Harmonic maps, inverse problems, and related topics

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Abstract

In this thesis we consider Calderón's problem for harmonic maps in real-analytic setting. In the first chapter we provide foundational and background material such as the existence and uniqueness of a solution to the Dirichlet problem for the connection Laplacian, and the existence and uniqueness of the Dirichlet Green kernel. In the second chapter we discuss the properties of the Dirichlet-to-Neumann operator associated to the connection Laplacian and prove a result on reconstruction of geometric data on the boundary from a given Dirichlet-to-Neumann operator. We then use this to prove a uniqueness result for Calderón's inverse problem for the connection Laplacian on a vector bundle. In the third chapter we generalise the notion of the Dirichlet-to-Neumann operator to maps between manifolds and discuss what kind of difficulties arise along the way. We conclude the chapter with the uniqueness result for Calderón's inverse problem for maps between real-analytic manifolds.

Notation

$\mathbb{R}^n$	n-dimensional Euclidean space
$\|g\|$	determinant of a metric g
Tr_g	trace with respect to a metric g
div_g	divergence operator with respect to a metric g
grad_g	gradient operator with respect to a metric g
Δ_g	$= -\mathrm{div}_g \circ \mathrm{grad}_g$, Laplace-Beltrami operator
$\Gamma(E)$	space of smooth sections of a vector bundle E over a manifold N
$\mathscr{D}(E)$	space of test sections of E, i.e. the subspace of $\Gamma(E)$ formed by smooth sections whose supports lie in the interior of the base N
L^p	p-Lebesgue space
$\mathscr{W}^s$	$(s,2)$-Sobolev space
∇^E	connection on a vector bundle E
Δ^E	connection Laplacian on a vector bundle E
$[X,Y]$	homotopy classes of maps from X to Y
$\pi_i(X)$	i-th homotopy group of X
$\pi_i(X,A)$	i-th relative homotopy group of a pair (X,A)
$H_i(X)$	i-th homology group of X
$H_i(X,A)$	i-th relative homology group of (X,A)
$H^i(X)$	i-th cohomology group of X
$H^i(X,A)$	i-th relative cohomology group of (X,A)
$K(\Gamma,i)$	Eilenberg-MacLane space with i-th homotopy group Γ
$a_i a^i$	denotes the sum $\sum_i a_i a^i$, i.e. the Einstein summation convention is assumed throughout the text

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Chapter 0

Introduction

In this thesis we consider the generalisation of the geometric Calderón problem to maps between manifolds. Let us state the classical geometric Calderón problem. Recall that the Laplace-Beltrami operator Δ_g on a Riemannian manifold (N, g) is defined by

$$\Delta_g = -\mathrm{div}_g \circ \mathrm{grad}_g,$$

where div_g and grad_g are the divergence and gradient operators associated with the metric g. In local coordinates the Laplace-Beltrami operator is given by

$$\Delta_g u = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial u}{\partial x^j} \right),$$

where g^{ij} represent elements of the inverse matrix to g in the associated coordinate frame, and $|g|$ denotes the determinant of the matrix g in this frame. Let N be a compact manifold with boundary ∂N. Consider the following Dirichlet problem

$$\begin{cases} \Delta_g u = 0 & \text{in } N, \\ u = f & \text{on } \partial N, \end{cases} \tag{1}$$

and the DtN operator

$$\Lambda_g(f) = \nu_i g^{ij} \frac{\partial u}{\partial x^j} \sqrt{|g|}.$$

The *geometric Calderón problem* then is to reconstruct a Riemannian metric g from the DtN map Λ_g. The problem in this form is a geometric restatement [31] of an inverse problem in physics, which has important applications in the real world. One of these applications is Electrical Impedance Tomography (EIT). In short, the process of EIT goes as follows. Electrodes are attached to the surface of an object of study; some voltage is applied to the electrodes and the current created in the electrodes in response to the applied voltage is measured; the inverse problem of reconstructing the data (such as conductivity) inside the object from the voltage-to-current measurements is solved; different conductivities in various parts of the object are treated as representing different tissues, materials, or irregularities. The mathematical part of this process is the solution of the inverse problem. It was first studied by Alberto Calderón [4, 5] and it is customary to call it Calderón's problem.

Let us briefly discuss the underlying physics of Calderón's problem. It is well known that the static electric field $\vec{E}$ can be written in the form

$$\vec{E} = \vec{\nabla} u,$$

where $\vec{\nabla}$ denotes the gradient operator and u is a function called the potential of the electric field $\vec{E}$. Note that we consider the opposite (to the one used in physics) sign convention. Let us assume first, for simplicity, that the conductivity inside the object is isotropic. Using Ohm's law we obtain the following expression for the electric current density

$$\vec{j} = \gamma \vec{E} = \gamma \vec{\nabla} u, \tag{2}$$

where the function γ represents the conductivity. Now, assuming that no charges are created or destroyed inside the object, the continuity equation should hold for the electric current. Mathematically this is represented by the formula

$$\vec{\nabla} \cdot \vec{j} = 0, \tag{3}$$

where $\vec{\nabla}\cdot$ denotes the divergence operator. Combining this relation with (2) we obtain the following equation for the potential of the electric field

$$\vec{\nabla} \cdot \gamma \vec{\nabla} u = \frac{\partial}{\partial x^i}\left(\gamma \frac{\partial u}{\partial x^i}\right) = 0. \tag{4}$$

Here and throughout the text we assume the summation convention over repeated indices. When we apply some potential (voltage) f to the surface of the object an electric field is created, such that the corresponding potential satisfies (4). Since the potential is a continuous function, the condition $u = f$ on the surface should be satisfied. Summarising everything, we obtain the following Dirichlet problem

$$\begin{cases} \vec{\nabla} \cdot \gamma \vec{\nabla} u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Here the domain Ω represents the interior of the object, and the boundary $\partial\Omega$ represents its surface. Now, using Ohm's law again we can write the expression for the current flux through the surface of the object

$$\Lambda_\gamma f = \vec{j}\Big|_{\partial\Omega}^{\perp} = \gamma\,\vec{E}\Big|_{\partial\Omega}^{\perp} = \gamma \frac{\partial}{\partial\,\nu} u\Bigg|_{\partial\Omega}, \tag{6}$$

where ν is the outward unit normal vector to the boundary $\partial\Omega$ and $\perp$ indicates the projection to the normal vector ν. The map Λ_γ which sends a (voltage) function on the boundary to a (current flux) function on the boundary is usually simply called in physics the *voltage-to-current map*. In mathematics this map is usually called the *Dirichlet-to-Neumann (DtN) map (or operator)* associated with the operator

$$L_\gamma = \vec{\nabla} \cdot \gamma \vec{\nabla}, \tag{7}$$

The name comes from the fact that it maps Dirichlet boundary data to Neumann boundary data. Suppose we are given a voltage-to-current (DtN) map on the surface of an object (boundary of a domain). The Calderón problem is then to decide if the conductivity γ is uniquely determined by this voltage-to-current (DtN) map Λ_γ and if it is, then to reconstruct γ from it. This isotropic inverse problem was studied by A. Calderón in [4]. Note that the author used an equivalent formulation in terms of the quadratic form

$$Q_\gamma(f) = \int_\Omega \gamma \left|\vec{\nabla} u\right|^2 dV$$

instead of the map Λ_γ. The physical meaning of $Q_\gamma(f)$ is the power necessary to maintain the potential f on the boundary $\partial\Omega$. The author showed that in the case where γ is sufficiently close to a constant it can be nearly determined by Q_γ and in some cases it can be calculated with a relatively small error. This pioneering work of Calderón led to many developments in inverse problems. See [43] for a comprehensive survey on the topic of EIT and Calderón's problem.

In principle, the conductivity γ inside the object of study need not be isotropic. One important example of an anisotropic conductor is the muscle tissue in the human body. Mathematically, anisotropic conductivity can be represented as a positive definite, smooth, symmetric matrix function $\gamma = \left(\gamma^{ij}\right)$ on Ω. The Dirichlet problem (in Euclidean coordinates) in this case is

$$\begin{cases} \frac{\partial}{\partial x^i}\left(\gamma^{ij}\frac{\partial u}{\partial x^j}\right) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

and the voltage-to-current (DtN) map is defined by

$$\Lambda_\gamma f = \nu_i \gamma^{ij} \frac{\partial u}{\partial x^j}\bigg|_{\partial\Omega}.$$

Unfortunately, in this case Λ_γ does not determine γ uniquely. Namely, any change of variables in Ω that leaves $\partial\Omega$ fixed gives rise to a new conductivity with the same voltage-to-current map. Refer to [43] for a more detailed discussion on this topic. Let us now point out how this problem is related to the geometric Calderón problem discussed above. If N is an open, bounded subset of $\mathbb{R}^n$ with smooth boundary it was noted in [31] that

$$\Lambda_g = \Lambda_\gamma,$$

where

$$\left(g_{ij}\right) = |\gamma|^{\frac{1}{n-2}}\left(\gamma^{ij}\right)^{-1}, \text{ and } \left(\gamma^{ij}\right) = \sqrt{|g|}\left(g_{ij}\right)^{-1}.$$

So in this case the geometric problem is equivalent to the one that arises in physics.

Let us return to the geometric Calderón problem. Note that in general one has

$$\Lambda_{\psi^* g} = \Lambda_g,$$

where ψ is any diffeomorphism of N which is the identity on ∂N. Here $\psi^* g$ denotes the pull-back metric along ψ. This fact poses an obstacle to the unique reconstruction of a metric from the DtN map. In the two-dimensional case, the Laplace-Beltrami operator is conformally contravariant and, as a result, the DtN operators associated with conformal metrics that are equal on the boundary coincide, i.e.

$$\Lambda_{cg} = \Lambda_g,$$

where c is a conformal factor equal to 1 on the boundary. Hence, in two dimensions this is an additional obstacle to the unique reconstruction of the metric. The natural question to ask is if those are the only obstructions to the unique recovery of the metric from the DtN operator. The affirmative answer to this question was given in [27]. Namely, the authors showed that the DtN operator given on an open subset $W \subset \partial N$ determines a Riemannian surface N and the conformal class of metrics on N that coincide in W. In the same paper the authors also showed a similar result for higher dimensional manifolds (≥ 3) assuming that the structures are real analytic. More precisely, the authors proved that the DtN operator given on an open subset $W \subset \partial N$, in which ∂N is real analytic, determines a connected compact real analytic Riemannian manifold (N, g). This result was further generalised in [28] by relaxing the compactness hypothesis to a completeness hypothesis, assuming that ∂N remains compact. At the moment the problem for general compact connected Riemannian manifolds remains open. We refer the reader to the survey [43] for the history of the problem and a more detailed review of the results.

One way to generalise the geometric Calderón problem is to consider an inverse problem for harmonic forms similar to the inverse problem in Proposition A.2.2 which we use to prove Theorem 3.2.3 in Subsection 3.2.4. This problem was considered in [25]. The other way to extend this results is to consider an inverse problem for the connection Laplacian on a vector bundle. We discuss and study this problem in Chapter 2.

There is another way to define the Dirichlet-to-Neumann operator Λ, which is also common in the literature. We will use this definition throughout the text and call the DtN operator defined in this way the *classical (or scalar) DtN operator*. Namely, we define the classical DtN operator by the formula

$$\Lambda f = \left.\frac{\partial u}{\partial \nu}\right|_{\partial N}, \tag{8}$$

where ν is the outward unit normal vector field and u is the unique solution to the Dirichlet problem (1). It is well known (and will be shown later for more general situation) that this operator has nice properties. More precisely, it is a classical elliptic self-adjoint pseudodifferential operator of order one.

Let us talk briefly about the structure and contents of this thesis. In the first chapter we provide foundational and background material. We start with the necessary definitions and results in the theory of pseudodifferential operators, especially acting on sections of vector bundles. For more detailed exposition we refer the reader to the books [40, 42]. We continue with the definition and discussion of the connection

Laplacian operator, which is a natural generalisation of the Laplace-Beltrami operator to vector bundles with a connection. Alongside, we define and discuss the Schrödinger (type) operators, which play an important role in the proof of the main result in Chapter 3. We provide some properties of these operators, such as the symmetry and ellipticity, and then use them to show the existence and uniqueness of the solution to the Dirichlet problem. Exploiting this we define the associated Dirichlet-to-Neumann operator and discuss its properties. Our next section is devoted to the construction of the Dirichlet Green kernel and discussion of its properties. Here we utilise a slight modification of the classical method for the construction of Green's function [cf. 2]. After this we give a brief explanation of a more general approach to the construction of Green's kernels based on the theory of pseudodifferential operators. The next section describes a link between the Dirichlet-to-Neumann operator and single- and double-layer potentials [41].

In the second chapter we discuss the properties of the Dirichlet-to-Neumann operator associated with the connection Laplacian and prove a result on reconstruction of geometric data on the boundary from a given Dirichlet-to-Neumann operator. This part follows the method presented in [31], which was also used in [7] to obtain similar results for the case of Hermitian vector bundles, though our main result for the Schrödinger operators differs because of the way we want to apply it. We then use this local reconstruction to prove a uniqueness result for the Calderón inverse problem for the connection Laplacian on a vector bundle. This part follows the idea of immersions by Green's functions (kernels) exploited in the paper [28] for the scalar Dirichlet-to-Neumann operator. Note, that in their paper the authors consider the manifold to be at least 3-dimensional complete real-analytic (possibly) non-compact with compact boundary. In [27] the authors used a different approach, sheaves of analytic functions, to obtain the result for compact manifolds, including 2-dimensional ones.

In the third chapter we generalise the notion of the Dirichlet-to-Neumann operator to maps between manifolds and discuss the difficulties that arise in this case. Namely, the Dirichlet-to-Neumann operator is not always well defined. The first issue arises when we try to find harmonic extension of a map from the boundary to the whole manifold. In contrast to the classical case where the target manifold is the Euclidean real line we can have non-trivial homotopy type of the target manifold. Because of this, on the one hand, it is not always possible to extend a map from the boundary to a map from the whole manifold. On the other hand, when this extension is possible there may be more than one homotopy class of them. These issues are illustrated in the counterexamples section. In the next section we discuss the arising general topological extension problem and particular cases such as the case of target manifolds of the Eilenberg-MacLane type, the case of maps between surfaces, and the case of maps to a circle, which has a link to harmonic 1-forms. After this we give examples of generalisation of Calderón's problem for maps from domains of dimensions 1 and 2. These examples are then followed by the statement of the general Calderón problem for maps between manifolds. In contrast to the classical case, this problem is non-linear and, therefore, we proceed by obtaining results on its linearisation. We then show that the latter linearised problem is equivalent to Calderón's problem for Schrödinger type operators discussed in Chapter 2. We conclude the third chapter with a uniqueness result for the Calderón inverse problem for maps between real-analytic manifolds.

Chapter 1

Background material

1.1 Pseudodifferential operators

In this section we will recall the definition of a (standard) pseudodifferential operator (PDO) on vector bundles. We follow the exposition by Treves in [42]. Let Ω be an open subset of $\mathbb{R}^n$. We need first to define the special class of functions called *amplitudes*. Let m be any real number. We shall denote by $S^m(\Omega,\Omega)$ the linear space of C^∞ functions in $\Omega\times\Omega\times\mathbb{R}^n$, $a(x,y,\xi)$, which have the following property:

To every compact subset $\mathscr{K}$ of $\Omega\times\Omega$ and to every triplet of n-tuples α,β,γ, there is a constant $C_{\alpha,\beta,\gamma}(\mathscr{K})>0$ such that

$$\left|D_\xi^\alpha D_x^\beta D_y^\gamma a(x,y,\xi)\right|\leq C_{\alpha,\beta,\gamma}(\mathscr{K})(1+|\xi|)^{m-|\alpha|},\ \forall(x,y)\in\mathscr{K},\text{and }\forall\xi\in\mathbb{R}^n,\quad(1.1)$$

where $D_{z_i}:=-i\partial_{z_i}$. The elements of $S^m(\Omega,\Omega)$ are called amplitudes of degree $\leq m$ (in $\Omega\times\Omega$). The space $S^m(\Omega,\Omega)$ is endowed with a natural locally convex topology: denote by $\mathfrak{p}_{\mathscr{K};\alpha,\beta,\gamma}(a)$ the infimum of the constants $C_{\alpha,\beta,\gamma}(\mathscr{K})$ such that (1.1) is true.

One can see then that the function $\mathfrak{p}_{\mathscr{K};\alpha,\beta,\gamma}$ is a seminorm on $S^m(\Omega,\Omega)$ and defines the topology of this space when $\mathscr{K}$ ranges over the collection of all compact subsets of Ω and α,β,γ over that of all n-tuples. Thus topologised, $S^m(\Omega,\Omega)$ is a Fréchet space. We denote the subspace of amplitudes independent of y by $S^m(\Omega)$ and regard its elements as smooth functions in $\Omega\times\mathbb{R}^n$ (rather than in $\Omega\times\Omega\times\mathbb{R}^n$). The topology on $S^m(\Omega)$ is the induced subspace topology from $S^m(\Omega,\Omega)$. Hence, $S^m(\Omega)$ is a Fréchet space, as a closed linear subspace of the Fréchet space. The intersection of all $S^m(\Omega)$ is denoted by $S^{-\infty}(\Omega)$. The quotient vector space $S^m(\Omega)/S^{-\infty}(\Omega)$ is denoted by $\dot{S}^m(\Omega)$ and its elements are called *symbols* of degree $\leq m$. We use the term symbol for a representative ($\in S^m(\Omega)$) of an equivalence class in $\dot{S}^m(\Omega)$ as well. We shall also introduce the notion of a formal symbol. By a *formal symbol* we mean a sequence of symbols $a_{m_j}\in S^{m_j}(\Omega)$ whose orders m_j are strictly decreasing and converging to $-\infty$. It is standard to represent it by the formal series

$$\sum_{j=0}^{+\infty} a_{m_j}(x,\xi). \tag{1.2}$$

From such a formal symbol one can build true symbols, elements of $S^{m_0}(\Omega)$ in the present case, which all belong to the same class modulo $S^{-\infty}(\Omega)$. We will denote this class by (1.2). In order to construct a true symbol one may proceed as follows:

First, one may state that a symbol $a(x,\xi)$ belongs to the class (1.2) if, given any large positive number M, there is an integer $J\geq 0$ such that

$$a(x,\xi)-\sum_{j=0}^{J} a_{m_j}(x,\xi)\in S^{-M}(\Omega).$$

Second, one can construct such a symbol $a(x,\xi)$ as a sum of a series

$$\sum_{j=0}^{+\infty} \chi_j(\xi)\, a_{m_j}(x,\xi),$$

where $\chi_j(\xi)$ are suitable cutoff functions. By suitable we mean such that the above series converges in the space $S^{m_0}(\Omega)$. Note that since we are using the cutoff functions we are able to deal with terms $a_{m_j}(x,\xi)$ that are not "true" elements of $S^{m_j}(\Omega)$, e.g. the functions that are non-smooth or even not defined in neighborhoods of the origin $\xi = 0$. (If such neighborhoods depend on x we may consider the cutoff functions that also depend on x). The most important examples of formal symbols (1.2) with terms $a_{m_j}(x,\xi)$ that are not C^∞ functions of ξ at the origin are the classical symbols. The formal symbol (1.2) is called the *classical symbol* if each term $a_{m_j}(x,\xi)$ is a positive-homogeneous function of degree m_j of ξ and if differences $m_j - m_{j+1} \in \mathbb{Z}^+$, for all j. Recall that a function $f(\xi)$ is called *positive-homogeneous* of degree d with respect to ξ in $\mathbb{R}^n \setminus \{0\}$ if $f(\rho\xi) = \rho^d f(\xi)$ for every $\rho > 0$ (but not necessarily for every real ρ). For instance, the Heaviside function on $\mathbb{R}\setminus\{0\}$ is positive-homogeneous, but not homogeneous, of degree zero.

As is customary in the literature, we denote by $\mathscr{D}'(\Omega)$ and $\mathscr{E}'(\Omega)$ the spaces of distributions and compactly supported distributions, i.e. the continuous duals to $C_0^\infty(\Omega)$ and $C^\infty(\Omega)$, respectively, both assumed to be equipped with the weak topology. Let us recall the Schwartz kernel theorem [22].

Theorem. *Every $K \in \mathscr{D}'(\Omega\times\Omega)$ defines a continuous linear map A from $C_0^\infty(\Omega)$ to $\mathscr{D}'(\Omega)$ via*

$$A\varphi(\psi) = K(\varphi\otimes\psi), \quad \varphi,\psi \in C_0^\infty(\Omega). \tag{1.3}$$

Conversely, to any continuous linear map $A : C_0^\infty(\Omega) \to \mathscr{D}'(\Omega)$, *there corresponds a unique distribution* $K_A(x, y)$ *in* $\Omega \times \Omega$ *such that* (1.3) *holds. The distribution* K_A *is called the Schwartz kernel of* A.

Let us continue with the definition of a (standard) PDO.

Definition. A linear operator $A : C_0^\infty(\Omega) \to \mathscr{D}'(\Omega)$ is called a (standard) pseudodifferential operator of order m if there is an element $a \in S^m(\Omega)$ such that A can be represented in the form

$$Au(x) = (2\pi)^{-n} \int \int e^{i(x-y)\cdot\xi} a(x, \xi) u(y)\, dy\, d\xi, \tag{1.4}$$

where it is understood that the integration with respect to y is effected first and the one with respect to ξ last. The function $a(x, \xi)$ is called a *(complete) symbol of* the PDO A. A PDO is called a *classical pseudodifferential operator* if it has a classical symbol.

The set of standard pseudodifferential operators of order m will be denoted by $\Psi^m(\Omega)$. The union of these sets for all $m \in \mathbb{R}$ will be denoted by $\Psi(\Omega)$ and their intersection by $\Psi^{-\infty}(\Omega)$. Let us give an example of PDOs from which the motivation to consider them originates.

Example. Consider a linear differential operator A with smooth coefficients acting in an open subset Ω of $\mathbb{R}^n$. Namely, let

$$A = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \tag{1.5}$$

where α is a multiindex, e.g. $\alpha = (\alpha_1, \dots, \alpha_n)$ and α_j is a non-negative integer, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $a_\alpha(x)$ are smooth functions of $x \in \Omega$, $D^{\alpha_j} = (-i)^{\alpha_j}\, \partial^{\alpha_j}/\partial x_j^{\alpha_j}$, and $D^\alpha =$

$D^{\alpha_1}\dots D^{\alpha_n}$. Using the Fourier transform, we see that

$$D^{\alpha}u(x)=(2\pi)^{-n}\int\int\xi^{\alpha}e^{i(x-y)\cdot\xi}u(y)\,dy\,d\xi,$$

where $\xi\in\mathbb{R}^n$, $\xi^{\alpha}=\xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}$, and $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$ is the standard Euclidean product on $\mathbb{R}^n$. Combining this with the expression (1.5) we obtain

$$Au(x)=(2\pi)^{-n}\int\int e^{i(x-y)\cdot\xi}\sigma_A(x,\xi)u(y)\,dy\,d\xi, \tag{1.6}$$

where the polynomial on ξ function $\sigma_A(x,\xi)=\sum_{|\alpha|\leq m}a_{\alpha}(x)\xi^{\alpha}$ is the usual symbol of the differential operator A. One can see that $\sigma_A(x,\xi)\in S^m(\Omega)$ and, hence, the operator A is a pseudodifferential operator of order m. Recall that the part of a (complete) symbol corresponding to the highest derivatives, i.e. $p_A(x,\xi)=\sum_{|\alpha|=m}a_{\alpha}(x)\xi^{\alpha}$, is called the *principal symbol* of A.

Note that we can readily generalise the notion of a principal symbol to classical pseudodifferential operators. By a *principal symbol* of a classical PDO $A\in\Psi^m(\Omega)$ with formal symbol given by (1.2) we mean $a_{m_0}(x,\xi)$, which is the positive-homogeneous term of highest degree $m_0=m$. The following properties of PDOs are well known [40, 42].

Proposition 1.1.1. *Let A be a pseudodifferential operator. Then its Schwartz kernel K_A is smooth off the diagonal, i.e. $K_A\in C^{\infty}(\Omega\times\Omega\backslash diag(\Omega))$, where the diagonal is defined by $diag(\Omega):=\{(x,x)\in\Omega\times\Omega\}$, and A defines continuous linear maps*

$$A:C_0^{\infty}(\Omega)\to C^{\infty}(\Omega),$$

$$A:\mathscr{E}'(\Omega)\to\mathscr{D}'(\Omega).$$

Remark. It is customary in the literature to define the PDOs using amplitudes $a \in S^m(\Omega, \Omega)$ in (1.4) instead of the symbols, but these two definitions in fact coincide [40, 42]. Although the definition involving amplitudes has considerable expository advantages in the general theory of PDOs it is redundant in this text, since we will be exploiting only the symbols of PDOs.

We shall say that a PDO A is smoothing (regularizing) if it maps $\mathscr{E}'(\Omega)$ to $C^\infty(\Omega)$. In order for this to be the case, it is necessary and sufficient for the associated Schwartz kernel $K_A(x,y)$ to be C^∞ in $\Omega \times \Omega$. In terms of symbols, any smoothing operator has a symbol $a \in S^{-\infty}(\Omega)$. The following theorem [42, Theorem 2.1] gives a more precise description of the "regularizing" properties of a general PDO.

Theorem 1.1.2. *Let A be a pseudodifferential operator in Ω, of order $\leq m$. Given any real number s, the mapping $u \mapsto Au$ can be extended as a continuous linear mapping $\mathscr{W}^s_c(\Omega) \to \mathscr{W}^{s-m}_{loc}(\Omega)$ of Sobolev spaces.*

Note that we are using the definition of Sobolev spaces according to [42], that is by $\mathscr{W}^s$ we denote the $(s,2)$-Sobolev space, i.e. the space of functions square integrable together with first s derivatives (when s is a non-negative integer). For example, $\mathscr{W}^0$ denotes L^2. For a compact subset $K \subset \mathbb{R}^n$ we denote by $\mathscr{W}^s_c(K)$ the subspace of $\mathscr{W}^s(\mathbb{R}^n)$ consisting of a distributions having their support in a compact subset K. By $\mathscr{W}^s_c(\Omega)$ we denote the union of the spaces $\mathscr{W}^s_c(K)$ for K ranging over the collection of all compact subsets of Ω. Finally, $\mathscr{W}^s_{loc}(\Omega)$ denotes the space of distributions u in Ω such that $\phi u \in \mathscr{W}^s(\mathbb{R}^n)$ for any smooth function ϕ compactly supported in Ω.

Now we are able to define pseudodifferential operators acting on vector-valued distributions. Let $\mathbb{F}^r$ be a r-dimensional vector space over field $\mathbb{F}(=\mathbb{R},\mathbb{C})$. Let $\mathscr{D}'(\Omega;\mathbb{F}^r)$ ($\mathscr{E}'(\Omega;\mathbb{F}^r)$) denote the space of (compactly supported) distributions in Ω. Note that if we chose basis in $\mathbb{F}^r$ then a $\mathbb{F}^r$-valued distribution is a vector with coordinates consisting of r scalar distributions. Using this we can give the following natural definition. A linear operator $A : \mathscr{E}'(\Omega; \mathbb{F}^r) \to \mathscr{D}'\left(\Omega; \mathbb{F}^l\right)$ is called a pseudodifferential operator of order m if in any bases of $\mathbb{F}^r$ and $\mathbb{F}^l$ it is represented by a matrix of scalar pseudodifferential operators of order m.

Let F_1 and F_2 be two vector bundles over a smooth manifold N with fibres $\mathbb{F}^r$ and $\mathbb{F}^l$, respectively. Note that any F_i-valued distribution ($i = 1, 2$) in any local trivialisation is represented by a vector with coordinates being scalar distributions. This allows one to represent any linear operator $\Gamma(F_1) \to \Gamma(F_2)$ in any local trivialisations as an $l \times r$-matrix of linear operators $C^\infty(N) \to C^\infty(N)$. Therefore, it is natural to give the following definition.

Definition. A linear operator $A : \mathscr{E}'(F_1) \to \mathscr{D}'(F_2)$ is called a pseudodifferential operator (of order m) if in any pair of local trivialisations it is represented by a matrix of scalar pseudodifferential operators (of order m). The space of such operators is denoted by $\Psi^m(F_1, F_2)$.

If vector bundles F_1 and F_2 are equal to the vector bundle F, then we say that A is a PDO (acting) on a vector bundle F and write $A \in \Psi^m(F)$. Due to the above definition most of the theory for scalar PDOs generalises to the case of vector bundles naturally, e.g. the symbol calculus. In particular, if A and B are two PDOs on a vector bundle F, then in a local trivialisation the symbol of the composition $A \circ B$ is defined by the formal symbol

$$\sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi), \tag{1.7}$$

where $a(x,\xi)$ and $b(x,\xi)$ are the local symbols of A and B, respectively. Note that the notion of a classical PDO in $\Psi^m(F_1, F_2)$ and its principal symbol make sense [40, 42].

Moreover, the latter can be seen as a smooth section of $\pi^*\mathrm{Hom}_N(F_1, F_2)$ over $T^*N\backslash 0$, positive-homogeneous of degree m on $\xi \in T^*N_x \backslash \{0\}$, where $\pi : T^*N\backslash 0 \to N$ is the canonical projection to the base.

We shall introduce the notion of ellipticity for a classical PDO. We say that a (classical) PDO A acting on a vector bundle F over a manifold N is *elliptic* if its principal symbol $p(x,\xi)$ maps a fiber F_x to F_x bijectively for all $x \in N$, $\xi \in T_x^*N \backslash \{0\}$. Note that in any chosen basis of F_x this is equivalent to the requirement $\det p(x,\xi) \neq 0$ for all $x \in N$, $\xi \in T_x^*N \backslash \{0\}$. An important property of a classical elliptic PDO is that it has a two-sided *parametrix*. More precisely, the following theorem can be found in [42, Chapter II, Theorem 2.4].

Theorem 1.1.3. *Let A be a classical elliptic pseudodifferential operator of order m on a closed (i.e. compact without boundary) manifold M acting on sections of a vector bundle F, and $a(x,\xi)$ be its principal symbol. Then there is an elliptic pseudodifferential operator B of order $-m$ on N, with principal symbol $a^{-1}(x,\xi)$ (the inverse of $a(x,\xi)$), such that*

$$A \circ B - Id_F \text{ and } B \circ A - Id_F$$

are smoothing. This operator B is called the (two-sided) parametrix of A.

In conclusion of this section, let us note that the Schwartz kernel theorem generalises to the case of operators on vector bundles, but to make it precise one should consider the distributional densities or currents on manifolds [42].

1.2 Dirichlet-to-Neumann operator

1.2.1 Definition of the Dirichlet-to-Neumann operator

Let (N,g) be a compact connected Riemannian manifold with boundary ∂N and $\left(E,\nabla^E\right)$ be a vector bundle over N of rank r with a connection $\nabla^E:\Gamma(E)\to\Gamma(E\otimes T^*N)$. We assume that E is equipped with a compatible inner product $\langle\cdot,\cdot\rangle_E$, i.e. the relation

$$d\,\langle u,v\rangle_E=\left\langle\nabla^E u,v\right\rangle_E+\left\langle u,\nabla^E v\right\rangle_E$$

holds for any pair of smooth sections $u,v\in\Gamma(E)$. Note that both sides of the above identity are differential forms. We use the standard notation $\Gamma(E)$ for C^∞-smooth sections of the vector bundle E, $\mathscr{D}(E)$ for smooth compactly supported in N^{int} (the interior of N) sections of E, and $\mathscr{W}^s(E)$ for $\mathscr{W}^s$-smooth sections, where $\mathscr{W}^s=\mathscr{W}^{s,2}$ and $\mathscr{W}^{s,p}$ denotes the (s,p)-Sobolev space. Note that in this notation $\mathscr{W}^0(E)=L^2(E)$. The continuous dual of $\mathscr{D}(E)$ is denoted by $\mathscr{D}'(E)$ and assumed to be endowed with weak topology.

We can define the L^2—inner product of sections by

$$\langle u,v\rangle_{L^2}=\int\limits_N\langle u,v\rangle_E\,dV_g,$$

where dV_g is the Riemannian volume measure of (N,g). Similarly, we can define the L^2—inner product in $\Gamma(E\otimes T^*N)$ by

$$\langle\alpha,\beta\rangle_{L^2(E\otimes T^*N)}=\int\limits_N Tr_g\,\langle\alpha,\beta\rangle_E\,dV_g,$$

where the sections $\alpha, \beta \in \Gamma(E \otimes T^*N)$ considered as the E—valued 1—forms, and Tr_g denotes the trace with respect to g.

Let us consider the connection Laplacian Δ^E defined by

$$\Delta^E = -Tr_g \bar{\nabla}^E \nabla^E,$$

where $\bar{\nabla}^E = \nabla^E \otimes \nabla^{LC} : \Gamma(E \otimes T^*N) \to \Gamma(E \otimes T^*N \otimes T^*N)$ and ∇^{LC} is the Levi-Civita connection on (N, g). Note that we have the following equality [11]

$$\Delta^E = \left(\nabla^E\right)^* \nabla^E,$$

where $\left(\nabla^E\right)^*$ is the adjoint of ∇^E with respect to the L_2—inner products defined above. We will occasionally omit the word "connection" and call this operator the Laplacian for brevity. When it will not make any confusion, we will also sometimes omit the superscript E in the notation of the connection ∇^E.

Let us consider the Dirichlet problem for the Laplacian on a Euclidean vector bundle E over a compact connected Riemannian manifold (N, g) with boundary ∂N:

$$\begin{cases} \Delta^E u = 0 & on\ N, \\ u|_{\partial N} = \sigma & u \in \Gamma(E), \sigma \in \Gamma(E|_{\partial N}). \end{cases} \tag{1.8}$$

It has a unique solution for every $\sigma \in \Gamma(E|_{\partial N})$. We briefly explain why this is true in the next subsection. Now this allows us to introduce the *Dirichlet-to-Neumann operator* $\Lambda_{g,\nabla^E} : \Gamma(E|_{\partial N}) \to \Gamma(E|_{\partial N})$ associated with the connection Laplacian ∇^E by

$$\Lambda_{g,\nabla^E}\sigma = \nabla^E_\nu u\big|_{\partial N},$$

where u is the solution to (1.8), and ν is the outward unit normal vector field on ∂N. A slight variation occurs when we consider the case of the connection Laplacian plus a symmetric potential. Namely, when we adjust the Laplacian as

$$L_P = \Delta^E + P,$$

where P is a (Hermitian) symmetric endomorphism of a (Hermitian) vector bundle E. Recall, that the endomorphism P of a (Hermitian) vector bundle E is called (Hermitian) symmetric if for any two sections $u, v \in \Gamma(E)$ the following relation holds

$$\langle Pu, v\rangle_E = \langle u, Pv\rangle_E .$$

This means, in particular, that in a local orthonormal frame the matrix P^α_β representing P is (Hermitian) symmetric. We assume that 0 is not in the Dirichlet spectrum of L_P. This holds, for example, when P is positive semi-definite, i.e. when

$$\langle Pu, u\rangle_E \geq 0,$$

for any $u \in \Gamma(E)$. We will call this type of operator L_P a *Schrödinger (type) operator*. In this case the Dirichlet problem

$$\begin{cases} L_P u = 0, & u \in \mathscr{W}^{s+\frac{1}{2}}(E), \\ u|_{\partial N} = \sigma, & \sigma \in \mathscr{W}^s(E|_{\partial N}), \end{cases} \tag{1.9}$$

also has a unique solution for $s \geq \frac{1}{2}$. Therefore, we can similarly define the Dirichlet-to-Neumann operator $\Lambda_{g,P,\nabla^E} : \Gamma(E|_{\partial N}) \to \Gamma(E|_{\partial N})$ associated with the operator L_P

by

$$\Lambda_{g,P,\nabla^E}\sigma = \nabla^E_\nu u\Big|_{\partial N},$$

where u is the solution to (1.9), and ν is the outward unit normal vector field on ∂N. The results for Schrödinger operators will be of importance in the third chapter. For this reason, for each statement concerning the connection Laplacian we will discuss its Schrödinger counterpart alongside or afterwards.

1.2.2 Discussion on the Dirichlet problem

We start with well-known Green's identities. Let $w \in \Gamma(E \otimes T^*N)$ and $v \in \Gamma(E)$. Then we have the first Green's identity

$$\langle \nabla^* w, v\rangle_{L^2} - \langle w, \nabla v\rangle_{L^2(E\otimes T^*N)} = \int_{\partial N} \langle \iota_\nu w, v\rangle \, dS_g,$$

where dS_g is the associated volume form on the boundary ∂N, and ν is the outward unit normal vector field on ∂N. Now let $w = \nabla u$, where $u \in \Gamma(E)$. Then we obtain the following identity for the Laplacian

$$\left\langle \Delta^E u, v\right\rangle_{L^2} = \langle \nabla^* \nabla u, v\rangle_{L^2} = \langle \nabla u, \nabla v\rangle_{L^2(E\otimes T^*N)} + \int_{\partial N} \langle \iota_\nu \nabla u, v\rangle \, dS_g, \tag{1.10}$$

which gives us the second Green's identity

$$\left\langle \Delta^E u, v\right\rangle_{L^2} - \left\langle u, \Delta^E v\right\rangle_{L^2} = \int_{\partial N} \langle \iota_\nu \nabla u, v\rangle \, dS_g - \int_{\partial N} \langle u, \iota_\nu \nabla v\rangle \, dS_g. \tag{1.11}$$

It follows from this identity that the connection Laplacian is *symmetric*, namely, the equality

$$\left\langle \Delta^E u, v \right\rangle_{L^2} = \left\langle u, \Delta^E v \right\rangle_{L^2} \tag{1.12}$$

holds for all $u, v \in \mathscr{D}(E)$.

Let us show the uniqueness of the solution to (1.8). Suppose u_1 and u_2 are two solutions of (1.8). Then their difference $u = u_1 - u_2$ is a solution to (1.8) with the boundary condition $u|_{\partial N} = 0$. From (1.10) we obtain

$$0 = \left\langle \Delta^E u, u \right\rangle_{L^2} = \langle \nabla u, \nabla u \rangle_{L^2(E \otimes T^*N)} + \int_{\partial N} \langle \iota_\nu \nabla u, u \rangle \, dS_g =$$

$$= \langle \nabla u, \nabla u \rangle_{L^2(E \otimes T^*N)} = \|\nabla u\|^2_{L^2(E \otimes T^*N)},$$

which implies

$$\nabla u = 0,$$

and therefore

$$d \langle u, u \rangle_E = 2 \langle \nabla u, u \rangle_E = 0.$$

From this we conclude that $\langle u, u \rangle_E$ is constant on N and since u vanishes on the boundary it has to be identically zero on the whole manifold N. Hence, we have $u_1 - u_2 = u = 0$, which shows that u_1 and u_2 are equal. This completes the proof of uniqueness.

Remark. Since the potential of the Schrödinger operator is symmetric the identities (1.11) and (1.12) hold also for L_P. Moreover, since it is positive semi-definite we also have

$$0 = \langle L_P u, u \rangle_{L^2} = \|\nabla u\|^2_{L^2(E \otimes T^*N)} + \langle P u, u \rangle_E,$$

which gives the uniqueness of the solution to the Dirichlet problem for L_P.

Existence of the solution can be shown in two steps. First, we use the Lax-Milgram theorem to prove the existence of a weak solution $u \in \mathscr{W}^1(E)$ to the inhomogeneous Dirichlet problem

$$\begin{cases} \Delta^E u = -\Delta^E \tilde{\sigma}, \\ u|_{\partial N} = 0, \end{cases} \tag{1.13}$$

where $\tilde{\sigma} \in \mathscr{W}^1(E)$ in the right hand side is a given function. Then we use the elliptic regularity to show that this solution is smooth if $\tilde{\sigma}$ is. Note that the connection Laplacian and Schrödinger operator are elliptic, which is shown in Corollary 1.3.9. Below we briefly discuss both steps.

Let us recall the Lax-Milgram theorem [15].

Theorem (Lax-Milgram)**.** *Let V be a Hilbert space and $a(\cdot,\cdot)$ a bilinear form on V, which is*

1. *Bounded:* $|a(u,v)| \leq C\,\|u\|\,\|v\|$ *and*

2. *Coercive:* $|a(u,u)| \geq c\,\|u\|^2$.

