by

$$\langle (x_1, \ldots, x_n), (y_1 \ldots, y_m) \rangle = \begin{pmatrix} \langle x_1, y_1 \rangle_{\ell^2} & \cdots & \langle x_1, y_m \rangle_{\ell^2} \\ \langle x_2, y_1 \rangle_{\ell^2} & \cdots & \langle x_2, y_m \rangle_{\ell^2} \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle_{\ell^2} & \cdots & \langle x_n, y_m \rangle_{\ell^2} \end{pmatrix}.$$

The final thing to check is that for each $n \in \mathbb{N}$, the space $\mathcal{D}(n)$ is complete with respect to the norm induced by this inner product. It's probably not that hard to check this directly, however we'll proceed as follows; for $n \in \mathcal{D}(n)$, similarly to [2, Lemma 11] we see that $\mathcal{D}(n)$ is a $\mathcal{K} - \mathbb{M}_n(\mathbb{C})$ imprimitivity bimodule. Consequently, the two norms induced by the left and right Hilbert module structures coincide, and the $\mathbb{M}_n(\mathbb{C})$ valued inner product is precisely the one we've defined above, so we have the required completeness.

This gives us the behaviour of our functor F on objects; the single object of $\mathcal{K}$ will be sent to $\mathcal{D}$. Now we'll show how a compact operator $T \in \mathcal{K}$ defines a bounded adjointable operator $F(T) : \mathcal{D} \Rightarrow \mathcal{D}$. The obvious guess for the components of such an operator is to put $F(T)_n(x_1, \dots, x_n) = (Tx_1, \dots, Tx_n)$, then we have

$$\begin{aligned}\langle (x_1, \dots, x_n), F(T)_m(y_1, \dots, y_m)\rangle &= \langle (x_1, \dots, x_n), (Ty_1, \dots, Ty_m)\rangle \\ &= \begin{pmatrix} \langle x_1, Ty_1\rangle & \cdots & \langle x_1, Ty_m\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, Ty_1\rangle & \cdots & \langle x_n, Ty_m\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle T^*x_1, y_1\rangle & \cdots & \langle T^*x_1, y_m\rangle \\ \vdots & \ddots & \vdots \\ \langle T^*x_n, y_1\rangle & \cdots & \langle T^*x_n, y_m\rangle \end{pmatrix} \\ &= \langle (T^*x_1, \dots, T^*x_n), (y_1, \dots, y_m)\rangle.\end{aligned}$$

So if we define $F(T)^*_n(x_1, \dots, x_n) = (T^*x_1, \dots, T^*x_n)$ then we get what we need. Note further that $F(T) = F(T^*)$ and that F respects compositions. This shows that F is a *-functor and so is a right Hilbert $\mathcal{K} - \mathbb{C}_\oplus$ bimodule. Now we can start to check that the extra properties required of an equivalence bimodule are satisfied. The left action of this bimodule is the induced functor ${}_{\mathcal{K}}F : \mathbb{C}_\oplus^{\mathrm{op}} \to \mathcal{K} - \mathbf{Mod}$ where $n \in \mathcal{O}\mathrm{b}(\mathbb{C}_\oplus)$ is sent to the functor with object map

$$*_{\mathcal{K}} \mapsto [F_{\mathbb{C}_\oplus}(*_{\mathcal{K}})](n) = \bigoplus_{k=1}^{n} \ell^2(\mathbb{N}),$$

and with the action given by component-wise application of a given compact operator $T \in \mathcal{K}$,

$$T \mapsto F_{\mathbb{C}_\oplus}(T)_n = (T(-), \dots, T(-)).$$

We can define an inner product on this module by

$${}_{\mathcal{K}}\langle (x_1, \dots, x_n), (y_1, \dots, y_n)\rangle = \sum_{k=1}^{n} {}_{\mathcal{K}}\langle x_k, y_n\rangle,$$

showing that the left action ${}_{\mathcal{K}}F$ at least sends the objects of $\mathbb{C}_\oplus$ to left Hilbert $\mathcal{K}$-modules. We also need to check that each morphism $(a_{ij}) \in \mathbb{C}_\oplus(m, n)$ defines a bounded adjointable operator ${}_{\mathcal{K}}F(n) \Rightarrow {}_{\mathcal{K}}F(m)$, but this is straightforward. The operator ${}_{\mathcal{K}}F(a_{ij})$ is simply the map $\mathcal{D}(a_{ij})$ defined earlier, which is an operator between Hilbert spaces is is automatically adjointable, and is trivially bounded.

6.2.1 Pre-equivalence Bimodules

He we introduce some tools so that we can construct equivalence bimodules via completions. We will closely follow the exposition in [26, Chapter 3.1]. First we will need some supplementary terminology and definitions.

Definition 6.2.7. If $\mathcal{A}$ is a C*-category then we will call a wide *-subcategory $\mathcal{A}^0$ dense if for each pair $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$, the space $\mathcal{A}^0(A, A')$ is dense in $\mathcal{A}(A, A')$. We define dense ideals similarly.

Remark 6.2.8. Note that in the above definition, we are talking about purely algebraic subcategories and ideals, we are *not* assuming that they be complete.

Definition 6.2.9. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-categories with dense *-subcategories $\mathcal{A}^0 \subseteq \mathcal{A}$ and $\mathcal{B}^0 \subseteq \mathcal{B}$. Suppose that we have a functor $F_{(-)} : \mathcal{A}^0 \to \mathbf{Mod} - \mathcal{B}^0$ such that each right $\mathcal{B}^0$ module F_A carries a $\mathcal{B}^0$-valued inner product. We define

$$\mathcal{J}^0_{\mathcal{B}^0}(B, B') = \mathrm{Span}\{\langle y, x\rangle_{\mathcal{B}};\ A \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ y \in F_A(B'),\ x \in F_A(B)\}.$$

This gives us an ideal $\mathcal{J}^0_{\mathcal{B}^0}$ in $\mathcal{B}^0$ and we will call the functor F *full* if this ideal is dense in $\mathcal{B}$. Similarly, we can define what it means for a functor of the form ${}_{(-)}F : (\mathcal{B}^0)^{\mathrm{op}} \to \mathcal{A}^0 - \mathbf{Mod}$ to be full.

Now we present a lemma which will give us a slightly different characterisation of equivalence bimodules that we will make use of in defining our pre-equivalence bimodules.

Lemma 6.2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-categories. Suppose that we have a functor*[1] *$F_{(-)} : \mathcal{A} \to$ **Hilb**-$\mathcal{B}$ such that*

(i) $F_{(-)}$ is full,

*(ii) The associated functor ${}_{(-)}F : \mathcal{B}^{op} \to \mathcal{A} -$ **Mod** is a functor into $\mathcal{A}$-**Hilb**, and is full,*

(iii) For all $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$ and $B, B' \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we have the identity

$${}_{\mathcal{A}}\langle \xi, \eta\rangle \cdot \zeta = \xi \cdot \langle \eta, \zeta\rangle_{\mathcal{B}},$$

where

$$\xi \in F_{A'}(B'), \qquad \eta \in F_A(B'), \qquad \zeta \in F_A(B).$$

[1] Note we don't say *-*functor* here.

Then $F_{(-)}$ is an equivalence bimodule[2] *if and only if for all $a \in \mathcal{A}(A, A')$, $b \in \mathcal{B}(B, B')$ and $\eta \in {}_{B'}F(A)$ we have*

$$\langle a \cdot \eta, a \cdot \eta \rangle_{\mathcal{B}} \leq \|a\|^2 \langle \eta, \eta \rangle_{\mathcal{B}}, \qquad {}_{\mathcal{A}}\langle \eta \cdot b, \eta \cdot b \rangle \leq \|b\|^2 {}_{\mathcal{A}}\langle \eta, \eta \rangle.$$

Proof. If $F_{(-)}$ is an equivalence bimodule, then using similar arguments to Claim 4.2.6 and [26, Corollary 2.22] then the condition stated above is automatically satisfied.

On the other hand, if we take

$$\xi \in F_A(B'), \qquad \eta \in F_A(B), \qquad \mu \in F_{A'}(B'), \qquad \nu \in F_A(B),$$

then we have

$$\begin{aligned} {}_{\mathcal{A}}\langle \mu \cdot \langle \xi, \eta \rangle_{\mathcal{B}}, \nu \rangle &= {}_{\mathcal{A}}\langle {}_{\mathcal{A}}\langle \mu, \xi \rangle \cdot \eta, \nu \rangle \\ &= {}_{\mathcal{A}}\langle \mu, \xi \rangle_{\mathcal{A}}\langle \eta, \nu \rangle \\ &= {}_{\mathcal{A}}\langle \mu, {}_{\mathcal{A}}\langle \nu, \eta \rangle \cdot \xi \rangle \\ &= {}_{\mathcal{A}}\langle \mu, \nu \cdot \langle \eta, \xi \rangle_{\mathcal{B}} \rangle. \end{aligned}$$

From this, it follows that we have ${}_{\mathcal{A}}\langle \mu \cdot b, \nu \rangle = {}_{\mathcal{A}}\langle \mu, \nu \cdot b^* \rangle$, whenever $b \in \mathcal{J}^0_{\mathcal{B}}(B, B')$. Our assumed inequality implies that $\|\eta \cdot b\| \leq \|\eta\| \|b\|$, so from the Cauchy-Schwarz inequality we deduce that

$$\|{}_{\mathcal{A}}\langle \mu \cdot b, \nu \rangle\| \leq \|\mu\| \|b\| \|\nu\|,$$

where b is still an element of $\mathcal{J}^0_{\mathcal{B}}(B, B')$. This means that the identity ${}_{\mathcal{A}}\langle \mu \cdot b, \nu \rangle = {}_{\mathcal{A}}\langle \mu, \nu \cdot b^* \rangle$ extends by continuity to all $b \in \mathcal{B}(B, B')$. One may run a similar argument for the other case, and we are done. $\square$

Now we can make a definition.

Definition 6.2.11. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-categories with dense *-subcategories $\mathcal{A}^0 \subseteq \mathcal{A}$ and $\mathcal{B}^0 \subseteq \mathcal{B}$. We will call a functor $F_{(-)} : \mathcal{A}^0 \to \mathbf{Mod} - \mathcal{B}^0$ an $\mathcal{A}^0 - \mathcal{B}^0$ pre-equivalence bimodule if

(i) Each right $\mathcal{B}^0$-module F_A is equipped with a right $\mathcal{B}^0$-valued inner product,

(ii) Each left $\mathcal{A}^0$-module ${}_BF$ is equipped with a left $\mathcal{A}^0$-valued inner product,

[2] Equivalently, the functors $F_{(-)}$ and ${}_{(-)}F$ are *-functors.

(iii) The functors ${}_{(-)}F$ and $F_{(-)}$ are full,

(iv) For all $a \in \mathcal{A}^0(A, A')$, $b \in \mathcal{B}^0(B, B')$ and $\xi \in {}_{B'}F(A)$ we have

$$\langle a \cdot \xi, a \cdot \xi \rangle_{\mathcal{B}^0} \leq \|a\|^2 \langle \xi, \xi \rangle_{\mathcal{B}^0}, \qquad {}_{\mathcal{A}^0}\langle \xi \cdot b, \xi \cdot b \rangle \leq \|b\|^2 {}_{\mathcal{A}^0}\langle \xi, \xi \rangle,$$

in the categories $\mathcal{B}$ and $\mathcal{A}$ respectively,

(v) For all $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A}^0)$ and $B, B' \in \mathcal{O}\mathrm{b}(\mathcal{B}^0)$, we have

$${}_{\mathcal{A}^0}\langle \xi, \eta \rangle \cdot \zeta = \xi \cdot \langle \eta, \zeta \rangle_{\mathcal{B}^0},$$

where

$$\xi \in F_{A'}(B'), \qquad \eta \in F_A(B'), \qquad \zeta \in F_A(B).$$

Lemma 6.2.12. *If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories with dense *-subcategories $\mathcal{A}^0 \subseteq \mathcal{A}$ and $\mathcal{B}^0 \subseteq \mathcal{B}$, and if $F(-) : \mathcal{A}^0 \to \boldsymbol{Mod} - \mathcal{B}^0$ is an $\mathcal{A}^0 - \mathcal{B}^0$ pre-equivalence bimodule, then for each pair $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we have*

$$\|{}_{\mathcal{A}^0}\langle \eta, \eta \rangle\| = \|\langle \eta, \eta \rangle_{\mathcal{B}^0}\|,$$

for all $\eta \in F_A(B) = {}_BF(A)$. Hence the two norms on each space $F_A(B)$ induced by the left and right inner products coincide.