Then for any continuous linear functional $f \in V'$ there is a unique solution $u \in V$ to the equation

$$a(u,v) = f(v), \ \forall v \in V$$

and the following inequality holds

$$\|u\| \leq \frac{1}{c}\,\|f\|_{V'}.$$

Let $\mathscr{W}_0^1(E)$ be the Sobolev space of sections vanishing on the boundary ∂N. Consider a bilinear form $a(\cdot,\cdot)$ which acts on $u, v \in \mathscr{W}_0^1(E)$ as

$$a(u,v) = \langle \nabla u, \nabla v \rangle_{L^2(E \otimes T^*N)}.$$

We see that $a(u,v)$ is a bounded bilinear form on $\mathscr{W}_0^1(E)$. For a given $\tilde{\sigma} \in \mathscr{W}^1(E)$ let us take the functional

$$f_{\tilde{\sigma}}(v) = -\langle \nabla \tilde{\sigma}, \nabla v \rangle_{L^2(E \otimes T^*N)}.$$

Clearly, this is a bounded linear functional on $\mathscr{W}_0^1(E)$. So it is left to show that $a(u,v)$ is coercive, i.e. there exists c such that

$$\frac{\|\nabla u\|^2_{L^2(E \otimes T^*N)}}{\|u\|^2_{L^2}} \geq c > 0. \tag{1.14}$$

This follows, for instance, from the Sobolev embedding theorem [35]. We showed that all the conditions of the Lax-Milgram theorem are satisfied, therefore, there is a unique weak solution $u \in \mathscr{W}_0^1(E)$ to (1.13). Note that in a similar manner we can show the existence and uniqueness of a weak solution to the inhomogeneous Dirichlet problem

$$\begin{cases} \Delta^E u = \varphi, \quad \varphi \in L^2(E), \\ u|_{\partial N} = 0. \end{cases} \tag{1.15}$$

For this we have to choose the bounded linear functional f to be

$$f(v) = \langle \varphi, v \rangle_{L^2(E)}.$$

Let us now state the elliptic regularity theorem (see, for example, [35]).

Theorem (Elliptic regularity). *Let $\mathscr{L}$ be an elliptic operator acting on E. Suppose $\varphi \in \mathscr{W}^s(E)$ (smooth), and $u \in \mathscr{D}'(E)$ solves the equation*

$$\mathscr{L}u = \varphi.$$

Then $u \in \mathscr{W}^{s+2}(E)$ (respectively smooth).

Using this theorem we conclude that if $\tilde{\sigma}$ or φ is smooth then the solution u to (1.13) or (1.15), respectively, is also smooth. Note that the elliptic regularity theorem is a direct consequence of the existence of a parametrix for elliptic operators Theorem 1.1.3.

To obtain the solution to the boundary value problem (1.8) we take a smooth extension $\tilde{\sigma} \in \Gamma(E)$ of σ, and write $u = w + \tilde{\sigma}$. We see that

$$\begin{cases} \Delta^E u = \Delta^E w + \Delta^E \tilde{\sigma}, \\ u|_{\partial N} = w|_{\partial N} + \tilde{\sigma}|_{\partial N} = w|_{\partial N} + \sigma. \end{cases}$$

Now if w is the solution to (1.13), then u is the solution to (1.8).

Remark. The existence of a solution to the inhomogeneous Dirichlet problem

$$\begin{cases} L_P u = -L_P \tilde{\sigma}, \\ u|_{\partial N} = 0, \end{cases}$$

is similar. In the Lax-Milgram theorem we consider the bilinear form

$$a_P(u, v) = a(u, v) + \langle Pu, v \rangle_{L^2} = \langle \nabla u, \nabla v \rangle_{L^2(E \otimes T^*N)} + \langle Pu, v \rangle_{L^2},$$

which is clearly bounded. It is also coercive, since a is coercive and P is positive semi-definite. The linear functional in this case will be

$$f_{\tilde{\sigma}}(v) = -\langle \nabla\tilde{\sigma}, \nabla v\rangle_{L^2(E\otimes T^*N)} - \langle P\tilde{\sigma}, v\rangle_{L^2},$$

which is also bounded since P is bounded. The existence of a solution to the inhomogeneous Dirichlet problem

$$\begin{cases} L_P u = \varphi, \quad \varphi \in L^2(E), \\ u|_{\partial N} = 0, \end{cases}$$

can be obtained similarly. Since L_P is elliptic one obtains also the smoothness results from the elliptic regularity theorem. Note that the requirement that P be positive semi-definite is not necessary, in general, for the existence and uniqueness of the solution to the Dirichlet problem. But if it is dropped then one has to use some alternative to the Lax-Milgram theorem.

Note that the restriction to the boundary extends to the trace map

$$\mathscr{T} : \mathscr{W}^s(E) \to \mathscr{W}^{s-\frac{1}{2}}(E|_{\partial N}), \tag{1.16}$$

which has a right inverse

$$\mathscr{E} : \mathscr{W}^{s-\frac{1}{2}}(E|_{\partial N}) \to \mathscr{W}^s(E), \tag{1.17}$$

i.e. the map such that the equality $\mathscr{T} \circ \mathscr{E} = Id$ holds; both maps are linear bounded operators [3]. Using this we can show existence and uniqueness of the solution to the boundary value problem

$$\begin{cases} \Delta^E u = 0, & u \in \mathscr{W}^{s+\frac{1}{2}}(E), \\ \mathscr{T}(u) = \sigma, & \sigma \in \mathscr{W}^s(E|_{\partial N}), \end{cases} \tag{1.18}$$

(and its Schrödinger counterpart) for $s \geq 1/2$. Note that from the elliptic regularity the solution u is smooth in the interior of N.

Consider the complexification $\mathbb{C} \otimes E \cong E \oplus iE$ of the Euclidean vector bundle E. It has a natural structure of Hermitian vector bundle and a compatible connection $\nabla^{\mathbb{C}\otimes E} = \nabla^E \oplus \nabla^E$. The associated Laplacian is

$$\Delta^{\mathbb{C}\otimes E} = \Delta^E \oplus \Delta^E,$$

and we see that the complex analogue of the problem (1.15) decomposes into two real ones. The same holds for the Schrödinger operator. So all the results on the existence and uniqueness are straightforward from the real case. In the next chapter we will consider the complexified objects (vector bundle, connection, Laplacian) and use the same notation for them as for their real counterparts.

1.3 Green's kernel

In this section we discuss Green's kernel for the connection Laplacian and Schrödinger operator on a compact manifold. Let us define Green's kernel for the connection Laplacian. Suppose we have a Euclidean (or Hermitian) vector bundle E over N. Then we can define the vector bundle $E \boxtimes E$ over $N \times N$ whose fiber at the point $(x, y) \in N \times N$ is $E_x \otimes E_y$.

Definition. A smooth section G of the vector bundle $E \boxtimes E$ defined away from the diagonal $\operatorname{diag}(N) = \{(x,x) \in N \times N\}$ is called the *Green kernel* of the connection Laplacian on N if:

1. the integral of the function $y \mapsto |G(x,y)|_{E \boxtimes E}$ is finite for all $x \in N$;

2. the relation
$$\int_N \left\langle G(x,y), \Delta^E s(y) \right\rangle_E dV_{g,y} = s(x),$$
holds for all $s \in \mathscr{D}(E)$;

It is called the *Dirichlet Green kernel* if in addition it satisfies $G(x,y) = 0$ for all $x \in N$, $y \in \partial N$, $x \neq y$. Let us define a point analogue of this definition and prove the lemma which implies the uniqueness of the Dirichlet Green kernel.

Definition. For any interior point $x \in N$ we call a smooth section $F(y)$ of $E_x \otimes E$ defined away from the point x an *x-point potential* if:

1. the integral of the function $|F(y)|$ is finite;

2. the relation
$$\int_N \left\langle F(y), \Delta^E s(y) \right\rangle_E dV_{g,y} = s(x),$$
holds for all $s \in \mathscr{D}(E)$;

3. $F(y) = 0$ for $y \in \partial N$.

The above definitions are naturally generalised to the case of a Schrödinger operator.

Lemma 1.3.1. *For any interior point $x \in N$ the x-point potential $F(y)$ is unique.*

Proof. Suppose $H(y)$ is another x-point potential. Then

$$\int_N \left\langle F(y), \Delta^E s(y)\right\rangle_E dV_{g,y} - \int_N \left\langle H(y), \Delta^E s(y)\right\rangle_E dV_{g,y} = s(x) - s(x) = 0,$$

for any $s \in \mathscr{D}(E)$. If $K(y)$ denotes the difference $F(y) - H(y)$, then we have

$$\int_N \left\langle K(y), \Delta^E s(y)\right\rangle_E dV_{g,y} = 0,$$

$$K(y) = 0,\ y \in \partial N.$$

Clearly, this difference is a smooth section of $E_x \otimes E$ defined away from the point x. We see that the distribution defined by $K(y)$ is a weak solution to (1.15) with $\varphi = 0$. Since $\varphi = 0$ is smooth, the elliptic regularity implies that $K(y)$ extends smoothly to the point $y = x$. Therefore, $K(y)$ is smooth on the whole of N and from the uniqueness of a solution we have $K(y) = 0$ for all $y \in N$. It follows that $F(y) - H(y) = 0$, which implies $H(y) = F(y)$. □

Corollary 1.3.2. *The Dirichlet Green kernel of the connection Laplacian is unique.*

Proof. For the interior points x it follows from Lemma 1.3.1, since $G(x, y)$ is the x-point potential. For $x \in \partial N$, $x \neq y$ it follows from the smoothness of G and the fact that the interior of N is dense in N. □

The main ingredient in the proof of the above results is the elliptic regularity theorem. Therefore, the same results continue to hold for the Schrödinger operator.

We will be dealing mostly with the Dirichlet Green kernel. Therefore, we will occasionally omit the word "Dirichlet" for brevity. Hopefully, this will not make any confusion. One can see that in the sense of distributions the Green kernel can be thought of as a fundamental solution of the connection Laplacian, which is customary to write as the symbolic identity

$$\Delta_y^E G(x,y) = Id \cdot \delta_x(y),$$

where $\delta_x(y)$ denotes the Dirac delta centered at x. Note that this means, in particular, that the left hand side vanishes for $x \neq y$. This is also true for the x-point potential and justifies its name, since it can be thought of as "the potential created by an elementary charge placed at x". In the sense of operators, $G(x,y)$ can be thought of as the Schwartz kernel of the left inverse to the connection Laplacian. (Note that the necessary condition for a linear operator L to have a left inverse is $\ker L = 0$, which is satisfied for the Dirichlet Laplacian.) Taking this into account and the fact that Δ^E is symmetric (1.12) we may expect that G is also symmetric. And this is indeed the case, which we will prove using the following lemma.

Lemma 1.3.3. *Let x_1 and x_2 be two interior points of N with corresponding point potentials $F_1(y)$ and $F_2(y)$. Then*

$$F_1(x_2) = \tau \circ F_2(x_1),$$

where $\tau : E_{x_2} \otimes E_{x_1} \to E_{x_1} \otimes E_{x_2}$ is the canonical isomorphism.

Proof. Let us pick any two vectors $\xi_{x_1} \in E_{x_1}$, $\eta_{x_2} \in E_{x_2}$ and define two sections $u(y) = \left\langle \xi_{x_1}, F_1(y) \right\rangle_{E_{x_1}}$ and $v(y) = \left\langle \eta_{x_2}, F_2(y) \right\rangle_{E_{x_2}}$ of E. Note that $\Delta^E u(y)$ and $\Delta^E v(y)$ vanish for $y \neq x_1$ and $y \neq x_2$, respectively, and both vanish at the boundary ∂N. Using the identity (1.11) we can write

$$0 = \lim_{\varepsilon \to 0} \left[\int\limits_{N \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))} \left\langle u(y), \Delta^E v(y) \right\rangle_E dV_{g,y} - \right.$$
$$\left. - \int\limits_{N \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))} \left\langle \Delta^E u(y), v(y) \right\rangle_E dV_{g,y} \right] =$$
$$= -\lim_{\varepsilon \to 0} \left[\int\limits_{\partial B_\varepsilon(x_1) \cup \partial B_\varepsilon(x_2)} \langle u(y), \iota_\nu \nabla v(y) \rangle_E \, dS_{g,y} - \right.$$
$$\left. - \int\limits_{\partial B_\varepsilon(x_1) \cup \partial B_\varepsilon(x_2)} \langle \iota_\nu \nabla u(y), v(y) \rangle_E \, dS_{g,y} \right].$$

From this we conclude that

$$\lim_{\varepsilon \to 0} \left[\int\limits_{\partial B_\varepsilon(x_1)} \left(\langle u(y), \iota_\nu \nabla v(y) \rangle_E - \langle \iota_\nu \nabla u(y), v(y) \rangle_E \right) dV_{g,y} \right] =$$
$$\lim_{\varepsilon \to 0} \left[\int\limits_{\partial B_\varepsilon(x_2)} \left(\langle \iota_\nu \nabla u(y), v(y) \rangle_E - \langle u(y), \iota_\nu \nabla v(y) \rangle_E \right) dV_{g,y} \right].$$

Let us now multiply the section $v(y)$ in the first integral by $\chi_{x_1}(y)$ and the section $u(y)$ in the second integral by $\chi_{x_2}(y)$, where $\chi_{x_1}(y)$ and $\chi_{x_2}(y)$ have disjoint supports and are equal to 1 in a small neighborhood of x_1 and x_2, respectively. The first integral then

$$\lim_{\varepsilon\to 0}\left[\int\limits_{\partial B_\varepsilon(x_1)} \left(\left\langle u(y), \iota_\nu \nabla\left(\chi_{x_1}(y)v(y)\right)\right\rangle_E - \left\langle \iota_\nu \nabla u(y), \chi_{x_1}(y)v(y)\right\rangle_E\right) dV_{g,y}\right] =$$

$$\lim_{\varepsilon\to 0}\left[\int\limits_{N\setminus B_\varepsilon(x_1)} \left\langle u(y), \Delta^E\left(\chi_{x_1}(y)v(y)\right)\right\rangle_E dV_{g,y} - \right.$$

$$\left. - \int\limits_{N\setminus B_\varepsilon(x_1)} \left\langle \Delta^E u(y), \chi_{x_1}(y)v(y)\right\rangle_E dV_{g,y}\right] =$$

$$\lim_{\varepsilon\to 0}\int\limits_{N\setminus B_\varepsilon(x_1)} \left\langle u(y), \Delta^E\left(\chi_{x_1}(y)v(y)\right)\right\rangle_E dV_{g,y} =$$

$$= \lim_{\varepsilon\to 0}\left\langle \xi_{x_1}, \int\limits_{N\setminus B_\varepsilon(x_1)} \left\langle F_1(y), \Delta^E\left(\chi_{x_1}(y)v(y)\right)\right\rangle_E dV_{g,y}\right\rangle_{E_{x_1}} =$$

$$\left\langle \xi_{x_1}, v(x_1)\right\rangle_E.$$

Similarly, the second integral is equal to $\left\langle \eta_{x_2}, u(x_2)\right\rangle_E$, so we get the equality

$$\left\langle \xi_{x_1}, \left\langle \eta_{x_2}, F_2(x_1)\right\rangle_E\right\rangle_E = \left\langle \eta_{x_2}, \left\langle \xi_{x_1}, F_1(x_2)\right\rangle_E\right\rangle_E,$$

for any two vectors $\xi_{x_1} \in E_{x_1}$, $\eta_{x_2} \in E_{x_2}$. This implies the desired equality

$$F_1(x_2) = \tau \circ F_2(x_1),$$

where $\tau : E_{x_2} \otimes E_{x_1} \to E_{x_1} \otimes E_{x_2}$ is the canonical isomorphism. □

Corollary 1.3.4. *The Dirichlet Green kernel is symmetric, i.e. it satisfies*

$$G(x,y) = \tau \circ G(y,x),$$

where $\tau : E_y \otimes E_x \to E_x \otimes E_y$ *is the canonical isomorphism. In particular,* $G(x,y) = 0$ *whenever* x *or* y *lies on the boundary of* N, $x \neq y$.

Proof. For distinct interior points $x, y \in N$ it follows from Lemma 1.3.3. If at least one of the points x, y lies on the boundary ∂N it follows from the smoothness of G and the fact that the interior of N is dense in N. More precisely, let $x \in \partial N$, y be any interior point of N and $\{x_k\}$ be a sequence of interior points disjoint with y and converging to x. Then

$$|G(x,y)| = \lim_{k\to\infty} |G(x_k,y)| = \lim_{k\to\infty} |G(y,x_k)| = |G(y,x)| = 0,$$

which implies $G(x,y) = 0$. Based on this we can similarly prove the result for both x and y lying on the boundary. □

The proof of the above results relies on the identity (1.11), which holds also for the Schrödinger operator. Therefore, there are Schrödinger counterparts for these results.

The definition of the Green kernel for the connection Laplacian is just the generalisation of the notion of Green's function for the Laplace-Beltrami operator. The existence and uniqueness as well as the properties (e.g. symmetry) of Green's function are well known (see, for example, Aubin 2). The classical approach to the construction of Green's function is to take Green's function in $\mathbb{R}^n$ as the first approximation locally, and then iteratively regularise the difference with the true Green's function until this difference becomes regular enough to use the Lax-Milgram theorem. In our considerations we will shortcut this path by taking a local fundamental solution as a first approximation to the global Green's kernel. The difference between these two will then be a smooth section straight away, and we will deal with it by the use of the Lax-Milgram theorem.

We shall start with the results on the existence of a local fundamental solution for general elliptic systems in $\mathbb{R}^n$, and then we restrict ourselves to the case of the connection Laplacian.

1.3.1 Local fundamental solution for general elliptic systems

Consider a linear differential equation in an open subset U of $\mathbb{R}^n$

$$\mathfrak{M}u = f,$$

where $\mathfrak{M}$ is a differential operator acting on vector-valued functions, and u, f are functions valued in $\mathbb{F}^r$ (real or complex r-dimensional vector space). This is the same as the system of r linear differential equations in $U \subset \mathbb{R}^n$

$$\mathfrak{M}^\alpha_\beta u^\beta (x) = f^\alpha (x), \tag{1.19}$$

where $\alpha, \beta = 1, \ldots, r$, $x = \left(x^1, \ldots, x^n\right)$, and $\mathfrak{M}^\alpha_\beta$ are linear differential operators. Note that here and throughout the text we assume the Einstein summation convention, i.e. whenever an index in an expression repeats the sum over this index is assumed. The system is called *elliptic in the sense of Petrowsky* if

$$\det p (x, \xi) \neq 0 \, \text{for} \, \xi \neq 0,$$

where $p(x,\xi)$ is the principal symbol of $\mathfrak{M}$, i.e. if the operator $\mathfrak{M}$ is elliptic. Let L be a second or first order linear differential operator with smooth coefficients given in a domain of $\mathbb{R}^n$ by

$$Lu = \frac{\partial}{\partial x^i}\left(a^{ik}\frac{\partial u}{\partial x^k}\right) + b^i \frac{\partial u}{\partial x^i} + cu,$$

where a^{ik} assumed to be identically zero for a first order operator. Then its formal *formal adjoint* R is defined as

$$Rv = \frac{\partial}{\partial x^k}\left(\overline{a^{ik}}\frac{\partial v}{\partial x^i}\right) - \frac{\partial}{\partial x^i}\left(\overline{b^i} v\right) + \bar{c}v.$$

Given this we say that the system of operators $\mathfrak{R}$ is the formal adjoint to $\mathfrak{M}$ if the linear differential operator $\mathfrak{R}_{\alpha\beta}$ is the formal adjoint of $\mathfrak{M}_{\beta\alpha}$ for each pair $\alpha, \beta = 1, \ldots, r$. Note that the formal adjoint of the formal adjoint of an operator L is L itself, and a system is called *formally self-adjoint* if it is equal to its formal adjoint. One can see that in the definition of a formal adjoint the indices of $\mathfrak{R}$ and $\mathfrak{M}$ are lowered. This brings us to the idea that we should consider $\mathfrak{R}$ and $\mathfrak{M}$ as bilinear forms. Indeed, the motivation to consider this definition unfolds in the following proposition.

Proposition 1.3.5. *For any smooth $\mathbb{F}^r$-valued functions u, v compactly supported in an open subset $V \subset U$ the following relation holds*

$$\int\limits_V \langle \mathfrak{M}u, v\rangle_{\mathbb{F}^r}\, dx = \int\limits_V \langle u, \mathfrak{R}v\rangle_{\mathbb{F}^r}\, dx,$$

where $\langle,\rangle_{\mathbb{F}^r}$ is the standard Hermitian (Euclidean) product in $\mathbb{F}^r$ and dx is the standard measure in $\mathbb{R}^n$.

Proof. Let us consider the Hermitian case, the Euclidean one will follow. The matrix of $\langle , \rangle_{\mathbb{F}^r}$ is given by the Kronecker delta $\delta_{\gamma\beta}$, so we can write

$$\int\limits_V \langle \mathfrak{M}u, v\rangle_{\mathbb{F}^r}\, dx = \int\limits_V \left(\mathfrak{M}^\gamma_\alpha u^\alpha\right)\delta_{\gamma\beta}\overline{v^\beta}dx = \int\limits_V \left(\mathfrak{M}_{\beta\alpha}u^\alpha\right)\overline{v^\beta}dx =$$
$$= \int\limits_V \left(\mathfrak{M}_{\beta\alpha}u^\alpha\right)\overline{v^\beta}dx = \int\limits_V \left(\frac{\partial}{\partial x_i}\left(a^{ik}_{\beta\alpha}\frac{\partial u^\alpha}{\partial x_k}\right) + b^i_{\beta\alpha}\frac{\partial u^\alpha}{\partial x_i} + c_{\beta\alpha}u^\alpha\right)\overline{v^\beta}dx,$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Taking into account that u and v are identically zero near the boundary of V we can integrate by parts to get

$$\int\limits_V u^\alpha\left(\frac{\partial}{\partial x_k}\left(a^{ik}_{\beta\alpha}\frac{\partial}{\partial x_i}\overline{v^\beta}\right) + \frac{\partial}{\partial x_i}\left(b^i_{\beta\alpha}\overline{v^\beta}\right) + c_{\beta\alpha}\overline{v^\beta}\right)dx =$$
$$= \int\limits_V u^\alpha\overline{\left(\frac{\partial}{\partial x_k}\left(\overline{a^{ik}_{\beta\alpha}}\frac{\partial}{\partial x_i}v^\beta\right) + \frac{\partial}{\partial x_i}\left(\overline{b^i_{\beta\alpha}}v^\beta\right) + \overline{c_{\beta\alpha}}v^\beta\right)}dx =$$
$$= \int\limits_V u^\alpha\overline{\mathfrak{N}_{\alpha\beta}v^\beta}dx = \int\limits_V u^\alpha\delta_{\alpha\gamma}\overline{\mathfrak{N}^\gamma_\beta v^\beta}dx = \int\limits_V \langle u, \mathfrak{N}v\rangle_{\mathbb{F}^r}\, dx.$$

□

We shall say that an $r \times r$ matrix $G(x, y)$ is a *fundamental matrix* in a domain $W \subset U$ of the system (1.19) if for every sufficiently differentiable vector function v compactly supported in the interior of W we have

$$v(x) = \int\limits_W G(x, y)\overline{\mathfrak{N}v(y)}dy,$$

where $\mathfrak{N}$ is the formal adjoint of $\mathfrak{M}$.

There are many results concerning fundamental system of solutions to elliptic system of equations with coefficients of different regularities. We restrict ourselves to the case of elliptic systems with smooth coefficients. The following proposition summarises known results on the existence of a fundamental matrix for elliptic systems with smooth and analytic coefficients.

Proposition 1.3.6. *Let* $\mathfrak{M}^{\alpha}_{\beta}$ *be an elliptic (in the sense of Petrowsky) system of* r *linear differential operators with smooth (analytic) coefficients in an open neighborhood* $U \in \mathbb{R}^n$ *of the origin. Then there exists an open neighborhood* W *of the origin and a fundamental matrix* $G(x,y)$ *in* W *of the system* $\mathfrak{M}$*. The system of functions* $G^{\beta}_{\gamma}(x,y)$ *form a smooth (analytic) solution to*

$$\mathfrak{M}^{\alpha}_{\beta} G^{\beta}_{\gamma}(x,y) = 0$$

for $y \neq x$*, where* $\mathfrak{M}$ *acts on* y *coordinate. Moreover, there is the following representation of the fundamental matrix*

$$G^{\beta}_{\gamma}(x,y) = r^{m_{\underline{\gamma}}-n}\left[A^{\beta}_{\underline{\gamma}}(x,y,r,\zeta) + B^{\beta}_{\underline{\gamma}}(x,y,r,\zeta)\log r\right], \tag{1.20}$$

where we do not sum over $\underline{\gamma}$, $x - y = r\zeta$ with $r \in \mathbb{R}_{\geq 0}$, $|\zeta| = 1$, $A^{\beta}_{\underline{\gamma}}(x,y,r,\zeta)$ and $B^{\beta}_{\underline{\gamma}}(x,y,r,\zeta)$ are smooth (analytic) in their arguments, $B^{\beta}_{\underline{\gamma}} = 0$ for odd n.

The existence of a fundamental matrix for elliptic systems with smooth coefficients was proven in [32, 33]. Note that Petrowsky in [36] proves the analyticity of a solution to general (possibly non-linear) analytic elliptic systems. These two works imply the result in the case of analytic coefficients. The latter is also obtained by the use of a separate self-contained approach utilizing plane waves in [23].

Note that a restriction of a fundamental matrix in W to an open subset $W' \subset W$ is a fundamental matrix in W'.

We conclude this section with the notion of a strongly elliptic system. The system of equations (1.19) is called *strongly elliptic in the sense of Petrowsky* if

$$\operatorname{Re}\left(p^{\alpha}_{\beta}(x,\xi)\,\eta_{\alpha}\overline{\eta}^{\beta}\right)\neq 0,$$

for real ξ and real or complex η, both not equal to zero. Note that this notion of ellipticity for systems is the strongest one. One may refer to [34] for an overview of the different notions of ellipticity for systems and related results.

1.3.2 Local Green's kernel for the connection Laplacian

Here we specialise the results of the previous section to the case of the connection Laplacian and Schrödinger operators. Throughout this and the next sections we suppose that the connection Laplacian is acting on a real vector bundle, though we should note that there is a straightforward generalisation to the complex case. Consider the connection Laplacian in local coordinates and local orthonormal frame and define a system of linear differential operators $L=\sqrt{|g|}\Delta^{E}$, where $|g|$ denotes the determinant of a metric g in these coordinates. We will show next that this system is elliptic and formally self-adjoint. Let $p\in N$, $\left(x^{1},\dots,x^{n}\right)$ be local coordinates and $(\varepsilon_{1},\dots,\varepsilon_{r})$ be a smooth local orthonormal frame in a neighborhood U of p in N, such that $x(p)=0$. We can write

$$\left(\sqrt{|g|}\Delta^{E}s\right)^{\alpha}=L^{\alpha}_{\beta}s^{\beta}=-\left[\delta^{\alpha}_{\beta}\partial_{x^{i}}\left(\sqrt{|g|}g^{ij}\partial_{x^{j}}\right)+2\left(b^{\alpha}_{\beta}\right)^{j}\partial_{x^{j}}+c^{\alpha}_{\beta}+d^{\alpha}_{\beta}\right]s^{\beta}, \qquad (1.21)$$

where δ^α_β is the Kronecker delta, $s = s^\beta \varepsilon_\beta$ is any local section, and the coefficients are

$$\left(b^\alpha_\beta\right)^j = \sqrt{|g|} g^{jk} \left(\omega^\alpha_\beta\right)_k, \tag{1.22}$$

$$c^\alpha_\beta = \partial_{x^j} \left(b^\alpha_\beta\right)^j, \tag{1.23}$$

$$d^\alpha_\beta = \sqrt{|g|} g^{jk} \left(\omega^\alpha_\gamma\right)_j \left(\omega^\gamma_\beta\right)_k, \; j,k = 1,\dots,n, \tag{1.24}$$

where ω^α_β denote the connection form of ∇^E in this frame, i.e. we have $\nabla^E \varepsilon_\beta = \omega^\alpha_\beta \varepsilon_\alpha$ and $\left(\omega^\alpha_\beta\right)_k = \omega^\alpha_\beta(\partial_{x^k})$.

If a connection ∇^E is compatible with an inner product $\langle \cdot, \cdot \rangle_E$ on E, we have the following relation

$$d \langle u, v \rangle_E = \left\langle \nabla^E u, v \right\rangle_E + \left\langle u, \nabla^E v \right\rangle_E. \tag{1.25}$$

Note that in the orthonormal frame $(\varepsilon_1, \dots, \varepsilon_r)$ the connection form of a compatible connection is skew-symmetric. This can be seen by applying (1.25) to the orthonormal frame

$$0 = d\delta_{\alpha\beta} = d \left\langle \varepsilon_\alpha, \varepsilon_\beta \right\rangle_E = \left\langle \varepsilon_\tau \omega^\tau_\alpha, \varepsilon_\beta \right\rangle_E + \left\langle \varepsilon_\alpha, \varepsilon_\tau \omega^\tau_\beta \right\rangle_E =$$
$$= \delta_{\tau\beta} \omega^\tau_\alpha + \delta_{\alpha\tau} \omega^\tau_\beta = \omega_{\beta\alpha} + \omega_{\alpha\beta}, \text{ for } i = 1, \dots, n,$$

and concluding that $\omega = -\omega^T$, where the superscript T denotes the transposed matrix. Using this we can prove the following simple lemma.

Lemma 1.3.7. *The coefficients* $\left(b_{\alpha\beta}\right)^j$ *and* $c_{\alpha\beta}$ *are skew-symmetric while the coefficient* $d_{\alpha\beta}$ *is symmetric.*

Proof. Skew-symmetry of $\left(b_{\alpha\beta}\right)^j$ and $c_{\alpha\beta}$ follows directly from the skew symmetry of the connection form ω. For $d_{\alpha\beta}$ we have

$$d_{\beta\alpha} = \sqrt{|g|}g^{jk}\left(\omega_{\beta\gamma}\right)_j \delta^{\gamma\rho}\left(\omega_{\rho\alpha}\right)_k = \sqrt{|g|}g^{jk}(-1)\left(\omega_{\gamma\beta}\right)_j \delta^{\rho\gamma}(-1)\left(\omega_{\alpha\rho}\right)_k = \sqrt{|g|}g^{jk}\left(\omega_{\alpha\rho}\right)_k \delta^{\rho\gamma}\left(\omega_{\gamma\beta}\right)_j = \sqrt{|g|}g^{kj}\left(\omega_{\alpha\rho}\right)_k\left(\omega^{\rho}_{\beta}\right)_j = d_{\alpha\beta},$$

where we used the fact that g^{jk} is symmetric. □

For the Schrödinger operator the only difference is in the coefficient $d_{\alpha\beta}$, which will have an additional symmetric term $\sqrt{|g|}P_{\alpha\beta}$, and, hence, the result also holds in this case.

Now we are ready to prove an important proposition.

Proposition 1.3.8. *The system L^{α}_{β} associated to the connection Laplacian (or the Schrödinger operator) is (strongly) elliptic in the sense of Petrowsky and formally self-adjoint.*

Proof. One can see that the principal symbols of L is

$$p^{\alpha}_{\beta}(x,\xi) = \delta^{\alpha}_{\beta}\sqrt{|g|}g^{kl}(x)\,\xi_k\xi_l,$$

and since $\sqrt{|g|}$ is a positive function this guarantees that the system L is strongly elliptic in the sense of Petrowsky. Indeed, we see that

$$\operatorname{Re}\left(p^{\alpha}_{\beta}(x,\xi)\,\eta_{\alpha}\overline{\eta}^{\beta}\right) = \operatorname{Re}\left(\delta^{\alpha}_{\beta}\sqrt{|g|}g^{kl}(x)\,\xi_k\xi_l\eta_{\alpha}\overline{\eta}^{\beta}\right) = \operatorname{Re}\left(\sqrt{|g|}\,\|\xi\|^2_g\,\|\eta\|^2\right) = \sqrt{|g|}\,\|\xi\|^2_g\,\|\eta\|^2 \neq 0,$$

for non-zero vectors ξ and η. Let us now show the formal self-adjointness of L^{α}_{β}. For this we find its formal adjoint R to be

$$R_{\alpha\beta} = -\left[\delta_{\beta\alpha}\partial_{x^j}\left(\sqrt{|g|}g^{ij}\partial_{x^i}\right) - 2\partial_{x^j}\circ\left(b_{\beta\alpha}\right)^j + c_{\beta\alpha} + d_{\beta\alpha}\right].$$

Using the symmetry of g^{ij} and Lemma 1.3.7 we obtain

$$R_{\alpha\beta} = -\left[\delta_{\alpha\beta}\partial_{x^i}\left(\sqrt{|g|}g^{ij}\partial_{x^j}\right) + 2\partial_{x^j}\circ\left(b_{\alpha\beta}\right)^j - c_{\alpha\beta} + d_{\alpha\beta}\right].$$

It is left to see that the operator in the second term acts as

$$2\partial_{x^j}\circ\left(b_{\alpha\beta}\right)^j = 2\left(b_{\alpha\beta}\right)^j\partial_{x^j} + 2\left(\partial_{x^j}\left(b_{\alpha\beta}\right)^j\right) = 2\left(b_{\alpha\beta}\right)^j\partial_{x^j} + 2c_{\alpha\beta},$$

so we have

$$\begin{aligned} R_{\alpha\beta} &= -\left[\delta_{\alpha\beta}\partial_{x^i}\left(\sqrt{|g|}g^{ij}\partial_{x^j}\right) + 2\left(b_{\alpha\beta}\right)^j\partial_{x^j} + 2c_{\alpha\beta} - c_{\alpha\beta} + d_{\alpha\beta}\right] = \\ &= -\left[\delta_{\alpha\beta}\partial_{x^i}\left(\sqrt{|g|}g^{ij}\partial_{x^j}\right) + 2\left(b_{\alpha\beta}\right)^j\partial_{x^j} + c_{\alpha\beta} + d_{\alpha\beta}\right] = L_{\alpha\beta}, \end{aligned}$$

showing that L is formally self-adjoint. □

Corollary 1.3.9. *The connection Laplacian Δ^E and the Schrödinger operator L_P are elliptic (pseudo)differential operators.*

Proof. The principal symbols of both operators coincide and are equal to

$$p^\alpha_\beta(x,\xi) = \delta^\alpha_\beta g^{kl}(x)\xi_k\xi_l,$$

which has a determinant equal to

$$\left(g^{kl}(x)\,\xi_k\xi_l\right)^r = |\xi|_g^{2r} > 0,$$

for all $x \in N$, $\xi \in T_x^*N \setminus \{0\}$. □

In order to construct the global Green kernel on N we need to show that there is a *local Green kernel* near the interior point $p \in N$. Namely, there is an open subset $U \ni p$ trivializing E and a Green kernel G on U. This means that for any $v \in \mathscr{D}(E|_U)$ we should have the identity

$$v(x) = \int_U \left\langle G(x,y), \Delta^E v(y)\right\rangle_E dV_{g,y}.$$

Using the results in the previous section we prove the following lemma.

Lemma 1.3.10. *For any interior point p of N there is a local Green kernel (for the connection Laplacian or the Schrödinger operator) near p.*

Proof. By Proposition 1.3.8 the system L_β^α is elliptic and formally self-adjoint, therefore, by Proposition 1.3.6 there is an open neighborhood W of $x(p)$ and a fundamental matrix $G_\beta^\alpha(x,y)$ satisfying

$$v^\alpha(x) = \int_W G_\beta^\alpha(x,y)\,L_\gamma^\beta v^\gamma(y)\,dy,$$

where dy is the standard measure on $\mathbb{R}^n$, and we used the fact that L is formally self-adjoint. Note that this can be rewritten as

$$v^{\alpha}(x) = \int_W G^{\alpha}_{\beta}(x,y)\sqrt{|g(y)|}\left(\Delta^E v(y)\right)^{\beta} dy =$$

$$= \int_W G^{\alpha\gamma}(x,y)\,\delta_{\gamma\beta}\left(\Delta^E v(y)\right)^{\beta}\sqrt{|g(y)|}dy =$$

$$= \int_W \left\langle G^{\alpha}(x,y), \Delta^E v(y)\right\rangle_E dV_g, \quad (1.26)$$

where $\delta_{\gamma\beta}$ represents the fiber inner product in the chosen orthonormal frame. From (1.26) we see that $G^{\alpha\beta}(x,y)$ represents the local Green kernel near p. □

Note that this follows also from Theorem 1.1.3 and the existence of a solution to the Dirichlet problem for the connection Laplacian. Let us point out that a restriction of a Green kernel on U to an open subset $U' \subset U$ is a Green kernel on U'. In conclusion, let us prove the lemma which will be used in the next section.

Lemma 1.3.11. *Let $p \in N$, $\left(x^1,\dots,x^n\right)$ be local coordinates and $(\varepsilon_1,\dots,\varepsilon_r)$ be a smooth local orthonormal frame in a neighborhood U of p in N. Let $X = X^i\partial_i$ be a smooth vector field in U. Then $\mathscr{L} = \left|\sqrt{g}\right|\nabla^E_X$ is a first order linear differential operator (system) with smooth coefficients and its formal adjoint is*

$$\mathscr{R} = |\sqrt{g}|\,\nabla^E_X - |\sqrt{g}|\,Id\cdot divX,$$

where $divX = \frac{1}{|\sqrt{g}|}\partial_{x^i}\left(\left|\sqrt{g}\right|X^i\right)$ is the divergence of X. In particular, the relation

$$\int_U \left\langle \nabla^E_X u, v\right\rangle_E dV_g = -\int_U \left\langle u, \nabla^E_X v\right\rangle_E dV_g - \int_U (divX)\langle u,v\rangle_E\, dV_g$$

holds for all $u, v \in \mathscr{D}(E|_U)$.

Proof. For any local section $s = s^\beta \varepsilon_\beta$ we have

$$\left(|\sqrt{g}|\,\nabla_X^E s\right)^\alpha = \mathscr{L}_\beta^\alpha s^\beta = \left(\delta_\beta^\alpha \cdot |\sqrt{g}| X^i \partial_{x^i} + |\sqrt{g}| X^i \left(\omega_\beta^\alpha\right)_i\right) s^\beta,$$

so $\left|\sqrt{g}\right| \nabla_X^E$ is indeed a first order linear differential operator with smooth coefficients. To find its formal adjoint we write

$$\mathscr{R}_{\alpha\beta} = -\delta_{\beta\alpha} \cdot \partial_{x^i} \circ \left(|\sqrt{g}| X^i\right) + |\sqrt{g}| X^i \left(\omega_{\beta\alpha}\right)_i.$$

Note that the first term acts as

$$-\delta_{\alpha\beta} \cdot |\sqrt{g}| X^i \partial_{x^i} - \delta_{\alpha\beta} \cdot \left(\partial_{x^i}\left(|\sqrt{g}| X^i\right)\right),$$

and ω is skew-symmetric. Therefore, we obtain

$$\begin{aligned}\mathscr{R}_{\alpha\beta} &= -\delta_{\alpha\beta} \cdot |\sqrt{g}| X^i \partial_{x^i} - |\sqrt{g}| X^i \left(\omega_{\alpha\beta}\right)_i - \delta_{\alpha\beta} \cdot \left(\partial_{x^i}\left(|\sqrt{g}| X^i\right)\right) = \\ &= -\mathscr{L}_{\alpha\beta} - |\sqrt{g}|\, \delta_{\alpha\beta} \cdot \frac{1}{\left|\sqrt{g}\right|}\left(\partial_{x^i}\left(|\sqrt{g}| X^i\right)\right) = -\mathscr{L}_{\alpha\beta} - |\sqrt{g}|\, \delta_{\alpha\beta} \cdot \mathrm{div} X,\end{aligned}$$

which proves the desired

$$\mathscr{R} = |\sqrt{g}|\, \nabla_X^E - |\sqrt{g}|\, Id \cdot \mathrm{div} X.$$

Finally, the integral relation follows from Proposition 1.3.5. □

1.3.3 Green's kernel for the connection Laplacian

This section is devoted to the proof of the existence of the global Dirichlet Green kernel for the connection Laplacian and the Schrödinger operator.

Proposition 1.3.12. *Let N be a compact smooth (analytic) Riemannian manifold with boundary, E be a smooth (analytic) Euclidean bundle over N with a compatible (analytic) connection ∇^E. Then there exists a unique (analytic) Dirichlet Green kernel G for the associated connection Laplacian Δ^E (or the Schrödinger operator with (analytic) potential).*

Proof. We showed the uniqueness of G in Corollary 1.3.2. Let us now prove the existence of the Green kernel for the connection Laplacian. Let us fix a point p in the interior of N. By Lemma 1.3.10 there is an open neighborhood $U \ni p$ and a local Green kernel $\tilde{G}(x, y)$. Let $\mu(y)$ be a smooth cutoff function on N, such that $\mu(y)$ is equal to 1 in some open neighborhood $V \subset U$ of p and has a compact support in U. Consider a smooth section $\mu(y)\tilde{G}(x, y)$ defined on $V \times N$ away from the diagonal. For any section $s \in \mathscr{D}(E)$ we have

$$\int_N \left\langle \mu(y)\tilde{G}(x, y), \Delta^E s(y) \right\rangle_E dV_{g,y} =$$

$$= \int_U \left\langle \mu(y)\tilde{G}(x, y), \Delta^E s(y) \right\rangle_E dV_{g,y} =$$

$$= \int_U \left\langle \tilde{G}(x, y), \mu(y)\Delta^E s(y) \right\rangle_E dV_{g,y},$$

where the first equality is due to μ being compactly supported in U. Note that the following identity holds

$$\mu\Delta^E s = \Delta^E(\mu s) + \left(\Delta_g\mu\right)s - 2\nabla^E_{\mathrm{grad}_g\mu}s,$$

where $\Delta_g = -\mathrm{div}_g \circ \mathrm{grad}_g$ is the positive Laplace-Beltrami operator. Plugging it into the integral we have

$$\int\limits_U \left\langle \tilde{G}(x,y), \Delta^E(\mu s) - \left(\Delta_g\mu\right)s - 2\nabla^E_{\mathrm{grad}_g\mu}s\right\rangle_E dV_{g,y} =$$
$$= \mu(x)s(x) + \int\limits_U \left\langle \left(\Delta_g\mu\right)(y)\tilde{G}(x,y), s(y)\right\rangle_E dV_{g,y} -$$
$$- \int\limits_U \left\langle \tilde{G}(x,y), 2\nabla^E_{\mathrm{grad}_g\mu}s(y)\right\rangle_E dV_{g,y}.$$

Using Lemma 1.3.11 we see that the last integral is equal to

$$-2\int\limits_U \left\langle \nabla^E_{\mathrm{grad}\,\mu}\tilde{G}(x,y), s(y)\right\rangle_E dV_{g,y} + 2\int\limits_U \left(\Delta_g\mu\right)(y)\left\langle \tilde{G}(x,y), s(y)\right\rangle_E dV_{g,y},$$

where we used the fact that μ is compactly supported and equals to 1 on $V \ni x$. Therefore we obtain

$$\int\limits_N \left\langle \mu(y)\tilde{G}(x,y), \Delta^E s(y)\right\rangle_E dV_{g,y} =$$
$$= s(x) - \int\limits_N \left\langle \left(\Delta_g\mu(y) + 2\nabla^E_{\mathrm{grad}_g\mu}\right)\tilde{G}(x,y), s(y)\right\rangle_E dV_{g,y} =$$
$$= s(x) - \int\limits_N \langle R(x,y), s(y)\rangle_E \, dV_{g,y}.$$

Note that the section

$$R(x,y)=\left(\Delta_g\mu(y)+2\nabla^E_{\mathrm{grad}_g\,\mu(y)}\right)\tilde{G}(x,y)$$

is well defined and smooth on $V\times N$ due to the properties of μ. Now let us define $G_p(x,y)=\mu(y)\tilde{G}(x,y)+\check{G}(x,y)$, where $\check{G}(x,y)$ is the smooth solution to (1.15) with $\varphi(y)=R(x,y)$, i.e. we have

$$\begin{cases}\Delta^E_y\check{G}(x,y)=R(x,y) & y\in N,\\ \check{G}(x,y)=0 & y\in\partial N.\end{cases}$$

We argue that $G_p(x,y)$ is the desired Green kernel for x near p. Indeed, we have

$$\begin{aligned}&\int_N\left\langle G_p(x,y),\Delta^E s(y)\right\rangle_E dV_{g,y}=\\ &\qquad=\int_N\left\langle\mu(y)\tilde{G}(x,y)+\check{G}(x,y),\Delta^E s(y)\right\rangle_E dV_{g,y}=\\ &=s(x)-\int_N\langle R(x,y),s(y)\rangle_E\,dV_{g,y}+\int_N\left\langle\Delta^E_y\check{G}(x,y),s(y)\right\rangle_E dV_{g,y}=s(x).\end{aligned}$$

In addition, in light of Proposition 1.3.6 and our construction of G_p we conclude that it is smooth on $V\times N\backslash\mathrm{diag}(V)$, $G_p(x,y)=0$ for $y\in\partial N$, and the integral of $y\mapsto\left|G_p(x,y)\right|$ is finite. The global Dirichlet Green kernel $G(p,y)$ for any interior point p can be defined now as $G_p(p,y)$. This will not depend on the choices we made when constructed G_p due to Lemma 1.3.1. Note that $G(x,y)=G_p(x,y)$ for $x\in V$, $y\in N$ by Lemma 1.3.1, which gives the smoothness of $G(x,y)$ for $x\in N^{int}$, $y\in N$. Now by symmetry from Lemma 1.3.3 we can continue $G(x,y)$ smoothly to $x\in\partial N$,

$x \neq y \in N$. Finally, in the analytic case the analyticity of G is due to the fact that

$$\Delta_y^E G(x,y) = 0,$$

for $y \neq x$, and the analyticity of a solution to elliptic system with analytic coefficients [23, 36]. One can see that the proof for the Schrödinger operator is a straightforward generalisation. □

We know that in the sense of distributions we have for the connection Laplacian (or the Schrödinger operator)

$$\Delta_y^E G(x,y) = Id \cdot \delta_x(y). \tag{1.27}$$

In other words $G(x,\cdot) \in \mathscr{D}'(E)$ is a distribution for each $x \in N$. It is well known that the right hand side of (1.27) belongs to the Sobolev space $\mathscr{W}^{-k}(E)$ with $k > \frac{n}{2}$, where $n = \dim N$. Therefore, from the elliptic regularity theorem we have

$$G(x,\cdot) \in \mathscr{W}^{2-k}(E), \tag{1.28}$$

where $k > \frac{n}{2}$.

1.3.4 Construction of the Green kernel using parametrix

In this section we sketch a construction of the Green kernel using a general theory of PDOs. Let us consider $\left(N, g, E, \nabla^E, \langle,\rangle_E\right)$ as the restriction of the structure $\left(\tilde{N}, g, E, \nabla^E, \langle,\rangle_E\right)$ to N, where $N \subset \tilde{N}$ and $\left(\tilde{N}, g\right)$ is a closed (i.e. compact without boundary) Riemannian manifold. The connection Laplacian Δ^E is an elliptic differential operator. Using Theorem 1.1.3 we see that it has a parametrix $\mathscr{G}_0$, i.e. we have the equality

$$\mathscr{G}_0 \Delta^E = Id + \mathscr{R},$$

in $\tilde{N}$, where $\mathscr{R}$ is a smoothing operator. Since $\mathscr{R}$ is a smoothing operator it has a smooth kernel $R(x,y) \in C^\infty \left(\tilde{N} \times \tilde{N}; E \boxtimes E\right)$, meaning that

$$\mathscr{R}u(x) = \int_{\tilde{N}} \langle R(x,y), u(y)\rangle_{E_y} \, dV_{g,y}.$$

Now, we know that there is a smooth section $K(x,y) \in C^\infty (N \times N; E \boxtimes E)$ such that

$$\begin{cases} \Delta^E_y K(x,y) = R(x,y) & \text{in } N, \\ K(x,y) = 0 & \text{on } \partial N. \end{cases}$$

This section defines a smoothing operator

$$\mathscr{K} u(x) = \int_N \langle K(x,y), u(y)\rangle_{E_y} \, dV_{g,y}.$$

Let us consider the difference $\mathscr{G}_0 - \mathscr{K}$. Its composition with the Laplacian acts on $u \in \mathscr{D}(E|_N)$ as

$$\begin{aligned}(\mathscr{G}_0 - \mathscr{K})\Delta^E u = \mathscr{G}_0 \Delta^E u - \mathscr{K}\Delta^E u = u + \mathscr{R}u - \mathscr{K}\Delta^E u = \\ = u + \mathscr{R}u - \int_N \left\langle K(x,y), \Delta^E u(y)\right\rangle_{E_y} dV_{g,y},\end{aligned}$$

and using the symmetry of the connection Laplacian we get

$$u + \mathscr{R}u - \int\limits_N \left\langle \Delta_y^E K(x,y), u(y) \right\rangle_{E_y} dV_{g,y} + \\ + \int\limits_{\partial N} \left\langle \nabla_{\nu_y}^E K(x,y), u(y) \right\rangle_{E_y} dS_y = u.$$

Therefore, we see that the pseudodifferential operator $\mathscr{G} = \mathscr{G}_0 - \mathscr{K}$ is the left inverse to the connection Laplacian. Its kernel $G(x,y)$ is a Green kernel of the connection Laplacian.

1.4 Method of the layer potentials for the scalar DtN operator

1.4.1 Single- and double- layer potentials

By the *scalar Dirichlet-to-Neumann* operator we mean the DtN operator associated with the Laplace-Beltrami operator Δ. In this section we obtain some properties of the scalar DtN operator using the method of layer potentials. We start by introducing the layer potentials and their relation to the DtN operator. This material is well known and we follow closely the exposition in [41, Chapter 7]. Let $\overline{\Omega}$ be a compact n-dimensional Riemannian manifold with boundary. Suppose $\overline{\Omega} \subset M$, where M is an n-dimensional Riemannian manifold without boundary, and on M there is a fundamental solution $E(x,y)$ to the equation

$$\Delta_x E(x,y) = \delta_y(x), \tag{1.29}$$

where $E(x,y)$ can be thought of as the Schwartz kernel of an operator $E(x,D) \in \Psi^{-2}(M)$. Assume that $E(x,y) = E(y,x)$. Then we have as $x \to y$

$$E(x,y) \sim c_n \text{dist}(x,y)^{2-n} + \dots, \quad n \geq 3, \tag{1.30}$$

$$E(x,y) \sim c_2 \log \text{dist}(x,y) + \dots, \quad n = 2,$$

where $c_n = -[(n-2)Area(S^{n-1})]^{-1}$ for $n \geq 3$, $c_2 = 1/2\pi$, and $\text{dist}(x,y)$ denotes the distance between x and y [2, 41]. We define the single- and double-layer potentials of function f on $\partial\Omega$ as follows

$$\mathscr{S}l\, f(x) = \int\limits_{\partial\Omega} f(y) E(x,y) dS(y), \tag{1.31}$$

and

$$\mathscr{D}l\, f(x) = \int\limits_{\partial\Omega} f(y) \frac{\partial E}{\partial \nu_y}(x,y) dS(y), \tag{1.32}$$

for $x \in M \setminus \partial\Omega$. Given a function v on $M \setminus \partial\Omega$, for $x \in \partial\Omega$, let $v_+(x)$ and $v_-(x)$ denote the limits of $v(z)$ as $z \to x$, from $z \in \Omega$ and $z \in M \setminus \overline{\Omega} = \mathscr{O}$, respectively, when these limits exist. Then we can write the following properties of these layer potentials [41, Chapter 7, Proposition 11.1]. For $x \in \partial\Omega$ we have

$$\mathscr{S}l_+ f(x) = \mathscr{S}l_- f(x) = Sf(x), \tag{1.33}$$

$$\mathscr{D}l_\pm f(x) = \pm\frac{1}{2} f(x) + \frac{1}{2} N f(x), \tag{1.34}$$

where $\mathscr{S}l_{\pm}f(x)$ and $\mathscr{D}l_{\pm}f(x)$ denote $(\mathscr{S}l\, f)_{\pm}(x)$ and $(\mathscr{D}l\, f)_{\pm}(x)$, respectively, and for $x \in \partial\Omega$

$$Sf(x) = \int_{\partial\Omega} f(y)E(x,y)dS(y), \tag{1.35}$$

$$Nf(x) = 2\int_{\partial\Omega} f(y)\frac{\partial E}{\partial \nu_y}(x,y)dS(y). \tag{1.36}$$

By [41, Chapter 7, Proposition 11.2] we have the following

$$S, N \in \Psi^{-1}(\partial\Omega),\ S \text{ is elliptic.} \tag{1.37}$$

Now, recalling the definition of the Dirichlet-to-Neumann operator Λ, we can write

$$\int_{\partial\Omega} \left[f(y)\frac{\partial E}{\partial \nu_y}(x,y)dS(y) - \Lambda f(y)E(x,y) \right] dS(y) = \mathscr{D}l\, f(x) - \mathscr{S}l\, \Lambda f(x), \tag{1.38}$$

for $x \in M \setminus \partial\Omega$, and using Green's formula, we conclude

$$\mathscr{D}l\, f(x) - \mathscr{S}l\, \Lambda f(x) = \begin{cases} u(x), & x \in \Omega, \\ 0 & x \in M \setminus \overline{\Omega}, \end{cases} \tag{1.39}$$

where u is a harmonic extension of f to $\overline{\Omega}$. Taking the limit of (1.39) from within Ω, and using (1.33) and (1.34), we get $f = (1/2)f + (1/2)Nf - S\Lambda f$, which implies the identity

$$S\Lambda = -\frac{1}{2}(I - N). \tag{1.40}$$

Now, by Green's formula,

$$(\Lambda f, g)_{L^2(\partial\Omega)} = -(du, dv)_{L^2(\Omega)} = (f, \Lambda g)_{L^2(\partial\Omega)}, \tag{1.41}$$

where u and v are harmonic extensions to Ω of f and g, respectively. In other words, we see that Λ is symmetric.

Proposition 1.4.1. *The Dirichlet-to-Neumann operator* Λ *in* $\partial\Omega$ *is an elliptic pseudodifferential operator of order* 1.

Proof. By (1.37) the operator S is elliptic and so admits a left parametrix R, i.e. an operator R such that $R \in \Psi^1(\partial\Omega)$ and $RS = I + R_{-\infty}$ with $R_{-\infty} \in \Psi^{-\infty}(\partial\Omega)$. Multiplying (1.40) by R from the left we get $\Lambda = -\frac{1}{2}(R - RN) - R_{-\infty}\Lambda$. Therefore, we see that $\Lambda \in \Psi^1(\partial\Omega)$. The ellipticity of Λ follows from the ellipticity of R, as a parametrix of the elliptic (1.37) operator S. □

1.4.2 Neumann Green's function

Let us take a closer look at the relation (1.40). We see that if we could make N to be identically zero, then the operator $-2S$ would be the left inverse to Λ. This is equivalent to the requirement that E is the Green function on Ω with Neumann boundary condition, i.e. we should have

$$\begin{cases} \Delta_y E(x,y) = \delta_x(y) & \text{in } \Omega, \\ \frac{\partial E}{\partial \nu_y}(x,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

Unfortunately, this boundary value problem does not have a solution. Indeed, due to Green's identity we ought to have

$$\int_{\partial\Omega} \frac{\partial E}{\partial \nu_y}(x,y)dS(y) = \int_{\Omega} \Delta_y E(x,y)\,dS(y) = \int_{\Omega} \delta_x(y)\,dS(y) = 1, \tag{1.42}$$

which contradicts $\frac{\partial E}{\partial \nu_y}(x,y)$ being identically zero on the boundary. On the other hand, if we require

$$\frac{\partial E}{\partial \nu_y}(x,y) = \frac{1}{Vol(\partial\Omega)}$$

for any $x \in \Omega$ and any $y \in \partial\Omega$, then the condition (1.42) will be met. Therefore, it is natural to define the *Neumann Green's function* as a solution to the boundary value problem

$$\begin{cases} \Delta_y G(x,y) = \delta_x(y) & \text{in } \Omega, \\ \frac{\partial G}{\partial \nu_y}(x,y) = \frac{1}{Vol(\partial\Omega)} & \text{on } \partial\Omega. \end{cases} \tag{1.43}$$

If this problem has a solution then it is unique up to a constant if we add the symmetry condition. More precisely, we have the following result.

Proposition 1.4.2. *Let Ω be a compact Riemannian manifold with boundary $\partial\Omega$. Suppose there is a symmetric Neumann Green's function $G(x,y)$ on Ω. Then it is unique up to a constant. In particular, if we add the normalisation condition, e.g.*

$$\int_{\partial\Omega} G(x,y)\,dS(y) = 0,$$

then the normalised symmetric Neumann Green's function is unique.

Proof. Suppose $\tilde{G}(x,y)$ is another Neumann Green's function. Then the difference

$H(x,y) = G(x,y) - \tilde{G}(x,y)$ satisfies

$$\begin{cases} \Delta_y H(x,y) = 0 & \text{in } \Omega, \\ \frac{\partial H}{\partial \nu_y}(x,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

By elliptic regularity $H(x,y)$ is smooth on y variable in Ω. Using the boundary condition, we see that $H(x,y)$ belongs to the kernel of the Dirichlet-to-Neumann operator, which consists of constants. Hence, H does not depend on y, and by symmetry it does not depend on x also, i.e. H is constant. The last statement follows immediately. □

Now, if we define the single- and double- layer potentials using this Neumann Green's function, then from the identity (1.40) we obtain

$$-2S\Lambda = I$$

on the subspace of functions orthogonal to constants. From this identity we conclude that the Dirichlet-to-Neumann operator defines the restriction of the normalised Neumann Green's function to the boundary. In particular, if there are two compact manifolds with diffeomorphic boundaries such that the DtN operators are naturally equivalent under this diffeomorphism, then the restrictions of the normalised Neumann Green's functions to the boundary are naturally equivalent under this diffeomorphism.

1.4.3 Cases when S is invertible

Let us look closer at the operator S defined before. We can try to solve the problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega, \end{cases} \tag{1.44}$$

in terms of single-layer potential. If we consider

$$u(x) = \mathscr{S}l\, h(x), \quad x \in \Omega, \tag{1.45}$$

then (1.44) is equivalent to

$$f = Sh. \tag{1.46}$$

The single-layer potential has the following important property [41, Chapter 7, Proposition 11.3]. For $x \in \partial\Omega$, we have

$$\frac{\partial}{\partial \nu}\mathscr{S}l_+ h(x) - \frac{\partial}{\partial \nu}\mathscr{S}l_- h(x) = -h. \tag{1.47}$$

Now we can prove the following theorem concerning the kernel of S.

Theorem 1.4.3. *Let M be a complete simply connected n-dimensional manifold with $E(x,y) \to 0$ as dist$(x,y) \to \infty$, where $n \geq 3$ and $E(x,y)$ is a fundamental solution of the Laplace operator on M. If $\overline{\Omega}$ is a compact connected domain of M with non-empty smooth boundary and connected complement, then S has a trivial kernel.*

Proof. Suppose $h \in C^\infty(\partial\Omega)$ belongs to the null space of S. Then, by (1.46) and the uniqueness of the solution to (1.44) with $f = 0$, we have $\mathscr{S}l\, h(x) = 0$ on Ω. By (1.47), the jump of $(\partial/\partial \nu)\mathscr{S}l\, h(x)$ across $\partial\Omega$ is $-h$, so we have for $w = \mathscr{S}l\, h|_{\mathscr{O}}$ on the complement $\mathcal{O} = M \setminus \overline{\Omega}$ that

$$\Delta w = 0 \text{ on } \mathcal{O}, \quad \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega} = h. \tag{1.48}$$

Note also that $w(x) \to 0$ as $\text{dist}(x, \partial\Omega) \to +\infty$, this is a consequence of $w = \mathscr{S}l\,h$ and the definition (1.31) of the single-layer potential. From (1.33) we see that $\mathscr{S}l\,h$ does not jump across $\partial\Omega$, and since, by supposition, $Sh = 0$, we also have $w = 0$ on $\partial\Omega$. The maximum principle for harmonic functions [17, Chapter 8] forces $w = 0$ on $\mathcal{O}$, so $h = 0$, which completes the proof. □

This theorem is a slight generalisation of the result in [41, Proposition 11.5] concerning the case $M = \mathbb{R}^n$.

Remark. Using Theorem 1.4.3 we can give a different definition of the Dirichlet-to-Neumann operator on manifolds satisfying the conditions of the theorem. We simply multiply (1.40) on the left by the inverse of S to get

$$\Lambda = -\frac{1}{2}S^{-1}(I - N). \tag{1.49}$$

Note that for an arbitrary compact manifold Ω with boundary we can always define the DtN operator via (1.49) by an appropriate choice of M and $E(x, y)$. Namely, we can chose M to be a compact manifold with boundary ∂M, such that $\overline{\Omega}$ is contained in the interior of M, and $E(x, y)$ to be the Dirichlet Green's function on M. The proof of the invertibility of S in this case is almost identical to the proof of Theorem 1.4.3. Indeed, instead of the convergence of $w(x)$ to zero at infinity we have $w(x) = 0$ on

∂M, which follows from

$$w(x) = \mathscr{S}l\,h(x) = \int_{\partial\Omega} h(y)E(x,y)dS(y),$$

and the fact that $E(x,y)$ is the Dirichlet Green's function, i.e. $E(x,y) = 0$ for $x \in \partial M$.

In the next two sections we discuss the possible circumstances under which the required asymptotic behavior in Theorem 1.4.3 takes place. We show that these can be non-positivity of the sectional curvature or non-negativity of the Ricci curvature. In the former case we derive the required behavior from the heat kernel comparison, and in the latter case from the Li-Yau estimates for Green's function, when it exists.

1.4.4 Comparison theorem for heat kernel on manifolds with non-positive sectional curvature

We know from [37] that if M is a complete Riemannian manifold, then there exists a heat kernel $H(x,y,t) \in C^\infty(M \times M \times \mathbb{R}^+)$ such that

$$H(x,y,t) = H(y,x,t), \quad \lim_{t\to 0} H(x,y,t) = \delta_x(y), \tag{1.50}$$
$$\left(\Delta - \frac{\partial}{\partial t}\right)H = 0, \quad H(x,y,t) = \int H(x,z,t-s)H(z,y,s)dz.$$

The last equality is a heat semigroup property. If M is compact with non-empty boundary then there exists a heat kernel subject to Dirichlet or Neumann boundary conditions [17, 37]. We will need the following additional properties of the heat kernel for the proof of the heat kernel comparison theorem. Let us state them according to [37].

Lemma 1.4.4. *On a complete Riemannian manifold the heat kernel $H(x,y,t)$ is a strictly* *positive function for all t.*

Lemma 1.4.5. *Let $B(x,R)$ be a geodesic ball of radius R with center x in a space form (i.e. complete connected Riemannian manifold of constant sectional curvature), then its heat kernel $H(x,y,t)$ is only a function of $r = dist(x,y)$ and t, moreover, $\frac{\partial H}{\partial r} < 0$.*

Using these Lemmas, we can prove the following heat kernels comparison theorem for manifolds of non-positive sectional curvature, which is a counterpart of Cheeger-Yau comparison theorem for manifolds with non-negative Ricci curvature [37, Chapter III, Theorem 2].

Theorem 1.4.6. *Let M be a complete Riemannian manifold with sectional curvature $\kappa_M(\sigma) \leq \kappa \leq 0$. Fixing an arbitrary point $x \in M$ and a $r_0 > 0$, the heat kernel $H_{r_0}(x,y,t)$ of $B(x,r_0)$ and the heat kernel $\mathcal{E}_{r_0}(r(x,y),t)$ of $V(\kappa,r_0)$ satisfy*

$$\mathcal{E}_{r_0}(r(x,y),t) \geq H_{r_0}(x,y,t), \tag{1.51}$$

where $B(x,r_0) \subset M$ is the ball of radius r_0 centered at x, $V(\kappa,r_0)$ is the ball of radius r_0 in the space form of curvature κ, and boundary conditions will either be Dirichlet or Neumann.

Proof. Let us follow the proof of [37, Chapter III, Theorem 2] adjusting it to our case. Using the properties of the heat kernels (1.50) we can write the following sequence of equalities

$$\begin{aligned} H(x,y,t)-\mathscr{E}(r(x,y),t) &= \int_0^t \int_{B(x,r_0)} \frac{d}{ds}(\mathscr{E}(x,z,t-s)H(z,y,s))dzds = \\ &= -\int_0^t \int_{B(x,r_0)} \frac{\partial}{\partial(t-s)}[\mathscr{E}(r(x,z),t-s)]H(z,y,s)dzds + \\ &\quad + \int_0^t \int_{B(x,r_0)} \mathscr{E}(r(x,z),t-s)\frac{\partial}{\partial s}H(z,y,s)dzds = \\ &= -\int_0^t \int_{B(x,r_0)} \tilde{\Delta}\mathscr{E}(r(x,z),t-s)H(z,y,s)dzds + \\ &\qquad + \int_0^t \int_{B(x,r_0)} \mathscr{E}(r(x,z),t-s)\Delta H(z,y,s)dzds, \end{aligned}$$

where $\tilde{\Delta}$, Δ are respectively the Laplacian operators on the space form and M. Using the second Green's identity and either Dirichlet or Neumann boundary conditions, we get

$$\int_{B(x,r_0)} \mathscr{E} \cdot \Delta H = \int_{B(x,r_0)} \Delta\mathscr{E} \cdot H.$$

Hence,

$$H(x,y,t)-\mathscr{E}(r(x,y),t) = \int_0^t \int_{B(x,r_0)} (-\tilde{\Delta}\mathscr{E} + \Delta\mathscr{E}) \cdot H dzdt.$$

By Lemma 1.4.4, we know that $H > 0$. Thus, for the proof of the theorem it is sufficient to show that $(\Delta - \tilde{\Delta})\mathscr{E} < 0$. Let (r,ξ), where $r \in (0,r_0)$, $\xi \in S^{n-1}$, be geodesic spherical coordinates on the ball $B(x,r_0)$. Then the operators $\tilde{\Delta}$ and Δ have the following forms:

$$\tilde{\Delta} = \frac{\partial^2}{\partial r^2} + \phi(r)\frac{\partial}{\partial r}, \qquad \phi(r) = \frac{d\log\sqrt{\det \tilde{g}}}{dr},$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \phi(r,\xi)\frac{\partial}{\partial r}, \qquad \phi(r,\xi) = \frac{d\log\sqrt{\det g}}{dr},$$

where g and $\tilde{g}$ are Riemannian metrics on the ball $B(x,r_0) \subset M$ and the ball $V(\kappa,r_0)$ in the space form of curvature κ, respectively, and $\det g$ is the determinant of the metric g. Since sectional curvature $\kappa_M(\sigma) \leq \kappa \leq 0$ by the Volume Comparison Theorem of P. Gunther and R. L. Bishop (see, for instance, Theorem III.4.1 in [8]) we have $\phi(r,\xi) \geq \phi(r)$. Therefore $(\Delta - \tilde{\Delta})\mathscr{E} = (\phi(r,\xi) - \phi(r))\frac{\partial \mathscr{E}}{\partial r} \leq 0$, where we have used the inequality $\frac{\partial \mathscr{E}}{\partial r} < 0$ provided by Lemma 1.4.5. □

Let $G_{r_0}(x,y)$ be the Dirichlet Green's function for the ball $B(y,r_0) \subset M$. From [17, 37] we know that it can be defined by

$$G_{r_0}(x,y) = \int_0^\infty H_{r_0}(x,y,t)dt,$$

where $H_{r_0}(x,y,t)$ denotes the Dirichlet heat kernel on the ball $B(y,r_0)$, assuming that the integral on the right converges. Using Theorem 1.4.6 we can derive the comparison for Dirichlet Green's functions on balls in M and $\mathbb{R}^n$. Indeed, it follows easily from the theorem and (1.50) that for a fixed $y \in M$, $x \neq y$, and all $r_0 > 0$ we have

$$G_{r_0}(x,y) = \int_0^\infty H_{r_0}(x,y,t)dt \leq \int_0^\infty \mathscr{E}_{r_0}(r(x,y),t)dt = G_{r_0;0}(r(x,y)), \tag{1.52}$$

where $G_{r_0;0}(r(x,y))$ is the Dirichlet Green's function for the ball of radius r_0 in $\mathbb{R}^n$. By [2, Theorem 4.4] the spectrum of the Dirichlet Laplacian on a ball is strictly positive. Combining this with [17, Theorem 13.4] we see that both integrals in (1.52) converge for distinct x and y. As a consequence of properties (1.50) Dirichlet Green's function is smooth for $x \neq y$, and satisfies

$$\Delta_x G_{r_0}(x,y) = \delta_y(x),\ G_{r_0}(x,y) \geq 0, \tag{1.53}$$
$$G_{r_0}(x,y) = 0, \text{ for all } x \in \partial B(y, r_0).$$

Now, assuming that M is simply connected, we have the following useful corollary.

Corollary 1.4.7. *Let M be a simply connected complete n-dimensional Riemannian manifold with sectional curvature $\kappa_M(\sigma) \leq 0$, $n \geq 3$. Then there exists a positive fundamental solution with the following asymptotic behavior*

$$E(x,y) \to 0 \text{ as } dist(x,y) \to \infty, \tag{1.54}$$

where $E(x,y)$ is a fundamental solution to the Laplace equation (1.29) (an entire Green's function on M).

Proof. Fix an arbitrary point $y \in M$. Let $R_2 \geq R_1 > 0$ and $x \in B(y,R_1)$. By the maximum principle for harmonic functions [17, Chapter 8], we have

$$G_{R_2}(x,y) \geq G_{R_1}(x,y)\ \forall x \in B(y,R_1) \setminus \{y\}. \tag{1.55}$$

The Dirichlet Green's function for a ball $B(y,R)$ in $\mathbb{R}^n$ is $G_{R;0}(r(x,y)) = c_n |x-y|^{2-n} - c_n R^{2-n}$. Proceeding to the limit $R \to \infty$, we get

$$E_0(r(x,y)) = c_n |x-y|^{2-n} \tag{1.56}$$

the entire Green's function on $\mathbb{R}^n$. From (1.55) and (1.52) we have

$$G_R(x,y) \leq E_0(r(x,y)), \tag{1.57}$$

for any $R > 0$. This allows us to define

$$E(x,y) = \lim_{R\to\infty} G_R(x,y) \tag{1.58}$$

and check that $E(x,y)$ is an entire Green's function on M. This can be done by using the bound (1.57), the monotonicity (1.55), and Harnack's principle (see, for example, [17, Corollary 13.13]). Now, varying $y \in M$, we get $E(x,y)$ as a smooth symmetric function of $(x,y) \in M \times M \setminus \operatorname{diag}(M)$. From (1.56), (1.57), and (1.58) we get the desired behavior, since $|x-y|^{2-n} \to 0$ as $\operatorname{dist}(x,y) \to \infty$. □

1.4.5 Li-Yau estimates for Green's function on manifolds with non-negative Ricci curvature

The second class of manifolds that admit vanishing at infinity entire Green's function is complete Riemannian manifolds with non-negative Ricci curvature. We define Green's function on a complete Riemannian manifold as before by

$$G(x,y) = \int_0^\infty H(x,y,t)dt,$$

if the right hand side converges. According to [17] it is not always the case. For example, if M is compact then the right hand side is infinite everywhere and there is no fundamental solution of the Laplace operator on M. On the other hand, if $G(x,y)$

is finite (for all distinct $x, y \in M$), then it is positive by Lemma 1.4.4 and represents a fundamental solution of the Laplace operator. In particular, we see that for the Euclidean plane the Green function defined in the above way does not exist (not finite), since the integral diverges, see details in [17, p. 342].

Under the above assumptions on a manifold we have the following estimate [37, Theorem 4.13].

Theorem 1.4.8. *Let M be a complete Riemannian manifold without boundary and with $Ric(M) \geq 0$, if $G(x,y)$ exists, then*

$$C_1(n)\int_{r^2(x,y)}^{\infty}\frac{dt}{V_x(\sqrt{t})} \leq G(x,y) \leq C_2(n)\int_{r^2(x,y)}^{\infty}\frac{dt}{V_x(\sqrt{t})}$$

and

$$C_1(n)\int_{r^2(x,y)}^{\infty}\frac{dt}{\sqrt{V_x(\sqrt{t})V_y(\sqrt{t})}} \leq G(x,y) \leq C_2(n)\int_{r^2(x,y)}^{\infty}\frac{dt}{\sqrt{V_x(\sqrt{t})V_y(\sqrt{t})}},$$

where $V_x(\sqrt{t}) = Vol(B_x(\sqrt{t}))$ and $C_1(n), C_2(n)$ are positive constants depending only on the dimension n of M.

From the above theorem we get the following result for an asymptotic behavior of Green's function.

Corollary 1.4.9. *Let M be a complete Riemannian manifold without boundary and with $Ric(M) \geq 0$, if $G(x,y)$ exists, then $G(x,y) \to 0$ as $dist(x,y) \to +\infty$.*

Using Theorem 1.4.8, we can also find an examples of manifolds that do not admit a positive Green's function. Indeed, let us take, for example, a cylinder $S^{n-1} \times \mathbb{R}$ with a product metric of canonical metrics on S^{n-1} and $\mathbb{R}$. We have $\mathrm{Ric}(S^n \times \mathbb{R}) \geq 0$. Note that for a point $x = (p, \tau) \in S^{n-1} \times \mathbb{R}$ the ball $B_x(\sqrt{t})$ is contained in the product $S^{n-1} \times [\tau - \sqrt{t}, \tau + \sqrt{t}]$. Hence, we conclude that $V_x(\sqrt{t}) \leq 2Vol(S^{n-1})\sqrt{t}$. Then for any $\varepsilon > 0$ we have

$$\int_{\varepsilon}^{\infty} \frac{dt}{V_x(\sqrt{t})} \geq \frac{1}{2Vol(S^{n-1})} \int_{\varepsilon}^{\infty} \frac{dt}{\sqrt{t}} = +\infty.$$

This leads to a contradiction if we suppose that $G(x, y)$ exists and then apply Theorem 1.4.8. So we conclude that there is no positive Green's function on $S^{n-1} \times \mathbb{R}$ with the canonical metric.

Chapter 2

Calderón's problem for the connection Laplacian

2.1 Introduction

Let (N, g) be a compact connected Riemannian manifold with non-emtpy boundary ∂N, and let $E \to N$ be a Euclidean vector bundle endowed with a compatible connection ∇^E. Consider the connection Laplacian Δ^E associated with the connection ∇^E. It is a natural generalisation of the Laplace-Beltrami operator. We define the corresponding Dirichlet-to-Neumann (DtN) operator Λ_{g,∇^E} by sending a section σ on the boundary ∂N to the outward normal covariant derivative of its harmonic extension.

In this chapter we study the DtN operator Λ_{g,∇^E} as a pseudodifferential operator on the boundary. We follow the strategy in the paper [31] by Lee and Uhlmann for the Laplace-Beltrami operator. They showed that the Riemannian metric on the boundary can be recovered from a given DtN operator. In addition, if the dimension of the manifold is greater than 2, they showed that all the normal derivatives of the metric can be recovered as well. This result was used in [31] to recover a Riemannian manifold from the DtN operator under some assumptions on the geometry and topology of a manifold, and subsequently in [27, 28] under the assumption of real-analyticity. This method also appears in the work of Cekić [7] on Calderón's problem for Yang-Mills connections.

It is well known that a Riemannian metric on a manifold can be recovered from the Laplace-Beltrami operator. This can be done by considering the principal symbol of the Laplacian which is equal to $|\xi|^2_{g(x)}$, where $\xi \in T^*_x N$, $x \in N$. The Dirichlet-to-Neumann operator is a classical elliptic pseudodifferential operator of order one on the boundary. Therefore, it is natural to use the same idea for the recovery of the geometric data on the boundary from the DtN operator. There is a local factorisation of the Laplacian into the composition of two operators near the boundary which establishes the relationship between the symbols of the DtN operator and Laplacian. In particular, it turns out that the principal symbol of the DtN operator is the (minus) square root of the principal symbol of the boundary Laplacian. Therefore, the principal symbol of the DtN operator is equal to $-|\xi|_{g|_{\partial N}}$, and it is straightforward to determine the metric from it. The rest part of the symbol is expressed in terms of the local geometric data in a more sophisticated way. By analysing these expressions we are able to recover the geometric data on the boundary from the full symbol of the DtN operator. More precisely, we prove the following result.

Theorem 2.1.1. *Suppose $\dim N = n \geq 3$. Let $\left(x^1, \dots, x^{n-1}\right)$ be any local coordinates for an open set $W \subset \partial N$ and $(\epsilon_1, \dots, \epsilon_r)$ any local frame of E over W, and let $\left\{\lambda_j,\ j \leq 1\right\}$ be the full symbol of the DtN operator Λ in these coordinates and local frame. For any $p \in W$, the full Taylor series of g and ∇^E at p in boundary normal coordinates and boundary normal frame is given by an explicit formula in terms of the matrix functions*

$\{\lambda_j\}$ *and their tangential derivatives at* p.

On surfaces, the DtN operator naturally scales under the conformal changes of a metric. As a consequence, we cannot recover the normal derivatives of a metric and a connection at the boundary. So the result in this case is a bit weaker.

Theorem 2.1.2. *Let* (N,g) *be a Riemannian surface. Let* (x^1) *be any local coordinate for an open set* $W \subset \partial N$ *and* $(\epsilon_1,\ldots,\epsilon_r)$ *any local frame of* E *over* W. *Let* $\{\lambda_j,\, j \le 1\}$ *be the full symbol of the DtN operator* Λ *in these coordinate and local frame. Then for any* $p \in W$ *the metric* g *and the connection* ∇^E *at* p *in these coordinate and frame is given by an explicit formula in terms of the matrix functions* $\{\lambda_j\}$ *and their tangential derivatives at* p.

2.1.1 Well-posedness of the generalised heat equation and regularity of its solution.

In this subsection we describe the result by Trèves [42, III.1]. In order to be precise we introduce the original setting. Let X be a smooth manifold; $n = \dim X$; t be the variable in the closed half-line $\overline{\mathbb{R}}_+$; T be some positive real number.

We shall deal with functions and distributions valued in a finite-dimensional Hilbert space H over $\mathbb{C}$. The norm in H will be denoted by $|\cdot|_H$, whereas the operator norm in $L(H)$, the space of (bounded) linear operators in H, will be denoted by $\|\cdot\|$. The inner product in H will be denoted by $(\cdot,\cdot)_H$. The space of H-valued distributions in X will be denoted by $\mathscr{D}'(X;H)$.

Let $A(t)$ be a pseudodifferential operator of order 1 in X, valued in $L(H)$, depending smoothly on $t \in [0,T)$. If we fix basis in H, then $A(t)$ is a matrix whose entries are scalar pseudodifferential operators in X. This means that in every local chart $(\Omega, x_1, \ldots, x_n)$, $A(t)$ is congruent modulo smoothing operators which are C^∞-functions of t to an operator

$$A_\Omega(t)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a_\Omega(x,t,\xi)\hat{u}(\xi)\,d\xi, \quad u \in C_c^\infty(\Omega; H),$$

where $a_\Omega(x,t,\xi)$ is a smooth function of $t \in [0,T)$ valued in $S^1(\Omega; L(H))$, the space of symbols valued in $L(H)$.

According to Trèves [42, III.1], the heat equation for $A(t)$ is well-posed and possesses a regularity property described below if the following conditions are satisfied:

1. Let $(\Omega, x_1, \ldots, x_n)$ be any local chart in X. There is a symbol $a_\Omega(x,t,\xi)$ satisfying

$$a_\Omega(x,t,\xi) \text{ is a } C^\infty \text{ function of } t \in [0,T) \text{ valued in } S^1(\Omega; L(H)),$$

   and defining the operator $A_\Omega(t)$ congruent to $A(t)$ modulo smoothing operators in Ω depending smoothly on $t \in [0,T)$, such that

2. to every compact subset K of $\Omega \times [0,T)$ there is a compact subset K' of the open half-plane $\mathbb{C}_- = \{z \in \mathbb{C}; \operatorname{Re} z < 0\}$ such that

3. the map

$$z \cdot Id - \frac{a_\Omega(x,t,\xi)}{\left(1+|\xi|^2\right)^{1/2}} : H \to H$$

   is a bijection (hence also a homeomorphism), for all $(x,t) \in K$, $\xi \in \mathbb{R}^n$, $z \in \mathbb{C}\backslash K'$.

The regularity property that we are interested in is described in the following theorem [42, III, Theorem 1.2].

Theorem 2.1.3. *Let $\mathscr{O}$ be an open subset of X, u a C^∞ function of t in $[0,T)$ valued in $\mathscr{D}'(X;H)$.*

Suppose that $u(0) \in C^{\infty}(\mathscr{O};H)$ *and that*

$$\frac{\partial u}{\partial t} - A(t)u \in C^{\infty}(\mathscr{O}\times[0,T);H).$$

Then $u \in C^{\infty}(\mathscr{O}\times[0,T);H)$.