Proof. We take $\eta \in F_A(B)$, then compute;

$$\begin{aligned}
\|\langle \eta, \eta \rangle_{\mathcal{B}^0}\|^2 &= \|\langle \eta, \eta \rangle_{\mathcal{B}^0} \langle \eta, \eta \rangle_{\mathcal{B}^0}\| \\
&= \|\langle \eta, \eta \cdot \langle \eta, \eta \rangle_{\mathcal{B}^0} \rangle_{\mathcal{B}^0}\| \\
&= \|\langle \eta, {}_{\mathcal{A}^0}\langle \eta, \eta \rangle \cdot \eta \rangle_{\mathcal{B}^0}\| \\
&\leq \|\langle \eta, \eta \rangle_{\mathcal{B}^0}\|^{\frac{1}{2}} \|\langle {}_{\mathcal{A}^0}\langle \eta, \eta \rangle \cdot \eta, {}_{\mathcal{A}^0}\langle \eta, \eta \rangle \cdot \eta \rangle_{\mathcal{B}^0}\|^{\frac{1}{2}} \\
&\leq \|\langle \eta, \eta \rangle_{\mathcal{B}^0}\|^{\frac{1}{2}} \|{}_{\mathcal{A}^0}\langle \eta, \eta \rangle\| \|\langle \eta, \eta \rangle_{\mathcal{B}^0}\|^{\frac{1}{2}}.
\end{aligned}$$

After cancelling, we are left with the inequality $\|\langle \eta, \eta \rangle_{\mathcal{B}^0}\| \leq \|{}_{\mathcal{A}^0}\langle \eta, \eta \rangle\|$. If we run through the argument for the other case, we get what we want. □

This Lemma tells us that if we were attempt to complete a pre-equivalence bimodule it doesn't matter which induced norm we use to complete the spaces.

Corollary 6.2.13. *If $\mathcal{A}$ and $\mathcal{B}$ are C^*-categories with dense *-subcategories $\mathcal{A}^0 \subseteq \mathcal{A}$ and $\mathcal{B}^0 \subseteq \mathcal{B}$, and if $F(-) : \mathcal{A}^0 \to \boldsymbol{Mod} - \mathcal{B}^0$ is an $\mathcal{A}^0 - \mathcal{B}^0$ pre-equivalence bimodule,* *then by completing each space $F_A(B)$ with respect to the induced norm, we end up with an $\mathcal{A}-\mathcal{B}$ equivalence bimodule.*

The positivity condition in point (iv) of a pre-equivalence bimodule is a bit awkward to work with. When are considering bimodules over C*-categories where we have completeness of the categories, but possibly not the bimodule, then we have the following alternative.

Lemma 6.2.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-categories and let $F_{(-)} : \mathcal{A} \to \boldsymbol{Mod} - \mathcal{B}$ be a functor satisfying points (i), (ii), (iii) and (v) in Definition 6.2.11, then F is a pre-equivalence bimodule if and only if for all $a \in \mathcal{A}(A, A')$, $\xi, \in F_A(B')$ and $\eta \in F_{A'}(B)$ we have*

$$\langle a \cdot \xi, \eta\rangle_{\mathcal{B}} = \langle \xi, a^* \cdot \eta\rangle_{\mathcal{B}},$$

and for all $b \in \mathcal{B}(B, B')$, $\nu \in {}_{B'}F(A')$ and $\zeta \in {}_{B}F(A)$ we have

$$_{\mathcal{A}}\langle \nu \cdot b, \zeta\rangle = {}_{\mathcal{A}}\langle \nu, \zeta \cdot b^*\rangle.$$

Proof. First assume that the two new equalities stated in this Lemma are satisfied. Writing $\mathcal{A}^+$ for the minimal unitisation[3] of $\mathcal{A}$ we will extend $F_{(-)}$ to a functor $F_{(-)} : \mathcal{A}^+ \to \mathbf{Mod} - \mathcal{B}$. Recall that when constructing $\mathcal{A}^+$ we only add new morphisms to the endomorphism sets of $\mathcal{A}$. We define our extension to do the same thing on objects, and send

$$\mathcal{A}^+(A, A') \ni a \mapsto F_{(a)}, \qquad \mathcal{A}^+(A, A) \ni (a, \lambda) \mapsto F_{(a)} + \lambda \operatorname{id}_{F_A}.$$

If $(a, \lambda) \in \mathcal{A}^+(A, A)$, then note that

$$\langle (a, \lambda) \cdot \xi, \eta\rangle_{\mathcal{B}} = \langle a \cdot \xi + \lambda\xi, \eta\rangle_{\mathcal{B}} = \langle a \cdot \xi, \eta\rangle_{\mathcal{B}} + \lambda\langle \xi, \eta\rangle_{\mathcal{B}} = \langle \xi, a^* \cdot \eta + \overline{\lambda}\eta\rangle_{\mathcal{B}},$$

so it follows that for all $a \in \mathcal{A}^+(A, A')$, $\xi, \in F_A(B')$ and $\eta \in F_{A'}(B)$ we have

$$\langle a \cdot \xi, \eta\rangle_{\mathcal{B}} = \langle \xi, a^* \cdot \eta\rangle_{\mathcal{B}}.$$

In particular, we have $\langle \xi, a^*a \cdot \eta\rangle_{\mathcal{B}} = \langle a \cdot \xi, a \cdot \eta\rangle_{\mathcal{B}}$ whenever $a \in \mathcal{A}(A, A')$, $\xi \in F_A(B')$ and $\eta \in F_A(B)$. With a as in the previous sentence, a similar argument to that used in Claim 4.2.6 implies that $\|a\|^2 \operatorname{id}_{\mathcal{A}^+(A,A)} - a^*a$ is positive in $\mathcal{A}^+(A, A)$, and hence we

[3] See for example [23].

have

$$\|a\|^2\langle\xi,\xi\rangle_{\mathcal{B}} - \langle a\cdot\xi, a\cdot\xi\rangle_{\mathcal{B}} = \langle\xi, (\|a\|^2\,\mathrm{id}_{\mathcal{A}^+(A,A)} - a^*a)\cdot\xi\rangle_{\mathcal{B}} \geq 0.$$

Similarly by considering the related functor ${}_{(-)}F : \mathcal{B}^{\mathrm{op}} \to \mathcal{A} - \mathbf{Mod}$ one proves the other identity. The converse follows by re-using the proof of the second half of Lemma 6.2.10. □

Let's conclude this section with an example of a pre-equivalence bimodule. Recall that so far we have seen that Morita equivalence is a symmetric and reflexive relation. This example will show that it is transitive, and hence that Morita equivalence defines an equivalence relation. This will be a crucial tool for us.

Example 6.2.15. Suppose that we have equivalence bimodules ${}_{\mathcal{A}}F_{\mathcal{B}}$ and ${}_{\mathcal{B}}G_{\mathcal{C}}$. We will use Lemma 6.2.14 to show that their algebraic tensor product $F \odot_{\mathcal{B}} G$ is an $\mathcal{A} - \mathcal{C}$ pre-equivalence bimodule. We have already seen how to construct a right $\mathcal{C}$-valued inner product on each right $\mathcal{C}$-module $(F \odot_{\mathcal{B}} G)_A$, and how to construct a left $\mathcal{A}$-valued inner product on each left $\mathcal{A}$-module ${}_C(F \odot_{\mathcal{B}} G)$. This verifies the first two points required of a pre-equivalence bimodule. Now we will show that the relevant functors are full, so for a pair $C, C' \in \mathcal{O}\mathrm{b}(\mathcal{C})$, we must show that the set

$$\mathrm{Span}\{[\xi,\eta]_{\mathcal{C}};\ A \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ \xi \in (F \odot_{\mathcal{B}} G)_A(C'),\ \eta \in (F \odot_{\mathcal{B}} G)_A(C)\},$$

is dense in $\mathcal{C}(C, C')$. Note that because $G_{(-)}$ is full, we know that the set

$$\mathrm{Span}\{\langle\alpha,\beta\rangle_{\mathcal{C}};\ B \in \mathcal{O}\mathrm{b}(\mathcal{B}),\ \alpha \in G_B(C'),\ \beta \in G_B(C)\},$$

is dense in $\mathcal{C}(C, C')$. We will now proceed as follows;

Claim 6.2.16. *For $C, C' \in \mathcal{O}\mathrm{b}(\mathcal{C})$, the set*

$$\mathit{Span}\{[\xi,\eta]_{\mathcal{C}};\ A \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ \xi \in (F \odot_{\mathcal{B}} G)_A(C'),\ \eta \in (F \odot_{\mathcal{B}} G)_A(C)\},$$

is dense in

$$\mathit{Span}\{\langle\alpha,\beta\rangle_{\mathcal{C}};\ B \in \mathcal{O}\mathrm{b}(\mathcal{B}),\ \alpha \in G_B(C'),\ \beta \in G_B(C)\}.$$

Proof of claim: If we take $\epsilon > 0$, $\alpha \in G_B(C')$ and $\beta \in G_B(C)$, then we can use non-degeneracy of $G_{(-)}$ to find $\sum_{i=1}^m G_{(b_i)B}\delta_i$, where $b_i \in \mathcal{B}(B,B)$ and $\delta_i \in G_B(C)$, such that

$$\frac{\epsilon}{\|\alpha\|} > \|\beta - \sum_i G_{(b_i)B}\delta_i\|.$$

We can now use fullness of $F_{(-)}$ to find, for each i, $\sum_{j_i=1}^{n_j}\langle\gamma_{j_i},\zeta_{j_i}\rangle_{\mathcal{B}}$, where $\gamma_{j_i}\in F_{A_{j_i}}(B)$ and $\zeta_{j_i}\in F_{A_{j_i}}(B)$, such that

$$\frac{\epsilon}{m\|\alpha\|\|\delta_i\|} > \|b_i - \sum_{j_i}\langle\gamma_{j_i},\zeta_{j_i}\rangle_{\mathcal{B}}\|.$$

Now we have

$$\begin{aligned}
\|\langle\alpha,\beta\rangle_{\mathcal{C}} - \sum_i\sum_{j_i}[\gamma_{j_i}\otimes\alpha,\zeta_{j_i}\otimes\delta_i]_{\mathcal{C}}\| &= \|\langle\alpha,\beta\rangle_{\mathcal{C}} - \sum_i\sum_{j_i}\langle\alpha,\langle\gamma_{j_i},\zeta_{j_i}\rangle_{\mathcal{B}}\cdot\delta_i\rangle_{\mathcal{C}}\| \\
&= \|\langle\alpha,\beta - \sum_i\sum_{j_i}\langle\gamma_{j_i},\zeta_{j_i}\rangle_{\mathcal{B}}\cdot\delta_i\rangle_{\mathcal{C}}\| \\
&\le \|\alpha\|\|\beta - \sum_i\sum_{j_i}\langle\gamma_{j_i},\zeta_{j_i}\rangle_{\mathcal{B}}\cdot\delta_i\| \\
&\le \|\alpha\|\|\beta - \sum_i b_i\cdot\delta_i\| + \|\alpha\|\|\sum_i b_i\cdot\delta_i - \sum_i\sum_{j_i}\langle\gamma_{j_i},\zeta_{j_i}\rangle_{\mathcal{B}}\cdot\delta_i\| \\
&< \epsilon + \|\alpha\|\sum_i\|(b_i - \sum_{j_i}\langle\gamma_{j_i},\zeta_{j_i}\rangle_{\mathcal{B}})\cdot\delta_i\| \\
&< \epsilon + \sum_i\|\alpha\|\frac{\epsilon}{m\|\alpha\|\|\delta_i\|}\|\delta_i\| \\
&= 2\epsilon.
\end{aligned}$$

■

It follows from this claim that the functor $(F\odot_{\mathcal{B}}G)_{(-)}$ is full.

Claim 6.2.17. *For objects $A,A'\in\mathcal{O}\mathrm{b}(\mathcal{A})$, $C,C'\in\mathcal{O}\mathrm{b}(\mathcal{C})$ and $\sum_i\xi_i\otimes\eta_i\in(F\odot_{\mathcal{B}}G)_A(C')$, $\sum_j\nu_j\otimes\zeta_j\in(F\odot_{\mathcal{B}}G)_{A'}(C)$ and $a\in\mathcal{A}(A,A')$, we have*

$$\langle a\cdot\sum_i\xi_i\otimes\eta_i,\sum_j\nu_j\otimes\zeta_j\rangle_{\mathcal{C}} = \langle\sum_i\xi_i\otimes\eta_i, a^*\cdot\sum_j\nu_j\otimes\zeta_j\rangle_{\mathcal{C}}.$$

Proof of claim: We verify this claim directly;

$$\begin{aligned}\langle a\cdot \sum_i \xi_i\otimes\eta_i, \sum_j \nu_j\otimes\zeta_j\rangle_{\mathcal{C}} &= \sum_{i,j}\langle (a\cdot\xi_i)\otimes\eta_i, \nu_j\otimes\zeta_j\rangle_{\mathcal{C}}\\ &= \sum_{i,j}\langle \eta_i, \langle a\cdot\xi_i,\nu_j\rangle_{\mathcal{B}}\cdot\zeta_j\rangle_{\mathcal{C}}\\ &= \sum_{i,j}\langle \eta_i, \langle \xi_i, a^*\cdot\nu_j\rangle_{\mathcal{B}}\cdot\zeta_j\rangle_{\mathcal{C}}\\ &= \sum_{i,j}\langle \xi_i\otimes\eta_i, (a^*\cdot\nu_j)\otimes\zeta_j\rangle_{\mathcal{C}}\\ &= \langle \sum_i \xi_i\otimes\eta_i, a^*\cdot\sum_j \nu_j\otimes\zeta_j\rangle_{\mathcal{C}}.\end{aligned}$$

■

Similarly, one may verify that the functor ${}_{(-)}(F\odot_{\mathcal{B}}G)$ is full, and that the other equality demanded in Lemma 6.2.14 is satisfied.