This theorem is one of the main ingredients in the proof of Proposition 2.2.4, which relates the symbol of the DtN operator to the symbol of the connection Laplacian.

2.2 Reconstruction of the geometric data on the boundary

In this section we find the relation between full symbols of the DtN operator and the connection Laplacian, and then use this relation to prove Theorem 2.1.1. We follow the general strategy used in [31] for the DtN operator associated with the Laplace-Beltrami operator.

2.2.1 Local factorisation of the connection Laplacian

Let us recall the construction of geodesic coordinates with respect to the boundary. For each $q \in \partial N$, let $\gamma_q : [0,\epsilon) \to N$ denote the unit-speed geodesic starting at q and normal to ∂N. If $\{x^1,\dots,x^{n-1}\}$ are any local coordinates for ∂N near $p \in \partial N$, we can extend them smoothly to functions on a neighborhood of p in N by letting them be constant along each normal geodesic γ_q. If we then define x^n to be the parameter along each γ_q, it follows that $\{x^1,\dots,x^n\}$ form coordinates for N in some neighborhood of p, which we call the *boundary normal coordinates* determined by $\{x^1,\dots,x^{n-1}\}$. In these coordinates $x^n > 0$ in the interior of N, and ∂N is locally characterised by $x^n = 0$. The metric in these coordinates has the form

$$g = \sum_{i,j=1}^{n-1} g_{ij}\left(x^1, \ldots, x^n\right) dx^i dx^j + (dx^n)^2 .$$

Let $(\epsilon_1, \ldots, \epsilon_r)$ be a smooth local frame of $E|_{\partial N}$ near $p \in \partial N$, we can extend it to a smooth local frame $(\varepsilon_1, \ldots, \varepsilon_r)$ in a neighborhood of p in N by means of parallel transport along each γ_q, i.e. for each q we find the unique solution to the parallel transport equation

$$\nabla^E_{\dot{\gamma}_q} \varepsilon_\alpha = 0,$$
$$\varepsilon_\alpha|_{\gamma_q(0)} = \epsilon_\alpha, \text{ for } \alpha = 1, \ldots, r.$$

We call this frame the *boundary normal frame* determined by $(\epsilon_1, \ldots, \epsilon_r)$. In boundary normal coordinates we have then

$$\nabla^E_{\partial/\partial x^n} \varepsilon_\alpha = 0. \tag{2.1}$$

In local frame a section u is represented as a vector-valued function on N and the connection ∇^E acts as

$$\nabla^E_{\cdot} u = du(\cdot) + \omega(\cdot) u,$$

where $\omega = \omega_k dx^k$ denotes the matrix of the connection form of ∇^E. From (2.1) we have then

$$\omega_n = \omega\left(\frac{\partial}{\partial x^n}\right) = 0.$$

Remark 2.2.1. If the connection ∇^E is compatible with an inner product $\langle \cdot, \cdot \rangle_E$ on E, we have the following relation

$$d\left\langle u,v\right\rangle_E=\left\langle \nabla^E u,v\right\rangle_E+\left\langle u,\nabla^E v\right\rangle_E. \tag{2.2}$$

Note that if the frame $(\epsilon_1,\ldots,\epsilon_r)$ is orthonormal then the associated boundary normal frame is also orthonormal, since we have

$$\frac{\partial}{\partial t}\left\langle \varepsilon_\alpha,\varepsilon_\beta\right\rangle_E=\left\langle \nabla^E_{\dot\gamma}\varepsilon_\alpha,\varepsilon_\beta\right\rangle_E+\left\langle \varepsilon_\alpha,\nabla^E_{\dot\gamma}\varepsilon_\beta\right\rangle_E=0$$
$$\left\langle \varepsilon_\alpha,\varepsilon_\beta\right\rangle_E\Big|_{t=0}=\left\langle \epsilon_\alpha,\epsilon_\beta\right\rangle_E=0,\ \alpha\neq\beta,$$
$$\left\langle \varepsilon_\alpha,\varepsilon_\alpha\right\rangle_E\Big|_{t=0}=\left\langle \epsilon_\alpha,\epsilon_\alpha\right\rangle_E=1.$$

We will use further the following notation, $x=(x',x^n)$, $x'=\left(x^1,\ldots,x^{n-1}\right)$, $\partial_{x^j}=\partial/\partial x^j$, $D_{x^j}=-i\partial_{x^j}$, and $D_x=(D_{x^1},\ldots,D_{x^n})$, with similar definitions for $D_{x'}$, ∂_x, and $\partial_{x'}$. As usual the Einstein summation convention will be assumed throughout.

In boundary normal coordinates and boundary normal frame, the connection Laplacian is

$$\Delta^E u=\Delta u+g^{ij}\left[2\omega_i\partial_{x^j}u+\left(\left(\nabla_{\partial_{x^i}}\omega\right)(\partial_{x^j})+\omega_i\omega_j\right)u\right],$$

where $\left(g^{ij}\right)$ is the inverse of the matrix $\left(g_{ij}\right)$, Δ is the (scalar) Laplace-Beltrami operator on N. We can write

$$Lu:=\Delta^E u=\left[\Delta+i\sum_{j=1}^{n-1}V^jD_{x^j}+\tilde Q\right]u,$$

where

$$V^j = 2g^{jk}\omega_k,\ j = 1,\dots,n-1$$
$$\tilde{Q} = \sum_{i,j=1}^{n-1} g^{ij}\left[\left(\nabla_{\partial_{x^i}}\omega\right)(\partial_{x^j}) + \omega_i\omega_j\right].$$

The Laplace-Beltrami operator in boundary normal coordinates can be written as

$$\Delta u = \sum_{i,j=1}^{n} \varrho^{-1/2}\partial_{x^i}\left(\varrho^{1/2}g^{ij}u\right) = \partial_{x^n}\partial_{x^n}u + \frac{1}{2}\left(\partial_{x^n}\log\varrho\right)\partial_{x^n}u + \\ + \sum_{i,j=1}^{n-1}\left(g^{ij}\partial_{x^i}\partial_{x^j}u + \frac{1}{2}g^{ij}\left(\partial_{x^i}\log\varrho\right)\partial_{x^j}u + \left(\partial_{x^i}g^{ij}\right)\partial_{x^j}u\right),$$

where $\varrho = \det\left(g_{ij}\right)$. Using this we can write

$$-L = -\Delta - iV^l D_{x^l} - \tilde{Q} = D_{x^n}^2 + iF(x)D_{x^n} + Q(x, D_{x'}), \tag{2.3}$$

where

$$F(x) = -\frac{1}{2}\sum_{k,l=1}^{n-1} g^{kl}(x)\,\partial_{x^n}g_{kl}(x),$$

$$Q(x, D_{x'}) = \sum_{k,l=1}^{n-1} g^{kl}(x)D_{x^k}D_{x^l} - \\ - i\sum_{k,l=1}^{n-1}\left(\frac{1}{2}g^{kl}(x)\,\partial_{x^k}\log\varrho(x) + \partial_{x^k}g^{kl}(x) + V^l\right)D_{x^l} - \tilde{Q}.$$

The Dirichlet-to-Neumann operator in boundary normal coordinates and boundary normal frame is

$$\Lambda_{g,\nabla^E}\sigma = \nabla^E_\nu u\big|_{\partial N} = \left(\frac{\partial}{\partial x^n} + \omega_n\right)u\Big|_{\partial N} = \frac{\partial u}{\partial x^n}\Big|_{\partial N}, \tag{2.4}$$

where u is the harmonic extension of σ. The next proposition shows that there is a useful local factorisation of the Laplacian into a composition of two first-order pseudodifferential operators.

Proposition 2.2.2. *There exists a pseudodifferential operator $A(x, D_{x'})$ of order one in x' depending smoothly on $x^n \in [0, T]$, for some $T > 0$, such that*

$$-L \equiv (D_{x^n} + iF(x) - iA(x, D_{x'})) \circ (D_{x^n} + iA(x, D_{x'})) \tag{2.5}$$

modulo a smoothing operator.

Proof. We use the symbol calculus to construct such an operator $A(x, D_{x'})$. From (2.3) we get

$$L + (D_{x^n} + iF - iA) \circ (D_{x^n} + iA) = -D^2_{x^n} - iFD_{x^n} - Q + D^2_{x^n} + iFD_{x^n} -$$

$$- iAD_{x^n} + iD_{x^n}A - FA + AA = AA - Q + i[D_{x^n}, A] - FA. \tag{2.6}$$

Let a denote the full symbol of $A(x, D_{x'})$ and q denote the full symbol of $Q(x, D_{x'})$. Then, by (1.7) the full symbol of (2.6) is

$$\sum_K \frac{1}{K!}\partial^K_\xi a D^K_x a - q + \partial_{x^n} a - Fa,$$

and q splits into three terms

$$
\begin{aligned}
q\left(x,\xi'\right) = & \sum_{k,l=1}^{n-1} g^{kl}(x)\,\xi_k\xi_l - \\
& -i\sum_{l=1}^{n-1}\left[\sum_{k=1}^{n-1}\left(\frac{1}{2}g^{kl}(x)\,\partial_{x^k}\log\delta(x)+\partial_{x^k}g^{kl}(x)\right)+V^l\right]\xi_l-\tilde{Q}= \\
& = q_2\left(x,\xi'\right)+q_1\left(x,\xi'\right)+q_0(x),
\end{aligned}
$$

where $q_2(x,\xi')$, $q_1(x,\xi')$ and $q_0(x)$ are the quadratic, linear and constant in ξ' parts of $q(x,\xi')$, respectively. Let us write

$$
a\left(x,\xi'\right)\sim\sum_{j\leq 1}a_j\left(x,\xi'\right),
$$

where a_j are positive-homogeneous of degree j in ξ', that is we will define A by a formal symbol. We shall determine a_j recursively so that (2.6) is zero modulo symbols of smoothing operators.

The homogeneous terms of degree two in (2.6) give us

$$
a_1a_1-q_2=0,
$$

so we can choose

$$
a_1=-\sqrt{q_2}. \tag{2.7}
$$

Note that q_2 and, therefore, also a_1 are scalar matrices. The terms of degree one in (2.6) give us

$$
a_0a_1+a_1a_0+\sum_{l}^{n-1}\partial_{\xi^l}a_1D_{x^l}a_1-q_1+\partial_{x^n}a_1-Fa_1=0,
$$

and using relation (2.7), we get

$$-2\sqrt{q_2}a_0 + \sum_{l}^{n-1} \partial_{\xi^l}\sqrt{q_2}D_{x^l}\sqrt{q_2} - q_1 - \partial_{x^n}\sqrt{q_2} + F\sqrt{q_2} = 0,$$

thus we have

$$a_0 = \frac{1}{2\sqrt{q_2}}\left[\sum_{l}^{n-1} \partial_{\xi^l}\sqrt{q_2}D_{x^l}\sqrt{q_2} - q_1 - \partial_{x^n}\sqrt{q_2} + F\sqrt{q_2}\right]. \tag{2.8}$$

The terms of degree zero in (2.6) give us

$$a_{-1}a_1 + a_1 a_{-1} + \sum_{l}^{n-1} \partial_{\xi^l}a_1 D_{x^l}a_0 + \sum_{l}^{n-1} \partial_{\xi^l}a_0 D_{x^l}a_1 + \\ + \frac{1}{2}\sum_{k,l}^{n-1} \partial_{\xi^k}\partial_{\xi^l}a_1 D_{x^k}D_{x^l}a_1 - q_0 + \partial_{x^n}a_0 - Fa_0 = 0,$$

and using relation (2.7) again, we get

$$a_{-1} = \frac{1}{2\sqrt{q_2}}\left[-\sum_{l}^{n-1} \partial_{\xi^l}\sqrt{q_2}D_{x^l}a_0 - \sum_{l}^{n-1} \partial_{\xi^l}a_0 D_{x^l}\sqrt{q_2} + \right. \\ \left. + \frac{1}{2}\sum_{k,l}^{n-1} \partial_{\xi^k}\partial_{\xi^l}\sqrt{q_2}D_{x^k}D_{x^l}\sqrt{q_2} - q_0 + \partial_{x^n}a_0 - Fa_0\right], \tag{2.9}$$

where a_0 is given by (2.8). Continuing the recursion for the terms of degree $m \leq -1$ we have

$$-2\sqrt{q_2}a_{m-1} + \sum_{\substack{j,k,K \\ m\leq j,k\leq 1 \\ |K|=j+k-m}} \frac{1}{K!}\partial_\xi^K a_j D_x^K a_k + \partial_{x^n}a_m - Fa_m = 0.$$

Therefore we get

$$a_{m-1} = \frac{1}{2\sqrt{q_2}} \left[\sum_{\substack{j,k,K \\ m \le j,k \le 1 \\ |K| = j+k-m}} \frac{1}{K!} \partial_\xi^K a_j D_x^K a_k + \partial_{x^n} a_m - F a_m \right]. \tag{2.10}$$

□

Remark 2.2.3. Let $p \in \partial N$. Note that we can extend $\left(N, g, E, \nabla^E\right)$ along the boundary ∂N near p. This means that there is a vector bundle $\left(\tilde{E}, \nabla^{\tilde{E}}\right)$ over a Riemannian manifold $\left(\tilde{N}, \tilde{g}\right)$ such that N is included in $\tilde{N}$ isometrically, the restriction of $\left(\tilde{E}, \nabla^{\tilde{E}}\right)$ to N coincides with $\left(E, \nabla^E\right)$, and the point p lies in the interior of $\tilde{N}$. Clearly, near $p \in \tilde{N}$ there is an extension of boundary normal coordinates (so that $x^n \in (-\epsilon, \epsilon)$) and boundary normal frame. Due to the construction of $A(x, D_{x'})$ one sees that the factorisation in Proposition 2.2.2 extends to a neighborhood of p in $\tilde{N}$, i.e. there exists a PDO $\tilde{A}(x, D_{x'})$ of order one in x' depending smoothly on $x^n \in \left[-\tilde{T}, \tilde{T}\right]$, for some positive $\tilde{T} < T$, such that it coincides with $A(x, D_{x'})$ for $x^n \in \left[0, \tilde{T}\right]$.

2.2.2 The full symbol of the Dirichlet-to-Neumann operator Λ_{g,∇^E}.

Our next step is to relate the operator $A(x, D_{x'})$ with the Dirichlet-to-Neumann operator Λ_{g,∇^E}. It turns out that this relation is quite simple.

Proposition 2.2.4. *The operator A satisfies the following relation*

$$A(x, D_{x'})|_{\partial N} \sigma \equiv \partial_{x^n} u|_{\partial N} = \Lambda_{g,\nabla^E} \sigma$$

modulo a smoothing operator, where u is the harmonic extension of σ.

Proof. Let $p \in \partial N$. Using Remark 2.2.3 we consider an extension $\left(\tilde{N}, \tilde{g}, \tilde{E}, \nabla^{\tilde{E}}\right)$ along ∂N near p. Choose a coordinate chart (Ω, x') in ∂N containing p and denote by (x', x^n) the corresponding boundary normal coordinates in $\tilde{N}$. Let $\Omega' \subset \Omega$ be a precompact open subset and σ a section in $\mathscr{W}^{1/2}(E|_{\partial N})$ compactly supported in Ω'. Consider a solution $u \in \mathscr{W}^1\left(\tilde{E}\right) \subset \mathscr{D}'\left(\tilde{N}; \tilde{E}\right)$ to

$$\begin{cases} \Delta^{\tilde{E}} u = 0, & \text{in } \tilde{N} \\ u|_{\partial N} = \sigma. \end{cases}$$

By Proposition 2.2.2 and Remark 2.2.3, this problem is locally equivalent up to some smooth section h to the following system of equalities for u, v:

$$\left(Id \cdot D_{x^n} + i\tilde{A}(x, D_{x'})\right) u = v, \quad u|_{x^n=0} = \sigma,$$
$$\left(Id \cdot D_{x^n} + iF(x) - i\tilde{A}(x, D_{x'})\right) v = h \in C^\infty\left(\left[-\tilde{T}, \tilde{T}\right] \times \Omega'; \mathbb{C}^r\right).$$

The second equation above can be viewed as a backwards generalised heat equation; making the substitution $t = \tilde{T} - x^n$, it is equivalent to

$$Id \cdot \partial_t v - \left(\tilde{A} - F\right) v = -ih, \;\; t \in \left[0, 2\tilde{T}\right] \tag{2.11}$$

Since h is smooth and $\tilde{A} - F$ depends smoothly on t, by the transposed Leibniz formula we conclude that $v \in C^\infty\left(\left[-\tilde{T}, \tilde{T}\right]; \mathscr{D}'(\Omega'; \mathbb{C}^r)\right)$ (cf. [42, Remark 1.2]). By elliptic regularity for the Laplacian Δ^E, u (and therefore also v) is smooth in the interior of N, and so $v|_{x^n=T}$ is smooth. Now, if we were to show that the solution to (2.11) is smooth for $t \in [0, 2\tilde{T})$ then we are done. Indeed, we would have

$$Id \cdot D_{x^n} u + i\tilde{A}(x, D_{x'})u = v \in C^\infty\left(\left(-\tilde{T}, \tilde{T}\right], \times \Omega'; \mathbb{C}^r\right),$$

and in particular, the restriction to the boundary $v|_{x^n=0}$ is smooth. Now, if we set $R\sigma = v|_{\partial N}$, then

$$Id \cdot D_{x^n} u|_{\partial N} = -i\,\tilde{A}u\big|_{\partial N} + R\sigma = -i\,\tilde{A}\big|_{\partial N}\,\sigma + R\sigma = -i\,A|_{\partial N}\,\sigma + R\sigma,$$

and we will get the desired result since R is a smoothing operator. So in order to conclude the proof it is left to show that v is smooth for $t \in [0, 2\tilde{T})$. We will do this in the subsequent lemma. □

Lemma. *There is an operator B in the congruence class of $\tilde{A} - F$ which satisfies the conditions for a well-posed heat equation in Section 2.1.1. As a result, the solution v to the equation (2.11) is smooth for $t \in [0, 2\tilde{T})$.*

Proof. We will start by checking the conditions for the operator $\tilde{A} - F$. If some of them will not be satisfied then we will adjust the symbol of $\tilde{A} - F$ to obtain B. The first condition is satisfied due to the construction of $\tilde{A}$. Denote by $a_1 = -Id \cdot \sqrt{q_2}$ and $a_{\leq 0}$ the principal part and the reminder part, respectively, of the full symbol of $\tilde{A} - F$ (F has order zero). Let $\|\|$ be the (operator) norm on complex $r \times r$-matrices induced from the Hermitian norm on $\mathbb{C}^r$. Note that for any matrix M we have $\|M\| \geq |\lambda|$, where λ is any eigenvalue of M, which implies that the matrix

$$Id \cdot z - M$$

is non-degenerate when $|z| > \|M\|$. Indeed, its eigenvalues are equal to $z - \lambda$ and we have

$$|z - \lambda| \geq |z| - |\lambda| \geq |z| - \|M\| > 0.$$

Since $\tilde{A}$—F is an elliptic PDO of order 1 in Ω and Ω' is precompact we have the following uniform in $\left[-\tilde{T}, \tilde{T}\right] \times \Omega'$ bounds

$$c\,|\xi| \leq |a_1| \leq C_1 \left(1 + |\xi|^2\right)^{1/2}; \tag{2.12}$$

$$\left|a_{\leq 0}\right| \leq C_0, \tag{2.13}$$

where c, C_0, and C_1 are some positive constants. Using this we see that the matrix

$$Id \cdot z - \frac{a_1 + a_{\leq 0}}{\left(1 + |\xi|^2\right)^{1/2}} \tag{2.14}$$

is non-degenerate for $|z| > C_1 + C_0$. Indeed, from (2.12),(2.13) the norm of the quotient term is bounded from above by the constant $C_1 + C_0$. It is left to check if (2.14) is non-degenerate when $z = x + iy$ with $-\epsilon < x$, for some sufficiently small $\epsilon > 0$. We have

$$Id \cdot \left(x + iy + \frac{\sqrt{q_2}}{\left(1 + |\xi|^2\right)^{1/2}} \right) - \frac{a_{\leq 0}}{\left(1 + |\xi|^2\right)^{1/2}} = Id \cdot \rho - A_{\leq 0}, \tag{2.15}$$

where

$$\rho = x + iy + \frac{\sqrt{q_2}}{\left(1 + |\xi|^2\right)^{1/2}},$$

and

$$A_{\leq 0} = \frac{a_{\leq 0}}{\left(1 + |\xi|^2\right)^{1/2}}.$$

We know that the matrix (2.15) is non-degenerate when $|\rho| > \left\|A_{\leq 0}\right\|$. Since q_2 can be arbitrarily small for ξ close to 0 we cannot guarantee that ρ will not vanish for any small $\epsilon > 0$. Therefore, we have to adjust the symbol of $\tilde{A} - F$. From (2.12) we obtain

$$\frac{\sqrt{q_2}}{\left(1+|\xi|^2\right)^{1/2}} \geq \frac{c}{2},$$

when $|\xi| \geq 1$. Hence, when $\epsilon < \frac{c}{2}$ we have

$$|\rho|^2 = y^2 + \left(x + \frac{\sqrt{q_2}}{\left(1+|\xi|^2\right)^{1/2}}\right)^2 > \left(\frac{c}{2} - \epsilon\right)^2,$$

for $|\xi| \geq 1$. From (2.13) we see that

$$\left\|A_{\leq 0}\right\|^2 \leq \frac{C_0^2}{1+|\xi|^2},$$

which is less than $\left(\frac{c}{2}-\epsilon\right)^2$ when $|\xi| \geq C_0\left(\frac{c}{2}-\epsilon\right)^{-1}$. Let $R = \max\left(1, C_0\left(\frac{c}{2}-\epsilon\right)^{-1}\right)$, then the matrix (2.15) is non-degenerate for $|\xi| \geq R$. On the other hand, for $|\xi| < R$ we know that $\left\|A_{\leq 0}\right\| \leq C_0$. Now let us consider the congruent operator $B(t, x', D_{x'})$ by adjusting the full symbol as

$$a_1 + a_{\leq 0} - Id \cdot \psi(\xi),$$

where $\psi(\xi) = Ce^{-\frac{|\xi|^2}{R^2}}$ and the constant C is equal to $e\left(1+R^2\right)^{1/2}\left(C_0 + \frac{c}{2}\right)$. One sees that now for $|\xi| < R$ we have

$$|\rho|^2 = y^2 + \left(x + \frac{\sqrt{q_2} + \psi(\xi)}{\left(1+|\xi|^2\right)^{1/2}}\right)^2 > \left(C_0 + \frac{c}{2} - \epsilon\right)^2 > C_0^2 \geq \left\|A_{\leq 0}\right\|,$$

so the matrix (2.14) for the adjusted operator is non-degenerate in this case. In the other cases it is clear that it remains non-degenerate, which guarantees that the operator $B(t,x',D_{x'})$ satisfies the conditions for a well-posed heat equation. The second part of the lemma follows immediately. Indeed, since B is congruent to $\tilde{A}-F$ we have

$$Id\cdot\partial_t v - B(t)v \in C^{\infty}\left(\left[-\tilde{T},\tilde{T}\right]\times\Omega';\mathbb{C}^r\right)$$

and, therefore, by Theorem 2.1.3 the solution v is smooth for $t\in\left[0,2\tilde{T}\right)$. □

Corollary 2.2.5. *The full symbol of the DtN operator Λ_{g,∇^E} is the same as the full symbol of $A(x,D_{x'})|_{\partial N}$. In particular, the DtN operator is a classical elliptic pseudodifferential operator of order* 1.

We are now in a position to recover the geometric data (the metric g and the connection ∇^E) on the boundary ∂N from the given DtN operator Λ.

2.2.3 Proof of Theorem 2.1.1

Let $\left\{x^1,\ldots,x^n\right\}$ denote boundary normal coordinates associated with $\left\{x^1,\ldots,x^{n-1}\right\}$ and $(\varepsilon_1,\ldots,\varepsilon_r)$ denote boundary normal frame defined by $(\epsilon_1,\ldots,\epsilon_r)$. Note that since

$$\partial_{x^n} g_{kl} = -\sum_{\eta,\mu} g_{k\eta}\left(\partial_{x^n} g^{\eta\mu}\right) g_{\mu l},$$

it also suffices to determine the inverse matrix $\left(g^{kl}\right)$ and its normal derivatives instead of (g_{kl}) and its normal derivatives. Using Corollary 2.2.5 and (2.7) we get

$$\lambda_1 = -\sqrt{q_2},$$

and we have at any $p \in \partial N$,

$$q_2(p,\xi') = \sum_{k,l=1}^{n-1} g^{kl}(p)\,\xi_k\xi_l = \left(\frac{\text{Trace}\,\lambda_1}{r}\right)^2,$$

where r is the dimension of the vector bundle E. Thus, the principal symbol of the DtN operator determines g^{kl} at each boundary point p.

Next from Corollary 2.2.5 and (2.8) we have

$$\lambda_0 = \frac{1}{2\sqrt{q_2}}\left[\sum_{l}^{n-1} \partial_{\xi^l}\sqrt{q_2}D_{x^l}\sqrt{q_2} - q_1 - \partial_{x^n}\sqrt{q_2} + F\sqrt{q_2}\right] =$$
$$= -\frac{1}{2\sqrt{q_2}}\partial_{x^n}\sqrt{q_2} + \frac{i}{2\sqrt{q_2}}\sum_{l=1}^{n-1} V^l\xi_l - \frac{1}{4}\sum_{k,l=1}^{n-1} g^{kl}(x)\,\partial_{x^n}g_{kl}(x) + T_0, \quad (2.16)$$

where

$$T_0 = \frac{1}{2\sqrt{q_2}}\sum_{l}^{n-1} \partial_{\xi^l}\sqrt{q_2}D_{x^l}\sqrt{q_2} + \frac{i}{2\sqrt{q_2}}\sum_{k,l=1}^{n-1}\left(\frac{1}{2}g^{kl}(x)\,\partial_{x^k}\log\delta(x) + \partial_{x^k}g^{kl}(x)\right)\xi_l$$

is an expression involving only g_{kl}, g^{kl}, and their tangential derivatives along ∂N. Note that $\sum g^{kl}g_{kl} = n-1$, and so

$$-\sum_{k,l=1}^{n-1} g^{kl}(x)\,\partial_{x^n}g_{kl}(x) = \sum_{k,l=1}^{n-1} g_{kl}(x)\,\partial_{x^n}g^{kl}(x).$$

If we set $h^{kl} = \partial_{x^n}g^{kl}$, $h = \sum g_{kl}h^{kl}$, and $\|\xi'\|^2 = \sum g^{kl}\xi_k\xi_l = q_2$, we can rewrite (2.16) in the form

$$\lambda_0(\xi) = -\frac{1}{4\|\xi'\|^2}\sum_{k,l=1}^{n-1}\left(h^{kl} - hg^{kl}\right)\xi_k\xi_l + \frac{i}{2\|\xi'\|}\sum_{l=1}^{n-1} V^l\xi_l + T_0. \quad (2.17)$$

From antisymmetric part $\lambda_0(\xi) - \lambda_0(-\xi)$ we obtain

$$\frac{i}{\|\xi'\|}\sum_{l=1}^{n-1} V^l \xi_l,$$

which allows us to determine V^l and multiplying by $1/2 g_{kl}$ we get

$$\frac{1}{2} g_{kl} V^l = g_{kl} g^{lj} \omega_j = \delta_k^j \omega_j = \omega_k,$$

so we determined the connection matrix ω_k for $k = 1, \ldots, n-1$ on the boundary ∂N. Let us look at the remaining terms. Only the first term in (2.17) is unknown yet. But we can recover it from all the other terms. Thus, we can recover the quadratic form

$$\kappa^{kl} = h^{kl} - h g^{kl}.$$

When $n > 2$, this in turn determines $h^{kl} = \partial_{x^n} g^{kl}$, since

$$h^{kl} = \kappa^{kl} + \frac{1}{2-n}\left(\sum_{p,q} \kappa^{pq} g_{pq}\right) g^{kl}. \tag{2.18}$$

Now let us look at (2.9). We have

$$\begin{aligned}
\lambda_{-1} = \frac{1}{2\sqrt{q_2}} \Bigg[& -\sum_{l}^{n-1} \partial_{\xi^l} \sqrt{q_2} D_{x^l} \lambda_0 - \sum_{l}^{n-1} \partial_{\xi^l} \lambda_0 D_{x^l} \sqrt{q_2} + \\
& + \frac{1}{2} \sum_{k,l}^{n-1} \partial_{\xi^k} \partial_{\xi^l} \sqrt{q_2} D_{x^k} D_{x^l} \sqrt{q_2} - q_0 + \partial_{x^n} \lambda_0 - F \lambda_0 \Bigg] \\
& = -\frac{1}{2\sqrt{q_2}} q_0 + \frac{1}{2\sqrt{q_2}} \partial_{x^n} \lambda_0 + T_{-1}(g_{kl}, \omega_l) = \\
& -\frac{1}{8\|\xi'\|^3} \sum_{k,l=1}^{n-1} \partial_{x^n} \kappa^{kl} \xi_k \xi_l + \frac{i}{4\|\xi'\|^2} \sum_{l=1}^{n-1} \partial_{x^n} V^l \xi_l + T_{-1}(g_{kl}, \omega_l),
\end{aligned} \tag{2.19}$$

where $T_{-1}(g_{kl},\omega_l)$ is an expression involving only g_{kl}, g^{kl}, their first normal derivatives and the boundary values of ω_l. Here again looking at the antisymmetric part of λ_{-1} we determine $\partial_{x^n} V^l$ and subsequently $\partial_{x^n}\omega_k$ for $k=1,\dots,n-1$. The first term of (2.19) is again determined by the other terms, which allows us to recover

$$-\frac{1}{8\,\|\xi'\|^3}\sum_{k,l=1}^{n-1}\partial_{x^n}\kappa^{kl}\xi_k\xi_l,$$

and hence also $\partial_{x^n}\kappa^{kl}$. Due to (2.18) the latter determines $\partial_{x^n}h^{kl}=\partial^2_{x^n}g^{kl}$.

Proceeding by induction, let $m\le -1$, and suppose we have shown that, when $-1\ge j\ge m$,

$$\lambda_j=-\frac{1}{\|2\xi'\|^{2-j}}\sum_{k,l=1}^{n-1}\left(\partial_{x^n}^{|j|}\kappa^{kl}\right)\xi_k\xi_l+\frac{i}{\|2\xi'\|^{1-j}}\sum_{l=1}^{n-1}\partial_{x^n}^{|j|}V^l\xi_l+T_j(g_{kl},\omega_l),$$

where $T_j(g_{kl},\omega_l)$ involves only the boundary values of g_{kl}, g^{kl}, their normal derivatives of order at most $|j|$, and also for $j\le -1$ involves the boundary values of ω_l, and their normal derivatives of order at most $|j|-1$. From Corollary 2.2.5 and (2.10) we get

$$\lambda_{m-1}=\frac{1}{2\sqrt{q_2}}\left[\partial_{x^n}\lambda_m+\sum_{\substack{j,k,K\\ m\le j,k\le 1\\ |K|=j+k-m}}\frac{1}{K!}\partial_\xi^K\lambda_j D_x^K\lambda_k-F\lambda_m\right]=$$

$$=-\frac{1}{\|2\xi'\|^{3-m}}\sum_{k,l=1}^{n-1}\left(\partial_{x^n}^{|m-1|}\kappa^{kl}\right)\xi_k\xi_l+\frac{i}{\|2\xi'\|^{2-m}}\sum_{l=1}^{n-1}\partial_{x^n}^{|m-1|}V^l\xi_l+T_{m-1}(g_{kl},\omega_l).$$

Taking the antisymmetric part of λ_{m-1} we can determine $\partial_{x^n}^{|m-1|}V^l$ and, therefore, also $\partial_{x^n}^{|m-1|}\omega_k$. The first term is again determined by the other terms. So we can recover

$\partial_{x^n}^{|m-1|}\kappa^{kl}$ and thus $\partial_{x^n}^{|m-2|}g^{kl}$ also. This completes the induction step.

2.2.4 The case of Schrödinger type operators

One can see that the Schrödinger operator L_P differs from the connection Laplacian only in order zero. So we can think of the former locally as of the latter but with $\tilde{Q}$ adjusted by P. Therefore, it is straightforward to adapt the Propositions 2.2.2 and 2.2.4 to this case. Namely, the operator L_P will have the similar factorisation

$$-L_P \equiv (Id \cdot D_{x^n} + iF(x) - iA_P(x, D_{x'})) \circ (Id \cdot D_{x^n} + iA_P(x, D_{x'})),$$

and the following equality will hold

$$A_P(x, D_{x'})|_{\partial N}\,\sigma = \Lambda_{g,P,\nabla^E}\sigma.$$

So the full symbols of these two operators coincide.

Example 2.2.1. The operator

$$L_g = \Delta_g + \frac{n-2}{4(n-1)}S_g$$

is called the *conformal Laplacian*, where $\Delta_g = -\mathrm{div}_g \circ \mathrm{grad}_g$ is the Laplace-Beltrami operator and S_g is the scalar curvature of g. Clearly, it is equal to L_P with the (symmetric) potential

$$P = \frac{n-2}{4(n-1)}S_g. \tag{2.20}$$

The conformal Laplacian has the following conformal scaling property

$$L_{cg}u = c^{-\frac{n+2}{4}} L_g \left(c^{\frac{n-2}{4}} u \right)$$

for any conformal factor c, i.e. any smooth positive function on N. As a result, the Dirichlet-to-Neumann operator associated with L_g is invariant under the conformal transformations of the metric with conformal factor satisfying

$$\begin{cases} c|_{\partial N} = 1, \\ \partial_\nu c|_{\partial N} = 0. \end{cases} \tag{2.21}$$

Namely, for these transformations we have

$$\Lambda_{cg} = \Lambda_g,$$

where Λ_g is the DtN operator associated with L_g. Because of this conformal invariance we can only expect to recover the metric up to conformal scaling by conformal factors satisfying (2.21). Let $c = e^{2\rho}$ for some smooth function ρ, then the conditions (2.21) are equivalent to

$$\begin{cases} \rho|_{\partial N} = 0, \\ \partial_\nu \rho|_{\partial N} = 0. \end{cases} \tag{2.22}$$

Assuming these conditions are satisfied the formula for the transformation of the scalar curvature on the boundary is

$$S_{e^{2\rho}g} = S_g - 2(n-1)\Delta_g \rho = S_g - 2(n-1)\partial_\nu^2 \rho.$$

Since there is no restriction on $\partial_\nu^2 \rho$ on the boundary we see that the potential (2.20) cannot be uniquely determined from the DtN operator in this case. Moreover, it cannot be uniquely determined even in the real-analytic setting, since there exist real-analytic functions satisfying (2.22). For further details on this topic we refer the reader to [29].

This leads us to the extension of our main result to the case of the added potential.

Theorem 2.2.6. *Suppose* $dim N = n \geq 3$. *Let* $\left(x^1, \ldots, x^{n-1}\right)$ *be any local coordinates for an open set* $W \subset \partial N$ *and* $(\epsilon_1, \ldots, \epsilon_r)$ *be any local frame of* E *over* W, *and let* $\left\{\lambda_j,\ j \leq 1\right\}$ *denote the full symbol of the DtN operator* Λ_{g,P,∇^E} *in these coordinates and local frame. For any* $p \in W$, *the metric on a boundary* g_{kl}, *its normal derivative* g_{kl} *and the connection matrix* ω_k *are given by explicit formulae in terms of principal* (λ_1) *and subprincipal* (λ_2) *symbols of* Λ_{g,P,∇^E}. *In addition,*

1. *If the full Taylor series of* P *at* p *is known then the full Taylor series of* g *and* ∇^E *at* p *in boundary normal coordinates and boundary normal frame are given by an explicit formula in terms of the matrix functions* $\left\{\lambda_j\right\}$, *their tangential derivatives and the full Taylor series of* P *at* p.

2. *If all normal derivatives of order* ≥ 2 *of* g *at* p *are known then the full Taylor series of* g, P, *and* ∇^E *at* p *in boundary normal coordinates and boundary normal frame are given by an explicit formula in terms of the matrix functions* $\left\{\lambda_j\right\}$, *their tangential derivatives and the normal derivatives of order* ≥ 2 *of* g *at* p.

The proof of this result is similar to that of Theorem 2.1.1.

Proof of Theorem 2.2.6. The proof follows the same steps as the proof of Theorem 2.1.1. In particular, one can see that the first two symbols of Λ_{g,P,∇^E} and Λ_{g,∇^E} coincide. Therefore, we can recover the metric g^{kl}, its normal derivative $\partial_{x^n} g^{kl}$, and the connection matrix ω_k on the boundary. Now, the symbol of order -1 is slightly different and is equal to

$$\lambda_{-1} = -\frac{1}{8\|\xi'\|^3}\sum_{k,l=1}^{n-1}\partial_{x^n}\kappa^{kl}\xi_k\xi_l + \frac{i}{4\|\xi'\|^2}\sum_{l=1}^{n-1}\partial_{x^n}V^l\xi_l + \frac{1}{2\|\xi'\|}P + T_{-1}(g_{kl},\omega_l).$$

Now, depending on what is known (normal derivatives of g or the full Taylor series of P), in addition to what was recovered in the proof of Theorem 2.1.1 we can also recover the potential P or the second normal derivative of g on the boundary. Proceeding by induction, let $m \leq -1$, and suppose we have shown that, when $-1 \geq j \geq m$

$$\lambda_j = -\frac{1}{\|2\xi'\|^{2-j}}\sum_{k,l=1}^{n-1}\left(\partial_{x^n}^{|j|}\kappa^{kl}\right)\xi_k\xi_l + \\ + \frac{i}{\|2\xi'\|^{1-j}}\sum_{l=1}^{n-1}\partial_{x^n}^{|j|}V^l\xi_l + \frac{1}{\|2\xi'\|^{-j}}\partial_{x^n}^{|j+1|}P + T_j(g_{kl},\omega_l,P),$$

where $T_j(g_{kl},\omega_l,P)$ involves only the boundary values of g_{kl}, g^{kl}, their normal derivatives of order at most $|j|$, and also for $j \leq -1$ involves the boundary values of ω_l and P, and their normal derivatives of order at most $|j|-1$ and $|j+1|-1$, respectively. Thus, we get

$$\lambda_{m-1} = \frac{1}{2\sqrt{q_2}}\left[\partial_{x^n}\lambda_m + \sum_{\substack{j,k,K \\ m\leq j,k\leq 1 \\ |K|=j+k-m}}\frac{1}{K!}\partial_\xi^K\lambda_j D_x^K\lambda_k - F\lambda_m\right] = \\ = -\frac{1}{\|2\xi'\|^{3-m}}\sum_{k,l=1}^{n-1}\left(\partial_{x^n}^{|m-1|}\kappa^{kl}\right)\xi_k\xi_l + \frac{i}{\|2\xi'\|^{2-m}}\sum_{l=1}^{n-1}\partial_{x^n}^{|m-1|}V^l\xi_l + \\ + \frac{1}{\|2\xi'\|^{1-m}}\partial_{x^n}^{|m|}P + T_{m-1}(g_{kl},\omega_l,P),$$

which allows us to determine $\partial_{x^n}^{|m-1|} V^l$, $\partial_{x^n}^{|m-1|} \kappa^{kl}$ (and, therefore, also $\partial_{x^n}^{|m-2|} g^{kl}$) or $\partial_{x^n}^{|m|} P$. This completes the induction step. □

Remark 2.2.7. In general, if we do not want to assume that the metric or potential are known, then from the symbol λ_j we can recover the expression

$$\sum_{k,l=1}^{n-1} \left(\partial_{x^n}^{|j|} \kappa^{kl}\right) \xi_k \xi_l + 4 \left\|\xi'\right\|^2 \partial_{x^n}^{|j+1|} P = \sum_{k,l=1}^{n-1} \left(\partial_{x^n}^{|j|} \kappa^{kl} + 4 g^{kl} \partial_{x^n}^{|j+1|} P\right) \xi_k \xi_l,$$

and, consequently, the expression

$$\partial_{x^n}^{|j|} \kappa^{kl} \cdot Id + 4 g^{kl} \partial_{x^n}^{|j+1|} P.$$

Note that since the first term is scalar we can recover the off-diagonal components of P. Taking the traces both with respect to the metric and inner product on fibers we can see that modulo all the lower order derivatives we can recover the expression

$$(2-n)\, \partial_{x^n}^{|j|} h + 4(n-1) Tr\, \partial_{x^n}^{|j+1|} P = (2-n)\, \partial_{x^n}^{|j|} h + 4(n-1) \partial_{x^n}^{|j+1|} Tr\, P. \tag{2.23}$$

It is clear that we cannot say anything about the first and the second term of the sum separately. This fact agrees with what we saw in Example 2.2.1 with the potential given by scalar curvature. So, for each j in order to recover one of the terms in the sum (2.23) we should know the other one. For example, if the potential is given in the form

$$P = g^{kl} F_{lk} \tag{2.24}$$

with some known functions F_{lk}, then the second term of (2.23) will be known from the previously obtained derivatives of g^{kl} up to order $|j+1|$. This will allow to determine $\partial_{x^n}^{|j|}h$ and, therefore, the derivatives of g up to order $|j|+2$. So, for this type of potentials the full Taylor series of g (and, therefore, of P) on the boundary can be recovered from the full symbol of the DtN operator.

2.2.5 Gauge equivalence of the reconstructed connection

Let $\pi_E : E \to X$ and $\pi_F : F \to Y$ be two smooth vector bundles over smooth manifolds. We say that a morphism of vector bundles $\phi : E \to F$ covers a smooth map $\psi : X \to Y$ if the relation $\pi_F \circ \phi = \psi \circ \pi_E$ holds, that is the following diagram is commutative

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \\ \Big\downarrow{\scriptstyle \pi_E} & & \Big\downarrow{\scriptstyle \pi_F} \\ X & \xrightarrow{\psi} & Y \end{array}$$

Note that any morphism ϕ covers a unique underlying smooth map of the bases. We say that an isomorphism ϕ *intertwines* with linear operators A_E and B_F acting on vector bundles E and F, respectively, if

$$A_E\left(\phi^{-1}\circ s\circ\psi\right) = \phi^{-1}\circ B_F(s)\circ\psi,$$

for any section $s \in \Gamma(F)$. One of the corollaries of the local reconstruction result is the following proposition on gauge equivalence.

Proposition 2.2.8. *Let $\left(N_i, g_i, E_i, \nabla^i\right)$, where $i = 1, 2$, be two Euclidean smooth vector bundles defined over connected compact Riemannian manifolds with boundary. Suppose that for some open subsets $\Sigma_i \subset \partial N_i$ there exists a vector bundle isomorphism $\phi : E_1|_{\Sigma_1} \to E_2|_{\Sigma_2}$ that intertwines with the corresponding Dirichlet-to-Neumann operators Λ_{Σ_1} and Λ_{Σ_2}. Then the isomorphism ϕ is a gauge equivalence, $\phi^*\nabla^2 = \nabla^1$, and covers an isometry*

$\psi : (\Sigma_1, g_1) \to (\Sigma_2, g_2)$.

Proof. Clearly, the isomorphism ϕ intertwining the DtN operators Λ_{Σ_1} and Λ_{Σ_2} is equivalent to having the equality of operators

$$\Lambda_{\Sigma_1}(s) = \phi^{-1} \circ \Lambda_{\Sigma_2}\left(\phi \circ s \circ \psi^{-1}\right) \circ \psi.$$

The operator on the right hand side is a natural pull-back of the operator Λ_{Σ_2} along ϕ. Therefore, the metric and connection on (E_1, Σ_1) reconstructed from its full symbol are equal to $\psi^* g_2$ and $\phi^* \nabla^2$, respectively. On the other hand the full symbols of the above two operators coincide. Hence, we have $g_1 = \psi^* g_2$ and $\nabla^1 = \phi^* \nabla^2$, which completes the proof. □

Remark 2.2.9. In order to reconstruct the metric and connection we only need to know the principal and subprincipal symbols of the DtN operator. Therefore, since these symbols coincide for the connection Laplacian and the Schrödinger operator, Proposition 2.2.8 holds for the latter one as well. Though, due to Remark 2.2.7 we can conclude that there should be imposed some additional requirements on a potential (e.g. (2.24)) in order to recover it on the boundary from the full symbol of the DtN operator.

2.2.6 The case of surfaces

In two dimensions, we can only aim to reconstruct a conformal class of metrics from the DtN operator. This is because the connection Laplacian is conformally contravariant in two dimensions. Indeed, from the definition of the connection Laplacian we have

$$\Delta^E_{e^\mu g} = e^{-\mu}\Delta^E_g,$$

where the subscript indicates the metric used to define the connection Laplacian. Clearly, $\Delta^E_g u = 0$ if and only if $\Delta^E_{e^\mu g} u = 0$. The unit normal vector at the boundary for the conformal metric is equal to

$$e^{-\mu/2}\big|_{\partial N} \cdot \frac{\partial}{\partial \nu}.$$

Therefore, we have the following identity for the DtN operators

$$\Lambda_{e^\mu g, \nabla^E} = e^{-\mu/2}\big|_{\partial N} \cdot \Lambda_{g, \nabla^E}.$$

This identity shows that the DtN operators constructed using conformal metrics coincide if a conformal factor e^μ equals to 1 at the boundary. This fact poses obstacles to the recovery of the normal derivatives of the geometric data at the boundary of surfaces. However, we can still recover the metric and the connection on the boundary from the symbol of the DtN operator. This follows immediately from the proof of the Theorem 2.1.1.

2.3 Global reconstruction of the geometric data

In this section we prove a uniqueness result for the Calderón inverse problem for the connection Laplacian on a vector bundle. Our main hypothesis is that the geometry of a vector bundle, that is a connection, a compatible inner product, and a Riemannian metric on the base manifold, are real-analytic. This Calderón problem is motivated by the Aharonov-Bohm effect that says that different gauge equivalence classes of electromagnetic potentials have different physical effects that can be detected by experiments. Thus, our uniqueness result shows that different gauge equivalence classes of connections have different boundary data, that is such classes are detectable by boundary measurements.

We discuss some of the related literature on this problem in due course, but now say a few words about the classical Calderón problem. Recall that the classical result by Lassas and Uhlmann [27], see also [28, 30], says that the topology and geometry of a real-analytic Riemannian manifold with boundary can be recovered from the Dirichlet-to-Neumann map for the Laplace-Beltrami operator. The main result of this chapter can be viewed as a version of the Lassas-Uhlmann theorem in the setting of vector bundles, which allows us to recover additional topological and geometric structures. We proceed with the statement of related hypotheses and conclusions in more detail.

2.3.1 Main result

Let $(N_i g_i)$, where $i = 1, 2$, be two connected compact Riemannian manifolds with boundary, and let E_i be vector bundles over N_i. We assume that each E_i is equipped with a connection ∇^i and a Euclidean structure, that is a compatible inner product $\langle\cdot,\cdot\rangle_{E_i}$. For open subsets $\Sigma_i \subset \partial N_i$ we denote by Λ_{Σ_i} the corresponding Dirichlet-to-Neumann operators defined on a compactly supported sections of $E_i|_{\Sigma_i}$. Recall that a vector bundle isomorphism is called *Euclidean*, if it preserves Euclidean structures. Our main result in this section is the following theorem.

Theorem 2.3.1. *Let* $\left(N_i, g_i, E_i, \nabla^i\right)$*, where* $i = 1, 2$*, be two Euclidean real-analytic vector bundles defined over connected compact real-analytic Riemannian manifolds with* *boundary, equipped with real-analytic connections. Suppose that* $\dim N_i \geq 3$ *for each* $i = 1, 2$*, and for some open subsets* $\Sigma_i \subset \partial N_i$ *there exists a real-analytic Euclidean vector bundle isomorphism* $\phi : E_1|_{\Sigma_1} \to E_2|_{\Sigma_2}$ *that intertwines with the corresponding Dirichlet-to-Neumann operators* Λ_{Σ_1} *and* Λ_{Σ_2}*. Then the bundles* E_1 *and* E_2 *are isomorphic, and moreover, there exists a real-analytic Euclidean vector bundle isomorphism* $\Phi : E_1 \to E_2$ *that covers an isometry* $\Psi : (N_1, g_1) \to (N_2, g_2)$*, such that* $\Phi^* \nabla^2 = \nabla^1$ *and* $\Phi|_{\Sigma_1} = \phi$*.*

We will see further (Remark 2.3.12) that this result continues to hold in the case of a Schrödinger operator with a real-analytic potential of the form (2.24).

We note that the presence of Euclidean structures on vector bundles E_i in Theorem 2.3.1 plays an auxiliary, but important role. On the one hand, neither the connection Laplacian nor the associated DtN operator depend on them. On the other, we do not know whether the assumption that the isomorphism ϕ in Theorem 2.3.1 is Euclidean, and the conclusion that so is its extension Φ, can be dropped.

It is an open problem whether the conclusions in Theorem 2.3.1 hold for arbitrary smooth geometric data, connections and Riemannian metrics, of vector bundles. Under different conditions a similar problem has been considered by Cekić [6, 7]. Let us also mention that in [26] the authors consider the Calderón problem for the wave operator of the connection Laplacian on Hermitian vector bundles, and obtain conclusions similar to the ones in Theorem 2.3.1 without any hypotheses on the geometry of vector bundles.

Note that the hypothesis on the dimension of the base manifolds N_i in our results is essential for the conclusions to hold. In dimension two the connection Laplacian behaves differently when a Riemannian metric on the base changes conformally, see 2.2.6, and the Riemannian metric on the base cannot be recovered. Results on related problems in dimension two can be found in [1, 19].

The proof of Theorem 2.3.1 builds on an elegant idea in [28]. Using Green kernels for the connection Laplacian, we construct immersions of our vector bundles into some function space, and recover the geometry and topology from their images. We believe that some technical details of our argument in the setting of vector bundles might be of independent interest, and the improvements give a more streamlined proof of the original results in [28]. We believe that the result in Theorem 2.3.1 can be extended to vector bundles over non-compact complete manifolds with compact boundaries. We will also explain why the conclusions of Theorem 2.3.1 continue to hold for the Dirichlet-to-Neumann operators associated with Schrödinger operators of special type, that is connection Laplacians with symmetric real-analytic potentials of the form (2.24). The result for these operators will play an important role in the proof of the main theorem in Chapter 3.

2.3.2 Preliminaries

Let us briefly discuss the behavior of the connection Laplacian when a Riemannian metric or a connection change. First, if $\tilde{g} = \exp(2\varphi)\, g$ is another Riemannian metric, then a direct computation shows that

$$\Delta^E_{\tilde{g}} s = \exp(-2\varphi)\left(\Delta^E_g s - (n-2)\,\nabla^E_X s\right),$$

where $X = \operatorname{grad}(\varphi)$ is the gradient vector field with respect to a metric g, and n is the dimension of N. In particular, when $n = 2$, the operator Δ^E_g is conformally contravariant, and a section $s \in \Gamma(E)$ is harmonic or not with respect to g and $\tilde{g}$ simultaneously. Second, consider another connection $\tilde{\nabla}^E$ on E, and denote by $\tilde{\Delta}^E_g$ the corresponding connection Laplacian. Recall that a vector bundle isomorphism

$\Phi : E \to E$ is called a *gauge equivalence* if $\Phi^* \tilde{\nabla}^E = \nabla^E$, that is

$$\nabla^E_X s = \Phi^{-1} \circ \tilde{\nabla}^E_X (\Phi \circ s)$$

for any section $s \in \Gamma(E)$. In a local frame for E, this relation is equivalent to

$$\omega = \gamma^{-1} \tilde{\omega} \gamma + \gamma^{-1} d\gamma,$$

where ω and $\tilde{\omega}$ are the connection forms of ∇^E and $\tilde{\nabla}^E$ respectively, and γ is the matrix of Φ. A straightforward calculation shows that the connection Laplacians of gauge equivalent connections are related by the formula

$$\Delta^E_g = \Phi^{-1} \circ \tilde{\Delta}^E_g (\Phi \circ s) \tag{2.25}$$

for any section $s \in \Gamma(E)$. These properties determine the behavior of other quantities closely related to the Laplacian, such as its Green kernel and Dirichlet-to-Neumann operator.

Now let $\left(N_i, E_i, \nabla^i\right)$, where $i = 1, 2$, be two vector bundles over compact Riemannian manifolds with boundary $(N_i g_i)$, equipped with connections ∇^i. Suppose that these data are *gauge equivalent* in the following sense: there exists a vector bundle isomorphism $\Phi : E_1 \to E_2$ that covers an isometry $\Psi : (N_1, g_1) \to (N_2, g_2)$ such that $\Phi^* \nabla^2 = \nabla^1$. Then, using relation (2.25) it is straightforward to conclude that the corresponding Dirichlet-to-Neumann operators Λ_1 and Λ_2 intertwine, that is

$$\Lambda_1 \left(\phi^{-1} \circ s \circ \psi\right) = \phi^{-1} \circ \Lambda_2 (s) \circ \psi$$

for any smooth section s of $E_2|_{\partial N_2}$, where $\psi = \Psi|_{\partial N_1}$ and $\phi = \Phi|_{\partial M_1}$. Recall that the converse of this is given by Proposition 2.2.8 and can be viewed as the boundary version of Theorem 2.3.1. Note that there is no restriction on dimension of N in Proposition 2.2.8. Similar results continue to hold for the Dirichlet-to-Neumann operator associated with the connection Laplacian with a symmetric real-analytic potential.

2.3.3 Immersions by Green kernels

Let E be a Euclidean real-analytic vector bundle over a connected compact real-analytic manifold N with boundary, equipped with a real-analytic connection ∇^E. In this section we assume that $n = \dim N \geqslant 3$, and describe how one can reconstruct E from the Dirichlet-to-Neumann operator Λ_Σ, where $\Sigma \subset \partial N$ is an open subset. Our argument develops the ideas from [28] to the setting of vector bundles, and we attempt to make the related technical details to be rather explicit.

Fix a point $p \in \Sigma$. First, note that we may consider N as a subset of a larger real-analytic manifold $\tilde{N}$. More precisely, choosing boundary normal coordinates $(x^1, \ldots, x^n)$ around p, we may identify a neighborhood of p in N with the Euclidean half-ball

$$B^+(0,\rho) = \left\{(x^1, \ldots, x^n) \in B^n(0,\rho) : x^n \geqslant 0\right\},$$

where $B^n(0,\rho)$ is an open Euclidean ball of radius $\rho > 0$ in $\mathbb{R}^n$. Then, as the manifold $\tilde{N}$ one can take the manifold obtained by gluing $B^n(0,\rho)$ to N such that points in $B^+(0,\rho)$ are identified with points in N by means of boundary normal coordinates. Below by U we denote the open set $\tilde{N} \backslash \bar{N}$. For the sequel it is important to note that the set U does not really depend on N. In other words, if there are two manifolds N_i of the same dimension and two points $p_i \in \Sigma_i \subset \partial N_i$, where $i = 1, 2$, then choosing a sufficiently small $\rho > 0$ we may assume that the sets $\tilde{N}_1 \backslash \bar{N}_1$ and $\tilde{N}_2 \backslash \bar{N}_2$ coincide.

It is straightforward to see that a real-analytic metric g on N extends to a real-analytic metric $\tilde{g}$ on $\tilde{N}$, if ρ is sufficiently small. Similarly, the above construct shows that a real-analytic vector bundle E over N extends to a real-analytic vector bundle $\tilde{E}$ over $\tilde{N}$ such that $\tilde{E}\big|_U$ is trivial. Making $\rho > 0$ smaller if necessary, we may also assume that a real-analytic Euclidean structure and a real-analytic connection ∇ on E extend to a Euclidean structure and a connection $\tilde{\nabla}$ on $\tilde{E}$. If the former were compatible on E, then by real-analyticity so are the latter on $\tilde{E}$. Note that in the case of Schrödinger operators with real-analytic potentials we may also assume that the potential P extends to a real-analytic potential $\tilde{P}$ on $\tilde{N}$. Below by $\tilde{G}$ we denote the Dirichlet Green kernel (for the connection Laplacian or Schrödinger operator) on $\tilde{E}$.

Denote by $\mathscr{E}$ the trivial vector bundle $\tilde{E}\big|_U$. For a given integer $\ell < 2 - n/2$, where n is the dimension of N, we define the map $\mathscr{G} : \tilde{E} \to \mathscr{W}^{\ell}(\mathscr{E})$ by setting

$$\tilde{E}_x \ni v_x \longmapsto \left\langle v_x, \tilde{G}(x, \cdot)\right\rangle_{x,\tilde{E}} \in \mathscr{W}^{\ell}(\mathscr{E}), \tag{2.26}$$

where $x \in \tilde{N}$. The condition on ℓ guarantees that the space $\mathscr{W}_0^{2-\ell}(\mathscr{E})$ embeds into the Hölder space $C^{0,\alpha}(\mathscr{E})$ for some $\alpha > 0$, and hence, the dual space $\mathscr{W}^{\ell-2}$ contains the delta function. Then, by elliptic regularity we conclude that $\tilde{G}(x, \cdot)$ lies in $\mathscr{W}^{\ell}(\mathscr{E})$. In addition, it is straightforward to show that

$$\left|\tilde{G}(x_1, \cdot) - \tilde{G}(x_2, \cdot)\right|_{\mathscr{W}^{\ell}} \leqslant C_1 \mathrm{dist}(x_1, x_2)^{\alpha}$$

for some constant $C_1 > 0$, where for simplicity we may assume that the points x_1 and $x_2 \in \tilde{N}$ lie in the same chart. Thus, we conclude that the map $\mathscr{G}$ is continuous.

Note that the map (2.26) can be also defined for Schrödinger operators and the above results are true in this case as well. Similarly, we have the following statement.

Lemma 2.3.2. *Let ℓ be an integer such that $\ell < 1-n/2$. Then the map $\mathscr{G} : \tilde{E} \to \mathscr{W}^{\ell}(\mathscr{E})$ defined by (2.26) is C^1-smooth.*

Proof. Since the map $\mathscr{G} : \tilde{E} \to \mathscr{W}^{\ell}(\mathscr{E})$ is linear on each fibre, for a proof of the lemma it is sufficient to show that the map that sends a point $x \in \tilde{N}$ to the function $\tilde{G}(x,\cdot)$, viewed as an element in the Sobolev space $\mathscr{W}^{\ell}$, is smooth. Below we assume that x ranges in a chart on $\tilde{N}$ where the vector bundle $\tilde{E}$ is trivial. First, we claim that for any section $\varphi \in \mathscr{W}_0^{-\ell}(\mathscr{E})$ the section

$$\tilde{\psi}(x) = \int_{\tilde{N}} \left\langle \tilde{G}(x,y), \tilde{\varphi}(y) \right\rangle_y dVol(y),$$

is differentiable, where $\tilde{\varphi}$ is an extension of φ by zero, and for any $h \in \mathbb{R}^n$ the linear functional

$$\varphi \longmapsto \sum_i h_i \frac{\partial}{\partial x^i} \tilde{\psi}(x) = \sum_i h_i \frac{\partial}{\partial x^i} \int_{\tilde{N}} \left\langle \tilde{G}(x,y), \tilde{\varphi}(y) \right\rangle_y dVol(y) \qquad (2.27)$$

defines an element in $\mathscr{W}^{\ell}(\mathscr{E})$. Indeed, by standard theory the section $\tilde{\psi}$ can be viewed as the solution to the Dirichlet problem

$$\Delta^{\tilde{E}} \tilde{\psi} = \tilde{\varphi}, \qquad \tilde{\psi}\big|_{\partial \tilde{N}} = 0,$$

in the Sobolev space $\mathscr{W}_0^{-\ell+2}$, and since $\ell < 1-n/2$, it lies in the Hölder space $C^{1,\alpha}$ for some $\alpha > 0$. To show that the functional defined by (2.27) lies in $\mathscr{W}^{\ell}(\mathscr{E})$, it is sufficient to show that

$$\left|\sum_i h_i \frac{\partial}{\partial x^i}\tilde{\psi}(x)\right| \leqslant C\,|h|\,|\varphi|_{-\ell}$$

for some constant $C > 0$, where $|\cdot|_{-\ell}$ stands for the Sobolev norm. The latter is a direct consequence of the inequality $\left|\tilde{\psi}\right|_{-\ell+2} \leqslant C'\,|\tilde{\varphi}|_{-\ell}$, which follows from standard theory, together with the Sobolev embedding theorem. Thus, we obtain the linear operator

$$L_x : \mathbb{R}^n \ni h \longmapsto \sum_i h_i \frac{\partial}{\partial x^i}\tilde{G}(x,\cdot) \in \mathscr{W}^{\ell}(\mathscr{E}),$$

and claim that it is the differential of the map $x \mapsto \tilde{G}(x,\cdot)$. In other words, we claim that for any $\varepsilon > 0$ the inequality

$$\left|\tilde{G}(x+h,\cdot) - \tilde{G}(x,\cdot) - L_x(h)\right|_{\ell} \leqslant \varepsilon\,|h|$$

holds, for any $h \in \mathbb{R}^n$ such that $|h| < \delta$ for an appropriate $\delta > 0$. In the notation above, for the latter it is sufficient to show that

$$\left|\tilde{\psi}(x+h) - \tilde{\psi}(x) - \sum_i h_i \frac{\partial}{\partial x^i}\tilde{\psi}(x)\right| \leqslant C\,|h|^{1+\alpha}\,|\varphi|_{-\ell} \tag{2.28}$$

for some positive constants C and α, and arbitrary $h \in \mathbb{R}^n$. Recall the so-called Hadamard formula:

$$\tilde{\psi}(x+h) - \tilde{\psi}(x) = \sum_i \gamma_i(x) h_i, \quad \text{where} \quad \gamma_i(x) = \int_0^1 \frac{\partial \tilde{\psi}}{\partial x^i}(x+th)\,dt,$$

where we use a trivialisation of $\tilde{E}$ to view sections around x as vector-functions. Using this relation, we obtain

$$\left|\tilde{\psi}(x+h)-\tilde{\psi}(x)-\sum_i h_i\frac{\partial}{\partial x^i}\tilde{\psi}(x)\right| = \left|\sum_i\left(\gamma_i(x)-\frac{\partial}{\partial x^i}\tilde{\psi}(x)\right)h_i\right|$$
$$\leqslant |h|\left(\int_0^1 \left|D\tilde{\psi}(x+th)-D\tilde{\psi}(x)\right|^2 dt\right)^{1/2} \leqslant \left|\tilde{\psi}\right|_{C^{1,\alpha}}|h|^{1+\alpha} \leqslant C''\left|\tilde{\psi}\right|_{-\ell+2}|h|^{1+\alpha}$$
$$\leqslant C'C''|\tilde{\varphi}|_{-\ell}|h|^{1+\alpha} \leqslant C|\varphi|_{-\ell}|h|^{1+\alpha},$$

where in the second inequality we estimate the integral via the Hölder norm and $|h|^\alpha$, and in the third we use the Sobolev embedding theorem. Thus, relation (2.28) is demonstrated, and we conclude that the map $x \mapsto \tilde{G}(x,\cdot)$ is differentiable. Finally, for a proof that it is smooth, it remains to show that the map $x \mapsto L_x$ is continuous. The latter is a consequence of the inequality

$$\left|\sum_i\left(\frac{\partial}{\partial x^i}\tilde{\psi}(x_1)-\frac{\partial}{\partial x^i}\tilde{\psi}(x_2)\right)h_i\right| \leqslant C|h||x_1-x_2|^\alpha|\varphi|_{-\ell}$$

for some positive constants C and α, which can be proved in a fashion similar to the one above. Thus, we are done. □

One can see that the main ingredient of the proof is that Δ^E is an elliptic operator of order two, and as a result $\tilde{G}(x,\cdot)$ belongs to the Sobolev space $\mathscr{W}^\ell$. As we mentioned before, Schrödinger operators have the same properties. Therefore, Lemma 2.3.2 is true in the Schrödinger setting also. It allows us to study the map $\mathscr{G}$ from a viewpoint of differential geometry. As we shall see below, the map $\mathscr{G}$ is a linear embedding on each fibre $\tilde{E}_x$ for $x \notin \partial\tilde{N}$, and collapses the fibre to the origin for $x \in \partial\tilde{N}$. Further, it maps the base manifold $\tilde{N}$, viewed as the image of the zero section, to zero in $\mathscr{W}^\ell(\mathscr{E})$. To avoid these degeneracies we often restrict it to the open set $\tilde{E}^0$ obtained by considering $\tilde{E}$ on the interior of $\tilde{N}$ and removing the zero section. The following statement shows that the map $\mathcal{G}$ is well-behaved on $\tilde{E}^0$.

Lemma 2.3.3. *Let ℓ be an integer such that $\ell < 1-n/2$. Then the map $\mathcal{G} : \tilde{E} \to \mathcal{W}^{\ell}(\mathcal{E})$ defined by* (2.26) *is a linear embedding on each fibre $\tilde{E}_x$, where $x \notin \partial\tilde{N}$. Moreover, it is an injective immersion on the set $\tilde{E}^0$, obtained by removing the image of the zero section from $\tilde{E}$ over the interior of $\tilde{N}$.*

Proof. First, we show that the map $\mathcal{G}$ is a linear embedding on each fibre $\tilde{E}_x$. For otherwise, there exists a point x in the interior of $\tilde{N}$ and a non-zero vector $v_x \in \tilde{E}_x$ such that the product $\left\langle v_x, \tilde{G}(x,\cdot)\right\rangle_x$ equals zero in $\mathcal{W}^{\ell}(\mathcal{E})$. The latter in particular implies that

$$\left\langle v_x, \tilde{G}(x,y)\right\rangle_x = 0 \qquad \text{for all} \quad y \in U \setminus \{x\}. \tag{2.29}$$

Since the left-hand side above is real-analytic, we conclude that relation 2.29 continues to hold on $\tilde{N}\setminus\{x\}$. Now let $s \in \mathcal{D}\left(\tilde{E}\right)$ be a compactly supported section such that $s(x) = v_x$. Then, we obtain

$$0 = \int_{\tilde{N}} \left\langle \left\langle v_x, \tilde{G}(x,y)\right\rangle_x, \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y) =$$

$$= \left\langle v_x, \int_{\tilde{N}} \left\langle \tilde{G}(x,y), \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y) \right\rangle_x = \langle v_x, v_x\rangle,$$

where we changed the order of operations in independent variables x and y in the second relation, and used the definition of the Dirichlet Green kernel in the third. Thus, we conclude that the vector v_x has to vanish, and the kernel of a linear operator given by (2.26) is trivial, that is the map $\mathcal{G}$ is indeed a linear embedding on each fibre.

A similar argument shows that the map $\mathcal{G}$ is injective everywhere on $\tilde{E}^0$. Indeed, suppose that there exist two points x_1 and x_2 in the interior of $\tilde{N}$ and non-zero vectors v_{x_1} and v_{x_2} in the fibres over them such that $\left\langle v_{x_1}, \tilde{G}(x_1, \cdot)\right\rangle$ and $\left\langle v_{x_2}, \tilde{G}(x_2, \cdot)\right\rangle$ coincide in $\mathscr{W}^\ell(\mathscr{E})$. Then, it is straightforward to see that

$$\left\langle v_{x_1}, \tilde{G}(x_1, y)\right\rangle_{x_1} = \left\langle v_{x_2}, \tilde{G}(x_2, y)\right\rangle_{x_2} \qquad \text{for all} \quad y \in U \setminus \{x_1, x_2\}. \tag{2.30}$$

As above, by unique continuation we may assume that relation 2.30 holds for all $y \in \tilde{N} \setminus \{x_1, x_2\}$. In addition, since the map $\mathscr{G}$ is injective on fibres, we may assume that $x_1 \neq x_2$. Let $s \in \mathscr{D}\left(\tilde{E}\right)$ be a compactly supported section such that $s(x_1) = v_{x_1}$ and $s(x_2) = 0$. Then, we obtain

$$\begin{aligned}\left\langle v_{x_1}, v_{x_1}\right\rangle = \left\langle v_{x_1}, \int_{\tilde{N}} \left\langle \tilde{G}(x_1, y), \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y)\right\rangle_{x_1} &= \\ = \int_{\tilde{N}} \left\langle \left\langle v_{x_1}, \tilde{G}(x_1, y)\right\rangle_{x_1}, \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y)&,\end{aligned}$$

and due to 2.30 this is equal to

$$\begin{aligned}&\int_{\tilde{N}} \left\langle \left\langle v_{x_2}, \tilde{G}(x_2, y)\right\rangle_{x_2}, \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y) = \\ &= \left\langle v_{x_2}, \int_{\tilde{N}} \left\langle \tilde{G}(x_2, y), \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y)\right\rangle_{x_2} = \left\langle v_{x_2}, s(x_2)\right\rangle = \left\langle v_{x_2}, 0\right\rangle = 0.\end{aligned}$$

Thus, the vector v_{x_1} vanishes, and we arrive at a contradiction.

Finally, to show that the map $\mathscr{G}$ is an immersion we analyse its differential $D_{v_x}: T_{v_x}\tilde{E} \to \mathscr{W}^\ell(\mathscr{E})$. First, note that a connection on the vector bundle $\tilde{E}$ defines the decomposition of the tangent space $T_{v_x}\tilde{E}$ as the direct sum $H_{v_x} \oplus \tilde{E}_x$, where H_{v_x} is the so-called horizontal subspace, see [18]. Since the differential of the projection $\tilde{\pi} : \tilde{E} \to \tilde{N}$ establishes an isomorphism $D_{v_x}\tilde{\pi} : H_{v_x} \to T_x\tilde{N}$, we may view tangent vectors from $T_{v_x}\tilde{E}$ as pairs (X, ξ), where $X \in T_x\tilde{N}$, and $\xi \in \tilde{E}_x$. With these identifications, it is straightforward to show that

$$D_{v_x}\mathscr{G}(X,\xi) = \left\langle v_x, \nabla_X \tilde{G}(x,\cdot)\right\rangle_{x,\tilde{E}} + \left\langle \xi, \tilde{G}(x,\cdot)\right\rangle_{x,\tilde{E}}, \tag{2.31}$$

where by the covariant derivative $\nabla_X \tilde{G}(x,\cdot)$ we mean the derivative with respect to the variable x on $\tilde{E} \boxtimes \tilde{E}$, that is given by $\nabla_X \left(u_x \otimes u_y\right) = \nabla^{\tilde{E}}_X u_x \otimes u_y$. Now choosing appropriate test-sections in the fashion similar to the one above, it is straightforward to show that the differential $D_{v_x}\mathscr{G}$ is injective. In more detail, assume that the right-hand side of relation 2.31 equals zero for some $X \in T_x\tilde{N}$ and $\xi \in \tilde{E}_x$. Then, by unique continuation we may assume that

$$\left\langle v_x, \nabla_X \tilde{G}(x,y)\right\rangle_{x,\tilde{E}} + \left\langle \xi, \tilde{G}(x,y)\right\rangle_{x,\tilde{E}} = 0 \qquad \text{for all} \quad y \in \tilde{N} \setminus \{x\}. \tag{2.32}$$

Now choosing a compactly supported section $s \in \mathscr{D}\left(\tilde{E}\right)$ such that $s(x) = \xi$ and $\nabla^{\tilde{E}}_X s\big|_x = 0$, we obtain

$$0 = \int_{\tilde{N}} \left\langle \left\langle v_x, \nabla_X \tilde{G}(x,y)\right\rangle_x, \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y) + \\ + \int_{\tilde{N}} \left\langle \left\langle \xi, \tilde{G}(x,y)\right\rangle_x, \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y),$$

and changing the order of scalar product and integration we get

$$\left\langle v_x, \nabla_X \int_{\tilde{N}} \left\langle \tilde{G}(x,y), \Delta^{\tilde{E}} s(y) \right\rangle_y dVol(y) \right\rangle_x + \\ + \left\langle \xi, \int_{\tilde{N}} \left\langle \tilde{G}(x,y), \Delta^{\tilde{E}} s(y) \right\rangle_y dVol(y) \right\rangle_x = \left\langle v_x, \nabla_X^{\tilde{E}} s \right\rangle + \langle \xi, \xi \rangle = 0 + \langle \xi, \xi \rangle .$$

Thus, the vector $\xi \in \tilde{E}_x$ vanishes, and by relation 2.32 we conclude that the term $\left\langle v_x, \nabla_X \tilde{G}(x,\cdot) \right\rangle_x$ equals zero. Now choosing a test-section $s \in \mathscr{D}\left(\tilde{E}\right)$ such that $\nabla_X s|_x = v_x$, it is straightforward to see that the vector v_x equals zero as well. Thus, the differential $D_{v_x}\mathscr{G}$ is indeed injective, and we are done. □

The main ingredients in the proof of this lemma are the definition of the Dirichlet Green kernel and unique continuation due to real-analyticity. Both of these are also present in the setting of the Schrödinger operators with real-analytic potentials. Hence, Lemma 2.3.3 holds in this setting as well. Note that the image of the total space $\tilde{E}$ under $\mathscr{G}$ can be viewed as the cone whose link is the image of the subset $S_1\tilde{E}$ that is formed by vectors of unit length. Then the image of $\mathscr{G}\left(\tilde{E}^0\right)$ is precisely the set obtained by removing the origin from this cone. By Lemma 2.3.2 and Lemma 2.3.3 it is straightforward to see that the set $\mathscr{G}\left(\tilde{E}^0\right)$ is a C^1-smooth submanifold of $\mathscr{W}^\ell(\mathscr{E})$. The main idea behind the proof of Theorem 2.3.1 is to recover the topology and geometry of $\tilde{E}$ from this image.

We end this discussion with a lemma that describes another property of the image of $\mathscr{G}$.

Lemma 2.3.4. *For given two distinct points q_1 and q_2 in the interior of $\tilde{N}$ let $V_\oplus$ be the direct sum $\mathscr{G}\left(\left(\tilde{E}\right)_{q_1}\right) \oplus \mathscr{G}\left(\left(\tilde{E}\right)_{q_2}\right)$, viewed as subspace of $\mathscr{W}^\ell(\mathscr{E})$. Suppose that for some* *point $x \in \tilde{N}$ the intersection $\mathcal{G}\left(\left(\tilde{E}\right)_x\right) \cap V_\oplus$ is non-trivial. Then the point x has to coincide with one of the points q_1 or q_2.*

Proof. Suppose the contrary, the point x does not coincide neither with q_1 nor with q_2. Then there are vectors $v_x \in \tilde{E}_x$, $w_{q_1} \in \tilde{E}_{q_1}$, $w_{q_2} \in \tilde{E}_{q_2}$ such that v_x is non-zero and satisfies the equality

$$\left\langle v_x, \tilde{G}(x,y)\right\rangle_{x,\tilde{E}} = \left\langle w_{q_1}, \tilde{G}(q_1,y)\right\rangle_{q_1,\tilde{E}} + \left\langle w_{q_2}, \tilde{G}(q_2,y)\right\rangle_{q_2,\tilde{E}} \tag{2.33}$$

for any $y \in U$. Since both parts of this identity are real-analytic functions of y, we conclude that it continues to hold for all y in the complement of the points x, q_1, and q_2 in $\tilde{N}$. Since x does not coincide neither with q_1 nor with q_2, there exists a smooth section $s \in \mathcal{D}\left(\tilde{E}\right)$ whose support does not contain q_1 and q_2, and such that $s(x) = v_x$. Then, by definition of the Dirichlet Green kernel we obtain

$$\langle v_x, v_x\rangle = \left\langle v_x, \int_{\tilde{N}} \left\langle \tilde{G}(x,y), \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y)\right\rangle_x = \\ = \int_{\tilde{N}} \left\langle \left\langle v_x, \tilde{G}(x,y)\right\rangle_x, \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y),$$

applying 2.33 we get

$$\int_{\tilde{N}} \left\langle \left\langle w_{q_1}, \tilde{G}(q_1,y)\right\rangle_{q_1}, \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y) + \int_{\tilde{N}} \left\langle \left\langle w_{q_2}, \tilde{G}(q_2,y)\right\rangle_{q_2}, \Delta^{\tilde{E}} s(y)\right\rangle_y dVol(y)$$

and changing the order of scalar product and integration we have

$$\left\langle w_{q_1}, \int\limits_{\tilde{N}} \left\langle \tilde{G}(q_1,y), \Delta^{\tilde{E}} s(y) \right\rangle_y dVol(y) \right\rangle_{q_1} + \\ + \left\langle w_{q_2}, \int\limits_{\tilde{N}} \left\langle \tilde{G}(q_2,y), \Delta^{\tilde{E}} s(y) \right\rangle_y dVol(y) \right\rangle_{q_2} = \\ = \left\langle w_{q_1}, s(q_1) \right\rangle + \left\langle w_{q_2}, s(q_2) \right\rangle = 0,$$

where we used the fact that the section s is chosen so that it vanishes at q_1 and q_2 in the last equality. Thus, we conclude that v_x equals zero and arrive at a contradiction. □

This proof again uses only the definition of the Dirichlet Green kernel and unique continuation, which are present also in the case of Schrödinger operators with real-analytic potentials. Thus, Lemma 2.3.4 continues to hold in this case as well.

Remark 2.3.5. Summarising the comments on Schrödinger setting throughout this section we see that all of the results in this section continue to hold if we replace the connection Laplacian by the Schrödinger operator with real-analytic potential.

2.3.4 Proof of the main result

Now let E_i be two real-analytic vector bundles over real-analytic manifolds N_i, where $i = 1, 2$, and suppose that for some open sets $\Sigma_i \subset \partial N_i$ there exists a vector bundle isomorphism $\phi : E_1|_{\Sigma_1} \to E_2|_{\Sigma_2}$ that intertwines with the Dirichlet-to-Neumann operators Λ_{Σ_1} and Λ_{Σ_2}. Suppose that ϕ covers a diffeomorphism $\psi : \Sigma_1 \to \Sigma_2$. For a fixed point $p_1 \in \Sigma_1$ we set $p_2 = \psi(p_1)$, and choose local coordinates on the Σ_i's around these points that are related by ψ. Note that by Theorem 2.1.1 the metrics g_i coincide in such coordinates. Thus, making the Σ_i's smaller if necessary, we see that the map $\psi : \Sigma_1 \to \Sigma_2$ is an isometry. Since the metrics are real-analytic, by Theorem 2.1.1 we also conclude that their extensions $\tilde{g}_i$ coincide in neighbourhoods of the points p_i in $\tilde{N}_i$. In other words, the isometry $\psi : \Sigma_1 \to \Sigma_2$ extends to a real-analytic isometry $\Psi : W_1 \to W_2$, defined by identifying boundary normal coordinates, where W_i is a neighbourhood of the point p_i in $\tilde{N}_i$. In the sequel, we also identify the sets $W_1 \backslash \bar{N}_1$ and $W_2 \backslash \bar{N}_2$, and denote them by U.

Similarly, choosing frames related by ϕ, we may identify the trivialisations of $E_1|_{\Sigma_1}$ and $E_2|_{\Sigma_2}$ around the points p_1 and $p_2 = \psi(p_1)$. They extend to trivial vector bundles, which we may assume are defined over W_1 and W_2, and the isomorphism ϕ extends to the isomorphism $\Phi : \tilde{E}_1\big|_{W_1} \to \tilde{E}_2\big|_{W_2}$, defined by identifying the corresponding boundary normal frames. Note that Φ covers the isometry $\Psi : W_1 \to W_2$. By Theorem 2.1.1 the real-analytic connection matrices of ∇^1 and ∇^2 coincide in such frames, and we conclude that the isomorphism Φ is a gauge equivalence, that is $\Phi^* \tilde{\nabla}^2 = \tilde{\nabla}^1$. We continue to use the notation $\mathscr{E}$ for the vector bundles $\tilde{E}_i\big|_U$.

In the presence of a potential as we saw in Theorem 2.2.6 it is not possible to recover it simultaneously with a metric even if it is real-analytic. Due to this fact we have to restrict the Schrödinger setting to the real-analytic potentials of the form (2.24) for which the reconstruction result on the boundary continues to hold as was noted in Remark 2.2.7. One can see then that in this *restricted Schrödinger setting* the potentials $\tilde{P}_1$ and $\tilde{P}_2$ coincide in U.

Theorem 2.3.1 is a consequence of the following statement. Below by $\tilde{E}_i^0$ we denote the vector bundles $\tilde{E}_i$ with removed zero sections over the interiors of $\tilde{N}_i$, where $i = 1, 2$.