Claim 6.2.18. *For all $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$, $C, C' \in \mathcal{O}\mathrm{b}(\mathcal{C})$ we have the identity*

$$_{\mathcal{A}}\langle x, y\rangle\cdot z = x\cdot\langle y, z\rangle_{\mathcal{C}},$$

for all

$$x \in (F\odot_{\mathcal{B}} G)_{A'}(C'), \qquad y\in (F\odot_{\mathcal{B}} G)_{A}(C'), \qquad z\in (F\odot_{\mathcal{B}} G)_{A}(C).$$

Proof of claim: Given $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$, $C, C' \in \mathcal{O}\mathrm{b}(\mathcal{C})$, itis enough to verify the claim for simple tensors

$$\xi\otimes\eta \in (F\odot_{\mathcal{B}} G)_{A'}(C'), \qquad \xi'\otimes\eta' \in (F\odot_{\mathcal{B}} G)_{A}(C'), \qquad \xi''\otimes\eta''\in (F\odot_{\mathcal{B}} G)_{A}(C).$$

To this end, we compute;

$$\begin{aligned}
(\xi \otimes \eta) \cdot \langle \xi' \otimes \eta', \xi'' \otimes \eta'' \rangle_{\mathcal{C}} &= (\xi \otimes \eta) \cdot (\langle \eta', \langle \xi', \xi'' \rangle_{\mathcal{B}} \cdot \eta'' \rangle_{\mathcal{C}}) \\
&= \xi \otimes (\eta \cdot \langle \langle \xi'' \xi' \rangle_{\mathcal{B}} \cdot \eta', \eta'' \rangle_{\mathcal{C}}) \\
&= \xi \otimes ({}_{\mathcal{B}}\langle \eta, \langle \xi'', \xi' \rangle_{\mathcal{B}} \cdot \eta' \rangle \cdot \eta'') \\
&= (\xi \cdot {}_{\mathcal{B}}\langle \eta, \langle \xi'', \xi' \rangle_{\mathcal{B}} \cdot \eta' \rangle) \otimes \eta'' \\
&= (\xi \cdot {}_{\mathcal{B}}\langle \eta, \eta' \rangle \langle \xi', \xi'' \rangle_{\mathcal{B}}) \otimes \eta'' \\
&= (\xi \cdot \langle \xi' \cdot {}_{\mathcal{B}}\langle \eta', \eta \rangle, \xi' \rangle_{\mathcal{B}}) \otimes \eta'' \\
&= ({}_{\mathcal{A}}\langle \xi, \xi' \cdot {}_{\mathcal{B}}\langle \eta', \eta \rangle \rangle \cdot \xi'') \otimes \eta'' \\
&= ({}_{\mathcal{A}}\langle \xi \cdot {}_{\mathcal{B}}\langle \eta, \eta' \rangle, \xi' \rangle) \cdot (\xi'' \otimes \eta'').
\end{aligned}$$

This might be difficult to digest in one sitting, but each of these steps follows directly from standard facts, such as the axioms satisfied by inner products and the definition of the inner product on a tensor product. ■

6.3 Relationship With Category Algebras

It turns out that a C*-category is always Morita equivalent to its category algebra. In [12] this is proved[4] for unital categories with countably many objects by directly constructing an equivalence between the relevant categories of Hilbert modules. Here we will prove our claim by constructing a pre-equivalence bimodule and using the theory in the previous section.

We let $\mathcal{B}$ be a C*-category and let $\mathcal{I}$ be the right Hilbert $\mathcal{B}$-module that we've made use of a few times by now. We have the identification $A(\mathcal{B}) \cong \mathcal{K}(\mathcal{I})$ and in particular we have a *-functor

$$A(\mathcal{B}) \to \textbf{Hilb}\text{-}\mathcal{B},$$

which sends the single object of $A(\mathcal{B})$ to $\mathcal{I}$. This of course induces a functor $\mathcal{B}^{\mathrm{op}} \to A(\mathcal{B}) - \textbf{Mod}$, which is given on objects by $B \mapsto \mathcal{I}(B)$. By a small abuse of notation, we will also write $\mathcal{I}$ for the bimodule. We can now say more about this bimodule.

Theorem 6.3.1. *When $\mathcal{B}$ is a C*-category, the right Hilbert $A(\mathcal{B}) - \mathcal{B}$ bimodule $\mathcal{I}$:*

[4] Strictly speaking this isn't proved in the cited paper, since Morita equivalence isn't defined there. However using the results in our final chapter, we see that Joachim is showing that certain C*-categories are Morita equivalent to their category algebras.

*$A(\mathcal{B}) \to$ **Hilb**-$\mathcal{B}$ is an equivalence bimodule.*

Proof. Our plan for this proof is to show that the non-completed bimodule $\mathcal{I}^0$ is a pre-equivalence bimodule, so once we complete it we get what we want. We of course already have the $\mathcal{B}$-valued inner products, so our first step is to define, for each $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, a left $A^0(\mathcal{B})$ valued inner product on the left $A^0(\mathcal{B})$-module ${}_B\mathcal{I}$, and we define this by

$$\begin{aligned} &\mathcal{I}^0(B) \times \mathcal{I}^0(B) \to A(\mathcal{B}) \\ &(\sum_i a_i, \sum_j b_j) \mapsto \sum_i \sum_j a_i b_j^*. \end{aligned}$$

It is straightforward to check that this is indeed an inner product, for example we have

$$\left\langle \sum_i a_i, \sum_j b_j \right\rangle^* = \left(\sum_{i,j} a_i b_j^* \right)^* = \sum_{i,j} b_j a_i^* = \left\langle \sum_j b_j, \sum_i a_i \right\rangle.$$

For fullness of the functor $A(\mathcal{B}) \to$ **Hilb**-$\mathcal{B}$, we fix $B, B' \in \mathcal{B}(B, B')$ and let (e_λ) be a self adjoint approximate unit for $\mathcal{B}(B, B')$, then for any $b \in \mathcal{B}(B, B')$ we have

$$\langle e_\lambda, b \rangle_{\mathcal{B}} = e_\lambda b,$$

so that b belongs to the closure of $\mathcal{J}^0_{\mathcal{B}}(B, B')$ in $\mathcal{B}(B, B')$. For fullness of the functor $\mathcal{B}^{\mathrm{op}} \to A(\mathcal{B}) - \mathbf{Mod}$, we take $f \in A^0(\mathcal{B})$. We view f as a sum

$$f = \sum_{B' \in \mathcal{O}\mathrm{b}(\mathcal{B})} \sum_{B \in \mathcal{O}\mathrm{b}(\mathcal{B})} f_{B'B},$$

where $f_{B'B} \in \mathcal{B}(B, B')$, then for $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we let f_B be the sum

$$f_B = \sum_{B' \in \mathcal{O}\mathrm{b}(\mathcal{B})} f_{B'B}.$$

This is a sum of morphisms into the object B, so is an element of $\mathcal{I}^0(B)$. For each $B' \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we let $(e_\lambda^{B'})$ be a self adjoint approximate unit for $(B)(B', B')$, then observe that

$${}_{A^0(\mathcal{B})}\langle f_B, e_\lambda^{B'} \rangle = f_{B',B} e_\lambda^{B'},$$

so when we pass to the closure of the relevant ideal, we recover each morphism $f_{B',B}$, then summing them all together recovers the original f, hence we get fullness. To check the required positivity properties, note that the fact we have a *-functor $\mathcal{I} : A(\mathcal{B}) \to$

Hilb-$\mathcal{B}$ means that $A(\mathcal{B})$ acts on $\mathcal{I}$ as adjointable operators, so by standard results we have the inequality

$$\langle f \cdot \xi, f \cdot \xi \rangle_{\mathcal{B}} \leq \|f\|^2 \langle \xi, \xi \rangle_{\mathcal{B}},$$

which holds when ξ is from $\mathcal{I}$, so certainly holds when ξ is from $\mathcal{I}^0$. Next we will show that

$$_{A(\mathcal{B})}\langle u \cdot b, v \rangle = {}_{A(\mathcal{B})}\langle u, v \cdot b^* \rangle,$$

for $u \in \mathcal{I}^0(X')$, $v \in \mathcal{I}^0(X)$ and $b \in \mathcal{B}(X, X')$. Writing $u = \sum_i u_i$ and $v = \sum_j v_j$ we have

$$_{A(\mathcal{B})}\langle u \cdot b, v \rangle = \sum_{i,j} (u_i b) v_j^* = \sum_{i,j} u_i (v_j b^*)^* = {}_{A(\mathcal{B})}\langle u, v \cdot b \rangle.$$

Using the proof of Lemma 6.2.14 we see that this identity implies the inequality demanded by the definition of a pre-equivalence bimodule.

For the final point, we refer back to part (ii) of Lemma 5.1.4. Hence we have that $\mathcal{I}^0$ is a pre-equivalence bimodule, so it's completion $\mathcal{I}$ must be an $A(\mathcal{B}) - \mathcal{B}$ equivalence bimodule. □

Corollary 6.3.2. *C^*-categories $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent if and only if their category algebras $A(\mathcal{B})$ and $A(\mathcal{C})$ are Morita equivalent.*

Proof. Writing $\simeq_M$ for the equivalence relation of Morita equivalence; if $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent, then we have

$$A(\mathcal{B}) \simeq_M \mathcal{B} \simeq_M \mathcal{C} \simeq_M A(\mathcal{C}),$$

so by transitivity $A(\mathcal{B}) \simeq_M A(\mathcal{C})$. Conversely, if $A(\mathcal{B})$ and $A(\mathcal{C})$ are Morita equivalent, then we have

$$\mathcal{B} \simeq_M A(\mathcal{B}) \simeq A(\mathcal{C}) \simeq_M \mathcal{C},$$

so again by transitivity, $\mathcal{B} \simeq_M \mathcal{C}$. □

6.4 Properties of Equivalence Bimodules

We begin this section with the following theorem, more or less stating that equivalence bimodules are invertible.