Theorem 2.3.6. *Under the hypotheses of Theorem 2.3.1, consider the maps* $\mathscr{G}_i : \tilde{E}_i \to$

$\mathscr{W}^{\ell}(\mathscr{E})$ defined by (2.26), where $i = 1, 2$. Suppose that the vector bundle isomorphism $\Phi : \mathscr{E} \to \mathscr{E}$, described above, intertwines with the $\mathscr{G}_i$'s, that is

$$\mathscr{G}_2 \circ \Phi = \Phi \circ \mathscr{G}_1 \qquad on \quad \mathscr{E}. \tag{2.34}$$

Then the images $\mathscr{G}_2\left(\tilde{E}_2^0\right)$ and $\Phi \circ \mathscr{G}_1\left(\tilde{E}_1^0\right)$ coincide as subsets in $\mathscr{W}^{\ell}(\mathscr{E})$, and the map $\mathscr{G}_2^{-1} \circ \Phi \circ \mathscr{G}_1 : \tilde{E}_1^0 \to \tilde{E}_2^0$ extends to a real-analytic vector bundle isomorphism $J : \tilde{E}_1 \to \tilde{E}_2$ that covers an isometry $j : \tilde{N}_1 \to \tilde{N}_2$ such that $J^* \tilde{\nabla}^2 = \tilde{\nabla}^1$.

Now we show how Theorem 2.3.6 implies Theorem 2.3.1.

Proof of Theorem 2.3.1. First, note that if a vector bundle isomorphism $\phi : E_1|_{\Sigma_1} \to E_2|_{\Sigma_2}$ preserves inner products on E_1 and E_2, then so does its extension $\Phi : \mathscr{E} \to \mathscr{E}$. This statement follows directly from the definition of Φ as an isomorphism that identifies boundary normal frames. Now we claim that the conclusion of Theorem 2.3.6 implies Theorem 2.3.1. Indeed, by relation (2.34), we see that the vector bundle isomorphism $J : \tilde{E}_1 \to \tilde{E}_2$ coincides with Φ on the set $\tilde{E}_1\big|_U$, and the isometry $j : \tilde{N}_1 \to \tilde{N}_2$ coincides with Ψ on U. Thus, they are genuine extensions of the isomorphism ϕ and the isometry ψ from the boundary, and satisfy the conclusions of Theorem 2.3.1. Since Φ preserves the inner products, we conclude that the products $\langle \cdot, \cdot \rangle_{\tilde{E}_1}$ and $J^* \langle \cdot, \cdot \rangle_{\tilde{E}_2}$ coincide on $\tilde{E}_1\big|_U$, and hence, by unique continuation coincide everywhere on $\tilde{E}_1$. Thus, the isomorphism J preserves inner products, and its restriction to E_1 satisfies all conclusions of Theorem 2.3.1.

For a proof of Theorem 2.3.1 we need to prove relation (2.34), that is the vector bundle isomorphism $\Phi : \mathscr{E} \to \mathscr{E}$ intertwines with the immersions $\mathscr{G}_i$'s. Since Φ

preserves Euclidean structures, for the latter it is sufficient to show that

$$\tilde{G}_2(\Psi(x),\Psi(y)) = \Phi^{\boxtimes}\tilde{G}_1(x,y) \qquad \text{for all} \quad (x,y)\in U\times U.$$

Choosing coordinates on W_1 and W_2 related by Ψ, we may assume that $\Psi : W_1 \to W_2$ is the identity. Similarly, choosing local trivialisations of the $\tilde{E}_i\big|_{W_i} \simeq \mathscr{E}$ related by Φ, we assume that so is Φ. Thus, it remains to show that the Green matrices $\tilde{G}_1$ and $\tilde{G}_2$, viewed as sections of the trivial bundle $\mathscr{E}\boxtimes\mathscr{E}$, coincide. For classical Green functions, that is when the rank of $\mathscr{E}$ equals one, this statement is well known, see [28, Lemma 2.1]. It is a consequence of standard regularity theory together with uniqueness of Dirichlet Green functions. Below we outline a version of this argument in our setting.

First, since under our assumptions the isomorphism Φ is the identity on $\mathscr{E}$, the hypothesis in Theorem 2.3.1 means that the Dirichlet-to-Neumann operators Λ_1 and Λ_2 restricted to sections supported in $W_1\cap\partial N_1$ and $W_2\cap\partial N_2$, respectively, coincide. Pick a point $x\in U$, and for a non-zero vector v_x in the fibre $\mathscr{E}_x$ consider a solution s to the Dirichlet problem

$$\Delta^{E_2}s = 0, \qquad s|_{\partial N_2} = \left\langle v_x, \tilde{G}_1(x,\cdot)\right\rangle_{x,\tilde{E}_1},$$

on N_2. We define a continuous section $\tilde{s}$ of $\tilde{E}_2$ away from x by extending s as $\left\langle v_x, \tilde{G}_1(x,\cdot)\right\rangle_{x,\tilde{E}_1}$ on $U\setminus\{x\}$. Note that the section $\left\langle v_x, \tilde{G}_1(x,\cdot)\right\rangle_{x,\tilde{E}_1}$ solves the Dirichlet problem

$$\Delta^{E_1}s = 0, \qquad s|_{\partial N_1} = \left\langle v_x, \tilde{G}_1(x,\cdot)\right\rangle_{x,\tilde{E}_1},$$

and since the Dirichlet-to-Neumann operators coincide, we conclude that so do the normal derivatives of s and $\left\langle v_x, \tilde{G}_1(x,\cdot)\right\rangle_{x,\tilde{E}_1}$ on the boundary $W_2\cap\partial N_2$. Thus, the section $\tilde{s}$ is C^1-smooth, and the standard application of Green's formulae shows that $\tilde{s}$ is weakly harmonic on $\tilde{N}_2\setminus\{x\}$, and hence, is smooth. Since a vector $v_x \in \mathcal{E}_x$ is arbitrary, and the Euclidean structures agree, this construction yields a smooth section $H(x,y) \in \left(\tilde{E}_2\right)_x \otimes \left(\tilde{E}_2\right)_y$ such that:

- $\Delta_y^{E_2} H(x,y) = 0$ for $y \in N_2$;
- $H(x,y) = \tilde{G}_1(x,y)$ for $y \in U$, $y \neq x$;
- $H(x,y) = 0$ for $y \in \partial\tilde{N}_2$.

In particular, we see that $\Delta_y^{E_2} H(x,\cdot) = \delta_x$ on $\tilde{N}_2$, and the standard argument used to prove uniqueness of the Dirichlet Green kernel shows that $H(x,y)$ coincides with the Dirichlet Green kernel $\tilde{G}_2(x,y)$ for all $y \in \tilde{N}_2$. Thus, the Green matrices $\tilde{G}_1(x,\cdot)$ and $\tilde{G}_2(x,\cdot)$ indeed coincide on the set $U\setminus\{x\}$, and we are done. □

Remark 2.3.7. Theorem 2.3.6 continues to hold in the restricted Schrödinger setting with an additional relation

$$\tilde{P}_1 = J^{-1}\tilde{P}_2 J.$$

Indeed, the proof essentially uses real-analyticity and the existence and uniqueness of a solution to the Dirichlet problem for the connection Laplacian, and as we saw previously those are the features of the restricted Schrödinger setting as well.

2.3.5 Proof of Theorem 2.3.6

We start with outlining the general strategy of a proof. Let $B_1 \subset \tilde{N}_1$ be the largest connected open set containing the fixed point $p_1 \in \partial N_1$ and such that for any $x \in B_1$ there exists a unique $j(x) \in \tilde{N}_2$ such that the images of fibres $\Phi \circ \mathcal{G}_1\left(\left(\tilde{E}_1\right)_x\right)$ and

$\mathscr{G}_2\left(\left(\tilde{E}_2\right)_{j(x)}\right)$ coincide and the operator

$$J_x = \mathscr{G}_2^{-1} \circ \Phi \circ \mathscr{G}_1 : \left(\tilde{E}_1\right)_x \longrightarrow \left(\tilde{E}_2\right)_{j(x)} \tag{2.35}$$

is an isometry with respect to the inner products on the fibres. Note that if the subspaces $\Phi \circ \mathscr{G}_1\left(\left(\tilde{E}_1\right)_x\right)$ and $\mathscr{G}_2\left(\left(\tilde{E}_2\right)_{j(x)}\right)$, by Lemma 2.3.3 the map J_x, defined in 2.35, is automatically an isomorphism of the fibres, and defines a fibre preserving map $J : \tilde{E}_1\big|_{B_1} \to \tilde{E}_2$. Note that the set B_1 contains the neighbourhood W_1 of p_1 constructed above. Indeed, since Φ intertwines with the maps $\mathscr{G}_1$ and $\mathscr{G}_2$ on $\mathscr{E}$, we have

$$\Phi_y \left\langle v_x, \tilde{G}_1(x,y)\right\rangle_x = \left\langle \Phi_{\Psi(x)} v_{\Psi(x)}, \tilde{G}_2(\Psi(x), \Psi(y))\right\rangle_x \tag{2.36}$$

for all $x, y \in U$, and $v_x \in \left(\tilde{E}_1\right)_x$. Choosing a real-analytic non-zero section v on W_1, since both sides in the relation above are real-analytic, we conclude that this relation continues to hold for all $x \in W_1$, $y \in U$. Since v_x may take arbitrary values, it is straightforward to see that for any $x \in W_1$ we may choose $\Psi(x)$ as the point $j(x)$ in the definition of the set B_1. Indeed, relation (2.36) implies that for any $x \in W_1$ the operator J_x coincides with $\Phi : \left(\tilde{E}_1\right)_x \longrightarrow \left(\tilde{E}_2\right)_{j(x)}$, which is an isometry by its own definition. By Lemma 2.3.3 it is a unique point that satisfies this condition. Thus, the set W_1 indeed lies in B_1. This is true in the restricted Schrödinger setting also since as we saw before Lemma 2.3.3 remains valid.

Our main aim is to show that the set B_1 coincides with $\tilde{N}_1$. Once this statement is proved, we shall show that the map $J : \tilde{E}_1 \to \tilde{E}_2$, defined on each fibre by relation 2.35, is a vector bundle isomorphism that satisfies the conclusions of the theorem.

Suppose the contrary, $B_1 \neq \tilde{N}_1$. Then there exists a point $x_1 \in \partial B_1$ that lies in the interior of $\tilde{N}_1$, that is $x_1 \notin \partial \tilde{N}_1$. Since $W_1 \subset B_1$, the point x_1 lies in the complement $\tilde{N}_1 \backslash W_1$, and in particular, we see that $x_1 \notin \bar{U}$.

Step 1. First, we claim that the map J can be extended to the fibre $\left(\tilde{E}_1\right)_{x_1}$ over x_1.

Lemma 2.3.8. *Let $x_1 \in \partial B_1$ be a point such that $x_1 \notin \partial \tilde{N}_1$. Then there exists a unique point x_2 in the interior of $\tilde{N}_2$ such that the images of fibres $\Phi \circ \mathcal{G}_1\left(\left(\tilde{E}_1\right)_{x_1}\right)$ and $\mathcal{G}_2\left(\left(\tilde{E}_2\right)_{x_2}\right)$ coincide, and the corresponding operator J_{x_1} is an isometry. Moreover, for any non-zero vector $v_{x_1} \in \left(\tilde{E}_1\right)_{x_1}$ there exists a unique non-zero vector $w_{x_2} \in \left(\tilde{E}_2\right)_{x_2}$ such that*

$$\Phi \circ \mathcal{G}_1\left(v_{x_1}\right) = \mathcal{G}_2\left(w_{x_2}\right), \quad \left|v_{x_1}\right|_{\tilde{E}_1} = \left|w_{x_2}\right|_{\tilde{E}_2},$$

and for any converging sequence $v_{p_k} \to v_{x_1}$, where $p_k \to x_1$, we have $J\left(v_{p_k}\right) \to w_{x_2}$ as $k \to +\infty$.

Proof. Let $p_k \in B_1$ be a sequence of points that converges to the point $x_1 \in \partial B_1$, and q_k be the corresponding sequence of points such that the images of fibres $\Phi \circ \mathcal{G}_1\left(\left(\tilde{E}_1\right)_{p_k}\right)$ and $\mathcal{G}_2\left(\left(\tilde{E}_2\right)_{q_k}\right)$ coincide. Since $\tilde{N}_2 \cup \partial \tilde{N}_2$ is compact, then choosing a subsequence, which we denote by the same symbol q_k, we may assume that $q_k \to q_0 \in \tilde{N}_2$ as $k \to +\infty$. For a non-zero vector $v_{x_1} \in \left(\tilde{E}_1\right)_{x_1}$ pick a sequence $v_k \in \left(\tilde{E}_1\right)_{p_k}$ that converges to v_{x_1}, and let $w_k \in \left(\tilde{E}_2\right)_{q_k}$ be the corresponding sequence such that

$$\Phi \circ \mathcal{G}_1\left(v_k\right) = \mathcal{G}_2\left(w_k\right) \quad \text{and} \quad \left|v_k\right|_{\tilde{E}_1} = \left|w_k\right|_{\tilde{E}_2}.$$

Since the sequence w_k is bounded, we may assume, again after choosing a subsequence, that w_k converges to some vector $w_{q_0} \in \left(\tilde{E}_2\right)_{q_0}$ as $k \to +\infty$. It is straightforward to see that the norm of w_{q_0} equals the one of v_{x_1}. Now for a proof of the lemma it remains to show that $q_0 \notin \partial \tilde{N}_2$. If the latter holds, then we may take q_0 as x_2, and the statement follows directly by continuity of $\Phi \circ \mathscr{G}_1$ and $\mathscr{G}_2$.

Suppose the contrary, $q_0 \in \partial \tilde{N}_2$. Then by continuity we obtain

$$\Phi \circ \mathscr{G}_1\left(v_{x_1}\right) = \lim_{k \to +\infty} \Phi \circ \mathscr{G}_1\left(v_k\right) = \lim_{k \to +\infty} \mathscr{G}_2\left(w_k\right) = \mathscr{G}_2\left(w_{q_0}\right).$$

Since the point x_1 lies in the interior of $\tilde{N}_1$, by Lemma 2.3.3 the left-hand side above is non-zero, while since $q_0 \in \partial \tilde{N}_2$, the right-hand side vanishes. Thus, we arrive at a contradiction.

Let us prove the uniqueness. Suppose there is another pair $\left(\hat{x}_2, w_{\hat{x}_2}\right) \in \tilde{E}_2$ such that

$$\Phi \circ \mathscr{G}_1\left(v_{x_1}\right) = \mathscr{G}_2\left(w_{\hat{x}_2}\right),$$

where $w_{\hat{x}_2}$ is non-zero. Then we have the equality

$$\mathscr{G}_2\left(w_{\hat{x}_2}\right) = \mathscr{G}_2\left(w_{x_2}\right),$$

and the last part of Lemma 2.3.3 implies that $\hat{x}_2 = x_2$ and $w_{\hat{x}_2} = w_{x_2}$. □

Step 2. Now we analyse the images $\mathscr{R}_i$ of the maps $\mathscr{G}_i$ in $\mathscr{W}^\ell(\mathscr{E})$, where $i = 1, 2$. Take a non-zero vector $v_{x_1} \in \left(\tilde{E}_1\right)_{x_1}$, and let $x_2 \in \tilde{N}_2$ and $w_{x_2} \in \left(\tilde{E}_2\right)_{x_2}$ be a point and a vector respectively that satisfy the conclusions of Lemma 2.3.8. In particular, the vectors $\Phi \circ \mathscr{G}_1\left(v_{x_1}\right)$ and $\mathscr{G}_2\left(w_{x_2}\right)$ coincide in $\mathscr{W}^\ell(\mathscr{E})$, and we denote this value by u. By Lemma 2.3.3 we see that locally the sets $\Phi(\mathscr{R}_1)$ and $\mathscr{R}_2$ are submanifolds in $\mathscr{W}^\ell(\mathscr{E})$, whose tangent spaces can be viewed as the images of the differentials $D(\Phi \circ \mathscr{G}_1)$ and $D\mathscr{G}_2$. Combining this with Lemma 2.3.8, we conclude that the tangent spaces $T_u\Phi(\mathscr{R}_1)$ and $T_u\mathscr{R}_2$ coincide as subspaces in $\mathscr{W}^\ell(\mathscr{E})$. Using the inverse function theorem we may represent $\Phi(\mathscr{R}_1)$ and $\mathscr{R}_2$ locally near u as graphs of smooth functions defined on an open subset in

$$\mathscr{V} = T_u\Phi(\mathscr{R}_1) = T_u\mathscr{R}_2.$$

In more detail, let Π be orthogonal projection onto $\mathscr{V}$ in $\mathscr{W}^\ell(\mathscr{E})$, and consider the map

$$\Pi \circ \mathscr{G}_2 : \tilde{E}_2 \to \mathscr{V}, \qquad v_x \longmapsto \Pi\left(\left\langle v_x, \tilde{G}_2(x,\cdot)\right\rangle\right).$$

By Lemma 2.3.3, its differential is an isomorphism near u, and hence, there exists a C^1-smooth inverse map $H_2 : \mathscr{O} \to \tilde{E}_2$, defined in the neighbourhood $\mathscr{O}$ of $\Pi(u)$ in $\mathscr{V}$. Then, it is straightforward to see that $\mathscr{R}_2$ is the graph of the map

$$F_2 : \mathscr{O} \to \mathscr{V}^\perp, \qquad \upsilon \longmapsto \mathscr{G}_2(H_2(\upsilon)) - \upsilon.$$

Similarly, one shows that there exists a C^1-smooth map $H_1 : \mathscr{O} \to \tilde{E}_1$, which we may assume is defined on the same set $\mathscr{O}$, such that $\Phi(\mathscr{R}_1)$ is the graph of the map

$$F_1 : \mathscr{O} \to \mathscr{V}^\perp, \qquad \upsilon \longmapsto \Phi \circ \mathscr{G}_1(H_1(\upsilon)) - \upsilon.$$

From this construction we see that the vectors $v_{x_1} \in \left(\tilde{E}_1\right)_{x_1}$ and $w_{x_2} \in \left(\tilde{E}_2\right)_{x_2}$ are precisely the images $H_1 \circ \Pi(u)$ and $H_2 \circ \Pi(u)$, and the isomorphism J has the form $H_2 \circ H_1^{-1}$ on the open subset

$$\Omega_1 = H_1(\mathscr{O}) \cap \tilde{\pi}_1^{-1}(B_1) \subset \tilde{E}_1, \tag{2.37}$$

where $\tilde{\pi}_1 : \tilde{E}_1 \to \tilde{N}_1$ is the vector bundle projection.

For the sequel we need the following lemma.

Lemma 2.3.9. *The maps $H_i : \mathscr{O} \to \tilde{E}_i$ constructed above, where $i = 1, 2$, are real-analytic in a neighbourhood of $\Pi(u)$ in $\mathscr{V}$. In particular, there exists a neighbourhood of v_{x_1} in $\tilde{E}_1$ such that the map $H_2 \circ H_1^{-1}$ is real-analytic on it.*

Proof. Choosing an orthonormal basis (φ_i) in $\mathscr{V}$, where $i = 1, \ldots, m$, we may identify the vector space $\mathscr{V}$ with $\mathbb{R}^m$. First, we claim that the map $\Pi \circ \mathscr{G}_2 : \tilde{E}_2 \to \mathscr{V} \simeq \mathbb{R}^m$ is real-analytic in a neighbourhood of w_{x_2}, that is the coordinate functions, given by products

$$(\Pi \circ \mathscr{G}_2, \varphi_i)_\ell = (\mathscr{G}_2, \varphi_i)_\ell, \qquad \text{where} \quad i = 1, \ldots, m,$$

and $(\cdot, \cdot)_\ell$ stands for the scalar product in $\mathscr{W}^\ell(\mathscr{E})$, are real-analytic. By definition of $\mathscr{G}_2$ for the latter it is sufficient to show that the sections

$$x \longmapsto \left(\tilde{G}_2(x, \cdot), \varphi_i\right)_\ell \in E_x \qquad \text{where} \quad i = 1, \ldots, m,$$

are real-analytic in a neighbourhood of x_2. Let $f_i \in \mathscr{W}_0^{-\ell}(\mathscr{E})$ be a vector dual to φ_i, that is such that $\varphi_i(s) = (s, f_i)_{-\ell}$ for any $s \in \mathscr{W}_0^{-\ell}(\mathscr{E})$. Since the canonical map $f \mapsto (\cdot, f)_{-\ell}$ preserves scalar products, we conclude that

$$\left(\tilde{G}_2(x, \cdot), \varphi_i\right)_\ell = \int_U \left\langle \tilde{G}_2(x, y), f_i(y)\right\rangle_{y, \tilde{E}_2} dVol_g(y).$$

Recall that the point x_1 does not lie in the closure $\bar{U} \subset \tilde{N}_2$. Then, by properties of the Green kernel, it is straightforward to see that the integral on the right-hand side above defines a harmonic section in any neighbourhood of x_1 that is disjoint with U. Any harmonic section is real-analytic under our hypotheses, and we conclude that so is the integral above. Thus, the coordinate functions $(\mathscr{G}_2, \varphi_i)_\ell$ are real-analytic in a neighbourhood of x_1 for all $i = 1, \ldots, m$. Further, we conclude that the map H_2, as the inverse map to $\Pi \circ \mathscr{G}_2$, is also real-analytic in a neighbourhood of $\Pi(u)$.

A similar argument shows that the maps $\Pi \circ \Phi \circ \mathscr{G}_1$ and H_1 are real-analytic as well. Hence, the map $H_2 \circ H_1^{-1}$ is real-analytic as the composition of real-analytic maps. □

Step 3. Now we claim that the images of $\Phi(\mathscr{R}_1)$ and $\mathscr{R}_2$ coincide around the point u. This is the consequence of the following lemma.

Lemma 2.3.10. *The maps $F_i : \mathscr{O} \to \mathscr{V}^\perp$ constructed above, where $i = 1, 2$, coincide in a neighbourhood of $\Pi(u)$ in $\mathscr{V}$.*

Proof. Fix an orthonormal basis (φ_j) in $\mathscr{V}^\perp$, where $j = 1, 2, \ldots, \infty$. For a proof of the lemma it is sufficient to show that the coordinate functions $(F_1, \varphi_j)_\ell$ and $(F_2, \varphi_j)_\ell$ coincide for all $j = 1, 2, \ldots, \infty$, where $(\cdot, \cdot)_\ell$ is the scalar product in $\mathscr{W}^\ell(\mathscr{E})$. Note that

$$(F_2, \varphi_j)_\ell(v) = (\mathscr{G}_2, \varphi_j)_\ell \circ H_2(v) - (v, \varphi_j)_\ell \tag{2.38}$$

for any $v \in \mathscr{O}$. The argument used in the proof of Lemma 2.3.9 shows that the function $(\mathscr{G}_2, \varphi_j)_\ell$ is real-analytic in some neighbourhood of v_{x_1}, and by Lemma 2.3.9 we also know that the map H_2 is real-analytic in a neighbourhood of $\Pi(u)$. Since the second term on the right-hand side of (2.38) is linear in v, we conclude that the function $(F_2, \varphi_j)_\ell$ is real-analytic in a neighbourhood of $\Pi(u)$, which we may also denote by $\mathscr{O}$. This statement holds for all values $j = 1, 2, \ldots, \infty$, with the same set $\mathscr{O}$.

Similarly, one shows that all functions

$$(F_1, \varphi_j)_\ell(v) = (\Phi \circ \mathscr{G}_1, \varphi_j)_\ell \circ H_1(v) - (v, \varphi_j)_\ell \tag{2.39}$$

are also real-analytic on the same set $\mathscr{O}$. Without loss of generality, we may assume that the open set $\mathcal{O}$ is connected. Now by the choice of the point x_1, we know that the maps $\Phi \circ \mathcal{G}_1$ and $\mathcal{G}_2 \circ J$ coincide on an open subset $\Omega_1 \subset \tilde{E}_1$, defined in 2.37, whose closure contains v_{x_1}. Recall that the map J coincides with $H_2 \circ H_1^{-1}$ on Ω_1, and hence, the maps $\Phi \circ \mathcal{G}_1 \circ H_1$ and $\mathcal{G}_2 \circ H_2$ coincide on $H_1^{-1}(\Omega_1) \subset \mathcal{O}$. Combining the latter with relations (2.38) and (2.39), we conclude that the real-analytic functions $\left(F_1, \varphi_j\right)_\ell$ and $\left(F_2, \varphi_j\right)_\ell$ coincide on an open subset $H_1^{-1}(\Omega_1) \subset \mathcal{O}$, and hence, coincide on $\mathcal{O}$ for all $j = 1, 2, \ldots, \infty$. Thus, we are done. □

Due to the conical structure of the images $\Phi(\mathcal{R}_1)$ and $\mathcal{R}_2$, from the above we conclude that there are conical neighbourhoods of u, that is neighbourhoods invariant under multiplication by $t > 0$, that coincide. In fact, as the following lemma shows, even a stronger statement holds.

Lemma 2.3.11. *There is a neighbourhood O_1 of the point $x_1 \in \tilde{N}_1$ such that for any $x \in O_1$ there exists $z \in \tilde{N}_2$ such that the images of fibres $\Phi \circ \mathcal{G}_1\left(\tilde{E}_1\right)_x$ and $\mathcal{G}_2\left(\tilde{E}_2\right)_z$ coincide.*

Proof. Choose a neighbourhood O_1 of $x_1 \in \tilde{N}_1$ such that $O_1 \subset \tilde{\pi}_1 \circ H_1(\mathcal{O})$, where $\tilde{\pi}_1 : \tilde{E}_1 \to \tilde{N}_1$ is the vector bundle projection. We intend to show that for any $x \in O_1$ there exists $z \in \tilde{N}_2$ such that the image $\Phi \circ \mathcal{G}_1\left(\tilde{E}_1\right)_x$ lies in $\mathcal{G}_2\left(\tilde{E}_2\right)_z$. Since these images are vector spaces of the same dimension, the statement of the lemma follows immediately.

First, for a given point $x \in O_1$ and a vector $v_x \in \left(\tilde{E}_1\right)_x$, the considerations above show that the image $\Phi \circ \mathcal{G}_1(v_x)$ lies in the set $\mathcal{G}_2(CH_2(\mathcal{O}))$, where $CH_2(\mathcal{O})$ is a conical open set,

$$CH_2(\mathcal{O}) = \left\{tw \in \tilde{E}_2 : \ t \in \mathbb{R}, \ t > 0, \text{ and } w \in H_2(\mathcal{O})\right\}.$$

Thus, there exists $z \in \tilde{N}_2$ such that $\Phi \circ \mathscr{G}_1(v_x)$ lies in $\mathscr{G}_2\left(\tilde{E}_2\right)_z$. We claim that for any $w_x \in \left(\tilde{E}_1\right)_x$ its image $\Phi \circ \mathscr{G}_1(w_x)$ lies in $\mathscr{G}_2\left(\left(\tilde{E}_2\right)_z\right)$.

Suppose the contrary, that is there exists a non-zero vector $w_x \in \left(\tilde{E}_1\right)_x$ such that its image $\Phi \circ \mathscr{G}_1(w_x)$ lies in $\mathscr{G}_2\left(\left(\tilde{E}_2\right)_y\right)$, where $z \neq y$. Then, we see that

$$\Phi \circ \mathscr{G}_1(w_x - v_x) \in \mathscr{G}_2\left(\left(\tilde{E}_2\right)_z\right) \oplus \mathscr{G}_2\left(\left(\tilde{E}_2\right)_y\right).$$

Since the vectors v_x and w_x are different, arguing as above, we may find another point $q \in \tilde{N}_2$ such that $\Phi \circ \mathscr{G}_1(w_x - v_x)$ lies in $\mathscr{G}_2\left(\left(\tilde{E}_2\right)_q\right)$. Now by Lemma 2.3.4 we conclude that the point q coincides with either z or y, and in each case it is straightforward to arrive at a contradiction. For example, if $q = z$, we immediately conclude that the vector

$$\Phi \circ \mathscr{G}_1(w_x) = \Phi \circ \mathscr{G}_1(w_x - v_x) + \Phi \circ \mathscr{G}_1(v_x)$$

lies in the image $\mathscr{G}_2\left(\left(\tilde{E}_2\right)_z\right)$, and by Lemma 2.3.3, the points z and y coincide. □

Using this lemma we can extend the map J to an open neighbourhood O_1 of x_1. By the argument in Step 2 it is real-analytic. Moreover, it is an isometry on fibres over the open set $O_1 \cap B_1$ and since Euclidean structures are real-analytic we conclude that J is an isometry on fibres over the whole set O_1. This means that $O_1 \subset B_1$ and we arrive at a contradiction with the assumption $B_1 \neq \tilde{N}_1$, since the point $x_1 \in O_1 \subset \tilde{N}_1$ has been chosen on the boundary ∂B_1. Thus, we conclude that the set B_1 coincides with the whole manifold $\tilde{N}_1$.

Step 4. Now we collect final conclusions. First, relation 2.35 defines the fibre preserving map $J : \tilde{E}_1 \to \tilde{E}_2$. By the argument in Step 2 we see that locally it can be written in the form $H_2 \circ H_1^{-1}$, and hence, is smooth, and by Lemma 2.3.9 is real-analytic. Since by definition it is an isomorphism on each fibre, we conclude that it is a real-analytic vector bundle isomorphism. In particular, it covers a real-analytic map $j : \tilde{N}_1 \to \tilde{N}_2$.

Note that the isomorphism J coincides with the isomorphism Φ on fibres over $W_1 \subset \tilde{N}_1$. Since the latter is a gauge equivalence, the connections $J^*\tilde{\nabla}^2$ and $\tilde{\nabla}^1$ coincide on W_1, and since they are real-analytic and $\tilde{N}_1$ is connected, they coincide on $\tilde{N}_1$. Similarly, the map $j : \tilde{N}_1 \to \tilde{N}_2$ coincides with the isometry Ψ on W_1, that is the real-analytic metrics $j^*\tilde{g}_2$ and $\tilde{g}_1$ coincide on W_1, and hence, they coincide everywhere on $\tilde{N}_1$. Thus, the vector bundle isomorphism J is indeed a gauge equivalence that covers an isometry.

Remark 2.3.12. The results of this section remain valid in the restricted Schrödinger setting. Let us explain step by step why this is the case. As we already discussed the results up to Step 1 remain valid. The main ingredient of Step 1, Lemma 2.3.8, is valid since its proof relies on the definition of the Dirichlet Green kernel and Lemma 2.3.3, which still holds as was noted in Remark 2.3.5. Step 2 remains valid because it essentially uses Lemma 2.3.3 and the real-analyticity of the solutions to the equation

$$L_P = 0.$$

Step 3 is still valid because it uses real-analyticity again, Lemma 2.3.3, and Lemma 2.3.4, which are still valid as was discussed in Remark 2.3.5. Finally, Step 4 relies on the previous results and, therefore, also remains valid. Note that due to the local reconstruction result Remark 2.2.7 the potentials over W_1 are related by the formula

$$\tilde{P}_1 = \Phi^{-1} \circ \tilde{P}_2 \circ \Phi.$$

Now again the isomorphism J coincides with the isomorphism Φ over W_1, and since $\tilde{P}_1$ and $J^{-1} \circ \tilde{P}_2 \circ J$ are real-analytic we have the relation

$$\tilde{P}_1 = J^{-1} \circ \tilde{P}_2 \circ J.$$

Therefore, the result in Theorem 2.3.1 remains valid in the restricted Schrödinger setting.

Chapter 3

Calderón's problem for harmonic maps

3.1 Dirichlet problem for harmonic maps

3.1.1 Harmonic maps

Let us recall the definition of a harmonic map. Let (N, g) be a connected compact orientable Riemannian manifold and (M, h) be a connected complete orientable Riemannian manifold. We assume manifolds have dimensions n and m, respectively. Given a smooth map $u : N \to M$, we define the *tension field* operator τ as

$$\tau(u) = Tr_g \nabla du, \tag{3.1}$$

where $du \in \Gamma(T^*N \otimes u^*TM)$ is the differential of u, and $\nabla = \nabla^{T^*N \otimes u^*TM}$ is the induced connection on the vector bundle $T^*N \otimes u^*TM$.

Note that the tension field can be equivalently defined as $\tau(u) = -d^*du$, see [11].

Definition 3.1.1. A map $u : N \to M$ is called *harmonic* if its tension field vanishes $\tau(u) = 0$.

A straightforward calculation [cf. 11] shows that in local coordinates the equation $\tau(u) = 0$ takes the form of the following system of non-linear equations

$$\Delta u^i + g^{\alpha\beta}\Gamma^i_{jl}\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^l}{\partial x^\beta} = 0,$$

where $i, j, l = 1, \ldots, m$, $\alpha, \beta = 1, \ldots, n$, and we assume the summation over the repeated indices. Here Γ^i_{jl} are the Christoffel symbols of the Levi-Civita connection on M, Δ is the Laplace–Beltrami operator, and $g^{\alpha\beta}$ is the inverse matrix to $g_{\alpha\beta}$.

For a smooth map $u : N \to M$ we define its *energy* (functional) as

$$E(u) = \frac{1}{2}\int_N |du|^2 Vol_g,$$

where $|du|^2$ is the *energy density* of the map u defined by

$$|du|^2(x) \equiv Tr_g(u^*h)(x),$$

which in local coordinates has the form

$$g^{\alpha\beta}(x)h_{ij}(u(x))\frac{\partial u^i}{\partial x^\alpha}\frac{\partial u^j}{\partial x^\beta}(x).$$

The direct calculation of the first variation of the energy functional [cf. 11] gives the formula

$$\left.\frac{dE(u_t)}{dt}\right|_{t=0} = -\int_N \langle v, \tau(u)\rangle Vol_g,$$

where u_t is a family of maps $N \to M$ depending smoothly on t, such that $u_0 = u$ and $v = \frac{\partial u_t}{\partial t}\Big|_{t=0}$.

We see that $\tau(u) = 0$ is the Euler-Lagrange equation for the energy functional. This gives us the well-known variational characterisation of harmonic maps. Namely, harmonic maps are critical points of the energy functional.

The second variation of the energy functional is also well-known and leads to the definition of the Jacobi operator.

Proposition 3.1.2. *[14] The second variation of the energy functional is given by*

$$\frac{\partial^2 E(u_{s,t})}{\partial s \partial t}\bigg|_{s,t=0} = \int_N \langle J_u v, w \rangle \, Vol_g,$$

where $v = \frac{\partial u_{s,t}}{\partial s}\Big|_{s,t=0}$, $w = \frac{\partial u_{s,t}}{\partial t}\Big|_{s,t=0}$, *and* $J_u = \Delta^u - Trace_g R^M(du, \cdot)du$ *is the Jacobi operator,* R^M *is the Riemann curvature tensor on* M, *and* $\Delta^u := \Delta^{u^*TM}$ *is the connection Laplacian acting on sections of the pull-back bundle* u^*TM.

An important example of harmonic maps arises when we take M to be $\mathbb{R}$. In this case the tension field operator is the Laplace-Beltrami operator Δ on N and harmonic maps are just harmonic functions. Harmonic maps and, in particular, harmonic functions have been studied extensively, see [12] and references there.

In the Introduction we defined the classical DtN operator (8). Clearly, the uniqueness of the solution to (1) is crucial for the definition of the DtN operator and the existence of solutions defines its domain. Our aim is to generalise the definition of the DtN operator to maps between manifolds. First of all we need to consider the Dirichlet problem for harmonic maps. There are two different variants of the Dirichlet problem for harmonic maps. One of them fixes the homotopy type of a map and the other is not.

- **Dirichlet problem.** Assume N is compact with smooth boundary ∂N. Given $\varphi : N \to M$, find a harmonic map u with $u \equiv \varphi$ on ∂N.

- **Homotopy Dirichlet problem.** Assume N is compact with smooth boundary ∂N. Given $\varphi : N \to M$, find a harmonic map u with $u \equiv \varphi$ on ∂N that is homotopic to φ relatively ∂N, i.e. there exists a family of maps $u_t : N \to M$, with $0 \le t \le 1$, such that $u_0 = \varphi$, $u_1 = u$, and $u_t \equiv \varphi$ on ∂N.

Note that for a closed compact case there is a similar homotopy problem which reads as follows.

- **Homotopy problem.** Assume N is closed and compact. Given a smooth map $\varphi : N \to M$, find a harmonic map u homotopic to φ.

Following the definition of the classical DtN operator we consider the following setting. Let N be a smooth connected Riemannian manifold with boundary ∂N and M be a smooth Riemannian manifold, then (under some assumptions) to a smooth map $\varphi : \partial N \to M$ we can assign its unique harmonic extension $u : N \to M$ and get $du|_{\partial N} : \mathcal{N}_p \partial N \to T_{\varphi(p)}M$, for any $p \in \partial N$, where $\mathcal{N}_p \partial N \subset T_p N$ is the normal line for ∂N in N at the point $p \in \partial N$. In this setting the Dirichlet-to-Neumann operator sends a map $\varphi \in C^\infty(\partial N, M)$ to a section $\Lambda[\varphi]$ of the pull-back bundle $\varphi^* TM$, where $\Lambda[\varphi](x) = du(\nu_x)$ and $\nu_x \in \mathcal{N}_x \partial N$ is the outward unit normal vector at the point $x \in \partial N$. We will call this (non-linear) operator the *Dirichlet-to-Neumann map* in order to distinguish between the DtN operator for harmonic maps and linear DtN operator on vector bundles.

3.1.2 Existence and uniqueness theorems

Of course, the definition of the DtN operator for harmonic maps is valid only under the assumption that the Dirichlet problem for harmonic maps has a solution for any map $\varphi \in C^\infty(\partial N, M)$ and this solution is unique. This is not the case in general, but there are some results which give us the conditions under which the above assumption holds. For the homotopy Dirichlet problem we have the following existence result of Hamilton [20].

Theorem 3.1.3 (Hamilton)**.** *Suppose N is compact with (nontrivial) boundary, and M is complete with non-positive sectional curvature. Let $u_0 : N \to M$ be a smooth mapping. Then there exists a smooth harmonic mapping $u : N \to M$ such that $u = u_0$ on ∂N and u homotopic (relative to ∂N) to u_0.*

In addition Hartman proved the following uniqueness theorem [21].

Theorem 3.1.4 (Hartman)**.** *Suppose N is compact with (nontrivial) boundary, M is complete with non-positive sectional curvature, and $u_0 : N \to M$ is harmonic. Then any harmonic map $u_1 : N \to M$ homotopic to u_0 relative to ∂N must coincide with u_0.*

Example. If we consider $M = \mathbb{R}$ and a real valued function $\varphi \in C^\infty(\partial N)$, then the classical Dirichlet problem arises. It is well known [cf. 2] that in this case there exists a unique harmonic map (real valued function) $u : N \to \mathbb{R}$ such that $u = \varphi$ on ∂N. This result also follows from the discussion in Chapter 1 since the Laplace-Beltrami operator is a connection Laplacian for the trivial connection on the trivial vector bundle of functions.

In light of these theorems it is natural to restrict ourselves to the case when the target manifold M is complete and has non-positive sectional curvature.

In order to define the Dirichlet-to-Neumann map we need to use both existence and uniqueness results for the homotopy Dirichlet problem. These results may not hold in general, so we continue our discussion with some counterexamples.