Theorem 6.4.1. *If ${}_{\mathcal{A}}F_{\mathcal{B}}$ is an $\mathcal{A} - \mathcal{B}$ equivalence bimodule, then we have bimodule isomorphisms*

$$F \otimes_{\mathcal{B}} \widetilde{F} \cong \mathcal{A}(-,=), \qquad \widetilde{F} \otimes_{\mathcal{A}} F \cong \mathcal{B}(-,=).$$

Proof. We will prove the existence of the second isomorphism, and the other follows from a similar argument. We will use our characterisations of bimodule isomorphisms, and the first step in this is to construct a family of maps

$$\{{}_{B}\Phi_{B'} : \widetilde{F}_{B'} \otimes_{\mathcal{A}} {}_{B}F \to \mathcal{B}(B,B');\ B, B' \in \mathcal{O}\mathrm{b}(\mathcal{B})\}.$$

We take $B, B' \in \mathcal{O}\mathrm{b}(\mathcal{B})$, and recall that

$$\widetilde{F}_{B'} \otimes_{\mathcal{A}} {}_{B}F = \widetilde{F}_{B'} \boxtimes_{\mathcal{A}} {}_{B}F,$$

so our first move is to use the universal property of coends to define a linear map $\widetilde{F}_{B'} \boxdot_{\mathcal{A}} {}_{B}F \to \mathcal{B}(B,B')$. We fix $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, and we note that

$$\widetilde{F}_{B'}(A) \times {}_{B}F(A) = \widetilde{{}_{B'}F(A)} \times F_A(B) = \widetilde{F_A(B')} \times F_A(B).$$

We define $h_A : \widetilde{F}_{B'}(A) \times {}_{B}F(A) \to \mathcal{B}(B,B')$ to be the composite

$$\widetilde{F_A(B')} \times F_A(B) \xrightarrow{(\flat,\mathrm{id})} F_{B'}(A) \times F_B(A) \xrightarrow{\langle -,- \rangle} \mathcal{B}(B,B').$$

This gives us a family of maps $\{h_A :\ A \in \mathcal{O}\mathrm{b}(\mathcal{A})\}$, and for each $a \in \mathcal{A}(A,A')$ we need the following diagram to commute,

$$\begin{tikzcd}
\widetilde{F_{A'}(B')} \times F_A(B) \arrow[r, "{(id,F_{(a)})}"] \arrow[d, "{(\widetilde{F}_{(a)},\mathrm{id})}"'] & \widetilde{F_{A'}(B)} \times F_{A'}(B) \arrow[d] \arrow[ddr, bend left, "h_{A'}"] & \\
\widetilde{F_A(B')} \times F_A(B) \arrow[r] \arrow[drr, bend right, "h_A"'] & F_{B'} \boxdot_{\mathcal{A}} {}_{B}F \arrow[dr, dashed] & \\
 & & \mathcal{B}(B,B').
\end{tikzcd}$$

checking this is straightfoward; we take $(\flat(\alpha),\beta)$ from the top left corner, then

$$\begin{aligned} h_A(\widetilde{F}_{(a)}\flat(\alpha),\beta) &= h_A(\flat(F_{(a^*)}\alpha,\beta) \\ &= \langle F_{(a^*)}\alpha,\beta\rangle \\ &= \langle \alpha, F_{(a)}\beta\rangle \\ &= h_{A'}(\flat(\alpha),F_{(a)}\beta). \end{aligned}$$

So the universal property of coends gives us a unique linear map ${}_B\Phi_{B'}$ which fills in the dashed arrow of the diagram, so that the whole thing still commutes. We need to check that our maps extend to the completed coend, and for this we show that the inner product on $\widetilde{F} \odot_{\mathcal{A}} F$ is preserved.

$$\begin{aligned} \langle {}_B\Phi_{B'}\left(\sum \flat(\alpha_A)\otimes\beta_A\right), {}_B\Phi_{B'}\left(\sum \flat(\gamma_{A'})\otimes\delta_{A'}\right)\rangle &= \sum\sum \langle {}_B\Phi_{B'}(\flat(\alpha_A)\otimes\beta_A), {}_B\Phi_{B'}(\flat(\gamma_{A'})\otimes\delta_{A'})\rangle \\ &= \sum\sum \langle\langle\alpha_A,\beta_A\rangle,\langle\gamma_{A'},\delta_{A'}\rangle\rangle \\ &= \sum\sum \langle\beta_A,\alpha_A\rangle\langle\gamma_{A'},\delta_{A'}\rangle \\ &= \sum\sum \langle\beta_A,\alpha_A\cdot\langle\gamma_{A'},\delta_{A'}\rangle\rangle \\ &= \sum\sum \langle\beta_A,\langle\alpha_A,\gamma_{A'}\rangle\cdot\delta_{A'}\rangle \\ &= \sum\sum \langle\langle\flat(\gamma_{A'}),\flat(\alpha_A)\rangle\cdot\beta_A,\delta_{A'}\rangle \\ &= \left[\sum \flat(\alpha_A)\otimes\beta_A, \sum\flat(\gamma_{A'})\otimes\delta_{A'}\right]. \end{aligned}$$

One may verify each of the steps in the above calculation by carefully picking back through things like the definition of the inner product on a tensor product and properties of equivalence bimodules.

It follows from this that we have linear maps ${}_B\Phi_{B'} : \widetilde{F}_{B'} \otimes_{\mathcal{A}} {}_BF \to \mathcal{B}(B,B')$ which preserve the right and left $\mathcal{B}$-valued inner products. Our final task is to check that these maps have dense image, but this follows immediately from the fullness conditions we impose on equivalence bimodules. □

6.4.1 An Alternative Characterisation of Equivalence Bimodules

For equivalence bimodules over C*-algebras, there exists the following alternative characterisation of equivalence bimodules.

Proposition 6.4.2. *If A and B are C^*-algebras, $X \in$ **Hilb**-B and $\varphi : A \to \mathcal{L}(X)$ is* *a *-homomorphism, then X is an equivalence bimodule if and only if it is a full right Hilbert B-module, and φ induces an isomorphism $A \cong \mathcal{K}(X)$.*

In this section we will show that a similar result holds for our equivalence bimodules between C*-categories.

Lemma 6.4.3. *Let ${}_{\mathcal{A}}F_{\mathcal{B}}$ be an $\mathcal{A}-\mathcal{B}$ equivalence bimodule. Then the right action $F_{(-)}$ is a faithful *-functor.*

Proof. Fix any pair $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$, we need to show that the map

$$F_{(-)} : \mathcal{A}(A, A') \to \mathcal{L}(F_A, F_{A'}),$$

is injective. We assume that we have $a \in \mathcal{A}(A, A')$ for which $F_{(a)} = 0$ and towards a contradiction, further assume that $a \neq 0$. We let (u_λ) be an approximate unit for the C*-algebra $\mathcal{A}(A, A)$, then given $\epsilon > 0$ we use Lemma 1.2.14 to find λ such that

$$\|a - au_\lambda\| < \frac{\epsilon}{2}.$$

Condition (ii) in the definition of an equivalence bimodule means that we can find a sum $\sum_{i=1}^{n} {}_{\mathcal{A}}\langle \xi_i, \eta_i \rangle$, where ξ_i and η_i belong to $F_{A_i}(B)$, such that

$$\left\| u_\lambda - \sum_i {}_{\mathcal{A}}\langle \xi_i, \eta_i \rangle \right\| < \frac{\epsilon}{2\|a\|},$$

which implies that

$$\left\| au_\lambda - a\sum_i {}_{\mathcal{A}}\langle \xi_i, \eta_i \rangle \right\| < \frac{\epsilon}{2}.$$

Now we combine our inequalities, to see that

$$\begin{aligned} \epsilon = \frac{\epsilon}{2} + \frac{\epsilon}{2} &> \|a - au_\lambda\| + \left\| au_\lambda - a\sum_i {}_{\mathcal{A}}\langle \xi_i, \eta_i \rangle \right\| \\ &\geq \left\| a - a\sum_i {}_{\mathcal{A}}\langle \xi_i, \eta_i \rangle \right\|. \end{aligned}$$

The assumption that $F_{(a)} = 0$ now tells us that

$$a\sum_i {}_{\mathcal{A}}\langle \xi_i, \eta_i \rangle = \sum_i {}_{\mathcal{A}}\langle a \cdot \xi_i, \eta_i \rangle = \sum_i {}_{\mathcal{A}}\langle F_{(a)B}\xi_i, \eta_i \rangle = 0.$$

So that $\|a\| < \epsilon$ for each $\epsilon > 0$, and therefore $\|a\| = 0$. $\square$

Corollary 6.4.4. *Let ${}_{\mathcal{A}}F_{\mathcal{B}}$ be an $\mathcal{A} - \mathcal{B}$ equivalence bimodule. For each pair $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$, the functor $F_{(-)}$ gives an isomorphism*

$$\mathcal{A}(A, A') \cong \mathcal{K}(F_A, F_{A'}).$$

Proof. The previous result tells us that $F_{(-)}$ is faithful, so for $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$, the map $F_{(-)} : \mathcal{A}(A, A') \to \mathcal{L}(F_A, F_{A'})$ is an isometry and hence has closed image. To finish the proof, we need to show that the map is surjective onto the relevant set of compact operators. We take $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$ and vectors $\xi \in F_{A'}(B)$, $\eta \in F_A(B)$. Then we have

$$F_{({}_{\mathcal{A}}\langle \xi, \eta \rangle)} = {}_{\mathcal{A}}\langle \xi, \eta \rangle \cdot (-) = \xi \cdot \langle \eta, - \rangle_{\mathcal{B}},$$

where the second equality follows because F is an equivalence bimodule. This shows that $F_{({}_{\mathcal{A}}\langle \xi, \eta \rangle)}$ is a rank one operator in $\mathcal{K}(F_A, F_{A'})$. Fullness of the equivalence bimodule means that

$$\mathrm{Span}\{{}_{\mathcal{A}}\langle \xi, \eta \rangle;\ B \in \mathcal{O}\mathrm{b}(\mathcal{B}),\ \xi \in {}_BF(A'),\ \eta \in {}_BF(A)\},$$

is dense in $\mathcal{A}(A, A')$. This tells us that $F_{(-)}$ is a map

$$\mathcal{A}(A, A') \to \mathcal{K}(F_A, F_{A'}),$$

and moreover $F_{(-)}$ maps the linear span stated above into the space of finite rank operators $F_A \Rightarrow F_{A'}$, which is dense in $\mathcal{K}(F_A, F_{A'})$. This shows that we have the claimed isomorphism. $\square$

Remark 6.4.5. The previous two results can be distilled, leaving us with the statement that an equivalence bimodule $F_{(-)} : \mathcal{A} \to$ **Hilb**-$\mathcal{B}$ is in fact a full and faithful *-functor

$$F_{(-)} : \mathcal{A} \to \mathbf{Hilb}_{\mathcal{K}}\text{-}\mathcal{B}.$$

As a converse to the last few results, we have the following.

Proposition 6.4.6. *Suppose that $F_{(-)} : \mathcal{A} \to$ **Hilb**-$\mathcal{B}$ is a full right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule such that for each pair $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$ the functor F induces an isomorphism*

$$\mathcal{A}(A, A') \cong \mathcal{K}(F_A, F_{A'}).$$

Then F is an equivalence bimodule.

Proof. First we shall define, for each $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$ an $\mathcal{A}$-valued inner product on ${}_BF$ to be the following composite,

$$
\begin{array}{ccccc}
{}_BF(A') \times {}_BF(A) & \longrightarrow & \mathcal{K}(F_A, F_{A'}) & \xrightarrow{F^{-1}} & \mathcal{A}(A, A') \\
\\
(\xi, \eta) & \longmapsto & \xi \cdot \langle \eta, - \rangle_{\mathcal{B}} & &
\end{array}
$$

The computation

$$
{}_{\mathcal{A}}\langle \xi \cdot b, \eta \rangle = F^{-1}((\xi \cdot b) \cdot \langle \eta, - \rangle_{\mathcal{B}}) = F^{-1}(\xi \cdot \langle \eta \cdot b^*, - \rangle_{\mathcal{B}}) = {}_{\mathcal{A}}\langle \xi, \eta \cdot b^* \rangle,
$$

where $b \in \mathcal{B}(B, B')$, $\xi \in {}_BF(A')$ and $\eta \in {}_{B'}F(A)$ shows that the functor ${}_{(-)}F$ is a *-functor into $\mathcal{A}$-**Hilb**, hence F is a bi-Hilbert $\mathcal{A} - \mathcal{B}$ bimodule. By definition, the set

$$
\mathrm{Span}\{\xi \cdot \langle \eta, - \rangle_{\mathcal{B}};\ B \in \mathcal{O}\mathrm{b}(\mathcal{B}),\ \eta \in F_A(B),\ \xi \in F_{A'}(B)\},
$$

is dense in $\mathcal{K}(F_A, F_{A'})$, so the bimodule ${}_{\mathcal{A}}F$ is also full. Finally, we observe that

$$
{}_{\mathcal{A}}\langle \xi, \eta \rangle \cdot \zeta = F_{(F^{-1}({}_{\mathcal{A}}\langle \xi, \eta \rangle))}\zeta = \xi \cdot \langle \eta, \zeta \rangle_{\mathcal{B}},
$$

so that $F_{(-)}$ satisfies all the properties required of an equivalence bimodule. ☐

One immediate consequence is the following.

Corollary 6.4.7. *If $F_{(-)} : \mathcal{A} \to$ **Hilb**-$\mathcal{B}$ is a right Hilbert $\mathcal{A} - \mathcal{B}$ bimodule, then F is an equivalence bimodule if and only if F is full and induces an isomorphism*

$$
\mathcal{A}(A, A') \cong \mathcal{K}(F_A, F_{A'}),
$$

for each pair $A, A' \in \mathcal{O}\mathrm{b}(\mathcal{A})$.

Remark 6.4.8. Note how this corollary gives us an alternative proof that we have an equivalence bimodule between a C*-category and it's category algebra.

Chapter 7

The Bicategory of Correspondences

Here we will introduce the bicategory of correspondences. This will be an analogue of the bicategory of C*-algebras introduced by Landsman [17]. Later will will show that the category algebra construction is a pseudofunctor from our bicategory to Landsman's bicategory of C*-algebras. First we will fix an extra piece of notation that we will make frequent use of in this chapter.

Notation 7.0.1. If $\mathcal{A}$ and $\mathcal{B}$ are C*-categories and we have a correspondence $F_{(-)} : \mathcal{A} \to \textbf{Hilb}\text{-}\mathcal{B}$, then we will write

$$\mathcal{A} \xrightarrow{F} \mathcal{B}$$

to indicate that $F_{(-)}$ is a correspondence from $\mathcal{A}$ to $\mathcal{B}$.

7.1 Constructing The Bicategory

The goal of this section is to show that we have a bicategory whose objects are small C*-categories, whose 1−cells are correspondences, and whose 2−cells are homomorphisms of correspondences. For the definition of a bicategory, one may consult [19, XII.6], or the short document of Leinster [18]. The first step of constructing our bicategory is showing that the collection of correspondences between two C*-categories forms a category.

Lemma 7.1.1. *For C^*-categories $\mathcal{A}$ and $\mathcal{B}$, we have a category $\mathfrak{Corr}(\mathcal{A},\mathcal{B})$ whose objects are $\mathcal{A}-\mathcal{B}$ correspondences, and whose morphisms are correspondence homomorphisms.*

Proof. Given $F, G, H \in \mathfrak{Corr}(\mathcal{A},\mathcal{B})$, we need to define the composition law

$$\hom(F,G) \times \hom(G,H) \to \hom(F,H).$$

Given $\theta : F \Rightarrow G$ and $\psi : G \Rightarrow H$, we define $\psi \circ \theta$ to have components

$$\{F_A(B) \overset{\theta_{A,B}}{\to} G_A(B) \overset{\psi_{A,B}}{\to} H_A(B);\ A \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ B \in \mathcal{O}\mathrm{b}(\mathcal{B})\}.$$

It is straightforward to check that $\psi\circ\theta$ is a homomorphism of correspondences. Furthermore, this composition is associative, and given $F \in \mathfrak{Corr}(\mathcal{A},\mathcal{B})$, we have an identity homomorphism with whose components are the relevant identity maps,

$$\{\mathrm{id} : F_A(B) \to F_A(B);\ A \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ B \in \mathcal{O}\mathrm{b}(\mathcal{B})\}.$$

$\square$

Now we need to construct the composition bifunctors for our bicategory.

Proposition 7.1.2. *For each trio of C^*-categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, we have a bifunctor*[1]

$$c_{\mathcal{A},\mathcal{B},\mathcal{C}} : \mathfrak{Corr}(\mathcal{A},\mathcal{B}) \times \mathfrak{Corr}(\mathcal{B},\mathcal{C}) \to \mathfrak{Corr}(\mathcal{A},\mathcal{C}).$$

Proof. We define the bifunctor on objects by

$$c_{\mathcal{A},\mathcal{B},\mathcal{C}}(F,G) = F \otimes_{\mathcal{B}} G.$$

We have previously shown that the interior tensor product of two correspondences is again a correspondence, so this at least makes sense. Now we need to define the behaviour of $c_{\mathcal{A},\mathcal{B},\mathcal{C}}$ on morphisms, so suppose that we have the following setup,

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F", "/"{marking}] \arrow[r, bend right, "F'"', "/"{marking}] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{B} \arrow[r, bend left, "G", "/"{marking}] \arrow[r, bend right, "G'"', "/"{marking}] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{C} \end{tikzcd}$$

[1]When we say *bifunctor*, we simply mean a functor whose domain is the product of two categories.

We will write $\theta \otimes \psi$ for the homomorphism $c_{\mathcal{A},\mathcal{B},\mathcal{C}}(\theta, \psi)$, the $(A, C) - th$ component of which we define as follows; for $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we consider the following map,

$$\begin{aligned} F_A(B) \times {}_C G(B) &\to F'_A \boxdot_{\mathcal{B}} {}_C G' \\ (\xi, \eta) &\mapsto \theta_{A,B}(\xi) \otimes \psi_{B,C}(\eta). \end{aligned}$$

This gives us a family of bilinear maps, and we can check that for each $b \in \mathcal{B}(B, B')$ the following diagram commutes,

$$\begin{tikzcd} F_A(B') \times {}_C G(B) \arrow[r, "(\mathrm{id},(G_b)_C)"] \arrow[d, "(F_A(b)\text{,id})"'] & F_A(B') \times {}_C G(B') \arrow[d] \\ F_A(B) \times {}_C G(B) \arrow[r] & F'_A \boxdot {}_C G' \end{tikzcd}$$

Appealing to the universal property of coends, this presents us with a well defined linear map

$$F_A \boxdot {}_C G \to F'_A \boxdot {}_C G'.$$

We now check that these maps extend to ones between the completed coends, so let $x = \sum_i \xi_i \otimes \eta_i \in F_A \boxdot {}_C G$, then make the following computation,

$$\begin{aligned} \|\theta\|\|\psi\|\langle x, x\rangle - \langle \theta \otimes \psi(x), \theta \otimes \psi(x)\rangle &= \langle x, \|\theta\|\|\psi\|x - \theta^*\theta \otimes \psi^*\psi(x)\rangle \\ &= \langle x, \sum_i (\|\theta\|\|\psi\|\xi_i \otimes \eta_i - \theta^*\theta(\xi_i) \otimes \psi^*\psi(\eta_i))\rangle \\ &= \langle x, \sum_i (\|\theta\| \operatorname{id}_{F_A} - \theta^*\theta)\xi_i \otimes (\|\psi\| \operatorname{id}_{{}_C G} - \psi^*\psi)\eta_i\rangle \\ &\stackrel{!}{=} \langle x, \sum_i S^*S\xi_i \otimes T^*T\eta_i\rangle \\ &= \langle \sum_i S(\xi_i) \otimes T(\eta_i), \sum_i S(\xi_i) \otimes T(\eta_i)\rangle \\ &\geq 0. \end{aligned}$$

The equality marked ! uses a similar argument to that employed in Claim 4.2.6 to see that the operators $\|\theta\| \operatorname{id}_{F_A} - \theta^*\theta$ and $\|\psi\| \operatorname{id}_{{}_C G} - \psi^*\psi$ are positive. It follows that $\|\theta \otimes \psi(x)\| \leq \|\theta\|\|\psi\|\|x\|$ so we get a well defined linear map

$$F_A \boxtimes {}_C G \to F'_A \boxtimes {}_C G'.$$

Running this argument through with θ^* and ψ^* then gives us a well defined linear map

$$F'_A \boxtimes {}_C G' \to F_A \boxtimes {}_C G.$$

We need to check that for each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, we can assemble these maps to get a bounded adjointable operator

$$(F \otimes_{\mathcal{B}} G)_A(-) \Rightarrow (F' \otimes_{\mathcal{B}} G')_A(-).$$

To this end, we take sums $\sum_i \xi_i \otimes \eta_i \in F_A \boxdot {}_{C'} G$ and $\sum_j \xi'_j \otimes \eta'_j \in F_A \boxdot {}_C G$, then we have the following,

$$\begin{aligned}
[\sum_i \xi_i \otimes \eta_i, \theta \otimes \psi(\sum_j \xi'_j \otimes \eta'_j)] &= \sum_{i,j} \langle \eta_i, \langle \xi_i, \theta(\xi'_j)\rangle_{\mathcal{B}} \cdot \psi(\eta'_j)\rangle_{\mathcal{C}} \\
&= \sum_{i,j} \langle \eta, \psi(\langle \theta^*(\xi), \xi'_j\rangle_{\mathcal{B}} \cdot \eta'_j)\rangle_{\mathcal{C}} \\
&= \sum_{i,j} \langle \psi^*(\eta), \langle \theta^*(\xi), \xi'_j\rangle_{\mathcal{B}} \cdot \eta'_j\rangle_{\mathcal{C}} \\
&= [\theta^* \otimes \psi^*(\sum_i \xi \otimes \eta), \sum_j \xi'_j \otimes \eta'_j].
\end{aligned}$$

Note that for the second equality, we have used point (ii) in the definition of ψ being a correspondence homomorphism. This implies that we have

$$[\alpha, \theta \otimes \psi\beta] = [\theta^* \otimes \psi^*, \beta],$$

for all $\alpha \in F_A \boxtimes {}_{C'} G$ and $\beta \in F_A \boxtimes {}_C G$, so that we have an adjointable operator as required. Earlier we obtained the bound

$$\|\theta \otimes \psi(x)\| \leq \|\theta\|\|\psi\|\|x\|,$$

when $x \in F_A \boxdot_{\mathcal{A}} {}_C G$, which continues to hold true when $x \in F_A \boxtimes_{\mathcal{A}} {}_C G$ so that

$$\sup_{\|x\| \leq 1} \|\theta \otimes \psi(x)\| \leq \|\theta\|\|\psi\|,$$

which shows that our operator is a bounded adjointable operator. It is straightforward to check that we also have module homomorphisms

$${}_C(F \otimes_{\mathcal{B}} G)(-) \Rightarrow {}_C(F' \otimes_{\mathcal{B}} G)(-),$$

for each $C \in \mathcal{O}\mathrm{b}(\mathcal{C})$. Furthermore, it is evident that the assignment $(\theta, \psi) \mapsto \theta \otimes \psi$ is functorial. $\square$

Now we shall construct unitors.

Proposition 7.1.3. *For each C^*-category $\mathcal{B}$, there is $\mathrm{id}_{\mathcal{B}} \in \mathfrak{Corr}(\mathcal{B}, \mathcal{B})$ such that we have natural isomorphisms of functors*

$$c_{\mathcal{B},\mathcal{B},\mathcal{C}}(\mathrm{id}_{\mathcal{B}}, -) \simeq \mathrm{id}_{\mathfrak{Corr}(\mathcal{B},\mathcal{C})}, \qquad c_{\mathcal{B},\mathcal{C},\mathcal{C}}(-, \mathrm{id}_{\mathcal{C}}) \simeq \mathrm{id}_{\mathfrak{Corr}(\mathcal{B},\mathcal{C})}\,.$$

Proof. We let $\mathrm{id}_{\mathcal{B}}$ be the equivalence bimodule $\mathcal{B}(-,=)$. For the first isomorphism, we need to construct, for each $F \in \mathfrak{Corr}(\mathcal{B}, \mathcal{C})$ an isomorphism of correspondences

$$\mathrm{id}_{\mathcal{B}} \otimes_{\mathcal{B}} F \Rightarrow F.$$

For this, we need to define, for each $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$ and $C \in \mathcal{O}\mathrm{b}(\mathcal{C})$ a map

$$(\mathrm{id}_{\mathcal{B}} \otimes_{\mathcal{B}} F)_B(C) \to F_B(C).$$

With B, C as above, we define a map for each $B' \in \mathcal{O}\mathrm{b}(\mathcal{B})$ by

$$\begin{aligned} \mathcal{B}(B', B) \times {}_C F(B') &\to F_B(C) \\ (b, \xi) &\mapsto F_{(b)C}(\xi). \end{aligned}$$

This gives a family of bilinear maps, which make the following squares commute

$$\begin{array}{ccc} \mathcal{B}(B'', B) \times {}_C F(B') & \longrightarrow & \mathcal{B}(B'', B) \times {}_C F(B') \\ \downarrow & & \downarrow \\ \mathcal{B}(B', B) \times {}_C F(B') & \longrightarrow & F_B(C), \end{array}$$

because $(bb') \cdot \xi = b \cdot (b' \cdot \xi)$. This gives us well defined linear maps

$$\mathcal{B}(-, B) \boxdot_C F \to F_B(C),$$

and we can use non-degeneracy of F to see that the image of each of these maps is dense. The equality

$$[b \otimes \xi, b' \otimes \xi'] = \langle \xi, \langle b, b' \rangle_{\mathcal{B}} \cdot \xi' \rangle_{\mathcal{C}} = \langle \xi, b^* b' \cdot \xi' \rangle_{\mathcal{C}} = \langle b \cdot \xi, b' \cdot \xi' \rangle_{\mathcal{C}},$$

may be used to show that the maps preserve the right $\mathcal{C}$-valued inner product. This proves that for each $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we have a unitary isomorphism

$$(\mathrm{id}_{\mathcal{B}} \otimes_{\mathcal{B}} F)_B(-) \simeq F_B(-).$$

Finally we can easily check that for each $C \in \mathcal{O}\mathrm{b}(\mathcal{C})$, the maps we constructed give a module isomorphism

$$_C(\mathrm{id}_{\mathcal{B}} \otimes_{\mathcal{B}} F)(-) \simeq {}_C F(-).$$

So far we have constructed, for each $F \in \mathfrak{Corr}(\mathcal{B}, \mathcal{C})$ an isomorphism

$$\mathrm{id}_{\mathcal{B}} \otimes_{\mathcal{B}} F \simeq F.$$

To check that this gives a natural isomorphism as claimed, we take $F, F' \in \mathfrak{Corr}(\mathcal{B}, \mathcal{C})$, $\theta : F \Rightarrow F'$ and need to show that for each $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, $C \in \mathcal{O}\mathrm{b}(\mathcal{C})$, the following diagram commutes,

$$\begin{tikzcd}
(\mathrm{id}_{\mathcal{B}} \otimes_{\mathcal{B}} F)_B(C) \arrow[r] \arrow[d, "\mathrm{id} \otimes \theta"'] & F_B(C) \arrow[d, "\theta"] \\
(\mathrm{id}_{\mathcal{B}} \otimes_{\mathcal{B}} F')_B(C) \arrow[r] & F'_B(C),
\end{tikzcd}$$

which is easily done and we omit the details. Proving that we have the other natural isomorphism is similar, we begin by fixing $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$ and $C \in \mathcal{O}\mathrm{b}(\mathcal{C})$, then define for each $C' \in \mathcal{O}\mathrm{b}(\mathcal{C})$ a map

$$\begin{aligned} F_B(C') \times \mathcal{C}(C, C') &\to F_B(C) \\ (\eta, c) &\mapsto \eta \cdot c \end{aligned}$$