3.1.3 Counterexamples

In this section we discuss some counterexamples to the existence and uniqueness of a solution to the homotopy Dirichlet problem. The easiest counterexample to the uniqueness of the solution to the homotopy Dirichlet problem is probably given by the maps from the interval $N = [0, \pi]$ to a unit sphere $M = S^n$, $n > 1$. If we pick $\varphi : [0, \pi] \to S^n$ to be a geodesic joining antipodal points p and q in S^n, then any geodesic joining these two points will represent a harmonic map homotopic to φ relative to ∂N. Now, it is well known that there is infinite number of geodesics joining antipodal points on a sphere and since $n > 1$ all of them are homotopic to φ relative ∂N. Hence, we have infinite number of solutions to the homotopy Dirichlet problem in this case. One can obtain a generalisation of this example to higher dimensions as follows. Let N be a unit hemisphere

$$S^k_+ \overset{j^k_+}{\hookrightarrow} \mathbb{R}^{k+1}$$

defined in coordinates by

$$S^k_+ = \left\{(x_0, x_1, \ldots, x_k) \in \mathbb{R}^{k+1} : x_0^2 + x_1^2 + \cdots + x_k^2 = 1, x_k \geq 0\right\},$$

and let M be a unit sphere $S^n \subset \mathbb{R}^{n+1}$ defined in coordinates by

$$S^n = \left\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \cdots + x_n^2 = 1\right\}.$$

Suppose that $k \geq 1$ and $n \geq k+1$. Let $i_k^n : \mathbb{R}^{k+1} \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion defined in coordinates by

$$(x_0, x_1, \dots, x_k) \mapsto (x_0, x_1, \dots, x_k, 0, \dots, 0).$$

The boundary ∂N is then S^{k-1} given in coordinates by

$$S^{k-1} = \left\{ (x_0, x_1, \dots, x_k) \in \mathbb{R}^{k+1} : x_0^2 + x_1^2 + \dots + x_{k-1}^2 = 1, x_k = 0 \right\},$$

and the following inclusion holds

$$\partial N = S^{k-1} \subset i_{k-1}^k \left(\mathbb{R}^k \right) \subset \mathbb{R}^{k+1}.$$

One can see that the map $\varphi = i_k^n \circ j_+^k$ defines an inclusion of S_+^k into S^n, this inclusion is totally geodesic and, hence, harmonic. Therefore, the homotopy Dirichlet problem for φ has a solution given by $u = \varphi$. Let us now show that this solution is not unique. Consider the subgroup $SO(n-k+1)$ of $SO(n+1)$ which stabilises the elements of the subspace $i_{k-1}^k \left(\mathbb{R}^k \right) \subset \mathbb{R}^{n+1}$. Its elements can be represented as a block diagonal matrix

$$R_A = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix},$$

where I is $k \times k$ identity matrix and $A \in SO(n-k+1)$. Clearly, each R_A defines an isometric diffeomorphism $S^{n+1} \to S^{n+1}$ and, thus, a totally geodesic map. Since there is no retraction of S_+^k onto its boundary there is at least one point $y \in S_+^k$ such that $\left(\varphi(y)_k, \dots, \varphi(y)_n \right) \neq (0, \dots, 0) \in \mathbb{R}^{n-k+1}$. Now, it is well known that $SO(n-k+1)$ acts transitively on spheres of constant radius in $\mathbb{R}^{n-k+1}$. Therefore, there will be infinitely many different maps $R_A \circ \varphi$. Moreover, due to the choice of a subgroup

$SO(n-k+1)$ we see that $R_A \circ \varphi|_{\partial N} = \varphi|_{\partial N}$. Finally, all the maps $R_A \circ \varphi$ are harmonic (in fact totally geodesic) as a composition of a harmonic map with a totally geodesic map [11]. Therefore, we see that there is an infinite number of harmonic maps homotopic to φ relative to ∂N. Note that the fact that they all homotopic to φ follows from their definition and path-connectedness of $SO(n-k+1)$.

The following example from [38] shows that the existence and uniqueness of a solution to the Dirichlet problem for harmonic maps may depend on the dimension of manifolds. Let $\mathbb{R}^n \to \mathbb{R}^{n+1}$ be the injection $i(x_1,\ldots,x_n) = (x_1,\ldots,x_n,0)$. We consider the following Dirichlet problem: Find a rotationally symmetric harmonic map $u : B^n \to S^n \subset \mathbb{R}^{n+1}$ such that $u|_{\partial B^n} = i|_{\partial B^n}$, i.e. u maps the boundary B^n to the equator of S^n.

Let (r,θ) be polar coordinates on B^n and (ρ,φ) geodesic coordinates on S^n such that ρ is a distance from the north pole of S^n and $\varphi \in S^{n-1}$. With respect to these coordinates, the metric on B^n is $dr^2+r^2d\theta^2$ and the metric on S^n is $d\rho^2+\left(\sin^2\varphi\right)d\varphi^2$. Clearly we may identify the θ and φ coordinates. Thus, we are looking for a solution of the form

$$u(r,\theta) = (\rho(r),\theta), \quad u \text{ harmonic}, \quad \rho(1) = \frac{\pi}{2}.$$

Note that

$$u^*\left(d\rho^2 + \sin^2\rho\, d\varphi^2\right) = \left(\rho'(r)\right)^2 dr^2 + \sin^2\rho(r)\, d\theta^2.$$

Choose an orthonormal frame $\theta_1,\ldots,\theta_{n-1}$ on S^{n-1} such that

$$d\theta^2 = \sum_{i=1}^{n-1} \theta_i^2, \quad \theta_n = dr.$$

With respect to this frame, the metric on B^n is

$$r^2\theta_1^2 + \ldots + r^2\theta_{n-1}^2 + \theta_n^2,$$

and the pull-back of the metric of S^n is

$$\sin^2\rho\,\theta_1^2 + \ldots + \sin^2\rho\,\theta_{n-1}^2 + \left(\rho'(r)\right)^2\theta_n^2.$$

Hence $|du|^2 = (\rho')^2 + (n-1)\frac{\sin^2\rho}{r^2}$ and

$$E(u) = \int_{S^{n-1}}\int_0^1 \left[\left(\rho'\right)^2 + (n-1)\frac{\sin^2\rho}{r^2}\right] r^{n-1}dr d\theta =$$
$$= Vol\left(S^{n-1}\right)\int_0^1 \left[\left(\rho'\right)^2 + (n-1)\frac{\sin^2\rho}{r^2}\right] r^{n-1}dr.$$

For any $\eta(r) \in C_c^\infty((0,1))$,

$$0 = \frac{d}{dt}E(\rho + t\eta)\bigg|_{t=0} = Vol\left(S^{n-1}\right)\int_0^1 \left[\rho'\eta' + 2(n-1)\sin\rho\,\cos\rho\cdot r^{-2}\eta\right] r^{n-1}dr$$
$$= Vol\left(S^{n-1}\right)\int_0^1 \eta\left[-2\left(r^{n-1}\rho'\right)' + (n-1)\sin 2\rho\cdot r^{-2}r^{n-1}\right]dr,$$

where we used integration by parts to obtain the first term. Hence the Euler-Lagrange equation becomes

$$-2\left(r^{n-1}\rho'\right)' + (n-1)\sin 2\rho\cdot r^{n-3} = 0,$$

or

$$\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{d\rho}{dr}\right) - \frac{n-1}{2}\frac{\sin 2\rho}{r^2} = 0.$$

Substituting $t = \log r$ and $\alpha = 2\rho$, we get the equation

$$\frac{d^2\alpha}{dt^2} + (n-2)\frac{d\alpha}{dt} - (n-1)\sin\alpha = 0,$$

where $t \in (-\infty, 0]$ and $\alpha(0) = \pi$ since $r \in [0,1]$ and $\rho(1) = \pi/2$. We assume $u(0)$ to be the north pole, or $\rho(0) = 0$. Then $\lim_{t\to-\infty} \alpha(t) = 0$. Thus the Dirichlet problem for rotationally symmetric maps reduces to the following ODE problem with boundary conditions:

$$\begin{cases} \frac{d^2\alpha}{dt^2} + (n-2)\frac{d\alpha}{dt} - (n-1)\sin\alpha = 0, \\ \alpha(0) = \pi, \quad \lim_{t\to-\infty} \alpha(t) = 0. \end{cases}$$

The solution to this problem depends on the dimension n, therefore, we continue with three separate cases.

1. Case $n = 2$. The desired solution is $\alpha(t) = 4\tan^{-1}(e^t)$. Hence $\rho(r) = 2\tan^{-1} r$. It can be seen that the map u is the inverse of stereographic projection. We see that we have existence and uniqueness in this case.

2. Case $3 \le n \le 6$. We have infinitely many solutions in this case. Which gives us an example of non-uniqueness of a solution to the Dirichlet problem.

3. Case $7 \le n$. We do not have a solution in this case. Which gives us an example of non-existence of a solution to the Dirichlet problem. Strictly speaking, only in the class of rotationally symmetric maps.

Let us finish this section with a counterexample to the existence of a solution to a homotopy problem for closed manifolds. It was noted in [13] that there is no harmonic map from a two-dimensional torus (T^2, g) to a two-dimensional sphere (S^2, h) of degree ± 1, whatever the metrics g, h.

3.2 Topological extension problem and DtN operator for maps to manifolds with non-positive sectional curvature

3.2.1 General topological extension problem

As we saw above, under some assumptions we can define the Dirichlet-to-Neumann map following the definition of the classical DtN operator. One of these assumptions is that there is an extension of the map from the boundary to the whole manifold. Note that there is an important difference when we define the classical DtN operator and the DtN map. Namely, since $\mathbb{R}$ is contractible every map $\partial N \to \mathbb{R}$ (real valued function) extends to a map $N \to \mathbb{R}$ and all such maps are homotopic to each other relatively ∂N, i.e. there is only one homotopy class of maps $N \to \mathbb{R}$ relative to ∂N. This is not true in general (and, in particular, in the case of non-positively curved target manifold), because not every map of the boundary $\partial N \to M$ can be extended to a map $N \to M$, and, in contrast, there can be more than one homotopy class of such extensions. Indeed, if we take N to be a disk and M to have non-trivial fundamental group, then the map of the boundary $\partial N = S^1 \to M$ representing a non-trivial loop clearly does not have an extension to the whole disk N. On the other hand, if we also take M to be S^2 and the map $\partial N = S^1 \to S^2 = M$ to be the standard inclusion of the equator, then this map can be extended as an inclusion of the upper or lower hemispheres, and these maps are clearly not homotopic. It means that we can define the DtN map not for any map of the boundary ∂N and we have to deal with the topological extension problem. In general it can be formulated as follows.

Topological extension problem. Let N and M be topological spaces and W be a subspace of N. Suppose we have a continuous map $\varphi : W \to M$. Then we can ask the following questions. When the map φ extends to a continuous map $\Phi : N \to M$, namely, when there is a map $\Phi : N \to M$, such that $\Phi|_W = \varphi$? If such extensions exist, then what are the homotopy classes of these extensions?

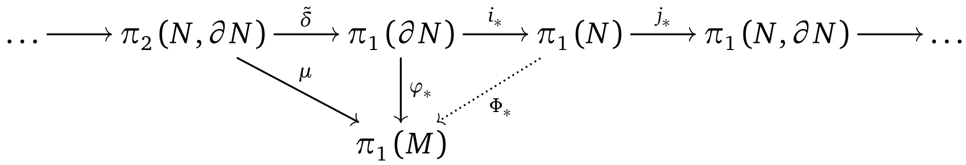
The topological extension problem is quite hard to deal with in general, but it becomes somehow easier if we consider the target manifolds M of the Eilenberg-MacLane type. We actually consider the case of target manifolds of non-positive sectional curvature, that are Eilenberg-MacLane spaces $K(\pi_1(M), 1)$. Indeed, from the well known Cartan–Hadamard theorem we conclude that such manifolds have contractible universal cover, which implies that they can only have the first homotopy group as a non-trivial one.

If we consider the case of the abelian fundamental group $\pi_1(M)$ then it is well known fact that the Eilenberg-MacLane spaces $K(\pi_1(M), 1)$ are classifying spaces for the first cohomology group with coefficients in $\pi_1(M)$, i.e. every element ϑ_f of the first cohomology group $H^1(N, \pi_1(M))$ of the space N is represented by the homotopy class $[f]$ of the maps $N \to K(\pi_1(M), 1)$. This fact allows us to restate the topological extension problem by means of the long exact sequence of the pair $(N, \partial N)$

$$\ldots \longrightarrow H^1(N, \partial N, \pi_1(M)) \xrightarrow{\ j^*\ } H^1(N, \pi_1(M)) \xrightarrow{\ i^*\ }$$

$$\xrightarrow{\ i^*\ } H^1(\partial N, \pi_1(M)) \xrightarrow{\ \delta\ } H^2(N, \partial N, \pi_1(M)) \longrightarrow \ldots$$

where the map i^* is induced by the natural inclusion $i : \partial N \hookrightarrow N$, the map j^* is induced by the natural inclusion $j : (N, \emptyset) \hookrightarrow (N, \partial N)$, and δ is the connecting map. We see that an element $[\varphi] \in H^1(\partial N, \pi_1(M))$ lifts to $H^1(N, \pi_1(M))$ (to give rise to an extension of φ to N) if and only if its image $\delta([\varphi]) = 0 \in H^2(N, \partial N, \pi_1(M))$. Moreover, this lifting is unique if and only if j^* is trivial. Although, the answer to the topological extension problem is now in computation of cohomology groups and maps between them, it is very hard to do this in any particular case. Moreover, the fundamental groups of the Cartan–Hadamard manifolds are usually highly non-abelian. Therefore, for the most part of the Cartan–Hadamard manifolds we do not have the long exact cohomology sequence described above. In contrast, we always have the long exact sequence



of homotopy groups of a pair. From this sequence we see that the homomorphism φ_* lifts to the homomorphism Φ_* only if the composition $\mu = \varphi_* \circ \tilde{\delta}$ is trivial, but it is not a sufficient condition for the lift.

3.2.2 Special case of topological extension problem

Our main result in this section concerns manifolds with simply connected boundary and is stated in the following proposition. The proof of this proposition is based on some technical results from algebraic topology. We provide these results and all the necessary definitions in Section A.1.

Proposition 3.2.1. *Let N be a compact connected manifold with non-empty boundary ∂N, and M be a manifold that is a $K(\Gamma, 1)$-space. Suppose that ∂N is simply connected. Then for any continuous map $\varphi : \partial N \to M$ and any homomorphism $h : \pi_1(N) \to \Gamma$*

there is a continuous extension $\tilde{\varphi} : N \to M$ *such that* $\tilde{\varphi}|_{\partial N} = \varphi$*,* $\tilde{\varphi}_*|_{\pi_1(N)} = h$*, and every two such extensions are homotopic relative to* ∂N*.*

Proof. Using Fact A.1.1 we take the associated finite CW structure on N so that $(N, \partial N)$ is a CW pair. This CW pair is homotopic to a CW pair with only one 0-cell, and hence, the 1-skeleton of the latter space can be viewed as the bouquet (wedge sum) of a finite number of circles S^1. Since ∂N is simply connected we may assume without loss of generality that the 1-skeleton of ∂N has no 1-cells (see [16, p. 58]). Now take the generators γ_i, $i \in I$ of the fundamental group of the 1-skeleton $N^1 = \bigvee_{i\in I} S^1$ such that the map $\gamma_i : S^1 \to \bigvee_{i\in I} S^1 = N^1$ is a natural inclusion of i-th circle. Note that $[\gamma_i] \in \pi_1(N)$. Let us send these generators to some representatives $\tilde{\gamma}_i$ in $\pi_1(M)$, such that $[\tilde{\gamma}_i] = h[\gamma_i]$. This defines $\tilde{\varphi}$ on the 1-skeleton N^1. The attaching maps $\delta^2_\alpha : D^2 \supset S^1 \to N^1$ for 2-cells are of the form $\delta^2_\alpha = \gamma^{\varepsilon_1}_{k_1} \cdot \ldots \cdot \gamma^{\varepsilon_n}_{k_n}$, where $\varepsilon_i = \pm 1$ and $k_j \in I$. The images of these maps are $\tilde{\delta}^2_\alpha = \varphi \circ \delta^2_\alpha = \tilde{\gamma}^{\varepsilon_1}_{k_1} \cdot \ldots \cdot \tilde{\gamma}^{\varepsilon_n}_{k_n}$. Since we attach 2-cells via boundary maps δ^2_α the classes $\left[\delta^2_\alpha\right] \in \pi_1\left(N^2\right) = \pi_1(N)$ (the last equality follows from the Cellular Approximation Theorem, see [16, p.52]) are trivial, and it follows that their images $\left[\tilde{\delta}^2_\alpha\right] \in \pi_1(M)$ under the homomorphism h are also trivial. Using Lemma A.1.4 we can define $\tilde{\varphi}$ on 2-skeleton N^2. Since all higher homotopy groups $\pi_i(M)$, $i \geq 2$ vanish the same procedure with the use of Lemma A.1.4 can be implemented to higher-dimensional cells, which allows to extend $\tilde{\varphi}$ to all of N by induction on l-skeletons N^l. Thus, we get the extension $\tilde{\varphi} : N \to M$ such that $\tilde{\varphi}_*|_{\pi_1(N)} = h$. Now it suffices to get $\tilde{\varphi}|_{\partial N} = \varphi$. It follows from Lemma A.1.3 for ∂N and Corollary A.1.1. If there is another extension $\tilde{\varphi}' : N \to M$ such that $\tilde{\varphi}'_*\big|_{\pi_1(N)} = h$, then it is homotopic to $\tilde{\varphi}$ on 1-skeleton N^1, by the construction of $\tilde{\varphi}$. Using Borsuk's Theorem for the CW pair $\left(N, N^1\right)$ we obtain a homotopy between $\tilde{\varphi}'$ and $\tilde{\varphi}$ on whole N. □

Example 3.2.1. The main example in this setting is a manifold N obtained by the excision of an n-dimensional open ball from an n-dimensional compact connected manifold X, where $n \geq 3$. The boundary of the obtained manifold will be S^{n-1}, and it is well-known fact that it is simply connected for $n \geq 3$.

If we restrict ourselves to the case of domain manifolds being compact surfaces we get an additional result.

3.2.3 Extension problem for maps of surfaces

As we already discussed, in general, the topological extension problem for maps from smooth manifolds with boundary to Eilenberg-MacLane spaces $K(\pi_1(M), 1)$ can be reformulated in terms of the group theory. Namely, the question of the classification of topological extensions can be reformulated as the question of the classification of group homomorphism extensions for the fundamental groups of the associated spaces. To get some examples and explicit solutions we will consider $N = \Omega$ to be a compact connected orientable surface with connected boundary $\partial\Omega \underset{\gamma}{\overset{\sim}{\longleftarrow}} S^1$, s.t. $i : \partial\Omega \hookrightarrow \Omega$ is a natural inclusion. Let $\tilde{\gamma} = i_*([\gamma]) \in \pi_1(\Omega, *)$.

We argue that even in this setting the topological extension problem cannot be simply solved. Let Ω be endowed with a normal structure of a CW complex, i.e. Ω is obtained by attaching the disk D^2 to the 1-skeleton of Ω

$$Sk_1(\Omega) = \bigvee_{i=1}^{g} S^1_{a_i} \bigvee_{i=1}^{g} S^1_{b_i} \bigvee S^1_{\gamma}$$

by a loop

$$l = \tilde{\gamma} b_g a_g b_g^{-1} a_g^{-1} \ldots b_1 a_1 b_1^{-1} a_1^{-1} \in \pi_1(\Omega, *),$$

where g is the genus of the surface Ω. The fundamental group $\pi_1(\Omega,*)$ of this surface then isomorphic to $\left\langle \tilde{\gamma}, a_1, b_1, \dots, a_g, b_g | \gamma b_g a_g b_g^{-1} a_g^{-1} \dots b_1 a_1 b_1^{-1} a_1^{-1} = 1 \right\rangle$. We see, that we can actually express γ in terms of other generators, since $l = 1$ we get $\tilde{\gamma} = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$. So we may assume that $\pi_1(\Omega,*)$ is isomorphic to a free group $F_{2g} = \left\langle a_1, b_1, \dots, a_g, b_g \right\rangle$ on $2g$ generators. It is actually true, and the best way to see this is probably to note that the surface Ω is homotopically retracts to its CW subcomplex $\bigvee\limits_{i=1}^{g} S^1_{a_i} \bigvee\limits_{i=1}^{g} S^1_{b_i}$, which has the needed fundamental group. It is more convenient to consider this bouquet instead of Ω. The useful feature of free groups is that every homomorphism from a free group is completely defined by the images of its generators. In this sense the topological extension problem reads as follows. Assume that we have a homomorphism of fundamental groups $\varphi_* : \pi_1(\partial\Omega,*) \to \pi_1(M,*)$. When does this homomorphism extend to a homomorphism $\Phi_* : \pi_1(\Omega,*) \to \pi_1(M,*)$? Suppose we have an extension Φ_* and let us look at the image of the generator $[\gamma] \in \pi_1(\partial\Omega,*) \simeq \langle [\gamma] \rangle$. We have

$$\Phi_*([\gamma]) = \varphi_* \circ i_*([\gamma]) = \varphi_*(\tilde{\gamma}) = \varphi_*\left([a_1,b_1][a_2,b_2]\dots\left[a_g,b_g\right]\right) = \\ = [\varphi_*(a_1),\varphi_*(b_1)]\dots\left[\varphi_*\left(a_g\right),\varphi_*\left(b_g\right)\right],$$

where $[x,y] = xyx^{-1}y^{-1}$ is the commutator of elements x and y. Since the fundamental group of Ω is free, the images $\varphi_*(a_k), \varphi_*(b_k)$, $k = 1,\dots,g$ can be arbitrary elements of $\pi_1(M,*)$. Hence the extension exists if and only if the element $\varphi_*(\tilde{\gamma})$ can be expressed as a product of $\leq g$ commutators in $\pi_1(M,*)$. This leads us to the question "what elements in $\pi_1(M,*)$ can be expressed as a product of $\leq g$ commutators?" In general, it is not clear how to answer this question. Thus, we will further specify our setting to get the following proposition.

Proposition 3.2.2. *Let Ω be a compact orientable surface of genus g with boundary $\partial\Omega \simeq S^1$ as above, and M be a closed manifold with the abelian fundamental group. Let $\varphi : \partial\Omega \to M$. Then φ extends to a map $\Phi : \Omega \to M$ if and only if $[\varphi]$ represents a trivial element of $\pi_1(M, *)$.*

Proof. We have seen above that the map φ extends to a map Φ if and only if the class $[\varphi]$ can be expressed as a product of $\leq g$ commutators in $\pi_1(M, *)$. Since the fundamental group $\pi_1(M, *)$ is abelian all commutators vanish. Thus, the map φ extends to a map Φ if and only if $[\varphi] = 0 \in \pi_1(M, *)$. □

Example 3.2.2. Let M in the above proposition be a 2-dimensional torus T^2. It is well known that the fundamental group of T^2 is the abelian group $\mathbb{Z} \oplus \mathbb{Z}$. Using the above proposition we see that a map from the boundary of a surface to a torus extends if and only if it is null-homotopic.

Example 3.2.3. Let M be a circle S^1. It is well known that the fundamental group of S^1 is the abelian group $\mathbb{Z}$. Then a map from the boundary $\partial\Omega$ extends if and only if it is null-homotopic.

If we consider the extension problem for maps from a general manifold N to a circle then there is an approach exploiting the connection to harmonic 1-fields.

3.2.4 Harmonic maps into S^1 and harmonic 1-fields

It is well known fact that there is 1-1 correspondence between the space $[N, S^1]$ of homotopy classes of maps from a topological space N to a circle S^1 and the integral cohomology $H^1(N)$ of N. Eells and Sampson in [14, Example (D), p.128] show that for the case of N being a manifold this correspondence can be established through the relation between harmonic 1-forms on N and harmonic maps to S^1. In this subsection we use this correspondence to find out when a map from the boundary ∂N to a circle S^1 extends to a map from the whole N. We start with a construction of the correspondence.

Fact 3.2.1 ([14, Example (D), p.128]). *Suppose N is connected. Fix a point $P_0 \in N$. Given any integral harmonic 1-field ω (i.e. closed and coclosed harmonic 1-form with integral periods) and any smooth path γ_P from P_0 to a point $P \in N$, we define the number*

$$f(P) = \int_{\gamma_P} \omega.$$

A different choice $\tilde{\gamma}_P$ of γ_P may give a different number $\tilde{f}(P)$, but

$$\tilde{f}(P) - f(P) = \int_{\tilde{\gamma}_P - \gamma_P} \omega$$

is an integer since the periods of ω are integral. Hence, ω determines a well defined map $f_\omega : N \to S^1$ by letting $f_\omega(P)$ be the residue class modulo 1 of $f(P)$.

Now since ω is harmonic, every $P \in N$ has a neighborhood U in which $df = \omega$. Thus $\Delta f = \delta d f + d\delta f = \delta\omega = 0$ in U, i.e. the map f_ω is harmonic. In the case of a manifold without boundary it is easy to see that $\omega \mapsto f_\omega$ establishes an isomorphism $[N, S^1] \cong H^1(N, \mathbb{Z})$.

Our main result in this subsection is the following theorem.

Theorem 3.2.3. *A map $\varphi : \partial N \to S^1$ extends to a map $\Phi : N \to S^1$ if and only if the* *corresponding cohomology class* $[\alpha_\varphi] \in H^1(\partial N, \mathbb{Z})$ *satisfies*

$$\int_{\partial N} \alpha_\varphi \wedge i^* \lambda = 0 \text{ for any } \lambda \in H^{n-2}(N), \tag{3.2}$$

where $i : \partial N \to N$ *is the natural inclusion of the boundary and* α_φ *is any representative of the cohomology class* $[\alpha_\varphi] \in H^1(\partial N, \mathbb{Z})$*.*

Moreover, if φ *is extendable then all its extensions are classified by the relative integral de Rham cohomology* $H^1_r(N, \mathbb{Z})$*.*

The relative integral de Rham cohomology is the lattice in the space of the relative de Rham cohomology consisting of elements with integral periods. Note that the condition (3.2) in the above theorem is really on a cohomology class $[\alpha_\varphi] \in H^1(\partial N, \mathbb{Z})$ since

$$\int_{\partial N} (\alpha_\varphi + d\beta) \wedge i^* \lambda = \int_{\partial N} \alpha_\varphi \wedge i^* \lambda + \int_{\partial N} d\beta \wedge i^* \lambda = 0 + \int_{\partial N} \beta \wedge i^*(d\lambda) = 0,$$

for every 0-cochain β, and it follows that this condition is topological. To prove the above theorem we shall use the results based on Hodge-Morrey and Friedrichs decompositions and analogues of the de Rham theorems for manifolds with boundary. All the additional technical results and definitions can be found in Section A.2.

Note that we have the following commutative diagram

$$\begin{array}{ccccc} H^1_r(N, \mathbb{Z}) & \xrightarrow{e} & H^1(N, \mathbb{Z}) & \xrightarrow{i^*} & H^1(\partial N, \mathbb{Z}), \\ & & \downarrow \cong & & \downarrow \cong \\ & & [N, S^1] & \xrightarrow{r} & [\partial N, S^1] \end{array}$$

where r is the restriction map and e is the natural inclusion of Dirichlet forms. This diagram allows us to obtain harmonic extensions of a map $\varphi : \partial N \to S^1$ in few steps. Firstly, using the Fact 3.2.1 we lift φ to a corresponding 1-form $\alpha_\varphi \in H^1(\partial N, \mathbb{Z})$. Secondly, using Proposition A.2.3 and Corollary A.2.8 we extend this form to an integral harmonic 1-field $\omega_\varphi \in H^1(N, \mathbb{Z})$ (note that by Proposition A.2.3 there are obstacles (A.13) to the extension, meaning that it does not always exist). Finally, using the Fact 3.2.1 again we define a harmonic map $\Phi = f_{\omega_\varphi} : N \to S^1$, which is by the construction clearly restricts to φ on the boundary. Note that by Corollary A.2.8 the extension ω_φ is defined up to an integral Dirichlet harmonic 1-field. In other words by Theorem A.2.6 all such extensions ω_φ and correspondingly all the extensions Φ are classified by the relative integral de Rham cohomology $H^1_r(N, \mathbb{Z})$.

Proof of Theorem 3.2.3. The part "if" follows from the above correspondence and Proposition A.2.3. The last part of the theorem follows from Corollary A.2.8. For the proof of the part "only if" suppose we have an extension Φ. Then we can represent it by the integral harmonic 1-field ω_Φ. From the exactness of ω_Φ and Green's formula we see that conditions (A.13) hold for the whole cohomology class $[i^*\omega_\Phi]$ and in particular for α_φ. □

3.3 Inverse problems of Calderón's type for the Dirichlet-to-Neumann operator on maps between manifolds

3.3.1 Calderón's problem for Dirichlet-to-Neumann map

Our aim is to generalise the Calderón problem to the setting of harmonic maps, but let us first discuss couple of low-dimensional examples of the DtN map.

Example 3.3.1. The lowest dimension we can take for N is 1. Under the assumption of connectedness of N we have that N should be a closed interval $[0,l]$, where l is the length of N. The boundary of N consists of two points 0 and l, and the map $u:\partial N \to M$ is basically equivalent to the choice of two (possibly non-distinct) points $u(0)$ and $u(l)$ in M. If M is complete, then for any two points $p_0, p_1 \in M$ there is a harmonic map $\overline{u}:[0,l]\to M$ such that $\overline{u}(0)=p_0$ and $\overline{u}(l)=p_1$, which is just a geodesic with a constant speed parametrisation joining p_0 and p_1. This means that we have

$$\left|\frac{\partial \overline{u}}{\partial t}\right| = v,$$

where $t \in [0,l]$ is the (normal) coordinate on N and $v \in \mathbb{R}$ is a constant (speed). Assuming that the DtN map is well defined (for two fixed endpoints and a homotopy class of paths joining them) we obtain

$$\Lambda[u](0) = d\overline{u}(v_0) = d\overline{u}\left(-\frac{\partial}{\partial t}\right)\bigg|_{t=0} = -\frac{\partial \overline{u}}{\partial t}\bigg|_{t=0} \in T_{u(0)}M,$$

and

$$\Lambda[u](l) = d\overline{u}(v_l) = d\overline{u}\left(\frac{\partial}{\partial t}\right)\bigg|_{t=l} = \frac{\partial \overline{u}}{\partial t}\bigg|_{t=l} \in T_{u(l)}M.$$

So the DtN map assigns to two points in M the outward speed at these points. In the classical case of functions, i.e. when $M=\mathbb{R}$ we have the equation

$$\tau(\overline{u}) = \frac{\partial^2}{\partial t^2}\overline{u} = 0.$$

Let $u(0)=a$ and $u(l)=b$. Then the harmonic extension of u is equal to

$$\overline{u}(t) = a\left(1-\frac{t}{l}\right) + b\frac{t}{l} = \frac{b-a}{l}t + a,$$

and the DtN operator acts as

$$\Lambda[u](l) = \left.\frac{\partial \overline{u}}{\partial t}\right|_{t=l} = \frac{b-a}{l} = -\Lambda[u](0).$$

Therefore, we can find the length of the interval via

$$l = \frac{b-a}{\Lambda[u](l)} = \frac{u(l)-u(0)}{\Lambda[u](l)} = \frac{a-b}{\Lambda[u](0)} = \frac{u(0)-u(l)}{\Lambda[u](0)}.$$

Let us mention briefly the geometric meaning of the DtN map in this case. Fix a point $p_0 \in M$. Let $r(p_0)$ be the injectivity radius at p_0 and pick $0 < \varepsilon < r(p_0)$. Suppose that $u(\varepsilon) = p_1$ runs over the points in the geodesic sphere in M of radius ε centered in p_0. Assume that the DtN map is well defined for $N = [0, \varepsilon]$ and harmonic extensions being homotopic to geodesics joining points $u(0) = p_0$ and $u(\varepsilon) = p_1$. Then the map

$$p_1 = u(\varepsilon) \to \Lambda[u](\varepsilon) = \left.\frac{\partial \overline{u}}{\partial t}\right|_{t=\varepsilon} \in T_{p_1}M$$

is a Gauss map on the considered geodesic sphere, i.e. it sends a point on a geodesic sphere to the outward unit normal vector at this point.

Example 3.3.2. Now we want to discuss the case of surfaces, i.e. the manifolds N of dimension 2. The energy functional is N-conformally invariant, i.e. it does not depend on metric in a fixed conformal class on N [38]. As a result, if there are two conformal metrics on N then the map $\overline{u}$ is harmonic with respect to one of them if and only if it is harmonic with respect to the other one. Similarly to what we saw in 2.2.6 because of this conformal invariance the DtN maps are related as

$$\Lambda_{e^{\mu}g} = e^{-\mu/2}\big|_{\partial N} \cdot \Lambda_g,$$

where g and $e^{\mu}g$ are two conformal metrics on N, the maps Λ_g and $\Lambda_{e^{\mu}g}$, respectively, are two DtN maps associated with these metrics. On can see that if the conformal factor is equal to 1 on the boundary, then the DtN maps are equal. Therefore, we can only try to recover from the DtN map the metric on N up to a conformal class. Our conjecture in this 2-dimensional case is that one can recover a conformal class of a metric on N from a given DtN map.

We want to consider the inverse geometric problem of Calderón's type which utilise the DtN map instead of the classical DtN operator. First of all we should ensure that the DtN operator is well defined. For this reason we need the following hypothesis.

Hypothesis 3.3.1 (H*)**.** The topological extension problem is solvable for the data

$$(N, \partial N, M, [u], [\bar{u}]),$$

where $[u] \in [\partial N, M]$ is a fixed homotopy class and a homotopy calss $[\bar{u}] \in [N, M]$ is its extension, i.e. $[\bar{u}]|_{\partial N} = [u]$.

If in addition to Hypothesis H* the Homotopy Dirichlet Problem has a unique solution for all smooth maps $\tilde{u} : N \to M$ in a homotopy class $[\bar{u}]$, then the Dirichlet-to-Neumann operator $\Lambda_{N,g,M,[u],[\bar{u}]}$ is well-defined. This is the case assuming N is compact and M is complete with non-positive sectional curvature, due to Theorem 3.1.3 and Theorem 3.1.4.

Definition 3.3.1. Let N_1 and N_2 be connected compact Riemannian manifolds with boundaries ∂N_1 and ∂N_2, respectively. Let $\psi : \partial N_1 \to \partial N_2$ be a diffeomorphism and M be a complete Riemannian manifold. Suppose that the Dirichlet-to-Neumann maps $\Lambda_1 := \Lambda_{N_1,g_1,[u_1],[\bar{u}_1]}$ and $\Lambda_2 := \Lambda_{N_2,g_2,[u_2],[\bar{u}_2]}$ are well-defined for $[u_1] = [\psi \circ u_2]$

and some $[\bar{u}_1], [\bar{u}_2]$. Then we say that the Dirichlet-to-Neumann maps Λ_1 and Λ_2 *intertwine* if

$$\phi_u \circ \Lambda_1 (u \circ \psi) = \Lambda_2 (u) \circ \psi,$$

for all smooth $u \in [u_2]$, where ϕ_u is defined by the following commutative diagram

$$\begin{array}{ccc} \psi^* (u^* TM) & \xrightarrow{\phi_u} & u^* TM \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi_2} \\ \partial N_1 & \xrightarrow{\quad \psi \quad} & \partial N_2 \end{array}$$

i.e. ϕ_u is the morphism from the pullback bundle $\psi^* (u^* TM) = (u \circ \psi)^* TM$ covering ψ. In other words, Λ_1 and Λ_2 intertwine if for any smooth $u \in [u_2]$ and any point $x \in \partial N_1$ we have

$$d\bar{u} \left(\nu_{\psi(x)} \right) = d\overline{u \circ \psi} (\nu_x),$$

where $\overline{u \circ \psi} \in C^\infty (N_1, M)$ is the unique harmonic extension of $u \circ \psi \in C^\infty (\partial N_1, M)$, $\nu_x \in T_x N_1$ is the outward unit normal vector to ∂N_1 at x, and $\nu_{\psi(x)} \in T_{\psi(x)} N_2$ is the outward unit normal vector to ∂N_2 at $\psi (x) \in \partial N_2$.

The inverse problem that we want to consider leads us to the following conjecture.

Conjecture 3.3.2 (Weak). *Let N_1 and N_2 be compact Riemannian manifolds with boundaries ∂N_1 and ∂N_2, respectively. Let M be a complete Riemannian manifold with non-positive sectional curvature. Let $\phi : \partial N_1 \to \partial N_2$ be a diffeomorphism. Assume that Hypothesis H* holds for data*

$$(N_1, \partial N_1, M, [u_1], [\bar{u}_1]) \text{ and } (N_2, \partial N_2, M, [u_2], [\bar{u}_2]),$$

with $[u_1] = [\phi \circ u_2]$ and some $[\bar{u}_1], [\bar{u}_2]$. If $\phi^* \circ \Lambda_{N_1, g_1, [u_1], [\bar{u}_1]} (u \circ \phi) = \Lambda_{N_2, g_2, [u_2], [\bar{u}_2]} (u)$

for all $u \in C^{\infty}(\partial N_2, M)$ *in a homotopy class* $[u_2] \in [\partial N_2, M]$*. Then there is an isometry* $\bar{\phi} : N_1 \to N_2$ *such that* $\bar{\phi}\big|_{\partial N_1} = \phi$*.*

This weak conjecture is a generalisation of the conjecture for the classical DtN operator. In other words, we are asking if it is possible to determine a Riemannian manifold from a given Dirichlet-to-Neumann map on its boundary. Since the DtN map defined on some homotopy class of extensions we may be able to also determine this homotopy class. Which leads us to the following strong conjecture.

Conjecture 3.3.3 (Strong)**.** *In addition to the statement of Conjecture 3.3.2 we have* $[\bar{u}_1] = [\bar{u}_2 \circ \bar{\phi}]$.

There is no equivalent of this conjecture in classical setting, because, as we mentioned before, all the extensions of (functions) maps to $\mathbb{R}$ are homotopic. In the previous chapter we obtained the uniqueness result for the linear Calderón's problem on vector bundles. Thus, in order to solve the generalised Calderón's problem for maps between manifolds it is natural to consider the linearisation of the DtN map and inverse problem for it. Therefore, we continue with the discussion of this linearisation.

3.3.2 Linearisation and Jacobi operator

Let u be a smooth mapping of N to M, and let v be a smooth vector field on M along u. There always exist $\varepsilon > 0$ and a smooth mapping U of the product $N \times (-\varepsilon, \varepsilon)$ to M such that the family $u_t(\cdot) = U(\cdot, t)$, $t \in (-\varepsilon, \varepsilon)$ has the following properties

$$u_0 = u, \; \frac{\partial u_t}{\partial t}\bigg|_{t=0} = v. \tag{3.3}$$

Let A be a pseudo-differential operator (on smooth mappings between manifolds). We define the linearisation of a pseudo-differential operator as follows.

Definition 3.3.4. The ∇-linearisation of an operator A at a map $u \in C^\infty(N, M)$ is the linear operator $A_*(u)$ on a vector bundle u^*TM, defined by the formula

$$A_*(u)v = \left(\nabla^*_{\frac{\partial}{\partial t}} Au_t\right)\Big|_{t=0},$$

where v is a section of the vector bundle u^*TM, a family of mappings u_t is defined using u and v in the conditions (3.3), and $\nabla^* \equiv \nabla^{U^*TM}$ is a connection on the bundle U^*TM induced by the connection ∇ on TM.

The linearisation has the following natural properties. If A is a differential operator, then A_* is also a differential operator, see [24]. Later, see Proposition 3.3.6, we will show that the linearisation of the DtN map gives a pseudodifferential operator the vector bundle u^*TM.

Note that for the connection on the pull-back bundle u^*TM we have the following useful formula [14]

$$\nabla_X^{u^*TM} du(Y) - \nabla_Y^{u^*TM} du(X) = du([X, Y]), \tag{3.4}$$

where X and Y are vector fields on N.

If we consider the Levi-Civita linearisation of the tension field operator we get the following result.

Proposition 3.3.5. *If ∇ is the Levi-Civita connection on M then ∇-linearisation $\tau_*(u)$ of the tension field operator τ at a map u is the Jacobi operator J_u.*

Proof. By the definition

$$\tau_*(u)v = \left(\nabla^*_{\frac{\partial}{\partial t}} \tau(u_t)\right)\Big|_{t=0} = \left(\nabla^*_{\frac{\partial}{\partial t}} Tr_g \nabla du_t\right)\Big|_{t=0}.$$

The covariant derivative $\nabla^*_{\frac{\partial}{\partial t}}$ commutes with Tr_g. Thus we have

$$\tau_*(u)v = \left(Tr_g \nabla^*_{\frac{\partial}{\partial t}} \nabla du_t\right)\Big|_{t=0} = Tr_g\left(\nabla^*_{\frac{\partial}{\partial t}} \nabla du_t\right)\Big|_{t=0}.$$

For a vector field $X \in \Gamma(TN)$ which is constant on t when seen as a vector field on $N \times (-\varepsilon, \varepsilon)$ we have $\nabla_X s(t) = \left(\nabla^*_X s\right)(t)$ for any $t \in (-\varepsilon, \varepsilon)$. By the definition of the Riemann curvature tensor R we have

$$\nabla^*_{\frac{\partial}{\partial t}} \nabla^*_X du_t(Y) = \nabla^*_X \nabla^*_{\frac{\partial}{\partial t}} du_t(Y) + R^*\left(X, \frac{\partial}{\partial t}\right) du_t(Y) - \nabla^*_{\left[X, \frac{\partial}{\partial t}\right]} du_t(Y),$$

and we see that the last term vanishes since $\left[X, \frac{\partial}{\partial t}\right] = 0$. Next, using (3.4) we see that

$$\nabla^*_{\frac{\partial}{\partial t}} du_t(Y) = \nabla^*_Y du_t\left(\frac{\partial}{\partial t}\right) + du_t\left(\left[\frac{\partial}{\partial t}, Y\right]\right),$$

where the last term vanishes since $Y \in \Gamma(TN)$ is constant on t when seen as a vector field on $N \times (-\varepsilon, \varepsilon)$. So we have

$$\tau_*(u)v = Tr_g\left(\left(\nabla^*_\cdot \nabla^*_\cdot du_t\left(\frac{\partial}{\partial t}\right) + R^*\left(\cdot, \frac{\partial}{\partial t}\right) du_t(\cdot)\right)\right)\Big|_{t=0} =$$
$$= Tr_g\left(\nabla_\cdot \nabla_\cdot v + R^M(du(\cdot), v)\, du(\cdot)\right) = \Delta^u v - Tr_g R^M(du(\cdot), v)\, du(\cdot) = J_u v.$$

Thus, the proposition is proved. □

Using the above proposition we obtain the following result for the Levi-Civita linearisation of the DtN operator.