We can then run a similar argument to the one used earlier, except rather than using non-degeneracy of F to show that our maps have dense image, we use the fact that each right Hilbert $\mathcal{C}$-module F_B is non-degenerate. $\square$

Proposition 7.1.4. *There exists a natural isomorphism α as depicted in the following diagram.*

$$\begin{tikzcd}
& \mathfrak{C}orr(\mathcal{A},\mathcal{B})\times\mathfrak{C}orr(\mathcal{B},\mathcal{C})\times\mathfrak{C}orr(\mathcal{C},\mathcal{D}) \arrow[dl, "c_{\mathcal{A},\mathcal{B},\mathcal{C}}\times\mathrm{id}"'] \arrow[dr, "\mathrm{id}\times c_{\mathcal{B},\mathcal{C},\mathcal{D}}"] & \\
\mathfrak{C}orr(\mathcal{A},\mathcal{C})\times\mathfrak{C}orr(\mathcal{C},\mathcal{D}) \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, "c_{\mathcal{A},\mathcal{C},\mathcal{D}}"'] & & \mathfrak{C}orr(\mathcal{A},\mathcal{B})\times\mathfrak{C}orr(\mathcal{B},\mathcal{D}) \arrow[dl, "c_{\mathcal{A},\mathcal{B},\mathcal{D}}"] \\
& \mathfrak{C}orr(\mathcal{A},\mathcal{D}) &
\end{tikzcd}$$

Proof. The first thing we need is to show that for suitable F, G, H we have an isomorphism of correspondences

$$(F \otimes_{\mathcal{B}} G) \otimes_{\mathcal{C}} H \cong F \otimes_{\mathcal{B}} (G \otimes_{\mathcal{C}} H).$$

So we need a family of maps

$$\{[F \otimes_{\mathcal{B}} (G \otimes_{\mathcal{C}} H)]_A(D) \to [(F \otimes_{\mathcal{B}} G) \otimes_{\mathcal{C}} H]_A(D);\ A \in \mathcal{O}\mathrm{b}(\mathcal{A}),\ D \in \mathcal{O}\mathrm{b}(\mathcal{D})\}.$$

These maps are constructed in stages. Fix $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ and $D \in \mathcal{O}\mathrm{b}(\mathcal{D})$. For $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we fix a choice of $\zeta \in F_A(B)$, then for $C \in \mathcal{O}\mathrm{b}(\mathcal{C})$ define the following map

$$\begin{aligned} T^{\zeta}_C : G_B(C) \times {}_D H(C) &\to (F_A \otimes_{\mathcal{B}} G) \boxdot_{\mathcal{C}} {}_D H \\ \xi \otimes \eta &\mapsto (\zeta \otimes \xi) \otimes \eta. \end{aligned}$$

The family $\{T^{\zeta}_C\}_{C \in \mathcal{O}\mathrm{b}(\mathcal{C})}$ is a collection of bilinear maps, and by construction, if $c \in \mathcal{C}(C, C')$ then

$$T^{\zeta}_C(\xi \cdot c \otimes \eta) = (\zeta \otimes \xi \cdot c) \otimes \eta = ((\zeta \otimes \xi) \cdot c) \otimes \eta = (\zeta \otimes \xi) \otimes c \cdot \eta = T^{\zeta}{}_{C'}(\xi \otimes \eta),$$

so that by the universal property of coends, we get a well defined linear map

$$T^{\zeta} : G_B \boxdot_{\mathcal{C}} {}_D H \to (F_A \otimes_{\mathcal{B}} G) \boxdot_{\mathcal{C}} {}_D H.$$

Let's make a computation;

$$\begin{aligned}
\langle T^\zeta(\sum_i \xi_i \otimes \eta_i), T^\zeta(\sum_i \xi_i \otimes \eta_i)\rangle &= \langle \sum_i (\zeta \otimes \xi_i) \otimes \eta_i, \sum_i (\zeta \otimes \xi_i) \otimes \eta_i\rangle \\
&= \sum_{i,j} \langle (\zeta \otimes \xi_i) \otimes \eta_i, (\zeta \otimes \xi_j) \otimes \eta_j\rangle \\
&= \sum_{i,j} \langle \eta_i, \langle \zeta \otimes \xi_i, \zeta \otimes \xi_j\rangle \cdot \eta_j\rangle \\
&= \sum_{i,j} \langle \eta_i, \langle \xi_i, \langle \zeta, \zeta\rangle \cdot \xi_j\rangle \cdot \eta_j\rangle \\
&= \sum_{i,j} \langle \xi_i \otimes \eta_i, (\langle \zeta, \zeta\rangle \cdot \xi_j) \otimes \eta_j\rangle \\
&= \sum_{i,j} \langle \zeta \otimes (\xi_i \otimes \eta_i), \zeta \otimes (\xi_j \otimes \eta_j)\rangle \\
&= \langle \zeta \otimes (\sum_i \xi_i \otimes \eta_i), \zeta \otimes (\sum_i \xi_i \otimes \eta_i)\rangle.
\end{aligned}$$

Together with the Cauchy-Schwartz inequality, this shows that the map T^ζ is continuous, so extends to a continuous linear map

$$T^\zeta : G_B \boxtimes_{\mathcal{C}\ D} H \to (F_A \otimes_{\mathcal{B}} G) \boxtimes_{\mathcal{C}\ D} H,$$

which by the above must satisfy $\langle T^\zeta(x), T^\zeta(x)\rangle = \langle \zeta \otimes x, \zeta \otimes x\rangle$. Each mapping

$$\begin{gathered}
F_A(B) \times {}_D(G \otimes H)(B) \to (F_A \otimes_{\mathcal{A}} G) \boxtimes_{\mathcal{C}\ D} H \\
(\zeta, x) \mapsto T^\zeta(x),
\end{gathered}$$

is bilinear and $\mathcal{B}$-balanced, so we get a well defined linear map

$$T : F_A \boxdot_{\mathcal{B}\ D} (G \otimes H) \to (F_A \otimes_{\mathcal{B}} G) \boxtimes_{\mathcal{C}\ D} H.$$

We make another computation, making full use of the previous work

$$\begin{aligned}
\langle T(\sum_i \zeta_i \otimes x_i), T(\sum_i \zeta_i \otimes x_i)\rangle &= \sum_{i,j} \langle T(\zeta_i \otimes x_i), T(\zeta_j \otimes x_j\rangle \\
&= \sum_{i,j} \langle T^{\zeta_i}(x_i), T^{\zeta_j}(x_j)\rangle \\
&= \sum_{i,j} \langle \zeta_i \otimes x_i, \zeta_j \otimes x_j\rangle \\
&= \langle \sum_i \zeta_i \otimes x_i, \sum_i \zeta_i \otimes \ x_i\rangle,
\end{aligned}$$

and this shows that the map T extends to a continuous linear map

$$F_A \boxtimes_{\mathcal{B}\ D}(G \otimes H) \to (F_A \otimes_{\mathcal{B}} G) \boxtimes_{\mathcal{C}\ D} H.$$

This gives us our required family of maps. A similar computation to the one above shows that the maps are inner product preserving, and it is almost trivial that they have dense image, so we get unitary operators as required for our correspondence isomorphism. It is also straightforward to verify that the maps assemble to give isomorphisms between the relevant left modules, so that we have a correspondence isomorphism.

In order to check that we have a natural isomorphism between the composition functors, we need to check that the following square commutes,

$$\begin{tikzcd}
F \otimes_{\mathcal{B}} (G \otimes_{\mathcal{C}} H) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (F \otimes_{\mathcal{B}} G) \otimes_{\mathcal{C}} H \arrow[d, Rightarrow] \\
F' \otimes_{\mathcal{B}} (G' \otimes_{\mathcal{C}} H') \arrow[r, Rightarrow] & (F' \otimes_{\mathcal{B}} G') \otimes_{\mathcal{C}} H'
\end{tikzcd}$$

which requires checking that the following square commutes, for each pair $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$ and $D \in \mathcal{O}\mathrm{b}(\mathcal{D})$,

$$\begin{tikzcd}
{[F \otimes_{\mathcal{B}} (G \otimes_{\mathcal{C}} H)]_A(D)} \arrow[r] \arrow[d] & {[(F \otimes_{\mathcal{B}} G) \otimes_{\mathcal{C}} H]_A(D)} \arrow[d] \\
{[F' \otimes_{\mathcal{B}} (G' \otimes_{\mathcal{C}} H')]_A(D)} \arrow[r] & {[(F' \otimes_{\mathcal{B}} G') \otimes_{\mathcal{C}} H']_A(D).}
\end{tikzcd}$$

If we follow a tensor of the form $\zeta \otimes (\xi \otimes \eta)$ around this diagram, then it trivially commutes, and then to extend the result we can appeal to things like the linearity and continuity of all the maps involved. □

We omit the details for checking that the triangle and pentagon axioms are satisfied, but they are straightforward to verify.

7.2 The $A(\mathcal{B})$ Construction is a Pseudofunctor

One of the drawbacks of the category algebra construction is that it is, in general, not functorial for *-homomorphisms [12, p.652]. However, notice that if ${}_{\mathcal{B}}F_{\mathcal{C}}$ is a correspondence, then we can consider the composite

$$A(\mathcal{B}) \longrightarrow \mathcal{B} \xrightarrow{F} \mathcal{C} \longrightarrow A(\mathcal{C}),$$

where the unlabelled correspondences are equivalence bimodules. Consequently, we see that each correspondence between two C*-categories defines, in a very natural way, a correspondence between their category algebras.

First we need a definition of a bicategorical nature.

Definition 7.2.1. Let $S : \mathcal{C} \to \mathcal{B}$ be a pseudofunctor between bicategories. For $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we say that $X \in \mathcal{O}\mathrm{b}(\mathcal{C})$ and $u \in \mathcal{B}(B, S(X))$ are a *biuniversal arrow* from B to S if for every $C \in \mathcal{O}\mathrm{b}(\mathcal{C})$, the functor

$$\begin{aligned} \mathcal{C}(X, C) &\xrightarrow{\phi} \mathcal{B}(B, S(C)) \\ f &\mapsto S(f) \circ u \\ \theta &\mapsto S(\theta) * \mathrm{id}_u, \end{aligned}$$

is an equivalence of categories.

Now let's show that in our setting we have biuniversal arrows, where the pseudofunctor under consideration is the inclusion from the bicategory of C*-algebras to the bicategory of C*-categories.

Lemma 7.2.2. *Let $\mathcal{B}$ be a C*-category, and consider the pair $(A(\mathcal{B}), \widetilde{\mathcal{I}_{\mathcal{B}}})$ consisting of the category algebra of $\mathcal{B}$ and the equivalence bimodule $\widetilde{\mathcal{I}_{\mathcal{B}}}$. Then for any C*-algebra C, the functor*

$$\begin{aligned} \mathfrak{Corr}(A(\mathcal{B}), C) &\xrightarrow{\phi} \mathfrak{Corr}(\mathcal{B}, C) \\ F &\mapsto \widetilde{\mathcal{I}_{\mathcal{B}}} \otimes F \\ \theta &\mapsto \mathrm{id} \otimes \theta, \end{aligned}$$

is an equivalence of categories.

Proof. We define our candidate for an inverse functor by

$$\begin{gathered}\mathfrak{Corr}(\mathcal{B}, C) \xrightarrow{\psi} \mathfrak{Corr}(A(\mathcal{B}), C)\\ G \mapsto \mathcal{I}_{\mathcal{B}} \otimes G\\ \omega \mapsto \mathrm{id} \otimes \omega.\end{gathered}$$

We consider the composite $\phi \circ \psi$, noting that for each $G \in \mathfrak{Corr}(\mathcal{B}, C)$ we have a chain of invertible 2-cells

$$\widetilde{\mathcal{I}_{\mathcal{B}}} \otimes (\mathcal{I}_{\mathcal{B}} \otimes G) \Rightarrow (\widetilde{\mathcal{I}_{\mathcal{B}}} \otimes \mathcal{I}_{\mathcal{B}}) \otimes G \Rightarrow \mathrm{id}_{\mathcal{B}} \otimes G \Rightarrow G.$$

We use these compositions of 2-cells to define the components of our proposed natural isomorphism $\phi \circ \psi \Rightarrow \mathrm{id}$. We now need to check that for each $\theta : G \Rightarrow G'$, we have commutativity in the following square,

$$\begin{tikzcd} \phi \circ \psi(G) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & G \arrow[d, Rightarrow] \\ \phi \circ \psi(G') \arrow[r, Rightarrow] & G'. \end{tikzcd}$$

We expand this square using our construction of the horizontal arrows, and add some additional vertical arrows,

$$\begin{tikzcd} \widetilde{\mathcal{I}_{\mathcal{B}}} \otimes (\mathcal{I}_{\mathcal{B}} \otimes G) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\widetilde{\mathcal{I}_{\mathcal{B}}} \otimes \mathcal{I}_{\mathcal{B}}) \otimes G \arrow[r, Rightarrow] \arrow[d, Rightarrow, dashed] & \mathrm{id}_{\mathcal{B}} \otimes G \arrow[r, Rightarrow] \arrow[d, Rightarrow, dashed] & G \arrow[d, Rightarrow] \\ \widetilde{\mathcal{I}_{\mathcal{B}}} \otimes (\mathcal{I}_{\mathcal{B}} \otimes G') \arrow[r, Rightarrow] & (\widetilde{\mathcal{I}_{\mathcal{B}}} \otimes \mathcal{I}_{\mathcal{B}}) \otimes G' \arrow[r, Rightarrow] & \mathrm{id}_{\mathcal{B}} \otimes G' \arrow[r, Rightarrow] & G'. \end{tikzcd}$$

The left and right squares commute by naturality of associators and unitors in a bicategory, and the central square commutes by direct computation, hence our original square commutes and we have a natural isomorphism $\phi \circ \psi \Rightarrow \mathrm{id}$. Checking that we have a natural isomorphism $\psi \circ \phi \Rightarrow \mathrm{id}$ is similar; if $F \in \mathfrak{Corr}(A(\mathcal{B}), C)$, then we use the following chain of 2-cells to define the relevant component of our natural isomorphism

$$\mathcal{I}_{\mathcal{B}} \otimes (\widetilde{\mathcal{I}_{\mathcal{B}}} \otimes G) \Rightarrow (\mathcal{I}_{\mathcal{B}} \otimes \widetilde{\mathcal{I}_{\mathcal{B}}}) \otimes F \Rightarrow \mathrm{id}_{A(\mathcal{B})} \otimes F \Rightarrow F.$$

Checking the rest of the details follows much the same as we did above. □

We can say a little bit more about the functors involved in the previous result.

Lemma 7.2.3. *If $\mathcal{B}$ is a C^*-category, then for each C^*-algebra C the functors*

$$\mathfrak{Corr}(A(\mathcal{B}), C) \xrightarrow{\phi} \mathfrak{Corr}(\mathcal{B}, C), \qquad \mathfrak{Corr}(\mathcal{B}, C) \xrightarrow{\psi} \mathfrak{Corr}(A(\mathcal{B}), C),$$

are an adjoint pair.

Proof. We'll check the characterisation of adjoint functors via universal arrows to check that ψ is right adjoint to ϕ. If $F \in \mathfrak{Corr}(A(\mathcal{B}), C)$, then because ϕ and ψ constitute an equivalence of categories, we have an invertible 2-cell

$$\eta_F : F \Rightarrow \psi\phi(F),$$

and we will check that the pair $(\phi(F), \eta_F)$ are a universal arrow from ψ to F. Suppose that we have $G \in \mathfrak{Corr}(\mathcal{B}, C)$ and a 2-cell $\theta : F \Rightarrow \psi(G)$, then the composite $\theta \circ \eta_F^{-1}$ is a 2-cell $\psi\phi(F) \Rightarrow \psi(G)$. Because ψ is an equivalence of categories, we have a unique 2-cell $\psi^{-1}(\theta\eta_F^{-1}) : \phi(F) \Rightarrow G$ which makes the following triangle commute,

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\eta_F"] \arrow[dr, Rightarrow, "\theta"'] & \psi\phi(F) \arrow[d, Rightarrow, "\psi(\psi^{-1}(\theta\eta_F^{-1}))"] \\ & \psi(G) \end{tikzcd}$$

So it follows that ψ is right adjoint to ϕ. Similarly we can check that ϕ is left adjoint to ψ. If $G \in \mathfrak{Corr}(\mathcal{B}, C)$, then we have an invertible 2-cell

$$\epsilon_G : \phi\psi(G) \Rightarrow G.$$

Given $F \in \mathfrak{Corr}(A(\mathcal{B}), C)$ and a 2-cell $\theta : \phi(F) \Rightarrow G$, then look at the composite $\epsilon_G^{-1} \circ \theta : \phi(F) \Rightarrow \phi\psi(G)$. Because ϕ is an equivalence, we have a unique 2-cell $\phi^{-1}(\epsilon_G^{-1}\theta) : F \Rightarrow \psi(G)$ making the following diagram commute,

$$\begin{tikzcd} \phi(F) \arrow[d, Rightarrow, "\phi(\phi^{-1}(\epsilon_G^{-1}\theta))"'] \arrow[dr, Rightarrow, "\theta"] & \\ \phi\psi(G) \arrow[r, Rightarrow, "\epsilon_G"'] & G. \end{tikzcd}$$

$\square$

Remark 7.2.4. At this point, one might like to say that since we have these biuniversal arrows, then there exists a pseudofunctor that is left biadjoint to the inclusion functor from the bicategory of C*-algebras, to the bicategory of C*-categories. However we couldn't find any reference where it is shown that the existence of biuniversal arrows implies the existence of such a biadjoint. The closest we could find was [9, Theorem 9.16] where it is proved for pseudofunctors between 2-categories. The following result is the best progress we could make on this matter.

Lemma 7.2.5. *The assignment*

$$\begin{gathered}\mathcal{B} \mapsto A(\mathcal{B}) \\ \mathfrak{C}orr(\mathcal{B},\mathcal{C}) \ni F \mapsto A(F) := (\mathcal{I}_{\mathcal{B}} \otimes F) \otimes \widetilde{\mathcal{I}_{\mathcal{C}}} \\ \hom(F,G) \ni \theta \mapsto A(\theta) := (\mathrm{id} \otimes \theta) \otimes \mathrm{id},\end{gathered}$$

defines a pseudofunctor from the correspondence bicategory of C-categories to the correspondence bicategory of C*-algebras.*

Proof. Consider the inclusion pseudofunctor from the bicategory of C*-algebras to the bicategory of C*-categories. This pseudofunctor is essentially surjective on objects, because we have for each C*-category $\mathcal{B}$, a C*-algebra $A(\mathcal{B})$ and adjoint equivalence between $\mathcal{B}$ and $A(\mathcal{B})$. Trivially it gives an equivalence (equality!) between the relevant hom-categories, so that the bicategorical Whitehead theorem [13, Theorem 7.4.1] tells us that there is a pseudofunctor from the bicategory of C*-categories to the bicategory of C*-algebras. Picking through the proof in the referenced paper, one can check that the inverse pseudofunctor constructed there is defined on objects, 1-cells and 2-cells by the rules stated in this lemma. ☐

7.3 Equivalences in the Bicategory

To finish this chapter, we include the following result, classifying the equivalences in our bicategory.

Lemma 7.3.1. *A correspondence*

$$\mathcal{B} \xrightarrow{F} \mathcal{C}$$

is an equivalence in our bicategory of C-categories if and only if it is an equivalence bimodule.*

Proof. If F is an equivalence bimodule, then Theorem 6.4.1 shows that it is an equivalence in the bicategory. Conversely, if we assume that F is an equivalence, then we note that the pseudofunctor $A(-)$ automatically preserves equivalences, and that the equivalences in the bicategory of C*-algebras are precisely Morita equivalences. Consequently, the correspondence $A(F)$ is an equivalence bimodule between $A(\mathcal{B})$ and $A(\mathcal{C})$, so that $\mathcal{B}$ are Morita equivalent. This isn't quite the statement that we needed to prove, however by transitivity of Morita equivalence, the following tensor product is an equivalence bimodule,

$$\widetilde{\mathcal{I}_{\mathcal{B}}} \otimes A(F) \otimes \mathcal{I}_{\mathcal{C}}.$$

Moreover, we have the following chain of isomorphisms,

$$\widetilde{\mathcal{I}_{\mathcal{B}}} \otimes A(F) \otimes \mathcal{I}_{\mathcal{C}} \cong (\widetilde{\mathcal{I}_{\mathcal{B}}} \otimes \mathcal{I}_{\mathcal{B}}) \otimes F \otimes (\widetilde{\mathcal{I}_{\mathcal{C}}} \otimes \mathcal{I}_{\mathcal{C}}) \cong \mathcal{B} \otimes F \otimes \mathcal{C} \cong F,$$

which proves that F is an equivalence bimodule. $\square$

Chapter 8

Strongly Continuous *-functors

In this chapter we are going to be considering *-functors between categories of Hilbert modules, which satisfy an additional continuity condition. Our goal is to demonstrate that Morita equivalent C*-categories have equivalent categories of Hilbert modules.

We'll begin by introducing strongly continuous *-functors. Our definitions are adapted from [4][1].

Definition 8.0.1. If $\mathcal{A}$ is a C*-category and $\mathcal{E}, \mathcal{F} \in$ **Hilb**-$\mathcal{A}$, then a net of operators (T_λ) in $\mathcal{L}(\mathcal{E}, \mathcal{F})$ is said to converge *strongly* to $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ if for each $A \in \mathcal{O}\mathrm{b}(\mathcal{A})$, we have

$$\|(T_\lambda)_A \xi - T_A \xi\| \to 0,$$

for all $\xi \in \mathcal{E}(A)$. The net (T_λ) is said to converge **-strongly* to T if additionally we have

$$\|(T_\lambda^*)_A \xi - T_A^* \xi\| \to 0.$$

Definition 8.0.2. We say that a *-functor F : **Hilb**-$\mathcal{A} \to$ **Hilb**-$\mathcal{B}$ is strongly continuous if whenever (T_λ) is a bounded net in $\mathcal{L}(E, F)$ converging strongly to $T \in \mathcal{L}(E, F)$, the net $(F(T_\lambda))$ converges strongly to $F(T)$.

Perhaps the above definition feels slightly too weak, since it doesn't explicitly take into account any sort of continuity with respect to adjoints. This isn't a problem.

Proposition 8.0.3. *For a *-functor F : **Hilb**-$\mathcal{A} \to$ **Hilb**-$\mathcal{B}$, the following are equivalent;*

[1] It is worth noting that [4] uses different notations to our exposition, due to the fact they are working with categories of Hilbert modules over C*-algebras but with all bounded module maps (not just adjointables) as morphisms.

(i) *F is strongly continuous,*

(ii) *F is *-strongly continuous, where *-strong continuity is defined by taking Definition 8.0.2 and swapping strongly with *-strongly,*

(iii) *The restriction $F : \boldsymbol{Hilb}_{\mathcal{K}}\text{-}\mathcal{A} \to \boldsymbol{Hilb}\text{-}\mathcal{B}$ is a non-degenerate *-functor.*

Proof.

(i) $\Rightarrow$ *(ii)* If (T_λ) is a bounded net in $\mathcal{L}(\mathcal{E}, \mathcal{F})$ which converges *-strongly to $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, then we have the identity

$$\|F(T_\lambda)^* x - F(T)^* x\| = \|F(T_\lambda^*)x - F(T^*)x\|,$$

the right hand side of which converges to 0 because $F(T_\lambda^*)$ converges strongly to $F(T^*)$.

(ii) $\Rightarrow$ *(iii)* Let $\mathcal{E} \in \textbf{Hilb}\text{-}\mathcal{A}$ and let (e_λ) be an approximate unit for $\mathcal{K}(\mathcal{E})$. Then (e_λ) converges strongly to $\mathrm{id}_{\mathcal{E}}$ and so $F(e_\lambda)$ converges strongly to $\mathrm{id}_{F(\mathcal{E})}$ and it follows that $F : \mathcal{K}(\mathcal{E}) \to \mathcal{L}(F(\mathcal{E}))$ is a non-degenerate *-homomorphism.