Proposition 3.3.6. *If ∇ is the Levi-Civita connection on M then ∇-linearisation of the Dirichlet-to-Neumann operator Λ at a map u is the Dirichlet-to-Neumann operator $DtN(\tau_*(\bar{u}))$ associated with the linearisation $\tau_*(\bar{u})$, i.e.*

$$\Lambda_*(u) = (DtN(\tau))_*(u) = DtN(\tau_*(\bar{u})) = DtN(J_{\bar{u}}),$$

where $\bar{u} : N \to M$ is a harmonic extension of a map $u : \partial N \to M$, and $DtN(A)$ denotes the DtN map associated with an operator A.

Proof. We have

$$\Lambda_*(u)v = (DtN(\tau))_*(u)v = \left(\nabla^*_{\frac{\partial}{\partial t}} DtN(\tau)u_t\right)\Big|_{t=0} = \\ = \left(\nabla^*_{\frac{\partial}{\partial t}} \Lambda u_t\right)\Big|_{t=0} = \left(\nabla^*_{\frac{\partial}{\partial t}} d\bar{u}_t\left(\frac{\partial}{\partial \nu}\right)\right)\Big|_{t=0}$$

and using (3.4) we obtain

$$\left(\nabla^*_{\frac{\partial}{\partial n}} d\bar{u}_t\left(\frac{\partial}{\partial t}\right)\right)\Big|_{t=0} = \left(\nabla^*_{\frac{\partial}{\partial n}} \frac{\partial}{\partial t}\bar{u}_t\right)\Big|_{t=0} = \\ = \nabla_{\frac{\partial}{\partial n}} \bar{v} = DtN(J_{\overline{u}})v = DtN(\tau_*(\overline{u}))v,$$

which proves the proposition. □

We see that the linearisation of the DtN operator on harmonic maps gives us the DtN operator associated to the Dirichlet problem for the Jacobi operator. Now if we omit the term with the Riemannian tensor in the definition of the Jacobi operator we get the inverse problem for the connection Laplacian. Thus it is natural to consider the latter problem first.

The Jacobi operator

$$J_{\overline{u}}v = \Delta^{\overline{u}}v - Tr_g R^M\left(d\overline{u}\left(\cdot\right), v\right) d\overline{u}\left(\cdot\right)$$

is the connection Laplacian with a potential P acting on sections of the pull-back vector bundle $\overline{u}^* TM$, where

$$P\left(v\right) = -Tr_g R^M\left(d\overline{u}\left(\cdot\right), v\right) d\overline{u}\left(\cdot\right).$$

When the Jacobi operator is considered acting on smooth sections vanishing at the boundary we call it the *Dirichlet Jacobi operator.*

Proposition 3.3.7. *The potential P is symmetric. In addition, if the target manifold M has non-positive sectional curvature at the points of the image $\overline{u}\left(N\right)$, then the Dirichlet Jacobi operator $J_{\overline{u}}$ is positive. In particular,* 0 *is not in the Dirichlet spectrum of $J_{\overline{u}}$.*

Proof. The symmetry of P follows from the symmetry of the Riemann tensor. Namely, for any two sections $v, w \in \Gamma\left(\overline{u}^* TM\right)$ we have

$$\begin{aligned}\langle P\left(v\right), w\rangle_{\overline{u}^*TM} &= \left\langle -Tr_g R^M\left(d\overline{u}\left(\cdot\right), v\right) d\overline{u}\left(\cdot\right), w\right\rangle_{\overline{u}^*TM} = \\ &= -Tr_g \left\langle R^M\left(d\overline{u}\left(\cdot\right), v\right) d\overline{u}\left(\cdot\right), w\right\rangle_{\overline{u}^*TM} = -Tr_g \left\langle R^M\left(d\overline{u}\left(\cdot\right), v\right) d\overline{u}\left(\cdot\right), w\right\rangle_{\overline{u}^*TM} = \\ &= -Tr_g \left\langle R^M\left(d\overline{u}\left(\cdot\right), w\right) d\overline{u}\left(\cdot\right), v\right\rangle_{\overline{u}^*TM} = \langle P\left(w\right), v\rangle_{\overline{u}^*TM},\end{aligned}$$

where the penultimate equality is due to the symmetry of the Riemann tensor. For the proof of the second part we consider the expression

$$\begin{aligned} \langle J_{\overline{u}}v, v\rangle_{\overline{u}^*TM} &= \left\langle \Delta^{\overline{u}}v - Tr_g R^M\left(d\overline{u}(\cdot), v\right)d\overline{u}(\cdot), v\right\rangle_{\overline{u}^*TM} = \\ &= \left\langle \Delta^{\overline{u}}v, v\right\rangle_{\overline{u}^*TM} - \left\langle Tr_g R^M\left(d\overline{u}(\cdot), v\right)d\overline{u}(\cdot), v\right\rangle_{\overline{u}^*TM}, \end{aligned} \tag{3.5}$$

for any non-zero section $v \in \mathscr{D}\left(\overline{u}^*TM\right)$. The first term of (3.5) is positive by (1.10) and (1.14). Using the symmetry of the Riemann tensor again we see that the second term of (3.5) is equal to

$$-Tr_g\left\langle R^M\left(d\overline{u}(\cdot), v\right)d\overline{u}(\cdot), v\right\rangle_{\overline{u}^*TM} = Tr_g\left\langle R^M\left(v, d\overline{u}(\cdot)\right)d\overline{u}(\cdot), v\right\rangle_{\overline{u}^*TM}.$$

This expression is non-negative due to the non-negativity of the sectional curvature along the image $\overline{u}(N)$. Hence, the Dirichlet Jacobi operator is positive as a sum of positive and non-negative terms. □

Taking this into account we see that the Jacobi operator $J_{\overline{u}}$ is of the Schrödinger type considered in the previous chapter.

Let us prove the proposition concerning the linearisations of intertwining operators.

Proposition 3.3.8. *Let N_1 and N_2 be connected compact Riemannian manifolds with boundaries ∂N_1 and ∂N_2, respectively. Let $\psi : \partial N_1 \to \partial N_2$ be a diffeomorphism and M be a complete Riemannian manifold. Suppose that the Dirichlet-to-Neumann maps $\Lambda_1 := \Lambda_{N_1,g_1,[u_1],[\bar{u}_1]}$ and $\Lambda_2 := \Lambda_{N_2,g_2,[u_2],[\bar{u}_2]}$ are well-defined for $u_1 = u_2 \circ \psi$ and some $[\bar{u}_1], [\bar{u}_2]$. Let ∇ be the Levi-Civita connection on M and L_1 and L_2 be ∇-linearisations of the DtN maps Λ_1 and Λ_2 at maps $u_1 = u_2 \circ \psi$ and u_2, respectively. Suppose Λ_1 and Λ_2*

intertwine. Then L_1 *and* L_2 *also intertwine, that is for any* $v \in \Gamma\left(u_2^* TM\right)$ *we have*

$$L_1\left(\phi^{-1} \circ v \circ \psi\right) = \phi^{-1} \circ L_2(v) \circ \psi,$$

where the bundle morphism $\phi : u_1^* TM = \psi^*\left(u_2^* TM\right) \to u_2^* TM$ *covers* ψ.

Proof. Let v be a section of $u_2^* TM$ and u_t be defined by (3.3) with $u_0 = u_2$ and

$$\left.\frac{\partial u_t}{\partial t}\right|_{t=0} = du_t\left(\frac{\partial}{\partial t}\right)\bigg|_{t=0} = h_2 \circ v,$$

where the bundle morphism h_2 covers u_2, i.e. the following diagram is commutative

$$\begin{array}{ccc} u_2^* TM & \xrightarrow{h_2} & TM \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi} \\ \partial N_2 & \xrightarrow{u_2} & M \end{array}$$

Let $\tilde{u}_t(x) = u_t(\psi(x))$. Then $\tilde{u}_0(x) = u_0(\psi(x)) = u_2(\psi(x)) = u_1(x)$ and

$$\left.\frac{\partial \tilde{u}_t(x)}{\partial t}\right|_{t=0} = d\tilde{u}_t(x)\left(\frac{\partial}{\partial t}\right)\bigg|_{t=0} = d\left(u_t(\psi(x))\right)\left(\frac{\partial}{\partial t}\right)\bigg|_{t=0} =$$
$$= du_t(\psi(x))\left(\frac{\partial}{\partial t}\right)\bigg|_{t=0} = h_2 \circ v(\psi(x)),$$

which gives

$$\left.\frac{\partial \tilde{u}_t}{\partial t}\right|_{t=0} = h_2 \circ v \circ \psi.$$

Note that since $u_2 \circ \psi = u_1$ we have $h_2 = h_1 \circ \phi^{-1}$, where the bundle morphism h_1

covers u_1, i.e. the following diagram is commutative

$$\begin{array}{ccc} u_1^* TM & \xrightarrow{h_1} & TM \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ \partial N_1 & \xrightarrow{u_1} & M \end{array}$$

Therefore, we have

$$\left.\frac{\partial \tilde{u}_t}{\partial t}\right|_{t=0} = h_1 \circ \phi^{-1} \circ v \circ \psi.$$

Now, by the definition of linearisation, we have

$$\phi^{-1} \circ L_2(v) \circ \psi = \phi^{-1} \circ \left.\left(\nabla^*_{\frac{\partial}{\partial t}} \Lambda_2(u_t)\right)\right|_{t=0} \circ \psi, \tag{3.6}$$

and

$$L_1\left(\phi^{-1} \circ v \circ \psi\right) = \left.\left(\nabla^*_{\frac{\partial}{\partial t}} \Lambda_1(\tilde{u}_t)\right)\right|_{t=0}$$

Since Λ_1 and Λ_2 intertwine for any smooth $u \in [u_2]$ we have

$$\Lambda_2(u) = \phi_u \circ \Lambda_1(u \circ \psi) \circ \psi^{-1}.$$

Substituting this into (3.6) we obtain

$$\begin{aligned} L_2(v) &= \left.\left(\nabla^*_{\frac{\partial}{\partial t}} \phi_u \circ \Lambda_1(u_t \circ \psi) \circ \psi^{-1}\right)\right|_{t=0} = \left.\left(\nabla^*_{\frac{\partial}{\partial t}} \phi_{u_t} \circ \Lambda_1(\tilde{u}_t) \circ \psi^{-1}\right)\right|_{t=0} = \\ &= \left.\left(\nabla^{U^*TM}_{\frac{\partial}{\partial t}} \phi_U \circ \Lambda_1(\tilde{u}_t) \circ \psi^{-1}\right)\right|_{t=0} = \left.\phi_U \circ \left(\nabla^{\tilde{U}^*TM}_{\frac{\partial}{\partial t}} \Lambda_1(\tilde{u}_t)\right) \circ \psi^{-1}\right|_{t=0} = \\ &= \phi_{u_0} \circ \left(\left.\nabla^{\tilde{U}^*TM}_{\frac{\partial}{\partial t}} \Lambda_1(\tilde{u}_t)\right|_{t=0}\right) \circ \psi^{-1} = \phi \circ L_1\left(\phi^{-1} \circ v \circ \psi\right) \circ \psi^{-1}, \end{aligned}$$

which is equivalent to the equality

$$\phi^{-1} \circ L_2(v) \circ \psi = L_1\left(\phi^{-1} \circ v \circ \psi\right)$$

proving the proposition. □

3.3.3 Main result

In this section we state and prove a uniqueness result for Calderón's problem for harmonic maps between real-analytic manifolds. The following theorem is an analogue of Theorem 2.3.1, the uniqueness result for Calderón's problem for the connection Laplacian.

Theorem 3.3.9. *Let N_1 and N_2 be real-analytic compact Riemannian manifolds with real-analytic boundaries ∂N_1 and ∂N_2, respectively. Let M be a complete real-analytic Riemannian manifold with non-positive sectional curvature. Let $\psi : \partial N_1 \to \partial N_2$ be a real-analytic diffeomorphism. Assume that Hypothesis H* holds for the data*

$$(N_1, \partial N_1, M, [u_1], [\bar{u}_1]) \text{ and } (N_2, \partial N_2, M, [u_2], [\bar{u}_2]),$$

with $[u_1] = [\psi \circ u_2]$ and some $[\bar{u}_1], [\bar{u}_2]$. Suppose that $\dim N_i \geq 3$, for each $i = 1, 2$, and ψ intertwines with the corresponding Dirichlet-to-Neumann operators. Then there is a real-analytic isometry $\bar{\psi} : N_1 \to N_2$ such that $\bar{\psi}\big|_{\partial N_1} = \psi$.

Proof. Let us fix a map $u_2 : \partial N_2 \to M$ and denote by L_1 and L_2 the ∇-linearisations of $\Lambda_1 := \Lambda_{N_1, g_1, [u_1], [\bar{u}_1]}$ and $\Lambda_2 := \Lambda_{N_2, g_2, [u_2], [\bar{u}_2]}$ at the maps $u_1 := \psi \circ u_2$ and u_2, respectively. Since ψ intertwines with DtN operators Λ_1 and Λ_2 we have by Proposition 3.3.8 that the bundle isomorphism $\phi : u_1^* TM = \psi^* \left(u_2^* TM\right) \to u_2^* TM$ covering ψ intertwines with the linearisations L_1 and L_2. As usual we will be working in boundary normal coordinates and boundary normal frame. Using Theorem 2.2.6 and Proposition 2.2.8 we conclude that isomorphism ϕ is a gauge equivalence, the diffeomorphism ψ is an isometry, and the first normal derivatives of the metrics g_1 and g_2 coincide. Now let us take a closer look at a potential. As we saw above the potential for the ∇-linearisation L of the DtN map $\Lambda_{N,g,[u],[\bar{u}]}$ at a map $u : \partial N \to M$ is given by

$$P = -Tr_g R^M(d\bar{u}, \cdot) d\bar{u},$$

where $\bar{u}$ is a unique harmonic extension of u, R^M is the Riemannian tensor on M. Note that the harmonic extension $\bar{u}$ solves the elliptic quasilinear equation

$$\tau(\bar{u}) = \Delta \bar{u}^i + g^{\alpha\beta} \Gamma^i_{jl} \frac{\partial \bar{u}^i}{\partial x^\alpha} \frac{\partial \bar{u}^j}{\partial x^\beta} = 0,$$

and by the standard theory [e.g. 36] it is analytic. In local coordinates the potential is given by

$$P^\alpha_\beta(x) = -g^{kl}(x) R^\alpha_{\delta\gamma\beta}(\bar{u}(x))(d\bar{u}(x))^\gamma_k (d\bar{u}(x))^\delta_l = g^{kl}(x) F^\alpha_{lk\beta}(\bar{u}(x)),$$

where $F^\alpha_{lk\beta}(\bar{u}(x)) := -R^\alpha_{\delta\gamma\beta}(\bar{u}(x))(d\bar{u}(x))^\gamma_k (d\bar{u}(x))^\delta_l$ depends only on Riemannian tensor on M and the map $\bar{u}$. One can see that the potential is analytic as a combination of analytic functions. Note that for a given map u there is a unique harmonic extension $\bar{u}$, so we can assume that the map $\bar{u}$ and therefore the functions $F^\alpha_{lk\beta}(\bar{u}(x))$ are known on the whole manifold N. Therefore, using Remark 2.2.7 we see that the full Taylor series of P and g at the boundary are given in terms of the full symbol of the linearisation L. The rest of the proof is similar to the proof of the main result in

Chapter 2 (Theorem 2.3.1). Let us discuss each step of the latter and mention the required modifications in order to generalise it to the present setting. As was mentioned in Remark 2.2.9, we can apply the Schrödinger version of Proposition 2.2.8 in our case. Namely, we have that the bundle isomorphism $\phi : u_1^*TM \to u_2^*TM$ is a gauge equivalence which covers an isometry $\psi : \partial N_1 \to \partial N_2$, and the potentials are related via

$$P_1(x) = \phi^{-1} \circ P_2(\psi(x)) \circ \phi, \tag{3.7}$$

for any $x \in \partial N_1$, i.e. P_1 is a natural pull-back of P_2 along ϕ. Now, the real-analytic manifolds N_i can be extended to larger real-analytic manifolds $\tilde{N}_i$, and all the geometric structures including the potentials can be extended to real-analytic ones over $\tilde{N}_i$, $i = 1, 2$. Due to Proposition 1.3.12 there is a unique real-analytic Dirichlet Green kernel $\tilde{G}$ on $\tilde{E}$ associated with the operator L_P. Hence, we can still define the map $\mathscr{G} : \tilde{E} \to \mathscr{W}^\ell(\mathscr{E})$ by (2.26). This map has the same properties as before, i.e. the Lemmas 2.3.2, 2.3.3, and 2.3.4 continue to hold as was mentioned in Remark 2.3.5. Now, due to Theorem 2.2.6 and Remark 2.2.7 the setting in 2.3.4 generalises to the present case with an addition that the potentials coincide in $\mathscr{E}$, i.e. the relation (3.7) extends to the relation

$$P_1(x) = \Phi^{-1} \circ P_2(\Psi(x)) \circ \Phi,$$

for any $x \in W_1$. Finally, the result follows from Theorem 2.3.1 and Remark 2.3.12 by taking $\overline{\psi} = \Psi$. □

Appendix A

Appendix

A.1 Topological facts

In this subsection we present all the necessary definitions and technical details from algebraic topology. We mainly refer to [16].

Definition. A *triangulation* of a topological space X is a simplicial complex K, homeomorphic to X, together with a homeomorphism $\kappa : K \to X$.

We will use the same notation X for a simplicial complex representing a triangulation of a topological space X. Throughout this section B^n and S^n denote the standard n-dimensional Euclidean unit (closed) ball and sphere, respectively.

Definition. A *CW complex* is a Hausdorff space X with a fixed partition $X = \bigcup_{q=0}^{\infty} \bigcup_{i \in I_q} e_i^q$ of X into pairwise disjoint set (*cells*) e_i^q such that for every cell e_i^q there exists a continuous map $f_i^q : B^q \to X$ (a characteristic map of the cell e_i^q) whose restriction to $\operatorname{Int} B^q$ is a homeomorphism $\operatorname{Int} B^q \approx e_i^q$ whose restriction to $S^{q-1} = B^q - \operatorname{Int} B^q$ maps S^{q-1} into the union of cells of dimensions $< q$ (the dimension of the cell e_i^q, $\dim e_i^q$ is, by definition,

q). The following two axioms assumed satisfied.

(C) The boundary $\dot{e}_i^q = \overline{e}_i^q - e_i^q = f_i^q\left(S^{q-1}\right)$ is contained in a finite union of cells.

(W) A set $F \subset X$ is closed if and only if for any cell e_i^q the intersection $F \cap \overline{e}_i^q$ is closed (in other words, $\left(f_i^q\right)^{-1}(F)$ is closed in B^q).

A *CW subcomplex* of a CW complex X is a closed subset composed of whole cells. It is clear that a CW subcomplex of a CW complex is a CW complex. An example of CW subcomplex of a CW complex X is the *n-skeleton* X^n or $sk_n X$ which is the union of all cells e_i^q with $q \leq n$.

Definition. A pair of topological spaces (X, Y) is called a CW pair if X is a CW complex and Y is its CW subcomplex.

We will be using sometimes the term *vertices* instead of 0-cells of a CW complex.

Definition. Let (X, Y) be a pair of spaces. Two maps $X \to Z$ are called homotopic relative to Y (or relatively Y) if there exists homotopy between these maps which is constant on the subspace Y.

Fact A.1.1. *Every smooth (compact) manifold admits a (finite) triangulation. Moreover, if a compact manifold has a boundary then it admits a compatible finite triangulation, i.e. such that the restriction of this triangulation to the boundary is a triangulation of the boundary.*

This is a classical result which can be found in [44]. Note that a triangulated manifold has a natural structure of a CW complex, and if the manifold has a boundary then it forms a CW pair with its boundary.

Definition. A pair (X,A) of topological spaces is called a *Borsuk pair* if for every topological space Y, every continuous map $F : X \to Y$, and every homotopy $f_t : A \to Y$ such that $f_0 = F|_A$, there exists a homotopy $F_t : X \to Y$ such that $F_0 = F$ and $F_t|_A = f_t$.

Theorem (Borsuk)**.** *Every CW pair is a Borsuk pair.*

For the proof of this theorem we refer to [16].

Corollary A.1.1. *Let (N,W) be a CW pair. Let $\varphi : W \to M$ and $\tilde{\varphi} : W \to M$ be homotopic. If the map φ has an extension $\Phi : N \to M$ then the map $\tilde{\varphi}$ also has an extension $\tilde{\Phi} : N \to M$ which is homotopic to Φ.*

Proof. Follows from the direct implication of Borsuk's Theorem. □

Definition. A topological space X is called n-connected if for $q \leq n$ the set $[S^q, X]$ of homotopy classes of maps from S^q to X consists of one element (that is, any two continuous maps $S^q \to X$ with $q \leq n$ are homotopic).

A 1-connected CW complex is also said to be *simply connected*.

Theorem A.1.2. *[16, p.58]Let n be a non-negative integer. An n-connected CW complex is homotopy equivalent to a CW complex which has one 0-cell and no cells of dimensions $1, 2, \dots, n$. In particular, every path connected CW complex is homotopy equivalent to a CW complex with only one vertex.*

Theorem. *[16, p. 138] Let n be a positive integer, and let Γ be a group which is supposed to be commutative if $n > 1$. Then there exists a CW complex Y such that*

$$\pi_q(Y) = \begin{cases} \Gamma, & \text{if } q = n, \\ 0, & \text{if } q \neq n, \end{cases}$$

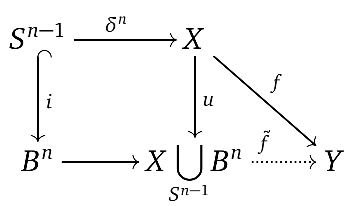
where $\pi_q(Y)$ *denotes qth homotopy group of* Y.

The spaces in the above theorem are called *Eilenberg-MacLane spaces* or $K(\pi, n)$-*spaces*.

Lemma A.1.3. *Let* X *be a simply connected CW complex. Then any map from* X *to a* $K(\Gamma, 1)$*-space is null-homotopic.*

Proof. Using Theorem A.1.2 we get a CW complex $\tilde{X}$ which is homotopy equivalent to X and has only one vertex as its 1-skeleton. Let us prove that any map $f : \tilde{X} \to K(\Gamma, 1)$ is homotopic to a trivial map. We use the induction by the dimension of skeletons of $\tilde{X}$. Clearly, the map f is trivial on the 1-skeleton. Now it suffices to prove that if the map f is homotopic to a trivial map on the n-skeleton $\tilde{X}^n$, then it is homotopic to a trivial map on the $(n+1)$-skeleton $\tilde{X}^{n+1}$, where $n \geq 1$. The CW pair $(\tilde{X}^{n+1}, \tilde{X}^n)$ is a Borsuk pair by Borsuk's Theorem. Hence, there is a homotopy of $f|_{\tilde{X}^{n+1}}$ to map $g_0^{n+1} : \tilde{X}^{n+1} \to K(\Gamma, 1)$ such that the restriction of g_0^{n+1} to $\tilde{X}^n$ is a constant map. This means that g_0^{n+1} factorises through a map to a bouquet $\bigvee_{i \in I} S^{n+1} = \tilde{X}^{n+1}/\tilde{X}^n$. Every map from a bouquet $\bigvee_{i \in I} S^{n+1}$ to $K(\Gamma, 1)$ represents an element of $\pi_{n+1}(K(\Gamma, 1)) = 0$, $n \geq 1$, and thus it is null-homotopic. This gives us a homotopy of g_0^{n+1} with a constant map. Using induction we conclude that the set $\left[\tilde{X}, K(\Gamma, 1)\right] \simeq [X, K(\Gamma, 1)]$ has only one element - homotopy class of a constant map. □

Lemma A.1.4. *Suppose we have the following diagram*



,

where X and Y are topological spaces (for example X can be an $(n-1)$-skeleton of a CW complex with some n-cells attached), δ^n is an attaching map of an n-cell B^n to the space X, i is the standard inclusion of the boundary. Then the map f extends to a map $\tilde{f}$ if and only if the composition $f \circ \delta^n$ is null-homotopic.

Proof. The part "only if" follows from the induced map on homotopy groups. Namely, the map $u \circ \delta^n$ is null-homotopic, since it shrinks to the centre of the corresponding n-cell B^n. Thus, the homotopy class $[u \circ \delta^n] \in \pi_{n-1}(X \bigcup_{S^{n-1}} B^n)$ is trivial. Suppose we have an extension $\tilde{f}$. Then by the conditions of the Lemma we have $\tilde{f} \circ u = f$ and $f \circ \delta^n = \tilde{f} \circ u \circ \delta^n$, which gives us the required result $[f \circ \delta^n] = \left[\tilde{f} \circ u \circ \delta^n\right] = \tilde{f}_*([u \circ \delta^n]) = \tilde{f}_*(0) = 0 \in \pi_{n-1}(Y)$.

The part "if" follows from the direct construction. Let $h : S^{n-1} \times [0,1] \to Y$ be a null-homotopy of the map $f \circ \delta^n$, i.e. $h(x,1) = f \circ \delta^n$ and $h(x,0) = const$. Define an extension $\tilde{f}$ as follows. Let the map $\tilde{f}$ coincides with f on X, and on B^n let it be defined in a spherical coordinates by the equality $\tilde{f}(x,r) = h(x,r)$. Clearly, the map $\tilde{f}$ is a well-defined extension of the map f. □

Theorem A.1.5 (Poincaré). *For an arbitrary path connected space X, the Hurewicz homomorphism [16, p. 179] $h : \pi_1(X) \to H_1(X)$ is an epimorphism whose kernel is the commutator subgroup $[\pi_1(X), \pi_1(X)]$ of the group $\pi_1(X)$. Thus,*

$$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)],$$

or in other words $H_1(X) \cong (\pi_1(X))_{ab}$, where $(\pi_1(X))_{ab}$ is the abelianisation of the group $\pi_1(X)$.

A.2 Harmonic forms on compact manifolds with boundary

In this subsection we present all the necessary definitions and technical details on harmonic forms (fields).

Let $\Omega^k(N)$ be the space of k-forms on a Riemannian manifold N. Using a Riemannian metric on a manifold N we can split a vector field $X \in \Gamma(TN|_{\partial N})$ on the boundary ∂N into its tangential and normal parts $X = X^{\parallel} + X^{\perp}$. With this in mind we define the operation $\mathbf{t}$ on $X \in \Gamma(\Omega^k N\big|_{\partial N})$

$$\mathbf{t}\omega(X_1,\ldots,X_k) = \omega(X_1^{\parallel},\ldots,X_k^{\parallel}),\ \forall X_1,\ldots,X_k \in \Gamma(TN|_{\partial N}),$$

and operation $\mathbf{n}$ by

$$\mathbf{n}\omega = \omega|_{\partial N} - \mathbf{t}\omega,$$

for $k \geq 1$, and $\mathbf{t}\omega = \omega$ for $k = 0$. These forms are called the *tangential component* and the *normal component*, respectively. The tangential component $\mathbf{t}\omega$ is uniquely determined by the pull-back $i^*\omega$ under the inclusion $i : \partial N \to N$. This gives the relation

$$i^*\omega = i^*\mathbf{t}\omega = \mathbf{t}\omega. \tag{A.1}$$

The tangential and normal components of a differential form have the following useful commutation relations.

Proposition A.2.1. *[39, p.27]*

1. *The normal and tangential components are Hodge adjoint to each other*

$$*(\mathbf{n}\omega) = \mathbf{t}(*\omega) \text{ and } *(\mathbf{t}\omega) = \mathbf{n}(*\omega). \tag{A.2}$$

   Here $*(\mathbf{n}\omega)$ and $*(\mathbf{t}\omega)$ are understood by the action of $*$ on an arbitrary extension of $(\mathbf{n}\omega)$ and $(\mathbf{t}\omega)$, respectively, followed by the restriction to ∂N.

2. *The exterior derivative commutes with the tangential projection $\mathbf{t}$, and the co-differential with the normal projection $\mathbf{n}$ of $\omega \in \Omega^k(N)$ in the following sense*

$$i^*(\mathbf{t}(d\omega)) = d(i^*\mathbf{t}\omega) \text{ and } i^*(*(\mathbf{n}\delta\omega)) = (-1)^{(k+1)(n-k+1)}d(i^*(*\mathbf{n}\omega)). \tag{A.3}$$

   Under the identification (A.1) these relations become

$$\mathbf{t}(d\omega) = d(\mathbf{t}\omega) \text{ and } \mathbf{n}(\delta\omega) = \delta(\mathbf{n}\omega). \tag{A.4}$$

3. *For differential forms $\omega \in \Omega^k(N)$ and $\eta \in \Omega^{k+1}(N)$ let $\chi = \mathbf{t}\omega \wedge *\mathbf{n}\eta$. Then*

$$\chi = \langle \omega, \iota_\nu \eta \rangle_{\Lambda^k} \mu_\partial, \tag{A.5}$$

   where $\mu_\partial \in \Omega^{n-1}(\partial N)$ is the Riemannian volume form on ∂N. Note that

$$\mu_\partial = \iota_\nu \mu|_{\partial N},$$

   where $\mu \in \Omega^n(N)$ is the Riemannian volume form on N.

We use the following notations for a number of different spaces.

Definition. $\Omega^k_{\mathscr{D}}(N) = \{\omega \in \Omega^k(N) | \, \mathbf{t}\omega = 0\}$ is the space of Dirichlet forms.

$\Omega^k_{\mathscr{N}}(N) = \{\omega \in \Omega^k(N) | \, \mathbf{n}\omega = 0\}$ is the space of Neumann forms.

$\mathscr{H}^k(N) = \{\lambda \in \Omega^k(N) | \, d\lambda = 0 \, and \, \delta\lambda = 0\}$ is the space of harmonic fields.

$\mathscr{H}^k_{\mathscr{D}}(N) = \Omega^k_{\mathscr{D}}(N) \cap \mathscr{H}^k(N)$ is the space of Dirichlet fields.

$\mathscr{H}^k_{\mathscr{N}}(N) = \Omega^k_{\mathscr{N}}(N) \cap \mathscr{H}^k(N)$ is the space of Neumann fields.

$\mathscr{E}^k(N) = \{d\alpha | \, \alpha \in \Omega^{k-1}_{\mathscr{D}}(N)\}$ is the space of exact forms.

$C^k(N) = \{\delta\beta | \, \beta \in \Omega^{k+1}_{\mathscr{N}}(N)\}$ is the space of co-exact forms.

$\mathscr{H}^k_{ex}(N) = \{\kappa \in \mathscr{H}^k(N) | \, \kappa = d\varepsilon\}$ is the space of exact harmonic fields.

$\mathscr{H}^k_{co}(N) = \{\kappa \in \mathscr{H}^k(N) | \, \kappa = d\gamma\}$ is the space of co-exact harmonic fields.

An important result - the analogue of the Hodge decomposition for manifolds with boundary.

Theorem (Hodge-Morrey decomposition [39, p.81])**.** *Let N be a compact Riemannian manifold with boundary ∂N. There is the L^2-orthogonal decomposition*

$$\Omega^k(N) = \mathscr{E}^k(N) \oplus C^k(N) \oplus \mathscr{H}^k(N). \tag{A.6}$$

The proof of this theorem is based on Green's formula for manifolds with boundary

$$\langle\langle d\omega, \eta\rangle\rangle = \langle\langle \omega, \delta\eta\rangle\rangle + \int_{\partial N} \mathbf{t}\omega \wedge *\mathbf{n}\eta, \tag{A.7}$$

where $\langle\langle \alpha, \beta\rangle\rangle = \int_N \alpha \wedge *\beta$. From this formula we can see that the space of harmonic fields is not the same as the space of harmonic forms (i.e. solutions of $\Delta\omega = 0$) on a manifold with boundary. The last part in this decomposition can be further decomposed as follows.

Theorem (Friedrichs decomposition [39, p.86]). *Let N be a compact Riemannian manifold with boundary ∂N. There is the L^2-orthogonal decompositions*

$$\mathcal{H}^k(N) = \mathcal{H}^k_{\mathcal{D}}(N) \oplus \mathcal{H}^k_{co}(N), \tag{A.8}$$

$$\mathcal{H}^k(N) = \mathcal{H}^k_{\mathcal{N}}(N) \oplus \mathcal{H}^k_{ex}(N). \tag{A.9}$$

Let us now look at 1-forms. It is clear that we have the orthogonal decomposition

$$\omega = \mathbf{t}\omega \oplus \mathbf{n}\omega,\ \omega \in \Gamma(T^*N|_{\partial N}) \tag{A.10}$$

and in this decomposition $\mathbf{t}\omega$ can be naturally identified with $i^*\omega$.

We want to know when a 1-form on the boundary ∂N extends to a harmonic 1-field on a manifold N. For this we need the following proposition.

Proposition A.2.2 ([39, p.129]). *Let N be a compact manifold with boundary, and $\psi \in \Omega^k(N)\big|_{\partial N}$. There exists a harmonic field $\omega \in \mathcal{H}^k(N)$ obeying the boundary condition $\mathbf{t}\psi = \mathbf{t}\omega$, if and only if*

$$\mathbf{t}d\psi = 0 \text{ and } \int_{\partial N} \mathbf{t}\psi \wedge *\mathbf{n}\lambda_{\mathcal{D}} = 0\ \forall \lambda_{\mathcal{D}} \in \mathcal{H}^{k+1}_{\mathcal{D}}(N). \tag{A.11}$$

In other words the boundary value problem

$$\begin{cases} d\omega = \delta\omega = 0 & \omega \in \Omega^k(N) \\ \mathbf{t}\omega = \mathbf{t}\psi & \psi \in \Omega^k(N)\big|_{\partial N} \end{cases}$$

is solvable if and only if the conditions (A.11) satisfied.

We actually want to solve a slightly different boundary value problem (BVP). Namely, we want to replace the boundary condition $\mathbf{t}\omega = \mathbf{t}\psi$ by $i^*\omega = \bar{\alpha}$, $\bar{\alpha} \in \Omega^1(\partial N)$.

Proposition A.2.3. *The boundary value problem*

$$\begin{cases} d\omega = \delta\omega = 0 & \omega \in \Omega^1(N) \\ i^*\omega = \bar{\alpha} & \bar{\alpha} \in \Omega^1(\partial N) \end{cases} \tag{A.12}$$

is solvable if and only if

$$d\bar{\alpha} = 0 \text{ and } \int_{\partial N} \bar{\alpha} \wedge i^*(*\lambda_{\mathscr{D}}) = 0 \text{ for all } \lambda_{\mathscr{D}} \in \mathscr{H}^2_{\mathscr{D}}(N). \tag{A.13}$$

Proof. First, we choose any continuation $\alpha \in \Omega^1(N)$ of the form $\bar{\alpha} \in \Omega^1(\partial N)$. From the relations A.1, A.10 we have $i^*\alpha = i^*\mathbf{t}\alpha = \bar{\alpha}$. Next want to check the conditions (A.11) and to use Proposition (A.2.2) with $\psi = \alpha$. Let us look at the first condition $\mathbf{t}d\alpha = 0$. By (A.3) and (A.1) we have

$$i^*(\mathbf{t}d\alpha) = d(i^*\mathbf{t}\alpha) = d(i^*\alpha) = d\bar{\alpha} = 0.$$

By (A.2) and (A.1) the second condition can be rewritten as

$$\int_{\partial N} \mathbf{t}\psi \wedge *\mathbf{n}\lambda_{\mathscr{D}} = \int_{\partial N} (i^*\mathbf{t}\psi) \wedge i^*(\mathbf{t} * \lambda_{\mathscr{D}}) = \int_{\partial N} \bar{\alpha} \wedge i^*(*\lambda_{\mathscr{D}}) = 0 \text{ for all } \lambda_{\mathscr{D}} \in \mathscr{H}^2_{\mathscr{D}}(N),$$

since $i^*\mathbf{t}\psi = i^*\mathbf{t}\omega = i^*\omega = \bar{\alpha}$. We see that the conditions do not depend on the chosen continuation α of $\bar{\alpha}$. □

Let us continue by establishing an extent of the uniqueness of the solution.

Proposition A.2.4. *When the solution to the boundary value problem (A.12) exists it is unique up to Dirichlet harmonic fields.*

Proof. Let ω_1 and ω_2 be two solutions to (A.12). Then $\tilde{\omega} = \omega_1 - \omega_2$ is a solution to the boundary value problem

$$\begin{cases} d\tilde{\omega} = \delta\tilde{\omega} = 0 & \tilde{\omega} \in \Omega^1(N) \\ i^*\tilde{\omega} = 0 & (or\ \mathbf{t}\tilde{\omega} = 0) \end{cases}$$

which defines the Dirichlet harmonic fields. □

Proposition A.2.5. *Let ω be a solution to the boundary value problem (A.12). Then cohomologous to ω form $\upsilon = \omega + d\beta$, $d\beta \in \mathscr{H}^1(N)$ is the solution to the boundary value problem*

$$\begin{cases} d\upsilon = \delta\upsilon = 0 & \upsilon \in \Omega^1(N) \\ i^*\upsilon = \bar{\alpha}_1 & \bar{\alpha}_1 \in \Omega^1(\partial N) \end{cases}, \tag{A.14}$$

where $\bar{\alpha}_1$ cohomologous to $\bar{\alpha}$.

Proof. It is clear that υ is a harmonic field. For the boundary condition we have

$$i^*\upsilon = i^*\omega + i^*d\beta = \bar{\alpha} + d(i^*\beta) = \bar{\alpha}_1,$$

where $\bar{\alpha}_1$ is cohomologous to $\bar{\alpha}$ for any $d\beta \in \mathscr{H}^1(N)$. □

Example. If the boundary value problem (A.12) is solvable, then the boundary value problem

$$\begin{cases} d\omega = \delta\omega = 0 & \omega \in \Omega^1(N) \\ i^*\omega = \bar{\alpha}_1 & \bar{\alpha}_1 \in \Omega^1(\partial N) \\ \mathbf{n}\omega = 0 \end{cases}$$

is solvable for some $\bar{\alpha}_1$ cohomologous to $\bar{\alpha}$. It follows from the previous Proposition and the Friedrichs decomposition.

Let us now provide some useful isomorphisms for cohomology.

Theorem A.2.6 ([39, Theorem 2.6.1, Corollary 2.6.2]). *For a compact Riemannian manifold N with boundary ∂N there are isomorphisms*

$$H^k(N,d) \cong \mathcal{H}^k_{\mathcal{N}}(N) \cong H^k_a(N)$$
$$H^k(N,\delta) \cong \mathcal{H}^k_{\mathcal{D}}(N) \cong H^k_r(N),$$

where $H^k_a(N)$ is a cohomology of a complex $\left(\Omega^k_{\mathcal{N}}(N),\delta\right)$ and $H^k_r(N)$ is a relative de Rham cohomology defined by a complex $\left(\Omega^k_{\mathcal{D}}(N),d\right)$.

Our next aim is to connect the obtained results on 1-forms to circle-valued maps. For this we need the following theorem.

Theorem A.2.7 ([9]). *There are non-singular bilinear pairing between the space of relative k-cycles $R_{n-k}(N) = R_k(N)$ and the space of Dirichlet harmonic k-fields on N given by periods map*

$$R_k(N) \times \mathcal{H}^k_{\mathcal{D}}(N) \to \int\limits_{R_k} \omega^k$$

Proof. The proof is a direct consequence of [10, Theorem 3, Corollary 8.1]. □

This, for example, means that we have an integral lattice in the space $\mathscr{H}^k_{\mathscr{D}}(N)$ (k-th relative de Rham cohomology group) which is formed by fields (forms) with integral relative periods. Let us call it integral Dirichlet harmonic k-fields (integral relative de Rham cohomology).

We need the following corollary.

Corollary A.2.8. *By the addition of sufficient Dirichlet harmonic* 1*-field we can get a solution of BVP* (A.12) *with integral relative periods. All the other solutions with integral relative periods are given by the addition of integral Dirichlet harmonic* 1*-fields. We call such solutions the integral harmonic fields.*

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