(iii) $\Rightarrow$ *(i)* If we take a bounded net (T_λ) in $\mathcal{L}(\mathcal{E}, \mathcal{F})$ which converges strongly to $T \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, then it is easy to check that for a compact operator $K \in \mathcal{K}(\mathcal{E})$ we have $T_\lambda K \to TK$. For each $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$ and $\xi \in F(\mathcal{E})(B)$ we use Lemma 2.6.5 to write

$$\xi = K_B(\eta),$$

for some $K \in \mathcal{K}(F(\mathcal{E}))$ and $\eta \in F(\mathcal{E})(B)$. Given $\epsilon > 0$, we use our non-degeneracy assumption to find $K_i \in \mathcal{K}(\mathcal{E})$ and $T_i \in \mathcal{K}(F(\mathcal{E}))$ for $i = 1, \ldots, k$ such that

$$\|K - \sum_i F(K_i)T_i\| < \frac{\epsilon}{3\|\eta\|M},$$

where M is a constant satisfying $\|T_\lambda\| < M$ for all λ. Now we can make estimates as follows,

$$\begin{aligned}
\|F(T_\lambda)\xi - F(T)\xi\| &= \|F(T_\lambda)K_B(\eta) - F(T)K_B(\eta)\| \\
&\leq \begin{aligned}\|F(T_\lambda)(K_B(\eta) - \sum_i F(K_i)T_i(\eta))\| + \|\sum_i F(T_\lambda K_i - TK_i)T_i(\eta)\| \\ + \|F(T)(\sum_i F(K_i)T_i(\eta) - K_B(\eta))\|\end{aligned} \\
&\leq \|T_\lambda\|\frac{\epsilon}{3M} + \sum_i \|F(T_\lambda K_i - TK_i)\|\|T_i(\eta)\| + \|T\|\frac{\epsilon}{3M} \\
&\leq \frac{2\epsilon}{3} + \sum_i \|F(T_\lambda K_i - TK_i)\|\|T_i(\eta)\|.
\end{aligned}$$

Each operator K_i is compact, so by our opening remarks and automatic continuity of F, we know that $F(T_\lambda K_i) \to F(TK_i)$. So for each i, we can find λ_i such that $\lambda \geq \lambda_i$ implies

$$\|F(T_\lambda K_i - TK_i)\| < \frac{\epsilon}{3k\|T_i(\eta)\|},$$

If we majorize all the λ_i by λ_0, we see that $\lambda \geq \lambda_0$ implies

$$\sum_i \|F(T_\lambda K_i - TK_i)\|\|T_i(\eta)\| < \sum_i \|T_i(\eta)\|\frac{\epsilon}{3k\|T_i(\eta)\|} = \frac{\epsilon}{3}.$$

Hence it follows that $\lambda \geq \lambda_0$ implies

$$\|F(T_\lambda)\xi - F(T)\xi\| < \epsilon,$$

so we have strong convergence of $(F(T_\lambda))$ to $F(T)$ as required.

□

Example 8.0.4. Let F be an $\mathcal{A} - \mathcal{B}$ correspondence. We have a *-functor $(-) \otimes_{\mathcal{A}} F :$ **Hilb**-$\mathcal{A} \to$ **Hilb**-$\mathcal{B}$ defined on objects by

$$\mathcal{E} \mapsto \mathcal{E} \otimes_{\mathcal{A}} F,$$

and on morphisms by

$$T \mapsto T \otimes \mathrm{id}\,.$$

Checking that this is actually a *-functor can be done via similar arguments to those we used when constructing the composition bifunctors for the bicategory of correspondences. We will show that this *-functor is strongly continuous, so let (T_λ) be a bounded net in $\mathcal{L}(\mathcal{D},\mathcal{E})$ which converges strongly to $T \in \mathcal{L}(\mathcal{D},\mathcal{E})$. For $B \in \mathcal{O}\mathrm{b}(\mathcal{B})$, we first take a sum of simple tensors $x = \sum_{i=1}^{k} \xi_i \otimes \eta_i \in (\mathcal{E} \otimes_{\mathcal{A}} F)(B)$. Note here that each $\xi_i \in \mathcal{E}(A_i)$ and $\eta_i \in F_{A_i}(B)$ for $A_i \in \mathcal{O}\mathrm{b}(\mathcal{A})$. Now we can make an estimate like so,

$$\|(T_\lambda \otimes \mathrm{id})x - (T \otimes \mathrm{id})x\| = \|\sum_i (T_\lambda \xi_i - T\xi_i) \otimes \eta_i\| \leq \sum_i \|T_\lambda \xi_i - T\xi_i\|\|\eta_i\|.$$

For each i, we use the strong convergence of (T_λ) to T to find λ_i such that $\lambda \geq \lambda_i$ implies

$$\|T_\lambda \xi_i - T\xi_i\| < \frac{\epsilon}{k\|\eta_i\|}.$$

Now we majorize all the λ_i by λ_0, and we see that $\lambda \geq \lambda_0$ implies

$$\sum_i \|T_\lambda \xi_i - T\xi_i\|\|\eta_i\| < \sum_i \frac{\epsilon}{k\|\eta_i\|}\|\eta_i\| = \epsilon.$$

Now we let $x \in (\mathcal{E} \otimes_{\mathcal{A}} F)(B)$ be arbitrary. Our assumption that the net (T_λ) is bounded means that we can find some constant M such that $\|T_\lambda \otimes \mathrm{id}\,\| \leq M$ for all λ and $\|T \otimes \mathrm{id}\,\| \leq M$. For $\epsilon > 0$ we estimate x by a sum of simple tensors, so we find $\xi_i \in \mathcal{E}(A_i)$, $\eta_i \in F_{A_i}(B)$, such that

$$\|x - \sum_i \xi_i \otimes \eta_i\| < \frac{3\epsilon}{M}.$$

For clarity in the following estimates, we define $y = \sum_{i=1}^{k} \xi_i \otimes \eta_i$. Then,

$$\begin{aligned}
\|(T_\lambda \otimes \mathrm{id})x - (T \otimes \mathrm{id})x\| &\leq \|(T_\lambda \otimes \mathrm{id})(x-y)\| + \|((T_\lambda \otimes \mathrm{id})y - (T \otimes \mathrm{id}))y\| + \|(T \otimes \mathrm{id})(y-x)\| \\
&\leq \|T_\lambda \otimes \mathrm{id}\,\|\|x-y\| + \|T \otimes \mathrm{id}\,\|\|x-y\| + \|\sum_i (T_\lambda - T)\xi_i \otimes \eta_i\| \\
&\leq \frac{2\epsilon}{3} + \sum_i \|(T_\lambda - T)\xi_i \otimes \eta_i\|.
\end{aligned}$$

From our earlier work, we can find λ_0 such that $\lambda \geq \lambda_0$ implies

$$\|(T_\lambda - T)\xi_i \otimes \eta_i\| < \frac{\epsilon}{3},$$

which implies that the net $(T_\lambda \otimes \mathrm{id})$ converges strongly to $T \otimes \mathrm{id}$ and we are done.

8.1 Theorems Involving Strongly Continuous *-Functors

The goal of this section is to prove that a pair of C*-categories are Morita equivalent if and only their categories of right Hilbert modules are equivalent, with the equivalence implemented by a pair of strongly continuous *-functors.

Lemma 8.1.1. *Let $\mathcal{B}$ be a C*-categories, and write* $\mathrm{id}_{\mathcal{B}}$ *for the correspondence* $\mathcal{B}(-,=)$*. Then we have a unitary isomorphism of *-functors*

$$(-) \otimes_{\mathcal{B}} \mathrm{id}_{\mathcal{B}} \Rightarrow \mathrm{id}_{\textbf{\textit{Hilb}}\text{-}\mathcal{B}}\,.$$

Proof. This is similar to the construction of unitors in the bicategory of correspondnces, so we will only sketch it. To start, we need to construct, for each right Hilbert $\mathcal{B}$-module $\mathcal{E}$ a unitary isomorphism

$$\mathcal{E} \otimes_{\mathcal{B}} \mathrm{id}_{\mathcal{B}} \Rightarrow \mathcal{E},$$

and the starting point of this isomorphism is to define the following map, for each pair $B, B' \in \mathcal{O}\mathrm{b}(\mathcal{B})$,

$$\begin{aligned} \mathcal{E}(B') \times \mathcal{B}(B, B') &\to \mathcal{E}(B) \\ (\eta, b) &\mapsto \eta \cdot b. \end{aligned}$$

Then we use the universal property of coends to deduce that we have a well defined linear map

$$\mathcal{E}(B) \boxdot_{\mathcal{B}} \mathcal{B}(B, -) \to \mathcal{E}(B).$$

Lemma 2.3.1 explains that the image of each of these maps is dense, and one can verify that they are inner product preserving via direct computation. Our earlier characterisation of unitary isomorphisms then shows that we have isomorphisms

$$\mathcal{E} \otimes_{\mathcal{B}} \mathrm{id}_{\mathcal{B}} \Rightarrow \mathcal{E},$$

as required. Verifying naturality is also straightforward from the observation that any adjointable operator $T : \mathcal{E} \Rightarrow \mathcal{E}'$ satisfies

$$T(\eta \cdot b) = T(\eta) \cdot b.$$

$\square$

Combining this lemma with Theorem 6.4.1 yields the following.

Corollary 8.1.2. *If F is an $\mathcal{A}-\mathcal{B}$ equivalence bimodule, then there is an equivalence of categories*

$$\textbf{\textit{Hilb}}\text{-}\mathcal{A} \simeq \textbf{\textit{Hilb}}\text{-}\mathcal{B},$$

implemented by the strongly continuous *-functors $(-)\otimes_{\mathcal{A}} F$ and $(-)\otimes_{\mathcal{B}} \widetilde{F}$.

Proof. Using the notation from the previous Lemma, we know from Theorem 6.4.1 that $F \otimes_{\mathcal{B}} \widetilde{F} \cong \mathrm{id}_{\mathcal{A}}$. To verify our claim, we need to show that we have a natural isomorphism

$$((-)\otimes_{\mathcal{A}} F)\otimes_{\mathcal{B}} \widetilde{F} \simeq \mathrm{id}_{\textbf{Hilb-}\mathcal{A}}\,.$$

For any bounded adjointable operator $T : \mathcal{E} \Rightarrow \mathcal{E}'$ we consider the following diagram

$$\begin{tikzcd}
(\mathcal{E}\otimes_{\mathcal{A}} F)\otimes_{\mathcal{B}} \widetilde{F} \arrow[r] \arrow[d] & \mathcal{E}\otimes_{\mathcal{A}} (F\otimes_{\mathcal{B}} \widetilde{F}) \arrow[r] \arrow[d] & \mathcal{E}\otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A}} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
(\mathcal{E}'\otimes_{\mathcal{A}} F)\otimes_{\mathcal{B}} \widetilde{F} \arrow[r] & \mathcal{E}'\otimes_{\mathcal{A}} (F\otimes_{\mathcal{B}} \widetilde{F}) \arrow[r] & \mathcal{E}'\otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A}} \arrow[r] & \mathcal{E}'
\end{tikzcd}$$

The rightmost square commutes by the previous Lemma, and the other two commute by direct computation. This confirms that we have a natural isomorphism

$$((-)\otimes_{\mathcal{A}} F)\otimes_{\mathcal{B}} \widetilde{F} \simeq \mathrm{id}_{\textbf{Hilb-}\mathcal{A}},$$

as required, and the other case is similar. □

To finish this chapter, we present the converse.

Proposition 8.1.3. *If $\mathcal{B}$ and $\mathcal{C}$ are C^*-categories, and we have a strongly continuous *-functor*

$$F : \textbf{\textit{Hilb}}\text{-}\mathcal{B} \to \textbf{\textit{Hilb}}\text{-}\mathcal{C},$$

which is an equivalence of categories, then $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent.

Proof. Consider the following diagram of strongly continuous *-functors.

$$\begin{tikzcd}
\textbf{Hilb-}\mathcal{B} \arrow[r, "F"] & \textbf{Hilb-}\mathcal{C} \arrow[d, "(-)\otimes_{\mathcal{C}}\widetilde{\mathcal{I}_{\mathcal{C}}}"] \\
\textbf{Hilb-}A(\mathcal{B}) \arrow[u, "(-)\otimes_{A(\mathcal{B})}\mathcal{I}_{\mathcal{B}}"] & \textbf{Hilb-}A(\mathcal{C}).
\end{tikzcd}$$

The composition of these three functors gives a strongly continuous *-functor $\mathbf{Hilb}\text{-}A(\mathcal{B}) \to \mathbf{Hilb}\text{-}A(\mathcal{C})$ which moreover is an equivalence of categories, since each of the functors involved is an equivalence. It now follows from [4, Theorem 5.5] that $A(\mathcal{B})$ and $A(\mathcal{C})$ are Morita equivalent, hence $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent. $\square$

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